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CONTENTS, VOL. II. PART II.

EXTRACTS *from Communications made to the* SOCIETY :

I. *On the Distribution of the Colouring Matter, and*

ERRATA.

Page 24 line 24, *for* Egglestone bank *read* Foggerthwaite bank.
30 —— 8, *for* Fopping *read* Topping.
40 —— 20, *for* east *read* south-east.
43 —— 5 from the bottom, *for* porpheroidal *read* porphyroidal.

farther than to investigate the absorbent power of the latter, and the changes which were superinduced by heat on the matter with which they were coloured.† The Brazilian Topazes which I used in these experiments were all crystallized, and did not exhibit any of the phenomena which I have since discovered during the examination of a large quantity of these crystals, which accident put into my possession.

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* *Phil. Trans. Lond.* 1814. p. 202.

† Id. Id. 1819. pp. 19, 125, &c.

I. *On the Distribution of the Colouring Matter in Topaz.*

In order to obtain results that might be considered as general, I ground and polished the flat summits of a great number of Brazilian Topazes, and having exposed them to polarized light, I observed the following structures.

1. The internal part of the crystal was almost always of a different colour from the external part, as will be afterwards more particularly described, but the pink colouring matter was confined to two small rhomboidal prisms placed at the acute angles of the prism of Topaz as shewn in Fig. 1. When the shorter diagonal of the rhomb is in the plane of primitive polarization, these small rhomboidal prisms, which are of a *pale pink* tint by common light, assume a deep and *brilliant pink* colour, and they become of a faint blue approaching to a *lilac* colour when the longer diagonal is in the same plane. In some specimens this complementary tint, in place of being *lilac*, is *yellow* of different characters; but in every specimen which I have examined, the pink, whenever it exists, is the colour which appears when the shorter diagonal is in the plane of primitive polarization.

2. In some crystals the red colouring matter is confined to a triangular prism as in Fig. 2. sometimes occupying the apex of the acute angles of the outer rhomb, and sometimes the same angles of the inner rhomb. In the specimen represented in Fig. 2, about one half of the pink portions becomes lilac before the other half, and before the longer diagonal comes into the plane of primitive polarization; thus indicating a species of hemitropism, or a want of parallelism in the principal sections or neutral axes of the different portions of the thick prism. This effect is shewn in Fig. 9, where no fewer than *four* different colours appear at the same time.

3. In some specimens the colouring matter is deposited on the faces of a rhomb parallel to the natural faces of the prism, as shewn in Fig. 3; and this space sometimes extends outwards to the natural faces as in Fig. 4, and sometimes inwards to the centre, as in Fig. 5.

4. The pink colouring matter frequently occupies only a part of the external rhomboidal space, as in Fig. 6; and at other times it is arranged as in Fig. 7, both the obtuse and the acute angles being free from the pink colour.

5. In some crystals which are very perfect, the pink colouring matter is uniformly distributed as in Fig. 8.

6. Among the mass of Topazes, amounting to some thousands, which I inspected, there were only *two* which possessed a decidedly *green* colouring matter. These crystals are represented in Figs. 10. and 12. In Fig. 10. the outer portion is of an orange yellow colour, and the inner prism is a mixture of green and pale pink. In Fig. 12. the outer portion is a mixture of green and *pale pink*, and the inner portion a light *orange yellow*. When the longer diagonals of both these prisms are in the plane of primitive polarization, the green tint is a maximum as shewn in the figures. For the sake of comparison, I have represented in Fig. 11. the distribution of the colouring matter in some of the finest and most valuable of the Scotch Topazes.

II. *On the Tesselated Structure of the Brazilian Topaz, and the singular Superposition of its external Laminæ.*

In the year 1819 Mr. Herschel informed me that he had observed in some Brazilian Topazes a hemitrope or tesselated structure, "It is a structure he remarked which never occurs "in fine crystals; but when properly examined, inferior spe-"cimens often present the phenomenon represented in the figure "(Fig. 13.), the principal sections of the different parts making

"angles of $20^\circ \pm$. In these the central portion is often of a
"high yellow colour, and manifests the phenomenon of absorp-
"tion of polarized pink and yellow rays alternately, while the
"external border is nearly colourless in all positions, and pre-
"sents no such phenomenon."

The tesselated structure is so common in the Brazilian Topaz, that I am more disposed to regard it as an essential character of that mineral, than as an accidental formation. Out of the great numbers which I have examined, there is not one in an hundred which is free from this hemitropism; and in some very fine crystals, in the examination of which this structure had escaped my notice, I have since detected it by more careful methods of observation. In these cases, however, the crystal is rather to be considered as a compound than as a hemitrope crystal, for the separated portions have their principal sections much nearer to coincidence, than in less perfect specimens.

The hemitropism of the Brazilian Topaz is of a very singular kind. The tesselæ are not turned round one half or any determinate portion of a circle, as this term implies, but the principal sections of different laminæ form different angles with one another, and hence we may distinguish these specimens by the more appropriate name of *polytrope* crystals.

This curious formation, which has not hitherto been observed in doubly refracting structures, will be understood from Fig. 14, where *ABED*, *CBEF* are the two external tesselæ at one of the obtuse angles of the rhomboid. If we suppose that these tesselæ are divided into four laminæ, 1, 2, 3, 4, and that *MN* is the principal section, or one of the neutral axes of the central portion of the crystal contiguous to *DEF*, then the laminæ 1, 1, have their principal section in the direction *aa′* forming a very small angle with *MN*; the laminæ 2, 2 have their principal section in the line *bb′*, and so on to the superficial laminæ 4, 4, which have their principal section in the direction dd' inclined from $10^\circ \pm$ to $22^\circ \pm$ to MN, the inclination differing in different crystals. The lines aa', bb', cc', dd' are also the principal sections of the corresponding laminæ on the side NC. In like manner the principal sections $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ of the laminæ in $BCFE$ are the principal sections of the corresponding laminæ on the other side AN. As the laminæ however are infinite in number, the principal sections have every possible direction between MN and dd'.

III. *On the Optical Structure and Properties of Brazilian Topaz.*

Having found that the Brazilian Topaz exercised a superficial action upon light different from the colourless Topaz of New Holland, I was induced to compare the relative intensities of their polarizing axes. In the blue Topaz of Aberdeenshire, and the colourless Topaz of New Holland, the inclination of the resultant axes of double refraction is about 65°, and the system of coloured rings round each axis deviates from the tints of Newton's scale, the red ends of the rings being outwards*. In the Topazes of Brazil, the inclination of the resultant axes varies in different specimens; I have found it in some crystals 50° 5′, and even so low as 43°; and, what is very remakable, the one resultant axis is often more inclined than the other to the natural surfaces of the laminæ, an effect which no doubt arises from the peculiarities of crystallization already described. In one specimen where the axes formed an angle of 50° 5′, the one axis was inclined only 22° 37′ to the axis of the prism, while

* In the *Phil. Trans.* 1814, p. 204. and Plate VII. Fig. 1. I have described the order of the tints, from which it will be seen, that the rings are *red* at one end and *blue* at the other, and that the colours do not originate at the centre of the rings.

the other had an inclination of 27° 28′. The tints of the Brazilian Topaz deviate more than those of the other crystals from the colours of Newton's scale, and are produced by a polarizing force of inferior intensity.

The phenomena of the coloured rings in Scotch Topaz which I have described in the *Phil. Trans.* for 1814, are neither seen with the same distinctness, nor to the same extent, in the Topaz of Brazil. This circumstance arises from the difference in the inclination of their resultant axes. In the Scotch Topaz the angle formed by their axes is such, that when light incident along the one axis is reflected in the direction of the other, the angle of reflection is almost the same as that of maximum polarization for Topaz, and hence the rings appear with peculiar brilliancy, and exhibit themselves under the new modifications which I have distinguished by the names of the *third* and *fourth* Set* in the Paper already quoted. In the Brazilian Topaz, however, the inclination of the resultant axes deviates considerably from twice the polarizing angle, and consequently the preceding phenomena are very indistinctly displayed.

The Brazilian Topazes are in general phosphorescent when placed upon a heated iron, although I have found several, especially among the finer crystals, that do not possess this property in the slightest degree. The tesselated crystals display their phosphorescence in a very singular manner. Sometimes it is of a rich *orange red* colour, and in many cases the external border is phosphorescent, while the interior nucleus discharges no light at all. This phosphorescent light is, in general, most brilliant in the outer laminæ, though I have seen in some crystals the greatest intensity of light at the boundary of the

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* The *fourth* set of rings which has a peculiar character, is a combination of the *first* and *second* set, or of the *direct* and *complementary* systems.

nucleus and the external tesselæ. In one specimen, a faint phosphorescent glow appeared and vanished at intervals in the nucleus, while the light shone in the outer border with a bright and permanent lustre.

IV. *On Substances found in the Brazilian Topaz.*

In a very great number of Brazilian Topazes there is a white pulverulent substance, which must have been formed at the same time with the mineral, as the most powerful Microscopes cannot detect any aperture or crevice through which it could have been admitted. Upon breaking up the specimens where it occurred, and examining the surfaces of the laminæ between which it had lain, I found that they had been acted upon by the substances, as they exhibited that superficial disintegration which is produced by the action of a solvent, and which is identically the same with what is found on many of the summit planes of the crystal, as shewn in Fig. 15. This circumstance proves in the clearest manner the contemporaneous formation of the white powder.

M. Berzelius, to whom I had transmitted some specimens of this substance had the goodness to analyse it for me, and found it to be a sort of *Marle* consisting of *silex*, *alumina*, *lime*, and *water*. "This substance, he remarks, if it were crystallized, would belong to Cronstedt's family of the Zeolites." From the frequency with which this matter occurs in the Brazilian Topaz, and the imperfect character of the crystallization which accompanies it, I cannot help thinking, that it is nothing more than the uncrystallized ingredients, and that *lime* is one of the constituents of the mineral*.

* When the crystal is placed upon a hot iron, the white portions are more phosphorescent than the other parts.

In many specimens of the Brazilian Topazes, I have observed another substance of a very singular kind. It is of a *brilliant red* colour, and in general perfectly transparent. Sometimes it appears in thin plates between the laminæ, and sometimes in long stripes parallel with the axis of the prism. By holding the neutral axis of the crystal in the plane of primitive polarization, and examining these red portions with polarizing Microscopes, I have found parts of them crystallized so as to produce four sectors of light round a black cross. When the crystal is broken, the surfaces of these red films have a high metallic lustre like Realgar or Cinnabar; but I have not been able to collect enough of the substance to determine what it is.

V. *On the probable Difference in the Chemical composition of the Brazilian and other Topazes.*

When I had ascertained the very marked difference between the optical properties of the Brazilian Topazes, and those of Saxony, New Holland and Scotland, I could not for a moment doubt that a difference would be found in their chemical composition. I accordingly sent specimens, that I had examined, to M. Berzelius, with the request that he would favour me with an analysis of them. This distinguished chemist, however, had previously analysed other specimens of the same minerals, and he informed me, "that his analyses gave exactly the same re-
" sults for the New Holland and Brazilian Topazes, with the
" exception of a small quantity of hydrate of iron, with which
" the latter was coloured, and which, when decomposed by heat,
" gave the fine pink colour to burned Topaz in consequence of
" the oxide of iron being set at liberty."

High as M. Berzelius's authority undoubtedly is, I cannot avoid placing the most unbounded confidence in the general principle, that every difference in optical structure is accompanied with a difference in chemical composition; and I am therefore disposed to ascribe the present apparent exception to the imperfection of chemical analysis.

This opinion, indeed, may be placed upon a firmer basis by a reference to some experiments of that excellent Chemist, the late Rev. William Gregor, which have recently come to my knowledge. In a letter to the Rev. John Rogers, dated Nov. 21, 1811, he states that some pieces of Topaz from St. Michael's Mount consisted of silica and alumina in the proportion of 3.5 to 3.1, and a small proportion of *lime*. "The proportion of "silica and alumina," he adds, "does not agree with that in the "Saxon Topaz; but still I see a great difference between the "Saxon and Brazilian Topaz, as to the relative proportion of "those ingredients. In neither of them do I see any lime re-"cognized; but I must say, notwithstanding the analysis of "Klaproth, *that I extracted a portion of* LIME from the *Bra-"zilian Topaz,* by means of a *simple acid.*"

In another letter to the same gentleman, dated April 24, 1812, he says, that "subsequent experiments confirmed him in "the opinion, that *Brazilian Topaz contains a small proportion "of Potash;*" and Dr. Paris, from whose biographical Memoir of Mr. Gregor these particulars are taken, states that the experiments by which he established this fact were the last he performed; and that he himself was enabled to bear testimony to the presence of crystallized *alum,* which he saw Mr. Gregor produce by the action of sulphuric acid upon the pulverized mineral. In order to avoid every possible source of error, the mineral was reduced to powder in a steel mortar, and the sulphuric acid employed was carefully tested, and found to be perfectly free from any impurity.

Edinburgh,
March 4, 1822.

Fig 1.

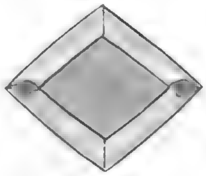


Fig 2.

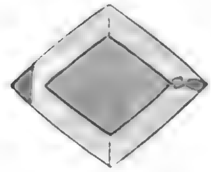


Fig 3.

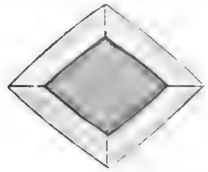
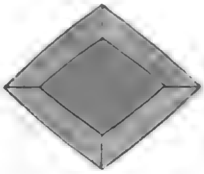


Fig. 4.



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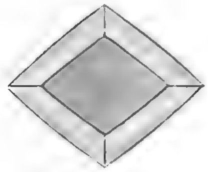


Fig 6.

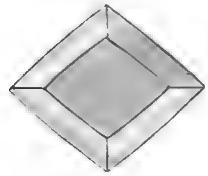


Fig 7.

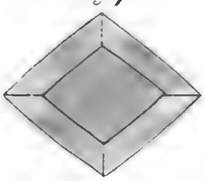


Fig 8.

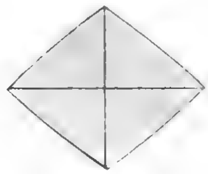


Fig 9.

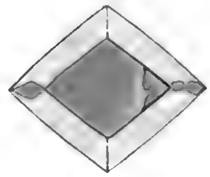


Fig. 10.

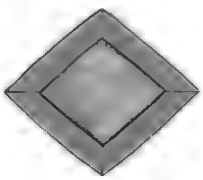


Fig. 11.

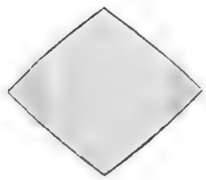


Fig 12.

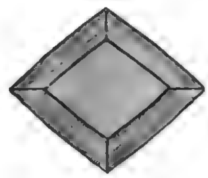


Fig. 13.

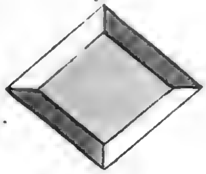


Fig 14.

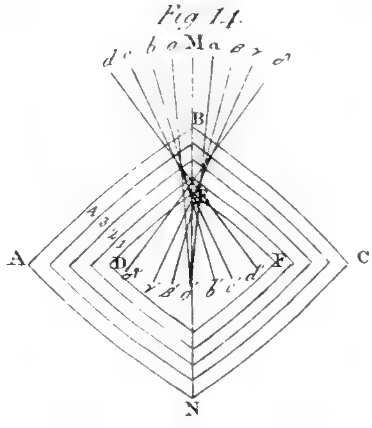
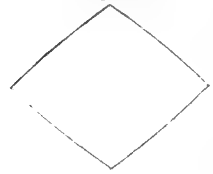


Fig 15.



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II. *On the Rotatory Motion of Bodies.*

By WILLIAM WHEWELL, M.A. F.R.S.

FELLOW OF TRINITY COLLEGE.

[Read *May* 6, 1822.]

THE mechanical Problem of the rotation of a body of any form under given circumstances, is one of some difficulty. It is remarkable, not only for the errors into which Mathematicians of great eminence have been led in treating it, but for being almost the only instance where there has been a permanent difference of opinion among writers, with respect to the results of our elementary mechanical laws in a particular case. In fact, it seems to present the most striking impeachment of the certainty of mathematical investigations which can be found, since the opposite conclusions are not obtained by an abstruse and complicated process, but arise immediately from different methods of applying the same fundamental principles. It may, therefore, be of service to solve the question in a manner which reduces it to a class of problems about which no doubt was ever entertained, and such is the object of the present paper.

It is easily seen that the motion of a body in any manner whatever about a centre, leads to considerations somewhat complicated. At any point of time the body may be conceived to be moving about some axis or other; but the position of this *instantaneous axis*, as it is called, both in the body and in fixed space, may be perpetually varying, as well as the angular velocity about it; and the forces exerted by each particle will vary with these changes. The first solution of the problem, taken thus generally, is due to Dalembert, who published it in 1749 in his *Researches on Precession and Nutation*. Euler, in the *Berlin Memoirs* shortly after, put the solution in a more symmetrical and simple form, acknowledging at the same time Dalembert's prior claim to it. The equations however to which the conditions of rotation are reduced seem to have first appeared in the form in which they are now generally presented, in the *Berlin Memoirs* for 1758. The same subject was pursued in other Memoirs, and more extensively in Euler's "*Theoria Motus Corporum Solidorum et Rigidorum*," which appeared in 1767. Lagrange, in the *Memoirs of the Berlin Society* for 1773, considered the subject on principles more general than his predecessors. The results, however, of all these different authors, as well as of Frisi and others who afterwards treated the question, agreed, though obtained by a variety of methods. In the *Philosophical Transactions* for 1777, Mr. Landen gave "A new Theory of the Rotatory Motion of Bodies, affected by forces disturbing such motion." In this Memoir, he expresses himself dissatisfied with the explanations which he had seen on the subject; but it does not appear that he was at that time acquainted with the researches of Dalembert and Euler, and he has not examined particularly the cases in which his conclusions are at variance with theirs. He afterwards read the solutions of his precursors in this path of enquiry, and convinced himself that they were false; and in the *Transactions of the Royal Society* for 1785 he stated this, and gave his own method applied to the case of bodies of any figure. A leading difference in the results was this: that while according to Euler, Dalembert, &c. in an irregular body not acted on by any extraneous forces, the angular velocity about the instantaneous axis is variable, as well as the position of the axis itself; Mr. Landen found, that the angular velocity is constant, and that the instantaneous axis in changing its place in the body, can assume only a series of positions, for all of which the rotatory inertia is the same; so that the trace of the instantaneous pole upon a concentric spherical surface would generally be an oval, and its projection an ellipse or hyperbola. In the second part of his "*Mathematical Memoirs,*" published in 1789, he resumed the subject, and having occasion to establish propositions analogous to those in Euler's "*Theoria,*" the most regular treatise on the subject which had then appeared, he obtains theorems at variance with those of that author, and expresses rather strongly his astonishment, that Mathematicians so celebrated as his opponents, could fall into mistakes so gross. His death, which took place shortly afterwards, closed the controversy; but it is said that his opinions remained unaltered. Mr. Wildbore, in the *Philosophical Transactions* for 1790, re-considered the subject in a new point of view, and declared against Landen; and since that time it does not appear that any person has revived his ideas, and the foreign elementary treatises all proceed according to the method which Mr. Landen declared erroneous. Perhaps, therefore, it may seem that the question is already sufficiently settled, and that there is no farther necessity to prove the truth of the conclusions of Euler, Dalembert, Lagrange, &c. As, however, the subject is both important and curious, and as Mr. Landen's mathematical talents are deservedly highly estimated, I may be excused for making an attempt to place in a clearer point of view the falsity of his results. The Memoir of

Mr. Wildbore will not, I think, be found so simple and easily intelligible as might be wished; and the following method appears to me to possess these advantages in a great degree.

It would be taking up too much time to trace Mr. Landen's mistakes from their first principles; the general foundation of them may be stated to be the assumption which he makes, that if one force will produce the same effect as another in affecting the motion *round the axis*, it may be substituted for that other; not observing that forces equivalent in that respect will not necessarily produce the same effect in other respects.

The error of principle with which he charges other writers is "the resolving a force productive of rotatory motion into three forces, and considering each of these forces as acting separately on the body impelled." (*Math. Mem.* xiv, p. 79). Now there can be no doubt that with respect to a *single point* in motion, we may resolve the forces which act upon it in the direction of three co-ordinates, and consider the motion of the body in the direction of each co-ordinate as affected only by the force in that direction; and we shall find that this reasoning will lead us to conclusions on the subject of rotation.

The case which I shall take is where the system is acted upon by no extraneous forces whatever. Let three lines at right angles to each other pass through a point O, and at the extremities of the lines, suppose three material points m, m', m''. Let these points be joined by lines mm', mm'', $m'm''$. If now the six lines Om, Om', Om'', mm', mm'', $m'm''$, be supposed to be rigid rods without weight, the system will be perfectly unalterable in form, and the whole mass will be collected at the points m, m', m''. Let this system be supposed to move any how about the point O, being left to itself, and not acted on by any force; then the only forces which act on the points m, m', m'', are the tensions of the six rods just mentioned; and by resolving these, we may obtain the motion of the system, by applying to m, m', m'', the formulæ for the motion of a point.

Let the three lines Om, Om', Om'', be equal; then mm' will make angles of 45° with Om and Om'; and the tension of mm', which acts as an equal moving force on m and m', being resolved, it will be seen that the force on m parallel to Om' is equal to the force on m' parallel to Om. In the same manner, we may obtain two other equations, by considering the tension of mm'' and of $m'm''$. And these three equations, when put in terms of the quantities which are generally used in expressing rotatory motion, give us the three equations of motion.

Let m, m', m'', be referred to Ox, Oy, Oz, three *fixed* coordinates.

$$\begin{array}{llll} \text{The co-ordinates of } m & \text{are} & x, & y, & z, \\ \qquad\qquad\qquad m' & & x', & y', & z', \\ \qquad\qquad\qquad m'' & & x'', & y'', & z''; \end{array}$$

and it is easily seen, that these quantities are also

$$\begin{array}{lll} = \cos.\, mOx, & \cos.\, mOy, & \cos.\, mOz, \\ \cos.\, m'Ox, & \cos.\, m'Oy, & \cos.\, m'Oz, \\ \cos.\, m''Ox, & \cos.\, m''Oy, & \cos.\, m''Oz; \end{array}$$

$$(\text{because } Om = Om' = Om'' = 1).$$

We shall then have these equations (Poisson, *Tr. de Mec.* No. 361*.);

$$\therefore \left.\begin{array}{l} 0 = xx' + yy' + zz' \\ 0 = xx'' + yy'' + zz'' \\ 0 = x'x'' + y'y'' + z'z'' \end{array}\right\} (a). \qquad \text{Also,} \quad \left.\begin{array}{l} x^2 + y^2 + z^2 = 1 \\ x'^2 + y'^2 + z'^2 = 1 \\ x''^2 + y''^2 + z''^2 = 1 \end{array}\right\} (b).$$

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* And in the same manner we might obtain these equations,

$$\left.\begin{array}{l} xy + x'y' + x''y'' = 0 \\ xz + x'z' + x''z'' = 0 \\ yz + y'z' + y''z'' = 0 \end{array}\right\} (a'). \qquad \left.\begin{array}{l} x^2 + x'^2 + x''^2 = 1 \\ y^2 + y'^2 + y''^2 = 1 \\ z^2 + z'^2 + z''^2 = 1 \end{array}\right\} (b').$$

The only forces which act upon the points, are the ***tensions***, or reactions of the six rods Om, Om', Om'', mm', mm'', $m'm''$. And if these be resolved at each point m, m', m'', into three rectangular components X, Y, Z, X', Y', Z', X'', Y'', Z'', we shall have

$$\left.\begin{array}{lll} \dfrac{d^2x}{dt^2} = \dfrac{X}{m}, & \dfrac{d^2y}{dt^2} = \dfrac{Y}{m}, & \dfrac{d^2z}{dt^2} = \dfrac{Z}{m} \\ \dfrac{d^2x'}{dt^2} = \dfrac{X'}{m'}, & \dfrac{d^2y'}{dt^2} = \dfrac{Y'}{m'}, & \dfrac{d^2z'}{dt^2} = \dfrac{Z'}{m'} \\ \dfrac{d^2x''}{dt^2} = \dfrac{X''}{m''}, & \dfrac{d^2y''}{dt^2} = \dfrac{Y''}{m''}, & \dfrac{d^2z''}{dt^2} = \dfrac{Z''}{m''} \end{array}\right\} (c).$$

Now let the force which acts on m, and which is composed of X, Y, Z, be resolved parallel to Om, Om', Om''. And since x is the cosine of the angle which Om makes with X, we shall have Xx for the part of the force arising from X, acting on m in the direction Om. Similarly, Yy and Zz, are the parts arising from Y and Z.

Hence, the whole force on m,

in direction Om is $Xx + Yy + Zz$,

. . . . Om' is $Xx' + Yy' + Zz'$,

. . . . Om'' is $Xx'' + Yy'' + Zz''$.

Similarly, the force on m'

in direction Om is $X'x + Y'y + Z'z$,

. . . . Om' is $X'x' + Y'y' + Z'z'$,

. . . . Om'' is $X'x'' + Y'y'' + Z'z''$.

And the force on m''

in direction Om is $X''x + Y''y + Z''z$,

. . . . Om' is $X''x' + Y''y' + Z''z'$,

. . . . Om'' is $X''x'' + Y''y'' + Z''z''$.

But the force on m parallel to Om' arises entirely from the tension of mm', and is $= \frac{\text{tension of } mm'}{\sqrt{2}}$. Similarly, the force on m' parallel to Om is $= \frac{\text{tension of } mm'}{\sqrt{2}}$. Hence, we have

$$Xx' + Yy' + Zz' = X'x + Y'y + Z'z.$$

$$\text{Similarly, } X'x'' + Y'y'' + Z'z'' = X''x' + Y''y' + Z''z',$$

$$\text{and } X''x + Y''y + Z''z = Xx'' + Yy'' + Zz''.$$

Or, putting for X, Y, Z, X', &c. their values from (c),

$$\left.\begin{array}{l} m\,(x'd^2x + y'd^2y + z'd^2z) = m'\,(xd^2x' + yd^2y + zd^2z') \\ m'\,(x''d^2x' + y''d^2y' + z''d^2z') = m''\,(x'd^2x'' + y'd^2y'' + z'd^2z'') \\ m''\,(xd^2x'' + yd^2y'' + zd^2z'') = m\,(x''d^2x + y''d^2y + z''d^2z) \end{array}\right\} (d),$$

which are three equations of motion.

Differentiating the first of equations (a), we have

$$xdx' + ydy' + zdz' + x'dx + y'dy + z'dz = 0.$$

Hence, if we make*

$$\left.\begin{array}{lll} & x'dx + y'dy + z'dz = rdt, \text{ we have} & xdx' + ydy' + zdz' = -rdt \\ \text{Sim}^{\text{ly}}.\text{ if} & x''dx' + y''dy' + z''dz' = pdt \;\ldots\ldots\ldots & x'dx'' + y'dy'' + z'dz'' = -pdt \\ \text{and if} & xdx'' + ydy'' + zdz'' = qdt \;\ldots\ldots\ldots & x''dx + y''dy + z''dz = -qdt \end{array}\right\} (e).$$

Now, take the three equations,

$$\left.\begin{array}{l} xdx + ydy + zdz = 0 \\ x'dx + y'dy + z'dz = rdt \\ x''dx + y''dy + z''dz = -qdt \end{array}\right\} (f).$$

Multiply by x, x', x'', and add; we then have, observing equations (a') and (b'),

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* This notation is the same as that used by Lagrange, Poisson, &c. with very little alteration.

$$dx = (x'r \quad - x''q)\,dt.$$

Similarly, $dy = (y'r \quad - y''q)\,dt,$

and $dz = (z'r \quad - z''q)\,dt.$

Hence, $d^2x = (x'dr - x''dq + rdx' - qdx'')\,dt,$

$$d^2y = (y'dr - y''dq + rdy' - qdy'')\,dt,$$

$$d^2z = (z'dr - z''dq + rdz' - qdz'')\,dt;$$

And multiplying these equations by x', y', z', and adding, observing the simplifications which result from (a), (b), and from the equations

$$\left.\begin{array}{l} xdx \;\; + ydy \;\; + zdz \;\; = 0 \\ x'dx' \;\; + y'dy' \;\; + z'dz' \;\; = 0 \\ x''dx'' + y''dy'' + z''dz'' = 0 \end{array}\right\} \text{(g)},$$

which arise from differentiating (b); we shall find

$$\therefore\; x'd^2x + y'd^2y + z'd^2z = \{dr - q(x'dx'' + y'dy'' + z'dz'')\,dt\} = (dr + pq\,dt)\,dt.$$

Similarly, we should find

$$x''d^2x' + y''d^2y' + z''d^2z' = (dp + qr\,dt)\,dt,$$

$$x\,d^2x'' + y\,d^2y'' + z\,d^2z'' = (dq + pr\,dt)\,dt.$$

Also,

$$x\,d^2x' \; + y\,d^2y' \; + z\,d^2z' \; = (-dr + pq\,dt)\,dt,$$

$$x'd^2x'' + y'd^2y'' + z'd^2z'' = (-dp + qr\,dt)\,dt,$$

$$x''d^2x \; + y''d^2y \; + z''d^2z \; = (-dq + pr\,dt)\,dt.$$

Hence, equations (d) become

$$m\,(dr + pq\,dt) = m'\,(-dr + pq\,dt),$$

$$m\,(dp + qr\,dt) = m''\,(-dp + qr\,dt);$$

$$m\,(dq + pr\,dt) = m''\,(-dq + pr\,dt),$$

$$\therefore\; \left.\begin{array}{l} (m + m')\,dr + (m - m')\,pqdt = 0 \\ (m' + m'')\,dp + (m' - m'')\,qrdt = 0 \\ (m'' + m)\,dq + (m'' - m)\,prdt = 0 \end{array}\right\} (h).$$

Let $m + m' = C$, $m'' + m = B$, $m' + m'' = A$*;

$$\therefore\ m - m' = B - A,\ m - m'' = C - B,\ m'' - m = A - C;$$

$$\therefore \left.\begin{aligned} C\,dr + (B - A)\,pq\,dt &= 0 \\ A\,dp + (C - B)\,qr\,dt &= 0 \\ B\,dq + (A - C)\,pr\,dt &= 0 \end{aligned}\right\}\ (k),$$

which are the equations of Euler†. And hence it appears, that in this case the equations coincide with those which are always given for the motion of a solid body by foreign Mathematicians, and differ from the results which Mr. Landen obtained by his method.

The equations above obtained agree with the general equations for the motion of any solid body of which A, B, C, are the moments of inertia with respect to the three principal axes. Hence it appears, that whatever the body be which revolves about a given centre, we may always take a system consisting of three material points, such, that their motion shall be exactly similar to that of the body. Mr. Landen had observed, that it is always possible to substitute for a solid body a system of *eight* material points placed at the angles of a parallelepiped, whose centre is the centre of motion; but I am not aware of its having been noticed, that these may be replaced by three points. If we wished, in our system of points, to have their centre of gravity coincident with the centre of motion, we may conceive each of the lines mO, $m'O$, $m'O''$ to be produced, and an equal distance and an equal point to be taken on the oppo-

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* It is manifest that $m+m'$ or C is the moment of the system round the axis Om''; and similarly, B and A are the moments round Om and Om.

† Euler, *Theoria Motus Corp. Rig.* Prop. 90. Poisson, *Dynamique,* Art. 383.

site side. The three new points may of course be made to move in exactly the same manner as m, m', m'', and these three pairs of points may represent any solid body whatever.

Having thus obtained the equations of motion, the deductions from them follow in a manner which has been sufficiently explained by many authors who have treated on the subject, and on which, therefore, it will not be necessary here to dilate.

W. WHEWELL.

Trinity College,
May 1, 1822.

III. *On the Phenomena connected with some Trap Dykes in Yorkshire and Durham.*

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[Read *May* 20, 1822.]

Introduction.

The various phenomena presented by trap rocks have long engaged the attention of Geologists. Different ages have been assigned to them, founded on their union with older or newer strata, and distinctive characters have been pointed out by which it has been attempted to separate the several formations from each other. As observations have become more widely extended, many of the conclusions founded on such characters have proved to be fallacious; and it is now generally admitted, that the mineralogical composition of any system of trap rocks gives us little information respecting its antiquity or probable associations. When strata rest conformably upon each other, in such a way as to indicate a continued succession of depositions, we can immediately determine, at least, their *relative* antiquity, and may often adopt some natural or artificial arrangement which will greatly facilitate their description. But formations, which appear as dykes and overlying masses, afford no such facilities for correct classification; and the only general conclusion which we can arrive at respecting them is, that they are newer than the beds into which they have intruded. It is on this account that different observers have formed completely different views respecting the classification of certain formations of trap; each, in ambiguous cases, having adopted that opinion which happened to fall in with his favorite theory.—In determining the origin of any one of these formations, it seems essential to inquire, (1) In what manner it is associated with other rocks. (2) What minerals enter into its composition. (3) What effects are produced by its presence. Satisfactory answers to these questions have been obtained from so many quarters, that the discussions in which they have originated will perhaps soon terminate. It is my intention in this communication to bring together some facts, connected with the subject, which fell under my observation during last summer.

Trap dykes in the coal-fields.

Dykes and overlying masses of trap are of such ordinary occurrence in many of our coal-fields, that they have sometimes been regarded as true members of the great coal formation. Should it, however, appear, that they have not originated in the same causes which formed those innumerable layers of sandstone, shale, ironstone, &c. which enter into the composition of the coal strata; but that they have been subsequently driven in among these beds by the irregular action of powerful disturbing forces; we shall then be compelled to regard them, not as the subordinate members, but as the intrusive associates of the great coal formation. In confirmation of this opinion it may be stated; (1) That in many extensive coal-fields there are no traces of any beds or dykes of trap. (2) That in other places, such beds or dykes pass beyond the bounds of the coal-fields, and traverse indifferently all the newer strata which cross their line of direction. The facts presented by the north coast of Ireland afford several illustrations of the truth of this assertion.

Mr. Winch, in the fourth volume of the *Geological Transactions,* has given many interesting details respecting the dykes* which intersect the great coal basin of Northumberland and Durham. They are in some instances filled with clay and rounded pebbles or shattered fragments of sandstone, mixed with other materials derived from the neighbouring rocks, and their whole appearance plainly indicates the violent nature of the forces by which the solid strata have been cleft asunder. In other instances, the fissures are filled with a variety of basalt, which rises like a great partition wall through all the beds of the formation. (*Geol. Trans.* vol. IV. p. 21—30.) It is the opinion of Mr. Winch that these basaltic dykes never pass up into the magnesian limestone which reposes immediately on the coal strata. Thus, for example, the cliff of Tynemouth castle is intersected by a basaltic dyke which does not penetrate the capping of magnesian limestone.

Every one who is acquainted with the details of English Geology must have remarked, that our newer strata, down to the magnesian limestone inclusive, are generally unconformable to all the older rocks. Thus in numberless instances, more especially in the West of England, we find some of the newer strata filling up the inequalities, or resting on the inclined edges, of the coal measures. In all such cases, the fractures and contortions of the lower formation must have taken place prior to the deposition of the superincumbent horizontal beds. Now if it appear, that masses of trap are not only the common associates of such fractures and dislocations, but sometimes the very instruments by which they have been produced; it follows, almost of necessity, that the dykes we have been describing will not generally be found among the horizontal beds which repose upon the disturbed strata. Such a rule as this

* In the North of England the term *dyke* is not confined to the description of those fissures which have been filled with trap, but is extended to all the great faults and dislocations which intersect the strata in a nearly vertical direction. A want of attention to this extended use of the word has given rise to occasional mis-statements and false inferences.

may, however, admit of many exceptions. For no reason can be given *à priori*, why the same forces, which produced the great fissures in our coal formations, should not again come into action in successive epochs in the natural history of the earth. Accordingly, it is found that basaltic dykes are not confined to any particular set of strata, but may occasionally appear among the newest secondary rocks. The facts exhibited by the north coast of Ireland have been already alluded to. The great dyke which starting from Cockfield Fell, in the county of Durham, crosses the plain of Cleveland, and terminates in the eastern moors of Yorkshire, leads us to a similar conclusion.

Cockfield Fell and Cleveland dykes.

This dyke, which preserves such an extraordinary continuity, forms a striking feature in all the geological maps of the district. Some good general descriptions have already been given of it*. My principal object in this paper will be, to place before the Society, in a connected point of view, those facts which appear to bear on the question of its origin. I shall afterwards notice some phenomena which are exhibited in High Teesdale, and seem to throw light on the same question.

Dykes near Egglestone in Upper Teesdale

A mass of trap occupies the lower part of the left bank of the river Tees exactly opposite to the entrance of the Lune. It may be traced without difficulty for three or four hundred feet, close to the edge of the water; and it at length disappears under Egglestone bank; where it rests upon, or abuts against a bed of slate clay. The prolongation of the trap to the other side of the Tees is rendered highly probable by the appearance of a bed of similar character in the left bank of the Lune immediately under Lonton Chapel. But the accumulation of diluvium prevents this connexion from being established by direct evidence. The imperfect denudation on the left bank of the Tees did not

* See the Geological Survey of the Yorkshire Coast by Young and Bird, p. 171.
† *Geological Transactions*, vol. IV. p. 76.

allow me to ascertain the exact relation which the trap on that side of the water has to the contiguous strata. Above Egglestone bank another mass of trap, to all appearance immediately connected with that which has been described, crosses the road about a mile to the north-west of the village. It there assumes the unequivocal characters of a dyke, ranges (as nearly as I could discover from very imperfect data) E. by N. and a few hundred yards above the road crosses the western branch of the rivulet which runs past Egglestone. A quarter of a mile farther up the same branch of the rivulet, a second dyke crosses its bed and seems to range about S.E. by S. From what has been stated it appears probable that these two dykes unite, or intersect each other. Their concourse will probably be found on the moor above the new smelting-house. The former, where it is seen above Egglestone, is about forty feet wide, and cuts through a bed of coarse grit, provincially called firestone. The latter is about sixty feet wide, and is associated with gritstone and a band of indurated shale which has been much quarried for whetstones.

It would certainly be very interesting to trace these dykes as far as possible through the eastern moors, as there can be little doubt of their connexion with some of those masses of trap which traverse the great coal-field. My own observations were much too limited to complete this task. I however found on Woolly Hills, in the Woodland Fells, several quarries opened in a dyke which, from its position as well as in its structure, seemed to form a connecting link between the trap of High Teesdale and some of the dykes which traverse the country near Cockfield Fell*.

* It is stated by Mr. Winch (*Geological Transactions*, vol. IV. p. 76.) That " at Egglestone, three miles below Middleton, a very strong vein of basalt may be seen crossing the Tees in a diagonal direction." I suspect that he here alludes to the mass of basalt above-mentioned, which appears on the left bank of the Tees opposite to the entrance of the Lune,

as

Cockfield Fell dyke.

Proceeding some miles farther to the S. E. we come to the north-western termination of Cockfield Fell dyke, which is seen in a quarry by the side of the brook which runs past Gaundlass Mill. In that single locality it assumes a compound form, being made up of three distinct and nearly vertical masses of trap alternating with a variety of indurated slate-clay. The following is a transverse horizontal section of the whole dyke. (1) On the south-west side, common coal shale, which, as it approaches the dyke, becomes much indurated and has a vertical cleavage. In this state it is provincially termed *pencil.* (2) Trap one yard. (3) *Pencil* about four or five yards, but of variable thickness and much shattered. (4) Trap two yards. (5) *Pencil* half a yard. (6) Trap about seven yards. (7) Coal shale resembling No. (1). These entangled masses of coal shale are probably not prolonged far beyond the quarry, as they are seen in no other section.

The dyke afterwards ranges through the coal works which are opened in Cockfield Fell about half a mile to the north of the village; and its farther progress in a direction about E. S. E. is marked in Blackburn quarry and Crag-wood. Near the former place it is intersected by a cross course, and heaved several yards out of the line of its direction. To the S. E. of Crag-wood, it would perhaps be impossible to trace it at the surface; but the vein of trap which runs along the high ridge of coal strata between Bolam and Houghton-le-side, agrees so well in character and direction with the masses above-mentioned, that it has generally been assumed as the prolongation of them.

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as I in vain endeavoured to discover the traces of a dyke farther down the river. If this conjecture be right, it will be necessary to remove the dyke (which in the map accompanying Mr. Winch's Memoir is made to cross the Tees below Egglestone) to a place considerably to the N. W. of its present position. When so represented, it will be seen, by an inspection of the map, that the basalt in Teesdale and the neighbourhood of Cockfield Fell are much more nearly in a straight line than they have been represented.

In the quarries which they are now excavating near Bolam, Bolam quarry. the vertical dyke is unusually contracted in its dimensions; but on reaching the surface, it undergoes a great lateral extension, especially on the south-west side, so that the works are conducted in a perpendicular face of columnar trap more than two hundred feet wide. The changes produced by this overlying columnar mass are highly instructive, and will be described in their proper place*. The old excavations, in the direction of Houghton-le-side, shew that the trap is there confined to a fissure nearly forty feet wide, which, with a slight undulation in its direction, bears to a point about S.E. by E.

There is another locality, the mention of which must not be Sandstone on the trap. omitted, though I think it probable that it is not in the line of the great dyke. In this opinion I may, however, have been misled by the maps of the district, in which many of the places are laid down entirely out of their true bearings. At Wackerfield-lane-end, half a mile W.N.W. of Hilton, a mass of trap appears to range east and west, and may therefore join the leading dyke which intersects the country still farther to the east. The excavations in that place would not deserve any particular attention, were it not for the important fact, that at their western termination horizontal beds of sandstone are seen to rest immediately upon the upper surface of the dyke. I have been informed that masses of trap occur on the north-east side of the quarries of Bolam; but I had no opportunity of examining them with a view of ascertaining their probable connexion with the principal dyke.

From all these facts we may infer—(1) That from Gaundlass Mill to Houghton-le-side, a distance of about ten miles, the dyke of trap is uninterrupted—(2) That it may be connected with other dykes, which appear still farther to the north-west nearly in the

* See Plate II. Fig. 4.

same line of direction, and through them with the dykes in Upper Teesdale—(3) That it probably gives out some lateral branches connecting it with other masses of trap in the same district. It may farther be observed, that all this portion of the dyke, however modified by local circumstances, dips towards a point on the north-eastern side of its general line of direction, so as to make with the horizon an angle perhaps in no instance less than eighty degrees.

The high ridge of coal strata, extending from Bolam to Houghton-le-side, forms a kind of abutment which encroaches considerably on the line of the magnesian limestone. The present collocation of the two formations might lead to a conjecture that a great fault, ranging along the line of demarcation, had thrown the magnesian limestone down below its natural level. But the supposition is not necessary; for the appearance of the limestone below the level of the ridge may be only an indication of its unconformable position.

Dyke in Lower Teesdale.

In the low region of the magnesian limestone we lose all traces of the basalt from Houghton-le-side to Coatham Stob. From the last mentioned place it may be traced through the quarries of Preston across the Tees; and very large excavations have been made in a corresponding quarry at Barwick on the right bank of the river. The mineralogical character of this dyke, its direction, and its dip, agree so well with the one which ranges through Cockfield Fell; that no one has, I believe, denied the probability of their being continuous*. The great distance between Houghton-le-side and Coatham Stob in which no trap has been discovered; and still more the fact, that the basaltic veins in the great coal-field do not generally pass up

* Should any one maintain that the dykes of Cockfield Fell and the plain of Cleveland have a distinct origin; he may, perhaps, draw an argument in favour of his own opinion, from the great thickness of the vein of trap in the quarries of Preston, Barwick, Langbargh, &c. In this one respect there is undoubtedly a considerable difference between them.

into the magnesian limestone; have led some to imagine, that the prolongation of the dyke of Cockfield Fell is for several miles concealed beneath the beds of that formation. These basaltic veins, which do not penetrate the magnesian limestone, prove one of two things. Either that they took their present form before the deposition of the limestone; or that they were injected from below, but not with sufficient energy to break through the superincumbent limestone. Neither of these suppositions can apply to a great dyke intersecting an enormous mass of secondary strata which are newer than the magnesian limestone, and probably rest upon it. If therefore we admit the identity of the Cockfield Fell and Cleveland dykes; we must suppose that in the whole interval, between Houghton-le-side and Coatham Stob, it is concealed by a thick covering of diluvium: an opinion which no one will have much difficulty in admitting who has observed the enormous accumulation of transported materials in all the neighbouring district*.

Range of the dyke through the Eastern Moors.

At Preston the trap emerges from beneath nearly fifty feet of diluvian brick earth; and would probably have remained concealed, had it not been laid bare in the bank of the river. On both sides of the Tees it is more than seventy feet wide, and ranges through horizontal strata of sandstone in a direction about S.E. by E. These horizontal strata must be referred to the new red sandstone formation, though they exhibit but faint traces of the usual ferruginous tinge. From Barwick the dyke passes through the quarries of Stainton, Nunthorp, and Langbargh, to the foot of the Cleveland hills; making in its progress a considerable flexure to the north. At Stainton the north face of the dyke is interrupted by a fissure about five feet wide, which is filled with light coloured argillaceous materials, with a transverse

* See Plate II. Fig. 1.

slaty texture. These substances bear no resemblance either to the sound or decomposing specimens of the dyke itself*.

On the east side of Nunthorp it gradually rises above the level of the neighbouring country, and might be mistaken for a gigantic artificial mound, had not the quarries exposed its interior structure. A well defined ridge, about four hundred feet above the level of the neighbouring plains, marks its passage over the south flank of Rosebury Fopping. Still farther to the east it is traced by a gap in the outline of the moors: for the upper beds of sandstone appear to have been shattered and carried off, and the dyke only rises to the highest level of the great bed of alum-shale. After passing through this gap and descending into Lownsdale, we found the trap forming a mass of bare rock which rose twenty or thirty feet above the vegetable soil. From thence it may be followed without difficulty many miles down the valley of the Esk, in a line bearing about E. S. E. Afterwards, by the turn of the valley at Egton Bridge, it is once more brought through the high moorlands; and its course is marked in that desolate region by a low ridge resembling an ancient Roman road. A quarry which is opened at Silhoue, near the seventh milestone on the road from Whitby to Pickering, proves the whole thickness of the dyke to be about forty feet, and its inclination and direction nearly the same as in the other localities. Beyond this place it continues to thin off, but it may be traced, though not without some difficulty, as far as a small rivulet about two miles to the east of the road. The exact point of its termination has perhaps not been ascertained; but there does not seem to be any good reason for supposing that it is continued to the German ocean; as no vestige of it has been seen in any part of the cliff where it might be expected to appear.

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* See Plate II. Fig. 3.

Extent and Position.

No other dyke has, I believe, been yet described, which intersects so many secondary formations, and preserves such an extraordinary uniformity of direction and inclination. The whole length, reckoning from the quarry at Gaundlass Mill, is more than fifty miles: and if any one should object to this, as including a considerable space in which the continuity is not apparent; there will still remain from Coatham Stob a distance of about thirty-five miles, through which it is almost certain that the trap ranges without any break or interruption. Perhaps it might with more justice be objected, that the first computation falls below the truth; in consequence of the probable extension of the dyke to the N. W. through the Woodland Fells and Egglestone Burn to the banks of the Tees. Should this supposition be admitted, we shall have an uninterrupted dyke extending from High Teesdale to the confines of the eastern coast; a distance of more than sixty miles.

The angle at which it cuts the strata is of course variable, and in many places cannot possibly be ascertained. At Barwick, near the Tees, its inclination to the horizontal beds of sandstone is more than eighty degrees; and the angle at which it intersects the beds of shale and sandstone in the eastern moors is still greater; occasioned, perhaps, by the south-eastern dip, which generally prevails among the strata in that region*.

Secondary formations, when interrupted in the manner above described, seldom preserve the same level on the opposite sides of their line of separation. Thus at Cockfield Fell, the coal-beds on the north side of the dyke are eighteen feet below the corresponding beds on the south side. In the excavations at Preston and Barwick there is no indication of any great change having been produced in the relative level of the beds of sandstone: nor can any conclusive evidence be obtained on this

* See the Survey of the Yorkshire coast by Young and Bird.

subject from the obscure sections exhibited by the quarries in the eastern moorlands. Perhaps, as a general rule, the greatest dislocations are produced by those fissures into which trap is not intruded: such at least appears to be the case in the great coal-field of Northumberland and Durham. The injected masses of trap may be supposed to have acted as a kind of support, and to have partially hindered the broken ends of the strata from sliding past each other.

Structure of the dyke.

Notwithstanding the great length of the Cleveland dyke, and the different nature of the rocks with which it is associated, it undergoes very little modification in its general structure. Its prevailing character is that of a fine granular trap rock of a dark bluish colour. This colour is indeed, with some unimportant exceptions, so constant in all the sound specimens, that the dyke is provincially termed blue-stone by the men who are employed in working the quarries. It breaks into irregular, sharp, angular fragments; and on a recently exposed surface there generally may be seen a number of minute brilliant facets: but the constituent parts are never sufficiently distinguished from each other to give it the appearance of a green-stone. The essential ingredients of the rock are, if I mistake not, pyroxène and felspar, in which respect it agrees with the greater number of trap dykes which have been carefully examined, as well as with a great many varieties of recent lava. The principal modifications, of course, arise from the variable proportions of these essential ingredients. Among the prevailing and nearly compact portions of the dyke, there are some larger crystals of felspar and carbonate of lime; very rarely, however, in such abundance or order of arrangement, as to give any decided appearance of porphyritic structure. Good specimens of amygdaloid are not common; where they do occur the nodules are chiefly composed of carbonate of lime. In one or two instances we found chalcedony filling the hollows of an imperfect amygdaloid. Iron pyrites may be mentioned among the minerals frequently associated with the dyke. It is found disseminated through the substance of some decomposing varieties in considerable abundance; and small spangles of it may occasionally be seen in the sound specimens, especially among the larger crystals of felspar before mentioned. All the dark sonorous specimens act strongly on the magnet; but some of the light-coloured varieties, which contain a great excess of decomposing felspar, do not sensibly affect it.

The dyke is generally separated by a number of natural partings into large blocks, which are amorphous, prismatic, or globular. Near the centre they are sometimes of such entire irregularity as to defy all description. Not unfrequently, however, in the midst of this confusion we may observe traces of a prismatic form; and where this arrangement is most complete the prisms are always transverse to the dyke. Good examples of this form may be seen in the quarry of Preston, and in other localities above described*. The sides of the dyke are generally occupied by clusters of minute horizontal prisms, which are often seen in great perfection even where the central mass is amorphous. In the great quarry of Bolam, where the trap has extended laterally over the horizontal beds of sandstone and coal shale, the capping of basaltic rock is divided into rude columns which are perpendicular to the strata on which they rest; and, therefore, nearly at right angles to the prismatic blocks which lie across the leading dyke. This arrangement is exactly similar to that which takes place among some masses of ancient lava near Mount Vesuvius†.

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* See Plate II. Fig. 1.

† Altered beds of coal in contact with trap sometimes exhibit a similar arrangement. Thus at Coley Hill (*Geological Transactions*, vol. IV. p. 23.) a small bed of coal abuts against a dyke of basalt, and near this contact, the coal is deprived of its bitumen, and arranged in beautiful

Traces of the globular structure are often visible, especially where the trap passes into an earthy state: for many of the larger blocks, whether prismatic or amorphous, decompose in concentric crusts, which easily fall off and expose the hard spherical *nuclei*.

These balls are particularly abundant in the old quarry of Coatham Stob, and are associated with some blocks of a light grey colour, which have an earthy fracture. Both these varieties are interesting. Some of the balls contain a considerable quantity of olivine, which is, if I mistake not, a very rare mineral in all the other localities. The light-coloured blocks have a superabundance of decomposing felspar, and are partially porphyritic. Carbonate of lime exists in the form of distinct crystals, and is also disseminated through the mass; and in some instances small spherical concretions of compact felspar are found in a congeries of very minute crystals, giving to such specimens the appearance of an amygdaloidal structure. In other cases the concretions effervesce when first plunged into acids, are opaque from the admixture of impurities, and do not possess the characters of a simple mineral.

Effects of decomposition.

In this dyke, as in almost every similar formation, the effects produced by decomposition are exceedingly varied. The component parts, from the centre to the surface, are in some quarries hard and sonorous. In others, the sides are invested with a ferruginous earthy matter which only penetrates to the depth of a few inches, and gradually passes into a sonorous granular rock. Not unfrequently, a decomposing crust of more considerable thickness covers the surface even of the blocks which are derived from the center of the dyke. A number of white spots, probably

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beautiful small *horizontal prisms.* Under the overlying mass in the quarry of Bolam, the carbonaceous shale is rudely prismatic; and in one or two places where this structure is best exhibited, the prisms are nearly *vertical.*

resulting from decomposing felspar, are often disseminated through these earthy masses, and enable us to separate them from other argillaceous materials with which they are sometimes in contact. It would be a laborious, and not a very profitable task, to attempt a minute account of phenomena like these, which vary with every different locality.

Effects produced by contact of the dyke.

It now remains to describe some of the effects produced by the intrusion of the dyke. These effects will of course vary with the substances which are acted on. In some of the quarries which have been already described, the trap passes through horizontal beds of slate-clay, and the changes produced by its presence are in all these cases strikingly similar. At Nunthorp and Langbargh* these beds of slate-clay belong to the great *allum-shale formation* (*Lias*), and are easily identified by the belemnites, pectinites and other characteristic fossils which are imbedded in them. On approaching the dyke they become much indurated, and are divided by a great many vertical fissures, which, when combined with the ordinary cleavage, separate the strata into rhomboidal fragments. In all such cases the rifts and fissures are coated over with oxide of iron. In other instances, the true horizontal cleavage entirely disappears; and the indurated masses might then be easily mistaken for beds which had been tilted out of their original position. The alteration produced in the coal-shale at Gaundlass Mill is exactly analogous to what has been described, though not so strikingly exhibited.

In the quarry at Barwick, on the right bank of the Tees, the vein of trap is well denuded, and the south side of the section exposes a great many horizontal beds of sandstone, which are separated into prismatic blocks by a number of natural transverse fissures. Close to the dyke this structure disappears;

* See Plate II. Fig. 2.

the sandstone is much more compact, and breaks into amorphous fragments.

It must however be allowed that in some other localities the sandstone did not, under similar circumstances, appear to have undergone any modification.

Perhaps, as a general rule, none of the changes above described are well exhibited, where the portion of the dyke, in contact with the horizontal beds, assumes the appearance of a wacké. Should this observation be sufficiently verified, it would seem to indicate, that the earthy texture of the dyke is, in some cases, rather due to its original mode of aggregation, than to any subsequent decomposition. I may, however, assert unequivocally, that I never saw any beds which are easily susceptible of modification (such as coal or carbonaceous shale) in immediate contact with the trap, without having undergone a remarkable change.

The overlying trap at Bolam bears no resemblance to a substance which has been tranquilly deposited on the inferior strata; for it is separated from them by a broken indented superficies which has exposed many distinct beds to its immediate action. Some of the massy columns rest on a bed of shale partially converted into a substance resembling Lydian stone, which rings under the hammer, or flies in all directions into a number of sharp splinters. Others are supported by a bed of impure coal or carbonaceous shale, in the upper part of which are found shapeless masses in various states of induration, mixed irregularly with angular pieces of trap, and an earthy substance like soot or pounded charcoal. Where the carbonaceous ingredients are most abundant, the parts of the bed in immediate contact assume the appearance of coke derived from the artificial distillation of impure coal, and not unfrequently separate into a number of minute prisms*. An impure carbonaceous powder is

* See the Note to p. 33.

sometimes found in the crevices between the basaltic columns, several feet above the beds on which they rest.

In addition to the substances above described, I found beneath the trap some thin white porcellanous fragments, which appeared to be derived from an indurated bed of *fire-clay*—a well known associate of the great coal formation.

All these phenomena so exactly resemble the effects produced by fire, that I am unable to describe them without using language which may be thought hypothetical by those who deny the igneous origin of trap dykes.

In Cockfield Fell the coal-works have been conducted on both sides of the dyke, and the extraordinary changes produced by its influence have been recorded by practical men who had no theory to support, and who founded their opinions upon actual observation. The works are not now carried on in the immediate neighbourhood of the dyke; but I procured so many specimens of the substances which had been taken from the altered coal-beds, that I have no doubt of the general accuracy of the accounts which have been published.

Close to the dyke, the *main coal* is converted into a substance resembling *soot*, and at some distance it passes into a more solid substance, which the miners call *cinder*. At a still greater distance it retains a part of its bitumen, and about thirty yards from the trap it does not differ from the ordinary pit-coal of the district. It is stated, (*Hutchinson's History of Durham*) " that immediately above the *cinder* there is a great deal of sulphur in angular forms of a bright yellow colour. The *cinder* burns clear, without smoke, and affords very little sulphurous effluvia."

Were there no other examples of corresponding phenomena it would perhaps be unsafe to draw any direct conclusions from the facts which have been stated. But in different parts of the British Isles, similar effects appear, in instances almost without number, to have been produced by the operation of similar

Igneous origin of the dyke.

causes: so that the igneous origin of a large class of trap dykes seems to be established by evidence which is almost irresistible.

It is urged to no purpose, that Lydian stone and glance-coal occur in places which have never been influenced by volcanic action. The assertion may be true, but is of no value in determining the question; unless it can be shewn, that substances, similar to those derived from the sides of the dykes, are found in other parts of the same district which are removed from their influence. This however is not the case, for the enormous excavations which have been carried on in the great coal-basin of Northumberland and Durham have, with one ambiguous exception*, made us acquainted with no similar substances excepting those which appear to have been produced by similar agents.

General summary.

It may be proper briefly to enumerate some of the facts which are established by a detailed examination of the great dyke, and which will, perhaps, be considered to place its origin out of all doubt.

(1) It is more recent than the formations which it traverses. For it occupies the interval between beds which were evidently once continuous; but have been subsequently broken up and severed by some great convulsion.

(2) It was consolidated prior to the last great catastrophe which formed the beds of superficial gravel, and excavated the secondary vallies. In proof of this we need only state, that it partakes of all the inequalities of the districts through which it passes, rising with the hills and falling with the vallies, so that

* See the *Geological Transactions*, vol. IV. p. 27. The case is obviously ambiguous, because the effect of a large mass of trap on a bed of coal may be propagated to a considerable distance. The very change described by Mr. Winch *may* therefore have been effected by a mass of trap which is not exposed in the workings. We must carefully distinguish between the phenomena here described, and the effects of those dislocations which so commonly intersect the coal strata. In these latter instances the coal beds are often deteriorated on both sides of the line of *fault* by the mere *mechanical* effects of the rupture.

in many of the lower regions it is buried in diluvium. On this subject there is, I believe, no difference of opinion.

(3) There is every reason to believe that it has been filled from below. For there exists no trace of any upper bed from which its materials could have been supplied; and in one place, horizontal beds of sandstone rest on the top of a mass of trap which is probably connected with the dyke. We may further state, that many dykes of similar origin wedge out before they reach the surface*.

(4) The dyke has once been in a fluid state. For it is moulded to all the flexures of the chasm which it fills up. The same assertion is also proved by its crystalline texture.

(5) The materials of which it is composed are the same with those which abound in a great many varieties of recent lava. On this subject there is perhaps no difference of opinion. For the Wernerians at one time asserted, that recent lava was derived from the igneous fusion of trap rocks of aqueous origin.

(6) The effects produced by the dyke are such as might be expected from the intrusion of a great mass of ignited matter. This assertion is fully established by the facts which have been already stated.

If, therefore, similar effects have originated in similar causes, we must conclude, that this dyke, as well as all the other similar masses in the great Durham coal-field, are the undoubted monuments of ancient volcanic action.

Conclusion.

It is a matter of fact, which is independent of all theory, that an enormous mass of strata has been rent asunder; and it is probable that the rent has been prolonged to the extent of fifty or sixty miles. If we exclude volcanic agency, what power in nature is there capable of producing such an effect? By sup-

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* See Professor Henslow's paper on the Isle of Anglesea; Dr. Mac Culloch on the Hebrides, &c. &c.

posing such phenomena the effects of volcanic action, we bring into operation no causes but those which are known to exist, and are adequate to effects even more extensive than those which have been described.

Combining this observation with the facts described with minute detail in the preceding parts of this paper, we obtain a chain of evidence, in favour of the igneous origin of a certain class of trap dykes, not one link of which appears to be defective. It is not to be denied, that the associations of trap rocks may in other cases present great difficulties to the igneous theorist. But these difficulties are not the present subject of consideration. I have confined myself, as far as possible, to a statement of facts, and I have only attempted to record such conclusions as a review of those facts appeared fully to justify.

Trin. Coll. March 12, 1823.

P. S. Before this paper was sent to the press, I received two letters from my friend Mr. Wharton, of Oswald House, near Durham, communicating some very interesting facts connected with the appearance of a basaltic dyke; which ranges from the escarpment of the magnesian limestone (at Quarrington Hill, a few miles to the east of Durham) through the great coal-field, in a direction about W.S.W. It is found along this line at Crowtrees, Tarsdale, Hett, Tudhoe, Whitworth, and Constantine farm. From the last mentioned place, it passes along the same line of bearing, through the collieries of Bitchburn and Hargill Hill, to a spot near the confluence of Bedburn Beck and the river Wear, where it is well exposed on the surface of the ground; and it is known to pass up the Bedburn Beck valley towards Egglestone Moor. If prolonged a few miles in the same direction, it must meet the line of the Cockfield Fell dyke within a short distance of Egglestone; and may, perhaps, be a prolongation of one of the masses of trap described in a former part of this paper.

This dyke is laid down in none of our Geological maps. Indeed its existence was probably unknown before Mr. Wharton ascertained its continuity, by examining the thickness, the dip, and the bearing, of several masses of trap, which appeared in separate quarries, but in the same general line of direction. That its further extension towards Egglestone Moor, and its probable connexion with the trap of High Teesdale, should be correctly determined, is certainly an object of considerable interest.

The following facts appear of most importance in illustrating the natural history of this dyke.

(1) The trap, in colour, fracture, and external form, is similar to that of Cockfield Fell. It often parts into irregular prismatic blocks with well defined angles, and four or five plane sides covered with an ochreous crust.

(2) The width of the dyke appears to increase in its progress westward. Thus, at Crowtrees quarry it is six feet and a half wide—at Tarsdale quarry nine feet and a half—at Bitchburn bank fifteen feet—and still farther west it is seventeen feet wide.

(3) It dips to the north at an angle which brings it up in a direction which is nearly perpendicular to the coal strata; which, on the north side of the dyke, are found about twenty-four feet above the level of the corresponding beds on the south side.

(4) In the collieries situate in its line of direction (viz. Crowtrees, Bitchburn, and Hargill Hill) the seams of coal near the dyke are charred, or converted into a hard mass of cinders; in consequence of which, the works have in some cases been partially abandoned.

(5) The dyke appears to decrease in width as it rises towards the surface. Thus, in Crowtrees colliery, the width of the dyke, where it is cut through at the depth of fifteen fathoms, is nearly twice as great as at the surface.

(6) It does not appear at Quarrington Hill to cut through a bed of sand and pebbles, which lies between the highest beds of the coal-formation and the magnesian limestone.

The importance of these facts in confirming the theoretical views given in the preceding paper, is too obvious to need any explanation.

Mr. Winch asserts (*Geological Transactions*, vol. IV. p. 25.) "that he has never been able to trace any of these basaltic veins "into the magnesian limestone, and is almost certain that, with "other members of the coal-formation, they are covered by it." The dyke just described affords some additional evidence in support of this opinion. Moreover, it appears, in its general relations, to agree so exactly with the Cockfield Fell dyke; that I now cannot help suspecting, that this latter also belongs to the class of "basaltic veins" which do not pass up into the magnesian limestone, though I inclined to a different opinion when the preceding paper was written.

Respecting the prolongation of the Cockfield Fell dyke through the region of the magnesian limestone, there are conflicting probabilities which lead to directly opposite conclusions. The near agreement in the direction and dip of the Cockfield Fell and Cleveland dykes, has generally been supposed to afford sufficient evidence for their continuity. If this opinion be adopted, we must, I think, be compelled to admit the existence of a dyke through all the intermediate district*.—On the contrary, there is no direct evidence for the existence of any trap associated with

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* See the observations at p. 29 of this paper.

the magnesian limestone; and the relations of all the analogous formations in the coal district seem to prove, that the Cockfield Fell dyke cannot pass out of the limits of the coal-formation.

If we adopt this latter opinion, we must admit, that the dykes of Cockfield Fell and Cleveland (notwithstanding the agreement in their line of direction) belong to two distinct epochs. After all, the question is only one of local interest; and as far as regards the leading object of this paper, of no importance whatsoever.

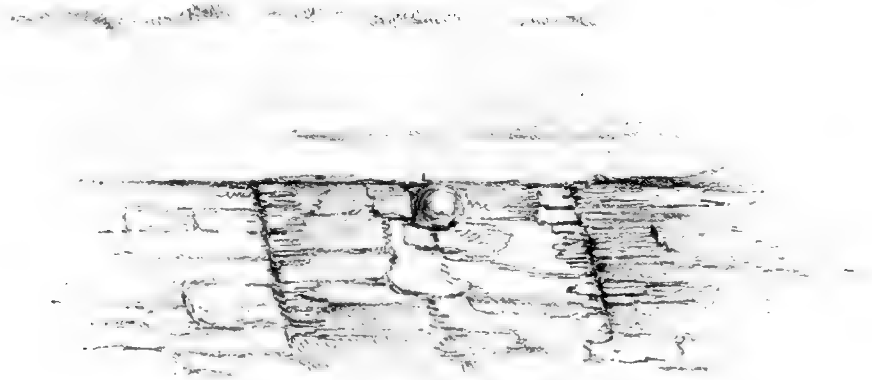
Through the kindness of T. R. Underwood, Esq. of Paris, I have become acquainted with the results of an examination of specimens from several English trap dykes by Professor Cordier. I will subjoin his description of such specimens as were derived from localities alluded to in the preceding paper.

No. 1. From Preston quarry in the Cleveland dyke. *Mimosite,* fine grained, imperfectly porpheroidal from the salient crystals of Pyroxène. It is a ***Basalt*** of the ancient mineralogists. The specimen contains a great abundance of dark-greenish grey Felspar, mixed with a very small quantity of Pyroxène and titaniferous Iron. Some points of Pyrites are to be seen. The *paste* also envelops laminar crystals of Felspar, having a considerable lustre, which give the *paste* a scaly appearance which distinguishes it from ***Basalt.***

No. 2. From Coaly Hill dyke near Newcastle. *Mimosite,* small grained, passing into *Xerasite.* Many of the cavities contain *green-earth.* It is imperfectly porpheroidal. The crystals of Felspar very brilliant.

No. 3. From Walbottle Dean dyke. This has a more decided character of a ***Dolerite,*** very fine grained, the Felspar whiter than in the others.

As these distinctive terms are not generally adopted by English mineralogists; it may be proper to state that *Mimosite* and *Dolerite* are granular rocks. *Xerasite* and *Basalt* are composed of the same elements, but microscopic, and having the appearance of a *paste*.



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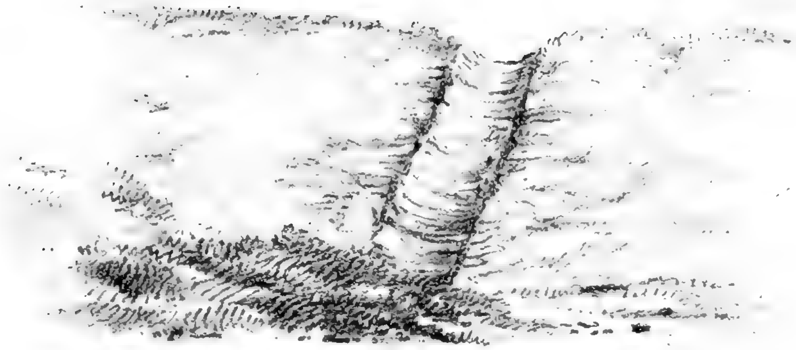

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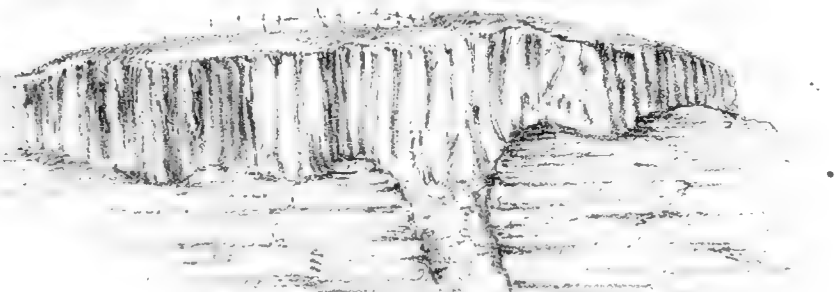

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IV. *A new Demonstration of the Parallelogram of Forces.*

By JOSHUA KING, Esq. M.A.

FELLOW AND TUTOR OF QUEEN'S COLLEGE.

[Read *April* 14, 1823.]

Let two equal forces, each of which is represented by p, act upon a material point inclined to each other at an angle 2θ, and let r be their resultant; which will evidently bisect the angle 2θ; for there is no reason why it should be inclined to one force at a greater angle than to the other.

Now for every value of θ, r will vanish when p vanishes, but only upon that supposition. Again, for every value of p, r will vanish when $\theta = \pm\frac{\pi}{2}$, or $\pm\frac{3\pi}{2}$, or $\pm\frac{5\pi}{2}$, or &c. $\pm\frac{2n+1}{2}\cdot\pi$, but upon no other supposition: Hence the factors p, $\left(1-\frac{2^2\theta^2}{\pi^2}\right)$, $\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)$ &c. will enter into the expression for r, but no other, except quadratic factors having impossible roots: we may therefore suppose

$$r = k \cdot p \cdot \left(1-\frac{2^2\theta^2}{\pi^2}\right)\cdot\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\cdot\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\cdot \text{\&c.}$$

$$= k \cdot p \cdot \cos.\theta.$$

Now (1). k cannot involve p, for the angle of inclination remaining the same, the resultant must necessarily be proportional to the component.

(2). k cannot involve θ: for if it does, it must evidently be such a function of θ as has no possible root: it must moreover involve no negative powers of θ, for then would r become infinite when θ vanished: nor yet any odd powers of θ, for then would r be altered by changing the sign of θ: it must therefore involve the even powers of θ only; and as all the roots are impossible, the terms must all have the same sign, and consequently such a value of θ may easily be assumed as will make $k \,.\, p \,.\, \cos.\, \theta$ greater than $2p$, which is impossible.

(3). $\cos.\, \theta$ cannot be raised to any power as $\overline{\cos.\, \theta}|^m$, for then the equation could not be made identical both when $\theta = 0$ and when $\theta = \dfrac{\pi}{3}$.

(4). the factors composing $\cos.\, \theta$ cannot be raised to different powers, for the equation could not then be made identical when θ equals any term of the series $\pi,\ 3\pi,\ 5\pi \ldots\ldots \overline{2n+1}\,\pi$.

To determine k, let $\theta = 0$, in which case r ought to be equal to $2p$;

$$\therefore\ 2p = kp, \text{ and } \therefore\ k = 2.$$

$$\text{Hence } r = 2p \,.\, \cos.\, \theta.$$

The resultant of two unequal rectangular forces, and consequently of any two forces whatever inclined to each other at any angle, may easily be deduced from the expression for the resultant of two equal forces by processes already known, which require neither elucidation nor abbreviation.

V. *On the Developement of Electro-Magnetism by Heat.*

BY THE REV. J. CUMMING, M.A. F.RS. M.G.S.

PROFESSOR OF CHEMISTRY

IN THE UNIVERSITY OF CAMBRIDGE.

[Read *April* 28, 1823.]

THE property which the Tourmaline and a few of the crystallized gems possess, of exhibiting the opposite electricities by the action of heat alone, has hitherto been considered as peculiar to those bodies; but a recent experiment by Dr. Seebeck of Berlin, has proved that this power, so far from being confined to non-conductors, as from analogy might have been suspected, is possessed by one, at least, amongst a class of substances, which are, comparatively, perfect conductors both of heat and electricity.

The experiment is described in these words: "Take a bar "of antimony about eight inches long and half an inch thick, "connect its extremities by twisting a piece of brass wire round "them, so as to form a loop, each end of the bar having several "coils of the wire. If one of the extremities be heated a short "time by a spirit lamp, electro-magnetic phenomena may be ex-"hibited in every part of it."

On repeating this experiment, it appeared to me highly probable that other metals might possess the same property, and in prosecuting this enquiry, I have been led to some results, which, I hope, will be both as new, and as interesting, to this Society, as they have been to myself.

My first object was to ascertain whether in this, as in some late electro-magnetic experiments, the effect depended on the wire being *coiled round* the bar; but by repeated trials, both on antimony and other metals, it was found to be indifferent, in what manner the wire and bar were connected, provided that, in all cases, the metallic contacts were complete: I have therefore, in general, made the connection, either by soldering, rivetting, or casting the bar upon the wire. The result was, that not only all the metals, including fluid mercury, but likewise plumbago and charcoal, and some, at least, of the metallic sulphurets, possess the property of exhibiting electro-magnetism by heat, differing, however, both in quantity and quality. If, for instance, a bar of bismuth, having copper wires at each end, be heated at one extremity; on placing the wires in the mercurial caps of the galvanoscope, the heated end produces a deviation on the compass needle, in the same direction as a wire from the silver disk, in the common galvanic circuit: with antimony it is the reverse. (Fig. 1 and 2). These metals may, therefore, so far be considered, as positive and negative with respect to each other.

On examining the other metals with *copper wires*, I found that they might be distinguished into two classes, the heated end of the one, and the cooler end of the other exhibiting the silver or positive electricity. (Table I). When wires of other metals are used, there are modifications, which appear to me some of the most singular circumstances in these experiments. If the bar be of copper, the deviation becomes negative or positive, accordingly as the wires are platina or silver; or if the extremities be considered as positive and negative with the one, they are negative and positive with the other. The same effect was produced with a bar of zinc, and zinc or copper wires; and with silver, platina, and palladium, as the wires were silver or platina. In these instances it seems remarkable, not only, that the bar appeared to change its electrical states with different wires, but that electricity or rather electro-magnetism should be exhibited when the bar and wire were of the *same* metal. In the first case, it might be supposed that the electricity was excited by the contact of dissimilar* metals; as, in the galvanic circuit, copper is positive to zinc, but negative to silver. But this hypothesis is inapplicable to cases of the second description. If the effect depended on the contact of dissimilar metals, it would be greatest between those which are opposed in the galvanic circuit, and would cease when the bar and wire were of the same substance. On making the trial with a bar of zinc and silver wires, the deviation was not greater than that by the same bar and wires of zinc. Again, platina and silver are both positive with reference to copper, yet the deviations were opposite; and silver and copper bars acted strongly with silver and copper wires respectively. As in all these instances, to prevent ambiguity, the wires were not soldered, but rivetted to the bars, I cannot but conclude, that the hypothesis of electricity being excited by the contact of dissimilar metals, is, whatever plausibility it may possess in other circumstances, inapplicable to the case of its developement by heat. The following experiment is, I think, decisive. — Two wires, each composed, the one half of platina, the other of silver, soldered together in the middle, were rivetted into a bar of brass. When the silver ends were connected with the brass, the deviation was positive; on reversing each wire, and therefore connecting the platina and brass, it was negative; still retaining the platina contact, but shortening the platina wire to about half an inch, the deviation again became positive. In every case the brass was in contact with a metal highly positive with respect

* Dissimilar as to their galvanic relations.

to itself, yet the deviations in the two last were in opposite directions, though the contact of brass and platina was the same in both.

If these experiments be referred to the hypothesis which accounts for electrical excitation by the oxidation of the metals, they seem equally adverse to it. Not to repeat the instances of its production where the heated bar and the wires were of the same metal, and in consequence, similarly, if at all, oxidated; it can scarcely be imagined, that an elevation of temperature of not more than two or three degrees, should cause a difference of oxidation; and it is to be remarked, that the effect is produced, whether the temperature be elevated or depressed. On placing one end of a bar of bismuth in a freezing mixture, or even by allowing a few drops of ether to evaporate from its surface, there was produced a considerable deviation on the needle of the galvanoscope; that extremity which remained at the temperature of the room acting as the heated end of the bar in the other instances.

Having ascertained that in *all* the perfect conductors of electricity, electro-magnetism may be excited by the unequal distribution of heat, my next endeavour was to determine the direction in which this peculiar influence is exerted, and the mode of its propagation. If a bar of antimony, *AB* (Fig. 3.), having its ends connected by copper wire *A a b B*, be heated at one extremity and presented to the compass, the deviation, in every part both of the bar and the wire, is of the same nature, and therefore the current of electricity (if there be such a current) is throughout in the same direction. The effect is similar, whatever metal be employed; but, as will be seen by reference to Table I, the direction of the current in some is opposite to that in others. If two bars of the same or of similar* metals, equal in power, be

* Similar, as to their developement of electro-magnetism by heat.

connected at their heated extremities; or, which is the same thing, if a single bar be heated at the middle, no effect is produced, the equal and opposite currents counteracting each other: if under the same circumstances, the metals are dissimilar, the effect is that arising from the joint action of their conspiring currents. In some respects this arrangement is analogous to that of the galvanic circuit; heat in one case acting the part of an acid in the other; but there is one material difference between them. In the first, the metallic circuit is complete, and the current is, as has been already observed, in the same direction throughout every part of it. In the second, the circuit is interrupted, and the current through the acid is opposite to that through the wire.* (Fig. 4 and 5).

If the bar of antimony *AB* above mentioned be broken unequally into two parts *ab, cd,* (Fig. 6.), and these be connected by a copper wire; on heating one part and cooling the other regularly throughout, no effect is produced, however short the interval may be between them.† If the parts *ab, cd* be (Fig. 6.) again

* This may be readily shewn by connecting *S* and *Z* in Fig. 5. by a fine wire.

If two wires, the one platina, the other bismuth, be connected with the galvanoscope, on immersing their other extremities (not in contact) for a short time in nitric acid, electro-magnetic effects are produced by their galvanic action; on making the contact the effect still continues, and in the same direction, but arises from heat; if platina and iron be used the second action is contrary to the first.

† As electricity is excited by the contact and separation of two polished disks of zinc and copper, Mr. Herschel suggested to me, that it might be desirable to try what effect would be produced, by heating one of the disks previously to its application to the other. If their thickness be inconsiderable in comparison with their surface, the electro-magnetism elicited is scarcely, if at all perceptible; if they be of considerable thickness, and both bismuth or both antimony, it is greatly increased; and again is materially diminished if one be antimony and the other bismuth. The reason is obvious: in the first case each disk receives almost instantaneously an uniform temperature throughout; the second case becomes that of two bars of the same metal, having their extremities at unequal temperatures in contact, and their electric currents in the same direction; in the third case they are opposed. All these instances are decidedly unfavourable to the supposition of electro-magnetism being evolved by the *contact* of dissimilar metals.

soldered together, with a thin plate of copper interposed, they no longer act as one, but as two distinct bars. When heat is applied at the extremity *a*, the deviation is, as usual, negative; at *c* the same; but if at *b*, the deviation is positive, the extremity *a* becoming the cooler end, and the part *cd* merely conducting the electricity: but as the bar cools, *ab*, and *c* the extremity of the other part, gradually assume the same temperature, and consequently the bar acts negatively as at first. It appears then, that when the bar was entire, the heat was not merely conducted from one extremity to the other, but, by some means modified in its progress, and that, for the production of this species of electricity, there is required the juxta-position of two particles of the same metal at different temperatures. If therefore, a cylindrical bar, unequally heated, be supposed to be divided into an indefinite number of circular laminæ, each will act, as a layer of hot particles upon the lamina on one side, and of cold upon that on the other, and the total effect of the bar will arise from the aggregate action of these laminæ.

By soldering wires to a long rod of bismuth (Fig. 7.), the parts of which were alternately hot and cold, it was found that the action of the whole exceeded that of any two portions taken separately, and as the only condition appears to be, that there should exist a difference of temperature between two adjoining particles, it may be inferred, that if it were possible to increase these divisions *sine limite*, each bar would act as an assemblage of an indefinite number of small plates; as the common magnet may be conceived to be composed (if the expression may be allowed) of an indefinite number of atomic magnets.*

* The connection of the wires may be either 1 and 2, 2 and 3, 3 and 4, or 1 and 4, or 1 and 3 may be placed in one cup of the galvanoscope, and 2 and 4 in the other, in which case the effect is the greatest. The deviation caused by connecting 1 and 4 is not affected by connecting at the same time 2 and 3; this seems unfavourable to the supposition of circulating currents.

In the beginning of this paper it was mentioned that all the metals possess the power of exhibiting electro-magnetism by heat, but in different degrees. Even when these are the greatest, they are much inferior to what can be readily produced in the common galvanic apparatus; it was consequently necessary, in detecting and forming a comparison of their relative powers, to use the delicate instrument described in the First Volume of our Transactions*; by which some, though not in all cases, an accurate measure of them could be obtained. For the more minute effects a compass was employed in the galvanoscope, having its terrestrial magnetism neutralized, which gave a deviation of from 10° to 20° with two disks of zinc and copper of $4\frac{1}{2}$ inches diameter, excited by spring water, and which was readily sensible to the galvanic action of zinc and copper wires, excited by nitric acid, whose diameters were less than $\frac{1}{20}$, and whose excited surfaces were consequently between $\frac{1}{400}$ and $\frac{1}{500}$ of an inch. With this instrument, two rods of zinc and copper of $\frac{3}{10}$ inch diameter and $\frac{3}{10}$ apart, excited by equal parts of muriatic acid and water, gave a deviation of 40°; a bar of bismuth $4\frac{1}{2}$ inches long by $\frac{1}{2}$ broad, and $\frac{2}{5}$ thick, gave 70° of deviation at the melting point of the bismuth, and 10° when the difference of the temperatures of its extremities was 12° of Fahrenheit. The slip of palladium before mentioned, which was $2\frac{1}{2}$ inches long by $\frac{6}{10}$ broad, and weighed 35 grains, gave a deviation of 70° positive with silver, and 10° negative with platina wires, when heated red hot by a spirit lamp: a slip of platina

* The spiral wire of the instrument was originally of copper, but silver wire of the *same* diameter is more efficacious. If the spiral parallelogram be vertical, and the wire of silver of $\frac{1}{25}$ inch diameter, it acts very powerfully, and presents little or no impediment to a view of the needle; for this reason I prefer the vertical to the horizontal form of the spiral. The needle is neutralized by placing a powerful magnet North and South on a line with its centre; and another, which is much weaker, East and West at some distance above it: by means of the first, the needle is placed nearly at right angles to the meridian, and the adjustment is completed by the second.

of the same dimensions, and under the same circumstances, gave deviations of 65° positive, and 4° negative respectively. These are selected as being some of the most remarkable results, and as serving, at the same time, to form a comparison between the effects of electro-magnetism as excited by heat, and by the usual process. The results of similar experiments on some metals and metallic alloys are given in Table II.

The metals which were found to be most powerful in their action, and at the same time most readily prepared for experiment, were bismuth and antimony. These were sufficiently energetic to have their effects estimated by a compass needle 4 inches in length, the terrestrial magnetism of which was not neutralized, and therefore the deviations caused by different increments of temperature may be employed as data for the corresponding electro-magnetic action. With this compass, rods of zinc and copper, each $\frac{7}{10}$ inch diameter, and $\frac{4}{10}$ apart, excited by equal parts of muriatic acid and water, gave a deviation of 27°. A circular rod of bismuth $4\frac{1}{2}$ inches long and $\frac{1}{2}$ inch diameter, gave a deviation of 21° at the melting point of bismuth, 12° at the temperature of 180°, and 5° at 100°, the cooler extremity being in water at 60°. A similar bar of antimony gave 19° with the utmost heat of a spirit lamp, and a bar of platina which weighed 565 grains, and was 7 inches long by $\frac{1}{2}$ broad, gave 19° at a red heat. Hence it appears, that, at the same temperature, bismuth is more powerful than antimony, and antimony more powerful than platina: the other metals, with the exception perhaps of iron, were inferior to platina.

Were it possible to acquire the same accumulation of power in this species of electro-magnetism as is obtained by the use of large plates in the galvanic, or by numbers in the voltaic apparatus, it would be an interesting object of research to compare the electricities thus differently excited. For this purpose bars were cast, differing both in length, breadth, and thickness, but an increase in either of these dimensions was not attended with an equivalent increase of power. A cylindrical rod of bismuth, 9 inches long by $\frac{3}{4}$ diameter was *rather* more powerful than another of $2\frac{1}{2}$ inches by 2; but, between this, and a thin plate $4\frac{3}{4}$ inches long by 1 broad and $\frac{1}{4}$ thick, the difference was scarcely perceptible. By increasing the surfaces in contact, or rather, as it afterwards appeared, by adding to the *conducting* surface of the connecting wire, there was a slight addition to the effect produced. The cylinder of bismuth of $2\frac{1}{2}$ inches by 2, having a plate of copper of the same diameter soldered upon it, with four connecting wires, (Table III.) was equally, if not more powerful, than the longer cylinder, and on the whole, this seems to be the best form; yet the gain of power, by increasing either the diameters of the bars or of the surfaces in contact, is not such as to promise any advantages by the use of large metallic bars, analogous to those obtained by the employment of large plates in the galvanic apparatus. When two metallic rods in connexion, were heated at the same time, there was some accumulation of power; which appeared to be greater when they were in sequence than when the wires from both the heated ends were placed in one cup of the galvanoscope, and the wires from the cooler ends in the other. (Fig. 8 and 9). A bar *AB* which gave a deviation of 16°, when placed in sequence, as Fig. 8, with another *CD*, whose deviation was 21°, gave a deviation of 25°, but when connected, as in Fig. 9, gave only 23°.

As in this experiment there was some, though not a considerable increase of power, a battery was formed of eight plates, four of antimony, and four of bismuth, placed alternately, and connected in sequence, (Fig. 10.). This (Table IV.) at the mean temperatures of 175° and 100°, gave 17° and $7\frac{1}{2}$° of deviation; a single plate of bismuth, at the same temperatures, gave the deviations $7\frac{3}{4}$° and $2\frac{3}{4}$°, and one of antimony gave $6\frac{1}{4}$° and 2°; the effect therefore of the eight plates is but little more than double that of the single plate of bismuth*. A battery of 6 plates of bismuth, (Fig. 11.) gave at 140° a deviation of only $8\frac{1}{4}$°; but two plates alone of the same battery, gave a deviation of between 9° and 10°. This diminution of power arose, as appears by Table V. from some of the plates being inferior to the others; as is the case in the voltaic battery, when the plates are of different dimensions or differently excited, the weaker having a tendency to reduce the others to their own standard. The greatest deviation I have as yet produced was by a double bar 7 inches long, composed of two bars, the one antimony, the other bismuth, soldered together at the middle. This, at the melting point of bismuth gave a deviation of 36° with the same compass which shewed a deviation of 28° with the zinc and copper rods excited by dilute muriatic acid. As these rods were capable, though slightly, of magnetizing a needle inclosed in a spiral wire, and of exciting the limbs of a frog, I endeavoured, but without success, to effect the same by the double bar of antimony and bismuth. Whether this failure were owing to a want of sufficient action, or to some peculiarity in the electro-magnetism excited by heat must be determined by the application of a more powerful apparatus; but it does not seem improbable that it may be owing to some peculiarity in this mode of excitation†. There is something perhaps analogous in common electricity; a large battery may be dis-

* The bars of bismuth and antimony, having their extremities connected by copper wires, were fixed in a wooden trough, in such a manner that one half of their length projected below it. The trough was then filled with hot water or sand, and placed upon another vessel containing cold water. The mean of four thermometers in the upper part was assumed as the temperature of the heated extremities, and those experiments alone retained, in which the thermometers did not differ materially from each other. As the melting point (475°) of bismuth, the more fusible metal, is at nearly the temperature (500°) of boiling linseed oil, it is evident that a considerable increase of power may be obtained, if the trough be filled with ice, and placed upon a vessel of oil heated nearly to ebullition.

† Possibly from its low intensity.

charged through a spiral wire, without magnetising a needle inclosed within it, provided the discharge be made gradually, and without a shock. The tourmaline, which has been mentioned as analogous to the metals, in receiving polarity by heat, I found, even when strongly excited, to have no magnetic action, either when two silver wires were coiled round its extremities and connected with the galvanoscope, or when the same wire was continued throughout; but it has not, I believe, been noticed, though it might have been expected, that in the last case the wire prevents the tourmaline from acquiring the opposite polarities. There is a singularity in the electric properties of the tourmaline, which was pointed out to me by Professor Henslow in the Abbè Hauy's *Traitè de Mineralogie*: if it be exposed to a *low* temperature, its extremities assume the opposite electricities; on increasing the temperature, the electric polarities diminish, at length altogether cease, and are afterwards resumed, but in the opposite states; that end which was positive becoming negative, and *vice versa*. Very unexpectedly, I discovered a similar phenomenon in the metallic electro-magnetism. As bismuth and antimony are oppositely affected by heat, and nearly to the same extent, it seemed probable that in certain proportions they might neutralize each other, and a compound bar would be produced, in which the electro-magnetic effects would cease. This is certainly the case, but these bars possess, what may be called a moveable zero. I have four bars, with different proportions of antimony and bismuth: the first slowly exhibits with the large compass $\frac{1}{2}^{\circ}$ deviation, as antimony, then, on continuing the heat, returns to zero, passes through it, and deviates 5° as bismuth; the second deviates 3° as antimony, returns to zero, deviates 4° as bismuth, and again returns to zero as the bar begins to melt; the third deviates through 7°, and just returns to zero at its melting point; the fourth deviates 7° in the same manner, but returns only to 4°. These results were obtained with copper wires; if wires of silver or platina be used, the effects are similar, but the positions of the zero are changed. (Table VI.)* The proportions of these bars are between four and six of bismuth to one of antimony; but the metals require so many repeated castings before they are thoroughly incorporated, and their union is so much affected by fusion, that it is nearly impossible, without destroying and analyzing them, to determine accurately their composition.

If it be supposed that the formulæ, which express the relation between the increase of temperature and the magnetic effects, be different in the two metals, it is obvious that though in equilibrio at one temperature, they will not be so at any other. This would account for the phenomena of the first, third and fourth bars, supposing them to be rather mixtures than alloys; and perhaps the anomaly of the second bar may be resolved by supposing the metals in this to be partly mixed and partly forming an alloy; in which case there would be three substances, each following a different law. With the same metal, when the differences of temperature at its extremities are not considerable, they correspond very nearly with the deviations; yet proceeding in an increasing ratio, which augments as it approaches the melting point; but not so rapidly with bismuth as with antimony. The deviations of two similar rods of antimony and bismuth, from 65° to 240° of heat, were 10° and 14° respectively, (Table VII.) Taking the deviations at intervals of $3\frac{1}{3}°$ in the one, and $4\frac{2}{3}°$ in the other, the corresponding differences of temperature will be 42, 47, 81, and 47, 60, 68. If the melting point of bismuth be assumed 475°, and that of antimony 800°, the deviations corresponding to the last 235° of

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* The order of deviations with copper wires, as shewn by the small compass, was (Table VI).. 0 . 36 neg. 0 . 25 pos. 0 .

With platina wires they were .... 0 . 35 neg. 30 neg. 65 neg.

bismuth will be only 7°, and 9° for the last 560 of antimony; or, in other words, the deviation in bismuth by 175°, reckoning from 65° to 240°, is double that by 235°, from 240° to 475°; and, in antimony, the deviation by the first 175° exceeds that by the last 560°.

The experiments which have hitherto been detailed, rest upon the supposition that the agency of heat in exciting the magnetic electricity is effective on the *bars* alone, and that the action of the *wires* is solely to *conduct* the electricity thus excited. This is not the fact. When the extremity of a bar is heated in connexion with a wire, the wire is itself in the same state with the bar, having its extremities at different temperatures: the total effect is therefore the sum or difference of that of the bar and wire, accordingly as their relations to heat and electricity are different or the same. If a bar of bismuth be connected with the galvanoscope by wires of antimony, it is obvious, from the experiment of the double bars, that its effect is increased by the conspiring action of the antimony; had the wires been of platina, the effects would, for the same reason, have been diminished. This would be the case, not merely from platina being either a better or worse conductor of heat than bismuth or antimony, but because its electrical properties, as developed by heat, are similar, though inferior to those of bismuth; and therefore when heated in contact with bismuth, its action is in the opposite direction. As the metals differ materially, not only in the nature, but in the strength of their action, it may happen, when the energy of the bar is weak in comparison with that of the wire, that their joint action will be nearly the same as that of the *wire* alone. This was the case in the experiment with the bar of brass, heated with silver and platina wires*. For, though when brass and silver, or brass and platina wires of the same size are

* This experiment would be yet more decisive with wires of gold and platina.

used, the brass is negative with respect to both, yet, the electric energies of silver and brass differ so little, that, by increasing the size of the brass, it may be made to overcome the action of the silver, and give a positive deviation, as in the first case of the experiment. The action of platina is so much more energetic than that of brass, as not to be overcome by the increased dimensions of the bar, and therefore the deviation is negative. In the last case, where the platina was shortened to half an inch, it became heated throughout, and communicated heat to the silver wire in contact with it; consequently the effect of the platina disappeared, and this case became similar to the first. To have made the series (Table I.) given in the first part of this paper, an accurate representation of the electro-magnetic relations of the metals, the bars should have been connected with the galvanoscope, by a substance (if there be such an one) that should merely conduct, without modifying the electricity*. The electric energy of copper is so much inferior to that of many of the metals, that, when the dimensions of the bars considerably exceed those of the copper wires, the series thus obtained may be considered as tolerably accurate; but, with others, as lead and tin for instance, it is manifestly imperfect. The most complete scale seems to be that, which may be formed by using bars and wires of the same dimensions, i. e. by heating equal wires of the different metals in contact, taken two and two together. The series given in Tables VIII. and IX. was formed in this manner, so far as it was practicable. The ends of the wires to be examined, were placed, one in each cap of the galvanoscope; the other ends were then connected and dipped into a capsule filled with boiling mercury;

* If two small rods of antimony and bismuth, properly adjusted, were soldered together longitudinally, they might, perhaps, at a fixed temperature, be considered as a neutral conductor.

by this means all the metals which can be procured in the form of wires may be compared in a few minutes. The others were connected with similar pieces of those metals which had been already examined. The series, Table IX. is so constructed, that every substance may be considered as positive to all below, and negative to all above, and therefore any metal in the upper part of the series will form a circuit with all below it, similar to, though less powerful than that of bismuth and antimony. Platina and iron form an arrangement of this description: equal wires of these metals of $\frac{1}{10}$ inch diameter gave, with the large compass, a deviation of 16° by the heat of a single spirit lamp*. As these metals admit of being welded together, and of sustaining an intense heat, I at first imagined, that an instrument might have been constructed on this principle, as a pyrometer; but, the slow increase of deviation at elevated temperatures, appears to present an insuperable objection.

If either series, whether that formed by metallic bars with copper wires, or that made by the comparison of *equal* wires of different metals, be referred to the common galvanic series, to those of the conductors of electricity or heat, or to the order of specific gravities, it will be found that there is no correspondence between them: the electro-magnetic relations of the metals, as developed by heat, can be determined by experiment alone†. As this property cannot be previously inferred from any known distinctions of the metals considered collectively, so it does not appear to be dependent upon any peculiarity, such as crystallization, in

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* If silver and iron wires be heated in connection, the deviation attains a maximum, diminishes on increasing the heat, and again attains the former maximum in cooling.

† The series is consistent with itself, that is, if on experiment, a metal be found to be positive to one, and negative to another substance in the series, it may be inferred, without further trial, that it is positive to all below the first, and negative to all above the second.

each taken separately. If it were, it would be affected by altering the internal structure. As bismuth is a metal easily cast, and which has a strong tendency to crystallization, I cast two bars of the same dimensions; one in a charcoal mould, which was cooled as slowly as possible; the other in iron placed in cold water, and therefore cooled almost instantaneously: but I could discover no difference either in the nature or quantities of the deviations they produced, when heated to the same temperature. Again, a glass tube eight inches long was filled with mercury, and copper wires being passed through corks at each end it was placed, the one half in hot sand at 170°, and the other in water at 60°*. The deviation was 9° positive, which at 150°, and at 115°, became 6° and 3°. It appears then, that, in the same metal, the magnetic effects are not varied, whether the crystallization be more or less perfect; and that they may be exhibited in a fluid metal, where, of course, there can be no crystallization. When, however, two metals are combined, the change of structure thus produced, is attended with a change of electro-magnetic properties. The alloy of bismuth and tin in Table I. is negative with copper wires, though each metal separately is positive, and one of them to a high degree; in other instances, as zinc with lead, the deviation of the predominating metal seems to be increased rather than diminished, by its union with another, whose deviation, separately considered, is opposite.

Leaving these theoretical considerations, I wish to call the attention of those who may be disposed for a further investigation of this subject, to the facilities which this mode of exciting electro-magnetism affords, for examining some points, which are as yet undecided.

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* In this and other instances it appeared to be immaterial whether one half of the bar were heated, or merely the extremity.

In determining the effects of the common galvanic apparatus, an almost insuperable difficulty occurs from the loss of power, arising from the gradual saturation of the acid, and oxidation of the plates. In the electro-magnetism excited by heat, provided that the extremities of the bars are kept at an uniform temperature, (which there is no difficulty in doing) the power remains unaltered. There is therefore, by this process, no impediment in determining the relative conducting powers of *different* metals, or the effects of different dimensions of the *same* metal.

Some experiments, I had previously made, led me to imagine that the conducting powers of metals were materially affected, not only by the diameters of the wires, but by their lengths; the law of which it would be desirable to ascertain. For this purpose, the circuit from a bar of antimony kept at a steady temperature, was made through a connecting copper wire 32 feet in length, the deviation was 7°; 16 feet of the same wire gave 10°; 8 and 4 feet gave $15\frac{1}{2}$° and 20°. Allowing for the unavoidable inaccuracies of an experiment, in which the divisions of the compass were estimated by the eye, it seems that the conducting power of the wire diminishes in a much lower ratio than its length increases. A slight change in the deviations making them 6°, 11°, 16°, 21°, would form an arithmetical series corresponding to the diminutions of length in geometrical progression.

The diameter of the wire used in this experiment was $\frac{1}{20}$ inch; when 8 feet of wire of $\frac{1}{37}$ was used, the deviation, which had been $15\frac{1}{2}$°, was reduced to $6\frac{1}{2}$°, and with platina wire of $\frac{1}{100}$ it became certainly not more than $\frac{1}{2}$°. When the minute quantity of electricity developed in this experiment is considered, it might have been expected that the platina, and much more the copper wire of $\frac{1}{37}$ inch, would have conveyed it without loss; yet I found that, even the larger wire of $\frac{1}{20}$ was not sufficient, but that the deviation was still augmented by employing wire of $\frac{1}{12}$. This seemed the limit, for the effect was not increased by using two such wires at the same time; but a silver wire of half the diameter was, in consequence of its superior conducting power, at least equally efficacious, (Table X.) These experiments are confessedly given as mere approximations, which, hereafter, I hope to rectify, by the aid of a more powerful and accurate apparatus.

The new subjects and modes of experiment arising from this hitherto untried department of science, and the light it promises to throw upon all enquiries connected with heat and electricity, have, I fear, led me to encroach too much upon the indulgence of those who may not have the same inducement as myself to excite their attention: I shall therefore conclude, with one quære, which suggested itself to me whilst writing this Paper, and which, whether true or false, seems to have at least as much plausibility, as some theories that have been recently advanced upon this subject. Magnetism, and that to a considerable extent, it appears is excited by the unequal distribution of heat amongst metallic, and possibly amongst other bodies. Is it improbable that the diurnal variation of the needle, which follows the course of the sun, and therefore seems to depend upon heat, may result from the metals and other substances which compose the surface of the earth, being *unequally* heated, and consequently suffering a change in their magnetic influence?

TABLE I.

*A List of Subtances heated at one extremity, in contact with Copper Wires; the Wires being small in comparison with the Substance examined, excepting in the cases marked *.*

Positive.	Negative.
Bismuth.	Antimony.
Mercury.	Iridium and Osmium*.
Nickel.	Rhodium.
Platina.	Gold.
Palladium.	Zinc.
Cobalt.	Iron.
Silver.	Arsenic.
Tin.	1 Bismuth + 1 Zinc†.
Lead.	1 Bismuth + 1 Tin.
Copper.	1 Zinc + 1 Tin.
Brass.	1 Zinc + 1 Lead.
1 Nickel + 1 Iron*.	4 Zinc + 1 Antimony.
1 Tin + $\frac{1}{4}$ Antimony.	1 Nickel + 1 Palladium*.
Solder (common).	1 Nickel + 2 Platina*.
Pewter.	Printer's Type.
Galena §.	Fusible Metal.
	1 Ditto + 1 Arsenic.
	Zinc + Tin + Copper ‡.
	Sulphuret of Antimony §.
	Plumbago.
	Charcoal.

* None of these specimens weighed more than $\frac{1}{2}$ a grain.

† Not an alloy but a mixture, yet was negative whether the heated part appeared to be zinc or bismuth.

A magnetic compound, capable of polarity, composed of copper two atoms, zinc and tin each one atom.—From Mr. G. Spillsbury.

§ The sulphurets of lead and antimony alone were examined, probably other sulphurets, phosphurets, &c. would give similar results.

TABLE II.

Deviations observed with the small Compass neutralized.

Metal.	Wire.	Dev.	Heat.
Platina Rod, 18 inches long, $\frac{1}{5}$ diam.	Silver.	70 pos.	Spirit lamp.
	Brass.	60 pos.	Ditto.
Bar 7 inches by $\frac{1}{2}$, weight 565 grains.	Silver.	70 pos.	Ditto.
	Copper.	70 pos.	Ditto.
Foil 30 grains.	Silver.	70 pos.	Ditto.
	Platina.	4 neg.	Ditto.
Palladium 35 grains.	Silver.	70 pos.	Spirit lamp.
	Platina.	20 neg.	Ditto.
Silver Bar 7 inches by $\frac{1}{2}$, wt. 300 grs.	Platina.	50 neg.	Spirit lamp.
	Silver.	35 pos.	Ditto.
Silver wire $\frac{1}{34}$.	Copper $\frac{1}{10}$.	10 neg.	Ditto.
Ditto $\frac{3}{40}$.	Ditto $\frac{1}{40}$.	10 pos.	Ditto.
Mercury in glass tube, 6 inches by $\frac{1}{2}$.	Copper.	16 pos.	170°
		12 pos.	150
		6 pos.	115
Brass bar*, 17 inches by $\frac{1}{2}$ and $\frac{1}{8}$.	Silver.	60 pos.	Spirit lamp.
	Silver and Platina, Silver in contact.	40 pos.	Ditto.
	Ditto Platina in contact.	20 neg.	Ditto.
	Ditto Platina of $\frac{1}{2}$ inch in contact.	35 pos.	Ditto.
	Platina.	20 neg.	Ditto.
	Copper.	35 pos.	Ditto.
	Zinc.	65 pos.	Ditto.
	Brass.	45 pos.	Ditto.

Metal †.	Wire.	Dev.	Heat.
Copper.	Copper.	16 pos.	Spirit lamp.
	Platina.	18 neg.	Ditto.
	Silver.	40 pos.	Ditto.
	Zinc.	30 pos.	Ditto.
Zinc.	Zinc.	4 pos.	Spirit lamp.
	Iron.	6 pos.	Ditto.
	Silver.	4 pos.	Ditto.
	Copper.	45 neg.	Ditto.
	Platina.	50 neg.	Ditto.
Tin.	Copper.	10 pos.	Solder melted.
	Zinc.	27 pos.	Ditto.
Lead.	Copper.	25 pos.	Solder melted.
Iron.	Copper.	45 neg.	Spirit lamp.
	Iron.	0——	2 Ditto.
	Brass.	40 neg.	Solder melted.
Nickel impure.	Copper.	35 pos.	Solder melted.
Pewter.	Copper.	25 pos.	melted.
Plumber's solder.	Ditto.	24 pos.	Ditto.
Printer's type.	Ditto.	32 neg.	Spirit lamp.
Fusible metal.	Ditto.	5 neg.	melted.
Ditto + Arsenic ‡.	Ditto.	20 neg.	Ditto.
1 do. + 1 Type metal.	Ditto.	60 neg.	Ditto.
1 Bismuth + 1 Tin.	Ditto.	60 neg.	Solder melted.
Antimony + Tin ‡.	Ditto.	8 pos.	Ditto.
1 Zinc + 1 Tin.	Ditto.	22 neg.	Ditto.
1 Zinc + 1 Lead.	Ditto.	40 neg.	Ditto.
Zinc + Antimony ‡.	Ditto.	50 neg.	Ditto.
Copper + Zinc + Tin, magnetic.	Ditto.	10 pos.	Ditto.

* The deviation was not affected by heating either the middle of the bar, or of the silver wire by another spirit lamp at the same time.

† The metals in this column were square bars 6 inches long by $\frac{1}{2}$ inch.

‡ Proportions uncertain.

TABLE III.

With small Compass.

Temperatures.	A	B	B 1	B 2	B 2*
180	35	65	—	—	68
160	30	60	48	50	61
130	25	50	40	39	52
120	22	42	35	28	47
105	19	32	27	23	33
	Deviations.				

A and B, cylinders of antimony and bismuth, each 9 inches long by ¾. B 1, plate of bismuth 4¾ inches by 1 inch and ¼. B 2, cylinder of bismuth 2½ inches by 2. B 2*, the same with four connecting wires. The cold extremities in water at 60°.

TABLE IV.

Battery of Antimony and Bismuth with the large Compass.

Temperatures.	8 plates.	Ditto Small Compass.	6 plates.	4 plates.	2 plates.	1 plate, Bismuth.	1 plate, Antimony.
175	17°	—	—	—	9½	7¾	6¼
170	15½	—	13¼	—	—	—	—
160	14½	64	11¾	—	7¾	6	5
150	13¼	—	10¾	—	—	—	—
140	12¼	—	10½	10¼	—	—	—
135	11¼	—	9¾	9½	7¼	5	4¼
130	10¼	51	9	9	6¾	—	—
125	9½	47	8½	—	6¼	—	—
120	9¼	—	—	8	—	—	—
115	8¾	—	7¾	7½	5¾	4½	3¾
110	7¾	43	—	6¾	—	—	—
100	7¼	—	6¼	5¾	4¼	2¾	2
95	6½	35	—	5	—	—	—
90	6	—	—	—	—	—	—
	Deviations.						

The cold extremities of the plates in water at 57°.

TABLE V.

Battery of Bismuth, with the large Compass.

	6 plates.	plate 6.	plate 5.	plate 4.	plate 3.	plate 2.	plate 1.	2+5.	5+6.	4+5.	2+3.	2+4+5+6.	2+3+5.
140	$8\frac{1}{4}$	7	$9\frac{1}{2}$	$7\frac{1}{2}$	$3\frac{1}{2}$	$8\frac{1}{2}$	5	—	—	—	—	—	—
130	$7\frac{1}{4}$	—	—	—	—	—	—	$9\frac{1}{4}$	—	—	—	$8\frac{3}{4}$	$7\frac{3}{4}$
116	6	—	$7\frac{1}{2}$	—	—	7	—	$8\frac{1}{2}$	—	—	—	—	—
108	$5\frac{1}{4}$	4	$6\frac{1}{2}$	5	$2\frac{1}{2}$	$6\frac{1}{2}$	$3\frac{1}{2}$	—	6	$6\frac{1}{2}$	5	—	—
Temperatures.	Deviations.												

The cold extremities in water at 60°.

TABLE VI.

Alloy of Bismuth and Antimony, with the small Compass.

	Copper Wire.	Silver.	Copper and Silver.
Melting.	0°	—	—
—	25 pos.	—	—
280	1 pos.	6 pos.	—
260	0	5 pos.	—
250	5 neg	2 pos.	0
235	7 neg.	0	—
200	27 neg.	16 neg.	25 neg.
190	32 neg.	—	—
160	36 neg.	27 neg.	37 neg.
140	32 neg.	26 neg.	36 neg.
105	24 neg.	20 neg.	28 neg.
60	0	0	0
Temperatures.	Deviations.		

The colder extremity in water at 60°.

TABLE VII.

Deviations of Cylindrical rods of Antimony and Bismuth, each $4\frac{1}{2}$ inches long by $\frac{1}{2}$ inch diameter, taken separately and jointly, with the large Compass.

Temperatures.	B	A	A + B
240	14	10	$16\frac{1}{4}$
220	$12\frac{3}{4}$	$9\frac{1}{2}$	$14\frac{3}{4}$
210	—	9	14
200	11	$8\frac{1}{2}$	$13\frac{1}{4}$
190	—	8	$12\frac{1}{4}$
180	$9\frac{1}{4}$	$7\frac{3}{4}$	$11\frac{3}{4}$
170	—	$7\frac{1}{4}$	$11\frac{1}{4}$
160	$8\frac{1}{2}$	$6\frac{3}{4}$	$10\frac{1}{2}$
150	$7\frac{1}{4}$	$6\frac{1}{4}$	$9\frac{1}{2}$
140	7	$5\frac{3}{4}$	$8\frac{3}{4}$
130	6	5	8
120	5	$4\frac{1}{4}$	7
110	$4\frac{1}{2}$	$3\frac{3}{4}$	6
100	3	$2\frac{1}{2}$	$4\frac{3}{4}$
90	$2\frac{1}{4}$	$1\frac{1}{2}$	$3\frac{1}{2}$

Deviations.

The greatest deviation of Bismuth beginning to melt......21°.
Ditto of Antimony with two Spirit lamps ....19.
The bar of Platina, Ditto Silver, } (Table II.) red heat .... { 19. 5

The colder extremities in water at 65°.

TABLE VIII.

Relative Series of Electro-magnetics by Heat.

	Bismuth.	Nickel.	Platina.	Palladium.	Silver.	Tin.	Lead.	Brass.	Copper.	Gold.	Zinc.	Plumbago.	Iron.	Antimony.
Bismuth...	0	+	+	+	+	+	+	+	+	+	+	+	+	+
Mercury...			+		+				+	+	+		+	+
Nickel....	−	0	+	+	+	+	+	+	+	+	+	+	+	+
Platina....	−	−	0	+	+	+	+	+	+	+	+	+	+	+
Palladium..	−	−	−	0	+	+	+	+	+	+	+	+	+	+
Cobalt.....			−		+				+	+			+	
Silver.....	−	−	−	−	0	+	+	+	+	+	+	+	+	+
Tin......	−	−	−	−	−	0	+	+	+	+	+	+	+	+
Lead......	−	−	−	−	−	−	0	+	+	+	+	+	+	+
Rhodium...	−		−	−	−				−	+			+	+
Brass.....	−	−	−	−	−	−	−	0	+	+	+	+	+	+
Copper....	−	−	−	−	−	−	−	−	0	+	+	+	+	+
Gold.....	−	−	−	−	−	−	−	−	−	0	+	+	+	+
Zinc	−	−	−	−	−	−	−	−	−	−	0	+	+	+
Charcoal ..	−	−	−						−				+	+
Plumbago..	−	−	−	−	−	−	−	−	−	−	−	0	+	+
Iron	−	−	−	−	−	−	−	−	−	−	−	−	0	+
Arsenic....	−	−	−	−	−	−	−	−	−	−	−		−	+
Antimony..	−	−	−	−	−	−	−	−	−	−	−	−	−	0

TABLE IX.

Comparative Series.

Series of Electro-magnetics, by Heat (*a*).	Voltaic Series (*b*).	Series of Conductors	
		of Electricity (*c*).	of Heat (*d*).
Bismuth			
Mercury.	Charcoal.	Silver.	Silver.
Nickel.			
Platina.	Platina.	Copper.	Gold.
Palladium.	Gold.	Lead.	Tin.
Cobalt.	Silver.	Gold.	Copper.
Manganese.	Antimony *.	Brass.	Platina.
Silver.	Copper.	Zinc.	Iron.
Tin.	Lead.	Tin.	Lead.
Lead.	Tin.	Platina.	
Rhodium.	Iron.	Palladium.	
Brass.	Bismuth *.	Iron.	
Copper.	Zinc.		
Gold.			
Zinc.			
Charcoal.			
Plumbago.			
Iron.			
Arsenic.			
Antimony.			

[a] From Table VIII.
[b] Sir H. Davy, Elements of Chemistry: (*) being inserted from experiment.
[c] From Table X. and Sir H. Davy, Phil. Trans. 1821.
[d] Thomson's Chemistry. Vol. I.

TABLE X.

Conducting powers of Copper Wires of different diameters. Double bar of Antimony and Bismuth with the large Compass; the colder extremity in Water at 65°.

Temperature.	Diameters.						
	$\frac{1}{8}$ inch.	$\frac{1}{13}$	$\frac{1}{23}$	$\frac{1}{40}$	$\frac{1}{69}$	$\frac{1}{130}$	
230	16.	14.5	13.	8.5	6.5	2°.	Length of wires $16\frac{1}{2}$ inches.
175	12.25	10.8	9.8	6.3	4.9	1.5	
160	10.3	9.25	8.4	5.6	—	—	
215	15.8	15.6	15.3	14.3	13.7	8.7	Length of wires $1\frac{1}{2}$ inches.
	Deviations.						

When 33 inches of the wires of $\frac{1}{23}$ and $\frac{1}{40}$ were used, the observed deviations were 6°.4 and 3°; with $16\frac{1}{2}$ inches of the same wires, they were 6.4, and 4.27; and with $1\frac{1}{2}$ inches were 6.4, and 5.98; therefore, the difference of their conducting powers, which is inconsiderable with short wires, increases very rapidly with their length.

Hence it is obvious that wires which do not exceed $\frac{1}{50}$ inch in diameter, are improper in the construction of instruments for detecting minute Galvanic action; since their loss of conducting power may be such, as to counterbalance any advantage arising from the multiplying action of the spiral.

$$\left.\begin{array}{l}\text{Six wires of } \tfrac{1}{40}\\ \text{Four of .. } \tfrac{1}{130}\\ \text{Three of ... } \tfrac{1}{40}\end{array}\right\}\text{ gave the same deviations as one of }\left\{\begin{array}{l}\left.\begin{array}{l}\tfrac{1}{23}\\ \tfrac{1}{60}\end{array}\right\}\text{ each } 16\tfrac{1}{2}\text{ inches long.}\\ \tfrac{1}{23}\text{ each } 1\tfrac{1}{2}\text{ inches long.}\end{array}\right.$$

TABLE XI.

Conducting powers of different Metals, with the same Bar and Compass.

Temperature.	Silver.	Copper.	Gold.	Zinc.	Brass.	Platina.	Iron.	
220	13.	12.	—	9.	8.75	7.75	7.	Diameters of the wires $\frac{1}{23}$ inch. Length $16\frac{1}{2}$ inches.
195	12.	10.5	—	—	—	—	—	
170	10.	8.9	—	—	—	—	—	
160	9.9	9.	7.65	7.4	7.4	6.4	5.6	
140	7.9	7.2	—	—	—	—	—	
180	12.44	12.1	11.8	11.45	11.3	9.9	9.7	Diameters $\frac{1}{23}$ inch. Length $1\frac{1}{2}$ inches.
	Deviations.							

Hence the order of conductors is Silver, Copper, Gold, Zinc, Brass, Platina, Iron.

TABLE XII.

Deviations by equal differences of Heat, at different Temperatures.

Temp. of extremities.		Diff. of Temp.	Deviations.	Deviations.	Diff. of Temp.	Temp. of extremities.	
Hot.	Cold.					Cold.	Hot.
190	110	80	10.7	9.4	80	67	147
180	107	73	10.	9.	73		140
170	104	66	9.3	8.6	66		133
153	100	53	7.8	7.27	53		120
140	99	41	7.3	5.8	41		108

The deviations produced by given differences of heat are therefore increased by raising the temperature of the whole bar.

APPENDIX.

SINCE this paper was read to the Society, it occurred to me, that, as the juxta-position of two particles of the same metal at different temperatures, was the sole condition requisite for eliciting electro-magnetism, it might be exhibited by the minutest metallic specimens. Portions of bismuth and antimony, each weighing one grain, were therefore placed on a silver disk connected with the galvanoscope; on touching the upper surfaces of each separately, with one end of a heated silver wire, the other extremity of which was placed in the other cup of the galvanoscope, the needle deviated through 90°, positive and negative respectively. By this method I was enabled to examine the compound ore of iridium and osmium, of which the largest specimen did not exceed $\frac{1}{10}$ of a grain, and to verify, in a few minutes, results, for which the laborious process of casting bars of the different metals had been previously requisite*.

As the effect of the electro-magnetism developed by heat, is perfectly analogous to that caused by galvanic excitation, in its tendency to produce the rotation of a magnetic bar, it is evident, that, if the magnet be fixed and the apparatus at liberty to revolve, it will, in like manner, exhibit the *converse* experiment. The instruments formed of Platina and Silver wires, which are represented by Fig. 12 and 13, were constructed for this

* If a heated metallic wire be applied to a plate of the *same* metal, there is in all cases a deviation; the nature of which seems to depend upon some peculiar property in the metal itself. Copper, Zinc, Bismuth, for instance, are positive; Platina, Silver, Iron, Brass, Antimony, are negative.

purpose. Platina and iron are obviously unsuitable; and taking into consideration the fact of small wires being far more energetic than larger in proportion to their bulk, I believe no other arrangement of metals would be found so efficacious.

The wire parallelogram *AbcB*, Fig. 12, deviates from right to left, or *vice versa*, accordingly as the north or south poles of the magnet are presented to it; its weight, together with that of the magnet, is rather less than seven grains. The apparatus, Fig. 13*, when heated by a Spirit lamp, exhibits a perpetual rotation, and is analogous to the instrument invented by Ampìre for producing the same effect by galvanic excitation.

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* A small wire is soldered at *d* in the opposite direction to *B, D, C* to preserve the equilibrium: notice of this was accidentally omitted in the plate.

ADDITIONS AND CORRECTIONS

To the preceding Paper.

Page 14. after line 8. *add,* If one part of antimony and six of bismuth, both in powder, be mixed together and inclosed in a glass tube, they exhibit the same phenomena as the alloy.

P. 17. lines 5, 6. *for* positive, negative; *read* negative, positive.

*Addition to Note * same page.* If gold, silver, copper, brass or zinc wires be heated in connection with iron, the deviation which is at first positive, becomes negative at a red heat. If the experiment be made by dipping wires not previously connected, in boiling mercury, the deviation at the instant of contact, depends, in some cases, upon the order in which they are immersed. This effect is produced by copper with gold, silver, zinc, brass and plumbago, but not with platina, tin or iron; by zinc with gold, silver, brass, iron and plumbago, but not with platina or tin; and by brass with gold, silver, and tin, but not with platina or iron.

Note † same page, for below, above; *read* above, below.

Table VIII. *dele* the vertical and horizontal columns marked silver, and in the columns for gold *read* silver.

In the horizontal column for rhodium, *insert* tin and brass negative, and copper positive.

Table IX. *add* galena above bismuth; after manganese *dele* silver; *insert* brass between lead and rhodium; *for* brass *read* gold, and *for* gold *read* silver.

P. 31. line 5. *for* deviates *read* revolves.

line 8. after exhibits *read* another arrangement for producing.

Fig 1

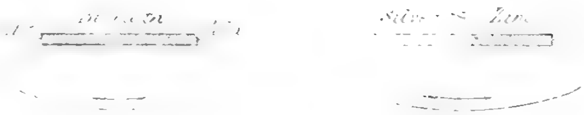


Fig 3

Fig 2

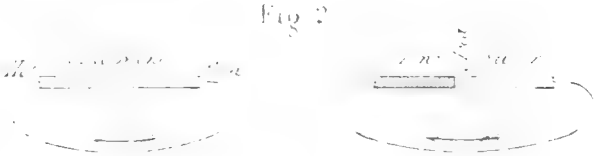


Fig 4

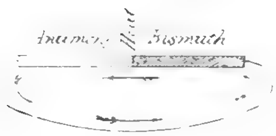


Fig 5.

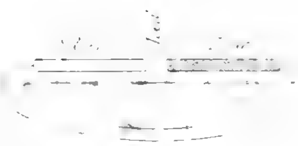


Fig 6

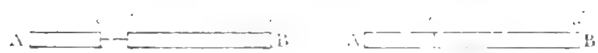


Fig 7

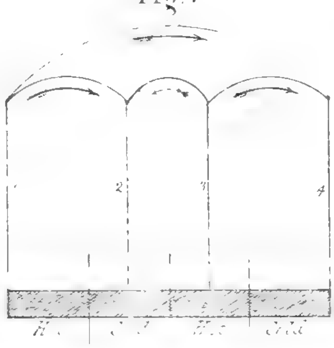


Fig 8

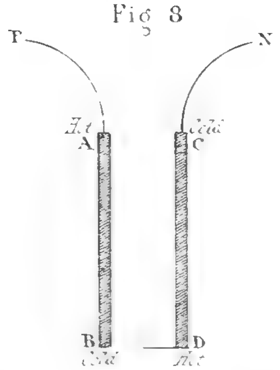


Fig 9

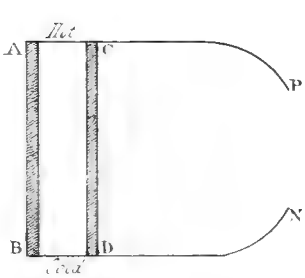


Fig 10

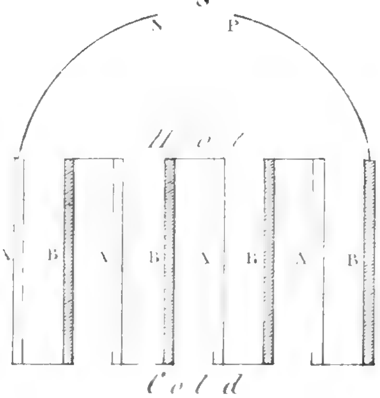


Fig 11

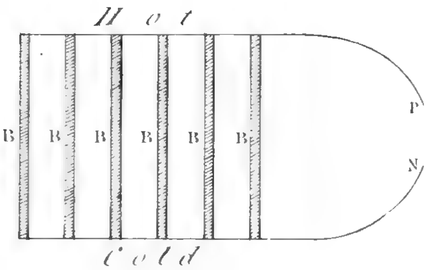
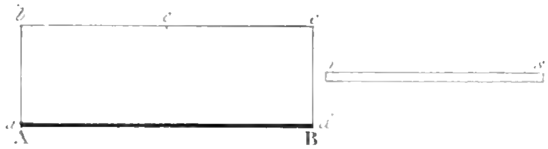
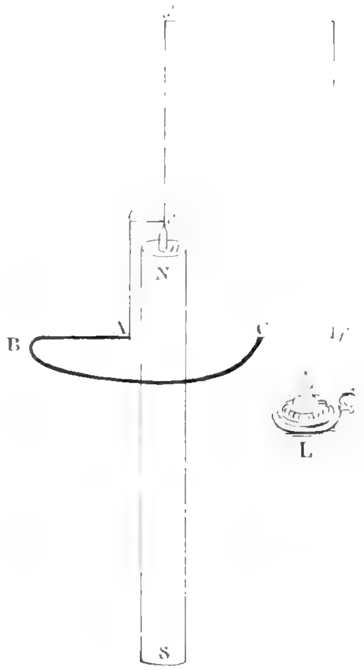


Fig 12



A B *Platina*
a b, c d Silver
e Support
n, s Magnet

Fig: 13.



A.B.C	*Platina*
A *b.c.d.e.f.*C..	*Silver*
c. n	*Support*
N.S	*Magnet*
L	*Lamp*

C. Hullmandel's Lithography.

VI. *Extract from a Memoir on a Peculiar Connexion which exists between the Magnetism evolved by a Single Galvanic Combination, and the relative Magnitude of the Opposing Surfaces of that Combination.*

By FRANCIS GYBBON SPILSBURY.

[Read *Nov.* 25, 1822.]

Anxious to repeat some of the experiments, detailed by Professor Cumming in the two Papers published on this subject in the Society's Transactions, (Vol. I.) we proceeded to construct an instrument similar to the galvanoscope* there described, though upon

* The term *galvanoscope* appears objectionable, as implying that magnetism may be made a measure of galvanism; a supposition unsupported by facts. The galvanic intensity of any combination has no relation to the magnetic. In a series of these combinations, magnetism is scarcely to be detected, yet is the galvanism intense. In a single combination composed of two unconnected plates, a quantity of magnetism is evolved, sufficient to fulfil all the phenomena of a powerful magnet; and it is on the first instant of immersion that this is most intense. On the opposite hand, galvanism is scarcely to be detected in such a combination, and not at all until the plates have been immersed some minutes. It has been indeed alleged that the cause of this seeming difference between a single and compound combination arises from the non-conductibility of a liquid to magnetism; yet we have discovered a combination composed of two plates only, from which galvanism is evolved, without magnetism. If a slice of medullary matter, and one of coagulated arterial blood, be placed on each other, and platina wires proceeding from each of these be immersed in dilute sulphuric acid, the wires will quickly give out gas; which will not cease, until the arterial blood is converted on its surface next the medullary matter into venous. By exposure to air the blood will become again oxygenated, and upon applying it again to the medullary matter, galvanism will again become apparent by the decomposition of the dilute acid. But, neither oxygenated nor disoxygenated, can any effect be perceived by connecting this apparatus with the galvanoscope.

a more delicate scale. Extraordinary as is the action evinced by the contact of an iron and steel wire, our curiosity was not a little increased, on finding that even two pieces of iron were capable of producing an equal, nay, a greater effect. As even the purest iron is, from the very nature of the methods employed to procure it, combined with a portion of carbon, the phenomenon in question might be attributed to its presence. If this supposition were correct, all metallic alloys would furnish similar results.

To decide this question, a wire of brass, $\frac{1}{10}$ inch diameter, was divided into two equal parts; one of these was connected with the one cup of the galvanoscope, and the other with the opposite cup. On making the connection between the two pieces of wire by nitric acid, the needle turned through a half circle.

The laws, by which this singular phenomenon was regulated, were not at first sight very apparent. That they were not arbitrary or accidental, appeared evident; because, as often as the experiment was repeated with the same pieces of wire, the wire which was in the first instance *positive*, continued so throughout, and *vice versa*. After some ineffectual attempts, we succeeded in developing them; that wire, which was the larger or exposed the greater surface to the action of the oxidating medium, was always *positive*; the smaller wire of course *negative**. Thus the same wire became alternately *positive* or *negative*, as a greater or less

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* The terms *positive* and *negative* are taken throughout this Paper in a sense contrary to the received one. By experiment it has been proved by Œrsted, and also Mole, (Edinburgh Journal, Vol. V. 352), and we have verified it, that in a battery, composed of two unconnected plates of copper and zinc, the latter is negative and the former positive. As these simple batteries decompose water, in a similar manner with those of a compound series (it is true in a much smaller degree), but reverse the phenomena, will it not be better to consider henceforward compound batteries merely in the light of an assemblage of single pairs, whose active surfaces are those which are opposed and unconnected: consequently that the hydrogen is evolved in either species from the zinc side or negative, and the oxygen from the copper or positive?

quantity of its surface was exposed to the action of the acid medium.

Results, however, were sometimes observed, which scarcely coincided with the foregoing law; yet the experiments, in which these anomalies were observed, did not appear to have been conducted in a manner, different from those, in which they were absent. This was more particularly the case, where the two surfaces of the wires under examination differed not greatly in magnitude. It was a considerable time before the cause of this anomaly was unravelled; fortunately it struck us, that the results might be materially affected by the *position* of the surfaces towards each other.

To verify this conjecture, two brass wires were hung so as to admit of being placed either parallel to, or across each other; remaining at the same time in contact with the galvanoscope. The solution of our difficulties was then immediately apparent. When the wires were parallel, or placed in any position, not crossing, the greater surface was always positive, as formerly observed: but if they were placed across each other, and, at the same time, nearly in contact at the point of intersection; then that wire which has the greater surface under oxidation at the point of contact, and above it, is *positive;* though it have the smaller total quantity of surface under action. This law has however a limit; when the difference of the two surfaces is very considerable, the larger total surface is positive, whether crossed or parallel.

The diagrams in Plate V. will render these laws clearer, where ***P*** signifies positive, ***N*** negative, the blank space representing the oxidating medium.

In Fig. 1, the wires α, β are parallel, the greater surface α is therefore positive; but in Fig. 2, though the wire β is less in total quantity of surface; yet as by its oblique position, there is a greater surface exposed to oxidation at the point of contact, and above it, it will become positive; the former positive wire being negative.

The next enquiry was whether the same phenomena were observable in other alloys. The first tried was pewter, which was regulated by similar laws; but the intensity of magnetism produced was considerably less, than when brass was employed. An alloy of silver and copper, called by the platers *coarse silver*, (a little finer than standard), had a much greater effect; standard silver had about the same power as the last; but in an alloy of zinc, copper, and silver, the intensity was very much increased. It is however worthy of remark, that no alloy, not even brass, was capable of exciting so intense an effect as iron. May not this be taken as a further argument in favor of carbon being the oxide of a metal, which has an affinity for oxygen surpassing all other metals, as Dr. Mac Culloch has attempted to prove in a late very able paper printed in the Edinburgh Journal?

All these alloys are regulated by the law of the larger surface being positive to the less. The following may be taken as the order of the intensity, in which magnetism is produced by each of those tried.

Iron wire (called binding.)
Brass.
Silver solder, (brass, copper and zinc.)
Coarse silver, (copper and silver.)
Standard silver.
Pewter, (tin, lead and antimony.)
Tin of commerce.
Copper of commerce.

From the foregoing list it will appear, that some effect was produced by the simple metals themselves, as met with in commerce. To ascertain, if this arose from accidental impurities combined with them, or from a property hitherto undiscovered, two pieces of pure platina foil were subjected to examination; but whatever was the difference of surface between the two plates (and in some experiments this was very considerable), or whether the acid was nitric or nitro-muriatic, assisted by heat, in neither case was the slightest action on the instrument to be detected. *Coarse silver* (which is finer than standard) gave evidence of magnetism, yet pure silver had no such effect. May we not then justly conclude, that, in those cases where magnetism is produced by two bars of the same metal, it arises from that metal containing other substances in combination with it? This conjecture is further supported by this fact; the copper of commerce is incapable of acting upon the galvanoscope, except when the surfaces under oxidation are very extensive, or at least when one surface very materially exceeds the other in extent. The grain-tin of commerce also requires a similar extension of surface; though not in so great a degree as copper. Upon chemical examination this tin was found to contain a minute portion of zinc and manganese, both metals highly electro-negative to tin. The copper was not examined; the most probable impurity would be antimony, arsenic, or lead, neither certainly very electro-negative metals. If this supposition be correct, the different degrees of intensity in the two may be easily accounted for.

Should future experiments confirm the truth of these views, we may probably have it in our power to construct an instrument, by which the relative quantities of two metals in any alloys shall be discovered; without having recourse to the more laborious method of humid analysis. The infinite service this would be in many departments of the arts, particularly in that of assaying, is evident. Also by submitting in turn each of the known metals, to the action of this test, very possibly we shall have it in our power to shew, as far as analogy can shew, the compound nature of some, which have hitherto been considered simple undecomposed bodies. It is scarcely possible, certainly improbable, that all should be simple: and the time is perhaps not very far distant, when our views in reference to these bodies will be very considerably modified.

To discover in what way the above phenomena were produced, two brass wires were placed in connexion with the galvanoscope, one containing a much greater extent of surface than the other; the fluid medium was common water. Of course no action was apparent. A few drops of nitric acid were then dropped on the less wire, which instantly shewed evidence of being in a positive state to the larger, continuing so until the acid became intimately mixed with the water, when it suddenly changed its state, and became during the remainder of the experiment negative. This proves that the larger wire acts by, in some way, attracting to it a greater proportion of acid.

Such are the facts which Professor Cumming's invaluable instrument has brought to light. In these experiments, it was always necessary to neutralize the earth's magnetism by acting on the needle in the way described by the late Abbé Hauy; it was also equally necessary to see, that the ends of the wires, by which they were connected with the galvanoscope, were coated with mercury; this is rather difficult with iron wires, but may be accomplished by immersing them first in a solution of sulphate of copper, and afterwards in one of nitrate of mercury. As the galvanoscope employed by us was from its construction much more delicate than the one described by Professor Cumming, a figure of it is annexed, (Fig. 3).

A, A, A, A is the section of an ivory box containing the spiral parallelopipedon of wire, *a, a, a, a*; on the top it is glazed, that the motion of the needle may be observed. ***B, B*** is a tube also of ivory, containing a screw *C,* which moves easily but without shake; to the point ***D*** of this screw is cemented a fibre of silk drawn from the silk-worm pod, to which is attached the magnetic needle $\epsilon, \epsilon, \epsilon$. *E* is an ivory gradulated table supported by the pedestal *G*. *F* is one of the connecting cups containing mercury. The intention of the screw *C* is to regulate the height of the needle $\epsilon, \epsilon, \epsilon$, which otherwise might be difficult to accomplish. The mode of using this instrument is of course similar to that described by Professor Cumming.

Fig 1

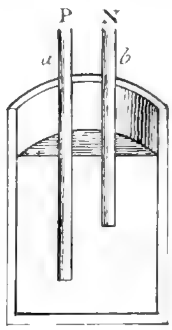


Fig 2

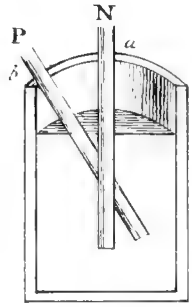
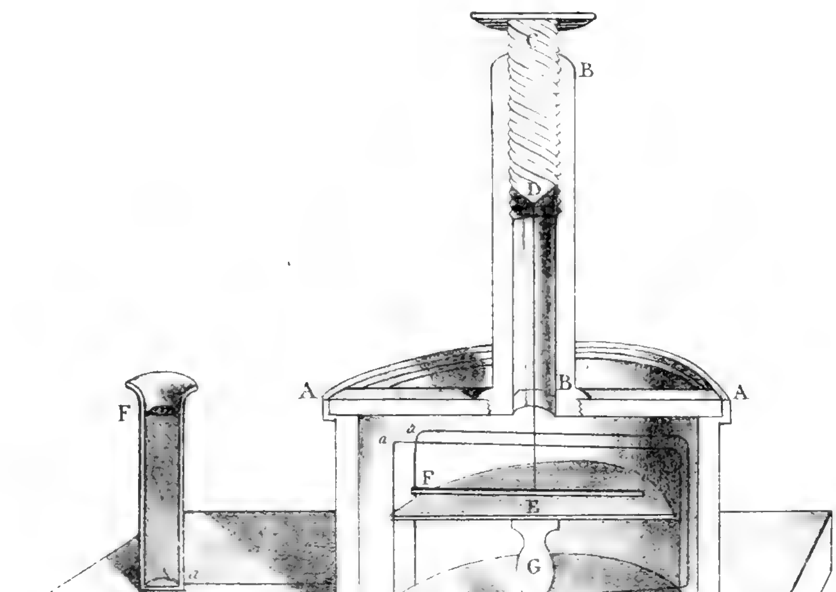


Fig: 3.



VII. *On an Apparatus for Grinding Telescopic Mirrors and Object Lenses.*

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[Read *Dec.* 11, 1822.]

The grinding of telescopic mirrors by machinery has been considered nearly a hopeless attempt; partly from the degree of accuracy which is required, and partly from the supposed necessity of a parabolic figure. The condition of accuracy is indeed indispensable. If a reflecting telescope be not superior to the best achromatic refracting telescope that can be made, it may be considered, for the purposes of science, nearly useless. The inferior sort of reflecting telescopes are adopted chiefly on account of the great ease with which they may be constructed*.

* One of the easiest telescopes to construct is that of Sir Isaac Newton, which requires very little accuracy except in the figure of the object mirror. The figure of the small plane mirror is quite indifferent for *distinctness* if only the polish be good: For if the image be thrown nearly on the surface of the small mirror, the foci of incident and reflected rays will coincide at the surface; and the rays will diverge accurately from the several points of the image, whatever be the form of the reflecting surface. To prevent *distortion* of the image, it is indeed necessary that the small mirror should be nearly plane; but the *distinctness* of the several parts of the image will depend only on the polish. Yet, as the best polish must be imperfect, when its defects are magnified by the eye-piece, it is usual to throw the image at some short distance from the small mirror.

If spherical surfaces only be employed, the extreme accuracy required does not exclude the use of machinery, but rather calls for the aid of machinery, provided that we are able to imitate or improve upon the methods used in grinding mirrors by the hand. With respect to the spherical aberrations, it has been proved in a paper lately read to the Society, that they may be effectually corrected by a combination of spherical surfaces affording two refractions and two reflections: and even where the parabolic figure is supposed necessary, as in the Newtonian telescope, (there being only one surface employed in forming the first image,) yet this figure is not attempted except in the last stage of the process, while the mirror is being polished, and after a spherical surface has been first obtained. To obtain only the spherical form is a very unpleasant and laborious operation, sufficient to discourage any but professed mechanics. To relieve this part of the work is the design of the machine about to be described: and the mirror may be entirely finished on such a machine, if it be thought that a correct sphere with a moderate aperture is preferable to a paraboloid formed by methods purely tentative and very uncertain. An experiment was made, some years ago, with a machine of this kind, moved by a small steam engine; and the result was favourable. The figure of the mirror appeared correct, and would take high magnifying powers, with as much distinctness as could be expected from the quality of the metal.

The machines employed for grinding lenses with a short focus will not at all assist us in grinding reflectors and object glasses. The greater degree of accuracy necessary to the latter, requires a totally different method of grinding. A brief description of this method will lead to an explanation of the machine by which the operation may be performed.

We have first to procure two gages of plate brass, one convex and the other concave, and adapted to the intended curvature of the mirror. With these gages are formed a wooden model, if required, for casting the mirror; and also a pair of spherical tools, made of pewter, consisting of five parts of lead and one of tin; the convex tool having the larger surface. The proportion of width usually adopted is that of ten to eight and a half; but this depends partly on the length of the stroke which the mirror is intended to perform; and on some other data which cannot be estimated in theory. The concave leaden tool is similar to the mirror, and may be cast in the same mould. But for grinding *lenses,* it is the opinion of the workmen that this tool should be made of brass. The composition of speculum metal is of copper and tin, mixed in the proportion of fourteen and a half ounces of grain tin to two pounds of copper, with a small addition of arsenic. This produces a very brilliant metal, but is said to be sometimes porous. This inconvenience is remedied by first pouring the metal into an ingot. Other proportions of tin are given by different authors; to which are sometimes added brass and silver in small quantities. The copper must be first melted by itself, and the tin poured upon it in a state of fusion; a flux may then be added, and the whole well stirred with a wooden stick. The last portions of tin are added gradually; and between each addition a few drops of the metal are cooled, to observe the brilliancy by fracture, which is called *tasting* the metal.

The extreme brittleness of this composition makes the casting very difficult and hazardous. Of several mirrors cast successively, with every attention to the printed directions, scarcely any were found perfect enough to repay the trouble of grinding. The difficulty of casting and annealing may however be greatly lessened by pouring the metal, carefully skimmed, into a mould of cast iron, heated below redness. The mould may be heated over a small stove, and continue, in the same position, gradually cooling for about twelve hours. The mould is formed of two iron plates of proper curvature, (the concave plate having a hole in the centre), and an iron ring, all accurately turned. The mirror may be cast in a horizontal position, with its face downwards. Some of the best workmen cast mirrors vertically, in an iron mould, having a circuitous git entering at the bottom. In this case great caution is necessary to confine the melted metal from escaping through the joints of the mould.

The gages are first to be made perfect by rubbing them length-ways, one upon the other, with fine emery: whereby the prominent parts, undergoing most friction, will be worn away, and the receding parts will advance, till the contact becomes uninterrupted in all positions, that is, till both gages become truly circular with the same curvature.

The leaden tools also may be fitted more perfectly to the gages in a common lathe; after which, the grinding may commence, by rubbing the concave leaden tool upon the convex tool, with a rectilineal motion across the centres. Considering the whole spherical surface as made up of parallel circular arcs, each arc will thus be rubbed length-ways, upon a similar arc, great circles upon great circles, and small circles upon small; so that each arc will become truly circular, as in the gages: and, by changing the motion in every possible direction, the whole surface will become truly spherical. The concave leaden tool and the mirror are ground alternately upon the convex surface of the larger tool, till all three surfaces become spherical with the same curvature.

The substances employed for grinding upon lead, are first sand, when the surface requires to be much reduced; then emery of different degrees of fineness; the coarsest kind used being superfine corn emery; then a spherical surface of hone or Turkey stone; which is intersected with several trenches, at right angles to each other, to receive particles of sand, emery, and other adventitious substances; and lastly, are employed, the oxides of iron or tin (colcathar or putty) upon a surface of common pitch purified by straining.

This is a brief outline* of the process for grinding a spherical mirror; we will now consider the mechanical conditions necessary to render this process as accurate as possible.

The first condition is a due proportion between the surfaces of the two leaden tools. If these surfaces be made equal, it is observed in practice, that the radius of the sphere formed by grinding will continually diminish: and if the convex surface be much larger than the concave, it is manifest that the radius of the sphere must continually increase; till at length the curvature will become inverted. There is therefore a certain proportion between the surfaces, which will keep the curvature invariable; the length of the stroke being also given. This proportion is not very easily determined by theory, but for practical purposes may be taken as above stated; that is, ten for the diameter of the lower or convex tool, and eight and a half for the diameter of the concave. Still the curvature may be made to increase, or to decrease, or to remain constant, merely by adjusting the length of the stroke. This circumstance furnishes a simple but important means for adjusting the focal length, which will be considered afterwards. It also points out the necessity of proper limits to the surfaces of the leaden tools.

A considerable vertical pressure must be applied to the mirror, at least till the polishing commences; and this pressure, not attended with the inertia of a moving weight, must take place wholly at the centre of the mirror, so as to be equally effective on all parts of the surface.

* For more exact particulars of casting and polishing metallic specula, see Smith's Optics, Newton's Optics, also Mudge's Paper in the Philosophical Transactions for the year 1777.

The mirror must be carried backwards and forwards across the centre of the leaden tool with an equal length of stroke. The force which produces the rectilineal motion should be applied horizontally, in a plane nearly coinciding with the surface of the mirror, so that the friction on the surface and the force which overcomes it may destroy each other, both being nearly in the same plane.

During the whole process, both the mirror and the leaden tool should be made to revolve uniformly, but very slowly, about their own axes, in opposite directions; that all parts may be ground equally. It is also desirable that the angular velocity of the mirror and of the leaden tool should not be the same, but one double of the other, and in opposite directions. For if the mirror be somewhat elliptical, having its opposite diameters of different curvature, the leaden tool being also similarly affected, by this contrivance the diameters which are least adapted to each other will be ground together, and will approximate rapidly to an equal curvature: this would never be the case if the angular velocities were equal*. Opposite motions are preferable as producing the greatest relative motion with the least absolute; that is, with the least friction and inertia. It is also very convenient that the mirror should be capable of being easily taken up, to be cleaned or examined;

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* Fig. 1. Let *AB, CD,* be the major and minor axes of the leaden tool.
— *ab, cd* ———————————— mirror.
MN the direction of the stroke.

The upper figure may represent the position which results from grinding; where the major axis of the mirror coincides with the major axis of the leaden tool, and both with the direction of the stroke.

First, let the angular velocities be equal and opposite; and let *AB, ab,* revolve 90°. Then the second figure represents the new situation, in which the axes minores coincide, and rub length-ways upon each other; without any tendency to correct the elliptical form. But if the inner wheel revolve twice as fast as the other, the lowest figure represents the new situation; in which the axis major of the small wheel, (with its extremities inverted), rubs length-ways upon the axis minor of the large wheel; whereby they will speedily be reduced to a mean and common curvature.

and be restored to its place in the same relative situation, as before.

All these conditions may be easily accomplished by machinery, and perhaps with far greater accuracy than by the hand of a workman. The convex leaden tool is fixed into the middle of a wooden wheel, (*abc*, Fig. 2,) having a groove in its circumference. To the under side of the wheel is annexed a circular iron plate, of less diameter, confined horizontally by three knobs, upon which the wheel rests. The knobs, (*a*, *b*, *c*,) fixed in position, are screwed to a board; one only being moveable for the purpose of adjustment. Fig. 3, is a vertical section of the wooden wheel, (*a*, *a*), with the leaden tool, (*b*, *b*), and the circular iron plate, (*c*, *c*).

The mirror and the concave leaden tool are each inserted in a circular box, having likewise a groove in the upper part of its circumference; the lower part being smooth and flat. Fig. 4, is a vertical section of the box (*a*, *a*), containing the mirror or concave leaden tool (*b*, *b*) One of these boxes, containing the mirror or leaden tool, is placed (Fig. 2), on the convex leaden tool, and between two thin laths inclined at a small angle, and united into a triangle by a cross bar of iron. The smallest angle (*d*), of the triangle (*d*, *e*, *f*,) is made the centre of motion, and the opposite iron side is produced both ways to pass through two fixed holes (*g*, *h*), confining the triangle to move in its own horizontal plane. The sides of the triangle press only horizontally on the lower parts of the box where it is smooth and flat, (i. e. cylindrical). The motion is communicated to the triangular frame by a rod (*u v*), connected with a crank and fly wheel. The velocity of the mirror, moved in this manner, is greatest in passing the centre, and increases gradually from the stationary points: such a motion is peculiarly adapted for steadiness and equality of stroke; and the momentum of the fly wheel prevents all sticking and jerking. But it is necessary continually to take up and examine the mirror during the polishing, and to wipe off whatever is found adhering to its surface.

The centre of the mirror is confined to move in a given horizontal line, or small circular arc, by a lever (k, l, m), having at one extremity (k) the same centre of motion as the triangle, the other extremity (m) resting, by an iron point, upon a brass socket in the centre of the box containing the mirror. Any degree of pressure may be produced on the centre of the mirror, without the inertia of a moving weight, by extending a string from any point (l) of the lever (k, l, m), to the extremity (n) of the lever (n, o), moveable about the point (o), and bearing a weight (W).

When it is wanted to remove or examine the mirror, the lever (k, l, m), which sustains the weight (W) may be raised by the point (m) and laid on one side; the mirror may then be taken up, or turned upon its back: the string remains in the groove as before, and is kept stretched by the moveable weight (W'). An alteration has been suggested with respect to this lever which may seem an improvement, though in practice it is less expeditious for exchanging the mirror. The extremity (m) of the lever (k, m), is attached, by a joint, to a small iron upright, fixed into the middle of the iron bar (g, h). The other extremity (k) is left at liberty, and bears a small weight; which presses, at a mechanical advantage, on the centre of the mirror. This weight, being near the centre of motion remains nearly stationary while the mirror is moved horizontally, and therefore takes effect by pressure without inertia. The vertical pressure, in the former case, does not any way interfere with the triangular frame; but on this latter construction, there will be some increase of friction at the holes (g) and (h), by a constant pressure upwards.

The rotatory motion is communicated in the following manner. The point (p) of the triangular frame is connected, by the cross bar

(q, r), with the double lever (r, s), which moves about (s) in a vertical plane, perpendicular to the fixed axis (s, t). The double lever (r, s), has between its two sides a tongue of spring steel, which in one direction catches the cogs of the cogged wheel (s), and in the contrary motion passes over them. To prevent any retrograde motion of the cogged wheel, there is another steel tongue attached to the iron pillar (h).

By the oscillation of the lever (r, s), the wheel moves over one or more cogs for every stroke of the mirror; and by a worm-screw (t), with a series of wheels and axles*, turns a large wooden pully ($\boldsymbol{P}$) very slowly, but with great mechanical advantage. A string passes round three-fourths of the circumference of this pully, and from the upper side of it passes under the moveable pully ($\boldsymbol{Q}$), over the fixed pully ($\boldsymbol{R}$), quite round the groove of the box containing the mirror, over the fixed pully ($\boldsymbol{S}$), quite round the groove of the large wheel containing the leaden tool, from whence it is gathered directly on to the first pully ($\boldsymbol{P}$). The weight (W') and moveable pully ($\boldsymbol{Q}$) are to produce friction on the first pully ($\boldsymbol{P}$); and to gather up the loose string as fast as it is delivered off from that pully. The same string passing over both wheels, causes the mirror and the leaden tool to move round slowly, in opposite directions, with angular velocities inversely proportioned to the diameters of the wheels; that is, in the required proportion of two to one.

To prevent slipping over any of the pullies, the string should be rubbed with resin: also any degree of friction may be produced on the first pully by increasing the weight attached to the moveable pully ($\boldsymbol{Q}$): or the first pully may be a solid block with several grooves of equal radius; the string being made to pass several

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* Nothing can answer better for this part of the machine than a common kitchen jack; and as the lever is applied to the worm-screw, and acts at a great mechanical advantage, the most defective instrument will serve the purpose as well as the most perfect.

times round it; and, to prevent confusion, to pass also as many times, wanting one, round a small beaded roller fixed in position. But the first construction is quite sufficient in practice to prevent the string from slipping, all grooves being made in the form of a V: the latter may be used when several mirrors are to be ground at once on the same machine.

Another objection to this manner of producing the circular motion by a string, may seem to arise from the greatness of the resistance, by which the string might seem liable to be broken; though the resistance, being overcome at a great mechanical advantage, is hardly sensible as opposed to the moving power. This inconvenience is not found to arise in practice: on the contrary it is remarkable how thin a string will sustain the work during the whole formation of a mirror. The reason seems to be, that the tension of the string, which causes circular motion, enters into composition with the larger force which produces the rectilinear stroke: and it is obvious that two forces applied to the same body in different directions, may both take effect when compounded, though either force would be ineffective by itself. Much more if one force, namely the rectilineal, be adequate to produce motion, the other, however small, will enter into composition with it, and will give an oblique motion to all points of the mirror except the centre.

The sufficiency of the string in point of strength, depends chiefly on the comparative velocities of the circular and rectilineal motions, which are extremely disproportionate. The lower wheel with the convex leaden tool, is moved round once every ten minutes; in which time the mirror revolves twice, and performs three or four thousand rectilineal strokes.

While the mirror is being ground, the curvature is to be frequently examined by the gages. If the focal length is very far from what is wanted, the convex leaden tool may be corrected with a file, and the grinding renewed till the marks of the file are erased. When the focal length is brought within a few inches of what is desired, it may be made perfect by a very simple process. It is observed in practice that the focal length will increase or decrease, or remain constant, according to the length of the stroke which the mirror performs. This depends wholly on the length of the crank, which may be adjusted at pleasure. A focal length of two feet may be varied several inches, in a short time, simply by grinding the leaden tools together in this manner*.

It is one advantage of grinding mirrors by machinery, that the same workman may grind several mirrors at one time: and this even upon the same machine, provided the mirrors be of different dimensions. Thus in grinding two mirrors, for a telescope of Gregory's or Cassegrain's construction, the smaller mirror may be ground at a proportional distance from the centre of motion, on the same machine with the larger mirror, and both will be ground in the same time and with the same accuracy. It is however the opinion of professional persons, that the rectilineal stroke is chiefly adapted for *object* mirrors, which have small curvature. The practice for grinding spherical mirrors of greater curvature is by a spiral stroke from the centre of the leaden tool towards the circumference: without much regard to any exact proportion of surfaces.

It is obvious that the same machine may be employed in forming object glasses for refracting telescopes. A piece of plane glass being cemented upon a block similar to the convex leaden tool, the edges of the glass are worn off by grinding with sand in a spherical basin of cast iron. It is then ground by the brass tool with emeries of different fineness, beginning with superfine corn

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* If the figure be impaired by this means, it may speedily be restored by adjusting the crank to produce constant curvature.

emery; and lastly, the surface is polished upon woollen cloth filled with putty. When the lens is finished on one side, it is separated by striking the block edgeways with a hammer: It is again cemented upon the block, and the other side finished as before. The detail of this operation is omitted for the sake of brevity.

As the parabolic curve is generally considered necessary for the object mirror of a reflecting telescope, we will now shew in what manner it may be attempted. The common method of grinding paraboloids is altogether tentative, and depends on the skill and experience of the workman. A perfect sphere being first obtained, it is ground away toward the centre, by a few circular strokes upon the polishing tool; till the centre and circumference are found to reflect parallel rays to the same point. It may then be supposed that the whole figure more nearly resembles a paraboloid than a sphere. But a method founded upon calculation, especially when improved by practice, may produce a more perfect figure without requiring any manual skill; and such a method may be adopted on the machine already described. If a paraboloid and a sphere be made to touch internally at the vertex of the paraboloid, where they have the same curvature, the thickness intercepted between them, measured parallel to the axis, varies, for small arcs, nearly as the fourth power of the distance from the vertex of the paraboloid. The same law holds also for the other conoids, affected only with different constant quantities*.

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* Let AV, AV' (Fig. 7,) be two conic sections, having the same axis AZ, and the same curvature at their common vertix (A); and let their equations be

$$y^2 = ax + mx^2; \qquad y'^2 = ax' + mx'^2;$$

in which (a), the latus rectum, is common to both, their curvatures at the vertex having been assumed equal.

From the first equation,

$$x = \frac{\sqrt{a^2 + 4my^2} - a}{2m}$$

It is possible therefore to form a paraboloid by a regular method, by taking first a perfect sphere and grinding it *toward*

$$= \frac{a}{2\,m} \cdot \left\{\left(1 + \frac{4\,m}{a^2} . y^2\right)^{\frac{1}{2}} - 1\right\}$$

$$= \frac{a}{2\,m} \cdot \left\{1 + \frac{2\,m}{a^2}\,y^2 - \frac{2\,m^2}{a^4} . y^4 + \frac{4\,m^3}{a^6} . y^6 - \&c. - 1\right\}$$

$$= \frac{y^2}{a} - \frac{m}{a^3} . y^4 + \frac{2\,m^3}{a^5} y^6 - \&c.;$$

similarly, $x' = \frac{y'^2}{a} - \frac{n\,y'^4}{a^3} + \frac{2\,n^3}{a^5} . y^6 - \&c.$

Let $PP' = z$; $\therefore (x + z)$ is the value of x' when $y' = y$;

$$\left.\begin{aligned} \therefore\ x + z &= \frac{y^2}{a} - \frac{n\,y^4}{a^3} + \frac{2\,n^3}{a^5}\,y^6 - \&c. \\ \text{and by first equation,}\quad x &= \frac{y^2}{a} - \frac{m\,y^4}{a^3} + \frac{2\,m^3}{a^5}\,y^6 - \&c. \end{aligned}\right\}$$

$$\therefore \text{ by subtraction,}\quad z = \frac{m - n}{a^3} y^4 - 2 . \frac{(m^3 - n^3)}{a^5} . y^6 + \&c.$$

Therefore the limit of $\frac{z}{y^4} = \frac{m-n}{a^3}$ a constant quantity; or $z \propto y^4$ ultimately; that is, the thickness, parallel to the common axis, intercepted between the two conic sections varies, for small arcs, nearly as the fourth power of the distance from the point of contact. Hence, to grind a conoid from a sphere, the quantity of friction applied to every point of the surface should vary as the fourth power of the distance from the centre of the mirror, and the parts to be left on the surface of the pitch tool should be bounded by a curve whose equation is $z \propto y^4$. The specific conoid will depend on the value of the constant quantity $\left(\frac{m-n}{a^3}\right)$.

EXAMPLE I. (Fig. 7.) Let AV be a parabola; AV' a circle always.

In the parabola, $y^2 = a\,x$; $\therefore m = 0$,

——— circle, $y^2 = a\,x - x^2$; $\therefore n = -1$;

$$\therefore \frac{m-n}{a^3} = \frac{1}{a^3}, \text{ or } z = \frac{y^4}{a^3}, \text{ the equation to the curve.}$$

Fig. 5., represents on a reduced scale the whole grinding surface to be left for a mirror three inches in aperture, and two feet focus.

Ex. II. Let AV be an hyperbola; its semiaxes being v and b;

$$\therefore y^2 = \frac{2\,b^2}{v}\,x + \frac{b^2}{v^2}\,x^2;\ \therefore m = + \frac{b^2}{v^2}.$$

the circumference, according to the law above mentioned: i. e. as the fourth power of the distance from the centre of the mirror. The quantity of friction applied to the surface of the mirror may be made to vary according to any direct law of the distance from the centre of the mirror by cutting away certain parts from the face of the polishing tool, which is formed of pitch, covered with an oxide of iron or tin. The surface of the pitch tool after removing the superfluous parts, is represented on a scale half the real size, in Fig. 6; which is adapted to a three-inch aperture with a two-feet focus. This figure, cut out in a piece of pasteboard or tin, may be applied to the pitch surface and traced upon it. To prevent the pitch from extending by pressure into the void spaces, those parts may be cut away or depressed in the tool itself, as well as in the pitch surface.

In the circle $n = -1$;

$$\therefore \frac{m-n}{a^3} = \frac{1+\frac{b^2}{v^2}}{a^3} = \frac{v^2+b^2}{a^3};$$

$$\therefore z = \frac{v^2+b^2}{a^3} \times y^4,$$

the equation to the curve to be traced on the pitch tool.

Ex. III. Let AV be an ellipse;

$$\therefore y^2 = \frac{2b^2}{v} x - \frac{b^2}{v^2} x^2; \therefore m = -\frac{b^2}{v^2}.$$

In circle $n = -1$;

$$\therefore \frac{m-n}{a^3} = \frac{1-\frac{b^2}{v^2}}{a^3} = \frac{v^2-b^2}{a^3};$$

$$\therefore z = \frac{v^2-b^2}{a^3} \times y^4;$$

Therefore, if the point under consideration is the extremity of the major axis, v is greater than b, and the construction is possible; but if the point be at the extremity of the minor axis, the construction is impossible.

If the original figure be a paraboloid with latus rectum (A), and it be required to form it into a paraboloid with latus rectum (a), it may be shewn nearly in the same manner that the curve traced upon the pitch tool must be a common parabola with latus rectum $\left(\frac{A-a}{Aa}\right)$.

The chief difficulty is to find a suitable motion for grinding. To grind one paraboloid upon another by a rectilineal motion, or by any motion largely compounded with a rectilineal motion, is evidently impossible. And to grind only by a rotatory motion about the axes, would entirely destroy the surface, by producing rings. Still it may be possible to combine these motions, sufficiently for practical purposes, by making the crank, and therefore the rectilineal motion, extremely short, and the circular motion more rapid. The rotatory motion is increased by removing the cogged wheel (*s*), and putting in its place a small pully with a string passing round the circumference of the fly wheel. If the cogged wheel had twenty cogs, and the pully now substituted be one-fifth of the diameter of the fly wheel, the rotatory motion will be increased a hundred times; or the leaden tool will revolve ten times in a minute, and the mirror twenty times; hence their relative motion is thirty revolutions in a minute. The quantity of rectilineal motion should be the least possible, that is sufficient to prevent the formation of annular streaks on the mirror.

As far as respects the grinding of conoids, this subject is only theoretical; so that appropriate improvements and directions must be sought from a few experiments.

NOTE

On the Attempts to grind LENSES *and* MIRRORS *by Machinery, and to give them a Parabolic Form.*

WE find very early notices of attempts to apply machinery to grinding the object-glasses of telescopes. In the first number of the Philosophical Transactions (1665), mention is made of glasses produced this way by Campani in Italy. Campani at Bologna, and Devini at Rome, were about that time disputing on the relative merits of the lenses which they produced, and the preference was generally given to the former. A machine employed by him is said to be still preserved in the apartments of the Institute at Bologna, and was found there by M. Fougeroux; it is not mentioned what peculiarities it had, but it was something of the nature of the common lathe, and probably, like almost all that have been since invented, was used only to assist, and not to supersede the work of the hand. Hooke in his Micrographia, published 1665, describes a machine for performing the whole work. In this, the glass and the tool turn each about its axis, these axes being inclined at a small angle to each other, and consequently, the motions of each particle of the glass and the tool being oblique with respect to each other. This method, however, appears liable to the objection which we shall find to apply to almost all those which have been proposed without experimental proof of their sufficiency. Namely, that each particle of the glass is ground by a surface whose motion, relatively to it, is, at similar points of the revolution, always the same, so that the inequalities of friction would act in the same manner in each successive revolution; and these cycles of action would probably cause the friction to affect the parts differently, according to their distance from the axis. It was also objected to by

Auzout, himself a constructor of telescopes in France, on the ground that it was impossible to give that steadiness and firmness to its structure which were requisite for grinding glasses of any long focus; an effect which Hooke promised as a consequence of its adoption. In 1668 we find a description of machine by Mancini, which consists, however, only of a long arm, one end of which is fastened to the ceiling, while the glass is fixed to the other end worked on a plane. In 1676 Borelli was celebrated as a constructor of telescope-glasses, but we are not aware of the method which he employed. To 1719 belongs a work of Leutman's, entitled, "Remarks on Glass-Polishing, in which are described improved machines for bringing glasses to greater perfection by *the help of three motions*." This Work I have not seen, and cannot therefore be certain what is meant by the three motions here mentioned in the title. In 1741 Mr. Jenkins published in the Philosophical Transactions his method of grinding glasses spherical. It consists in making the glasses, fixed on the surface of a sphere, of which they are segments, revolve round one axis, while a hemispherical cap, which is fitted to the globe, and which grinds them, revolves on an axis at right angles to the former. This, though in appearance much different from Hooke's machine, is nearly the same in principle, differing only in having the axis inclined at a different angle, and seems to be liable to the same objections. Huyghens's machine is described in Smith's Optics. It only serves to produce pressure, and to reduce the action of moving the glass backwards and forwards on the tool, to the action of turning a winch; the care of avoiding inequalities, by changing the relative position of the glass and tool, is left to the operator. Nollet's machine, among the *Machines Approuvées par l'Academie Royale des Sciences* for 1733, (tom. VI. p. 127), is merely to give a rotatory motion to the tool by the feet, while the glass is ground upon it by the hand.

Another machine for grinding glasses is described in the *Machines Approuvées* for 1736. It is by M. de Parcieux, and seems more likely than

Addition to NOTE, p. 96.

THE different conclusions in this note for the grinding surface necessary to obtain the different conic sections, are reducible to the simple condition of the *absolute value of the ordinate* in the curve $(z \propto y^4)$ traced upon the pitch tool: or what is the same thing, to the *time of grinding*, the absolute value of the ordinate being given. It will be most convenient to assume the ordinate so as to make the grinding surface continuous at the circumference of the pitch tool, (see Fig. 5.) The figure of the mirror, which is at first truly spherical, will migrate successively into an elliptical spheroid, a paraboloid, a hyperboloid, according to the *time of grinding*. The times of producing these effects will be respectively proportional to $v^2 - b^2$, v^2, $v^2 + b^2$.

W. C.

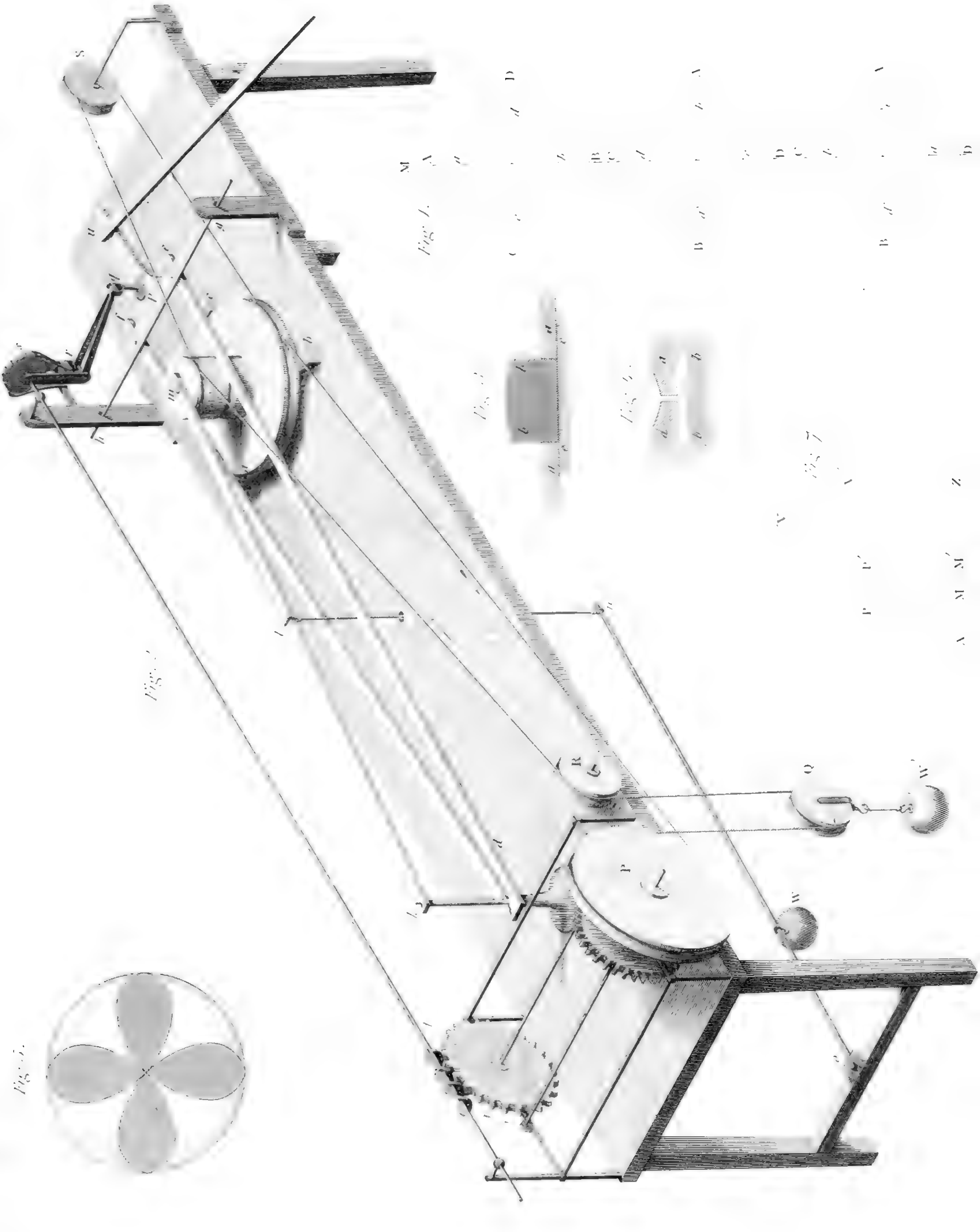
ERRATA.

Page 96, Line 9. *For* Fig. 6. *read* Fig. 5.

—— — Note. Line 5 from the bottom, for $a x' + m x'^2$ read $a x' + n x'^2$.

—— 98, Note. Lines 7, 8, 13, 14 from the bottom,

for $\frac{v^2 + b^2}{a^3}$ read $\frac{v^2 \pm b^2}{v^2 a^3}$.



VIII. *On the use of Silvered Glass for the Mirrors of Reflecting Telescopes.*

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THE idea which probably first occurred to the inventors of reflecting telescopes was that of constructing their reflectors of silvered glass; but it appears to have been immediately rejected. The difficulty of grinding and polishing metallic specula was then so great, and the formation of glass reflectors so easy, that some serious disadvantages must have appeared to be connected with the use of the latter, or Newton would not have bestowed so much labour on the construction of metallic mirrors. The principal objections appear to have been these: if the surfaces were ground to equal radii there would be a confusion of the images produced by reflection at the two surfaces of the glass: if ground to different radii, refraction would be introduced, and consequently dispersion, which it was intended by the construction of the reflecting telescope to avoid. Besides these, the loss of light by reflection from glass is perhaps greater than that by reflection from polished metal. For these reasons, it would seem, the use of glass reflectors has been entirely neglected, and opticians have endeavoured to improve the reflecting telescope only by improving the composition of the speculum-metal.

It appears surprising that no attempt has been made to remove the inconveniences attached to the use of glass reflectors. It occurred to me some time since that in Gregory's or Cassegrain's telescope, supposing the mirrors constructed of silvered lenses the radii of whose surfaces were different, the chromatic aberration of one mirror might be corrected by that of the other. For example, suppose the great mirror to be a meniscus silvered on its convex side: since the convergence of the rays is caused partly by the refraction of the meniscus through which they pass twice, and partly by the reflection at the silvered surface, the violet rays will converge sooner than the red, but the aberration will not be so great as if the convergence were occasioned entirely by refraction. Now if the small mirror be so constructed that its chromatic aberration may be equal to that of the great mirror but in the opposite direction, that is, so that the focal length for violet rays may be greater than that for red rays, as much as the focal length of the great mirror for red rays is greater than that for violet rays, the rays of all colours will after the second reflection converge to the same distance from the small mirror, and will therefore form an image free from chromatic aberration. This may be effected by using for the small mirror a concavo-convex lens, silvered on its convex side. For since the refraction tends to make the rays diverge, and the reflection to make them converge, the convergence produced, being the excess of the latter above the former, is greater for red than for violet rays, and consequently the focal length is greater for violet than for red rays. In this manner by a proper adjustment of the surfaces, the rays of all colours in each pencil may always be made to converge and form an image at the same distance from the small mirror.

It is yet desirable to correct if possible the spherical aberration by the use of spherical surfaces only. The method by which the chromatic aberration is corrected naturally suggested a mode of correcting the spherical aberration also: namely, by making the spherical aberration of one mirror equal and opposite to that of the other. From the investigation it appears that the determination of the radii of the surfaces proper for this purpose depends on the solution of a cubic equation, and therefore the correction of the spherical aberration is always possible. This will enable us to employ so large an aperture that the objection founded on the loss of light will be entirely removed.

In the construction of the telescope it is necessary to attend to another consideration, the nature of which may be thus explained. The apparent distance of an object seen through a telescope from the center of the field of view, depends on the angle made by the axis of the pencil of rays, when it enters the eye, with the axis of the telescope. Now when the axis of the pencil which is reflected from the great mirror is incident on the small mirror, the refraction of the concavo-convex lens, which forms the small mirror in the Gregorian construction, will disperse the axes of the differently coloured pencils: the axis of the red pencil will meet the first eye-glass at a greater distance from the axis of the telescope than that of the violet pencil. The refraction of the first eye-glass would make these axes intersect at a distance greater than its focal length. If then no other eye-glass were interposed, the axes of red and violet pencils would enter the eye in different directions, and the image formed by the red rays would appear more distant from the center of the field of view, than the image formed by the violet rays: that is, objects near the edge of the field of view would be coloured. This will be removed by placing a concave lens near the eye, which will cause the axes of pencils of all colours to enter the eye parallel to each other. By similar reasoning a combination of eye-glasses may always be found, which will produce an image perfectly free from colour.

These investigations were prepared for the notice of the Philosophical Society, when I discovered that the same idea had been published in the Philosophical Transactions for 1740, by Mr. Caleb Smith. He has given without demonstration some formulæ for the correction of colour, but the correction of spherical aberration he leaves, as leading into too intricate investigations; the achromatism of the eye-piece does not seem to have occurred to him. He speaks of having made one experiment which gave him the greatest hopes of success. As the construction has not been remarked by any of the friends to whom I have mentioned it, and as the subject is one of considerable interest, I am induced to think that a statement of the principles, and an investigation of the formulæ for the construction of telescopes on this plan will be not unacceptable to the Philosophical Society.

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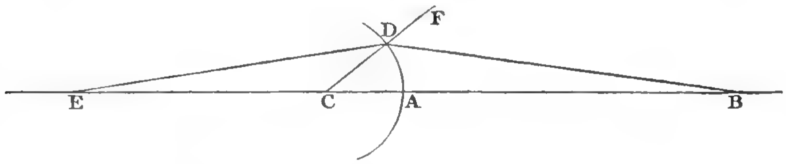
(1). In the following articles we shall always consider the radii of convex surfaces as positive. We shall denote by n the ratio of the sine of incidence to the sine of refraction out of air into glass for mean rays, and by δn the variation of this ratio arising from unequal refrangibility. For the alteration in focal lengths, &c. arising from chromatic aberration we shall use δ, and for those produced by spherical aberration we shall use d. Our approximation will be extended to the first power of δn, and the second power of the apertures; and all our investigations will be made on the supposition that the telescope is Gregorian.

(2). A mirror is formed of a double convex lens silvered on one surface: rays are incident from a given point in the axis; to find the focus of reflected rays.

(3). 1st. For first refraction. Let C be the center of the surface AD; B the given point; BD an incident ray, DE the refracted ray; join CD and produce it to F; let $\frac{1}{BA} = D$; $\frac{1}{CD} = r$; $\frac{1}{AE} = x$;



the angle $ACD = \theta$; $AD = \alpha$. Then $\sin BDF = n \,.\, \sin EDC$; or

$$\frac{BC}{BD} \sin\theta = n \,.\, \frac{EC}{ED} \sin\theta; \text{ or } BC \,.\, ED = n \,.\, BD \,.\, EC.$$

$$\text{Now } BD^2 = \left(\frac{1}{D} + \frac{1}{r}\right)^2 + \frac{1}{r^2} - \frac{2}{r}\left(\frac{1}{D} + \frac{1}{r}\right) . \left(1 - \frac{\theta^2}{2}\right)$$

(neglecting in the expression for $\cos\theta$ the powers above the second)

$$= \frac{1}{D^2} + \frac{2}{r}\left(\frac{1}{D} + \frac{1}{r}\right) . \frac{\theta^2}{2}; \ \therefore\ BD = \frac{1}{D} + \frac{D}{r}\left(\frac{1}{D} + \frac{1}{r}\right)\frac{\theta^2}{2} = \frac{1}{D} + (r+D)\frac{a^2}{2},$$

$$\text{since } \alpha = AC \times \theta = \frac{\theta}{r}.$$

$$\text{Similarly, } ED = \frac{1}{x} - (r-x)\frac{a^2}{2}.$$

Hence the equation becomes

$$\left(\frac{1}{D} + \frac{1}{r}\right)\left(\frac{1}{x} - \overline{r-x} \,.\, \frac{a^2}{2}\right) = n\left(\frac{1}{x} - \frac{1}{r}\right)\left(\frac{1}{D} + \overline{r+D} \,.\, \frac{a^2}{2}\right),$$

$$\text{or } (r+D)\left(1 - x \,.\, \overline{r-x} \,.\, \frac{a^2}{2}\right) = n\,(r-x)\left(1 + D \,.\, \overline{r+D} \,.\, \frac{a^2}{2}\right).$$

Let $\alpha = 0$; $r + D = n\,(r-x)$; $r - x = \frac{r+D}{n}$, and $x = \frac{n-1}{n}r - \frac{D}{n}$.

Substituting these approximate values in the coefficient of $\frac{a^2}{2}$,

$$(r+D)\left(1 - \frac{1}{n^2} . (r+D) . (\overline{n-1} \,.\, r - D) . \frac{a^2}{2}\right) = n\,(r-x)\left(1 + D\,(D+r)\frac{a^2}{2}\right),$$

$$\text{hence } r - x = \frac{r+D}{n}\left\{1 - \frac{a^2}{2} . \frac{(r+D)\,(\overline{n-1}\ r - D) + (r+D) . D . n^2}{n^2}\right\}$$

$$= \frac{r+D}{n} - \frac{a^2}{2} \cdot \frac{(r+D)^2 \cdot (\overline{n-1}\, r + \overline{n^2-1}\, D)}{n^3},$$

and $x = \frac{n-1}{n} r - \frac{D}{n} + \frac{a^2}{2} \cdot \frac{n-1}{n^3} \cdot (r+D)^2 \cdot (r + \overline{n+1}\, D)$.

(4). 2nd. For the reflection; let $\frac{1}{y}$ be the distance of the focus after reflection from the reflecting surface, measured in the same direction; $\frac{1}{R}$ the radius of that surface; then in the formula just found making $n = -1$, putting $-R$ for r (since the rays are incident on the concave surface) and $-x$ for D, we find

$$y = -2R - x + \frac{a^2}{2} \cdot 2(R+x)^2 \cdot (-R) = -2R - x - \frac{a^2}{2} \cdot 2R \cdot (R+x)^2.$$

(5). 3rd. For refraction at emergence; let $\frac{1}{z}$ be the distance of the focus of emergent rays, measured in the opposite direction; in the same formula put $\frac{1}{n}$ for n, $-r$ for r, y for D;

$$\text{then } z = -\frac{\frac{1}{n} - 1}{\frac{1}{n}} r - \frac{y}{\frac{1}{n}} + \frac{a^2}{2} \cdot \frac{\frac{1}{n} - 1}{\frac{1}{n^3}} (y-r)^2 \cdot \left(-r + \overline{\frac{1}{n} + 1} \cdot y\right)$$

$$= \overline{n-1} \cdot r - ny - \frac{a^2}{2} \cdot n \cdot \overline{n-1} \cdot (y-r)^2 \cdot (\overline{n+1} \cdot y - nr).$$

(6). From (3) $R + x$ approximately $= \frac{n-1}{n} r + R - \frac{D}{n}$;

therefore by (4), $y = -2R - \frac{n-1}{n} r + \frac{D}{n}$

$$- \frac{a^2}{2} \cdot \left\{ \frac{n-1}{n^3} (r+D)^2 \cdot (r + \overline{n+1}\, D) + 2R \cdot \left(\frac{\overline{n-1} \cdot r + nR - D}{n} \right)^2 \right\}.$$

(7). From (6), $y - r$ approximately $= -2R - \frac{2n-1}{n} r + \frac{D}{n}$;

$$\overline{n+1} \cdot y - nr = -2 \cdot \overline{n+1} \cdot R - \frac{2n^2 - 1}{n} r + \frac{n+1}{n} D;$$

hence by (5), $z = 2\,.\,\overline{n-1}\,.\,r + 2n\,R - D + \frac{a^2}{2}\left\{\frac{n-1}{n^2}.(r+D)^2.(r+\overline{n+1}\,.\,D)\right.$

$$+\,2R\,\frac{(\overline{n-1}\,.\,r+nR-D)^2}{n} + n\,.\,\overline{n-1}\,.\left(2\,R + \frac{2n-1}{n}\,r - \frac{D}{n}\right)^2 \times$$

$$\left.\left(2\,.\,\overline{n+1}\,.\,R + \frac{2n^2-1}{n}\,r - \frac{n+1}{n}\,D\right)\right\}.$$

(8). The thickness is neglected, as its introduction would not sensibly alter either term of the expression for z.

(9). Let a be the distance of the focus of emergent rays from the surface; then $\frac{1}{a} = z$; therefore $da = -\,a^2dz$, and $\delta a = -\,a^2\delta z$, nearly. Let Z and F denote the values of z and a, when the incident rays are parallel; then $dF = -\,F^2dZ$, and $\delta F = -\,F^2\delta Z$.

(10). That the chromatic aberration in a telescope may be destroyed, it is necessary that the images formed by rays of different colours after reflection from the small mirror be equally distant from that mirror. If now we suppose rays of all colours diverging from the place at which the second image is formed, to be incident on the small mirror, the rays of each colour after reflection will converge to the point in which an image is formed by rays of the same colour reflected from the great mirror. Let a be the distance of the first image from the small mirror, F the focal length of the great mirror, b the distance between the mirrors, $\frac{1}{D}$ the distance of the second image from the small mirror, $\frac{1}{\rho}$ and $\frac{1}{\rho'}$ the radii of the unsilvered and silvered surfaces of the small mirror; since $b = a + F$ we have $\delta a + \delta F = 0$. But by (7), $\frac{1}{a} = 2\,.\,\overline{n-1}\,.\,\rho + 2n\rho' - D$, neglecting at present spherical aberration; therefore $\delta a = -\,2a^2\,(\rho+\rho')\,\delta n$, since by supposition D is the same for all colours: similarly $\delta F = -\,2F^2\,(r+R)\,\delta n$. Substituting these values in the equation $\delta a + \delta F = 0$, and dividing by $2\delta n$ it becomes $a^2(\rho + \rho') + F^2(r + R) = 0$.

When this condition is satisfied the second image is free from colour.

(11). That the spherical aberration may be destroyed, we must have for the same reason $da + dF = 0$, or $a^2 dz + F^2 dZ = 0$. Substituting for these their values from (7), and putting β for the semi-aperture of the small mirror, we find

$$a^2 \cdot \frac{\beta^2}{2}\left\{\frac{n-1}{n^2}\cdot(\rho+D)^2\cdot(\rho+\overline{n+1}\,D) + 2\rho'\,\frac{(\overline{n-1}\,.\,\rho+n\rho'-D)^2}{n}\right.$$
$$\left.+\,n\,.\,\overline{n-1}\,.\left(2\rho' + \frac{2n-1}{n}\rho - \frac{D}{n}\right)^2\cdot\left(2\,.\,\overline{n+1}\,.\,\rho' + \frac{2n^2-1}{n}\rho - \frac{n+1}{n}D\right)\right\}$$
$$+\,F^2\,.\,\frac{a^2}{2}\left\{\frac{n-1}{n^2}\,.\,r^3 + 2R\,\frac{(\overline{n-1}\,.\,r+nR)^2}{n}\right.$$
$$\left.+\,n\,.\,\overline{n-1}\,.\left(2R + \frac{2n-1}{n}r\right)^2\cdot\left(2\,.\,\overline{n+1}\,.\,R + \frac{2n^2-1}{n}r\right)\right\} = 0.$$

But it is easily seen that for the same ray $\beta = \frac{a}{F}\alpha$; substituting

$$a^4\left\{\frac{n-1}{n^2}(\rho+D)^2\,.(\rho+\overline{n+1}\,D) + 2\rho'\,.\,\frac{(\overline{n-1}\,.\,\rho+n\rho'-D)^2}{n}\right.$$
$$\left.+\,n\,.\,\overline{n-1}\,.\left(2\rho' + \frac{2n-1}{n}\rho - \frac{D}{n}\right)^2\cdot\left(2\,.\,\overline{n+1}\,.\,\rho' + \frac{2n^2-1}{n}\rho - \frac{n+1}{n}D\right)\right\}$$
$$+\,F^4\left\{\frac{n-1}{n^2}r^3 + 2R\,.\,\frac{(\overline{n-1}\,.\,r+nR)^2}{n}\right.$$
$$\left.+\,n\,.\,\overline{n-1}\,.\left(2R + \frac{2n-1}{n}r\right)^2\left(2\,.\,\overline{n+1}\,.\,R + \frac{2n^2-1}{n}r\right)\right\} = 0.$$

When this equation holds, the spherical aberration is corrected.

(12). We have found then the conditions which must be satisfied in order that all the rays in each pencil may converge to the same distance from the small mirror, and that an object in the center of the field of view may be distinctly seen. We must now consider the combination of eye-glasses, which will cause the axes of the differently coloured pencils from the same point to enter the eye parallel, and with which objects near the edge of the field of view will not be coloured. The process which we shall employ, and which is easily applicable to achromatic eye-pieces in general, consists in finding the tangent of the angle made by the axis of any pencil with the axis of the telescope, and making its chromatic variation equal to nothing.

(13). Let the focal length of the small mirror $=f$; the distance from the small mirror to the first eye-glass $=p$; from the first eye-glass to the second (or that nearest the eye) $= q$; let the focal lengths of the first and second eye-glasses be g and k. Then the axis of any pencil after reflection from the small mirror crosses the axis of the telescope at a point whose distance from the small mirror $= \dfrac{1}{\frac{1}{f} - \frac{1}{b}} = \dfrac{fb}{b-f}$; and whose distance from the first eye-glass

$= p - \dfrac{fb}{b-f} = \dfrac{pb-(p+b)f}{b-f}$. It crosses again at distance from first eye-glass $= \dfrac{1}{\frac{1}{g} - \frac{b-f}{pb-(p+b)f}} = \dfrac{pbg-(p+b)fg}{pb-(p+b)f-bg+fg}$; distance from second eye-glass

$$= q - \frac{pbg-(p+b)fg}{pb-(p+b)f-bg+fg} = \frac{pbq-q(p+b)f-b(q+p)g+(p+b+q)fg}{pb-(p+b)f-bg+fg}.$$

It finally crosses at a distance from the second eye-glass

$$= \frac{1}{\frac{1}{k} - \frac{pb-(p+b)f-bg+fg}{pbq-q(p+b)f-b(q+p)g+(p+b+q)fg}}$$

$$= \frac{k\{pbq-q(p+b)f-b(q+p)g+(p+b+q)fg\}}{pbq-q(p+b)f-b(q+p)g-pbk+(p+b+q)fg+(p+b)fk+b.gk-fgk} (=A).$$

(14). Let m be the distance from the axis of the small mirror at which the axis of any pencil is incident upon it; $m \times \dfrac{b-f}{fb}$

$\times \dfrac{pb-(p+b)f}{b-f}$ (by similar triangles) or $m \times \dfrac{pb-(p+b)f}{fb}$ is the distance from the axis at which it is incident on the first eye-glass;

$$m \times \frac{pb-(p+b)f}{fb} \times \frac{pb-(p+b)f-bg+fg}{pbg-(p+b)fg}$$

$$\times \frac{pbq-q\,(p+b)f-b\,(q+p)\,g+(p+b+q)\,fg}{pb-(p+b)f-bg+fg},$$

or $m \times \dfrac{pbq-q\,(p+b)f-b\,(q+p)\,g+(p+b+q)fg}{bfg}$ $(=B)$ is the distance from the axis at which it is incident on the second eye-glass. Hence the tangent of the angle with the axis of the telescope $\left(=\dfrac{B}{A}\right)$ is

$$m \times \frac{pbq-q(p+b)f-b(q+p)g-pbk+(p+b+q)fg+(p+b)fk+bgk-fgk}{bfgk}\ (=C).$$

(15). Now by (12), we must have $\delta C=0$, or (since m, b, p, q, are constant),

$$-\frac{pbq}{fgk}\left(\frac{\delta f}{f}+\frac{\delta g}{g}+\frac{\delta k}{k}\right)+\frac{q\,(p+b)}{gk}\left(\frac{\delta g}{g}+\frac{\delta k}{k}\right)+\frac{b\,(q+p)}{fk}\left(\frac{\delta f}{f}+\frac{\delta k}{k}\right)$$

$$+\frac{pb}{fg}\left(\frac{\delta f}{f}+\frac{\delta g}{g}\right)-\frac{p+b+q}{k}\cdot\frac{\delta k}{k}-\frac{p+b}{g}\cdot\frac{\delta g}{g}-\frac{b}{f}\cdot\frac{\delta f}{f}=0.$$

But $g=\dfrac{h}{n-1}$, $k=\dfrac{l}{n-1}$, where h and l are independent of n:

$$\therefore\ \frac{\delta g}{g}=-\frac{\delta n}{n-1},\ \frac{\delta k}{k}=-\frac{\delta n}{n-1};\ \text{also}\ \frac{1}{f}=2n\rho'+2\overline{n-1}\,\rho;$$

$\therefore\ \dfrac{\delta f}{f}=-2f\,(\rho'+\rho)\,\delta n.$ Substituting and dividing by δn we get at length

$$k=\frac{\left(\frac{2pb}{f}-2\,.\,\overline{p+b}+2bp\,.\,\overline{n-1}\,.\,\overline{\rho'+\rho}\right)q+\left(-\frac{pb}{f}+\overline{p+b}-2bp\,.\,\overline{n-1}\,.\,\overline{\rho'+\rho}\right)g+\left(1-\frac{b}{f}-2b\,.\,\overline{n-1}\,.\,\overline{\rho'+\rho}\right)qg}{\frac{pb}{f}-(p+b)+2pb\,.\,\overline{n-1}\,.\,\overline{\rho'+\rho}-2b\,.\,\overline{n-1}\,.\,\overline{\rho'+\rho}\,.\,g}.$$

When k has this value, objects near the edge of the field of view will not be coloured.

(16). Since $\frac{1}{f} = 2n\rho' + 2.\overline{n-1}.\rho = 2\rho' + 2.\overline{n-1}.\overline{\rho+\rho'}$, we may change this expression into the following:

$$k = \frac{\left(\frac{3pb}{f} - 2pb.\rho' - 2\overline{p+b}\right)q + \left(-\frac{2pb}{f} + 2pb.\rho' + \overline{p+b}\right)g + \left(-\frac{2b}{f} + 2b\rho' + 1\right)qg}{\frac{2pb}{f} - 2pb.\rho' - \overline{p+b} - \left(\frac{b}{f} - 2b\rho'\right)g}.$$

which is rather more convenient for practical application.

(17). To construct a telescope on these principles it will be most convenient to assume the values of F and a; $\frac{1}{D}$ will generally be rather greater than the distance of the two mirrors; and f will be found by the equation $D + \frac{1}{a} = \frac{1}{f}$. To determine ρ, ρ', r, R, we have then the four equations

$$2n\rho' + 2.\overline{n-1}.\rho = \frac{1}{f} \ldots (7)$$

$$2nR + 2.\overline{n-1}.r = \frac{1}{F} \ldots (7)$$

$$a^2(\rho + \rho') + F^2(r + R) = 0 \ldots (10)$$

$$a^4\left\{\frac{n-1}{n^2}(\rho + D)^2.(\rho + \overline{n+1}D) + 2\rho'\frac{(\overline{n-1}.\rho + n\rho' - D)^2}{n}\right.$$
$$\left. + n.\overline{n-1}.\left(2\rho' + \frac{2n-1}{n}\rho - \frac{D}{n}\right)^2.\left(2.\overline{n+1}.\rho' + \frac{2n^2-1}{n}\rho - \frac{n+1}{n}D\right)\right\}$$
$$+ F^4\left\{\frac{n-1}{n^2}r^3 + 2R\frac{(\overline{n-1}.r + nR)^2}{n}\right.$$
$$\left. + n.\overline{n-1}\left(2R + \frac{2n-1}{n}r\right)^2.\left(2.\overline{n+1}.R + \frac{2n^2-1}{n}r\right)\right\} = 0 \ldots (11).$$

From the three first equations

$$\rho' = \frac{1}{2nf} - \frac{n-1}{n}\rho,$$

$$r = -\frac{F + \frac{a^2}{f}}{2F^2} - \frac{a^2}{F^2}\rho,$$

$$R = \frac{nF + \overline{n-1}\,.\,\frac{a^2}{f}}{2nF^2} + \frac{n-1}{n}\,.\,\frac{a^2}{F^2}\rho.$$

Substituting these values in the fourth, observing that $D + \frac{1}{a} = \frac{1}{f}$, and multiplying by $\frac{n^2 F^2}{n-1}$, we get for the determination of ρ the following equation:

$$2a^4 . (F^2 - a^2) . \rho^3 + \left.\begin{matrix} \frac{(\overline{n+3}\,.\,F^2 - 3a^2)\,a^4}{f} \\ + n\,.\,a^4 F \end{matrix}\right\}\rho^2 \left.\begin{matrix} + \overline{2n+3}\,.\,\overline{a^2 D^2 + 1}\,.\,a^2 F^2 \\ - 2a^4 F^2 \left(\frac{1}{2f} - D\right)^2 \\ - \frac{3a^2\left(F + \frac{a^2}{f}\right)^2}{4} \\ + \frac{F^2 a^2}{2} \\ - \frac{\left(\overline{4n+3}\,.\,F - \frac{3a^2}{f}\right) . a^2 . \left(F - \frac{a^2}{f}\right)}{4} \end{matrix}\right\}\rho \left.\begin{matrix} + \overline{n+1}\,.\,\overline{D^3 + \frac{1}{a^3}}\,.\,a^4 F^2 \\ + \frac{\left(\overline{2n+1}\,.\,F - \frac{a^2}{f}\right)\left(F - \frac{a^2}{f}\right)^2}{8} \\ - \frac{\left(F + \frac{a^2}{f}\right)^3}{8} + \frac{a^2 F^2}{4f} \\ + \frac{1}{n-1}\left\{\frac{1}{f} . \left(\frac{1}{2f} - D\right)^2 . a^4 F^2 + n\frac{F^3}{4}\right\} \end{matrix}\right\} = 0.$$

(18). This equation being a cubic has at least one possible root; and ρ', r, R, are then determined by the expressions

$$\rho' = -\rho + \frac{1}{n}\left(\frac{1}{2f} + \rho\right),$$

$$r = -\frac{F + \frac{a^2}{f}}{2F^2} - \frac{a^2}{F^2}\rho,$$

$$R = -r - \frac{a^2}{F^2}(\rho + \rho').$$

Thus it is always possible to correct the spherical and chromatic aberrations, except $F^2 = a^2$; a case which can never occur in practice.

(19). The coefficients of the equation of (16), may then be found, and by assuming values of g and q, the corresponding values of k may be found.

(20). The place of the second image may now be found; and if the value assumed for $\frac{1}{D}$ was not sufficiently accurate, the process may be repeated, and a more correct value found for ρ, &c. The equation of the eye-piece will probably not require alteration.

(21). It is scarcely necessary to observe that the small mirror must not be changed as in telescopes with metal reflectors; the power must be altered only by changing the eye-piece.

(22). The investigations have all been made on the supposition that the telescope was Gregorian. For one of Cassegrain's construction it is merely necessary to take a and f negative.

The advantages which telescopes on this construction might be expected to possess over those in use may be estimated from the following statement. The principal difficulty in the construction of the achromatic lens arises from the irregularities in flint-glass, which render it almost impossible to make an object-glass of large diameter. As only one kind of glass is necessary for the construction here proposed, the artist evidently has the power of selecting that which will be most easily found free from irregularities. From the different laws of dispersion in different kinds of glass the exact correction of colour by the achromatic lens is impossible; but as in the proposed construction the same kind of glass is used in every part of the telescope, it is not liable to the same objection. Considerable difficulty is found in adapting to each other two lenses of crown and flint; for this telescope the radii, &c. might be calculated with the certainty of removing all aberration. For the object-metal in reflecting telescopes of large aperture, the parabolic form is absolutely necessary, and this can be given only by the best workmen. The advantage of this construction in which none but spherical surfaces are used is sufficiently obvious. The difference in the quantity of light reflected, if there is any, appears so small that it offers a very slight objection; especially as we can increase the aperture without fear of indistinctness for aberration.

I have constructed two Cassegrain's telescopes on this plan, whose object-mirrors are 4 inches in aperture, and 20 inches in focal length. From some cause with which I am unacquainted the image of a star or planet is surrounded with radiations which make the telescope quite useless for practical purposes, and render it extremely difficult to pronounce any thing on the success of the principle. I have not however been able to observe the slightest appearance of colour: of the spherical aberration I cannot, in consequence of the radiation, speak so decidedly, but I am certain that if there is any it is very small. But the success of a principle of this kind is not to be determined from one or two experiments; several trials should be made, and every endeavour used to overcome the difficulties which always occur in instruments made on a new construction; and even if it should appear at present to fail, some improvement in the theoretical principles or in the practical application may make it useful hereafter. It is my intention to make new trials; and to attempt the correction of the defect at present existing; and the results of my experiments will be communicated to the Society should this paper appear worthy of their attention.

G. B. AIRY.

Trinity College,
Nov. 25, 1822.

IX. *An Account of some Experiments made in order to determine the Velocity with which Sound is transmitted in the Atmosphere.*

By OLINTHUS GREGORY, LL.D.

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[Read *Dec.* 8, 1823.]

THE theoretical investigations of different philosophers, in order to ascertain the velocity with which Sound is transmitted through the atmosphere, however ingenious and elegant some of them may be, seem to rest too much upon gratuitous assumptions, to allow any cautious enquirer after physical truth, to receive them unhesitatingly, except so far as they may be confirmed by accurate experiment. Unfortunately, too, the results of experiment present irregularities both formidable and perplexing; since many of them cannot well be imputed to any want of skill, or caution, in the conductors of the enquiry.

	Feet per Second.
Thus, Mr. Roberts assigns a velocity of	1300
Mr. Boyle	1200
Mr. Walker, and Duhamel	1338
Mersenne, in his Treatise de Sonorum Natura, Causis et Effectibus	1474
The Florence Academy	1148
Cassini de Thury (*Mem. Paris Acad.* ann. 1738)	1107
Meyer	1105
Derham	1142
Muller	1109
Pictet	1130
Arago, &c. from experiments in June 1822, give 337.2 metres, at the temperature of + 10° Centigrade	1106.32*

The theoretical formula most generally adopted, especially by Continental philosophers, is this:—

Velocity in horizontal direction $= 333.44$ met. $\sqrt{1+.00375t}$; the metre being $= 3.2809$ English feet, and t denoting the indication of the temperature upon the centigrade thermometer.

I am inclined, however, to think that this can only be regarded as an approximative formula; and that we are not yet in a state to receive otherwise, than as an approximation, *any* theorem which simply includes the variations of temperature. The air is subject to various classes of changes, indicated by the barometer, thermometer, hygrometer, and anemometer respectively, as well as others probably, for the ascertaining of which we have not yet any appropriate instrument. If we could select these, one by one, *ad libitum*, and carry experiments first through a moderate

* This is the last result of which I had heard, previously to the commencement of my own experiments.

range upon the barometric scale, all the other probable elements of modification remaining constant; then, through a sufficiently extensive range upon the thermometric scale, the others remaining invariable, and so on; the question would soon be set at rest: but this is impossible. It becomes desirable, therefore, to augment the number of recorded facts, as they result from accurate experiments, in order that, at some future (and it is hoped, no very remote) time, a cautious investigator may so select, compare, and classify them, as to deduce a more comprehensive and accurate theorem than is yet known.

With a view to contribute, though in a small degree, to this purpose, I now present an account of a few experiments made by myself in the course of the present year.

My objects were, to ascertain the velocity with which the sound passed over the surface of the earth, over the surface of water; under different temperatures; in a quiescent state of the atmosphere, and in windy weather; by day and by night; the velocities of direct and reflected sound; and the velocities of sounds of different intensities and produced by different means. As yet the experiments have not been carried to their projected extent; but while I record the results thus far obtained, I look forward with hope, that, in another year or two, I shall be able to complete them satisfactorily.

The instrument with which I measured the intervals of time, was one invented and made by *Mr. Hardy*, by means of which, with a little previous practice, I could measure an interval *accurately* to a tenth of a second, and *approximatively* to a twentieth of a second. The velocity of the wind was ascertained by means of an anemometer; and the barometer and thermometer were of the best construction.

I employed no hygrometer, (much as I wished it); for as yet, I am not acquainted with any in whose results I should be inclined to confide. With regard to the distances between the stations at which the sound was emitted and heard, they were in some cases taken from the Ordnance Map of Kent, and verified by new operations; in others they were determined by actual and careful measurement: in others by trigonometrical operations with accurate instruments. The whole were conducted with care; and it would be useless to enter into the detail of them.

FRIDAY, *January* 3, 1823.

A musquet was fired from the battery near the Royal Artillery Barracks, and the interval of time between the flash and sound was observed at two different distances on the mortar-range, direction nearly north and south.

January 3, half past 2, P.M. barom. 29.7 inches, Fahr. therm. 45°, rather moist atmosphere, but no rain; very gentle wind blowing in direction nearly perpendicular to that of the range.

Distance from musquet to my station 3600 feet. *Six rounds fired:* in one the interval of time employed by the sound in passing over the 3600 feet was doubtful: in the other five the intervals were 3″.25, 3″.3, 3″.25, 3″.2, 3″.26, the mean of these is 3″.252.

$$\frac{3600}{3.252} = 1107 \text{ feet, velocity of sound; therm. } 45^{\circ}.$$

Same day, three o'clock P.M. barom. 29.64 inches, Fahr. therm. 45°, atmosphere, wind and weather as before.

Distance from musquet to station 3600 feet. *Five rounds fired,* intervals, 3″.2, 3″.2, 3″.3, 3″.3, 3″.25; their mean 3″.25.

$$\frac{3600}{3.25} = 1108 \text{ feet, velocity of sound; therm. } 45^{\circ}.$$

Same day, half past 3, P.M. barom. 29.64 inches, Fahr. therm. 45°, atmosphere, wind, and weather, as before.

Distance from musquet to station 2100 feet. *Eight rounds fired*

Interval between flash and report in one case doubtful: the others were 1″.88, 1″.88, 1″.9, 1″.9, 1″.9, 1″.9, 1″.91; the mean 1″.896,

$$\frac{2100}{1.896} = 1108 \text{ feet, velocity of sound; therm. } 45^{\circ}.$$

Thursday, January 9, three quarters past 7, P. M. dark, but clear, star-light, frosty night. Barom. 29.82 inches, Fahr. therm. 27°. Dry; no wind. Musquets fired from the battery, as before, distance 3600 feet.

Six rounds fired, one doubtful. The other intervals between observing the flash and hearing the report, were 3″.25, 3″.28, 3″.3, 3″.3, 3″.32; mean 3″.29,

$$\frac{3600}{3.29} = 1094.2 \text{ feet, velocity of sound; therm. } 27^{\circ}.$$

The sound of the same charge, fired from the same musquet, was heard much more intensely, on this clear frosty night than in the day-time of January 3, at the same distance 3600 feet.

Same day, January 9. Being anxious to extend the experiments to greater distances, I had previously applied to General Ramsey, of the Royal Artillery, the Commandant of the Garrison here, for the use of cannons as well as musquets, these, with his accustomed courtesy and kindness, he immediately ordered to be at my disposal, whenever I should need them in the course of my experiments.

On the morning of this day, therefore, I chose a station for the gun, on the side of Shooter's Hill, between Severn-Droog Castle and the 8 mile-stone on the Dover road. I selected three other stations from which the gun could be seen with a good Theodolite telescope; one of these was at the entrance of the lane turning from the Dover road to *Charlton,* between "the Sun in the Sands," and the 7 mile-stone; the second in the *Kidbrook Lane* which turns off from the Dover road between the 6 mile-stone and "the Sun in the Sands;" and the third on *Blackheath,* nearly in a continuation of the western wall of Greenwich Park towards the windmills. These three stations are probably 200 feet above the high water-mark in the Thames at Woolwich; and the station at which the gun was placed is still more elevated.

The distances, as accurately measured, were, from the Shooter's Hill station to that in Charlton Lane 6550 feet; from Shooter's Hill to that in Kidbrook Lane 8820 feet; from the Shooter's Hill station to that on Blackheath, 13440 feet.

The gun employed was a six pounder, the charge of powder eight ounces. The serjeant-major who remained at the gun, was directed to order the men to commence firing at a certain minute by his watch, (which was previously made to agree with mine) and then to fire regularly a certain number of rounds at intervals of *two minutes*: this was the practice throughout the experiments, the gun was always pointed towards me, at a very small elevation, except it be otherwise expressed.

January 9th, *noon*. Barom. 29.92 inches; Fahr. therm. 33°; weather dry, wind scarcely perceptible, a clear cloudless frosty day.

Six rounds fired. Interval of passage of sound from Shooter's Hill to Charlton Lane, 5″.9, 6″.0, 5″.9, 6″.0, 6″.0, 6″.0, their mean 5″.9$\frac{2}{3}$, distance 6550,

$$\frac{6550}{5.9\frac{2}{3}} = 1098 \text{ feet, velocity of sound; therm. } 33°.$$

Same day, January 9, half past 12, barom. 29.86 inches; Fahr. therm. 33°; weather dry, wind scarcely perceptible. *Six rounds fired*. Result in reference to one, very doubtful. Intervals of passage of sound from Shooter's Hill to Kidbrook Lane, were 7″.95, 8″.0. 8″.0, 8″.0, 8″.05; their mean 8″. Distance 8820 feet,

$$\frac{8820}{8} = 1102\tfrac{1}{2} \text{ feet, velocity of sound; therm. } 33°.$$

Same day, January 9, quarter past 1, P. M. barom. 29.82 inches; Fahr. therm. 33°; weather dry, wind scarcely perceptible. *Five*

rounds fired, intervals of the passage of sound between the stations at Shooter's Hill and Blackheath, 12″.2, 12″.25, 12″.3, 12″.24, 12″.26; mean 12″.25, distance 13440 feet,

$$\frac{13440}{12\frac{1}{4}} = 1097 \text{ feet, velocity of sound; therm. } 33°.$$

$\frac{1}{3}$ $(1098 + 1102\frac{1}{2} + 1097) = 1099\frac{1}{6}$ feet, *mean velocity* from the sixteen rounds; therm. 33°.

Monday, February 17, *noon.* Barometer 29.98 inches; Fahr. therm. 35°. Air humid; but neither rain nor sleet; very gentle wind, N. E. by E. Employed *bells* on the mortar-range on Woolwich Common, lying nearly north and south.

A bell rung at the north station, was heard by a soldier at the south station, who immediately rang another bell, having his arm elevated for the purpose. I stood by the soldier who rang the first bell, and measured the interval of time between the sound of the first bell, and the sound of the second bell, when transmitted from the other station.

By several preceding experiments, I estimated the time which elapsed between the moment, when the man with the second bell heard the sound from the other, and struck the clapper against his own bell, finding it to be *one-fifth of a second,* this, therefore, I deducted from the intervals which marked the passage of sound, before I recorded them, as below.

Distance between the two bells 1350 feet; whole distance traversed by the sound 2700 feet. Intervals elapsed (corrected as above) in five experiments; 2″.5, 2″.48, 2″.44, 2″.46, 2″.42; mean 2″.46.

$$\frac{2700}{2.46} = 1098 \text{ feet, velocity of sound; therm. } 35°.$$

Same day, quarter past 12, barom. therm. wind and weather as before.

Distance between the two bells 1650 feet; whole distance 3300 feet. Intervals elapsed in four experiments; 3″.0, 3″.0, 3″.0, 3″.0,

$$\frac{3300}{3}=1100$$ feet, velocity of sound; therm. 35°.

Same day, half past 12, barom. therm. wind and weather as before. Distance between the two bells 1800 feet; whole distance 3600 feet. Intervals elapsed in five trials, 3″.25, 3″.24, 3″.26, 3″.25, 3″.25; mean 3″.25,

$$\frac{3600}{3.25}=1108$$ feet, velocity of sound; therm. 35°.

$\frac{1}{3}(1098+1100+1108)=1102$ feet, *mean velocity* from this day's experiments; therm. 35°.

Friday, May 23. This morning there was a tolerably brisk wind blowing from the S.W. by W. nearly in the direction of my Charlton and Kidbrook stations from Shooter's Hill. Of this I gladly availed myself, as the morning was in other respects favourable, in order to ascertain what would be the effect of such a wind upon the velocity. Cloudy, air humid, but no rain.

I measured the velocity of the wind frequently with an anemometer, and found it vary between 22 and 26 feet, the mean 24 feet.

The gun a six pounder, charge 8 oz. of powder. 11, A.M. *gun at Shooter's Hill, sound heard at Charlton Lane,* distance 6550 feet. barom. 29.66 inches. Fahr. therm. 58°, air humid. *Six rounds fired;* the intervals were 6″.1, 6″.05, 6″.0, 6″.05, 6″.0, 6″.04; their mean 6″.037.

$$\frac{6550}{6.037}=1085$$ feet, velocity of sound, when *opposed* by the wind.

Same day, quarter past 1, P.M. barom. 29.67 inches, Fahr. therm. 60°; air dryer. *Gun at Charlton Lane. Sound heard at Shooter's Hill.* Distance 6550 feet. *Five rounds fired:* the intervals were, 5″.65 doubtful, 5″.8, 5″.78, 5″.76, 5″.78, omitting the first, the mean interval of the other four is 5″.78.

$\frac{6550}{5.78} = 1133\frac{1}{2}$ feet, velocity of sound, when *aided* by the wind.

$\frac{1}{2}$ $(1085 + 1133\frac{1}{2}) = 1109\frac{1}{4}$ feet inferred velocity of *sound* independently of the wind; therm. 59°.

And $\frac{1}{2}$ $(1133\frac{1}{2} - 1085) = 24\frac{1}{4}$ inferred velocity of the wind at the times of the experiment, supposing it to be nearly the same at both times. This agrees quite as nearly as could be expected with the mean velocity of the wind determined by the anemometer.

Same day, May 23, half past 11 A.M. barom. 29.67 inches. Fahr. therm. 58°: air humid; wind as before.

Before the gun was removed from *Shooter's Hill, six rounds* more were fired. The intervals in which the sound reached *Kidbrook Lane,* were 8″.1, 8″.125, 8″.13, 8″.15, 8″.1, and one very doubtful. The mean of these is 8″.121. Distance 8820 feet.

$\frac{8820}{8.121} = 1086$ feet, velocity of sound, opposed by the wind.

Here the sound was but just audible, the wind diminishing its intensity exceedingly.

Same day, therefore, the gun was removed to *Kidbrook Lane,* while I went back to *Shooter's Hill.*

Half past 12, barom. 29.67 inches; Fahr. therm. 60°; air dryer; wind as before. *Six rounds were fired.* The intervals between the flash and the report were 7″.8, 7″.7, 7″.8, 7″.78, 7″.78, and one very doubtful; mean 7″.77.

$\frac{8820}{7.77} = 1136$ feet, velocity of sound, when *aided* by the wind.

$\frac{1}{2}$ $(1086 + 1136) = 1113$ feet, inferred velocity of the *sound* independent of the wind; therm. 59°. $\frac{1}{2}$ $(1136 - 1086) = 25$ feet inferred velocity of the *wind,* nearly as before.

The same day, May 23, in the afternoon, the wind subsided, so as not to exceed 6 or 8 feet per second, while the temperature of the air remained nearly the same. I anxiously availed myself of this opportunity to ascertain the velocity of the sound, when scarcely affected by the wind. Mortars and howitzers were firing from the battery, the former at an angle of 45°, the latter at low angles for Ricochet practice. At $3\frac{1}{2}$ P. M. when the barom. was at 29.68 inches, Fahr. therm. at 60°, the sun shining, I took a station 3100 feet from the battery, and in a direction nearly perpendicular to that of the wind, then gently blowing. I observed the intervals between the flash and the report, for *six rounds*, of which the first three were with howitzers, the next three with mortars; these were successively 2″.77, 2″.76, 2″.79, 2″.79, 2″.8, 2″.8; their mean 2″.786.

$$\frac{3100}{2.786} = 1112 \text{ feet, velocity of sound; therm. } 60^\circ.$$

In these latter experiments the sound was very distinct and sharp: the result, though drawn from a short distance serves to confirm the preceding results on the same day.

Thursday, August 7. On this day, which was cloudy, but with intervals of sunshine, I employed the same 6 pounder as before, sometimes with charges of 8 oz. of powder, at others, when the distance required it, with 12 oz. The wind was quite brisk, varying in velocity from 30 to 35 feet, as determined by an anemometer.

At eleven o'clock A. M. barom. 29.80 inches; Fahr. therm. 66°; air dry, cloudy, but sun shining; wind nearly *opposing* the motion of the sound, and having a velocity of 30 feet. *Six rounds* were fired from *Shooter's Hill*. The intervals occupied in the passage of sound from thence to Kidbrook Lane, distance 8820 feet, were 8″.1, 8″.15, 8″.16, 8″.13, 8″.13, 8″.12; their mean, 8″.13.

$$\frac{8820}{8.13} = 1085 \text{ feet, velocity of sound, when } \textit{opposed} \text{ by the wind.}$$

Same day, August 7, quarter past 1, P. M. barom. therm. wind and weather as before.

The gun being placed in Kidbrook Lane, I went to the station on Shooter's Hill. Six rounds were fired, and the intervals occupied in the transmission of the sound were $7''.7$, $7''.75$, $7''.68$, $7''.67$, $7''.72$, $7''.68$; their mean $7''.7$.

$\frac{8820}{7.7} = 1145\frac{1}{2}$ feet, velocity of sound, when *aided* by a wind of about the same velocity as the former.

$\frac{1}{2}(1085 + 1145\frac{1}{2}) = 1115\frac{1}{4}$ feet, velocity of *sound*; therm. 66°,

$\frac{1}{2}(1145\frac{1}{2} - 1085) = 30\frac{1}{4}$ feet, velocity of the *wind*.

Same day, August 7, half past 11, A.M. barom. 29.80 inches; Fahr. therm. 64°, the wind blowing in the same direction as before, with (an estimated) velocity of 30 feet; air dry, cloudy, no sun. The same 6 pounder gun was fired from the Shooter's Hill station with a charge of 12 oz. of powder, and I took a station on Blackheath 20 feet farther than on January 9, its distance being 13460 feet from the gun.

Six rounds were fired, one of the intervals was very doubtful; the others were $12''.4$, $12''.38$, $12''.42$, $12''.38$, $12''.4$, $12''.4$; their mean $12''.396$.

$\frac{13460}{12.396} = 1085.8$ feet, velocity of sound when opposed by the wind.

Being fearful of bringing the gun to Blackheath, in the vicinity of so many carriages as were incessantly passing, I could not *here* avail myself of the benefit of comparing the above intervals with those in which the direction of the transmission should be reversed. I venture, therefore, to *add* the velocity of the wind to that of the sound, as obtained by the experiment, and thus obtain $1085.8 + 30 = 1116$ feet nearly, for the velocity of sound, the therm. standing at 64°.

Monday, August 18. On this day, the same 6 pounder gun was placed upon the wharf by the side of the Thames in the *Royal Arsenal,* and I took a station at the opposite extremity of the *Gallion's Reach,* not far from the mouth of *Barking Creek;* the distance from the gun was 9874 feet, the time of high water there, on that day, was about 11 o'clock, A. M.

At half past 11, A. M. barom. 29.84 inches; therm. 66° ; air dry, sky rather cloudy; very gentle wind nearly perpendicular to the line of transmission of the sound: *Six rounds* were fired with the muzzle of the gun towards me: the intervals were 8″.8, 8″.84, 8″.86, 8″.86, 8″.83, 8″.85; their mean, 8″.84.

At three quarters past 11, A. M. barom. &c. as before, *six more rounds* were fired, the gun muzzle being directed *from* us (up the river) in a horizontal angle of about 140 degrees: the intervals were 8″.86, 8″.84; 8″.82, 8″.82, 8″.85, 8″.86; their mean 8″.841,

$\frac{9874}{8.84} = 1117$ feet, vel. of sound; therm. 66° over a surface of water.

Although there was no perceptible difference in the mean *intervals* occupied by the transmission of sound, in the two different directions of the gun, yet there was a considerable modification of the *intensity;* the sound being *much* weaker when the gun muzzle was directed westerly, up the river, than when it was pointed down Gallion's Reach, towards the place where I stood. In the former case, too, besides the first report, which was marked and distinct, though comparatively feeble, there was a series of audible re-percussions, at intervals of about a tenth of a second, and gradually dying away: these, I conjecture, were reflected sounds from the faces of stone houses and other buildings standing on, or near the side of the river, at Woolwich.

Same day, August 18, one o'clock, P. M. barom. 29.82 inches, Fahr. therm. 66°, fair, but cloudy; scarcely any wind: I took a station on the Essex bank of the Thames perpendicularly opposite the large storehouse on *Roff's Wharf* at Woolwich, in order to ascertain the interval occupied by both the direct and the reflected transmission of the sound from a musquet fired by my side, and returned in an echo from the front of the said storehouse. The distance from my station to the front of the storehouse, determined carefully by a trigonometrical operation, was 1523 feet.

Of *eight rounds* fired from the musquet, I failed twice in the appreciation of the interval between the sound and the returning echo, from a very wrong estimate of its probable duration; and that from an erroneous impression as to the time observed by *Dr. Derham* in a similar experiment.* Of the remaining *six rounds*, the musquet pointed *across* the river, the intervals were 2″.7, 2″.75, 2″.74, 2″.72, 2″.75, 2″.74; their mean 2″.73.

Next, *three rounds* were fired, the musquet being pointed directly *from* the river; the intervals were 2″.7, 2″.73, 2″.76; mean as before.

Lastly, *four rounds* were fired *along the bank*, at an elevation of about 45°; the intervals were 2″.75, 2″.7, 2″.73, 2″.74; mean as before.

Distance occupied by the direct and the reflected sounds 3046 feet.

$\frac{3046}{2.73} = 1116$ feet, velocity of sound, across a surface of water, half direct, half reflected; therm. 66°.

The near agreement of this with the former result on the same day, serves to confirm the opinion that *direct* and *reflected* sounds move with the same velocity.

Thursday, August 21, three o'clock P. M. barom. 29.86 inches, Fahr. therm. 64°; clear sunshine; wind scarcely perceptible, westerly.

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* *He* made it 3 seconds, by means of a *half-second* pendulum. My erroneous recollection of his experiment led me to anticipate an interval of between 4 and 5 seconds. I could not account for the supposed discrepance, until after my return home, when on examining Derham's paper, and computing the real breadth of the river from my trigonometrical operation, I found the correspondence of the two experiments to be quite as great as could be expected, considering the different natures of the chronometers employed, and the varying breadth of the river.

Mortars were firing from the battery, and I took a station 3900 feet south of it. I observed the intervals between the flash and the report in six successive rounds: they were 3″.5, 3″.5, 3″.48, 3″.52, 3″.5, 3″.5, respectively; the mean being 3″.5.

$\frac{3900}{3.5}=\frac{7800}{7}=1114\frac{2}{7}$ feet, velocity of sound, therm. 64°.

These are all the experiments in reference to the velocity of sound, as transmitted through the atmosphere, which I have yet been able to make. Their chief results may be brought into one view as below.

		Feet.
Velocity of sound, Fahr. therm.	27°	1094.2
——— ditto ———	33°	$1099\frac{1}{6}$
——— ditto ———	35°	1102
——— ditto ———	45°	$1107\frac{2}{3}$
——— ditto ———	59°	$1109\frac{1}{4}$
——— ditto ———	60°	1112
——— ditto ———	64°	$\begin{cases}1114\frac{2}{7}\\1116\end{cases}$
——— ditto ———	66°	$\begin{cases}1116\\1117\end{cases}$

Of these results, some have been obtained in the day-time, others in the night; some when the sound has been transmitted over the surface of the earth, others when it has been transmitted over the surface of water; some are the result of direct sound, others of both direct and reflected sound; some from the report of cannons, others of musquets, others from the sound of bells.

Were these the only experiments on the subject that had ever been made, I should not regard them sufficiently extensive to justify me in deducing from them even an *approximative* rule. But as they have been made with great care, I may at least venture to present a rule, which, while it includes with only slight discrepancies all the preceding results, is simple enough to be easily recollected by practical men; and may, perhaps, be employed in our own climate. It is this:—

At the temperature of freezing, 33°, *the velocity of sound is* 1100 *feet per second.*

$$\left.\begin{array}{l}\textit{For lower temperatures deduct}\\ \textit{For higher temperatures add}\end{array}\right\}\ \textit{half a foot.}$$

$$\left.\begin{array}{r}\text{From the 1100}\\ \text{to the 1100}\end{array}\right\}\ \textit{for every degree of difference from } 33^{\circ} \textit{ on}$$

Fahr. therm.; the result will show the velocity of sound, very nearly, at all such temperatures.

Thus, at the temperature of 50°, the velocity of sound is,

$$1100 \times \tfrac{1}{2}(50-33) = 1108\tfrac{1}{2} \text{ feet.}$$

At temperature 60°, it is $1100+\frac{1}{2}(60-33)=1113\frac{1}{2}$ feet; agreeing with the experimental result quite within the limits of a practical rule.

The theorem 333.44 met. $\sqrt{1+00375\ t}$, before cited, gives nearly 1094 feet for the velocity at the freezing point; and 1114 feet for the temperature 10° centigrade, or 50° Fahrenheit: thus occasioning a greater augmentation to the velocity in the higher temperatures, than my experiments seem to indicate.

The above practical rule, so far as it may be entitled to confidence, may be useful, 1st, to the military man in determining the distance of an enemy's camp, of a fortress, a battery, &c. 2d, to the sailor, in determining the distance of another ship, &c. 3d, to the land surveyor in ascertaining the length of base lines, &c. in conducting the survey of a lordship or county; 4th, to the philosophic observer, in appreciating the distances of thunder-clouds during a storm. Yet, in either of these applications, the rule must be regarded as *approximative* only; because, few practical men can be expected to possess a time-measurer for less intervals than *tenths* of seconds (if indeed, so small): and an error of a tenth of a second, will occasion a mistake of from 37 to 40 yards in the estimate of the distance. Beyond this, however, the error need scarcely ever extend; because a mean of 5 or 6 careful experiments will usually give the interval to a degree of correctness far within the limits just specified. Indeed, an error of from 30 to 40 yards in a distance of three or four miles, will, on most occasions, where such approximative estimates are required, be of but small consequence. When the distance exceeds four miles, this method of approximating to it can only be employed under favorable circumstances of a very quiescent atmosphere, &c.: on which account, I felt scarcely any desire to extend my own experiments to stations more remote from each other, than those which I selected on Shooter's Hill and Blackheath.

Combining the results of experiments here recorded with those which have been formerly deduced by Derham and others, we may, I think, conclude unhesitatingly:

1st, That sound moves *uniformly;* at least, in a horizontal direction, or one that does not deviate greatly from horizontality.

2d, That the difference in intensity of a sound makes no appreciable difference in its velocity.*

3d, Nor, consequently, does a difference in the instrument from which the sound is emitted.

4th, That wind greatly affects sound in point of *intensity;* and that it affects it, also, in point of *velocity.*

5th, That when the direction of the wind *concurs* with that of the sound, the *sum* of their separate velocities gives the *apparent*

* The consecution of the notes in a tune, notwithstanding the difference in their intensity, being uninterrupted when heard at a distance, furnishes an elegant and decisive confirmation of this proposition.

velocity of sound; when the direction of the wind *opposes* that of the sound, the *difference* of the separate velocities must be taken.

6th, That in the case of echoes, the velocity of the *reflected* sound, is the same as that of the *direct* sound.

7th, That, therefore, distances may frequently be measured by means of echoes.

8th, That an augmentation of temperature occasions an augmentation of the velocity of sound; and *vice versa.*

(See Newton, Principia, Lib. 2. Prop. 50. Parkinson's Mechanics, Vol. II. p. 148.)

The enquiries with regard to the transmission of sound in the atmosphere,* which notwithstanding the curious investigations of *Newton, Laplace, Poisson,* and others, require the farther aid of experiment for satisfactory determination, are, I think, the following: viz.

1st, Whether hygrometric changes in the atmosphere have much or little influence on the velocity of sound?

2d, Whether barometric changes in the atmosphere have much or little influence?

3d, Whether, as *Muschenbroek* conjectured, sound have not different degrees of velocity, at the same temperature, in different regions of the earth? And whether *high* barometric pressures would not be found (even independently of temperature) to produce greater velocities?

4th, Whether, therefore, sound would not pass more slowly between the summits of two mountains, than between their bases?

5th, Whether sound, independently of the changes in the air's elasticity, move quicker or slower near the earth's surface, than at some distance from it?

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* I say nothing in this Paper, of the transmission of sound through the gases, along metallic conductors, &c. These furnish a most interesting department of separate enquiry.

(See *Savart's* interesting papers on the communication of sonorous vibration.)

6th, Whether sound would not employ a longer interval *in passing over a given space, as a mile, vertically upwards,* than in a *horizontal* direction? and, if so, would the formula which should express the relation of the intervals include more than thermometric and barometric coefficients?

7th, Whether, or not, the principle of the parallelogram of forces may be employed in estimating the effect of wind upon sound, when their respective velocities do not aid, or oppose each other in the same line, or nearly so?

8th, Whether those eudiometric qualities, generally, (whether hitherto detected or not) which affect the elasticity of the air, will not proportionally affect the velocity of sound? and if so, how are the modifications to be appreciated?

To the experimental solution of some of these enquiries I hope to devote myself at no very remote period: but others of them, it is evident, can only be satisfactorily answered, if ever, by means of a cautious classification of skilful experiments made by various philosophers in different parts of the globe.

OLINTHUS GREGORY.

Royal Military Academy, Woolwich,
October 25, 1823.

Postscript. Since the above paper was drawn up, a friend has favoured me with the perusal of *Mr. Goldingham's* account of his experiments in reference to the velocity of sound, made at Madras. From this very interesting Dissertation I shall venture to transcribe the following Table of the mean motion of sound for each month of the year, at Madras.

Month.	Mean height of Barometer.	Mean height of Thermometer.	Mean height of Hygrometer.	Velocity in a Second.
	Inches.	Degrees.	Dry.	Feet.
January	30.124	79.05	6.2	1101
February	30.126	78.84	14.70	1117
March	30.072	82.30	15.22	1134
April	30.031	85.79	17.23	1145
May	29.892	88.11	19.92	1151
June	29.907	87.10	24.77	1157
July	29.914	86.65	27.85	1164
August	29.931	85.02	21.54	1163
September . . .	29.963	84.49	18.97	1152
October	30.058	84.33	18.23	1128
November . . .	30.125	81.35	8.18	1101
December . . .	30.087	79.37	1.43	1099

These results serve, as far as they go, to confirm the suspicions which I have long entertained, that the velocity of sound is somewhat different in different climates; and that hygrometric changes have more influence than has usually been imputed to them by theorists. The velocity varies from 1099 to 1164 feet, while the barometric range does not exceed a quarter of an inch, and the thermometer varies only from about 78° to 88°. But the indications of hygrometric change are considerable, passing from 1 to nearly 28 degrees. Unfortunately, however, we are not able to make such satisfactory deductions from these curious experiments as they might have furnished, had Mr. Goldingham described the construction of his hygrometer, and the fixed points, or the extent of its scale.

Royal Military Academy, Nov. 8, 1823.

X. *On the Association of Trap Rocks with the Mountain Limestone Formation in High Teesdale, &c.*

By the Rev. A. SEDGWICK, M.A. F.R.S. & M.G.S.

WOODWARDIAN PROFESSOR, AND FELLOW OF TRINITY COLLEGE,

[Read *May* 12, 1823; *March* 1, and *March* 15, 1824.]

Section I.

On the great Calcareous Chain of the North of England.

In a paper which was read to the Society last year*, I described some of the principal phenomena exhibited by the great dykes of Cockfield Fell and Cleveland. I also brought forward some facts which made it probable, that either the dykes above-mentioned, or other masses similarly associated with the coal formation, were prolonged into High Teesdale, and connected with the beds of trap which form so extraordinary a feature in that region. Trap of High Teesdale.

Having concluded, on evidence which seemed quite irresistible, that this intrusive class of rocks was of igneous origin, it became the more necessary to examine the trap of High Teesdale, provincially termed the great *Whin Sill* †: especially since

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* The paper alluded to is printed in this Volume, p. 21, &c.

† The word *Sill,* is, in some of the mining districts of the north of England, synonymous with *stratum.* By the *Whin Sill* is, therefore, understood a large tabular mass of trap imbedded in, and nearly parallel to, the other strata.

it has often been triumphantly appealed to, as affording a proof that such formations, may, at least in some instances, be of aqueous origin. With this view, I ascended High Teesdale in the autumn of 1821, and, although interrupted by torrents of rain which rendered the banks of the river in many places inaccessible, succeeded in examining some localities which promised to throw light on the true relations of the trap. Last summer I again passed through that valley, and had an opportunity of verifying the observations of the preceding year. My time was, however, too limited to enable me to complete the task which I had undertaken. What is now offered to the Society, must, therefore, be considered as an imperfect sketch drawn from observations, directed, almost entirely, to the elucidation of the true relations of the trap to the contiguous strata. Under such circumstances it will be important carefully to separate such appearances as are doubtful or hypothetical from those which are plainly exhibited; and to draw our conclusions, respecting the origin of the basaltic rocks, from those facts only which are established on the clearest evidence*.

Calcareous Chain of the North.

Those who are acquainted with the geological features of England, must have remarked the chain of calcareous hills which runs nearly north and south through a part of Yorkshire, Durham, and Northumberland. The rocks forming all the higher parts of this chain are composed of limestone, sandstone, and shale (slate-clay) repeated in numerous alternations. From their continuity and prevailing characters, they are undoubtedly members of one formation—the *mountain* or *metalliferous limestone*. Yet, in districts considerably remote from each other, it is only

* Since the Paper was first read to the Society, the author has made a third visit to High Teesdale, and as all his previous observations have been carefully revised, he has considerable confidence in the general accuracy of the details which are now offered to the public.

possible to institute a general comparison of the subordinate beds; as all the minuter features are modified or changed by the action of local causes.

From one end of the mountain ridge to the other, the beds on its eastern flank, dip to a variable point between north-east and south-east, and gradually pass under some deposits connected with a part of the great coal formation. Between the mountain limestone and coal formation, there is not, however, any precise line of demarcation; as in many instances, they are decidedly interlaced with each other; the beds of coal and carbonaceous shale appearing among the upper beds of the lower series. It is, in part, to this cause that we must attribute certain discrepancies in our best geological maps. Thus in the map to which I referred in a former Paper (*supra*, p. 25.) Mr. Winch extends the region of the limestone to Winston on the Tees, while Mr. Greenough places the demarcation near Eglestone, more than ten miles farther up the river. The line marked out by Mr. Greenough agrees better with the general features of the district. But on the other hand, it may be contended, that thin beds of limestone alternate with the sandstone and shale in the immediate vicinity of Winston Bridge*.

The eastern flank of the calcareous chain is intersected by many transverse vallies, in which the waters of the mountain-torrents unite and fall down into the lower carboniferous region; and from which they either find an immediate passage into the sea, or descend into the great plain of the new red sandstone. Vallies which traverse beds of the same formation, nearly in the Transverse Vallies.

* The discrepancy is not, however, so great as might appear at first sight, for Mr. Greenough places some of the lower beds of the *Coal-measures*, and some of the upper beds of the *Lead-measures*, in a separate class under the name of the *Millstone Grit*. This classification may be good for a general map of England; but no one would, I think, have adopted it, who had taken his type of the great carboniferous series from this part of the north of England.

same direction, must necessarily agree in some of their characteristic features. Amidst many beauties and many objects of interest which are common to them all, each possesses some peculiarities of structure which would well deserve a separate illustration. They all agree in exhibiting the most obvious traces of powerful denuding forces. The vallies are all nearly perpendicular to the general bearing of the strata, and often exhibit, on their opposite sides, a succession of beds which tally with each other even in their minutest subdivisions. In not one of them is there an indication of dislocation, contortion, or subsidence, sufficient to account for the present inequalities of the surface. On the contrary, the salient and re-entering angles which determine the present directions of the descending waters, and still more the various ramifications of the rivulets near the higher parts of the chain, plainly indicate the action of diluvian torrents. Any one who considers such a conclusion as doubtful or hypothetical, has only to examine the enormous beds of transported materials which are accumulated on the sides of the transverse vallies, and near the gorges where they first enter into the plains. He will there find the broken fragments of the rocks which once filled up the inequalities of the higher region, not only agreeing, in every respect, with the strata from which they have been separated, but heaped up in those very places which first offered a lodgement for them, after being propelled by the descending currents.

Stratification and Structure.

There are other phenomena, originating in the physical structure of the country, which are common to almost all the vallies before alluded to. For some distance above the places where the rivers first enter on the lower region, the waters descend down planes which are less inclined than the neighbouring strata. Hence, in ascending any of these transverse vallies, we pass over the outgoings of a succession of strata; and where the denudations are considerable, sometimes reach the lower part of the series of beds, which forms the foundation of the mountains. But on ascending towards the last ramifications of the rivers, the inclination of the vallies always begins to exceed that of the strata, so that we again cross some of the beds, which, by their natural rise, had ascended from the region we had left behind. For example:—In following the course of the Tees towards its source, we first traverse the plain of the new red sandstone, and then cross the line of the magnesian limestone, and enter on the carboniferous series. Continuing to ascend, we pass over the lower beds of the coal formation, and at length reach the beds associated with the metalliferous limestone. The successive members of the new series occupy the banks of the river for several miles. But on reaching the upper part of High Teesdale, we may ascend, by one of the ramifications of the river, towards the top of the chain, and find a part of the same series of strata which we had before passed over, presented to us in an inverted order.

The structure we have described, is evidently favourable to a minute examination of the geological relations of each district, and the existence of a great many rich metalliferous veins in the same region, has directed the attention of practical men to the phenomena of stratification. Hence, many excellent sections, both of the calcareous, and carboniferous series, which form the eastern flank of the mountain chain, have been already given to the public*. Sections of this kind, made in situations which are considerably remote from each other, give us every information respecting the analogies of structure and composition; but, as was before observed, seldom enable us to identify the subordinate members of the prevailing formations.

* See Forster's Section of the Lead-measures. Winch's Paper on the Geology of Northumberland and Durham, *Geological Transactions*, vol. IV. &c.

SECTION II.

On the Structure of High Teesdale.

High Teesdale.

THE descriptions given in the preceding Section, are sufficiently general to be applied to the whole system of the vallies which traverse the eastern flank of the great calcareous chain of the north of England. I shall now direct my attention, almost exclusively, to the phenomena which are exhibited in High Teesdale. As a matter of convenience, this part of the valley will be supposed to commence near the village of Eglestone. For above that place the dale begins to assume an austere aspect, which differs greatly from the softer and more picturesque features of the lower banks of the river. There also commences a series of phenomena to which I wish principally to direct the attention of the Society.

Denudation.

That High Teesdale has been formed by denudation, is proved unequivocally, by the whole contour of the valley, by the ramifications of the tributary streams, and by the accumulations of diluvial matter, in almost every place, where it was possible for it to find a lodgement*. This assertion is, however, by no means intended to exclude the supposition of pre-existing inequalities and fractures, which may have enabled the diluvian torrents to pass in one direction rather than another. But facts of this kind are, for the most part, too much removed from direct observation to be deserving of much attention.

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* A person, practised in observations of this kind, would, from the mere external form of the country, generally succeed in pointing out the places where diluvial-gravel has been much accumulated. Transported materials, containing large blocks of trap, limestone and sandstone, are in great abundance near the junction of the Lune and the Tees. On the right bank of the Tees, between Lonton and Eglestone, the gravel is in some places not less than a hundred feet thick; but it gradually becomes comminuted, and in some places passes into a coarse sand.

Among the circumstances of most importance in the structure of High Teesdale, appear to be the following:

(1.) The want of correspondence in the strata, on the two sides of all that part of the valley, which extends five or six miles above Eglestone.

(2.) The manner in which masses of trap are, in this part of the valley, associated with the other strata; more especially the appearance of a great *bed* of trap (the great *Whin-Sill* of the mining district) first on the south side of the valley, and afterwards in the banks, and in the bed of the river.

(3.) The appearance of a great transverse *fault*, which intersects the whole valley about a mile above the High Force (in a direction which is about N. N. W. and E. S. E.), and throws the whole system of strata on the south-west side of its range, twenty or thirty fathoms above their natural level: thus exhibiting in the highest part of Teesdale, a repetition of the phenomena which appear at the junction of the trap with the other strata.

I. To these facts I now proceed to call the attention of the Society, in the order just pointed out. In the hope of making these complex phenomena better understood, I shall subjoin a detailed section of the strata in High Teesdale; and shall add a section of the upper beds of the same series, obtained from the lead-works of Old Langdon, in the High Moors, which extend to the north-west of Middleton. General Section.

(1.) *A general Section of the principal Strata in High Teesdale, commencing with the highest Bed of the Series*.*

No.		Fath.	Ft.	No.		Fath.	Ft.
1.	Millstone Grit	6	0	28.	Plate	1	3
2.	Plate	3	0	29.	Hazle	2	0
3.	High Slate Sill	3	0	30.	Plate	3	0
4.	Plate	2	0	31.	Limestone	1	0
5.	Low Slate Sill	4	0	32.	Plate and Coal	1	0
6.	Plate	4	0	33.	High Brigstone Hazle	6	0
7.	Ironstone	2	0	34.	Plate	2	0
8.	Plate	5	0	35.	Limestone	1	3
9.	Firestone	8	0	36.	Plate	0	3
10.	Plate	5	0	37.	Low Brigstone Hazle	6	0
11.	Pattison's Hazle	1	3	38.	Plate	6	0
12.	Plate	6	0	39.	Limestone	3	0
13.	Little Limestone	1	0	40.	Hazle Plate and Grey Beds	7	0
14.	Plate	1	0	41.	Limestone	1	3
15.	White Sill	1	3	42.	Hazle Plate and Grey Beds	5	0
16.	Plate	0	3	43.	Limestone	2	0
17.	High Coal Sill	2	0	44.	Hazle	1	3
18.	Plate	2	0	45.	Hazle Plate and Grey Beds	6	0
19.	Low Coal Sill	2	0	46.	Plate	6	0
20.	Plate	1	0	47.	Scar Limestone	6	0
21.	Great Limestone	8	0	48.	Hazle	2	0
22.	Tuft Sill	2	0	49.	Hazle Plate and Grey Beds	4	0
23.	Plate	2	0	50.	Limestone	1	0
24.	Quarry Hazle	3	0				
25.	Plate	6	0				
26.	Limestone	4	0				
27.	Hazle	4	0				

* The mountains on the north side of Teesdale exhibit several beds superior to the Millstone Grit, (No. 1.) which are not here enumerated.

	Fath.	Ft.
51. Plate and Grey Beds....	4	0
52. Hazle	1	3
53. Plate	2	0
54. Limestone	3	0
55. Plate	0	3
56. Hazle	1	3
57. Plate	1	0
58. Limestone and Hazle ...	1	3
59. Plate	1	0
60. Limestone	2	0
61. Plate	1	0
62. Hazle Plate and Grey Beds	5	0
63. Limestone	1	0
64. Hazle Plate and Grey Beds	3	0
65. Limestone	0	3
66. Plate	4	0
67. Limestone	3	0
68. Hazle	4	0
69. Plate	4	0
70. Limestone	8	0
71. Whin Sill	12	0
72. Hazle	0	1
73. Limestone	2	0
74. Hazle Plate and Grey Beds	5	0
75. Unknown	4	4

(2.) *Section of the Strata, as they have been proved in the Works of Old Langdon, commencing with the highest.*

No.	Fath.	Ft.	In.
1. Plate	3	2	8
2. Grindstone Sill	3	3	8
3. Plate	9	5	4
4. Millstone Grit......	5	3	11
5. Plate	14	5	10
6. High Slate Sill	13	2	1
7. Plate	9	0	10
8. Coal (a variety of Carbonaceous Shale).	2	4	0
9. Plate	3	2	10
10. Grey Beds	2	2	1
11. Low Slate Sill......	2	3	5
12. Plate	2	5	9
13. Hazle or Slate......	0	5	0
14. Plate	3	0	0
15. Ironstone	1	2	0
16. Plate	4	0	0
17. Firestone	7	3	0
18. Plate	3	3	0
19. Pattison's Sill	1	1	0
20. Plate	3	0	0
21. Little Limestone	1	1	0
22. Grey Beds	1	3	0
23. Plate	1	0	0
24. White Sill	1	2	0
25. Plate	1	1	0
26. High and Low Coal Hazle	13	3	0
27. Great Limestone....	7	3	0
28. Tuft, or Water Sill..	1	3	0
29. Plate	2	0	0

For these two sections, I am indebted to Mr. Walton, a mine agent at Middleton, in Teesdale. It may, perhaps, not be improper to add, that the word *sill* is synonymous with *stratum*—that *plate* designates all varieties of *slate clay*—that *grey beds* are composed of slate clay and siliceous grit—that *hazle* and *firestone* designate varieties of siliceous grit. The different *slate sills* are composed of fine schistose varieties of siliceous grit, which are sometimes used for roofing-slate. The *coal sills* of the district are principally composed of carbonaceous shale, with a few thin layers of impure coal, much impregnated with sulphur. Connected with this subject, see Forster's Section of the Strata from Newcastle upon Tyne to Cross Fell, and Transactions of the Geological Society, Vol. IV. p. 63—66.

Discrepancies in the Sections.

It will be seen (by comparing the first section, from No. 1. to No. 21, with the second section, from No. 4. to No. 27.) how difficult it is to identify all the subordinate parts of the formation, even in the same district. In addition to the faults and dislocations, which throw great difficulties in the way of a correct comparison of the successive strata; the beds of grit and shale perpetually change their characters, and often replace each other. The beds of limestone are, on the whole, more regular and continuous; and it is to their presence that all of the transverse vallies above described, owe their most remarkable features*.

Great Teesdale fault.

On commencing an examination of High Teesdale near Eglestone Bridge, we soon discover that want of correspondence in the strata on the opposite sides of the river, to which I before alluded.

* The elaborate section of the strata from Newcastle upon Tyne to Cross Fell, published by Mr. Westgarth Forster, is, I believe, partly compiled from various registers of the Lead-works which are conducted on Aldstone Moor, and in the higher parts of Weardale. For the purpose of further explaining the true relations of the great *Whin Sill*, to the inferior or superior strata, I will subjoin a short extract from that Section.

EXTRACT

The *great limestone* (Teesdale Section, No. 21) appears in the north bank of the river above Eglestone Bridge; and after traversing some ground where the relative position of the strata is much disturbed by *faults,* it is again well exposed about a mile and a half above Eglestone, in the side of the high bank called Foggerthwaite, near which place a large quarry is opened in it. From thence it may be followed without difficulty through the mid region of the mountains on the north side of the valley.

On the south side of the Tees, the *great limestone* is seen, at a considerably higher elevation, gradually rising above the village of Mickleton into the brow of the hill which overlooks the

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Extract *from* Forster's Section, *2d Edit. p.* 169.

No.	Yds.	Ft.	In.	No.	Yds.	Ft.	In.
169. Scar Limestone	10	0	0	183. Plate	0	1	0
170. Plate	1	0	0	184. Hazle	3	0	0
171. Hazle	1	0	0	185. Plate	1	2	0
172. Coal	0	0	6	186. Single Post Limestone	2	0	0
173. Plate	2	1	6	187. Plate	1	0	0
174. Hazle	4	0	0	188. Greystone	1	0	0
175. Plate	1	0	0	189. Plate and Grey Beds	18	0	0
176. Hazle	0	2	0	190. Tyne Bottom Limestone	8	0	0
177. Plate	3	0	0	191. Whetstone Bed	1	0	0
178. Hazle	0	2	0	192. Whin Sill	40	0	0
179. Plate	0	2	0	193. Plate	3	0	0
180. Hazle	0	1	0	194. Hazle	3	1	0
181. Cockle Shell Limestone	0	2	0	195. Plate	3	2	0
182. Hazle	0	2	6	196. Hazle	3	2	6

By comparing this extract with the latter part of the general section of the strata of High Teesdale, we see at once the impossibility of reconciling all the successive terms of each series. It is undoubtedly true, that we find in both sections, a great bed of trap interposed between the other strata. The details do not, however, by any means demonstrate that this bed of trap has a given place in a regular series of deposits. Those who have contended for the aqueous origin of the great Whin Sill, because it occupies a fixed place in a regular succession of aqueous deposits, found their chief argument on an assumption, which is not justified by any of the sections which have been published.

lower part of Lunedale. This want of agreement in the corresponding parts of the same formation is not accounted for by their direction and their dip; but is supposed to originate in a longitudinal *fault*, (the great *Teesdale fault*) which commences near Eglestone Bridge, and throws the whole system of strata on the north side of the valley below their original level. This *fault* is not interposed hypothetically for the mere purpose of meeting the difficulty; for we have direct evidence for its existence in the bed of the river above Eglestone Bridge, and in the right bank of the Lune very near its junction with the Tees. Between these two places, the line of *fault* passes on the south side of the Tees through the low grounds under the village of Mickleton*.

Dislocations in Lunedale.

Before I proceed to trace the progress of this fault farther up Teesdale, it may be proper to remark, that in many of the lower parts of Lunedale, more especially on the west side of the rivulet near Greengate and Saddle Bow, the strata are in a state of inextricable confusion. The whole country appears to be externally modified by disturbing forces, which have rent asunder the beds of rock and heaved them entirely out of their natural position and inclination. From this part of the valley of the Lune, a great break in the strata ranges towards Teesdale, and

* In the mining districts of Durham, every species of nearly vertical fissure, by which the continuity of the strata has been interrupted, is called a *vein*. The corresponding beds, on the opposite sides of these fissures, are very seldom on the same level; and the interval is sometimes filled up with materials (provincially called *riders*) which are highly metalliferous.

In some parts of the Lodge Syke vein, which is worked in the hill immediately north of Middleton, the *rider* is not less than twenty-four feet wide. It consists of shattered fragments of the neighbouring strata mixed with oxide of iron; the whole being cemented and held together by quartz, fluor-spar, carbonate of lime, and great ribs of galena.

But there are many dislocations in the district where the *rider* is altogether of a different character. The masses on opposite sides of the fracture are in some places in almost immediate contact, in other places they are separated from each other by thin beds with an imperfect vertical cleavage, which were probably formed at the time of the fracture by the motion of the broken edges past each other. It is from indications of this kind that the great *Teesdale fault* has been traced in the bed of the river near Eglestone.

intersects the *fault* above described not far from the junction of the two rivers. Some complex phenomena result from this double dislocation, especially at Leeky Hill (in the angle between the Lune and the Tees) where the *great limestone* (No. 21.) is brought in contact with the high *brigstone hazle* (No. 33.)

Want of correspondence in the opposite sides of Teesdale.

Above the entrance of the Lune, the true order of the strata on the north side of Teesdale may be ascertained without much difficulty. The low *brigstone hazle* (No. 37.) occupies the bed of the river near Middleton, and the *great limestone* is, therefore, to be sought at a considerable elevation on the side of the neighbouring mountain. To pass the outcrop of the other beds (below No. 37.) we must ascend still higher up the valley, and we may then observe the whole series (to No. 71. inclusive) rising in regular succession from beneath the upper parts of the formation.

Since the mean direction of all this part of Teesdale is nearly transverse to the general bearing of the strata; it is obvious that the same rocks might be expected to appear in both sides of the river. If, however, we cross to the right bank of the Tees, we find a bed of trap (to which there is no counterpart on the left bank) commencing near Lonton, and forming a very grand escarpment near the base of the mountain, until it stretches over the bed of the river about four miles above the place of its first appearance.

The strata which form the support of the trap, before it extends over the bed of the river, are generally concealed by diluvial gravel and vegetable soil; but the strata which rest upon it may be conveniently examined in many places on the flanks of the mountains which range on the south side of High Teesdale. Those who are best acquainted with the features of the country assert that they have in this way detected all the principal beds of the general section. It appears that the mountain, on the S.W. side of the valley, opposite Middleton, is crowned by the *four fathom limestone* (No. 26.) and a thin bed of grit. If, therefore, we assume that the great bed of trap is the representative of the *Whin-Sill* (No. 71.); it follows that the beds we pass over in ascending to the top of the mountain, must tally with those of the general section between No. 26. and No. 71. I confess that I have considerable doubts of the entire accuracy of this statement, as it assumes a consistency in the relative situation of the bed of trap, which we do not find in other localities. There can, however, be no doubt that many strata on the south side of the valley, are higher, perhaps by three or four hundred feet, than the corresponding strata on the north side*.

Range of the Fault.

This derangement in the neighbouring parts of the same formation, can only be accounted for by the prolongation of the great *Teesdale fault* in a direction nearly parallel to the escarpment of the trap. The attention of several practical men has been directed to this phenomenon, under the expectation of finding metalliferous deposits along the line of dislocation. Though disappointed in this expectation, they have found direct evidence for the existence of the *fault* in several places where the strata are laid bare by the river. I shall proceed to point out a few of the localities.

The indications of the great longitudinal *fault* in the bed of the river above Eglestone Bridge and on the right bank of the Lune, have already been pointed out. Similar appearances are seen farther up the valley on the road leading from Middleton to Lonton. The line of *fault*, though unquestionably continued on the south side of the river, is for some way lost in diluvium; but it is afterwards distinctly exposed in two places on the side

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* See the section of Teesdale, Pl. VIII. Fig. 1. The dislocation in this part of the valley is many times greater than that which is exhibited below the junction of the Lune with the Tees.

of the river, (opposite Breckholm and Low Houses) where the beds are set on edge in a form which unequivocally indicates the presence of a fault, and where the contiguous strata are tilted exactly in that way which would be occasioned by a *downcast* on the north side of the line of dislocation.

That the great Teesdale fault is continued far beyond the places last-mentioned cannot admit of doubt, though the exact line of its direction is in some measure hypothetical. Perhaps it may be considered to terminate in the bed of the river at Holwick Head, about half a mile below the High Force. For the trap there puts on the appearance of thin vertical beds, which so often mark the presence of a *fault;* though the quantity of depression on the north side must be incomparably less than it is in some of the lower parts of the valley.

By some of the practical men of the country, the great *Teesdale fault* is supposed to cross the river at Holwick Head, to range on the south-west side of High and Low Hag, and from thence to be continued along a depression of the country as far as Hunt Hall; where it is supposed to meet the great *transverse dyke* to be described in a subsequent part of this Paper. I am unwilling to admit the truth of this hypothesis. First, because it does not seem adequate to the explanation of some of the intricate phenomena exhibited in this part of the valley. Secondly, because the strata on the north side of the line, extending from Holwick Head to Hunt Hall, do not appear (as in the case of the great *Teesdale fault*) to be at all points thrown down below the level of the corresponding beds on the south side.

Before I proceed to consider the manner in which the trap is associated with the other strata, it may be proper to remark, that there are many other *veins* or *faults* which produce a considerable effect in modifying the relative position of the strata in the parts of the valley above described. To attempt to enumerate the whole of Other Veins and Dislocations.

them would lead me into details quite foreign to the objects of this Paper. Among the most remarkable may be mentioned the two following:

(1.) A vein which cuts through the escarpment of the trap (the great *Whin Sill*) and forms a great ravine near the village of Holwick. From thence it crosses the bed of the river and ranges through the hill immediately behind Middleton.

(2.) A great vein, or system of veins, which ranges across the river, in a direction N.N.W. and S.S.E., about two hundred yards below Holwick Head. It has been traced through a considerable extent of country, especially on the south side of the Tees; and all the beds on the N.E. side of its course are thrown up above their proper level*.

II. I now proceed to consider the manner in which masses of trap are associated with the other strata in certain parts of High Teesdale.

Trap on the North Bank of the Tees.

(1.) In a Paper which was read to the Society in 1822, I described a mass of trap, which occupies the left bank of the Tees, nearly opposite to the entrance of the Lune; and extends about a quarter of a mile to a high bank called Foggerthwaite, where it is seen abutting against a bed of slate clay†.

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* Where this vein intersects the great *Whin Sill*, it contains much sulphate of barytes, but does not appear to be metalliferous.

Though the great metalliferous veins of Teesdale intersect, indifferently, all the strata of the district, yet they are by no means equally rich in all parts of their course. Some particular parts of the formation are eminently productive (as for example, almost all the beds from No. 6. to No. 27. of the general section); but the great *Whin Sill* is not commonly regarded as one of the *bearing beds*. To this rule there are, however, some exceptions, for productive lead-veins have, near the top of High Teesdale, been worked in the *Whin Sill*. In one or two parts of High Teesdale, I have seen minute veins, containing sulphate of barytes and small cubes of galena, passing through the trap.

† This mass of trap is of considerably greater extent than I supposed when I wrote my former paper (*supra*, p. 24.) In the same part of the paper Eglestone bank is erroneously written for Foggerthwaite bank.

My last visit to Teesdale did not enable me to make out the true relations of this mass of trap; but I think there is no doubt of its immediate connexion with the dyke, which, after crossing the road, ranges about E. by N. and is seen again in the western branch of the rivulet which runs down to Eglestone.

In the course of last summer I also traced the second trap-dyke from the bed of the same rivulet (where it first appears about a quarter of a mile above the preceding) to the top of the ridge of hills which extend on the north side of the smelting-house: and I was enabled to ascertain that its mean bearing is very nearly magnetic E. and W. The two dykes do not, therefore, probably unite or intersect each other in their course through the eastern moors as I erroneously stated in my former paper, (*supra*, p. 25.); but on the contrary, appear to diverge from each other in that direction. With respect to the probable connexion of these two dykes with the Cockfield-fell and Quarringdon Hill dykes, I have nothing to add to what is already before the Society (*supra*, p. 40).

(2.) It will be proper in the next place to notice the trap which is associated with the shattered and dislocated strata on the west side of Lunedale. The first appearance of this kind occurs about a mile above Lonton, where the trap rests upon a highly inclined bed of indurated slate clay. Two or three similar and probably connected masses, may be traced on the north side of the road towards a conical hill called Saddle-bow. The upper part of this hill is composed of a great protruding mass of trap which appears to be the centre of the confusion which marks the neighbouring parts of the valley. From the cone of Saddle-bow to the ridge of hills on the side of Greengate farm, there are at least three places where the trap again appears at the surface; but of its true relation to the other strata, it is hardly possible to form any correct estimate.

Trap of Saddle-bow.

In the other direction, to the S.W. of Saddle-bow, it passes into the form of a regular *dyke*, the range and extent of which I had, however, no opportunity of examining.

I think it quite impossible to convey a correct notion of the physical structure of this part of Lunedale by mere verbal description. I have, therefore, endeavoured to shew the manner in which the intrusive rocks appear at the surface by help of a section (Pl. VIII. Fig. 3.) passing through Saddle-bow in a direction from N.E. to S.W. The filling up of this section is partly hypothetical. It is, however, sufficiently founded on direct observation to prove, that the trap can not have been originally interstratified with the other beds which form a regular part of the Teesdale series (Section 1. p. 8).

Whin-Sill.

(3.) I next proceed to consider the manner in which the great *Whin-Sill* appears among the strata of Teesdale. It is first seen in the bed of the Lune immediately under Lonton Chapel, where its whole thickness does not amount to more than eleven or twelve feet. In my former paper (p. 24.) I remarked the probable connexion between this bed of trap, and the mass, of the same mineralogical character, which appears in the north bank of the Tees. I still think this connexion very probable; but I must at the same time observe, that the trap on the north bank of the river has little appearance of being a regular bed, and is in contact with rocks which belong to a higher part of the general Teesdale section. The trap under Lonton Chapel is exactly parallel to the beds which are above it and below it; and by the gradual rise of the whole system of strata it is brought to the top of the escarpment on the left bank of the river, about two hundred yards from the former place (Pl. IX. Fig. 1.). Beyond this place it is lost under a great mass of gravel, but it again breaks out in the hill side, on the road leading from Lonton to Middleton. From thence it is continued, as has already been stated, for several miles on the south side of Teesdale, forming in many places a magnificent escarpment, and not unfrequently putting on a rude columnar form. That the *Whin-Sill* is here imbedded among the other strata and nearly parallel to them cannot be doubted. Its upper and lower surfaces are not, however, exactly parallel, as it continually increases in thickness in its range up the valley towards Holwick. Near that place its whole thickness is, perhaps, more than thirty fathoms.

As the mean inclination of the bed of the Tees, between Middleton and the High Force, is considerably less than the dip of the strata on the north side of the great longitudinal *fault;* it follows that in advancing up this part of Teesdale by the side of the river, we must necessarily cross a series of beds of the general section presented in a descending order. In this way (after passing all the beds from No. 38 to No. 70.) we find the *Whin-Sill* (No. 71.) rising up in the bed of the river about half way between Bowlee Beck and Winch Bridge. Not far from the last-mentioned place it comes into immediate contact with the trap on the south side of the great *fault*—a fact which seems to prove, that the dislocation, produced by the rupture of the strata, is much less there, than it is farther down the valley. The collocation of the strata will be best understood by a reference to the accompanying map (Pl. VII.) and to a transverse section of the strata (Pl. VIII. Fig. 2.) From the place of its first appearance, between Bowlee Beck and Winch Bridge, the *Whin-Sill* extends on the north side of the *fault* for more than a mile through a low flat region, which, had it not been intersected by a deep chasm affording a passage to the river, would have been devoid of any geological interest. The only direct communication between the opposite sides of this portion of the dale, is over Winch Bridge, which is formed of a single plank of wood suspended by chains from the two escarpments of columnar rock which rise on the opposite sides of the river.

Trap in the Bed of the River.

Dislocation below Winch Bridge.

The chasm through which the Tees descends may have been considerably modified by the long continued action of the waters, but its general form must have originated in some more powerful cause. A little way below the bridge there are indications of a considerable dislocation, by which the inferior beds have been brought out, from beneath the trap, into the right bank of the river. The appearance of the rocks in that locality is represented Pl. IX. Fig. 2.

Strata, inferior to the Trap.

I think it unnecessary to describe the phenomena exhibited in the bed of the river between Winch Bridge and Holwick Head, because they throw no light upon the relation of the trap to the inferior and superior strata. At Holwick Head the line of *fault* (as was stated above, p. 153.) crosses the bed of the Tees, and a few hundred feet above that place the inferior strata begin to rise from beneath the *Whin-Sill,* and gradually occupy the lower part of the lofty escarpments which extend on both sides of the river to the High Force. They are composed of limestone, slate-clay, and sandstone, and are exactly parallel to the lower surface of the super-incumbent *whin-stone.* The relations of the several beds to each other, are finely exhibited in different natural sections, but more especially at the High Force, where the whole system of strata forms one grand escarpment, over which the waters descend by a single plunge of sixty feet. Of the picturesque features of a place which has been so long and so justly celebrated, it is not my intention now to speak; but I may observe, that the interest of the scene is greatly heightened by the singular contrast presented by the horizontal beds which form the base, and the prismatic masses of trap which form the crown of the escarpment.

Forcegarth Hill.

For two or three hundred yards above the great water-fall, the trap again occupies the bed of the river. It is then thrown out into the side of Forcegarth-hill, partly by the natural rise of the inferior strata, which present themselves in the same order in which they appeared below the High Force; partly also by the intervention of a small *fault* which elevates the beds on the west side of its range.

The west side of Forcegarth-hill exhibits, under a different form, a repetition of the beds, which are laid bare in the natural sections below the High Force. The relations of this part of the valley, will be best understood by a reference to the accompanying section, (Pl. IX. Fig. 3.) the line of which passes from Forcegarth-hill to the bed of the river below the High Force.

III. I have now only to consider the phenomena produced by the great transverse *fault*, and the geological relations of the highest parts of Teesdale.

Great transverse *fault*.

Above Forcegarth-hill, the Tees is no longer confined to a narrow channel, by lofty precipitous banks, but finds its way through an open valley, the lower parts of which are, in some places, so much occupied by turf-bog or diluvial gravel, as to give us little information respecting the internal structure of the country. One or two secondary vallies, which branch out from this part of Teesdale, may, at the time of the great denudation, have assisted in producing this striking alteration in its external character. The change is, however, principally to be ascribed to a great transverse *fault*, which raises the trap to a great elevation (perhaps not less than forty or fifty fathoms) above its former level, and exposes the more yielding beds of shale limestone and sandstone to the immediate action of the waters.

This *fault* crosses the Tees, about a mile and a half above High Force, ranges past Hunt Hall, and then ascends by the right bank of a rivulet towards Middleton Fell. In its further range towards the north, it is said to be prolonged to the Burtreeford *dyke*, an assertion which may admit of doubt, since the dislocations produced by that *dyke*, are, according to the statement of Mr. Winch, of an opposite kind to those in Teesdale, though both may have been produced by forces nearly allied to each other*.

Escarpment of Low Cronkley.

On the west side of the great *fault* above-mentioned, the dale has a singularly wild and desolate character, and at first sight presents no object which would arrest attention, except the rocks of Low Cronkley, which form a kind of breast-work in front of the mountains ranging on the south side of the valley. The upper part of the escarpment of Low Cronkley, is composed of a great bed of trap. This is supposed to represent the *Whin-Sill;* and from its elevation above all the rocks of the same kind, between Forcegarth-hill and High Force, the quantity of dislocation produced by the great *transverse fault* has been estimated. The escarpment which commences at Low Cronkley, is prolonged, without interruption, for about four miles, and forms the most magnificent feature in this part of Teesdale. The north side of the river is comparatively devoid of interest. By following the brook which descends from Harwood Chapel, we may, however, discover the *whin-stone* at an elevation which is not lower than the summit of Cronkley Fell.

Glen between High Cronkley and Widdy Bank.

About two miles above the place where this brook joins the Tees, the valley sweeps round to the south-west, and the river once more descends through a narrow glen, being hemmed in by the rocks of Widdy Bank on one side, and the basaltic terrace of High Cronkley on the other. No other part of Upper Teesdale can bear a comparison with this glen, either for the grandeur of its features, or for the interest which arises out of its physical structure. The lowest parts of it are formed by alluvial gravel and vegetable soil, resting on nearly horizontal beds of limestone, sandstone, and shale. Immediately upon these stratified rocks, there rests, on each side of the river,

* See the Transactions of the Geological Society, Vol. IV. p. 78.

a great bed of trap, which, in some places, appears to be more than two hundred feet thick. The two escarpments which are thus formed opposite to each other, are often broken into the finest forms, and marked by clusters of columns, which are grouped with much more regularity than in any other part of Teesdale. The trap can on neither side of the valley be considered as an overlying formation; for the higher portion of the metalliferous series rests immediately upon its upper surface. At first, also, it appears to partake of the general inclination of the other strata. We might, therefore, conclude, that it was a regular bed inter-stratified with the metalliferous limestone. But, if we follow the part of the valley above-mentioned, from the commencement of Widdy Bank to Caldron Snout, a distance of about two miles, we have repeated opportunities of verifying its true relations to the subjacent strata: and on the north side of the river, not more than two or three hundred yards below Caldron Snout, we find the base of the trap gradually sweeping over the broken ends of the stratified rocks, and descending into the bed of the river.

Natural Sections.

These phenomena will be better understood, by referring to the accompanying sections (Pl. IX. Fig. 4, 5.) than by any verbal description. The first section (Pl. IX. Fig. 4.) is transverse to the valley, and shews the apparent relation of the *Whin-Sill* to the whole formation of metalliferous limestone. The second is longitudinal, and shews the true relation of the trap to the inferior strata, as it is exhibited in the natural sections below Caldron Snout. These two sections completely demonstrate, that the *Whin-Sill* (at least in this region) has not been regularly deposited with the other stratified rocks, and that it is not limited to a given place in the general section of the Teesdale strata. There is, indeed, an evident impropriety in designating these masses of trap by the name of *Whin-Sill*: for the word *Sill*, as I have before remarked, is synonymous with the word *stratum*, and the *Whin-Sill* is theoretically assumed (in the general section, No, 71.) to be parallel to the upper and lower beds of the great formation of mountain-limestone.

There are some other sections which confirm, in the strongest manner, the conclusion I am endeavouring to establish. But they are so intimately connected with the changes produced in the other beds, by the contact of the trap, that they will be better described in another part of this Paper.

Caldron Snout, &c.

At Caldron Snout, the waters of the Tees are once more precipitated over the escarpment of the trap, not as at High Force, in one perpendicular descent, but in a rapid succession of cascades, which together form a scene of great magnificence. We may ascend, without difficulty, among the broken prismatic blocks, which are strown about the edge of the cataract, and soon after we reach the upper surface of the trap, the river is found to expand into a deep and nearly stagnant pool of water, called the Weel. Beyond this place it branches out into several torrents, which form the drainage of one side of Cross Fell. Not many hundred yards to the east of the Weel, a bed of *granular limestone*, and a bed of *whetstone-slate*, are seen to rest on a very irregular surface of trap: and still farther up the hill, we find the regular parallel beds belonging to the higher parts of the formation of metalliferous limestone.

The trap may be also followed up the denudation of Birkdale, where it is surmounted by a bed of *granular limestone*, in external character, exactly like that which was last-mentioned. It was not, however, my object to trace the range of the trap in that direction, or to ascertain its probable connexion with the rocks of the same formation which appear on the west side of the Cross Fell chain.

The preceding description, with the accompanying sections, will, I hope, have made the principal peculiarities in the structure of Teesdale sufficiently understood. In regard to the imbedded masses of trap (commonly called the great *Whin-Sill*) it appears:

(1.) That immediately below Caldron Snout, they are not parallel to the strata between which they are interposed. Summary.

(2.) That between Forcegarth Hill and Holwick, where the inferior surface of the trap appears to be nearly parallel to the lower strata, the country is intersected by numerous fractures which have greatly changed the level of the corresponding parts of the different formations.

(3.) That a fracture passes down the valley, and produces a great *down-cast* which conceals the trap on the north side of the river.

(4.) That the trap on the south side of the valley descends among the strata in the form of a great wedge, which diminishes in thickness from thirty or forty fathoms to about twelve feet.

(5.) That, near the apparent termination of the *Whin-Sill*, a mass of trap breaks out on the opposite bank of the Tees, and is probably prolonged in the form of a *dyke*, or system of *dykes*, through the eastern moorlands.

The importance of these facts cannot be properly understood, without first considering the effects produced by the contact of the trap with the other rocks of the district.

Section III.

Composition of the Whin-Sill, &c.—Effects produced by its Contact with the other Rocks of the District.

The trap rocks of High Teesdale do not exhibit any great variety of external character or of structure. They are often External Character of the Escarpments of Trap.

coated over with innumerable lichens, and their surface is sometimes obscured by a brown ochreous incrustation. Where they are best exposed they appear, according to the circumstances of their position, of various shades of colour between light-grey and dark-brown. From the other rocks of the district they are easily distinguished, even at considerable distances, by the harsh and angular appearance of their escarpments. This phenomenon arises, partly from the prismatic structure of the trap, partly also from its indestructible nature. Throughout the district it seems admirably formed for resisting the decomposing powers of the elements; so that the blocks generally present their rude angular forms in great perfection.

There is not a single remarkable escarpment of the trap in which we cannot discover traces of a prismatic structure, and in some places the vertical lengthened prisms give the rock a fine columnar aspect. The best examples of this structure are seen below Caldron Snout, in the bed of the river near Winch Bridge, and in some of the escarpments near the village of Holwick. In the arrangement and dimensions of the prisms there is not in general any approach to regularity; neither did I discover any traces of those natural joints which are so common in the columns of overlying basalt.

A globular structure is very common in the beds of trap, associated with the mountain-limestone, in Derbyshire and some other parts of England; but in Teesdale there is not a single good example of that arrangement. It is, indeed, most frequent in rocks of the class we are now describing, which are in a state of disintegration, or are naturally of an earthy texture. But the trap of High Teesdale, with a very few exceptions, is of a highly crystalline texture, and perhaps on that account the globular form is less frequent.

Composition of the Trap.

All the sound specimens are very sonorous, and offer great resistance to the blows of the hammer. The fracture of the finer grained varieties is often imperfectly conchoidal, in the coarser specimens it is almost always uneven. A newly exposed surface is generally of a dark grey colour, and exhibits a granular texture, in which the light coloured feldspathic portion of the rock is distinctly separated from the brilliant dark crystals of pyroxene. In this respect, the *Whin-Sill* greatly differs from the *dykes* of the Durham coal-field, in which the component elements are so intimately blended, that the naked eye can seldom distinguish their separation. The feldspathic portion, notwithstanding the granular texture of the trap of High Teesdale, seldom appears in the form of distinct crystals. When it is exhibited in greatest perfection, it may be detached from the general mass in minute fragments, which melt with a slight ebullition, in the flame of the blowpipe, into a light semi-transparent glass, and it appears, in this respect, not to differ from common feldspar. Not unfrequently this portion is made up of minute opaque grains, which do not fuse without great difficulty; and in the superficial parts of the rock, it rarely passes into an earthy substance, which is harsh to the touch, adheres to the tongue, and is infusible. These different states of what I have called the feldspathic part of the trap, may not always arise out of the progress of decomposition. It may, therefore, in some cases admit of doubt, whether the white earthy constituent is to be regarded as a true simple mineral.

Crystals of Pyroxene.

In all the varieties of the trap of High Teesdale, pyroxene is very abundant, and it forms, occasionally, the principal constituent. It is sometimes amorphous, but generally crystalline, and exhibits, through the surface exposed by a recent fracture, a number of minute and very brilliant facets of a black or brownish black colour, and the foliated structure of the mineral is occasionally seen even in the small-grained varieties of the rock. Among the hard and almost indestructible masses, there may be found a few concretions or irregular veins, of a much coarser and more decomposing variety of rock, in which the crystals of pyroxene are large and abundant. This mineral, in such cases, often puts on the form of irregular prisms, or lengthened tabular crystals, the planes of which are bent and undulating. As I had some doubts respecting the true nature of these crystals, some of which reach the length of two inches, I requested my friend Mr. W. Phillips, to undertake their examination, and he determined, by their cleavage, and by the reflecting goniometer, that they are pyroxene—that they cleave easiest parallel to the plane *P*, which is uncommon, and that the broad surfaces of the long crystals, are not primary planes, but represent the plane *h*. (See *Phillips' Mineralogy*, 3d Edit. p. 59. Fig. 2.)

Magnetic.

In the specimens last described, the pyroxene is of a black or brownish black colour, rarely of a yellowish brown. The surfaces are, in some cases splendid, in others they have an imperfect pseudo-metallic lustre or tarnish, and not unfrequently the crystalline faces are disguised by a coating of oxide of iron.

Both the small-grained and the large-grained varieties of the trap, act vigorously on the magnetic needle, an effect probably due to very small grains of titaniferous iron, disseminated through the mass. I did not, however, observe any instances of those large concretions of this mineral, which so constantly occur in rocks, composed of diallage and feldspar, belonging to formations of serpentine.

Veins in the Trap.

The passage of a vein or *fault* through the trap (as well as through the other strata of the country) is marked by the presence of a number of thin vertical beds; some portions of which, when struck with the hammer, fall into irregular solids, terminated by plane surfaces, which are partially coated over with oxide of iron. But a clean fracture of most of these masses, exposes a very fine granular texture, and a bluish black colour, in which the distinction between the light and dark constituents of the rock almost entirely disappears. This change in external character appears to arise only from the different size and arrangement of the same component elements.

Minute fragments of all the preceding varieties of the rock, fuse readily into a bright glass bead of an uniform black colour. On this circumstance alone, some mineralogists have attempted to found a distinction between trap of the kind I am now describing, and greenstone, i. e. a granular compound of feldspar and hornblende.

Other varieties of Trap.

Every thing which has been hitherto said on the trap of Teesdale, is derived from observations made at those localities where the beds are of considerable thickness. At the eastern termination of the *Whin-Sill* (in the bed of the Lune,) where its whole thickness is not more than eleven or twelve feet, it has lost its well defined texture, and very nearly resembles the dykes described in my preceding Paper (*supra*, p. 21.) The same observation applies to the other masses of trap in Lunedale, and to the mass on the north bank of the Tees, opposite to the mouth of the Lune. The two dykes in Eglestone burn, seem not to differ in any essential respect from the dykes of the Durham coal-field.

From one of the last-mentioned localities, I obtained a specimen which had a globular structure, and contained a great quantity of pyroxene, some of which was granular, semi-transparent, and of an olive colour. It was not, however, necessary to apply the flame of the blowpipe in order to separate these grains from olivine; because some crystals exhibited the ordinary characters of pyroxene gradually passing (probably from the effect of decomposition,) into a substance exactly resembling the olive-coloured grains above-mentioned*.

* It is stated in my former Paper (*supra*, p. 34.) that parts of the decomposing dyke of Coatham

Contemporaneous Veins, &c.

The *Whin-Sill* rarely contains spangles of iron pyrites. It is sometimes traversed by thin, and probably contemporaneous veins of carbonate of lime; but I did not find a single example of amygdaloidal concretions of that mineral, such as are seen in the dyke of Cockfield Fell. Large veins, containing sulphate of barytes, blende, galena, and other minerals, pass through the rock in one or two places; but they are of a different class from the former, and belong to the great metalliferous veins which traverse all the strata of the country. Such appear to be the principal facts connected with the structure of the great *Whin-Sill* in the district here described*.

Coatham Stob, contain a considerable quantity of olivine. As I did not discover any trace of this mineral in High Teesdale, and as I have not found it in any of the dykes in the North of England, which I have since examined, I am disposed to believe, that, in consequence of having acted with the blowpipe on over-large fragments, I may have concluded erroneously, that the olive-coloured grains of the decomposing trap of Coatham Stob were infusible; and on that account, I may have mistaken granular decomposing pyroxene for olivine.

* While this Paper was passing through the press, I received from Mr. W. Phillips, the result of his examination of some specimens of trap rocks derived from the districts described in this and in my preceding Paper. I select from his descriptions, those which bear on the present subject.

No. 1. (From High Teesdale,) contains augite with yellow iron pyrites—the white and grey specks not probably felspar, for they immediately pass into a white powder under the point of the knife.

No. 2. (From High Teesdale,) same structure with the preceding, and includes minute octohedrons of oxidulous iron. The light grey substance is soft and granular.

No. 3. One of the varieties (from High Teesdale) with large crystals of augite. In it is a greyish or yellowish substance, which is translucent, or even transparent, which has cleavages, but not sufficiently brilliant for the goniometer. The substance cannot be feldspar, because it crumbles, with a moderate pressure, into a grey powder, and its translucency, and even transparency, proves that it cannot be in a state of decomposition.

In the greyish white granular or earthy substance, derived from one specimen, Mr. Phillips discovered, by help of a glass, very numerous minute slender crystals, which were colourless, and from their form and hardness, appeared to be feldspar. In none of the specimens, did he discover any hornblende. In two specimens, one taken from the basaltic dyke of Lower Teesdale, and the other taken from a similar dyke at Coley Hill, near Newcastle, he discovered well defined crystals of cleavelandite. The reader will remember, that when my preceding Paper was written, this mineral had not been separated from feldspar.

I now proceed to consider the effects apparently produced by its contact with the other strata. These effects are of two kinds—mechanical and chemical. By mechanical effects, are understood an alteration in the position of certain beds, or the parts of certain beds, by the agency of the trap: by chemical effects, are understood, some changes produced by it in the texture and mineralogical character of the contiguous rocks. Effects produced by the Trap.

1. To the mechanical action exerted by the trap, I venture to attribute the shattered appearance of the strata on the crest of the hill above the new plantations north of Eglestone. It was this very appearance, seen from the opposite side of the valley, which first led me to suspect, that the upper *dyke* passed considerably to the north of the direction which I had given it in my former Paper (*supra*, p. 25.) A subsequent examination confirmed this suspicion, and enabled me to ascertain, that the upper dyke (see the Map, Pl. VII.) ranges through the center of the shattered beds which belong to the *mill-stone grit* (general Section, No. 1,) and that it has apparently protruded some masses of slate-clay from their true situation. These masses of slate-clay are in a state of considerable induration; but it is not clear that their present state is to be ascribed to any direct agency of the trap dyke. Eglestone Burn.

2. By the operation of similar causes, we may readily account for the extraordinary appearance of the strata near Greengate-farm, in Lunedale, (See the Map, Pl. VII.) The beds of limestone and sandstone, unquestionably belong to the general Teesdale section, but they are broken and tilted in every possible direction, (See the Section, Pl. VIII. Fig. 3.) Had the trap been regularly deposited along with the other strata, I cannot conceive any system of disturbing forces capable of producing the present arrangement. If, on the contrary, we suppose it to have been forcibly driven up amongst the other beds after their partial con- Effects of the Trap in Lunedale.

solidation, the present position of the dislocated and highly inclined masses of rock is at once accounted for. I might here mention the granular texture of the limestone, and the induration of the other beds in contact with the trap: but I will not attempt to describe phenomena which are exhibited more perfectly and more unequivocally in other localities.

Near Lonton.

3. The sections exhibited in the banks of the Lune, near Lonton, afford very curious illustrations of the mèchanical action of the trap. In the first section, (Pl. VIII, Fig. 4.) it rests on a variety of hard siliceous sandstone (provincially called *hazle*) and it is surmounted by beds of slate clay and limestone. The beds in contact with it do not appear to be much changed in their texture. It is imperfectly prismatic, and its lower surface shew, a tendency to a globular arrangement. The only fact of importance brought to light by this natural section, is the following: A portion of the hard sandstone is lifted up, at a considerable angle from its native bed, and entangled in the superincumbent trap. I am utterly unable to comprehend any explanation of this phenomenon, which does not include the supposition of a distinct mechanical action. Without the support of the trap, the projecting mass of sandstone could not for a moment retain its present position.

In the next section, (Pl. IX. Fig. 1.) we have a repetition of the same phenomenon, a portion of the inferior *hazle* is bent up into the trap which forms the top of the escarpment. An appearance very similar to that which is exposed in each of the two last-mentioned sections, is described by Dr. Mac Culloch, in the second volume of the Geological Transactions, p. 305.

In the Bed of the River below Winchbridge.

4. I forbear to speak, in this place, of the enormous *faults* by which the whole of High Teesdale seems to be traversed; because it is impossible, by direct evidence, to establish their connexion with the mechanical operations of the trap.

It has been stated above (p. 157.) that the *Whin-Sill* rises out in the bed of the river, about half-way between Bowlee Beck foot and Winch Bridge. The beds resting on the *Whin*, appear in the following order, beginning with the lowest. (1) Shale and hazle alternating; the shale in a state of extreme induration. (2) Limestone, in texture entirely granular. (3) Indurated shale. (4) Limestone less granular than the preceding, and part of it unchanged. (5) Shale, soft, earthy, and in every respect like the beds of slate-clay in other parts of the district. I have thought it proper to give this section, because it is taken from the only place below the great transverse *fault*, (the Burtreeford dyke,) where I had an opportunity of examining the effects produced by the *Whin-Sill* upon the superior strata.

5. The phenomena exhibited at the junction of the trap with the inferior strata at Forcegarth-hill, and at the High Force, are well deserving of attention, (See the Section, Pl. IX. Fig. 3.) As the several escarpments in this part of the dale, only give a repetition of the same general facts, I shall confine myself to a notice of the mineralogical character of the beds which appear under the trap on the south-east side of Forcegarth-hill. Sections at Forcegarth-hill and High Force.

(1) The upper part of the escarpment (where the inferior beds are first seen on the side of the hill,) is composed of prismatic trap. (2) Under the trap appears a bed of sandstone nine feet thick. It is in a state of extreme induration, difficult of fracture, and in its texture differs entirely from the ordinary sandstone beds of the district. (3) Under the sandstone is a bed about 18 feet thick, which, from its hardness and texture, might at first sight be mistaken for greywacké slate. I do not, however, believe that it differs in its composition in any respect from slate clay, especially as it contains nodules of ironstone arranged exactly in the same manner in which they appear among the soft argillaceous beds of the metalliferous series. We may, therefore, conclude,

that the unusual induration of this bed, has been produced by the agency of the trap*. (4) Under the indurated shale, appear a number of beds of dark encrinite limestone. The highest portion of them seems to have been slightly acted on, but the lower portion is in its ordinary unaltered state. The two preceding sections do not give any direct proof of the mechanical action of the trap. Its position on the west side of Forcegarth-hill, seems, however, to shew that its lower surface is not there parallel to the strata on which it rests.

Junction below Caldron Snout.

6. It appears from the section, (Pl. IX. Fig. 5.) that immediately below Caldron Snout, the trap is not parallel to the beds on which it rests. By this arrangement, it is brought into immediate contact with a succession of the inferior strata.

Some of the phenomena produced by the junction, are exhibited in a natural section (See Pl. X. Fig. 1.) about 200 yards below Caldron Snout. The base of this section is formed by two beds of granular limestone, separated by a thin indurated argillaceous bed of a light brown colour, and in texture resembling some varieties of whet-slate. Over these comes a thin and very irregular argillaceous bed like the preceding, and the whole series is surmounted by an unconformable mass of columnar trap. It appears, by a comparison of specimens, that these argillaceous beds are exactly analogous to certain portions of the beds of slate-clay which abut against the trap dykes in Anglesea†. The two beds of limestone are as granular as Parian marble, and do not contain a vestige of organic remains. Except where they are stained by impurities, they are of a dull white colour. The grains, which are highly crystalline, adhere very imperfectly to each other, so that projecting portions of the rock often shiver

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* I have been informed, that this indurated shale has been occasionally used for roofing-slate.

† See Professor Henslow's Paper on Anglesea, Vol. I. p. 401.

to pieces under the blows of the hammer. When thrown upon a plate of iron, at a temperature a little below red heat, the grains give out a beautiful pale yellowish phosphorescent light.

7. About half a mile below the last-mentioned locality, and immediately above a remarkable escarpment called the Falcon-clint, the trap on the north bank of the Tees appears to be nearly parallel to the inferior beds, and the effects produced by it are illustrated by several fine natural sections, some of which I proceed to mention*. Other natural Sections.

SECT. 1.

No.		Feet.
(1.)	Columnar trap, forming a high escarpment.	
(2.)	An indurated mass, variable in colour and texture, and full of cells and cavities	3
(3.)	Granular limestone	4
(4.)	Hard whet-slate	1
(5.)	Granular limestone........................	8

The inferior portion of the escarpment is concealed.

SECT. 2.

No.		Feet.
(1.)	Prismatic trap, nearly forty fathoms thick, forming a succession of escarpments to the top of Widdy Bank.	
(2.)	Hard irregular cellular bed	7
(3.)	Indurated slate-clay	$2\frac{1}{2}$
(4.)	Cellular bed	1
(5.)	Granular limestone........................	10

The lower beds covered by alluvial matter.

* Each of these sections is given in descending order, No. (1.) forming the highest part of the escarpment.

SECT. 3.

No.		Feet.
(1.)	Trap forming a fine lofty escarpment.	
(2.)	A bed, every portion of which appears in the greatest confusion. Some parts as full of irregular cells and pores as a piece of slag: other parts resembling hornstone, chert, or porcelain jasper, mixed with nodules, beds, and irregular concretions of granular limestone. Whole thickness	18
(3.)	Slate-clay in a state of partial induration ..	10
(4.)	Softer slate-clay, apparently decomposing..	4
(5.)	Dark impure limestone	1
(6.)	Various beds which appear to have been slightly acted on	9
(7.)	Dark encrinite limestone in the ordinary state	4
(8.)	Slaty sandstone.............................	6
(9.)	Hard sandstone	6
(10.)	A bed appearing at the base of the cliff, and in the edge of the river, composed of impure slate-clay and argillaceous limestone, mixed with grains and small pebbles of quartz.	

The three preceding sections are copied from memoranda, made upon the spot in the year 1822. The numbers are not the result of any actual measurement, and are only to be considered as approximations. If the sections convey only a correct general notion of the strata immediately below the great escarpment of the trap, they will answer the purpose intended.

Mineralogical Character of the Beds under the Trap.

I now proceed to describe some of the specimens derived from the three last-mentioned localities. The beds immediately under the trap, are of the greatest interest, and they so nearly resemble substances acted on by fire, that the practical men by whom I was accompanied, described them to me by the name of the *slaggy beds.* The confused mass under the trap, in Section 3, seems to be formed by the irregular blending of two or three strata, by an action partly mechanical, and partly chemical. In the cellular beds, the most solid portions differ from each other in hardness and in colour, so that in one specimen we may find substances resembling hornstone, chert, or porcelain jasper; sometimes appearing rudely arranged in concentric layers which run in irregular curves round the cavities of the mass. All these substances are hard, and under the hammer fly into splinters, which are translucent at the edges. Small fragments of them readily fuse into a whitish glass with minute air-bubbles, or into a light coloured enamel. The cells are very irregular in size and shape, varying from the fraction of an inch to three or four inches in diameter; and they are lined and partly filled with a rough spongy mass, which, in some specimens, does not appear to differ from the more solid part of the rock; but in others, is mixed with a dark greenish substance, which readily yields to the knife, and fuses into a dark coloured glass. It is generally crystalline in structure, but no distinct cleavages could be obtained from it, and it sometimes passes into a compact mass of the same colour. I do not think that it is allied to green earth.

In the same cells are also found minute garnets with perfect rhombic faces, which are sometimes of an olive-brown, but more frequently of a dark olive-green colour. By breaking up a great many masses on the spot, I obtained a few crystals, which it was impossible to mistake; and some almost microscopic specimens, have been since examined by Mr. Phillips, who found that they exhibited the angles of garnet. The existence of garnets in this singular locality, was first discovered by the Rev. J. Harriman, (*Sowerby's Brit. Min.* Vol. II. p. 37.) And under exactly similar circumstances of association, very fine garnets

Garnets.

were found by Professor Henslow, in the Isle of Anglesea. For an elaborate description of the accompanying phenomena, I must refer to the preceding volume of the Transactions of the Society, (p. 407—410.)

On a portion of one of the indurated cellular masses containing garnets, I found the impression of a madrepore of a species very common in the mountain limestone.

Organic Remains, &c.

In Section 3. No. 2, there are balls and concretions of limestone, associated irregularly with the hard cellular substances above described. In a part of the same anomalous mass, the limestone assumes the character of a distinct subordinate bed, penetrated irregularly by crystalline matter, resembling that which lines the cavities of the splintery jaspideous rock. The limestone is always crystalline and granular, and sometimes uniformly white; but we may generally trace many cloudy bluish spots through the mass, which seem to indicate the partial presence of the ordinary colouring principle. In those portions of the limestone which are nearest to the trap, and most granular, impressions of organic remains are very rare. Under such circumstances, we may, however, sometimes trace the stems, and even the stellated structure of madrepores in certain parts of the rock, which are always of a darker colour where organic remains are present. The fact, that such discoloration arises from the presence of organic remains, can, in many cases, be established only by a comparison of a large suite of specimens. Associated with the granular limestone, I found in one place, a considerable portion of mica, and a substance resembling compact feldspar, so that a hard specimen might easily have been mistaken for a fragment of a primitive rock.

The lower beds of granular limestone are often of a lead colour, and not unfrequently the remains of madrepores, encrinites, and other fossils, appear through the granular mass in the same abundance and arrangement in which they are seen in ordinary calcareous beds of the country.

Globular Structure.

Of the whet-slate (Section 1. No. 4.) it is not necessary to speak: but there is a peculiarity in the indurated slate-clay (Section 2. No. 3.) well deserving of notice. Parts are in a state of great induration, and exhibit distinct traces of a globular structure; and through the substance, may be seen many yellowish white spots, some of which are solid, but generally they are of a friable earthy texture, and do not effervesce in acids. Other parts of the same bed are imperfectly indurated, consisting of irregular layers of a hard substance like the preceding, blended with masses which are soft and earthy. Through every part of such specimens, we may often trace a number of light coloured globular concretions, about a tenth of an inch in diameter, which are rather harder than their matrix. Some masses of indurated shale in the Isle of Anglesea, put on an exactly similar appearance (See *Professor Henslow's Paper*, Vol. I. p. 407.): but the globules were there of much greater interest, because they were sometimes of larger dimension, and on approaching a trap *dyke*, passed by a regular progression of changes into distinct crystals.

Beds above the Trap.

8. If we ascend to the top of the escarpment between Caldron Snout and Widdy Bank, we reach the upper surface of the trap, and by advancing northwards, we soon find it surmounted by a bed of granular limestone. The plane of separation is not, I think, parallel to the stratification of the limestone; and in one place, the upper and lower beds appear on the same level, abutting against each other. In consequence, perhaps, of this irregularity in the upper surface of the trap, the limestone appears to be of very variable thickness. It is of a granular structure throughout, and from its texture and colour, might, without examination, be easily mistaken for coarse siliceous sandstone. Each grain is, however, composed of crystalline carbonate of lime.

Specimens from different parts of the bed vary from a dirty white to a dark lead colour, and many of them, especially at some distance from the trap, contain the organic remains of the metalliferous limestone in great abundance. Over this limestone, comes a bed of whet-slate of a yellowish brown colour. By the action of the mountain streams, some parts of this bed appear to be decomposing, and gradually returning to their original soft earthy state. On the brow of the hill, still farther to the north, may be seen many beds of limestone, sandstone, and shale, in their common unaltered state*.

Top of High and Low Cronkley.

At the top of the great terrace of High and Low Cronkley, on the south bank of the Tees, the relation of the trap to the incumbent strata is still more perfectly exposed. Here, as well as in the preceding locality, the trap is surmounted by a bed of limestone, which, in some places, is five or six yards thick: but the line of separation is so irregular, that it occasionally rises through the limestone, and brings the trap into immediate contact with a still higher bed of hard white siliceous sandstone. In consequence of this arrangement, we find on the *plateau* of High Cronkley, masses of sandstone and of limestone lying like irregular patches on the surface, sometimes partially filling up the hollows, and sometimes crowning the protuberances of the inferior bed of trap.

In one place above Low Cronkley, the horizontal section exposes a mass of limestone about four yards wide, passing like a vein between two nearly perpendicular cheeks of trap. The appearance is, I have no doubt, deceptive; as there is no trace

* Two nearly north and south lead-veins, run through the beds above described. They were formerly worked down into the whet-slate and granular limestone; but, if I have not been misinformed, they became unproductive on approaching the trap. They are now worked with advantage in the upper unaltered beds. The *vein-stuff* is principally composed of sulphate of barytes.

of any vein of carbonate of lime in the neighbouring escarpments; and the phenomena may be easily conceived to have arisen from a long trough-shaped depression in the upper surface of the fundamental rock.

At all the last-mentioned places, the mineralogical phenomena are of no common interest. The beds of limestone externally resemble coarse masses of millstone-grit which has been bleached by the action of running water. When struck with the hammer, they generally shiver to pieces, and fall down in the form of large grains, which are externally dull and amorphous, but which easily split into transparent crystalline plates. No traces of organic remains are to be discovered in these large-grained masses of limestone, and in many of them, all colouring matter and impurity seems to be completely driven off. They are all beautifully phosphorescent. I did not ascend from High Cronkley to the side of Mickle Fell, but I was informed, that the upper beds of the general section appear there with their usual external characters.

9. At White Force, near the eastern extremity of Low Cronkley, a tributary mountain-stream is precipitated over one of the finest escarpments in Teesdale. The quantity of water was inconsiderable at the time I visited this spot; but when the stream is swollen by rain, the effect of the cascade must, in some respects, be superior both to High Force and Caldron Snout. The upper part of the precipice consists of a great tabular mass of prismatic trap, the whole thickness of which must be very considerable, as it extends some way above the precipice, and a perpendicular face of more than sixty feet is laid bare in a part of the natural section. Immediately behind the water-fall, the trap rests upon a single bed of granular limestone, more than thirty feet thick; but the eastern escarpment, a few yards below, is much more complex. The trap is, on that side, supported by two beds of

Junction at White Force.

limestone separated by a thin bed of indurated *shale*, and the three beds are, for some distance, kept separate from each other by two wedges of trap, which are connected with, and undoubtedly form a part of the superincumbent mass. I have endeavoured, by the accompanying section (Pl. x. Fig. 2.) to convey a correct notion of this geological fact, which seems to prove the mechanical agency of the trap in a manner the most unequivocal and convincing. A good representation of the features of the place would require the pencil of a skilful artist.

Indurated *Shale* and Granular Limestone.

The upper bed of limestone, especially that part of it which protrudes into the trap, is white and perfectly granular. The indurated argillaceous bed is of a light grey colour, and when struck with the hammer, flies into sharp splinters or irregular angular fragments. The form of the fragments is partly due to a number of flaws or natural partings, coated over with minute crystalline plates, exactly like those observed by Professor Henslow in the hardened shale in contact with the Plas-Newydd dyke in the Isle of Anglesea*. The splinters are all translucent at the edges, and so hard that the knife makes very little impression upon them. They are nearly infusible, but very minute fragments, when urged to the utmost in the flame of the blowpipe, exhibit a slight superficial glaze without melting down so as to lose their form. The lower part of the great bed of limestone, which forms the visible base of the escarpment, is less granular than the upper part, and is more discoloured by a number of cloudy blue spots. From analogy, I should conclude, that organic remains may be found in it; though I have not discovered any in the specimens brought away from this locality. It is not divided into subordinate beds by any natural partings, but may be regarded as one great mass of granular marble. All the

* See the preceding Volume of the Society's Transactions, p. 405.

parts of it which are exposed, are, however, too crumbling and incoherent to be of any use in ornamental architecture.

Chemical action of the Trap.

I have now detailed to the Society all the important facts with which I became acquainted, during my examination of the physical structure of High Teesdale*. Of the mechanical agency of the trap, it appears to me, that we have, in two or three instances, a perfect demonstration. As far as regards its chemical action, some one may perhaps object, that the beds of granular limestone and hard shale are formations *sui generis*, which took place immediately before and after the deposition of the great *Whin-Sill* (General Section, No. 71.) and that they have not been modified by any external action subsequent to their original deposition. This theory not only leaves the mechanical action unaccounted for, but appears to involve the supposition, that the *Whin-Sill* must occupy a given place in a regular succession of strata, a supposition which is at direct variance with some of the facts described above, as will be seen at once by a comparison of the general section (*supra*, p. 8.) with the sections below Caldron Snout and White Force.

If the granular limestone be independent of the trap, it might be expected to appear in other parts of the metalliferous formation where no trap is present; but under such circumstances, it has never been met with. Again, if the hard and granular beds only indicate a natural alteration in the deposits immediately preceding the precipitation of the trap, we should naturally expect to find some analogy between the component elements of these beds, and of the trap. But I could discover no such analogy; nor can I comprehend how a bed of pure granular limestone,

* As I was accompanied in almost all my excursions by Mark Watson and J. Dent, two intelligent practical men well acquainted with the mineralogical features of the district; I venture to hope, that very few facts, connected with the objects of this paper, escaped my observation.

can in any way be considered as a connecting link between the ordinary beds of the carboniferous series, and a mass of crystalline rock composed of pyroxene feldspathic earth and magnetic oxide of iron.

The strata of the metalliferous formation are essentially composed of limestone, of sandstone, and of slate-clay. The strata immediately above and below the *Whin-Sill*, exhibit the same alternations, differing from other beds only in being more granular or more indurated; and the change of texture is greatest in those which are in immediate contact with the *Whin*. In such localities, the beds of crystalline limestone are identified as true members of the great formation by their organic remains, which often exist in the same order and abundance in which they are found in other unaltered beds, only disappearing very near the trap where the limestone becomes completely and coarsely granular. The beds of sandstone never lose their original texture, although, in some instances, they have undergone considerable modification. Some of the hard flinty beds are identified with the shale, by their slaty structure, and by other external characters; occasionally also by imbedded nodules of iron-stone, and by organic remains. Close to the trap, the beds of shale are sometimes so much modified, that they entirely lose their identity, and it would be impossible to pronounce upon them by the help of any series of hand specimens. To understand their relations, it is absolutely necessary to examine them on the spot.

In addition to what has already been stated, I think it important to observe, that the changes above described, are the most remarkable, where the masses of trap, in contact with the other beds, are the greatest. Above Middleton, I did not examine a single locality where the strata, immediately above or below the *Whin-Sill*, had not undergone a very striking modification: but the *dykes* in Eglestone Burn, and the thin bed in the banks of the Lune, near Lonton, have hardly produced any perceptible change in the texture of the contiguous rocks.

Combining all these facts together, I do not hesitate to conclude, that many of the mineralogical phenomena above-mentioned, are to be accounted for exclusively by the chemical action of the trap upon the contiguous beds of the metalliferous formation.

Analogous action of Trap *Dykes*.

If there were any doubt about the justness of the preceding conclusion, it is, I think, set at rest by examining the effects produced by trap *dykes* upon beds of the same general character with those in High Teesdale. In the case of certain *dykes*, the fact of a chemical change produced by the action of the upfilling trap upon the contiguous edges of the strata, is proved by evidence which amounts to demonstration. For a detail of some of this evidence, I must refer to my preceding paper (*supra*, p. 35, &c.)

I think the University fortunate, in possessing a beautiful series of specimens, derived from beds in contact with trap *dykes* in the Isle of Anglesea. Those which Professor Henslow brought from the side of the Plas-Newydd *dyke*, not only resemble, but are almost identical with many of the specimens which I found near the *Whin-Sill* in High Teesdale. Now the Anglesea specimens are taken from nearly horizontal beds of limestone, slate-clay, &c. which abut against a vertical *dyke* of trap filling up a chasm in the strata. Whatever may have been its origin, it must have been formed after all the horizontal strata were deposited; and as those portions which abut against the dyke are changed in texture, but gradually, at some distance from it, recover their character, the change can only be ascribed to the action of the trap*. If then we be allowed to argue from similarity of

* For all details on this subject, I must refer to Professor Henslow's Paper in the preceding Volume, p. 401—424.

effects, I should conclude (as I have already done on other evidence) that many of the phenomena in High Teesdale, have been produced by the chemical action of the *Whin-Sill*.

Objections of Dr. Boué considered.

Before I conclude the details of this section, I think it proper to notice some remarks of Dr. Boué, in his *Essai Géologique sur L'Écosse*. He admits the igneous origin of all trap rocks of the class I have been describing, but in almost every instance denies the fact of their having modified the beds with which they are in contact. I am not sure that I comprehend his theoretical views connected with this subject: but in speaking of a series of beds in character exactly similar to those which are found close to the trap in High Teesdale, he says, that some of them may perhaps have been formed by a deposit of volcanic matter suspended in a liquid (“*formées peut-être par un dépôt de matières* “*volcaniques suspendues dans un liquide*,” p. 246.) It is unquestionably possible that the waters might have become turbid in the vicinity of a submarine eruption of lava; and that the suspended particles might afterwards have formed a peculiar deposit. But such a deposit could never consist of alternating beds like those in High Teesdale, of pure limestone containing, in every part of its mass, organic remains of the same kind with those which are found in the ordinary strata, of hard siliceous sandstone, and of hard argillaceous and siliceous slate with organic remains and with layers and nodules of iron-stone. The supposition of Dr. Boué is not only gratuitous, but utterly inadequate to the explanation of such phenomena as are detailed in this Paper.

In regard to certain substances, which, from his description, appear to resemble parts of the indurated *shale* in contact with the *Whin-Sill*, he states his opinion in the following words: “*Leur fusibilité et leur degré de dureté suffisent pour les distinguer du schiste siliceux ou lydien de Werner, qui leur ressemble au premier abord; leur dureté et leurs autres caractères empêchent* *de les confondre avec les argiles schisteuses ou les schistes argileux,*" *Essai*, p. 249. He afterwards adds, "*L'on peut donc conclure que ces roches feldspathiques sont liées intimement aux dépôts basaltiques,*" *&c. &c.* p. 252. To such remarks as these, I would observe by way of reply—(1.) That hand specimens of slate-clay, indurated by the action of trap, sometimes resemble true Lydian-stone which is subordinate to transition-slate, but may generally be separated from it by the help only of external characters. (2.) That the fusibility of such indurated specimens into a light coloured enamel, or into glass of various shades between light grey and brownish black, affords no test whatsoever by which we can separate them geologically from the soft unaltered beds of slate-clay, and arrange them with feldspathic rocks of a different family; because specimens from the soft unaltered argillaceous beds (especially those which alternate with limestone and contain a portion of calcareous matter) generally melt into an enamel, and sometimes into a light transparent glass. Lastly, the hypothesis of Boué affords no solution of, and is contradicted by, the phenomena which are so constantly seen in the vicinity of trap *dykes* *.

His account of the trap *dykes* of the Northumberland and Durham coal-fields, is, throughout, either defective or erroneous. For example: he states that some of those dykes are separated from the neighbouring rocks, by a *salbande* of grit, containing vegetable fossils. There is no such *salbande;* and he has been led into the mistake, by misinterpreting a passage in the paper of Mr. Winch †.

Dykes in the Coal-field.

* In confirmation of this assertion, I may refer to passages without number in Dr. Mac Culloch's Geological Description of the Hebrides; to Professor Henslow's Paper on the Geology of Anglesea; and to certain passages in the description of the trap dykes in the Durham coal-field (*supra*, p. 35, &c.)

† Geological Transactions, Vol. IV. p. 21.

2. He asserts, that they are filled from above, whereas it is demonstrable that some of them, and highly probable that all of them, have been formed by injection from below.

3. He states, that they have, in some places, converted columnar anthracite into cellular anthracite. But the fact is, that they have converted bituminous coal into anthracite, which is partly cellular and partly columnar.

4. He asserts, that in the lower part of the English coal strata, there are beds of columnar anthracite—that anthracite is found at a distance from trap—and that the bad quality of coal in the vicinity of trap dykes, is not generally to be accounted for by the action of the trap. Now, I do not believe that there is a single bed of columnar anthracite in any part of the great coal-basin of Durham and Northumberland; and the universal fact is, that wherever a trap dyke passes through a bed of bituminous coal, such coal is, for a certain distance on each side of the dyke, converted into a more or less pure anthracite, which is sometimes columnar. In short, there is not a single remark in the *Essai Géologique sur l'Écosse*, which makes me in any doubt about the correctness of the conclusions I have endeavoured to establish, either in this or in my preceding Paper (*supra*, p. 21—44*.)

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* My only object in making the preceding remarks, was to take away the force of certain objections which might be urged against the conclusions I wished to draw from the phenomena observed in High Teesdale. I think that the descriptive part of Boué's Work is often excellent, and, whatever inaccuracies there may be in certain parts of its details, that it contains much highly interesting and valuable information.

Section IV.

On the Origin of the Trap Rocks of High Teesdale.

Having, in the two preceding Sections, considered the principal facts connected with the structure of High Teesdale, and the phenomena arising out of the association of trap rocks with the regular strata of the district; it only remains for me to consider the probable origin of this anomalous class of rocks. Origin of the Trap, igneous.

1. In the first place, they are, in mineralogical composition, all nearly identical with undoubted volcanic products. This assertion is sufficiently proved by the details given in the early part of the preceding Section. On the contrary, the regular strata of the district are unquestionably of aqueous origin, many of them consisting, almost exclusively, of the petrified remains of zoophytes, shells, and corals, and they differ altogether in composition and structure, from every variety of trap. This entire contrast makes it probable that the different formations have had a different origin.

2. The lower *dyke* which crosses Eglestone Burn, is connected, at one extremity, with a mass of trap on the north bank of the Tees, and in the other direction, it is prolonged into the eastern moors. In its structure, and in its position among the regular beds, it differs in no essential respect from the *dykes* described in a preceding Paper (*supra*, p. 21.) and it is probably connected with some of them. If then we have good evidence for the igneous origin of the *dykes* in the coal-field, it follows, that the lower *dyke* in Eglestone Burn, and the mass of trap on the north bank of the Tees, must be of igneous origin.

3. The connexion of this lower *dyke*, with the mass of trap on the north bank of the Tees, seems to prove, that it must have been formed by injection from below.

4. The upper *dyke* in Eglestone Burn, and the *dyke* in Lunedale, must have originated in a cause of the same kind with that which produced the *dyke* last-mentioned.

5. The *dyke* in Lunedale, is probably connected with the protruding masses of trap near Saddle-bow, (See Pl. VIII. Fig. 3.) and these protruding masses cannot possibly have been formed by infiltration after the deposition of the other strata; from which it follows, that the *dyke* in Lunedale was probably injected from below. I may add, that there is no fact which makes the contrary opinion in any way probable.

6. It would be highly unphilosophical, to class these *dykes* and masses of trap among igneous rocks, and at the same time, to class the beds, and tabular masses of trap in the higher part of the dale (the *Whin-Sill*) among aqueous deposits.

7. The beds and tabular masses of trap, have produced mechanical effects which it is impossible to account for, on the supposition, that the strata originated in a succession of aqueous deposits.

8. These tabular masses have produced, in the texture of the neighbouring rocks, many important changes which cannot be explained by any hypothesis, which rejects their igneous origin. On the contrary, many of the changes may be imitated, by direct experiment, and almost all of them may be explained on the hypothesis which admits the igneous origin of the trap, and supposes it to have acted, both chemically and mechanically, on pre-existing strata.

The observations on which these conclusions are founded, have been given with sufficient detail, and various illustrative specimens have been brought from the different localities, which the Members of the Society may have an opportunity of examining in the cabinets of the Woodwardian Museum.

Assuming, that the trap rocks of High Teesdale are of volcanic origin, a question may arise respecting the time of their irruption among the strata. It is clear, that they took their present form posterior to the deposition of those members of the metalliferous formation with which they are in contact. It is also clear from what was stated in a former Paper (*supra*, p. 42.) that most of the *dykes* in the coal-field existed before the deposition of the magnesian limestone. Now, I think it perfectly certain, that the basaltic rocks of High Teesdale, were not formed after the *dykes* of the coal-field. It therefore follows, that these rocks must have existed in their present form before the deposition of the magnesian limestone. Thus, the time of their irruption, is limited to one great geological era. This era may, however, have been of long duration, and it by no means follows, that all the basaltic *dykes* of the coal-field are strictly contemporaneous; still less does it follow, that these basaltic *dykes* were formed at the same time with the imbedded masses of trap in High Teesdale (the *Whin-Sill.*) Probable Time of the Irruption

I think, from the facts stated in the previous Sections of this Paper, that the *Whin-Sill* cannot have been formed before the beds which rest immediately upon it; but it may, perhaps, have been formed at the same time with the trap in Lunedale, and with the *dykes* in Eglestone Burn, in which case, it must probably have been injected among the metalliferous strata, after the deposition of, at least, a part of the coal formation.

If this last conclusion be admitted, we must allow, that the *Whin-Sill* has been produced by a lateral injection of volcanic matter in a state of igneous fusion. That such a mode of formation is possible, has been unequivocally demonstrated in Dr. Mac Culloch's excellent Work on the Hebrides (See the Plates to the Geological Description of Sky). Lateral Injection.

Our reluctance in admitting this explanation, arises from the difficulty of conceiving any powers in nature capable of producing such an effect. But all the phenomena of Geology shew, that the great disturbing forces by which the crust of the globe has been modified, acted in former times with incomparably more energy than they do at present. Volcanic forces are now employed in lifting a column of melted lava to the lips of a crater. The same kind of forces acting with more energy, and through a wider region, may, in the early history of the globe, have been employed in lifting islands, and even continents, from the bottom of the ocean. During an operation like this, the elastic forces acting from below, may often have driven masses of fluid lava among the superincumbent strata; and, in every case, the lava would naturally be propelled through those portions which were most easily penetrated. In such a state of things, a great lateral injection would not only be a possible, but a probable circumstance. For if the lava acting on the superincumbent strata were in a fluid state, the lateral pressure must, at every point, have been exactly equal to the vertical pressure. The expansive forces may not, at any point, have been able to drive a column of lava through all the solid unbroken beds; but the lateral forces may have driven a portion of the fluid between the partings of two horizontal beds, and when a penetration of this kind was once effected, the lava would act like a wedge at a mechanical advantage, and rush in a horizontal stream to a distance proportioned to the elastic forces which were in action.

To break through a mass of solid strata, might require a force almost infinite, but to produce a lateral injection between two horizontal beds, would only require the lava to be fluid, and the expansive force to be greater than a given pressure. It is, I think, perfectly clear, that lateral injections must generally take place when volcanic forces act upon unbroken stratified rocks. If, in such a case, the pressure upwards becomes greater than the weight of the incumbent beds, an upheaving motion must take place, some of the solid masses will become broken or disjointed, the lava will be propelled between them, and the lateral pressure will produce a lateral injection. Neither can I see any assignable limit to the extent of such an injection, as long as the superior beds remain unbroken, and the elastic forces are in undiminished action. If, however, the continuity of the upper beds were once broken, the melted lava would instantly occupy the fissure with a velocity proportioned to the pressure, the elastic fluids would find a new vent, and the horizontal motion of the lava would cease altogether.

If volcanic forces ever have acted on a great scale upon unbroken and nearly horizontal strata, especially while such strata were under the pressure of the sea, the formation of tabular and vertical masses of lava (nearly resembling the *Whin-Sill* and *dykes* described in this Paper) appears to me to be a natural consequence of such action. Where the pressure of the sea is removed, and the crust of the earth is broken through, volcanic fluids find a ready escape, eruptions of lava are confined to one spot, and the operations are of a class altogether distinct from those which produced the phenomena of High Teesdale. Though the preceding statements are purely hypothetical, yet, if they shew the possibility of explaining the appearances in Teesdale by any modification of volcanic action, they will answer the purpose intended. For, after the facts adduced in the preceding Sections of this Paper, there cannot, I think, be any reasonable doubt that the trap rocks of that region are of igneous origin.

Obscurity in Geological Questions.

The earth's surface has been modified by so many disturbing forces, since the deposition of the great carboniferous formation, and the whole face of nature has been so much changed by the last great catastrophe which formed the superficial gravel, and excavated most of the secondary vallies, that the conditions of any great geological problem are seldom completely before us. From the composition of a rock (like the *Whin-Sill*) and from its obvious effects, we may refer it to a certain class of operations: but it may be utterly impossible to point out the modifying forces by which it has been brought into its existing form. In such a case, it is enough for us to shew, that the phenomena are not physically incompatible with the causes we assign for them.

Other greater Dislocations.

The fractures and dislocations which intersect the mountain-limestone and coal formations, are not confined to the region above-described. Similar dislocations abound in every part of the mountain-chain which borders on the different branches of the Wear and the Tyne. Though striking and important in themselves, they are as nothing compared with the rupture which has severed the calcareous chain of Cross Fell from the zone of metalliferous limestone, which sweeps round the northern side of the slate mountains of Cumberland. This enormous *fault*, has not only rent asunder the whole formation, through a distance of twenty or thirty miles, but appears, in some places, to have elevated the beds of metalliferous limestone on the north-east side of its range, more than two thousand feet above the level of the corresponding beds on the other side. The range of this *fault* is, in a great measure, concealed by the *new red sandstone*, which extends from the foot of Stainmoor to Solway Frith: but its termination may be seen in the neighbourhood of Brough, where mountain masses of the strata have been rent asunder and brought down with an inverted dip into the bottom of the valley.

Causes of them.

It is hardly possible to help speculating on the mechanical forces which have produced such gigantic effects. Some such modification of volcanic agency, as I have before alluded to, may, perhaps, have produced them; and the supposition seems confirmed by the presence of volcanic rocks along the whole escarpment of the calcareous chain *. I cannot help regarding these prolonged masses of trap, as the broken ends of the great levers which nature employed in severing the metalliferous beds, and bringing the chain of Cross Fell to its present elevation. In this point of view, we must consider the intrusive rocks which have so much modified the structure of High Teesdale, as originating in a system of disturbing forces, which have acted, perhaps, simultaneously, upon all that portion of the great calcareous chain which stretches on the north-east side of the valley of the Eden.

Whatever may be thought of these speculations, the facts detailed in the previous Sections of this Paper, still retain their importance, and, as far I comprehend them, completely establish the volcanic origin of the trap rocks, associated with the *mountain-limestone formation* in High Teesdale.

* The Reader may form a general notion of the extent of the trap, by consulting Mr. Greenough's Geological Map.

EXPLANATION OF THE PLATES.

Plate VII.

Map of the part of Teesdale described in the preceding Paper. The course of the river here represented is about 14 miles in length.

Note. It was not originally the intention of the Author to have given any map of the district; but he found that some of the preceding descriptions would not be intelligible without it. Professor Henslow kindly undertook to construct a map on the proper scale, and, it is hoped, that the principal phenomena described in the Paper, are laid down with sufficient geographical accuracy to answer the purpose intended.

Plate VIII.

Fig. 1. Section transverse to the valley, a little above Middleton. (1.) Beds of the general section above the *Whin-Sill.* (2.) The *Whin-Sill.* (3.) Beds of the general section under the *Whin-Sill.*

Fig. 2. Section transverse to the valley near Winch Bridge, shewing the diminished effect of the great Teesdale *fault.*

Fig. 3. Section through some of the masses of trap near Greengate Farm, in Lunedale. (1.) Grit. (2.) Trap of Saddle Bow. (3.) Dislocated beds of limestone and grit. (4.) Trap. (5.) Grit-stone. (6.) Trap. (7.) Beds of grit-stone, shale, &c.

Fig. 4. Section exposed on the right bank of the Lune, near Lonton. (1.) Diluvium. (2.) Limestone. (3.) Slate-clay. (4.) Argillaceous limestone. (5.) Slate-clay. (6.) Trap, about 11 feet thick. (7.) Hard sandstone (*hazle*) part of it bent up into the trap. (8.) Bed of the river Lune.

Plate IX.

Fig. 1. Section exposed on the left bank of the Lune, near Lonton. (1.) Trap about 11 feet thick. (2.) Hard sandstone. (3.) Shale, with nodules of iron-stone, and impure beds of limestone. (4.) Shale, with bands of hard sandstone.

Fig. 2. Appearance of the trap on the right bank of the Tees below Winch Bridge. The mass of trap is about sixteen feet high, and rests upon thin beds of indurated shale and sandstone.

Fig. 3. Section, shewing the position of the strata between Forcegarth Hill and the bed of the river below High Force. (1.) Trap. (2.) Hard sandstone and slate-clay. (3.) Thin beds of dark blue encrinite limestone.

Fig. 4. Section transverse to the course of the Tees, a mile below Caldron Snout. (1.) Beds of the general section above the *Whin-Sill.* (2.) A great irregular bed of trap, about 200 feet thick, supposed to represent the *Whin-Sill* (General Section, No. 71.) (3.) Beds of limestone, sandstone, and slate-clay.

Fig. 5. Section, commencing immediately below Caldron Snout, and exhibiting the relation of the trap to the inferior strata on the left bank of the Tees. (1.) Trap. (2.) Beds of limestone, sandstone, and slate-clay.

Plate X.

Fig. 1. Junction of the trap with the inferior strata, about 200 yards below Caldron Snout. The length of the exposed beds, is about 40 feet. (1). Trap. (2.) Indurated slate-clay. (3.) Granular limestone. (4.) Indurated slate-clay. (5.) Granular limestone.

Fig. 2. A section exposing the junction of the trap with the inferior strata at White Force. (1.) Trap. The escarpment, in some places, not less than sixty or eighty feet. (2.) Granular limestone. (3.) A bed of highly indurated shale. (4.) Granular limestone. The face of this rock appears to be, in some places, nearly forty feet. Large wedge-shaped masses of trap are interposed between Nos. 2, 3, and 4.

Note. It will be seen by a reference to the Map (Pl. VII.) that the preceding sections are not constructed upon any regular scale. The only wish of the Author was, to convey, by their help, a correct notion of certain geological facts, and to make the descriptions in the text intelligible.

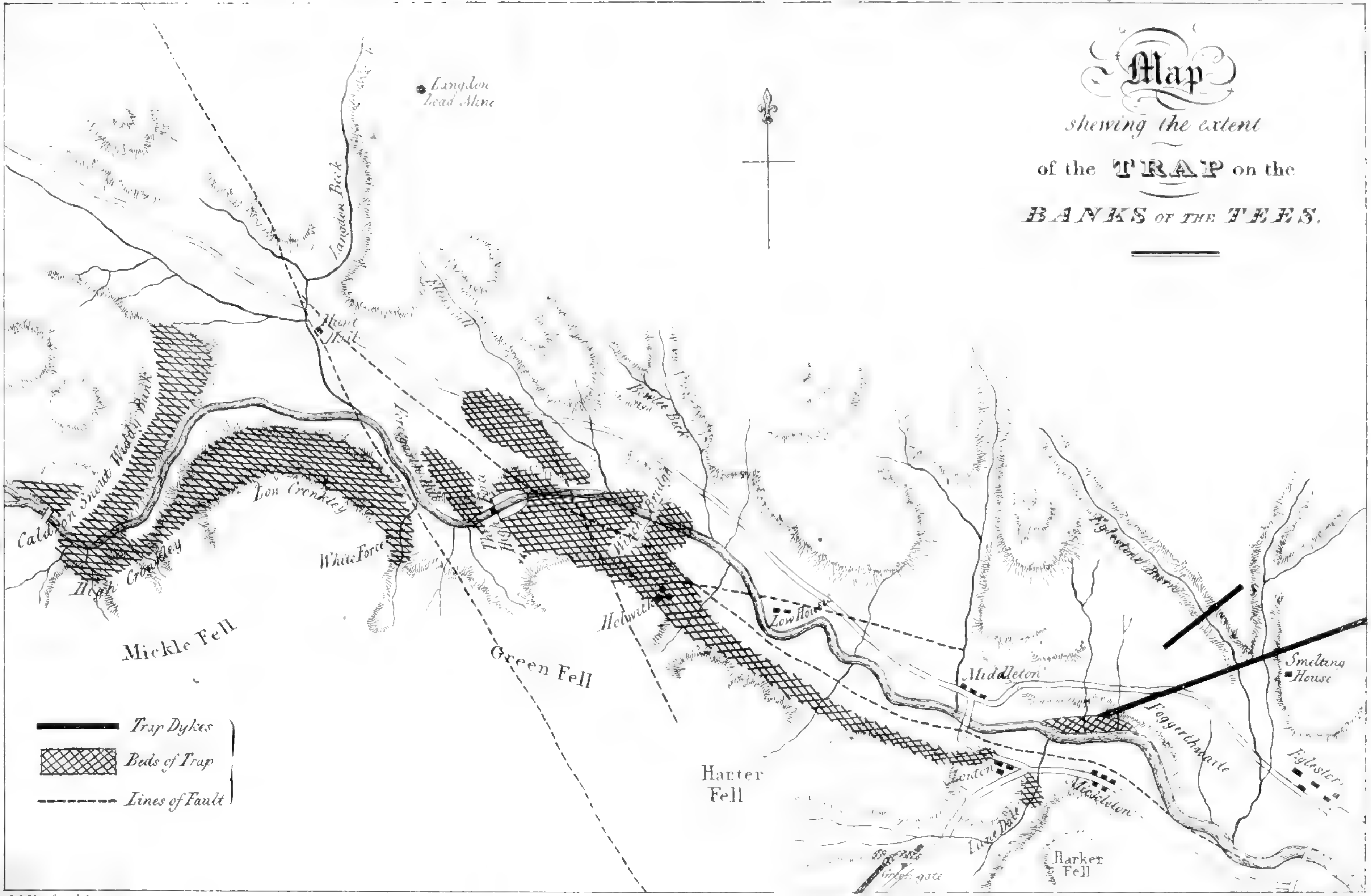

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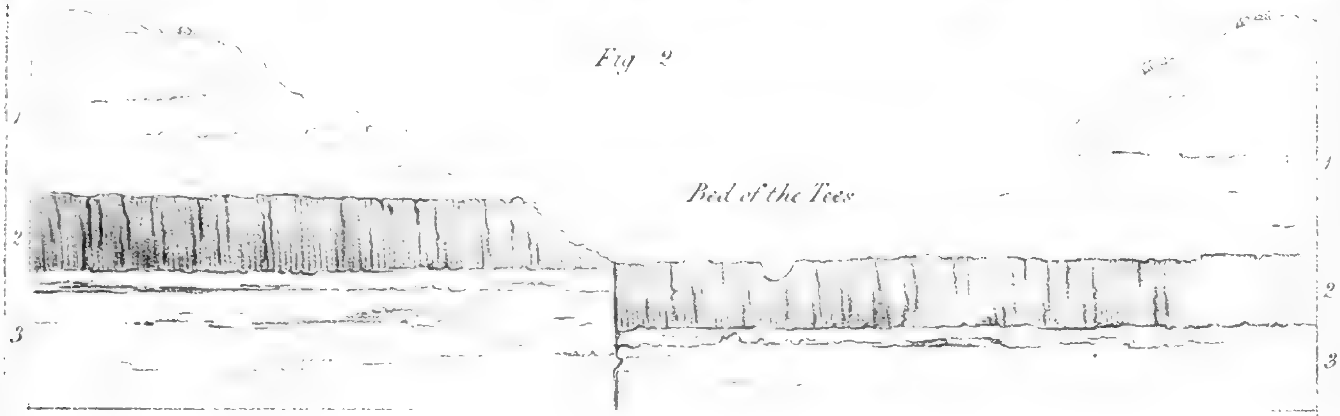
Fig 1

Fig 2



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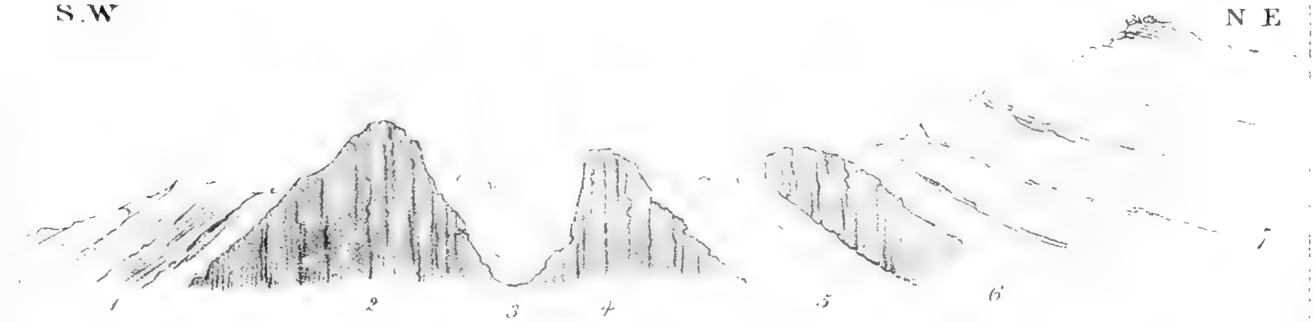

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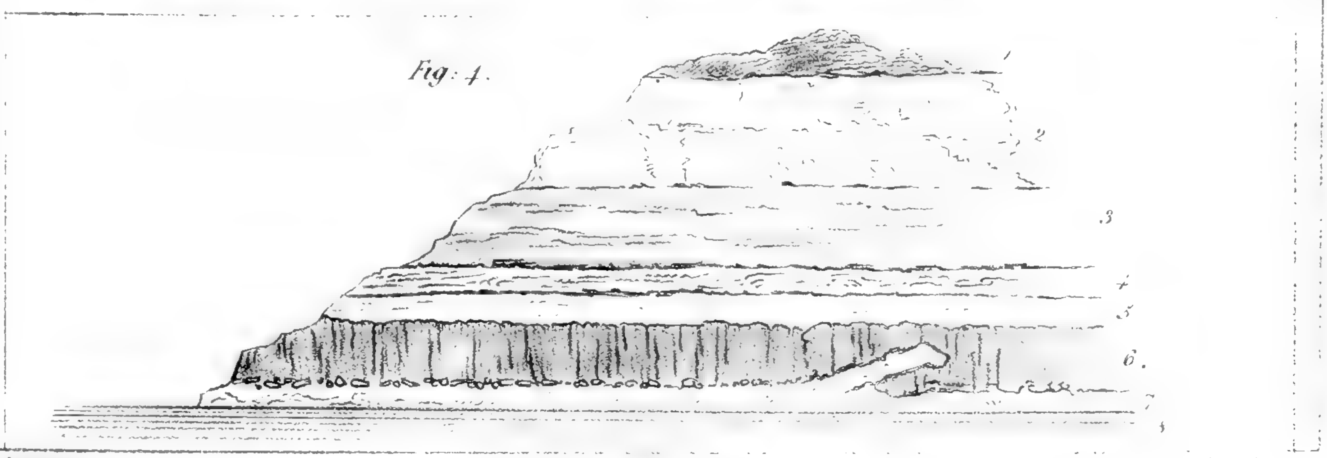

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Fig 1

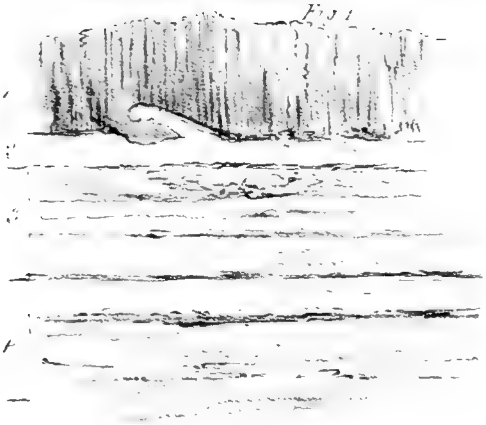

Fig 2

Fig 3

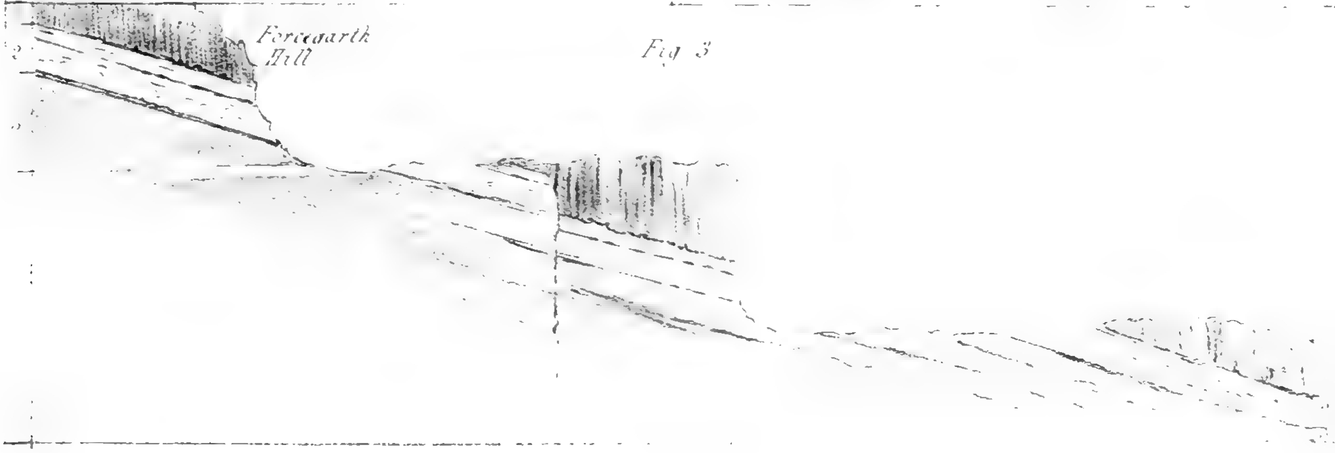

Fig 4

Fig 5

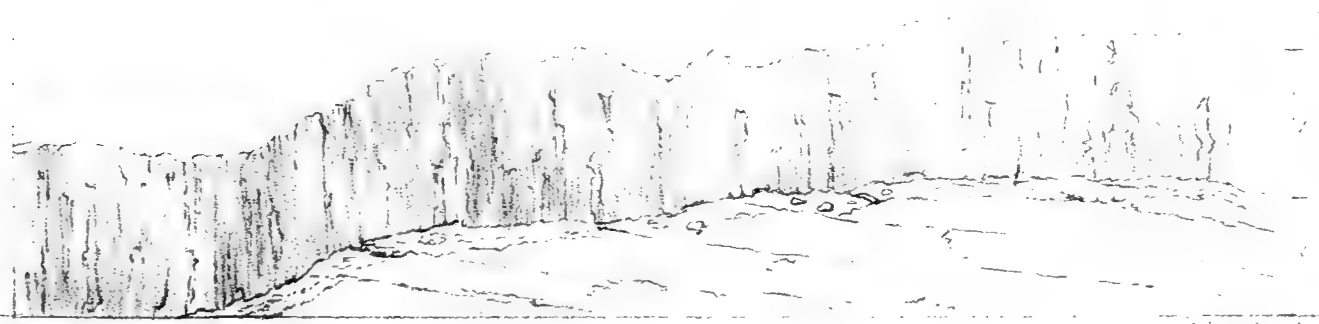

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Fig. 1

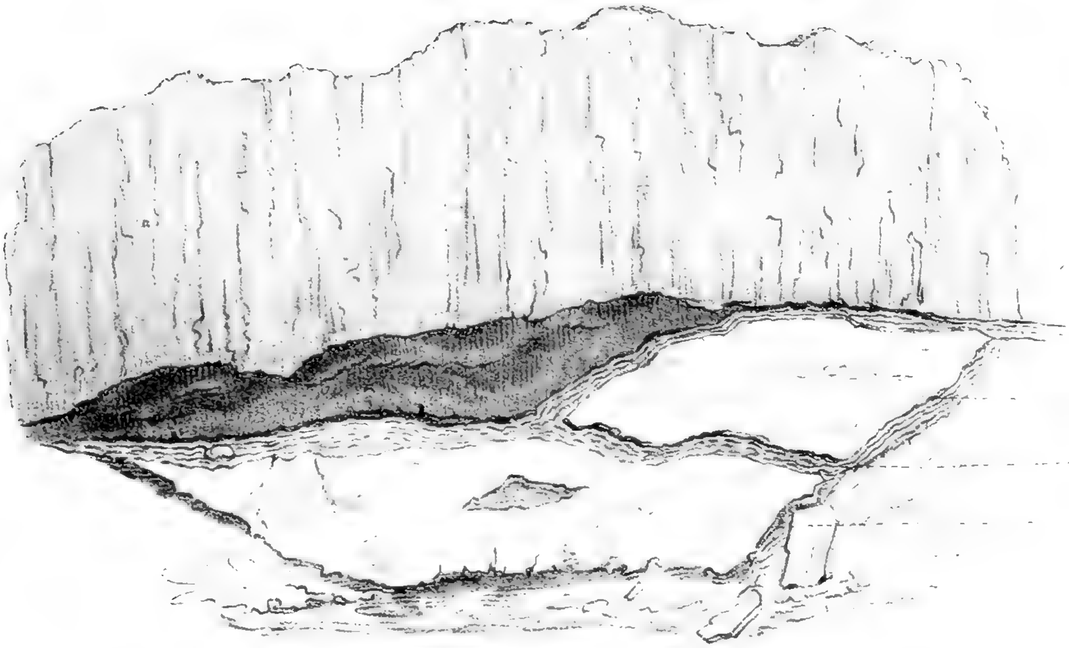
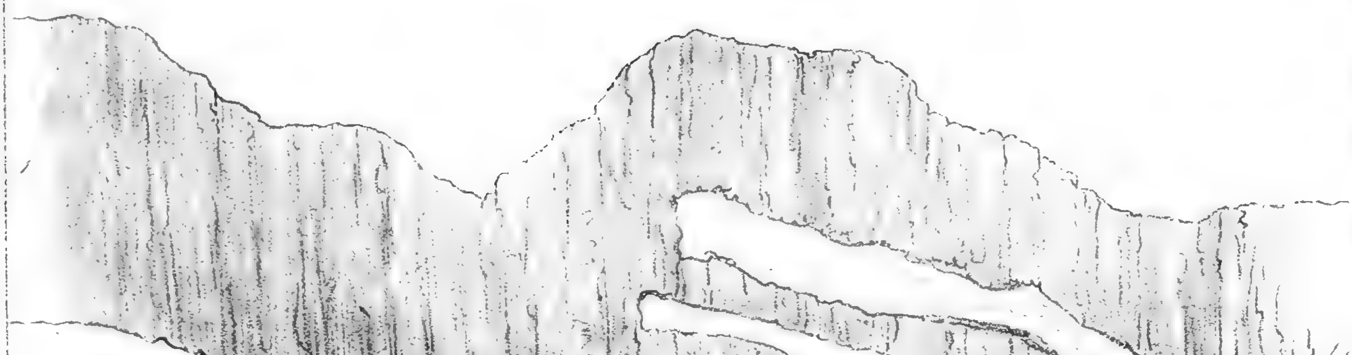


Fig. 2



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XI. *On the Angle made by two Planes, or two straight Lines, referred to three oblique Co-ordinates.*

COMMUNICATED BY W. WHEWELL, A.M. F.R.S.

FELLOW AND TUTOR OF TRINITY COLLEGE.

[Read *Nov.* 24, 1823.]

1. LET there be three co-ordinates, x, y, z, making any angles with each other, and let the dihedral angle, made at the axis of x by the planes xy and xz, be α; that at the axis of y be β, and that at the axis of z, γ. Let there be two planes referred to these co-ordinates, and let their equations be

$$Ax + By + Cz = m, \quad A'x + B'y + C'z = m';$$

it is required to find θ, the angle contained by these planes.

Let three *rectangular* co-ordinates, having the same origin as the others, be assumed, and let x_1, y_1, z_1, be the rectangular co-ordinates of the point of which the oblique co-ordinates are x, y, z. Let A be the origin, and AM, MN, NP, the rectangular co-ordinates, and through P let a plane be drawn parallel to the plane of yz, and meeting the axis of x in L: and let planes, parallel to this, be drawn also through M and N, meeting Ax in H, K. Let Al be a line perpendicular to the plane yz, and let the planes parallel to yz meet this line in h, k, l. Then AL is x. And if a, a', a'', be the cosines of the angles which x_1, y_1, and z_1, make with Al, we shall have

$$Al = Ah + hk + hl = ax_1 + a'y_1 + a''z_1.$$

And, if the cosine of the angle LAl, which x makes with a perpendicular to yz, be d, we shall have,

$$AL = \frac{Al}{d}, \text{ or } x = \frac{1}{d}(ax_1 + a'y_1 + a''z_1) \ldots\ldots (1).$$

Similarly, if a perpendicular to the plane xz make with y an angle whose cosine is e, and with x_1, y_1, z_1, angles whose cosines are b, b', b''; and, if a perpendicular to the plane xy make with z an angle whose cosine is f, and with x_1, y_1, z_1, angles whose cosines are c, c', c'', we shall have

$$y = \frac{1}{e}(bx_1 + b'y_1 + b''z_1) \ldots\ldots (1).$$

$$z = \frac{1}{f}(cx_1 + c'y_1 + c''z_1) \ldots\ldots (1).$$

Also, we have

$$a^2 + a'^2 + a''^2 = 1,\ b^2 + b'^2 + b''^2 = 1,\ c^2 + c'^2 + c''^2 = 1 \ldots\ldots (2).$$

And the equation, to the rectangular co-ordinates,

$$\begin{aligned} \text{of the plane } yz \text{ is } & ax_1 + a'y_1 + a''z_1 = 0, \\ \text{of } xz \ldots & bx_1 + b'y_1 + b''z_1 = 0, \\ \text{of } xy \ldots & cx_1 + c'y_1 + c''z_1 = 0. \end{aligned}$$

The formula for the angle made by two planes referred to rectangular co-ordinates is known, and since α is the angle of the two latter planes, we have, by applying the formula,

$$\left.\begin{aligned} -\cos.\alpha &= \frac{bc + b'c' + b''c''}{\sqrt{(b^2 + b'^2 + b''^2)(c^2 + c'^2 + c''^2)}} = bc + b'c' + b''c''. \\ &\text{Similarly,} \\ -\cos.\beta &= ac + a'c' + a''c'' \\ -\cos.\gamma &= a'b' + a'b' + a''b'' \end{aligned}\right\} \ldots (3).$$

Also $Ax + By + Cz = m$, $A'x + B'y + C'z = m'$, being the equations to the two planes, whose angle (θ) is required, if we put for x, y, z, their values from (1) these equations become

$$\left.\begin{array}{l} \left(\frac{Aa}{d}+\frac{Bb}{e}+\frac{Cc}{f}\right)x_1+\left(\frac{Aa'}{d}+\frac{Bb'}{e}+\frac{Cc'}{f}\right)y_1+\left(\frac{Aa''}{d}+\frac{Bb''}{e}+\frac{Cc''}{f}\right)z_1=m \\ \left(\frac{A'a}{d}+\frac{B'b}{e}+\frac{C'c}{f}\right)x_1+\left(\frac{A'a'}{d}+\frac{B'b'}{e}+\frac{C'c'}{f}\right)y_1+\left(\frac{A'a''}{d}+\frac{B'b''}{e}+\frac{C'c''}{f}\right)z_1=m' \end{array}\right\}\dots\dots(4).$$

Hence the value of cos. θ will be given by the formula

$$-\cos.\theta = \frac{PP' + QQ' + RR'}{\sqrt{(P^2 + Q^2 + R^2)(P'^2 + Q'^2 + R'^2)}},$$

P, Q, R, P', Q', R', being the coefficients in equations (4).

By developing the numerator $PP' + QQ' + RR'$ we obtain

$$\frac{AA'a^2}{d^2}+\frac{BB'b^2}{e^2}+\frac{CC'c^2}{f^2}+(AB'+A'B)\frac{ab}{de}+(AC'+A'C)\frac{ac}{df}+(BC'+B'C)\frac{bc}{dg}$$

$$+\frac{AA'a'^2}{d^2}+\frac{BB'b'^2}{e^2}+\frac{CC'c'^2}{f^2}+(AB'+A'B)\frac{a'b'}{de}+(AC'+A'C)\frac{a'c'}{df}+(BC'+B'C)\frac{b'c'}{df}$$

$$+\frac{AA'a'^2}{d^2}+\frac{BB'b''^2}{e^2}+\frac{CC'c''^2}{f^2}+(AB'+A'B)\frac{a''b''}{de}+(AC'+A'C)\frac{a''c''}{df}+(BC'+B'C)\frac{b''c''}{df}.$$

And reducing, by equations (2) and (3), this becomes

$$\frac{AA'}{d^2}+\frac{BB'}{e^2}+\frac{CC'}{f^2}-\frac{(AB'+A'B)}{de}\cos.\gamma-\frac{(AC'+A'C)}{df}\cos.\beta-\frac{(BC'+B'C)}{ef}\cos.\alpha.$$

The denominator of $-\cos.\theta$ being transformed in the same manner, we shall find for $P^2 + Q^2 + R^2$ the expression

$$\frac{A^2a^2}{d^2}+\frac{B^2b^2}{e^2}+\frac{C^2c^2}{f^2}+\frac{2ABab}{de}+\frac{2ACac}{df}+\frac{2BCbc}{ef}$$

$$+\frac{A^2a'^2}{d^2}+\frac{B^2b'^2}{e^2}+\text{\&c.}$$

$$+\frac{A^2a''^2}{d^2}+\frac{B^2b''^2}{e^2}+\text{\&c.}$$

which, as before, is reduced to

$$\frac{A^2}{d^2}+\frac{B^2}{e^2}+\frac{C^2}{f^2}-\frac{2AB}{de}\cos.\gamma-\frac{2AC}{df}\cos.\beta-\frac{2BC}{ef}\cos.\alpha.$$

In the same manner we transform $P'^2 + Q'^2 + R'^2$, and the expression becomes

$$- \cos.\theta =$$

$$\frac{\frac{AA'}{d^2} + \frac{BB'}{e^2} + \frac{CC'}{f^2} - \frac{A'B + AB'}{de}\cos.\gamma - \frac{A'C + AC'}{df}\cos.\beta - \frac{B'C + BC'}{ef}\cos.\alpha}{\sqrt{\left\{\left(\frac{A^2}{d^2} + \frac{B^2}{e^2} + \frac{C^2}{f^2} - \frac{2AB}{de}\cos.\gamma - \frac{2AC}{df}\cos.\beta - \frac{2BC}{ef}\cos.\alpha\right)\left(\frac{A'^2}{d^2} + \frac{B'^2}{e^2} + \frac{C'^2}{f^2} - \frac{2A'B'}{de}\cos.\gamma - \frac{2A'C'}{df}\cos.\beta - \frac{2B'C'}{ef}\cos.\alpha\right)\right\}}}$$

If we suppose a sphere to be described with its center at the origin, it will cut the planes xy, xz, yz, in three arcs, which will form a triangle, whose angles will be α, β, γ. And, if perpendiculars be let fall from its angles on the opposite sides, d, e, f, will be the sines of these perpendiculars.

2. Let there be three oblique co-ordinates as before, and two lines of which the equations are respectively

$$\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\} \qquad \left.\begin{array}{l} x = a'z + \alpha' \\ y = b'z + \beta' \end{array}\right\};$$

it is required to find η the angle contained by these lines*.

Let $\cos. xz$, $\cos. vt$, &c. indicate the cosines of the angles which the directions of the lines x and z, v and t, &c. make with each other.

Now if D be the diagonal of any parallelepiped, t, u, v its sides,

$$D^2 = t^2 + u^2 + v^2 + 2uv\cos.uv + 2vt\cos.vt + 2tu\cos.tu \ldots\ldots (1).$$

The angles uv, &c. being measured at the solid angles to which the diagonal is drawn.

Let there be lines passing through the origin, parallel to the given lines; the equations to these lines will be respectively,

$$\left.\begin{array}{l} x = az \\ y = bz \end{array}\right\}, \text{ and } \left.\begin{array}{l} x' = a'z' \\ y' = b'z' \end{array}\right\}.$$

Let D be the distance of any two points from each other, which

* The investigation which follows, is by J. W. Lubbock, Esq. of Trinity College.

are situated, one on each of these lines; δ, δ' the distances of these points from the origin. Then

$$D^2 = \delta^2 + \delta'^2 - 2\delta\delta' \cos.\eta \ldots\ldots (2).$$

But by equation (1), we have

$$D^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 +$$
$$+ 2(x-x')(y-y')\cos.xy + 2(y-y')(z-z')\cos.yz + 2(z-z')(x-x')\cos.zx.$$

Also

$$\delta^2 = x^2 + y^2 + z^2 + 2xy\cos.xy + 2yz\cos.yz + 2zx\cos.zx,$$
$$\delta'^2 = x'^2 + y'^2 + z'^2 + 2x'y'\cos.xy + 2y'z'\cos.yz + 2z'x'\cos.zx;$$

putting these values in (2) and reducing,

$$xx' + yy' + zz' + (xy' + x'y)\cos.xy + (yz' + y'z)\cos.yz + (zx' + z'x)\cos.zx$$
$$= \cos.\eta \,.\, \sqrt{\{x^2 + y^2 + z^2 + 2xy\cos.xy + 2yz\cos.yz + 2zx\cos.zx\}} \,.\, \sqrt{\{x'^2 + \&c.\}}$$

And putting for x, y, x', y', their values in z, z', and dividing by zz', we shall find

$$\cos.\eta = \frac{aa' + bb' + 1 + (ab' + a'b)\cos.xy + (b + b')\cos.yz + (a + a')\cos.xz}{\sqrt{(a^2 + b^2 + 1 + 2ab\cos.xy + 2b\cos.yz + 2a\cos.xz)} \,.\, \sqrt{(a'^2 + \&c.)}},$$

the second factor of the denominator differing from the first only in having a', b', instead of a, b: and thus we have the required angle.

When the co-ordinates are rectangular, $\cos.xy$, $\cos.yz$, $\cos.xz$, are each 0, and

$$\cos.\eta = \frac{aa' + bb' + 1}{\sqrt{(a^2 + b^2 + 1)} \,.\, \sqrt{(a'^2 + b'^2 + 1)}},$$

which is the known formula.

From the above expression, we may deduce the fundamental formula of Spherical Trigonometry.

Let the two given lines be in the planes xz, yz, respectively, and both perpendicular to the axis of z. In this case, their angle will measure the inclination of the planes xz, yz, and their equations will be

$$\left.\begin{array}{l} x = az \\ y = 0 \end{array}\right\}, \text{ and } \left.\begin{array}{l} x' = 0 \\ y' = b'z' \end{array}\right\}.$$

But manifestly, in this case, the equations are

$$x = -\frac{z}{\cos. xz}, \text{ and } y' = -\frac{z'}{\cos. yz},$$

for the two lines respectively. Hence

$$a = -\frac{1}{\cos. xz}, \; b = 0, \; a' = 0, \; b' = -\frac{1}{\cos. yz},$$

and putting these values in cos. γ,

$$\cos. \eta = \frac{1 - 1 - 1 + \frac{\cos. xy}{\cos. xz \cos. yz}}{\sqrt{\left(\frac{1}{\cos.^2 xz} + 1 - 2\right)} \sqrt{\left(\frac{1}{\cos.^2 x^2 z} + 1 - 2\right)}}$$

$$= \frac{\cos. xy - \cos. xz \cos. yz}{\sin. xz \sin. yz}.$$

Now if a sphere be described about the origin, the co-ordinate planes will cut it, making a triangle, of which the sides are measured by the angles xy, xz, yz, and the angle opposite to xy is η. Hence, the above is the expression for the cosine of the angle of a spherical triangle in terms of the sides.

Fig. 1

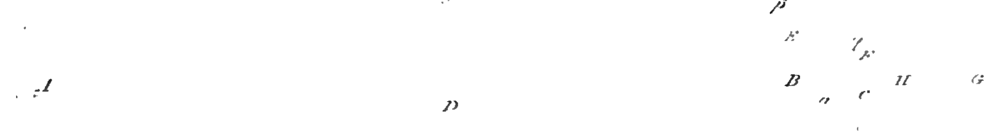


Fig. 2

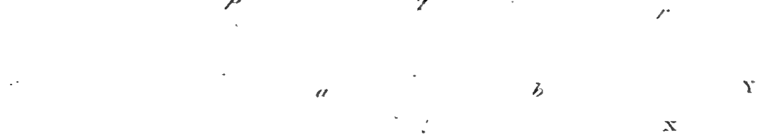


Fig. 3

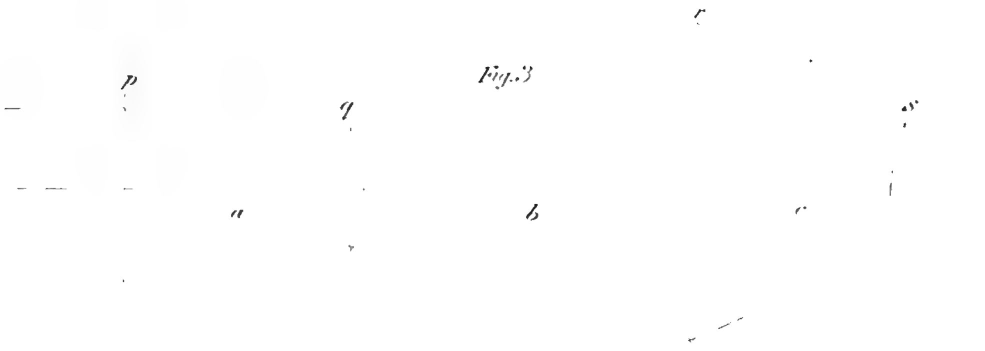


Fig. 4

Fig. 5

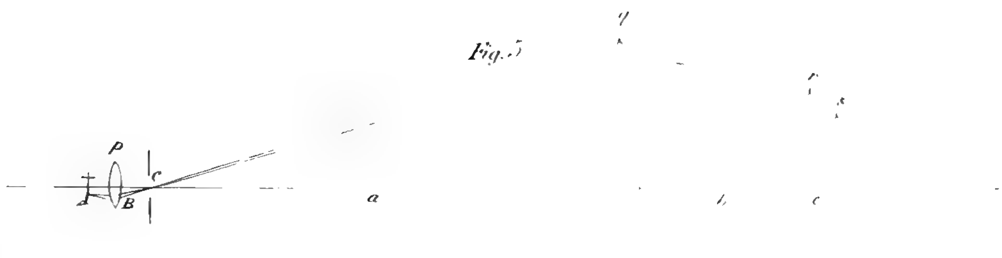
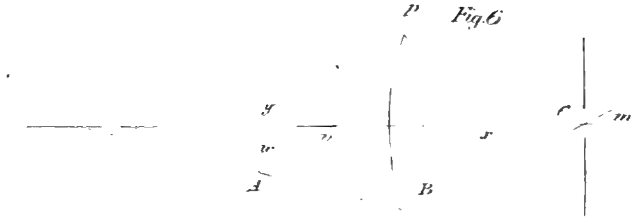


Fig. 6



XII. *On the Figure assumed by a Fluid Homogeneous Mass, whose Particles are acted on by their mutual Attraction, and by small extraneous Forces.*

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[Read *March* 15, 1824.]

The principal difficulty in the solution of this problem, consists in the investigation of the attraction of any spheroid (differing little from a sphere) upon a point in its surface. This has been found by Laplace, in a manner so general, and by an analysis so powerful, that any new investigations might seem entirely unnecessary. But the abstruse nature of that analysis, it must be acknowledged, is such as to make a more simple investigation desirable: and the obscurities which have led Laplace himself into error, serve to shew the value of a process which involves nothing more difficult than the common applications of the differential calculus. I venture to indulge in a hope that the solution which I have the honor to lay before this Society, imperfect as it may be, will tend to make this subject more accessible to those who have hitherto been deterred from pursuing it by the mass of analysis in the *Mecanique Celeste*.

With the exception of the proposition which reduces the discovery of the attraction in any direction to the investigation of the value of a single integral, the process pursued in this Paper is entirely different from that of Laplace. The theory given in detail, is restricted to the case in which the disturbing forces are symmetrical about an axis; but the extension of the same principle, which enables us to apply a similar process to any forces whatever, is shortly indicated at the termination. To shew the method of applying the theory to any given case, I have considered the effect which the ring of Saturn produces on his figure (supposing him homogeneous) and arrive at the important conclusion, that the attraction of the ring, on the hypothesis of gravitation, will not explain the peculiarity which has been observed in the figure of Saturn. It is not easy to assign any other cause for the singularity of form of that remarkable planet; and if the observations remain undisputed, the physical explanation of this phenomenon may exercise the ingenuity of future philosophers.

(1.) Let a, b, c, be the rectangular co-ordinates of any point of the mass parallel to the axes of x, y, z,: let X, Y, Z, be the forces acting upon that point in the same directions. Then if a function U can be found such that

$$\frac{dU}{da} = X, \ \frac{dU}{db} = Y, \ \frac{dU}{dc} = Z,$$

the equation to the surface of equal pressure passing through this point, will be $U = C$. The condition necessary to the existence of equilibrium is, that a function U can be found which will satisfy these equations. In the case which we propose to consider, the forces arise entirely from attractions and centrifugal force; and will therefore, by a well-known theorem, satisfy these equations. The equilibrium being possible, we have only to find the equation of the surface bounding the mass; which we shall attempt by the assistance of the following theorem of Laplace.

(2.) Let V be the sum of the products of every attracting particle into the reciprocal of its distance from the attracted point: x, y, z, the co-ordinates of any point: let the forces be considered positive when they tend to make a, b, c, increase: call U', X', Y', Z', those parts of U, X, Y, Z which arise from these attractions.

$$\text{Then } \frac{d^3 V}{dx\,dy\,dz} = \frac{1}{\sqrt{\{(x-a)^2 + (y-b)^2 + (z-c)^2\}}};$$

$$\therefore \frac{d^4 V}{da\,dx\,dy\,dz} = \frac{x-a}{\{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{3}{2}}}$$

$$= \frac{d^3 X'}{dx\,dy\,dz}, \text{ or } \frac{d^4 V}{dx\,dy\,dz\,da}; = \frac{d^3 X'}{dx\,dy\,dz}; \quad \therefore \frac{dV}{da} = X'.$$

Similarly $\frac{dV}{db} = Y'$, $\frac{dV}{dc} = Z'$, (the limits of integration being supposed independent of a, b, c,) and the value of U' is therefore V. If then we take the sum of the products of each attracting particle into the reciprocal of its distance from the point whose co-ordinates are a, b, c, and if we add that part of U which originates from the extraneous forces, and make the sum $= C$, we shall have the equation to the surface.

(3.) Our object then, at present, is to find the value of this sum for a point in the surface; and we might suppose the disturbing forces to be any whatever. But in nearly all the cases to which we can apply this investigation, the forces act symmetrically round an axis. Suppose, then, that the forces are symmetrical about an axis: the body will then be a solid of revolution; and, for the sake of simplicity, we will make $b = 0$. We proceed to investigate that part of V which arises from the attraction of the particles of the body.

(4.) First, for a small pyramid whose vertex is the attracted point. Let ρ be its whole length, p any variable length, A the area of a section perpendicular to its axis at distance 1 from the vertex: the area at distance p is Ap^2; the sum of the products of the masses into the reciprocals of their distances, for the slice included between the lengths p and $p + \delta p$ is ultimately $Ap\delta p$;

hence the sum for the pyramid is $\int_p A p^* = \frac{A p^2}{2}$; which for the whole pyramid is $\frac{A\rho^2}{2}$.

(5.) To find the length of ρ and the value of A, suppose c to be the ordinate parallel to the axis of the solid: suppose the solid divided into wedges by planes passing through the line c; let two of these planes make, with the plane of xz, the angles ϕ and $\phi + \delta\phi$: suppose the included wedge divided into pyramids by lines on these planes, drawn from the attracted point; let two of them make with c the angles θ and $\theta + \delta\theta$. Then $A = \sin.\theta\,.\,\delta\theta\,.\,\delta\phi$. Also $x = a - \rho \sin.\theta \cos.\phi$; $y = \rho \sin.\theta.\sin.\phi$; $z = c - \rho\cos.\theta$; $x, y, z,$ being the co-ordinates of the point at which ρ meets the surface again. Substituting these values in the equation to the surface, the value of ρ will be had in terms of θ and ϕ.

(6.) The disturbing forces being small, we will neglect the squares and higher powers of the disturbing forces and quantities dependent on them. And as the body, without this disturbance, would be a sphere, we will assume for its equation $x^2 + y^2 + z^2 = r^2 + \chi(z)$, where $\chi(z)$ is a function of z involving a small multiplier. Substituting for $x, y, z,$ the values found above,

$$a^2 + c^2 - 2\rho\,(c\cos.\theta + a\cos.\phi.\sin.\theta) + \rho^2 = r^2 + \chi\,(c - \rho\cos.\theta).$$

But a and c are co-ordinates of a point in the surface: hence

$$a^2 + c^2 \qquad\qquad = r^2 + \chi\,(c).$$

Subtracting, $\rho^2 - 2\rho\,(c\cos.\theta + a\cos.\phi\sin.\theta) = \chi\,(c - \rho\cos.\theta) - \chi\,(c)$;

$$\therefore\ \rho = 2\,(c\cos.\theta + a\cos.\phi\sin.\theta) + \frac{\chi\,(c-\rho\cos.\theta) - \chi\,(c)}{\rho}.$$

An approximate value of ρ is $2\,(c\cos.\theta + a\cos.\phi\sin.\theta)$: let this be v; substituting it in the small term,

$$\rho = v + \frac{\chi\,(c - v\cos.\theta) - \chi\,(c)}{v}. \quad \text{And } \frac{\rho^2}{2} = \frac{v^2}{2} + \chi\,(c - v\cos.\theta) - \chi\,(c).$$

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* By $\int_p A p$ is meant what is usually written $\int A p\, dp$, the quantity whose differential coefficient, taken with respect to p, is $A p$.

(7.) The sum of the products of each particle into the reciprocal of its distance, for one of the pyramids into which the mass is divided, has been found to be $\frac{A\rho^2}{2}$, or ultimately $\frac{\rho^2}{2}.\sin.\theta.\delta\theta.\delta\phi$. For one of the wedges, then, this will be $\delta\phi\int_\theta\frac{\rho^2}{2}\sin.\theta$. And it is plain, that one of the lines, which determine the limits of integration for the wedge, is in the direction of the ordinate c; and the other is the tangent of the curve, formed by the intersection of one of its planes with the surface, at the attracted point; the direction of which is determined by making $\rho=0$. Let Θ be the value of θ which makes $\rho=0$: then $\int_\theta\frac{\rho^2}{2}\sin.\theta$ must be taken between the limits $\theta=0$, $\theta=\Theta$. This must then be integrated with respect to ϕ through a whole circumference; and thus the value of that part of V which arises from the attraction of the particles, is $\int_\phi\int_\theta\frac{\rho^2}{2}\sin.\theta$, taken between the limits above-mentioned.

(8.) To determine Θ we observe that $\rho=2\,(c\cos.\theta+a\cos.\phi\sin.\theta)$

$$+\frac{-\chi'(c).\rho\cos.\theta+\frac{\chi''(c)}{2}\rho^2\cos.^2\theta-\&c.}{\rho}$$

$$=2\,(c\cos.\theta+a\cos.\phi\sin.\theta)-\chi'(c)\cos.\theta+\frac{\chi''(c)}{2}\rho\cos.^2\theta-\&c.$$

where $\chi'(c)$, $\chi''(c)$, &c. are the differential coefficients of $\chi'(c)$ taken with respect to c. In this expression make $\rho=0$, $\theta=\Theta$;

$$\text{then } 0=2\,(c\cos.\Theta+a\cos.\phi\sin.\Theta)-\chi'(c).\cos.\Theta;$$

$$\text{which gives }\tan.\Theta=-\frac{c-\frac{1}{2}\chi'(c)}{a\cos.\phi};\ \sin.\Theta$$

$$=\frac{c-\frac{1}{2}\chi'(c)}{\sqrt{\{\overline{a\cos.\phi}|^2+\overline{c-\frac{1}{2}\chi'(c)}|^2\}}};\ \cos.\Theta=\frac{-a\cos.\phi}{\sqrt{\{\overline{a\cos.\phi}|^2+\overline{c-\frac{1}{2}\chi'(c)}|^2\}}}$$

(9.) The first part of $\frac{\rho^2}{2}\sin.\theta$ is $\frac{v^2}{2}\sin.\theta$, or

$$2\sin.\theta\,\{c^2\cos.^2\theta + 2ac\cos.\phi.\sin.\theta.\cos.\theta + a^2\cos.^2\phi.\sin.^2\theta\}.$$

The general integral with respect to θ is

$$-\frac{2c^2}{3}\cos.^3\theta + \frac{4ac}{3}\cos.\phi.\sin.^3\theta - \frac{2a^2}{3}\cos.^2\phi\cos.\theta\,(\sin.^2\theta+2).$$

Upon giving to θ the value Θ, it will be found that each term involves an odd power of cos. ϕ, divided by an odd power of

$$\sqrt{\{\overline{a\cos.\phi}|^2 + \overline{c-\tfrac{1}{2}\chi'(c)}|^2\}}.$$

Now the sign with which this radical is taken, can never alter, for the radical never becomes $=0$; if then we put $\pi+\phi$ for ϕ, we shall have the same expression with a different sign. Hence, on performing the next integration, these terms will disappear by the opposition of signs, and we may, therefore, reject them at once. On giving to θ the value 0, the expression becomes

$$-\frac{2c^2}{3} - \frac{4a^2}{3}\cos.^2\phi:$$

the value of the integral is therefore

$$\frac{2c^2}{3} + \frac{4a^2}{3}\cos.^2\phi.$$

(10.) Integrating this with respect to ϕ through a circumference, we have, for the first part of the required expression,

$$\frac{4\pi}{3}(c^2+a^2).$$

(11.) The other part to be integrated is

$$\sin.\theta.\{\chi(c-v\cos.\theta)-\chi(c)\}.$$

If we expand $\chi(c-v\cos.\theta)$ by Taylor's Series, the m^{th} term

$$\text{will be}\ \frac{\frac{d^m.\chi(c)}{dc^m}}{1.2.3\ldots m}\times\overline{-1}|^m.v^m.\overline{\cos.\theta}|^m.\sin.\theta$$

$$=\frac{\overline{-2}|^m}{1.2.\ldots.m}\cdot\frac{d^m.\chi(c)}{dc^m}(c\cos.\theta + a\cos.\phi\sin.\theta)^m.\overline{\cos.\theta}|^m.\sin.\theta.$$

The $\overline{p+1}^{\text{th}}$ term of

$$(c\cos.\theta + a\cos.\phi\sin.\theta)^m.\overline{\cos.\theta}|^m.\sin\theta \text{ is}$$

$$\frac{m.\overline{m-1}\ldots\overline{m-p+1}}{1.2\ldots p}\,c^{m-p}.a^p.\overline{\cos.\phi}|^p.\overline{\sin.\theta}|^{p+1}.\overline{\cos.\theta}|^{2m-p}.$$

Now when p is odd

$$\textstyle\int_\theta \overline{\sin.\theta}|^{p+1}.\overline{\cos.\theta}|^{2m-p} = \overline{\sin.\theta}|^{p+2}.\left\{\frac{1}{2m+1}\overline{\cos.\theta}|^{2m-p-1}\right.$$

$$\left.+\frac{2m-p-1}{\overline{2m+1}.\overline{2m-1}}.\overline{\cos.\theta}|^{2m-p-3} + \&c. + \frac{\overline{2m-p-1}.\overline{2n-p-3}\ldots.4.2}{\overline{2m+1}.\overline{2m-1}\ldots\ldots\overline{p+4}.\overline{p+2}}\right\},$$

This vanishes when $\theta = 0$; when $\theta = \Theta$, every term involves an even power of cos. ϕ, which is multiplied by $\overline{\cos.\phi}|^p$, and therefore vanishes on being integrated through a circumference at the next operation. No term, therefore, results from the odd values of p.

(12.) When p is even, $\int_\theta \overline{\sin.\theta}|^{p+1}.\overline{\cos.\theta}|^{2m-p}$ is $\overline{\sin.\theta}|^{p+2}\times$

$$\left\{\frac{1}{2m+1}\overline{\cos.\theta}|^{2m-p-1} + \frac{2m-p-1}{\overline{2m+1}.\overline{2m-1}}\overline{\cos.\theta}|^{2m-p-3} + \&c.\right.$$

$$\left.+\frac{\overline{2m-p-1}.\overline{2m-p-3}\ldots.5.3}{\overline{2m+1}.\overline{2m-1}\ldots\ldots\overline{p+5}.\overline{p+3}}\cos.\theta\right\}$$

$$-\frac{\overline{2m-p-1}.\overline{2m-p-3}\ldots\ldots5.3}{\overline{2m+1}.\overline{2m-1}\ldots.\overline{p+5}.\overline{p+3}}.\cos.\theta\left\{\frac{1}{p+1}\overline{\sin.\theta}|^p\right.$$

$$+\frac{p}{\overline{p+1}.\overline{p-1}}\overline{\sin.\theta}|^{p-2} + \&c.$$

$$\left.+\frac{p.\overline{p-2}\ldots.4.2}{\overline{p+1}.\overline{p-1}\ldots\ldots3.1}\right\}.$$

When $\theta = \Theta$, every term involves an odd power of cos. ϕ, and vanishes on integration with respect to ϕ. When $\theta = 0$, the expression becomes

$$-\frac{\overline{2m-p-1}.\overline{2m-p-3}\ldots.3.1.p.\overline{p-2}\ldots4.2}{\overline{2m+1}.\overline{2m-1}\ldots\ldots\ldots3.1},$$

or the integral is $\dfrac{\overline{2m-p-1}.\overline{2m-p-3}\ldots.3.1.p.\overline{p-2}\ldots.4.2}{\overline{2m+1}.\overline{2m-1}\ldots\ldots\ldots3.1}$.

(13.) Multiplying this by $\overline{\cos.\phi}|^p$, and integrating with respect to ϕ through a circumference, since the definite integral of $\overline{\cos.\phi}|^p$ $=\frac{\overline{p-1}.\overline{p-3}....3.1}{p.\overline{p-2}....4.2}\,2\pi$, we have

$$\int_\phi\int_\theta \overline{\cos.\phi}|^p.\ \overline{\sin.\theta}|^{p+1}.\ \overline{\cos.\theta}|^{2m-p} = 2\pi.\frac{\overline{2m-p-1}.\overline{2m-p-3}....3.1}{\overline{2m+1}.\overline{2m-1}....\overline{p+3}.\overline{p+1}};$$

and the definite integral of the $\overline{p+1}|^{\text{th}}$ term of

$$(c\cos.\theta+a\cos.\phi\sin.\theta)^m.\ \overline{\cos.\theta}|^m.\ \sin.\theta \text{ is } 2\pi\times\frac{m.\overline{m-1}...\overline{m-p+1}}{1.2....p}$$

$$\times\frac{\overline{2m-p-1}.\overline{2m-p-3}.....3.1}{\overline{2m+1}.\overline{2m-1}......\overline{p+3}.\overline{p+1}}\times c^{m-p}.a^p.$$

Hence, we have the following rules for finding the value of

$$\int_\theta\int_\phi \sin.\theta\,(\chi(c-v\cos.\theta)-\chi(c)).$$

1. Expand $\overline{c+a}|^m$, and select those terms

$$\left(\text{as } \frac{m.\overline{m-1}..\overline{m-p+1}}{1.2....p}\,c^{m-p}.a^p\right)$$

in which the index of a is even or 0.

2. Multiply the term involving a^p by

$$\frac{\overline{2m-p-1}.\overline{2m-p-3}....3.1}{\overline{2m+1}.\overline{2m-1}......\overline{p+3}.\overline{p+1}}.$$

3. Collect the terms for different values of p with the same value of m, and multiply their sum by $\frac{\overline{-2}|^m}{1.2.....m}\cdot\frac{d^m.\chi(c)}{dc^m}$.

4. Collect the series found by giving to m the values 1, 2, 3, &c. call the sum $\psi(c)$: then $2\pi.\psi(c)$ is the integral required.

(14.) The expression for $\psi(c)$ may be put under the following form, which will probably be found more convenient,

$$\psi(c)=\frac{-\chi'(c)}{1}.\frac{2c}{3}+\frac{\chi''(c)}{1.2}.\frac{\overline{2c}|^2}{5}-\frac{\chi'''(c)}{1.2.3}.\frac{\overline{2c}|^3}{7}+\&c.$$

$$+\frac{2a^2}{1}.\left\{\chi''(c).\frac{1}{3.5}-\frac{\chi'''(c)}{1}.\frac{2c}{5.7}+\frac{\chi''''(c)}{1.2}.\frac{\overline{2c}|^2}{7.9}-\&c.\right\}$$

$$+\frac{\overline{2a^2}|^2}{1.2}\cdot\left\{\chi''''(c)\cdot\frac{1}{5.7.9}-\frac{\chi'''''(c)}{1}\cdot\frac{2c}{7.9.11}+\&c.\right\}$$

$$+\frac{\overline{2a^2}|^3}{1.2.3}\cdot\left\{\chi''''''(c)\cdot\frac{1}{7.9.11.13}-\&c.\right\}$$

$+ \&c.$

(15.) The whole integral of $\frac{\rho^2}{2}$ sin. θ is therefore

$$2\pi.\left\{\frac{2}{3}(c^2+a^2)+\psi(c)\right\}.$$

This forms part of the equation to the surface on the supposition that the attraction of the matter in volume 1 (collected into a point) at distance 1, is represented by 1; if it be represented by k, the part of the equation is $2\pi.k\left\{\frac{2}{3}(c^2+a^2)+\psi(c)\right\}$.

(16.) Suppose the part of U arising from the disturbing forces to be $2\pi.\epsilon(c)$, $\epsilon(c)$ being a given function of c. Then the equation to the generating curve will be

$$2\pi\left\{\frac{2}{3}k(c^2+a^2)+k\psi(c)+\epsilon(c)\right\}=C, \text{ or } \frac{2}{3}k(c^2+a^2)+k.\psi(c)+\epsilon(c)=C:$$

where the two last terms of the first side are small.

(17.) This must coincide with the equation $a^2+c^2=r^2+\chi(c)$. And, therefore, if by means of this equation we eliminate a from the former, the resulting equation must be identically true. The value of a^2 to be substituted in the large terms is $r^2-c^2+\chi(c)$: in the small terms r^2-c^2. Thus we get the equation

$$\frac{2}{3}k.\chi(c)+k\psi(c)+\epsilon(c)=C-\frac{2}{3}kr^2.$$

Assuming then for $\chi(c)$ a form with indeterminate coefficients, such as the given form of $\epsilon(c)$ appears to require, and determining $\psi(c)$, and eliminating a by putting $a^2=r^2-c^2$, every term of the equation multiplying a power of c must be made $=0$, and thus a number of equations will be obtained sufficient to determine all the constants, except that independent of c.

(18.) It will be observed that $\psi(c)$ is now expressed by this form

$$-\frac{\chi'(c)}{1}\cdot\frac{2c}{3}+\frac{\chi''(c)}{1.2}\cdot\frac{\overline{2c}|^2}{5}-\frac{\chi'''(c)}{1.2.3}\cdot\frac{\overline{2c}|^3}{7}+\&c.$$

$$+\frac{2(r^2-c^2)}{1}\left\{\chi''(c)\cdot\frac{1}{3.5}-\frac{\chi'''(c)}{1}\cdot\frac{2c}{5.7}+\frac{\chi''''(c)}{1.2}\cdot\frac{\overline{2c}|^2}{7.9}-\&c.\right\}$$

$$+\frac{\overline{2(r^2-c^2)}|^2}{1.2}\cdot\left\{\chi''''(c)\cdot\frac{1}{5.7.9}-\frac{\chi'''''(c)}{1}\cdot\frac{2c}{7.9.11}+\&c.\right\}$$

$$+\frac{\overline{2(r^2-c^2)}|^3}{1.2.3}\left\{\chi''''''(c)\cdot\frac{1}{7.9.11.13}-\&c.\right\}$$

$$+\&c.$$

(19.) If the disturbing forces were not symmetrical about an axis, it would be necessary to assume for the equation to the surface $x^2+y^2+z^2=r^2+\chi(x,y)$. The values of x, y, z, would be $a-\rho\sin.\theta.\cos.\phi$, $b-\rho\sin.\theta.\sin.\phi$, $c-\rho\cos.\theta$; a, b, c, being the co-ordinates of the attracted point. Substituting these in the assumed equation, a value would be found for ρ, as in the case we have considered: and the integral $\int_\phi\int_\theta\frac{\rho^2}{2}.\sin.\theta$ would be taken in the same manner. And we should arrive at a similar equation $\frac{2}{3}k.\chi(a,b)+k\psi(a,b)+\epsilon(a,b)=C-\frac{2}{3}kr^2$; and assuming a proper form for $\chi(a,b)$, forming $\psi(a,b)$, and eliminating c by means of the equation $c^2=r^2-a^2-b^2$, the coefficients would be determined by the comparison of similar terms. As the applications of this theory are few, we shall content ourselves with thus pointing out the course to be pursued in any given case.

(20.) Suppose $\epsilon(c)$ to be of the form $A+Bc^2+Dc^4+Ec^6+Fc^8$. Then $\chi(c)$ will evidently have the form $P+Qc^2+Rc^4+Sc^6+Tc^8$. From this, by the expression in (18), we find $\psi(c)$ (neglecting the constant term)

$$= \left\{\frac{1280}{11.13.15.17} Tr^6 + \frac{432}{7.9.11.13} Sr^4 + \frac{24}{5.7.9} Rr^2 - \frac{4}{5} Q\right\} c^2$$

$$+ \left\{\frac{1120}{11.13.15.17} Tr^4 + \frac{60}{9.11.13} Sr^2 - \frac{8}{9} R\right\} c^4$$

$$+ \left\{\frac{112}{13.15.17} Tr^2 - \frac{12}{13} S\right\} c^6 - \frac{16}{17} Tc^8.$$

Hence $\frac{2}{3} k \chi(c) + k \psi(c)$

$$= k\left\{\frac{1280}{11.13.15.17} Tr^6 + \frac{432}{7.9.11.13} Sr^4 + \frac{24}{5.7.9} Rr^2 - \frac{2}{3.5} Q\right\} c^2$$

$$+ k\left\{\frac{1120}{11.13.15.17} Tr^4 + \frac{60}{9.11.13} Sr^2 - \frac{6}{3.9} R\right\} c^4$$

$$+ k\left\{\frac{112}{13.15.17} Tr^2 - \frac{10}{3.13} S\right\} c^6 - k\frac{14}{3.17} Tc^8.$$

(21.) The equation $\frac{2}{3} k \chi(c) + k \psi(c) + \epsilon(c) = C - \frac{2}{3} kr^2$ gives the following,

$$k\left\{\frac{1280}{11.13.15.17} Tr^6 + \frac{432}{7.9.11.13} Sr^4 + \frac{24}{5.7.9} Rr^2 - \frac{2}{3.5} Q\right\} + B = 0,$$

$$k\left\{\frac{1120}{11.13.15.17} Tr^4 + \frac{60}{9.11.13} Sr^2 - \frac{6}{3.9} R\right\} + D = 0,$$

$$k\left\{\frac{112}{13.15.17} Tr^2 - \frac{10}{3.13} S\right\} + E = 0$$

$$-k \cdot \frac{14}{3.17} T + F = 0.$$

From these $T = \frac{3.17}{14} \cdot \frac{F}{k}$,

$$S = \frac{12}{25} \cdot \frac{Fr^2}{k} + \frac{3.13}{10} \cdot \frac{E}{k},$$

$$R = \frac{27.16}{11.13.5} \cdot \frac{Fr^4}{k} + \frac{9}{11} \cdot \frac{Er^2}{k} + \frac{9}{2} \cdot \frac{D}{k},$$

$$Q = \frac{96}{5.13} \cdot \frac{Fr^6}{k} + \frac{144}{7.11} \cdot \frac{Er^4}{k} + \frac{18}{7} \cdot \frac{Dr^2}{k} + \frac{15}{2} \cdot \frac{B}{k}.$$

(22.) Since C is indeterminate, it is manifest that there is nothing to determine the value of the constant term in $\chi(c)$, and it is, therefore, quite arbitrary. It may conveniently be taken so as to make $\chi(c)$ vanish when $c = r$. This gives

$$\chi(c) = -Q(r^2-c^2) - R(r^4-c^4) - S(r^6-c^6) - T(r^8-c^8) = -(r^2-c^2)$$
$$\times \{\overline{Q + Rr^2 + Sr^4 + Tr^6} + \overline{R + Sr^2 + Tr^4}.c^2 + \overline{S + Tr^2}.c^4 + Tc^6\}$$
$$= -(r^2-c^2) \times \left\{\left(\frac{310509}{11.13.14.25}\cdot\frac{Fr^6}{k} + \frac{5073}{7.10.11}\cdot\frac{Er^4}{k} + \frac{99}{14}\cdot\frac{Dr^2}{k} + \frac{15}{2}\cdot\frac{B}{k}\right)\right.$$
$$+ \left(\frac{236589}{11.13.14.25}\cdot\frac{Fr^4}{k} + \frac{519}{10.11}\cdot\frac{Er^2}{k} + \frac{9}{2}\cdot\frac{D}{k}\right)c^2$$
$$\left. + \left(\frac{1443}{14.25}\cdot\frac{Fr^2}{k} + \frac{39}{10}\cdot\frac{E}{k}\right)c^4 + \frac{51}{14}\cdot\frac{F}{k}c^6\right\}.$$

(23.) As an instance of the application of this formula, let it be required to find the figure of Saturn, as affected by his rotation about his axis, and by the attraction of his ring. Let T be the time of his diurnal revolution: the resolved parts of the centrifugal force on a point whose co-ordinates in the plane of his equator are a, b, will be $\frac{\overline{2\pi}|^2}{T^2}\,a$ and $\frac{\overline{2\pi}|^2}{T^2}\,b$. The part of U which arises from this, will be $\frac{2\pi^2}{T^2}(a^2+b^2) = \frac{2\pi^2}{T^2}(r^2-c^2)$, and, therefore, the part of $\epsilon(c)$ will be $\frac{\pi}{T^2}(r^2-c^2)$, or $-\frac{\pi}{T^2}c^2$, neglecting the constant term.

(24.) Suppose Saturn's ring to be a mathematical line, into which is collected a quantity of matter $= \frac{1}{n}$th of the matter in the body of Saturn $= \frac{4\pi r^3 k}{3n}$; let its radius $= R$. The part of U or of V which arises from this is

$$\frac{2r^3k}{3n}\int_\theta \frac{1}{\sqrt{\{R^2+a^2+2aR\cos.\theta+c^2\}}} = \frac{2r^3k}{3n}\int_\theta \frac{1}{\sqrt{\{R^2+r^2+2aR\cos.\theta\}}}.$$

Expanding this in a series of the form $A + B\cos.\theta + C\cos.2\theta + \&c.$ and integrating every term through a circumference, we get

$$\frac{4\pi r^3 k}{3n} \times \left\{ \frac{1}{\sqrt{R^2 + r^2}} + \frac{1^2.3}{2^2.4} \cdot \frac{4a^2 R^2}{(R^2 + r^2)^{\frac{5}{2}}} + \frac{1^2.3^2.5.7}{2^2.4^2.6.8} \cdot \frac{16a^4 R^4}{(R^2 + r^2)^{\frac{9}{2}}} \right.$$

$$\left. + \frac{1^2.3^2.5^2.7.9.11}{2^2.4^2.6^2.8.10.12} \cdot \frac{64a^6 R^6}{(R^2 + r^2)^{\frac{13}{2}}} + \frac{1^2.3^2.5^2.7^2.9.11.13.15}{2^2.4^2.6^2.8^2.10.12.14.16} \cdot \frac{256.a^8 R^8}{(R^2 + r^2)^{\frac{17}{2}}} \right\},$$

if we stop at a^8. Let e, f, g, h, be the coefficients of a^2, a^4, a^6, a^8; then putting $r^2 - c^2$ for a^2, dividing by 2π, and neglecting the term independent of c, we find that this contributes to $\epsilon(c)$ the following terms $\frac{2r^3k}{3n}\{-(e+2f+3g+4h)c^2 + (f+3g+6h)c^4 - (g+4h).c^6 + h.c^8\}$.

(25.) Our expression will not be very erroneous, if for R we put the mean radius of the ring. Suppose then $\frac{R}{r} = 2$. The last expression then becomes

$$\frac{k}{n}\left\{-.09233.c^2 + .04849.\frac{c^4}{r^2} - .01768.\frac{c^6}{r^4} + .00295.\frac{c^8}{r^6}\right\};$$

and adding the term for centrifugal force,

$$B = \frac{-.09233}{n} - \frac{\pi}{T^2};\ D = \frac{04849}{nr^2};\ E = -\frac{.01768}{nr^4};\ F = \frac{.00295}{nr^6}.$$

Substituting these values in the expression of (22),

$$\chi(c) = (r^2 - c^2)\left\{\frac{15}{2} \cdot \frac{\pi}{kT^2} + \frac{.4476}{n} - \frac{.1487}{n} \cdot \frac{c^2}{r^2} + \frac{.0568}{n} \cdot \frac{c^4}{r^4} - \frac{.0107}{n} \cdot \frac{c^6}{r^6}\right\}.$$

(26.) To exterminate k from the first term of this expression, let t be the periodic time of Saturn's 7th satellite, s its distance; the motion of this satellite is nearly the same as if all the matter of Saturn and his ring were collected into Saturn's center: hence

$$\frac{4\pi}{3} r^3 k\left(1 + \frac{1}{n}\right)\frac{1}{S^2} = 4\pi^2 \frac{S}{t^2};\ \text{hence} \frac{\pi}{kT^2} = \frac{1}{3} \cdot \frac{r^3}{S^3} \cdot \frac{t^2}{T^2}\left(1 + \frac{1}{n}\right);$$

And

$$\frac{15}{2} \cdot \frac{\pi}{kT^2} = \frac{5}{2} \cdot \frac{r^3}{s^3} \cdot \frac{t^2}{T^3}\left(1 + \frac{1}{n}\right) = \frac{5}{2}\left(\frac{1}{59.15}\right)^3 \cdot \left(\frac{79.33}{0.428}\right)^2 \cdot \left(1 + \frac{1}{n}\right) = \left(1 + \frac{1}{n}\right) \times .415.$$

Making this substitution in the expression for $\chi(c)$ we find

$$a^2 = r^2 - c^2 + \chi(c) = (r^2 - c^2)\left\{1.415 + \frac{.862}{n} - \frac{.149}{n}\cdot\frac{c^2}{r^2} + \frac{.057}{n}\cdot\frac{c^4}{r^4} - \frac{.01}{n}\cdot\frac{c^6}{r^6}\right\},$$

an approximate equation to the generating curve of Saturn supposing him homogeneous.

(27.) This gives an ellipticity = .185, independently of that produced by the attraction of the ring. This is so large a quantity, that the neglect of its square and higher powers must produce sensible errors; those terms, however, which arise from the attraction of the ring, and which it is our more immediate object to investigate, will be little affected by their rejection. And though our supposition of Saturn's homogeneity is highly improbable, yet if his density be variable, the aberration from the elliptic figure produced by the attraction of his ring, will be the same in kind (though differing in quantity) as that which would exist, were his density uniform.

(28.) An inspection of the equation at the end of (26) will shew, that the theoretical figure of Saturn is flattened between the poles and the equator. It is remarkable, that this deviation from the elliptic form, is exactly the opposite to that given by the observations of Dr. Herschel. This accurate observer, in the Philosophical Transactions for 1805 and 1806, has given a great number of his observations, which shew, that Saturn is protuberant between the poles and the equator, and that his longest diameter makes an angle of 43° with the plane of his equator. Here then is a complete discordance between theory and observation; nor is it easy, with our present knowledge of the planet, to suggest any thing by which they can be reconciled.

G. B. AIRY.

Trinity College,
March 15, 1824.

XIII. *On the Determination of the General Term of a New Class of Infinite Series.*

BY CHARLES BABBAGE, ESQ. M.A.

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[Read *May* 3, 1824.]

THE subject of investigation on which I have entered in the following Paper, had its origin in a circumstance which is, I believe, as yet singular in the history of mathematical science, although there exists considerable probability, that it will not long remain an isolated example of analytical enquiries, suggested and rendered necessary by the progress of machinery adapted to numerical computation. Some time has elapsed since I was examining a small machine I had constructed, by which a Table, having its second difference constant, might be computed by mechanical means. In considering the various changes which might be made in the arrangement of its parts, I observed an alteration, by which the calculated series would always have its second difference equal to the unit's figure of the last computed term of the series; other forms of the machine would make the first or the third, or generally any given difference equal to the unit's figure of the term last computed; and a further alteration would make the same difference equal to double, or generally to (a) times the digit in the unit's place: or if it were preferred, the digit fixed upon might be that occurring in the ten's place, or generally in the n^{th} place. I did not, at that time, possess the means of making these alterations which I had contemplated, but I immediately proceeded to write down one of the series which would have been calculated by the machine thus altered; and commencing with one of the most simple, I formed the series.

Series.	Diff.
2	2
4	4
8	8
16	6
22	2
24	4
28	8
.	.
.	.

If u_z represent any term of this series, then the equation which determines u_z is

$$\Delta u_z = \text{unit's figure of } u_z,$$

an equation of differences of a nature not hitherto considered, nor am I aware that any method has been pointed out for the determining u_z in functions of z from such laws. I shall now lay before the Society, the steps which I took for ascertaining the general terms of such series, and of integrating the equations to which they lead. I shall not, however, commence with the general investigation of the subject, but shall simply point out the paths through which I was led to their solution, conceiving this course to be much more conducive to the progress of analysis, although not so much in unison with the taste which at present prevails in that science.

If we examine the series, and its first differences, it will be perceived, that the terms of the latter recur after intervals of four, and that all the changes in the first differences, are comprised in the numbers 2, 4, 8, 6, which recur continually, and the series may be written thus:

	Series.	Diff.
	2	2
	4	4
	8	8
	16	6
5	22 = 20 + 2	2
	24 = 20 + 4	4
	28 = 20 + 8	8
	36 = 26 + 16	6
	42 = 40 + 2	2
10	44 = 40 + 4	4
	48 = 40 + 8	8
	56 = 40 + 16	6
	62 = 60 + 2	2
	64 = 60 + 4	4
15	68 = 60 + 8	8
	76 = 60 + 16	6
	82 = 80 + 2	2

If then z be of the form $4v + i$, the value of u_z will be $20v +$ one of four numbers 2, 4, 8, 16, according to the value of i, and if i always represents one of the numbers 1, 2, 3, 4, the value of u_z will be thus expressed,

$$u_z = 20v + 2^i.$$

As a second example, let us consider the series whose first term is 2, its first difference 0, and its second difference always equal the unit's figure of the next term; its equation will be

$$\Delta^2 u_z = \text{unit's figure of } u_z,$$

and the few first terms are

2	28
2	48
4	76
10	110
16	144
	182

This series may be put under the form

	Series.	1 Diff.
0	2	0
	2	2
	4	6
	10	6
	16	12
5	28	20
	48	28
	76	34
	110	34
	144	38
10	182	40 = 40 + 0
	222	42 = 40 + 2
	264	46 = 40 + 6
	310	46 = 40 + 6
	356	52 = 40 + 12
15	408	60 = 40 + 20
	468	68 = 40 + 28
	536	74 = 40 + 34
	610	74 = 40 + 34
	684	78 = 40 + 38
20	762	80 = 80 + 0
	842	82 = 80 + 2
	924	86 = 80 + 6

Table of $(\bar{a})$.

if $a = 0$	$(\bar{a}) = 2$
1	2
2	4
3	10
4	16
5	28
6	48
7	76
8	110
9	144

	Series.	1 Diff.
	1010	86 = 80 + 6
	1096	92 = 80 + 12
25	1188	100 = 80 + 20
	1288	108 = 80 + 28
	1396	114 = 80 + 34
	1510	114 = 80 + 34
	1624	118 = 80 + 38
30	1742	120 = 120 + 0
	1862	122 = 120 + 2

In this series it may be observed, that u_z when z is less than 10, is equal to the sum of the first differences of all the preceding terms, and if z be greater than 10, it will be composed of four terms, viz., first the sum of the ten first terms of the first difference, multiplied by the number of tens contained in z; secondly, of the sum of the series $40 + 80 + 120 +$ to as many terms as there are tens in z, this must be multiplied by 10, as each term is ten times added; and thirdly, of the number 40 multiplied by the same number of the tens, and also multiplied by the digit in the unit's place of z; and fourthly, of the sum of so many terms of the series as is equal to the unit's figure of z; this being expressed by $(\bar{a})$ signifying the number opposite a in the previous Table. These four parts, if $z = 10b + a$, are thus expressed,

1st $180b,$

2nd $40\frac{b.b-1}{2}10,$

3rd $40ba,$

4th $(\bar{a}).$

These added together produce

$$u_z = 20b(10b + 2a - 1) + (\bar{a}).$$

This value of u_z, if diminished by 2, is equal to the sum of $z-1$ term of the series which constitute the first difference.

This inductive process for discovering the n^{th} terms of such series, might be applied to others of the same kind, but it does not admit of an application sufficiently general or direct, to render it desirable that it should be pursued further.

If we consider any series in which the first difference is equal to the digit occurring in the unit's place of the corresponding term, as for example, the series

6	6
12	2
14	4
18	8
26	6
32	2

a slight examination will satisfy us, that the value of the digit occurring in the unit's figure of u_z, depends entirely on the value of u_z, at the commencement of the series, and also that whenever the same digit again occurs, there will, at that point, commence a repetition of the same figures which have preceded; consequently, the first difference at those two points will be equal.

In the first example which I have adduced of a series of this kind, it will be found, that this re-appearance of the terminal figure, happens at the 5th, at the 9th, at the 13th terms, &c. or that

$$\Delta u_1 = \Delta u_5 = \Delta u_9 = \Delta u_{13} = \ldots.$$

This gives for the equation of the series,

$$\Delta u_z = \Delta u_{z+4},$$

or by integrating

$$u_z = u_{z+4} + b,$$

but when $z = 1$, $u_1 = u$, therefore $b = 0$, and

$$u_{z+1} - u_z = 0,$$

whose integral is

$$u_z = a(-\sqrt{-1})^z + b(-\sqrt{-1})^{z+1} + c(-\sqrt{-1})^{z+2} + d(-\sqrt{-1})^{z+3} + 5z.$$

The four constants being determined, by comparing this value of u_z with the first four terms of the series, we shall find

$$a=0,\ b=-5,\ c=\tfrac{1}{2}-\sqrt{-1},\ d=\tfrac{1}{2}+\sqrt{-1},$$

and the value of u_z becomes

$$u_z = 5(z-1) + (\tfrac{1}{2}-\sqrt{-1})(\sqrt{-1})^z + (\tfrac{1}{2}+\sqrt{-1})(-\sqrt{-1})^z,$$

which expresses any term of the series

2, 4, 8, 16, 22, 24, 28, 36, 42, 44, 48.

It is necessary, for the success of this method, that we should have continued the given series until we arrive at some term whose unit's figure is the same as that of some term which has preceded it: now if we consider that this figure depends solely on that of the one which occupied the same place in the preceding term, it will appear that the same digit must re-appear in the course of ten terms at the utmost, since there are only ten digits, and that it may re-occur sooner. The same reasoning is applicable to the case of series whose first difference is equal to any multiple of the digits found in the unit's place of the corresponding term, or to those contained in the equation

$$\Delta u_z = a \times (\text{unit's figure of } u_z),$$

as also to those in which this is encreased by a given quantity, as

$$\Delta u_z = a\,(\text{unit's figure of } u_z) + b.$$

If the second difference is equal to some multiple of the figure occurring in the unit's place of the next term, as in the series

2, 2, 4, 10, 16,

already given, since the unit's figure must always depend on the same figure in the first term of the series, and its first difference

2	0
2	2
4	6
10	6
.	.
.	.

whenever those two figures are the same, a similar period must re-appear: now as there are only two figures concerned, they can only admit of 100 permutations, consequently, this is the greatest limit of the periods in such species of series.—In the one in question the period is comprized in ten terms. This reasoning may be extended to other forms of series in which higher differences are given in terms of the digits occurring in the unit's, ten's, or other places of u_z or u_{z+1} or elsewhere, but I am aware that it does not in its present form present that degree of generality which ought to be expected on such a subject: probably the attempt to solve directly that class of equations to which these and similar enquiries lead, may be attended with more valuable results.

As the term "*unit's figure of*" occurs frequently, it will be convenient to designate it by an abbreviation; that which I shall propose is the combination of the two initials, and I shall write the above equation of differences thus

$$\Delta u_z = a\,UF u_z \ldots\ldots\ldots (a).$$

This may be reduced to a more usual form by the following method. If S_x represent the sum of the x^{th} powers of unity, divided by ten; then

$$0 S_x + 1 S_{x+1} + 2 S_{x+2} + 3 S_{x+3} + 4 S_{x+4} + 5 S_{x+5} + 6 S_{x+6} + 7 S_{x+7} + 8 S_{x+8} + 9 S_{x+9},$$

will represent the figure which occurs in the unit's place of any number x: substituting u_z instead of x, we have

$$\frac{1}{a}\Delta u_z = 0 S_{u_z} + 1 S_{u_z+1} + 2 S_{u_z+2} + \ldots . 9 S_{u_z+9} \ldots\ldots\ldots (b).$$

an equation in which u_z enters as an exponent.

From the previous knowledge of the form of the general terms of the series we are considering, it would appear that the general solution of the equations (a) and (b) is

$$u = 9z + c S_z + c_1 S_{z+z} + c_2 S_{z+2} + \ldots\ldots c_9 S_{z+9}.$$

where the constants must be determined from the conditions. In the further pursuit of any enquiries in this direction, much assistance may be derived by consulting a Paper of Mr. Herschel's in the Philosophical Transactions for 1818. "On Circulating Functions."

Amongst the conditions for determining the general terms of series by some relation amongst particular figures, there occurs a curious class, in which, if we consider only whole numbers, the series becomes impossible after a certain number of terms.

Let the equation determining u_z be

$$\Delta u_z = \tfrac{1}{2}\,(UFu_{z-1} + UFu_{z+1}).$$

Then the following series conform to this law,

Series.	Diff.	Series.	Diff.	Series.	Diff.
1	3	4	6	1	9
4	5	10	4	10	1
9		14	4	11	1
		18		12	3
				15	

If the law is restricted to whole numbers, none of these series admit of any prolongation; nor have I, with that restriction, been able to discover any series of the kind possessing more than five terms.

C. BABBAGE.

Devonshire Street, Portland Place,
March 29, 1824.

XIV. *On the Principles and Construction of the Achromatic Eye-Pieces of Telescopes, and on the Achromatism of Microscopes.*

By GEORGE BIDDELL AIRY, B.A.

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[Read *May* 17, 1824.]

In the theory of Telescopes no part is more interesting, and in the practical construction scarcely any more important, than the Achromatic Eye-piece. The effects of a badly formed eye-piece are even more disagreeable to the eye than those of a defective object-glass: whether we consider the indistinctness near the extremity of the field of view, the distortion of the image, or the fringes of colours which surround an object when not observed in the center of the field. Important and interesting as the subject appears, we might expect, in works of the highest pretensions, to find it treated in the comprehensive manner it deserves: and simple as are its principles, we might imagine that they would be introduced into our elementary treatises on Optics. But it is remarkable that while all writers have given at great length the theory of the achromatic object-glass, there are not more than one or two books in the English language, in which the achromatic eye-piece is alluded to; and though not a telescope has been made except on this construction for many years, the artist is still obliged to work almost without rules, in most instances merely copying the constructions which more able opticians have found to succeed. I flatter myself that an attempt to investigate, on the simplest principles and in the most comprehensive manner, an optical theory of so great importance, avoiding at the same time all unnecessary refinements and useless generalities, will be received with encouragement by the Society, whom I have now the honor of addressing.

The first attempt at improvement on the single eye-glass of the first refracting telescopes was made by Huygens. Instead of a single eye-glass he used two convex lenses, whose focal lengths were in the proportion of 3 to 1, and which were placed at a distance equal to twice the shorter; the lens of greater focal length being that nearer to the object-glass; and the image being formed between the two eye-glasses. His intention was to diminish the spherical aberration of the extreme pencils by dividing the refraction into two parts: for, as he found the spherical aberration to vary nearly as the cube of the refracting angle, it was easy to see that the aberration would be reduced to $\frac{1}{4}$th of its former quantity, if the whole refraction, instead of being effected by one refracting angle, were effected by two, each equal to half the former. It will easily be seen that in his construction any pencil is refracted equally by both lenses, and the improvement which he aimed at was therefore completely attained. But there were other advantages of which he was ignorant, and which were not discovered until the construction had been many years employed. It is singular, that in his attempt to diminish spherical aberration, he should have hit upon a construction which completely removes also chromatic aberration. This was discovered when the unequal refrangibility of light was established, and the mode of obviating the inconveniences thence arising was invented: and the construction of Huygens, commonly known by the term of the

Huygenian eye-piece, is now in general use for astronomical and reflecting telescopes.

While the eye-piece of Huygens was used for astronomical purposes, the eye-piece of three lenses, each contiguous pair being placed at a distance equal to the sum of their focal lengths, as described in our elementary works, was still used for erecting the image. The first notice that we find of any alteration in the erecting eye-piece, is contained in a letter from Dollond to Short, printed in the *Phil. Trans.* for 1753. In this he stated that he had made achromatic eye-pieces with 4 and 5 lenses for some time, and that, finding some of them to have reached the Continent, he considered it expedient to assert his claim to the invention before it was likely to be disputed. It appears that his object at first was merely to diminish the spherical aberration, on the same principle on which it had been attempted by Huygens: but that, finding that the chromatic aberration might also be corrected by the same construction, he had combined 4 and 5 lenses, with particular attention to the correction of colour. It would seem from this letter, that he had then no theory which enabled him, generally, to make his eye-pieces achromatic: and it is not known whether he afterwards made use of any formula. It is remarkable that in this letter he speaks of an achromatic object-glass, as a thing totally to be despaired of.

The attention of philosophers was at this time very generally turned to the consideration of chromatic aberration. Six years before that letter was written, Euler, from a hint given by Newton, had conceived the idea of making a lens truly achromatic. Two years after its date, Dollond found that it was possible to make an achromatic lens. In the next year, 1756, Clairaut gave, in the Memoirs of the French Academy, his first investigations relative to achromatic object-glasses: which he continued for several years. They were simplified by D'Alembert, in some papers first published in the Memoirs of the Academy, and afterwards collected in his Opuscules Mathematiques, published 1764. And Euler, in the Berlin Memoirs for 1757, gave a great many theorems upon combinations of lenses: which, in a paper in the Turin Memoirs, Vol. III, published in 1766, are applied to the achromatic eye-piece. These papers, with considerable alterations, were afterwards embodied in his large work on Dioptrics, which was published in 1769 and the two following years.

It does not appear that any more original investigations were made, till the Abbé Boscovich in the Memoirs of the Academy of Bologna gave some new theorems which were collected in his Opuscula, published in 1785: those relating to the achromatic eye-piece are principally contained in the second Volume. The subject is treated with great mathematical skill, and the work bears strong evidence of considerable practical acquaintance with optical instruments: the rules given by him are in general well adapted to the use of the working optician: and are, in fact, the foundation of all that have since that time been published.

During forty years which have elapsed since the investigations of Boscovich were given to the world, no addition I believe has been made to his theory. The only English treatise on this subject that I have seen, is one published by Professor Robison in his Mechanical Philosophy, and in the Encyclopædia Britannica, *Art.* Telescope. Though this writer has not closely copied from Boscovich, yet he seems to have been guided entirely by his theory: his calculations appear to have been made in the same way: however I have seen no treatise, which on the whole seems better adapted to give clear ideas on the construction of telescopes, and useful rules for the assistance of workmen. From this treatise, and from the work of Boscovich, have been extracted the theorems given by Dr. Brewster in the appendix to his edition of Ferguson's Lectures. I ought not to omit that Robison mentions a translation in English of a work by Schærfer, which I have not been so fortunate as to meet with.

When it is known that much attention has been bestowed on this theory by Clairaut, Euler, D'Alembert, and Boscovich, it may perhaps appear presumptuous to attempt any improvement on what they have done. I will therefore briefly describe the manner in which each of these mathematicians has considered it, and thus, besides pointing out the defects of their operations, I shall have the advantage of more completely elucidating the subject.

To explain the principle of the achromatic eye-piece, I will take its simplest case. Suppose a telescope composed of an object-glass and a single eye-glass, placed in the manner usually described under the head of Astronomical Telescope in most of our elementary works. The axis of a pencil of rays coming from the extreme visible point of an object, will pass through the center of the object-glass, and impinge on the eye-glass near its circumference, whence it will be refracted to the eye. But the dispersion attendant on this refraction will separate the ray into its differently coloured primitive rays. The violet pencil will be more refracted than the red, and will enter the eye, making with the axis of the telescope a greater angle than is made by the red rays. The place of the image therefore, as seen by the violet rays, is more distant from the center of the field of view than its place as seen by the red rays; and the object therefore appears coloured. But if a second eye-glass be placed at some distance from the first, the violet rays, after refraction at the first eye-glass, will be incident on the second at a point nearer to the center than that at which the red rays are incident on it; and falling therefore on a smaller refracting angle, they will, by the proper adjustment of the lenses, be so much less refracted, as to emerge parallel to the red rays; and the object is now seen without any tinge of colour. In a way nearly similar it may be shewn, that any number of eye-glasses may be combined so as to form an achromatic eye-piece.

The investigations of Clairaut and D'Alembert do not relate to the achromatic eye-piece properly so called. These writers have shewn, that in certain cases a fault of the object-glass may be corrected by the eye-piece, and that all the rays of any one colour may be made to emerge parallel to each other. But the intention of the achromatic eye-piece is, not to make the rays of each colour emerge parallel to each other, (a small deviation from which is quite insensible to the eye), but to make the axes of the differently coloured pencils emerge parallel.

Euler in the Turin Memoirs has considered at great length the properties of eye-pieces. But he has paid most attention to the correction of spherical aberration, and his formula for the correction of chromatic aberration is not demonstrated. The greater part too of his work is occupied with the consideration of eye-pieces of five and six lenses; which now are seldom or perhaps never employed. In fact, little information can be gathered from this paper, that is either interesting to the theorist, or useful to the workman.

In his large work on Dioptrics he has treated the subject far more completely. His method (to which that which I have employed is in some degree similar) is this; he investigates the position and magnitude of every image: from the position and magnitude of the last image, and the position of the eye, he finds the visual angle; he then differentiates this, supposing the focal lengths of the lenses to vary in consequence of the variation of the refractive index for differently coloured rays, and he makes this variation $=0$. He then makes the position of the eye that which gives the greatest field of view, and eliminating it, he has an equation between the focal lengths of the lenses and their distance from each other. Though the principle of this process is general and not inelegant, yet the method of introducing and afterwards eliminating the distance of the eye from the last eye-glass, is not so good as might have been expected from Euler. In pursuing the theory, he has loaded it with generalizations and details, to such a degree as to make it almost useless. The thickness of the lenses, in some parts, makes his formulæ extremely complicated, though that might be safely neglected; in other parts, he has considered the effect of lenses of different sorts of glass, though no one would resort to that construction, when it is possible to avoid it. And I believe that I do not exaggerate when I say, that, after a general acquaintance with the principles of chromatic dispersion, it would be easier for any one to form a theory for himself, than to select the parts that are useful from the book of Euler.

The work of Boscovich is much better calculated to explain the principles of the achromatic eye-piece, than any of those I have before mentioned. And in his investigation he has pursued the natural course of tracing the axis of pencils of differently coloured rays through the eye-piece, and making them emerge parallel. But his method is greatly deficient in facility and power. He finds the quantity of mean refraction where a pencil is incident on the first eye-glass, from which he gets the dispersion of the violet and red rays: he then finds the distance at which they are incident on the next lens, and the refractions there; and, continuing this process, makes the last emerging rays parallel. In some places he has expressed the dispersion by an algebraic symbol; in others, by its numerical value; and in some he has calculated the progress of a ray by a laborious trigonometrical process. And this is the last theory pretending to any degree of generality that has yet appeared.

Professor Robison, in the books already mentioned, has given for the simpler forms of the eye-piece, a geometrical construction. That an optician should be able to apply an algebraical formula is not improbable; but that he should make a geometrical construction, and deduce a theorem, is almost impossible. In the more complicated cases, he has assumed a particular condition, namely, that the axes of the pencils shall meet on the field-glass. This puts a stop at once to all improvements: and even for the completion of this solution a geometrical construction is required, not less difficult than the former. The extracts which have been made from this work by other writers, it is not necessary to notice.

When engaged in some investigations respecting a peculiar construction of telescopes, the principles of which were laid before the Society about a year and half ago, I found it necessary to obtain a general formula for making it achromatic. At that time I was not acquainted with any of the works that I have described: and the difficulties of the case compelled me to use a method, which appears to me to be free from most of the objections that can be laid to them. In principle it consists in finding an expression for the visual angle by tracing the axis of a pencil of rays through the eye-piece, and, by a kind of differential process, making its variation, depending on the alteration of the index of refraction, = 0. I have here applied it to eye-pieces of two, three, and four lenses, and have pointed out some of the uses and peculiarities of each construction. The latter part, I hope, may not be without its value: it is upon the achromatism of Microscopes. This, I believe, has not been treated of by any author: and as it is not less important, as far as it extends, than the achromatism of Telescopes, and as its theory is singular, and (I think) not inelegant, I have introduced it in this paper.

The preliminary investigation for the variation of the power of a lens suggested a mode of treating of the achromatic object-glass, which is, I think, as simple as any that has yet appeared. And a law which I have assumed for the dispersive power of one medium in terms of that of another, reduces at once to mathematical investigation every thing relating to the irrationality of dispersion. Some speculations on achromatic object-glasses I have, therefore, given in the beginning.

I have been induced to press this subject on the notice of the Society from a conviction of its importance, and a knowledge of the little attention usually paid to it. In the date of its invention, the achromatic eye-piece is prior to the object-glass: in its application it is even more general; in its theory it is, I think, more interesting: and in the principles of that theory it is not more difficult. Yet while the object-glass continues to receive the attention of scientific men, the theory of the eye-piece in the space of 40 years has received no improvement: in the elementary works on Optics, the object-glass is described at length, but the eye-piece is entirely omitted. If it should appear that in this paper I have rendered the theory more generally accessible to the inquiring mathematician, or more easily adapted to the purposes of the practical optician, I trust that it will have the approbation of this Society.

(1) In the following articles, we shall use the letter n to express the index of refraction for rays of some one colour, which we shall call mean rays, passing out of glass or any other refracting medium into air; and shall denote the index of refraction for rays of any other colour by $n+\delta n$. The letter δ prefixed to any quantity dependent on n will be employed to denote the alteration produced in that quantity by changing n to $n+\delta n$.

(2) As we shall make great use of the expression for the variation of the power of a lens, we will investigate it here. Let

F be the focal length, $\frac{1}{F}$ the power, of a lens: the expression for $\delta\frac{1}{F}$ is required. It is known that $\frac{1}{F}=\frac{n-1}{G}$, G being a quantity independent of n;

$$\therefore \delta\frac{1}{F}=\frac{\delta n}{G}=\frac{1}{F}\cdot\frac{\delta n}{n-1}.$$

If δF be required, since

$$F+\delta F=\frac{1}{\frac{1}{F}+\delta\frac{1}{F}}=\frac{F}{1+\frac{\delta n}{n-1}}=F-F\left\{\frac{\delta n}{n-1}-\left(\frac{\delta n}{n-1}\right)^2+\&c.\right\};$$

$$\therefore \delta F=-F\left\{\frac{\delta n}{n-1}-\left(\frac{\delta n}{n-1}\right)^2+\&c.\right\}:$$

or rejecting the square, &c. of $\frac{\delta n}{n-1}$, $\delta F=-F.\frac{\delta n}{n-1}$. If the product $\frac{1}{Ff}$ should occur, F and f being the focal lengths of two lenses of the same sort of glass, since $\frac{1}{F}$ is changed into $\frac{1}{F}\left(1+\frac{\delta n}{n-1}\right)$, and $\frac{1}{f}$ into $\frac{1}{f}\left(1+\frac{\delta n}{n-1}\right)$, the variation of $\frac{1}{Ff}$ as far as the first power of $\frac{\delta n}{n-1}$ is $\frac{2}{Ff}\cdot\frac{\delta n}{n-1}$.

Similarly the variation of $\frac{1}{Fff'}$ is $\frac{3}{Fff'}\cdot\frac{\delta n}{n-1}$, &c.

(3) On the achromatic object-glass. The principle of this is so well known, that we shall not stop to describe it. Suppose then a convex lens of crown-glass, and a concave lens of flint-glass, to be placed in contact; let n and n' be the indices of refraction for mean rays, for crown and flint-glass respectively; F and f the focal lengths of the lenses: the power of the compound lens for mean rays is $\frac{1}{F}-\frac{1}{f}$: for any other rays it is

$$\frac{1}{F}+\delta\frac{1}{F}-\frac{1}{f}-\delta\frac{1}{f}=\frac{1}{F}-\frac{1}{f}+\frac{1}{F}\cdot\frac{\delta n}{n-1}-\frac{1}{f}\cdot\frac{dn'}{n'-1}.$$

This may be made the same for rays of all colours, or all may be made to converge to the same point, if $\frac{\delta n'}{n'-1}$ be always $= c \cdot \frac{\delta n}{n-1}$: for then by making

$$\frac{1}{F} - \frac{c}{f} = 0, \text{ or } f = cF,$$

the power of the compound lens for rays of all colours is $\frac{1}{F} - \frac{1}{f}$. It is found in practice that $\frac{\delta n'}{n'-1}$ can be very nearly, but not exactly, expressed by $c \cdot \frac{\delta n}{n-1}$ for rays of all colours, c being nearly $= 1,5$. Object-glasses therefore may be made on this plan very nearly, but not quite, achromatic.

(4) Instead of placing the lenses in contact, suppose them to be separated; let their distance $= a$. The rays which fall on the second lens are converging to a point whose distance is $F - a$: hence the power after refraction at the second lens is

$$\frac{1}{F-a} - \frac{1}{f} = \frac{\frac{1}{F}}{1 - \frac{a}{F}} - \frac{1}{f}$$

for mean rays. For rays of any other colour, it is

$$\frac{\frac{1}{F}\left(1 + \frac{\delta n}{n-1}\right)}{1 - \frac{a}{F} - \frac{a}{F} \cdot \frac{\delta n}{n-1}} - \frac{1}{f}\left(1 + \frac{\delta n'}{n'-1}\right) = \frac{1 + \frac{\delta n}{n-1}}{F - a - a\frac{\delta n}{n-1}} - \frac{1}{f}\left(1 + \frac{\delta n'}{n'-1}\right):$$

or, by expanding and neglecting $\left(\frac{\delta n}{n-1}\right)^2$ &c., the power is

$$\frac{1}{F-a} - \frac{1}{f} + \frac{F}{(F-a)^2} \cdot \frac{\delta n}{n-1} - \frac{1}{f} \cdot \frac{\delta n'}{n'-1}.$$

If $\frac{\delta n'}{n'-1}$ be $= c\,\frac{\delta n}{n-1}$, the compound lens, as before, will be achromatic, upon making

$$\frac{F}{(F-a)^2} - \frac{c}{f} = 0, \quad \text{or} \quad f = \frac{c\,(F-a)^2}{F}.$$

(5) The following remark is worthy of notice. In an achromatic object-glass we have found, when the lenses are in contact, $f = cF$; when the lenses are separated by the space a, $f = \frac{c\,(F-a)^2}{F}$: a quantity evidently less than the former. Hence if, when the lenses are in contact, the focal length of the concave lens be too short, or its refraction too great, or (in the language of opticians) the colour be over-corrected, the object-glass may be made achromatic by separating the lenses. This important theorem is well known to every scientific workman. If the focal length of the concave lens be too great, no alteration will make the object-glass achromatic.

(6) Upon comparing the dispersions of different media, it appears that $\frac{\delta n'}{n'-1}$ may be represented with an accuracy great enough for all practical purposes, by the form $c\,.\,\frac{\delta n}{n-1} + e\left(\frac{\delta n}{n-1}\right)^2$. Suppose an object-glass to be formed of three lenses of different substances: let n, n', n'', be their refractive indices for mean rays, and let

$$\frac{\delta n'}{n'-1} = c'\,\frac{\delta n}{n-1} + e'\left(\frac{\delta n}{n-1}\right)^2 : \quad \frac{\delta n''}{n''-1} = c''\,.\,\frac{\delta n}{n-1} + e''\left(\frac{\delta n}{n-1}\right)^2 :$$

then if f, f', f'', be the focal lengths for mean rays, the power of the object-glass for rays of any other colour will be

$$\frac{1}{f}\left(1 + \frac{\delta n}{n-1}\right) + \frac{1}{f'}\left(1 + \frac{\delta n'}{n'-1}\right) + \frac{1}{f''}\left(1 + \frac{\delta n''}{n''-1}\right)$$

$$= \frac{1}{f} + \frac{1}{f'} + \frac{1}{f''} + \left(\frac{1}{f} + \frac{c'}{f'} + \frac{c''}{f''}\right) \cdot \frac{\delta n}{n-1} + \left(\frac{e'}{f'} + \frac{e''}{f''}\right) \cdot \left(\frac{\delta n}{n-1}\right)^2.$$

This will be the same for all colours, if

$$\frac{1}{f}+\frac{c'}{f'}+\frac{c''}{f''}=0,\quad \frac{e'}{f'}+\frac{e''}{f''}=0,$$

which will determine f' and f'' from f.

(7) This is the principle of Dr. Blair's object-glass. A refracting fluid (a solution of some metal in muriatic acid) is inclosed between two lenses of crown and flint-glass. It is found in practice that object-glasses thus constructed do not last, owing to the chemical changes in the fluid, and its corrosive action on the glass.

(8) The separation of the lenses promises, in theory at least, to produce beneficial effects. Suppose a convex crown-glass lens whose focal length is F, and a concave flint-glass lens of focal length f to be separated to the distance a: then (see 4) the power of the combination is

$$\frac{1+\dfrac{\delta n}{n-1}}{F-a-a\dfrac{\delta n}{n-1}}-\frac{1}{f}\left(1+\frac{\delta n'}{n'-1}\right)$$

$$=\frac{1}{F-a}+\frac{F}{(F-a)^2}\cdot\frac{\delta n}{n-1}+\frac{Fa}{(F-a)^3}\cdot\left(\frac{\delta n}{n-1}\right)^2-\frac{1}{f}-\frac{1}{f}\cdot\frac{\delta n'}{n'-1}.$$

Putting for $\dfrac{\delta n'}{n'-1}$ the expression $c\dfrac{\delta n}{n-1}+e\cdot\left(\dfrac{\delta n}{n-1}\right)^2$, this combination is achromatic, if

$$\frac{F}{(F-a)^2}-\frac{c}{f}=0,\quad \frac{Fa}{(F-a)^3}-\frac{e}{f}=0.$$

From these equations $a=\dfrac{e}{c+e}F$: $f=\dfrac{c^3}{(c+e)^2}F$. The power of the compound lens

$$=\frac{1}{F-a}-\frac{1}{f}=\frac{1}{F}\cdot\frac{(c+e)\,(c^2-\overline{c+e})}{c^3}.$$

This must be positive; $\therefore\ e < c^2 - c < c\,(c-1)$. It appears that no combination of glasses, whose refractive powers are known, will satisfy this equation. We might make F negative, or suppose the crown-glass concave; but as we should in that manner lose one important advantage, viz., the diminution of the flint-glass, it is not worthy of consideration.

(9) By using three sorts of glass, and also separating the lenses, besides making the object-glass truly achromatic we may diminish the aperture necessary for the flint lenses. Suppose two lenses of different kinds of flint-glass to be placed in contact, at the distance a from a crown lens: let F be the focal length of the latter, f and f' those of the former: n, n', n'', the refractive indices for mean rays. Then the power for mean rays after the last refraction, is $\frac{1}{F-a} + \frac{1}{f} + \frac{1}{f'}$, which for other rays is changed to

$$\frac{1}{F-a} + \frac{F}{(F-a)^2} \cdot \frac{\delta n}{n-1} + \frac{Fa}{(F-a)^3} \cdot \left(\frac{\delta n}{n-1}\right)^2$$
$$+ \frac{1}{f} + \frac{1}{f} \cdot \frac{\delta n'}{n'-1} + \frac{1}{f'} + \frac{1}{f'} \cdot \frac{\delta n''}{n''-1}\,.$$

Let $\frac{\delta n'}{n'-1} = c\,\frac{\delta n}{n-1} + e\left(\frac{\delta n}{n-1}\right)^2 : \frac{\delta n''}{n''-1} = c'\,\frac{\delta n}{n-1} + e'\left(\frac{\delta n}{n-1}\right)^2 :$

then the power

$$= \frac{1}{F-a} + \frac{1}{f} + \frac{1}{f'} + \left(\frac{F}{(F-a)^2} + \frac{c}{f} + \frac{c'}{f'}\right) \cdot \frac{\delta n}{n-1}$$
$$+ \left(\frac{Fa}{(F-a)^3} + \frac{e}{f} + \frac{e'}{f'}\right) \left(\frac{\delta n}{n-1}\right)^2.$$

Making the latter parts each $= 0$,

$$\frac{F}{(F-a)^2} + \frac{c}{f} + \frac{c'}{f'} = 0, \qquad \frac{Fa}{(F-a)^3} + \frac{e}{f} + \frac{e'}{f'} = 0.$$

And the aperture necessary for the flint lenses would be that of the crown lens multiplied by $\frac{F-a}{F}$.

(10) As an example, let us take from the table of M. Frauenhofer, given in the Edinburgh Philosophical Journal, Vol. X, the refractive indices in the kinds of glass which he calls Crown 9, Flint 13, and Flint 23, for the rays *B*, *E*, *G*. They are as follows.

Ray		*B*	*E*	*G*
Refractive index in......	Crown 9	1,525832	1,533005	1,541657
	Flint 13	1,627749	1,642024	1,660285
	Flint 23	1,626580	1,640 570	1,658849

Let E be the mean ray: then the values of $\frac{\delta n}{n-1}$, for B and G, are $-,013457$, $+,016232$; those of $\frac{\delta n'}{n'-1}$ are $-,02223$, $+,02844$: those of $\frac{\delta n''}{n''-1}$ are $-,021840$, $+,028536$. Making the assumption above, we find $c = 1,516$, $e = 1,41$; $c' = 1,5039$, $e' = 2,363$. Solving the equations above,

$$f = F.\frac{\left(1-\frac{a}{F}\right)^3}{-1,616+2,645\,\frac{a}{F}}, \quad f' = F.\frac{\left(1-\frac{a}{F}\right)^3}{,964-2,001\,\frac{a}{F}},$$

(F, f, f' being the focal lengths for the ray E). Since the compound lens is to make the rays converge after refraction, we must have

$$\frac{1}{F-a}+\frac{1}{f}+\frac{1}{f'}, \quad \text{or} \quad \frac{,348-1,356\,\frac{a}{F}+\frac{a^2}{F^2}}{F\left(1-\frac{a}{F}\right)^3}$$

a positive quantity, and $\frac{a}{F}$ must not be greater than ,344. If however $\frac{a}{F}$ were $\frac{1}{4}$ or $\frac{1}{5}$, the aperture of the flint lenses would be less than that of the crown by the same part: an advantage by no means inconsiderable, as flint-glasses of 4 inches diameter can be very frequently obtained, while those of 5 inches cannot be procured once in several years.

(11) On the achromatic eye-piece. The defect in telescopes which occasioned the invention of the achromatic eye-piece, and the manner in which it removes that defect, have been explained in the Introduction to this Paper. We shall merely state in this place, that in all the following instances we shall pursue the same course; we shall find the distance from the last lens at which the axis of a pencil of rays meets the axis of the telescope: we shall then find the distance from the center of the lens at which the ray is incident upon it; and having found the tangent of the visual angle, by dividing the latter by the former, we shall make its variation, depending on the variation of n, $=0$.

(12) On the eye-piece with two eye-glasses. Let D be the distance of the first eye-glass from the object-glass $= AB$, Plate XI. Fig. 1: p its focal length: a the distance of the second eye-glass from the first $= BC$: q its focal length. Suppose the axis of a pencil of rays coming from the center of the object-glass, to fall upon the first eye-glass at E: it will, after refraction, tend to cross the axis of the telescope at the distance

$$\frac{1}{\frac{1}{p} - \frac{1}{D}} \text{ or } \frac{Dp}{D-p} = BG$$

beyond that lens, and therefore at the distance

$$\frac{Dp}{D-p} - a \text{ or } \frac{\overline{D+a}.p - Da}{D-p} = CG$$

beyond the second eye-glass. After refraction therefore at the second eye-glass, it will meet the axis of the telescope at the distance

$$\frac{1}{\dfrac{D-p}{\overline{D+a}\,.p-Da}+\dfrac{1}{q}} \text{ or } \frac{q\,(\overline{D+a}\,.p-Da)}{-pq+Dq+\overline{D+a}\,.p-Da}=CH.$$

And if k be the tangent of the angle BAE, then $BE=Dk$;

$$\therefore\ CF=BE.\frac{CG}{BG}=Dk.\frac{\overline{D+a}\,.p-Da}{Dp};$$

$$\therefore \tan FHC=\frac{FC}{CH}=k.\frac{-pq+Dq+\overline{D+a}\,.p-Da}{pq}$$

$$=k\left\{-1+\frac{D}{p}+\frac{D+a}{q}-\frac{Da}{pq}\right\}.$$

Taking the variation of this to the first power of $\frac{\delta n}{n-1}$, and making it $=0$, we have

$$\left(\frac{D}{p}+\frac{D+a}{q}-\frac{2\,Da}{pq}\right)\frac{\delta n}{n-1}=0,$$

whence $a=\dfrac{p+q}{2-\dfrac{p}{D}}$. Or if D be very great, $a=\dfrac{p+q}{2}$. This is the rule of opticians. If $p=3q$ then $a=2q$: this is the Huygenian construction before mentioned.

(13). If $a=q$, we must have

$$2q-\frac{pq}{D}=p+q, \text{ or } a=q=\frac{Dp}{D-p}=BG.$$

In this case the lens CF is placed at G: and since its focal length equals the distance between the two lenses, the image, to be distinctly seen, must be formed upon the lens BE. This is inadmissible; because the particles of dust on the glass, since the smallest substance will now intercept an entire pencil of rays, will prevent the object from being distinctly seen: and because it is impossible to apply a micrometer. Add to this, that the eye must be in contact with the lens CF, or it will not receive all the pencils from different points of the object.

(14). If a be greater than q, the image is formed between the two lenses; and, since there is always some distortion at refraction by a single lens, the micrometer cannot safely be applied. If a be less than q, we shall have, $\left(\text{since } q = 2a - \frac{ap}{D} - p\right)$

$$a - \frac{ap}{D} > p, \text{ or } a > \frac{Dp}{D-p} > BG.$$

In this case the pencils cross before they can be received by the eye: this construction therefore cannot be employed in any case. The achromatic eye-piece of two glasses can never therefore be used with a micrometer.

(15). In telescopes with a micrometer, the eye-piece generally consists of two lenses, and the image is formed between EB and the object-glass. As no part but the center of the field of view can be distinctly seen, there is an apparatus which enables the observer to slide the eye-piece across the end of the telescope, and thus move the center of the field to the object. This kind of eye-piece is called by artists the *positive* eye-piece in contradistinction to that in which the image is formed between the eye-glasses, which is called the *negative* eye-piece. With the positive eye-piece the lens EB should always be placed as near the first image, and the lens CF as near the intersection of the pencils with the axis, as convenience will allow.

(16). On the eye-piece with three eye-glasses. Let p, q, r, be the focal lengths of the lenses, beginning with that nearest to the object-glass; let a and b be their distances: and for simplicity, suppose the object-glass so distant, that the axes of the pencils incident on the first eye-glass, may be supposed parallel to the axis of the telescope. Then they will cross, or tend to cross, the axis at the distance p from the first lens, or $a-p$ from the second. They will again cross it at the distance

$$\frac{1}{\frac{1}{q}-\frac{1}{a-p}} \text{ or } \frac{q\,(a-p)}{a-p-q} \text{ from the second lens, or}$$

$$b-\frac{q\,(a-p)}{a-p-q}=\frac{ab-bp-\overline{a+b}.q+pq}{a-p-q} \text{ from the third.}$$

They will finally cross it at the distance

$$\frac{1}{\frac{1}{r}-\frac{a-p-q}{ab-bp-\overline{a+b}.q+pq}} \text{ or } \frac{r\,(ab-bp-\overline{a+b}\,.\,q+pq)}{ab-bp-\overline{a+b}.q-ar+pq+pr+qr}$$

from the third eye-glass. And if m be the distance from the axis at which a pencil is incident on the first eye-glass, $m\times\frac{a-p}{p}$ is the same on the second eye-glass; and

$$m\,.\,\frac{a-p}{p}\,.\,\frac{ab-bp-\overline{a+b}\,.\,q+pq}{a-p-q}\,.\,\frac{a-p-q}{q\,(a-p)} \text{ or } m\frac{ab-bp-\overline{a+b}\,.\,q+pq}{pq}$$

that on the third eye-glass. Hence the tangent of the visual angle

$$= m\left\{\frac{ab}{pqr}-\frac{b}{qr}-\frac{a+b}{pr}-\frac{a}{pq}+\frac{1}{r}+\frac{1}{q}+\frac{1}{p}\right\}.$$

Taking the chromatic variation of this, as far as the first power of $\frac{\delta n}{n-1}$, and making it vanish, we have

$$\frac{3ab}{pqr}-\frac{2b}{qr}-\frac{2\,(a+b)}{pr}-\frac{2a}{pq}+\frac{1}{r}+\frac{1}{q}+\frac{1}{p}=0,$$

$$\text{or } 3ab-2bp-2\,.\,\overline{a+b}\,.\,q-2ar+pq+pr+qr=0,$$

the general equation for an eye-piece with three eye-glasses. If $a, p, q, r,$ be given, then

$$b = \frac{2aq + 2ar - pq - pr - qr}{3a - 2p - 2q}.$$

(17). Suppose, for example, $a = p + q$. Then

$$b = \frac{2q^2 + pq + pr + qr}{p + q} = q + r + \frac{q^2}{p+q}.$$

This formula is given by Boscovich.

(18). The eye-piece with three eye-glasses is little used, for this reason. The refraction, which causes the final intersection of the pencils with the axis, must be effected entirely by the last eye-glass; and except its aperture be small, the spherical aberration distorts the image very much: so that it is impossible to have a large field of view. In the eye-piece instanced above, the course of the axis of a pencil is represented in Fig. 2. and it is evident that the inclination of the pencil to the axis of the telescope at Y, is occasioned entirely by the single refraction at X. On this account, the eye-piece with four eye-glasses is now universally employed.

(19). On the eye-piece with four eye-glasses. Taking up the investigation of (16), and supposing the focal length of the last eye-glass $= s$, and its distance from the third $= c$, the pencils cross the axis at a point whose distance from the last eye-glass

$$= c - \frac{r(ab - bp - \overline{a+b}.q + pq)}{ab - bp - \overline{a+b}.q - ar + pq + pr + qr}$$

$$= \frac{abc - bcp - \overline{a+b}.cq - \overline{b+c}.ar + cpq + \overline{b+c}.pr + \overline{a+b+c}.qr - pqr}{ab - bp - \overline{a+b}.q - ar + pq + pr + qr} \; (= t).$$

They finally cross at the distance

$$\frac{1}{\frac{1}{s}-\frac{1}{t}} = \frac{st}{t-s} =$$

$$\frac{s\,(abc-bcp-\overline{a+b}.cq-\overline{b+c}.ar+cpq+\overline{b+c}.pr+\overline{a+b+c}.qr-pqr)}{abc-bcp-\overline{a+b}.cq-\overline{b+c}.ar-abs+cpq+\overline{b+c}.pr+\overline{a+b+c}.qr+bps+\overline{a+b}.qs+ars-pqr-pqs-prs-qrs}\ (=v).$$

Also the distance from the center of the lens at which the axis of a pencil is incident on it is

$$m\,.\frac{ab-bp-\overline{a+b}.q+pq}{pq}.\frac{t}{c-t} = \frac{m}{pqr}$$

$$\times\{abc-bcp-\overline{a+b}.cq-\overline{b+c}.ar+cpq+\overline{b+c}.pr+\overline{a+b+c}.qr-pqr\}.$$

Hence the tangent of the visual angle, found by dividing this by v, is equal to

$$m\left\{\frac{abc}{pqrs}-\frac{bc}{qrs}-\frac{\overline{a+b}.c}{prs}-\frac{a.\overline{b+c}}{pqs}-\frac{ab}{pqr}+\frac{c}{rs}+\frac{b+c}{qs}+\frac{a+b+c}{ps}\right.$$

$$\left.+\frac{b}{qr}+\frac{a+b}{pr}+\frac{a}{pq}-\frac{1}{s}-\frac{1}{r}-\frac{1}{q}-\frac{1}{p}\right\}.$$

Making its variation $=0$ as before, and reducing, we have the general equation of the eye-piece with four glasses,

$$4abc-3bcp-3.\overline{a+b}.cq-3a.\overline{b+c}.r-3abs+2cpq+2.\overline{b+c}.pr$$

$$+2.\overline{a+b+c}.qr+2bps+2.\overline{a+b}.qs+2ars-pqr-pqs-prs-qrs=0.$$

If a, b, p, q, r, s, be given,

$$c = \frac{(b.\overline{a+2q}-\overline{2a-p}.\overline{2b-q}).(r+s)+rs\,(2a-\overline{p+q})}{p\,(3b-2.\overline{q+r})+q\,(3.\overline{a+b}-2r)-a\,(4b-3r)}.$$

(20). In a common perspective-glass it was found, that

$a = 1{,}9$ inch, $b = 2{,}2$, $c = 1{,}8$, $p = 1{,}2$, $q = 1{,}8$, $r = 1{,}8$, $s = 1{,}2$.

The following numbers have been given for a good eye-piece:

$p = 14$, $q = 21$, $r = 27$, $s = 32$, $a = 23$, $b = 44$, $c = 40$.

In one of Dollond's

$$p = 14{,}25 \text{ lines}, \ q = 19, \ r = 22{,}75, \ s = 14, \ a = 22{,}48,$$
$$b = 46{,}17, \ c = 21{,}45.$$

In one of Ramsden's small telescopes,

$$p = 0{,}77 \text{ inch}, \ q = 1{,}025, \ r = 1{,}01, \ s = 0{,}79, \ a = 1{,}18,$$
$$b = 1{,}83, \quad c = 1{,}1.$$

It appears from the formula, that the values of c in the several cases should have been

$$2{,}31, \quad 37{,}88, \quad 19{,}37, \quad 1{,}12.$$

The course of a ray in the last of these is represented in Fig. 3.; the advantages which it possesses over the eye-piece with three lenses are sufficiently obvious.

(21). If it be required to find where the axes of the differently coloured pencils intersect each other: let z be the distance from the center of any lens, at which the axis of the pencil of mean rays is incident upon it $= LM$, Fig. 4: t the tangent of the angle made by that axis with the axis of the telescope $=$ tan. MNL, and let x be the distance from the lens at which the differently coloured axes cross: then it is plain from the figure that $x\delta t = \delta z$, or $x = \dfrac{\delta z}{\delta t}$. Thus, to find where the rays intersect after refraction at the second lens, we have, by (12), putting m for Dk, and supposing D very great,

$$z = m\left(1 - \frac{a}{p}\right): \quad t = m\left(\frac{1}{p} + \frac{1}{q} - \frac{a}{pq}\right):$$

hence

$$\delta z = -\frac{ma}{p} \cdot \frac{\delta n}{n-1}: \quad \delta t = m\left(\frac{1}{p} + \frac{1}{q} - \frac{2a}{pq}\right) \cdot \frac{\delta n}{n-1}: \quad \therefore\ x = \frac{aq}{2a - p - q}.$$

To find the intersection after refraction at the third lens, we have, by (16),

$$z=m\left(\frac{ab}{pq}-\frac{b}{q}-\frac{a+b}{p}+1\right):\ t=m\left(\frac{ab}{pqr}-\frac{b}{qr}-\frac{a+b}{pr}-\frac{a}{pq}+\frac{1}{r}+\frac{1}{q}+\frac{1}{p}\right):$$

$$\text{hence } x=\frac{\dfrac{2ab}{pq}-\dfrac{b}{q}-\dfrac{a+b}{p}}{\dfrac{3ab}{pqr}-\dfrac{2b}{qr}-\dfrac{2(a+b)}{pr}-\dfrac{2a}{pq}+\dfrac{1}{r}+\dfrac{1}{q}+\dfrac{1}{p}}$$

$$=\frac{r(bp+\overline{a+b}\,.\,q-2ab)}{-3ab+2bp+2\,.\,\overline{a+b}\,.\,q+2ar-pq-pr-qr}.$$

(22). Robison has proposed as a general rule in constructing eye-pieces of four glasses, to unite all the differently coloured pencils on the third lens or field-glass. For this purpose, it is merely necessary to make $b=\dfrac{aq}{2a-p-q}$.

(23.) These investigations, it is evident, can be extended to five or a greater number of lenses, and the formulæ, though troublesome, will be as simple as the nature of the subject will allow. And any other problems, relating to the intersection of the rays &c., can be solved on the same principles. On this subject therefore we shall not stop any longer.

(24.) On the Achromatism of Microscopes. In the construction of the common microscope there is no part similar to the achromatic object-glass of the telescope, for the purpose of making the rays of any one colour from a point of the object enter the eye parallel to each other. The aperture of the object-glass is so small, that the colour arising from this chromatic aberration is not perceptible. The only endeavour is to make the axes of the differently coloured pencils enter the eye parallel to each other: which is effected by properly placing the diaphragm that determines the quantity of light received from the object. Fig. 5. represents the path of the axis of a pencil of rays in the common microscope: at C is the diaphragm which admits a cone of rays smaller than the object-glass. Since the position of this diaphragm determines the part of the object-glass, through which the rays coming from any point A of the object must pass, it is evident that the refraction, and consequently the dispersion of the differently coloured rays will depend upon its position.

(25). We shall begin the investigation by finding the effect produced by variation of refrangibility upon the directions of the axes of the pencils from A, passing through C. Let v be the distance of the object from the object-glass: suppose BA produced, Fig. 6, to meet the axis of the microscope at a point whose distance from the object-glass $= y$. Let m be the tangent of the angle made by the pencil, after refraction at the object-glass, with the axis of the microscope. Then it is easily seen that (if w be the distance of the point of the object from that axis,) $\frac{v}{y} + \frac{w}{xm} = 1$: and since a similar expression is true for rays of all colours,

$$v\left(\frac{1}{y} + \delta\frac{1}{y}\right) + \frac{w}{xm}\left(1 - \frac{\delta m}{m}\right) = 1.$$

Eliminating w,

$$v\left(\frac{1}{y} + \delta\frac{1}{y}\right) - \frac{v}{y}\left(1 - \frac{\delta m}{m}\right) = \frac{\delta m}{m}: \quad \text{or } v\,.\,\delta\frac{1}{y} = \left(1 - \frac{v}{y}\right)\cdot\frac{\delta m}{m}.$$

$$\text{Now } \frac{1}{y} = \frac{1}{p} - \frac{1}{x}; \quad \therefore \delta\frac{1}{y} = \delta\frac{1}{p} = \frac{1}{p}\cdot\frac{\delta n}{n-1};$$

$$\therefore \frac{v}{p}\cdot\frac{\delta n}{n-1} = \left(1 - \frac{v}{p} + \frac{v}{x}\right)\cdot\frac{\delta m}{m}.$$

And v is always very nearly $= p$;

$$\therefore \frac{v}{p}\cdot\frac{\delta n}{n-1} = \frac{v}{x}\cdot\frac{\delta m}{m}, \quad \text{or } \delta m = \frac{mx}{p}\cdot\frac{\delta n}{n-1}.$$

(26). Now the axes of the pencils after refraction at q will tend to cross the axis at a distance $\dfrac{1}{\frac{1}{q}-\frac{1}{a}}$ or $\dfrac{aq}{a-q}$ from q, or when incident on r they are converging to a point whose distance from it $=\dfrac{aq}{a-q}-b=\dfrac{\overline{a+b}\,.\,q-ab}{a-q}$; they then tend to cross the axis at the distance

$$\frac{1}{\frac{1}{r}+\frac{a-q}{\overline{a+b}\,.\,q-ab}} \text{ or } \frac{r\,(\overline{a+b}\,.\,q-ab)}{-ab+\overline{a+b}\,.\,q+ar-qr}$$

from r, or the distance

$$\frac{r\,(\overline{a+b}\,.\,q-ab)}{-ab+\overline{a+b}\,.\,q+ar-ab}-c, \text{ or } \frac{\overline{a+b+c}\,.\,qr-\overline{a+b}\,.\,cq-\overline{b+c}\,.\,ar+abc}{-qr+\overline{a+b}\,.\,q+ar-ab}$$

$(=g)$ from s. They finally cross the axis at the distance

$$\frac{1}{\frac{1}{s}+\frac{1}{g}}=\frac{gs}{g+s}$$

$$=\frac{s\,\{\overline{a+b+c}\,.\,qr-\overline{a+b}\,.\,cq-\overline{b+c}\,.\,ar+abc\}}{-qrs+\overline{a+b+c}\,.\,qr+\overline{a+b}\,.\,qs+a\,.\,rs-\overline{a+b}\,.\,cq-\overline{b+c}\,.\,ar-abs+abc}$$

from s. And the distance from the axis, at which a pencil of rays meets q, is am: that at r is $m\times\dfrac{\overline{a+b}\,.\,q-ab}{q}$: that at s is

$$m\,.\,\frac{\overline{a+b+c}\,.\,qr-\overline{a+b}\,.\,cq-\overline{b+c}\,.\,ar+abc}{qr}.$$

Hence the tangent of the visual angle is

$$m\left\{-1+\frac{a+b+c}{s}+\frac{a+b}{r}+\frac{a}{q}-\frac{\overline{a+b}\,.\,c}{rs}-\frac{a\,.\,\overline{b+c}}{qs}-\frac{ab}{qr}+\frac{abc}{qrs}\right\}:$$

its variation

$$= m\left\{\frac{a+b+c}{s}+\frac{a+b}{r}+\frac{a}{q}-\frac{2.\overline{a+b}.c}{rs}-\frac{2a.\overline{b+c}}{qs}-\frac{2ab}{qr}+\frac{3abc}{qrs}\right\}\frac{\delta n}{n-1}$$

$$+\delta m\left\{-1+\frac{a+b+c}{s}+\frac{a+b}{r}+\frac{a}{q}-\frac{\overline{a+b}.c}{rs}-\frac{a.\overline{b+c}}{qs}-\frac{ab}{qr}+\frac{abc}{qrs}\right\}.$$

Making this = 0, and putting the value of δm in (25),

$$x=p\,.\,\frac{\overline{a+b+c}.qr+\overline{a+b}.qs+ars-2.\overline{a+b}.cq-2.\overline{b+c}.ar-2abs+3abc}{qrs-\overline{a+b+c}.qr-\overline{a+b}.qs-ars+\overline{a+b}.cq+\overline{b+c}.ar+abs-abc}.$$

(27). It is usual in practice to place the diaphragm in the situation which we have supposed. If it should be placed on the other side of the object-glass, m would be invariable, and the investigation would be in every respect similar to those for eye-pieces of telescopes.

(28). In a common microscope, in which the diaphragm was at a little distance from p towards q, it was found that $a=5$, $b=2$, $c=,3$, $q=2,8$, $r=1,9$, $s=1,6$. The equation above gives $x=-p\times,191$; hence the microscope was not properly constructed.

To complete the theory of eye-pieces, and to shew what are the most advantageous arrangements of their lenses, there remain to be considered the effects of spherical aberration. This part is equally important with that we have treated, and far more difficult of investigation: and may perhaps form the subject of a future communication.

G. B. AIRY.

TRINITY COLLEGE,
April 26, 1824.

XV. *An Account of a Whale of the Spermaceti Tribe, cast on shore on the Yorkshire Coast, on the 28th of April,* 1825.

By JAMES ALDERSON, B.A.

FELLOW OF PEMBROKE COLLEGE, CAMBRIDGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *May* 16, 1825.]

So little is still known with respect to the natural history and anatomy of whales, that any opportunity of contributing a few facts to the information already ascertained, is extremely desirable.

It is this which has induced me to communicate what I have seen.

The subject of the following remarks was seen on the afternoon of Thursday the 28th of April, drifting up from the southward with the ebb tide, off Tunstall in Holderness, and in the course of the afternoon was landed low in the tide. At the ensuing flood it was floated higher up on the beach, where it was left during the early part of the ebb. It may seem extraordinary, but it is no less true, that it was not generally known in Hull to be on shore until the Tuesday following. Nothing can be more contrasted than the view of the animal perfect, and its skeleton. The enormous and preposterous mass of matter upon its cheeks and jowl bearing no proportion to that of any other animal whatever, when compared with the bones of the head

On looking over the several works that have been published relating to these animals, there are evidently so many contradictions, that it is very difficult to fix on any specific point on which to rest, for forming those distinctions on which the pleasure derived from knowledge depends.

This is remarkably the case with the Physeter Macrocephalus, the external orifice of whose breathing tube has been described, as to its termination, so variously *, that it is very uncertain where to place the animal we have had the opportunity of examining; its form and connexion too, its final cause, its mode of action, are all hitherto unascertained, and although in the following account every thing has not been done, which more favourable circumstances might have afforded, yet I trust that something will have been done worth recording.

External and essential characters of the animal.

† Length of the animal from the snout to the division of the tail $58\frac{1}{2}$ feet.

† Distance of the eye from the snout 20 feet 8 inches.

† Circumference of the head from the sand on the one side to the sand on the other, taken midway between the eye and the snout, 31 feet 4 inches, (this of course does not include the lower jaw.)

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* Essential character of the Genus Physeter of Shaw;

Dentes in maxilla inferiore
Fistula in Capite s. fronte.

P. Macrocephalus. P. dorso impinni, fistula in cervice[a].

† These measurements were taken on Friday the 29th of April, by the Rev. Christopher Sykes, Rector of Roos; an ardent promoter of science. They were taken by means of a tape.

[a] This expression, according to Fabricius, is not quite correct. Shaw's *Zoology*, Vol. II. P. 2.

Dr. Schwediawer, in *Phil. Trans.* for the year 1783, has given a much more correct account of the external

† Circumference of the body at the spring of the tail 8 feet.

The orifice of the spiracle, or breathing tube, was at the extremity of the snout, and rather on the left side of the median line, somewhat of an *ʃ* form, and 2 feet 4 inches in length, externally.

There were two pectoral fins, in length 5½ feet, in breadth 2 feet 9 inches, at the broadest part, with distinct furrows between the phalanges.

The glenoid cavity of the scapula, (the chord of the arc) measured 9 inches, and the articulating surface of the humerus measured 10 inches.

Dorsal fin. This was only the rudiment of a fin; it was composed of the cutis and adipose cellular membrane, like the rest of the external covering of the body, and projected at its greatest height not more than a foot, commencing gradually, and terminating abruptly in a sort of hook-like process posteriorly.

Sex. Male.

Tail, horizontal; the span from tip to tip measured 14 feet; there were several transverse parallel bands distinctly seen upon it, where the epidermis was removed.

The Eye was placed at nearly the greatest lateral projection, a little inferiorly; the eye-lids were formed of a duplicature of the outer covering, about an inch in thickness; the upper lid projected more than the under, somewhat like a flap, and the opening was about seven inches in length.

There was no external *ear,* but simply a small circular opening about nine inches posteriorly to the posterior canthus of the eye, which just admitted the finger.

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ternal marks of this animal. "It is observed," he says, "that this species has but one spout (fistula). This spout is not, as has been generally hitherto asserted, in the neck (cervix) of the fish, but in its front, and on the very edge of the head, bending obliquely on the left side, so that whenever he spouts it is always on that side only." His description, however, of the "peculiar bony triangular cavity, or trunk, which contains the spermaceti, and which is lodged near the brain, and occupies nearly the whole of the upper part of the head," in no way agrees with my observations on the animal cast on shore at Tunstall.

The Vent was situated at a distance of 12 feet from the spring of the tail, and the penis at a distance of 19 feet from the same place.

This latter organ protruded about $1\frac{1}{2}$ feet from the body, and was surrounded by a shaggy process of the cuticle. The urethra admitted the point of the finger.

Lower Jaw. This is of the form of a Y, though all its parts are not in the same plane. The bifurcated parts lying in a plane which cuts that in which the stalk of the letter lies, when produced.

From the symphysis to the bifurcation it measured 11 feet.
———————————— articulation ———————— 16 feet.
The number of teeth in the jaw 47 (visible). Two more were found on cutting down upon the gum on the right side. In the skeleton, therefore, there will be 49 teeth.

I should hence infer the animal to be young, though, as they that were uncut were the most posterior of the teeth, it is possible it had reached its full growth. There is a remarkable difference in the posterior teeth compared with the others; they were much smaller, and rather hooked; particularly the last but one, and last but two on the left side, and the last but two and last but three on the right side. The two teeth at the symphysis were much smaller than those near them; they were front teeth, and were only three inches asunder. The teeth projected about two inches from the gum, and were, with the exception of the last five or six, *blunt*, with concentric lines on the worn surfaces.

The relative position of the teeth, in the two sides of the jaw, varies in different parts of it. Thus, the last tooth on the right side is without a fellow on the left; the succeeding seven correspond with and are opposite to, the last seven on the left side; the six nearest the symphysis correspond in like manner; the intermediate teeth are alternate.

The tenth tooth from the symphysis is nearly vertical, and the others, with the exception of a few of the last teeth, *all tend towards it*; that is, those anterior to it look towards the articulation; those posterior, towards the symphysis. The distance between the second pair of teeth nearest the symphysis (measured within the teeth at the gum) is $5\frac{1}{2}$ inches. The distance between the teeth at the bifurcation 13 inches.

In a vertical plane, the greatest depth of the jaw, near to the articulation, measured 2 feet 2 inches; the depth at the symphysis $2\frac{1}{2}$ inches only.

The *upper jaw* presented no teeth, but cavities lined with the mucous membrane of the mouth, and very firm; into these cavities the teeth of the lower jaw fitted, when the mouth was closed. The upper lip appeared to overhang considerably the under.

The *epidermis* was black, and varied in thickness in different parts, in no place exceeding one-third of an inch. When cut into layers, it still preserved its color.

The Cutis appeared intimately connected with what may be termed the adipose cellular membrane of the body; the cells filled with liquid fat and oil readily expressed. The thickness of this layer was different too in different places, reaching to 15 inches on the ridge, whilst at the sides it was not more than 9 or 10. On the head this covering appeared of a more fibrous nature, and had lost its cellular structure as well as its oil; it seemed towards the snout solely to afford a surface for the insertion of the numerous tendons which were found in the head: it was here much thinner, and was almost too dense to be cut through by the spade*.

The *head* formed a very considerable part of the animal, as far as regards its bulk.

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* A tool in use at the fisheries.

Its commencement, at what may be termed the snout, was very abrupt, increasing in magnitude posteriorly, as far as the junction of the atlas with the os occipitis.

No measurement, that I could hear of, was taken in this part, and it is the more to be regretted, as it was by far the most prominent part of the animal.

There was a considerable contraction behind the head, corresponding to the cervical vertibræ. A small * portion of the spear or tooth of the sword-fish, about 5 inches in length, was found enveloped in the adipose cellular membrane, near the ridge of the back, anteriorly to the rudimentary dorsal fin; there appeared too, near this same place, a wound, a fistulous-like opening in the cutis, supposed by the labourers to have been made by a harpoon.

Examination of the internal parts.

The interior of the head † contained, on the right side, a cavity or sac, or several sacs, holding spermaceti; the left was occupied by the breathing tube, and nearly all the surrounding and intermediate parts by long tendons.

The most posterior part of this mass, filling up the large basin, formed by the bones of the cranium, as depicted in Plate XI, Fig. 2, contained a large cavity, lined posteriorly with a membrane, in color yellowish white, and in structure cellular; the convex surfaces of the cells being *towards* the cavity, and about an inch in diameter. On being cut into, all the cells appeared to communicate with each other. The structure of the lining membrane of the anterior wall was very different; it consisted of transverse

* This is in the possesion of Mr. Hickney of Ridgemont, one of the Society of Friends, and a great agriculturist.

† Still exterior to the cranium.

folds, exactly similar to the lining membrane of the fourth stomach in sheep, in which the secretion of the gastric juice takes place.

I traced the communication from this cavity forwards, towards the snout, first passing on the left side of the head, *under* the breathing tube; then crossing over to the right side of the head, and joining with the sac or sacs in which the spermaceti was found; here the lining membrane was altered in appearance; it was more that of a continued mucous surface.

The breathing tube commenced at the posterior nares, on the left side of the head, and proceeded on this side to the snout, where it opened externally. Its length was 20 feet, 3 inches; its internal circumference, when cut open, measured 3 feet, 1 inch, near to the external orifice; it, however, became much smaller at its entrance into the foramen allotted to it, leading to the posterior nares. Near to the external orifice, the tube made a turn outwards, so as to propel fluid perpendicularly to the way of the animal, unless acted on by some of the tendons before mentioned. It was single throughout its whole length, and was lined with a continuation of the epidermis, much thinner, however, than in the snout, and becoming still thinner, as well as losing its black color, as it proceeded to the posterior origin of the tube, where it is connected with the trachea.

As the whole of this mass was *torn* from within the basin formed by the bones of the cranium, by the assistance of horses, this communication of the breathing tube and the trachea, could not be made out satisfactorily.

I am not convinced that there was not a communication between the posterior nares, and the peculiar cavity before described; in which cavity, it is probable, the secretion of the spermaceti takes place.

I am not, however, disposed to make any further *conjectures* on this point: dubia pro falsis adhibenda.

Along the sides of the sacs containing the spermaceti, as well as those of the breathing tube, ran innumerable thick and strong tendons*, terminating in the snout; evidently an apparatus for moving the snout, and the lips of the external orifice of the breathing tube.

The snout overhung considerably the lower jaw. I am not aware that this was measured; indeed from the very first, the lower jaw was dislocated, and projected on the left side of the animal, from under the upper lip, at nearly a right angle to the body.

The Eye.—This organ was not examined until the 8th of May: the parts offered for dissection consisted simply of the ball of the eye, together with the sheath which contains the straight muscles of the eye, enveloping the optic nerve, and were in no state to give any satisfactory result. The connexions with the adjoining parts had all been removed, through the ignorance of the workmen. The cornea was very much sunk and flaccid, and the conjunctiva had a dull bluish cast.

On clearing away the insertions of the cut muscles, the ball of the eye† was measured.

Its transverse diameter measured 2.35 inches.

Diameter in the direction of the axis of the eye measured 2.25 inches. This latter measurement taken on the supposition that the cornea was originally flat.

The color of the iris, as well as could be judged from its decomposed state, was bluish-brown; very dark; the pupil was transverse, as in ruminating animals.

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* Some of these were drawn out of the head by the by-standers, upwards of nine feet in length.

† These measurements were all taken by means of a pair of graduated callipers.

The Chrystalline lens was nearly spherical, and rather more convex anteriorly than posteriorly.

The transverse diameter measured .45 inch,

Diameter in direction of the axis .375 . .

The sclerotic coat was cut into about its middle, and there measured .7 inch in thickness, becoming thinner as it proceeded forwards to join the cornea, where it turned inwards.

Where the optic nerve enters the sclerotic coat, this latter only measured .5 inch in thickness, differing in this point from the same part in the balæna, which, according to Cuvier, is an inch and a half in thickness.

The density of this coat is very great, so much so, as previously to its being cut into, to give the idea of its being formed of cartilage, or corn of bony matter.

The internal cavity, containing the humours and the iris, was very small, and not of the same shape as the exterior of the eye-ball, on account of the varying thickness of the sclerotic coat; the depth of the interior cavity, from the anterior part of the sclerotic coat, measured only .8 of an inch.

We learn from Hunter, that the pigmentum nigrum covers the back part of the iris, and the corpus ciliare, but that it does not extend farther back. I should have felt great difficulty in stating its limits, as the sclerotica was certainly tinged with it posteriorly to this part.

The tapetum presented a very beautiful appearance; its color was a green, formed by an admixture of blue and yellow, with a slight predominance of blue; it was speckled with lighter colored spots throughout.

Posteriorly to the transverse axis of the eye, was inserted into the sclerotica a thick mass of muscle, enveloping the optic nerve. This mass was itself enclosed in a dense fibrous sheath,

($\frac{1}{4}$ inch in thickness), and was of great length, owing to the extraordinary breadth of the head at this part. The diameter of the mass, including the sheath, was about two inches at its insertion into the sclerotica, but gradually decreased as it approached the bony canal in the frontal* bone, through which the nerve passes to the brain.

The sole use of these muscles must be to draw in the eye.

The optic nerve itself appeared to have a vascular tunic.

The Heart.—This viscus was furnished with a pericardium, and in structure was exactly similar to that of man. It was very flaccid, and the parietes, when examined on the 9th of May, lay in contact.

Its weight was 171lbs.

The descending cava measured 9 inches in diameter†: the ascending cava was not with the heart, for it was not examined in situ‡.

Diameter of the pulmonary artery $12\frac{2}{3}$ inches; thickness of the coat $\frac{1}{4}$ inch.

Breadth of one of the semi-lunar valves of the pulmonary artery, 5 inches; its length 17 inches.

There was no corpus sesamoideum apparent.

The diameter of the aorta was $12\frac{1}{6}$ inches.

Thickness of the coat of the artery $\frac{7}{16}$ inch.

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* However misplaced the orbit, it is still the frontal bone which dips down to form it: the sutures are well marked in Fig. 1. Plate XIV.

† As these diameters were obtained through the medium of the half circumferences, it is probable, from the more yielding nature of the coats of veins, the measurements of the veins will rather exceed the truth, and those of the arteries rather fall short of it.

‡ The heart was examined at the house of Mr. Sawyer, Surgeon, at Hedon, to whom I have to acknowledge myself indebted for his kind and skilful assistance in its examination.

In the sinus, behind the valves, the thickness was not greater than that of the pulmonary artery.

Length of the heart, from the apex to the valves of the aorta, 3 feet 10 inches.

The columnæ carneæ were very large, and one of the cordæ tendineæ in the tricuspid valve, measured 7 inches in length.

Near the middle of the left ventricle, the wall of the ventricle measured about 3 inches.

The diameter of the coronary artery was $1\frac{2}{3}$ inches.

On the left ventricle being laid open, its capacity was guessed, by some farming gentlemen present, to contain from 8 to 10 gallons. The heart was destitute of fat.

Larynx and Œsophagus.—These parts, of great interest in this tribe, were supposed to have been sent perfect with the heart, but, on examination, it was ascertained that the œsophagus was wholly missing; and, in consequence, the parts could not be satisfactorily examined.

There remained only a part of the trachea, with the cricoid and thyroid cartilages as in man, nearly, together with what has been termed the pyramid, the connexion of the trachea with the breathing tube.

The *urinary and genital* organs were not examined; they were removed, with other parts of the animal, during the time I was occupied in the examination of the head.

The Stomach.—This organ was cut open by the labourers, not with a view to the examination of its structure, but in search of ambergris, of which none was found.

Near the termination of the œsophagus was found about a bucket-full of the beaks of one of the cuttle fishes. See Plate XIV. Fig. 4.

The lungs and liver were but cursorily examined; indeed, the viscera were so quickly removed, with a view to clearing the bones of the animal, that it was impossible to examine every organ.

Mr. Hunter, in his paper on this tribe, has made some curious remarks on the mode in which these animals suck, "which," he says, "would appear to be very inconvenient for respiration, as either the mother or the young one will be prevented from breathing at the time, their nostrils being in opposite directions; therefore the nose of one must be under water, and the time of sucking can only be between each respiration." Now, as the external orifice of the breathing tube is on the left side of the median line, and the mammæ near the anus, supposing both the mother and the sucker on their right sides respectively, but reversed in position, I see nothing to prevent the sucker continuing his occupation, without any interruption whatever to his respiration, or to that of the mother.

It only remains for me to state, that the skeleton will be articulated and preserved at Burton Constable, the seat of Sir Thomas Constable, Bart., to whom the animal belongs, as Lord Paramount of Holderness.

The bones are now macerating in pits, where they will have to remain a considerable time. In the Autumn, probably, the process of articulating will be commenced, and from the known zeal of the steward* of the estate, there is every probability of its being completed in a skilful manner.

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* Mr. Iveson of Hedon.

Kingston upon Hull,
May 14, 1825.

EXPLANATION OF THE PLATES.

Plate XII.

Fig. 1. A VENTRAL view of the Whale, shewing the jaw in its dislocated position; the animal lying on its right side.

Fig. 2. A dorsal view of the Whale, shewing the external orifice of the spiracle or breathing tube; also the rudimentary dorsal fin.

Plate XIII.

Fig. 1. The lower jaw, seen from the articulation, after having been removed from the body.

Fig. 2. The cranium without the lower jaw, seen from the anterior part of the head, to shew the basin formed by the bones, for the reception of the enormous mass composing the head and throat.

The width of this basin measured 5 feet, 11 inches.

The width across the orbits —— 8 feet, 8 inches.

The length* from the summit of the cranium to the anterior extremity, 20 feet.

Plate XIV.

Fig. 1. A side view of the cranium without the lower jaw. Shewing the articulating surface of the os occipitis, which is received into the concave surface of the atlas.

Horizontal diameter of this articulating surface, including, between the condyles, the foramen magnum, 2 feet, 5 inches.

Horizontal diameter of the foramen magnum, 8 inches.

Vertical diameter of ditto, ... 9 inches.

This gradually became smaller as it approached the cavity for the brain, when it did not exceed 4 inches in diameter.

Transverse diameter of orbit, (horizontal) . . . 8 inches.

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* The hypothenuse of a triangle formed by these two points, and the bottom of the basin.

Fig. 2*. The eye, in the state in which it was delivered to us for dissection.

Fig. 3*. The eye, after a transverse section had been made, shewing the magnitude of the internal cavity of the eye, the thickness of the sclerotic coat, the choroid having been separated from the sclerotic, to which it closely adhered, and turned aside, containing within it the retina.

Fig. 4*. A representation of a beak of one of the cuttle fishes; it was found in the stomach of the animal.

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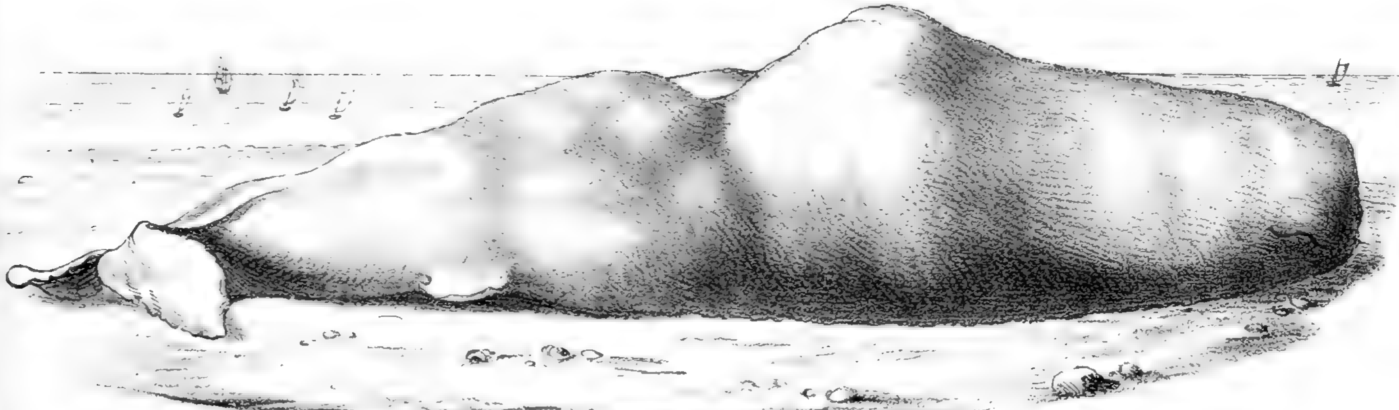
* These are very nearly of the natural size.

Fig 1

Fig 2



Drawn by C. R. Alderson, M.D.

C. Hullmandel's Lithography

Fig. 1

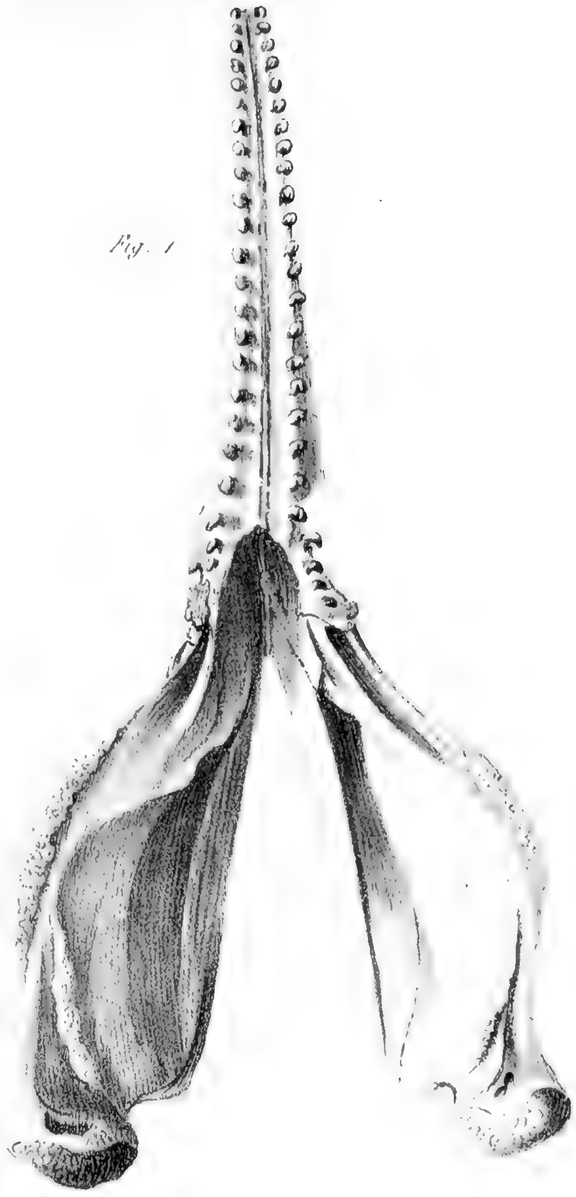


Fig. 2

Drawn by C. R. Atkinson M.D

On Stone by C. Hullmandel

Fig: 1

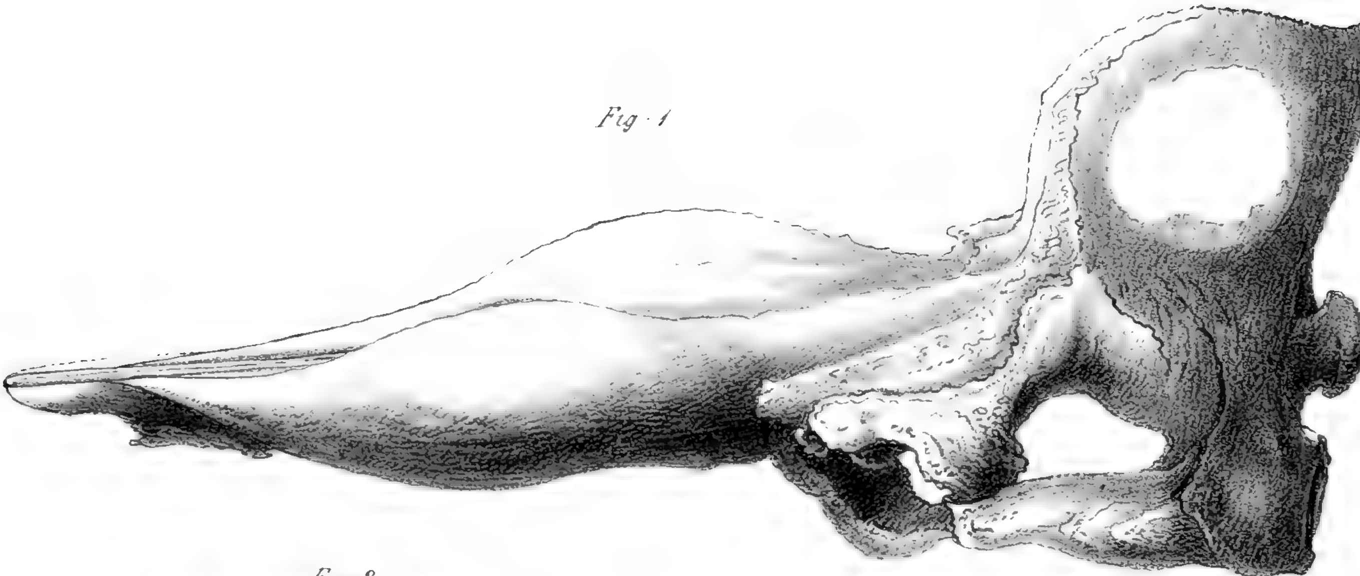


Fig: 2

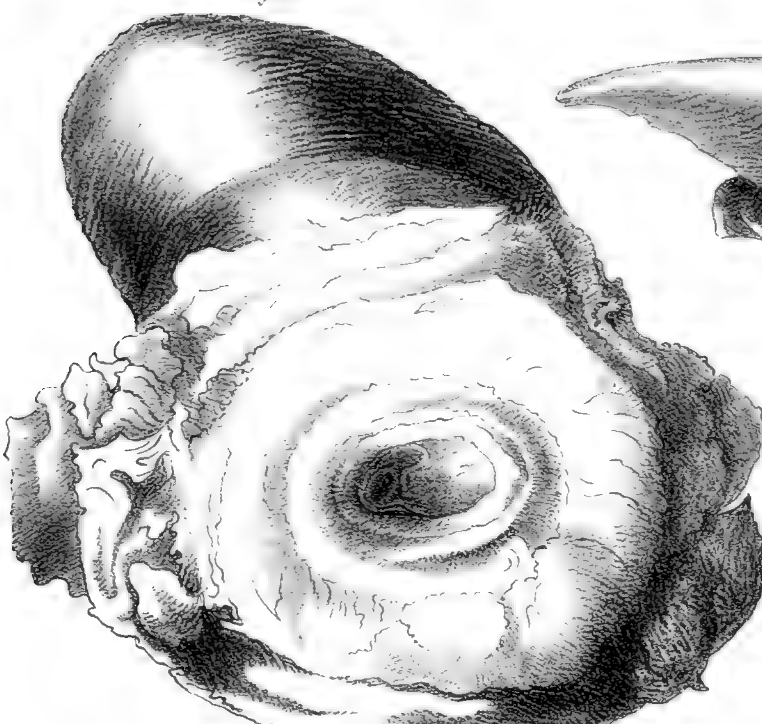


Fig. 4

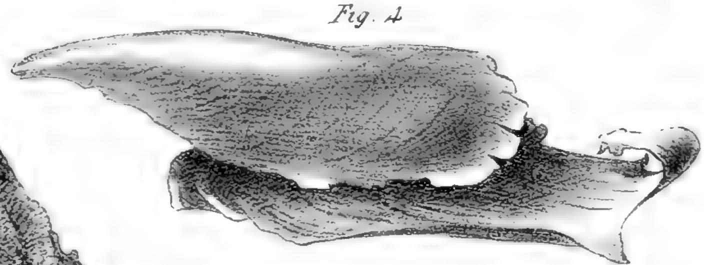
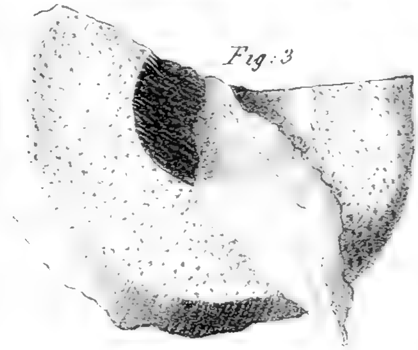


Fig: 3



Drawn by C. R. Alderson M. D.

C Hullmandel's Lithography.

XVI. *On a peculiar Defect in the Eye, and a mode of correcting it.*

By GEORGE BIDDELL AIRY, B. A.

FELLOW OF TRINITY COLLEGE, OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, AND CORRESPONDING MEMBER OF THE NORTHERN INSTITUTE.

[Read *Feb.* 21, 1825.]

THE communication which I have now the honour to make to this Society, relates to a peculiar defect of the eye, and the mode of correcting it. On a subject so important, I trust I shall be excused if I enter into details; as the mal-formation which I am about to describe, though hitherto unnoticed, is probably not uncommon.

Two or three years since, I discovered that in reading I did not usually employ my left eye, and that in looking carefully at any near object, it was totally useless: in fact, the image formed in that eye was not perceived except my attention was particularly directed to it. Supposing this to be entirely owing to habit, and that it might be corrected by using the left eye as much as possible, I endeavoured to read with the right eye closed or shaded, but found that I could not distinguish a letter, at least in small print, at whatever distance from my eye the characters were placed. No further remark suggested itself at that time, but a considerable time afterwards I observed, that the image formed by a bright point (as a distant lamp or a star) in my left eye, was not circular, as it is in the eye which has no other defect than that of being near-sighted, but elliptical, the major axis making an angle of about 35° with the vertical, and its higher extremity being inclined to the right. Upon putting on concave spectacles, by the assistance of which I saw distant objects distinctly with my right eye, I found that to my left eye a distant lucid point had the appearance of a well defined line, corresponding exactly in direction, and nearly in length to the major axis of the ellipse above-mentioned. I found also that if I drew upon paper two black lines crossing each other at right angles, and placed the paper in a proper position, and at a certain distance from the eye, one line was seen perfectly distinct, while the other was barely visible: upon bringing the paper nearer to the eye, the line which was distinct now disappeared, and the other was seen very well defined. All these appearances indicated that the refraction of the eye was greater in the plane nearly vertical, than in that at right angles to it, and that consequently it would not be possible to see distinctly by the assistance of lenses with spherical surfaces. I found, indeed, that by turning a concave lens obliquely, or by looking directly through a part near the edge, I could see objects without confusion; but in both cases, the distortion produced in their figure was such, that I could not hope to make any use of the left eye without some more effectual assistance.

My object now was to form a lens which should refract more powerfully the rays in one certain plane, than those in the plane at right angles to it; and the first idea was to employ one whose surfaces should be cylindrical and concave, the axes of the cylinders crossing each other at right angles, and their radii being different. To shew that this construction would effect my purpose, it is only necessary to imagine the lens divided into two lenses by a plane perpendicular to its axis; then it is easily seen that the refraction of one will not be perceptibly altered by that of the other, and that the whole refraction will be the combination of the two separate refractions. The rays in one plane will be made to diverge entirely by the refraction of one lens, and those in the other plane by that of the other lens. If then r and r' be the radii of the surfaces, and n the refractive index, and parallel rays be incident, the rays in one plane after refraction will diverge from a point whose distance is $\frac{r}{n-1}$, and those in another plane from a point whose distance is $\frac{r'}{n-1}$. This construction then was sufficient; but for the facility of grinding, and for the diminution of the curvatures, it appeared preferable to make one surface cylindrical, the other spherical; both concave. Let r be the radius of the cylindrical surface, R that of the spherical: then the refraction in the plane passing through the axis of the cylindrical surface, being entirely effected by the spherical surface, parallel rays in this plane after refraction will diverge from the distance $\frac{R}{n-1}$: while the refraction in the plane perpendicular to the axis being caused by both surfaces, parallel rays, in this plane, will on their emergence, diverge from the distance $\frac{1}{\overline{n-1}\left(\frac{1}{R}+\frac{1}{r}\right)}$.

To discover the necessary data, I made a very fine hole with the point of a needle in a blackened card, which I caused to slide on a graduated scale; then strongly illuminating a sheet of paper, and holding the card between it and the eye, I had a lucid point upon which I could make observations with great ease and exactness. Then resting the end of the scale upon the cheek-bone, and sliding the card on the scale, I found that the point at the distance of 6 inches, appeared a very well defined line inclined to the vertical about 35°, and subtending an angle of 2° (by estimation): at the distance of $3\frac{1}{2}$ inches it appeared a very well defined line at right angles to the former, and of the same apparent length. It was necessary therefore to make a lens, which, when parallel rays were incident, should cause those in one plane to diverge from the distance $3\frac{1}{2}$ inches, and those in another plane from the distance 6 inches. Making the expressions above equal to these numbers, and supposing $n = 1{,}53$, we find $R=3{,}18$, $r=4{,}45$. To prevent if possible the eye from becoming more short-sighted, I fixed upon the values $R = 3\frac{1}{3}$, $r = 4\frac{1}{2}$.

After some ineffectual applications to different workmen, I at last procured a lens to these dimensions from an artist named Fuller, of Ipswich. It satisfies my wishes in every respect. I can now read the smallest print at a considerable distance with the left eye, as well as with the right. I have found that vision is most distinct when the cylindrical surface is turned from the eye: and as when the lens is distant from the eye, it alters the apparent figure of objects by refracting differently the rays in different planes, I judged it proper to have the frame of my spectacles made so as to bring the glass pretty close to the eye. With these precautions I find that the eye which I once feared would become quite useless, can be used in almost every respect as well as the other.

The publication of this case, I imagine, may be not without utility. I believe it has generally been found, that where the direction of the axis of the eye is distorted, the sight of the eye is defective, but not lost: and the distortion is by many ascribed to the disuse of the eye, which is occasioned by this defect. If it should be found that the defect is at all similar to that which I have described, it can be perfectly corrected. The examination of the defect in the manner which I have detailed is very easy; and it is merely necessary to write down fully the appearance of the brilliant point at different distances, in order to enable the theoretical optician to invent a glass which shall make the vision of the eye distinct. If the defects arise from insensibility of the nerve, or opacity of the humours, they are beyond his power: but any fault in the refracting surfaces it is possible to correct.

Since I procured this lens, I have been informed that a foreign artist has made spectacle-glasses with cylindrical surfaces of different radii for general use. What his object can be I am quite unable to imagine; certainly no one whose eyes are not defective can see with them distinctly. With my right eye which (by the method of examination above described) I find to have no other defect than short-sightedness, I am unable to read any thing in the lens made for my left eye. After many inquiries I have not been able to discover that this construction has been used to correct any defect in the eye, or even that a defect similar to that which I have described, has ever been noticed. In laying before this Society the notices of a case which appears at once novel and important, I trust that I shall not be thought to have trespassed unprofitably upon their time.

G. B. AIRY.

Trinity College,
Feb. 5, 1825.

XVII. *A general Demonstration of the Principle of Virtual Velocities.*

BY THE REV. J. POWER, A.M.

FELLOW OF CLARE HALL, CAMBRIDGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *March* 21, 1825.]

CONCEIVE a machine, of any description whatever, to be kept at rest by forces P, P', P''....., and let m, m', m''..... denote the points of their application.

Call α, α', α''..... any indefinitely small spaces, subject to the conditions of the system, which these points are at liberty to run over in the same instant; and β, β', β''..... the corresponding spaces estimated in direction of the forces. Then $\frac{\beta}{\alpha}$, $\frac{\beta'}{\alpha'}$, $\frac{\beta''}{\alpha''}$..... will be the cosines of the angles at which P, P', P''..... are inclined to the spaces α, α', α''.....; so that if we resolve these forces into others, a part tangential, and the remainder normal to the above spaces, the former will be expressed by

$$\frac{P\beta}{\alpha},\ \frac{P'\beta'}{\alpha'},\ \frac{P''\beta''}{\alpha''}\ .....;$$

for which we may substitute the following: namely,

$$\text{at the point } m,\quad \frac{P\beta}{a},$$

$$\text{at } m',\quad \frac{P\beta+P'\beta'}{a'}-\frac{P\beta}{a'},$$

$$\text{at } m'',\quad \frac{P\beta+P'\beta'+P''\beta''}{a''}-\frac{P\beta+P'\beta'}{a''},$$

and so on.

But the equilibrium will remain undisturbed, if for the two forces $\frac{P\beta}{a}$ and $\frac{P\beta}{a'}$, (of which the former acts upon m in direction of α, and the latter upon m' in the direction opposed to α',) we substitute two strings stretched with these forces, and acting in the same directions: again, the strings so stretched, may be conducted over fixed pullies, in such a manner, that their other extremities may impel perpendicularly the arms of a straight lever, divided by its fulcrum in the ratio $\alpha : \alpha'$. The fulcrum will then re-act with a force equal to the sum of the two tensions, and the whole will remain at rest. Should $P\beta$ be negative, the strings stretched with the positive forces, must proceed from the two points in directions opposite to what we have just supposed, and be attached to the lever as before.

In the same manner we may substitute for the forces

$$\frac{P\beta+P'\beta'}{a'} \text{ and } \frac{P\beta+P'\beta'}{a''}$$

the re-action of a second lever, divided in the ratio $\alpha' : \alpha''$.

If we make similar substitutions throughout the system, at the same time substituting for the normal part of the resolved forces the re-action of so many fixed surfaces intersecting them at right angles; there will at length remain a single tangential force, whose numerator is the sum of all the terms $P\beta+P'\beta'+P''\beta''+\ldots$, making equilibrium with the machine, which results from uniting with the original system the series of levers and surfaces we have just introduced.

Now it is easy to perceive, that the new connexion we have established, leaves the points at liberty to describe, ***backwards or forwards***, the same indefinitely small spaces, which were allowed them by the original machine, provided they were previously at liberty to describe spaces proportional to $a, a', a'' \ldots$ in the opposite directions: but this will evidently be the case; for $a, a', a'' \ldots$ being indefinitely small, the equations to which they are subject, must be homogeneous with respect to these variables, or may be considered so; they will consequently remain equally satisfied when these quantities simultaneously change their signs.

From hence it follows, that, in the combined machine, the slightest tangential force applied to one alone of the points, on either side of it, will necessarily disturb the equilibrium. The equilibrium is, therefore, impossible, unless

$$P\beta + P'\beta' + P''\beta'' + \ldots = 0,$$

which is the symbolic enunciation of the principle in question.

The converse of this proposition is equally true; namely, that if the equation

$$P\beta + P'\beta' + P''\beta'' + \ldots = 0,$$

be satisfied for every indefinitely small variation in the position of the system, there will be equilibrium.

For if an initial motion were possible, the equilibrium might evidently be restored, by applying new forces in the opposite direction to those points *only*, whose variations, taken along the initial spaces, (but not necessarily identic with them,) may be regarded arbitrary and independent in other respects, without interfering with those points, whose variations are altogether determined, and their motions constrained, by those of the former. Let α, α', α''. . . . . be the variations of the first class, and $\alpha_{,}$, $\alpha_{,,}$, ... those of the second. Then β, β', β''. . . . $\beta_{,}$, $\beta_{,,}$. . . . being the corresponding spaces estimated in direction of the forces, and F, F', F'', ... the forces requisite to maintain the equilibrium, we shall have, in consequence of what has been already proved,

$$\left.\begin{array}{l} P\beta+P'\beta'+P''\beta''+\ldots\ldots+P_{,}\beta_{,}+P_{,,}\beta_{,,}+\ldots\ldots \\ -F\alpha-F'\alpha'-F''\alpha''-\ldots\ldots \end{array}\right\}=0.$$

This equation is reduced by the hypothesis to

$$F\alpha+F'\alpha'+F''\alpha''+\ldots\ldots=0.$$

But α, α', α''. . . . being arbitrary and independent, we must have, separately, $F=0$, $F'=0$, $F''=0$, &c., and, consequently, the forces

$$P,\ P',\ P''.\ldots.\ P_{,},\ P_{,,}\ldots.$$

will be in equilibrium by themselves.

J. POWER.

Clare Hall,
March 19, 1825.

XVIII. *On the Forms of the Teeth of Wheels.*

By GEORGE BIDDELL AIRY, B.A.

FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, AND CORRESPONDING MEMBER OF THE NORTHERN INSTITUTE.

[Read *May* 2, 1825.]

THE investigation of the forms proper for the teeth of wheels is a useful and interesting inquiry. The mechanical principles are very simple, and the geometrical propositions on which it is immediately made to depend, admit of being put in an elegant form. But all the theories which have yet been given, are, I believe, very imperfect. Euler in the New Petersburgh Commentaries for 1760 has treated the subject with great generality; but the analytical method which he has used is very unfavourable for the discovery of the most obvious properties of the curves. In all the other theories that I have seen, no forms are mentioned but the involute of a circle, and the epicycloid and hypocycloid. In this paper I propose to consider generally the figures which must be given to the teeth of wheels to insure uniformity of action. The curves above alluded to, though probably the most convenient of all, I shall shew are particular cases of a very general construction: and the demonstration which has usually been given for them, I shall apply to every other case.

That the mechanical effect which one wheel produces upon another, may in all positions be the same, it is necessary that the line perpendicular to the surfaces of the teeth, at the point of contact, intersect the line joining the centers at a fixed point, which divides that line into two parts, the ratio of which is the mechanical power. When this holds, the proportion of the angular velocities will be constant. For let A and B (Plate XV. Fig. 1.) be the centers of the wheels, C the point through which the line of action passes: D the point of contact: upon moving the wheels with the teeth still in contact through a very small angle, D in one tooth will be carried to F, and in the other to G, FG being ultimately parallel to the tangent at D, or perpendicular to CD, and DF, DG, perpendicular to AD, BD respectively. Then,

$$FD : GD :: \sin G : \sin F :: \sin BDC : \sin ADC :: \frac{BC}{BD} : \frac{AC}{AD};$$

therefore the angular velocities, which are as $\frac{FD}{AD} : \frac{GD}{BD}$, will be as $BC : AC$, a constant ratio. If then with centers A and B circles be described passing through C, and these circles revolve so as to make the velocities of their circumferences equal, the teeth of the wheels, if properly formed, will be in contact, and the normals to both will pass through C. These circles we shall call the principal circles of the wheels.

If the normals from every point of the tooth should be equally inclined to the tangents of the circle at the points where they meet the circle, they evidently would if produced be tangents to a circle, whose radius : radius of circle described :: cosine of inclination of normal with tangent of circle described : 1. In this case both teeth would be involutes of circles. If the inclinations are not equal, we must make use of the following theorem. It is always possible to find a curve which by revolving upon a given curve, shall by some describing point, in the manner of a trochoid, generate a second given curve: provided that the normals from all points of the second curve meet the first.

To prove this let AB, (Fig. 2.) be the first curve, AC the second; from the points C and E, which are very near, draw the normals CD, EF; if a describing point P be taken, and PQ, PR, be made respectively equal to CD, EF, and QR equal to DF, and this process be continued, a curve will be formed, which by revolving upon BA, will, by the describing point P, generate the curve AC. For if Q coincide with D, then R will afterwards coincide with F, and so on for all succeeding points, since $QR = DF$. Also $DC = QP$, &c. And the angles made by these with the tangents are equal. For the cosines of these angles, drawing DG, QS, perpendicular to EF, PR, are $\frac{FG}{FD}$ and $\frac{RS}{RQ}$, in which the numerators are the differences of equal lines, and the denominators are equal. Hence P will describe AC. And the formation of the curve RQ is always possible, because RQ is greater than RS; for FD is necessarily greater than FG. As an example of this, suppose it were required to find the curve, which revolving on one straight line AB, (Fig. 3.) would generate another straight line AC. Since the angles made by the line PQ with the tangent, must be constant, it follows, that the curve would be the logarithmic spiral, P being its pole.

The entire theory of the teeth of wheels, may now be included in this proposition. If the tooth HD, (Fig. 4.) be generated by the revolution of any curve on the outside of the circle HC, and if DK be generated by the revolution of the same curve in the same direction, in the inside of the circle KC, then the normal at the point of contact of the teeth, will pass through C. For let the generating curve be brought to the position LC, so as to touch the circle HC at C; DC will be the normal of HD at D; and that the teeth may be in contact, the same generating curve in the other circle must touch KC at C; in which case it will coincide with this; D therefore will be in the surfaces of both of the teeth, and CD the normal of both at that point; therefore they will touch at D, and the line of action CD, will pass through the fixed point C. If now we give equal velocities to the circumferences CH, CK, the same will be found at all times to be true. These forms then are proper for the teeth of wheels.

Suppose then this problem proposed. Given the form of the teeth of one wheel, to find the form of those of another, that they may work together correctly. The following is the obvious solution. Divide the line joining the centers of the circles at C, into two parts, whose proportion is the mechanical power. Describe the circles CH, CK. Find the curve which by revolving upon CH, will generate the given tooth HD. Make the same curve revolve in CK, and with the same describing point let it generate KD; KD is the form required.

The usual construction of the involute of a circle, would seem to require that the circles AH, and BK, should be separated. If however DH be the involute formed in the usual way from the circle MN, (Fig. 5.) the normal CM will be inclined at a constant angle to CA, $\left(\text{since its sine} = \frac{AM}{AC}\right)$, and the construction given before shews that the involute HD may be generated by the revolution of a logarithmic spiral upon CH, the describing point being the pole of the spiral, and the angle between its radius and tangent, the same as the angle made by MC, with the tangent of the circle at C. In the same way the revolution of this spiral in the second circle will generate another involute; and hence if the teeth of one wheel be involutes, those of the other wheel must also be involutes. The generating circles of the involutes must have radii proportional to AC, BC.

It will be seen immediately, that we may if we please suppose successive parts of the curve described by different generating curves; or we may make one curve revolve on the outside of the circle CH, and another on the inside, making the same curves revolve on the inside and outside of CK respectively, and thus an infinite variety of curves may be found. The construction last mentioned gives forms approximating most nearly to the usual forms of teeth. We may even give different forms to different teeth; but this probably would not be desirable.

It may be desirable to know when the nature of the teeth will admit of an alteration in the distance of the centers of the wheels. Suppose then DL and FP, (Fig. 6.) to be the principal circles when the wheels are in the first position; KS and HR, the principal circles when the distance of the centers is increased. Suppose in the first position C was in contact with E, and M with O; suppose in the second position, G and Q are in contact with E and O; draw normals to all these points as in the figure. Since the wheels in the first position work correctly, by supposition, the angles at D and N will equal those at F and P. And if they work correctly in the second position, HG will $= KE$, &c. HR will $= KS$, and the angles at H and R will equal those at K and S. By attending to this condition, when the tooth EO is given, we can always form a tooth CQ, which will work with it in two positions of the wheels. Since the angles at H and R equal those at K and S, the angles at L and T will equal those at F and P; and therefore will equal those at D and N. It is evident that this condition will always be satisfied, if CQ be the involute, and therefore if the teeth be involutes, the distance of the centers may be altered to any degree, allowing the teeth to act on each other.

In all, however, that has yet been stated, we have only considered the mathematical conditions of the contact of two curves. That these forms may be applicable in practice, it is necessary that the curvature of the convexity of one tooth, should be greater than that of the concavity of the other, or else that both should be convex. For this purpose we must investigate the curvature at any point.

Take then two points on the circle near each other, and the two points of the generating curve which will touch them; join these with the center of curvature of the generating curve, and with the describing point; let ϕ, θ, ψ, (Fig. 7.) be the small angles at the center of curvature, the describing point, and the center of the circle; suppose the lines from the describing point, when in contact with the circle, to be produced respectively, and let the angle at their point of intersection $= \chi$. Also let α and β be the angles which those lines make with the radii of the circle. Then we shall have

$$\theta - \phi = \alpha - \beta; \ \psi - \chi = \alpha - \beta; \ \therefore \ \chi = \psi + \phi - \theta.$$

But calling R the radius of the circle, r the radius of curvature, s the distance of the describing point, x the distance of the point of intersection,

$$\psi = \frac{\text{arc}}{R}; \quad \phi = \frac{\text{arc}}{r}; \quad \theta = \frac{\text{arc}\,.\cos\alpha}{s}; \quad \chi = \frac{\text{arc}\,.\cos\alpha}{x};$$

$$\therefore \ \frac{\cos\alpha}{x} = \frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}; \quad \therefore \ x = \frac{\cos\alpha}{\frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}};$$

$$\therefore \ x + s = s\frac{\frac{1}{R} + \frac{1}{r}}{\frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}} = \text{rad. of curvature of tooth};$$

$$\therefore \ \text{curvature} = \frac{1}{s}\,.\,\frac{\frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}}{\frac{1}{R} + \frac{1}{r}} = \frac{1}{s} - \frac{\frac{\cos\alpha}{s^2}}{\frac{1}{R} + \frac{1}{r}}.$$

From an examination of this expression, it appears, that when α is $< 90^\circ$, r may be positive or negative, but must be less than the radius of the circle in the same direction; when

α is $>90^0$, r may be positive or negative, and must be greater than the radius in the same direction.

If then, as is the case in general, α be $<90^0$, that part of the tooth which is without the circle, must be formed by the revolution of some curve upon the circle, and that which is within it by the revolution of some curve within the circle. This kind of tooth is represented in Fig. 4. But if α may be $>90^0$, the whole of the teeth may be formed by the revolution of a single curve; an instance of this is represented in (Fig. 8.) where the teeth GH and KL are formed by the motion of MN, carrying the describing point P. In the last case, if the curve be a circle equal to one of the circles, one tooth will be reduced to a point, the other will be an epicycloid or epitrochoid, according as the describing point is in the circumference of the circle, or in any other part.

It will easily be seen, that where the acting surface of the driving tooth is above the circle, the action takes place after passing the line joining the centers; when below the circle, it is before passing that line. Now practical men always think it proper, that the action should take place only after passing the line of centers. It is thought necessary that the direction of the friction should be such as to wipe off the dust, &c. from the teeth. For this purpose then, the curve which has been found for the lower part of the teeth, must be considered as a limit which that tooth must not reach. In the case in which the whole is formed by the revolution of one curve, the whole action takes place after passing the line of centers.

To find what the friction really amounts to, we have merely to observe, that in Fig. 1. if D be brought to G in one tooth, and to F in the other, GF is the friction, and if $BDC=\alpha$, FG : FD :: $\sin ADB$: $\sin \alpha$; therefore frictional motion $\propto \dfrac{\sin ADB}{\sin \alpha}$, $\propto \dfrac{\sin ADB}{\sin BCD}$ nearly, (the teeth being so small, that DF may be considered as nearly representing the motion of the circumference.) Also the pressure occasioned by a given force in given circumstances $\propto \frac{1}{\sin BCD}$; and the mechanical effect of friction is proportional to the pressure by which it is caused multiplied by the velocity of the rubbing surfaces; and therefore $\propto \frac{\sin ADB}{\sin^2 BCD}$ nearly. The numerator is proportional to the distance from the line of centers; and therefore will be the same for all teeth, when that distance is the same. But the denominator is largest when the face of the tooth is parallel to the radius of the circle. I imagine then that it is advisable to make the teeth work a little before as well as a little after the line of centers. And I should think that a tooth similar to that formed by the union of the epicycloid and hypocycloid, is preferable to any other form whatever. For the line of action is always very nearly perpendicular to the radius; by which means not only is the friction made much less, but also the strain upon the axes is considerably diminished.

If it be thought desirable to prevent back-lashing, this can be done by giving proper forms on the same principles to the faces of the teeth, which are not the working faces. But the chance of very greatly increasing the friction, makes the propriety of this consideration very doubtful.

The whole of what has been stated with regard to circles, it is evident will apply equally to straight lines. Thus the teeth of rack-work may be formed of a combination of cycloids, in which case those of the wheel must consist of epicycloids, and hypocycloids; they may be straight, which will make those of the wheel the involutes of a circle, (both being generated by the revolution of a logarithmic spiral;) they may be mere pins, in which case the teeth of the wheel will be involutes, or curves described in nearly the same manner as involutes. In this case, and in the case of trundles, if it be required to take account of the diameter of the pins, this will be done by taking a curve, whose normal distance from the curve found by considering them as points, shall at all parts be equal to the radius of the pin. Or the form of the teeth may be found by the general theorem.

For crown wheels, as the contrate wheel of a watch, the teeth without sensible error may have the same form as for rack-work. The theory may be extended to bevelled wheels, without any difficulty.

There is one case which ought to be mentioned particularly. It may be desired that the teeth of one wheel have plane surfaces passing through the axis of the wheel. Since a straight line is the hypocycloid, in which the radius of the generating circle is half that of the fixed circle, the teeth of the other wheel must be epicycloids, the radius of the generating circle being half that of the first wheel. The action here takes place entirely after the line of centers, and the direction of the action is nearly perpendicular to that line. I imagine this to be a good construction for pinions with a small number of teeth driven by a large wheel. If each tooth consist of a line within the principal circle, and an epicycloid without it, the radius of the generating circle of each epicycloid, being half that of the other principal circle, a very good form will be produced. The action takes place before as well as after the line of centers, and is always nearly perpendicular to that line. The figure usually given to the teeth of watch-wheels approaches very nearly to this.

I have confined my attention entirely to uniformity of action, and uniformity of motion, as I conceive them to be of far greater consequence than the diminution of friction. The friction can never be made = 0, except the point of contact be always in the line of centers; a condition which may be satisfied by an infinite number of curves, and amongst others by two logarithmic spirals.

But the mechanical action and the motion would be dreadfully irregular.

I am informed by engineers, that this question is now little more than one of mere curiosity. In consequence of the very extensive use of iron, where wood was formerly employed, the teeth of wheels are now made so small, that it is of little consequence whether they have, or have not, the exact theoretical form. Almost all teeth are now made with plane faces passing through the axis of the wheel, and are expected to wear themselves in a short time into proper forms. This is the case with nearly all the modern iron wheels that I have examined; in the wheels of clock and watch-work, some attention to the figure is however thought necessary.

G. B. AIRY.

Trinity College,
April 30, 1825.

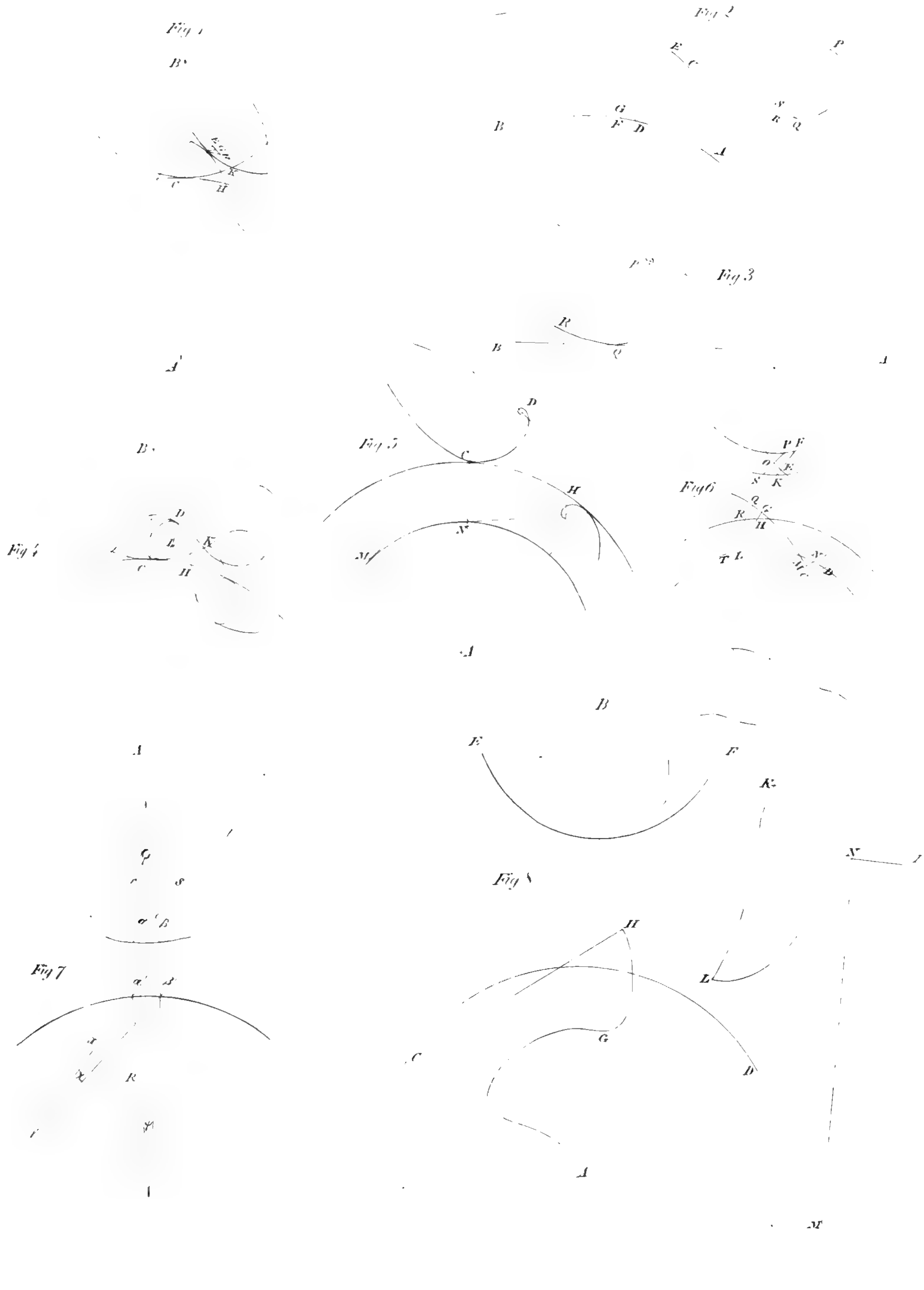


Fig 1
Fig 2
Fig 3
Fig 4
Fig 5
Fig 6
Fig 7
Fig 8

XIX. *Observations on the Ornithology of Cambridgeshire.*

By the Rev. LEONARD JENYNS, M. A. F. L. S.

AND FELLOW OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *Nov.* 28, 1825.]

In drawing up the following paper, it has been my object to collect a few particulars respecting the Ornithology of Cambridgeshire. A first attempt towards any undertaking of this nature must necessarily be very imperfect. From this circumstance I desire that the present may not be considered as a complete catalogue of the birds which are found in this county, especially as the greater part of the observations from which it has been chiefly compiled, have been confined to the neighbourhoods of Cambridge and Bottisham. With the view, however, of rendering it as extensive as possible, and in some measure of supplying the deficiency arising from this cause, I have added from our English authors whatever notices I could find in their works relating to species which though formerly met with in this district have not lately occurred to my knowledge. There is the greater interest attached to these, as there is reason to believe that many of them have become extremely rare, if not wholly extinct, in consequence of the striking change which has of late years taken place in the face of this county from drainage and enclosure. I have also occasionally benefitted from the information of my friends. Such are the sources from whence I have drawn my materials. Under many of the species I have inserted, from personal observation, a few remarks illustrative of their habits and manners; particularly distinguishing such as are indigenous from such as are only periodical or occasional visitants. Possibly some of these observations are not new, but it is perhaps of advantage to the science to have them confirmed in different parts of the country. I have judged it unnecessary in a paper of this nature to give any synonyms or full descriptions, but have annexed to each species a reference to Temminck, in whose excellent Manuel d'Ornithologie* ample information on these subjects will be found. In the systematic arrangement, with the exception of one or two instances, I have uniformly followed that author.

ORDER I. Rapaces.

Genus I. FALCO. *Linn.*

Sp. 1. F. peregrinus, *Temm. Man. d'Ornith.* p. 22.

Peregrine Falcon.—There is a specimen of this bird in the Museum of the Cambridge Philosophical Society, which was shot near Cambridge in the spring of 1823. I have since heard of others that have been observed at Coton.

Sp. 2. F. Subbuteo, *Temm. Man. d'Ornith.* p. 25.

Hobby.—A nest of these birds was once found at Cottenham.

Sp. 3. F. Æsalon, *Temm. Man. d'Ornith.* p. 27.

Merlin.—Inserted on the authority of Graves, who in his British Ornithology has figured a specimen which was killed near Cambridge.

* I refer to the second edition in two volumes octavo, published at Paris in 1820.

Sp. 4. F. Tinnunculus, ***Temm. Man. d'Ornith.*** p. 29.

Kestril.—This is by far the most common hawk we have. The nest, which consists of little else than a few sticks loosely put together, is often placed on the tops of the tallest spruce firs. The eggs are four or five in number, of a reddish brown colour, stained with darker spots and blotches. These are hatched the latter end of April.

Sp. 5. F. Nisus, ***Temm. Man. d'Ornith.*** p. 56.

Sparrow-Hawk.—The males of this species occur much less frequently than the females.

Sp. 6. F. Milvus, ***Temm. Man. d'Ornith.*** p. 59.

Kite.—Not so abundant as the two preceding species.

Sp. 7. F. Buteo, ***Temm. Man. d'Ornith.*** p. 63.

Common Buzzard.

Sp. 8. F. Lagopus, ***Temm. Man. d'Ornith.*** p. 65.

Rough-legged Buzzard.—A specimen of this bird, shot in the vicinity of Cambridge, is in the collection of Dr. Thackeray, Provost of King's College.

Sp. 9. F. rufus, ***Temm. Man. d'Ornith.*** p. 69.

Moor Buzzard. — This species is entirely confined to the fens and low grounds, in which situations however it is very plentiful, building its nest amongst the tall grass and rushes. I have had the newly fledged young brought me from Burwell fen, the second week in May: these have uniformly wanted the yellow patch on the crown of the head, so conspicuous in the adult bird. It is at all times a variable species with respect to plumage, being sometimes found with the lower half of the abdomen entirely white, and the other parts of the body here and there spotted with that colour. This and the Common Buzzard bear indiscriminately the provincial name of ***Puttock.***

Sp. 10. F. cyaneus, ***Temm. Man. d'Ornith.*** p. 72.

Hen Harrier, Male.
Ring-Tail, Female. } This species seems also to be most partial to marshy districts; at least it always breeds in such situations, placing its nest on the ground. In the young birds the difference of plumage between the two sexes is not discernible.

Genus II. STRIX. *Linn.*

* With ears.

Sp. 11. S. Otus, *Temm. Man. d'Ornith.* p. 102.

Long-eared Owl.—This is a rare species. Some years ago a female was taken out of a hollow tree at Bottisham, and was kept alive for a few days, during which time it layed one egg of a dull white colour. It has this year (1825) been shot at Swaffham Prior.

Sp. 12. S. Brachyotos, *Temm. Man. d'Ornith.* p. 99.

Short-eared Owl.—This is only seen with us during the autumnal and winter months, retiring northward in the spring to breed. Though unknown in many parts of England, it is not uncommon throughout the low grounds of Cambridgeshire, where it makes its first appearance towards the latter end of September. I have been informed that in the fens, in the neighbourhood of Littleport, these birds are sometimes found in astonishing plenty, particularly after those seasons which have been most productive of field mice, which appear to be their favourite food and a great object of attraction. In those districts they are known by the name of *Norway Owl*, being supposed to come over to us from that country*. Their usual haunts are fields of coleseed and turnips, in which situations they may often be put up one after another to the number of fifty or more; but they are never observed in stubbles or amongst trees during the day, though they resort to these last to roost at night, and at such times seem much attached to plantations of spruce firs.

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* Montagu in his Ornithological Dictionary appears to have been of the same opinion.

* * With smooth heads.

Sp. 13. S. flammea, *Temm. Mann. d'Ornith.* p. 91.

White Owl.—The food of this species is entirely confined to fresh field mice, which are devoured whole. During the breeding season, which continues throughout the summer, I have observed that it will often catch shrew mice, and bring them home to its young, but it is worthy of note that these were uniformly rejected afterwards, (probably on account of their strong musky odour,) and might be found entire at the foot of the nest. In one instance that occurred at Ely, I noticed amongst these rejectamenta a mutilated specimen of the rare species, the Watershrew (Sorex fodiens).

Sp. 14. S. Aluco, *Temm. Man. d'Ornith.* p. 89.

Brown Owl. — Unlike the preceding this is a very general feeder, preying upon rats, moles, rabbits, small leverets, &c. and is consequently destructive to game. It builds in old trees, and is a very early breeder, frequently hatching by the end of March. It is the only British species that hoots.

ORDER II. Omnivori.

Genus III. CORVUS. *Linn.*

Sp. 15. C. Corax, *Temm. Man. d'Ornith.* p. 107.

Raven.—Not so plentiful as formerly.

Sp. 16. C. Corone, *Temm. Man. d'Ornith.* p. 108.

Carrion Crow.

Sp. 17. C. Cornix, *Temm. Man. d'Ornith.* p. 109.

Royston Crow. — Plentiful on our downs from October to April.

Sp. 18. C. frugilegus, *Temm. Man. d'Ornith.* p. 110.

Rook. — Varieties of this bird, more or less spotted with grey and white, not unfrequently occur at Bottisham. There are two specimens of this kind from that neighbourhood, in the Museum of the Cambridge Philosophical Society.

Sp. 19. C. Monedula, *Temm. Man. d'Ornith.* p. 111.

Jackdaw.

Sp. 20. C. Pica, *Temm. Man. d'Ornith.* p. 113.

Magpie.

Sp. 21. C. glandarius, *Temm. Man. d'Ornith.* p. 114.

Jay.—This is a rare bird at Bottisham, and only an occasional visitant of that district; though more plentiful in the woodlands. At Gamlingay I have observed them in abundance.

Genus IV. BOMBYCIVORA. *Temm.*

Sp. 22. B. garrula, *Temm. Man. d'Ornith.* p. 124.

Bohemian Wax-wing.—I have been informed that some few years back two flights of these birds were at different times observed near Cambridge.

Genus V. STURNUS. *Linn.*

Sp. 23. S. vulgaris, *Temm. Man. d'Ornith.* p. 132.

Starling.—Towards autumn these birds congregate in immense flocks.

ORDER III. Insectivori.

Genus VI. LANIUS. *Linn.*

Sp. 24. L. Excubitor, *Temm. Man. d'Ornith.* p. 142.

Cinereous Shrike.—Has been observed to visit Cambridgeshire in the autumnal and winter months. There is a specimen in the collection of Dr. Thackeray, which was found dead at Melbourn in the year 1824, and was supposed to have been recently killed by a hawk.

Sp. 25. L. Collurio, *Temm. Man. d'Ornith.* p. 147.

Red-backed Shrike.—This species has occasionally been shot at Cherry-Hinton.

Genus VII. MUSCICAPA. *Linn.*

Sp. 26. M. grisola, *Temm. Man. d'Ornith.* p. 152.

Spotted Fly-catcher.—This is one of our latest summer visitants, never appearing before the middle, and often not till the end, of May. Its food consists entirely of insects, taken on the wing. The method which it adopts for this purpose is somewhat singular, and as I believe peculiar to itself. Taking its station generally on the top of a post, it watches till an insect passes by, when it suddenly darts forward, hovers for a moment in order to secure its prey, and then returns to the same spot again. This operation it will often repeat for a considerable length of time without changing its place.

Genus VIII. TURDUS. *Linn.*

Sp. 27. T. viscivorus, *Temm. Man. d'Ornith.* p. 160.

Missel Thrush.—Tolerably plentiful in the neighbourhood of Bottisham, where it remains all the year round. Its song is very powerful, and in mild weather is often heard as early as the beginning of January, but wholly ceases by the end of May.

Sp. 28. T. pilaris, *Temm. Man. d'Ornith.* p. 163.

Fieldfare.—This is one of our winter visitants. It is seldom seen before November, but it remains with us till very late in the spring. At Anglesea Abbey in particular, I have for some years back noticed individuals as late as the middle of May.

Sp. 29. T. musicus, *Temm. Man. d'Ornith.* p. 164.

Song-Thrush.—Sings from the end of January to the middle of July.

Sp. 30. T. iliacus, *Temm. Man. d'Ornith.* p. 165.

Redwing.—Migratory like the Fieldfare; but generally preceding that species in its arrival.

Sp. 31. T. torquatus, *Temm. Man. d'Ornith.* p. 166.

Ring-Ouzel.—I have been informed that a single bird of this species was shot on the borders of the county, near Great Chesterford, but I could not learn at what season of the year.

Sp. 32. T. Merula, *Temm. Man. d'Ornith.* p. 168.

Blackbird.—Sings from the beginning of February to the end of July.

Genus IX. SYLVIA. *Lath.*

Sp. 33. S. Phragmitis, *Temm. Man. d'Ornith.* p. 189.

Sedge-warbler.—This and all the other species of this genus, with the exception of the Redbreast (S. Rubecula,) are birds of passage, appearing with us in the spring and departing either before or at the approach of autumn. The Sedge-warbler is first seen the last week in April. It is very plentiful throughout the fens and low grounds of Cambridgeshire, especially where there are osiers and other covert, in which situations it remains closely concealed, rarely exposing itself to view. The nest is suspended at a small height from the ground between the stems of the Arundo Phragmites. During the breeding season it sings incessantly night and day in a somewhat hurried and confused manner, often imitating the notes of other birds.

Sp. 34. S. Luscinia, *Temm. Man. d'Ornith.* p. 195.

Nightingale.—This species is seldom heard with us before the 16th of April. After the young broods are hatched, which usually takes place by the end of the first week in June, its song wholly ceases.

Sp. 35. S. atracapilla, *Temm. Man. d'Ornith.* p. 201.

Blackcap.—The note of this bird much resembles, and is only inferior to, that of the Nightingale. It is usually first heard about the middle of April, but in very mild seasons I have noticed it as early as the 29th of March. It continues in full song till August.

Sp. 36. S. hortensis, *Temm. Man. d'Ornith.* p. 206.

Greater Pettychaps.—This species is not very unfrequent in gardens, copses, and high hedges; though more plentiful some years than others. Its note is soft, possessing much variety, and particularly pleasing; but the individual which utters it, from its extreme shyness and its manner of concealing itself in the thickest parts of the wood, is not often seen. I never heard it before the 1st of May nor after the 18th of July.

Sp. 37. S. cinerea, *Temm. Man. d'Ornith.* p. 207.

Whitethroat.—Towards the end of April, this species resorts to our hedges in great quantities, where it must often have attracted notice by its very peculiar manners. For the most part it sings concealed, but every now and then it may be observed to rise suddenly from its retreat to a considerable height in the air, and without desisting from its song, to shoot about with some rapidity, accompanying its flight all the while with singular jerks and gesticulations of the wing. After continuing these movements for a greater or less interval, it returns slowly to the bush from whence it sprung, and resumes its former station.

I cannot forbear mentioning in this place, that I have at different times been much inclined to suspect, that under the name of Whitethroat, there have been two species hitherto confounded together. What has chiefly led me to this opinion, is the circumstance of my having occasionally noticed amongst these birds certain individuals, which not only differed strikingly from the above in habits and manners, but also in note, and which invariably preceded the others in their arrival by a week or a fortnight. This year in particular, I observed some of these last as early as the first week in April. Their haunts were much the same as those of the common sort, being generally in thick hedges and close copses of underwood: in these situations however they were oftener heard than seen, as they always sculked about in the most concealed spots, and never rose into the air with that peculiarity of gesture which I have attempted to describe above. Their song too was very different, being much superior to that of the common sort, more melodious and varied in its notes, though so soft and inward as to be scarcely noticed unless near: moreover, this was never exerted on wing. That these birds are really distinct from the others, I will not at present presume to decide, as I have not hitherto had an opportunity of comparing specimens of each sort together, which would afford the only means of detecting a specific difference if such exist between them. I find myself however somewhat corroborated in my suspicions, by the following observation of Montagu. In his Ornithological Dictionary, (Art. Whitethroat) he mentions having more than once killed a bird whose plumage differed in some respects from that of the common Whitethroat, and in one instance from off the nest, which contained four eggs almost entirely white, not nearly so much speckled with brown and ash-colour as those of this bird generally are: and whose weight was also greater. He confesses himself to have been much puzzled on this occasion, and concludes by hinting at the possibility of its being proved hereafter that there are two distinct species.

Sp. 38. S. Curruca, *Temm. Man. d'Ornith.* p. 209.

Lesser Whitethroat.—This bird, the Lesser Whitethroat of Latham and Montagu, corresponds so exactly in every particular with the Sylvia Curruca of Temminck, that I have accordingly referred it to that species, though it is very doubtful whether it be the Motacilla Curruca of Linnæus. In this country it does not appear to have been generally noticed, nor at all known till Latham first described it in the Supplement to his Synopsis, which circumstance is probably owing to its being of very local occurrence, and almost entirely confined to the eastern parts of the kingdom. In Cambridgeshire, it is far from uncommon, making its first appearance in the last week of April. Like the rest of its tribe it is extremely shy and very difficult to get sight of, though when near easily recognized by its note, which consists of a shrill shivering cry repeated at intervals from the thickest parts of the wood. It resides for the most part in copses and gardens, building its nest in some low shrub at the height of about four feet from the ground. This is of a very loose and flimsy structure, and composed of dry bents with the addition of a small quantity of wool placed in patches on its exterior surface; within, it is lined with a scanty supply of white hairs. The eggs are five in number, white, spotted chiefly towards the greater end with small dots of brown, and larger irregular stains of the same colour. Incubation commences about the 20th of May, and the young broods are fledged in June, but the note of the parent birds is continued till the middle or even till the end of July. Montagu has stated very accurately the several points of difference between this species and the preceding, which, if attended to, will always serve to distinguish them from each other. Latham's figure, in his first supplement, is incorrect in representing the upper parts of the plumage of a deep brown, whereas they are wholly cinereous.

Sp. 39. S. Rubecula, *Temm. Man. d'Ornith.* p. 215.

Redbreast.—This species continues in song the whole year round, excepting in times of severe frost.

Sp. 40. S. Phœnicurus, *Temm. Man. d'Ornith.* p. 220.

Redstart.—A very abundant species throughout Cambridgeshire where it arrives the middle of April. It is particularly constant in the time of its first appearance, perhaps more so than any other bird, as I do not ever remember to have noticed its arrival before the twelfth or later than the sixteenth of this month.

Sp. 41. S. Hippolais, *Temm. Man. d'Ornith.* p. 222.

Lesser Pettychaps.—Of all our summer visitants this is undoubtedly the earliest, often arriving by the middle, or at latest by the end of March. Although I have generally observed it to be diffused in tolerable plenty over most other parts of the county, yet, in the neighbourhood of Bottisham, it is of very uncertain appearance, as in some seasons not a single individual is seen there, whilst in others they are abundant. It is a restless and an active bird, and is much attached to spruce firs and other tall trees, from the tops of which it issues its incessant but monotonous song, consisting only of two loud piercing notes, which it continues throughout the summer and even till late in September. By this and by its early arrival, it may readily be distinguished from the following species, but as far as respects plumage, the two are so extremely similar, that it is difficult to discriminate between dead specimens. Most authors represent this as being of less size and of a paler colour in its under parts, but I am of opinion, that little reliance can be placed on these marks, as from an examination of a great many specimens of each, I have found them very variable. The only constant character that I have observed, resides in the colour of the legs, which in this are dusky, whereas in the following they are pale brown.

Sp. 42. S. Trochilus, *Temm. Man. d'Ornith.* p. 224.

Willow Wren.—This is a great deal more plentiful than the preceding, and not so much confined to large trees and woods, being a general inhabitant of hedges, underwood, and a variety of other situations. It appears about the same time as the Redstart, and, as is the case with many of this tribe, the males invariably precede the females, by an interval of several days. Its song consists of seven or eight notes which are modulated in a soft and particularly pleasing, though somewhat plaintive, manner. This is continued without intermission during the breeding season, but generally ceases by the beginning of July.

Genus X. REGULUS. *Cuv.*

Sp. 43. R. aurocapillus, *Selby.*

Sylvia Regulus, *Temm. Man. d'Ornith.* p. 229.

Golden-crowned Wren.—These birds from their diminutive size and solitary habits are not often noticed, and may be easily overlooked, but I believe them to be very plentiful wherever there are plantations of spruce firs, to which trees they seem extremely partial, hanging their nests to the under-surface of the lower branches. Though apparently of so delicate a nature, they remain with us all the winter, and appear to suffer less from severe cold than even many of our hard-billed species. It is not at all improbable that at this season they may derive their chief support from the smaller tribes of Tipulidæ, many of which are to be found on wing and in a state of activity at all times of the year, and even occasionally when the ground is covered with snow.

Genus XI. TROGLODYTES. *Cuv.*

Sp. 44. T. europæus, *Cuv.*

Sylvia Troglodytes, *Temm. Man. d'Ornith.* p. 233.

Common Wren.—Like the Robin, this bird sings throughout the year, but its note in the winter months is very weak compared to what it is in the spring.

Genus XII. SAXICOLA. *Bechst.*

Sp. 45. S. Œnanthe, *Temm. Man. d'Ornith.* p. 237.

Wheat-ear.—I have occasionally observed these birds on the Devil's Ditch and the open parts about Newmarket heath, but from their not being in any great plenty, I am unable to say at what period of the year they first visit those districts, or when they withdraw. They breed on the first-mentioned place, depositing their nest in an old rabbit-burrow, or some other hole under ground.

Sp. 46. S. rubetra, *Temm. Man. d'Ornith.* p. 244.

Whin-chat.—Like the preceding a bird of passage, appearing in the middle of April, and departing in the autumn.

Sp. 47. S. rubicola, *Temm. Man. d'Ornith.* p. 246.

Stone-chat.—This is plentiful, and resides with us all the year on fens and other open grounds.

Genus XIII. ACCENTOR. *Bechst.*

Sp. 48. A. alpinus, *Temm. Man. d'Ornith.* p. 248.

Alpine Accentor.—The discovery of this addition to the Ornithology of Great Britain is due to Dr. Thackeray, who observed a pair of these birds in the open space immediately under the east window of King's College chapel, on the twenty-third of November, 1822: one of them, which proved to be a female, was shot, and is at present in his collection. I am not aware that any others have since been met with in this country, where indeed it can only be looked upon as an accidental visitant. According to Temminck its native haunts are the Swiss Alps and the mountainous parts of Germany and France.

Sp. 49. A. modularis, *Temm. Man. d'Ornith.* p. 249.

Hedge-Accentor.—One of the few soft-billed birds that remain with us the whole year, singing at all seasons if the weather be mild.

Genus XIV. MOTACILLA. *Linn.*

Sp. 50. M. alba, *Temm. Man. d'Ornith.* p. 255.

Pied Wagtail.—I had often observed that we see greater numbers of these birds in the autumn than in any other season of the year, but was not aware of the cause till I learnt from Selby's Illustrations of British Ornithology, that in the north of England this species is a regular migrant, retiring southward in October, and not re-appearing till February or the beginning of March. This circumstance renders it highly probable that at the time above-mentioned the birds of our own neighbourhood are joined by those which arrive from the higher parts of the country.

Sp. 51. M. Boarula, *Temm. Man. d'Ornith.* p. 257.

Grey Wagtail.—This is the least plentiful of the three British species of Wagtail, and is only seen in Cambridgeshire during the autumnal and winter months, appearing first in October or earlier. In the spring it retires northward to breed. About Bottisham I have noticed it most frequently in January.

Sp. 52. M. flava, *Temm. Man. d'Ornith.* p. 260.

Yellow Wagtail. — This species visits us in the spring and departs in the autumn. It does not appear to be uncommon in many parts of the county, though much more so than the Motacilla alba. I have occasionally seen them in considerable plenty upon the arable lands bordering on Bottisham and Swaffham fens, and likewise in the low meadows about Quy Water.

Genus XV. ANTHUS. *Bechst.*

Sp. 53. A. pratensis, *Temm. Man. d'Ornith.* p. 269.

Tit-pipit.—Equally abundant on the low and fenny as well as on the high and heathy parts of the county, in which situations it is to be found all the year. In the autumn it appears to be subject to a considerable change of plumage, from which circumstance some authors have erroneously made two species of this bird.

Sp. 54. A. arboreus, *Temm. Man. d'Ornith.* p. 271.

Tree-pipit.—This species very strongly resembles the last in plumage, but may always be distinguished by the curvature of the hind claw, and the greater dilatation of the bill towards its base. In its haunts it is widely different, being entirely confined to woods and plantations of tall trees, and never frequenting the open parts of the country; nor does it remain with us through the winter, but makes its first appearance about the third week in April, and departs at the approach of autumn. Its song, which is delivered on wing in its descent, is heard till the middle of July.

ORDER IV. Granivori.

Genus XVI. ALAUDA. *Linn.*

Sp. 55. A. arvensis, *Temm. Man. d'Ornith.* p. 281.

Sky-lark.—These birds get together in small companies at the approach of winter, but the flocks are not considerable except in very severe weather.

Genus XVII. PARUS. *Linn.*

Sp. 56. P. major, *Temm. Man. d'Ornith.* p. 287.

Great Titmouse.—In hard weather this and the two following species leave their native woods and resort to the immediate vicinity of dwelling-houses, in order to avail themselves of what they can pick up. At such times I have observed that they will devour flesh with greediness, and may be caught in great numbers by a trap baited with suet.

Sp. 57. P. cœruleus, *Temm. Man. d'Ornith.* p. 289.

Blue Titmouse.

Sp. 58. P. palustris, *Temm. Man. d'Ornith.* p. 291.

Marsh Titmouse.

Sp. 59. P. ater, *Temm. Man. d'Ornith.* p. 288.

Cole Titmouse.—Less frequent with us than any of the other species, though probably often overlooked from its strong resemblance to the preceding. It may however be easily distinguished by its peculiar note independently of other characteristic marks.

Sp. 60. P. caudatus, *Temm. Man. d'Ornith.* p. 296.

Long-tailed Titmouse.—Very common in woods, constructing its singular nest in cedars, small firs, and trees of that kind. The young broods do not disperse when fledged, but follow the parent-birds through the autumn and winter.

Genus XVIII. EMBERIZA. *Linn.*

Sp. 61. E. citrinella, *Temm. Man. d'Ornith.* p. 304.

Yellow Bunting.—In some places this species is known by the name of *Writing Lark*, from the peculiar markings on the egg, which have somewhat the appearance of written characters.

Sp. 62. E. Miliaria, *Temm. Man. d'Ornith.* p. 306.

Common Bunting.—These birds being much attached to open cultivated ground and extensive corn lands, are extremely plentiful in Cambridgeshire where they are called Bunting Larks. Towards the approach of winter they collect together in large flocks, and do not separate till the ensuing spring.

Sp. 63. E. Schœniculus, ***Temm. Man. d'Ornith.*** p. 307.

Reed Bunting.—Common in fens and low meadows, but confined to such situations. As far as I have observed, the nest is always placed on the ground, and never suspended between the stems of aquatic plants, as described by Bewick and some other authors, who have strangely confounded the manners of this bird with those of the Sedge Warbler (Sylvia Phragmitis.) This error has probably arisen from the circumstance of the two species frequenting the same haunts, and being in a general way both called *Reed Sparrows.*

Genus XIX. PYRRHULA, *Briss.*

Sp. 64. P. vulgaris, ***Temm. Man. d'Ornith.*** p. 338.

Bulfinch.—This is generally reckoned a very common bird; but I have rarely noticed it in the neighbourhood of Bottisham. Perhaps it is attached to more wooded districts.

Genus XX. FRINGILLA, *Illig.*

Sp. 65. F. Chloris, ***Temm. Man. d'Ornith.*** p. 346.

Greenfinch.—Collect together in large flocks in the winter.

Sp. 66. F. domestica, ***Temm. Man. d'Ornith.*** p. 350.

House Sparrow.—White varieties of this bird have been occasionally observed near Bottisham.

Sp. 67. F. montana, ***Temm. Man. d'Ornith.*** p. 354.

Tree Sparrow.—I have inserted this species on the authority of Selby, who, in his Illustrations of British Ornithology, mentions having received specimens from the neighbourhood of Cambridge.

Sp. 68. F. cœlebs, ***Temm. Man. d'Ornith.*** p. 357.

Chaffinch.—This species continues in full song from the first week in February to the end of June, after which time it is silent till September, when it reassumes its note for a few weeks if the weather be mild. As far as I have observed, both sexes remain with us all the year, and do not appear to separate at the approach of winter, as they are said to do in other parts of England.

Sp. 69. F. cannabina, *Temm. Man. d'Ornith.* p. 364.

Common Linnet.—These birds begin to assemble in flocks about the middle of October, which increase in numbers as the weather becomes more severe.

Sp. 70. F. Spinus, *Temm. Man. d'Ornith.* p. 371.

Siskin.—Only an occasional visitant during the winter months. Large flocks appeared at Bottisham the last week in January of the present year (1825) and many specimens were killed both male and female *.

Sp. 71. F. Carduelis, *Temm. Man. d'Ornith.* p. 376.

Goldfinch.

ORDER V. Zygodactyli.

Genus XXI. CUCULUS, *Linn.*

Sp. 72. C. canorus, *Temm. Man. d'Ornith.* p. 381.

Cuckow.—This species visits us in the middle of April, and is heard till the beginning of July, when it again departs; but the young birds appear to remain for a much longer period, as I have occasionally observed them in September. I never found the egg of this bird myself, but have seen one which was taken at Great Swaffham from the nest of a Hedge Accentor, (Accentor Modularis.)

Genus XXII. PICUS, *Linn.*

Sp. 73. P. viridis, *Temm. Man. d'Ornith.* p. 391.

Green Woodpecker.

* This species was also noticed in Cambridgeshire by Turner. See his work, *De Avibus*, p. 56.

Sp. 74. P. major, *Temm. Man. d'Ornith.* p. 395.

Great Spotted Woodpecker.—Much less common than the preceding, but has been occasionally shot at Bottisham.

Sp. 75. P. minor, *Temm. Man. d'Ornith.* p. 399.

Lesser Spotted Woodpecker. — I have at different times known several instances in which this bird has been met with in Cambridgeshire, but it must be esteemed a rare species. The last specimen which occurred to my knowledge was shot at Anglesea Abbey in March 1824, and is now in the Museum of the Cambridge Philosophical Society.

Genus XXIII. YUNX, *Linn.*

Sp. 76. Y. Torquilla, *Temm. Man. d'Ornith.* p. 403.

Wryneck.—A few of these birds visit us regularly every spring, but they are never plentiful.

ORDER VI. Anisodactyli.

Genus XXIV. SITTA, *Linn.*

Sp. 77. S. europæa, *Temm. Man. d'Ornith.* p. 407.

Nuthatch.—Not uncommon in the neighbourhood of Bottisham. During a certain portion of the year, these birds feed chiefly upon nuts which they break with their bill, after having firmly fixed them in the crevices of the bark of trees. For this purpose they appear to resort frequently to the same spots, as I have observed some old trees in particular whose clefts are full of broken shells, whilst in others not one is to be seen.

Genus XXV. CERTHIA, *Illig.*

Sp. 78. C. familiaris, *Temm. Man. d'Ornith.* p. 410.

Common Creeper.—This bird frequently builds its nest under the loose and decayed bark of old trees; it consists of little else than a few twigs and small sticks piled rudely together with a layer of feathers upon the top of them. The eggs are very numerous, often as many as nine or ten.

ORDER VII. Alcyones.

Genus XXVI. ALCEDO, *Linn.*

Sp. 79. A. Ispida, *Temm. Man. d'Ornith.* p. 423.

Common King's-Fisher.—Tolerably plentiful in the neighbourhood of streams and clear waters. During its flight, which is very rapid, it utters a shrill piercing note that may be heard to a great distance.

ORDER VIII. Chelidones.

Genus XXVII. HIRUNDO, *Linn.*

Sp. 80. H. rustica, *Temm. Man. d'Ornith.* p. 427.

Chimney Swallow.—The arrival of this species in the neighbourhood of Bottisham usually takes place about the fifteenth of April, as I have found by many years' observation, but has been occasionally deferred till the twenty-second, which is the latest that I ever noticed. The first broods are fledged early in August, and towards the middle of that month they begin to collect into large flocks, which increase in numbers as the season advances and the time of departure draws near. This, with respect to the majority, takes place in the beginning of October, but stragglers may be seen a week or two longer. I once observed a white variety of this bird at Ely.

Sp. 81. H. urbica, *Temm. Man. d'Ornith.* p. 428.

House Martin.—This appears about a week after the swallow, but is seldom in great plenty before the beginning of May. It however remains with us later than that species, and is occasionally seen through the first week in November *, though the greater part withdraw before that time. Previously to migration they congregate upon the roofs of houses and churches.

Sp. 82. H. riparia, *Temm. Man. d'Ornith.* p. 429.

Sand Martin.—The only places where I have hitherto observed these birds in Cambridgeshire, are the chalky banks by the side of the road near Quy Water, and some gravel-pits in the neighbourhood of Bourn Bridge. In the former of these situations I have noticed them regularly every year, and have found the time of their arrival to be about the middle of April, but I am unable to say when they leave us, though I am inclined to suspect that this takes place at a much earlier period than with either of the preceding species.

Genus XXVIII. CYPSELUS, *Illig.*

Sp. 83. C. murarius, *Temm. Man. d'Ornith.* p. 434.

Swift.—This species is by far the latest in its arrival of all the Swallow tribe, as I never remember to have seen it before the seventh of May. Generally speaking it is equally remarkable for its early departure, withdrawing from most places by the beginning of August, and often by the end of July. A singular exception, however, to this last mentioned circumstance takes place with respect to these birds in the neighbourhood of Ely, where the bulk of them hardly ever retire till quite the end of August, and a few individuals may often be observed through the first week in September †. From what cause they are induced to pro-

* Dr. Thackeray informs me that he has known three or four individuals stay about the south side of Clare Hall till the eighteenth of this month.

† A single bird has also been noticed by Dr. Thackeray about King's College Chapel for many successive days during the early part of September.

tract their stay at that place so much beyond its usual limit I am unable to say, but the fact itself I regularly noticed during a period of several years that I was in the habit of residing there for the summer months. Possibly they may in some measure be influenced by the cathedral and other old buildings adjacent, in the holes and crannies of which these birds meet with a retreat peculiarly congenial to their habits, as appears by the immense numbers that annually resort thither in the early part of the season *.

GENUS XXIX. CAPRIMULGUS, *Linn.*

SP. 84. C. europæus, *Temm. Man. d'Ornith.* p. 436.

GOATSUCKER.—I have occasionally observed these birds about Ely, and also in the neighbourhood of Bottisham, but at the last mentioned place they have not of late years appeared in such plenty as formerly. Like the rest of this order they are migratory, arriving about the beginning of June and departing in September. In the dusk of the evening they utter a singular chattering noise somewhat resembling that of a spinning-wheel, by which they may easily be distinguished. From an examination of the stomach, their food appears to consist of the larger night-flying Phalænæ, particularly those belonging to the Linnæan section Noctua, and the various species of Phryganea. It is also probable that during a short part of the season they derive much of their support from the Midsummer Dor (Melolontha solstitialis) as I have seen them hawking about in places where these insects were abundant.

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* White, in his Natural History of Selborne, observes that swallows are seen later at Oxford than elsewhere, and enquires whether it may not be owing to the vast massy buildings of that place. See his twenty-third Letter to Pennant.

ORDER IX. Columbæ.

Genus XXX. COLUMBA, *Linn.*

Sp. 85. C. Palumbus, *Temm. Man. d'Ornith.* p. 444.

Ring-Dove. — These birds are exceedingly abundant in Cambridgeshire, where they do an incredible deal of mischief by devouring pease, beans and other leguminous plants. They are well known by their cooing notes which are heard incessantly from February to October. After that time they begin to collect together into enormous flocks, which disperse themselves over the country during the day-time to feed, but return regularly home in the evening to roost in their native woods and plantations. Some of these flocks do not wholly separate till very late in the spring, though the greater part pair off for the purpose of breeding by the beginning of March. In the Autumn I have observed that they subsist chiefly upon acorns and beech-mast.

Sp. 86. C. Œnas, *Temm. Man. d'Ornith.* p. 445.

Stock-Dove.—White, in his Natural History of Selborne, mentions the Stock-Dove as being seen there during the winter months only, appearing in large flocks about the end of November, and departing in February. Whatever may be the case with respect to these birds in the southern counties, with us they certainly remain the whole year, as I have noticed them at all seasons and repeatedly found their nests. They are considerably less plentiful than the Ring-Dove, but have much the habits of that species with which they frequently associate in hard weather. Like them they breed very early in the spring. The nest which is flat and shallow, consists merely of a few sticks put loosely together in the hollow of some old tree. The eggs are two in number, white like those of the Ring-Dove, but somewhat smaller and rather more rounded. As far as I have observed, the Stock-Dove never cooes, but utters only a hollow rumbling note during the breeding season, which may be heard to a considerable distance. Montagu in his Ornithological Dictionary has evidently confounded this species with the Rock-Dove, (Columba livia, Temm.) which is supposed to be the origin of our dove-house pigeon, and is found in a wild state upon some of the steep shores and cliffs of Great Britain, but is not a native of Cambridgeshire. The Stock-Dove and Ring-Dove are indiscriminately called *Woodpigeons* by the country people.

Sp. 87. C. Turtur, *Temm. Man. d'Ornith.* p. 448.

Turtle-Dove.—Some few individuals of this species visit the plantations in the neighbourhood of Bottisham regularly every spring, and are first seen towards the latter end of May, but they are never numerous, and do not stay with us long, departing again soon after the breeding season is over. The young birds, however, appear to remain for a longer period, as I have had them shot in the month of September. I have also noticed this species at Stetchworth and Wood-Ditton.

ORDER X. Gallinæ.

Genus XXXI. PHASIANUS, *Linn.*

Sp. 88. P. colchicus, *Temm. Man. d'Ornith.* p. 453.

Common Pheasant.—Instances have now and then occurred at Bottisham in which the hen of this species had partially assumed the plumage of the cock. This singular change has only been observed in individuals which had reached an advanced age. Such are termed by sportsmen *mule-birds*.

Genus XXXII. PERDIX, *Lath.*

Sp. 89. P. rubra, *Temm. Man. d'Ornith.* p. 485.

Red-legged Partridge.—One of these birds was shot near Anglesea Abbey on the twenty-seventh of September 1821, and is at present in my possession.

Sp. 90. P. cinerea, *Temm. Man. d'Ornith.* p. 488.

Common Partridge.—A covey of these birds were bred in the neighbourhood of Clayhithe, of which a considerable number were perfectly white.

Sp. 91. P. Coturnix, *Temm. Man. d'Ornith.* p. 491.

Quail.—This species is one of our latest summer visitants as I have seldom noticed it before the beginning of June; but it remains with us till the end of October. In the present year (1825) two specimens were killed at Bottisham so late as on the eleventh of November.

ORDER XI. Cursores.

Genus XXXIII. OTIS, *Linn.*

Sp. 92. O. Tarda, *Temm. Man. d'Ornith.* p. 506.

Great Bustard.—Formerly these birds were plentiful in the open tracts about Newmarket Heath, and till within a few years single individuals have occasionally been seen in that neighbourhood, but they are supposed to be now almost extinct. Ray and Willoughby mention also Royston Heath as a place frequented in their time by this species.

Sp. 93. O. Tetrax, *Temm. Man. d'Ornith.* p. 507.

Little Bustard.—Bewick has figured a specimen of this bird which was taken alive on the edge of Newmarket Heath. So far as I am aware no other instance has occurred of its having been met with in this county.

ORDER XII. GRALLATORES.

GENUS XXXIV. OEDICNEMUS, *Temm.*

SP. 94. O. crepitans, *Temm. Man. d'Ornith.* p. 521.

STONE-CURLEW.—This species does not appear to be plentiful in these districts. I have seen specimens that were killed in the vicinity of Cambridge, and am informed by Dr. Thackeray, that about two years since he had brought him a young bird which was bred very near that place, but I never observed any myself. It is migratory, and only met with during the summer months.

GENUS XXXV. CHARADRIUS, *Linn.*

SP. 95. C. pluvialis, *Temm. Man. d'Ornith.* p. 535.

GOLDEN PLOVER.—Common in the fens as well as the high lands, but appear to breed generally in the last mentioned situations.

SP. 96. C. Morinellus, *Temm. Man. d'Ornith.* p. 537.

DOTTEREL.—This baits with us for a short time in its passage to and from the North where it probably breeds, being seen here in the spring and autumn only. The largest flocks occur about the middle of September. They frequent the same situations as the preceding species.

SP. 97. C. Hiaticula, *Temm. Man. d'Ornith.* p. 539.

RINGED PLOVER.—Great quantities of these birds appeared in Bottisham and Swaffham fens in the months of June and July 1824, which was a remarkably wet season. They are by no means regular visitants of those districts.

Montagu asserts in the Supplement to his Ornithological Dictionary that this species resides on the sea shore the whole year, but from the above circumstance it is probable that they occasionally retire inland to breed.

Genus XXXVI. VANELLUS, *Briss.*

Sp. 98. V. cristatus, *Temm. Man. d'Ornith.* p. 550.

Lapwing.—A very abundant species. In the autumn they collect into large flocks.

Genus XXXVII. GRUS, *Pallas.*

Sp. 99. G. cinerea, *Temm. Man. d'Ornith.* p. 557.

Crane.—In the time of Ray,* these birds appear to have visited our fens in large flocks regularly during the winter months, but they have long since deserted them; nor is it likely, from the altered state of the country in consequence of the improved system of drainage which is now carried on, that they will ever return thither. According to Pennant (*Brit. Zool.* Vol. II. p. 629.) a single specimen was killed near Cambridge about the year 1773. This I believe to be the latest instance on record in which the species has been met with.

Genus XXXVIII. ARDEA, *Linn.*

Sp. 100. A. cinerea, *Temm. Man. d'Ornith.* p. 567.

Heron.—In hard weather this species resorts to those brooks and running streams which seldom or never freeze, and at such times is met with in great numbers; but in other seasons it is only occasionally noticed in the neighbourhood of Bottisham, there being no place within a considerable distance where these birds are known to breed.

Sp. 101. A. stellaris, *Temm. Man. d'Ornith.* p. 580.

Bittern.—These birds are met with in Burwell fen and occasionally on the moors about Cambridge, but they appear to be getting more scarce every year. Formerly they were plentiful.

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* "In palustibus Lincolniensibus et Cantabrigiensibus magni horum greges hyberno tempore inveniuntur." *Raii Syn. Meth. Avium,* p. 95. Art. The *Crane.*

Genus XXXIX. RECURVIROSTRA, *Linn.*

Sp. 102. R. Avocetta, *Temm. Man. d'Ornith.* p. 590.

Scooping Avoset.—I have inserted this species on the authority of Donovan, who, in his History of British Birds, (*Pl.* 66.) speaks of it as being common in the breeding season in the fens of Cambridgeshire. It is very probable that this was formerly the case, when our marshes were more extensive than they are at present, but I have not been able to learn that it is ever met with now.

Genus XL. NUMENIUS, *Briss.*

Sp. 103. N. arquata, *Temm. Man. d'Ornith.* p. 603.

Common Curlew.—Sometimes seen in small flocks about Newmarket Heath.

Sp. 104. N. Phæopus, *Temm. Man. d'Ornith.* p. 604.

Whimbrel.—Has been occasionally exposed for sale in Cambridge.

Genus XLI. TRINGA, *Briss.*

Sp. 105. T. variabilis, *Temm. Man. d'Ornith.* p. 612.

Dunlin.—These birds now and then visit our fens during the summer months, and it is not improbable that they may breed in those situations. In the beginning of July 1824, they were very abundant. Several which were then killed and came under my observation, I found to answer, in most particulars, pretty correctly to Temminck's description of this species, as it appears in its summer plumage. The black, however, on the under parts was very variable in different specimens, some of which were only faintly spotted with this colour, whilst in others the whole of the belly and abdomen were thickly blotched over with large irregular patches of the same, but in no case without some mixture of white. From this last circumstance it is likely, that at the time above-mentioned, the season of incubation was just over, as, according to Temminck, so long as that lasts, the belly is wholly of a deep black, but immediately after a change takes place in the colour of that part, and the white begins to reappear. Those birds which were the darkest in this respect, were also a trifle larger than the others, and had the bill somewhat longer. These last characters indicate I believe the female sex.

Sp. 106. T. minuta, *Temm. Man. d'Ornith.* p. 624.

Little Sandpiper.—I suppose this to be the *Little Sandpiper* of Pennant, (*Brit. Zool.* Vol. II. p. 473.) which he described from a specimen shot near Cambridge in the month of September, and which is the only one I ever heard of. At the same time there is some doubt attached to its identity, as it appears from Temminck, that under that name two species have been often confounded together, the *Tringa Temminckii* and the *Tringa Minuta* of the Manuel d'Ornithologie, and Pennant's description is so short and imperfect, that it is not easy to pronounce with certainty which of these two was before him when it was drawn up, though from the circumstance of the legs being mentioned as black, I am inclined to think it was the latter. Add to this, that I am not aware of the *Tringa Temminckii* having ever been met with in England, except the *Little Sandpiper*, described by Montagu in the Appendix to his Ornithological Dictionary, be of that species, which I think not improbable. That however was shot in Devonshire.

Sp. 107. T. cinerea, *Temm. Man. d'Ornith.* p. 627.

Knot.—According to Montagu these birds were formerly found in the Isle of Ely during the autumnal months, and I have heard of specimens being occasionally met with still.

Sp. 108. T. pugnax, *Temm. Man. d'Ornith.* p. 631.

Ruff, Male. }
Reeve, Female. } This species visits us in the spring, remains through the summer and departs at the approach of autumn, but is much more plentiful some years than others. It is found in the Isle of Ely, and occasionally in Bottisham and Swaffham fens. The male bird loses the ruff so soon as the season of incubation is over.

Genus XLII. TOTANUS, *Bechst.*

Sp. 109. T. fuscus, *Temm. Man. d'Ornith.* p. 639.

I insert this species with considerable hesitation. According to Temminck it is synonymous with the *Scolopax Cantabrigiensis* of Gmelin and Latham, and the *Cambridge Godwit* of Pennant, (*Brit. Zool.* Vol. II. p. 447.) which was originally described by this last Author from a stuffed specimen shot in the vicinity of Cambridge. Since that time it appears to have been a great question whether the *Cambridge Godwit* was admissible as a distinct species. Some have supposed it to be the same with the *Greenshank*, whilst Montagu, in his Supplement to the Ornithological Dictionary, suspects it to be only the young of the *Redshank*. Where the truth lies I shall not presume to decide at present.

Sp. 110. T. Calidris, *Temm. Man. d'Ornith.* p. 643.

Redshank.—These birds were formerly plentiful in the fens, particularly during the summer months, but are rarely met with now. Bewick has figured a specimen which had been sent him from Cambridge.

Sp. 111. T. ochropus, *Temm. Man. d'Ornith.* p. 651.

Green Sandpiper.—This is a very rare species. The only specimen that ever came under my observation was shot in the Isle of Ely between Downham and the Hundred-foot River, on the 28th of August, 1821.

Sp. 112. T. Hypoleucos, *Temm. Man. d'Ornith.* p. 657.

Common Sandpiper.—These birds are occasionally met with on the banks of the river below Cambridge.

Sp. 113. T. Glottis, *Temm. Man. d'Ornith.* p. 659.

Greenshank.—This is another species which, since the drainage of the greater part of our fens, has become very rare in Cambridgeshire. There is a specimen, however, in the collection of Dr. Thackeray which was killed in the county.

Genus XLIII. LIMOSA, *Briss.*

Sp. 114. L. Melanura, *Temm. Man. d'Ornith.* p. 664.

Common Godwit.—Formerly plentiful throughout the fens, where according to Willoughby, it was known by the name of *Yarwhelp.* It is not very often met with now.

Sp. 115. L. rufa, *Temm. Man. d'Ornith.* p. 668.

Red Godwit.—I have given this species on the authority of Bewick, who has figured and described a specimen that had been sent him from Cambridge.

Genus XLIV. SCOLOPAX, *Illig.*

Sp. 116. S. Rusticola, *Temm. Man. d'Ornith.* p. 673.

Woodcock.—This bird usually makes its first appearance in the neighbourhood of Bottisham about the end of October, but was once killed as early as on the 18th of that month. It remains with us till the middle or occasionally till the end of March.

Sp. 117. S. major, *Temm. Man. d'Ornith.* p. 675.

Great Snipe.—A mutilated specimen of this bird was brought to the Cambridge market some time since.

Sp. 118. S. Gallinago, *Temm. Man. d'Ornith.* p. 676.

Common Snipe.—Many of these birds, if not all of them, remain with us the whole year, and breed constantly in Burwell and Swaffham fens.

Sp. 119. S. Gallinula, *Temm. Man. d'Ornith.* p. 678.

Jack Snipe.—This species is less plentiful than the preceding, and is a regular migrant, appearing first about the end of September. I never heard of an instance of its breeding with us.

Genus XLV. RALLUS, *Linn.*

Sp. 120. R. aquaticus, *Temm. Man. d'Ornith.* p. 683.

Water Rail.—Occasionally met with in the neighbourhood of Bottisham.

GENUS XLVI. GALLINULA, *Lath.*

SP. 121. G. Crex, ***Temm. Man. d'Ornith.*** p. 686.

LAND RAIL.—This is a migratory species which visits us in the spring and departs in the autumn, but is by no means plentiful.

SP. 122. G. Porzana, ***Temm. Man. d'Ornith.*** p. 688.

SPOTTED GALLINULE.—Montagu supposes this species likewise to be migratory, and not found in England during the winter, but if so, it must visit us very early in the year, as it has been killed near Bottisham in the middle of March. It frequents the same situations with the Water Rail, (*Sp.* 120.) but occurs much more rarely.

SP. 123. G. Baillonii, ***Temm. Man. d'Ornith.*** p. 692.

Caught alive at Melbourn in January 1823, and is in the collection of Dr. Thackeray. This is the only instance on record in which this species has been met with in England.

SP. 124. G. chloropus, ***Temm. Man. d'Ornith.*** p. 693.

COMMON GALLINULE.—This species generally builds on the ground, but I have occasionally found the nest in trees. In one instance it was constructed amongst the ivy encircling a large elm which hung over the water's edge, at the height of at least ten feet from the ground.

ORDER XIII. PINNATIPEDES.

GENUS XLVII. FULICA, *Briss.*

SP. 125. F. atra, ***Temm. Man. d'Ornith.*** p. 706.

COMMON COOT.—These birds were formerly plentiful in the fens between Ely and Littleport. They are probably still to be met with in other parts of the county, as they are of frequent occurrence in the Cambridge market.

GENUS XLVIII. PHALAROPUS, *Briss.*

SP. 126. P. platyrhinchus, *Temm. Man. d'Ornith.* p. 712.

GREY PHALAROPE.—Three specimens of this rare bird were shot in the fens near Cambridge in the hard winter of 1819-20.

GENUS XLIX. PODICEPS, *Lath.*

SP. 127. P. auritus, *Temm. Man. d'Ornith.* p. 725.

EARED GREBE.—In the collection of Dr. Thackeray: from the Cambridge market.

SP. 128. P. minor, *Temm. Man. d'Ornith.* p. 727.

LITTLE GREBE.—Common every where in the neighbourhood of streams, ponds and other pieces of water.

ORDER XIV. PALMIPEDES.

GENUS L. STERNA, *Linn.*

SP. 129. S. Hirundo, *Temm. Man. d'Ornith.* p. 740.

COMMON TERN.—Found in the Isle of Ely during the summer months.

SP. 130. S. nigra, *Temm. Man. d'Ornith.* p. 749.

BLACK TERN.—Immense flocks of these birds appeared in Bottisham and Swaffham fens in the summer of 1824. Many of the specimens which came under my observation differed considerably from each other in their plumage, particularly with respect to the colours about the head and throat. According to Temminck, these parts, which in the winter are much varied with pure white, become in the breeding season wholly black, or at least of a very dark ash-colour like the rest of the body; but in some of these individuals no such alteration had taken place, the forehead, space between the bill and the eyes, throat, and forepart of the neck being as white as at other times of the year, so that this periodical change of plumage cannot be looked upon as constant. Possibly however it may be confined to one sex. On the 8th of July a nest of this species was taken, which was perfectly flat, placed on the ground, about six inches in diameter, and composed of roots and dry grass, which appeared to have been trodden down so as to be rendered quite firm and compact. The eggs were two in number, of an olive-green colour, thickly spotted and blotched with deep brown, especially towards the larger end. These had been incubated some days. Montagu observes that this bird is known in some parts of Cambridgeshire by the name of *Car-swallow*.

Genus LI. LARUS, *Linn.*

Sp. 131. L. marinus, ***Temm. Man. d'Ornith.*** **p. 760.**

Great Black-backed Gull.—There is an adult bird of this species in the collection of Dr. Thackeray, which was procured in the Cambridge market.

Sp. 132. L. argentatus, ***Temm. Man. d'Ornith.*** **p. 764.**

Silvery Gull.—Towards the middle of December 1824, several Gulls in immature plumage were shot at Overcote near Swavesey in this county, which I believe to have been the *Larus argentatus* of Temminck, which is synonymous with the *Herring Gull* of Latham and Montagu: but owing to the strong resemblance between the young of this species, and those of the preceding, and of the *L. fuscus* Temm. it was impossible to identify them with complete certainty. One of these specimens is preserved in the Museum of the Cambridge Philosophical Society.

Sp. 133. L. canus, ***Temm. Man. d'Ornith.*** **p. 771.**

Common Gull.—Met with occasionally in the fens, but chiefly during the autumnal and winter months. Its provincial name is *Coddy-Moddy*.

Sp. 134. L. ridibundus, ***Temm. Man. d'Ornith.*** p. 780.

Black-headed Gull.—In some seasons these birds frequent our fens in great plenty. Specimens shot near Bottisham in the beginning of October wanted the black head; from whence it appears that the periodical change which takes place in the colour of that part, is completed before that time.

Genus LII. LESTRIS, *Illig.*

Sp. 135. L. pomarinus, ***Temm. Man. d'Ornith.*** p. 793.

A specimen of this rare bird (which has been only very lately discovered in this country) is in the collection of Dr. Thackeray, and was shot near Cambridge.

Genus LIII. ANAS, *Linn.*

Sp. 136. A. Anser ferus, ***Temm. Man. d'Ornith.*** p. 818.

Common Wild Goose.—Bewick observes that many of these birds are known to remain in the fens of Cambridgeshire and to breed there. This may have been the case formerly, but I never heard of an instance myself.

Sp. 137. A. Segetum, ***Temm. Man. d'Ornith.*** p. 820.

Bean Goose.

Sp. 138. A. albifrons, ***Temm. Man. d'Ornith.*** p. 821.

White-fronted Goose.

Sp. 139. A. Bernicla, ***Temm. Man. d'Ornith.*** p. 824.

Brent Goose.—This and the three preceding species are indiscriminately called *Wild Geese* by the country people, and occasionally appear in the Cambridge market under that name, more particularly the A. Segetum. They are all found in our fens during the winter months, in greater or less plenty according to the severity or the mildness of the season. The earliest flocks which I ever noticed were seen on the twentieth of October; this, however, is not much before the usual time of their first arrival.

Sp. 140. A. Cygnus, *Temm. Man. d'Ornith.* p. 828.

Wild Swan.—Seen occasionally in small flocks about our streams and ditches in very severe winters.

Sp. 141. A. Tadorna, *Temm. Man. d'Ornith.* p. 833.

Shieldrake.—Not uncommon.

Sp. 142. A. Boschas, *Temm. Man. d'Ornith.* p. 835.

Common Wild Duck.—First seen about the middle of October.

Sp. 143. A. Strepera, *Temm. Man. d'Ornith.* p. 837.

Gadwall. Two of these birds were exposed for sale in the Cambridge market on the twenty-fifth of February 1824.

Sp. 144. A. acuta, *Temm. Man. d'Ornith.* p. 838.

Pin-tail Duck. — Shot near Cambridge, and in the collection of Dr. Thackeray.

Sp. 145. A. Penelope, *Temm. Man. d'Ornith.* p. 840.

Wigeon.

Sp. 146. A. clypeata, *Temm. Man. d'Ornith.* p. 842.

Shoveller.—This species appears to be of not unfrequent occurrence in the fens. I have seen specimens from the Isle of Ely, and also from the vicinity of Cambridge. In the market at the latter place it may often be met with.

Sp. 147. A. Querquedula, *Temm. Man. d'Ornith.* p. 844.

Garganey.—In the collection of Dr. Thackeray: from the Cambridge market.

Sp. 148. A. Crecca, *Temm. Man. d'Ornith.* p. 846.

Teal.

Sp. 149. A. nigra, *Temm. Man. d'Ornith.* p. 856.

Scoter.—Montagu observes in his Ornithological Dictionary that this species never visits our rivers and inland waters; but I have been informed by Dr. Thackeray, that large flocks of these birds have been occasionally seen in the fens near Cambridge, from whence he has specimens in his collection.

Sp. 150. A. ferina, *Temm. Man. d'Ornith.* p. 868.
Pochard.

Sp. 151. A. Clangula, *Temm. Man. d'Ornith.* p. 870.
Golden Eye.—Occasionally met with in the fens near Cambridge. Willoughby mentions in his Ornithology (p. 28.) having had one sent him from thence by the name of *Shelden.*

Sp. 152. A. Fuligula, *Temm. Man. d'Ornith.* p. 873.
Tufted Duck.—I have seen specimens of this bird which were killed in the neighbourhood of Cambridge. It is not unfrequent in the market.

Sp. 153. A. Leucopthalmos, *Temm. Man. d'Ornith.* p. 876.
Ferruginous Duck.—Dr. Thackeray has a specimen of this very rare duck in his collection, which was procured in the Cambridge market.

Genus LIV. MERGUS, *Linn.*

Sp. 154. M. Merganser, *Temm. Man. d'Ornith.* p. 881.
Goosander, Male. }
Dun Diver, Female. }

Sp. 155. M. Serrator, *Temm. Man. d'Ornith.* p. 884.
Red-breasted Merganser. — Both this and the preceding species are in the collection of Dr. Thackeray, from the Cambridge market. It is probable that they only visit this part of the country in very severe weather.

Sp. 156. M. albellus, *Temm. Man. d'Ornith.* p. 887.
Smew.—This likewise can only be looked upon as an occasional visitant. I find mention in Willoughby's Ornithology (p. 337.) of a specimen that was sent to the author of that work from Cambridge; and have myself seen another, a male bird, in the collection of Dr. Thackeray, which was bought in the market at this place in April 1825.

Genus LV. CARBO, *Meyer*.

Sp. 157. C. Cormoranus, *Temm. Man. d'Ornith.* p. 894.

Cormorant.—On the seventeenth of August in the present year (1825) one of these birds alighted on the top of King's College chapel, and was there shot. It is now in the collection of Dr. Thackeray. I have been informed that it is not unusual for this species to follow the course of rivers to a great distance from the sea.

Genus LVI. SULA, *Briss.*

Sp. 158. S. alba, *Temm. Man. d'Ornith.* p. 905.

Gannet.—This is a much more extraordinary instance of a bird's being noticed so far from its usual haunts. Two specimens were killed in Cambridgeshire during the autumn of 1824. The first of these was shot near Fulbourn on the eleventh of October, and is now in the Museum of the Cambridge Philosophical Society. About a week afterwards, the second was killed near Southery fen in the Isle of Ely. Montagu observes that in the autumn these birds leave our northern islands where they breed, journeying southward, and may be seen during their winter migration in every part of the British channel, but that generally they keep far out at sea. I cannot indeed find mention in any author of their being found inland. The above therefore appears to be a solitary instance, and must have been occasioned by some very peculiar accident.

It will be readily seen in the foregoing catalogue, that one or two species supposed to be of general occurrence are not inserted, as well as others, which it is not improbable may occasionally visit this county; but as these have never fallen under my own observation, and I have been unable to learn any thing respecting them, they are necessarily omitted. I trust, however, that what was stated in the Introduction to this Paper, will sufficiently apologize for its imperfection, and shall conclude by requesting from the Members of this Society, any further information on the subject they may chance to possess.

XX. *On the Influence of Signs in Mathematical Reasoning.*

By CHARLES BABBAGE, Esq. M. A. Trin. Coll.

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[Read *Dec.* 16, 1821.]

It can scarcely excite our surprise that the earlier geometers, engaged in successfully employing the most powerful instrument of discovery which human thought has yet contrived, and seduced by the splendour of the view their science had opened to them, should press with earnestness to enlarge its boundaries by new applications, rather than exert their genius in explaining the causes which have combined to advance it to such unrivalled eminence. On the discovery of those branches which have so completely altered the face of the science, the use of the new acquisitions was too inviting to allow time for any very scrupulous enquiry into the principles on which they were founded: satisfied with the accuracy of the results at which they arrived, the desire of multiplying them naturally prevented any return on their steps for the purpose of applying themselves to the less promising task of establishing on secure foundations, principles of whose truth they felt confident.

These efforts to extend the reach rather than fix the basis of the new calculus, were undoubtedly to be admired at the period to which we refer: an acquaintance with its extensive bearings ought justly to have no inconsiderable influence on the form in which its elements should be delivered; hence the lapse of nearly a century has been required to fix permanently the foundations on which the calculus of Newton and of Leibnitz shall rest.

Time which has at length developed the various bearings of the differential calculus, has also accumulated a mass of materials of a very heterogeneous nature, comprehending fragments of unfinished theories, contrivances adapted to peculiar purposes, views perhaps sufficiently general, enveloped in notation sufficiently obscure, a multitude of methods leading to one result, and bounded by the same difficulties, and what is worse than all, a profusion of notations (when we regard the whole science) which threaten, if not duly corrected, to multiply our difficulties instead of promoting our progress.

As a remedy to the inconveniences which must inevitably result from the continued accumulation of new materials, as well as from the various dress in which the old may be exhibited, nothing appears so likely to succeed as a revision of the language in which all the results of the science are expressed, and the establishment of general principles which shall curtail its exuberance, and regulate that which has hitherto been considered as arbitrary—the contrivance of a notation to express new relations. Previous however to this, some observations on the nature of that assistance which signs lend to our reasoning faculties, and on the causes which give such certainty to the conclusions of analysis, may render our future enquiries more intelligible.

The nature of the quantities with which the mathematical sciences are conversant, is undoubtedly one of the first of those causes: in Geometry it has been well remarked* that its foundations rest on definition, and if this do not altogether hold in algebraical enquiries, at least the meaning of the symbols employed

* Elements of the Philosophy of the Human Mind, Vol. II. p. 150.

must be regulated by definition; and here arises one of the great differences which characterise this science, the definitions themselves being exceedingly simple, comprising but few ideas, whilst in other sciences they are usually much more complicated. In Geometry, definition is the beginning of any enquiry; in metaphysical science, it is frequently the result of one: thus that a triangle is a figure formed by three sides, is a convention on which many of Euclid's propositions rest, and from this, as a point of departure, numerous deductions are made: on the other hand, our idea, and consequently our definition of beauty, is only the result of considerable thought and enquiry.

In the language of analysis, it is very rare that any symbol possesses more than one meaning; in ordinary language, it is as rare to find a word having but one signification: nor is this the only difference; when an algebraical symbol has more than one meaning, they are always well defined and distinct, and should there exist several signs for the same operation, the only difference is in their external form, not the slightest in their meaning; whilst in common language, the meanings of words shade away into each other, and it is frequently difficult, even on mature consideration, to assign the precise limits of the signification of words which are nearly synonymous.

Now if this be the case when the words themselves are the especial objects of our thoughts, how open must all reasoning be to inadvertencies when the mind is compelled to occupy itself at once on the various meanings of the signs it uses and on the train of consequences which it endeavours to deduce by them.

The multitude of significations which attach to many of the words that compose our ordinary language, is a disadvantage which is completely removed from that of analysis. In our reasoning concerning any objects even of a moderately complicated nature, we are obliged to make use of the words attached to those objects, which consequently recall to the mind the variety of particulars of which they consist, some with more, others with less vividness according to our previous habits of thought; from this cause it sometimes happens that the real ground on which our reasoning depends, is with difficulty kept in view by a laborious effort of the attention, and is in many instances very indistinctly perceived.

In the use of algebraic signs this inconvenience entirely vanishes; we can always so arrange them, that that quality on which the whole force of our reasoning turns shall be visible to the eye, whilst the numerous others which contribute to form the expression we are considering, although thrown into the back ground, are still by no means excluded. This species of insulation of the property whose consequences we wish to trace, enables the mind to apply that attention, which must otherwise be exerted in keeping it in view, to the more immediate purpose of tracing its connection with other properties that are the objects of our research. As an example of these ideas, I would mention the word government, upon which we may reason in many different directions, either as it secures domestic liberty, or protects from foreign attacks, as it discourages vice or promotes commerce: in these and in numerous other courses, our reasoning may be pursued, and the word government will constantly recur without the possibility of avoiding it but by the most tedious circumlocution, or of restricting the view in which it is regarded but by the most unwearied efforts of attention. The word function in analysis possesses a still more extensive signification than that which has been just mentioned:

$$\text{The sign } \psi\left(\overline{x}, \overline{\frac{1}{x}}\right)$$

signifies any symmetrical combination of the two quantities x

and $\frac{1}{x}$. That combination is in this mode of expressing it left arbitrary and undefined. The same function ψ may at the same time be a function of

$$\frac{x^2+1}{x}, \quad \frac{x}{x^2+1}, \quad \frac{x^4+x^2+1}{x^2},$$

or of a thousand other quantities; all which circumstances although deducible from the original expression, are not presented to the eye, because in the consequences which it is proposed to deduce, they are entirely immaterial.

If two circumstances in the nature of the function are jointly the ground on which any of its properties depend, they may be separated from the rest and made prominent by several methods. Thus the index n subjoined to a function

$$\psi(\overline{x}, \overline{y})_n$$

may be defined to mean that it is homogeneous with respect to x and y and of the dimensions n: these two circumstances are the causes on which the truth of the following property depends.

xt being substituted for y

$$\psi(\overline{x}, \overline{xt})_n = x^n \psi(\overline{t}, \overline{1})_n.$$

In all our attempts at mathematical generalization, it is of great importance to discover and distinguish these immediate causes of successful operations; in almost all cases they lead us at once to the highest point of generality, and very frequently contribute in no inconsiderable degree to simplify the processes of the investigation. This advantage so peculiar to algebraic signs, has been remarked by M. Degerando, from whose writings I have derived much satisfaction by observing the support which many of those views that I had taken previous to my acquaintance with them, received from the reflections of that distinguished philosopher. "La troisième raison," observes M. Degerando, "est dans la propriété qu' à l'algèbre de ne saisir, dans les idées des quantités, que certains rapports généraux, de ne présentir ainsi à notre esprit que les considérations qui lui sont vraiment utiles dans les recherches auxquelles il se livre. De là il arrive que notre attention se trouve débarassée d'un grand nombre d'idées accessoires, qui etrangères au but de ses méditations, n'auroient servi qu' à la distraire *."

The quantity of meaning compressed into small space by algebraic signs, is another circumstance that facilitates the reasonings we are accustomed to carry on by their aid. The assumption of lines and figures to represent quantity and magnitude, was the method employed by the ancient geometers to present to the eye some picture by which the course of their reasonings might be traced: it was however necessary to fill up this outline by a tedious description, which in some † instances even of no peculiar difficulty became nearly unintelligible, simply from its extreme length: the invention of algebra almost entirely removed this inconvenience, and presented to the eye a picture perfect in all its parts, disclosing at a glance, not merely the conclusion

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* Des Signes et l'art de Penser, p. 214. tom. II.

† The difficulty which many students experience in understanding the propositions relating to ratios as delivered in the fifth book of Euclid, arises entirely from this cause, and the facility of comprehending their algebraic demonstrations forms a striking contrast with the prolixity of the geometrical proofs.

A still better illustration of this fact is noticed by Lagrange and Delambre, in their report to the French Institute on the translation of the works of Archimedes by M. Peyrard.

It occurs in the ninth proposition of the 2nd book on the equilibrium of planes, on which they observe, "La demonstration d'Archimede a trois énormes colonnes in-folio, et n'est rien moin que lumineuse." Eutochius commence sa note "en disant que le theorême est fort peu clair, et il promet de l'expliquer de son mieux. Il emploie quatre colonnes du même format et d'un charactère plus serré sans reussir d'avantage; au lieu que quatre lignes d'algebre suffisent a M. Peyrard pour mettre la verité du theorême dans le plus grand jour." Ouvrages d'Archimede traduites par M. Peyrard, p. 415. tom. II.

in which it terminated, but every stage of its progress. At first it appeared probable that this triumph of signs over words would have limits to its extent: a time it might be feared would arrive, when oppressed by the multitude of its productions, the language of signs would sink under the obscurity produced by its own multiplication: had these expectations been realized, still its utility would have been extensive, and mankind, whilst they felt grateful for the many stages it had advanced them, must have sought some more powerful auxiliary for their ulterior progress. Fortunately however such anticipations have proved unfounded; in whatever department of analysis the number of symbols has encreased to a troublesome extent, contrivances have soon occurred for diminishing it without any sacrifice of perspicuity: the inconvenience has always been temporary, the advantage permanent.

In later times the generalization and contraction introduced by the use of signs, seems even to have outstepped the discoveries which have resulted from them; and reasoning from the past course of science to its future advances, we may fairly presume that our power of condensing symbols will at least keep pace with the demands of the science.

Examples of the power of a well contrived notation to condense into small space, a meaning which would in ordinary language require several lines or even pages, can hardly have escaped the notice of most of my readers: in the calculus of functions this condensation is carried to a far greater extent than in any other branch of analysis, and yet instead of creating any obscurity, the expressions are far more readily understood than if they were written at length: the instance I shall choose as an example is the equation

$$\psi^{\overline{9,9}}(x, y) = \psi(x, y)\,*.$$

* Transactions of Cambridge Philosophical Society, Vol. I. p. 68.

To any person acquainted with the notation belonging to the calculus, it is instantly intelligible; yet if it were written out at length, the letters x and y would be each repeated 257 times, the letter ψ would be found 512 times, whilst the expression would also contain 257 commas and 512 pairs of parentheses; thus comprising in the whole 2307 symbols; and it may be added that it would require a much longer time to understand the meaning of the equation written out at length, than it would to find its general solution.

The power which we possess by the aid of symbols of compressing into small compass the several steps of a chain of reasoning, whilst it contributes greatly to abridge the time which our enquiries would otherwise occupy, in difficult cases influences the accuracy of our conclusions: for from the distance which is sometimes interposed between the beginning and the end of a chain of reasoning, although the separate parts are sufficiently clear, the whole is often obscure. This observation furnishes another ground for the preference of algebraical over geometrical reasoning, and is one which had not escaped the notice of Lagrange. "Chaque membre de phrase est claire et très intelligible a le considerer seul, mais le tout est si long qu'on a souvent oublié le commencement quand on arrive a l'endroit ou le sens est complet."

The closer the succession between two ideas which the mind compares, provided those ideas are clearly perceived, the more accurate will be the judgement that results; and the rapidity of forming this judgement, which is a matter of great importance, inasmuch as the quantity of knowledge we can acquire in a great measure depends on it, will be proportionably encreased. M. Degerando has clearly stated this advantage in comparing the decimal arithmetic with that of the Romans. "La rapidité d'une operation intellectuelle est toujours en raison inverse des efforts qu'on demande a l'attention et a la memoire. Cette operation qui consiste a fixer les rapports de ses idées pour leur appliquer les même judgements s'executera donc autant plus promptement qu'il nous sera facile de nous rappeler et de remarquer ces rapports *."

The almost mechanical nature of many of the operations of Algebra, which certainly contributes greatly to its power, has been strangely misunderstood by some who have even regarded it as a defect. When a difficulty is divided into a number of separate ones†, each individual will in all probability be more easily solved than that from which they spring. In many cases several of these secondary ones are well known, and methods of overcoming them have already been contrived: it is not merely useless to re-consider each of these, but it would obviously distract the attention from those which are new: something very similar to this occurs in Geometry; every proposition that has been previously taught is considered as a known truth, and whenever it occurs in the course of an investigation, instead of repeating it, or even for a moment thinking on its demonstration, it is referred to as a known datum. It is this power of separating the difficulties of a question which gives peculiar force to analytical investigations, and by which the most complicated expressions are reduced to laws and comparative simplicity. One of the most elegant illustrations of this opinion I shall at present briefly allude to, as a more detailed account of it will be given in a subsequent essay. Among the papers left by the late Mr. Spence, is one on a method of solving certain equations of differences: elimination is the means by which he proposed to

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* Degerando sur les signes, Tom. II. p. 196.

† Of so much importance is this maxim, that it has been adopted by Des Cartes as one of his principles of philosophizing. "Diviser chacune des difficultés en autant des parcelles qu'il se pourrait et qu'il serait requises pour les resoudre." *Discours de la Methode.*

accomplish his object, but the results soon became so complicated that little expectation could be formed of succeeding by that means. In this difficulty Mr. Spence introduced into the equation to be solved an arbitrary quantity a, which is merely employed as a letter by whose powers the resulting series may be arranged: if the attempt is now made by continually eliminating, a series arises proceeding according to the powers of a, and equations are found for determining their coefficients: finally, the arbitrary quantity a having performed its office is made equal to unity, and the result is the solution of the equation. The success of this plan depends entirely on breaking into a number of separate parts a very complicated expression, each of these portions being separately reducible to known laws.

On resolving into their separate parts a vast variety of questions which have occurred, it has been found that the number of individual difficulties is by no means so large as had been originally supposed; many of very different kinds have been found to depend perhaps on the same integral, or on the solution of the same equation. In proportion to the number of questions which are reduced to these new difficulties, they themselves assume importance, and the celebrity which always attaches to those who remove obstacles regarded as insuperable by their predecessors, induces many to attempt the solution of these purely abstract questions. Perhaps these ultimate points of reference may not from their nature admit of a comparison with, or reduction to, existing transcendents: the labour and ingenuity employed in the attempt are not however thrown away; relations are discovered by which, from a certain number of particular cases numerically given, all others may be readily calculated, approximations are discovered for determining the cases which are required as data, and, finally, they are arranged in tables and accompanied by rules for their employment, by which, as far as results in pure numbers are required, all questions that are made to depend on them may be considered as solved.

The power which language gives us of generalizing our reasonings concerning individuals by the aid of general terms, is no where more eminent than in the mathematical sciences, nor is it carried to so great an extent in any other part of human knowledge. In the transition from Arithmetic to Algebra, when letters began to be substituted for numbers, the first step consisted rather in the circumstance of the possibility of operating on a quantity determined but unknown. Thus if it were proposed to discover such a number, that its square added to three should be equal to four times the number itself; we commence by supposing the number to be represented by x: now it is quite certain, as soon as the question is stated, that there can only exist two numbers fulfilling the condition; x therefore must in reality mean either of these two, and the rest of the process is

$$x^2 + 3 = 4x,$$
$$x^2 - 4x + 4 = 1,$$
$$x - 2 = \pm 1,$$
$$x = 3 \text{ or } 1,$$

To point out more clearly the force of this observation, we adopt the plan which Vieta introduced into Algebra, that of denoting known quantities by letters: instead of the numbers 3 and 4, let us use the letters a and b; then the process is as follows:

$$x^2 + a = bx,$$
$$x^2 - bx = -a,$$
$$x^2 - bx + \frac{b^2}{4} = \frac{b^2}{4} - a,$$
$$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a}:$$

here it is true that a and b meant 3 and 4, but as no part of the reasoning employed in any manner depended on their numerical value, the result must be independent of it, and is consequently true for all possible values. It may perhaps be contended that by the assumption of x for the number to be found, it was meant to represent number in the abstract, and that such was also the meaning of a and b; but there exists this difference, that it is not in our power to alter the value of x, but we may give to those of a and b any numerical magnitude we may please*.

The utility of the unknown quantities in Algebra, arises from their capability of being operated on without reference to the determined values for which they are placed, the advantage of employing letters for the known quantities, consists in their similarity to general terms in language, and the consequent extension of the reasoning from an individual case to a numerous species. The light in which this question has been regarded, is purely arithmetical, it may however be placed in another point of view, in which without any change in the quantities concerned, it is still more general in its nature; instead of restricting the equation

$$x^2 - bx = -a$$

to number, it may be considered as indicating that x is composed of a and b in such a manner, that when its value is substituted in that equation, all the terms shall mutually destroy each other. This signification, it is true, is not contained in the original question, but arises from the equation into which it is translated: the language of signs is far more general that that of arithmetic, a circumstance which is not perhaps sufficiently attended to in the application of it to questions of pure number. In one respect this generality is not so unexpected, for if a number is required satisfying a certain condition, and if it should happen

* There is in truth one restriction, namely, that a must always be less than $\frac{b^2}{4}$; but this will be removed when the question is viewed in an algebraical light, and does not in the least affect the argument.

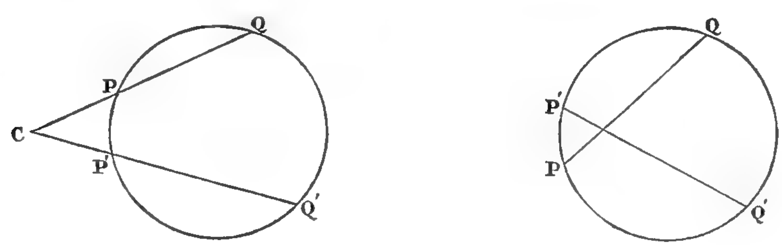
that more numbers than one fulfil that condition, there is no reason why the answer should produce one of these numbers rather than another, it must therefore contain them all.

The reasoning which is carried on in Geometry is of a general nature, and applies to a species, although it is impossible that the picture presented to the eye can be any thing else than that of an individual; hence, it not unfrequently happens that some peculiarity in the figure which is actually employed, either leads us into erroneous conclusions, or when the results are correct, they are supposed to be limited by the individual nature of the figure we have employed. If a line is made use of to represent number, since some other line is the standard unit, it is impossible by such means to represent number in the abstract, but if number is denoted by a letter, there is nothing in the sign which at all indicates the magnitude of that which it represents: it is evident therefore that a property which might lead us into error in the first case, is removed from our view in the second. It will perhaps be objected that the standard unit need not be visible to the eye, since the force of the demonstration is in no way affected by its magnitude, this observation is perfectly correct, and if only one line be considered, and no unity of linear measure be stated, that line may represent length in general, and is to all purposes an arbitrary sign: but the moment any other line is introduced into the diagram, although the unit should not be mentioned, the generality of the former sign is diminished, a relation is instantly established, and whatever may be the unit of length, the ratio of those two lines is fixed and determinate.

The position of a line is another circumstance in Geometry which must always remain particular, and this brings with it that of the points formed by its intersection, as well as that of the angles formed by it with other lines, and the attention which the mind must exert to perceive that no part of the reasoning it is pursuing rests on any of these individualities, itself requires a considerable effort. The substances of these observations may be expressed in this conclusion. ***The reasonings employed in Geometry and in Algebra are both of them general, but the signs which we use in the former, are of an* individual nature, whilst those which are employed in the latter, are as abstract as any of the terms in which the reasoning is expressed.***

The signs used in Geometry, are frequently merely *individuals* of the *species* they represent; whilst those employed in Algebra having a connection purely arbitrary with the species for which they stand, do not force on the attention one individual in preference to any other.

An example of the limitation which geometrical considerations introduce, we shall select from a very well known author. In determining the relation between the rectangle under the parts of two lines intersecting each other and cutting a circle, Euclid considers separately the two cases of the point of intersection being situated within and without the circle, and he shows that in the two figures



the rectangle under CP and CQ, is in both cases equal to that under $CP_{,}$ and $CQ_{,}$: the case of one of the lines becoming a

* Halley's paper on the determination of the foci of lenses, would furnish a very apposite example of this principle, and probably few of my readers will fail to recollect instances where the same identical words of a proposition, and the same letters apply to two, three, or more different geometrical figures.

tangent is also a separate investigation. Now in the algebraic mode of treating these questions, the three cases are comprised in one formula*.

The indication of the extraction of roots by means of an appropriate sign, instead of actually performing the operation, is one of the circumstances which add generality to the conclusions of Algebra, and the same principle of indicating operations, instead of executing them, when employed with judgement, contributes frequently in no small degree to the perspicuity of the result, and sometimes enables us to read in the conclusion every stage which has been passed through in the progress towards it. Any general rules to direct us in the application of this principle will be difficult to form, because they ought in a great measure to depend on the objects we have in view: it may, however, be stated generally that it is improper to adhere to it, when by an opposite course any reduction or contraction can be made in the formula; thus generally speaking it would be better to write

$$y = \sqrt{(a-x)^2 + b^2},$$

than

$$y = \sqrt{(a+x)^2 - 4ax + b^2},$$

and on the other hand, wherever in the course of any reasoning the actual execution of operations would add to the length of the formula, it is preferable merely to indicate them.

Some of the advantages which arise from the use of letters to denote known quantities, have already been adverted to; but there are others of considerable value which may now be noticed, and which relate in a great measure to the higher departments of analysis. If a player bet a certain sum of money u, he may either win it and become possessed of $+u$, or he may lose it and possess $-u$. If we now suppose that he regulates the amount of his

* Book III. Prop. 36. and 37.

second stake by the result of the first, and that he makes it $u-v$ if he had won the first, but $u+v$ if he had lost it; on this second bet he will either win or lose

$$u+v, \text{ or } u-v.$$

Supposing him to determine his third stake from his second, in the same manner as he fixed his second from his first, it is clear that according to the determination of his previous bets, he may stake on the third event either of the three sums

$$u+2v, \; u, \; u-2v.$$

And generally on the nth event, if he proceed according to this law, there are n different bets which he may make according to the order in which the previous ones were decided. Now in any question in which such a mode of play entered, it would be exceedingly tedious to consider separately all these cases, and to repeat the same or nearly the same reasoning for each individual case. This may be avoided by rendering the events indeterminate, for we then find his first profit may be denoted by

$$u(-1)^a,$$

in which the letter a represents any whole number whatever; if it is an even number he wins, and if an odd one he loses; the same artifice applied to his second stake gives for it

$$u-v(-1)^a,$$

as a is still undetermined, this will represent that stake truly, whichever event has happened on the first.

The result of this second stake may be represented by

$$\{u-v(-1)^a\}(-1)^b$$

whether it is lost or gained, and this is still kept undecided by means of the letter b.

The third stake will be

$$u-v(-1)^a-v(-1)^b,$$

and the profit or loss by that bet will be

$$\{u - v(-1)^a - v(-1)^b\}(-1)^c$$

a new undetermined letter being again introduced.

By these means his nth stake will be represented by

$$u - v(-1)^a - v(-1)^b - v(-1)^c - \ldots v(-1)^m;$$

and thus all the n cases become reduced into one expression.

The influence of this indeterminateness in contributing to the success of the solution of the following problem is very apparent.

Supposing a player to regulate his stakes in the manner just described, and that his chances of winning and losing are equal, what will he have profited after $p+q$ events, p of which have been favorable, and q have been unfavorable?

At a first view it might perhaps be considered that the data are insufficient for the solution, and that the order in which the events take place ought to be given: the result however of the investigation will prove that the profit or loss is entirely independent of that consideration.

It has already been shewn that his profit on each event will be as follows:

$$\begin{aligned}
&u(-1)^a,\\
&u(-1)^b - v\{(-1)^a\}(-1)^b,\\
&u(-1)^c - v\{(-1)^a + (-1)^b\}(-1)^c,\\
&u(-1)^d - v\{(-1)^a + (-1)^b + (-1)^c\}(-1)^d,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&u(-1)^{(n)} - v\{(-1)^a + (-1)^b + \ldots\ldots(-1)^m\}(-1)^{(n)}.
\end{aligned}$$

On examining the composition of that collection of terms which multiply v it will appear that the sum of all the above quantities is equal to

$$u\{(-1)^a + (-1)^b + \ldots(-1)^{(n)}\},$$

$$-v \times \left\{\begin{matrix}\text{the sum of all the products two by two which can}\\ \text{be formed out of the quantities } (-1)^a,\ (-1)^b, \ldots (-1)^{(n)}\end{matrix}\right\}$$

and since p of the quantities a, b, c, d, n are even and q are odd, this last expression is equal to

$$(p-q)\,u-v\times\{\text{the coefficient of the third term of } (x-1)^p\,(x+1)^q\}$$

$$=(p-q)\,u-v\left\{\frac{p.\overline{p-1}}{2}-pq+\frac{q.\overline{q-1}}{2}\right\}$$

$$=(p-q)\,u-\frac{p^2-p-2pq+q^2-q}{2}\,v$$

$$=(p-q)\,u-\frac{(p-q)^2-(p+q)}{2}\,v.$$

If the order in which the events happened, or what corresponds to it, if the quantities $a, b, c, \ldots n$ had been given, the process we have gone through could not have been completed, for the perception of the nature of the quantity, which in the sum of all the profits multiplies v, depends entirely on preserving those quantities distinct and unconnected with each other; a relation quite impossible, had each been an individual number.

The remarkable influence of signs in the successful termination of this process of reasoning, claims our particular attention: abstract number, from its very nature, admits of amalgamation when subjected to the various operations expressed by algebraic signs: hence all trace of the mode in which it originally entered is completely lost; or if this inconvenience be studiously avoided, by merely indicating instead of executing the arithmetical operations, still the individual nature of the several numbers presents so many points to which the attention is attracted, that it would be almost impossible, even for the most attentive observer, to seize that general view in which they all agree. This influence is still more remarkable in investigations, where characteristics of operation occur, and when letters are used to the exclusion of number, the relations are not merely more apparent, but the results, although attained with difficulty, are more worthy of confidence: the reason of which, is to be found in this circumstance, that when letters only are employed, the functional characteristics convey no meaning except that on which the force of the reasoning depends; but, if numbers are used, they convey, besides this signification, a multitude of others, which distract the attention, although they are quite insignificant in producing the result.

This principle of preserving undetermined* until the conclusion, the quantities on which we are reasoning, seems to be the only one, which promises success in questions, where two parties successively make choice either of things or of situations: of this nature are many of those questions which relate to games dependent on skill. In these cases, the number of things, out of

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* The profound remark of Lagrange, that the true secret of analysis, consists in the art of seizing the various degrees of indeterminateness, of which quantity is susceptible, has been so beautifully illustrated by M. Carnot, and so well accords with my own ideas on the subject, that I should be unwilling to present it to my readers in any language but his own.

J'ai ouï dire plusieurs fois à ce profond penseur, (M. Lagrange) que le véritable secret de l'analyse consistait dans l'art de saisir les divers degrés d'indetermination dont la quantité est susceptible; idée dont je fus toujours pénétré, et qui m'a fait regarder la méthode des indéterminées de Descartes comme le plus important des corollaires de la méthode d'exhaustion.

Un nombre abstrait est moins déterminé qu'un nombre concret, parceque celui-ci spécifie non seulement le combien du nombre, mais encore la qualité de l'object soumis au calcul; les quantités algébriques sont plus indéterminées que les nombres abstraits parce qu'elles ne spécifient pas même le combien: parmi ces dernières les variables sont plus indéterminées que les constantes, parce que celles-ci sont considérées comme fixes pendant un plus longue période de calcul; les quantités infinitésimales sont plus indéterminées que les simples variables, parce quelles demeurent encore susceptible de mutation, lors même qu'on est déjà convenu de considérer les autres comme fixes; enfin les variations sont plus indéterminées que les simples differentielles, parce que celles-ci sont assujéties à varier suivant une loi donnée, au lieu que la loi suivant laquelle on fait changer les autres est arbitraire. Rien ne termine cette échelle de divers degrés d'indétermination, et c'est précisément dans cet assemblage de quantités plus au moins definies, plus au moins arbitraires, qu'est le principe fécond de la methode génerale des indéterminées, dont le calcul infinitésimal n'est véritablement qu'une heureuse application. *Carnot, Reflexions sur la Metaphysique du Calcul Infinitesimal,* 2d edit. p. 207.

which the choice is to be made, is always finite, and the second player can only make his selection out of that number diminished by unity. The first player, at his second stroke, can only choose out of the same number diminished by two, and so on. Now it is evident, that if the individual actually chosen at each step, were fixed permanently, the reasoning must diverge into an immense multitude of cases; whereas, if any one indifferently is chosen by the first, and again, any one indifferently out of the $k-1$ remaining ones by the second, and so on, the things chosen by the two parties would all be comprehended in two expressions.

It is this power of representing any one of $k-p$ things, where p have already been taken out of k, (subject only to the condition that the expression for it can never represent any one of those already chosen,) which enables us to delay the decision of the individuals actually selected until the conclusion; and thus by their means, to satisfy the other conditions of the problem. Several instances of such questions, will be noticed in a future paper: The only means that have hitherto been employed for this purpose, are the roots of unity, and the sines, and other similar functions of submultiples of π, and from the great length of the formulæ in which they occur, I am by no means sanguine in my expectation of much success in those enquiries, until some more condensed method of indicating and to a certain extent also of executing such operations, shall have been contrived. The principle in discussion, appears to me to be the only one, by which any general and complete solutions can be arrived at: more partial views may doubtless be taken, more adapted to the present state of symbolic language, and even these become extremely valuable, in such difficult enquiries, not merely from the advances they themselves introduce, but as the ground-work of generalization to more perfect methods.

The principle of representing any one quantity indifferently, out of a given number, has been employed on several occasions, and occurs in some mechanical questions. Lagrange has made use of it in a memoir relative to the theory of Sound*, and M. Poisson has employed it on a similar occasion †. The cause, on which its successful application depends, seems to be the power which it gives of uniting together a number of cases totally distinct, and of expressing them all by the same formula, the consequence of which is, that one single investigation frequently supersedes the necessity of a multitude.

Many examples of the successful application of this principle, are to be found in a paper on Circulating Functions ‡, in which it is applied to the solution of a peculiar class of equations of finite differences: other instances will come under our notice, in a future communication.

An examination of the various stages, by which, from certain data, we arrive at the solution of the questions to which they belong, ought to constitute a prominent feature, in any work devoted to the philosophical explanation of analytical language. It is a subject, the consideration of which is too frequently omitted altogether, and as a correct view of it tends materially to advance the science, and also by pointing out more clearly the nature of the difficulties it has to contend with, and also by guarding us against those openings by which errors are most frequently introduced, I shall enter into the question at some length.

In the application of analysis to the various questions which are submitted to it §, there are three distinct stages, each subject

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* Memoirs de Turin, vol. I. † Journal de l'Ecole Polytecnique Cah. 18.

‡ Phil. Trans. p. 144. 1818. J. F. W. Herschel, Esq.

§ Of course questions already in an algebraic form are not here alluded to, such as the integration of equations, &c.

to its particular rules, and each liable to its peculiar difficulties. The order in which these succeed each other, will prescribe that which will be followed in the remarks upon them.

I. The first stage consists in translating the proposed question into the language of analysis.

II. The second, comprehends the system of operations necessary to be performed, in order to resolve that analytical question into which the first stage had transformed the proposed one.

III. The third and last stage consists in retranslating the results of the analytical process into ordinary language.

I. In the first step, which consists in translating the proposed question into the language of analysis, much caution is requisite; for unless this is correctly done, it is quite manifest that the labour bestowed on the remaining stages is useless. It is at this point that the principles applicable to the question are employed for its solution; in all the succeeding parts they are kept totally out of sight: the termination of this stage is usually marked by the circumstance of the difficulty being reduced to one or more equations*, which is the case in most of those questions that arise in the application of analysis to physics: but this however does not always happen, since very many questions relating to chances are reduced to the finding of the coefficient of a certain term in the developement of a given function into a series, and although most of these may be reduced to equations of differences, yet it is not universally the case. It is of some importance to observe, that those errors most difficult to detect and remove,

* The term equation in this place must be understood to comprehend such expressions as $A - B >$ or $< C$.

usually occur in this first stage; and it cannot be too strongly recommended, that every part even of the most difficult problems should be fully translated into the language of analysis, before any attempts at simplification are made. In the first stage, it is scarcely possible to see clearly in what degree the results will be affected by a proposed omission; whilst in the second, any quantity which it is conjectured will have little influence on the result, although it adds greatly to the difficulty of calculation, may be kept separate, and the operations to which it is submitted, may be indicated rather than performed. In many of the applications of analysis, and particularly in its treatment of mechanical questions, the principles which regulate the first stage of the process are completely known, and little difficulty is experienced in translating them into the language of signs, the difficulties when they occur, usually taking their rise in the solutions of the equations thus produced. A similar remark is applicable to optical questions, and indeed to by far the greater part of those which occur in the mixed sciences.

II. The second stage in the solution of any problem, generally begins with the equations into which it has been translated, and terminates with their solution. The point at which it commences is not always so well defined as that at which it ends, and this is more particularly the case when the question relates to geometrical figures, where in some instances, the first and second stages are much intermixed.

The difficulties which now occur are purely analytical, and are generally such as have been treated of in works devoted to the subject. The solution of one or more algebraic equations is frequently the object to be obtained: differential equations, or equations of finite differences are another class of analytical expressions to which physical problems are often reduced; many of these can only be resolved approximately, but in proportion to the interest these questions have excited, the variety and accuracy of the approximations have been multiplied. This is strongly exemplified in the problem of the three bodies, as applied to determining the place of the Moon: the great importance of the question has caused the approximations to be pushed to such an extent, that they have arrived at a degree of inelegance and complexity, which would long since have caused them to be rejected from any other question on the exact solution of which less important interests depended. But on this second stage in the solution of a question, it is less necessary to add many observations; the operations which are concerned in it, and the modes of effecting them, being more fully treated of in works of instruction than either of the others.

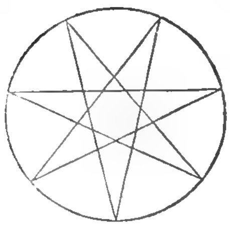
III. The last of the stages into which the resolution of a question has been distributed, has been more neglected than any other. It may perhaps appear singular that the answer to a question, which is of course the great object of research, should have been passed over without sufficient attention. It is not however of any errors in those results which are usually arrived at that I complain; but it is, that sufficient instruction is not given in elementary works, as to the full meaning of all the different circumstances which are contained in the result that analysis has presented. In those questions which lead to algebraic equations, it is not unfrequently the case, that some one or perhaps two roots are taken as the answer, whilst all the remaining ones are completely neglected. Now a question can never be said to be fully answered until every root of the equation to which it has conducted has been discovered, and its signification with reference to the data of that question been explained. It sometimes happens that superfluous roots have been introduced by the algebraic operations that were necessary to arrive at the final equation: these ought to be pointed out, and the step at which they were introduced should be noticed, and also whether they can admit of any translation into the language of the problem considered. Imaginary roots are very frequently introduced; they sometimes imply impossibility or contradiction amongst the data; their origin ought to be carefully traced, and such a course will frequently make us acquainted with the maxima and minima which belong to the question. It is still more necessary to attend to all the real roots whether positive or negative, and to explain the various circumstances in the solution to which they refer.

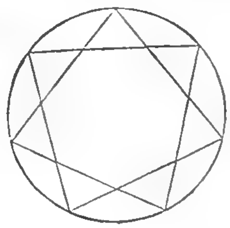
It is by no means uncommon with algebraical authors, when they have led their readers through a process which terminates in an equation, to select that root which gives the answer they require, without explaining the signification of the other roots that are equally comprised in it; and this incomplete mode of solution, which is censurable from revealing only a part of the truth, has in some instances caused the most interesting circumstances attending a question to be entirely overlooked. A singular example of this occurs in several authors who have sought analytically the side of a heptagon inscribed in a circle, or the radius of a circle which would circumscribe a regular heptagon whose side is given. In neither of these questions can the equation to which we are led, be reduced below the third degree, and the three roots of the cubic are always real: the largest of the positive roots gives the answer to the latter of these questions for the common heptagon of Euclid: but no reason is stated why this root should be considered as the true answer to the question in preference to either of the others. In the Analytical Institutions of M. Agnesi*, where the first problem is solved, no

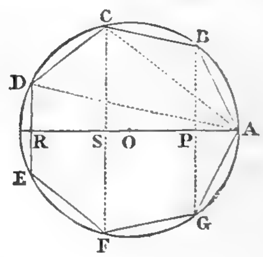
* Vol. I. p. 168. English Translation.

notice is taken of the fact that all the three roots are possible, nor am I aware of its being noticed by any author who has treated of this question: had it been observed and enquired into, the existence of three species of heptagons answering strictly to the definition, and the knowledge of the star-shaped polygons which were discovered by M. Poinsot, could not have remained so long unknown.

If $x = OP$, denote part of the diameter intercepted between the







centre, and a perpendicular from the extremity of the first side of the heptagon, then the usual trigonometrical formulæ give

$$x^6 - \frac{5}{4}x^4 + \frac{3}{8}x^2 - \frac{1}{64} = 0,$$

this contains six roots, of which the three positive are (abstracting the signs) equal to the three negative: it may be resolved into the two factors

$$\left(x^3 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}\right)\left(x^3 + \frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{8}\right) = 0,$$

the second of which is only the first with the signs of its roots changed. The three roots are real and are represented by

$$x = OP, \; x = OS, \text{ and } x = OR,$$

the first of these gives AB for the side of the heptagon, this is the same as that which has long been known; the other two roots give AC and AD, as the sides of the polygon, and by carrying them round the circle, the two star-shaped heptagons, (Figures 2 and 3,) are produced which have no re-entering angles, and which are in fact comprehended in Euclid's definition of regular polygons. The sum of the interior angles of the first of these heptagons is ten right angles, the sum of the interior angles of the second is six right angles, and that of the third species is two right angles. These new species of polygons were first noticed by M. Poinsot, in a highly interesting memoir on subjects connected with the *Geometry of Situation*, read before the Institute in 1809, and subsequently printed in the *Journal de l'Ecole Polytecnique*, 10e Cah.

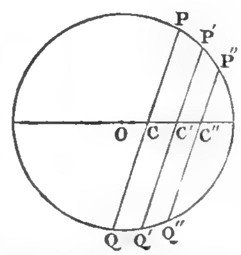
Another important business which belongs to this stage of the question, is to examine carefully what changes will ensue from supposing any peculiar relations amongst the data; or from any of the constant quantities becoming infinite or evanescent, such circumstances frequently introduce great simplicity, and when they refer to geometrical questions, are sometimes the means of making us acquainted with general properties, by which the construction of the problem is greatly facilitated.

A careful and laborious attention to all the possible modifications of a problem which might result from any relation amongst its data, was considered by the ancient Geometers as an indispensible part of its investigation, and the manner in which this was accomplished, was generally little else than a repetition of the whole process under the altered circumstances: when the data are numerous, the length of such a system of operations becomes intolerable, and if more rapid methods had not been contrived, Geometry must have become stationary from the accumulation of the details with which it was thus encumbered. Many instances of the extreme length to which a full investigation of comparatively a very simple problem will lead, occur in the treatises *De Sectione Rationis*, &c. The advantage of Algebraic language is in this respect very striking: all the data of the questions are embodied in the equation in which its solution terminates, and without repeating any part of the process by which that was produced, we can examine with ease all those modifications which any differences in the actual magnitude of the data can introduce into the question under consideration: and moreover the equation itself will suggest to us such relations amongst those quantities as will have the effect of lowering the number of its dimensions, or of rendering it the product of two or more factors.

It sometimes happens that by a peculiar relation amongst the data of a question, the number of solutions instead of being limited becomes infinite: thus, if the position of a line is determined by two points, when those points coincide, any line passing through the point in which they coalesce, will satisfy the conditions of the question which becomes to a certain extent indeterminate: this gives rise to a class of propositions in Geometry which are called porisms. When the data on which questions depend are numerous, it is by no means so easy to discover by Geometrical considerations that relation amongst them in which the question becomes indeterminate, as it is by an Algebraical inquiry where the solution is presented in its most condensed form: one consequence of this is, that such cases have frequently escaped the notice of those who have treated the problems to which they belong in a Geometrical manner. One celebrated and important oversight of this kind occurred in a problem which Newton solved in order to determine the orbit of a Comet.

Having four lines given in position, it was required to draw a fifth line which should be cut by the other four into segments, having a given ratio to each other. Of this question Wren, Wallis, and Newton had given solutions, but when Zanotti, Boscovich and other Astronomers made use of them, employing the observed places of a Comet, the results were found greatly erroneous. Boscovich inquired into the reason of this singular result, and having first assured himself of the accuracy of the solutions, he discovered that in a particular relation between the given lines the problem became indeterminate, and admitted of an infinite number of solutions, and that the case of a Comet approached extremely near to this, and consequently that any very small error in observations must produce an extremely large one in the result.

As an instance of the curious and elegant properties to which such an examination of the relations of the data contained in the final equation sometimes leads, I shall propose the following problem.



A circle whose radius is r *being given, and also three points in one of its diameters, at what angle must three parallel chords be drawn through these points so that the sum of the squares of two of them shall be equal to a given multiple* n *of the square of the remaining one?*

Let the distance of the three points in the diameter from the centre be

$$v,\ v_1 \text{ and } v_2,$$

and calling the angle which is sought θ, we have

$$CP = -v\cos\theta + \sqrt{r^2 - v^2(\sin\theta)^2},$$

and

$$CQ = +v\cos\theta + \sqrt{r^2 - v^2(\sin\theta)^2},$$

hence

$$PQ = 2\sqrt{r^2 - v^2(\sin\theta)^2},$$

and similarly for the other two chords

$$P_1Q_1 = 2\sqrt{r^2 - v_1^2(\sin\theta)^2},$$

$$\text{and } P_2Q_2 = 2\sqrt{r^2 - v_2^2(\sin\theta)^2}.$$

These values of the chords being employed give

$$4\,(r^2 - v \sin\theta^2) + 4\,(r^2 - v_1^2 \sin\theta^2) = 4\,n\,(r^2 - v_2^2 \sin\theta^2)$$

At this step the first of the three stages which have been described terminates; the question is now translated into the language of Algebra, and must be treated according to its rules: the following reductions must then be made

$$r^2 - v^2 \sin\theta^2 + r^2 - v_1^2 \sin\theta^2 = nr^2 - nv_2^2 \sin\theta^2$$

$$(nv_2^2 - v_1^2 - v^2) \sin\theta^2 = nr^2 - 2\,r^2,$$

$$\sin\theta = \pm r \sqrt{\frac{n-2}{nv_2^2 - v_1^2 - v^2}},$$

The second stage is here concluded by the solution of the equation to which the first conducted us, and we have now to explain the meaning of its two roots, and the modifications which may arise from any peculiar relations amongst the data.

The two signs signify that the angle θ, may be measured either above the diameter or below it, as is apparent on inspecting the figure. As the result contains an even radical, we must enquire if in all cases a solution is possible, and if not, what are the conditions of possibility. For this purpose we observe that the numerator and denominator must be both positive or both negative, consequently

$$n - 2 > 0, \text{ and } nv_2^2 - v_1^2 - v^2 > 0,$$

or

$$n - 2 < 0, \text{ and } nv_2^2 - v_1^2 - v^2 < 0, \text{ and also } \sin\theta < 1,$$

are the two sets of conditions; in all other cases the question is impossible.

From this, however, must be excepted the case of the numerator and denominator simultaneously vanishing in consequence of the following relations taking place amongst the data,

$$n - 2 = 0, \text{ and } nv_2^2 - v_1^2 - v^2 = 0,$$

whence

$$n = 2, \text{ and } v_2 = \pm \sqrt{\frac{v_1^2 + v^2}{2}},$$

under which circumstance the value of sin θ, becomes really indeterminate, not depending even on the value of r the radius of the circle.

This indeterminate case suggests the following porism:

Any three points in a straight line being given, another point may be found about which as a centre if a circle with any radius be drawn, and if through the three given points, three chords be drawn in any direction, but parallel to each other; then the sum of the squares of two of them shall be always equal to the square of the third.

It is to be observed, that the origin of the lines denoted by v, v_1 v_2, may be changed by removing it to the distance a, then the latter of the two conditions which rendered the problem indefinite becomes

$$2(v_2 + a)^2 = (v + a)^2 + (v_1 + a)^2,$$

whence

$$a = \frac{2 v_2^2 - v_1^2 - v^2}{2 v + 2 v_1 - 4 v_2}.$$

Before I conclude my observations on this subject, which may perhaps be considered as a digression from that which the title prefixed to this Essay would seem to imply, I shall offer one more illustration of the division of a problem into the several stages which I have pointed out.

This examination of all the circumstances attending the equation containing the solution, is still more necessary when that equation is a differential one: if it be only capable of integration by means of transcendents or by approximating series, it sometimes happens that some relation amongst the data may be assumed, by which in the one case the transcendents shall disappear, and in the other, that the series shall terminate. Euler has taken advantage of the former of these circumstances, to discover curves whose indefinite quadrature should depend on a given species of transcendent, whilst the areas of particular portions of them are susceptible of an Algebraic expression*. The integration of the equation is not always sufficient for a complete analysis of a question, for in some cases besides the general integral, there exists another not included in it, which is known by the name of a particular solution; in order to be secure of not overlooking any such, it must be observed that a change in the magnitude of an index may cause the introduction of such a solution. When the complete integral as well as all the particular solutions are found, the interpretation of them according to the circumstances of the question is not always an easy task, nor are any general rules yet established to which we can refer for information. In the theory of curves the interpretation of particular solutions is sufficiently well known: they represent the curve which touches all those formed by the complete integral when its parameter varies. In mechanical questions a considerable degree of uncertainty prevails relative to these kind of solutions, as in some instances they seem to have no reference to the problem which gave rise to them, whilst in other cases its solution can only be fully represented by their assistance; some light has been thrown on this subject by M. Poisson† in a memoir in which he has explained the theory of particular solutions with great perspicuity.

Of whatever kind the equation to which our question conducts us, may be, it ought to be regarded, merely in an analytical point of view; and all its various roots or solutions, should

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* (Euler Acta Acad. Petrop.)

† Journal de l'Ecole Polytecnique, Cah. 13. p. 60.

be sought after; out of these, by means of some peculiarity in the problem, we must select that individual, which more immediately satisfies the particular view of it which we have taken, and the other solutions must be explained if possible, by means of the data, from which we commenced the process; or should that be impossible, their entrance must be traced to some generalization in signs, to which the language of the question was incapable of adapting itself. In the demonstration of the composition of forces, given in the Mecanique Cœleste, which has sometimes been unjustly censured on account of its analytical nature, this does not appear to have been completely attended to. In the enquiry, to which I refer, x one of the forces is assumed equal to $z\phi(\theta)$, where z is the resultant force, and $\phi(\theta)$ some function of the angle between it and the force x, which function it is required to determine; by changing x into y and θ into $\frac{\pi}{2}-\theta$, the two values of x and y are found to be

$$x=z\phi(\theta), \quad y=z\,\phi\left(\frac{\pi}{2}-\theta\right),$$

and the equation

$$x^2+y^2=z^2$$

is arrived at: this equation in fact amounts to

$$[\phi(\theta)]^2+\left[\phi\left(\frac{\pi}{2}-\theta\right)\right]^2=1,$$

which results* from it, by merely substituting for x and y, their values.

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* This equation is one of that class whose general solution I have ascertained, and it may be exhibited in either of the following forms

$$\phi(\theta)=\sqrt{\frac{\phi_1(\theta)}{\phi_1(\theta)+\phi_1\left(\frac{\pi}{2}-\theta\right)}}$$

$$\phi(\theta)=\sqrt{\tfrac{1}{2}+\left(\frac{\pi}{2}-2\theta\right)\bar{\chi}\left(\theta,\frac{\pi}{2}-\theta\right)},$$

From this equation it appears to me, to be the direct course to deduce the form of ϕ: its general solution should first be shown, and then from the peculiar circumstances of the problem, that particular one which belongs to it should be pointed out. In the work, to which I refer, the particular form of ϕ has been deduced at once by properties peculiar to the problem, without any reference to the general solution of the equation. A similar objection may be made to other demonstrations of this celebrated theorem: the equation to which the investigation conducts, is usually solved in a manner not sufficiently general. This is the case in a work devoted to the analytical exposition of the elements of Geometry, pp. 53, 54;* the substitutions employed, although satisfying the conditions, not containing all possible solutions. M. Poisson has given an investigation of this theorem not quite so open to the objections just stated; by the introduction of two variables and the employment of one sign of function, the solution is necessarily more restricted in its extent. Equations of that class are frequently contradictory, although in the case referred to, a fortunate property leads directly to the solution. See Poisson, *Mecanique*, p. 14. I cannot conclude this slight criticism on a detached passage of the Mecanique Cœleste, without expressing that respect for its illustrious author, which is shared with all those, who are capable of appreciating the important additions he has made to mathematical science, or who have the happiness of being personally acquainted with him.

When any question leads to an algebraic equation, it is usual to resolve it generally, and then to point out amongst its roots that particular one which is sought; if the individual root re-

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where ϕ_1 is perfectly arbitrary, and $\overline{\chi}$ is any symmetrical function of θ and $\frac{\pi}{2} - \theta$.

* Precis d'une nouvelle methode pour reduire à de simples procedès Analytiques la demonstration des principaux theorêmes de Geometrie. Par. I. G. C. Paris, An. vi.

quired were discovered by some artifice without the solution of the equation, we could not feel assured that it alone fulfilled all the conditions, and we should, by arriving at an equation and rejecting its use, have the semblance of generality without its reality; nor do I perceive any reasons which should induce us to change our course, when we have to consider equations of a more comprehensive nature.

Any enumeration of the causes which contribute to give such extensive power to the employment of algebraic signs, would be justly considered incomplete, if no notice were bestowed on the symmetry that ought always to prevail, where the calculations in which it is employed are in any degree complicated.

In its least restricted signification, symmetry is applied to two things, which although sometimes connected, are yet, in many instances, totally independent: it either refers to a resemblance between the systems of characters assumed to represent the data of a question, or it implies a similarity of situation, between certain of the letters, which are found in an analytical formula. An attention to it in either of these senses, has a direct and very beneficial influence in relieving the memory from a considerable burthen. In the first case, its precepts would direct us to assume similar letters as the representatives of similar things: thus, if we have two series, and propose to find another, which consists of the product of the corresponding terms of the two former; if we assume for the first of two series

$$a+b+c+d+\ldots$$

the terms of the second ought to be

$$A+B+C+D+\ldots$$

or which is still more convenient

$$a'+b'+c'+d'+$$

and the series, whose sum we wish to find is then denoted by

$$aA+bB+cC+dD+..$$

or by

$$aa'+bb'+cc'+dd'+..$$

The assumption of

$$A_1+A_2+A_3+..$$
$$A_1'+A_2'+A_3'+..$$

would have been equally proper, and the result equally clear: but had we assumed for the two series

$$a+b+c+d+..$$
$$A_1+A_2+A_3+A_4+..$$

the resulting series

$$aA_1+bA_2+cA_3+dA_4+..$$

would have been devoid of that symmetry, which forms so prominent a character in the former cases.

The plan of accenting letters, in order to represent quantities which stand in similar relations, adds, when employed with discretion, much to the perspicuity of the formulæ in which it is used; but like many other innovations, whose tendency is on the whole decidedly beneficial, an attempt to extend it beyond its proper limits, has been productive of inconveniences as considerable as those which its introduction was proposed to remove. Indices in various positions have been substituted in many cases for the system of accentuation, and the admirers of this scheme, pursuing it with equal ardor, have not been more fortunate in avoiding the confusion, which a multitude of signs, differing but by the slightest shades, can scarcely fail of producing. The taste of the geometer is not less strongly tried by the choice of the letters in which he conducts his reasoning, than his skill and ingenuity are by the artifices he invents to surmount the difficulties opposed to him; in the one case, the elegance which a judicious selection produces, carries the reader pleasantly and almost imperceptibly through an abstruse calculation, whilst the latent cause, which gives facility to his progress, is rarely appreciated, because it is spread uniformly over the whole question: in the other, whose essence often consists in some happy substitution, which is always concentrated in some point, the effect is too remarkable to escape the least attentive enquirer, and its success too striking not to command his admiration.

Unlimited variety in the use of signs, is as much to be deprecated as too great an adherence to one class of them; the one conceals the appearance of relations that really exist, whilst the other, affecting to display them too clearly, fails from its want of distinctness. It is difficult to point out models of imitation, even amongst the most eminent; the same writer, who at one time might be safely trusted as a pattern of correct judgement, has indulged at another in the most unexampled innovation; completely setting at defiance many of those principles, whose authority I have endeavoured to establish. The fate, which has attended the greatest of these proposed reformations, though sanctioned by the illustrious name of Lagrange, is no slight testimony of the validity of those views, which it is the object of several of these Essays to advocate*. Having delivered a theory of the differential calculus, which rested on principles entirely new, he introduced it to the world, clothed in a language of his own creation. The value of the present was too great to allow of its utility being impeded by the garb with which it was encumbered; and the labor of acquiring the language was compensated by the truths which it revealed. Time, however, and experience, convinced even its immortal author, that the language of signs rests

* This relates more particularly to an Essay on the Principles of Notation, which is not yet published.

on principles, which cannot be neglected without danger, or violated with impunity;—the authority of the greatest geometer of the age, failed to make converts to the language he had invented, whilst the justice of the view he had taken, was admitted, and his explanations almost universally adopted. Whilst the language, in which the Theory and the Calculus of functions are conveyed, is pointed out as a warning, not to be neglected by the most successful, that of the Mechanique Analytique of the same author, may perhaps be held up to imitation, with fewer limitations than any other work of equal magnitude. In returning to the notation of Leibnitz, Lagrange has in this work, reduced the whole theory of mechanics to the dominion of pure analysis, and in the choice of his symbols has frequently displayed that happy selection, which so much facilitates the process of reading and comprehending analytical formulæ.

The value of symmetrical symbols is greater in proportion to the complexity of the operations, and the number of quantities, which are concerned; but unless their selection is attended to at the outset of our studies, it is not to be expected that a correct taste can be acquired, I would therefore recommend a degree of attention to this subject, which is not usually bestowed on it by elementary writers. Some instances I shall select from the simplest applications of Algebra to Geometry. The equation of a right line is usually written thus:

$$y = a x + b,$$

which is sufficiently convenient. M. Biot in his Geometrie Analytique * has employed this notation, as also has M. Hachette in his Introduction to the admirable work of Monge †: both authors in treating of lines referred to three co-ordinates have denoted it thus:

$$y = a x + \alpha,$$

* P. 30. † Application de l'Analyse a la Geometrie, 4^{me} ed. p. 2.

the difference is apparently trivial, but the convenience or inconvenience of notation frequently depends on differences as trifling. It may be observed, that in the first equation, a denotes the tangent of an angle, and b an absolute line; two things which have no relation to each other, and which are therefore justly represented by dissimilar signs. In the second equation, the line and the angle, are both represented by the *same letter* of different alphabets; a circumstance, which will infallibly suggest some idea of a relation that does not exist. When two straight lines enter into the question, other reasons present themselves: they may be represented in any of these four ways;

$$\begin{array}{llll} y = ax + b & y = ax + \alpha & y = ax + b & y = ax + b \\ y = a'x + b' & y = bx + \beta & y = \alpha x + \beta & y = cx + d \end{array}$$

and if we seek the ordinate of their point of intersection it will be

$$y = \frac{ab' - a'b}{a - a'}, \quad y = \frac{a\beta - b\alpha}{a - b}, \quad y = \frac{a\beta - \alpha b}{a - \alpha}, \quad y = \frac{ad - cb}{a - c}:$$

the latter of these expressions is quite devoid of all symmetry in regard to its letters, and the larger the number of lines about which we reason, the more confused will such a mode of expressing them render the result. In the first and third mode, it is sufficient to remember that the letter a, under all its forms, represents the tangent of an angle, and that the letter b, in every form, always represents a particular ordinate: with this principle in our mind, we can see at a glance, however numerous the lines introduced, to what property of them each individual letter refers; whereas in the last method, we must, in order to discover the meaning of any letter, refer back for each individual one, to the original translation into algebraic language. The third plan will suffice, where only a few different lines are concerned, but its application is limited by the smallness of the number of different alphabets we can command. The second method may be defended on the ground, that the tangents are denoted by one class of letters;

namely, Italics, whilst the Greek letters are reserved for lines; perhaps it might still be improved by interchanging these significations of the two alphabets.

Before I pass on to the consideration of the second species of symmetry, I shall select from the Arithmetica Universalis, an example, in which the choice of the letters employed seems to have been made without any rule; and shall subjoin to it, the same problem expressed in a language consistent with the views I am illustrating. This course will render more apparent the advantages of attending even to the letters which we select to represent the quantities.

"The velocities of two moving bodies A and B being given, and also their distance, and the difference of the times of the commencement of their motion, to determine the point in which they will meet.

Let A have such a velocity that it will pass over the space c in the time f; and let B have such that it will pass over the space d in the time g, and let the interval between the two bodies be e, and that of the times when they begin to move be h.

Case I. Then if both move in the same direction, and if A is farther distant from the point of meeting, call that distance x, from this take away the interval e, and there will remain $x-e$ for the distance of B, from the same point. And since A passes over the space c in the time f, the time in which it will pass over the space x will be $\frac{fx}{c}$.

And so also, since B passes over the space d in the time g, the time in which it will pass over the space $x-e$, will be $\frac{gx-ge}{d}$. Now since the difference of these is supposed to be x, in order that they may become equal, add h to the smaller time; namely, to the time $\frac{fx}{c}$ (if B moved first) and it will become

$$\frac{fx}{c} + h = \frac{gx - ge}{d},$$

and by reduction

$$x = \frac{ceg + cdh}{cg - df} = \frac{ge + dh}{g - \frac{d}{c}f},$$

but if A began to move first, add h to the time $\frac{gx - ge}{d}$ and it will become

$$\frac{gx - ge}{d} + h = \frac{fx}{c},$$

and by reduction

$$x = \frac{cge - cdh}{cg - df}.$$

Case II. If the bodies move in opposite directions, and if x is, as before, the distance of the body A from the point of meeting, then $e - x$ will be the distance of B from the same point, and $\frac{fx}{c}$ the time in which A will pass over the distance x; and $\frac{eg - gx}{d}$ will be the time in which B will perform the distance $e - x$. To the less of these times add the difference h, namely, to $\frac{fx}{c}$ if B first began to move, and thus we shall have

$$\frac{fx}{c} + h = \frac{eg - gx}{d},$$

and by reduction

$$x = \frac{cge - cdh}{cg + df}.$$

If A began to move first, add h to the time $\frac{eg - gx}{d}$, and it will become

$$\frac{eg - gx}{d} + h = \frac{fx}{c},$$

and by reduction

$$x = \frac{ceg + cdh}{cg + df}.$$

This same question with the following data may be solved in nearly the same way,

v = velocity per second of A,

v' = ... B,

s = the space one (A) is distant from the other B,

t = the time in seconds one (B) starts before the other.

Since the velocities are both positive, the bodies move in the same direction, and x being the distance of the point where they meet from the place of A, the number of seconds which A will take to pass over it, will be found from the ratio

$$v : 1'' :: x : \frac{x}{v},$$

the distance of the other body from the same point will be $x - s$, and the time it takes to arrive will be $\frac{x - s}{v'}$, but the other body began to move t seconds before B, therefore t added to the time of its motion must equal $\frac{x - s}{v'}$, or

$$\frac{x}{v} + t = \frac{x}{v'} - \frac{s}{v'}:$$

hence

$$t + \frac{s}{v'} = x\left(\frac{1}{v'} - \frac{1}{v}\right),$$

and

$$x = \frac{\frac{s}{v'} + t}{\frac{1}{v'} - \frac{1}{v}} = v\,\frac{s + tv'}{v - v'}.$$

If A move in an opposite direction, its velocity must be accounted negative; hence in that case v' becomes $-v'$, and

$$x = v\frac{s - tv'}{v + v'}.$$

If A begin to move after B, the time t must be made negative, and then these two cases become

$$x = v\frac{s - tv'}{v - v'},$$

and

$$x = v\frac{s - tv'}{v + v'},$$

the former of these referring to the case of the bodies moving in the same direction, and the latter to that of their direction being opposite.

There are some restrictions which ought to be noticed, if the velocities of the two bodies are equal, or $v' = v$ the first and third cases show that the bodies can never meet. To this there is however an exception, if $v' = v$, and also $s = tv'$, then $x = \frac{0}{0}$, an indefinite expression and x may have any value: the signification of this is that both bodies move in the same direction and with equal velocities, since $v' = v$, and that the hindermost B, which starts t seconds before the other, is situated at such a distance s from it, that it arrives at the point where the other is, exactly as it begins to move; this appears from the equation $s = tv'$, it is therefore obvious that the two bodies will be at the same point at every part of their progress and for every value of x. Whenever x is negative they can never arrive at the same point in the direction in which they move. If however we conceive that they had been moving at the same rate prior to the point of time at which we consider them, the negative value assigned to x marks a point through which they both passed at the same moment.

These two modes of translating the same question into Algebra, and of re-translating the result into ordinary language, give rise to several observations. The assumption of v to represent the velocity of one body per second, instead of the plan pursued in the first solution, was productive of two advantages: first, it substituted one letter instead of two; and secondly, it is so usual in all mechanical problems to make that letter denote velocity, that it is in such cases associated with it in the mind. The next assumption of v' for the velocity of the other body, possessed both these advantages, and tended to make the result more apparently symmetrical in case it was susceptible of that species of arrangement. The selection of the letters s and t, to represent space and time, was adopted with a similar view of making the signs recal the thing signified. In pure analysis there is but little room for taking advantage of this species of connection, but in all the mixed questions to which it is applied, it may be extensively employed. The general principle is, that either the initial or some prominent letter should be selected from the word which denotes the thing we wish to represent. The beneficial effect of this arrangement is felt a little in the first stage of the solution; it has no influence on the second; but in the last stage it saves considerable trouble by obviating the necessity of constantly referring back to enquire what particular letters represent.

In the first solution, there are in fact two distinct cases, in which the reasoning is repeated from the beginning to the end; and each of these cases has two sub-species: so that there are, in fact, four cases treated of in the Arithmetica Universalis. This defect has been remedied in the second solution, where it has been shown that the four cases are included in one formula, to which it is only necessary to give the proper interpretation, and every circumstance connected with the problem is brought to light. Two causes seem to have contributed to produce this separation into cases: in the first place, the extreme generality of the language of Algebra may not have been sufficiently noticed: the ancients were accustomed to divide their problems into cases, and the habit of treating these separately, may have produced the same cautious treatment of a question when resolved by methods of a far more general and comprehensive nature: such indeed would naturally be the case, until the degree of generalization introduced by the new method, was fully ascertained. In some instances, the action of another cause may be traced, and one that is more easily removed than that which arises from a want of confidence in the method employed: it may be referred to the imperfect manner, in which the very first elements of the application of Algebra to Geometry and to Mechanics, have been communicated to us: in order to explain clearly the result, it is quite indispensible that we should be perfectly familiar with the principles which regulate these: without them, we may frequently put the question into an algebraic form, because we view it only in one light; and from the nature of the language made use of, that one, whichever it may be, virtually embodies the whole; but when we require to retranslate our conclusions, as they contain every possible case, we must of course be able to translate each individual. To display in a more prominent point of view, the reason why there are so few failures in expressing the conditions of a problem in Algebra, and so many in giving the full account of all which the solution informs us, let us imagine a problem proposed that admits of ten cases; if we are capable of translating any one of these into an equation, such is the comprehensive nature of Algebraic language, that though they are all contained in that single expression, we may be ignorant of the manner of treating nine cases, and competent to manage one of them. If this happened, we might succeed in resolving the equation, and discovering all its roots, which would answer all the ten cases; yet it is scarcely probable, with such moderate knowledge, we should succeed in explaining in common language, the meaning of many of them.

The advantage of selecting in our signs, those which have some resemblance to, or which from some circumstance are associated in the mind with the thing signified, has scarcely been stated with sufficient force: the fatigue, from which such an arrangement saves the reader, is very advantageous to the more complete devotion of his attention to the subject examined; and the more complicated the subject, the more numerous the symbols and the less their arrangement is susceptible of symmetry, the more indispensible will such a system be found. This rule is by no means confined to the choice of the letters which represent quantity, but is meant to extend, when it is possible, to cases where new arbitrary signs are invented to denote operations.

In the formation of some of the most common algebraic signs, this maxim has been attended to; but although in many individual instances it has been admitted, it is still desirable that it should be recognised as a general principle. The sign of equality was obviously adopted from the circumstance of the same relation existing between its two parts, as that which it indicates between the two quantities which it separates, and the propriety of this selection has confirmed its use, although Girard employed = to denote difference, and Descartes used $\propto$ to represent equality. In the two signs representing greater and less than

$$a > b \text{ and } b < a\,*,$$

the prevalence of the same motive of choice is equally apparent, for it is immediately seen, that these signs are so contrived, that the largest end is always placed next to the largest quantity, and

* That this principle operated in inducing Harriot, who first used these signs, to adopt them, I have now very little doubt: I may however, remark, that on my first initiation into Algebra, finding some difficulty in remembering their distinction, I formed the association above alluded to as a kind of artificial memory, a purpose, which it effectually answered.

consequently, the smallest end next to the smallest quantity. At a period when the more frequently occurring signs were not permanently settled, one method of denoting equality was by the following sign, 2|2, and the same author P. Herigone employed 3|2, and 2|3 to represent respectively, greater than, and less than. In each of these, the principle we have been contending for, has undoubtedly had its influence, and the signs themselves are objectionable, on entirely different grounds. The same writer made use of the common sign of equality to denote parallelism; a purpose for which it was then well adapted; since that time, however, its long continued use in another sense, has compelled geometers to change its position; and when it is proposed to state that two lines AB and CD are parallel, instead of putting $AB = CD$ as Herigone would have done, they merely change the position, and write $AB \parallel CB$; thus preserving the advantage, without infringing another rule, which ought never to be violated, that of avoiding the use of any sign in two senses.

In the doctrine of triangles, Lagrange has introduced a species of symmetry, which has been found productive of very advantageous results; it consists in denoting the three sides by the letters a, b, c, and the angles respectively opposite to them by the capitals A, B, C: by this arrangement, not merely the quantities themselves are indicated, but in some measure also their position, and the transition from any relation between one side and given data, to other sides and the corresponding data, are made with the greatest ease.

The more complicated the enquiries on which we enter, and the more numerous the quantities which it becomes necessary to represent symbolically, the more essentially necessary it will be found to assist the memory by contriving such signs as may immediately recal the thing which they are intended to represent. The notation which M. Carnot has contrived, for the purpose of illustrating his view of the application of Algebra to Geometry, possesses in an eminent degree the qualifications we are now considering; and although I cannot altogether agree with the conclusions, which it is the object of the author of that highly ingenious work, the "Geometry of Position," to establish, yet I am happy to acknowledge the instruction I have derived from the very original view which he has taken.

To denote a line, M. Carnot places a bar over the letters which represent it, thus, $\overline{AB}$: an arc of a circle or other curve, will naturally be represented by $\overset{\frown}{AB}$. To indicate the point where two lines meet, $\overline{AB}\cdot\overline{CD}$ is used, and similarly the points formed by the meeting of two arcs, or an arc and a circle, are denoted respectively,

$$\overset{\frown}{AB}\cdot\overset{\frown}{CD},\ \text{and}\ \overline{AB}\cdot\overset{\frown}{CD};$$

according to these principles, the line drawn from the point F, to the intersection of AB and CD, will be represented thus,

$$\overline{F\ \overline{AB}\cdot\overline{CD}},$$

and the two expressions,

$$\overline{\overline{AB}\cdot\overline{CD}\ \ \overline{FG}\cdot\overline{HK}}, \qquad \overline{\overline{AB}\cdot\overline{CD}\ \ \overline{FG}\cdot\overline{HK}}\cdot\overset{\frown}{LM}$$

denote the first, the line which joins the points, where AB, CD, and FG, HK respectively intersect each other: and the second, the point where that line cuts the arc LM,

$\pm$ is used to signify coincidence with, as

$$\overline{AB}\cdot\overline{CD} \pm \overset{\frown}{FG}\cdot\overset{\frown}{HK},$$

indicates, that the two points formed by the intersection of the lines AB, CD, and by the arcs FG and HK, coincide.

The angle ABC, is indicated thus, $A\hat{B}C$, and if it is formed by the two lines AB and CD, it is thus expressed,

$$\hat{ABC} = \overline{AB} \wedge \overline{CD}.$$

The angle formed by the intersection of arcs, will therefore stand thus,

$$\overset{\frown}{AB} \wedge \overset{\frown}{CD}.$$

Surfaces are represented by the position of two lines above the letters, which indicate them, as

$$\overline{\overline{AB\ CD}},$$

and the angles which are formed by these, with other surfaces, or with lines, can be expressed on the principles already explained; as for example,

$$\overline{AB} \wedge \overline{CDEF},$$

is the angle formed by the line *AB,* with the surface *CDEF.* When single letters are employed to denote lines, as *a, b, c,* the angles formed by them are represented thus $a^\wedge b$, $a^\wedge c$, $b^\wedge c$.

The effect of these few abbreviations, which, when once explained, need no effort to fix them in the memory, is to render unnecessary the almost endless repetitions of the same words, and to convey the writer's meaning in the fewest terms. In the calculation of the values of annuities dependent on lives, if the question is in any degree complicated, the number of independent data is very considerable, and the letters which have been employed to denote them, have in many instances been selected, without the least regard to assisting the reader's memory. The symbols made use of are far too numerous to be extracted, and the immense advantages which attend a judicious selection, can only be appreciated by those, who have had occasion to study the subject, in the writings of Price and Morgan, and have afterwards perused those of Mr. Baily or Mr. Milne; the former gentleman has had the merit of explicitly stating the principle *, and the use, which he has continually made of it, has had the effect of giving great perspicuity to his formulæ.

In Astronomy, this principle has been adopted with much success, and signs ☉ and ☽ for the Sun and Moon constantly occur. It is rather singular, that a principle on which the earliest and most imperfect written language rested, should be found to add so essentially to the value of the most accurate and comprehensive: yet the language of hieroglyphics, is but the next stage in the progress of the art, to the mere picture of the event recorded; and the signs it employs, in most cases, closely resemble the things they express. In Algebra, although the principle has not been pushed to its extreme limits, the grounds of its observance are the same; the associations, are by its assistance, more easily and more permanently formed, and the memory most effectually assisted †.

I have entered into more detail respecting these causes, than the importance of the subject, may in the opinion of some appear

* "When compound quantities are represented by more simple expressions, those characters ought to be preferred, which will most readily, and with least effort of memory, bring to our recollection the original quantity intended to be expressed." *Baily on Life Annuities*, Pref. p. 39.

† The benefit derived from a proper choice of signs, or from a judicious mode of presenting them, is not entirely confined to mathematical studies; wherever the multiplicity of particulars renders it of importance to assist the memory, or to give quickness to the apprehension of the terms employed, such a principle, if it can be adopted, will be found of considerable value. Amongst the authors who have availed themselves of this principle, in treating other subjects, than those, in which it is so eminently useful, the name of M. Cuvier, may be mentioned. In the preface to his work, Le Règne Animal, he remarks, "Partout les noms des divisions superieures sont en grandes majuscules; ceux des familles, des genres et de sous genres, en petit majuscules, correspondentes aux trois caractéres employés dans le texte; ceux des espèces en Italiques; le nom Latin est a la suite du nom Français, mais entre deux parentheses.——Ainsi l'œil distinguera d'avance l'importance de chaque chose et l'ordre de chaque idée, et l'imprimeux aura secunder l'auteur de tous les artifices que son art peut preter à la mnémonique. *Cuvier, Le Règne Animal*, tom. I. Preface, p. 18.

to justify: the reasons which have induced me to do so, are to be found in the incomplete manner, in which these parts of the subject are treated by the generality of our elementary writers, a circumstance which impedes the subsequent progress of the student more than is perhaps usually allowed. To those who may hereafter employ themselves in supplying a considerable desideratum, in English Mathematics, by composing an introductory treatise on the application of Algebra to Geometry, I may be permitted to recommend a copious developement of its very first elements, not merely a detailed account of the changes of the situation or direction of lines, by changes in the signs, or magnitudes of quantity, or by the circumstances of the radicals, in which they may be involved: but by way of giving practical illustrations of the precepts so delivered, they should be accompanied by a series of examples, in which every circumstance, attending the section of two or more right lines, should be explained with scrupulous minuteness: the influence of such a course of reading, will be sensibly felt by the student as he proceeds; and the uninviting dryness of the results will be relieved by a judicious selection, which may render them valuable points of reference, in his future reading.* How much such a system is calculated to assist all our enquiries in theoretical mechanics will be allowed by those, who have had, in some measure, to form it for themselves, at the moment when the natural difficulties of the subject were sufficient to require their undivided attention. I am indeed inclined to refer a considerable portion of those difficulties, which are so much complained of by English students, well versed in the mathematical science of their own country, when they first open the works of the continental geometers, to

* These observations were written prior to the publication of the Analytical Geometry of Mr. Lardner, and to the still more recent work of Mr. Hamilton. I am happy that one considerable impediment to the progress of the English student is at length removed.

the want of such a previous course of instruction, as that which I have now pointed out. I would however guard myself from being supposed to imagine, that this is by any means the sole obstacle; I mention it as one, which appears to me of some weight, and which might, without much difficulty, be removed.

The other, and more general, acceptation of the word symmetry, applies to the position, as well as the choice of the letters, employed in an enquiry: in this sense, it can scarcely exist, without a previous attention to that which has just been explained, for however regularly and analogously two series of quantities may be arranged, unless the signs, by which they are represented, are so constructed, as mutually to excite the idea of their correlatives, it is impossible, that the symmetry can be apparent to the eye. By employing the first species of symmetry, we assist the memory in remembering the ideas indicated by signs; by the use of the second, we enable it more easily to retain the form in which our investigation has arranged those signs, as well as facilitate the processes, by which that final arrangement was accomplished. By the happy union of the two, our formulæ acquire that wonderful property of conveying to the mind, almost at a single glance, the most complicated relations of quantity, exciting a succession of ideas, with a rapidity and accuracy, which would baffle the powers of the most copious language.

The mode of expressing an angle of a triangle, in terms of the radii of three circles, which touch respectively each side, and the other two prolonged, will furnish the first example. Calling a, b, c those radii, and θ the angle opposite a, we have

$$\cot\frac{\theta}{2} = \sqrt{\frac{b}{a} + \frac{c}{a} + \frac{bc}{a^2}},$$

an unsymmetrical expression. This can be improved by a very trifling change, for it is equivalent to

$$\cot\frac{\theta}{2} = \frac{\sqrt{ab+ac+bc}}{a},$$

in which the numerator is instantly perceived to be the square root of the sum of the products of the radii, two by two. In as far as regards its form, it does not admit of any further improvement; but if this expression were to be employed in any enquiry into the properties of a triangle, it would be much more convenient to use the letters A, B, C, and a, b, c, to denote respectively the angles and sides, and to employ for the radii of these circles, the letters ρ', ρ'', ρ''', thus,

$$\cot\frac{A}{2} = \frac{\sqrt{\rho'\rho''+\rho'\rho'''+\rho''\rho'''}}{\rho'},$$

the convenience of so employing the letters A, B, C, a, b, c, has been already noticed, and the letters ρ', ρ'', ρ''', cannot fail, after a very short use, to recal the idea of radii, as well as fix the particular side, which each circle touches.

I have now enumerated what appear to me to be the principle causes which exert an influence on the success of mathematical reasoning, and have illustrated, with examples, those which were susceptible of it. They may be recapitulated in few words. The nature of the quantities which form the subject of the science, together with the distinctness of its definitions — the power of placing in a prominent light, the particular point on which the reasoning turns — the quantity of meaning condensed into small space—the possibility of separating difficulties, and of combining innumerable cases,—together with the symmetry, which may be made to pervade the reasoning, both in the choice, and in the position of the symbols, are the grounds of that pre-eminence, which has invariably been allowed to the accuracy of the conclusions deduced by mathematical reasoning.

XXI. *On Laplace's Investigation of the Attraction of Spheroids differing little from a Sphere.*

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[Read *May* 8, 1826.]

IN two papers printed in the Philosophical Transactions for 1812, and in a third in the Transactions for 1822, Mr. Ivory has objected to some parts of Laplace's investigation of the attraction of spheroids differing little from a sphere. That there are difficulties in that theory cannot be denied, but that Mr. Ivory has pointed out correctly the errors from which the obscurities arise appears to me quite doubtful. After considering the subject attentively, I have come to the conclusion, that in the part to which Mr. Ivory has most strongly objected, Laplace's investigation may, by a slight alteration, be made free from error; but that an assertion of Laplace which Mr. Ivory has admitted without scruple, is absolutely unsupported by any demonstrative evidence whatever. The result of my examination is however the same as that of Mr. Ivory's; namely, that the method of Laplace may be applied without error, when the elevation of the spheroid above the sphere can be expressed by a rational function of μ, $\sqrt{1-\mu^2}.\cos\omega$, and $\sqrt{1-\mu^2}.\sin\omega$, and that it is not demonstrated for any other case. In the present communication I propose to point out those parts of the investigation which appear to me to be unsatisfactory, and to state them in the form to which, as I conceive, the objections will not apply.

It may perhaps be thought an arrogant attempt in me to correct at once the errors of the two most distinguished analysts of the age. On a mathematical subject, however, in which the value of an assertion depends entirely on the proofs by which it is supported, it is at least allowable for any one to give the reasons which have led him to a conclusion different from that at which another has arrived. For the transcendent powers of Laplace, and for the analytical skill of Mr. Ivory, I have the most profound admiration; in particular the investigations of attractions given in the *Mecanique Celeste,* I consider as the most wonderful theory in the whole compass of mathematical science. But I am induced by this very circumstance to think, that the efforts of mathematicians are well directed when they endeavour to correct any errors which may have crept into such abstruse investigations. That a theory so complicated should be free from defects, is hardly to be expected; and the discussion of the faults which are to be ascribed only to inadvertence, or oversight, will always serve to display more clearly the excellencies which are the offspring of nothing but genius and labour.

The attraction of a spheroid in any direction is made by Laplace to depend on the function V, which expresses the sum of the products of every particle, into the reciprocal of its distance from the attracted point. If b be the radius of a sphere, r the distance of the attracted point from its center, this sum for the sphere is $\frac{4\pi . b^3}{3r}$, (Liv. II. No. 12. and Liv. III. No. 6.) and for a spheroid, the attracted point being supposed exterior to it,

$$V = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c.$$

where the function $U^{(i)}$ satisfies the equation

$$0 = \frac{d}{d\mu}\left\{\overline{1-\mu^2} \cdot \frac{dU^{(i)}}{d\mu}\right\} + \frac{1}{1-\mu^2} \cdot \frac{d^2 U^{(i)}}{d\omega^2} + i \,.\, \overline{i+1} \,.\, U^{(i)}.$$

Now suppose the attracted point to be upon the surface of the spheroid, and suppose b to be the radius of a sphere differing little from the spheroid, and passing through the attracted point. The value of V will consist of two parts, one of which (B) is the sum of the products of each particle of the sphere by the reciprocal of its distance from the attracted point, and the other (C) is the similar sum for all the particles of the excess of the spheroid above the sphere, that excess being supposed negative when the surface of the spheroid is below that of the sphere. Suppose now the point to be raised δr above the surface; B will be increased by a quantity which is ultimately proportional to δr, and which we shall call $B' . \delta r$; and C will be increased by a quantity ultimately equal to $\frac{dC}{dr}\,\delta r$. To find the value of $\frac{dC}{dr}$, let f be the distance of a particle δm of the spheroidal excess, from the attracted point; if γ be the angle made by the radii drawn to the attracted point and the particle δm, $f^2 = 2b(1-\cos\gamma)$. When the point is raised δr above the surface, $f'^2 = \overline{b+\delta r}^{\,2} - 2b \,.\, \overline{b+\delta r} \,.\, \cos\gamma + b^2$; or if we reject the term $\overline{\delta r}|^2$,

$$f'^2 = 2b^2(1-\cos\gamma) + 2b \,.\, \delta r\,(1-\cos\gamma) = f^2\left(1 + \frac{\delta r}{b}\right),$$

and consequently,

$$\frac{1}{f'} = \frac{1}{f}\left(1 - \frac{\delta r}{2b}\right),$$

and therefore

$$\int \frac{\delta m}{f'} = \left(1 - \frac{\delta r}{2b}\right) . \int \frac{\delta m}{f}, \quad \text{or } C + \frac{dC}{dr}\,\delta r = \left(1 - \frac{\delta r}{2b}\right) . C.$$

Hence $\frac{dC}{dr} = -\frac{C}{2b}$; and putting for C its value $V-B$, since $\frac{dB}{dr}$ is the same with B', we have

$$\frac{dV}{dr} - B' = -\frac{V-B}{2b}, \quad \text{or} - b\frac{dV}{dr} = -bB' - \tfrac{1}{2}B + \tfrac{1}{2}V.$$

Now B, as we have stated above, is generally $\frac{4\pi b^3}{3r}$, and therefore $\frac{dB}{dr}$ is $-\frac{4\pi b^3}{3r^2}$; and making $r = b$, we have in the present case

$$B = \frac{4\pi}{3}b^2, \quad bB' = -\frac{4\pi}{3}b^2.$$

The equation becomes by this substitution,

$$-b\frac{dV}{dr} = \frac{2\pi}{3}b^2 + \tfrac{1}{2}V,$$

b being the radius of the sphere passing through the given point, whose center is the origin of co-ordinates. This is precisely the substance of Laplace's investigation (Liv. III, No. 10,) with a slight change of notation, and putting b instead of a, for a reason which we shall presently mention.

It is to this investigation that Mr. Ivory's objections are made. The equation

$$\int\frac{\delta m}{f'} = \left(1 - \frac{\delta r}{2b}\right)\int\frac{\delta m}{f},$$

he allows is quite correct except for the particles very near the attracted point, for which f is very small. In that case, taking the complete value of f'^2, or $f^2\left(1+\frac{\delta r}{b}\right) + \delta r^2$, it is evident that δr^2 may be comparable with f^2; and therefore the value of $\frac{1}{f'}$, when it is largest, will be erroneous, not only to the greatest amount, but also to the greatest proportional amount. And

Mr. Ivory proceeds to shew that the equation $\frac{dC}{dr} = -\frac{C}{2b}$, cannot be considered to be demonstrated by Laplace's process, and that in a particular instance it is actually erroneous.

Perhaps it is doubtful whether in the elucidation of nice and difficult points like that before us, it is advisable for any one to confine himself entirely to analysis. By considering the expressions with reference to their geometrical signification, I think we shall be able to satisfy ourselves that there can be no error in the process of Laplace.

It is allowed that the expression for the variation of C is correct, for all that part of the excess of the spheroid above the sphere, whose distance from the attracted point is much greater than δr. Suppose then a small circle described on the sphere with a radius very much smaller than the radius of the sphere, and which may be made as small as we please: and suppose δr to be taken very much smaller than the radius of this circle. Since there is no doubt that the equation is true for that part of C which is not included within this circle, it will be proved to be true for the whole if we can shew—1st that for the matter included within this circle, C may be made as small as we please, by diminishing the radius of the circle, and the magnitude of δr: —2nd that the variation of C dependent on δr, or its proportion to δr, may be made as small as we please by the same process. Now if we suppose two planes making a small angle to pass through r, cutting out a small pyramid from the matter between the sphere and spheroid; since the angle which is made by the tangent planes of the sphere and spheroid, though small, is finite, their distance from each other, or the thickness of the included matter, is nearly proportional to the distance from the attracted point; the breadth also is proportional to the same distance: hence a section of the pyramid is proportional to the square of the distance; and in a small frustrum of the pyramid, the mass × the reciprocal of the distance is proportional to the distance × the increment of the distance: and therefore that whole sum for the pyramid is proportional to the square of its length. Since the same may be proved for every other pyramid thus formed, it follows that the value of C for the matter included within the circle varies nearly as half the square of the radius. And since this radius may be taken as small as we please, this part of C may be made to bear as small a proportion as we please to the whole. Now suppose the point to be raised δr: by common integration, it is found that the value of C is altered by a quantity proportional to

$$\frac{f}{2}\left(\sqrt{f^2+\overline{\delta r}|^2}-f\right)-\frac{\delta r^2}{2}\log \overline{f+\sqrt{f^2+\delta r^2}}+\frac{\delta r^2}{2}\log \delta r.$$

When δr is small, the two first terms are proportional to $\overline{\delta r}|^2$, and the third $=\frac{\delta r}{2}\delta r \,.\log\delta r$, of which the latter factor, it is well known, may be made as small as we please. Consequently the proportion which this part of the variation of C bears to δr, may be made as small as we please: or the part of $\frac{dC}{dr}$ dependent on the matter included within the circle may be made as small as we please. Since then the equation $\frac{dC}{dr}=-\frac{C}{2b}$ is true for all the matter without the circle, and since for that within it, C and $\frac{dC}{dr}$ may be made as small as we please, it follows that the equation may be applied to the whole of the matter: or for the whole matter between the sphere and the spheroid $\frac{dC}{dr}=-\frac{C}{2b}$.

Mr. Ivory has mentioned as an example of an erroneous conclusion, deduced from Laplace's process, the case of a uniform thickness of matter diffused over the sphere. But it is evident that this is not included in the conditions of the investigation. It is distinctly stated by Laplace, that the attracted point is to be supposed at the intersection or point of contact of the sphere and spheroid, (supposons de plus que la sphère touche le spheroide, et que le point attiré soit au point de contact des deux surfaces), and the thickness of matter at that point must therefore be nothing.

The explanation given above, seems to be that which Laplace has sketched out in the *Mecanique Celeste,* Liv. II. Chap. II. But it appears that it is not necessary to suppose, as he has there stated, that the sphere is a tangent to the spheroid.

The accuracy of the investigation in Liv. III. No. 10. being supposed to be established, I proceed to No. 11. In this article as it stands, there is certainly an obscurity (attended however with no erroneous results) which a small change in the notation will entirely remove. In the preceding article Laplace has taken $r = a$ at the attracted point, he now supposes $r = a\,(1 + \alpha y)$: and whether this be an inaccuracy, or the origin of co-ordinates be supposed to be changed, it is equally incomprehensible to the reader, and equally likely to lead him into error. I have thought it better therefore in the preceding investigation to suppose the origin of co-ordinates the same, and to take b for the radius of the sphere passing through the attracted point, b being $= a\,(1 + \alpha y)$. With this alteration the investigation in No. 11 will stand thus.

It has been previously shewn that

$$V = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c.$$

$U^{(i)}$ being subject to the equation

$$0 = \frac{d}{d\mu}\left\{\overline{1-\mu^2}\,\frac{dU^{(i)}}{d\mu}\right\} + \frac{1}{1-\mu^2}\cdot\frac{d^2U^{(i)}}{d\omega^2} + i\,.\,\overline{i+1}\,.\,U^{(i)}.$$

Differentiating with respect to r,

$$-\frac{dV}{dr} = \frac{U^{(0)}}{r^2} + \frac{2\,U^{(1)}}{r^3} + \frac{3\,U^{(2)}}{r^4} + \&c.$$

Now if we neglected quantities of the order α, or neglected the difference between the sphere whose radius is a, and the spheroid (the radius of the spheroid at this point being $a\,(1 + \alpha y)$), we should have $V = \frac{4\pi a^3}{3r}$. Hence in the preceding expression for V, $U^{(0)} = \frac{4\pi a^3}{3}$ + a small quantity of the order α, which we shall call $U'^{(0)}$: and $U^{(1)}$, $U^{(2)}$, &c., are small quantities of the order α. Substituting $a\,(1 + \alpha y)$ for r in the expressions above, and neglecting quantities of the order α^2, we have

$$\frac{1}{2}\,V = \frac{2\pi a^2}{3}\,(1 - \alpha y) + \frac{U'^{(0)}}{2a} + \frac{U^{(1)}}{2a^2} + \frac{U^{(2)}}{2a^3} + \&c.$$

$$-b\,\frac{dV}{dr} = \frac{4\pi a^2}{3}\,(1 - \alpha y) + \frac{U'^{(0)}}{a} + \frac{2\,U^{(1)}}{a^2} + \frac{3\,U^{(2)}}{a^3} + \&c.$$

Substituting these values in the equation,

$$-b\,\frac{dV}{dr} = \frac{2\pi}{3}\,b^2 + \frac{1}{2}\,V,$$

we have at length

$$4\alpha\pi a^2 y = \frac{U'^{(0)}}{a} + \frac{3\,U^{(1)}}{a^2} + \frac{5\,U^{(2)}}{a^3} + \&c.$$

Hence it follows that y is of the form $Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$, $Y^{(i)}$ being subject to the equation

$$0 = \frac{d}{d\mu}\left\{\overline{1-\mu^2}\,.\,\frac{dY^{(i)}}{d\mu}\right\} + \frac{1}{1-\mu^2}\,.\,\frac{d^2Y^{(i)}}{d\omega^2} + i\,.\,\overline{i+1}\,.\,Y^{(i)}:$$

taking it for granted then, that y can be expanded only into one such series, and supposing y so expanded, we shall have

$$U^{(i)} = \frac{4a\pi}{2i+1}\, a^{i+3} \,.\, Y^{(i)}:$$

substituting this in the general expression for V,

$$V = \frac{4\pi\,.\,a^3}{3r} + \frac{4a\pi\,.\,a^3}{r}\left\{Y^{(0)} + \frac{a}{3r}\,Y^{(1)} + \frac{a^2}{5r^2}\,.\,Y^{(2)} + \&\text{c.}\right\}.$$

In those parts of this investigation which are here set down, I think it is plain that there is no material error. The only thing which is taken for granted, is, that the expression for y can be resolved into a series of the form $Y^{(0)} + Y^{(1)} + \&c.$ in one manner only. In the proof of this important proposition which Laplace has offered, and which Mr. Ivory has mentioned several times without making any objection to it, there is, I apprehend, a radical defect. In fact, Laplace has made with the utmost confidence a statement, for which there does not appear to me to be the slightest evidence. I will now describe the nature of Laplace's demonstration, and point out the parts which seem to be defective.

Laplace first shews, in the most satisfactory manner, that the double integral of $Y^{(i)}\,.\,Z^{(k)}$ with respect to μ and ω, (the limits of μ being -1 and $+1$, and those of ω being 0 and 2π,) is 0 except $k = i$. The remainder of his demonstration is in substance as follows. If possible, let y be resolved into the series $Y^{(0)} + Y^{(1)} + \&c.$ and also into the series $Y'^{(0)} + Y'^{(1)} + \&c.$ Multiply y by $Z^{(i)}$ and integrate between the limits mentioned above: then, in consequence of the proposition already demonstrated, the integral in one case will be reduced to the integral of $Y^{(i)}\,.\,Z^{(i)}$, and in the other case to the integral of $Y'^{(i)}\,.\,Z^{(i)}$. If then y can be resolved into two such series, we shall have the integral of $Y^{(i)}.Z^{(i)}$ equal to the integral of $Y'^{(i)}.Z^{(i)}$, or the integral of $\{Y^{(i)} - Y'^{(i)}\}\, Z^{(i)}=0$. But, says Laplace, it is easily seen that if we take for $Z^{(i)}$, the most general function of its kind, this equation cannot be true except $Y^{(i)} - Y^{(i)} = 0$, in which case (as the same applies to every other term) the two developments are the same. And this is all he offers as a proof of this important theorem.

Now I would ask whether there is for this assertion the slightest evidence whatever? and whether all that has been demonstrated before, if such vague reasoning may be allowed at all, is not entirely opposed to it? It has been clearly shewn that the integral of $\{Y^{(i)} - Y'^{(i)}\} Z^{(k)}$ is always $= 0$ when k differs from i, whatever be the value of $Y^{(i)} - Y'^{(i)}$: and why it is so evident that it is not $= 0$ when $k = i$ it is not very easy to say. If any thing but positive demonstration could be permitted, the analogy of the preceding cases would tend to shew that this integral is $= 0$ when $k = i$. For on the ground of generality the function $Z^{(k)}$ may be made even more comprehensive than $Z^{(i)}$: if, for instance, $k = 2i$, it is well known (by what follows in No. 16.) that the number of arbitrary constants in its expression will be about twice as great as in the expression for $Z^{(i)}$, and consequently it admits of far more variation of form than $Z^{(i)}$. And if in every one of these numerous variations of form, the integral of $\{Y^{(i)} - Y'^{(i)}\} Z^{(2i)}$ is 0, much more probable would it seem, that as $Z^{(i)}$ admits of far less variation, the integral of $\{Y^{(i)} - Y'^{(i)}\} Z^{(i)}$ is constantly $= 0$.

I think it is evident from this statement, that the theorem alluded to rests at present entirely on Laplace's unsupported assertion. It is necessary now to consider in what manner it can be demonstrated. I have not been able to discover any shorter or more general method than the following.

Suppose y to be a rational function of μ, $\sqrt{1-\mu^2}\,.\cos\omega$, and $\sqrt{1-\mu^2}\,.\sin\omega$, of s dimensions. Laplace has shewn (Liv. III. No. 16,) that if $Y^{(i)}$ be a rational function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\,.\sin\omega$, it is the sum of a series of quantities represented by

$$\overline{1-\mu^2}|^{\frac{n}{2}}\left\{\mu^{i-n}-\frac{(i-n)\,(i-n-1)}{2\,.\,(2i-1)}\,\mu^{i-n-2}+\&\text{c.}\right\}\,.\,\{A^{(n)}\sin n\omega+B^{(n)}\cos n\omega\},$$

upon giving to n the values 0, 1, 2, &c. Now since the first term of the series multiplying $\sin n\omega$, or $\cos n\omega$ is always $\mu^{i-n}\,.\,(1-\mu^2)^{\frac{n}{2}}$, it is evident that no combination, by addition, or subtraction, of the series, multiplying $\cos n\omega$ in the expressions for $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}, \ldots Y^{(i-1)}$ can be equal to that in the expression for $Y^{(i)}$. If then we resolve y into a number of such terms as $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, &c by putting it into the form

$$\left.\begin{array}{l} P+Q\cos\omega+R\cos 2\omega+\&\text{c.}+T\cos s\omega \\ \quad +\,q\sin\omega+\,r\sin 2\omega+\&\text{c.}+\,t\sin s\omega \end{array}\right\},$$

and first determine $Y^{(s)}$, so that, upon subtracting it from this expression for y, the terms $\cos s\omega$ and $\sin s\omega$, and the first term of each of the series multiplying the cosines of the other multiples of ω, may be taken away; then determine $Y^{(s-1)}$ in the same manner; and so on to $Y^{(0)}$; as $Y^{(s)}$ cannot be expressed by any multiples of $Y^{(s-1)}$, $Y^{(s-2)}$, &c. the constants entering into its expression will be determinate, and will admit of only one value. The same reasoning then applies to the constants entering into the expression for $Y^{(s-1)}$, $Y^{(s-2)}$, &c. and thus it appears that if y be a rational function of μ, $\sqrt{1-\mu^2}\,\cos\omega$, and $\sqrt{1-\mu^2}\,\sin\omega$, it can always be resolved into a series of the form $Y^{(0)}+Y^{(1)}+$&c. and can be so resolved only in one manner.

Since the theory of the third book of the *Mecanique Celeste* depends entirely on this proposition, I conclude with Mr. Ivory, that that theory applies only to spheroids in which the elevation of the spheroid above the sphere is expressed by a rational function of μ, $\sqrt{1-\mu^2}\,\cos\omega$, and $\sqrt{1-\mu^2}\,\sin\omega$. If any other function can be expanded into a converging series of this form, it is plain that it must be included in this statement.

I shall conclude this paper by briefly mentioning the order in which, in conformity with the preceding remarks, the principal parts of this theory ought to be taken.

First, it must be shewn, that the expression for V is of the form

$$\frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \text{\&c.}$$

Secondly, the equation

$$-b\frac{dV}{dr} = \frac{2\pi b^2}{3} + \tfrac{1}{2}V \text{ must be proved.}$$

Thirdly, by the application of this equation, the equation

$$4\,a\pi a^2 . y = \frac{U'^{(0)}}{a} + \frac{3\,U^{(1)}}{a^2} + \frac{5\,.\,U^{(2)}}{a^3} + \text{\&c.}$$

must be demonstrated.

Fourthly, the form of $V^{(1)}$, supposed a rational function of μ, $\sqrt{1-\mu^2}\,.\cos\omega$, and $\sqrt{1-\mu^2}\,.\sin\omega$, must be found.

Fifthly, it must be shewn that y, if a rational function of the same quantities, can be resolved into such a series, and only into one.

Lastly, by substituting this in the equation above, the values of $U'^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c. are to be found.

The theory, after this point, presents no difficulties which it is necessary to consider here.

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Trinity College,
April 29, 1826.

XXII. *On the Classification of Crystalline Combinations, and the Canons by which their Laws of Derivation may be investigated.*

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INTRODUCTION.

It is possible so to classify crystalline forms, and so to consider them with respect to certain fundamental forms, that our reasonings, with respect to the laws by which their planes are determined, shall be greatly simplified and facilitated.

Classification of Forms.

For this purpose all crystalline forms are supposed to be derived from right pyramids as *fundamental forms;* and they are classified into four *systems,* according to the fundamental form.

1. The *Rhombohedral,* in which forms are derived from a pyramid, having for its base an equilateral triangle.

2. The *Square-Pyramidal,* in which forms are derived from a pyramid, having for its base a square.

3. The *Oblong-Pyramidal,* in which forms are derived from a pyramid, having for its base a rhombus.

4. The *Octahedral,* in which forms are derived from a regular octahedron.

In all cases all the planes are supposed to be formed which belong to the symmetry of the figure.

Laws of Derivation.

The *Laws of* crystalline *derivation* are two; the *first,* or *law of right derivatives;* and the *second,* or *law of scalene derivatives.*

First. By the law of right derivatives a form is deduced by retaining the base, and changing the axis in any ratio ($p : 1$).

The number p is called the index of right derivation, or the first index.

Second. By the law of scalene derivatives a form is deduced as follows. A plane is made to pass through one angle of the base, parallel to the slant side passing through the next angle, and is determined by the manner in which it cuts the diagonal, or perpendicular drawn in the base, through the next angle. In the square and oblong-pyramidal systems the plane would intercept, in the diagonal produced, a line which is in a given ratio ($m : 1$) to the semi-diagonal. In the rhombohedral system it is in such a position, that it would intercept in the perpendicular produced a line which is in a given ratio ($m : 1$) to the perpendicular.

Thus in the square and oblong-pyramidal system (Fig. 2.) let $MC' = m \,.\, MC$, and let $C'V'$ be parallel to CV. The figure bounded by all such planes as $BC'V'$ will be the one derived by the second law from BCV.

In the rhombohedral system let CM be the perpendicular of the triangle (Fig. 1.). Let $MC' = m \,.\, MC$, and $C'V'$ parallel to CV. The figure bounded by all such planes as $BC'V'$ is derived by the second law from BCV.

In the octahedral system the process is the same as in the square pyramidal, with the understanding, that any one of the lines drawn from the center to the angles may be made the axis.

In all these cases the number m is the index of scalene derivation, and the form is said to be derived according to m.

Notation. By means of these two laws of derivation may be obtained any form or face whatever, of which these respective systems are susceptible.

Let O, P, Q, R represent the fundamental form in the octahedral, oblong-pyramidal, square-pyramidal, and rhombohedral systems respectively.

Let pO, pP, pQ, pR, represent forms derived by the first law from each of these forms, according to the index p.

Let pOm, pPm, pQm, pRm, represent forms derived from pO, pP, pQ, pR, by the second law, according to the index m.

In the rhombohedral system, if R be turned round its axis through half a circumference, it is said to be in a ***transverse position***, and is designated by R′. And the derivatives of R′ are marked pR′, pR′m, &c.

In the oblong-pyramidal system, if one diagonal (*MC*, Fig. 2.) be called the principal diagonal, the other (*MB*) is called the transverse diagonal. And when the derivation is made on the transverse diagonal, it is marked with P′, as pP′m.

It is easy to see that $m\mathrm{R}\,\frac{2\,m}{3\,m'}$, when $m=0$, is an equilateral six-sided pyramid with the same axis as the fundamental form. In the same manner $m\mathrm{O}\,\frac{1}{m}$ and $m\mathrm{Q}\,\frac{1}{m}$, when $m=0$ are equilateral four-sided pyramids, in a diagonal position to O and Q: $m\mathrm{P}\,\frac{1}{m}$, when $m=0$, is a prism with its axis in the direction of the principal diagonal.

These forms are, for the sake of abbreviation, marked thus,

Rr, Or, Qr, Pr.

Sometimes one half of the number of faces resulting from any law is omitted, according to some principle of alternate selection. In this case the former is said to be ***hemihedral***, and is indicated by writing h before its symbol.

Hemihedral forms are sometimes ***plagihedral***, and the letters r and l are used to indicate the obliquity to the right or the left, as will be shewn.

It is easily seen that pR is a triangular pyramid, and, when its faces are repeated, a rhombohedron; that pRm is a six-sided pyramid, the hexagonal base of which has alternately equal angles (*V′AUBSCT*, Fig. 1.). Also pP and pQ are four-sided pyramids: pPm is a four-sided pyramid (*V′A′BC′*, Fig. 2.), and pQm an eight-sided one, with the angles of its base alternately equal (*V′BSC*, &c. Fig. 2.). The derivations of O are less obvious, but need not here be explained.

Theorems of Crystallometry.

In the combinations of different forms, the intersections of the planes bounding the figure are called the *Edges of Combination*. These lines have various parallelisms and relations, which, with other properties of the forms, are enunciated in the following theorems:

A. If two forms are derived by the first law according to different indices, their edges of combination are horizontal (the axis being vertical).

B. If two forms are derived from the same form by the second law, their edges of combination are parallel to the slant edge of that form.

C. In the following series, each form truncates the edges of the following one, making parallel edges of combination;

$$\ldots\tfrac{1}{8}\mathrm{R}',\ \tfrac{1}{4}\mathrm{R},\ \tfrac{1}{2}\mathrm{R}',\ \mathrm{R},\ 2\mathrm{R}',\ 4\mathrm{R},\ 8\mathrm{R}'\ldots$$

$$\ldots\tfrac{1}{8}\mathrm{R},\ \tfrac{1}{4}\mathrm{R}',\ \tfrac{1}{2}\mathrm{R},\ \mathrm{R}',\ 2\mathrm{R},\ 4\mathrm{R}',\ 8\mathrm{R}\ldots$$

$$\tfrac{1}{2}\mathrm{Qr},\ \tfrac{1}{2}\mathrm{Q},\ \mathrm{Qr},\ \mathrm{Q},\ 2\mathrm{Qr},\ 2\mathrm{Q},\ 4\mathrm{Qr},$$

$$\mathrm{Pr},\ \mathrm{P},\ \text{or}\ \mathrm{P'r},\ \mathrm{P}.$$

D. If a be the axis of O, P, Q, R, the axis of pO, pP, pQ, pR will be pa.

E. If b be the principal semi-diagonal of the base of O, P, Q, pOm, pPm, pQm will have for their axes pma, and mb for the intercepted part of the principal semi-diagonal.

F. The bases of pOm, pQm, are octagons with alternately equal angles. The radii of the circles passing through the original and the derived angles (*MB* and *MS*, Fig. 2.), are b and $\frac{mb\sqrt{2}}{m+1}$.

G. The axis of $p\text{R}m$ is $\frac{3m-1}{2}\,pa$.

H. If b be the radius of the circle circumscribing the base of R, the base of $p\text{R}m$ is a hexagon with alternately equal angles, and the radii of the circle passing through the original and the derived angles (*OB* and *OS*, Fig. 1.), are b and $\frac{3m-1}{3m+1}\,b$.

I. The acuter and obtuser edges of $p\text{R}m$ make with the axis angles of which the tangents are respectively

$$\frac{2}{3m-1}\,\frac{b}{pa} \text{ and } \frac{2}{3m+1}\,\frac{b}{pa}.$$

The demonstrations of the preceding theorems will be given in another place. To complete the subject, it will also be requisite to give methods of transforming the symbols, which are employed in other systems of notation hitherto proposed, (those of Hauy, Weiss, Mohs, &c.) into this system. This will be done by means of certain formulæ, which we may call ***formulæ of transformation***.

At present our object is to shew how, by means of the theorems above enunciated, with the additional assistance of a few ***subsidiary theorems*** derived from them, we may reason from the parallelisms and other properties of crystalline forms, so as to obtain the indices which belong to those forms; and the rules for drawing these inferences may be called CANONS OF DERIVATION.

Our object, therefore, will be to divide the combinations of crystalline faces into classes, so that each class may offer some peculiarity in the position of its edges and faces, by which we may recognize the laws from which they result. These peculiarities of combination will be exemplified in the figures referred to; and we shall also give instances where these combinations, either alone or united with others, have been observed among minerals. For this latter purpose, the references will be made to Mohs's Crystallography, the name of the mineral and the number of the figure being mentioned.

Hence, in the following tables, the principal columns are those two which, for the sake of convenience in printing, are put the last. The one containing the symbol of the combination, and the other the resulting properties. The columns which are placed before these, contain the *Class* and *Number* of the Combination of which we speak, the examples of its occurrence with their symbols, and such other observations as are requisite for the purpose of distinguishing among resembling combinations, or any other circumstances proper to be remarked.

As the words Edges of Combinations occur very frequently, the abbreviation E. C. is used to represent them.

The preceding Theorems are referred to by means of the letters (*A*), (*B*), &c. by which they are here distinguished. The Subsidiary Theorems are marked with the class and number of the combination to which they belong.

SUBSIDIARY THEOREMS.

Rhombohedral System.

Class II. *Comb.* 4. Let $p\mathrm{R}'$ and $q\mathrm{R}m$ be the forms. Then, in order that $p\mathrm{R}'$ may truncate the acute terminal edges of $q\mathrm{R}m$, we must have the truncating pyramid $p\mathrm{R}'$, equivalent to one which has its axis (OV', Fig. 1.) equal to that of the truncated pyramid qRm, and which has the edges of its base passing through the points A, B, C, from which the derivation of qRm proceeds. Or, we may reduce the axis and the linear dimensions of the truncating pyramid to one half of these dimensions, and its form will remain the same. Hence, its base being that of the fundamental pyramid (in a transverse position, R'), its axis will be half that of the derived pyramid qRm. Now the axis of qRm is $\frac{3m-1}{2} qa$;

$$\therefore\ pa = \tfrac{1}{2} \frac{3m-1}{2} qa, \text{ and } p = \frac{3m-1}{4} q.$$

Thus, if $m = 3$, $p = 2q$; if $m=1$, $p=\frac{1}{2}q$; if $m=2$, $p=\frac{5}{4}q$.

II. 5. Let pR and qRm be the forms. In order that pR may truncate obtuse edges of qRm, the pyramid pR must be equivalent to one which has the edges of its base passing through the points S, and its axis equal to that of the truncated pyramid OV (Fig. 1.). Now, diminish the linear dimensions of pR in the ratio of $OS : OK$, or of $\frac{b(3m-1)}{3m+1} : \frac{b}{2}$ (see H). Its base then becomes that of the fundamental form R, and its axis becomes $\frac{3m+1}{2(3m-1)}$ into the axis of qRm. But the axis of the pyramid qRm is $\frac{3m-1}{2} qa$ (see G). Therefore, $pa = \frac{3m+1}{4} qa$, and $p = \frac{3m+1}{4} q$. Thus, if $m = 2$, $p = \frac{7}{4}q$; if $m = 3$, $p = \frac{5}{2}q$; if $m = 5$, $p = 4q$.

II. 6. Let pR and qRm be the forms. In order that the more acute terminal edges of the pyramid qRm, may coincide with the edges of the rhombohedron pR, the axes of the two forms must be equal;

$$\therefore\ pa = \frac{3m-1}{2} qa, \text{ and } p = \frac{3m-1}{2} q.$$

II. 7. Let pR′ and qRm be the forms. In order that the obtuse edges of the pyramid qRm may replace the edges of the rhombohedron, we must have the axes coincident, and the base of a rhombohedron equivalent to pR' must have its angles at those obtuse edges (at S, Fig. 1.). Or, increasing the linear dimensions of the rhombohedron in the ratio of OS to OC, or (see H), $3m-1 : 3m+1$, its axis becomes $\frac{3m+1}{3m-1}$ into the axis of qRm. The axis of qRm is $\frac{3m-1}{2} qa$. Hence, $pa = \frac{3m+1}{2} qa$, $p = \frac{3m+1}{2} q$.

IV. 5. Let pRm and qR′n be the forms. If we suppose the angle which the acute terminal edge of pRm makes with the axis to be the same as the angle which the obtuse terminal edge of qR′n makes, and the forms to be in a transverse position, it is manifest, that the faces meeting at the last-mentioned edge will truncate the faces meeting at the former edge, and will make parallel edges of combination. Now, the tangents of these angles are $\frac{2}{3m+1} \frac{b}{pa}$ and $\frac{2}{3n-1} \frac{b}{qa}$ (see I): and these will be equal if

$$(3m+1)p = (3n-1)q.$$

Hence, we have these corresponding values of these indices:

$$\left.\begin{matrix} \left.\begin{matrix} p=1 \\ q=2 \end{matrix}\right\} & \begin{matrix} m=3, & 5, & 7 \\ n=2, & 3, & 4 \end{matrix} \end{matrix}\right\} \left.\begin{matrix} \left.\begin{matrix} p=2 \\ q=1 \end{matrix}\right\} & \begin{matrix} m=2, & 3, & 4 \\ n=5, & 7, & 9 \end{matrix} \end{matrix}\right\} \left.\begin{matrix} \left.\begin{matrix} p=1 \\ q=5 \end{matrix}\right\} & \begin{matrix} m=8 \\ n=2 \end{matrix} \end{matrix}\right\}.$$

V. 1. Let pRm, qRr be the forms. In the isosceles pyramid qRr, the tangent of the angle which the terminal edge makes with the axis, is $\frac{1}{q} \cdot \frac{OB}{OV}$ (Fig. 1.). Therefore, in order that the obtuse terminal edges of pRm may truncate these, we must have (see I)

$$\frac{1}{q} \cdot \frac{OB}{OV} = \frac{2}{(3m+1)p} \frac{OB}{OV}; \therefore (3m+1)p = 2q.$$

V. 2. Let pRm, qRr be the forms. In order that the acute terminal edges of pRm may truncate the edges of qRr, we must have

$$\frac{1}{q}\,\frac{OB}{OV} = \frac{2}{(3m-1)\,p}\,\frac{OB}{OV}, \quad (3m-1)\,p = 2q.$$

Square Pyramidal System.

CLASS II. *Comb.* 2. In order that the faces of pQr, replacing the summit of qQn, may be rhombs, we must have qQn coincident with an eight-sided pyramid, derived from pQr (see B). Now, if this deduction from pQr be according to m, the form will be the same as one derived from pQ, thus, $\frac{m-1}{2}\,pQ\,\frac{m+1}{m-1}$. Hence, this form and qQn will coincide if $\frac{m+1}{m-1} = n$, and $\frac{m-1}{2}\,p = q$: whence $\frac{2}{m-1} = n-1$, and $p = q\,(n-1)$, and the combination is $(n-1)\,qQr$, qQn.

II. 6. In order that the faces of the four-sided pyramid pQ may truncate the *scalene* terminal edges of the pyramid qQn, they must make the same angle with the axis which those edges do. But the tangent of the angle which the faces of the pyramid pQ make with the axis is manifestly

$$\frac{b}{\sqrt{2}\,.\,pa}; \quad \therefore \text{ (see } F\text{) } \frac{b}{\sqrt{2}\,.\,pa} = \frac{nb\sqrt{2}}{\overline{n+1}\,.\,qa}, \text{ and } q = \frac{2np}{n+1}.$$

And the combination is pQ, $\frac{2np}{n+1}\,Qn$.

II. 7. In order that the faces of the four-sided pyramid pQr may truncate the ***principal*** terminal edges of the figure qQn, they must make the same angle with the axis which those edges make. But the angle which the faces of the first pyramid make, has for its tangent $\frac{b}{pa}$; and the angle which the edges of qQn make has for its tangent $\frac{b}{qna}$; $\therefore p = qn$. And the combination is $qnQr$, qQn.

II. 8. In order that the faces of the eight-sided pyramid qQn, meeting at the *scalene* edge, may replace the terminal edge of the pyramid pQr, the angles which these edges make with the axis must be equal; and equating their tangents, we have (see F)

$$\frac{b\sqrt{2}}{pa} = \frac{nb\sqrt{2}}{\overline{n+1}\,.qa}; \quad \therefore\ p = \frac{n+1}{n}\,q.$$

II. 9. In order that the faces of the eight-sided pyramid, meeting at the *principal* edge, may replace the terminal edge of the pyramid pQ, the angles which these two edges make with the axis must be equal; and, equating their tangents, we have

$$\frac{b}{pa} = \frac{b}{nqa}; \quad \therefore\ p = nq.$$

III. 4. In order that in the eight-sided pyramid pQm, the principal terminal edges may be replaced by the faces, in pairs, of the pyramid qQn, meeting at its principal terminal edges, we must have the angles which the principal terminal edges make with the axis, equal in the two pyramids. Hence, equating the tangents, $\frac{b}{pma} = \frac{b}{qna}$, $pm = qn$.

Oblong Pyramidal System.

Class II. *Comb.* 3, 4. When the terminal edges of the pyramid pP are replaced by the faces, in pairs, of qPn, we must have the proportion of the axes a and b the same in the two forms; ∴ by E, $p = qn$.

III. 3. When we have the prism qP′r, cutting pP and pPr so as to make the figure $APB'Q$, (Fig. 3.) it is required that this figure should be a rhomb. For this purpose it is requisite that the diagonals of the figure AB' and PQ should bisect each other in N; ∴ $AB' = 2\,AN$, and $= 2\,B'N$, and $B'M' = 2\,B'N'$. Hence $\frac{AM'}{M'B'} = \frac{1}{2}\,\frac{GN'}{N'B'}$. Now the prism which passes through the line $B'G$ is pP′r. Hence, the prism which passes through AB' is $\frac{1}{2}p$P′r.

TABLES

OF THE

COMBINATIONS OF CRYSTALS

WITH THE PARALLELISMS RESULTING,

AND THE

CANONS OF DERIVATION.

THE combinations of faces which can occur in crystals, are, in the following pages, tabulated and classified according to the parallelisms and other relations of edges which they exhibit. And these relations thus afford the means of recognizing the class to which the forms belong, and consequently the laws by which the faces are derived.

The *fourth* column contains the symbols for these combinations, according to the notation which has been explained.

The *last* column, which is on the opposite page, contains the corresponding properties; and these are to be looked for in a given crystal, in order to determine its class.

These combinations are exhibited in the *figures* in the Plates, where they are numbered according to the order of the classes in the *third* column: the letters in the figures refer to the second column.

In the *second* column are examples, in particular minerals, of these combinations; with references to the figures by which they are illustrated in Professor Mohs's Treatise on Mineralogy; and the nomenclature of that work is adopted in designating these minerals.

The *first* column contains such other remarks as may be useful for distinguishing and discriminating different forms.

Besides the theorems (*A*), (*B*), &c. the subsidiary theorems are referred to by the letters (*S. T.*)

RHOMBOHEDRAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class I.	pR, qR & pR, qR'
	Rh[1]. Kouphone-spar. $\frac{1}{2}R', R$ Mohs, Vol. II. 120. n P	No. 1.	$2^nR, 2^{n+1}R'$ $2^{n-1}R', 2^nR$ $pR, 2pR'$ or $\frac{1}{2}pR', pR$
	Rh[1]. Lime-haloide R, 4 R II. 115. P m	No. 2.	$2^nR, 2^{n+2r}R$ generally pR, qR
	Rh[1]. Alum-haloide 0R, R II. 111. o P	No. 3.	$0R, qR$
This prism is distinguishable from ∞Rr in III. 4. by the position of its faces.	Rh[1]. Lime-haloide R, ∞ R II. 114. P c	No. 4.	$pR, \infty R$
		Class II.	pR, qRm and pR', qRm
	Rh[1]. Lime-haloide R, R 3 II. 116. P r	No. 1.	pR, pRm
	0 R, R 3 o r	No. 2.	$0R, pRm$
The figure of the faces of the prism shews it to be ∞R, and not ∞ Rr, in which the tetragons are rhombs, see V. 4.	Rh[1]. Lime-haloide ∞ R, R 5 c y	No. 3.	$\infty R, pRm$
	Rh[1]. Ruby-blende $\frac{1}{2}R', \frac{1}{4}R\,3$ II. 126. z t	No. 4.	$\frac{5}{4}qR', qR2$ $2qR', qR3$ $\frac{11}{4}qR', qR4$ .... generally $\frac{3m-1}{4}qR', qRm$

RHOMBOHEDRAL SYSTEM.

Relations of Edges, &c.
The forms are in a transverse position. The first truncates the edges of the second (*C*). The Edges of Combination are parallel to one another; to the terminal edges of the acute rhombohedron; to the inclined diagonals of the obtuse one (*C*).
The forms are in a parallel position. E. C. are horizontal (*A*).
A face perpendicular to the axis. E. C. horizontal.
The second form is a regular six-sided prism. E. C. of its alternate faces with upper faces of pR, are horizontal (*A*); and of remaining alternate faces, with lower faces of pR. remaining E. C. are inclined. cos. hor. edge = 2 cos. incl. edge.
E. C. parallel to edges of rhombohedron pR (*B*); and to lateral edges of pyramid pRm. Faces of rhombohedron are rhombs contiguous to the apex.
E. C. horizontal. The face (0*R*) perpendicular to axis, is a hexagon with alternate angles equal.
The faces of the prism ∞R are irregular tetragons divisible by a horizontal line into two isosceles triangles of unequal heights. These have their more acute angles alternately upwards and downwards. The heights of the two isosceles triangles are as $m-1 : m+1$.
The forms are in a transverse position (being derived from R′ and R). Faces of the rhombohedron replace *acute* terminal edges of the pyramid. (*S. T.*) E. C. parallel to each other, ... to acute terminal edges of pyramid, ... to inclined diagonals of rhombohedron. The lateral edges (e) of the pyramid are inclined in a direction opposite to the edges of intersection (f) of the pyramid and rhombohedron.

RHOMBOHEDRAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class II.	$pR,\ qRm$
	Rh[1]. Lime-haloide $4R, R5$ Mohs, Vol. II. 116. $m\quad y$	No. 5.	$\frac{7}{4}qR,\ qR2$ $\frac{5}{2}qR,\ qR3$ $\frac{13}{4}qR,\ qR4$ $4qR,\ qR5$ .... generally $\frac{3m+1}{4}qR,\ qRm$
	Rh[1]. Ruby-blende $R, \frac{1}{4}R3$ II. 126. $P\quad t$	No. 6.	$\frac{5}{2}qR,\ qR2$ $4qR,\ qR3$ $\frac{11}{2}qR,\ qR4$ ...... generally $\frac{3m-1}{2}qR,\ qRm$
	$5R', RS$ $n\quad t$	No. 7.	$\frac{7}{2}qR', qR2$ $5qR', qR3$ $\frac{13}{2}qR', qR4$ $8qR', qR5$ .... generally $\frac{3m+1}{2}qR', qRm$
		Class III.	$pR,\ qRr$
If the angles of a regular prism with a pyram[l]. term[n]. be truncated by *rhombs*, the pyramid and truncating pyramid are pRr, pR, or it may be pRr, pR', or both.	Rh[1]. Quartz R, Rr II. 145. $s\quad z$	No. 1.	$pR,\ pRr$ $pR, pRr, \infty Rr$
If alternate edges of a regular pyramid be truncated, truncating rhombohedron and pyramid are $pR,\ 2pRr$.	Rh[1]. Corundum $R, 2Rr$ II. 121. $P\quad r$	No. 2.	$pR,\ 2pRr$

RHOMBOHEDRAL SYSTEM.

Relations of Edges, &c.

The forms are in a parallel position.
The faces of the rhombohedron replace *obtuse* terminal edges of pyramid. (*S. T.*)
E. C. parallel to each other,
.... to obtuse terminal edges of pyramid,
.... to inclined diagonals of rhombohedron.
The lateral edges of the pyramid (*e*) are inclined in the same direction as the edges of intersection (*g*) of the pyramid and rhombohedron.

The forms are in a parallel position.
The *acute* terminal edges of the pyramid in pairs replace the edges of the rhombohedron. (*S. T.*)
E. C. parallel to each other,
.... to edges of rhombohedron,
.... to obtuse terminal edges of pyramid.

The forms are in a transverse position.
The *obtuse* terminal edges of the pyramid in pairs replace the edges of the rhombohedron. (*S. T.*)
E. C. parallel to each other,
.... to edges of rhombohedron,
.... to obtuse terminal edges of pyramid.

The forms are co-ordinate.
The faces of the pyramid in pairs replace the edges of the rhombohedron.
E. C. parallel to the edges of the rhombohedron,
.... to the alternate terminal edges of the pyramid (*B*).
If we have also the prism ∞ Rr, the faces *p*R retain their rhombic form.

The faces of the rhombohedron replace the alternate edges of the pyramid.
The edges replaced are alternate at the two apices.
E. C. parallel to alternate terminal edges of pyramid,
.... to inclined diagonals of rhombohedron.

RHOMBOHEDRAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class III.	pR, qRr
	0R, Rr *o* *r*	No. 3.	0R, qRr
This position of the faces of the prism, easily distinguishable from I. 4. shews that the prism is ∞Rr, and not ∞R.	Rh[1]. Emerald-mal. 2R, ∞Rr Mohs, Vol. II. 118. *r* *s*	No. 4.	pR, ∞Rr
The form of the faces of the prism distinguishes it from ∞Rr, as in VI. 2.	∞R, Rr *e* *r*	No. 5.	∞R, pRr
	Rh[1]. Fluor-haloide ∞R, ∞Rr II. 149. *e* *M*	No. 6.	∞R, ∞Rr
		Class IV.	pRm, qRn
If edges of combination of two pyramids are parallel to lateral edges, the first index is the same for both.	Rh[1]. Lime-haloide, R3, R5 II. 116. *r* *y*	No. 1.	pRm, pRn
If edges of combination are horizontal, the second index is the same for both pyramids.	Rh[1]. Lime-haloide, $\frac{1}{4}$R3, R3 II. 129. *t* *r*	No. 2.	pRm, qRm
	R3, 0R *r* *o*	No. 3.	pRm, 0Rn
	Rh[1]. Fluor-haloide R$\frac{5}{3}$, ∞R$\frac{5}{3}$ II. 148. *u* *c*	No. 4.	pRm, ∞Rm

RHOMBOHEDRAL SYSTEM.

Relations of Edges, &c.
A horizontal face, which is a regular hexagon. E. C. horizontal.
A regular six-sided prism terminated by a rhombohedron. Faces of prism replace lateral edges of rhombohedron. E. C. are parallel to each other, and to lateral edges of rhombohedron. Faces of prism are rhombs with two sides vertical.
The lateral angles of the pyramid are truncated by the faces of a prism. E. C. are parallel to each other. Faces of prism are rhombs with a diagonal vertical.
An equiangular twelve-sided prism.
Forms are co-ordinate six-sided pyramids. Lateral edges of obtuse pyramid replaced by faces of acute pyramid. E. C. are parallel to lateral edges of both pyramids (*B*).
E. C. are horizontal (*A*). Two pyramids meeting in a plane perpendicular to the axis.
Face perpendicular to axis is a hexagon with alternate angles equal. E. C. horizontal.
A twelve-sided prism, terminated by a six-sided pyramid. E. C. of pyramid with alternate pairs of faces of prism, are horizontal. The pairs of faces with which this is the case, are alternate at the two apices. The remaining edges of combination are inclined.

RHOMBOHEDRAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class IV.	$p\,\mathrm{R}\,m,\ q\,\mathrm{R}'\,n$
	Rh^1. Lime-haloide $\underset{r}{\mathrm{R}\,3},\ \underset{x}{2\,\mathrm{R}'2}$	No. 5.	$p\,\mathrm{R}'5,\ 2\,p\,\mathrm{R}\,3$ $p\,\mathrm{R}\,3,\ 2\,p\,\mathrm{R}'\,2$ $2\,p\,\mathrm{R}'\,2,\ 2\,p\,\mathrm{R}\,5$ generally, if $(3m+1)\,p$ $=(3\,n-1)\,q.$
		Class V.	$p\,\mathrm{R}\,m,\ q\,\mathrm{R}\,\mathrm{r}$
	Rh^1. Iron-ore $\underset{g}{\tfrac{1}{4}\,\mathrm{R}\,5},\ \underset{n}{2\,\mathrm{R}\,\mathrm{r}}$	No. 1.	$p\,\mathrm{R}\,3,\ 5\,p\,\mathrm{R}\,\mathrm{r}$ $p\,\mathrm{R}\,5,\ 8\,p\,\mathrm{R}\,\mathrm{r}$ generally, if $(3\,m+1)\,p=2\,q$
	$\underset{f}{\mathrm{R}\,3},\ \underset{n}{4\,\mathrm{R}\,\mathrm{r}}$	No. 2.	$p\,\mathrm{R}\,3,\ 4\,\mathrm{R}\,\mathrm{r}$ $p\,\mathrm{R}\,5,\ 7\,\mathrm{R}\,\mathrm{r}$ generally, if $(3\,m-1)\,p=2\,q$
		No. 3.	$0\,\mathrm{R}\,m,\ q\,\mathrm{R}\,\mathrm{r}$ $p\,\mathrm{R}\,m,\ 0\,\mathrm{R}\,\mathrm{r}$
This position of the edges shews the prism not to be $\infty\,\mathrm{R}$, as in II. 3.	Rh^1. Ruby-blende $\underset{h}{\mathrm{R}\,3},\ \underset{n}{\infty\,\mathrm{R}}$ Mohs, Vol. II. 126.	No. 4.	$p\,\mathrm{R}\,m,\ \infty\,\mathrm{R}\,\mathrm{r}$
		Class VI.	$p\,\mathrm{R}\,\mathrm{r},\ q\,\mathrm{R}\,\mathrm{r}$
	Rh^1. Corundum $\underset{r}{2\,\mathrm{R}\,\mathrm{r}},\ \underset{b}{4\,\mathrm{R}\,\mathrm{r}}$ II. 122.	No. 1.	$p\,\mathrm{R}\,\mathrm{r},\ q\,\mathrm{R}\,\mathrm{r}$
		No. 2.	$p\,\mathrm{R}\,\mathrm{r},\ \infty\,\mathrm{R}\,\mathrm{r}$

RHOMBOHEDRAL SYSTEM.

Relations of Edges, &c.
The acute terminal edges of the more acute pyramid are replaced by the obtuse terminal edges of the more obtuse pyramid in pairs. E. C. parallel to each other and to these terminal edges. (*S. T.*)
The alternate terminal edges of the isosceles pyramid are replaced by the faces, in pairs, of the scalene pyramid, meeting at its *obtuse* terminal edges. E. C. are parallel to each other and to these terminal edges. (*S. T.*)
The alternate terminal edges of the isosceles pyramid are replaced by the faces, in pairs, of the scalene pyramid, meeting at its *acute* edges. E. C. are parallel to each other and to those terminal edges. (*S. T.*)
See III. 3.} Face perpendicular to axis. See IV. 3.} E. C. horizontal.
The lateral edges of the scalene pyramid are replaced by the faces of the prism. E. C. are parallel to each other and to these lateral edges.
A pyramid with its summit replaced by a more obtuse pyramid. E. C. horizontal.
A regular prism terminated by a regular pyramid. E. C. horizontal.

RHOMBOHEDRAL SYSTEM.

Dirhombohedral Combinations are those which contain corresponding derivatives of R and R′.

Remarks.	*Example and Figure.*	*Class.*	*Combinations.*
		Class VII.	Dirhombohedral.
This pyramid is different in position from pRr.	Rh[1]. Emerald R, R′ Mohs II. 150. *s*	No. 1.	pR, pR′
		No. 2.	pR, pR′, pRr
	Rh[1]. Emerald $R\frac{5}{3}$, $R'\frac{5}{3}$ II. 150. *a*	No. 3.	pR*m*, pR′*m*

Hemirhombohedral Combinations are those in which half the faces of a rhombohedral combination are omitted in an alternate manner. These are also *Plagihedral,* if the remaining faces are not symmetrical with respect to right and left.

		Class VIII.	Simple Plagihedral.
When plagihedral faces are thus distributed, the formula is $\frac{\text{r}}{\text{r}}$, or $\frac{\text{l}}{\text{l}}$.	Rh[1]. quartz. Rr, R, $\frac{\text{r}}{\text{r}}\{R\frac{7}{3}, R3, R\frac{11}{3}, R5\}$ *p s x y u v* II. 146.	No. 1.	$\frac{\text{r}}{\text{r}}\{p\text{R}m, p\text{R}n\}$
When plagihedral faces are thus distributed, the formula is $\frac{\text{r}}{\text{r}'}$, or $\frac{\text{l}}{\text{l}'}$.	Rh[1]. quartz. Rr, R, $\frac{\text{r}}{\text{r}'}\{R\frac{7}{3}, R3, R\frac{11}{3}, R5\}$ *p s x y u v* II. 147.	No. 2.	$\frac{\text{r}}{\text{r}'}\{p\text{R}m, p\text{R}n\}$

RHOMBOHEDRAL SYSTEM.

Relations of Edges, &c.

The form is an isosceles six-sided pyramid. If $a =$ edge of rhombohedron, $\cos.\ \theta = \frac{1}{3} (\cos.\ a - 2)$.

See III. 1. The form is an isosceles six-sided pyramid; its edges at the summit replaced by the faces of another six-sided pyramid (pRr), which are rhombs.

See IV. The form is a scalene twelve-sided pyramid.
E. C. are parallel to the lines joining the apices with the bisections of the lateral edges of each pyramid.
This form is called a Dirhombohedron.

In this class the plagihedral faces are turned the same way at both apices: that is, at both to the right, or at both to the left.

The plagihedral faces appear on the alternate angles, and alternately at the two apices.
Their edges of combination are parallel to each other and to the lateral edges of the pyramid pR (A). See IV. 1.

The plagihedral faces appear on the alternate angles, and simultaneously at the two apices.
Their edges of combination are parallel to each other: those at one apex to the lateral edges of the pyramid pR, and those at the other apex to the lateral edge of pR$'$.

RHOMBOHEDRAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class IX.	Double Plagihedral.
When plagihedral faces are thus distributed, the formula is $\frac{r}{l}$.		No. 1.	$\frac{r}{l}\{pRm, pRn\}$
$\frac{r}{l'}$.		No. 2.	$\frac{r}{l'}\{pRm, pRn\}$
When plagihedral faces are thus distributed, the formula is $\frac{rr'}{ll'}$.	Rh^1. Fluor-haloide. $\underset{x}{Rr}$, $\underset{s}{RR'}$, $\frac{rr'}{ll'}\{\underset{u}{R\tfrac{5}{3}}, \underset{b}{R\tfrac{7}{3}}\}$ Mohs II. 149.	No. 3.	$\frac{rr'}{ll'}\{pRm, pRn\}$

SQUARE-PYRAMIDAL COMBINATIONS.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class I.	pQ, qQ & pQ, qQr
	Square Pyr^1. Zircon $\underset{t}{Qr}$, $\underset{P}{Q}$ Mohs, Vol. II. 99.	No. 1.	$pQ, 2pQr$ or pQr, pQ
	Square Pyr^1. Garnet $\underset{c}{Q}$, $\underset{b}{2Q}$ II. 96.	No. 2.	$pQ, 2^n pQ$
	Sq. Pyr^1. Lead-baryt $\underset{b}{\tfrac{1}{3}Q}$, $\underset{a}{0Q}$ II. 92.	No. 3.	$pQ, 0Q$
	Sq. Pyr^1. Tin-ore $\underset{p}{Q}$, $\underset{l}{\infty Q}$ II. 102.	No. 4.	$pQ, \infty Q$

RHOMBOHEDRAL SYSTEM.

Relations of Edges, &c.
In this class the plagihedral faces are turned opposite ways at the two apices: that is, twisted to the right at one apex, and to the left at the other.
The plagihedral faces appear on the alternate angles, and alternately at the two apices.
The plagihedral faces appear on the alternate angles, and simultaneously at the two apices.
The plagihedral faces appear on all the angles, and at each apex. Their Edges of Combination are parallel to each other and to the lateral edges of the pyramids pR, pR′.

SQUARE-PYRAMIDAL COMBINATIONS.

Relations of Edges, &c.
The forms are in a diagonal position. The preceding form in these combinations truncates the edges of the succeeding. The Edges of Combination are parallel to each other and to the terminal edges of the more acute pyramid. .... also to the lines bisecting the lateral edges of the more obtuse pyramid.
The forms are in a diagonal position. The preceding form replaces the summit of the succeeding. E. C. horizontal (*A*).
E. C. horizontal (*A*). A face perpendicular to the axis.
A regular four-sided prism, terminated by a pyramid in a parallel position. E. C. horizontal (*A*).

SQUARE-PYRAMIDAL COMBINATIONS.

Remarks.	*Example and Figure.*	*Class.*	*Combinations.*
		Class I.	
Included in 4 by Mohs.	Sq. Pyr^1. Tin-ore Qr, ∞Q Mohs II. 102. *s* *l*	No. 5.	pQ, ∞Qr or pQr, ∞Q
Mohs, I. 5.	Sq. Pyr^1. Tin-ore ∞Q, ∞Qr II. 102. *l* *g*	No. 6.	∞Q, ∞Qr
So also Sq. Pyr^1. Garnet Q, Q2 II. 96. *c* *s* Q2 is marked $(P-1)^3$ in Mohs, being derived from Qr, by 3.		Class II.	pQ, qQn or pQr, pQn
	Sq. Pyr^1. Garnet Q, Q3, Q4 II. 96. *c* *s* *x*	No. 1. Mohs II. 1.	pQ, pQn
	Sq. Pyr^1. Garnet Qr, $\frac{1}{2}$Q3 II. 96. *o* *a*	No. 2. Mohs II. 1.	pQr, $\frac{p}{n-1}Qn$ $\overline{n-1}\,qQr$, qQn
		No. 3. Mohs II. 2.	$0Q$, qQn
	Sq. Pyr^1. Zircon ∞Q, Q3 II. 99. *l* *x*	No. 4. Mohs II. 3.	∞Q, qQn
If we have also qQ, the sides of the rhombs are parallel to the terminal edges of qQ.	Sq. Pyr^1. Zircon ∞Qr, Q3 II. 99. *s* *x*	No. 5. Mohs II. 3.	∞Qr, qQn

SQUARE-PYRAMIDAL SYSTEM.

Relations of Edges, &c.
A regular four-sided prism, terminated by a pyramid in a diagonal position. E. C. parallel to the rhombic section of the pyramid through its diagonal. Faces of pyramid are rhombs, and faces of prism, if they are complete. Faces of prism are rhombs if they do not meet: if they do, faces of pyramid are rhombs.
A regular eight-sided prism.
The forms are in a parallel position. The faces of the four-sided pyramid replace the apex of the eight-sided one, and are rhombs. E. C. parallel to the terminal edges of the four-sided pyramid. (*B*).
The forms are in a diagonal position. The faces of the four-sided pyramid replace the apex of the eight-sided one, and are rhombs. E. C. parallel to the terminal edges of the four-sided pyramid pQr (*B*). $\left\{\frac{p}{n-1}\right.$ Qn is the same as a form derived by the law of scalene derivation from pQr, according to $\left.\frac{n+1}{n-1}\right\}$. (*S. T.*)
A face perpendicular to axis. E. C. horizontal.
A regular square prism in a parallel position, with an eight-sided pyramid. The faces of the prism appear as rhombs replacing four lateral angles of the pyramid. The form of these rhombs depends on n.
A regular square prism in a diagonal position with an eight-sided pyramid. The faces of the prism appear as rhombs replacing four lateral angles of the pyramid. The form of these rhombs is similar to that of the section of the pyramid q*Q* through its diagonal, and E. C. parallel to this section.

SQUARE-PYRAMIDAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class II.	
	Sq. Pyr[1]. Titan-ore Mohs, Vol. II. 100. $\frac{1}{5}$ Q, $\frac{1}{4}$ Q $\frac{3}{5}$ *r* *s*	No. 6. Mohs II. 4.	$\frac{3}{2} pQ, pQ\,2$ $2 pQ, pQ\,3$ $\frac{5}{2} pQ, pQ\,4$ generally $\frac{n+1}{2} pQ, pQn$
P + n, (P + n − 3)[4], &c. of Mohs.	2 Qr, Q 2 *u* *z*	No. 7. Mohs II. 5.	$2 qQr, qQ2$ $3 qQr, qQ3$ $4 qQr, qQ4$ generally $nqQr, qQn$
P + n, (P + n − 3)[5], &c. of Mohs.	3 Qr, Q 2 *v* *z*	No. 8. Mohs II. 6.	$3 qQr, qQ2$ $4 qQr, qQ3$ $5 qQr, qQ4$ $6 qQr, qQ5$ generally $\overline{n+1}\, qQr, qQn$
These are included by Mohs in the preceding combination, because he derives Q 2 from Qr.	Sq. Pyr[1]. Garnet II. 96. 2 Q, Q 2 *b* *z* 4 Q, 2 Q 2 *r* *e* and 4 Q, Q 4 *r* *x*	No. 9. Mohs II. 7.	$2 qQ, qQ2$ $3 qQ, qQ3$ $4 qQ, qQ4$ $5 qQ, qQ5$ generally nqQ, qQn
		Class III.	pQm, qQn
From this parallelism of the edges of several pyramids we learn that they are coordinate (i. e. $q = p$).	Sq. Pyr[1]. Zircon Q 3, Q 4, Q 5 II. 99. *x* *y* *z*	No. 1.	pQm, pQn
	Sq. Pyr[1]. Garnet 2 Q 2, Q 2 II. 96. *e* *z*	No. 2.	pQm, qQm

SQUARE-PYRAMIDAL SYSTEM.

Relations of Edges, &c.
The forms are in a parallel position. The faces of the four-sided pyramid truncate the *scalene* edges of the eight-sided pyramid. (*S. T.*) The E. C. are parallel to themselves and to the truncated edges. . . . . and to the line bisecting the lateral edges of the four-sided pyramid.
The forms are in a diagonal position. The faces of the four-sided pyramid truncate the *principal* edges of the eight-sided pyramid. (*S. T.*) The E. C. are parallel to themselves and to the truncated edges. . . . . and to the line bisecting the lateral edges of the four-sided pyramid.
The forms are in a diagonal position. The faces of the eight-sided pyramid, adjacent to its *scalene* edges, appear in pairs in the place of the terminal edges of the four-sided pyramid. (*S. T.*) The E. C. are parallel to themselves and to the replaced edges. . . . . and to the scalene edges of the eight-sided pyramid.
The forms are in a parallel position. The faces of the eight-sided pyramid, adjacent to its *principal* edges, appear in pairs in the place of the terminal edges of the four-sided pyramid. (*S. T.*) The E. C. are parallel to themselves and to the replaced edges. . . . . and to the principal edges of the eight-sided pyramid.
The forms are co-ordinate pyramids in a parallel position. E. C. parallel to the terminal edges of pQ.
The forms are in a parallel position. E. C. horizontal.

SQUARE-PYRAMIDAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class III.	
	Sq. Pyr^1. Garnet $2Q2, \infty Q2$ Mohs, Vol. II. 96. e f	No. 3.	$pQm, \infty Qm$
In Mohs these are $(P)^4$ and $(P+1)^3$.	Sq. Pyr^1. Garnet $2Q2, Q4$ II. 96. x e	No. 4.	pQm, qQn when $pm=qn$ Thus $Q5, 3Q\frac{5}{3}$ $2Q3, 4Q\frac{3}{2}$
		Class IV.	Plagihedral.
In Mohs $\dfrac{\left(\frac{2\sqrt{2}}{2}P-2\right)^3}{2}$ a	Sq. Pyr^1. Scheel. Baryt. $\frac{r}{l}\ \frac{2}{3}Q2,\ \frac{l}{r}\ 2Q2$ II. 108. a b		$\frac{r}{l}\,pQm$ $\frac{l}{r}\,qQn$
		Class V.	Hemihedral.
In Mohs $\dfrac{\left(2\frac{\sqrt{2}}{3}P-3\right)^3}{2}$ e	Sq. Pyr^1. Copper pyr. $h'Q,\ h'\frac{1}{3}Q,\ h'\frac{1}{4}Q,\ h\frac{1}{6}Q3$ p' e d f II. 178.		$h\,pQm$ $h'\,q'Qn$

OBLONG-PYRAMIDAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combination.*
		Class I.	pP, qP
	Topaz $0P, \frac{2}{3}P, P, \infty P$ p s o m Mohs, Vol. II. 34.	No. 1.	pP, qP

SQUARE-PYRAMIDAL SYSTEM.

Relations of Edges, &c.
An eight-sided prism with an eight-sided pyramid. E. C. horizontal.
The faces of one eight-sided pyramid replace in pairs the principal terminal edges of the other pyramid. (*S. T.*) E. C. parallel to the edges replaced, and to each other.
Half the number of faces of eight-sided pyramids, taken alternately. The faces may be *corresponding* ones at the two summits, or *alternate* ones.
Half the number of faces of four-sided pyramids taken, and alternately at the two summits. (h and h'.) And, of eight-sided pyramids, the faces corresponding to such faces of the four-sided ones.

OBLONG-PYRAMIDAL SYSTEM.

Relations of Edges, &c.
A pyramid with its summit replaced by another pyramid. E. C. horizontal.

OBLONG-PYRAMIDAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combinations.*
		Class II. M.II,IV,V.	pP, qPm or pP, $qP'm$
P 2 is $(\breve{\mathrm{P}}\mathrm{r})^3$ in Mohs, and is ranged in Class IV.	Ob. Pyr^{1}. Serpentine P, P 2 Mohs, Vol. II. 33. *p* *n*	No. 1.	pP, pPm
	Ob. P^{1}. Mel.-glance P, P′ 3 II. 30. *p* *a*	No. 2.	pP, $pP'm$
N^{os}. 3 and 4 differ from 1 and 2 in that the faces replacing edges belong to pP in 1 and 2, to qPn and $q'Pn$ in 3 and 4.	P, $\frac{1}{2}$P 2 Fig. to No. 1. *n* *p*	No. 3.	qnP, qPn
	P, $\frac{1}{2}$P′2 Fig. to No. 2. *a* *p*	No. 4.	qnP, $qP'n$
		Class III. Mohs VI.	pP, qPr, $q'P'r$
	Ob. P^{1}. Olive-mal. P, P r II. 5. *p* *o*	No. 1.	pP, pPr
	Ob. P^{1}. Iron-ore P, P′r II. 4. *o* *p*	No. 2. M. VII. 1.	pP, $pP'r$
	P, Pr, $\frac{1}{2}$P′r *p* *o* *n*	No. 3. M. VII. 2.	pP, pPr, $\frac{1}{2}pP'r$
If the faces of a horizontal prism be rhombs, we can determine them.	∞P, Pr, P′r *z* *o* *p*	No. 4. M. VII. 3.	∞P, pPr, $pP'r$

OBLONG-PYRAMIDAL SYSTEM.

Relations of Edges, &c.
The *principal* terminal edges of the pyramid pPm are replaced by the faces in pairs of pP. E. C. parallel to each other and to the above principal edges.
The *transverse* terminal edges of the pyramid $pP'm$ are replaced by the faces in pairs of pP. E. C. parallel to each other and to the above terminal edges.
The *principal* terminal edges of the pyramid qnP are replaced by the faces in pairs of qPn. E. C. parallel to each other and to the above transverse terminal edges.
The *transverse* terminal edges of the pyramid qnP are replaced by the faces in pairs of $qP'n$. E. C. parallel to those edges.
The *transverse* terminal edges of the pyramid pP are replaced by the faces of the horizontal prism. E. C. parallel to each other and to these terminal edges.
The *principal* terminal edges of the pyramid pP are replaced by the faces of the horizontal prism. E. C. parallel to each other and to these terminal edges.
The faces of $\frac{1}{2}pP'r$ will appear as rhombs replacing the corners at the upper edge of the compound figure pP, pPr.
The form appears as a vertical rhombic prism terminated by a pyramid in a diagonal position. All the faces of the figure are rhombs.

OBLONG-PYRAMIDAL SYSTEM.

Remarks.	*Example and Figure.*	*Class.*	*Combinations.*
		Class III.	
	Pr, P′r o q	No. 5. M. VIII. 1.	*p* Pr, *p* P′r
	Pr, ∞ P′r x y	No. 6. M. VIII. 2.	∞ Pr, *q* P′r or ∞ P′r, *p* Pr
This is not to be confounded with square prisms 0Q, ∞ Q, and 0Q, ∞ Qr.	∞ Pr, ∞ P′r x y	No. 7. M. VIII. 3.	0P, ∞ Pr, ∞ P′r

The *Hemihedral* forms in this system, are those which contain the faces on one side of the axis only, at each summit. They are designated, as before, by the prefix h.

There are also *Tetartohedral* forms, which exhibit only one out of the four faces given by each law.

There are also forms in which the axis of the oblong pyramid is oblique, and these require other modes of investigation.

The Octahedral System, including the combinations of figures symmetrical in all directions (cubes, octahedrons, dodecahedrons, icositesserahedrons, &c.), requires to be treated in a manner somewhat different from the preceding, and will not be here considered.

The application of the preceding classification will be easily seen. In any proposed form, the relations of the faces and edges, being compared with the preceding descriptions, and with the figures here given, or with those referred to in Professor Mohs's Mineralogy, will shew to what Class and Number the combinations of its faces may be referred. This will give us data for discovering the symbols of its faces, and, by connecting such data, we are to obtain the laws of the derivation.

OBLONG-PYRAMIDAL SYSTEM.

Relations of Edges, &c.
The form is a right pyramid with a rectangular base.
A horizontal prism with vertical ends. E. C. parallel to terminal edges of the *finite* prism.
A right rectangular prism, the section of which is an oblong.

An example of this application in the *Rhombohedral System* is given in the Edinburgh Journal of Science for January, 1827. The following examples may shew its use in the Square-Pyramidal and Oblong-Pyramidal systems.

Fig. 4, (Mohs I. 67, 68.) belongs to the *Square-Pyramidal* system. The faces a alone would form some pyramid $p\mathrm{Q}$; the faces f, a pyramid $p''\mathrm{Q}$; the faces b alone, some pyramid in a diagonal position $p'\mathrm{Q}\mathrm{r}$; the faces c, an eight-sided pyramid $q\mathrm{Q}n$; the faces d and e, prisms $\infty\mathrm{Q}m$, and $\infty\mathrm{Q}m'$. Hence, the general designation of the form is

$$\begin{array}{cccccc} p\mathrm{Q}, & p'\mathrm{Q}\mathrm{r}, & q\mathrm{Q}n, & \infty\mathrm{Q}m, & \infty\mathrm{Q}m', & p''\mathrm{Q} \\ a & b & c & d & e & f. \end{array}$$

Let a belong to the fundamental pyramid: therefore, $p\mathrm{Q} = \mathrm{Q}$. The combination a, b agrees with p, l, Class I, No. 4. The combination b, e, would have horizontal edges; therefore, in the same manner,

$$\infty\mathrm{Q}m' = \infty\mathrm{Q}\mathrm{r}.$$

The edges of the pyramid b are truncated by the faces a, the edges of combination being parallel. Hence, by C, b is $2\,\mathrm{Qr}$.

The edges of combination of a and c are parallel to the terminal edges of a. Hence, by B, c is derived, by the second law, from a. Therefore,

$$q = p,\ q\mathrm{Q}n = \mathrm{Q}n.$$

If the faces a were removed, the combination b, c, would agree with c, a, in II. 2. Hence

$$\overline{n-1}\,q = 2, \quad \text{or } n = 2 + 1 = 3: \text{ and } c \text{ is } \mathrm{Q}\,3.$$

The combination c, f, supposing the other faces absent, would agree with x, r, II. 9. Hence, $p'' = 3$.

Therefore, the expression for the form is

$$\begin{array}{cccccc} \mathrm{Q}, & 2\,\mathrm{Qr}, & 3\,\mathrm{Q}, & \mathrm{Q}\,3, & \infty\mathrm{Q}, & \infty\mathrm{Qr} \\ a & b & f & c & d & e. \end{array}$$

Fig. 5, (Mohs I. 72.) is a form belonging to the *Oblong-Pyramidal* system. It is obvious that it may be thus represented

$$\begin{array}{cccccccc} p\mathrm{P}, & p'\mathrm{P}, & q\mathrm{P}n, & p''\mathrm{Pr}, & p'''\mathrm{P'r}, & \infty\,\mathrm{Pr}, & \infty\mathrm{P}m, & \infty\mathrm{P}m' \\ b & e & d & c & a & h & f & g. \end{array}$$

Since we have horizontal edges of combination between b, e, f, we will suppose the pyramid e to be the fundamental form. Therefore, as in I. 1, $\infty\mathrm{P}m$ is $\infty\mathrm{P}$.

Since c would truncate the edges of e, by III. 1, c is Pr, and $p' = 1$. The faces a would make rhombs with c and e, as in III. 3. Hence, a is $\frac{1}{2}\mathrm{P'r}$.

The faces a truncate the edges of b, as in III. 2. Hence, b is $\frac{1}{2}\mathrm{P}$.

The pyramid d would have its edges truncated by c and a respectively. Hence its edges make the same angles with the axis as the faces of those prisms do.

Now if a be the axis, b and c the semi-diagonals of the fundamental form, the axis and semi-diagonals of $q\mathrm{P}n$ are respectively qna, nb, c. And hence, we have

$$\frac{c}{qna} = \frac{c}{a}; \quad \frac{nb}{qna} = \frac{b}{\frac{1}{2}a}; \text{ whence } q = \tfrac{1}{2},\ n = 2.$$

Since g makes horizontal edges with d, and d is $\frac{1}{2}\mathrm{P}\,2$, q is $\infty\mathrm{P}\,2$. Hence, the expression for the form is

$$\begin{array}{cccccccc} \frac{1}{2}\mathrm{P}, & \mathrm{P}, & \frac{1}{2}\mathrm{P}\,2, & \mathrm{Pr}, & \frac{1}{2}\mathrm{P'r}, & \infty\mathrm{Pr}, & \infty\mathrm{P}, & \infty\mathrm{P}\,2 \\ b & e & d & c & a & h & f & g. \end{array}$$

W. WHEWELL.

TRINITY COLLEGE,
April 29, 1826.

XXIII. *Reasons for the Selection of a Notation to designate the Planes of Crystals.*

By the Rev. W. WHEWELL, M. A. F.R.S.

FELLOW AND TUTOR OF TRINITY COLLEGE, AND SECRETARY OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *Feb.* 11, 1826.]

In a communication read to the Society in the course of last year, I pointed out that crystals may be divided into four classes, according to their degree of symmetry. Being referred to an axis, they may consist of *three* similar and symmetrical portions about this axis, or of *four*, or of *two* opposite *pairs* of similar portions. They may also be symmetrical, with respect to *three axes at right angles* to each other. The merit of this classification has been a subject of controversy between Professor Mohs, and Professor Weiss, both of them persons to whom the science of Mineralogy is deeply indebted; but to whomsoever it is to be ascribed, it is a division so consonant to the nature of the case, and the mode in which we have to reason on crystalline forms, that there can be no doubt of its being, for the future, the proper and scientific view of the subject. Founded upon this view, it may be mentioned also, that Professor Mohs had given modes of deducing all derived crystalline forms, and of designating them by a notation, such as to render easy and general the process of deducing the laws according to which each is formed. Whether his mode of deduction is simpler than any previously proposed, is a question which must be determined by a consideration of the forms which really occur in nature, and the discussion of it must be reserved for another opportunity. That Professor Mohs' method does offer remarkable and valuable facilities for the determination of crystalline laws, is a fact which may be ascertained by examining the applications which have been made of it. The object of the following pages is to modify the notation, so that, besides answering this purpose, it may possess as much of the scientific beauties of simplicity and generality, as is consistent with the nature of the subject to which it is applied.

In speaking formerly of the above four classes of crystals, the terms were mentioned by which previous authors have designated them, and others were suggested. It is, however, reasonable that those who first navigate these new shores of science, should have the privilege of giving the names to the objects they discover, and I shall, therefore, adopt the denominations of preceding writers, with some alterations, which analogy seems to demand. The four classes will be termed the Rhombohedral, the Oblong-Pyramidal, the Square-Pyramidal, the Octahedral. And the fundamental forms which will be used in deducing the derived forms will be, in each instance, a pyramid. In the first class, an equilateral triangular pyramid; in the second, one with a rhombic base; in the third, one with a square base, and in the fourth, the half of a regular octahedron. The first class is called Rhombohedral both by Mohs, Weiss, and Breithaupt; the last, which I have called Octahedral, in order to refer it to a pyramid like the rest, is what is by these writers called Tessular, Hexahedral, &c. In the names of the second and third, Mohs and Breithaupt differ with each other, and neither appears to attend sufficiently to analogy. Mohs calls the square-pyramidal simply *pyramidal*, and the oblong-pyramidal, *prismatic*, in which nomenclature the two terms have in no way a relation corresponding to the relation of the forms, and might, with equal propriety be permuted. Breithaupt calls these two classes *rhombic* and *tetragonal*, or square, which is consistent as far as those two alone are concerned, but does not exhibit their relation to the others. I think the terms I have adopted ensure both these objects.

The selection of a notation to designate the relations of derived forms in crystallometry is by no means unimportant; for, if it be constructed with a proper symmetry, and a due regard to the analogies which prevail in these forms, it may be made the means of facilitating and compressing, in a remarkable degree, our reasonings with respect to the laws and connexions of crystallized forms. In fact, it has already been adapted and applied to this end, with great ingenuity and success, in the hands of Professor Mohs; and there is no one to whom the merit of making this great step in the science of mineralogy could more naturally fall, than to one possessed of his talents, perseverance, and extraordinary knowledge of the subject. In proportion as this process—I mean that of reasoning from our symbols—is valuable and important, it becomes desirable to perfect our notation as much as possible, and to carry to the greatest extent which they admit, the analogy and simplicity of our symbolical system. It is to be wished that perfection in this point should be attained as early as possible, in order that we may not have afterwards to alter a notation to which men's minds have become familiarized, and to learn afresh the alphabet of the science, which it is an irksome task to have even once to acquire. These considerations will excuse the attempt made in the following paper, and will also account for the liberty which I have allowed myself of pointing out some anomalies in Professor Mohs' System of Notation. Without some strong reasons, such as those above mentioned, any endeavour to alter a notation, in which so much information is embodied, as is contained in his treatise, would be inexcusable. Nothing, indeed, can be more likely to produce confusion and unnecessary labour than the levity and restlessness in proposing new symbols, which some writers allow themselves: and no notation at all deserves consideration, which does not by its structure exhibit the analogies of the things represented, so as either to facilitate our reasonings on them, or, at least, our understanding of them. I hope, however, to be able to shew that what I have proposed in the following pages introduces no alterations, except such as are requisite to maintain the symmetry, and, if I may so speak, the *homogeneity* of the notation: and, having made it my object to leave no anomalies which appeared capable of removal, I trust the evil of change will be more than compensated by the establishment of a system, which there will be no future necessity to remodel. The merit of this improvement will hardly diminish the obligation due to the first introducer of an efficient notation.

In order to point out the reasons for the proposed changes, it may be allowed to introduce, and refer to, some general principles which ought to regulate all scientific notation, and which, though they come with no authority beyond their own reasonableness and utility, will nevertheless, I think, be acknowledged as true by all who give the subject a scientific consideration. They will offer themselves as we review the different points of the system.

It will be seen that the indices which I have used, are generally the same as those used by Mohs, and that the difference lies principally in the mode of writing and combining them: hence, my system has the same means of obtaining results which his possesses, with so much additional symmetry as was consistent with the retention of this property.

1. Mohs designates the fundamental rhombohedron by *R*, and the principal series of rhombohedrons by

$$\ldots R-3,\ R-2,\ R-1,\ R,\ R+1,\ R+2 \ldots$$

In the same manner, in the rhombic and square-pyramidal systems, he designates the fundamental pyramid by *P*, and the principal series of derived pyramids by

$$P-2,\ P-1,\ P,\ P+1,\ P+2 \ldots$$

This notation violates the principle *that the signs + and – cannot properly be used to connect signs of form* (*R* or *P*) *with signs of quantity* (2, 3, &c.). Such ideas are altogether heterogeneous, and should not be combined as if they were capable of addition or subtraction.

It may be said that in this notation -2, -1, 1, &c. are not quantities added, but indices. If so, they should be so written and the principal series would be then

$${}_{-2}R,\ {}_{-1}R,\ R,\ {}_{1}R,\ {}_{2}R,\ \text{\&c.}$$

which would be free from the above objection.

This designation of the principal series is somewhat simpler than ours, which is

$$2^{-2}R,\ 2^{-1}R,\ R,\ 2^{1}R,\ 2^{2}R,\ \text{\&c.}:$$

where the exponent of the root 2, in this case, is the index in the system previously mentioned. And this simplicity would be a reason for preference, if the series here referred to contained the only forms which are to be taken into consideration. But, in using this notation, Mohs is compelled, when forms occur which are not in this series (subordinate series), to adopt a new convention for expressing them; whereas in our method, the principal, and the subordinate series, are cases of the same general notation. And the law of the principal series is sufficiently evident in our symbols.

2. This principal series

$$R-2,\ R-1,\ R,\ R+1,\ R+2\ldots$$

$$\text{and}\ P-2,\ P-1,\ P,\ P+1,\ P+2\ldots$$

in another system, are, by Mohs, indicated by the same notation, though they are derived in different ways. In the first series the axis is doubled, and the base turned through 180°. In the second, or oblong-pyramidal system, the axis is doubled simply. In the square-pyramidal system, the base is turned through 45°, and the axis increased in the ratio $1 : \sqrt{2}$.

The principle on which these series are constructed by Mohs is, that each member of the principal series shall truncate the preceding member. But in the oblong-pyramidal system this analogy is violated by omitting the alternate members.

3. The subordinate series, or forms whose axes do not agree with terms in the principal series, are represented thus, $mR+n$, which is used to indicate that the axis is m times that of the form $R+n$. It is manifest that since the m thus refers to the whole $R+n$, the symbol ought to be $m\,.\,\overline{R+n}$, or rather, as has been said, mR_n, or $m\,.\,{}_nR$. But, even with this alteration, it would follow that m and n, indicating operations of the same kind, viz. an alteration of the axis, are different in their mode of writing, and in their effect in calculation; m indicating an increase of the axis in the ratio $1 : m$; while n indicates an increase in the ratio $1 : 2^n$.

It is a principle to be adopted in notation, that *symbols which are identical in their meaning, should appear to be so by the application of the rules of algebraical operation;* thus, $2\,.\,2R$ and $2^2\,.\,R$ are the same in our notation. But, in Mohs' notation, we have expressions 2^2R+1 and $R+3$, which are identical without the identity appearing in the form of the expression. *Vice versâ,* we have, in the same rhombohedral system,

$(P+n)^1$, which is not the same form as $P+n$, to which the symbol is algebraically equal. It is this principle of identity of meaning, corresponding with identity of numerical value, which mainly assists us in employing symbols as instruments of reasoning. The algebraical reductions which we make, represent then different ways of considering the same form.

4. Forms derived from $R+n$ in a certain manner (namely, by the law of scalene derivatives,) are, by Mohs, represented by

$$(R+n)^2, \quad (P+n)^3, \quad \&c.$$

It may be observed, first, that there is no sufficient reason in this case for altering the fundamental letter R into P. Mohs does this because the Rhombohedron becomes a six-sided Pyramid, which change is marked by adopting the new initial letter: but it would have been more important to mark that this derivation was *from* the rhombohedron; and to leave it to be recollected that its form was a pyramid. The expressions would then be

$$(R+n)^2, \quad (R+n)^3, \quad \&c.$$

5. But, in this notation, the exponents 2, 3, are used merely as indices: and there is no propriety in writing them in such a manner as to represent the *powers* of the symbols $R+n$. To refer again to the principle in the third observation, this would have been proper, if, by this means, reduction had given us different but identical symbols: if, for instance, $(2R)^3$ had been the same as $2^3(R)^3$; but this is not at all the case. These indices, therefore, ought to be written where they have no algebraical meaning, and, therefore, cannot give wrong coincidences. If we write both these numbers and those mentioned in Observation 1. as indices before and after the foot of the letter, we shall have for

	$(R+2)^5$,	$(R-3)^2$, &c.
these	${}_2R_3$	${}_{-3}R_2$, &c. which agree very nearly with
	$2^2 R\,3$	$2^{-3} R\,2$, &c. the symbols here proposed.

6. A rule is given, Mohs, Sect. 96, to refer the members of the subordinate series to the members of the principal series, whose axis comes *nearest* to them. This rule introduces complicated expressions unnecessarily, because the subordinate form may have a very complicated ratio to the *nearest* principal form, though a very simple one to some other. Thus, we have a form in the square-pyramidal system, which is represented by

$$\frac{2\sqrt{2}}{3} P - 3;$$

its axis being, therefore,

$$\frac{2\sqrt{2}}{3} \cdot \frac{a}{2\sqrt{2}} = \frac{a}{3}.$$

Hence, if we refer it to P, the fundamental form of it will be $\frac{1}{3}P$. By this mode of designating the forms (particularly in the square-pyramidal system), we lose sight of the simplicity of the law by which they are deduced.

7. In the same manner in which Mohs changes R into P, when the rhombohedron becomes a pyramid, he changes, in the pyramidal system, P into Pr, when the pyramid becomes a prism. Here, however, some alteration is necessary, because the indices, in this case, become infinite, and, therefore, the symbol inconvenient. I have adopted, in some measure, this part of the notation, but with these advantages, that the r added indicates, in our method, a particular process, the analogy of which runs through all the systems; and, also, that the letter P, which recurs in the symbol, marks the fundamental form from which the deduction is made.

8. Analogous to this process is the deduction, in the rhombohedral system, of the isosceles six-sided pyramid (Rr) from the rhombohedron. This deduction, however, is, by Mohs, designated in a different manner, namely, by the change of R to P. And it may be again observed that this change from R to P is different from the same change made, by Mohs, in the manner mentioned in Observation 4, where both are in brackets, and indicates a process altogether different.

9. We may observe, also, that Mohs has several other signs, which offend against the rule of *having the symbols as homogeneous, and the conventions, from which they derive their meaning, as few and as general as is possible.* Thus, the prosodical marks ‒ and ◡ are introduced over the P and Pr to indicate the derivation being made from the extremity of the *long* or *short* diagonal. And it may be observed also here, that the comparative length of these diagonals has nothing to do with the derivation; but only the consideration of one as principal, and the other as transverse to it.

Again, the crotchets [ ] are used to mark that the limiting prism is in the diagonal position, which notation has no analogy with the other parts of the system, (Sect. 102.)

Again, the negative sign, Sect. 128, is used to indicate a second half of a derived form. This also is suggested by no analogy, except in the case where the half form is the same as $-1 . R$ according to the definitions, and may, therefore, be properly represented by $-R$. I have employed an accent (as in R') to indicate this change, which mark is also employed in an analogous manner in the next paragraph.

10. In such cases, where we subject a form to some conditions, after we have established the way in which it is deduced; it seems better that the part of the symbol which represents these conditions should be separate or separable from the rest. Thus,

to indicate the half of a form nRm, it will be better to write h before it, than to write it $\frac{nRm}{2}$, as is done in this part of Mohs' notation.

The notation $\frac{\mathrm{r}}{\mathrm{l}}$, $\frac{\mathrm{r}}{\mathrm{r}}$, &c. by which he designates the direction (right or left) in which hemihedral faces follow each other round the figure, both in the upper and under halves of the form, is convenient and significant, and seems not open to any sufficient objection.

As it is only in cases of hemihedral combinations that these symbols occur, it does not appear necessary to indicate this by the denominator 2. I have, therefore, written simply $\frac{\mathrm{r}}{\mathrm{r}}\, nRm$, and not $\frac{\mathrm{r}}{\mathrm{r}}\, \frac{nRm}{2}$, or $\frac{\mathrm{r}}{\mathrm{r}}\, \mathrm{h}nRm$.

If one of the hemihedral portions arise from the fundamental form in a transverse position, this circumstance is indicated by giving an accent to the corresponding letter of the symbols $\frac{\mathrm{r}}{\mathrm{r}}$, &c. Thus, $\frac{\mathrm{r}}{\mathrm{r}'}\, nRm$ indicates that half the faces of nRm, at one apex, are combined with half the faces of $nR'm$ at the other, both portions being turned to the right.

11. In the octahedral or tessular form, Mohs uses a different conventional letter for almost every different species of form. To this, perhaps, there is no great objection, as, from the regularity of these forms, the facility of reasoning does not depend much upon the generality of the symbols. It may be observed, however, that, according to our system, all these forms are provided with symbols, according to the analogy of the other systems of crystallization.

12. The designation of twin crystals by Mohs seems also

deficient in simplicity and symmetry; but this will be considered hereafter.

Removing then these objections; placing the indices in the simplest manner, namely, before and after the letter; altering the first index so that the same law shall obtain for principal and subordinate series of forms; indicating each of the four systems of crystallization by a distinct letter, and pursuing this in all the derived forms; we have the above system of notation.

13. We shall notice some of the properties which appear to recommend the preceding system of notation, by their simplicity and uniformity.

The fundamental forms are four. These are in our method designated by the successive letters O, P, Q, R, which appear in each of the derivatives of these respective systems, and thus indicate to which class each derived form belongs. These may be called *fundamental letters*.

Every derivative is represented by the letter designating the fundamental form, attended by two indices, or numbers, one of which is always written before and one after the letter.

In those cases in which we have only one number or index, as mP, $2R$, $Q3$, the other index is 1, and may be so written, if convenient, for the sake of analogy. Thus,

$$0O \text{ is } 0O1, \quad Q \text{ is } 1Q1, \text{ \&c.}$$

The index which is written before the fundamental letter always designates the same law of derivation in all the systems; namely, the derivation of one pyramid from another, by changing the axis in the ratio of the index.

The index which is written after the fundamental letter also designates a law of derivation, which is the same in all the systems.

This law may be thus expressed.

A diameter is drawn to the base of any pyramid; (*MC* in Figs. 1, 2, 3.)

This diameter is increased (or diminished) in the ratio of the index m (producing MC').

Through the angle (*B*,) and the extremity of this diameter, parallel to the edge of the original pyramid, (*CV*,) is drawn a plane ($BC'V'$).

The figure bounded by all such planes is taken.

By symbols thus constructed, we can represent any plane whatever, and it has been seen in a preceding Paper that the mode of intersection of different planes will afford us the means of recognizing their origin and law.

For the sake of convenience, two other symbols are introduced.

1st. The letter r after the fundamental letter.

For $\frac{2}{3\,\infty} R \infty$, we have put Rr,

for $\frac{1}{\infty} P \infty$ .... Pr,

for $\frac{1}{\infty} Q \infty$ .... Qr,

for $\frac{1}{\infty} O \infty$ .... Or.

The letter r is added to the fundamental letter, when m, the second index, becomes infinite, and the axis of the fundamental pyramid remains unaltered.

2d. An accent added to the fundamental letter in two cases.

R' is used to designate the fundamental pyramid of the rhombohedral system, its base being turned through two right angles, or placed in a *transverse position.*

P' is used to designate the fundamental pyramid of the oblong pyramidal system when the derivation is made from the *transverse angle* of the base.

Derivatives of R' and P' are obtained and designated in the same manner as for R and P.

In cases of hemihedral combinations, we have also the symbols h, $\frac{\text{r}}{\text{l}}$, $\frac{\text{r}}{\text{r}'}$, &c. as explained in paragraph 10.

W. WHEWELL.

TRINITY COLLEGE,
Feb. 11, 1827.

EXTRACTS

FROM COMMUNICATIONS MADE TO THE SOCIETY.

I. *On an Artificial Formation of Plumbago.*

EXTRACT of a LETTER from JAMES ALDERSON, Esq. to the Rev. Professor CUMMING.

[Read February 21, 1825.]

THE accompanying specimen of Carburet of Iron formerly composed part of a groove in which a patent perpetual log was made to slide. It was originally cast iron, and was nailed to the coppered stern post of the ship Zoroaster, of the port of Hull.

On the return of the ship from India, after an absence of about nine months, it was found as it now remains, converted into Plumbago, having lost very considerably in weight, but having perfectly retained its original form.

Pembroke College, Cambridge,
Jan. 22, 1825.

REMARKS BY PROFESSOR CUMMING.

ONE of the most striking properties of this artificial Plumbago, is its extreme levity as compared with that of the substance from which it is obtained.

The specific gravity of soft cast iron is about 6.94, of the hard 7.54; the average may therefore be considered as 7.24 containing about 5 per cent. of carbon; but the plumbago resulting from it, when perfectly dry, floated not only on water, but, for an instant, even on alcohol. With the view of freeing it entirely from the air included in its pores, a portion weighing 226.2. grains was boiled in distilled water, by which, notwithstanding it deposited two grains of ochre, it gained 214.8 grains. Its specific gravity was then taken, and was found to be 1.26. It appears therefore that nearly all the metallic iron had been gradually converted into a soluble muriate, which, as it was formed, was removed by the passage of the vessel through the water; leaving only the carbon in the interstices of which it originally existed. The average specific gravity of the Borrowdale graphite is 2.13 and it contains $4\frac{1}{2}$ per cent. of metallic iron; the inferior specific gravity of the artificial specimen made it probable that it contained still less, which was confirmed by deflagration, in the usual mode, with nitre. The exact quantity seems however of no importance; as there can be no doubt that the same process which had removed so great a portion of the iron, would, if continued, have left nothing but the carbon. For the agent being evidently Galvanism, exerted in precisely the same manner as in its recent application by Sir H. Davy, would have acted upon the iron, through the intermedium of the carbon, until it were entirely removed.

In regard to the useful application of this substance, there can be little doubt that in many cases it may be substituted for the coarser kind of plumbago; but it is so much softer that it cannot be used for the purposes of drawing, unless as a crayon. The colour of its streak on paper is not so dark as that of plumbago, and instead of presenting a continuous line, the streak appears, when examined by a lens, to be composed of minute detached portions of carbon, intermixed with specks of metallic iron.

In the 1st Volume of the Annals of Philosophy, Dr. Henry of Manchester has given an account of a similar substance found in a colliery near Newcastle, the water of which was strongly impregnated with the muriates of lime, magnesia and soda. The specific gravity of his specimen was 2.155, the decomposition had therefore evidently not been so complete as in the present instance. It had been known, so long since as in the time of Scheele, that though the affinity of iron for muriatic acid is less than that of soda, lime, or magnesia; yet when either of these substances, in the state of a muriate, is present in great excess, it is itself decomposed and a muriate of iron is formed. Hence Dr. Henry seems to have attributed the corrosion of the cast iron solely to the action of the muriates of lime and magnesia. But it has been recently proved by Dr. Marcet that muriate of lime does not exist in sea water; and though, by long continued action, the muriate of magnesia alone might possibly be adequate to the decomposition, without having recourse to galvanic agency; yet, as muriate of soda makes a powerful galvanic circuit between dissimilar metals, which undoubtedly existed in contact in the present instance, and probably in the case cited by Dr. Henry (as the corrosion seems to have taken place at the junction of two pipes) there appears to be no necessity for the rather improbable supposition that chemical affinity alone had effected so complete a decomposition in so short a time. How far the original formation of native graphite may be attributed to a similar cause, is so much a matter of hypothesis, that I shall not waste the valuable time of this Society by attempting to discuss it.

II. *Experiments on Percussion, made on a spring of Memel Deal, 20 feet long; Weight 16.87 lbs. Weight of falling body 3.222 lbs. Deflection by steady weight of 6 lbs. = 1 inch.*

By B. BEVAN, Esq.

Inches Fall.	Inches Deflection.	Squares.
2	1.33	1.7689
4	1.68	2.8224
8	2.08	4.3264
16	2.60	6.76
32	3.48	12.1104

After adding to the middle of the spring, 4.85lbs.

2	1.37
4	1.60
8	1.90
16	2.26
32	2.95

After adding 10lbs. to the spring instead of the above:

2	1.23
4	1.36
8	1.65
16	2.35
32	3.10

III. *Table of the Computed and Observed Variations of the Magnetic Intensity at the Earth's Surface.*

By R. W. ROTHMAN, Esq. M. A.

FELLOW OF TRINITY COLLEGE.

The formula used in the computation is $\sqrt{4-3\,(\text{sin. dip})^2}$.

	Dip.	Observed Intensity.	Computed Intensity.	Differences.
Peru	0° 0′ H	1.0000	1.0000	
J. Carlos del Rio Negro	20 22 H	1.0457	1.0484	+ .0027
Carichana	30 19 H	1.1574	1.1121	− .0453
Mexico	42 10 H	1.3155	1.2290	− .0865
Paris	68 38 H	1.3482	1.6914	+ .2432
London	70 33 S	1.4142	1.7326	+ .3184
Christiana	72 30	1.4959	1.7737	+ .2778
Arendahl	72 45	1.4756	1.7790	+ .3034
Brassa	74 21	1.4941	1.8118	+ .3177
Hare Island	82 49	1.6939	1.9547	+ .2608
Davis's Straits	83 88	1.6900	1.9584	+ .2684
Baffin's Bay	84 25	1.6685	1.9721	+ .3036
———	84 39	1.7349	1.9743	+ .2394
———	84 44	1.6943	1.9751	+ .2808
———	85 55.5	1.7383	1.9850	+ .2467
———	86 9	1.7606	1.9866	+ .2260

Nov. 10, 1825.

A

LIST OF DONATIONS

TO THE

LIBRARY AND MUSEUM

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY.

I. *Donations to the* LIBRARY.

1822.		DONORS.
May 6.	BERZELIUS on the Blowpipe	J. G. Children, Esq.
	On the Structure of the Apophyllite	Dr. Brewster.
Nov. 11.	Translation of Venturoli's Mechanics	Dr. Cresswell.
	On calculating Tables by Machinery	C. Babbage, Esq.
	Periculum ad Cippos Punicos	Prof. Reuvens.
	Diatribe de monumentis Punicis	Prof. Hamaker.
	Observations on the Nautical Almanack . . .	J. South, Esq.
	Tracts on Political Economy	W. Spence, Esq.
Nov. 25.	Art of Embalming	Dr. F. Thackeray.
	Sur la Lumière des Ondes	— Spooner, Esq.
1823.		
Feb. 17.	Monographia Apum Angliæ	Rev. W. Kirby.
Mar. 17.	Treatise on Dynamics	Rev. W. Whewell.
April 14.	Geological Transactions, new series, Vol. I. Part 1.	Geological Society.
April 28.	Winch's Geographical distribution of Plants of Northumberland, &c.	J. Hogg, Esq.

1823.		Donors.
Nov. 10.	Experiments at Hafod Copper Works. . . .	Swansea Society.
	Jerusalem Delivered, Book 4.	J. H. Wiffen, Esq.
	Treatise on Optics.	Rev. H. Coddington.
	Address to the Hull Philosophical Society . .	Dr. J. Alderson.
	Essai Historique sur le Probleme des trois Corps.	A. Gautier.
	Sur quelques Observations Astronomiques. .	———
	Transactions of Royal Society of Edinburgh, Vol. IX. Part 2.	Royal Soc. of Edin.
	Sur les Axes de Rotation.	M. Ampere.
	Sur L'Electro-dynamique.	———
	Sur la Theorie du jeu.	———
	Sur une Example de Reunion de Coquilles marines et fluviatiles.	C. Prevost.
	Sur le gres coquiller de Bernchamp.	———
	Essai sur le basin de Vienne.	———
Nov. 24.	Solution of the higher order of Equations. .	Rev. J. Buck.
Dec. 8.	24 Nos. of Tilloch's Philosophical Magazine	Rev. L. Jenyns.
	6 Early Nos. of the Transactions of the Royal Society.	T. Thorp, Esq.
1824.		
Mar. 1.	Classical Journal, No. 55.	Prof. Reuvens.
	On Slavery in the West Indies.	Jas. Stephen, Esq.
	History of Persia.	Sir J. Malcolm.
Mar. 15.	Geological Map of country round Bath. . . .	Rev. W. Conybeare.
May 3.	Transactions of Royal Society of Edinburgh, Vol. X. Part 1.	Royal Soc. of Edin.
	Astronomical Tables.	Dr. Pearson.
Nov. 15.	Treatise on Mechanics.	Rev. W. Whewell.
Dec. 13.	Museum Britannicum.	Rev. L. Jenyns.
1825.		
Feb. 21.	Memoirs of the Astronomical Society, Vol. I. Part 2.	Astronomical Society.
	On Flustra Arenosa.	J. Hogg, Esq.
	View of the College of Calcutta.	Rev. R. Duffield.

		Donors.
1825.		
Mar. 7.	Transactions of the Geological Society, new series, Vol. I. Part 2..................	Geological Society.
	Various Tracts.........................	Dr. Hagarth.
Mar 21.	Portrait of Saussure......................	Rev. D. Pettiward.
Nov. 28.	Transactions of the Royal Irish Academy, Vol. XIII..........................	Royal Irish Academy.
1826.		
Feb. 13.	Martyn's History of the Academy at Paris..	T. Thorp, Esq.
	No. 439 and Vol. XXXIX. of the Philosophical Transactions.....................	————
	Imperial Almanack for 1826.............	R. Sheepshanks, Esq.
	Dissertation sur la Theorie de la Vision ...	G. Maurice.
	Memoire sur les Apparences visibles.......	————
April 10.	On the Magnetizing Influence of the Sun's Rays.............................	Mrs. Somerville.
	Observations on the Longitude of Paris and Greenwich..........................	J.F.W. Herchel, Esq.
	Memoirs of the Astronomical Society of London, Vol. II. Part 1...............	Astronomical Society.
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1826.		DONORS.
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1824.		
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1825.		
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1827		
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The Author of the Paper "On Achromatic Eye-pieces," &c., has to apologize for an error in his statement of the principle of Dr. Blair's object-glass, (in Article 7 of that Paper, page 239). The construction finally adopted by Dr. Blair was, to inclose a mixture of muriatic acid with a muriatic salt between two lenses of the same sort of glass. By varying the proportions of the mixture it was found possible to form a fluid in which $\frac{\delta n'}{n'-1}$ was exactly expressed by the formula $c\,\frac{\delta n}{n-1}$.

Fig. 3

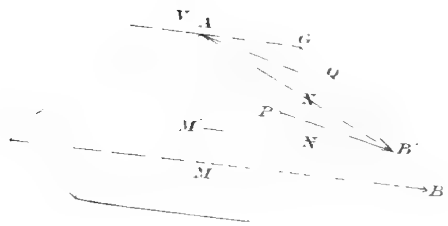


Fig. 1

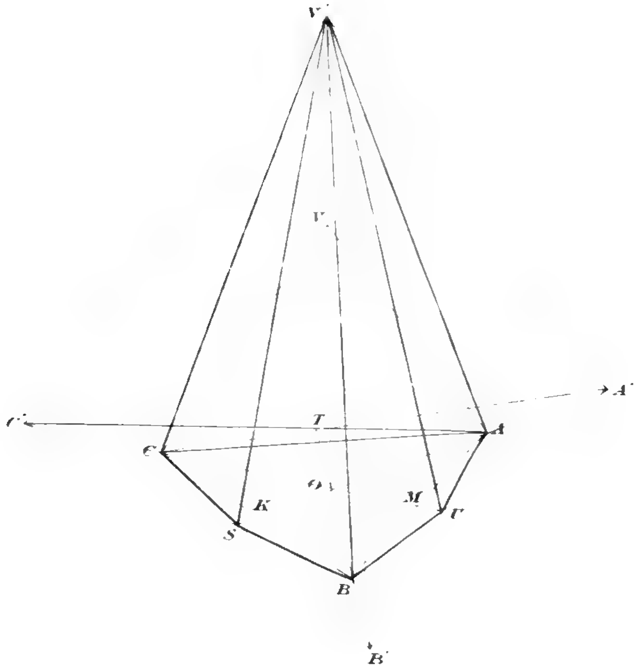
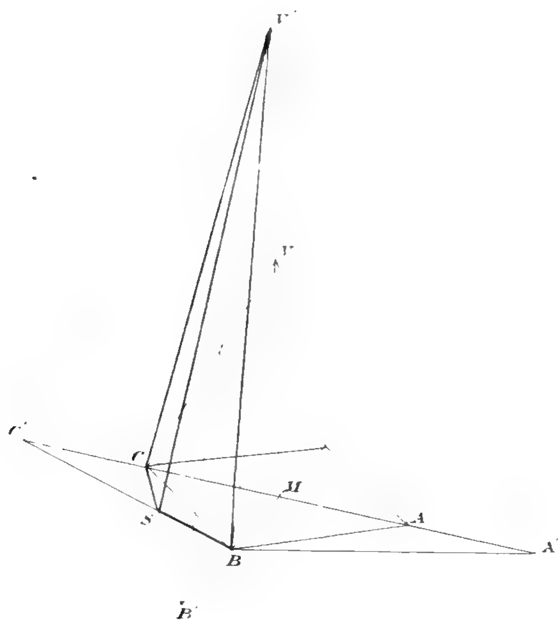


Fig. 2



RHOMBOHEDRAL SYSTEM.

CLASS 1. *CLASS 2.* PLATE 2

RHOMBOHEDRAL SYSTEM

CLASS III. *CLASS IV.*

OBLONG PYRAMIDAL SYSTEM.

CLASS I.

1

CLASS II.

1

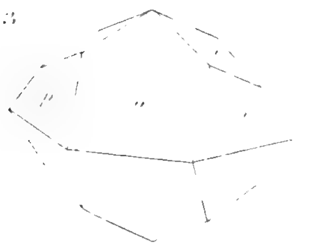
2

CLASS III.

1

2

3



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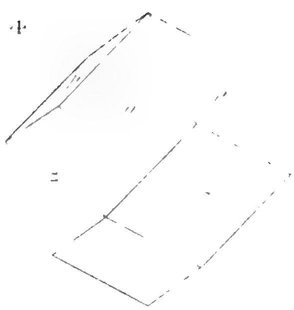

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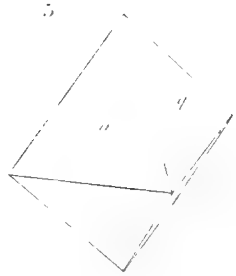

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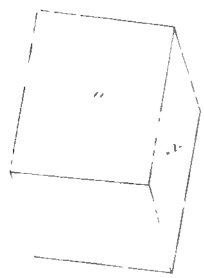

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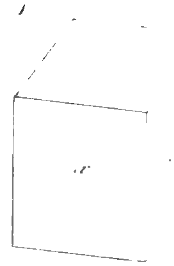


Fig 4

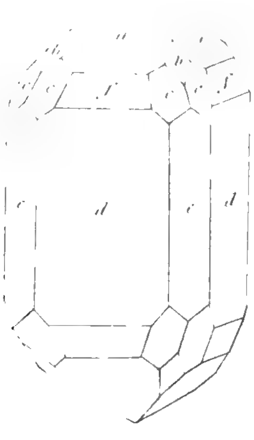
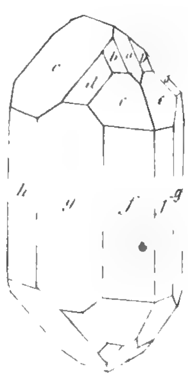


Fig 5



SQUARE PYRAMIDAL SYSTEM.

CLASS I. CLASS II. CLASS III. PLATE 5