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I. *On the arc of the meridian measured in Lapland.* By I. TODHUNTER, M.A., F.R.S.

[Read *May* 1, 1871.]

1. THE arc of the meridian measured in Lapland, although of small extent, and therefore unimportant when compared with the very large arcs which have been since measured, is of great interest in an historical point of view. The measurement of this arc in 1736 and 1737 settled a controversy which had been carried on for some time as to the figure of the earth. The eminent observers of the family and school of Cassini maintained that the polar diameter of the earth was larger than the equatorial diameter; while Newton and his followers maintained the contrary opinion. The result of the operations carried on in Lapland was decisive against the Cassinian theory; but it introduced another difficulty into the subject, and thus stimulated theorists and observers to further exertions. For the ellipticity of the earth obtained from the use of the measurement in Lapland seemed too large to harmonize with the result of other investigations. And accordingly the arc was remeasured at the beginning of the present century.

2. I have recently had occasion to study the details of the two measurements of the arc in Lapland in connexion with a work on the history of the Mathematical Theories of Attraction, and of the Figure of the Earth, which has for a long time engaged my attention. I have been surprised to find that the accounts of these operations, which are most easily accessible, although written by very distinguished astronomers, contain numerous and serious errors; and I have thought it would be interesting and instructive to point out these errors and supply the appropriate corrections. I do not propose to give a narrative of all the circumstances of the two operations, but only such a sketch as will render intelligible the critical remarks which form the substance of this memoir.

3. The controversy which was maintained between the Cassinians and the Newtonians as to the figure of the earth, induced the French Academy to procure the measurement of a degree of the meridian near the equator, in order that the question in dispute might be settled by comparing this length with what was considered to be the known length of a degree of the meridian in the latitude of Paris. Accordingly Bouguer, La Condamine, and Godin started in May, 1735, on this expedition for Peru.

Shortly afterwards another expedition was arranged for measuring an arc of the meridian as near as possible to the pole. This task was entrusted to four members of the Academy, Maupertuis, Clairaut, Camus, and Le Monnier. Moreover l'abbé Outhier, who was a correspondent of the Academy, and Celsius, who was professor of astronomy at Upsal, were associated with the four members of the Academy.

The Arctic expedition left Paris in April, 1736.

4. Two accounts of the French expedition were published by members of the party, Maupertuis and Outhier.

Maupertuis published *La Figure de la Terre déterminée*....Paris, 1738. The greater part of this work is reproduced in the *Histoire de l'Académie*...for 1737, which appeared in 1740. The work of Maupertuis was translated into various languages. See La Condamine *Journal du Voyage*,...page iii. I have a German translation which was published at Zürich in 1741, and a Latin translation which was published at Leipsic in 1742. I shall make my references to the original French edition of Maupertuis's work.

Outhier published *Journal d'un Voyage au Nord*....Paris, 1744. This work seems to be scarce and little known; I have used the copy which is fortunately in the University Library. I may mention that the list of plates does not quite correspond with the plates in this copy, which is the only one I have seen.

5. The calculations and the theoretical deductions are given most fully by Maupertuis; the details of the daily occupations of the members of the party, and the peculiarities of the country and of the inhabitants, are given most fully by Outhier.

6. It will be convenient to name some other important authorities to whom I shall have frequent occasion to refer.

The original account of the remeasurement of the arc at the beginning of the present century is that published by Svanberg under the title *Exposition des opérations faites en Lapponie*....Stockholm, 1805.

In various volumes of Zach's *Monatliche Correspondenz* papers will be found which relate to the Swedish survey. They are interesting from the fact that they are nearly contemporary with the operation to which they relate. In Vol. xii., page 425, of the Journal references are given to the preceding papers on the subject.

The *Connaissance des Tems* for 1808 contains in pages 466...479 what may be called a review of Svanberg's book. This review may be safely attributed to Delambre, who was then the editor of the *Connaissance des Tems*. In fact on the first page of the review the writer speaks of himself as having caused the repeating circle to be made; and from page xv of the preface to Svanberg's book we identify the writer with Delambre.

In the numbers 121 and 122 of the *Astronomische Nachrichten*, published in 1827, there is an elaborate discussion by Rosenberger of the French measure of the arc.

The well-known treatise on the Figure of the Earth in the *Encyclopædia Metropolitana*, by G. B. Airy, the present Astronomer Royal, is dated 1830.

7. In referring to the works of which I have spoken I shall give the initial letter of the author's name, followed by the page of the work to which I invite attention.

8. We return to Maupertuis and his companions. There was some doubt at first as to the most eligible place for measuring the arc of the meridian. M. 11; O. 3, 52. Finally Tornea was taken for the most southern station, and Kittis for the most northern station; these are both close to the river Tornea and nearly on the same meridian. The other stations were mountains not far from the river.

9. All the geodetical angles were observed in the space of about two months, between the beginning of July and the beginning of September, 1736. The observations were made with a quadrant of two feet radius. M. 33, 79; O. 204...219.

10. The step next taken was probably the most difficult and the most important of all, namely to determine the amplitude of the arc, that is the difference between the latitudes of the extreme stations. The star δ Draconis was selected, which passed the meridian very near to the zenith of both places. Observations of this star were made at Kittis on five days of October, and at Tornea on five days of November. The instrument employed was a zenith sector constructed by Graham, at London; it resembled that with which Bradley had made the observations which established the aberration of light. The instrument was not employed to determine the absolute zenith distance of a star, and hence the latitude of the place, but only the *difference* between the zenith distances of the same star at two places, and the consequent *difference* of latitudes. M. 37.

11. It is necessary to observe very carefully the fact recorded at the conclusion of the preceding Article. The contrary has indeed been stated: thus "The latitudes were observed with a sector made by Graham" "...... the latitudes were re-observed......" A. 169. These statements are of course erroneous, and, as we shall see, the error has exerted a fatal influence.

12. The observations for determining the difference of latitude required corrections for aberration, for precession, and for a third inequality which had been recently discovered by Bradley, and which is doubtless what we now call nutation. M. 44, 123. No correction was made for refraction. M. 125. The final result was that the difference of latitude appeared to be $57' \, 26''\cdot93$. M. 104, 124.

13. The base was measured on the frozen surface of the river Tornea, very nearly in the direction of the stream; the extremities of the base were on the land. The measurement of the base was begun on December 21st, and occupied a week. Eight rods of fir were employed each five toises long.

14. It has been stated however that the rods were each *four* toises long. A. 205. I am at a loss to account for this statement, as both the original authorities concur in giving *five* toises or thirty feet. M. 49; O. 137. It would I think be advantageous for the sake of accuracy to have the rods as long as possible.

15. The correct length of the rods of fir was determined by the aid of an iron toise which had been carefully adjusted to the length of the standard toise at Paris. M. 49; O. 137. This toise has since taken its place in the history of the subject as the *Toise du Nord.*

It is necessary to say a few words on this Toise. An iron toise had in like manner been taken by the party despatched to Peru, which has been since known as the *Toise du Pérou.* It is curious that neither Maupertuis nor Outhier alludes to the Toise of Peru. We know however that the Toise of Peru and the Toise of the North were made by the same artist, and that they were carefully compared and found equal before the expedition started for Peru. See La Condamine *Mesure des trois premiers degrés*...page 75.

On the return of Maupertuis from Lapland to France he was shipwrecked in the Gulf of Bothnia. He barely alludes to the misfortune himself, but we find from Outhier that the instruments were immersed, and were cleaned rather more than a month after the accident. M. 78; O. 169, 189.

La Condamine gives an account of these and other toises in the *Histoire de l'Académie* ...for 1772, second part, pages 482...501. He considers that the Toise of the North may be from $\frac{1}{30}$ to $\frac{1}{20}$ of a line shorter than the Toise of Peru; this he attributes to the results of the shipwreck. We learn from his memoir that the name of the artist was Langlois. This memoir is perhaps the authority which is required by Airy: see the note to his page 206.

The Toises of Peru and of the North were again compared in 1799 and found to be sensibly equal: see *Base du Syst. Métrique* III. 413.

The toises are referred to by Delambre in his *Rapport historique*...1810, page 69.

16. I cannot give any later references for the history of these famous toises. It is reported that one at least of them was damaged by a workman who used it as a crow-bar, being ignorant of its importance. From enquiries lately made at Paris by a distinguished member of the University it may be inferred that the toises cannot be found, or that they are in such a state that no safe determination could now be made of their length when first employed. Although it is much to be regretted that any accident should have befallen these famous toises, yet even if there had been no accident I should hesitate to believe that after the lapse of 130 years the lengths of such iron bars would be absolutely the same as they were originally.

17. It followed from the amplitude of the arc and the measured length of the base that a degree of the meridian at the Arctic circle was nearly 1000 toises larger than it should have been, according to the Cassinian theory. M. 58. The magnitude of this result astonished Maupertuis and his companions; and although they considered their operations to be incontestable they resolved to make most rigorous verifications. M. 68.

18. The angles of the triangles were supposed to admit of no doubt; these angles had been observed many times by various persons, and the three angles of every triangle had been observed.

19. The following statement has been made: "The result being somewhat different from what they had expected, the latitudes were re-observed, and some angles of the triangles, which had been before omitted, were observed." A. 169.

The first part of this sentence does not adequately represent the opinion of Maupertuis, as we see from his *différoit tant* and his *nous étonnoit*.

I have already drawn attention to the error involved in the statement that the *latitudes* were re-observed, Art. 11.

The statement with respect to "some angles of the triangles" is not in harmony with the evidence. It is true that *while* the base was being measured the altitude of one object, a tree, which had been neglected, and which was of very little importance, was observed. M. 53; O. 140. But no angles of triangles were observed *after* the base had been measured. Maupertuis obviously lays stress on this fact. M. 48.

20. The accuracy of the calculations was verified by combining the triangles in various ways; and also by investigating what the result would be on the assumption that errors had arisen in measuring the angles which all tended to make the length of the arc of the meridian greater than it should have been. But the length of the arc still continued without any important diminution. M. 63...65. The measurement of the base was considered to be above suspicion.

21. Thus it only remained to consider the amplitude of the arc. Maupertuis gives reasons for warranting confidence in this also; nevertheless he resolved to employ a most laborious verification, which would exhibit the accuracy of the instrument and the degree of precision which could be ascribed to the value of the amplitude. The verification consisted in determining the amplitude again by observations on another star: the star α Draconis was selected.

It should be remarked that another reason has been assigned for selecting the amplitude as the part of the whole work which should be verified; namely, that it was more easy to verify this than any other part. *Histoire de l'Académie*...for 1737, page 95. And Outhier seems to offer as a reason simply the fact that the long winter gave them time. O. 153. See also Art. 64 of the present memoir.

22. The star α Draconis was observed with the zenith sector at Tornea on the 17th, 18th, and 19th of March, 1737; and at Kittis on the 4th, 5th, and 6th of April.

The amplitude of the arc as determined from these observations, when corrected in the manner noticed in Art. 12, was $57' \, 30''\cdot 42$.

Thus the second determination of the amplitude *exceeded* the first by $3''\cdot 49$. Maupertuis remarks that this excess would be reduced to $2''\cdot 54$ by allowing for an inequality in the divisions of the sector; and that it would not surpass $2''$ if only the best observations were employed. M. 124. But he does not avail himself of these adjustments.

23. Maupertuis adopted the mean of the results furnished by the stars δ and α Draconis, namely, $57' \, 28''\cdot 67$, as the amplitude of the arc between his extreme stations;

and deduced 57437·9 toises for the length of the degree of the meridian which is bisected by the Arctic circle. M. 125.

24. It is reported that Maupertuis himself was not satisfied with the result obtained, and wished to repeat the operations at his own expense. See Lalande's *Bibliographie Astronomique*, pages 407 and 811; Zach's *Monatliche Correspondenz*, Vol. I. page 116.

25. The scrutiny to which the operations of the French expedition have since been subjected has resulted in the introduction of various small corrections; although these do not materially affect the result, yet they deserve some attention, and we will now consider them.

26. A correction for refraction was suggested at a very early period; see Bouguer's *Figure de la Terre*, page 290: it was considered that the amplitude of the arc ought to be increased by about a second, and the length of the degree diminished in consequence by about 16 toises. Bailly, in a note on page 39 of Vol. III. of the *Histoire de l'Astronomie Moderne*, has by a misprint sixteen *seconds* instead of sixteen *toises*. Rosenberger corrects the observations of each star for refraction; and on the whole his corrections on this account increase the amplitude by about $1''\cdot 03$. In the *Encyclopædia Metropolitana*, page 206, the correction of the amplitude for refraction is put at $0''\cdot 7$.

27. In the trigonometrical calculations Maupertuis in general reduced his angles to the horizon; but in one case he seems to have omitted to do so, which produced a slight error. R. 21...23. That this error is slight might have been anticipated from the fact that Outhier calculated the length of the arc without any reduction to the horizon, and obtained a result which did not differ by 5 toises from that of Maupertuis.

Maupertuis made no reduction for the elevation of his arc above the level of the sea; but this is of very small importance. Rosenberger allows for it. R. 22.

A more important point is the levelling of the base; Maupertuis makes no reduction on this account: we shall have to recur to it. See Art. 49.

28. Rosenberger determines what he considers the most probable values of the angles of the triangles according to a method proposed by Bessel. After making the necessary small corrections, Rosenberger finally obtains for the length of the arc between the extreme stations a result about 2·7 toises less than that of Maupertuis. M. 93; R. 26.

We may consider it to be certain, after Rosenberger's investigations, that the geodetical data supplied by Maupertuis cannot be made to produce a result sensibly different from that which Maupertuis himself obtained. Indeed, it seems to me that this conclusion might have been taken as obvious without Rosenberger's investigations, when we consider the various calculations made by Maupertuis himself, especially that on his Plate VIII., and also Outhier's calculation.

29. The astronomical part of the operation is however by far the most important. Maupertuis found for the amplitude, $57'\ 26''\cdot 93$ by δ Draconis, and $57'\ 30''\cdot 42$ by α Draconis.

M. 124. Rosenberger finds respectively, $57' \, 27''\cdot 60$ and $57' \, 31''\cdot 87$. Rosenberger allows for refraction, and he uses astronomical corrections which differ somewhat from those of Maupertuis. Both Maupertuis and Rosenberger allow for a fault in the sector which the artist Graham had detected; namely, that the arc was $3''\cdot 75$ too short. M. 104, 117; O. 231.

Thus, taking the mean of the two results, Maupertuis made the amplitude $57' \, 28''\cdot 67$; and Rosenberger $57' \, 29''\cdot 74$.

It seems to me that Rosenberger ought to have adopted the result he had thus found; but he alters it: and this brings us to a very troublesome matter.

30. Maupertuis records some observations which were made in order to verify the statement of Graham that the sector was too short. He selected two objects which subtended at a certain point an angle known by calculation to be $5^\circ \, 29' \, 50''$; and he found by using the sector horizontally, in a way which he explains, that the observed angle, *allowing for Graham's defect*, was $5^\circ \, 29' \, 48''\cdot 95$. The close agreement Maupertuis considers to shew the accuracy of the instrument. M. 117, 118.

There is, however, this peculiarity, namely, that in describing the operation Maupertuis says that the angle observed was found *plus grand*. Here *grand* must be a mistake for *petit*. Rosenberger detected the misprint in the German translation of Maupertuis, and ascertained that it occurred also in the original. I see that the misprint occurs also in the Latin translation and in page 457 of the *Histoire de l'Académie*...for 1737. It is still more extraordinary to find the same error in Outhier; he says, in fact, that the observed angle was $5^\circ \, 30' \, 7''\frac{3}{10}$, where he ought to say that it was $7''\frac{3}{10}$ *less* than $5^\circ \, 30'$. O. 231. There can be no doubt that it is a misprint; the sector indeed, could not have been used to measure an angle greater than $5^\circ \, 30'$.

According to Rosenberger the misprint is corrected in a work entitled *Degré du Méridien* . . . 1740. This work contains an account of some operations in which Maupertuis was engaged after he returned from Lapland: I have not seen the original, but in a German translation which I have the misprint is corrected.

The occurrence of this strange misprint in the accounts both of Maupertuis and Outhier is remarkable; more especially as Outhier's work appeared *after* the correction in 1740. I may state that Rosenberger seems not aware of the existence of Outhier's work.

31. We have however to continue our discussion of the irregularity in the zenith sector.

Maupertuis found by trial that the degree of the sector which was used with respect to δ Draconis was $0''\cdot 95$ larger than the degree which was used with respect to α Draconis; and he observes, that in consequence, the discrepancy between the two determinations of the amplitude is reduced by this amount. M. 70, 75, 118. It should be observed that one out of five observers differs considerably from the other four as to this amount.

Maupertuis does not however make any correction of his amplitude on account of this irregularity; and indeed it is plain that no correction could be made unless more information were furnished. For suppose it to be certain that the degree used with respect to δ Draconis was exact; then the degree used with respect to α Draconis must have been too small, and the amplitude determined from α Draconis must have been about a second greater than it ought to have been: thus the mean amplitude formerly obtained ought to be *diminished* by about half a second. Suppose however it to be certain that the degree used with respect to α Draconis was exact; then the degree used with respect to δ Draconis must have been too large, and the amplitude determined from δ Draconis must have been about a second less than it ought to have been: thus the mean amplitude formerly obtained ought to be *increased* by about half a second.

We see then from these cases that the correction to be applied to the amplitude is indeterminate without further information.

32. Rosenberger in fact seeks further information from Maupertuis's book, and arrives at the conclusion that the degree used with respect to α Draconis was $0''\cdot928$ too large, and that the degree used with respect to δ Draconis was $1''\cdot878$ too large: thus on the whole the mean amplitude formerly obtained is to be increased by half the sum of these two quantities, and so it becomes $57'\ 31''\cdot14$.

Finally he takes a mean between this and his former result, and thus obtains $57'\ 30''\cdot44$.

I venture to consider the whole of this process as arbitrary and unjustifiable.

33. Let us first examine the additional information which Rosenberger supposes that he obtains from Maupertuis's book.

The sector was furnished with a micrometer screw. Each revolution of the screw was divided into 44 parts: and each revolution of the screw was equivalent to $43''\cdot8$. Perhaps we may infer that the maker intended each *part* to correspond to one second, and that he had not exactly succeeded in his design.

34. Now Maupertuis gives a table of the results obtained in verifying the intervals of 15′ on the arc of the sector; see his page 120. This table records the number of turns and parts of a turn for successive intervals of 15′ as found by the French observers and as stated by Graham. The following is an extract from the table:

1° 30′ to 1° 45′	23·2	23·5
1° 45′ to 2° 0′	23·8	24·5
2° 0′ to 2° 15′	23·4	23·875
2° 15′ to 2° 30′	23·1	23·5
2° 30′ to 2° 45′	23·6	24·125

This signifies that for the interval from 1° 30′ to 1° 45′ on the limb of the sector, besides 20 complete revolutions of the screw the French observers found 23·2 parts, while Graham had found 23·5 parts. And so on.

35. Now on taking the mean of all the table it is found that according to Graham 15 minutes correspond to 20 revolutions and 23·636 parts; and according to the French observers to 20 revolutions and 23·354 parts. The mean to three places of decimals is 20 revolutions and 23·495 parts.

Rosenberger however puts the number of parts in the French mean as 23·4 instead of 23·354 and so obtains 23·518 instead of 23·495. But the difference is unimportant.

Thus according to Rosenberger 900 seconds correspond to 903·518 parts: from which we obtain $0''\cdot996106$ as the value of a *part*.

36. Now according to the table we have as the mean between Graham and the French observers 80 revolutions 95 parts for the degree of the limb between $3^\circ\,15'$ and $4^\circ\,15'$; that is 3615 parts. But from Art. 35 if there were no irregularity we ought to have $4 \times 903\cdot518$ parts, that is 3614·072 parts. Hence there is an excess of ·928 of a part; and this gives about $0''\cdot924$. Rosenberger says $0''\cdot928$; so that he takes a *part* to correspond to a *second*. There can be no objection to this identification of a part with a second here; but then it was quite unnecessary for Rosenberger to give the value of a part in seconds to six places of decimals which occurs at the end of Art. 35.

37. Having thus drawn from the table the inference that the degree used with respect to α Draconis was $0''\cdot928$ too large, Rosenberger does not consult the table again to ascertain the state of the degree used with respect to δ Draconis, but says that this degree was $1''\cdot878$ too great; he must therefore have added to his $0''\cdot928$ the $0''\cdot95$ which was mentioned in Art. 31.

But let us refer to the table; the part which we require is given in Art. 34. Assume that for the interval between $1^\circ\,37'\,30''$ and $1^\circ\,45'$ we have half the results given in the table for the interval between $1^\circ\,30'$ and $1^\circ\,45'$; and make a similar assumption with respect to the interval between $2^\circ\,30'$ and $2^\circ\,37'\,30''$. We shall then have for the degree between $1^\circ\,37'\,30''$ and $2^\circ\,37'\,30''$, according to Graham, 80 revolutions 95·6875 parts, and according to the French observers 80 revolutions 93·7 parts; the mean is 80 revolutions 94·69375 parts, which is *less* than the 80 revolutions 95 parts obtained for the degree between $3^\circ\,15'$ and $4^\circ\,15'$.

38. I conclude that the table is not to be held as accurate to the degree necessary for the use to which Rosenberger applies it. The table indeed is not thoroughly consistent with the statement made in Art. 31; and Rosenberger has made an arbitrary selection of the part of the table to which he trusts. Probably the French observers considered that the statement made in Art. 31 was very accurately established, but that the statements involved in the table were less certain. In fact, according to the table the French observers made the degree of the sector between 3° and 4° almost $3''$ longer than the degree between 4° and 5°, and we should naturally expect that they would have drawn attention to this circumstance if they had considered it to be really established by their observations.

93. I maintain therefore that Rosenberger was not justified in taking $57'\,31''\cdot14$ as the amplitude when allowance is made for the irregularity of the sector, because we have not the necessary trustworthy information as to this irregularity.

If however the information had been trustworthy he should have maintained this value, and not have taken a mean between this and the value obtained on the supposition that the sector was free from irregularity. Of course if there had been other observers equal in credit to Graham and the French observers, and they gave as the result of observation that the sector was quite regular, there would have been a ground for taking the mean; but in the absence of such evidence I see no possible reason for it.

Thus I should take $57' \, 29''\cdot74$ for the amplitude as deduced from the observations of Maupertuis. This allows for the refraction and depends on astronomical corrections which are doubtless more accurate than those used by Maupertuis. At the same time it is quite obvious that whether from irregularity in the graduations, or inequality in the screw, an error of a second was very likely to occur in using the sector.

40. Something remains to be said as to the table on page 120 of Maupertuis's work which has already engaged so much of our attention.

The mean of the French observations is, as we have stated, 20 revolutions 23·354 parts. Rosenberger takes the number of parts as 23·4, while Maupertuis himself takes it as 23·3. The table is reprinted in the *Degré du Méridien entre Paris et Amiens...* 1740. I have not seen this work, but only a German translation of it. In this translation the entry of the French observers opposite to the interval between $4^0 \, 15'$ and $4^0 \, 30'$ is 23·9 parts instead of 22·9 parts. I do not know whether this is a correction or a misprint. If we take 23·9 parts the mean becomes exactly 20 revolutions 23·4 parts.

41. It may be observed that Outhier throws no light on the matter which we have discussed in Arts. 37...40; he merely says that the divisions from degree to degree were verified by observations. O. 231.

42. Perhaps it will appear that I have given too much consideration to a point which does not affect the amplitude by much more than a second. My reply is that every second is important in the correct evaluation of the amplitude, if it can be really shewn that such an increase or diminution ought to be made. But I am principally desirous of drawing attention to the process as an example of what appears to me to be not unfrequently occurring in science, namely the attempt to extract trustworthy results from inadequate data by taking means.

I am of course aware that I shall seem presumptuous in putting my own opinion or arguments in opposition to such an eminent authority as Rosenberger, but that is a risk which I have to encounter repeatedly in the course of this memoir.

43. We must now pass from the French to the Swedish measure of the arc of the meridian in Lapland.

In 1799 on the recommendation of Melanderhjelm the Academy of Sciences at Stockholm was led to consider the advantage of a remeasurement of the arc of the meridian in Lapland, and Svanberg, who was born in the vicinity of Tornea, made a preliminary survey. It was resolved to undertake the work, and Ofverbom and Svanberg spent the time from April to October of the year 1801 in choosing stations and erecting signals and observatories as a preparation for the final voyage in which the measurement was to be effected. See the preface by Melanderhjelm to Svanberg's book. Lalande in 1801 used these words—"et nous aurons, en 1803, la solution de cette ancienne difficulté." *Bibliographie Astronomique*, p. 857.

It may be stated in general terms that while Svanberg obtained a decidedly shorter length for a degree of the meridian than that of Maupertuis, yet the difficulty can hardly be said to be solved.

44. The measurement of the base occupied from February 22 to April 11 of the year 1802. Svanberg adds, with a laxity of expression of which this is not the only example, that this work continued during two entire months. S. xix. The geodetical angles were observed in June, July and August of the year. The observations of the pole-star to determine the latitudes were made at Mallorn, the southern extremity of the arc, in October, and at Pahtavara, the northern extremity, in December. S. 141, 152.

45. All the angles, both geodetical and astronomical, were observed with the same instrument, which was a Borda's repeating circle that had been constructed by Lenoir at Paris under the eyes of Delambre. S. xviii. Svanberg does not record the dimensions of this instrument.

The base was measured by iron bars about six metres long; the lengths of these were adjusted by the aid of an iron bar two metres long, which was prepared at Paris by Lenoir, and the length of which was verified by Mechain and Delambre. This double metre was presented to the Swedish party by the French Academy, together with a copy of the toise of Peru. S. 1, 2.

46. Svanberg himself records his observations and his results in *centesimal* degrees, minutes, and seconds; but unless the contrary is stated or obviously implied I shall always use the sexagesimal scale.

47. Svanberg extended the arc of the meridian which the French had surveyed, a little more than a third of a degree towards the north, and a little less than a third of a degree towards the south. The two extreme stations of the French arc were nearly on the same meridian; Svanberg's extreme stations were both decidedly to the west of the meridian. See Svanberg's diagram of triangles.

Rosenberger says that the ends of Svanberg's arc are more than 20′ from those of the French arc; with respect to the southern ends this must mean not more than 20′ of latitude, but more than 20′ on a great circle passing through the two southern ends.

48. Svanberg chose the same locality for his base as Maupertuis; but unfortunately the bases did not exactly coincide. Svanberg admits that at the North end he did not discover the precise point at which the French base terminated; he states that at the South end he did discover the marks left by the French. S. 19, 20. But from a comparison of the diagrams given by Svanberg and by Outhier it is clear that the French and Swedish bases did not terminate precisely at the same point even at the South end.

See S. figure 26; O. 217; D. 469. As Delambre observes, it is to be regretted that the Swedish astronomers had no knowledge of Outhier's book, which would have guided them in their search and perhaps enabled them to discover the exact positions of the extremities of the French base.

49. There is another difficulty which arises in the comparison of the bases.

The French accounts leave the impression that, except at the descent from the land to the ice, the base was *level;* while on the other hand Svanberg speaks of a little mountain at the South end. O. 138; S. 17. Rosenberger finds that the South end of Svanberg's base was 8·4 toises above the North end. Rosenberger also speaks of a hill more than 50 feet high. R. 20.

I may observe that Rosenberger does not refer to Svanberg's own book, but to the account of it given in Zach's *Monatliche Correspondenz*. It seems to me that the writer of this account has fallen into one very serious error, and drawn Rosenberger after him; namely, the error of supposing that considerable cataracts occur in that part of the river Tornea over which the French base was measured. *Monat. Corresp.* XII. 430. What Svanberg himself says is, that the river Tornea has very considerable cataracts *entre Niemisby et la ville:* here *la ville* means Tornea, and Niemisby is at the South end of the base. Also the maps given by Maupertuis and Outhier mark several cataracts at various points of the stream; but none between the extremities of the base. Thus, in fact, these cataracts do not exercise any influence on the length of the base; although, of course, they supply information which may be useful when the length of the whole arc has to be reduced to the level of the sea. In Svanberg's first journey, although he noticed that the river had a decided slope in the neighbourhood of the French base, yet he says nothing about cataracts. Zach's *Monatliche Correspondenz*, Vol. I. page 142. Delambre draws attention to the fact that the French observers make no reference to the inequalities in level of the base, and to the reductions which would thus be necessary. D. 469.

50. According to Svanberg his base required to be diminished by about 1·41 toises, for error of level. S. 21; D. 469. Delambre proposes to diminish Maupertuis's base by ·7 of a toise on this account. D. 471. I do not know how he obtains this ·7 of a toise. Can it be possible that, since Maupertuis himself made no correction, while Svanberg made a correction of about 1·4 toises, Delambre took ·7 of a toise as a mean?

Rosenberger adopts what seems to me the most reasonable course, of leaving Maupertuis's result unchanged on this account.

51. The length of Svanberg's base, reduced to the level of the sea, is 7414·4919 toises of Peru; but there is a little uncertainty as to whether the proper allowance was made for temperature. S. 21, 191; A. 210.

Maupertuis's base was 7406 toises 5 feet 2 inches.

But, as the two bases did not quite coincide, no inference can be drawn from these figures as to the accuracy of the measurements. We must, therefore, make some comparison of the results deduced from the base lines.

52. Svanberg records the angles which he observed, and the lengths which were deduced by calculation, but he does not give a table of the triangles which were employed in these calculations. It is to be regretted, as Delambre says, that Svanberg deviated from the received custom in this matter. In the *Monat. Corresp.* Vol. XII. Svanberg's deficiencies are usefully supplied.

53. Svanberg says that the choice of the French stations was found to be very good. But it may be presumed that he could hardly be certain of the *exact* positions of any of the French stations except Tornea; and this must be borne in mind in the comparison of results. On this important point Svanberg is silent, as on many others: Delambre adverts to it. D. 471, 472.

54. The following table of results shews that there is a close agreement as to the values of the lengths calculated from the two bases. Maupertuis does not give the figures, so that they are taken from Outhier's book. It must be observed that Outhier did not reduce his triangles to the horizon.

The distances are in toises. The letters have the following meanings: Q stands for Kittis, P for Pullingi, N for Niemi, H for Horrilakero, A for Avasaxa, K for Kakama, C for Cuitaperi, b for the North end of the base, B for the South end, T for the church at Tornea.

It will be remembered that b and B do not denote the same points in the two surveys; and indeed we cannot be sure that any letter does except T.

	S.		O.
QP	10672·3	...	10676·0
QN	13549·1	...	13560·0
PN	8757·6	...	8768·8
PH	11522·9	...	11558·5
PA	14271·0	...	14277·3
NH	7028·4	...	7029·0
NK	25047·2	...	25053·5
HA	7447·9	...	7451·6
HC	13396·1	...	13402·0
HK	19066·5	...	19073·0
Ab	1186·0	...	1207·3
AB	7239·7	...	7242·8
AC	8656·9	...	8660·0
CK	11406·8	...	11411·5
KT	16688·5	...	16695·0

55. The preceding Table contains all the lengths which are common to Svanberg and Outhier. I have given them in the order in which they occur in Svanberg's pages 99, 100.

Delambre gives such a comparison on his page 471; there are, however, some points to be noticed.

He omits CK and KT.

His table is arranged in a different order; the last entry in it is NQ: and here he has an important misprint in Svanberg's value.

He says that *in general* the sides in Svanberg's series are too short; as this is the case with *all* the sides it is difficult to see the meaning of the words *in general*.

His table is separated into two parts by a line across the page: I do not see the meaning of this.

56. The whole length of the arc of the meridian which the French surveyed is according to Svanberg 54919·25 toises; while according to Maupertuis it is 54945·95 toises. S. 171.

Svanberg says that by taking the mean of the different results recorded by Maupertuis we get 54925·63 toises, which differs by only 6·38 toises from Svanberg's own result; and 5·355 toises of the difference can be attributed to the circumstance that Maupertuis's base was not levelled. This statement about the 5·355 toises is one of the many things in Svanberg's book which are not sufficiently explained. Svanberg himself diminished his base by about 1·41 toises for error of level: see Art. 50. This would correspond to about 11 toises for the whole length of the arc. D. 469. Can it be possible that Svanberg takes a mean between the zero which corresponds to the absence of any correction in Maupertuis's own process and the 11 toises which correspond to his own correction?

Rosenberger adverts to the difficulty. R. 20.

It is plain that there is a good general agreement between the geodetical parts of the two surveys. The French did not measure any base of verification: and indeed, owing to the shortness of the arc surveyed, it was not necessary to do so.

Svanberg in his book does not mention any base of verification; but we learn from another source that such a base, though smaller than the first, was measured with wooden measuring rods, and that it agreed perfectly with the first. See Zach's *Monatliche Correspondenz*, Vol. VII., page 564.

Delambre gives the preference to the Swedish operations: see his page 474. He asserts, as I understand him, that in the French operations the difference between the sums of the angles of a triangle and 180° ranged from 18″ to 40″: but according to the numbers supplied by Maupertuis, the greatest error of this kind is 29″·4; and according to the numbers supplied by Outhier, it is 31″. M. 87...90; O. 221. On the other hand, Delambre once expressed a different opinion as to the relative merits of the two surveys. For it is asserted by Zach, on the authority of Lalande, that Delambre wrote a long letter to Melanderhjelm full of objections to Svanberg's measure. It seemed to Delambre impossible that the French Academicians could have made an error of 12″ in their amplitude; and he did not admit as certain the superiority of the small repeating circle which he had himself used. See Zach's *Monatliche Correspondenz*, Vol. VIII., page 446.

57. But we will turn to the most important subject; namely, the amplitude.

Delambre says, on his page 477:

Prenons de même dans la table de M. Svanberg la latitude de Kittis ...	74·2324245
Celle de Tornea étant	73·1648380
La différence en degrés centésimaux sera	1·0675865
en degrés sexagésimaux	0° 57′ 39″·01
Suivant Maupertuis, par α du Dragon	57′ 30″·35
par δ	57′ 26″·9

d'où résulte une erreur de 10 à 11″ dans l'arc de 1736, à moins qu'on ne dise que le nouveau signal de Kittis était dans une position très-différente de l'ancien, ce qui ne paraît guères probable d'après l'arc terrestre et toutes les comparaisons rapportées ci-dessus. . . .

In the latitude of Tornea the last figure should be 3 instead of 0; but this is of no consequence. However, the result at which Delambre arrives requires consideration.

In the first place the latitude above assigned for Tornea is that of the spire of the church of Tornea; now the French observatory at Tornea was more than 73 toises to the south of this. M. 92; S. 171. Hence, as Svanberg calculates, the latitude of the French observatory was 73·1634058; and so the amplitude becomes, according to Svanberg, 1·0690187; and this in sexagesimal notation is 57′ 43″·62. In the second place, as to the French amplitude. Delambre quotes indeed results obtained by Maupertuis, but not the results which Maupertuis himself adopted, and which in fact involved corrections for what we now call nutation. M. 122, 123. Maupertuis himself estimated the amplitude at 57′ 28″·67. Thus the difference between the French and Swedish amplitudes is almost 15″, and not 10″ or 11″ as Delambre says.

Delambre having to compare the French and Swedish amplitudes, as we see, quoted neither of them correctly.

If we adopt Rosenberger's value 57′ 30″·44 for the amplitude, the difference is rather more than 13″; if we adopt Rosenberger's more trustworthy value 57′ 29″·74, the difference is nearly 14″.

58. We may observe that there are three points at Tornea which have to be carefully distinguished: the church of the town of Tornea, another church called the Finnish church, and the French observatory. The French made no use of the Finnish church for their triangles, so that in reading Maupertuis and Outhier we have only to distinguish between the church of the town and the French observatory. Svanberg made use of the Finnish church. M. 31; O. 53; S. 42; D. 467.

59. But we have by no means finished the subject of the amplitudes. Such strange errors have been made by distinguished astronomers that it is necessary to be very particular as to the facts.

Svanberg determined by *observation* with his repeating circle the *latitudes* of his extreme stations Mallorn and Pahtavara; he determined by *calculation* the latitudes of his intermediate stations. S. 170, 171.

Maupertuis determined by *observation* with his zenith sector the *difference* of latitude of Tornea and Kittis.

60. Maupertuis however did make observations to determine the latitude of Tornea, some with a quadrant of three feet radius, and some with a quadrant of two feet radius. From the former Maupertuis obtained $65^0\ 50'\ 50''$ for the latitude, and from the latter $65^0\ 50'\ 51''$. It was necessary for Maupertuis to know approximately the latitude of one of his stations in order to determine the situation of the degree of the meridian of which he had found the length, but an approximate latitude was sufficient. But whether the latitude which he adopted was exact or only approximate is a matter of no interest to us at this point, because it has *no influence whatever* on the amplitude of the French arc; this amplitude we know rests on differential observations made with the zenith sector.

But it may be observed that from the nature of the instruments employed we cannot give much confidence to Maupertuis's determination of the latitude of Tornea. Indeed the recorded observations themselves warn us of their character; as Maupertuis himself says, they give distances of Polaris from the pole which differ by as much as $14''$. See M. 138. The observations were criticised at an early period. See Article 8 of the preface to the Latin translation of Maupertuis's work.

Outhier records the latitude as $65^0\ 50'\ 50''$ for the observatory, $65^0\ 50'\ 54\frac{1}{2}''$ for the church, and says that Belberg had found $65^0\ 43'$ in 1695. See O. 233.

According to the Russian survey the latitude of Tornea is $65^0\ 49'\ 44''\cdot 7$. See *Ordnance Survey...Principal Triangulation...*page 752. I do not know at present what point in Tornea is taken, as Struve's great work on the Russian survey is not in the University Library. This differs much from Maupertuis's value, which is of small importance as regards the French survey, but of great importance as regards the Swedish survey.

61. I repeat for the sake of distinctness that *the amplitude of the French arc is in no way connected with the latitude which Maupertuis assigned to Tornea.*

62. Now Svanberg, as we have said in Art. 57, found for the latitude of the French observatory at Tornea 73·1631058, which is equivalent to $65^0\ 50'\ 49''\cdot 435$; then he adds: ce qui est précisément celle qui résulte des observations faites au commencement du Janvier l'an 1737; de sorte que dans cet élément nous ne différons point du tout des déterminations anciennes. S. 171.

It is obvious that the word *précisément* is improper here; the difference between Svanberg's result and the mean of Maupertuis's results is a trifle more than a second, and every second is of importance here. It is absurd to pretend to be accurate to a thousandth part of a centesimal second, and then to reject more than a sexagesimal second. Rosenberger by taking a stricter mean of Maupertuis's results and using other corrections for refraction makes the difference between Maupertuis and Svanberg to be $2''\cdot4$. See R. 26.

It is a grave fault in Svanberg to appeal to the rough determination of the latitude made by Maupertuis, which was comparatively unimportant in the French survey, as affording any real corroboration of the Swedish astronomical observations. It is curious that although Svanberg goes on to shew that the difference in toises between the French value and his own of the common arc is small, he makes no reference here to the great difference of amplitude. He avows however in his introduction his astonishment at the difference in the two operations. S. xxv.

63. Svanberg then goes so far as to consider that he and Maupertuis agree in one important element, namely the latitude of Tornea. Let us now see how Delambre states the matter:

"Calculant d'après sa formule la différence des parallèles entre Mallorn et tous les signaux, et l'église de Tornea en particulier, il arrive à ce résultat remarquable, que la latitude de cette église se trouve précisément la même qu'en 1736. Nous avons déjà vu que l'erreur ne tient pas à l'arc terrestre, tout le mal viendrait donc des observations faites à Kittis."

Thus Delambre adopts what was objectionable in Svanberg, and adds to it the fatal suggestion that the discrepancy between the French and Swedish results arose at Kittis.

64. In a note Delambre remarks that Maupertuis's observations of the latitude at Tornea were not made with the zenith sector; and that they were really unsatisfactory; but Delambre omits what is the essence of the matter, namely the fact stated in Art. 61.

The note is important on other grounds. Delambre had in his hands a manuscript by Le Monnier. This manuscript contains some observations not to be found in Maupertuis's book. Delambre says: "Parmi ces observations de la polaire, j'en ai trouvé qui donnent $57' \, 36''$ pour l'amplitude à $3''$ près, comme elle résulte des observations nouvelles [but see Art. 57]; et ce sont elles qui ont décidé les académiciens français, en mars 1737, à vérifier leur amplitude par a du dragon." This passage seems to shew what we should have thought impossible, that some attempt was made to determine the amplitude by the rude observations with the quadrants. Also we are furnished with a *fourth* reason quite distinct from the three which have been already assigned for the re-determination of the amplitude; and all four reasons rest on contemporary authority. See Art. 21.

65. The error of Delambre has been reproduced with greater distinctness. We read:

"The geodetic measures, as far as they went together, agree very well; the latitude of Tornea, as determined by Maupertuis, agrees perfectly well with the observations of Svanberg; the latitude of Kittis was not observed by Svanberg. It is much, very much, to be regretted that the Swedish astronomer did not repeat the observations at the only place where an important error could be feared." A. 173.

Here besides the mistake of supposing that Maupertuis's latitude of Tornea is connected with his amplitude, we have the distinct statement that the only error to be feared was at Kittis. Moreover it is implied that Svanberg *observed* the latitude at Tornea, but not at Kittis: whereas he *observed* neither, but determined both in the same way by calculations from the observed latitudes of his extreme stations.

66. Of course the practical consequences of the errors which we have pointed out in Arts. 63 and 65 might have been very serious. Any person who wished to settle the difference between the French and the Swedish surveys, would naturally rely on the authority of the eminent astronomers we have quoted; and so would conclude that all he had to do was to observe carefully the latitude of Kittis. The fact is that in order to confirm or correct Maupertuis the latitudes both of Tornea and of Kittis must be determined carefully, or rather the difference between the two latitudes. It is to be regretted that Svanberg did not carefully observe these latitudes; in fact he should have given his main attention to verifying the French arc rather than to extending it.

67. Of course the great difference in the amplitude of the common arc leads to a corresponding great difference in the length of a degree of the meridian. According to Maupertuis the length of a degree of the meridian bisected by the Arctic circle is 57437·9 toises. According to Svanberg the length of a degree of the meridian which has its middle point in latitude $66^0\ 20'\ 10''\cdot047$, which is at the middle point of his arc of the meridian, is 57196·159 toises. S. 192. It is curious that the amount of the difference should be something like the 1133 feet which was the excess of La Cuille's degree at the Cape of Good Hope above the recent determination by Sir Thomas Maclear. *Proceedings of the Royal Society*, XVIII. 110.

68. We may briefly advert to some opinions which have been held as to the measurements of the degree of the meridian in Lapland.

Laplace when the second volume of the *Mécanique Céleste* was published had only the French measure which he employed; taking a mean between the several series of triangles and correcting for the refraction.

Voiron says cautiously "Les résultats des dernières opérations faites en Laponie paroissent obtenir la préférence sur ceux de 1736." *Histoire de l'Astronomie*, 1810, page 292.

Rosenberger in 1827 maintained the trustworthiness of the French survey.

In the article in the *Encyclopædia Metropolitana* which is dated 1830 an agreement with Rosenberger is expressed. A. 206, 210. Both the French and the Swedish results are used in determining the Figure of the Earth. A. 218.

Rothman in his *History of Astronomy* also follows Rosenberger; see page 94 of the History. Perhaps Rothman yielded to the authority of the *Encyclopædia Metropolitana*, to which he refers in his page 108.

In 1832 Bowditch in the notes to his translation of the second volume of the *Mécanique Céleste* adopts the Swedish result.

In 1833 Narrien is perhaps in favour of the Swedish result. He speaks of Tornea as a *city;* it consisted of 70 wooden houses according to Outhier. Narrien's *History of Astronomy,* 479; O. 119.

Bessel in 1837 and 1841 in determining the Figure of the Earth employed the Swedish result and not the French, without any remark on the subject: *Astronomische Nachrichten,* numbers 333 and 438. Thus it seems that Rosenberger's advocacy of the French operations had not convinced his distinguished master.

Biot in 1845 seems to accept Svanberg's result as a matter of course. *Astronomie Physique,* III. 181.

69. It is not easy to suggest any explanation of the discrepancy between the French and the Swedish results. Before the Swedish expedition had been arranged, the opinion might have been held that the mountainous nature of the ground had caused a deviation of the plumb-line of the French sector. But Svanberg came to the conclusion that this was not the case; and that the attraction of the whole mountain chain could scarcely have caused a deviation of half a second. See Lalande's *Bibliographie Astronomique,* 811. Zach's *Monatliche Correspondenz,* XII. 424, and the preface by Melanderhjelm to Svanberg's work.

Svanberg, as usual omitting the most important matters, tells us nothing about his extreme stations: and he gives no map. Delambre thinks that the south points of Svanberg's arc are in the islands of the Gulf of Bothnia. D. 467. It is of course possible that the land may have been disposed unsymmetrically in the vicinity of the islands, and thus some deviation have been produced in the direction of the vertical.

70. It is natural to turn our attention to the instruments on which the determinations of the amplitude depended in the two surveys. The success which had attended Bradley in his employment of a zenith sector for his investigation of the aberration of light without doubt must have reflected great credit on the instrument, and probably gave the French observers much confidence. Speaking without any practical familiarity with a zenith sector, I may say that I should have been disposed to think much more highly of it when employed for such a purpose as Bradley's in a fixed observatory than when it had to be moved about in a difficult country and to remain mounted for only a short time at one place. However Rosenberger makes a powerful defence of the French zenith sector. Rosenberger shews from comparing the indications of the instrument with the known declinations of α and δ Draconis that the line of collimation did not change its position in the interval between the two sets of observations at Kittis; and during this interval the instrument was taken from Kittis to Tornea, and back again from Tornea to Kittis. Moreover the line of collimation did not change its position between the two sets of observations at Tornea; during this

interval the instrument remained in general at Tornea; this interval in fact forms part of the former interval.

71. The facts which Rosenberger thus establishes with respect to the sector are very important; but we must not over-estimate them. It is still *possible*, though not very likely, that a change may have occurred in the line of collimation while the instrument was brought from Kittis to Tornea, and a change of an opposite kind while the instrument was taken back from Tornea to Kittis. In fact what is essential is that no change should have taken place in either journey; and it is not quite enough to shew that no change was perceptible after the two journeys.

72. An error has been introduced into the following account of Rosenberger's results: "...... he has shewn that the line of collimation was in the same state *before the journey from and after the return to Tornea* as well as before the journey from and after the return to Kittis." A. 206. The words which I have put in italics imply a journey from Tornea to Kittis and back again; but there was not such a journey between the two sets of observations at Tornea which Rosenberger examines.

73. I have said in Article 70 that the instrument remained *in general* at Tornea between the two sets of observations which Rosenberger examines. For owing to want of acquaintance with Outhier's book Rosenberger does not notice a slight strengthening of his case to which he might have appealed. Outhier in fact tells us that during the interval spent at Tornea the zenith sector was taken to a place at a little distance, set up there and used for observations, and then brought back; and that no change was detected in its indications. O. 131.

74. Rosenberger shews on the testimony of the work to which we have referred in Arts. 30 and 40 that the zenith sector was used in France after its return from Lapland; and that its performance then was excellent. But this evidence is not quite decisive as to the certainty of the results obtained in Lapland for two reasons; the circumstances in France would be much more favourable for the safe transport of the instrument, and its proper adjustment; and moreover the observers would have rendered themselves more familiar with the instrument, and therefore would be probably able to use it with greater advantage. It must be remembered that the French observers had not used their instrument before they were in Lapland; the instrument in fact was sent after them, see M. 31; O. 91. I cannot help expressing the opinion that in such delicate operations the instrument ought to be finished and if possible tested by the observers who are to use it before it is applied to the work for which it is especially constructed. If any new form of zenith sector is devised it ought to be employed to verify the latitude of some known place or the difference of latitudes of two known places before it is trusted to determine the latitude of an unknown place or the difference of latitude of two unknown places.

75. The preceding Article was written before I had seen the paper by Lieut.-Col.

Strange in the *Monthly Notices of the Royal Astronomical Society,* vol. XXXI. pages 10...16, which is very valuable in connexion with our subject. I may refer especially to the opinion of the late Sir George Everest which is given on page 10; and to the remarks at the top of page 13 and at the top of page 16.

76. Svanberg himself considered that the discrepancy between the two surveys was due to the untrustworthiness of the observations with the sector; he draws attention to the great differences which were found by the observers in Peru to occur in their use of zenith sectors. S. XXV. See also Melanderhjelm's preface to Svanberg's work, VI. VII.

I may add to this the testimony of Maskelyne as recorded by Frisi. Maskelyne found at S. Helena that in sectors like that which Maupertuis used the friction of the thread suspended from the centre sometimes produced an error of $3''$ or $4''$. See Frisi *De Gravitate Corporum* page 140, or *Cosmographia* Vol. II. page 88.

77. Let us now turn to the instrument employed by Svanberg. Repeating circles seem now to have gone almost out of use; while zenith sectors retain their position. Since the geodetical parts of the French and Swedish operations agree well together we have so far a favourable testimony to Svanberg's repeating circle. But it must be remembered that a much higher degree of accuracy is necessary for the astronomical work than for the geodetical work. An error of $10''$ would be most serious in the amplitude; but such an error in the geodetical angles would be of little consequence. See M. 65. But unfortunately it is the opinion of well-qualified judges that the repeating circle is less satisfactory for the measuring of vertical angles than for the measuring of horizontal angles.

78. Rosenberger's conclusion as to the French amplitude is that there cannot be an error of $12''$ in it; and that the great difference between the French and the Swedish results must be explained in some other way.

79. It has been stated:

"In order to make this measure and that of Svanberg agree in all points, it is necessary to suppose an error of $12''$ or more in the French observations of latitude at Kittis. From the excellence of the instrument, the reputation of its maker, the care and fidelity of the observers, as shewn in the points that have been examined, and the circumstance of their having repeated the observations under the fear that some error had crept into the first set, we have no hesitation in expressing our opinion that this is *impossible.*" A. 206.

This involves the important error we have already pointed out; it assumes that the two operations agree at Tornea, and that the whole difference arises from the observations at Kittis. If however we take with this writer the erroneous hypothesis that the zenith sector and the repeating circle were in accurate agreement at Tornea the two instruments appear with equal testimony in their favour; and there seems no ground then for saying that it is impossible for the zenith sector to be wrong at Kittis since this implies that the repeating circle must be wrong there.

80. After all it may be observed that discrepancies of equal or greater amount have occured in other geodetical operations. See Delambre's *Rapport Historique*, page 74; also the article *Trigonometrical Survey* in the Penny Cyclopedia, page 224.

81. It would be a curious subject of speculation whether the theoretical opinions of persons engaged in geodetical surveys could have exercised any influence on their observations; I mean of course unconsciously, for it would be wrong to suspect any deliberate unfairness in any of the operations which I have examined. From a passage in the article *Figure de la Terre* by D'Alembert in the original *Encyclopédie* it would appear that the school of Cassini originally believed that in consequence of the oblate form of the earth, the length of a degree of the meridian *would decrease* from the equator to the pole. It seems strange perhaps now to suppose that such an error could be seriously maintained; but there can be no doubt of it: for example, the error was vehemently maintained by Keill, a man of some reputation, who was ultimately a Savilian professor at Oxford. See Keill's *Examination of Dr Burnet's Theory of the Earth*...page 140. It is certainly a remarkable coincidence that the school of Cassini starting with the erroneous theoretical notion that the degrees of the meridian *ought* to decrease from the equator to the pole arrived at the same result by observation and measurement.

There can, I think, be no doubt that at least Maupertuis and Clairaut, who were the most eminent of the French party, held the correct Newtonian theory as to the figure of the earth; and their result was rather too decided in its confirmation of this theory. Now the geodetical angles could scarcely be influenced by the theoretical opinions of the observers; because it would not be obvious in what way the result would be affected by an error in an angle. But in measuring the base it would of course be obvious that the larger was the value obtained, the stronger was the evidence for an oblate form. Similarly in estimating the amplitude, the smaller the value obtained the stronger was the evidence for the oblate form. In these two parts of the survey then it would be necessary to be on the watch lest the conviction of what the result ought to be should influence the impression of what the observation really gives.

It is curious that Maupertuis and his party seem to have thought at first that their success was too decided and therefore their amplitude too small; and that on their second determination they should have made it between 3″ and 4″ larger than at first.

Svanberg, I consider, was sent to Lapland with a strong expectation that he would obtain a less value of a degree of the meridian than that of Maupertuis. See Melanderhjelm's preface to Svanberg's book.

82. There is a passage in Maupertuis's book which seems intended to shew that he had taken precautions to free his operations from the influence of theoretical opinions. See his page 48 *Tout notre ouvrage* ...

It is not unlikely that the triangles were calculated, at least roughly, before the base was measured, assuming the base to be unity. The natural curiosity to become acquainted with the character of the result as soon as possible would suggest this course. This conjecture seems a little confirmed by a passage on page 57 of Maupertuis's work, "Nous vîmes..."

But on the other hand see Outhier's page 146, "Nous nous occupâmes ..." and page 219.

It is said that the French party kept the result very secret, in order to reflect at leisure on what had been little expected, and to have the pleasure of bringing the first intelligence of it to Paris. *Histoire de l'Académie* for 1737, page 94.

83. What is now really required is a good determination of the latitude of Kittis; if we may assume that the latitude of Tornea is known by the Russian survey: see Art. 60. It seems probable that by the aid of Outhier's book the exact position of the French station at Kittis might be ascertained. D. 477. Thus the French amplitude would be tested. As to the Swedish amplitude the primary object would be to determine the latitudes of the extreme stations Mallorn and Pahtavara; but unless some information about the precise position of these stations exists in unpublished papers there would be little hope of identifying them. Of course the latitudes of intermediate stations between the extreme stations would be valuable as testing the Swedish operations, and perhaps supplying hints as to local irregularity of attraction. The recent construction of an improved zenith sector brings it within the bounds of possibility that national or individual energy may again attempt to solve this ancient difficulty.

84. Svanberg's book has received high praise. Delambre calls it *bel*; Laplace calls it *excellent*; and Airy calls it *very elegant*. *Rapport Historique*, 73. *Théorie des Probabilités, Deuxième Supplément*, 32. *Encyclopaedia Metropolitana*, 173. The theoretical investigations are satisfactory; but do not seem to me very remarkable; they are printed in a very repulsive manner. The great defect of the book is the perpetual omission of important details which cannot be supplied from any other source. There are however only two other matters on which I shall here offer any criticism; and it is necessary to advert to them, because silence might be misunderstood. One of these matters is Svanberg's method of combining his observations; and the other is Svanberg's notions on a question of mathematical history.

85. The repeating circle is sufficiently well known to render it unnecessary to give a full description of it. A pointer we may say takes up in succession positions on a graduated circle for which the readings are A_0, A_1, A_2,...A_n. If there were no error in the instrument or in the observation the difference of any two consecutive readings, as $A_{r+1} - A_r$, would give the required angle. If n is very large it is considered that $\frac{A_n - A_0}{n}$ would represent very accurately the required angle. Thus the intermediate values A_1, A_2,...A_{n-1} would not be used, and in fact would not be read. This then is the common mode of using a repeating circle.

86. Svanberg however proposed to record and use the intermediate readings. *Every pair* of readings was to be employed. Thus $A_k - A_h$ would be taken as corresponding to $(k - h)$ times the original angle.

For a simple example, let us suppose there are five readings A_0, A_1, A_2, A_3, A_4. Then Svanberg's notion is to take the following:

$$\begin{array}{r} A_1 - A_0,\ A_2 - A_0,\ A_3 - A_0,\ A_4 - A_0, \\ A_2 - A_1,\ A_3 - A_1,\ A_4 - A_1, \\ A_3 - A_2,\ A_4 - A_2, \\ A_4 - A_3, \end{array}$$

then add all these together, and divide by the sum of

$$\begin{array}{r} 1 + 2 + 3 + 4 \\ 1 + 2 + 3 \\ 1 + 2 \\ 1 \end{array}$$

The result in this case would be

$$\frac{4A_4 + 2A_3 - 2A_1 - 4A_0}{20}$$

Of course general formulæ can easily be constructed. S. 30.

87. It seems to me that the idea of this method might have presented itself to any person, and would in most cases have been immediately dismissed. I should call it a commonplace blunder. The principle of the repeating circle is that even if the instrument be small and therefore the graduations not very fine yet by taking a *large* multiple of the required angle we may eliminate the errors, instrumental and observational. To use as Svanberg does *small* multiples is equivalent to surrendering the supposed advantages of the instrument.

88. Svanberg himself seems to have had some misgivings; for he proposes also a modification of his process: this amounts to rejecting every such pair as $A_k - A_h$ when $k - h$ is less than some fixed number, say c. See his page 31.

He gives an example on his page 32, which is taken from his actual calculations on page 44. In this case $n = 12$ and $c = 5$.

89. Svanberg speaks of the original method as that in which *all the observations have the right to vote for the determination of the angle;* and then proceeds to apply somewhat different language to the modified method. The language seems to me very unfortunate. Take the example we have used in Art. 86. The *observations* strictly speaking are A_0, A_1, A_2, A_3, A_4. Now it will be seen that in the result A_2 does not occur; while A_0 and A_4 occur with twice the coefficient that A_1 and A_3 have.

90. It is strikingly characteristic of the serious defect in Svanberg's book that although he records all the readings of his instrument, and gives in every case the angle which he has adopted, he never tells us whether he has calculated this angle by the original method or by the modified method, except in the single example which he gives in his page 32. Moreover if he used the modified method he ought to have told us the value of e.

I may invite the attention of any student of Svanberg's book to the curious method by which he treats some doubtful observations on his page 148. It requires some exertion to discover his meaning.

91. Laplace devotes a few pages in the second supplement to his Theory of Probabilities to the consideration of Svanberg's method. Laplace says that it is a new example of the illusions to which we are liable in these delicate subjects.

Laplace considers only the original method, and not the modified method. He comes to the conclusion that Svanberg's method requires more observations than the common method in the ratio of 6 to 5 in order to give equally good results. I am unable to reconcile this numerical statement with a passage which I find in Delambre, page 472. "M. Laplace, qui a analysé cette nouvelle méthode, a trouvé que l'erreur à laquelle elle expose, est à celle de la méthode ordinaire comme 3 : 2, ainsi la méthode commune est préférable à tous égards; M. Svanberg, en convenant de la vérité de cette remarque en certains cas, pense qu'il en est d'autres où le rapport serait au contraire celui de 1 : 3,85...."

92. It remains for me to speak on Svanberg's notions as to a question of mathematical history; see his pages iv. v.

Svanberg alludes to the researches of Huygens on centrifugal force; and then he says that if we suppose with Huygens that the force of gravity resides in the centre of the earth we obtain for the ellipticity $\frac{1}{578 \cdot 73}$. Then he proceeds with the strange statement that these discoveries were the precursors of the hypothesis which Newton afterwards proposed of universal gravitation. Thus Svanberg ascribes to Huygens and not to Newton the priority in investigations of the Figure of the Earth. Now the first edition of the Principia was published in 1686. Huygens published in 1690 his *Discours de la cause de la Pesanteur*. Although it appears from the preface that some part of this work had been long before communicated to the Academy at Paris, yet Huygens excepts the part which relates to the motion of pendulums as affected by the Figure of the Earth: and all that relates to the calculation of the Figure of the Earth he expressly says was added after the publication of Newton's Principia. Perhaps Svanberg was misled by a hasty glance at Bailly's *Histoire de l'Astronomie Moderne*, Vol. III., page 9. Bailly accidentally notices Huygens before Newton.

93. Again Svanberg says, "Newton ne tarda pas à remarquer, que l'hypothèse de Huygens d'une force unique tendante au centre de la terre, étoit équivalente à celle d'y supposer une densité infinie." I do not know what this means. Huygens really used the

hypothesis of a force constant in magnitude, and directed always to the centre of the earth; it is of course only as to the latter part of the hypothesis that we can assert the equivalence of an infinite density at the centre. I do not think that Newton drew attention to this equivalence: Clairaut did in the *Philosophical Transactions*, Vol. 40, pages 277...306.

Clairaut, I think, was the first who shewed that Huygens's value of the ellipticity follows approximately from the latter part of Huygens's hypothesis, namely, that the force always passes through the centre, without assuming any particular law of intensity of the force at different distances. See Clairaut's *Figure de la Terre*, page 141. See also Plana *Astronomische Nachrichten*, Number 839.

I. TODHUNTER.

April, 1871.

Addition. *March*, 1872. Just as this memoir has been sent to the press I have obtained access to the work by F. G. W. Struve, entitled *Arc du Méridien de 25° 20′ entre le Danube et la Mer Glaciale...* Part of the great arc to which this work relates extends from Tornea to the northern extremity of Norway. This part of the survey was executed by the Swedes and Norwegians; and a fuller account of it is to be published by the Academy of Stockholm.

With such attention as I have been able to give to Struve's work, I have not discovered any explanation of the discrepancy between Maupertuis and Svanberg. The latitude of Kittis is not given; and neither of Svanberg's extreme stations occurs in the new survey.

The latitude of the Finnish church at Tornea, which is called *église du district*, is given as 65° 49′ 44″·57: Struve, Vol. I., page lxvi. Svanberg made it 73·164838 grades, that is 65° 50′ 54″·07. This seems to shew a great error in the most important part of Svanberg's operation.

On the other hand, Struve gives for the latitude of the next station in the survey towards the North, namely Stuor-oivi, 68° 40′ 58″·40; and for the length of the arc of the meridian between these two latitudes 163221·904 toises. This gives about 57194 toises for the length of a degree at the middle part of the arc, corresponding very closely with Svanberg's conclusion.

Struve adverts to the difference between Maupertuis and Svanberg. He says on page xvi. of Vol. I.: "En considérant que les observations de latitude n'avaient pas été faites sur les mêmes lieux, il a fallu admettre ou des déviations locales du fil à plomb, relativement très-considérables, ou une incertitude dans l'une ou l'autre des 4 latitudes observées, par suite d'imperfections des instruments employés." In speaking of *the four observed latitudes* Struve falls into the same error as the other astronomers cited in the memoir.

The Toise of Peru is mentioned on page lxxiv. of Vol. I.; this refers to the year 1821, and implies that the Toise was then in a trustworthy state.

II. *A Monograph of Ebenaceæ.* *By* W. P. Hiern, M.A., *St John's College.*

[Read March 11, 1872.]

The family Ebenaceæ was first established by *Ventenat* in 1799 in his "Tableau du Regne Végétal," was revised by *Jussieu* in the "Annales du Museum," Vol. v. p. 417, in 1804, and was finally assigned in 1810 by *Brown* in his "Prodromus Floræ Novæ Hollandiæ et Van-Diemen" and briefly reduced to its present shape.

In 1837 *George Don* in his "General System of Gardening and Botany," Vol. iv., gave an account of the whole family as understood by him; he enumerated 83 species which he distributed amongst 8 genera. He however included the genus *Diclidanthera* with 2 species which is now placed in the family Styraceæ: he placed in Ilicineæ instead of Ebenaceæ *Leucoxylum buxifolium*, Blum.: and he described the new genus *Diplonema* which however has not been maintained by subsequent authors as distinct from *Euclea*.

In 1844 *Alphonse De Candolle* monographed the family, amongst the earliest of his works, in the "Prodromus Systematis Naturalis Regni Vegetabilis," Vol. viii., and produced 160 species and 8 genera, with the omission however of *Leucoxylum buxifolium*, Blum. Three of these 8 genera were first defined in this monograph.

No subsequent treatise of an original character on the whole number of species of the family has appeared.

In the present monograph 5 genera only are recognized, one of which (*Tetraclis*) is new, and amongst these are distributed about 250 species; an account is also given of the fossils that have been published as members of the family, but these are not included in the above-mentioned estimate.

For the purpose of preparing the present paper I have consulted all the materials within my reach; I may mention the following important collections which I have examined.

(i) The royal herbarium at Kew, well known to be amongst the largest in existence, where I have had the advantage of *Professor Oliver's* incidental assistance.

(ii) The herbarium of the British Museum, containing many valuable type-specimens and a large miscellaneous collection.

(iii) The herbarium belonging to the University of Cambridge, including the late *Dr Lindley's* herbarium and *Lehmann's* herbarium, the latter named for the University by *Mr Bentham.*

(iv) The herbarium of the University of Oxford.

(v) The *Wallichian* herbarium of East Indian plants, now the property of the Linnean Society of London.

(vi) The EBENACEÆ of the University of Dublin, extremely rich in South African plants, got together by the late *Dr Harvey.*

(vii) The EBENACEÆ of *Dr Sonder's* herbarium of Hamburg, also rich in South African plants.

(viii) The herbarium of *Dr Van Heurck* at Antwerp.

(ix) The royal herbarium belonging to the botanical garden at Brussels, containing the private collection of the late *Von Martius,* the editor of the "Flora Brasiliensis."

(x) The royal herbarium at Leiden, where is the best collection in Europe of plants indigenous to the Malay archipelago.

(xi) The imperial herbarium at Berlin, where also is the important type-collection of *Willdenow.*

(xii) The imperial herbarium at Vienna.

(xiii) The royal herbarium at Munich, which is especially rich in Brazilian plants.

(xiv) The type-herbarium of *De Candolle* at Geneva.

(xv) The *Delessert* herbarium also at Geneva.

(xvi) The herbarium of the Paris Museum, which contains the best collections from Madagascar and New Caledonia, the herbarium of *Jussieu,* and a very large general collection of plants.

(xvii) The EBENACEÆ of the fine Angolan collection made by *Dr Welwitsch* with extraordinary care and true scientific judgment in the expedition undertaken by the Portuguese government from 1853 to 1860.

(xviii) The Australian collection of the great botanist *Brown,* now the property of Mr Bennett, late of the British Museum.

I have also been favoured with the manuscript of the African genus *Royena* belonging to the late *Dr Harvey,* which he had prepared but not completed for the "Flora Capensis;" I have taken up some new species of *Royena* which *Dr Harvey* had briefly described in this manuscript.

Dr Thwaites, of the royal botanical garden at Peradenia in Ceylon, has with much kindness supplied me with fresh flowers in spirit, as well as dried flowers, belonging to Ebenaceous species indigenous to that island, and published by him in his "Enumeratio Plantarum Zeylaniæ."

ECONOMIC PRODUCTS, &c.

The economic properties of Ebenaceæ are principally connected with the wood and the fruit, though other parts in some species are of value and importance. The valuable wood known by the name of Ebony is a black hard and heavy wood, produced for the most part by members of this family. Other families, however, such as *Leguminosæ, Sterculiaceæ, Bignoniaceæ,* &c. supply different kinds of wood that are also called by the name of Ebony. *Bertolini* in *Miscellanea Botanica,* VIII. p. 1 (1849), discusses the various claims of different plants to represent the ebony of the ancients, and decides in favour of a Leguminous species, which he calls *Fornasinia ebenifera.* For an account of ebony and its varieties, a paper may be consulted which was contributed by Mr P. L. Simmonds in the *Art Journal* for 1872, pp. 66—68. Ebony is confined to the heart-wood of the trees that produce it and is chiefly found in older trees; the wood of the younger being often of a pale colour.

Ebony, as the term is used in commerce, is a close-grained and nearly black wood of high specific gravity, heavier than water, a cubic foot weighing from 1100 to 1330 oz., is susceptible of a high polish, and is chiefly used for inlaying and fancy-work. The price of the timber as imported into England varies from £8. 10*s.* to £9. 12*s.* 6*d.* per ton; from 700 to 1000 tons are annually imported.

The wood is of an acrid pungent taste, and gives off an aromatic smell when burnt; when dried at 100° C. it is said to contain 49·8 per cent. of carbon, 5·3 of hydrogen and 44·9 of oxygen; it is also said to contain ulmic acid (see Schacht, *Der Baum*, p. 198). The strength of the wood is illustrated by the following experiment, but as it was tried on a piece of inferior specific gravity the result is probably below the full strength of a better class of ebony. A piece planed to one inch square and 24 inches long was supported at each end by two props, the clear distance from prop to prop being 20 inches; it was then found that a weight of 2 cwt. 3 qu. 20 lbs. was required (when hung on the middle) to break the piece. (See *Transactions of the Society of Arts*, Vol. XLVIII.)

Sawdust of Ceylon ebony (? Diospyros Ebenum, L.) when treated with cold water produces in the latter a rich or reddish brown colour, and after boiling together for some time no further change of colour results; the sawdust retains its original very dark colour.

Ebony is employed to make pianoforte keys, the stringholder in violins, spear-points, &c.; and the best kind of ebony is very valuable on account of its maintaining a permanent shape and not warping, and is therefore used for rules and measures.

Many hard woods such as box-wood, pear-tree wood, &c. are now artificially dyed black, and are used in commerce as ebony.

The following species supply ebony:

Diospyros Ebenum, König. India, &c.
Diospyros melanoxylon, Roxb. India.
Diospyros Dendo, Welw. Angola, West tropical Africa.
Diospyros sylvatica, Roxb. India, &c.
Diospyros Gardneri, Thw. Ceylon.
Diospyros hirsuta, Linn. fil. Ceylon.
Diospyros discolor, Willd. Malaya, &c.
Diospyros Embryopteris, Pers. India, &c.
Diospyros Ebenaster, Retz. Malaya, &c.
Diospyros montana, Roxb. India, &c.
Diospyros insignis, Thw. Ceylon and S. India.
Diospyros Tupru, Buch.-Ham. India.
Diospyros mespiliformis, Hochst. Tropical Africa.
Diospyros truncata, Zoll. and Mor. Java.
Diospyros tessellaria, Poir. Mauritius.
Diospyros haplostylis, Boiv. Madagascar.
Diospyros microrhombus. Madagascar.
Diospyros ramiflora, Wall. N.E. India.
Maba buxifolia, Pers. India, Madagascar, &c.

Maba Mualala, Welw. Angola, West tropical Africa.
Euclea pseudebenus, E. Mey. South Africa.
&c. &c.

The following species also produce good wood.

Diospyros Malacapai, Alph. DC. Wood yellow with black spots. Philippine Islands.
Diospyros pilosanthera, Blanc. Ornamental wood. Philippine Islands.
Diospyros pilosa, Alph. DC. Timber fit for building purpòses. Cochin China.
Diospyros pentamera. Wood very hard, pale. Australia.
Diospyros australis. Wood close-grained, fit for turnery. Australia.
Diospyros chloroxylon, Roxb. Wood pale. Circars, India.
Diospyros Paralia, Steud. Wood white and hard. Guiana.
Diospyros foliolosa, Wall. Valuable light-coloured wood. S. India.
Diospyros leucomelas, Poir. White wood with black lines. Mauritius.
Diospyros lanceæfolia, Roxb. Hard and handsome wood. E. Indies.
Maba geminata, Br. Australia.
Royena lucida, L. Cape of Good Hope.
Euclea racemosa, L. and E. undulata, Thunb. Cape of Good Hope.

In New Caledonia the species of *Maba* and *Diospyros* furnish excellent woods for building.

Calamander or Coromandel wood, a finely variegated and scarce wood, is produced by *Diospyros quæsita*, Thw. and by *Diospyros oppositifolia*, Thw.

Black dyes are obtained from *Diospyros mollis* in Burmah, according to the Rev. *Dr. Mason;* and from *Diospyros Cunalon*, Alph. DC., according to *Blanco*.

Anchors for large boats are made, in the province of Tavoy in Burmah, of the wood of *Maba buxifolia*, Pers.

Birds are said to die soon after eating the fruit of *Diospyros toxicaria*; and *Diospyros multiflora*, Blum., *Diospyros Ebenaster*, Retz, *Diospyros samoensis*, A. Gray, and a Brazilian species of Diospyros are fish-poisoners (see *Allemão*, Considerações sobre as plantas medicinaes da flora Cearense, pp. 41, 43 [1862]).

A decoction of the bark of *Diospyros Paralea*, Steud. is valuable against fevers in French Guiana; also in North America *Diospyros virginiana*, L. is used for a similar purpose.

The juice of the fruit of *Diospyros Embryopteris*, Pers. is very glutinous and charged with tannic acid, and is used throughout South India for paying the seams of fishing boats and for preserving fishing lines and nets.

The fresh wood of *Diospyros Malacapai*, Alph. DC. is said to keep off bugs (see *Blanco*, "Flora de Filipinas," p. 303 [1837]).

A decoction of the leaves of *Maba buxifolia*, Pers. in Madagascar is employed in cases of gastritis.

The fruits of the following species are edible.

Diospyros Kaki, Linn. fil. China, &c.
Diospyros virginiana, L. North America.
Diospyros Lotus, L. Asia.
Diospyros chloroxylon, Roxb. Circars, India.

Diospyros decandra, Lour. Cochin China.
Diospyros melanoxylon, Roxb. S. India.
Diospyros Embryopteris, Pers. S. India and Ceylon, &c.
Diospyros Ebenaster, Retz. Malaya, &c.
Diospyros Kirkii. East tropical Africa.
Diospyros Tupru, Buch. India.
Diospyros mespiliformis, Hochst. Tropical Africa.
Diospyros australis. Australia.
Diospyros batocana. Tropical Africa.
Diospyros tessellaria, Poir. Mauritius.
Maba major, Forst. Friendly Islands.
Euclea undulata, Thunb. South Africa.
&c. &c.

According to *Dr Kirk* near Victoria Falls in Tropical Africa the shrub *Euclea divinorum* is the medicine of the diviners, being rubbed in the hands.

Geographical Distribution.

The head-quarters of this family is India where the species are numerous, but of the five genera which compose the family only two (though these are the largest genera) occur in the whole of the East Indian regions. Two genera are peculiar to the continent of Africa, and one, a new monotypic genus, is peculiar to the island of Madagascar. Not a single species is indigenous to Europe; one however is naturalized in the countries bordering on the Mediterranean sea.

The majority of the species are confined to the tropical regions of both the eastern and western hemispheres; several species are found in the subtropical regions, especially of South Africa; very few in temperate regions, and none in the colder regions of either hemisphere.

A specimen, apparently belonging to a tropical species (*Maba buxifolia*, Pers.), is stated to have been met with near the straits of Magellan; but this is probably an error.

For the better comprehension of the distribution, I have given below lists of species as they are known to occur in the different botanical regions into which the whole earth's surface has been divided by Grisebach.

Arabia, New Zealand, Tasmania, Western Australia, and the district along the Andes in South America are destitute of a single representative of the family.

Geographical distribution of *Ebenaceæ* with reference to *Grisebach's* regions. See "Die Vegetation der Erde." 2 vols. 8vo. Leipzic, 1872.

I. Arctic flora. 0.
II. Forest region of the Eastern continent. 0.
III. Mediterranean region. *Diospyros Lotus*, L. (Naturalized.)
IV. Steppes region. *Diospyros Lotus*, L.
V. China, Japan region. *Diospyros Lotus*, L.; *D. Kaki*, L. f.; *D. Morrisiana*, Hance; *D. eriantha*, Champ.; *D. vaccinioides*, Lindl.

VI. Indian monsoon region. *Diospyros*, 86 sp. *Maba*, 19 sp. Centre for *Ebenaceæ*, &c. "which have not a wide distribution beyond," II. p. 71.

VII. Sahara. 0.

VIII. Tropical Africa and Natal. *Diospyros*, 15 sp.; *Maba*, 7 sp.; *Euclea*, 11 sp.; *Royena*, 10 sp.

IX. Kalahari. *Royena*, 2 sp.; *Euclea*, 5 sp.

X. Cape flora. *Euclea*, 14 sp.; *Royena*, 8 sp.

XI. Australia. *Maba*, 10 sp.; *Diospyros*, 6 sp.

XII. Forest region of the Western continent. *Diospyros virginiana*, L.

XIII. Prairie region. *Diospyros texana*, Scheele; *Maba intricata*.

XIV. Californian coast region. 0.

XV. Mexican region. *Diospyros*, 5 sp.; *Maba*, 3 sp.

XVI. West Indies. *Diospyros*, 3 sp.; *Maba*, 3 sp.

XVII. South American region North of the Equator. *Diospyros*, 8 sp.; *Maba*, 3 sp.

XVIII. Hylæa, region of equatorial Brazil. *Diospyros*, 8 sp.; *Maba*, 2 sp.

XIX. Brazil. *Diospyros*, 11 sp.; *Maba*, 3 sp.

XX. Flora of the tropical Andes of South America. 0?

XXI. Pampas region. 0.

XXII. Chilian transition region. 0?

XXIII. Antarctic forest region. 0?

XXIV. Ocean Islands. *Diospyros*, 27 sp.; *Maba*, 12 sp.; *Tetraclis*, 1 sp.

1. Azores. 0. 2. Madeira. 0. 3. Canaries. 0. 4. Cape Verd I. 0.
5. Ascension. 0. 6. St Helena. 0.
7. Madagascar. *Maba*, 3 sp.; *Diospyros*, 19 sp.; *Tetraclis*, 1 sp.
8. Mascarene I. *Diospyros*, 6 sp.
9. Seychelles. *Diospyros*, 1 sp.; *Maba*, 1 sp.
10. Sandwich I. *Maba*, 2 sp.
11. Fiji I. *Maba*, 2 sp.
12. New Caledonia. *Diospyros*, 3 sp.; *Maba*, 7 sp.
13. Norfolk I. 0. 14. New Zealand. 0. 15. Galapagos. 0.
16. Juan Fernandez. 0. 17. Falkland I. 0. 18. Tristan da Cunha. 0.
19. Kerguelens-land. 0.

LISTS OF SPECIES IN ABOVE-MENTIONED REGIONS.

VI. INDIAN MONSOON REGION.

Maba acuminata. Ceylon.
Maba oblongifolia. Ceylon.
Maba ovalifolia. Ceylon.
Maba nigrescens, Dalz. India.
Maba buxifolia, Pers.
Maba Andersoni, Soland. Tonga Islands.
Maba major, Forst. Tonga Islands.
Maba elliptica, Forst. Amboina and Cochin China.

Maba sumatrana, Miq. Java and Sumatra. Andaman Islands (?).
Maba micrantha. India.
Maba lamponga, Miq. Sumatra.
Maba merguensis. Mergui archipelago.
Maba confertiflora. Labuan.
Maba punctata. Borneo.
Maba Teijsmanni. Java.
Maba hermaphroditica, Zoll. Java.
Maba javanica, Zoll. Java.
Maba Maingayi. Malacca.
Maba Motleyi. Borneo.
Diospyros insignis, Thw. Ceylon and S. India.
Diospyros Tupru, Buch.-Ham. India.
Diospyros melanoxylon, Roxb. India.
Diospyros decandra, Lour. Cochin China.
Diospyros affinis, Thw. Ceylon.
Diospyros crumenata, Thw. Ceylon.
Diospyros sylvatica, Roxb. India and Ceylon. Java (?).
Diospyros Kurzii. South Andaman.
Diospyros ehretioides, Wall. Tavoy, &c.
Diospyros hirsuta, Linn. fil.
Diospyros Korthalsiana. Borneo.
Diospyros oocarpa, Thw. India and Ceylon.
Diospyros truncata, Zoll. and Mor. Java.
Diospyros borneensis. Labuan.
Diospyros quæsita, Thw. Ceylon.
Diospyros Malacapai, Alph. DC. Philippine Islands.
Diospyros attenuata, Thw. Ceylon.
Diospyros acuta, Thw. Ceylon.
Diospyros Brandisiana, Kurz. Burmah.
Diospyros pruriens, Dalz. Bombay and Ceylon.
Diospyros apiculata. Penang.
Diospyros foliolosa, Wall. Madras.
Diospyros pilosula, Wall. Silhet.
Diospyros paniculata, Dalz. Bombay.
Diospyros Horsfieldii. Malacca and Java.
Diospyros densiflora, Wall. Moolmyne and Amherst.
Diospyros oppositifolia, Thw. Ceylon.
Diospyros Carthei. Manila.
Diospyros polyalthioides, Korth. Borneo.
Diospyros octandra. Burmah and Pegu.
Diospyros stricta, Roxb. East Bengal.
Diospyros eriantha, Champ. Borneo and Sumatra.
Diospyros dasyphylla, Kurz. Burmah.

Diospyros flavicans. Malacca, &c.
Diospyros aurea, Teijsm. et Binn. Java.
Diospyros nigricans, Wall. East Bengal.
Diospyros Ebenum, Kön. (Extensive range.)
Diospyros pellucida. Philippine Islands.
Diospyros maritima, Blum. Java, Celebes, Timor, Samoa Islands.
Diospyros philippinensis, Alph. DC. Philippine Islands.
Diospyros Gardneri, Thw. Ceylon.
Diospyros lanceæfolia, Roxb. East Bengal.
Diospyros undulata, Wall. Malacca, Tavoy, &c.
Diospyros multiflora, Blanc. Philippine Islands.
Diospyros buxifolia. Malacca, Java, and S. Canara.
Diospyros montana, Roxb. (Extensive range.)
Diospyros Zollingeri. Java.
Diospyros Kaki, Linn. fil. Khasia.
Diospyros chartacea, Wall. Burmah.
Diospyros variegata, Kurz. Pegu.
Diospyros chloroxylon, Roxb. Bombay and Madras.
Diospyros cauliflora, Blum. Java.
Diospyros ramiflora, Wall. Bengal.
Diospyros Diepenhorstii, Miq. W. Sumatra.
Diospyros sumatrana, Miq. Sumatra and Borneo.
Diospyros pendula, Hasselt. Java.
Diospyros biflora, Blanc. Philippine Islands.
Diospyros macrophylla, Blum. Java.
Diospyros oleifolia, Wall. Amhert, Pegu, and Java.
Diospyros samoensis, A. Gr. Friendly Islands.
Diospyros ovalifolia, Wight. Ceylon and Madras.
Diospyros rhodocalyx, Kurz. Siam.
Diospyros frutescens, Blum. Java.
Diospyros perforata. Ceram I., Oceania.
Diospyros burmanica, Kurz. Pegu.
Diospyros oblonga, Wall. Penang and Singapore.
Diospyros Ebenaster, Retz. Philippine Islands, Celebes, and Amboina.
Diospyros discolor, Willd. Philippine Islands, Malaya, &c.
Diospyros argentea, Griff. Malacca.
Diospyros Embryopteris, Pers. (Extensive range.)
Diospyros vaccinioides, Lindl. Malacca, S. Andaman, &c.
Diospyros Cunalon, Alph. DC. Philippine Islands.
Diospyros dodecandra, Lour. Cochin China.
Diospyros Toposia, Ham. Bengal and Ceylon.
Diospyros cystopus, Miq. S. Sumatra.
Diospyros glauca, Rottl. Madras.
Diospyros grata, Wall. Nepal.

Diospyros Hasseltii, Zoll. Java.
Diospyros Kuhlii, Zoll. Java.
Diospyros microcarpa, Span. Timor.
Diospyros orixensis, Wight. Courtallum.
Diospyros penduliflora, Zoll. Java.
Diospyros pilosa, Alph. DC. Cochin China.
Diospyros pilosanthera, Blanc. Philippine Islands.
Diospyros pyrrhocarpa, Miq. W. Sumatra.
Diospyros timoriana, Miq. Timor.

VIII. TROPICAL AFRICA.

Royena pallens, Thunb. Angola and Seshike, Manganja hills, &c.
Royena cistoides, Welw. Angola.
Euclea pseudebenus, E. Mey. Niger, Angola, &c.
Euclea fructuosa. Zambesia.
Euclea bilocularis. Zanzibar.
Euclea Kellau, Hochst. Abyssinia.
Euclea multiflora. Angola, &c.
Euclea divinorum. Victoria Falls (and Delagoa Bay).
Euclea lanceolata, E. Mey. Benguela, &c.
Maba buxifolia, Pers. var. Guinea and Angola.
Maba lancea. Sierra Leone.
Maba Mannii. Niger.
Maba abyssinica. Abyssinia.
Maba quiloënsis. Quiloa, Zanzibar coast.
Maba Mualâla, Welw. Angola.
Diospyros Barteri. Niger.
Diospyros batocana. Zambesia.
Diospyros crassiflora. Old Calabar.
Diospyros Heudelotii. Senegal.
Diospyros Kirkii. Zambesia.
Diospyros Loureiriana, G. Don. Mozambique and Angola.
Diospyros Mannii. Niger.
Diospyros mespiliformis, Hochst. E. and W. tropical Africa.
Diospyros platyphylla, Welw. Angola.
Diospyros senensis, Kl. Mozambique.
Diospyros squarrosa, Kl. Mozambique.
Diospyros tricolor. Guinea.
Diospyros verrucosa. Zambesia.
Diospyros Dendo, Welw. Angola.

NATAL, DELAGOA BAY, &c.

Royena lucida, L.
Royena cordata, E. Mey.
Royena scabrida, Harv.
Royena villosa, L.

Royena hirsuta, L.
Royena pallens, Thunb.
Royena parviflora.
Royena nitens, Harv.
Royena glandulosa, Harv.
Euclea lanceolata, E. Mey.
Euclea divinorum.
Euclea multiflora.
Euclea natalensis, Alph. DC.
Euclea daphnoides.
Euclea undulata, Thunb.
Euclea macrophylla, E. Mey.
Maba natalensis, Harv.
Diospyros rotundifolia.

IX. Kalahari.

Royena hirsuta, L.
Royena pallens, Thunb.
Euclea pseudebenus, E. Mey.
Euclea tomentosa, E. Mey.
Euclea lanceolata, E. Mey.
Euclea ovata, Burch.
Euclea undulata, Thunb.

X. Cape Flora.

Royena lucida, L.
Royena cordata, E. Mey.
Royena villosa, L.
Royena hirsuta, L.
? Royena sessilifolia.
Royena pallens, Thunb.
Royena ambigua, Vent.
Royena glabra, L.
Euclea polyandra, E. Mey.
Euclea tomentosa, E. Mey.
Euclea coriacea, Alph. DC.
Euclea acutifolia, E. Mey.
Euclea lancea, Thunb.
Euclea pseudebenus, E. Mey.
Euclea linearis, Zeyh.
Euclea lanceolata, E. Mey.
Euclea ovata, Burch.
Euclea multiflora.
Euclea macrophylla, E. Mey.
Euclea daphnoides.
Euclea racemosa, L.
Euclea undulata, Thunb.

XI. Australia.

Maba hemicycloides, F. Muell.
Maba rufa, Labill.
Maba laurina, Br.
Cfr. Maba buxifolia, Pers.
Maba obovata, Br.
Maba geminata, Br.
Maba humilis, Br.
Maba reticulata, Br.
Maba compacta, Br.
Maba fasciculosa, F. Muell.
Diospyros australis.
Diospyros hebecarpa, Cunn.
Diospyros maritima, Bl.
Diospyros montana, Roxb.
Diospyros mabacea.
Diospyros pentamera.

XV. Mexican Region.

Maba albens.
Maba acapulcensis.
(?) Maba Pavonii.
Diospyros velutina.
Diospyros ciliata, Alph. DC.
Diospyros Ebenaster, Retz. (introduced?).
Diospyros texana, Scheele.
Diospyros cuneifolia.

XVI. West Indies.

Maba Grisebachii. Cuba.
Maba caribæa.
Maba inconstans. Griseb.
Diospyros tetrasperma, Sw. Jamaica, St Domingo and Cuba.
Diospyros halesioides, Griseb. Cuba.
Diospyros laurifolia, Rich. Cuba.

XVII. South American region north of the Equator.

Maba inconstans, Griseb.
Maba cauliflora, Mart. Cayenne.
Maba Mellinoni. French Guiana.
Diospyros tetrandra. Guiana.
Diospyros velutina. New Granada.
Diospyros Sprucei. San Carlos, Columbia.
Diospyros cayennensis, Alph. DC. French Guiana.
Diospyros Paralea, Steud. Guiana.
Diospyros glomerata, Spruce. French Guiana.
Diospyros capræfolia, Mart. Surinam.
Diospyros Goudotii.
Diospyros (?) xylopioides, Mart. Guiana.

XVIII. Hylæa, region of equatorial Brazil.

Maba myrmecocarpa.
Maba myristicoides.
Diospyros Poëppigiana, Alph. DC.
Diospyros emarginata.
Diospyros subrotata.
Diospyros polyandra, Spruce.
Diospyros glomerata, Spruce.
Diospyros capræfolia, Mart.
Diospyros artanthæfolia, Mart.
Diospyros Paralea, Steud.

XIX. Brazil.

Maba inconstans, Griseb.
Maba sericea.
Maba Hilairei, sp. nov.
Diospyros velutina.
Diospyros spinosa.
Diospyros Ebenaster, Retz. (introduced ?).
Diospyros discolor, Willd. (introduced ?).
Diospyros ovalis. Pernambuco.
Diospyros hispida, Alph. DC.
Diospyros coccolobæfolia, Mart.
Diospyros gaultheriæfolia, Mart.
Diospyros Weddellii.
Diospyros capræfolia, Mart.
Diospyros apeibacarpos, Radd.

XXIV. OCEAN ISLANDS.

Madagascar.	Mascarene I.	Seychelles.	Sandwich I.	Fiji I.	New Caledonia.
Maba diffusa.		Maba Seychellarum.		Maba foliosa, Rich.	
Maba buxifolia, Pers.		Diospyros platycalyx.			Maba rufa, Labill.
Maba lanceolata.					Maba buxifolia, Pers.
Diospyros toxicaria.			Maba Hillebrandii, Seem.		Maba fasciculosa, F. Muell.
	Diospyros tessellaria, Poir.		Maba sandwicensis, Alph. DC.		Maba ruminata.
Diospyros haplostylis, Boiv.					Maba elliptica, Forst.
	Diospyros melanida, Poir.				Maba Vieillardi.
Diospyros nodosa, Poir.					Diospyros Ebenum, Kön.
	Diospyros anonæfolia, Alph. DC.				Diospyros macrocarpa.
Diospyros leucomelas, Poir.					Diospyros Olen.
	Diospyros chrysophyllos, Poir.				
Diospyros calophylla.					
Diospyros subacuta.					
Diospyros microrhombus.					
Diospyros gracilipes.					
Diospyros Pervillei.					
Diospyros Boivini.					
Diospyros parvifolia.					
Diospyros squamosa, Boj.					
Diospyros comorensis.					
Diospyros lævis, Boj.					
Diospyros Thouarsii.					
Diospyros Vescoi.					
Diospyros Bernieri.					
Diospyros leucocalyx.					
Diospyros pruinosa.					
Tetraclis clusiæfolia					

Lists arranged in Numerical Order of numbered Collections of Ebenaceæ made by various principal Botanical travellers.

For use with numbered sets of distributed or large collections of plants, I have drawn up lists arranged in numerical order, so that in the case of any Ebenaceous plant belonging to such collections the name of the species can be at once ascertained when the number of the plant in the set is known. It will also give a direct view of the whole number of species of the family obtained by each botanical traveller; and as travellers have in most cases limited their journeys with respect to each set of plants to a particular region or locality, it follows that such lists are calculated to throw much light on the geographical distribution of the family.

Alphabetical List of enumerated Collectors.

Atherstone, Cape of Good Hope and Namaqua-land.
Barber, Borneo.
Barter, Niger.
Beccari, Keren, N.E. Tropical Africa.
Berlandier, Mexico.
Bernier, Madagascar.
Blanchet, Brazil.
Bolus, Cape of Good Hope.
Bonpland, Mexico and Guayaquil.
Botteri, Mexico.
Brown, Australia.
Burchell, Cape of Good Hope and Brazil.
Burton, Congo.
Chapelier, Madagascar.
Claussen, Brazil.
Cooper, Cape of Good Hope.
Cuming, Philippine Islands.
Cunningham, Australia.
Deplanche, New Caledonia.
Drege, Cape of Good Hope.
Drummond, North America.
Ecklon, Cape of Good Hope.
Forbes, Delagoa Bay, South Africa.
Galleotti, Mexico.
Gardner, Brazil and Ceylon.
Gerard, Natal and Madagascar.
Glaziou, Rio de Janeiro.
Goudot, New Granada and Madagascar.
Griffith and Helfer, East Indies.
Haenke, Mexico.
Harvey, Cape of Good Hope, Friendly Islands and Australia.
Helfer (and Griffith), East Indies.
Heudelot, Senegambia.
Hillebrand, Sandwich Islands.
Hohenacker, Canara, India.
Horsfield, Java.
Hostmann, Surinam, South America.
Irvine, Abbeokuta, West tropical Africa.
Jenkins, Assam.
Junghuhn, Sumatra.
Kotschy, Sennar, East tropical Africa.
Krauss, South Africa and Natal.
Lindheimer, Texas.
M'Ken, Natal and Madagascar.
Mac Owan, Cape of Good Hope.
Maingay, Malay Peninsula.
Mann, Guinea.
March, Jamaica.
Miers, Brazil.
Motley, Borneo.
Niven, Cape of Good Hope.
Oldham, Japan and Formosa.
Pancher, New Caledonia.
Pervillé, Madagascar and Seychelles.
Plée, Martinique.
Poëppig, Brazil, &c.
Pohl, Brazil.
Regnell, Brazil.

Remy, Sandwich Islands.
Richard, Madagascar and Bourbon.
Ritchie, East India.
Rugel, Cuba.
Sagot, Cayenne.
Saint Hilaire, Brazil.
Sanderson, Cape of Good Hope and Natal.
Schimper, Abyssinia.
Schlim, New Granada.
Schomburgk, Siam and Guiana.
Schott, Brazil.
Schweinfurth, Gallabat, East tropical Africa.
Seemann, Fiji Islands and China.
Sello, Brazil.
Sieber, Cape of Good Hope and Mauritius.
Spruce, Brazil.
Thwaites, Ceylon.
Trecul, Texas.
Triana, New Granada.
Vieillard, New Caledonia.
Wallich, East India.
Wawra, Mexico.
Weddell, Brazil.
Welwitsch, Angola.
Wight, East India.
Wilford, China, &c.
Wright, China, Japan, Cape of Good Hope, Cuba, &c.
Xantus, Lower California.
Zeyher, Cape of Good Hope.
Zollinger, Java.

1828—1832. *Wallich's* NUMERICAL LIST OF EAST INDIA PLANTS.

No. 4115. Diospyros montana. Roxb.
4116. Diospyros cordifolia, Roxb.
4117. Diospyros sylvatica, Roxb.
4118. Diospyros chloroxylon, Roxb.
4119. Diospyros ramiflora, Roxb.
4120. Diospyros Ebenum, König.
4121. Diospyros stricta, Roxb.
4122. Diospyros lanceolata, Roxb.
4123. Diospyros Embryopteris, Pers.
4124. Diospyros oblonga, Wall.
4125. Diospyros frondosa, Wall. (*excluded*).
4126. Diospyros venosa, Wall. (*excluded*).
4127. Diospyros lucida, Wall.
4128. Diospyros oleifolia, Wall.
4129. Diospyros acuminata, Wall. (*excluded*).
4130. Diospyros vaccinioides, Lindl.
4131. Diospyros discolor, Willd.
4132. Diospyros pilosula, Wall.
4133. Diospyros tomentosa, Roxb.
4134. Diospyros Roylii, Wall.
No. 4135. Diospyros chartacea, Wall.
4136. Diospyros undulata, Wall.
4137. Diospyros ehretioides, Wall. and *var.* mollis, Wall.
4138. Diospyros heterophylla, Wall.
4139. Diospyros amœna, Wall.
4140. Diospyros densiflora, Wall.
4141. Diospyros macrophylla, Wall.
4142. Diospyros grata, Wall.
4143. Diospyros foliolosa, Wall.
4144. Diospyros multiflora, Wall.
4145. Maba buxifolia, Pers.
4406. Diospyros Wightiana, Wall.
4407. Diospyros dubia, Wall.
6350. Diospyros Sapota, Roxb.
6351. Diospyros nigricans, Wall.
7295. Diospyros flavicans.
7461. Maba buxifolia, Pers.
7535. Maba buxifolia, Pers.
9061. "Ebenacea" est Erycibe glomerata, Wall. (*excluded*).

Thwaites. Enumeration of Ceylon plants, 1858—1864.

(*The pages refer to Dr Thwaites' book.*)

No. 382. Diospyros hirsuta, Linn. fil. p. 181.
477. Maba buxifolia, Pers. p. 183.
1815. Diospyros ovalifolia, Wight. p. 181.
1816. Diospyros ovalifolia, Wight. pp. 459, 181.
1908. Diospyros Gardneri, Thw. p. 181.
1909. Diospyros cordifolia, Roxb. p. 178.
1910. Diospyros Embryopteris, Thw. γ. nervosa. p. 178.
1911. Diospyros Toposia, Ham. p. 179.
1912. Diospyros Ebenum, Retz. p. 180.
1913. Diospyros Ebenum, Retz. p. 180.
1914. Diospyros oocarpa, Thw. p. 180.
1915. Diospyros Embryopteris, Pers. p. 178.
1916. Maba buxifolia, Pers. β. microphylla. p. 183.
1917. Maba buxifolia, Pers. δ. angustifolia. p. 183.
2437. Diospyros Ebenum, Retz. p. 180.
2438. Diospyros crumenata, Thw. p. 179.
2439. Diospyros Ebenum, Retz. p. 180.
2514. Diospyros Toposia, Ham. pp. 462, 179.
No. 2533. Diospyros ovalifolia, Wight. p. 181.
2729. Diospyros sylvatica, Roxb. p. 178.
2730. Diospyros insignis, Thw. p. 180.
2731. Diospyros Embryopteris, Pers. β. atrata. p. 178.
2833. Diospyros Moonii, Thw. p. 182.
2836. Diospyros pruriens, Dalz. p. 423.
2924. Diospyros affinis, Thw. p. 179.
3010. Diospyros quæsita, Thw. p. 180.
3011. Diospyros oppositifolia, Thw. p. 181.
3394. Diospyros Candolleana, Wight. p. 181.
3395. Maba buxifolia, Pers. γ. Ebenus. p. 183.
3396. Macreightia oblongifolia, Thw. p. 183.
3476. Diospyros acuta, Thw. p. 182.
3477. Diospyros insignis, Thw. p. 180.
3478. Diospyros attenuata, Thw. p. 182.
3717. Macreightia ovalifolia, Thw. p. 424.
3718. Macreightia acuminata, Thw. p. 424.
3774. Diospyros montana, Roxb. p. 423.

1862—3. Kew Distribution. Hb. Griffith and Helfer. East Indies.

No. 423. Diospyros flavicans.
454. Diospyros flavicans.
3616. Diospyros Lotus, L.
3617. Diospyros chloroxylon, Roxb.
3618. Maba merguensis, H.
3619. Diospyros undulata, Wall.
3621. Diospyros Ebenum, König.
3622. Diospyros Toposia, Hamilt.
3623. Diospyros flavicans.
3624. Diospyros stricta, Roxb.
3625. Diospyros argentea, Griff.
3626. Diospyros Embryopteris, Pers.
3626 (1). Diospyros melanoxylon, Roxb.
3627. Diospyros Embryopteris, Pers.
3627 (1). Diospyros cordifolia, Roxb.
3628. Diospyros nigricans, Wall.
No. 3629.(?) Diospyros undulata, Wall.
3630. Diospyros melanoxylon, Roxb.
3631. Diospyros lanceæfolia, Roxb.
3632.(?) Diospyros hirsuta, Linn. fil.
3633. Diospyros Kaki, Linn. fil. Bootan (*cult.*).
3634. Diospyros lanceæfolia, Roxb.
3635. Diospyros Ebenum, König, *var.*
3636. Diospyros undulata, Wall.
3637. Diospyros hirsuta, Linn. fil.
3638. Diospyros octandra.
3639. Diospyros flavicans.
3640.(?) Diospyros flavicans.
3641. Maba buxifolia, Pers. *var.*
3643. Diospyros vaccinioides, Lindl.

1866—7. KEW DISTRIBUTION. HB. WIGHT. EAST INDIES.

No. 1711 *bis*. Diospyros Embryopteris, Pers. Ceylon. March, 1836.
1712. Diospyros chloroxylon, Roxb.
1713. Diospyros montana, Roxb. Courtallum. April, 1835.
1714. Diospyros Ebenum, König. Malacca.
1715. Diospyros hirsuta, Linn. fil. Dec. 1835.
1716. Diospyros auriculata, Wight.
1717. Diospyros montana, Roxb. 1835.
1718. Diospyros obovata, Wight (*excluded*).
1719. Diospyros chloroxylon, Roxb.
1720. Diospyros ovalifolia, Wight. Madras. 1836, 1845.
1721. Diospyros melanoxylon, Roxb.
1722. Diospyros orixensis, Wight.
1723. Diospyros melanoxylon, Roxb. Calicut. 1846.
1724. Diospyros montana, Roxb. 1849.
1725. Diospyros melanoxylon, Roxb.
1726. Diospyros montana, Roxb. 1835.
1727. Diospyros melanoxylon, Roxb. Subbulpore.
1728. Diospyros hirsuta, Linn. fil. Mangalore. March, 1852.
1729. Maba buxifolia, Pers.
1730. Maba buxifolia, Pers. δ.
1731. Maba buxifolia, Pers. δ. Courtallum and Coonmore. 1846.

DR MAINGAY'S MALAY PLANTS.

966. Diospyros buxifolia. Malacca.
967. Diospyros oblonga, Wall. Singapore.
968. Diospyros argentea, Griff. Malacca.
969. Diospyros hirsuta, Linn. fil. *var.* Malacca.
970. Diospyros sp.
970 (2). Diospyros discolor, Willd. Pulo Ticus.
971. Diospyros Ebenum, König, *var.* Malacca.
972. Diospyros flavicans. Malacca.
973. Diospyros hirsuta, Linn. fil. Malacca.
974. Diospyros undulata, Wall. Malacca.
975. Diospyros Ebenaster, Retz. Malacca (*cult.*).
976. Maba Maingayi. Malacca.
977. Diospyros undulata, Wall. Malacca.
978. Diospyros sp.
979. Maba buxifolia, Pers. Malacca.
1514. Diospyros apiculata. Penang.

HOHENACKER.

No. 389. Canara, India. 1849. Maba nigrescens, Dalz.
591. Canara, India. 1847 (or 1849). Diospyros hirsuta, Linn. fil.
869. Canara, India. 1851. Diospyros Embryopteris, Pers.

RITCHIE.

No. 85. Moollis, India. 1853. Maba nigrescens, Dalz.
96. Ram Ghaut and Phoondu Ghaut, India. 1850—3. Diospyros hirsuta, Linn. fil.
970. India. 1853. Diospyros montana, Roxb.
972. India. 1853. Diospyros montana, Roxb.
1108. Belgaum, India. Diospyros melanoxylon, Roxb.
1240. India. 1853. Diospyros montana, Roxb.
1831. India. Diospyros Embryopteris, Pers.
1833. Bombay. Diospyros pruriens, Dalz.
1884. Canara, India. 1853. Diospyros paniculata, Dalz.

JENKINS.

277. Upper Assam. Diospyros Embryopteris, Pers.
Assam. (See under Diospyros Zollingeri.)

IRVINE.

No. 141. Abbeokuta. Diospyros senensis, Kl.

KOTSCHY.

394. Sennar (1836—8). Diospyros mespiliformis, Hochst.
470. Sennar (1836—8). Diospyros mespiliformis, Hochst.

SCHWEINFURTH.

973. Gallabat. Diospyros mespiliformis, Hochst.
974. Gallabat. Diospyros mespiliformis, Hochst.

BARTER (1857—8).

No. 290. Niger. Diospyros senensis, Klotsch.
1208. Niger. Diospyros mespiliformis, Hochst.
1220. Niger. Maba Mannii.
1334. Niger. Diospyros mespiliformis, Hochst.
3250. Niger. Diospyros senensis, Kl.
3251. Niger. Diospyros senensis, Kl.
3390. Niger. Diospyros senensis, Kl.
20194. Niger. Diospyros Barteri.

BURTON. 1863.

Congo. Diospyros Loureiriana, G. Don.

MANN. 1861.

No. 839. Bagroo river, West Tropical Africa. Maba Mannii.
924. Gaboon river, West Tropical Africa. Diospyros Mannii.

HEUDELOT. 1835—7.

638. Senegambia. Diospyros Hendelotii.
Senegambia. Diospyros senegalensis, Perrott.

BECCARI.

55. Keren. Upper Nubia or Abyssinia. 1870. Maba abyssinica.

SCHIMPER.

Sect. i.	No. 159.	Abyssinia.	1837.	Euclea Kellau, Hochst.
ii.	655.	Abyssinia.	1840.	Diospyros mespiliformis, Hochst.
ii.	1078.	Abyssinia.	1838.	Euclea Kellau, Hochst.
ii.	1243.	Abyssinia.	1838.	Diospyros mespiliformis, Hochst.
ii.	1527.	Abyssinia.		Euclea Kellau, Hochst.
iii.	1919.	Abyssinia.	1842.	Euclea Kellau, Hochst.
	913.	Abyssinia.	1852.	Euclea Kellau, Hochst.
	1080.	Abyssinia.		Maba abyssinica.
	1334.	Abyssinia.	1854.	Maba abyssinica.
	80.	Abyssinia.	1862.	Euclea Kellau, Hochst.

GERARD AND M'KEN.

No. 12.	Natal.	Royena cordata, E. Mey.
28.	Madagascar.	Maba buxifolia, Pers. *var.*
30.	Natal.	Royena villosa, L.
33.		Euclea lanceolata, E. Mey.
92.	Natal.	Euclea multiflora.
99.	Natal.	Royena cordata, E. Mey.
110.	Natal.	Maba natalensis, Harv.
129.	Natal.	Royena pallens, Thunb.
190.	?	Cfr. Diospyros toxicaria.
528.	Natal.	Euclea lanceolata, E. Mey.
613.	Natal.	Royena villosa, L.
614.	Natal.	Royena villosa, L.
615.	Natal.	Royena pallens, Thunb.
673.	Natal.	Euclea macrophylla, E. Mey.
675.	Natal.	Maba natalensis, Harv.
699.	Natal.	Euclea multiflora.

No. 1155. Natal. Euclea lanceolata, E. Mey.
1156. Natal. Euclea lanceolata, E. Mey.
1157. Natal. Royena pallens, Thunb.
1158. Natal. Royena nitens, Harv.
1238. Natal. Royena pallens, Thunb.
1506. Natal. Euclea daphnoides.
1604. Tugela. Cfr. Euclea macrophylla, E. Mey.
1605. Tugela. Euclea lanceolata, E. Mey.
1607. Tugela. Royena pallens, Thunb.
1608. Tugela. Royena cordata, E. Mey.
1609. Tugela. Royena scabrida, Harv.
1610. Tugela. Royena pallens, Thunb.
1611. Tugela. Royena pallens, Thunb.
2013. Natal. Royena villosa, L.
2015. Zulu-land. Royena parviflora.

COOPER.

No. 1. Cape of Good Hope. 1860. British Kaffraria. Royena villosa, L.
35. Cape of Good Hope. 1860. British Kaffraria. Royena cordata, E. Mey.
44. Cape of Good Hope. 1860. Euclea multiflora.
186. Cape of Good Hope. 1860. British Kaffraria. Royena cordata, E. Mey.
212. Cape of Good Hope. 1860. Queenstown. Royena hirsuta, L.
272. Cape of Good Hope. 1860. Queenstown. Royena pallens, Thunb.
306. Cape of Good Hope. 1860. British Kaffraria. Royena cordata, E. Mey.
408. Cape of Good Hope. 1863. Eastern Frontier. Euclea undulata, Thunb.
418. Cape of Good Hope. 1860. Queenstown. Royena pallens, Thunb.
1062. Cape of Good Hope. 1862. Orange Free State. Royena lucida, L.
1157. Natal. 1862. Royena pallens, Thunb.
1238. Natal. 1862. Royena pallens, Thunb.
1253. Natal. 1862. Euclea multiflora.
Cape of Good Hope. 1862. Royena lucida, L.

MAC OWAN.

269.	Cape of Good Hope.	Comm. 1867.	Royena hirsuta, L.
309.	Cape of Good Hope.		Royena lucida, L.
429.	Cape of Good Hope.		Royena cordata, E. Mey.
516.	Cape of Good Hope.	Comm. 1865.	Royena villosa, L.
527.	Cape of Good Hope.	Comm. 1867.	Royena cordata, E. Mey.
902.	Cape of Good Hope.	Comm. 1867.	Euclea lanceolata, E. Mey.
1646.	Cape of Good Hope.	Comm. 1870.	Royena pallens, Thunb.
	Cape of Good Hope.	Comm. 1865.	Euclea undulata, Thunb.
	Cape of Good Hope.		Euclea multiflora.

NIVEN (1798—1803).

No. 46. Cape of Good Hope. Euclea pseudebenus, E. Mey.
47. Cape of Good Hope. Euclea polyandra, E. Mey.
48. Cape of Good Hope. Royena glabra, L.
51. Cape of Good Hope. Royena pallens, Thunb.
53. Cape of Good Hope. Euclea polyandra, E. Mey.

SIEBER.

94. Cape of Good Hope. 1824. Royena glabra, L.
Suppl. 29. Mauritius. Diospyros discolor, Willd.

ATHERSTONE.

2. Namaqualand. S. Africa. Euclea pseudebenus, E. Mey.
461. Cape of Good Hope. Euclea macrophylla, E. Mey.

HARVEY.

544. Cape of Good Hope. Royena pallens, Thunb.
574. Cape of Good Hope. Euclea racemosa, L.
575. Cape of Good Hope. Euclea lanceolata, E. Mey.
690. Cape of Good Hope. Euclea lanceolata, E. Mey.
Friendly Islands. 1855. Diospyros samoënsis, A. Gr.
New South Wales. Diospyros australis.

FORBES (1822—3).

34. Delagoa Bay. Diospyros rotundifolia.
56. Delagoa Bay. Euclea divinorum.

BOLUS.

638. Graaf Reinet. Euclea coriacea, Alph. DC.

ZEYHER (1840—).

767. Cape of Good Hope. Euclea multiflora.
778. Cape of Good Hope. Euclea multiflora.
3348. Cape of Good Hope. Royena pallens, Th.
3349. Cape of Good Hope. Royena glabra, L.
3350. Cape of Good Hope. Royena hirsuta, L.
3351. Cape of Good Hope. Royena hirsuta, L.
3352. Cape of Good Hope. Royena lucida, L.
3353. Cape of Good Hope. Royena pallens, Th.
3354. Cape of Good Hope. Royena pallens, Th.
3355. Cape of Good Hope. Euclea lanceolata.

No. 3356. Cape of Good Hope. Euclea racemosa, L.
3357. Cape of Good Hope. Euclea lanceolata, E. Mey.
3358. Cape of Good Hope. Euclea undulata, Th.
3359. Cape of Good Hope. Cfr. Euclea lanceolata. Leaves wide.
3360. Cape of Good Hope. Euclea polyandra, E. Mey. or E. sp.
3361. Cape of Good Hope. Euclea multiflora.
3362. Cape of Good Hope. Euclea polyandra, E. Mey.
3363. Cape of Good Hope. Euclea polyandra, E. Mey.
3364. Cape of Good Hope. Euclea polyandra, E. Mey.

ECKLON AND ZEYHER.

1123. Cape of Good Hope. Euclea lanceolata, E. Mey.
1124. Cape of Good Hope. Cfr. Euclea undulata, Thunb.
1125. Cape of Good Hope. Euclea linearis, Zeyh.
1126. Cape of Good Hope. Royena ambigua, Vent.
1127. Cape of Good Hope. Royena pallens, Th.

DREGE (1826—34).

9140. Cape of Good Hope. Euclea coriacea, Alph. DC.

SANDERSON.

140. Cape of Good Hope. 1860. Royena pallens, Thunb.
150. Natal. Royena villosa, L.
318. Cape of Good Hope. 1860. Royena pallens, Thunb.
511. Cape of Good Hope. 1860. Royena pallens, Thunb.
527. Natal. 1860. Royena pallens, Thunb.
613. Natal. 1864. Royena villosa, L.
715. Natal. Royena villosa, L.
717. Natal. Royena pallens, Thunb.

KRAUSS (1838—40).

226. Natal. Royena villosa, L.
423. S. Africa. Royena pallens, Thunb.
472. Natal. Royena villosa, L.
482. Natal. Royena villosa, L.
1719. S. Africa. Royena hirsuta, L.
1721. S. Africa. Royena pallens, Thunb.
S. Africa. Royena glabra, L.

DR WELWITSCH, *Angola*, 1853—1860.

No. 1247. Pungo Andongo. Cfr. Euclea lanceolata, E. Mey.
1255. Huilla. Cfr. Royena pallens, Thunb.
1257. Huilla. Euclea multiflora.
1258. Huilla. Euclea multiflora.
2527. Congo. Maba buxifolia, Pers. γ. Ebenus, Thw.
2528. Congo. Diospyros mespiliformis, Hochst.
2529. Golungo Alto. Diospyros mespiliformis, Hochst.
2530. Bumbo. Diospyros mespiliformis, Hochst.
2531. Pungo Andongo. Diospyros (?) platyphylla, Welw.
2532. Pungo Andongo. Royena cistoides, Welw.
2533. Huilla. Royena pallens, Thunb.
2534. Huilla. Royena pallens, Thunb.
2535. Golungo Alto. Diospyros Loureiriana, Don.
2536. I. St Thomé. Ebenacea (?).
2537. Golungo Alto. Diospyros Dendo, Welw.
2538. Golungo Alto. Diospyros Dendo, Welw.
2539. Golungo Alto. Maba Mualala, Welw.
2540. Golungo Alto. Maba Mualala, Welw.
2541. Golungo Alto. Maba Mualala, Welw.
2542. Loanda. Maba Mualala, Welw.
2543. Mossamedes. Euclea pseudebenus, E. Mey.
2544. Mossamedes. Euclea pseudebenus, E. Mey.
2545. Benguela. Euclea lanceolata, E. Mey. *a*.
2546. Mossamedes. Euclea lanceolata, E. Mey. *a*.
2547. Mossamedes. Euclea lanceolata, E. Mey. *a*.
2548. Bumbo and Golungo Alto. Euclea lanceolata, E. Mey. *a*.
2549. Mumpulla, Huilla. Euclea lanceolata, E. Mey. *a*.
2550. Mumpulla, Huilla. Euclea lanceolata, E. Mey. *β*.
2551. Huilla. Euclea lanceolata, E. Mey. *β*.
2552. Huilla. Euclea lanceolata, E. Mey. *β*.
2553. Huilla. Cfr. Euclea lanceolata, E. Mey.
2555. Huilla. Euclea multiflora.
2557 Huilla. Euclea multiflora.

BURCHELL. ENUMERATION OF SOUTH AFRICAN PLANTS, DEC. 5, 1810 TO MARCH 30, 1815.

No. 2. Royena glabra, L.
397. Euclea racemosa, Thunb.
745—1. Royena pallens, Th. (cult.)
807. Euclea racemosa, L.
808. Royena glabra, L.
987. ? Euclea tomentosa, E. Mey.
No. 1696. Royena hirsuta, L. (R. microphylla, Burch.)
1706. Euclea ovata, Burch.
1750. Royena pallens, Th. (R. decidua, Burch.)
1792. Euclea undulata, Th.

No. 2162. Euclea undulata, Th. *var.* (E. myrtina, Burch.)
2371. Royena pallens, Th. (R. decidua, Burch.)
2487—2. Euclea ovata, Burch.
2487—7. Euclea ovata, Burch.
2502. Royena hirsuta, L. (R. microphylla, Burch.)
2542. Euclea ovata, Burch.
2573. Euclea undulata, Th. *var.* (E. myrtina, Burch.)
2920. Euclea ovata, Burch.
2930. Royena pallens, Th.
2943. Euclea undulata, Th.
3058—1. Euclea ovata, Burch.
3058—2. Euclea ovata, Burch.
3080. Royena pallens, Thunb. ex Harv. (Sond.) mss.
3102. Euclea ovata, Burch.
3168. Euclea undulata, Th.
3219. Euclea racemosa, L.
3301. Royena pallens, Th.
3325. Royena pallens, Th.
3396. Royena pallens, Th.
3472. Royena pallens, Th.
3510. Euclea multiflora.
3572. Euclea multiflora.
3673. Royena villosa, L. (R. scandens, Burch. ms.)
3789. Royena pallens, Th.
3793. Royena villosa, L. (R. scandens, Burch. ms.)
3806. Euclea racemosa, L.
3980. Euclea multiflora.
4166. Royena cordata, E. Mey.
4184. Royena pallens, Th.
4186. Royena cordata, E. Mey.
No. 4356. Euclea daphnoides.
4501. Royena pallens, Th.
4506. Royena villosa, L.
4807. ? Euclea polyandra, E. Mey.
4835. Euclea multiflora.
4873. ? Euclea polyandra, E. Mey.
4880. Euclea lanceolata, E. Mey.
4898. Royena hirsuta, L.
4907. Royena cordata, E. Mey. (R. supracordata, Burch. ms.)
4909. Euclea daphnoides.
4938. Euclea lanceolata, E. Mey.
4998. ? Euclea polyandra, E. Mey.
5093. Royena glabra, L.
5256. Royena lucida, L.
5367. Royena glabra, L.
5415. Royena lucida, L.
5490. Royena pallens, Th.
5529. Royena pallens, Th.
5632. Royena pallens, Th.
5648. Euclea lanceolata, E. Mey.
5784. Royena glabra, L.
6054. ? Royena villosa, L. (Royena scandens ? Burch. ms.)
6490. Royena pallens, Th.
6788. Royena glabra, L.
6813. Royena pallens, Th.
6941. Euclea polyandra, E. Mey.
7186. Royena glabra, L.
7198. Euclea undulata, Th.
7208. Royena glabra, L.
7288. Royena glabra, L.
7446. Royena hirsuta, L.
7531. Royena hirsuta, L.
7537. Royena hirsuta, L.
8295. Euclea racemosa, L.

BURCHELL. ENUMERATION OF BRAZILIAN PLANTS, April 10, 1828—Dec. 30, 1829.

6970. Maba sericea.
6986—2. Maba sericea.
6994. Diospyros hispida, Alph. DC.
7437. Diospyros hispida, Alph. DC.
8396. Diospyros hispida, Alph. DC.
9275. Diospyros subrotata.
9923. Diospyros subrotata.
9952. Diospyros subrotata.

PERVILLÉ (1837—41).

No. 6. Madagascar. Tetraclis clusiæfolia.
36. Seychelles. 1841. Maba Seychellarum.
275. Nossibe, Madagascar, 1841. Diospyros gracilipes.
439. Nossibe, Madagascar. Diospyros haplostylis, Boiv.
505. Nossibe, Madagascar. Diospyros haplostylis, Boiv.
525. Nossibe, Madagascar. 1841. Diospyros Pervillei.
640. Seychelles and Ambongo. 1841. Diospyros platycalyx.
700. Nossibe, Madagascar. 1841. Maba buxifolia, Pers.
Madagascar. 1841. Maba diffusa.

BERNIER, 1834—5.

112. Madagascar. 1834. Maba buxifolia, Pers. *var.*
113. Madagascar. Diospyros Bernieri.

RICHARD (ABOUT 1837).

36. Madagascar. Diospyros calophylla.
388. Madagascar. Tetraclis clusiæfolia.
Bourbon. Diospyros melanida ? Poir.

CHAPELIER.

82. Madagascar. Diospyros gracilipes.
Madagascar. Diospyros leucomelas, Poir.
Madagascar. Diospyros squamosa, Boj.

XANTUS.

68. Lower California. Maba intricata.

HAENKE (1790—1817).

47. Mexico. Maba albens.

WAWRA.

168. Mexico. Maba albens.
226. Mexico. Cfr. Diospyros velutina.
249. Mexico. Cfr. Diospyros Ebenaster, Retz.
1029. Mexico. Diospyros Ebenaster, Retz.

BONPLAND (1799—1803).

3846. Guayaquil. Maba inconstans, Griseb.
3984. Mexico. Cfr. Diospyros Ebenaster, Retz.
Mexico. 1803. Maba acapulcensis.

Pohl (about 1817).

No. 455. Brazil. Diospyros coccolobæfolia, Mart.
1980. Brazil. Maba inconstans, Griseb.
4568 (Schott). Brazil. Diospyros brasiliensis, Mart.

Sello (1819).

1211. Brazil. Maba inconstans, Griseb.
1689. Brazil. Maba inconstans, Griseb.
2301. Brazil. Maba inconstans, Griseb.
Brazil. Maba capreæfolia, Mart.

Schott, 1817. Cfr. Pohl.

Botteri.

909. Mexico. 1855. Diospyros brasiliensis, Mart.

Galleotti.

4609. Mexico. (1835—40). Diospyros brasiliensis, Mart.

Weddell.

577. Brazil. Diospyros Weddellii.
Brazil. (1843—4). Maba inconstans, Griseb.

Regnell.

iii. 1516. Brazil (before 1843). Maba inconstans, Griseb.

Claussen (about 1812).

67. Brazil. Diospyros sericea, Alph. DC.
147. Brazil. Cfr. Diospyros velutina.
464. Brazil. Diospyros sericea, Alph. DC. ex Mart.
478. Brazil. Diospyros hispida, Alph. DC.
1062. Brazil. Diospyros sericea, Alph. DC.
1063. Brazil. Cfr. Diospyros velutina.

Berlandier, 1827—30.

3030. Mexico. Diospyros texana, Scheele.

Trecul.

1249. Texas. Diospyros texana, Scheele.

DRUMMOND (1825—35).

No. 204 bis. N. America. Diospyros virginiana, L.
ii. 201. Texas. Diospyros texana, Scheele.
iii. 329. Texas. Diospyros texana, Scheele.

LINDHEIMER (1843—).

iii. 451. Texas. Diospyros texana, Scheele.
iii. 452. Texas. Diospyros texana, Scheele.
iii. 453. Texas. Diospyros texana, Scheele.

RUGEL.

662. Cuba. Maba caribæa.

PLÉE (1820—7).

762. Martinique. Maba inconstans, Griseb.

WRIGHT.

64. Hong Kong. Diospyros eriantha, Champ.
189. Loo Choo I. Cfr. Diospyros Lotus, L. and Diospyros maritima, Blum.
312. Hong Kong. Diospyros vaccinioides, Lindl.
313. Hong Kong. Diospyros Morrisiana, Hance.
348. Japan. Diospyros Kaki, L. f. Cuba. D. tetrasperma, Sw.
423. New Mexico. Diospyros texana, Scheele.
1331. Cuba. Maba caribæa.
2936. Cuba. Diospyros halesioides, Griseb.
2937. Cuba. Diospyros halesioides, Griseb.
2938. Cuba. Maba Grisebachii.
Cape of Good Hope. Euclea racemosa, L.

SPRUCE. 1851—56.

1516. Brazil. 1851. Diospyros Paralea, Steud.
1528. Brazil. 1851. Diospyros polyandra, Spr.
1913. Brazil. 1851. Diospyros emarginata.
1938. Brazil. 1851. Diospyros Pöppigiana, Alph. DC.
2542. Brazil. 1852. Maba myristicoides.
2635. Brazil. 1852. Diospyros Pöppigiana, Alph. DC.
2701. Brazil. 1852. Diospyros glomerata, Spr.
3138. Columbia. 1853. Diospyros Sprucei.
3159. Brazil. 1853. Diospyros Paralea, Steud.
3166. Brazil. 1853. Diospyros polyandra, Spr.
4411. Peru. 1855—6. Diospyros peruviana.

Gardner.

No. 1412. Brazil. 1838. Diospyros gaultheriæfolia, Mart.
1511. Brazil. 1838. Diospyros coccolobæfolia, Mart.
1512. Brazil. 1838. Diospyros velutina.
2284. Brazil. 1839. Diospyros velutina.
2813. Brazil. 1839. Diospyros ovalis.
531. Ceylon. Diospyros Embryopteris, Pers.
532. Ceylon. Diospyros Gardneri, Thw.
533. Ceylon. Diospyros Toposia, Ham.
Mauritius. Diospyros nodosa, Poir. *var.*
Mauritius. Diospyros chrysophyllos, Poir.

Sagot.

1253. Cayenne. 1859. Diospyros Paralea, Steud.

Schomburgk.

115. Siam. Diospyros Embryopteris, Pers.
1492. British Guiana (1864). Diospyros Paralea, Steud.

Hostmann.

547. Surinam (1843). Diospyros Paralea, Steud.

March.

1190. Jamaica. 1858. Diospyros tetrasperma, Sw.

Poeppig (1827—32).

2266. Maynas. Diospyros artanthæfolia, Mart.
2639. Brazil. Diospyros Pöppigiana, Alph. DC.

Blanchet (before 1844).

1886. Brazil. Diospyros gaultheriæfolia, Mart.
3358. Brazil. Diospyros sericea, Alph. DC. ex Mart.

Schlim (1846—52).

698. New Granada. Diospyros peruviana.

Triana.

2612. New Granada. Diospyros velutina.
2613. New Granada. Maba inconstans, Griseb.

Goudot.

No. 1. S. Martha. Maba inconstans, Griseb.
3. New Granada. Diospyros Goudotii.
Madagascar. Diospyros leucocalyx.

Glaziou.

1560. Rio de Janeiro. Diospyros discolor, Willd. *var.*
1561. Rio de Janeiro. Diospyros discolor, Willd. *var.*

St Hilaire.

375. Brazil (1816—22). Maba Hilairei.

Miers (1831—8).

3709. Brazil. Cfr. Diospyros velutina.

Oldham.

299. Formosa. 1864. Diospyros Kaki, L. f. *var.*
528. Japan. 1862. Diospyros Kaki, L. f.
529. Japan. 1862. Diospyros japonica, Sieb. et Zucc.

Wilford.

423. China. 1858. Diospyros vaccinioides, Lindl.
756. Tsu-Sima I., Str. Corea. Diospyros Kaki, L. f.
Tsu-Sima I., Str. Corea. Diospyros Lotus, L.

A. Cunningham.

157. N. S. Wales. 1818. Diospyros australis.
284. Australia. 1818. Diospyros cordifolia, Roxb.
306. Rodds Bay, Australia. 1819. Maba geminata, R. Br.
Australia, Queensland. Diospyros hebecarpa, A. Cunn.
Australia, Brisbane River. 1828. Diospyros pentamera.

Horsfield (Ebenaceæ). 1802—18.

1. Java. Diospyros Horsfieldii.
2. Java. Diospyros Embryopteris, Pers.
3. Java. Cfr. Diospyros aurea, Teijsm. et Binn.
4. Java. Diospyros truncata, Zoll et Mor.
5. Java. Cfr. Diospyros maritima, Blume.
6. Java. Cfr. Diospyros aurea, Teijsm. et Binn.
7. Java. Diospyros Embryopteris, Pers.
8. Java. Diospyros Embryopteris, Pers.

MOTLEY.

No. 7. Borneo (Labuan). Diospyros borneensis.
205. Borneo (Labuan). Maba confertiflora.
721. Borneo (1857—8). Maba Motleyi.
766. Borneo. Maba punctata.

ZOLLINGER (1841—).

1156. Java. Diospyros truncata, Zoll. et Mor.
1516. Java. Diospyros frutescens, Blume.
1833. Java. Diospyros maritima, Blume.
2651. Java. Diospyros Zollingeri.
3247. Java. Diospyros buxifolia.
3438. Java. Diospyros buxifolia.
3467. Java. Maba hermaphroditica, Zoll.
3565. Java. Diospyros Embryopteris, Pers.

BARBER.

167. Borneo. Maba Motleyi.

JUNGHUHN (1835—6).

719. Sumatra. Maba sumatrana, Miq.

HILLEBRAND.

273. Sandwich Islands. Maba sandwicensis, Alph. DC.
274. Sandwich Islands. Maba sandwicensis, Alph. DC.
Sandwich Islands. Maba Hillebrandii, Seem.

REMY.

470.(?) Sandwich Islands. Maba sandwicensis, Alph. DC.
472. Sandwich Islands. Maba Hillebrandii, Seem.
473. Sandwich Islands. Maba sandwicensis, Alph. DC.

SEEMANN.

295. Fiji Islands. 1860. ?Maba sandwicensis, Alph. DC.
2454. S. China. 1850. Diospyros vaccinioides, Lindl.

DEPLANCHE (1866—).

31. I. Lifu, Oceania. Diospyros Olen.
48. New Caledonia. Maba fasciculosa, F. Muell.
206. New Caledonia. Maba fasciculosa, F. Muell.

No. 311. New Caledonia. Maba ruminata.
312. New Caledonia. Maba rufa, Labill. *var.*
446. New Caledonia. Maba rufa, Labill.
448. New Caledonia. Maba Vieillardi.
449. New Caledonia. Maba Vieillardi.

VIEILLARD (1855—67).

No. 890. Kanala, New Caledonia (1855—60). Diospyros macrocarpa.
891. New Caledonia. Maba rufa, Labill.
892. New Caledonia. Maba rufa, Labill.
893. New Caledonia. Maba elliptica, Forst.
894. New Caledonia. Maba rufa, Labill.
895. New Caledonia. Maba rufa, Labill. *var.*
896. New Caledonia. Maba rufa, Labill.
897. New Caledonia. Maba Vieillardi.
898. New Caledonia. Diospyros Ebenum, Kœnig.
899. New Caledonia (1855—60). Maba fasciculosa, F. Muell.
2864. New Caledonia (1861—6). Maba buxifolia, Pers.
2869. New Caledonia (1861—6). Diospyros hebecarpa, Cunn.
2872. New Caledonia (1861—6). Cfr. Maba rufa, Labill.
2873. New Caledonia (1861—6). Maba buxifolia, Pers.
2876. New Caledonia (1861—6). Maba revoluta, Vieill.
2877. New Caledonia (1861—6). Cfr. Maba buxifolia, Pers.
2880. New Caledonia (1861—6). Maba rufa, Labill.

PANCHER.

No. 249. New Caledonia. Maba buxifolia, Pers. *var.*
251. New Caledonia. Diospyros macrocarpa.
301. New Caledonia. Maba foliosa, Rich.
New Caledonia. 1862. Maba rufa, Labill.
New Caledonia. Cfr. Diospyros Ebenum, Kœnig.

R. BROWN. AUSTRALIA. 1802—5.

Diospyros rugosula, R. Br. Carpentaria. Groote Island. Jan. 15, 1803.
Cargillia laxa, R. Br. Carpentaria. Jan. 4, 5, 1803.
Cargillia australis, R. Br. Port Jackson, &c. Nov. 1804.
Maba laurina, R. Br. Cumberland Islands. Oct. 17, 1802.
Maba obovata, R. Br. Carpentaria Islands, &c. Nov. 17, 18, 1802.
Maba humilis, R. Br. Broad Sound. Nov. 14, 1802.
Maba geminata, R. Br. Keppel Bay. August 10—12, 1802.
Maba littorea, R. Br. N. Coast Bay. March 3—6, 1803.

Maba reticulata, R. Br. Prince of Wales Islands, &c. Nov. 2—4, 1802. Cumberland Islands. Oct. 16, 17, 1802.

Maba compacta, R. Br. N. Coast Island. Feb. 18, 21, 1803.

CUMING (1836—9).

No. 1142. Philippine Islands. 1841. Diospyros philippinensis, Alph. DC.

1496, 1506. Philippine Islands. Diospyros pellucida.

1694. Philippine Islands. 1841. Maba Cumingiana, Alph. DC.

1829. Philippine Islands. 1841. Diospyros multiflora, Blanco, non Wall.

Description of the Family.

EBENACEÆ. Vent. Tabl. regn. veg. ii. p. 443 (*excl. pler. gen.*) ann. vii. (1799),
Juss. in Ann. Mus. v. p. 417 (*part.*) (ann. xiii. 1804),
Br. Prodr. Fl. Nov. Holl. et Van Diem. p. 524 (1810),
Agardh, class. plant. p. 18 (*part.*) (1825),
Bartl. ord. nat. plant. p. 161 (1830),
Mart. conspect. regn. veg. p. 26 (1835),
Perleb, Clav. class. ord. et Fam. p. 24 (1838),
Endl. gen. plant. p. 741 (1836—40),
Alph. DC. Prodr. viii. p. 209 (1844),
Lindl. Veget. kingd. *edit.* iii. p. 595 (1853),
Griseb. Grundr. syst. bot. p. 141 (1854),
Agardh, Theor. syst. plant. p. 128 t. x. fig. 11—13 (1858),
Le Maout et Decaisne, Trait. Gen. bot. p. 222 (1868).

Vaccinia, sect. iii. (*part.*) Adans. fam. pl. ii. p. 165 (1763);
Guaiacanæ, Juss. Gen. plant. p. 155 (*excl. pler. gen.*) (1789);
Bicornes, Giseke, Prælect. p. 337 (*part.*) (1792);
Diospyri (*part.*), Trattinnick, gen. pl. meth. Nat. disp. p. 52 (1802);
Ebenaceæ, trib. I. *Diospyreæ*, DC. et Dub. Bot. Gall. i. p. 320 (1828);
Sapotaceæ, c. *Sapoteæ*, bb. *Mimusopeæ* (*part.*), Reich. Pflanz. p. 38 (1834);
Ebenaceæ (*part.*) et *Styraceæ* (*part.*), Meisn. gen. i. p. 250, ii. p. 159 (1836—43);
Ebeneæ, Horan. Tetract. Nat. p. 27 (1843);
Diospyraceæ, Voigt, Hort. Suburb. Calcutt. p. 343 (1845).

EBENACEÆ.

CHARACTER ORDINIS.

Flores *sæpius diœci, rarius hermaphroditi vel polygami, dichlamydei,* 3—7-*meri.*

Calyx *inferior, synsepalus, persistens, in fructu sæpe plus minus accretus.*

Corolla *sympetala, regularis, hypogyna, decidua; lobis in præfloratione sinistrorse contortis, rarissime valvatis.*

Flos masculus: stamina 3—∞, *distincta vel geminata vel ad basim plus minus connata, corollæ lobis alterna vel alterna atque opposita, imo corollæ inserta vel hypogyna* *vel partim corollâ partim toro inserta.* Antheræ *basi affixæ, liberæ, biloculares, sæpius lineari-lanceolatæ et longitudinaliter dehiscentes; connectivo apice sæpius producto. Pollen sphæricum vel ellipsoideum, læve.* Ovarium *sæpius abortivum vel nullum.*

Flos femineus: staminodia 0—∞, *sæpius effœta, quam in mare sæpius pauciora.* Ovarium *liberum, integrum,* 2—16- *loculare; loculis* 1-, *rarius* 2- *ovulatis.* Ovula *ex apice anguli interioris pendula, anatropa, numero duplici stylorum vel styli loborum.* Stigmata *parva vel paulim dilatata, emarginata.*

Fructus *baccatus, abortu sæpe pauci-locularis et tunc mono- vel oligo-spermus, carnosus vel coriaceus.*

Semina *pendula, albuminosa, nervis depressis a basi ad apicem* 2 *vel* 3 *percursa; testâ lævi, coriaceâ.*

Albumen *copiosum, cartilagineum, æquabile vel interdum ruminatum.*

Embryo *dicotyledoneus, axilis vel paulo obliquus, semine dimidio vel dodrante circiter brevior, rectus vel leviter curvatus; radiculâ superâ, cylindricâ; cotyledonibus foliaceis, ovatis vel lanceolatis, radiculam subæquantibus vel excedentibus.*

Arbores vel frutices, ligno sæpe denso gravi duro et interdum in centro nigro, succo non lacteo, foliis alternis vel rarius suboppositis vel rarissime in tribus subverticillatis, simplicibus, integerrimis, exstipulaceis, sæpius coriaceis. Flores axillares vel laterales, cymosæ vel solitarii, albi carnei flavescentes vel virides nunquam cærulei.

Trees or shrubs, never herbs, varying in height from a few inches to 100 feet or more. Bark various, sometimes quite smooth as in several species of *Royena* and *Euclea*, in other cases as in *Diospyros virginiana* deeply scored both longitudinally and transversely. Wood hard, heavy and durable; in several species, namely in those which supply ebony, very dark or black in the centre and paler towards the circumference. Sap limpid, not milky. Leaves in most cases alternate, often distichous or with an angular divergence of $\frac{2}{5}$ths, rarely opposite or sub-opposite as in some species of *Euclea* and *Diospyros*, very rarely verticillate in whorls of 3 as in a few species of *Euclea;* simple, quite entire, rarely somewhat sinuous and in *Euclea ovata* minutely crenulate; usually coriaceous and opaque, less commonly membranous or pellucid-punctate; in the majority of species elliptic or oblong and often acuminate at apex; midrib usually depressed on the upper surface, secondary veins pinnately arranged usually remote arching within the margin and anastomosing; tertiary veins obscure or manifest, often transverse to the midrib, or in various directions; petioles usually short, rarely long or obsolete. Leaves evergreen or deciduous, in most cases pubescent at least when young, often shining on the upper surface. The general appearance of the foliage places the family in that type of vegetation which Grisebach names after the Bay-laurel. Inflorescence cymose, usually in the axils of the younger leaves, sometimes with solitary flowers as in some species of the genus *Diospyros* in most species of the genus *Royena* and in the female plants of many other species of the family; or occasionally lateral on the older branches as in *Maba cauliflora, Diospyros cauliflora, Diospyros ramiflora* and *Diospyros Diepenhorstii;* in most species more or less pubescent or tomentose and often ferruginous. Bracts usually present, and in many cases bracteoles also; both of these organs are in most cases glabrous inside; sometimes the peduncle arises from a nest of imbricated bracts. Pedicels articulated at the apex to the flowers. Cymes in most cases few-flowered in both sexes, especially in the female sex in which the flowers are usually fewer or solitary. In those cases in which the flowers are solitary, the presence of bracts on the peduncles or at their base often indicates the tendency to a more numerously flowered cyme. In the genus *Euclea* the cymes are often racemose. In some species of *Diospyros* the short cymes are arranged close together towards the extremities of the branches and not in the axils of fully developed leaves, so that the inflorescence puts on the appearance of being terminal, as for example in *Diospyros discolor.*

Flowers in the great majority of species diœcious, but with an occasional tendency to a polygamous condition, and in the genus *Royena* chiefly hermaphrodite; nearly always regular, 3-7-merous but usually tetramerous or pentamerous, in the genus *Royena* generally pentamerous, in *Euclea* never trimerous, and in *Maba* mostly trimerous; fragrant or without scent.

In some cases that monstrous condition called *phyllomania,* in which imbricated bracts take the place and give the appearance of flower-buds, is met with, as for example in *Diospyros flavicans* and *D. Zollingeri;* and *D. platyphylla* is at present known only in this state. In other cases male flowers become double (*flore pleno*) by conversion of stamens into petaloid organs, as for instance in *Maba lamponga.*

Male flowers usually with a rudiment of an ovary which is hairy or glabrous in correspondence with the hairy or glabrous ovary which is developed in the female plant of the same species. Sometimes however in the male plant the ovary is completely obsolete and the receptacle is the only representative of it.

Female flowers usually thicker than the male, and in most species furnished with staminodes which however are commonly fewer in number than the stamens of the corresponding male plant, or without staminodes as in the great majority of species of the genus *Euclea* and in the section *Ferreola* of the genus *Maba.*

Calyx synsepalous (gamosepalous), inferior, lobed to various depths or indistinctly lobed or even in a few species of *Diospyros* and *Maba* truncate and entire, and in *D. Toposia* closed in (male) bud and bursting irregularly as the flower opens, persistent, commonly campanulate and not reflexed in the flower, often accrescent and either erect or spreading or reflexed and sometimes plicate in the fruit, in a few species as in *Diospyros Ebenum* with an internal elevated rim at the top of the tube in fruit and the lobes spreading or reflexed. Rarely the calyx is irregular, the lobing being chiefly on one side, as in *Maba ovalifolia.* Calyx usually greenish and when hairy usually clothed with a shorter indumentum than that of the corolla; as exceptional cases it is whitened inside in *Diospyros gracilipes,* and violaceo-pruinose in the fruit of *D. pruinosa.* Æstivation of calyx various, valvate imbricated or contorted, and when contorted sinistrorsely so (as seen from inside).

Corolla sympetalous (gamopetalous), hypogynous, usually isomerous with the calyx, lobed to various depths in different species, usually exceeding the calyx and often greatly so, hypocrateriform, tubular, campanulate, urceolate, globose or even rotate; often hirsute sericeous or otherwise pubescent, especially on the back of the lobes, but sometimes glabrous outside, commonly glabrous inside, but in a few species hairy on both sides; subcoriaceous or fleshy; deciduous or occasionally marcescent and detached at the top of the fruit or rarely in a fragmentary state at its base; white, flesh-coloured, greenish, or yellow, never blue; lobes equal, obtuse or rounded or in some species acute, usually spreading or reflexed in full flower, contorted sinistrorsely in æstivation as regarded from inside except *Diospyros oocarpa* in which the æstivation is variously imbricated and except the new genus *Tetraclis* in which the æstivation is valvate.

Stamens in male flowers all fertile, hypogynous or more commonly inserted at or near the base of the corolla-tube or by exception about the middle of the corolla in *Diospyros Dendo* and some at the middle of the corolla in *D. Cunalon;* often in two rows or combined by their filaments in pairs or otherwise; the inner ones usually shorter, or subequal; varying in number from 3 to about 100, the average or common number being 10 in *Royena,* 16 in *Euclea* and *Diospyros,* 9 in *Maba* and 30 in *Tetraclis;* when equal in number to the lobes of the corolla alternating with them. Anthers usually lanceolate linear or oblong, hairy or glabrous, erect, attached by their base, free, 2-celled, dehiscing at their sides by longitudinal slits or rarely by apical pores; pollen globular or ellipsoidal, smooth; connective usually produced at the apex beyond the anthers, apiculate, often hairy; filaments usually shorter than the anthers, glabrous or hairy, compressed or filiform. Staminodes in female flowers without anthers or barren, often glabrous, sometimes absent.

Ovary in male flower abortive or absent; in female flower free, sessile, subglobose ovoid or conical (or "stipitato-constricted at base" in *Diospyros Diepenhorstii*), not lobed, syncarpous, without a disk, hairy or glabrous, 2-16-celled, usually 3- or 6-celled in the genus *Maba,* 4-celled in *Euclea,* 4-, 6-, 8- or 10-celled in *Royena,* and 4-, 8- or 10-celled in *Diospyros,* never with 5 or an odd number of cells except 3; cells 1-ovuled, or 2-ovuled in the section *Ferreola* of Maba and in the section *Cargillia* of *Diospyros;* the septa however are sometimes incomplete, especially in the lower part, and the alternate ones, namely, those opposite the styles or lobes of the style, are often thinner. Styles 1—5, distinct or connate at the base; stigmas often bifid at apex. Ovules pendulous from the inner side of the top of the cell of the ovary, commonly twice as numerous as the styles or as the lobes of the style, anatropal, oblong or ovoid; raphe decurrent on the outer side to an inferior chalaza. Fruit coriaceous or fleshy, tomentose pubescent glandular glabrate or glabrous, globular ovoid oblong or conical (depressed in *Diospyros apeibacarpos,* compressed in *D. dodecandra,* obconical in *D. stricta*), varying from $\frac{1}{8}-3$ in. in diameter, usually small in the genus *Euclea,* of moderate size in *Royena* and *Maba* and rather large in *Diospyros* and *Tetraclis;* in several species edible; indehiscent or in a few species splitting in a valvate manner from the apex; with several or by abortion with few cells; pericarp coriaceous or in the edible species thin and membranous.

Seeds 1—10, pendulous, usually solitary in the cells of the fruit, usually oblong and laterally compressed or when sole globose, marked externally with 2 or 3 depressed longitudinal lines; hile small; testa smooth, thin or coriaceous; albumen cartilaginous, abundant, white, uniform or in some species ruminated by intrusion of the coriaceous testa or obscurely striate in radiating lines in a few species; embryo axile or slightly oblique, straight or somewhat curved especially in globular seeds, whitish, $\frac{1}{3}-\frac{3}{4}$ths of the length of the seed; cotyledons 2, equal, foliaceous, with or without veins, contiguous, ovate or lanceolate; radicle superior, cylindrical, not thick, $\frac{1}{3}-\frac{2}{3}$rds of the length of the embryo.

The family as here presented contains 5 Genera, namely,

Royena 13 species, *Diospyros* about 160 species,
Euclea 19 species, *Tetraclis* 1 species;
Maba 56 species,

in all about 250 species; besides several fossil species that have been described as members of the family.

EBENACEÆ.

KEY TO THE GENERA.

Corolla with contorted æstivation.

- Hermaphrodite or rarely sub-diœcious. Stamens in one row. I. ROYENA.
- Diœcious or rarely polygamous. Stamens usually in 2 or more rows, often in pairs.
    - Calyx not accrescent. Staminodes usually absent from the ♀ flower. Ovary 4- (or very rarely 2- or 6-) celled. Inflorescence usually racemose, rarely panicled. II. EUCLEA.
    - Calyx often accrescent. Staminodes usually present in the ♀ flower, except in the section *Ferreola* of MABA. Ovary usually 3-, 6-, or 8-celled, occasionally 4- or 10-16-celled. Inflorescence cymose or 1-flowered, not racemose.
        - Ovary 3- or 6-celled. Flowers usually trimerous. III. MABA.
        - Ovary 4- or 8-16-celled. Flowers rarely trimerous. IV. DIOSPYROS.

Corolla with valvate æstivation. V. TETRACLIS.

THE AFFINITIES OF EBENACEÆ.

The following families have the closest affinity to *Ebenaceæ*.

Olacineæ.

Points of approach:

Calyx often accrescent. Seeds usually pendulous; albumen usually copious. Leaves alternate or rarely opposite, simple, usually quite entire, exstipulate.

Points of departure:

Petals usually valvate in æstivation. Ovary usually 1-celled. Stamens sometimes in part sterile.

Styraceæ.

Points of approach:

Corolla sympetalous, with imbricated æstivation. Stamens definite or indefinite, arising from the tube of the corolla, of unequal length. Ovary several-celled. Leaves alternate, exstipulate, simple. Seeds albuminous with axile embryo.

Points of departure:

Filaments usually longer than the anthers. Ovary usually quite or partially inferior. Flowers hermaphrodite. Leaves often serrulate. Ovules 2—∞ in each cell of the ovary. Albumen fleshy.

Anonaceæ.

Points of approach:

Flowers usually trimerous (as in *Maba*). Stamens often indefinite. Albumen copious, ruminated (as in several *Ebenaceæ*). Leaves alternate, quite entire, exstipulate. Pistil superior.

Points of departure:

Corolla apopetalous. Pistil usually apocarpous. Ovules erect. Embryo minute.

Ternstrœmiaceæ.

Points of approach:

Stamens ∞ or equal to or double of the number of the parts of the corolla, hypogynous or inserted at the base of the corolla. Ovary usually free. Leaves alternate or very rarely opposite, usually simple and exstipulate.

Points of departure:

Corolla usually apopetalous. Flowers usually hermaphrodite. Stamens not in pairs. Fruits frequently many-seeded.

Sapotaceæ.

Points of approach:

Corolla sympetalous, deciduous, imbricated in æstivation. Ovary free, usually hairy. Ovules solitary. Leaves alternate or very rarely subverticillate, quite entire, shortly petioled, exstipulate.

Points of departure:

Sterile stamens usually present; fertile ones opposite the corolla-lobes. Testa of the seeds bony or crustaceous, with a high polish, albumen wanting or fleshy or oily. Radicle inferior. Flowers hermaphrodite. Sap milky. According to *Richard* the suspension of the seed distinguishes *Ebenaceæ* from *Sapotaceæ.*

Ilicineæ.

Points of approach:

Corolla sympetalous, imbricated in æstivation. Stamens inserted on the corolla. Ovary superior; ovules solitary, pendulous, anatropal. Albumen copious. Embryo straight. Radicle superior. Leaves evergreen, alternate or opposite, coriaceous, simple, exstipulate. Flowers often unisexual.

Points of departure:

Filaments usually exceeding the anthers. Albumen fleshy. Embryo small. Stamens equal in number to the parts of the flower.

The following natural orders also bear some affinity to EBENACEÆ, but in a less degree than the previously mentioned ones:

Ericaceæ.	Euphorbiaceæ.
Humiriaceæ.	Laurineæ.
Tiliaceæ.	Myrsineæ.
Bixineæ.	Convolvulaceæ (Erycibeæ).
Magnoliaceæ.	Celastrineæ.
Chailletiaceæ.	Oleaceæ.

The accompanying plan is intended to set forth the affinities of Ebenaceæ (see Plate I.).

Mr Miers in "Contributions to Botany," Vol. I. p. 24, makes some pertinent remarks on the affinities of EBENACEÆ. He questions their close alliance with STYRACEÆ, compares them with ANONACEÆ, and considers that they ought rather to be arranged among the polypetalous groups. There is no doubt that many South American species point plainly to such a position (though I have always found the corolla to be sympetalous, even if its partitions are only slightly connate at the base); but if it be necessary to choose between a polypetalous and a gamopetalous position, I certainly prefer the latter. Indeed, several species have the corolla lobed only near the apex, and the affinity to Sapotaceæ (a gamopetalous family) is as close as to any other. *Mr Miers* seems to me to be quite right in maintaining the affinity of the family to OLACINEÆ.

Choisy, in his "Mémoires des Ternstroemiaceæ," p. 9 (1855) compares EBENACEÆ with TERNSTROEMIACEÆ and points out their proximity.

ON THE GENERA OF EBENACEÆ.

The diagnostic characters of the genera of this family are not well defined; indeed it has been proposed to unite all into one genus. Two genera are endemic in Africa, namely ROYENA and EUCLEA and are chiefly found at the Cape of Good Hope; however both genera enter Tropical Africa south of the Equator, and one species of EUCLEA occurs in Abyssinia. One species each of DIOSPYROS and MABA occur in South-east Africa south of the Tropic.

TETRACLIS has at present been detected only in the Island of Madagascar.

ROYENA is mainly characterized by its hermaphrodite solitary peduncled and drooping flowers with the stamens in one row and comparatively small leaves; but the flowers are not always hermaphrodite, and hermaphrodite flowers occasionally occur in other genera especially in EUCLEA and DIOSPYROS; and the remaining characters occur in several cases among the species of the other genera of the family, nor are they constant in the genus Royena. The genus approaches the section *Gunisanthus* of DIOSPYROS, and *D. Loureiriana* Don is closely allied to *R. parviflora.*

EUCLEA approaches ROYENA on one hand and DIOSPYROS on the other. The racemose or cymose inflorescence and the diœcious flowers generally distinguish it from ROYENA; and its African habitat with small fruit and non-accrescent calyx help to separate it from DIOSPYROS.

MABA in the majority of its species is remarkable for the trimerous symmetry of the flower, and 3- or 6-celled ovary with 6 ovules. The flowers however are not always trimerous, and the section *Trichanthera* of MABA approaches closely the section *Rospidios* of DIOSPYROS. The geographical distribution of the two genera is nearly identical and co-extensive. MABA however is more frequent in New Caledonia and other islands in the South Pacific Ocean as well as in Australia, and one species (*M. buxifolia*, Pers.) seems to have the widest range of any species of the family.

The best character to distinguish MABA is its 3- or 6-celled ovary, but some species both of EUCLEA and DIOSPYROS occasionally, but not I think normally, possess this peculiarity; I have therefore made this the fundamental character of the genus, and feel no doubt, notwithstanding certain cases of perplexing variability, that it is convenient for the practical classification of the family to maintain the genus MABA. I have on the other hand merged with it Alphonse De Candolle's genus *Macreightia*, Dalzell's genus *Holochilus* (the only species however of which I have not seen), and Hasskarl's genus *Rhipidostigma*, none of which three genera by such incorporation weakens the main characters of MABA.

DIOSPYROS being the largest genus, and indeed including the majority of the species of the family, exhibits the greatest amount of variation, and possesses points of contact with all the other genera. I have united with it the following genera, *Cargillia* R. Br., *Leucoxylum* Blum., *Noltia* Schum., *Gunisanthus* Alph. DC., and *Rospidios* Alph. DC., inasmuch as I fail to find any good or even plausible ground for maintaining any separation amongst them. *Cargillia* with two species (*C. laxa* and *C. australis*) was made a genus by Robert Brown principally on account of having 2 ovules in each cell of a 4-celled ovary. The former species often and so far as my observation goes always has an 8-celled ovary with a solitary ovule in each cell, and it is identical with *Diospyros maritima* Blum.; and the second species (*C. australis*) has sometimes at least an imperfect septum, partially dividing each cell of the 4-celled ovary.

Again in certain other species of *Diospyros*, the septa of the ovary when 8-celled are alternately thinner, and are therefore difficult to discern; so much so that Dr Solander in his manuscript notes now in the British Museum described the immature fruit of *D. chloroxylon* as 4-celled with 2 seeds in each cell, although the ovary is really 8-celled.

In some cases in the lower part of the ovary the alternate septa are imperfect and do not reach the axis, and therefore a transverse section across this portion would give a 4-celled ovary; but in the middle part of the ovary the same septa are joined with the axis, and a transverse section there would shew an 8-celled ovary.

Failing also to detect any peculiarity of habit to distinguish the genus *Cargillia*, I am obliged notwithstanding the eminence and reputation of its inventor to treat it as a mere section of DIOSPYROS. For a similar reason I consider *Macreightia* to be a section of MABA.

The genus ROSPIDIOS was founded on the combination of a 3-celled ovary associated with a tetramerous symmetry in the flower; but this observation was made on a cultivated specimen, while the wild specimens since observed reveal an 8-celled ovary: the genus therefore lapses into DIOSPYROS; the name however I have retained for a section of the latter genus.

Several species of DIOSPYROS are remarkable for ruminated albumen in the seeds, and I have employed this character for the purpose of separating such species as a section from the remaining species of the genus. It must however be admitted that as the condition of the albumen is unknown in many of the remaining species, it is quite possible and indeed probable, that some of them will require when better known to be removed to this section. In cases where the albumen has not been observed, I have, with the exception of *D. Kurzii* which is evidently very near to *D. sylvatica* Roxb. and of *D. decandra* Lour., considered for the purposes of classification the albumen to be equable.

The new genus which I call TETRACLIS differs from the rest of the family by a strictly valvate æstivation of the corolla instead of a contorted one. In other respects its characters do not substantially differ from DIOSPYROS. Mr Bentham's Brazilian genus BRACHYNEMA, which he described as a doubtful member of this family, certainly differs remarkably from it in habit, especially in respect of the foliage and fruit. The structure of the seed is not clearly known, but seems to me not to agree with that of this family; on the whole it seems to shew an alliance with the family OLACINEÆ rather than with EBENACEÆ.

Again Zollinger's second genus DREBBELIA from Java, which was described as Ebenaceous, seems to me to have characters absolutely accordant with the family OLACINEÆ, and indeed may even belong to OLAX itself. However, I have not seen a specimen and therefore cannot speak with confidence about it.

There is in the Kew herbarium, collected by *Mann* (No. 1800), a specimen of a male plant from Mount John river, West equatorial Africa, which probably belongs to a new genus of Ebenaceæ; but as the female plant is unknown to me I do not venture to publish it with a new name. The characters are as follows:

Flores diœci. Flores masculi 1—3-ni, subsessiles, axillares. Bracteæ minutæ. Calyx inferior, 3—4-fidus, campanulatus, parvus. Corolla monopetala, gracillime tubulosa, apice 3—4-loba; lobis patentibus, in præfloratione sinistrorse contortis. Stamina 2—3, receptaculo inserta; filamentis brevibus hirsutis; antheris linearibus, lateraliter bilocularibus, dorso pubescentibus. Ovarii rudimentum nullum. Flores feminei et fructus ignoti. Arbor parva, foliis simplicibus integerrimis distichis obliquis subsessilibus firmiter sub-membranaceis exstipulaceis. Species unica, Africæ occidentalis æquatoriæ incola.

BRIEF HISTORY OF THE SPECIFIC NAMES.

At the time of the publication of the first edition of Linnæus' "Species Plantarum," in 1753, only 5 Ebenaceous species were known, 3 belonging to *Royena* and 2 to *Diospyros*. The first species of *Euclea* was published in the 13th edition of the "Systema" of Linnæus in 1774, and the first species of *Maba* in 1776 by the two *Forsters*.

Loureiro in 1790 published several new species in his "Flora Cochinchinensis," but most of them remain a puzzle to this day, as but few of his specimens have reached European botanists.

Several Indian species were described and figured by *Roxburgh* in the first volume of his work on Coromandel plants published in 1795; Dr Koenig concurrently described some of them in the same work. In the "Encyclopædia Méthodique" *Poiret* published many new species in 1804, chiefly from Mauritius, and in the following year *Willdenow* described others in his edition of the "Species Plantarum." In 1810 *Brown* published his Australian Flora, and in it several species of the genus *Maba,* the new genus *Cargillia,* &c.

In 1825-26 *Blume* published some Javan species of *Diospyros* and a new genus *Leucoxylum.*

From 1828 to 1832 *Wallich* circulated the chief part of his lithographed list with specimens of East Indian plants, and amongst them a large number of Ebenaceæ chiefly belonging to the genus *Diospyros.*

In 1837 *Blanco* published his "Flora de Filipinas," which contained a few new names of Ebenaceæ; also in the same year *G. Don,* in the 4th volume of his "General System of Gardening and Botany," described all the species of Ebenaceæ known to him, including a revision of the family; and *E. Meyer* published his catalogue of the plants of South Africa collected by *Drege,* containing the names of several new species of *Royena* and *Euclea,* but without descriptions.

In 1844 *Alphonse De Candolle* monographed the family in the "Prodromus" and added many new species and the three new genera *Gunisanthus, Rospidios,* and *Macreightia.* In the following year *Alexander Braun* published the first fossil species of the family.

The most important subsequent contributions to the family contain some plates of *Dr Wight* in 1850; fossil species by *Unger* in 1850, 1851, 1866, 1867; some Indian species by *Dalzell* in 1852 and 1861; Brazilian species by *Martius* and *Miquel* in 1856; Java and Sumatra species by *Zollinger, Miquel, Teijsmann, Hasskarl, &c.;* Ceylon species by *Dr Thwaites* in 1860 and 1864; fossil species by *Ettingshausen, Heer, Massalongo, &c.;* Mozambique species by *Klotzsch* in 1862; Australian species by *Dr F. Mueller* and *Mr Bentham* in 1864—1869; and Indian species by *Major Beddome* and *Mr S. Kurz* in 1871.

In the present paper between 80 and 90 new species are described and 1 new genus.

In the subjoined chronological list of specific names each name is given only on the first occasion of its publication, even though the same name may have been subsequently published for a different species.

EBENACEÆ.

Chronological List of published specific Names, with references and localities.

A.D.

1753. Royena lucida, Linn. Spec. Plant. (vol. I.) p. 397. Cape of Good Hope.

1753. Royena glabra, Linn. Spec. Plant. (vol. I.) p. 397. Cape of Good Hope.

1753. Royena hirsuta, Linn. Spec. Plant. (vol. I.) p. 397. Cape of Good Hope.

1753. Diospyros Lotus, Linn. Spec. Plant. (vol. II.) p. 1057. Mediterranean region.

1753. Diospyros virginiana, Linn. Spec. Plant. (vol. II.) p. 1057. N. America.

1763. Diospyros inconstans, Jacq. Amer. p. 276. t. 174. f. 67. S. America.

1767. Royena villosa, Linn. Syst. Nat. edit. XII. vol. II. p. 302. Cape of Good Hope.

1768. Royena scabra, Burm. Prodr. Fl. Cap. p. 13. Cape of Good Hope.

1771. Vaccinium pensylvanicum, Miller, Gard. Dict. edit. VI. Cfr. Aiton, Hort. Kew. ed. II. vol. III. p. 62 (1811).

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1774. Euclea racemosa, Linn. Syst. Veg. edit. XIII. p. 747. Cape of Good Hope.
1775. Dactylus trapezuntinus, Forskål, Fl. Ægypt.—Arab. p. xxxvi. Constantinople.
1775. Paralea guyanensis, Aubl. Plant. Guin. vol. I. p. 576. t. 231. Guiana, S. America
1776. Diospyros Ebenum, Koenig in Physiogr. Sålsk. Handl. vol. I. p. 176. Ceylon.
1776. Maba elliptica, J. R. and G. Forst. Charact. Gen. Pl. p. 122. Friendly Islands.
1781. Royena polyandra, Linn. fil. Supplem. p. 240. Cape of Good Hope.
1781. Diospyros Kaki, Linn. fil. Supplem. p. 439. China.
1781. Diospyros hirsuta, Linn. fil. Supplem. p. 440. E. Indies.
1783. Pisonia (?) buxifolia, Rottb. in Nye Saml. Kong. Danske Skrift. vol. II. p. 536. t. 4. f. 2. Malabar.
1783. Diospyros glaberrima, Rottb. in Nye Saml. Kong. Danske Skrift. vol. II. p. 540. tab. V. E. Indies.
1784. Euclea undulata, Thunb. Nov. Gen. Pl. (v) p. 86. Cape of Good Hope.
1786. Maba major, G. Forst. Pl. escul. insul. Ocean. Austr. n. 21. p. 54. Friendly Islands.
1788. Embryopteris peregrina, Gaertn. Fruct. vol. I. p. 145. t. 29. E. Indies.
1788. Diospyros tetrasperma, Swartz Prodr. p. 62. W. India Islands.
1789. Cavanillea philippensis, Desrouss. in Encyl. Méth. vol. III. p. 663. Philippine Islands.
1789. Garcinia malabarica, Desrouss. in Encycl. Méth. vol. III. p. 701. E. Indies.
1789. Diospyros Ebenaster, Retz. Observ. Bot. fasc. v. p. 31. n. 88. E. Indies.
1790. Diospyros lobata, Loureiro, Fl. Cochinchin. p. 227. Cochinchina.
1790. Diospyros decandra, Loureiro, Fl. Cochinch. p. 227. N. Cochinchina.
1790. Diospyros dodecandra, Loureiro, Fl. Cochinch. p. 228. Cochinchina.
1790. Ebenoxylum verum, Loureiro, Fl. Cochinch. p. 613. Cochinchina.
1790. Euclea pilosa, Loureiro, Fl. Cochinch. p. 629. Cochinchina.
1790. Euclea herbacea, Loureiro, Fl. Cochinch. p. 629. China.
1794. Diospyros concolor, Moench. Meth. p. 470. N. America.
1794. Royena pallens, Thunb. Prodr. Plant. Capens., pars prior, p. 80. Cape of Good Hope.
1794. Ehretia ferrea, Willd. Phytogr. I. p. 4. t. 2. f. 2. Malabar.
1795. Ferreola buxifolia, Roxb. Coromand. vol. I. p. 35. t. 45. Coromandel Coast.
1795. Diospyros melanoxylon, Roxb. Coromand. vol. I. p. 36. t. 46. Coromandel Coast.
1795. Diospyros montana, Roxb. Coromand. vol. I. p. 37. t. 48. Coromandel Coast.
1795. Diospyros sylvatica, Roxb. Coromand. vol. I. p. 38. t. 47. Coromandel Coast.
1795. Diospyros chloroxylon, Roxb. Coromand. vol. I. p. 38. t. 49. Coromandel Coast.
1795. Diospyros cordifolia, Roxb. Coromand. vol. I. p. 38. t. 50. Coromandel Coast.
1795. Embryopteris glutinifera, Roxb. Coromand. vol. I. p. 49. t. 70. Coromandel Coast.
1798. Diospyros obovata, Jacq. Hort. Schœnbr. vol. III. p. 34. t. 312. St Domingo.
1798. Diospyros digyna, Jacq. Hort. Schœnbr. vol. III. p. 35. t. 313. I. Celebes.
1799. Royena angustifolia, Willd. Spec. Plant. II. p. 633. Cape of Good Hope.
1800. Euclea lancea, Thunb. Prodr. Pl. Capens., pars posterior, p. 85. Cape of Good Hope.
1803. Diospyros glauca, Rottler in Gesellschaft Naturf. freunde zu Berlin. Neue Schriften. vol. IV. p. 221. Madras.
1803. Diospyros ambigua, Vent. Malm. t. 17. Cape of Good Hope.
1803. Royena ambigua, Vent. Malm. n. 17. Cape of Good Hope.

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1804. Diospyros tessellaria, Poir. in Encycl. Méth. vol. v. p. 430. Mauritius.
1804. Ebenus tessellaria, Commers. ex Poir. in Encycl. Méth. vol. v. p. 430. Mauritius.
1804. Diospyros melanida, Poir. in Encycl. Méth. vol. v. p. 431. Mauritius.
1804. Ebenus melanida, Commers. ex Poir. in Encycl. Méth. vol. v. p. 431. Mauritius.
1804. Diospyros leucomelas, Poir. in Encycl. Méth. vol. v. p. 432. Mauritius.
1804. Ebenus leucomelas, Commers. ex Poir. in Encyclop. Méth. vol. v. p. 432. Mauritius.
1804. Diospyros nodosa, Poir. in Encyclop. Méth. vol. v. p. 432. Mauritius.
1804. Diospyros chrysophyllos, Poir. in Encyclop. Méth. vol. v. p. 433. Mauritius.
1804. Diospyros angulata, Poir. in Encyclop. Méth. vol. v. p. 434. Mauritius.
1804. Diospyros lanceolata, Poir. in Encyclop. Méth. vol. v. p. 434. Madagascar.
1804. Diospyros revoluta, Poir. in Encyclop. Méth. vol. v. p. 435. S. America.
1804. Diospyros tomentosa, Poir. in Encyclop. Méth. vol. v. p. 436. Tranquebar, E. Indies.
1804. Royena cuneata, Poir. in Encyclop. Méth. vol. VI. p. 322. Cape of Good Hope.
1805. Diospyros lycioides, Desf. in Annal. Mus. vol. VI. p. 448. t. 62. f. 1. Cape of Good Hope.
1805. Diospyros Commersoni, Gaertn. fil. Carp. III. p. 136. t. 208. Madagascar.
1805. Diospyros rubra, Gaertn. fil. Carp. III. 138. Mauritius (?).
1805. Diospyros discolor, Willd. Spec. Plant. vol. IV. p. 1108. E. Indies.
1805. Diospyros reticulata, Willd. Spec. Plant. vol. IV. p. 1109. Mauritius.
1805. Diospyros orixensis, Klein ex Willd. Spec. Plant. vol. IV. p. 1110. E. Indies.
1805. Diospyros salicifolia, Humb. and Bonpl. ex Willd. Spec. Plant. vol. IV. p. 1112. S. America.
1805. Diospyros obtusifolia, Humb. and Bonpl. ex Willd. Spec. Plant. IV. p. 1112. Mexico.
1807. Diospyros Tupru, Buchan. Journ. vol. I. p. 183. E. Indies.
1807. Maba buxifolia, Pers. Synops. Plant. vol. II. p. 606. E. Indies.
1807. Diospyros Embryopteris, Pers. Synops. Plant. vol. II. p. 624. E. Indies.
1807. Diospyros pubescens, Pers. Synops. Plant. vol. II. p. 625. Cape of Good Hope.
1807. Diospyros guiacana, Rob. Voyages, vol. III. p. 417. Louisiana and Florida.
1800—1809. Annona microcarpa, Jacq. Fragm. Bot. p. 40. t. 44. f. 7. Australia.
1809. Royena pubescens, Willd. Berol. p. 457. Cape of Good Hope.
1810. Diospyros rugosula, R. Br. Prodr. Fl. Nov. Holl. p. 526. Australia.
1810. Cargillia laxa, R. Br. Prodr. Fl. Nov. Holl. p. 526. n. 1. Australia.
1810. Cargillia australis, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 2. Australia.
1810. Maba laurina, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 1. Australia.
1810. Maba obovata, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 2. Australia.
1810. Maba humilis, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 3.
1810. Maba geminata, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 4. Australia.
1810. Maba littorea, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 5. Australia.
1810. Maba reticulata, R. Br. Prodr. Fl. Nov. Holl. p. 528. n. 6. Australia.
1810. Maba compacta, R. Br. Prodr. Fl. Nov. Holl. p. 528. n. 7. Australia.
1813. Royena latifolia, Willd. Enum. Pl. Berol. Suppl. p. 23.
1813. Diospyros lanceæfolia, Roxb. Catal. Pl. class. XIII. Polyandr. monog. Silhet.
1813. Diospyros stricta, Roxb. Catal. Pl. class. XIII. Polyandr. monog. Tipperah, E. Indies.

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1813. Diospyros bracteata, Roxb. Catal. Pl. class. XIII. Polyandr. monog. Dooab, E. Indies.
1814. Diospyros glutinosa, Roxb. Hort. Bengal. p. 40. India.
1814. Diospyros Sapota, Roxb. Hort. Bengal. p. 40. Mauritius [cult. ?]
1814. Diospyros Mabola, Roxb. Hort. Bengal. p. 40. Philippine Islands.
1814. Diospyros racemosa, Roxb. Hort. Bengal. p. 40. Tipperah, E. Indies.
1814. Diospyros ramiflora, Roxb. Hort. Bengal. p. 40. Tipperah, E. Indies.
1814. Ferriola buxifolia, Roxb. Hort. Bengal. p. 72. Coromandel.
1816. Diospyros sapotanigra, DC. Ess. Prop. Med. Pl. p. 200. Mexico.
1816. Royena lycioides, [Desf.] Cat. Hort. Paris. ex Poir. in Encyclop. Méth. Suppl. vol. IV p. 435. Cape of Good Hope.
1817. Monodora microcarpa, Dunal Monogr. Anonac. p. 80. Australia.
1817. Diospyros caroliniana, Muhlenb. ex Rafin. Florula Ludovic. p. 139. Mississippi.
1818. Diospyros acapulcensis, Kunth in Humb. and Bonpl. Nov. Gen. III. p. 254. Mexico.
1818. Diospyros psidioides, Kunth in Humb. and Bonpl. Nov. Gen. III. p. 254. S. America.
1818. Diospyros conduplicata, Kunth in Humb. and Bonpl. Nov. Gen. III. p. 254. S. America.
1820. Celastrus crispus, Thunb. Fl. Cap. edit. ii. vol. ii. p. 115. Cape of Good Hope.
1820. Diospyros apeibacarpos, Raddi, Quar. nuov. del Bras. p. 12. n. 10. Brazil.
1821. Diospyros rubiginosa, Roth, Nov. pl. sp. p. 385. E. Indies.
1821. Royena myrtifolia, Wendl. ex Steud. Nomencl. Bot. p. 705. Cape of Good Hope.
1822. Royena decidua, Burch. Trav. int. S. Afric. vol. I. p. 317. South Africa.
1822. Royena microphylla, Burch. Trav. int. S. Afric. vol. I. p. 348. South Africa.
1822. Euclea ovata, Burch. Trav. int. S. Afric. vol. I. p. 387. South Africa.
1823. Diospyros chinensis, Blume, Cat. Hort. Buit. p. 110. China.
1823. Cavanillea Mabolo, Lamarck in Encyclop. Méth. *tab.* 454. Philippine Islands.
1824. Maba rufa, Labill. Sert. Austr. Caled. p. 33. t. 36. New Caledonia.
1824. Euclea myrtina, Burch. Trav. int. S. Afric. vol. II. p. 588. South Africa.
1825. Maba Ebenus, Spreng. Syst. Veg. vol. II. p. 126. Molucca Islands.
1825. Diospyros vaccinioides, Lindl. in Hook. Exot. Fl. t. 139. China.
1825. Diospyros serrata, Hamilt. ex D. Don, Prodr. Fl. Nep. p. 143. Nepal.
1825. Diospyros cerasifolia, D. Don, Prodr. Fl. Nepal. p. 144. Nepal.
1825. Diospyros cauliflora, Blume, Bijdr. Fl. Ned. Ind. p. 668. Java.
1825. Diospyros frutescens, Blume, Bijdr. Fl. Ned. Ind. p. 668. Java.
1825. Diospyros maritima, Blume, Bijdr. Fl. Ned. Ind. p. 669. Java.
1825. Diospyros macrophylla, Blume, Bijdr. Fl. Ned. Ind. p. 670. Java.
1826. Leucoxylum buxifolium, Blume, Bijdr. Fl. Ned. Ind. p. 1169. Java.
1827. Diospyros exculpta, Hamilt. in Trans. Linn. Soc. Lond. vol. XV. p. 110. E. Indies.
1827. Diospyros insculpta, Hamilt. in Trans. Linn. Soc. Lond. vol. XV. p. 112. E. Indies.
1827. Diospyros Toposia, Hamilt. in Trans. Linn. Soc. Lond. vol. XV. p. 115. Bengal.
1827. Noltia tricolor, Schum. and Thonn. Plant. Guin. p. 189. Guinea, Africa.
1827. Diospyros edulis, Lodd. ex Sweet. Hort. Brit. p. 270. E. Indies.
1828—32. Diospyros incisa, Hamilt. Hb. ex Wallich, list n. 4122 D. E. Indies.
1828—32. Diospyros glutinifera, Hb. Madr. ex Wallich, list n. 4123 B. Quilon. E. Indies.
1828—32. Diospyros oblonga, Wallich, list n. 4124. Penang, India.

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1828—32. Diospyros (?) frondosa, Wallich, list n. 4125. Penang, India.
1828—32. Diospyros venosa, Wallich, list n. 4126. Penang, India.
1828—32. Diospyros lucida, Wallich, list n. 4127. Singapore, India.
1828—32. Diospyros oleifolia, Wallich, list n. 4128. Amherst, India.
1828—32. Diospyros (?) acuminata, Wallich, list n. 4129. Singapore, India.
1828—32. Diospyros Mabolo, Wallich, list n. 4131 A.
1828—32. Diospyros (?) pilosula, Wallich, list n. 4132. Sillet, India.
1828—32. Diospyros Roylii, Wallich, list n. 4134. India.
1828—32. Diospyros (?) chartacea, Wallich, list n. 4135. Burmah, India.
1828—32. Diospyros undulata, Wallich, list n. 4136. Amherst, India.
1828—32. Diospyros ehretioides, Wallich, list n. 4137. Amherst, India.
1828—32. Diospyros heterophylla, Wallich, list n. 4138. Ava.
1828—32. Diospyros amoena, Wallich, list n. 4139. Sillet.
1828—32. Diospyros densiflora, Wallich, list n. 4140. Moulmyne and Amherst.
1828—32. Diospyros grata, Wallich, list n. 4142. Nepal.
1828—32. Diospyros (?) foliolosa, Wallich, list n. 4143. S. India.
1828—32. Diospyros multiflora, Wallich, list n. 4144. Sillet.
1828—32. Diospyros Wightiana, Wallich, list n. 4406. India.
1828—32. Diospyros dubia, Wallich, list n. 4407. India.
1828—32. Diospyros nigricans, Wallich, list n. 6351. Sillet.
1828—32. Guatteria (?) flavicans, Wallich, list n. 7295. Penang.
1832. Diospyros Schitze, Bunge, En. Chin. bor. n. 237. p. 42. N. China.
1834. Diospyros Persimon, Wikstr. Jahr. Schwed. 1830. pp. 92, 96. N. America.
1834. Diospyros punctata, Decaisne in N. Ann. Mus. Hist. Nat. vol. III. p. 407. Timor.
1834. Diospyros malabarica, Kosteletsky, Med. Pharmac. Flora (III.) p. 1099. India.
1835. Diospyros microcarpa, Spanoghe in Hook. Comp. Bot. Mag. vol. I. p. 348. Timor.
1835. Diospyros dioica, Spanoghe in Hook. Comp. Bot. Mag. vol. I. p. 348. Timor.
1835—36. Diospyros albens, Presl, Reliq. Haenk. II. p. 62. Mexico.
1836. Diospyros angustifolia, Lodd. Cat. ex Loudon, Arb. et Frut. Brit. II. p. 1197 (1838). N. America.
1836. Diospyros fertilis, Lodd. Cat. ex Loudon, Arb. et Frut. Brit. II. p. 1197 (1838). N. America.
1836. Diospyros ciliata, Rafin. New Flora and Bot. N. Amer. part III. p. 25. Florida, N. America.
1837. Diospyros biflora, Blanco, Fl. Filipin. p. 303. Philippine Islands.
1837. Diospyros pilosanthera, Blanco, Fl. Filipin. p. 304. Philippine Islands.
1837. Sapota nigra, Blanco, Fl. Filipin. p. 409. Philippine Islands.
1837. Diospyros pterocalyx, Bojer, Hort. Maurit. p. 200. Mauritius.
1837. Diospyros Loureiriana, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 39. E. Trop. Africa.
1837. Embryopteris gelatinifera, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 41. E. Indies.
1837. Embryopteris discolor, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 41. Philippine Islands.

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1837. Embryopteris racemosa, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 41. Sillet.
1837. Embryopteris Loureiriana, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 41. Cochin China.
1837. Embryopteris Kaki, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 41. Japan, China and Cochin China.
1837. Diplonema elliptica, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 42. Cape of Good Hope.
1837. Diplonema ambigua, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 42. Cape of Good Hope.
1837. Maba (?) Ebenoxylon, G. Don, Gen. Syst. Gard. and Bot. vol. IV. p. 43. Cochin China.
1837. Royena cordata, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Royena brachiata, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Royena cuneifolia, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Royena rugosa, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg, p. 7. Cape of Good Hope.
1837. Euclea rufescens, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea macrophylla, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea lanceolata, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea polyandra, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea tomentosa, E. Meyer, Cat. Pl. Exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea acutifolia, E. Meyer, Cat. Pl. exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea rigida, E. Meyer, Cat. Pl. exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Euclea pseudebenus, E. Meyer, Cat. Pl. exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1837. Leucoxilon laurinum, E. Meyer, Cat. Pl. exsicc. Afr. Austr. Dreg. p. 7. Cape of Good Hope.
1840. Myrsine Kellau, Schimper in Pl. Abyss. exsicc. sect. I. n. 159. Abyssinia.
1840. Diospyros mollis, Wall. ex Steud. Nomencl. Bot. edit. II. part 1. p. 514. Tavoy, E. Indies.
1840. Diospyros Paralea, Steud. Nomencl. Bot. edit. II. part. 1. p. 514. S. America.
1840. Diospyros phyllomegas, Steud. Nomencl. Bot. edit. II. part 1. p. 514. Java.
1840. Patonia Walkerii, Wight, Illustr. vol. I. p. 19. Ceylon.
1840. Diospyros calycina, Audib. Cat. Hort. Tonn. ex Spach, Hist. Végét. IX. p. 405. N. America.
1841. Royena media, Hort. ex Steud. Nomencl. Bot. edit. II. vol. II. p. 475. Cape of Good Hope.
1841. Diospyros tetrandra, Spanoghe, Prodr. Fl. Timor. in Linnæa XV. p. 336. Timor.
1842. Kellaua Schimperi, Alph. DC. in Ann. Sc. Nat. Ser. II. vol. XVIII. p. 209. Abyssinia.
1842. Diospyros mespiliformis, Hochst. in Pl. Schimp. Abyss. exsicc. sect. II. nn. 655, 1243. Abyssinia.
1842. Euclea Kellau, Hochst. in Pl. Schimp. Abyss. exsicc. sect. II. n 1078. Abyssinia.
1842. Diospyros intermedia, Hort. ex Loudon Enc. Tr. and Shr. p. 627. N. America.
1843. Royena rufescens, E. Meyer, Pflanzengeogr. Doc. Drèg. p. 154 in Flora. XXVI. ii. Cape of Good Hope.
1843. Royena opaca, E. Meyer, Pflanzengeogr. Doc. Drèg. p. 217 in Flora. XXVI. ii. Cape of Good Hope.

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1843. Royena falcata, E. Meyer, Pflanzengeogr. Doc. Drèg. p. 217 in Flora. XXVI. ii. Cape of Good Hope.

1843. Euclea ochrocarpa, E. Meyer, Pflanzengeogr. Doc. Drèg. p. 184 in Flora. XXVI. ii. Cape of Good Hope.

1844. Royena sericea, Bernh. in Flora. XXVII. ii. p. 824. Cape of Good Hope.

1844. Euclea Kraussiana, Bernh. in Flora. XXVII. ii. p. 824. Cape of Good Hope.

1844. Euclea ferruginea, Bernh. in Flora. XXVII. ii. p. 825. Cape of Good Hope.

1844. Royena ramulosa, E. Meyer ex Alph. DC. Prodr. VIII. p. 212. n. 6. Cape of Good Hope.

1844. Euclea elliptica, Alph. DC. Prodr. vol. VIII. p. 216. n. 1. Cape of Good Hope.

1844. Euclea Dregeana, Alph. DC. Prodr. vol. VIII. p. 216. n. 2. Cape of Good Hope.

1844. Euclea coriacea, Alph. DC. Prodr. vol. VIII. p. 216. n. 4. Cape of Good Hope.

1844. Euclea natalensis, Alph. DC. Prodr. VIII. p. 218. n. 10. Natal.

1844. Royena macrophylla, E. Meyer ex Alph. DC. Prodr. vol. VIII. p. 218. n. 10. Natal.

1844. Diospyros (?) pilosa, Alph. DC. Prodr. vol. VIII. p. 219. Cochinchina.

1844. Gunisanthus pilosulus, Alph. DC. Prodr. vol. VIII. p. 220. Sillet.

1844. Rospidios vaccinioides, Alph. DC. Prodr. vol. VIII. p. 220. China and Malacca.

1844. Macreightia caribæa, Alph. DC. Prodr. vol. VIII. p. 221. n. 1. St Domingo.

1844. Macreightia albens, Alph. DC. Prodr. vol. VIII. p. 221. n. 2. Mexico.

1844. Macreightia acapulcensis, Alph. DC. Prodr. vol. VIII. p. 221. n. 3. Mexico.

1844. Macreightia psidioides, Alph. DG. Prodr. vol. VIII. p. 221. n. 4? S. America.

1844. Macreightia conduplicata, Alph. DC. Prodr. vol. VIII. p. 221. n. 5. S. America.

1844. Macreightia inconstans, Alph. DC. Prodr. vol. VIII. p. 221. n. 6. New Granada.

1844. Macreightia Pavonii, Alph. Prodr. vol. VIII. p. 222. n. 7. America.

1844. Diospyros cayennensis, Alph. DC. Prodr. vol. VIII. p. 224. n. 8. Cayenne, &c.

1844. Danzleria axillaris, Bert. ex Alph. DC. Prodr. vol. VIII. p. 224. n. 8. Cayenne.

1844. Diospyros Pœppigiana, Alph. DC. Prodr. vol. VIII. p. 224. n. 9. Brazil?

1844. Diospyros mauritiana, Alph. DC. Prodr. vol. VIII. p. 226. n. 15. Mauritius.

1844. Diospyros macrocalyx, Alph. DC. Prodr. vol. VIII. p. 226, n. 17. Mauritius (or Bourbon ?).

1844. Diospyros capensis, Alph. DC. vol. VIII. p. 226. n. 19. Cape of Good Hope.

1844. Diospyros membranacea, Alph. DC. Prodr. vol. VIII. p. 227. n. 20. Mauritius.

1844. Diospyros anonæfolia, Alph. DC. Prodr. vol. VIII. p. 227. n. 21. Mauritius (or Bourbon ?).

1844. Diospyros Neraudii, Alph. DC. Prodr. vol. VIII. p. 227. n. 23. Mauritius.

1844. Diospyros philippinensis, Alph. DC. Prodr. vol. VIII. p. 231. n. 43. Philippine Islands.

1844. Diospyros squamosa, Bojer ex Alph. DC. Prodr. vol. VIII. p. 232. n. 49? Madagascar.

1844. Diospyros lævis, Bojer ex Alph. DC. Prodr. vol. VIII. p. 232. n. 50. Madagascar.

1844. Diospyros senegalensis, Perrottet ex Alph. DC. Prodr. vol. VIII. p. 234. n. 59? Senegambia.

1844. Diospyros Berterii, Alph. DC. Prodr. vol. VIII. p. 234. n. 61. New Granada.

1844. Diospyros citrifolia, Wallich ex Alph. DC. Prodr. VIII. p. 235. n 65. Burmah.

1844. Diospyros sericea, Alph. DC. Prodr. vol. VIII. p. 236. n. 67. Brazil.

1844. Diospyros hispida, Alph. DC. Prodr. vol. VIII. p. 236. n. 68? Brazil.

1844. Diospyros Boutoniana, Alph. DC. Prodr. vol. VIII. p. 236. n. 72. Mauritius (or Bourbon ?).

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1844. Diospyros Blancoi, Alph. DC. Prodr. vol. VIII. p. 237. n. 74. Philippine Islands.
1844. Diospyros Malacapai, Alph. DC. Prodr. VIII. p. 237. n. 75. Philippine Islands.
1844. Diospyros Canomoi, Alph. DC. Prodr. VIII. p. 237. n. 78. Philippine Islands.
1844. Diospyros (?) Cunalon, Alph. DC. Prodr. vol. VIII. p. 237. n. 79. Philippine Islands.
1844. Diospyros feminina, Hamilt. ex Alph. DC. Prodr. VIII. p. 238. n. 83. Nepal.
1844. Maba Cumingiana, Alph. DC. Prodr. vol. VIII. p. 241. n. 4. Philippine Islands.
1844. Maba madagascariensis, Alph. DC. Prodr. vol. VIII. p. 241. n. 7. Madagascar.
1844. Maba guineensis, Alph. DC. Prodr. vol. VIII. p. 241. n. 8. Guinea, Africa.
1844. Maba Smeathmanni, Alph. DC. Prodr. vol. VIII. p. 241. n. 9. Sierra Leone.
1844. Maba sandwicensis, Alph. DC. Prodr. vol. VIII. p. 242. n. 16. Sandwich Islands.
1844. Cargilia maritima, Hassk. Cat. Pl. Hort. Bot. Bogor. II. p. 159. Java.
1845. Vaccinium fragrans, Wall. ex Voigt Hort. Suburb. Calcutt. p. 345. n. 13. China.
1845. Diospyros grandifolia, Wall. ex Voigt Hort. Suburb. Calcutt. p. 345. n. 18. Mauritius.
1845. Diospyros nigra, Blanc. Flora de Filipinas, edit. ii. p. 211. Philippine Islands.
1845. Diospyros brachysepala, Alex. Braun in Leonhard and Bronn, Neues Jahrb. Mineral. p. 170. Germany.
1846. Diospyros japonica, Sieb. and Zucc. Fl. Jap. II. 12 in Abh. Bayer. Acad. IV. 3. p. 136. n. 459. Japan.
1846. Diospyros truncata, Zoll. and Mor. in Mor. Syst. Verz. Jav. Pflanzen. p. 43. Java.
1847. Brachycheila pubescens, Harv. ex Zeyh. in Linnæa XX. p. 192. Cape of Good Hope.
1847. Euclea pubescens, Eckl. and Zeyh. in Linnæa XX. p. 192. Cape of Good Hope.
1847. Euclea linearis, Zeyh. in Linnæa XX. p. 192. Cape of Good Hope.
1847. Euclea desertorum, Eckl. and Zeyh. in Linnæa XX. p. 192. Cape of Good Hope.
1847. Euclea humilis, Eckl. and Zeyh. in Linnæa XX. p. 192. Cape of Good Hope.
1848. Diospyros Umlovok, Griffith, Itinerary Notes, p. 355. India.
1848. Diospyros pendula, Hasselt ex Hassk. Plant. Javan. p. 468. Java.
1848. Diospyros hexasperma, Hasselt ex Hassk. Plant. Javan. p. 468. Java.
1848. Diospyros ferruginea, Spltgbr. in Vriese Ned. Kruidk. Arch. p. 327. Guiana.
1849. Euclea angustifolia, Benth. in Hook. Niger Fl. p. 441. W. Tropical Africa.
1849. Maba vacciniæfolia, Benth. in Hook. Niger Fl. p. 442. W. Tropical Africa.
1849. Diospyros texana, Scheele in Linnæa XXII. p. 145. Texas, N. America.
1850. Diospyros Candolleana, Wight, Icon. tt. 1221—2. India.
1850. Diospyros capitulata, Wight, Icon. tt. 1224, 1588 bis. India.
1850. Diospyros ovalifolia, Wight, Icon. t. 1227. Madras.
1850. Maba neilgherrensis, Wight, Ic. Pl. Ind. Or. nn. 1228—9. Neilgherries, India.
1850. Plumeria flos-Saturni, Unger, Gen. et Sp. Pl. Foss. p. 433. Croatia.
1850. Diospyros Wodani, Unger, Gen. et Sp. Pl. Foss. p. 435. Croatia.
1850. Diospyros Auricula, Unger, Gen. et Sp. Pl. Foss. p. 436. Croatia.
1850. Diospyros Myosotis, Unger, Gen. et Sp. Pl. Foss. p. 436. Croatia.
1850. Anona Lignitum, Unger, Gen. et Sp. Pl. Foss. p. 441. Europe.
1850. Celastrus europæus, Unger, Gen. et Sp. Pl. Foss. p. 459. Croatia.
1850. Tetrapteris Harpyiarum, Unger, Foss. Fl. Sotzka, p. 46. t. 29. ff. 9, 10. Europe.
1850. Getonia macroptera, Unger, Foss. Fl. Sotzka, p. 51. t. 33. ff. 6—8. Europe.

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1851. Diospyros amplexicaulis, Lindl. and Paxt. Fl. Gard. vol. II. p. 11. n. 271. f. 139. Mauritius.

1851. Diospyros Scheuzeri, Al. Br. ex Unger, Pflanzenwelt, p. 233. Europe.

1851. Diospyros lancifolia, Al. Br. ex Unger, Pflanzenwelt, p. 233. Europe.

1851. Diospyros pannonica, Ettingsh. Foss. Fl. Wien, p. 19. t. III. f. 8. Austria.

1851. Diospyros hæringiana, Ettingsh. Tert. Fl. Häring. p. 61. t. 21. f. 26. t. 22. f. 11. Tyrol.

1851. Diospyros longifolia, Stizenberger, Verzeichniss, p. 83. Europe.

1852. Diospyros paniculata, Dalzell in Kew Journ. Bot. vol. IV. p. 109. Bombay.

1852. Diospyros pruriens, Dalzell in Kew Journ. Bot. vol. IV. p. 110. Bombay.

1852. Diospyros Goindu, Dalzell in Kew Journ. Bot. vol. IV. p. 111. India.

1852. Holochilus micranthus, Dalzell in Kew Journ. Bot. vol. IV. p. 291. Bombay.

1852. Diospyros eriantha, Champion in Kew Journ. Bot. vol. IV. p. 302. Hong Kong.

1852. Diospyros Morrisiana, Hance ex Walpers Annal. vol. III. p. 14. Hong Kong.

1854. Diospyros argenteus, Griffith, Notulæ, vol. IV. p. 288. Malacca.

1854. Maba hermaphroditica, Zollinger, Syst. Verzeichniss Ind. Archip. p. 135. Java.

1854. Arbutus diospyrifolius, Massal. Lett. Scarab. p. 29. n. 203 in Ann. Sc. Nat. Bologn. Italy.

1845—55. Diospyros laurifolia, Rich. Fl. Cub. in Ramon de la Sagra, Hist. de Cuba, vol. XI. p. 86. tab. 55 ex Walp. Ann. bot. Syst. vol. V. p. 480 (1858).

1851—5. Diospyros sumatrana, Miq. Plant. Jungh. vol. I. p. 203. Sumatra.

1851—5. Maba sumatrana, Miq. Plant. Jungh. vol. I. p. 204. Sumatra.

1855. Diospyros aurea, Teijsm. and Binn. Pl. n. h. Bogor. in Nederl. Kruidk. arch. III. p. 405. Java.

1855. Diospyros laxa, Teijsm. and Binn. Pl. nov. hort. Bogor. in Nederl. Kruidk. arch. III. p. 406. Java.

1855. Rhipidostigma Zollingeri, Hassk. Retzia, I. p. 104. Java.

1855. Rhipidostigma Teijsmanni, Hassk. Retzia, I. p. 106. Java.

1855. Getonia truncata, Goëppert, Tert. Fl. v. Schossnitz, p. 37. t. 25. f. 11. Silesia.

1856. Diospyros gaultheriæfolia, Mart. Fl. Brasil. Eben. p. 5. t. 2. f. 1. Brazil.

1856. Diospyros brasiliensis, Mart. Fl. Brasil. Eben. p. 5. t. 2. f. 2. Brazil.

1856. Diospyros coccolobæfolia, Mart. Fl. Brasil. Eben. p. 6. t. 1. f. 1. Brazil.

1856. Diospyros artanthæfolia, Mart. Fl. Brasil. Eben. p. 7. Brazil.

1856. Diospyros (?) myrmecocarpus, Mart. Fl. Brasil. Eben. p. 7. Brazil.

1856. Diospyros (?) xylopioides, Mart. Fl. Brasil. Eben. p. 8. Guiana, S. America.

1856. Macreightia obovata, Mart. Fl. Brasil. Eben. p. 9. t. 2. f. 3. Brazil.

1856. Diospyros timoriana, Miq. Fl. Ind. Bat. vol. II. p. 1045. Timor.

1857. Maba javanica, Zollinger, Obs. Bot. Nov. p. 14 in Natuurk. tydschr. Neerl. Ind. vol. XIV. Java.

1857. Diospyros Kuhlii, Zollinger, Obs. Bot. Nov. p. 15 in Natuurk. tydschr. Neerl. Ind. vol. XIV. Java.

1857. Diospyros penduliflora, Zoll. Obs. Bot. Nov. p. 15 in Natuurk. tydschr. Neerl. Ind. vol. XIV. Java.

1857. Diospyros Hasseltii, Zollinger, Obs. Bot. Nov. p. 15 in Natuurk. tydschr. Neerl. Ind. vol. XIV. Java.

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1857. Drebbelia subarborescens, Zoll. Obs. Bot. Nov. p. 16 in Natuurk. tydschr. Neerl. Ind. XIV. Java.

1857. Brachynema ramiflorum, Benth. in Trans. Linn. Soc. Lond. vol. XXII. (part ii.) p. 126. t. 22. N. Brazil.

1858. Diospyros incerta, Massalongo, Synops. Fl. Foss. Senigall. p. 76. n. 197. Europe.

1859. Diospyros anceps, Heer, Fl. Tert. Helv. III. p. 12. t. CII. ff. 15—18. Oeningen, &c., Europe.

1859. Macreightia germanica, Heer, Fl. Tert. Helv. vol. III. p. 13. t. CIII. ff. 1, 2. Oeningen, &c., Europe.

1859. Cassia phaseolites, Heer, Fl. Tert. Helv. vol. III. tab. 138. f. 2 (*solum*). Europe.

1859. Diospyros laurina, Massalongo, Syllab. Pl. Foss. Tert. Venet. p. 77. Italy, Europe.

1859. Diospyros Weberii, Massal. Syllab. Pl. Foss. Tert. Venet. p. 77. Italy.

1859. Macreightia italica, Massalongo, Syllab. Pl. Foss. Tert. Venet. p. 77. Italy, Europe.

1859. Macreightia (?) umbellata, Massal. Syllab. Pl. Foss. Tert. Venet. p. 77. Italy.

1860. Diospyros pyrrhocarpa, Miq. Fl. Ind. Bat. Suppl. I. p. 583. W. Sumatra.

1860. Diospyros Diepenhorstii, Miq. Fl. Ind. Bat. Suppl. I. p. 583. W. Sumatra.

1860. Diospyros Teysmanni, Miq. Fl. Ind. Bat. Suppl. I. p. 583. S. Sumatra.

1860. Diospyros (?) cystopus, Miq. Fl. Ind. Bat. Suppl. I. p. 584. S. Sumatra.

1860. Maba (?) lamponga, Miq. Fl. Ind. Bat. Suppl. I. p. 584. S. Sumatra.

1860. Diospyros crumenata, Thwaites, Enum. Ceylon Pl. p. 179. n. 5. Ceylon.

1860. Diospyros affinis, Thwaites, Enum. Ceylon Pl. p. 179. n. 6. Ceylon.

1860. Diospyros quæsita, Thwaites, Enum. Ceylon Pl. p. 179. n. 7. Ceylon.

1860. Diospyros oocarpa, Thwaites, Enum. Ceylon Pl. p. 180. n. 9. Ceylon.

1860. Diospyros insignis, Thwaites, Enum. Ceylon Pl. p. 180. n. 10. Ceylon.

1860. Diospyros oppositifolia, Thwaites, Enum. Ceylon Pl. p. 181. n. 11. Ceylon.

1860. Diospyros Gardneri, Thwaites, Enum. Ceylon Pl. p. 181. n. 12. Ceylon.

1860. Diospyros Moonii, Thwaites, Enum. Ceylon Pl. p. 182. n. 16. Ceylon.

1860. Diospyros acuta, Thwaites, Enum. Ceylon Pl. p. 182. n. 17. Ceylon.

1860. Diospyros attenuata, Thwaites, Enum. Ceylon Pl. p. 182. n. 18. Ceylon.

1860. Maba angustifolia, Miq. ex Thwaites, Enum. Ceylon Pl. p. 183. Ceylon.

1860. Macreightia oblongifolia, Thwaites, Enum. Ceylon Pl. p. 183. Ceylon.

1860. Diospyros vetusta, Giebel, Flora Braunkohl. in Zeitschrift, vol. XVI. p. 57. Prussia.

1861. Maba nigrescens, Dalzell in Dalz. and Gibs. Bomb. Fl. p. 142. Bombay.

1861. Macreightia intricata, A. Gray in Proceed. Amer. Acad. vol. V. p. 163. Lower California.

1861. Ebenacites rugosus, Saporta, Exam. Anal. Fl. Tert. Prov. p. 31. S.E. France.

1862. Diospyros samoënsis, A. Gray in Proceed. Amer. Acad. vol. V. p. 326. Navigators' Island.

1862. Maba foliosa, Rich. ex A. Gray in Proceed. Amer. Acad. vol. V. p. 326. Feejee Islands.

1862. Diospyros senensis, Klotzsch in Peters Mossamb. I. p. 183. Mozambique.

1862. Diospyros squarrosa, Klotzsch in Peters Mossamb. I. p. 184. Mozambique.

1862. Diospyros bicolor, Klotzsch in Peters Mossamb. I. p. 184. Mozambique.

1862. Diospyros Waldemarii, Klotzsch in Prinz. Waldem. Preuss. p. 101. t. 55. India.

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1862. Diospyros rugosa, Saporta in Ann. Sc. Nat. ser. IV. vol. XVII. p. 264. t. 11. f. 3. S. E. France.

1863. Maba natalensis, Harvey, Thes. Capens. vol. II. 7. Natal.

1864. Maba inconstans, Griseb. Fl. Brit. W. Ind. p. 404. Tropical America.

1864. Diospyros Arnottiana, Miq. ex Thwaites, Enum. Ceylon Pl. p. 423. E. Indies.

1864. Macreightia ovalifolia, Thwaites, Enum. Ceylon Pl. p. 424. n. 2. Ceylon.

1864. Macreightia acuminata, Thwaites, Enum. Ceylon Pl. p. 424. n. 3. Ceylon.

1864. Cargillia pentamera, Woolls and F. Muell. in F. Muell. Fragm. IV. p. 82. Australia.

1865. Diospyros varians, Saporta in Ann. Sc. Nat. ser. V. vol. III. p. 111. t. iv. f. 14, t. vi. f. 4. S. E. France.

1866. Diospyros halesioides, Griseb. Cat. Pl. Cubens. p. 168. Cuba.

1866. Macreightia buxifolia, Griseb. Cat. Pl. Cubens. p. 169. E. Cuba.

1866. Cargillia mabacea, F. Muell. Fragm. V. p. 162. Australia.

1866. Maba quadridentata, F. Muell. Fragm. V. p. 162. Australia.

1866. Maba Cargillia, F. Muell. Fragm. V. p. 162. Australia.

1866. Maba pentamera, F. Muell. Fragm. V. p. 163. Australia.

1866. Cargillia megalocarpa, F. Muell. Fragm. V. p. 163. Australia.

1866. Maba megalocarpa, F. Muell. Fragm. V. p. 163. Australia.

1866. Maba interstans, F. Muell. Fragm. V. p. 163. Australia.

1866. Maba fasciculosa, F. Muell. Fragm. V. p. 163. Australia.

1866. Maba cupulosa, F. Muell. Fragm. V. p. 164. Australia.

1866. Maba sericocarpa, F. Muell. Fragm. V. p. 164. Australia.

1866. Maba Hillebrandii, Seem. Fl. Vit. p. 151. Sandwich Islands.

1866. Maba Andersoni, [Solander] ex Seem. Fl. Vit. p. 152. Tonga Islands.

1866. Euclea miocenica, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 25. t. viii. f. 8. Croatia.

1866. Euclea Apollinis, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 26. t. viii. f. 10. Croatia.

1866. Rhododendron Apollinis, Ettingsh. ex Ung. Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 26. Croatia.

1866. Diospyros Zollikoferi, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 27. t. ix. f. 6. Styria.

1866. Diospyros obliqua, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 29. t. ix. f. 17. Croatia.

1866. Diospyros Royena, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 29. t. ix. ff. 18, 19. Croatia.

1866. Diospyros Parthenon, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 29. t. ix. f. 8. Europe.

1866. Diospyros Lignitum, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 30. t. ix. f. 9. Europe.

1866. Diospyros lotoides, Unger, Syll. Pl. Foss., pug. iii., in Denkschrift. XXV. p. 30. t. x. ff. 1—12. Europe.

1867. Diospyros assimilis, Bedd. Rep. Ind. For. Madr. p. 20. "t. i." S. Canara, India.

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1867. Diospyros mabacea, F. Muell. Austral. Veg. in Intercolonial Exhibition Essays, 1866—67. p. 35. East Australia.

1867. Diospyros megalocarpa, F. Muell. *loc. cit.* p. 35. North Australia.

1867. Diospyros fasciculosa, F. Muell. *loc. cit.* p. 35. East Australia.

1867. Diospyros cupulosa, F. Muell. *loc. cit.* p. 35. Queensland.

1867. Diospyros sericocarpa, F. Muell. *loc. cit.* p. 35. Queensland.

1867. Diospyros Cargillia, F. Muell. *loc. cit.* p. 35. East Australia.

1867. Diospyros pentamera, Woolls and F. Muell. ex F. Muell. *loc. cit.* p. 35. East Australia.

1867. Diospyros humilis, F. Muell. *loc. cit.* p. 35. Queensland and North Australia.

1867. Diospyros geminata, F. Muell. *loc. cit.* p. 35. Queensland.

1867. Euclea relicta, Unger, Foss. Fl. Eub. in Denkschrift. XXVII. p. 68. t. xi. f. 39. Negropont.

1867. Royena græca, Unger, Foss. Fl. Eub. in Denkschrift. XXVII. p. 68. t. xi. ff. 40—51. Negropont.

1867. Royena Amaltheæ, Unger, Foss. Fl. Eub. in Denkschrift. XXVII. p. 69. t. xiv. f. 1. Negropont.

1867. Royena Euboea, Unger, Foss. Fl. Eub. in Denkschrift. XXVII. p. 69. t. xiv. ff. 2—4. Negropont.

1867. Royena Myosotis, Unger, Foss. Fl. Eub. in Denkschrift. XXVII. p. 69. t. xiv. ff. 5—8. Negropont.

1867. Royena Pentelici, Unger, Foss. Fl. Eub. in Denkschrift. XXVII. p. 70. t. xiv. f. 9. Negropont.

1868. Diospyros Loveni, Heer, Fl. Foss. Arct. p. 118. t. vii. ff. 7, 8. t. xlvii. f. 8. N. Greenland.

1868. Diospyros oligandra, Bedd. Rep. Forests Madras, 1867—68, p. 25. Madras, India.

1869. Diospyros hebecarpa, A. Cunn. ex Benth. Fl. Austr. IV. p. 286. Australia.

1869. Maba hemicycloides, F. Muell. ex Benth. Fl. Austr. IV. p. 290. Australia.

1869. Maba laxiflora, Benth. Fl. Austr. IV. p. 290. Australia.

1869. Diospyros speciosa, Wood, Rep. Forests Oudh, 1867—68, p. 33. Oudh, India.

1870. Diospyros costata, Carriere in Rev. Hortic. p. 134. China.

1870. Macreightia andamanica, Kurz, Rep. Veg. Andam. edit. II. p. 42. S. Andaman.

1871. Diospyros microphylla, Bedd. Ic. Pl. Ind. Or. p. 27. t. cxxxiii. S. India.

1871. Diospyros canarica, Bedd. Ic. Pl. Ind. Or. p. 27. t. cxxxiv. S. Canara.

1871. Diospyros Thwaitesii, Bedd. Ic. Pl. Ind. Or. p. 27. t. cxxxv. Ceylon.

1871. Diospyros nilagirica, Bedd. Ic. Pl. Ind. Or. p. 27. t. cxxxvi. S. India.

1871. Diospyros rhodocalyx, Kurz in Journ. Asiat. Soc. Bengal. XL. Pt. ii. p. 71. Siam.

1871. Diospyros dasyphylla, Kurz in Journ. Asiat. Soc. Beng. vol. XL. Pt. ii. p. 71. E. Indies.

1871. Diospyros Brandisiana, Kurz in Journ. Asiat. Soc. Beng. XL. ii. p. 72. Burmah.

1871. Diospyros burmanica, Kurz in Journ. Asiat. Soc. Beng. XL. ii. p. 73. Pegu.

1871. Diospyros variegata, Kurz in Journ. Asiat. Soc. Beng. XL. ii. p. 73. Pegu.

DESCRIPTION OF THE GENERA AND SPECIES, EXCLUSIVE OF FOSSILS.

I. ROYENA, Linn. Gen. Plant. p. 114. n. 325 (1737).

Flores sæpius hermaphroditi et pentameri.

Calyx plerumque accrescens, campanulatus vel urceolatus vel raro depresso-hemisphericus.

Corolla urceolata vel campanulata ; lobis in præfloratione sinistrorse contortis.

Stamina numero loborum corollæ dupla raro plura, in verticillum unicum disposita.

Ovarium hirsutum, 4-10-loculare ; ovula in loculis solitaria.

Frutices rarius arbores africani ; foliis alternis, plerumque coriaceis ; pedunculis axillaribus, sæpius unifloris.

Alph. DC. Prodr. VIII. p. 210 (1844); J. G. Agardh, Theor. Syst. Pl. tab. x. f. 13 (1858); Harv. MSS.; non Houston in Linn. Sp. Pl. p. 628 (1753) (= Loeselia).

Pistachia (sp.) Pluknet. Almag. p. 298. t. 63. f. 4. t. 317. f. 5 (1691, 1696).

Vitis Idæa (sp.) Plukn. Almag. p. 391. Phytogr. t. 321. f. 4 (1696).

Staphylodendrum, Commelin. Hort. Amstelod. I. p. 187. t. 96 (1697).

Staphylodendron, Hermann, Paradisus Batavus, p. 232 cum tab. (1698).

Arbutus (sp.) Linn. Hort. Cliff. p. 163 (1737).

Buxus (sp.) Linn. in Herb. Gronov.

Vaccinium (sp.) Mill. Gard. Dict. edit. VI. (1771).

Royenia, auct., non Houst.

Flowers usually hermaphrodite and pentamerous, in one species tetramerous, and in *R. ambigua* 5-7-merous.

Calyx 5-4-partite 5-fid or 5-toothed at apex, pubescent, usually accrescent in fruit.

Corolla usually 5-fid, urceolate or campanulate, with obtuse reflexed lobes.

Stamens 10, rarely 12—14, in one species 8; inserted in one row at the base of the corolla, usually 2 opposite each of its lobes; filaments very short, glabrous; anthers lanceolate-linear, hairy or in *R. sessilifolia* glabrous, dehiscing longitudinally by lateral slits, rarely in subhermaphrodite flowers barren.

Ovary pubescent, 4-10-celled; cells 1-ovuled; rarely abortive in male flowers.

Styles 2-5 or style 2-5-partite or -lobed.

Fruit coriaceous, globose ovoid or oblong, sometimes 5-sided and splitting by valves.

Seeds as in the family; albumen not ruminated.

Shrubs or small trees or even large trees (see Burchell, Trav. I. 390) mostly limited to South Africa, but two species (*R. pallens* and *R. cistoides*) reaching the tropics.

Leaves alternate, simple, entire, shortly petiolate or subsessile or in one species quite sessile, according to Dr Harvey evergreen. Bracts 1—5.

Flowers axillary, peduncled, solitary or in *R. glabra* 1-5 together or in *R. parviflora* in 3-5-flowered cymes.

Named after *Adrian* van *Royen,* Professor of Botany in the University of Leiden, who died in 1779 at the age of 74.

English name; African bladder-nuts.

Alph. De Candolle describes 10 small glands at the base around the ovary; I do not, however, notice any such in any of the species of the genus.

ROYENA.

Key to the Species.

Flowers 5—7-, usually 5-, merous. Fruit not glandular or rarely so.

- Calyx 5-lobed only at the apex. — 1. *R. lucida.*
- Calyx divided half way down or deeper.
  - Leaves cordate or sub-cordate or rarely rounded at base.
    - Style 2-lobed. Leaves subsessile.
      - Leaves smooth. Flowers hermaphrodite. — 2. *R. cordata.*
      - Leaves scabrous. ♂ Flowers with rudimentary ovary. — 3. *R. scabrida.*
    - Style 4—5-lobed. Leaves distinctly petiolate. — 4. *R. villosa.*
  - Leaves narrowed at base, not cordate.
    - Peduncles short, not or scarcely longer than the flowers.
      - Leaves subsessile. Anthers 10, hirsute. — 5. *R. hirsuta.*
      - Leaves sessile. Anthers 14, glabrous. — 6. *R. sessilifolia.*
    - Peduncles nearly as long as the leaves or much longer than the flowers.
      - Flowers solitary.
        - Calyx patent or reflexed in fruit.
          - Leaves more than $\frac{1}{2}$ in. long.
            - Flowers hermaphrodite, 5-, rarely 4-, merous. — 7. *R. pallens.*
            - Flowers polygamous, 5—7-merous. — 8. *R. ambigua.*
          - Leaves under $\frac{1}{2}$ in. long. — 9. *R. nitens.*
        - Calyx appressed to fruit. — 10. *R. cistoides.*
      - Flowers in 1—5-flowered cymes.
        - Leaves narrowly elliptical, $\frac{1}{2}$- 1 in. long. — 11. *R. glabra.*
        - Leaves obovate, 2—6$\frac{1}{2}$ in. long. — 12. *R. parviflora.*

Flowers tetramerous or rarely pentamerous. Fruit glandular. — 13. *R. glandulosa.*

1. ROYENA LUCIDA, Linn. Sp. Pl. p. 397. (1753).

R. foliis ellipticis vel ovatis, basi rotundatis vel cordatis, coriaceis, nitidis, breviter petiolatis; floribus hermaphroditis, pentameris; pedunculis unifloris; calyce campanulato, ampliato, utrinque hirsuto, apice breviter dentato, in fructu accrescente; stylo bifido; ovario 4-loculari, 4-ovulato.

Gærtn. Fruct. et Sem. pl. ii. p. 80. t. 94. f. 4 (1791).
Jacq. Fragm. Bot. p. 3. t. i. f. 6 (1800—1809).
Lam. Tabl. Encycl. ii. p. 492. t. 370. f. 1. (*not good*) (Anno. VIII. 1800 ?).
Desf. in Annal. Mus. VI. p. 445. t. 62. f. 3 (An. XIII.—1805).
Alph. DC. Prodr. VIII. p. 211. n. 1 (1844).
Lindl. in. Bot. Reg. XXXII. t. 40 (1846).
Pappe, Silva Capensis, p. 20 (1854).

Pistachia *africana* s. Staphylodendron *Æthiopicum* Μονολασιοκαλλῃνομενοφυλλον singulari hirsuto folio nitente, Pluknet! Almag. p. 298, Phytogr. Tab. 63, f. 4, tab. 317. f. 5 (1696, 1691).

STAPHYLODENDRUM *africanum* semper virens, foliis splendentibus, Commelin. Hort. Amstelod. i. p. 187. t. 96 (*bad*). (1697.)

STAPHYLODENDRON *africanum folio singulari lucido*, Herm. Parad. Batav. p. 232 cum tab. (1698), Linn. Hb. Hort. Cliff.!

Royena foliis ovatis scabriusculis, Linn. Syst. Veg. p. 410 (1784).

An evergreen shrub 5–12 feet high with numerous branches:

Stem 6–12 inches thick. Bark black, rather smooth. Wood hard, tough, yellowish with brownish stripes when polished, well adapted for furniture, tools, screws, &c., but chiefly used for wagon work (*Dr Pappe*). Young parts covered with subferruginous pubescence.

Leaves elliptical or somewhat ovate, usually pointed or apiculate at apex, obtuse or subacute, rounded or cordate very rarely somewhat narrowed at base, shortly petiolate, coriaceous, $\frac{1}{2}-2\frac{1}{4}$ inches long by $\frac{1}{3}-1\frac{1}{4}$ in. wide, glabrescent and shining above, hirsute especially on the midrib and margin or glabrate beneath; midrib in slight relief on both sides; lateral veins not conspicuous; petioles $\frac{1}{20}-\frac{1}{8}$ in. long, pubescent.

Bracts small or foliaceous, sericeous, caducous.

Peduncles axillary, solitary, 1-flowered, pubescent, patent or arching, $\frac{1}{4}-1$ in. long, on young branches, bearing $1-3$ bracts.

Flowers $\frac{1}{4}-\frac{1}{3}$ in. long, hermaphrodite.

Calyx urceolate, sericeous on both sides, $\frac{1}{5}-\frac{1}{4}$ in. long, 5-toothed at apex, much accrescent in fruit; teeth short, acute.

Corolla urceolate, 5-fid, with rounded lobes puberulous on both sides.

Stamens (9–) 10, inserted at the base 2 opposite or corresponding to each lobe of the corolla, $\frac{1}{6}$ in. long, equal; filaments very short, glabrous; anthers lanceolate-linear, hispid on upper half, glabrous below.

Ovary conical, pubescent, 4-celled, 4-ovuled; surmounted by bifid style, glabrous above; stigmas punctiform.

Fruit ovoid or subglobose, $\frac{1}{2}$—1 in. in diameter, enclosed by inflated pubescent or subglabrate calyx, 2—4-celled and -seeded, red and fleshy when ripe; flesh firm, whitish. Seeds glabrous, rather shining, yellowish; testa thin; albumen cartilaginous, hard, white; embryo half to two-thirds of the length of the albumen, somewhat curved inwards; cotyledons ovate, rather shorter than the radicle.

In Cape Colony known by the name of *Zwartbaste* (blackwood). See Burchell, Travels, vol. I. p. 317 (1822).

Grows in forests, stony places, on the sides of mountains, &c. South Africa. From Cape Town eastwards to Natal. *Reeves!*; *Ecklon!* 698 ("*R. hirsuta*," on the eastern side of Devil's Mountain); *Drège* A. (above the waterfall at Duivelsberg 1000—2000 ft. alt. May), B. (Boschrivier, in a wood, below 500 ft. alt. October), C! (Katrivier, in a wood on a hill, 1000—2000 ft. alt. November); *T. Cooper!* 1062 (Orange Free State); *Miller!*; *Bowie!*; *Dr Pappe!* (slope of Devil's Mountain); *Burchell!* 5256 (Hartebeest-Vlakte and Kaatje's Kraal), 5415 (in a forest close to Melkhout Kraal); *Roxburgh!*; *Harvey!*; *Alexander!*; *Mac Owan!* 309 Eastern districts; *Zeyher!* 3352.

Natal. *Dr Sutherland!* (a low scrubby bush growing among stones).

It is cultivated in St Helena, *Gen. Walker!* (stamens 5, abortive); and has long been introduced into Europe.

2. ROYENA CORDATA, E. Mey. Catal. Pl. Exs. Afr. Austr. Drèg. p. 7 (1837).

R. foliis ellipticis vel oblongis, basi cordatis, nitidis, coriaceis, apice obtusis vel subacutis, subsessilibus; floribus pentameris, hermaphroditis; pedunculis unifloris; calyce 5-partito, accrescente; stylo bilobo; ovario 4-loculari.

Alph. DC. Prodr. VIII. 211. n. 2 (1844).

R. opaca, E. Mey. Pflanz. Doc. Drèg. p. 217 in Flora XXVI. ii. (1843), Alph. DC. Prodr. VIII. p. 211. n. 3 (1844).

R. supra-cordata, Burch. MSS. in Hb. Afr. Austr. n. 4907 (1814).

A shrub with numerous branches, and a brown-ferruginous pubescence on young parts, quickly glabrescent and nitescent.

Leaves elliptical or oblong, cordate at base, usually obtuse-pointed mucronate or apiculate at apex, coriaceous, subsessile, often pubescent underneath, $\frac{1}{2}$—2 in. long by $\frac{2}{5}$—1 in. wide.

Peduncles $\frac{1}{6}$—$\frac{3}{4}$ in. long, arching, bearing 2 alternate caducous ovate acute ciliate bracts similar to the leaves in shape $\frac{1}{6}$ in. long. Flowers about double the length of the calyx, $\frac{1}{3}$ in. long.

Calyx 5-partite, villous on both sides, $\frac{1}{6}$ in. long; nearly glabrate in fruit; lobes ovate-lanceolate, acute, hirsute and ciliate; calyx much accrescent in fruit with wide ovate cordate or auricled lobes, often nearly an inch long.

Corolla 5-lobed, with a short cylindrical tube and reflexed rounded lobes; lobes oblong, $\frac{3}{4}$ length of corolla, puberulous on both sides.

Stamens 10, inserted at base of corolla, reaching to the mouth of the corolla, pilose.

Style 2-lobed; ovary 4-celled, pilose, cells 1-ovuled.

Fruit subglobose, half an inch or more in diameter.

Flowers in November and December. Grows by river-sides and among mountains. It reaches 4300 feet altitude.

South Africa. Eastern district of the Cape of Good Hope, and Natal.

Drège!; Zeyher! Uitenhage; *Mac Owan!* 429, 527, Mountains near Great Reynet; *Mrs Hutton!* Keiskamma, British Kaffraria; *T. Cooper!* 35, 186, 306, British Kaffraria; *Gueinzius!* Natal; *Gerrard and M'Ken!* 12, 18, 99, 1608, Natal; *Barber!* 307, Queenstown district, a shrub, grows amongst other bushes, blossoms in spring and summer, flower pale cream colour; *Burchell!* 4166, 4186, 4907.

3. ROYENA SCABRIDA, Harv. MSS.

R. foliis ovatis, basi cordatis, præsertim subtus scabris, coriaceis, subsessilibus; floribus pentameris, diœcis; pedunculis unifloris; calyce 5-partito; stylo in floribus masculis bifido, ovario abortivo.

A shrub with "branches simple, 8—15 feet high," pilose at the extremities with pale hairs.

Leaves ovate, cordate at base, acute or obtuse at apex, scabrous especially beneath, subsessile, shining above, sericeous when young, ranging up to $2\frac{1}{3}$ in. long by $1\frac{1}{3}$ in. wide; margins subrevolute.

♂ Flowers nearly $\frac{1}{2}$ in. long, white. Peduncles axillary, bracteate, much shorter than the leaves, 1-flowered. Bracts ovate, acuminate.

Calyx $\frac{2}{5}$ in. long, 5-partite, finely setose, erect; lobes ovate, acuminate, widened near base.

Corolla appressedly pilose, campanulate-urceolate, divided $\frac{3}{4}$ths way down into 5 ovate-oblong acute lobes.

Stamens 10, in one row, inserted at base of corolla, $\frac{1}{6}$ in. long; filaments very short, hairy at apex; anthers hairy at and near apex, linear, acute.

Ovary rudimentary, hairy; style bifid, hairy below, glabrous above.

Tugela, Natal. *Gerrard and M'Ken!* n. 1609. Grassy plains. Flowers in September and October.

Near *R. cordata,* E. Mey.

4. ROYENA VILLOSA, Linn. Systema Naturæ, ed. XII. tom. 2. p. 302 (1767).

R. foliis obovato-oblongis, basi cordatis, apice obtusis, villosis, petiolatis; floribus pentameris, hermaphroditis; pedunculis 1—3-floris; calyce 5-partito; stylo 4—5-lobo; ovario 8—10-loculari.

Alph. DC. Prodr. VIII. p. 213. n. 11 (1844).

R. scabra, Burm. Prodr. p. 13 (1768).

R. scandens, Burch. MSS. in Hb. Afr. Austr. nn. 3673, 3793 (1813).

Pubescent trailing shrub with patent branches, 5 to 40 feet long.

Leaves obovate-oblong with cordate base and rounded emarginate or shortly-pointed apex; pubescent especially beneath, glabrescent and dark green above, paler beneath, sometimes minutely pellucid-punctate, coriaceous, petiolate; edges recurved; veins distinct, depressed above; 1 to $4\frac{1}{2}$ in. long by $\frac{1}{2}$ to $2\frac{1}{2}$ in. wide. Petioles $\frac{1}{8}$—$\frac{3}{4}$ in. long, pubescent.

Peduncles axillary, either 1-flowered about $\frac{1}{5}$—$\frac{1}{2}$ in. long or 3-flowered longer and with pedicels about $\frac{1}{10}$ in. long, pubescent. Bracts leaf-like, but smaller and narrower than the leaves, caducous. Flowers densely pubescent.

Calyx with 5 ovate or lanceolate partitions, ovate and accrescent in fruit, closely pubescent on both sides.

Corolla with 5 oblong lobes reaching $\frac{2}{3}$rds down, tomentose outside except near base, glabrous inside.

Stamens 10, anthers densely villous.

Style 5-lobed; ovary 8- or 10-celled; stigmas punctiform.

Fruit globose-pentagonal, tomentose or hispid, $\frac{1}{2}$—$\frac{3}{4}$ in. long, surrounded by the widely ovate enlarged lobes of the calyx which reach nearly as high, sometimes dehiscing by 5 valves from apex.

Grows in woods. South Africa. Eastern districts and Natal.

Drège!; T. Cooper! 1, British Kaffraria, (in flower and fruit.) "Stem 30—40 ft., trailing or twining among trees. Flowers yellow;" *J. Sanderson!* 150, 613, 715, Natal (in flower); *W. T. Gerrard!* 30, Natal (in flower); *Krauss!* 226, 472, 482, Natal (in flower-bud); *Gueinzius!* Natal (in flower); *Dr Stuart!* (in flower); *P. Mac Owan!* 516, Grahamstown (in flower); *Bowie!* (in flower-bud); *Burchell!* 3673 (in flower), 3793 (in leaf), 4506 (in flower), 6054? (in leaf); *Masson! Ecklon and Zeyher!* 464, Uitenhage; *Gerrard and M'Ken,* 613, 614, 2013, Natal; *Alexander!*

5. ROYENA HIRSUTA, Linn. Sp. Pl. p. 397 (1753).

R. foliis oblanceolatis, basi cuneatis, subsessilibus, hirsutis; floribus pentameris, hermaphroditis; pedunculis brevibus, unifloris; calyce profunde 5-lobo; corollâ urceolatâ; staminibus 10; *stylo plerumque bilobo et ovario 4-loculari.*

Lam. Tabl. Encycl. II. p. 493. t. 370. f. 2 (anno VIII.—1800).

Alph. DC. Prodr. VIII. p. 212. n. 8 (1844), non Jacq. nec Eckl. nec Sieb.

R. angustifolia, Willd. Spec. Plant. II. p. 633 (1799), Alph. DC. *l. c.* n. 5.

? R. cuneata, Poir. in Encyclop. Méth. VI. p. 322 (1804), Alph. DC. *l. c.* p. 215. n. 18, non Spreng.

R. microphylla, Burch. Trav. Int. S. Afr. I. p. 348 (1822), Alph. DC. *l. c.* p. 212. n. 9.

R. rugosa, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Drèg. p. 7 (1837), Alph. *l. c.* n. 7.

Diospyros hirsuta, Desf. Ann. Mus. VI. p. 447. t. 62. f. 2 (1805), non L.

D. pubescens, Pers. Synops. II. p. 625 (1807), non Pursh.

Arbutus foliis lanceolatis integerrimis hirsutis, Linn. Hort. Cliff. p. 163 (1737).

A much-branched rigid shrub, 2—8 feet high, more or less downy-canescent or tomentose.

Leaves oblanceolate, obtuse or acute at apex, cuneate at base, subsessile, crowded, coriaceous, hairy and rugose with raised veins or pitted beneath, $\frac{1}{4}$—1 in. long by $\frac{1}{10}$—$\frac{1}{3}$ in wide; margins flattish or recurved.

Peduncles 1-flowered, arching, shorter than the flowers, $\frac{1}{10}$—$\frac{1}{5}$ in. long, usually bibracteate in the middle.

Calyx deeply 5-lobed, hairy on both sides; lobes ovate, accrescent, erect or reflexed in fruit. Corolla urceolate, 5-fid, grey-felted outside, puberulous inside; lobes rounded or obtuse.

Stamens usually 10; anthers hairy; filaments dilated.

Styles 2—4, usually 2; stigmas more or less dilated, glabrous. Ovary villous, 4-, 6-, 8-celled.

Fruit globose, about $\frac{1}{2}$ in. in diameter, more or less tomentose, often dehiscing by 2 or 3 valves. Fruiting calyx-lobes rounded, erect or reflexed.

Grows among mountains and rocks and along banks of rivers and reaches 5000 feet in Natal. Flowers in August. Cape of Good Hope, Kalihari region and Natal.

Dr Sutherland! Natal (in flower); *Dr Zeyher!* 3350, 3351, Uitenhage and Clanwilliam (in flower); *Burchell!* 7531 (in leaf), 7537 (in fruit), 7446 (in fruit), 1696 (in flower), 2502 (in fruit), 4898 (in fruit); *Drège!; P. Mac Owan!* 269, Humansdorp (in flower); *T. Cooper!* 212, Queenstown (in flower); *Verreaux!; Krauss*, 1719; S. Africa, *Masson Auge and Oldenburg!; Barber!* 311, Queenstown district, on stony hill-sides, flowers white, blossoms in spring.

6. ROYENA SESSILIFOLIA, sp. nov.

R. foliis oblongo-obovatis, membranaceis, sessilibus, obtusis, basi angustatis; floribus pentameris, diœcis; pedunculis unifloris, brevibus; calyce 5-partito; corollâ urceolatâ; staminibus in flore masculo 14, *glabris; ovario abortivo.*

A shrub with erect stem; branches pubescent, spreading at about 70°.

Leaves oblong-obovate, sessile, submembranous, pubescent beneath and on both sides when young, rounded or retuse at apex, narrowed to an obtuse base, $1\frac{1}{2}$—2 in. long by $\frac{1}{2}$—$\frac{3}{4}$ in. wide; veins inconspicuous, depressed on the upper surface.

Peduncles axillary, solitary, bearing one flower longer than itself, pubescent. Flowers fragrant, $\frac{1}{4}$ in. long.

Calyx pubescent outside, 5-partite, with lanceolate erect-patent 3-veined lobes $\frac{1}{6}$ in. long.

Corolla urceolate, 5-lobed at apex, glabrous inside, pubescent outside; lobes recurved, obtuse, $\frac{1}{10}$ in. long.

Stamens 14, glabrous; anthers dehiscing from apex; filaments short; pollen globular, smooth, $\frac{13}{10000}$ in. in diameter.

Ovary rudimentary, rounded, pubescent.

Pubescence whitish. Described from a living specimen cultivated in Hort. Kew! Approaches *R. ambigua*, Vent. by having more than 10 stamens, but differs from it by its shorter peduncles; differs also from all other species of *Royena* by its absolutely sessile leaves.

A specimen in the Leiden herbarium with sessile leaves, which however are coriaceous and usually pointed at the apex and have the veins in relief on both sides, may be the same species; it was cultivated in 1785.

Cfr. *R. latifolia*, Willd. Enum. Pl. Hort. Berol. Suppl. p. 23 (1813, sine descriptione), Alph. DC. Prodr. VIII. p. 215.

7. Royena pallens, Thunb. Prodr. Fl. Capens., pars prior, p. 80 (1794).

R. foliis anguste obovato-ellipticis, apice plerumque obtusis, basi in petiolum brevem angustatis, coriaceis; floribus hermaphroditis, plerumque pentameris; pedunculis plerumque unifloris flore longioribus; calyce 5-partito; stylo 3—5-fido; ovario 6—10-loculari.

Alph. DC. Prodr. viii. p. 213. n. 13 (1844); non Willd. Hb.! n. 8363.

R. hirsuta, Jacq. Collect. v. p. 110. t. 13. f. 1 (1796), et Fragm. bot. t. 1. f. 2 (1800—1809), non Linn. nec Eckl. nec Sieb.

Diospyros lycioides, Desf. in Annal. Mus. vi. p. 448. t. 62. f. 1 (An. xiii—1805).

R. pubescens, Willd. Hort. Berol. p. 457 (1809), Bot. Reg. t. 500 (1820), Alph. DC. *l. c.* n. 12.

R. lycioides, (Desf.) Cat. Hort. Paris ex Poir. in Encycl. Méth. Suppl. iv. p. 435 (1816), Alph. DC. *l. c.* p. 214. n. 17.

R. decidua, Burch. Trav. Int. S. Afr. i. p. 317 (1822).

R. cuneata, Speng. Syst. Vegetab. ii. p. 360 (1825), non Poir.

R. brachiata, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Dr g. p. 7 (1837), Alph. DC. *l. c.* p. 213. n. 10.

R. cuneifolia, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Drège. p. 7 (1837), Alph. DC. *l. c.* p. 214. n. 16.

R. ramulosa, E. Mey. ex Alph. DC. Prodr. viii. p. 212. n. 6 (1844).

R. sericea, Bernh. in Flora 1844, p. 824.

R. oleifolia, Desf. MSS. (1824) ex Gay MSS. in Herb.

R. hispidula, Harv. MSS.

A shrub or small tree, ranging up to 15 feet in height. Bark reddish brown. Branches silky-pubescent or often glabrescent, pale or cinereous.

Leaves more or less narrowly obovate-elliptical, obtuse or rarely acute at apex, narrowed at base into short petiole, silky especially beneath or glabrate, coriaceous, evergreen, $\frac{1}{2}$—2 in. long by $\frac{1}{6}$—$\frac{3}{4}$ in. wide; petioles $\frac{1}{20}$—$\frac{1}{5}$ in. long.

Peduncles $\frac{2}{5}$—$\frac{4}{5}$ in. long, longer than the flowers, usually considerably so, 1- (rarely 2-) flowered, arching, bearing 2—3 narrow bracts about or above the middle. Flowers white or yellowish, hermaphrodite, pentamerous (or rarely tetramerous), $\frac{1}{4}$—$\frac{2}{5}$ in. long.

Calyx partite; lobes ovate or lanceolate, acute, hirsute, accrescent, in fruit spreading or reflexed.

Corolla deeply lobed, hairy outside; lobes lanceolate, acuminate.

Stamens (9—) 10, about half the length of the flower; anthers hirsute.

Style 3—5-fid, hairy; stigmas punctiform, glabrous. Ovary 6-, 8-, 10-celled, hairy.

Fruit subglobose or ovoid, pubescent or rarely glabrate, $\frac{1}{2}$—1 in. in diameter, sometimes bursting in a valvate manner. Albumen of seeds not ruminated.

Reaches 5000 feet in Natal, 2500 at Graaf Reinet. Flowers in Sept., Oct. and Nov., Jan. Feb.

Grows at margins of woods, &c.

In South Africa on the banks of the Gariep it is called by the natives *Zwartebast* (black-wood). Cape of Good Hope, Kalihari region and Natal; also in West tropical Africa.

Drège!; Peddie! Natal; *Col. Bolton!* Grahamstown; *T. Williamson!* Albany; *Alexander!; Ecklon and Zeyher!* 1127; *Burchell!* 745—1, 1750, 2371, 2930, 3301, 3325, 3396, 3472, 3789, 4184, 4501, 5490, 5529, 5632, 6490, 6813; *Dr Zeyher!* 3348, 3353, 3354; *Burke!* Great

Fish River, and Crocodile River; *Harvey!* 544; Mac Owan, 1646; *W. T. Gerrard!* 129, 615, 1157, 1238, 1607, 1610, 1611, Natal; *T. Cooper!* 272, Queenstown; 418, Beaufort; 1238, 1157, Natal; *Hutton!* Howison's Port, Eastern Districts; *J. Sanderson!* 140, 318, 511, 527, 717, Natal; *Dr Sutherland!* Natal 3000—5000 feet alt.; *Bowker!* (103?) Albany; *Wyley!* 103, Namaqualand; *Bolus!* 128, Graaf Reinet (flowers in Oct., fruits in Nov.); *Krauss!* 423, Natal, 1721, Knysna; Cape. *Niven!* 51, large shrub 6 or 8 feet high, dry elevated plains near Goud river. Tropical Africa, *Dr Kirk!* Seshike (alt. 3000 feet); *C. J. Meller!* Manganja Hills (tree: always found by streams).

A form with leaves acute at both ends and turning black in drying and with globose fruits thinly sprinkled with rigid hairs is *R. hispidula*, Harv. MSS. Burchell! Cat. no. 3789 at the Kowi Station, 26 Sept. 1813; and no. 4501 at the Lead mine, 29 January, 1814.

Benguela. Distr. Huilla. *Dr Welwitsch!* no. 2533. A shrub 4—6 ft. high, rarely a small tree of 8 ft. Leaves broad. Flowers white, rather fleshy. Fruit puberulous. Fruiting calyx reflexed, not much increased. In woods and thickets between Lopollo and Monino.

Do. *Dr Welwitsch!* no. 2534. Leaves narrower.

Do. *Dr Welwitsch!* no. 1255. A small shrub, a few inches high, much branched. Leaves densely sericeous, with some species of *Æcidium* growing on them. A sickly specimen probably belonging to *R. pallens*.

8. ROYENA AMBIGUA, Vent. Jard. Malm. n. 17 (1803).

R. foliis obovato-ellipticis, obtusis, basi angustatis, coriaceis, breviter petiolatis; floribus 5—7-meris, diœcis; pedunculis unifloris, flore longioribus; calyce partito; corollâ urceolatâ; staminibus 10—14, *sterilibus* (?); *stylo* 5—7- (?) *lobo.*

Alph. DC. Prodr. VIII. p. 214. n. 14 (1844).

Diospyros ambigua, Vent. Malm. t. 17 (1803).

R. polyandra β. ambigua, Pers. Synops. I. p. 486 (1805).

Diplonema ambigua, G. Don, Gen. Syst. Gard. and Bot. IV. p. 42 (1837).

Shrub with numerous erect-patent or ascending branches tomentose-pubescent (at least in wild specimen) throughout, about 3 feet high when in cultivation.

Leaves obovate-elliptical, somewhat narrowed at base and rounded or apiculate at apex, dull green, sometimes minutely pellucid-punctate, coriaceous, shortly petiolate, 1 to 2 in. long by $\frac{1}{2}$ to $\frac{3}{4}$ in. wide. Petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long.

Peduncles (♀) arching downwards, 1-flowered, bearing 2 (or 3) alternate linear bracts about their middle, three times the length of the petiole in flower, $\frac{1}{2}$ to $\frac{3}{4}$ in. long in fruit. Flowers not hermaphrodite (?), drooping, orange-yellow, slightly scented.

Calyx with 5 (or 6?) lanceolate acute partitions.

Corolla urceolate, 5—7?-lobed: lobes rounded, shorter than the tube.

Stamens in ♀ flower 10—14? shorter than the tube of the corolla, barren.

Style 5—7?-lobed. Ovary with 5—7? external longitudinal furrows, 10-celled.

Fruit globular, bright pale brown, pubescent, nearly $\frac{1}{2}$ in. in diameter, sometimes dehiscing by 5 valves, in one case 3-seeded. Fruit-calyx accrescent, reflexed, with 5 oblong-lanceolate partitions $\frac{1}{2}$ in. long. Seeds oblong, $\frac{1}{4}$ in. long, pendulous.

South Africa. *Burke!* (in fruit); *Ventenat; Ecklon and Zeyher!* 1126, Magalisberg.

Perhaps ought to be united to *R. pallens*, Thunb., of which Dr Harvey considered it to be a garden variety.

9. ROYENA NITENS, Harvey MSS.

R. foliis anguste ellipticis, utrinque plus minus angustatis, coriaceis, subsessilibus, dense sericeis, parvis; pedunculis unifloris, semiuncialibus; calyce fructifero profunde 5-lobo, paulum aucto; fructu ellipsoideo, solitario.

A closely branched shrub about 4 feet high with young shoots and underside of leaves densely covered for the most part with close sericeous persistent pale hairs. Branches terete, ascending, with dark rather shining cuticle.

Leaves narrowly oval, crowded, narrowed more or less at both ends, coriaceous, dark and shining above, 1-veined, subsessile, $\frac{1}{3}$—$\frac{2}{5}$ in. long by $\frac{1}{10}$—$\frac{1}{8}$ in. wide.

Flowers unknown.

Fruit on the young branches, solitary, on arching pubescent peduncles nearly $\frac{1}{2}$ in. long. Fruiting calyx deeply 5-lobed, spreading, rather more than $\frac{1}{2}$ in. across, with lanceolate lobes which are about $\frac{1}{5}$ in. long. Fruit ellipsoidal, puberulous with very short inconspicuous hairs, splitting into 5 (?) parts at the apex, $\frac{1}{2}$—1 in. long by $\frac{1}{4}$—$\frac{2}{5}$ in. thick, 1-celled.

S. Africa. Natal. *W. T. Gerrard!* n. 1158, February.

10. ROYENA CISTOIDES, Welw. MSS.

R. foliis anguste obovatis, apice obtusis et mucronulatis, ad basim obtusam angustatis, utrinque incano-sericeis, breviter petiolatis, margine reflexo; fructibus solitariis; pedunculis fructum fere æquantibus; calyce fructibus appresso.

A low shrub, 1—1$\frac{1}{2}$ ft. high, branched from the base. Wood very hard, strong. Branches terete, ultimately glabrate; shoots softly pubescent, erect; the fruiting branches arcuate-ascending.

Leaves alternate, narrowly obovate, obtuse and mucronulate at apex, narrowed to an obtuse base, incano-sericeous on both sides, sub-coriaceous, $\frac{1}{2}$—1$\frac{1}{4}$ in. long by $\frac{1}{4}$—$\frac{3}{5}$ in. wide, shortly petiolate; margins reflexed; subvenose beneath.

Fruiting peduncles axillary, solitary, $\frac{1}{3}$—$\frac{5}{8}$ in. long, patent, hairy, 1-fruited. Fruiting calyx deeply 5-lobed, hairy on both sides, appressed to the fruit, $\frac{1}{4}$—$\frac{3}{8}$ in. long, articulated to the peduncle, with 10 little pits at the base on the concave surface of the articulation probably corresponding to the 10 cells of the ovary; lobes elliptical, obtusely pointed.

Fruit subglobose, puberulous, of shining golden colour, hard, 8—12-celled, $\frac{1}{2}$—$\frac{2}{3}$ in. in diameter, often bursting downwards from the apex, 3—5-seeded. Seeds $\frac{1}{4}$ in. long. Albumen of seeds white, cartilaginous, not ruminated.

Angola, W. Tropical Africa. Distr. Pungo Andongo, 3500 ft. altitude. *Dr Welwitsch!* no. 2532. In sandy thickets between Condo and Quisonde, near river Cuanza. Fruit ripe in March.

11. ROYENA GLABRA, Linn. Sp. Pl. p. 397 (1753).

R. foliis anguste ellipticis, utrinque angustatis, nitescentibus, subcoriaceis, subsessilibus, glabrescentibus; floribus pentameris, subhermaphroditis; pedunculis 1—5-floris; calyce partito, paulum accrescente; stylo bilobo; ovario 4-loculari.

Alph. DC. Prodr. VIII. p. 214. n. 15 (1844).

Vaccinium pensylvanicum, Miller, Gard. Dict. edit. vi. (1771).

R. myrtifolia, Wendl. ex Steud. Nomencl. Bot. p. 705 (1821), Alph. DC. *l. c.* p. 215.

R. hirsuta, Sieber! Fl. Cap. Exsicc. n. 94 (1824), non Linn. nec Jacq. nec Eckl.

R. falcata, E. Mey.! Zwei pflanz. doc. Drèg. p. 217 in Flora 1843, Alph. DC. *l. c.* p. 211. n. 4.

Vitis *Idæa æthiopica*, myrtinis folio, flosculis dependentibus, Plukn.! Almag. p. 391. Phytogr. t. 321. fig. 4 (1696).

Vitis Idæa æthiopica, buxi minoris folio, floribus albis, Commel. Hort. Amstelod. I. p. 125. t. 65 (1697).

Vitis Idæa foliis angustissimis longis alternis, Linn. in Hb. Hort. Cliff.!

? Buxus africana folio oblongiori non serrato, Linn. in Hb. Gronov.!

A shrub with erect or ascending branches, 2—6 feet high. Stem 5—6 in. thick. Bark thin, grey, smooth. Wood light, porous, little used except for fuel (*Dr Pappe*). Young parts pilose.

Leaves narrowly elliptical, usually narrowed at both ends, crowded, subsessile, at length glabrous, shining above, thinly coriaceous, $\frac{1}{2}$—1 in. long by $\frac{1}{7}$—$\frac{1}{3}$ in. wide.

Peduncles about as long as the leaves, bearing 1—5 flowers, hairy; equal to or longer than the pedicels, arching. Flowers subhermaphrodite, pentamerous. Bracts lanceolate.

Calyx partite, usually but little accrescent; lobes lanceolate or subulate, acute, hairy.

Corolla exceeding the calyx, glabrous; lobes reflexed.

Stamens (9—) 10, not always fertile. Style bilobed, hairy below. Ovary nearly glabrous, 4-celled.

Fruit oblong or globose, thinly glandular-pubescent, $\frac{1}{3}$—$\frac{2}{3}$ in. long, subtended by the usually reflexed calyx.

South Africa. Cape of Good Hope. Southern and Western districts.

Robertson!, *Drège!*, *Sieber!* 94, *Wallich!*, *Mund!*, *Ecklon!* 699, *Pappe!*, *Thom!*, *Mac Gillivray!* 610, *Krauss!*, *Masson!*, *Roxburgh!*, *Niven!* 48, *Hb. Ammann!*, *Nelson!*, *Forster!*, *Thunberg!*, *Oldenburg!*, *W. Elliot!*, *Zeyher!* 3349, *Harvey!* 572, *Burchell!* 2, 808, 5093, 5367, 5784, 6788, 7186, 7208, 7288, *Siekmann!*

12. ROYENA PARVIFLORA, sp. nov.

R. foliis obovatis, basi cuneatis, apice rotundatis vel ad apicem emarginatum brevissime et abrupte angustatis, membranaceis vel junioribus subcoriaceis, petiolatis; floribus pentameris, hermaphroditis, cymosis; calyce depresso-hemispherico, 5-fido, lobis deltoideis; stylo apice 5-lobo; ovario 10-loculari.

A large scandent shrub with terete branches. Young parts and inflorescence softly shortly and appressedly pubescent. Leaves alternate, obovate, cuneate at base, rounded or very shortly and abruptly narrowed to an emarginate apex, membranous or the smaller ones subcoriaceous, green when dry, glabrous and with inconspicuous veins above, somewhat paler delicately veined and puberulous beneath, 2—6$\frac{1}{2}$ in. long by 1—3$\frac{4}{5}$ in. wide; petiole $\frac{1}{3}$—$\frac{2}{3}$ in. long. Cymes axillary on the young shoots, $\frac{1}{2}$—$\frac{3}{4}$ in. long, bearing 3—5 flowers; common peduncle $\frac{1}{3}$—$\frac{1}{2}$ in. long; lateral pedicels $\frac{1}{6}$—$\frac{1}{4}$ in. long, with a narrow bract at base about as long as themselves. Flowers hermaphrodite, small, creamy-white, articulated at base to pedicel; in bud depresso-conical, about $\frac{1}{7}$ in. high and broad. Calyx depresso-hemispherical, short, 5-fid, with flat base, puberulous outside; lobes deltoid. Corolla much contorted sinistrorsely as regarded from within, shortly pubescent outside except on imbricated sides of the lobes, glabrous inside, 5-lobed; lobes obtuse, rounded, $\frac{1}{2}$ths of the depth of the corolla. Stamens 10, hairy, equal, in one row, inserted at base of corolla. Ovary covered with very short hair, depresso-conical, 10-celled, cells 1-ovuled; style 5-lobed at apex, shortly hairy.

S. Africa, Zulu-land, Incansla. *Gerrard and M^cKen!* no. 2015.

13. Royena glandulosa, Harvey MSS.

R. foliis ovato-ellipticis, obtusis, basi rotundatis, subcoriaceis, subsessilibus; floribus hermaphroditis, plerumque tetrameris; pedunculis unifloris; calyce 4-partito; corollâ urceolatâ; staminibus 8; stylo apice 4-lobo; ovario 8-loculari; fructibus ellipsoideis, glanduloso-hispidis.

A large shrub, "with pretty foliage and habit," 8—10 feet high. Young shoots, peduncles and fruit glanduloso-hispid, subferruginous. Branches spreading. Leaves alternate, ovate-elliptical, obtusely pointed at apex, rounded at or near base, thinly coriaceous or firmly membranous, ciliate and pilosulous beneath, $\frac{1}{2}$—1 in. long by $\frac{1}{4}$—$\frac{1}{2}$ in. wide; petioles about $\frac{1}{16}$ in. long, hirsute. Flowers hermaphrodite, axillary on the young shoots, about $\frac{1}{4}$ in. long, urceolate, articulated to the peduncle, tetramerous. Peduncles spreading, $\frac{2}{7}$ in. long, 1-flowered, solitary. Calyx pilose outside, pubescent inside, 4-partite; lobes $\frac{1}{7}$ in. long, lanceolate, acute, rather patent. Corolla urceolate, glabrous but margin minutely ciliate, deeply 4-lobed; lobes rounded, recurved above. Stamens 8, in one row, inserted at base of corolla, short, equal, 2 opposite each lobe of the corolla, pilose; filaments short. Ovary hairy (except perhaps at middle), 8-celled; style hairy, 4-lobed and glabrous at apex. Fruit ellipsoidal, scarcely $\frac{1}{2}$ in. long by $\frac{1}{3}$ in. thick, glandular-hispid. Fruiting calyx much enlarged, $\frac{1}{3}$ in. long, loosely enclosing the fruit or reflexed, 4-partite; lobes ovate-oblong, foliaceous, reddish when dry, about 8-nerved inconspicuously.

Rarely a flower is pentamerous.

S. Africa, Port Natal, Tugela. *Gerrard and M^cKen!* no. 1608.

PLATE II. Flowering and fruiting branches, *natural size*. *a.* Peduncle, *magnified 5 diameters*. *b.* Hair of peduncle, *magnified 30 diameters*. *c.* Flowering calyx, *magnified 5 diameters*. *d.* Interior of corolla with stamens, laid open, *magnified 5 diameters*. *e.* Stamen, *magnified 15 diameters*. *f.* Pistil, *magnified 5 diameters*. *g.* Transverse section of ovary, *magnified 5 diameters*.

EXCLUDED AND NOMINAL SPECIES.

Royena latifolia, Willd. Enum. pl. Berol. Suppl. p. 23 (1813). Name only. Cfr. *R. sessilifolia.*

Royena media, Hort. ex Steud. Nomencl. bot. edit. ii. vol. ii. p. 475 (1841). Name only. Cape of Good Hope.

Royena polyandra, Linn. fil. Suppl. p. 240 (1781) = *Euclea polyandra*, E. Mey.

Royena (sp.) n. 15, Eckl. and Zeyh. ex Harv. and Sond. Fl. Cap. i. p. 71 (1859—60) = *Aberia tristis*, Sond.

Royena 9140, Drèg. ex Alph. DC. Prodr. VIII. p. 216. n. 4 (1844) = *Euclea coriacea* Alph. DC.

II. EUCLEA, Linn. Syst. Nat. edit. XIII. p. 747 (1774), non Lour.

Flores diœci, rarius polygami, 4—7-meri, racemosi vel paniculati. Calyx non accrescens. Corolla campanulata vel urceolata, lobis in præfloratione sinistrorse contortis.

FLOS MASCULUS: *Stamina 10—30, sæpius geminata. Ovarium plerumque abortivum.*

FLOS FEMINEUS: *staminodia 0, rarius 2—4. Ovarium 4-loculare, rarius 2- vel 6-loculare; ovula in loculis solitaria, rarius bina in ovariis bilocularibus. Fructus parvus, sæpius 1-locularis et 1-spermus.*

Frutices vel rarius arbores Africani, foliis alternis vel oppositis vel rarius in tribus verticillatis, cymis axillaribus.

Alph. DC. Prodr. VIII. p. 215. n. II. (1844).

Padus (sp.) Burm. Rar. Afric. pl. p. 238. t. 84. f. 1. (1738).

Royena (sp.) Linn. fil. Suppl. p. 240 (1781).

Celastrus (sp.) Thunb. Fl. Cap., pars post., p. 115 (1800).

Diplonema G. Don, Dict. Gard. and Bot. IV. p. 42 (1837).

Myrsine (sp.) Hochst. in Pl. Schimp. Abyss. exsicc. sect. i. n. 159 (1840).

Rymia Endl. gen. pl. n. 4250 p. 743 (1835—40).

Kellaua Alph. DC. in Ann. Sc. Nat. ser. ii. vol. XVI. p. 96 (1841).

Brachycheila Harv. in Linnæa XX. p. 192 (1847).

Diœcious or occasionally polygamous. Calyx campanulate or small and flattish, 4—7-lobed. usually 4—5-fid; lobes lanceolate ovate or deltoid; not accrescent. Corolla campanulate or hemispherical, 4—7-lobed, 4—5-fid or -partite or -lobed only near the apex.

♂ Stamens 10—30, usually 12—20, either free or in pairs or combined at base of filaments, in one or two rows, inserted at base of interior of tube of corolla or hypogynous or partly in both ways, sometimes inserted on an hypogynous ring; anthers hairy or glabrous, oblong or lanceolate, 2-celled, dehiscing laterally; filaments short, usually slender and glabrous. Styles 1—2. Ovary usually abortive.

♀ Staminodes usually absent, sometimes 2—4, glabrous; anthers 0. Styles 2 (or 1, bifid), usually glabrous, rarely 3; stigmas emarginate or bifid at apex; ovary ovoid or globular, hairy or glabrous, usually 4-celled, rarely 2- or 6-celled; ovules 4, or rarely 6 when the ovary is 6-celled, pendulous. Fruit globular or rarely ovoid-conical, usually 1-celled and 1-seeded; pericarp fleshy. The fruit is edible and is called *Guarry.* Seed globular, usually marked outside by 3 longitudinal depressed lines. Albumen cartilaginous, usually with a normal intrusion of the testa at the micropyle, distinctly ruminated in a few species; embryo usually somewhat curved with its concavity towards the centre of the seed, tending to be incumbent; radicle superior, about as long as the foliaceous cotyledons. Flowers in axillary racemes or rarely in panicles or solitary.

African shrubs or trees with alternate or opposite leaves, or rarely verticillate 3 together. Leaves quite entire except *E. ovata* and *E. coriacea* in which they are sometimes minutely or obscurely crenulate; usually coriaceous, often obovate, not acuminate except in *E. ovata*, evergreen.

The name is derived from the Greek εὐκλεία, glory, in consequence of the beautiful evergreen foliage.

EUCLEA. KEY TO THE SPECIES.

Ovary hairy. Stamens 15—30.
- Corolla 4—7-lobed only at apex.
  - Leaves elliptical or obovate, flat or nearly so, not or very rarely cordate at base.
    - Stamens 20—30. ♂ racemes ½—1½ in. long. 1. *E. polyandra.*
    - Stamens 18. ♂ racemes short. 2. *E. tomentosa.*
  - Leaves ovate, subcordate, wavy. Stamens 16—17. 3. *E. coriacea.*
  - Leaves linear or lanceolate, flat, not cordate at base.
    - Flowers pentamerous or hexamerous. Leaves not falcate.
      - Leaves oblong-lanceolate, about ⅓ in. wide, apiculate. 4. *E. acutifolia.*
      - Leaves linear or linear-lanceolate, about $\frac{1}{10}$—$\frac{1}{4}$ in. wide.
        - Lower leaves obtuse, not apiculate. Flowers nearly glabrous. 5. *E. lancea.*
        - Leaves apiculate. Flowers hairy. 6. *E. pseudebenus.*
    - Flowers tetramerous or rarely pentamerous. Leaves falcate. 7. *E. linearis.*
- Corolla 4—5-fid or -partite.
  - Fruiting calyx-tube receiving the base of the fruit.
    - ♂ Flowers racemose, 3—9 together.
      - Leaves quite entire, obtuse or subacute. 8. *E. lanceolata.*
      - Leaves minutely crenulate or acutely apiculate. 9. *E. ovata.*
    - ♂ Flowers panicled or many together.
      - Leaves glabrous, subglaucous, opposite. 10. *E. divinorum.*
      - Leaves pubescent or not glaucous, alternate. 11. *E. multiflora.*
  - Fruiting calyx-tube consolidated and articulated to thickened pedicel.
    - Fruits many together. Albumen not ruminated. 12. *E. fructuosa.*
    - Fruits 3—4 together. Albumen ruminated. 13. *E. natalensis.*

Ovary usually glabrous or chiefly so. Stamens 10—18 usually about 12.
- Leaves flat or nearly so. Ovary quite glabrous or rarely pubescent all over.
  - Racemes dense. Leaves usually opposite or verticillate 3 together. Ovary 2—6-celled.
    - Leaves obovate. Ovary 2-celled. Staminodes 0. 14. *E. bilocularis.*
    - Leaves obvate-oblong. Ovary 2-, 4-, 6-celled. Staminodes 0. 15. *E. macrophylla.*
    - Leaves oblanceolate-oblong. Ovary 4-, 6-celled. Staminodes 0-4. 16. *E. daphnoides.*
  - Male racemes lax. Leaves subopposite or alternate. Ovary 4-celled.
    - Ovary glabrous. Staminodes 0. Abyssinian. 17. *E. Kellau.*
    - Ovary pubescent or rarely glabrous. Staminodes 2—4. S. African. 18. *E. racemosa.*
- Leaves wavy or very small. Ovary hairy at base, glabrous above. 19. *E. undulata.*

1. EUCLEA POLYANDRA, E. Mey. Cat. Pl. exsicc. Afr. Austr. Drège. p. 7 (1837).

E. foliis ellipticis, alternis vel suboppositis, obtusis, basi subangustatis rotundatis vel rarissime cordatis, breviter petiolatis, coriaceis, planis; cymis racemosis; floribus 5—7-meris, diœcis. corollâ apice lobatâ; staminibus 20—30, in floribus femineis 0; ovario hirsuto.

Royena polyandra, Linn. fil. Suppl. p. 240 (1781), non Willd. Hb. n. 8366;
Diplonema elliptica, G. Don, Gen. Syst. Gard. and Bot. IV. p. 42 (1837);
Rymia polyandra, Endl. Cat. hort. Acad. Vindob. II. p. 123, n. 4583 (1843);
E. elliptica, Alph. DC. Prodr. VIII. p. 216. n. 1 (1844);

E. Drègeana, Alph. DC. *l. c.* n. 2;

E. ferruginea, Bernh. in Flora XXVII. ii. p. 825 (1844);

E. pubescens, Eckl. et Zeyh. in Linnæa XX. p. 192 (1847);

Brachycheila pubescens, Harv. ex Eckl. et Zeyh. *l. c.*

A shrub 3—7 feet high, pubescent often ferruginous but sometimes glabrescent at least in the male plant, diœcious. Branches terete or subterete, alternate or subopposite. Leaves more or less elliptical, alternate or subopposite, more or less obtuse at apex, somewhat narrowed, rounded, or even in rare cases cordate at base, coriaceous, quite entire, flat, shortly petiolate, 1—3 in. long by $\frac{1}{2}$—$1\frac{1}{2}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{4}$ in. long.

♂. Cymes racemose, axillary, pubescent, 3—9-flowered, usually drooping, $\frac{1}{2}$—$1\frac{1}{2}$ in. long; pedicels $\frac{1}{10}$—$\frac{1}{2}$ in. long, the lower ones the longer; bracts lanceolate, caducous. Flowers $\frac{1}{5}$ in. long, urceolate, 5—7-merous, pubescent. Calyx $\frac{1}{20}$—$\frac{1}{10}$ in. long, glabrous inside; lobes lanceolate or deltoid. Corolla urceolate, lobed only near apex. Stamens 20—30, more or less united at base in pairs or otherwise, hairy. Ovary more or less abortive, with two slender styles.

♀. Cymes usually 3- rarely 4—5-flowered, axillary, $\frac{1}{10}$—$\frac{1}{4}$ in. long, pubescent or tomentose, usually drooping; pedicels short; bracts caducous. Flowers $\frac{1}{8}$ in. long, ellipsoidal, 5—7-merous. Calyx shorter than the corolla, 5—7-fid; lobes ovate or deltoid. Corolla shortly lobed at apex. Staminodes 0. Ovary ovoid-conical, hairy, 4-celled, 4-ovuled, $\frac{1}{10}$ in. long, surmounted by 2 short styles glabrous above which just appear at the mouth of the corolla. Stigmas emarginate. Fruit usually solitary, occasionally 2—3 together, tomentose, usually ferruginous, globular $\frac{1}{3}$—$\frac{1}{2}$ in. in diameter, 1-celled, 1-seeded. Seed globular; albumen somewhat ruminated.

The shrub is called *Kersse-bosch* by the natives in South Africa.

Frequent in S. and SW. districts of Cape Colony up to 2000 ft. alt. *Masson!*; *Niven!* 47, 53; *R. C. Alexander!*; *Burchell!* 4807?, 4873?, 4998?, 6941; *Ecklon!* 727; *Krauss*; *Zeyher!* 3362, 3363, 3364; *Drège!*

2. EUCLEA TOMENTOSA, E. Mey. Cat. Pl. exsicc. Afr. Austr. Drèg. p. 7 (1837).

E. foliis alternis, ellipticis, basi cuneatis, apice obtusiusculis vel obtuse angustatis, tomentosis, planis, coriaceis, breviter petiolatis; cymis breviter racemosis, 1—8-*floris; floribus* 5—7-*meris, diœcis; corollâ apice lobatâ; staminibus* 18, *in floribus femineis* 0; *ovario tomentoso.* 4-*loculari.*

Alph. DC. Prodr. VIII. p. 216. n. 3 (1844).

E. Kraussiana Bernh. in Flora XXVII. ii. p. 824 (1844).

A shrub about 4 feet high or more with dark brown bark and branches cinereotomentose at the extremities. Leaves alternate, elliptical, in most cases obtusely pointed or mucronate at apex and wedge-shaped or obtusely narrowed at base, coriaceous, tomentose, shortly petiolate, $1\frac{1}{4}$—$1\frac{3}{4}$ in. long by $\frac{1}{2}$—$\frac{5}{6}$ in. wide; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. long.

♂. Cymes racemose, axillary, few—8-flowered, much shorter than the leaves, pedicels rather longer than the flowers, crowded. Stamens 18, free or somewhat connate at the base.

♀. Cymes tomentose, densely racemose, axillary, 1—several-flowered, pendulous, shorter than the leaves; pedicels $\frac{1}{10}$ in. long. Flowers $\frac{1}{5}$ in. long, 5—7-merous, when solitary with numerous imbricated caducous bracts on the short peduncle. Corolla $\frac{1}{6}$ in. long, shortly lobed, villous outside, glabrous inside, urceolate or campanulate, nearly 3 times the length of the calyx. Staminodes 0. Ovary tomentose, 4-celled, 4-ovuled. Styles 2, nearly glabrous. Fruit solitary, on peduncle $\frac{1}{10}$—$\frac{1}{4}$ in. long, pubescent erect or erect-patent. Immature fruit ovoid, somewhat conical at apex, incano-tomentose, 4-celled, $\frac{2}{5}$—$\frac{1}{2}$ in. long by $\frac{1}{4}$—$\frac{3}{10}$ in. thick. Fruiting calyx 5—7-fid, very tomentose, shallow; lobes deltoid.

Called *Kersboschjes* also *Faxhàls-bosch* by the natives in South Africa.

Occurs in Western districts of Cape Colony up to 2000 ft. alt. *Masson!; Drège!; Krauss; (?) Burchell!* 987. Namaqualand, *Whitehead!*

3. EUCLEA CORIACEA, Alph. DC. Prodr. VIII. p. 216. n. 4 (1844).

E. foliis ovatis, alternis, plerumque acutis, apiculatis, basi latis et subcordatis, subglabrescentibus, breviter petiolatis, undatis; cymis densis, ♂ *1—3-floris,* ♀ *3—7-floris; floribus 5—6-meris, diœcis; corollâ apice lobatâ; staminibus 16—17; fructibus globosis, subglabratis.*

Euclea n. 9140, E. Mey. Zwei Pflanz. Doc. Drèg. in Flora XXVI. ii. p. 48(1843).

Royena n. 9140, Drèg. ex Alph. DC. *l.c.* (Hb. DC!).

A dense shrub with strong dark-cinereous branches. Young parts and inflorescence slightly pubescent. Leaves alternate, ovate, more or less acute, apiculate, wide and subcordate at base; coriaceous, pubescent, nearly glabrescent, without evident veins above, veined and duller beneath, 1—2 in. long by $\frac{1}{2}$—$1\frac{1}{2}$ in. wide; margins wavy, sometimes obscurely crenulate; petioles ranging up to $\frac{1}{8}$ in. long. Bracts ovate, small, caducous.

♂ Flowers $\frac{3}{20}$ in. long, axillary, 1—3 together, crowded; pedicels shorter than the flowers. Calyx 5—6-fid; lobes ovate, acute. Corolla urceolate, 4 times the length of the calyx, 5—6-lobed at the apex. Stamens 16—17, sometimes in pairs; anthers linear lanceolate, silky at the back. Ovary rudimentary.

♀. Flowers 3—7 together; peduncles very short; pedicels ranging up to $\frac{1}{10}$ in. long. Calyx (in fruit) 5—6-fid, nearly flat, stellate, $\frac{1}{5}$ in. in diameter; lobes ovate or lanceolate, acute. Fruit globose, $\frac{1}{4}$—$\frac{2}{5}$ in. in diameter, subglabrate or minutely puberulous, 1-celled, 1-seeded; seeds subglobose, about $\frac{1}{4}$ in. in diameter, marked outside with depressed curved lines; testa intruded into the hard ruminated albumen.

East-midland districts of Cape Colony, S. Africa. Tafelberg, *Drège!*, in moist and rocky places, 6000—7000 ft. alt. (in ♂ flower, December); side of Mount Oudeberg near Graaff Reinet, 4500 ft. alt., November, *Bolus!* n. 638 (in fruit).

4. EUCLEA ACUTIFOLIA, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Drèg. p. 7 (1837).

E. foliis alternis, oblongo-lanceolatis, apiculatis, coriaceis, glabris, basi cuneatis, subsessilibus; cymis fœmineis dense racemosis; calyce 6-lobato; corollâ apice lobatâ; ovario dense piloso; fructibus dense racemosis, globosis, glabrescentibus.

Alph. DC. Prodr. VIII. p. 217. n. 5 (1844).

Shrub with glabrous leaves and branches. Leaves oblong-lanceolate, apiculate, thickly coriaceous, alternate, cuneate at base, subsessile, erect, subglaucescent, $1\frac{1}{2}$—$2\frac{1}{2}$ in. in length by about $\frac{1}{3}$ in. in width.

♀. Fruit densely racemose on cymes $\frac{1}{4}$ in. long; pedicels very short, 3—7; flowers $\frac{1}{6}$ in. long, cylindrico-urceolate, pubescent, pentamerous. Calyx short. Corolla lobed at apex. Ovary densely pilose; styles 2, erect, glabrous; stigmas dilated. Fruit globose, glabrescent, finely netted, dark, $\frac{2}{5}$—$\frac{1}{2}$ in. in diameter. Fruit-calyx very small, with 6 or more lobes; seeds unequally divided by three depressed lines; albumen slightly ruminated.

South Western districts, Cape of Good Hope. I have seen this plant only in fruit. The flower is unknown. Between Vierentwintig-rivier and Pikenierskloof on the plain, under 500 feet, January, *Drège!*; *Ecklon and Zeyher!*

5. EUCLEA LANCEA, Thunb.! Prodr. Pl. Capens., pars posterior, p. 85 (1800).

E. foliis alternis, lineari-lanceolatis vel oblanceolatis, inæqualibus, inferioribus apice rotundatis superioribus acutis, subsessilibus, glabris; cymis 3-*floris; floribus subhermaphroditis* (?), 5—6-*meris; corollâ apice lobatâ, subglabrâ; staminibus* 15; *ovario hirsuto.*

Alph. DC. Prodr. VIII. p. 219. n. 16 (1844).

A glabrous shrub, erect, 3 ft. or more high. Branches alternate, terete, erect-patent Leaves alternate, subsessile, linear-lanceolate or -oblanceolate, unequal, the lower ones rounded, the upper acute at the apex, attenuate at base, coriaceous, 1—2 in. long by about $\frac{1}{4}$ in. wide, entire, inconspicuously reticulato-venose. Flowers axillary, "in 3-flowered cymes," very nearly glabrous, urceolate, $\frac{1}{7}$ in. long by $\frac{1}{9}$ in. wide (imperfectly hermaphrodite ?). Calyx short, $\frac{1}{30}$ in. high by $\frac{1}{12}$ in. wide, obscurely 5—6-lobed, coriaceous. Corolla 5—6-lobed at apex, shortly ciliate, imbricated in æstivation. Stamens 15, alternately (?) in pairs and single; the pairs consisting of 2 equal or unequal anthers placed back and front on a common filament or combined by their filaments, alternating with the corolla-lobes. Anthers pointed, with short patent pale setæ on upper half, dehiscing laterally; filaments dark glabrous slender, shorter than the majority of the anthers, mostly inserted at the base of the corolla. Ovary hairy, $\frac{1}{16}$ in. wide and long, ovoid-conical, rudimentary or 4-celled ? and -ovuled ? Styles 2, glabrous, erect; stigmas punctiform, emarginate at apex.

Cape of Good Hope. *Thunberg!*

Near *E. pseudebenus*, E. Meyer, but differs by its obtuse lower leaves and nearly glabrous corolla; it may possibly include *E. pseudebenus* as a form of the same species.

6. EUCLEA PSEUDEBENUS, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Drèg. p. 7 (1837).

E. foliis alternis, linearibus vel lineari-lanceolatis, apiculatis, glabris, breviter petiolatis; cymis masculis racemosis 3—7-*floris, femineis parvis* 1—3-*floris; floribus diœcis, plerumque* 5-*meris; corollâ pubescente, apice lobatâ; staminibus* 16—22, *in flore femineo* 0; *ovario pubescente,* 4-*loculari.*

Alph. DC. Prodr. VIII. p. 217. n. 7 (1844).

E. rigida, E. Mey. *l. c.*, Alph. DC. *l. c.* n. 6.

E. angustifolia, Benth. Niger Fl. p. 441 (1849). Leaves and branches glabrous or pubescent. Leaves linear or linear-lanceolate, apiculate, erect or patent, alternate, coriaceous, very shortly petiolate, crowded, 1—$2\frac{1}{4}$ in. in length by $\frac{1}{10}$—$\frac{1}{4}$ in. in width; petioles $\frac{1}{20}$—$\frac{1}{7}$ in. in length.

♂. Cymes racemose, hairy, bearing 3—7 flowers, erect or erect-patent, $\frac{1}{5}$—$\frac{2}{5}$ in. in length; pedicels slender, $\frac{1}{10}$—$\frac{1}{5}$ in. in length; flowers $\frac{1}{10}$—$\frac{1}{7}$ in. in length, puberulous or incano-pubescent, usually pentamerous, rarely hexamerous; calyx with deltoid lobes reaching half way down; corolla lobed at apex; stamens 16—22, with a few bristles on the lanceolate anthers or glabrous; filaments more or less combined at the base, inserted around base of rudimentary ovary.

♀. Flowers solitary or two or three together, or in small cymes, $\frac{1}{7}$ in. in length, pentamerous; peduncles $\frac{1}{10}$—$\frac{1}{4}$ in. in length, not drooping. Stamens 0; styles 2; ovary 4-celled, pubescent; fruit 1-celled, 1-seeded, glabrescent, globular, $\frac{1}{5}$ in. in diameter; albumen not or scarcely ruminated; fruit edible, fleshy, sweet and slightly astringent; seeds marked by three depressed lines.

There are three forms of this species according as the plant is glabrous with linear leaves, pubescent with linear leaves, or glabrous with linear-lanceolate glaucescent leaves. The two latter forms belong to *E. angustifolia*, Benth. and *E. rigida*, E. Mey. respectively.

It is known by the names of Orange river ebony, black ebony, zwartebbenhout, and sneezewood. It is a large shrub, 6—8 ft. high or a tree, the heart-wood of which is extremely hard and black. It occurs in the western districts of South Africa, up to an elevation of 4000 ft., and reaches the tropics. *Drege!*; *Niven!* n. 46. Namaqualand, *Dr Atherstone!* n. 2; *Wyley!* S.W. Tropical Africa, lat. 23°, *Chapman and Baines!*; *Curror!*; Angola, Distr. Mossamedes, shrub, 5—8 feet high, flowers white, diœcious, fruit the size of a pea, edible, glaucous-bluish (as in *Juniperus communis*), called by the natives (as also *Euclea lanceolata*) *Embolo*, quite frequent in thickets and woods in company with *Tamarix* and *Cordia* near the rivers Bero and Maiombo, *Dr Welwitsch!* nos. 2543, 2544.

NOTE. This species may prove identical with *E. lancea*, Thunb.

7. EUCLEA LINEARIS, Zeyher in Linnæa XX. p. 192 (1847, sine descriptione).

E. foliis alternis suboppositis vel oppositis, linearibus, acutis, falcatis, numerosis, sessilibus, glabris; cymis racemosis, 3—7-floris; floribus diœcis, tetrameris; corollâ breviter 4-fidâ; staminibus 16, *in flore femineo* 0; *ovario hirsuto.*

Plant quite glabrous and subglaucous, diœcious, $2\frac{1}{2}$—3ft. high. Branches numerous, at about 35° with stem. Leaves alternate opposite or subopposite, linear, acute, usually somewhat falcate, sessile, numerous, 1 to $2\frac{1}{2}$ in. in length by $\frac{1}{10}$ in. in width. Cymes racemose, bearing 3—7 flowers, ♂ $\frac{1}{4}$ to $\frac{1}{2}$ in. in length (excluding flower), usually drooping; ♀ $\frac{1}{8}$ to $\frac{1}{5}$ in. in length, pedicels not exceeding $\frac{1}{10}$ in. in length, less on the ♀ plant, opposite or alternate, falling short of or equalling the bracts; bracts at base of pedicels, caducous.

♂. Flower $\frac{1}{10}$ in. in length. Calyx small, flattish, 4-fid. with wide lobes. Corolla barrel-shaped, 4-lobed, many times higher than calyx; lobes about $\frac{1}{3}$ depth of corolla, not reflexed, semi-circular and imbricated in flower. Stamens 16, the few inner ones smaller, glabrous or nearly so, $\frac{1}{17}$—$\frac{1}{12}$ in. long; anthers oblong, 2-celled, dehiscing laterally, thick; filaments very short, thinner than anthers, inserted with corolla. Ovary rudimentary, slightly hairy, terminated by 1 or 2 styles.

♀. Bracts linear, rather longer than pedicel; flower about $\frac{1}{10}$ in. in length. Calyx campanulate, $\frac{1}{25}$ in. in height, 4-lobed; lobes not quite half the depth of the calyx, with intervening sinuses in the form of arcs of circles. Corolla openly campanulate, shortly 4-fid, with spreading not reflexed oval or ovate lobes, $\frac{1}{12}$ in. in length; stamens 0. Ovary ellipsoidal, hairy, terminated by 2 thick glabrous styles, $\frac{1}{13}$ in. in height; styles $\frac{1}{30}$ in. long, erect, contiguous, dilated, and emarginate at apex; ovary 4-celled, 4-ovuled, two of the septa being very slender, namely, those opposite the styles.

Rarely a flower is pentamerous.

Western districts of Cape Colony, South Africa. *Zeyher!* 1125, Windhoek, Olifant River; *Burke!* Great Fish River.

8. EUCLEA LANCEOLATA, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Drèg. p. 7 (1837).

E. foliis alternis vel oppositis, lanceolatis ovatis vel angustè ellipticis, apice obtusis vel subacutis, plerumque undulatis et basi in petiolum brevem angustatis, integerrimis; cymis racemosis, 3—9-floris; floribus diœcis, 4- raro 5- meris; calyce campanulato; corollâ 4—5-fidâ; staminibus 16—17, in flore femineo 0; ovario hirsutissimo.

Alph. DC. Prodr. VIII. p. 218. n. 12 (1844).

E. ochrocarpa, E. Mey. Zwei Pflanz. Doc. Drèg. p. 184; in Flora, 1843; Alph. DC. *l. c.* p. 217. n. 9.

E. desertorum, Eckl. and Zeyh. in Linnæa XX. p. 192 (1847).

Pubescent glabrous or glaucescent shrub or tree, ranging up to 20—25 feet high and trunk up to 10—15 inches thick; diœcious; branches terete, at 30°—45°; young shoots angular. Leaves lanceolate ovate or narrowly elliptical, alternate or opposite, coriaceous, obtuse or subacute at apex (very rarely acute), more or less undulating at the entire margins, often narrowed at base into the short petiole, 1—3 in. long by $\frac{1}{10}$—$1\frac{1}{10}$ in. wide; petioles $\frac{1}{14}$—$\frac{1}{5}$ in. long. Inflorescence racemose, often with leaf-like bracts.

♂. Racemes $\frac{1}{4}$—1 in. long, 3—9-flowered; occasionally two racemes proceed from the same axil; pedicels $\frac{1}{10}$—$\frac{3}{10}$ in. long. Flowers usually tetramerous, occasionally pentamerous, $\frac{1}{10}$—$\frac{3}{20}$ in. long. Calyx widely campanulate, small; lobes deltoid, about half the length of the calyx. Corolla campanulate; lobes ovate or oval, about half the length of the corolla, somewhat pubescent outside. Stamens 16—17 (very rarely 8—10), mostly inserted in pairs at base of corolla, shorter than the corolla; anthers hispid or nearly glabrous, as long as the slender filaments. Ovary rudimentary, hirsute; styles 2, glabrous.

♀. Racemes $\frac{1}{3}$ in. long, pubescent, 3—5-flowered; pedicels $\frac{3}{50}$ in. long. Flowers tetramerous or pentamerous. Calyx and corolla as in the male plant. Staminodes 0. Ovary subglobose, very hirsute, 4-celled; styles 2, glabrous, as high as the corolla. Fruit globular, $\frac{1}{4}$ in.

in diameter, pubescent or glabrescent, 1-celled, 1-seeded. Testa intruded some distance into the albumen.

A very variable species, and in some cases difficult to separate from *E. ovata*, Burch., to which possibly it ought to be united; it is also nearly related to *E. divinorum*. It is called *Omgwali* by the Kaffirs, according to *Dr Pappe.*

South Africa; Cape Colony, Namaqualand, Natal and Trans-Vaal; common. *Masson!*; *Burchell!* 4880, 4938, 5648; *Drège!*; *Ecklon!* 1123. Uitenhage, *Harvey!* 575, 690; *Zeyher!* 3355, 3357, 3359?; Bothasberg, in stony places at 2000 ft. alt. *Mac Owan!* 902; Bruintjies Hoghte, 4000 ft. alt. *Mac Owan!* 1740; Albany, *T. Williamson!*; Caffraria, *Bowker!* 324; Namaqualand, *Drège!*; Natal, *Gerrard!* 33, 528, 1155, 1156, 1605; Macalisberg, Trans-Vaal, *Burke!*

Dr Welwitsch has collected the following forms from Benguela:

α. Leaves glabrous and shining, young ones lepidote; branches spreading.

Benguela, Distr. Bumbo, 15° South Latitude, 2000 ft. altitude; shrub, 8 ft. high, in thickets; in male flower at end of October; *Dr Welwitsch!* n. 2548. Distr. Mossamedes; much branched shrub, 5—8 ft. high, branches occasionally 3 or 4 together; ♂ flowers of pale rose-colour; frequent in rocky places near the river Meriombo in company with *Tamarix articulata* and *Ximenia americana*, from Pedra de Rei almost to Bumbo; *Dr Welwitsch!* n. 2547. Distr. Huilla; shrub 4—6 ft. high, with rather rigid and tortuous ramification; ♀ flowers fallen; ovary hairy; at margins of woods between Mumpulla and Nene, at end of October; *Dr Welwitsch!* n. 2549. Distr. Benguela; shrub 4—6 ft. high, with virgate usually opposite branches; in maritime thickets near the city of Benguela; fruit in middle of June; *Dr Welwitsch!* n. 2545. Distr. Mossamedes; shrub, occasionally arborescent, 7—12 ft. high, evergreen; frequent in sandy and rocky thickets very near the river Bero; July; native name *Nboto* or *Emboto;* fruit edible, berries red; *Dr Welwitsch!* n. 2546.

β. Leaves and shoots pubescent. Branches ascending.

Benguela, Distr. Huilla; small shrub 1—1½ ft. high; flowers white; in somewhat stony thickets near Mumpulla, not unfrequent; male flower in October; *Dr Welwitsch!* n. 2550; Cfr. *E. ovata*, Burch. Distr. Huilla; small shrub 1—1⅓ ft. high, subcæspitose; frequent in steep pastures on right bank of river Lopollo in company with small myrtaceous plants; flowers in November, fruits in February; *Dr Welwitsch!* nn. 2551, 2552.

The following two specimens may also belong to this variable species:

Distr. Huilla; a small shrub, 6—8 in. high, from a woody base; fruit dark purple, edible, ⅙ in. in diameter; in moist sandy thickets on the right bank of the river Lopollo in company with small species of Eugenia and Celastrus; fruit in January; *Dr Welwitsch!* n. 2553. Arborescent shrub; in thickets on the sides of large rocks; Pedras de Guinga, Angola. Distr. Pungo Andongo; March, in young fruit; *Dr Welwitsch!* n. 1247.

9. EUCLEA OVATA, Burch. Trav. Int. S. Afr. I. p. 387 (1822).

E. foliis ellipticis vel acutè ovatis, oppositis vel alternis, plerumque apiculatis, breviter petiolatis, rigidis, margine planis et minute crenulatis vel undulatis et integerrimis; cymis racemosis, 3—7-floris; floribus sub-diœcis vel polygamis; calyce depresso-campanulato; corollà

4—5-fidâ; staminibus 16 vel 20, in floribus sub-hermaphroditis circiter 12; ovario hirsuto; fructibus globosis, primùm pubescentibus, demùm glabris.

Alph. DC. Prodr. VIII. p. 218. n. 13 (1844).

Celastrus crispus, Thunb. Fl. Cap. edit. ii. vol. II. p. 115 (1820). Cfr. Sond. in Harv. et Sond. Fl. Cap. I. p. 461 (1859—60).

E. rufescens, E. Mey. Cat. Pl. Exsicc. Afr. Austr. Drèg. p. 7 (1837).

Royena rufescens, E. Mey. Cat. Pl. Drèg. p. 154 (Flora, 1843).

A densely leafy shrub pubescent or sometimes glabrescent, 3—7 ft. high; branches terete, at 50°—60°. Leaves opposite or alternate, elliptical or narrowly ovate, usually apiculate, acute or obtuse, coriaceous, shortly petiolate, minutely crenulate or quite entire, flat or undulated, 1—2 in. long by $\frac{1}{4}$—1 in. wide; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. long. Racemes 3—7-flowered, at length drooping, $\frac{3}{10}$—$\frac{3}{5}$ in. long; pedicels $\frac{1}{20}$—$\frac{3}{50}$ in. long; flowers tetramerous or occasionally pentamerous, sub-diœcious or polygamous, pubescent, $\frac{1}{10}$—$\frac{1}{6}$ in. long. Calyx shortly campanulate; lobes deltoid. Corolla campanulate, 4—5-fid. Stamens 16 or 20, in subhermaphrodite flowers about 12, hairy; filaments slender, glabrous. Styles 2, glabrous; ovary hirsute, globose or ovoid, (2—) 4-celled.

Fruit globose, black, of the size of a pea, at first pubescent but at length glabrous.

The flavour of the fruit is pleasant with a little astringency and perfectly wholesome.

The variety with undulated leaves 20 stamens and less deeply divided corolla (*E. rufescens*) much resembles *E. coriacea*, Alph. DC.

Occurs in midland districts of Cape Colony and northwards into the Kalihari region of South Africa. *Burchell!* 1706, 2487—2, 2487—7, 2542, 2920, 3058—1, 3058—2, 3102; *Drege!*

10. EUCLEA DIVINORUM, sp. nov.

E. foliis ellipticis, oppositis, a medio utrinque angustatis, obtusis, breviter petiolatis, suprà glaucescentibus, undulatis; cymis masculis conferto-racemosis vel-paniculatis; floribus 4—5-meris, diœcis; corollâ profunde 4—5-lobâ; staminibus 16.

Shrub, nearly glabrous and somewhat glaucous, with opposite or subopposite leaves and branches; branches terete, making 30°—40° with the stem. Leaves elliptical, narrowed more or less from the middle towards both ends especially towards the base into the short petiole, obtuse, coriaceous, glaucescent above, reddish and somewhat farinaceous beneath; margins undulated; veins inconspicuous; $1\frac{1}{2}$—$2\frac{1}{4}$ in. long by $\frac{1}{2}$—$\frac{3}{4}$ in. wide; petioles about $\frac{1}{5}$ in. long. Bracts small, shorter than the pedicels, caducous.

Male flowers in crowded racemes or panicles, about 10 or more together, globular in bud, hemispherical when expanded, tetramerous or pentamerous. Cymes not exceeding $\frac{7}{10}$ in. long by $\frac{1}{3}$ in. broad, usually erect; pedicels about $\frac{1}{10}$ in. long, spreading. Calyx $\frac{1}{25}$ in. high, 4—5-fid; lobes small or depresso-deltoid. Corolla deeply 4—5-lobed, with a few whitish appressed hairs outside especially along the middle of the lobes, nearly glabrous inside; lobes rounded. Stamens 16, not in pairs, $\frac{1}{10}$ in. long, as high as the expanded corolla; anthers oblong, hairy; filaments shorter, glabrous. Ovary rudimentary, consisting of an ovoid bunch of hairs.

Female plant unknown. Called by the natives in Batoka country *Matlakula, Mosakola*, where it is the medicine of the diviners being rubbed in the hands.

South Tropical Africa, Victoria Falls, *Dr Kirk!*; Delagoa Bay, *Forbes!*

11. EUCLEA MULTIFLORA, sp. nov. PLATE III.

E. foliis ellipticis vel oblongis, apice rotundatis vel obtusis, basi subangustatis, alternis vel raro suboppositis; cymis præsertim masculis paniculatis, multifloris; floribus polygamis, 5- raro 6-meris; calyce campanulato; corollâ profunde lobatâ; staminibus numero corollæ loborum quadruplis, in flore femineo 0; *ovario hirsuto.*

Pubescent subglabrous or even subglaucous shrub, usually subferruginous, polygamous but usually diœcious, sometimes hermaphrodite, 2—10 ft. high. Branches usually angular near the extremities. Leaves elliptical or oblong, usually rounded or obtuse at apex and somewhat narrowed at base into the petiole, coriaceous, alternate or rarely sub-opposite, often dark and shining on the upper surface; veins usually not conspicuous; margins undulated or plane; 1—4 in. in length by $\frac{3}{10}$—1 in. in width; petiole $\frac{3}{10}$—$\frac{1}{2}$ in. in length. Flowers especially the male ones paniculate, sometimes as many as 30 in one panicle, variable in size, tetramerous or pentamerous or rarely hexamerous. Calyx campanulate, hairy, with ovate or deltoid lobes extending about half way down the calyx. Corolla about twice as long as the calyx, dark, deeply lobed; lobes oval usually with a hairy keel outside. Stamens 4 times as numerous as the lobes of the corolla in the male or hermaphrodite plants, none in the female plant, subglabrous or somewhat hairy, in pairs inserted at base of corolla or around base of ovary, outer ones longer, enclosed in corolla; filaments glabrous. Ovary in male flowers often abortive, in female or hermaphrodite flowers globular, hairy, 4-celled, cells 1-ovuled; styles 2, glabrous or nearly so, included within the corolla.

A variable and widely distributed plant. Flowers in August and fruits from September to October. Fruit at first pubescent, in most cases ferruginously so, subsequently black and glabrous, globular, $\frac{3}{10}$ in. in diameter, 1-celled, 1-seeded. Embryo curved and tending to be incumbent.

Cape of Good Hope, Natal and Angola.

Wallich!; *?Bergius!*; *Zeyher!* 767, 778, 3361; Grahamstown, *Mac Owan!*; *Burchell*, 3510, 3572, 3980 (seeds consumed by a species of *Apion*), 4835; Albany, *Miss Bowker!*; Eastern districts, *Hutton!*; British Kaffraria, *Cooper!* 44; Clanwilliam, *Zeyher!*; Natal, *Gueinzius!*, *Cooper!* 1253, *Gerrard!* 92, 699.

Benguela, Distr. Huilla, *Dr Welwitsch!* n. 2557, arborescent shrub 5—8, sometimes 10 ft. high, forming a dense dark green head, young fruit 1- rarely 2-seeded, hirsute-tomentose, in thickets, Matus de Monino, February. Do. *Dr Welwitsch!* n. 2555, bush 4—8 ft. high, in high thickets near Tâu, in bud, May. Do. *Dr Welwitsch!* n. 1258, handsome shrub 5—8 feet high, in thickets at the skirts of woods near Lopollo, leaves frequently attacked by a fungus (*Sphæria*). Angola, distr. Pungo Andongo, *Dr Welwitsch!* n. 1257, bush 5—7 feet high with erect trunk 2—2½ in. in diameter and spreading branches towards the top, branches and fruit tomentose, at first whitish, soon becoming rufous, leaves dark green with a high polish, in stony woods at Barrancos de Catele, young fruit in December.

Plate III. Fig. 1. Flowering branch, *natural size*. *a*. Flower unexpanded, *magnified* 5 diameters. *b*. Flower expanded, *magnified* 5 diameters. *c*. Interior of corolla with stamens, laid open, *magnified* 6 diameters. *d*. Pistil, *magnified* 5 diameters.

Fig. 2. Flowering branch of another form of the same species, *natural size*. *e*. Flower unexpanded, *magnified* 5 diameters. *f*. Flower expanded, *magnified* 5 diameters.

12. EUCLEA FRUCTUOSA, sp. nov.

E. foliis obovato-oblongis, basi in petiolum cuneatis, alternis vel suboppositis; cymis femineis racemosis vel paniculatis, 3—20-floris; calyce 4—5-lobo, tubo in fructu farcto, lobis deltoideis parvis; corollâ 4—5-fidâ (?), interdum ad fructus apicem marcescente; fructibus numerosis, fulvo-pubescentibus.

Varying in size from a small to an arborescent shrub with softly pubescent fulvous and terete branches spreading at 35°—40° with the stem. Leaves obovate-oblong, cuneate at base into the petiole, coriaceous, quickly glabrescent and nitescent, alternate or subopposite; margins reflexed, net-veins numerous and delicate; $1\frac{1}{2}$—$4\frac{1}{2}$ in. in length by $\frac{3}{4}$—$1\frac{1}{2}$ in. in width; petioles $\frac{3}{10}$—$\frac{2}{5}$ in. in length, pubescent.

♀ Racemes or panicles $\frac{1}{10}$—1 in. in length, bearing from 3—20 flowers and nearly as many fruits, pubescent; pedicels short not exceeding $\frac{1}{10}$ in. in length, dilated upwards in fruit to articulation with calyx. Fruit pale or darker, fulvo-pubescent, about $\frac{4}{5}$ in. in diameter, 1-celled, 1 seeded; embryo somewhat curved and tending to be incumbent; albumen not ruminated. Calyx 5-lobed; lobes deltoid, acute, small; tube consolidated in fruit and bearing fruit at its apex. Corolla 4—5-fid (?); sometimes marcescent; lobes ovate.

Known only in fruit. Grows in dry places, &c. East Tropical Africa. Zambesia, Luame river mouth, 8 Feb. 1861, *Dr Kirk!*; between Tette and the sea coast, 16 March, 1860, *Dr Kirk!*; Zanguebar, Dar Salam, October to December, 1868, *Dr Kirk!*

13. EUCLEA NATALENSIS, Alph. DC. Prodr. VIII. p. 218, n. 10 (1844).

E. foliis alternis, angustè ellipticis, basi cuneatis, petiolatis, undulatis, glabrescentibus; cymis femineis racemosis, 8—10-floris; calyce fructifero 4—5-fido, tubo farcto, lobis deltoideis; fructibus subglabris.

E. macrophylla, E. Mey. d, non a, b, Zwei Pflanz. Doc. Drèg. p. 184 in Flora 1843.

Royena macrophylla, E. Meyer, d! in Hb. DC. (Prodr. *l. c.*)

Young parts pubescent. Leaves alternate, erect, narrowly elliptical and cuneate at base into petiole, coriaceous, glabrescent, 2—4 in. in length by $\frac{1}{2}$—$\frac{4}{5}$ in. in width; margins undulated; petioles $\frac{3}{10}$—$\frac{1}{2}$ in. in length.

♀ Racemes solitary about $\frac{1}{2}$ in. in length, bearing 8—10 flowers and about 4 fruits. Pedicels very short, dilated upwards in fruit to articulation with calyx. Calyx 4—5-fid, glabrescent; lobes deltoid, acute; tube consolidated in fruit, with small spreading limb, and bearing fruit at its apex. Berry dark, sub-glabrous, globular $\frac{1}{4}$—[illegible] in. in diameter, 1-celled, 1-seeded; seed globose, black, marked outside with 3 longitudinal lines, albumen somewhat ruminated.

Port Natal. *Drège!*; *Peddie!*

14. EUCLEA BILOCULARIS, sp. nov.

E. foliis alternis oppositis et in tribus verticillatis, obovatis, apice rotundatis, basi cuneatis, breviter petiolatis; cymis femineis racemosis, sub-9-floris; pedicellis brevissimis; floribus tetrameris; calyce 4-dentato; corollâ breviter 4-lobâ; staminodiis 0; *ovario biloculari, glabro, loculis biovulatis.*

Glabrous. Branches at 50°, sometimes whorled 3 together. Leaves obovate, cuneate at base, rounded at apex, somewhat undulating, coriaceous, alternate opposite and in whorls of 3; veins inconspicuous, in relief on both sides, dark green above, ruddier beneath; 2—3 in. long by $\frac{4}{5}$—1$\frac{3}{5}$ in. wide, shortly petiolate; petioles about $\frac{1}{10}$ in. long.

Racemes of ♀ flowers (in bud) short, about $\frac{1}{3}$ in. long, bearing about 9 very short pedicels. Flower-buds $\frac{1}{12}$ in. long, tetramerous, with short cup-shaped 4-toothed calyx and corolla shortly 4-lobed. Staminodes 0. Ovary 2-celled, with 2 ovules in each cell, glabrous.

East tropical Africa, Zanzibar, *Dr Kirk!*

A male plant from Madagascar collected by *Bojer!* may belong to this species; it has 16—18 stamens.

15. E. MACROPHYLLA, E. Mey, Cat. Pl. Exsicc. Afr. Austr. Drèg. p. 7 (1837).

E. foliis in tribus verticillatis vel oppositis, obovato-oblongis, breviter petiolatis, subcoriaceis; cymis femineis 8—15-*floris, floribus tetrameris, calyce* 4-*fido, corollâ* 4-*fidâ, staminodiis* 0, *ovario* 4- *vel raro* 6-*loculari, glabro.*

Alph. DC. Prodr. VIII. p. 218. n. 11 (1844).

Glabrous. Stem nodose; branches at 60°, often verticillate three together, straight. Leaves obovate-oblong, rounded at apex, cuneate at base, shortly petiolate, sub-coriaceous, opposite or subverticillate three together; margins reflexed, plane or wavy; veins delicate; 2—4 in. in length by $\frac{1}{2}$—1$\frac{1}{2}$ in. in width; petiole $\frac{1}{10}$ to $\frac{1}{4}$ in. in length.

♀. Flowers in cymes which measure $\frac{1}{2}$—1 in. in length and bear 8—15 flowers; pedicels $\frac{1}{16}$ to $\frac{1}{6}$ in. in length; flowers tetramerous. Calyx 4-fid, shortly cup-shaped, with deltoid-pointed lobes. Corolla campanulate, 4-fid; lobes obtuse or mucronate. Stamens 0 or represented by a few hairs at circumference of disk. Ovary glabrous, 4- or rarely 6-celled, with one ovule in each cell; at the upper part the ovary is frequently 2-celled, according to Dr Atherstone, in consequence of two of the dissepiments being false; styles 2.

South Eastern districts, Cape of Good Hope. Enon in woods under 500 feet high, March, Uitenhage, *Drège!* in ♀ flower; Grahamstown, *Dr Atherstone!* 461.

16. EUCLEA DAPHNOIDES, sp. nov.

E. foliis alternis oppositis vel in tribus verticillatis, oblanceolato-oblongis, subsessilibus; cymis racemosis; floris 4—5-*meris; calyce* 4—5-*fido; corollâ profunde* 4—5-*lobâ; staminibus circiter* 12, *uniserialibus, in flore femineo* 0 *vel* 4 *effœtis; stylis* 2—3; *ovario glabro,* 4- *vel* 6-*loculari.*

Glabrous shrub, 2—4 feet high or more, or even a low tree. Stem shining and turning pale yellowish; branches at 40°—50° with stem, alternate opposite or subverticillate. Leaves alternate opposite or 3 in a whorl, varnished on surface, crowded, oblanceolate-oblong, thickly coriaceous, flat or wavy, subsessile with thick articulation, $1\frac{1}{2}$—3 in. in length by $\frac{1}{3}$ to $\frac{1}{2}$ in. in width. Cymes racemose, much shorter than the leaves; pedicels $\frac{1}{20}$ in. long; bracts small and slender.

♂. Flowers tetramerous, nearly glabrous, small. Calyx 4-fid. Corolla deeply 4-lobed. Stamens about 12, in one row. Ovary rudimentary.

♀. Flowers 11—21 in cyme, $\frac{1}{10}$ in. in length by $\frac{1}{15}$ in. in width, ovoid, glabrous. Calyx $\frac{1}{30}$ in. in height by $\frac{1}{15}$ in. in width, 4—5-fid. Corolla openly campanulate, with nearly erect lobes, $\frac{1}{15}$ in. in height by $\frac{1}{20}$ in. in width, 4—5-lobed. Staminodes 0 or 4, inserted at base of interior of corolla or around base of ovary, glabrous, without anthers. Styles 2—3, $\frac{1}{25}$ in. in length, somewhat concave as seen from inside; stigmas bilobed at apex, projecting beyond the corolla; ovary glabrous, ovoid, $\frac{1}{25}$ in. in length and width, 4—6-celled; ovules pendulous, solitary in the cells. Fruit globular, $\frac{1}{4}$ in. in diameter, dark, glabrous, 1-seeded, 1-celled; seed marked outside by 3 depressed longitudinal lines; fruiting calyx small; albumen not ruminated but testa introverted at apex; embryo slightly curved.

Nearly related to *E. racemosa* L. from which it differs by its oblanceolate-oblong and longer leaves and its longer and more numerously flowered racemes, by its ovary being sometimes 6-celled, and by its 12 stamens being in one row in the only ♂ specimen examined.

South Africa. South-western districts of Cape Colony and Natal. In a walk by the Baaken's river under Fort Frederick, Algoa Bay, 14 Dec. 1813, *Burchell!* 4356, in ♀ flower; on the rocky side of the mountain, also on the western bank of Wagenbooms river on the north side of Lange Kloof, 11 March 1814, *Burchell!* 4909, in ♀ flower; Cape of Good Hope, *Ecklon and Zeyher!*; Natal, *W. T. Gerrard!* 1506, 1606, in ♂ flower-bud.

17. EUCLEA KELLAU, Hochst. in pl. Schimp. Abyss. exsicc. sect. ii. n. 1078 (1842).

E. foliis subоppositis, obovatis vel oblanceolatis, apice rotundatis, basi cuneatis, breviter petiolatis; cymis racemosis; floribus diœcis, 4—5-meris; corollâ 4—5-fidâ; staminibus 12, *in flore femineo* 0; *stylis* 2; *ovario glabro, 4-loculari.*

Hochst. Nov. Gen. pl. Afr. in Flora 1843, p. 83; Alph. DC. Prodr. VIII. p. 672 (1844); Rich. Fl. Abyss. ii. p. 24. t. 66 (1847).

Myrsine Kellau, Hochst. in pl. Schimp. Abyss. exsicc. sect. i. n. 159 (1840).

Kellaua Schimperi, Alph. DC. in Ann. Sc. nat. ser. ii. vol. XVIII. p. 209 (1842), Prodr. VIII. p. 290 (1844).

Shrub or small tree, glabrous, diœcious; branches at 38°—45° with stem, subopposite, straight. Leaves obovate or oblanceolate, shortly petiolate, sub-coriaceous, rounded at apex, wedge-shaped at base, subopposite, shining and of a rich brown colour on upper face, paler beneath; veins delicate, flat or wavy at margins, spreading; 1—2 in. in length by $\frac{1}{4}$—1 in. in width; petiole $\frac{1}{10}$—$\frac{1}{8}$ in. in length. Flowers racemose with lanceolate bracts at base of pedicels, tetramerous or pentamerous.

♂ Racemes $\frac{3}{4}$—1 in. in length, bearing 9—13 flowers, spreading, dark; pedicels slender $\frac{3}{40}$—$\frac{1}{5}$ in. in length, the lower ones the longer, alternate or opposite, patent; flowers $\frac{1}{7}$—$\frac{1}{5}$ in. in length. Calyx short, 4-fid, with apiculate deltoid erect lobes. Corolla campanulata, 4-lobed; lobes $\frac{1}{3}$—$\frac{1}{2}$ depth of corolla, erect, oval. Stamens 12, free, 8 in one row and 4 interior and inserted lower at base of interior of corolla, included; anthers erect, with a few hairs at top or glabrous, 2-celled, dehiscing laterally from apex. Ovary rudimentary; styles 1—2.

♀ Racemes $\frac{2}{5}$—$\frac{1}{2}$ in. in length, bearing usually 11 flowers, dark; glabrous or glandular; pedicels $\frac{1}{50}$—$\frac{1}{20}$ in. in length, patent, sub-opposite; flower $\frac{1}{10}$ in. in length, campanulate. Calyx $\frac{1}{20}$ in. in height with 4 or 5 deeply divided erect deltoid acuminate lobes, persistent. Corolla campanulate, twice the height of the calyx, with 4—5 lobes divided more than half the depth of the corolla. Stamens 0. Ovary $\frac{3}{40}$ in. in height by $\frac{1}{20}$ in. in thickness, conical, glabrous, 4-celled, cells 1-ovuled; styles 2. Fruit globose, $\frac{1}{4}$ in. in diameter, glabrous, 1-celled, 1-seeded; seed filling the cavity of the fruit, marked externally by 3 longitudinal lines; albumen horny, not ruminated but with introversion of testa at apex; embryo slightly curved, tending to be incumbent.

Abyssinian name of the fruit; *Kĕllau.*

Abyssinia. *Schimper!* i. n. 159, among hills, valleys and low places near Adoa. In fruit, 1 June 1837.

„ „ ii. 1078. On mount Sina, near Adoa. In ♀ flower, 13 November 1838.

„ „ ii. 1527, iii. 1919. Near Axus. In ♂ flower.

„ „ 913. Agrima, 6000 ft. alt.; Legua, 5000 ft. alt. 1852.

„ „ 80. „ 5500—6500 ft. alt. October 1862.

„ *Quartin-Dillon and Petit.!* Scholoda.

18. EUCLEA RACEMOSA, Linn. Syst. veg. edit. XIII. p. 747. Cur. Murray (1774).

E. foliis alternis vel oppositis, obovatis vel oblongo-obovatis, apice rotundatis, basi cuneatis, breviter petiolatis; cymis racemosis; floribus diœcis, 4- raro 5—6-meris; corollâ profunde lobatâ; staminibus 12—18, *in flore femineo* 2—4 *effœtis; stylis* 2; *ovario toto pubescenti vel glabro, 4-loculari.*

Jacq. Fragm. t. 1, f. 5, t. 63, f. 3 (1800—9); Alph. DC. Prodr. VIII. p. 219. n. 15 (1844).

Padus foliis subrotundis, fructu racemoso, Burm. Afr. p. 238, t. 84, f. 1 (1739).

Glabrous shrub $2\frac{1}{2}$ to 6 feet high, or small tree 18 feet high. Branches making 30° with stem, purplish. Leaves obovate or oblong-obovate, coriaceous, alternate, subopposite, or opposite, marked with obscure transverse veins, green on the upper surface, pale beneath, margins somewhat reflexed, wavy or nearly flat, subsessile or shortly petiolate, rounded at apex, cuneate at base, $\frac{1}{2}$—$2\frac{1}{2}$ in. in length by $\frac{1}{3}$—$1\frac{1}{4}$ in. in width; petioles $\frac{1}{20}$—$\frac{3}{20}$ in. in length. Bracts narrow, at base of pedicels, solitary, linear-lanceolate.

♂ Racemes $\frac{1}{2}$—$1\frac{1}{4}$ in. in length, shorter than the leaves from the axils of which they spring, drooping; pedicels $\frac{1}{10}$—$\frac{1}{4}$ in. in length, 5—13 in cyme, articulated at apex; flowers $\frac{1}{10}$—$\frac{1}{7}$ in. in length, campanulate, 4- or rarely 5—6-merous, glabrous. Calyx short, lobes deltoid, about half length of calyx. Corolla campanulate, open, deeply lobed, much raised above the calyx; lobes oval, obtuse or acutish, spreading or erect but not reflexed, of a dirty white colour. Stamens 12—18, in two rows, inserted at base of interior of corolla or on an hypogynous ring; the inner ones smaller and often connate at the base with outer ones; anthers lanceolate, thick, 2-celled, with a few hairs or glabrous, included or exserted, erect, dehiscing laterally and widely from apex, $\frac{1}{20}$—$\frac{1}{15}$ in. in length; pollen white; filaments slender, $\frac{1}{50}$—$\frac{1}{20}$ in. in length, often united in pairs at base, glabrous. Ovary rudimentary, hairy or glabrous; styles 2, distinct, erect, terete, white.

♀. Racemose cymes $\frac{1}{2}$—$1\frac{1}{4}$ in. in length, usually shorter than the leaves but sometimes longer, drooping in fruit; pedicels about $\frac{1}{10}$ in. length, 9—13 in cyme. Flowers ovoid, rather smaller than the ♂ flowers, tetramerous or rarely pentamerous. Calyx hemispherical; lobes ovate, acute, about half depth of calyx. Corolla ovoid, deeply lobed; lobes not reflexed. Staminodes 2—4, glabrous. Ovary hairy or glabrous, 4-celled, cells 1-ovuled; styles 2; fruit globular, glabrescent or glabrous, black, 1-celled, 1-seeded, $\frac{1}{5}$ in. in diameter.

Bark grey, smooth. Wood hard, heavy, employed by wheelrights and turners; answers very well for wooden screws, but is chiefly used as fuel. (Dr Pappe, Silva Capensis, p. 21 (1854).)

The variety *Burchellii* with glabrous ovary may be a distinct species. It is a tree 18 feet high with erect trunk and ascending branches and oblong-obovate leaves; bark entire, turning white; ovary globose; styles 2, short; staminodes 2—4, inserted on corolla or around base of ovary.

Cape of Good Hope, southern and western districts. *Drege!; Talbot!; Reeves!; Wright!; Boivin!; Bowie!; Alexander-Prior!; Oldenburg!; Nelson!; Hove!; Zeyher!* 3356; *Harvey!* 574; *Burchell!* 397, 807, 3219 (var. *Burchellii*), 3806, 8295; Hondeklip Bay, Clanwilliam, *Rev. H. Whitehead!*

19. EUCLEA UNDULATA, Thunb. Nova Genera Plantarum (V.) p. 86 (1784).

E. foliis alternis vel oppositis, obovatis (vel in var. *oblanceolatis), apice obtusis, basi cuneatis, breviter petiolatis, undatis (in* var. *parvis et subplanis); cymis racemosis,* 3—8-*floris; floribus dioecis, tetrameris; staminibus* 10—15, *plerumque geminatis, in flore femineo* 0; *stylis* 2; *ovario basi subpubescente,* 2- *vel* 4-*loculari,* 4-*ovulato.*

Alph. DC. Prodr. VIII. p. 219. n. 14 (1844).

E. myrtina, Burch. Trav. Int. S. Afr. II. p. 588 (1824), Alph. DC. *l. c.* p. 217. n. 8.

E. humilis, Eckl. et Zeyh. in Linnæa, XX. p. 192 (1847).

Glabrous dense shrub, extremities and flowers glandular but not hairy, 4 to 9 feet in height or a moderate sized tree, diœcious. Branches alternate or opposite, at 40° to 60° with stem, numerous. Leaves obovate or oblanceolate, coriaceous, shortly petiolate, wavy or in var. *myrtina* nearly flat, opposite or alternate, veins inconspicuous, cuneate at base, rounded or nearly so at apex, evergreen, $\frac{1}{2}$—$1\frac{1}{2}$ in. in length by $\frac{1}{5}$—$\frac{3}{5}$ in. in width (or $\frac{1}{10}$—$\frac{1}{5}$ in. in width in variety *myrtina*); petioles $\frac{1}{25}$—$\frac{2}{25}$ in. in length. Bracts sometimes large and leaf-like, caducous; flowers racemose, with divisions of corolla reaching to level of apices of calycine lobes.

♂. Racemes $\frac{2}{5}$—$\frac{4}{5}$ in. in length, shorter than leaves, lax, bearing 5—7 flowers; pedicels $\frac{1}{10}$—$\frac{1}{4}$ in. in length, slender; flowers $\frac{1}{10}$ in. in length. Calyx openly cup-shaped, 4-fid, short; lobes deltoid, acute. Corolla hemispherical, 4-lobed, somewhat glandular outside; lobes oval, more than half the depth of the corolla. Stamens 10—15, mostly in pairs, inserted at base of interior of tube of corolla; anthers oblong, mucronate, with a few bristles near apex, dehiscing widely from apex; filaments slender.

♀. Racemes $\frac{1}{4}$—$\frac{1}{2}$ in. in length, nearly erect in flower, drooping in fruit, bearing 3—8 flowers; pedicels under $\frac{1}{10}$ in. in length; narrow bracteoles sometimes present on middle of pedicels. Flowers $\frac{3}{40}$ in. in length. Calyx short, campanulate, 4-lobed, lobes deltoid, extending less than half-way down the calyx. Corolla 4-partite, erect or spreading, in bud cylindrical, somewhat glandular outside along middle of lobes; lobes oval. Stamens 0. Ovary and 2 styles together rather longer than corolla; styles as long as ovary, at first erect and contiguous, glabrous, bifid at apex, deciduous; ovary somewhat hairy at base, hairs white, glabrous above, 2—4-celled, 4-ovuled; ovules oblong. Berry globular, $\frac{1}{10}$—$\frac{1}{5}$ in. in diameter, purple or red, glabrous, edible, 1—2-celled, ultimately only 1-celled, 1-seeded.

Bark whitish grey rough, wood brown hard close-grained and fit for joiners' fancy-work, veneering, &c. (Dr Pappe, Silva Capensis, p. 21 [1854]).

Var. *β. myrtina.* Leaves $\frac{1}{10}$—$\frac{1}{5}$ in. wide, oblanceolate, nearly plane; fruit black; about 4 ft. high. Known only in fruit, but probably a form of this species. (*E. myrtina*, Burch.)

The fruit is sweet, with some astringency; called, as well as other species of the genus, *guarribosches*, and the fruit *guarri*, by the Hottentots in South Africa.

Cape of Good Hope, Kalahari region and Trans-Vaal. *Drège!; Reeves!; Dr Pappe!; Burke!* (Trans-Vaal); *Masson!; Alexander-Prior!; Dr Thom!* 243, 386; *Cooper!* 408; *Mac Owan!; Zeyher!* 3358; *Ecklon and Zeyher!* 1124 (E. humilis, Eckl. et Zeyh.); *Burchell!* 1792 (2162, 2573, E. myrtina, Burch., Kalahari Region), 2943, 3168, 7198.

Excluded species of Euclea.

Euclea herbacea, Lour. Fl. Cochinch. p. 629 (1790). Cfr. Euphorbiaceæ.

Euclea pilosa, Lour. *loc. cit.* = *Diospyros pilosa,* Alph. DC.

III. MABA, J. R. et G. Forster, Characteres Generum Plantarum, p. 121. t. 61 (1776).

Flores diœci, rarissime monœci vel polygami, plerumque trimeri rarius 4—6-meri. Calyx campanulatus vel oblongus, non plicatus, lobatus vel truncatus; corolla campanulata vel tubulosa, lobis in præfloratione sinistrorse contortis.

Flos masculus; *stamina* 3—∞, *plerumque glabra rarius pilosa vel pubescentia. Ovarium abortivum.*

Flos femineus; *staminodia* 0—∞, *plerumque pauca; ovarium* 3- *vel* 6-*loculare,* 6-*ovulatum; fructus plerumque mediocris, baccatus.*

Arbores vel frutices, foliis alternis integerrimis, inflorescentiâ axillari vel rarius laterali. Alph. DC. Prodr. VIII. p. 240. n. VII. (1844).

Pisonia (sp.) Rottb. in Nye Saml. Kong. Danske Skrift. vol. II. p. 536. t. 4. f. 2 (1783)
Ehretia (sp.), Willd. Phytogr. I. p. 4. t. 2. f. 2 (1794).
Ferreola, Roxburgh, Pl. Coromandel, I. p. 35. t. 45 (1795).
Ferriola, Roxburgh, Hort. Bengal, p. 72 (1814), Fl. Ind. edit. 1832. vol. III. p. 790.
Macreightia, Alph. DC. Prodr. VIII. p. 220. n. v. (1844).
Holochilus, Dalzell in Kew Journal of Botany, IV. p. 290 (1852).
Rhipidostigma, Hasskarl, Retzia, I. p. 103 (1855).

Flowers diœcious or rarely polygamous or very rarely monœcious, usually trimerous, occasionally 4—6-merous. Calyx usually 3-fid, sometimes 4—6-fid or -partite or shortly lobed, rarely truncate and entire; more or less campanulate at least in flower, sometimes accrescent but less so than in many species of *Diospyros*, not plicate. Corolla usually 3-lobed, exceeding the calyx, campanulate or tubular; lobes contorted sinistrorsely as regarded from within. Stamens in ♂ flower 3—∞ usually about 9 and glabrous except in § *Trichanthera*, distinct or some or all united by their filaments in pairs or otherwise; anthers oblong or lanceolate-linear, dehiscing longitudinally by lateral slits; filaments inserted at base of corolla or hypogynous; staminodes in ♀ flower 0—∞, usually fewer than in ♂ flower, glabrous or hairy. Ovary in ♂ rudimentary, hairy or glabrous; in ♀ 3- or 6-celled, hairy or glabrous; style 3-lobed or styles 3; ovules 6, solitary in the cells or 2 together in 3-celled ovaries; rarely an ovary is 3-celled with 3 imperfect septa between the pairs of ovules not reaching the central axis of the ovary. Fruit usually globose or ovoid, glabrous or hairy, 1—6-celled and -seeded, usually not exceeding 1 in. long, baccate or dry; seeds as in the Order, in a few species with ruminated albumen. Fruiting calyx spreading or cupuliform. Trees or shrubs usually with hard wood, widely distributed in most countries where the Order is represented but absent from the Cape of Good Hope, though occurring in Natal and other parts of Africa.

Leaves always alternate simple and quite entire, smaller for the most part than in *Diospyros*, but reaching 10½ in. in length in *M. punctata*. Flowers solitary or in short cymes either axillary or very rarely lateral on the older branches.

The name is adopted from that locally used in the Friendly Islands for plants of this genus. *Maba* is also given by the natives in the vicinity of the Congo river, West tropical Africa, to the fruit of the oil-palm (*Elæis guineensis*).

MABA may be divided into the following sections, a key to which is subjoined.

Anthers glabrous or in a few species slightly hairy. Flowers trimerous or occasionally tetramerous or rarely in *M. lancea* pentamerous.
- Calyx-lobes not much imbricated.
  - Ovary densely hairy (except in *M. obovata*, R. Br.)
    - Staminodes 0. Ovary 3-celled. § 1. FERREOLA.
    - Staminodes 3—6. Ovary 6-celled. § 2. MACREIGHTIA.
  - Ovary glabrous (pubescent or nearly glabrous in *M. Seychellarum*).
    - Flowers sessile or subsessile. Ovary 3- or 6-celled. § 3. HOLOCHILUS.
    - Flowers crowded in short branched or fascicled cymes (♀ flowers solitary in *M. lamponga*). Ovary 6-celled. § 4. RHIPIDOSTIGMA.
- Calyx-lobes rounded and much imbricated so as to make the calyx appear subtruncate. § 5. BARBERIA.

Anthers pilose. Flowers 3—6-merous. Ovary hairy, 6-celled. § 6. TRICHANTHERA.

§ 1. FERREOLA.

Fruit reddish, brown, or dark-coloured.
  Fruiting calyx very small, usually not cupuliform, flat or reflexed. ♂ flowers tubular.
    Fruit subglabrous, shining. Fruiting calyx trifid.
      Fuliginous-hispid. Leaves about 1 in. long. — 1. *M. diffusa.*
      Glabrescent. Leaves 1½—4½ in. long.
        Fruit globose, ⅓ in. in diameter. Calyx glabrate. — 2. *M. Mualala.*
        Fruit ellipsoidal-oblique, ½ in. long. Calyx somewhat hairy. — 3. *M. hemicycloides.*
    Fruit tomentose. Fruiting calyx tripartite.
      Stamens 4—5. — 4. *M. acuminata.*
      Stamens 12—16.
        Leaves oblong. Stamens about 12. — 5. *M. oblongifolia.*
        Leaves oblong-ovate. Stamens 13—16. — 6. *M. ovalifolia.*
  Fruiting calyx accrescent or not very small, usually cupuliform (sometimes small in *M. buxifolia*).
    ♂ flowers subsessile in short cymes. ♀ flowers sessile or subsessile.
      Fruit hairy. ♂ flowers with tubular corolla.
        Stamens 3—6 (-7). Flowers trimerous.
          Leaves cordate at base, subsessile. — 7. *M. foliosa.*
          Leaves not cordate at base, shortly petiolate. — 8. *M. rufa.*
        Stamens 9 (in trimerous flowers, 4 in a tetramerous one).
          Leaves glabrous, elliptical. — 9. *M. laurina.*
          Leaves hairy, lanceolate-oblong. — 10. *M. nigrescens.*
      Fruit subglabrate. Corolla campanulate.
        Leaves without conspicuous net-veins.
          Calyx hairy. Leaves usually more than 1½ in. long.
            Branches nigro-verrucose. Stamens 15—17. — 11. *M. sandwicensis.*
            Branches smooth. Stamens 6—12.
              ♂ flowers 1—3 together. — 12. *M. buxifolia.*
              ♂ flowers several together.
                Leaves lanceolate, paler beneath. — 13. *M. lancea.*
                Leaves obovate, of same colour on both sides.
                  ♀ flowers solitary. — 14. *M. obovata.*
                  ♀ flowers 3 together. — 15. *M. geminata.*
          Calyx glabrous, at least in fruit. Leaves about 1 in. long. — 16. *M. humilis.*
        Leaves highly reticular.
          Bracts not much imbricated.
            Leaves elliptic-oblong, about 2 in. long. — 17. *M. reticulata.*
            Leaves elliptical, 3—4 in. long. — 18. *M. compacta.*
          Bracts much imbricated. — 19. *M. Hillebrandii.*
    ♂ flowers pedicelled in manifest cymes. ♂ corolla tubular. ♀ flowers stalked.
      Stamens 3—6. Leaves oval, obtuse, glabrescent. — 20. *M. elliptica.*
      Stamens 9. Leaves ovate-oblong, acuminate at apex, hairy. — 21. *M. sumatrana.*
Fruit covered with white efflorescence. — 22. *M. Vieillardi.*

Cfr. [23. *M. major.*
24. *M. Andersoni.*]

§ 2. MACREIGHTIA.

Glaucescent. (N. America and West Indian Islands.)

- Leaves rotund or ovate, spinulose-apiculate. 25. *M. Grisebachii.*
- Leaves obovate, not spinulose-apiculate.
  - Albumen deeply ruminated. Net-veins conspicuous. 26. *M. caribæa.*
  - Albumen not ruminated. Veins few. 27. *M. intricata.*

Dull or hairy. (S. America, Mexico, and West tropical Africa.)

- Albumen not ruminated. Stamens glabrous. Flowers campanulate. Fruiting calyx somewhat cupuliform, not very small. (S. America and Mexico.)
  - Cymes usually 3-flowered, $\frac{1}{20}$—$\frac{1}{3}$ in. long.
    - Leaves whitish beneath. 28. *M. albens.*
    - Leaves not whitish beneath.
      - Leaves oval or obovate. 29. *M. inconstans.*
      - Leaves obovate-lanceolate, membranous. 30. *M. acapulcensis.*
      - Leaves lanceolate-oblong, coriaceous. 31. *M. salicifolia.*
  - ♂. Cymes many flowered, $\frac{1}{2}$ in. long, ♀ flowers solitary. 32. *M. Pavonii.*
- Albumen ruminated. Stamens somewhat hairy. ♂ flowers tubular; fruiting calyx very small, flat. (Africa.) 33. *M. Mannii.*

§ 3. HOLOCHILUS.

Flowers campanulate or with short tube. (Africa.)

- Ovary 3-celled; cells 2-ovuled.
  - Calyx shortly 3-lobed, pubescent.
    - Ovary more or less hairy. Leaves narrowly elliptical, obtuse. 34. *M. Seychellarum.*
    - Ovary quite glabrous. Leaves lanceolate, acute. 35. *M. lanceolata.*
  - Calyx truncate entire, glabrous. 36. *M. natalensis.*
- Ovary 6-celled; cells 1-ovuled.
  - Leaves lanceolate-oblong; flowers several together. Branches dark. 37. *M. abyssinica.*
  - Leaves oval. ♀ flowers 3 together. Branches argenteo-cinereous. 38. *M. quiloënsis.*

Flowers tubular. (India.) 39. *M. micrantha.*

§ 4. RHIPIDOSTIGMA.

- Diœcious. Stamens 8—18.
  - Glabrous. Leaves not cordate at base.
    - ♂. Cymes rather lax.
      - ♀. Flowers solitary. Corolla-lobes acuminate. — 40. *M. lamponga.*
      - ♀. Flowers cymose. Corolla-lobes not acuminate.
        - Leaves submembranous. Stamens glabrous. — 41. *M. merguensis.*
        - Leaves coriaceous. Filaments often minutely ciliated. Albumen not ruminated. — 42. *M. fasciculosa.*
        - Leaves coriaceous. Albumen ruminated. — 43. *M. ruminata.*
    - ♂. Cymes dense. — 44. *M. confertiflora.*
  - Pubescent. Leaves cordate or subcordate at base. — 45. *M. punctata.*
- "Hermaphrodite. Stamens 4—5."
  - Cymes about ½ in. long. Leaves oblong. — 46. *M. Teijsmanni.*
  - Cymes very short. Leaves oblong-lanceolate. — 47. *M. hermaphroditica.*

Cfr. [48. *M. javanica.*]

§ 5. BARBERIA.

- Ovary glabrous.
  - Staminodes about 8. ♀ Cymes 3—5-flowered. — 49. *M. Maingayi.*
  - Staminodes about 16. Flowers subsolitary. — 50. *M. Motleyi.*
- Ovary shortly pubescent.
  - Glabrous. Leaves 2—4 in. long, more or less narrowed at base; petioles ⅙—⅖ in. long. — 51. *M. myrmecocalyx.*
  - Leaves ¾—2 in. long, rounded at base, with shortly tomentose midrib; petioles ⅛ in. long. — 52. *M. Beccarii.*

§ 6. TRICHANTHERA.

- Polygamous. Ovary ovoid-conical. Leaves appressedly flavo-sericeous beneath, not cordate. — 53. *M. sericea.*
- Ovary globose at base, narrowly conical above. Leaves supra-cordate — 54. *M. cordata.*
- Diœcious. Ovary subglobose. Leaves not flavo-sericeous beneath.
  - Flowers 3- (rarely 5-) merous.
    - Shoots with spreading hairs. — 55. *M. myrmecocarpa.*
    - Shoots with appressed hairs. — 56. *M. myristicoides.*
  - Flowers 5—6-merous.
    - Flowers arising from the old wood. — 57. *M. cauliflora.*
    - Flowers axillary from the young branches.
      - Staminodes 11—13, somewhat pilose — 58. *M. Hilairei.*
      - Staminodes 25—30, nearly glabrous. — 59. *M. Mellinoni.*

1. Maba diffusa, sp. nov.

M. ramulis patentibus, fuligineo-pubescentibus; foliis ovatis vel ovalibus, uncialibus, apice obtusè angustatis, basi subrotundis, glabris, nitentibus, subcoriaceis, breviter petiolatis; fructibus ellipsoideis, appressè subsericeis, trilocularibus, 1—2-spermis, brevissime pedunculatis; calyce fructifero minimo, trifido, non appresso.

Stem and branches terete, dark or cinereous; branches fuliginous-pubescent, patent, slender. Leaves ovate or oval, nearly rounded at base, obtusely narrowed at apex, glabrous and shining on both sides, subcoriaceous, margins thickened, somewhat wavy; midrib depressed on upper side; lateral veins patent very numerous and delicately raised on both sides; of a rich brown colour on both sides when dry; $\frac{1}{2}$ to 1 in. in length by $\frac{1}{3}$ to $\frac{2}{3}$ in. in width; petioles $\frac{1}{20}$ in. in length, pubescent. Known only in fruit. Fruit shortly pedunculate, near ends of branches, solitary; fruiting peduncle $\frac{1}{20}$—$\frac{1}{10}$ in. in length. Fruiting calyx loosely concave or horizontal, small, somewhat pubescent, about $\frac{1}{10}$ in. in length, roundedly 3-fid. Fruit somewhat appressedly silky, of rich brown colour, straight, ellipsoidal, $\frac{1}{3}$ to $\frac{1}{2}$ in. in height by $\frac{1}{4}$ to $\frac{1}{3}$ in. or more in thickness, 3-celled, 1—2-seeded; seeds black, $\frac{1}{4}$ in. in length; albumen not ruminated; embryo nearly flat.

N.W. Madagascar, *Pervillé!*

2. Maba Mualala, Welw. MSS.

M. glabra, foliis ellipticis, apice sæpius obtusè acuminatis, basi leviter angustatis vel sub-rotundis, tenuiter coriaceis, persistentibus, nitentibus, reticulatis, breviter petiolatis; fructibus solitariis vel binis, subsessilibus, globosis, glabris; calyce fructifero trifido, minimo, patente, glabrato.

A fine glabrous tree, 15—35 feet high in the interior of the country, or near the sea-coast scarcely more than a bush 3—5 feet high; very rarely flowering. Trunk strict; branches terete, leafy. Wood very hard, valuable, black in the centre but not always so. Leaves alternate, elliptical, in most cases obtusely acuminate, slightly narrowed at base or nearly rounded, thinly coriaceous, evergreen, deep green, highly polished, $1\frac{1}{2}$—$4\frac{1}{2}$ in. long by $\frac{4}{5}$—$1\frac{3}{4}$ in. wide, delicately reticulated; midrib depressed above; margins slightly undulated; petioles $\frac{1}{12}$—$\frac{1}{8}$ in. long. Flowers unknown, ♀ axillary, in very short 1—3-flowered cymes. Fruit solitary or two together, subsessile, globose, shining, glabrous, black-purplish, slightly nerved, about $\frac{1}{3}$ in. in diameter, 1-seeded; seed globose, nearly $\frac{1}{4}$ in. in diameter; albumen white, cartilaginous, not ruminated; fruiting calyx 3-fid, spreading, $\frac{1}{4}$—$\frac{1}{3}$ in. in diameter, glabrate; lobes ovate, subacute.

West tropical Africa, Distr. Golungo Alto, in dense woods, fruits in March, *Dr Welwitsch!* 2539, 2540, 2541; Do. Distr. Loanda, very rare, in thickets, *Dr Welwitsch!* 2542; native name *Mualála.*

3. Maba hemicycloides, F. Muell. ex. Benth. Fl. Austr. iv. p. 290. n. 3 (1869).

M. glabrescens, foliis ellipticis vel oblongis, utrinque plus minus angustatis, apice obtusis, subcoriaceis, breviter petiolatis; fructibus solitariis, brevissime pedunculatis, subglabris, obliquè ellipsoideis; calyce fructifero minimo, patente, trifido, leviter pubescente.

A small tree; branchlets slender, somewhat hirsute with dark hairs at extremities, quickly glabrescent, dark cinereous or brown, terete. Leaves elliptical or oblong, narrowed more or less at both ends, usually with an obtuse apex, thinly coriaceous, glabrous; margins with small undulations, just reflexed; midrib depressed above; lateral veins delicate, numerous, raised on both sides, at 60° to 70°; $2\frac{1}{4}$ to $4\frac{1}{2}$ in. in length by 1—$1\frac{4}{5}$ in. in width; petioles $\frac{1}{7}$—$\frac{1}{5}$ in. in length. Known only in fruit; fruiting peduncle $\frac{1}{20}$—$\frac{1}{10}$ in. in length, not thick, pubescent; fruit solitary, near ends of branches, glabrous or nearly so, pale brown, oblique, ellipsoidal, about $\frac{1}{2}$ in. in height by $\frac{2}{5}$—$\frac{9}{20}$ in. in thickness, tipped somewhat laterally with remains of style; fruiting calyx small, horizontal, 3-fid, $\frac{3}{20}$—$\frac{1}{5}$ in. in diameter, covered with scattered appressed short pale hairs; lobes deltoid.

Australia, Queensland, Rockingham Bay, *Dallachy!*

4. MABA ACUMINATA.

M. foliis ellipticis valde acuminatis, basi rotundatis vel parum angustatis, submembranaceis, breviter petiolatis; corollæ tubo quam calyce duplo longiore; staminibus 4—5; fructibus globosis, tomentosis et sparse pilosis; calyce fructifero tripartito, minimo.

Macreightia acuminata, Thw. Enum. Ceyl. Pl. p. 424. n. 3 (1864).

A moderate sized tree with terete erect-patent branches. Young parts pale brown sericeo-pubescent, afterwards becoming dark and glabrous. Leaves elliptical, long-acuminate, rounded or nearly so at base, in the dry state pale greenish glabrous and shining on upper side with scarcely raised veins, pale brown sericeous or subpubescent on under-side with raised clear lateral veins anastomosing near margin and sericeous prominent midrib, submembranous, shortly petiolate, 2—5 in. in length by $\frac{3}{4}$ to $1\frac{1}{2}$ in. in width; petioles $\frac{1}{10}$—$\frac{1}{7}$ in. in length, pubescent. Bracts imbricated, sericeous.

♂. Tube of the corolla twice as long as the calyx, $\frac{1}{3}$ in. in length; stamens 4—5; ovary pilose.

♀. Fruit globular, pale brown, appressedly sub-tomentose-pubescent, $\frac{2}{5}$—$\frac{3}{5}$ in. in diameter; fruiting calyx not auricled.

Ceylon, *Thwaites!* C.P. 3718.

5. MABA OBLONGIFOLIA.

M. foliis oblongis, acuminatis, subcoriaceis, basi rotundatis, subtus secus nervos cum petiolo brevi sub-ferrugineo-hispidis, denique glabris; floribus masculis solitariis crebris subsessilibus, calyce breviter lobato, staminibus 12 glabris; floribus femineis solitariis breviter pedunculatis, calyce tripartito, hispido, non accrescente, staminodiis 0, fructibus subglobosis tomentosis.

Macreightia oblongifolia, Thw. Enum. Ceyl. Pl. p. 183. n. 1. (1860), p. 423 (1864); non *Marcreightia oblongifolia,* Kurz.

A small tree; young parts very hispid, subferruginous; branches terete, quickly turning dark and glabrous, spreading at about 40°. Leaves oblong, acuminate, subcoriaceous; upper side brown (often of a rich deep colour) shining and glabrous when dry, midrib and lateral veins depressed; under-side palish brown, subpubescent, lateral anastomosing veins and especially midrib raised prominent and pubescent; 3 to $7\frac{1}{2}$ in. in length by $1\frac{1}{2}$ to 3 in. in width; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. in length, glabrescent, at first hispid.

♂. Flowers subsessile solitary crowded on short axillary densely pubescent branches; buds oblong, subferruginous, sericeous-pubescent, about $\frac{1}{5}$ in. in length. Calyx $\frac{1}{4}$ in. long, 3-lobed at apex. Stamens 12, glabrous, in several rows, unequal, partly hypogynous and partly at base of interior of tube of corolla. Ovary minute, hairy.

♀. Flowers solitary, ferruginous, shortly pedunculate, hispid; bracts imbricated, large, hispid; peduncle $\frac{1}{10}$ in. in length, hispid. Flowers $\frac{3}{10}$ in. in length (not expanded in specimens), ovoid-oblong. Calyx $\frac{1}{7}$ in. in length, with 3 deep diverging ovate acute lobes. Corolla 3-fid, glabrous inside. Stamens 0. Ovary covered with light ferruginous vertical hairs, 3-celled or, according to Dr Thwaites, 6-celled. Style divided at apex into 3 glabrous stigmas.

Fruit subglobose, ferruginous-tomentose, 1 in. in diameter, fruiting calyx not accrescent nor auricled; 2- or 3-seeded; seeds black, glabrous, about $\frac{1}{2}$ in. in length by $\frac{1}{3}$ in. in thickness, bounded by 2 plane contiguous sides and a curved surface, a horizontal section being a sector of a circle; a reddish raised line runs down middle part of outer surface of the seed; albumen not ruminated; radicle cylindrical, half as long again as the oblong cotyledons.

Ceylon, *Thwaites!* C.P. 3396.

6. MABA OVALIFOLIA.

M. foliis oblongo-ovatis, parum acuminatis, obtusiusculis, basi sæpius rotundatis, subcoriaceis, glabrescentibus, breviter petiolatis; floribus masculis solitariis, crebris, calyce inæqualiter tridentato, (corollâ 4-fidâ), staminibus 13—16, *glabris, ovarii rudimento hirsuto.*

Macreightia ovalifolia, Thw. Enum. Ceyl. pl. p. 424. n. 2 (1864).

Tree of moderate size; young parts pubescent, soon glabrescent and cinereous; branches terete, erect-patent. Leaves oblong-ovate, shortly acuminate, subcoriaceous, usually rounded at base, brown on both sides when dry, darker above, glabrescent, flat, margins just recurved, patent, shortly petiolate, midrib and lateral anastomosing veins raised beneath depressed above, 2 to $3\frac{1}{2}$ in. in length by 1 to $1\frac{3}{4}$ in. in width; petioles $\frac{1}{7}$ in. in length, stout. Bracts imbricated, large, caducous.

♂. Flowers solitary, crowded on young short branchlets, ferruginous sericeous, $\frac{7}{20}$ in. in length before expansion, oblong. Calyx $\frac{1}{4}$ in. in length, tubular, with 3 short acute teeth chiefly on one side, a deeper division being opposite. Corolla often bent sideways (closed in specimens), somewhat constricted about the middle, 4-fid, dark and glabrous inside. Stamens 13—16 (14 in one case examined), unequal, glabrous; ovary rudimentary, represented by a bunch of hairs.

Ceylon, *Thwaites!* C.P. 3717.

7. MABA FOLIOSA, Rich. ex Asa Gray in Proceedings of the American Academy of Arts and Sciences, Vol. V. p. 326 (1862).

M. foliis ovalibus vel ovatis, basi cordatis, coriaceis, confertis, subsessilibus; floribus masculis 3—5-*nis, brevissime cymosis, calyce campanulato-oblongo, breviter trifido, corollâ breviter trifidâ, staminibus* 3, *glabris; floribus femineis subsessilibus, cymis* 1—3-*floris, fructibus ferrugineo-tomentosis.*

Young parts rufous or fuliginous, hirsute; branches glabrescent, cinereous; leaves crowded, subsessile, oval or ovate, cordate at base, midrib depressed above, veins indistinct, coriaceous, rufous-hirsute when young, glabrescent except on margins and midrib beneath, 1—2½ in. long by ⅚—1 in. wide; petioles shorter than the emargination at the base of the leaves.

♂. Flowers on very short nodose pubescent 3—5-flowered cymes; flowers (in bud) ovoid-oblong, rufous-hirsute. Calyx ⅙ in. long, campanulate-oblong, shortly 3-fid, smooth inside; lobes deltoid; corolla shortly 3-fid, hirsute outside, glabrous inside; stamens 3, hypogynous, glabrous; filaments distinct; anthers linear, dehiscing laterally by longitudinal slits; ovary pubescent, small, rudimentary.

♀. Fruiting peduncles 1—3-flowered; calyx 3-lobed; fruit ferruginous-tomentose.

Feejee Islands, *Wilkes!*; New Caledonia, *Pancher!* 301; Muthuata and Ovolau, alt. 2000 feet, Feejee Islands, *Asa Gray, l. c.*

8. MABA RUFA, Labill. Sert. Austr. Caled. p. 33. t. 36 (1824).

M. foliis ovalibus vel oblongis, apice lanceolatis vel breviter et obtuse acuminatis vel rotundatis, basi angustatis vel rotundatis, junioribus utrinque rufo-sericeis, sæpius glabrescentibus, coriaceis, breviter petiolatis; inflorescentiâ et fructibus rufo-sericeis; floribus masculis 3—5-nis, brevissime cymosis, axillaribus, trimeris, corollâ tubulosâ, staminibus 3—6, glabris; floribus femineis solitariis, subsessilibus, staminodiis 0, ovario dense sericeo 3-loculari, fructibus subglobosis vel ellipsoideis, calyce fructifero cupuliformi.

Alph. DC. Prodr. VIII. p. 241. n. 10 (1844).

M. sericocarpa, F. Muell. Fragm. V. p. 164 (1866), Benth. Fl. Austral. IV. p. 289, n. 2 (1869).

M. cupulosa, F. Muell. Fragm. V. p. 164 (1866), VI. p. 253 (1868).

Diospyros sericocarpa, F. Muell. Austr. Veg. in Intercol. Exh. Ess., 1866—67, p. 35 (1867).

D. cupulosa, F. Muell. *l. c.*

M. revoluta, Vieill. MSS. in Hb. N. Caled. n. 2876.

A shrub or tree 20 feet high; branches terete, slender, spreading at about 45°—50°, rufous-sericeous when young, leafy. Leaves oval or oblong, lanceolate or shortly and obtusely acuminate or rounded at apex, narrowed or rounded at base, coriaceous, appressedly rufous-sericeous when young, usually glabrescent, 1—4¼ in. long by ⅔—2⅔ in. wide; midrib depressed on the upper surface, margins recurved (sides revolute in *M. revoluta*, Vieill.); petioles $\frac{1}{10}$—¼ in. long.

♂. Inflorescence rufous-sericeous, axillary on young branches; cymes 3—5-flowered; common peduncle ⅐ in. long; pedicels very short; flowers ovoid-oblong, ¼—½ in. long. Calyx tubular, shortly 3-lobed, ¼—⅓ in. long, crass, tomentose on both sides. Corolla tubular, shortly 3-lobed, sericeous outside, glabrous inside; lobes ovate. Stamens 3—6 (-7), glabrous, hypogynous; filaments slender. Ovary rudimentary, pilose.

♀. Flowers solitary, subsessile, about ⅓ in. long, ferruginous-hairy; bracts imbricated, caducous. Calyx campanulate, shortly 3-fid. Corolla tubular, 3-lobed at apex, with rounded imbricated lobes. Staminodes 0. Ovary 3-celled, densely sericeous; style 3-lobed at apex.

Fruit ellipsoidal or subglobose, ½—1 in. high, more or less sericeous, 3-celled, 1—3—4-seeded; fruiting calyx accrescent, cupuliform, trifid, reaching half way up fruit or higher, pubescent. Seeds oblong; albumen cartilaginous, not ruminated; embryo nearly straight.

Australia, Queensland, Rockingham Bay, *Dallachy!*;

New Caledonia, *Deplanche!* 312, 446; *Labillardiere!*; *Pancher!*; *Caldwell!*; *Vieillard!* 891, 892, 894, 895, 896, 2872 (?), 2876, 2880.

9. MABA LAURINA, R. Br. Prodr. Fl. Nov. Holl. p. 527. n. 1 (1810).

M. foliis ellipticis vel oblongis, apice rotundatis, glabris, nitentibus, tenuiter coriaceis, petiolatis; floribus trimeris, subsessilibus, calyce late campanulato, crasso, corolla tubulosá, staminibus 9, glabris; in floribus femineis staminodiis 0, ovario 3-loculari, subglabro, dense sericeo.

Alph. DC. Prodr. VIII. p. 241. n. 3 (1844), Benth. Fl. Austral. iv. p. 289 n. 1 (1869).

A small tree with smooth dark bark and quite glabrous shoots; buds and inflorescence rufous-hairy. Leaves elliptical or oblong, rounded at apex, thinly coriaceous, glabrous, shining especially above, 3—5 in. long by 1½—2½ in. wide, margins incrassato-recurved, veins slender, raised on both sides; petioles ⅛ in. long.

♂. Flowers few together subsessile (ex Benth. *l. c.*) solitary or sometimes 2 together very shortly peduncled (ex R. Br. MSS.), trimerous; calyx ¼ in. long, globose-campanulate, coriaceous, rather crass, with numerous soft subappressed cinereous-ferruginous hairs outside, glabrous inside; corolla yellowish white, tube cylindrical, twice the length of the calyx, hairy outside above the calyx, lobes rounded, one third the length of the corolla; stamens 9, glabrous, hypogynous, alternately in pairs and single, equal, pollen white; ovary subglobose, hairy, rudimentary (?); style and stigma wanting.

♀. Flowers solitary, subsessile, trimerous, rufous-tomentose, scarcely ½ in. long by ¼ in. thick; calyx ¼ in. long, semi-ellipsoidal, crass, appressedly hairy inside, shortly 3-fid, lobes obtuse; corolla urceolate-oblong, glabrous inside, lobes short spreading obtuse; staminodes 0; style 3-lobed at apex, stigma dilated; ovary subglobose, densely sericeous, rufous, 3-celled, cells 2-ovuled.

Cumberland Islands, Australia, *R. Brown!*, Oct. 17, 1802.

10. MABA NIGRESCENS, Dalz. in Dalz. et Gibs. Bomb. Fl. p. 142 (1861).

M. foliis lanceolato-oblongis, sub-coriaceis, undulatis, ciliatis, breviter petiolatis, nervis inconspicuis; floribus 1—5-nis, 3—4-meris, ferrugineo-pubescentibus, subsessilibus, staminibus 9 (vel in fl. 4-meris, 4—6) glabris; in floribus femineis staminodiis 0, ovario pubescente, 3-loculari, fructibus ellipsoideis, sericeis, calyce cupuliformi.

A tree from 15 to 35 feet high with dense ferruginous pubescence on the shoots petioles and flowers; older branches dark-cinerous; branches at about 50°, rigid. Leaves lanceolate-oblong, narrowed at least at apex, sometimes nearly rounded at base, coriaceous, 1—3½ in. long (including petiole $\frac{1}{10}$—¼ in. long) by ½—1½ in. wide, midrib depressed above, hairy beneath, margins ciliate, wavy. Flowers subsessile.

♂. Flowers 1—5 together in very short cymes, ⅓ in. long, trimerous or tetramerous;

calyx $\frac{1}{6}$ in. long, 3—4-lobed, lobes $\frac{1}{10}$ in. deep, deltoid acute; corolla campanulate-oblong, $\frac{1}{3}$ in. long, 3—4-fid, lobes spreading; stamens 9, 6 in 3 pairs and 3 distinct, or all in one row, or in tetramerous flowers 4—6, glabrous, hypogynous, anthers $\frac{1}{10}$ in. long, linear, acute, filaments slender; ovary rudimentary, hairy.

♀. Flowers 1—2 together, trimerous, $\frac{2}{7}$ in. long; calyx $\frac{1}{6}$ in. long, funnel-shaped, shortly 3-fid, lobes obtuse; corolla 3-fid, lobes somewhat spreading, rounded at apex; staminodes 0; ovary hairy, 3-celled, cells 2-ovuled. Fruit rufous, sericeous, ellipsoidal, obtuse, $\frac{1}{3}$ in. long in the specimens, often with the remains of the corolla at the apex which has been pushed forward during the growth of the fruit; fruiting calyx $\frac{1}{3}$ in. wide by $\frac{1}{7}$ in. high, somewhat accrescent and cup-shaped. Flowers in July, February; fruits in May.

India, Canara, Goa, *Dalzell!;* Moollis, *Dr Ritchie!* n. 85. Pretty common in the Ghaut jungles, native name "Ruktroora." The leaves turn black in drying, and appear quite veinless. Allied to *M. guineensis* ex Dalz. and Gibs. *l.c.* I have not seen an authentically named specimen.

11. MABA SANDWICENSIS, Alph. DC. Prodr. VIII. p. 242. n. 16 (1844).

M. ramis nigricantibus verrucosis, foliis ellipticis, obtuse acuminatis, basi angustatis, nervis inconspicuis; floribus subsessilibus plerumque trimeris, corollâ campanulatâ, staminibus 15—17, *glabris; fructibus solitariis ellipsoideis vel subglobosis, glabratis, calyce paulum aucto brevi.*

M. elliptica, Seem. Fl. Vit. p. 152 (1866), non Forst.

A tree or shrub, glabrous except the young parts and inflorescence which are pubescent; branches dark-cinereous, rough, verrucose; leaves elliptical, subacute or rounded at apex, coriaceous, glabrous, petiolate, 1—2$\frac{1}{4}$ in. in length by $\frac{1}{2}$ to 1$\frac{1}{2}$ in. in. width; petioles $\frac{1}{10}$—$\frac{9}{40}$ in. in length.

♂. Flowers subsessile; calyx 3-fid with deltoid acute lobes, hairy; corolla similar; stamens 15—17, glabrous, anthers of same length as filaments.

♀. Fruit ellipsoidal or subglobose but somewhat oblique, solitary, downy or subglabrous, $\frac{1}{2}$—$\frac{3}{4}$ in. in height, reddish; fruiting calyx, cup-shaped, not or scarcely accrescent, usually with rounded lobes, very rarely 4-lobed, somewhat hairy. Fruit-peduncle patent, $\frac{1}{8}$ in. in length or shorter.

Flora Hawaiiensis, no. 124, *H. Mann and W. T. Brigham!* 1867; in woods, Sandwich Islands, *Capt. Wilkes!* U.S. South Pacific Expl. Exp.; *Gaudichaud!;* Oahu and Numan, *Dr Hillebrand!* 273, *Remy!* 473; Hawaii, *Dr Hillebrand!* 274, *Remy!* 470 (?); Fiji Islands, *Dr Seemann!* 295.

12. MABA BUXIFOLIA, Pers. Synops. Plant. ii. p. 606 n. 2 (1807).

M. foliis ellipticis vel obovatis vel lanceolatis, apice obtusis, basi angustatis, coriaceis vel submembranaceis, glabris, breviter petiolatis; floribus 1—3-*nis subsessilibus trimeris pubescentibus, cymis brevissimis, calyce corollâque breviter trifidis, staminibus* 6—12 *glabris; in floribus femineis staminodiis* 0, *ovario hirsuto,* 3-*loculari; fructibus globosis vel ellipsoideis, glabratis, monospermis; albumine non ruminato.*

Wight, Ic. pl. Ind. Or. vol. iii. pt. i. p. 4. t. 763 (1843), Alph. DC. Prodr. VIII. p. 240. n. 2 (1844), Thw. En. Ceyl. pl. p. 183 (1860).

HIGHULHAENDA, Herm. Mus. Zeyl. p. 21 (1717).

Pisonia (?) *buxifolia*, Rottb. in Nye Saml. Kong. Danske Skrift. vol. II. p. 536. t. 4. f. 2 (1783).

Ehretia ferrea, Willd. Phytogr. I. p. 4. t. 2. f. 2 (1794).

Ferreola buxifolia, Roxb. Coromand. vol. I. p. 35. t. 45 (1795), Juss. in Ann. Mus. V. p. 418 (1804), Corréa de Serra in Ann. Mus. VIII. p. 399. t. 65. f. 2 (1806).

Maba littorea, R. Br. Prodr. Fl. Austral. p. 527. n. 5 (1810) [Mr Bentham unites this with *M. geminata*, R. Br.].

Ferriola buxifolia, Roxb. Hort. Bengal. p. 72 (1814).

Ferreola guineensis, Schum. Plant. Guin. p. 448 (1827), in Kong. Danske Vid. Selsk. iv. p. 222 (1829).

Maba Cumingiana, Alph. DC. Prodr. VIII. p. 241. n. 4 (1844).

M. madagascariensis, Alph. DC. l.c. n. 7.

M. guineensis, Alph. DC. l.c. n. 8.

M. Smeathmanni, Alph. DC. l.c. n. 9.

(?) *M. vacciniæfolia*, Benth. in Hook. Niger Fl. p. 442 (1849).

M. neilgherrensis, Wight, Ic. pl. Ind. Or. (iv.) nn. 1228—9 (1850), Illust. Ind. Bot. ii. p. 147. t. 148 bis E. (1850).

M. Ebenus, Wight. l.c. tt. 1228—9 (1850), non Spreng.

M. angustifolia, Miq. ex Thw. En. Ceyl. pl. p. 183 (1860).

A shrub or tree; young parts pubescent, glabrescent; branches terete, spreading at 35°—60°. Leaves elliptical obovate or lanceolate, obtuse at apex, more or less narrowed at base, coriaceous or submembranous, $\frac{1}{4}$—5 in. long by $\frac{1}{8}$—2 in. wide, margins usually thickened or reflexed and often undulated, veins inconspicuous, petioles $\frac{1}{20}$—$\frac{1}{4}$ in. long, sometimes hairy. Flowers subsessile, trimerous, pubescent, about $\frac{1}{5}$ in. long, 1—3 together, in very short axillary cymes, on the young branches. Calyx $\frac{1}{15}$ in. long, campanulate, with short deltoid lobes. Corolla campanulate-oblong, shortly 3-fid, lobes elliptical. Stamens 6—12 in male flower, 0 in female, hypogynous, glabrous; ovary rudimentary and hairy in male flower, 3-celled in female flower, style 3-lobed at apex. Fruit globose or ellipsoidal, glabrate, $\frac{1}{4}$—$\frac{1}{3}$ in. thick; fruiting calyx cupuliform, shorter than the fruit; seeds solitary; albumen white, cartilaginous, not ruminated.

Dr Thwaites, who has seen growing in Ceylon many forms of this polymorphic and widely distributed species, gives the following varieties:

Var. β. *microphylla*, foliis parvulis.

Var. γ. *Ebenus*, foliis majoribus membranaceis parum acuminatis vel retusis sæpe suborbiculatis.

Var. δ. *angustifolia*, foliis lanceolatis vel lineari-lanceolatis, obtusis.

Dr Thwaites *l.c.* adds: "I have devoted much time to the examination of the several very different-looking varieties of this plant, expecting to discover some sufficiently important constant characters to enable me to separate them specifically, but I find them so completely connected together by intermediate forms that I have no hesitation in considering them all as representing only one very variable species; variable it may truly be called, since the leaves in var. β. are sometimes not a quarter of an inch in length, whilst in var. δ. they reach to five inches in length."

East Indies, *Wallich!* list 4145, 7461, 7535; *Dr Wight!* 1729, 1730, 1731; *Koenig!; Perrottet!; Dr Abel!;* Malacca, *Chr. Smith!* 99, *Dr Maingay!* 979; *Helfer and Griffith!* 3641; Ceylon, *Walker!* 263, *Dr Thwaites!* 477, 1916, 1917, 3395; Philippine Islands, *Cuming!* 1694; Sooloo I., *Wilkes!*

New Caledonia, *Pancher!* 249, *Vieillard!* 2864, 2873, 2877 (?).

Australia, North Coast Bay, *R. Brown!*

Madagascar, *Gerard!* 28, *Bernier!* 112, *Pervillé!* 700.

Tropical Africa, Congo, *Chr. Smith!, Dr Welwitsch!* 2527; Sierra Leone, *Smeathmann!;* I. St Thomé, *Don!* (?); Guinea, *Leprieur!*

In Ceylon it is called *Kaloo-habaraleya-gass,* in Godaveri forests *Nella maddi,* and in Madagascar *Cacason mainti.*

The following specimens seem to me to belong to this widely-spread and variable species; namely, a plant in fruit from the Isle of Pines, Loyalty Islands, Oceania, collected by Sir E. Home (1853, Hb. Mus. Brit.) and Milne, n. 12 (1853, in Hb. Kew.); and a plant with subsessile ♂ flowers and fruit from the Fiji Islands collected by J. Storck, n. 898 in 1860, which Dr B. Seemann in Fl. Vit. p. 152 (1866) refers to *M. elliptica,* Forst. var. *glabrescens.*

A specimen stated to have been brought from the Straits of Magellan (but probably by mistake) in Herb. Commerson in fruit seems also to belong to this species.

According to Dr Roxburgh, this species among the mountains of the Coromandel coast of India grows to a small tree, but in the low countries it is only a shrub; it flowers during the hot season; the berries when ripe are there universally eaten and are very well tasted; the wood is dark-coloured, remarkably hard and durable, and when its size will allow it is employed for such uses as require the most durable and heavy wood.

13. MABA LANCEA, sp. nov.

M. foliis lanceolato-oblongis, apice acutè acuminatis, basi angustatis, subglabris, subtus pallidis, supra nervis inconspicuis, petiolatis; floribus masculis subsessilibus, dense cymosis, trimeris rarius pentameris, staminibus 5—6 (?), antheris basi pubescentibus, ovario 0.

Young parts and inflorescence puberulous; branches straight, terete, dark, spreading at about 50°. Leaves lanceolate-oblong, alternate, firmly submembranous, opaque, acutely acuminate at apex, somewhat narrowed at base, nearly glabrous except the veins beneath, dark green on upper side, pale beneath, with veins inconspicuous on upper side; 3—4 in. long by 1 in. or rather more wide; petioles $\frac{1}{10}$—$\frac{1}{8}$ in. long.

♂. Flowers small, several together, crowded on very short ferruginous-hairy axillary cymes, ferruginous hairy (closed in the specimen); bracts rounded; calyx openly campanulate, $\frac{1}{14}$ in. long, deeply 3-fid; with ovate acute lobes pubescent on both sides; corolla (closed) $\frac{1}{12}$ in. long, ovoid-conical, covered outside with pale ferruginous shining hairs, 3?-lobed, glabrous inside; stamens 5—6 (?), hypogynous, erect, anthers subsessile, hairy towards the base, subulate; ovary 0. Occasionally a calyx is pentamerous.

Africa, Sierra Leone, *Smeathman!*

14. MABA OBOVATA, R. Br. Prodr. Fl. Austr. p. 527. n. 2 (1810).

M. foliis obovatis, apice rotundatis vel retusis, basi cuneatis, breviter petiolatis, nervis inconspicuis; floribus masculis 3—7-nis, trimeris vel rarius tetrameris, brevissime cymosis, campanulatis, staminibus 6—12, sæpius 9, ovarii rudimento villoso; floribus femineis solitariis subsessilibus trimeris, staminodiis 0, ovario glabro triloculari.

Alph. DC. Prodr. VIII. p. 241. n. 5 (1844); Ettingsh. Blatt-skel. dikot. p. 90. t. 29. f. 6. t. 32. figs. 1, 2 (1861).

Young parts appressedly pubescent; branches terete, smooth. Leaves obovate, usually retuse or rounded at apex, cuneate at base, thinly coriaceous, about $1\frac{1}{2}$ in. long by 1 in. wide, veins inconspicuous, margins undulated, scarcely recurved, of same colour on both sides; petioles $\frac{1}{10}$ in. long.

♂. Flowers campanulate, $\frac{1}{6}$—$\frac{1}{5}$ in. long, 3—7 together, in very short axillary cymes crowded on the young shoots; calyx 3-fid or unequally 4-fid, somewhat pubescent outside, glabrous inside, lobes ovate; corolla whitish, exceeding the calyx, 3—4-fid, lobes obtuse, somewhat patent appressedly subsericeous outside; stamens 6—12, usually 9 and alternately in pairs, glabrous; pollen white; ovary rudimentary, hairy.

♀. Flowers solitary, axillary, subsessile, like ♂ but rather thicker; trimerous; staminodes 0; ovary glabrous, 3-celled, subglobose, cells 2-ovuled; style shorter than the ovary, stout, deeply 3-fid, glabrous; stigmas emarginate at apex, glabrous.

Australia, Carpentaria Islands, *R. Brown!*, flowers in November.

Mr Bentham unites this species with *M. humilis*, R. Br. The glabrous ovary in the ♀ is exceptional in this section of the genus, but the rudiment of the ♂ ovary is hairy; possibly the two sexes belong to different species, but the foliage is quite alike in both.

15. MABA GEMINATA, R. Br. Prodr. p. 527. n. 4 (1810).

M. foliis obovatis, apice subretusis vel obtusis, basi cuneatis, coriaceis, glabris, petiolatis; fructibus 1—3-nis, subsessilibus, subglabris, ellipsoideis; calyce fructifero breviter cupuliformi, trilobo, subglabro; floribus masculis 5—7-nis, subsessilibus, trimeris, campanulatis, calyce puberulo, staminibus 9, glabris.

Alph. DC. Prodr. VIII. 242. n. 13 (1844); Benth. Fl. Austr. IV. p. 291. n. 8 (1869), excl. syn.

Diospyros geminata, F. Muell. Austral. Veg. in Intercolonial Exhibition Essays, 1866—67, p. 35 (1867).

A tree, glabrous except the flowers and fruit, with a diffuse irregular head; branches terete, cinereous, smooth, spreading at 45°. Leaves obovate, coriaceous, subretuse or obtuse at apex, cuneate at base, $1\frac{1}{2}$ to 3 in. in length, by $\frac{3}{4}$ to 2 in. in width; petioles $\frac{1}{7}$—$\frac{1}{4}$ in. in length.

♂. Flowers in subsessile clusters, about 5 to 7 together, $\frac{1}{5}$ in. in length, oblong; calyx dark, with scattered short hairs, 3-lobed at apex, $\frac{1}{10}$ in. in length, lobes depresso-deltoid; corolla pale, sericeous, 3-fid. stamens 9, free, equal, glabrous, mostly hypogynous; ovary rudimentary.

♀. Flowers 3 together, subsessile; fruit 1 to 3 together, ellipsoidal, subglabrous, $\frac{2}{5}$ in. in length by $\frac{1}{4}$ in. in width, subsessile, 1—2-celled, terminated by remains of style, 1-seeded, straight, rarely 3-celled and 3-seeded; fruiting calyx, $\frac{1}{10}$—$\frac{1}{5}$ in. high, cup-shaped, with 3 broad and shallow lobes; seed $\frac{11}{40}$ in. in length by $\frac{2}{27}$ in. in thickness with a depressed longitudinal line; albumen not ruminated; radicle more than double the length of the cotyledons.

A slender tree attaining 50—60 feet in height and 9 to 12 inches in diameter, with dark scaly bark, found growing in the scrubs; wood soft and tough; fruit eaten by the natives (*Thozet*).

E. Australia, from Moreton Bay to Rockingham's Bay; Queensland, *Dallachy!*; Rodd's Bay, N. E. Australia, *A. Cunningham!* 306; Moretown Island, *Dr Mueller!*; Brisbane River, *Fraser!*, *Mueller!*; Queensland Woods, London Exhibition, 1862, no. 50, *Hill!*; Keppel Bay, Shoalwater Bay, Thirsty Sound, Broad Sound, *R. Brown!*

16. MABA HUMILIS, R. Br. Prodr. p. 527. n. 3 (1810).

M. foliis obovatis, parvis, apice rotundatis vel subretusis, basi cuneatis, coriaceis, glabris, subsessilibus; floribus masculis 3-nis, brevissime cymosis, trifidis, campanulatis, calyce subglabro, staminibus 8—9, glabris, ovarii rudimento hirsuto; floribus femineis solitariis, subsessilibus, trilobis; fructibus glabris, apice hirtellis, ellipsoideis, calyce fructifero cupuliformi, glabro.

Alph. DC. Prodr. VIII. p. 242. n. 12 (1844); Benth. Fl. Austr. IV. p. 291. n. 9 (1869); non Ettingsh. Blatt-skel. dikot. p. 90. t. 36. f. 8 (1861).

Diospyros humilis, F. Muell. Austral. Veg. in Intercolonial Exhibition Essays, 1866—67. p. 35 (1867).

An erect bush glabrous or puberulous except the flowers, 2—5 feet or sometimes 20 feet high, much branched; branches terete, subcinereous. Leaves obovate, rounded or retuse at apex, narrowed at base, coriaceous, $\frac{1}{2}$—$1\frac{1}{3}$ in. in length by $\frac{1}{4}$—$\frac{2}{3}$ in. in width; petioles $\frac{1}{5}$—$\frac{1}{10}$ in. in length; veins not conspicuous; young leaves with a few depressions on the lower surface which disappear from the older leaves.

♂. Cymes 3-flowered, $\frac{1}{27}$ in. in length; flowers not much exceeding $\frac{1}{10}$ in. in length; calyx in the dry state of a chestnut brown colour, 3-fid, subglabrous; corolla not much exceeding calyx, 3-fid, lobes straight, light-hairy outside; stamens 8 or 9, some in pairs, hypogynous, glabrous; ovary rudimentary, hairy.

♀. Flowers solitary, subsessile, $\frac{1}{5}$ in. in length, oval; calyx pubescent, subferruginous, with 3 shallow rounded lobes, turbinate; corolla not much exceeding calyx, hairy outside. Fruit solitary, 3-celled with cells 2-seeded, or 1—2 cells often abortive and seed solitary, glabrous except at apex, $\frac{3}{10}$ in. in length ellipsoidal; fruiting calyx between a half and a third of the length of fruit, cupshaped, at first with a short cylindrical base, glabrous; radicle longer than the cotyledons.

Australia, from Arnhem Land to the islands in the Gulf of Carpentaria and to the tropic in East Australia. Rockhampton, *Dallachy and O'Shanesy!*; Point Pear, *Mueller!*; Dawson River, *Mueller!*; Burnett River, *Mueller!*; Gilbert River, *Mueller!*; Cliffs on the entrance of the Victoria River, *Mueller!*; Sweers Island, *Henne!*; Broad Sound near upper head, in thickets not far from the shore, *R. Brown!*

17. MABA RETICULATA, R. Br. Prodr. p. 528. n. 6 (1810).

M. foliis oboratis vel ovalibus, apice emarginatis vel rotundatis, supra valde reticulatis, coriaceis, glabris, breviter petiolatis; floribus masculis 3—4-meris, 3—5-nis, brevissime cymosis, campanulatis, calyce subglabro, staminibus 7—14, glabris; floribus femineis solitariis, subsessilibus, corollâ 3—4-fidâ, staminodiis 0, ovario sericeo, 3-loculari; fructibus glabratis, subglobosis, calyce fructifero leviter aucto, intus breviter tomentoso, extus glabro.

Alph. DC. Prodr. VIII. p. 241. n. 6 (1844); Benth. Flora Austr. IV. p. 291. n. 7 (1869).

M. interstans, F. Muell. Fragm. bot. V. p. 163 (1866).

A shrub of 8 ft. or a tree from 20 to 30 feet in height, erect, glabrous or very quickly glabrescent, much branched; branches terete, spreading at about 45°—50°, bark cinereous, thinly rimose. Leaves oval or obovate, emarginate or rounded at apex, suddenly narrowed or rounded at base, margins often recurved, highly reticulated above, midrib depressed above, coriaceous, $1\frac{1}{2}$ to 4 in. in length by $\frac{1}{2}$ to $2\frac{1}{4}$ in. in width; petioles $\frac{1}{10}$—$\frac{2}{5}$ in. in length.

♂. Cymes 3—5-flowered, hairy, $\frac{1}{10}$—$\frac{1}{7}$ in. in length, crowded on young branches; pedicels very short, with oval ciliate caducous bract at base; flower $\frac{1}{5}$ in. in length, usually trimerous, occasionally tetramerous; calyx campanulate, dark, $\frac{1}{10}$ in. in height, with 3 or rarely 4 roundly deltoid lobes reaching about halfway down calyx, subglabrous; corolla 3—4-fid, argenteo-sericeous outside, narrowly urceolate; stamens 7—14, hypogynous, equal, glabrous, when numerous many in pairs, about $\frac{1}{6}$ in. in length; anthers about $\frac{1}{11}$ in. in length, narrow; ovary rudimentary, hairy.

♀. Flowers solitary, subsessile, thick, about $\frac{1}{5}$ in. in length; calyx 3-fid, nearly hemispherical, nearly glabrous outside; corolla urceolate, 3- or unequally 4-fid, silky; staminodes 0; style scarcely any; stigma 3-lobed; ovary globular-pointed, silky, pale, 3-celled, cells 2-ovuled; fruit globular or depresso-globular, $\frac{1}{2}$ in. thick, glabrate and shining; fruiting calyx 3-celled, 3-seeded, somewhat accrescent, finally recurved or spreading, covered inside with dense furlike hair, glabrate outside, $\frac{1}{3}$ in. across.

Australia. Cape York, Voyage of Rattlesnake, October 1848, *John Macgillivray!* 439; *Mr Daniel!* March 1868; Rockingham Bay, *Ferd. Mueller! Dallachy;* Prince of Wales and Cumberland Islands, *R. Brown!* Nov. 2, 1802, in male flower.

18. MABA COMPACTA, R. Br. Prodr. p. 528. n. 7 (1810).

M. foliis ovalibus, apice emarginatis vel rotundatis, coriaceis, glabris, reticulatis, breviter petiolatis; fructibus solitariis, subsessilibus, subglobosis, glabratis, nitentibus, 3-locularibus, 3-spermis; calyce fructifero patente vel reflexo, intus tomentoso, extus glabro.

Alph. DC. Prod. VIII. p. 242. n. 11 (1844); Benth. Fl. Austral. IV. p. 290. n. 6 (1869).

Known only in fruit; shrub 4—5 feet high, erect, branched; shoots terete, bark dark cinereous; glabrous except the inside of the spreading or recurved calyx. Leaves oval, suddenly narrowed or rounded at base, emarginate or rounded at base, coriaceous, highly reticulated, 2—4 in. long (including dark petiole $\frac{1}{4}$—$\frac{1}{2}$ in. long) by $1\frac{1}{8}$—$2\frac{1}{4}$ in. wide; midrib depressed above. Fruit subsessile, solitary, depresso-globose, yellow, about $\frac{1}{2}$ in. thick, glabrate and shining, 3-celled, 3-seeded; fruiting calyx $\frac{1}{3}$ in. across, spreading or recurved, densely covered on reflexed surface with short furlike tomentum, glabrous outside.

Differs from *Maba reticulata* by wider leaves and more spreading or reflexed not cupuli-form fruiting calyx.

Australia, North Coast Island, Feb. 18, 21, 1803, *R. Brown!*

19. MABA HILLEBRANDII, Seem. Fl. Vit. p. 151 (1866).

M. foliis oblongis vel ovato-oblongis, apice obtusis, basi rotundatis vel cordatis, glabris, tenuiter coriaceis, supra crebre reticulatis, breviter petiolatis; floribus solitariis sessilibus basi bracteatis, masculis 3-meris, femineis 3—4-meris; staminibus 9, glabris; fructibus oblongis subglabratis, calyce fructifero glabro, lobis deltoideis.

Glabrous except the inflorescence; branches dark cinereous. Leaves oblong or ovate-oblong, rounded or cordate at the base, usually obtuse at the apex, thinly coriaceous, 2—6 in. long by 1—$3\frac{1}{2}$ in. wide; veins except midrib in relief on both sides, remarkably prominent on the upper side, reticulated; petioles $\frac{1}{10}$—$\frac{3}{10}$ in. long. Flowers solitary sessile with several imbricated ciliate bracts at base.

♂. Flowers pubescent, trimerous; stamens 9, 6 in 3 pairs alternating with the corolla-lobes and 3 distinct opposite the corolla-lobes, all glabrous; ovary rudimentary, hairy.

♀. Fruit oblong, $\frac{3}{4}$ in. long by $\frac{1}{3}$—$\frac{2}{5}$ in. thick, subglabrate, somewhat oblique; fruiting calyx $\frac{1}{4}$ in. long by $\frac{2}{5}$—$\frac{1}{2}$ in. wide at apex, 3—4-fid, glabrous; lobes deltoid acute, somewhat spreading.

Sandwich Islands, Mountains, Oahu, *Dr Hillebrand!*, *Remy!* 472.

20. MABA ELLIPTICA, J. R. et G. Forst. Char. Gen. Pl. p. 122. t. 61 (1776).

M. foliis ellipticis vel oblongo-lanceolatis, apice obtusis, basi cuneatis, subcoriaceis, glabrescentibus, breviter petiolatis; cymis axillaribus, 3—8-floris, pubescentibus; floribus trimeris, campanulato-tubulosis; staminibus 3 vel 6; ovario 3-loculari, pubescente; fructibus ellipsoideis, pedunculatis, pubescentibus.

J. R. et G. Forst. Beschreib. Gatt. Pflanz. edit. Kerner, p. 127. t. XV. f. 61 (1779); Poiret in Lam. Encycl. Méth. Suppl. III. p 566. t. 803 (1813); Labill. Sert. Austro-Caled. p. 32. t. 35 (1824); Alph. DC. Prodr. VIII. p. 240. n. 1 (1844); Ettingsh. Blatt-skel. Dikot. p. 90. t. 40. f. 2 (1861); non Seem. Fl. Vit. p. 152 (1866).

Ebenus vulgaris, Rumph. Amb. Vol. III. p. 1. t. 1 (1750).

?*Ebenoxylum verum,* Lour. Fl. Cochinch. p. 613 (1790).

Maba Ebenus, Spreng. Syst. Veg. II. p. 126. n. 8 (1825), Alph. DC. *l. c.* p. 242. n. 17, Hassk. Retz. I. p. 107 (1855), non Wight.

? *Maba? ebenoxylon*, G. Don, Dict. Gard. and Bot. IV. p. 43. n. 10 (1837).

Diospyros hexasperma, Hasselt ex Hassk. Pl. Javan. p. 468. n. 353 (1848).

A shrub of 6 ft. or more or a moderate-sized tree or sometimes a lofty tree; branches slender, cinereous, terete, rather rough; shoots hairy; glabrescent; leaves elliptical or oblong-lanceolate, obtuse at apex, cuneate at base, glabrescent, subcoriaceous, $1\frac{1}{2}$—$4\frac{1}{2}$ in. long by $\frac{3}{4}$—$1\frac{3}{5}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long.

♂. Cymes longer than the petioles, $\frac{1}{5}$—$\frac{1}{3}$ in. long exclusive of the flowers, pubescent, 3—8-flowered, crowded on the young branches; common peduncle $\frac{1}{10}$—$\frac{1}{5}$ in. long; bracts linear, small, caducous; flowers trimerous, $\frac{1}{4}$ in. long, campanulate-tubular, pubescent; calyx campanulate, $\frac{1}{8}$ in. long, lobes deltoid-acute; corolla tubular, 3-fid, yellowish white, lobes acute, $\frac{1}{12}$ in. long, rather patent; stamens 3 or 6, hypogynous, glabrous, distinct; ovary rudimentary, hirsute.

♀. Cymes $\frac{1}{4}$—$\frac{1}{2}$ in. long; flowers as in ♂; staminodes 0; ovary hairy, ovoid, 3- (or according to Labillardière 4- or by abortion 2-) celled; cells 2-ovuled; style short; stigma 3 (—4)-lobed; fruit fleshy, pedunculate, crowded, greenish, ellipsoidal, scarcely 1 in. long by $\frac{1}{2}$ in. thick, pubescent or nearly glabrous, 2—3-celled; seeds triquetrous; albumen cartilaginous; plumule indistinct; fruiting calyx not accrescent, somewhat spreading, 3-fid, $\frac{1}{4}$—$\frac{1}{3}$ in. across; lobes deltoid.

Friendly Islands, *Forster!*, *Capt. Cook!*, *A. Matthews!* 144; Navigator's Islands, *Wilkes!* var. foliis acuminatis; Amboina, *Rumf*, *Teijsmann!*, *Hasskarl;* Java, *Hasselt;* Cochinchina (?), *Loureiro;* New Caledonia, *Labillardière!*, *Vieillard!* 893; "Amsterdam Insula Oceani pacifici" (= Tonga Tabboo, Friendly Islands), *J. R. and G. Forster!*. Called *Maba*, by the natives in the Friendly Islands, and *Kiharùpat* in Java. The plant called *Anúme* in Navigator's Islands (see Rev. Thomas Powell in Seemann's *Journal of Botany*, Vol. VI. p. 278, 1868) may belong to this species; it is eaten by children, and flowers in June or July and in January or February.

Difficult when young to distinguish from *M. rufa*, and approaching also *M. buxifolia*.

21. MABA SUMATRANA, Miq. Pl. Junghuhn. i. p. 204 (1851—55), Fl. Ned. Ind. vol. II. p. 1051, tab. XXXVI. B (1856).

M. foliis ovato- vel ovali-oblongis, acuminatis, basi rotundatis, costatis, subtus secus costas hirtellis; cymis masculis axillaribus, multifloris; calyce trilobo; corollâ ovoideo-tubulosâ; staminibus 9, glabris; ovarii rudimento pubescente.

A subferruginous, pubescent tree, about 30 feet in height. Branches terete. Leaves ovate- or oval-oblong, acuminate, rounded at base; margins flat, dark green, and with scattered appressed long hairs on upper face; velutinous and subferruginous, especially on veins beneath; lateral veins numerous (about 8), plain beneath; petiolate; subcoriaceous; $2\frac{1}{2}$—4 in. in length by $\frac{5}{6}$—$1\frac{1}{2}$ in. in width; petioles $\frac{1}{10}$—$\frac{1}{7}$ in. in length.

♂. Cymes pedunculate, many-flowered, $\frac{3}{4}$—1 in. in length; flower (in bud) $\frac{1}{4}$ in. in length, oblong, subferruginous, tomentose; calyx $\frac{1}{8}$ in. in length, 3-lobed at apex; corolla ovoid-tubular, with a slight constriction near middle, 3-fid; lobes cordate, sub-acute; stamens 9, 6 in 3 pairs, 3 distinct, glabrous; anthers as long as filaments; ovary rudimentary, hairy.

Sumatra, *Dr Fr. Junghuhn!* 719; in woods near Tobing, ex Miq. in Pl. Jungh. I. p. 204; Java, *De Vriese!*

Marcreightia andamanica, Kurz in Rep. Veg. Andam. I. edit. ii. p. 42 (1870), *M. oblongifolia*, Kurz *l. c.* edit. i. p. XI. (1867), is said by Mr Kurz in Journ. Asiat. Soc. Beng. vol. XL. pt. ii. p. 74 (1871) to belong to *Maba sumatrana*, Miq.; it is a dull dark green shrub, with oblong submembranous leaves 7—8 inches long by $2\frac{1}{2}$—3 in. wide, subcordate at base, and robust petioles $\frac{1}{5}$ in. long; it was collected in South Andaman by *Mr Kurz!* in which island he states that it is common.

22. MABA VIEILLARDI, sp. nov.

M. foliis obovato-ellipticis, apice rotundis vel retusis, basi cuneatis, coriaceis, glabris, undatis, breviter petiolatis; floribus masculis brevissime cymosis, monstrosis in speciminibus; floribus femineis solitariis breviter pedunculatis; fructibus glabratis, albido-pulverulentis, subglobosis, calyce trifido.

A tree of about 13 feet high; glabrous or on quite young parts slightly pubescent; branches numerous, terete, smooth; leaves oval or somewhat obovate, coriaceous, alternate, rounded or somewhat emarginate at apex, more or less narrowed at base, shining, of same metallic lustre when dry and without conspicuous veins on each side, coriaceous, 1—2 in. long by $\frac{1}{2}$—1 in. wide; petioles $\frac{1}{2}$—$\frac{1}{8}$ in. long, dark and rather stout; wavy (in the dry state) and with revolute margins.

♂. Cymes axillary on young branches, about $\frac{3}{16}$ in. long, recurved. puberulous; flowers about $\frac{1}{8}$ in. long, monstrous in the specimen (*Deplanche*, 449) by the stamens being petaloid, puberulous; calyx and corolla campanulate, about $\frac{1}{8}$ in. long, deeply 3-fid; ovary 0.

♀. Fruit solitary, on peduncles about $\frac{1}{4}$ in. long, puberulous or glabrate, subglobose, glabrous, covered with white efflorescence, nearly $\frac{1}{2}$ in. in diameter, 3-celled, 5—6-seeded; seeds about $\frac{1}{4}$ in. long; albumen scarcely ruminated, but with slight sinuous intrusion of the rather thick testa; fruiting calyx, puberulous outside, glabrous inside, not accrescent, appressed to base of r it, 3-fid, $\frac{1}{3}$ in. across.

New Caledonia, *Vieillard!* n. 897; *Deplanche!* 448 (in fruit), 449, Kanala; *Pancher!*, Iron Mountains of Kanala, 1862.

The following two species are very imperfectly known:

23. MABA ANDERSONI, Soland. MSS. in Herb. Mus. Brit., Seem. Fl. Vit. p. 152 (1866).

M. arborea, ramis cinereis glabris; foliis ellipticis, apice obtusis, basi subrotundis, petiolatis; floribus pubescentibus, subsessilibus, masculis glomeratis; fructibus solitariis.

A tree with cinereous branches, glabrous except the inflorescence, apparently diœcious. Leaves alternate, elliptical, obtuse at apex, rounded or nearly so at base, of uniform colour, with minute net-veins, $4\frac{3}{4}$—$5\frac{1}{4}$ in. long by $2\frac{1}{3}$—$3\frac{1}{4}$ in. wide; petioles about $\frac{1}{4}$ in. long.

♂ (?). Flowers subsessile, clustered several together on the young branches.

♀. Fruit solitary, subsessile, with wide articulation at base to the very short peduncle.

Tonga Islands, *Capt. Cook!*, third voyage.

Possibly identical with *M. major*, Forst. The foliage is somewhat like that of *M. compacta*, R. Br.

24. MABA MAJOR, G. Forst. Pl. Escul. Insul. Ocean. Austr. p. 54, n. 21 (1786).

M. arborea, fructibus edulibus bipollicaribus, ceterum M. ellipticæ similibus, 2—3-spermis; seminibus triquetris.

Cook, Voyage to the Pacific Ocean in 1776—80, edit. ii. p. 393 (1785); Alph. DC. Prodr. VIII. p. 242. n. 15 (1844).

A tree known only from its fruit, which is 2 in. long, "roundly oval," like that of *M. elliptica* Forst., but three times the size, tough, egg-shaped, and containing 2 or 3 triquetrous seeds in cells. The taste is insipid, but nevertheless is used by the natives of the Friendly Islands for food, and is frequently planted near their houses; they call it *Maba* or *Mabba.*

Tongatabu, Namoka, E-uwa, Hapa-i, and other of the Friendly Islands, *G. Forster, Capt. Cook.*

25. MABA GRISEBACHII.

M. glaucescens, foliis rotundato- vel ovali-ovatis, apice spinuloso-apiculatis, coriaceis, basi rotundis vel subcordatis, brevissime petiolatis, reticulatis; floribus femineis solitariis, axillaribus, brevissime pedunculatis, trimeris; corollæ lobis ovatis, acutis; staminodiis 6, glabris, uniserialibus; ovario ovoideo-conico, hirsuto, apice glabro, 6-loculari, 6-ovulato.

Macreightia buxifolia, Grisebach, Catal. Plant. Cubens. p. 169 (1866).

Pale glaucescent shining stiff (shrub?), with terete branches spreading at about 50°—60°, glabrous except the flowers. Leaves alternate, crowded, rotund, oval, or ovate, spinulose-apiculate, coriaceous, rounded or subcordate at base, shortly petiolate, average size $\frac{11}{20}$ in. long (including petiole and apiculus) by $\frac{3}{10}$ in. wide; petioles $\frac{1}{15}$ in. long by $\frac{1}{30}$ in. wide, dilatato-concave; veins reticulated, in relief on both sides, more conspicuous on under-side.

♀. Flowers solitary, crowded, in axils of upper leaves, shortly pedunculate, $\frac{2}{5}$ in. long, trimerous; peduncle equalling or slightly exceeding the petiole, hairy; calyx $\frac{1}{4}$ in. long, thickly coriaceous, covered outside with close short pale hairs and inside with denser hair except near base; lobes $\frac{1}{7}$ in. long, broadly ovate, suddenly acuminate at apex, with sides revolute and sub-auricular at base, somewhat concave within to make room for the ovary. Corolla $\frac{9}{25}$ in. long, hairy like the calyx outside except near base, glabrous inside; lobes $\frac{9}{50}$ in. long, ovate, acute, spreading; tube triangularly prismatic. Staminodes 6, $\frac{1}{10}$ in. long, glabrous, nearly equal, uniseriate, inserted near base of corolla. Ovary $\frac{1}{7}$ in. long (including style), ovoid-conical, continuous with the 3-lobed style, covered except at apex with short dense pale hair, 6-celled, cells 1-ovuled.

E. Cuba, near St Antonio, *Wright!* No. 2938.

26. MABA CARIBÆA.

M. glaucescens, foliis obovatis, apice rotundatis vel emarginatis, basi angustatis, coriaceis, glabris, reticulatis, breviter petiolatis; floribus masculis brevissime cymosis, pubescentibus, trimeris, staminibus 8; floribus femineis solitariis, sessilibus vel breviter pedunculatis, trimeris, staminodiis 3—6, ovario dense hirsuto, 6?-loculari, 6-ovulato; fructibus subglobosis, glabris, nitentibus; albumine ruminato.

Macreightia caribæa, Alph. DC. Prodr. VIII. p. 221. n. 1 (1844), non Griseb. Veg. Karab. Ins. Guadal. p. 91. n. 846 (1857, = *Casasia calophylla* Rich.).

Tree, glaucescent, glabrous except very young parts and flowers, which are pale fulvous and softly pubescent; branches making 60° with stem. Leaves obovate, rounded or emarginate at apex, somewhat narrowed at base, coriaceous, midrib depressed above, glabrous, plane but margins reflexed; net-veins very closely and clearly reticulated, raised on both sides; $1\frac{1}{2}$—3 in. in length by $\frac{2}{3}$—$1\frac{1}{3}$ in. in width, rather paler beneath; petioles $\frac{1}{10}$ in. in length.

♂. Cymes very short, usually 3-flowered, pubescent, pale fulvous; flowers narrowly oval; calyx tubular, with 3 shortly deltoid lobes at apex; corolla 3-fid; glabrous and dark inside; stamens 8, unequal; ovary rudimentary, hairy.

♀. Flowers solitary, sessile or on peduncles $\frac{1}{5}$—$\frac{2}{5}$ in. in length, pubescent, $\frac{2}{5}$—$\frac{1}{2}$ in. in height; bracts small, pubescent; calyx coriaceous, thick, with wide undulating diverging and auricled lobes; openly campanulate, deeply 3-fid, $\frac{2}{5}$ in. in width, hairy on both sides; corolla 3-fid, $\frac{2}{5}$ in. long, lobes acute, glabrous inside, hairy outside; staminodes 3—6, equal, inserted near base of corolla; ovary densely hairy, 6?-celled, 6-ovuled. According to Grisebach (Fl. Br. W. Ind. p. 404) the ovary is 3-celled, with 3 other incomplete dissepiments separating the geminate ovules. Fruit squarely subglobose, glabrous and shining, orange-coloured, about 1 in. in diameter; fruiting calyx nearly as wide, but not accrescent, horizontal; lobes with replicative sinuses; albumen deeply ruminated.

Cuba, *C. Wright!* 1331, near village called Monte Verde, E. Cuba; *Rugel*, 662; Haiti, *C. Ehrenberg!*; *Nectoux!*; Antilles!; "America meridionalis," *Richard!* in Hb. *Vahl.*

27. MABA INTRICATA.

M. glaucescens, intricato-ramosa, foliis obovatis, apice rotundatis, basi cuneatis, coriaceis, brevissime petiolatis; fructibus globosis, glabratis, uncialibus, breve pedunculatis, 6-spermis, albumine non ruminato, calyce fructifero patente, trilobo.

Macreightia intricata, A. Gray in Proceed. Amer. Acad. v. p. 163 (Jan. 1862).

Pale glaucescent (shrub?), with intricate branches spreading at 60°—80°; young parts weakly and appressedly pubescent. Leaves obovate, cuneate at base, rounded at apex, few-veined, appressedly and inconspicuously pubescent on midrib and beneath, about 1 in. long by $\frac{1}{2}$ in. wide; coriaceous; petioles very short. Fruiting peduncles arching-reflexed, $\frac{1}{4}$—$\frac{1}{3}$ in. long, tough, glabrous, solitary; fruiting calyx flat, $\frac{1}{2}$ in. in diameter, covered with very short inconspicuous and weak pale hairs, with 3 rounded lobes, $\frac{1}{5}$ in. long, reflexed at tip; fruit of bright orange colour, glabrate, globular, about 1 in. in diameter, 6-seeded; albumen not ruminated.

Lower California, Cape St Lucas, &c., *Xantus!* 68, Aug. 1859—Jan. 1860.

28. MABA ALBENS.

M. foliis obovato-oblongis, utrinque angustatis, confertis, molliter puberulis, subtus albentibus, subcoriaceis, breviter petiolatis; floribus masculis 3-nis, brevissime cymosis, 3—4—5-meris; staminibus 12—11, glabris; ovarii rudimento pubescente.

Diospyros albens, Presl, Reliq. Haenk. II. p. 62 (1835-6).

Macreightia albens, Alph. DC. Prodr. VIII. p. 221. n. 2 (1844); Ettingsh. Blatt-skel. Dikot. p. 89. t. 38. f. 11 (1861).

A shrub or tree with pallid or cinereous bark and dull leaves; branches terete, glabrescent; young parts pubescent; leaves obovate-oblong or lanceolate, more or less narrowed at both ends, crowded, softly puberulous, dull green above, paler beneath and with minute scales, subcoriaceous, midrib slightly depressed beneath, veins slender; $1\frac{1}{2}$—3 in. long by $\frac{3}{5}$—$1\frac{1}{5}$ in. wide; petiole $\frac{1}{10}$—$\frac{1}{6}$ in. long.

♂. Flowers arranged on short ($\frac{1}{20}$—$\frac{1}{8}$ in. long) pubescent 3-flowered cymes, which grow on the youngest shoots; $\frac{2}{5}$ in. long by $\frac{1}{5}$ in. wide; calyx campanulate or ovoid, $\frac{1}{4}$ in. long by $\frac{1}{5}$ in. wide, unequally 3-fid (occasionally 4—5-fid with lanceolate lobes), pubescent on both sides; lobes usually ovate; corolla shortly 3—4-lobed, urceolate-oblong, pubescent outside, glabrous inside, lobes oblique, imbricated sinistrorsely; stamens 12—11 (6 filaments, 2 together ex Presl *l. c.*) all or some inserted at the base of the corolla, glabrous; ovary rudimentary, pubescent.

Flowers in June.

Mexico, Acapulco, *Presl, Haenke!* 47; Soledad, *Dr Wawra!* 168.

29. MABA INCONSTANS, Grisebach, Fl. Brit. W. Ind. p. 404 (1864).

M. foliis oblongo-obovatis vel oblongis, apice obtusis, basi angustatis, subglabris vel subtomentosis, tenuiter reticulatis, subcoriaceis, interdum minute pellucido-punctatis, breviter petiolatis; floribus masculis breviter cymosis, 3—4-meris; staminibus 6—12, sæpius 9, inæqualibus, glabris; floribus femineis subsolitariis, 3—(4-)meris; staminodiis 3—4; ovario hirsuto, 6-loculari; fructibus solitariis, 6-locularibus, depresso-globosis, subglabratis; seminibus oblongis; albumine non ruminato.

Macreightia inconstans, Alph. DC. Prodr. VIII. p. 221. n. 6 (1844).

Diospyros inconstans, Jacq. Amer. p. 276, t. 174. f. 67 (1763).

Macreightia conduplicata, Alph. DC. Prodr. VIII. p. 221. n. 5 (1844).

Diospyros conduplicata, Kunth in Humb. et Bonpl. Nov. Gen. iii. p. 254 (1818).

Diospyros Berterii, Alph. DC. Prodr. VIII. p. 234. n. 61 (1844).

Diospyros obtusifolia, Bert. in Alph. DC. *l. c.*, non Humb. et Bonpl.

Macreightia obovata, Mart. in Fl. Bras. VII. Eben. p. 9. t. 2. f. 3 (1856).

Macreightia psidioides, Alph. DC. Prodr. VIII. p. 221. n. 4 (1844).

Diospyros psidioides, Kunth in Humb. et Bonpl. Nov. Gen. iii. p. 254 (1818).

A moderate-sized diœcious (monœcious, according to Jacquin) tree or shrub, with young parts and inflorescence fulvo- or ferruginous-pubescent, more or less glabrescent. Leaves alternate, oblong-obovate or oblong, subglabrous or subtomentose-pubescent, reticulated, subcoriaceous, somewhat narrowed at base, and more or less pointed or obtuse at apex; sometimes minutely pellucid-punctate; margins just recurved, $1\frac{3}{4}$—6 in. long, $\frac{2}{5}$—$2\frac{1}{8}$ in. wide; midrib depressed above; petioles $\frac{1}{8}$—$\frac{1}{3}$ in. long: cymes short, drooping, 3-flowered or 3-several-flowered in male plants, $\frac{1}{20}$—$\frac{1}{6}$—$\frac{1}{3}$ in. long; bracts small, caducous, acute, ovate or lanceolate or obovate-oblong.

♂. Flowers $\frac{3}{10}$—$\frac{2}{5}$ in. long, 3—4- (usually 3-) -merous, campanulate-oblong; calyx $\frac{1}{10}$—$\frac{1}{5}$ in. long, campanulate, pubescent; lobes ovate, somewhat spreading, about equalling the tube or exceeding it; corolla glabrous (villous, according to Jacquin) within, pubescent outside, conical at apex in bud; lobes ovate-lanceolate, about equalling the tube; stamens 6—12, usually 9 (3—10, according to Jacquin), unequal, either distinct or in pairs or 3 together, inserted at base of corolla or partly hypogynous, glabrous; ovary abortive; receptacle hairy.

♀. Cymes soon becoming 1-flowered by lapse of the lateral flowers, $\frac{1}{20}$—$\frac{1}{3}$ in. long. Calyx openly campanulate, 3- (4-)fid, with rounded lobes, about $\frac{1}{3}$ in. across, puberulous outside, tomentose inside; corolla widened below, the lobes extending only $\frac{1}{3}$ way down, densely ferruginous-pubescent outside; staminodes glabrous, (in one flower) 4, 2 being distinct and 2 combined by their filaments; in another flower 3, alternating with the corolla-lobes; ovary 6-celled, 6-ovuled, $\frac{1}{10}$ in. high, covered outside with short appressed drab hairs; style simple, columnar, $\frac{1}{25}$ in. high, trifid at apex, hairy at base; stigmas punctiform; fruit solitary, 6-celled, yellowish, with black bitter pulp, depresso-globose, subsessile or shortly stalked, $\frac{1}{3}$—$\frac{2}{3}$ in. thick, subglabrate and shining; fruiting calyx about $\frac{2}{3}$ in. in diameter, reflexed or nearly flat, with 3 (4) rounded or bifid lobes, tube thickened within; seeds oblong; albumen not ruminated.

The following varieties may be noticed:

a. obovata. A tree or shrub with obovate-oblong leaves.

β. granatensis. A shrub with oblong leaves. Occasionally the leaves are conduplicate (in the dry state).

Flowers in February, July, and September.

S. America, St Vincent, *Guilding!*; Martinique, *Plée!* 762; New Granada, Carthagena, *Jacquin*, *Triana!* 2613; Sabanilla, *Karsten!*; Sta Martha, *Purdie! Goudot!* No. 1; Guayaquil, *Bonpland!* 3846; Brazil, *Pohl!* 1980, *Sello!* 1211, 1689, 2301; Rio Janeiro, *Gaudichaud!*; Minas Geraes, *Weddell!*; *Dr Regnell!* iii. 1516.

30. MABA ACAPULCENSIS.

M. foliis obovato-lanceolatis, apice acutis, basi cuneatis, utrinque hirtellis, subtus subcanescentibus, reticulatis, submembranaceis, petiolatis; fructibus solitariis, subsessilibus, subglobosis, uncialibus; calyce fructifero patente, profunde 3-fido; albumine non ruminato.

Macreightia acapulcencis, Alph. DC. Prodr. VIII. p. 221. n. 3 (1844), excl. Syn. Diospyros salicifolia.

Diospyros acapulcensis, Kunth in Humb. et Bonpl. Nov. Gen. iii. p. 254 (1818).

Terminal buds oblong, sericeous-tomentose; the axillary ones smaller, pubescent; shoots glabrous, dark-cinereous, smooth; leaves obovate-lanceolate, acute, cuneate at the base, hirtellous on both sides, especially beneath where they are subcanescent, reticulato-venose, membranous, $2\frac{1}{2}$ in. long or more, by about $\frac{7}{8}$ in. wide, petiolate; fruit solitary, subsessile, subglobose, 1 in. in diameter; fruiting calyx flat, nearly 1 in. across, deeply 3-fid; lobes widely ovate, felted inside, puberulous outside; albumen not ruminated; cotyledons oblong, rather obtuse, double the length of the radicle.

Mexico, Acapulco, *Bonpland!*

31. Maba salicifolia.

M. ramis teretibus cinereis, junioribus pubescentibus; foliis lanceolato-oblongis, utrinque angustatis, apice obtusis, coriaceis, supra glabrescentibus, subtus pubescentibus, breviter petiolatis, nervis inconspicuis; fructibus solitariis, globosis, glabris, breviter pedunculatis; calyce fructifero trifido, utrinque puberulo, appresso.

Diospyros salicifolia, Humb. et Bonpl. ex Willd. Sp. Pl. iv. p. 1112. n. 18 (1805); Hb. Willd. n. 19250.

Young leaves and shoots pubescent; branches cinereous, terete. Leaves lanceolate-oblong, narrowed at both ends, obtuse at apex, glabrescent above, coriaceous, with inconspicuous veins, about $2\frac{1}{2}$ in. long by $\frac{3}{4}$ in. wide; petioles $\frac{1}{8}$ in. long, puberulous. Fruit solitary, globose, glabrous, shining, of a pale orange colour, about $1\frac{1}{2}$ in. in diameter; peduncles $\frac{1}{8}$ in. long, stout, puberulous; calyx 3-fid, appressed to base of fruit, puberulous on both sides, about 1 in. across, lobes semi-elliptical, with obscure parallel veins.

Equatorial America, *Humboldt and Bonpland!*

Alph. De Candolle unites this species with *M. acapulcensis*, but the foliage is sufficiently different.

32. Maba Pavonii.

M. foliis ovalibus, apice acutis, basi obtusis, supra subglabris, subtus velutinis, subcoriaceis, breviter petiolatis; floribus masculis cymosis, brevissime pedicellatis, 3-meris, pubescentibus; floribus femineis solitariis, breviter pedunculatis.

Macreightia Pavonii, Alph. DC. Prodr. viii. p. 222. n. 7 (1844).

Branches puberulous. Leaves oval, acute at apex, rather glabrous above, velutinous and paler beneath and on the petioles, 5—6 in. long by $2\frac{3}{4}$ in. wide; midrib puberulous above, thinly subcoriaceous; petioles $\frac{1}{4}$ in. long.

♂. Flowers $\frac{2}{5}$ in. long, several together on axillary fulvo-tomentose peduncles which are about $\frac{1}{2}$ in. long; pedicels scarcely $\frac{1}{12}$ in. long. Calyx $\frac{1}{4}$ in. long, ovoid, hairy on both sides; lobes ovate, acute. Corolla fulvo-sericeous outside except at base, glabrous outside, twice the length of the calyx.

♀. Flowers solitary, $\frac{5}{12}$ in. long; peduncles $\frac{1}{4}$ in. long; calyx deeply 3-fid; lobes oval, submucronate.

Local name *Orlaca*. Peru (?) or Mexico (?) ex Alph. DC., *Pavon!*

33. Maba Mannii, sp. nov.

M. glabrescens, foliis ovalibus, apice obtusis, basi rotundatis vel parum angustatis, subcoriaceis, breviter petiolatis; floribus masculis 3-nis, brevissime cymosis, trimeris, staminibus 6—9, leviter hirsutis, basi corollæ insertis; ovarii rudimento hirsuto; fructibus solitariis, subsessilibus, subglobosis, glabratis, 5—6-locularibus; calyce parvo, patente, leviter puberulo; seminibus 5—6, albumine ruminato.

A small tree, growing by rivers; glabrescent, dark when dried; branches terete, erect-patent. Leaves oval, browner beneath, subcoriaceous, spreading, midrib and lateral veins clear, raised beneath, depressed above, 3 to 5 in. in length by $1\frac{1}{2}$ to $2\frac{1}{2}$ in. in width; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. in length; flowers subsessile.

♂. Cymes very short, very slightly pubescent, dark, 3-flowered, thick. Flower trimerous, $\frac{1}{4}$—$\frac{2}{5}$ in. in length, slightly hairy, white when living, dark when dry. Calyx ciliate and slightly hairy, $\frac{1}{15}$ in. in height, with 3 rounded lobes about $\frac{1}{20}$ in. in length, campanulate, not appressed to corolla, not accrescent; corolla tubular, glabrous, 3-lobed near apex; lobes $\frac{1}{4}$ of the depth of the corolla, rounded. Stamens 8 (6—9), linear, acute, somewhat hairy, inserted at base of corolla, in 2 series (6 in outer series). Ovary rudimentary, hairy.

♀. Fruit glabrous, of bright orange colour when ripe (glabrescent), sub-globose, obscurely 5—6-sided, (5-) 6-celled, 5—6-seeded, nearly 1 in. in diameter. Fruiting calyx horizontal, small, $\frac{3}{10}$ in. across, faintly puberulous; albumen ruminated.

Flowers in April, near the Equator, West Africa, Niger Expedition, *Barter!* 1220; Bagroo River, *Mann!* 839; Quorra, *Vogel!*

34. MABA SEYCHELLARUM, sp. nov.

M. fruticosa, foliis anguste ellipticis, apice obtusis, glabris, subcoriaceis, distichis, subsessilibus; floribus fœmineis solitariis, subsessilibus, pubescentibus, trimeris, calyce breviter 3-*lobo, staminodiis* 3—6, *glabris, basi corollæ insertis, ovario ovoideo,* 3-*loculari, loculis biovulatis, stylo apice* 3-*lobo; fructibus ellipsoideis, glabris; calyce fructifero cupuliformi, appresso; seminibus solitariis, albumine non ruminato.*

Shrub 10—12 ft. high; branches dark-cinereous, terete, at 35°, with short patent hairs at extremities, glabrescent; terminal bud with light brown pubescence. Leaves narrowly elliptical, obtuse or notched at apex, slightly narrowed at base, subsessile, distichous, somewhat convex from above in dried state; midrib depressed above, other veins inconspicuous; subcoriaceous, glabrous, 1 to 2 in. in length by $\frac{1}{4}$ to $\frac{1}{2}$ in. in width; internodes $\frac{1}{5}$ to $\frac{1}{4}$ in. in length.

♀. Flowers solitary, subsessile, with light brown pubescence, $\frac{1}{5}$ in. long. Calyx campanulate, $\frac{1}{5}$ in. long, with 3 shallow depresso-deltoid apiculate lobes, pilose outside, glabrous within. Corolla 3-lobed, divided more than half-way down, hairy outside except near base, glabrous inside; lobes obtuse, imbricated. Staminodes 3 or 6, glabrous, inserted at base of corolla-tube. Ovary ovoid, glabrous except near apex or pubescent all over, 3-celled, cells 2-ovuled. Style erect, 3-lobed at apex, hairy except at apex. Fruit glabrous, ovoid, pallid, rather more than $\frac{1}{3}$ in. long by rather more than $\frac{1}{5}$ in. thick, 1 (?)-seeded. Albumen not ruminated, white; fruiting calyx 3-cornered, shortly cup-shaped, about $\frac{1}{6}$ in. high by $\frac{1}{4}$ in. wide.

Seychelles I., *Pervillé!* 36, mountains near the cascade; Mahé, 13 Febr. 1840.

A specimen with similar foliage but rather more slender branches and peduncles (spines?) $\frac{1}{5}$—$\frac{1}{6}$ in. long without flowers may belong here. Seychelles, *Boivin!* Mahé.

Fruit subsessile, solitary, axillary, ellipsoidal, $\frac{1}{3}$—$\frac{1}{2}$ in. long by $\frac{1}{6}$—$\frac{1}{4}$ in. thick, of a pale colour, shining, glabrous except at the apex where the remains of the hairy style project; fruiting calyx pubescent or glabrescent, cup-shaped, appressed to base of fruit, 3-lobed usually with short depresso-deltoid lobes, $\frac{1}{10}$—$\frac{1}{5}$ in. high; fruit 1-celled, 1-seeded; seed rather more than $\frac{1}{4}$ in. long (in one case) by $\frac{1}{8}$ in. thick; albumen not ruminated, bony. Seychelles, *Dr Percival Wright!* 1867, 30 May—23 Nov., n. 122.

35. Maba lanceolata.

M. foliis lanceolatis vel ovato-lanceolatis, utrinque acutis, glabris, coriaceis, breviter petiolatis; floribus masculis 1—3-nis, sessilibus, basi bracteatis, bracteis imbricatis, calyce hirsuto, breviter 3-lobo, corollâ 3—4-lobâ, staminibus 24—32, glabris, basi corollæ insertis; floribus femineis brevissime pedunculatis, ovario glabro, globoso, 3-loculari, loculis biovulatis; fructibus ovoideis, glabris, nitidis.

Diospyros lanceolata, Poir. Encycl. Méth. v. p. 434 (1804); Alph. DC. Prodr. VIII. p. 236. n. 69 (1844); non Wall.

A tree; with glabrous, lanceolate or ovate-lanceolate leaves, acute at both ends, especially at the apex, coriaceous, in the dry state brown on both sides, $1\frac{1}{2}$—$2\frac{3}{4}$ in. long by $\frac{5}{8}$—$1\frac{1}{16}$ in. wide; petioles spreading, $\frac{1}{5}$—$\frac{1}{10}$ in. long; veins confluent at the margin, shining above; margins recurved.

♂. Flowers 1—3 together, sessile, ovoid, acute in bud, $1\frac{1}{4}$ in. long; surrounded at the base by 7 imbricated rounded ciliolated unequal coriaceous bracts, glabrous except at margin, the inferior ones very short. Calyx nearly $\frac{1}{4}$ in. long, densely hirsute, ferruginous, shortly 3-lobed, 3-cornered, campanulate in flower (spreading in fruit much less hairy and $\frac{1}{3}$ in. across). Corolla glabrous but with broad hairy patches outside lobes, $\frac{1}{4}$—$\frac{3}{16}$ in. long, deeply 3—4-lobed; lobes oblong, emarginate, spreading and recurved. Stamens 24—32, glabrous, inserted at base of corolla, the outer ones shorter; filaments short. Ovary wanting.

♀. Peduncles very short, recurved; calyx urceolate, shortly 3-lobed, not accrescent; corolla narrowed at the throat, deeply 3—4-(?) lobed; staminodes ...; ovary quite glabrous, shining, spherical, 3-celled; cells 2-ovuled with imperfect septa in middle; style 3-lobed, erect; fruit ovoid, glabrous, shining.

Madagascar, *Commerson!*

The leaf described and figured by *Ettingshausen* in Blatt-skel. Dikot. p. 89. t. 37. fig. 12 (1861) is decidedly larger than in the specimens that I have seen of this species; it probably belonged to a different species.

36. Maba natalensis, Harv. Thes. Cap. II. p. 7. t. 110 (1863).

M. fruticosa, ramis gracilibus patentibus; foliis ovalibus, obtusis, glabris, supra nitentibus, subtus pallidioribus, breviter petiolatis; floribus femineis solitariis, brevissime pedunculatis; calyce cupuliformi, glabro, integro; corollâ trilobâ, extus sericeâ; staminodiis 6—9, glabris, uniserialibus; ovario conico, glabro, 3-loculari, loculis bi-ovulatis.

A quickly glabrescent shrub; branches pale, slender, spreading at 60°—65°; shoots flexuous, puberulous. Leaves oval, obtuse or mucronate at apex, submembranous, flat, $\frac{1}{2}$—1 in. in length by $\frac{1}{4}$—$\frac{7}{12}$ in. in width, veins delicate and inconspicuous, shining and dark green above, paler beneath; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. in length.

♀. Flower solitary, axillary, very shortly pedunculate, $\frac{1}{5}$ in. in length; peduncle $\frac{1}{25}$—$\frac{1}{20}$ in. in length. Calyx $\frac{1}{10}$ in. in length, truncate, entire, dark green, glabrous, semi-ellipsoidal. Corolla $\frac{2}{11}$ in. in length, argenteous-sericeous outside, 3-lobed; lobes $\frac{1}{8}$ in. in depth, diverging, oblong, acute. Staminodes 6—9, free, uniseriate, $\frac{1}{10}$ in. in length, glabrous. Ovary conical, drab, glabrous, terminated by a style 3-lobed at the apex, 3-celled, cells bi-ovuled; style as long as ovary, glabrous. Fruit ellipsoidal, glabrous, of pale chestnut colour, $\frac{1}{3}$ in. in height by $\frac{1}{5}$ in. in width; style persistent; fruiting calyx not increased in height, hemispherical, like an acorn-cup, 1-seeded.

S. Africa, Natal, *W. T. Gerrard!* 110; D'Urban, *Macken!* 675.

37. MABA ABYSSINICA, sp. nov.

M. fruticosa, foliis lanceolato-oblongis, plerumque apice obtusis et basi rotundatis, glabris, subcoriaceis, planis, breviter petiolatis, nervis inconspicuis; floribus masculis subsessilibus, aggregatis, 3—4-meris, calyce laxo, lobis rotundatis ciliatis, corollâ glabrâ, staminibus circiter 14, glabris; floribus femineis 3—5-nis, aggregatis, brevissime pedicellatis, 3—5- sæpius 3-meris, calyce campanulato, non accrescente, corollâ glabrâ, aperte campanulatâ; staminodiis 3—4, glabris; ovario ovoideo, glabro, 6-loculari, loculis uni-ovulatis, stylo apice 3-lobo; fructibus glabris, subglobosis; seminibus solitariis, albumine non ruminato.

A large shrub, glabrous except the inflorescence; shoots dark, terete. Leaves lanceolate-oblong, obtuse and often somewhat acuminate at apex, more or less narrowed at base, subcoriaceous, flat, of the same dull colour on each face, somewhat shining above, patent or erect-patent, 2—5 in. long by $\frac{1}{2}$—$1\frac{3}{8}$ in. wide; petioles $\frac{1}{8}$—$\frac{1}{6}$ in. long; midrib slightly depressed above, veins inconspicuous. Bracts pubescent, small, caducous; flowers subsessile, clustered, axillary; flowers mostly trimerous, sometimes 4—5-merous.

♂. Flowers in. long; calyx $\frac{1}{16}$ in. long, lax, usually 3-fid, lobes rounded, minutely ciliate; corolla widely campanulate, glabrous, 3—4-fid, lobes rounded; stamens about 14, glabrous, appearing at the mouth of the open corolla, mostly in pairs, inserted at the base of the corolla, filaments short; ovary rudimentary, glabrous.

♀. Flowers 3—5 together, on very short puberulous pedicels, $\frac{1}{4}$—$\frac{1}{3}$ in. long; calyx $\frac{1}{6}$ in. long, or after the fall of the corolla about $\frac{1}{4}$ in. long, glabrous or margins of the lobes minutely ciliate, deeply 3—5- usually 3-lobed, campanulate, lobes wide or cordate at base, $\frac{1}{6}$ in. long; corolla glabrous, openly campanulate, deeply 3—5-lobed, not always isomerous with the calyx, lobes spreading or recurved, oval-ovate, obtuse; staminodes 3—4, glabrous, alternate with the lobes of the corolla; ovary glabrous, ovoid, conical at apex, 6-celled, cells 1-ovuled; style thick, 3-lobed at apex; fruiting calyx lax, not accrescent; fruit glabrous, subglobose, shining, of dark coppery colour, $\frac{7}{16}$—$\frac{5}{8}$ in. long by $\frac{1}{3}$—$\frac{1}{2}$ in. thick, 1—few-seeded, bearing remains of style at apex; albumen of seeds cartilaginous, not ruminated.

Abyssinia, on the sides of the valley of Bellagass near Loegga, 5000 feet altitude, *Schimper!* 1854, n. 1080; Sila 5000—6000 feet altitude, *Schimper!* 1854, n. 1334; Keren, bank of river, *Beccari!* n. 55, May, 1870.

38. MABA QUILOËNSIS, sp. nov.

M. glabrescens, foliis ellipticis, apice obtusis, basi subcordatis, submembranaceis, breviter petiolatis, nervis lateralibus inconspicuis; floribus femineis subsessilibus, sub-3-nis, calyce trilobo, ovario glabro, ovoideo-conico, 6-loculari, loculis uni-ovulatis, stylo apice trilobo, glabro, calycem superante.

Glabrous shrub; branches terete, at about 60°, argenteous-cinereous, except when very young and then they become blackish in the dried state. Leaves alternate, elliptical, dark, without conspicuous lateral veins, submembranous, obtuse or rounded at apex, usually subcordate at base, 1 to 2 in. in length by $\frac{1}{3}$ to 1 in. in width; petioles scarcely $\frac{1}{10}$ in. in length.

♀. Flowers subsessile, about 3 together, dark when dry; calyx $\frac{1}{10}$ in. in height, with 3 ovate diverging lobes extending $\frac{2}{3}$ down the calyx; corolla fallen from specimens; ovary glabrous, ovoid-conical, 6-celled, cells 1-ovuled; style 3-lobed at apex, glabrous, higher than the calyx.

East Tropical Africa, Quiloa, *Dr Kirk!* Fl. Zangueb. n. 110, 10 January, 1867.

39. MABA MICRANTHA.

M. foliis ellipticis vel oblongis, basi attenuatis, apice obtuse acuminatis, glabris, coriaceis, petiolatis; floribus femineis solitariis, sessilibus, axillaribus; calyce tubuloso, integro, truncato, in squamis paucis bifariis imbricatis insidente; corollâ tubulosâ, trifidâ, quam calyce triplo longiore, lobis ovatis obtusis patentibus; staminodiis 6, basi corollæ insertis; ovario hemisphærico, glabro, 6-loculari; loculis uni-ovulatis; stylis 3, erectis.

Holochilus micranthus, Dalz. in Kew Jour. Bot. IV. p. 291 (1852), Dalz. et Gibs. Bomb. Fl. p. 142 (1861).

A middle-sized tree. Leaves elliptical or oblong, attenuate at base, obtusely acuminate at apex, coriaceous, glabrous, 4—5 in. long by 2 in. wide; petioles $\frac{1}{2}$ in. long. Flowers diœcious; ♂ flowers unknown.

♀. Flowers $\frac{1}{4}$ in. long, white, solitary, sessile, axillary; calyx tubular, entire, truncate, placed among a few bifarious scales; corolla tubular, 3 times the length of the calyx, 3-lobed nearly to the middle; lobes ovate, obtuse, spreading. Staminodes 6, inserted at the base of the corolla-tube, distinct, filaments double of the length of the barren anthers; styles 3, erect, rather thick, obtuse at the apex; ovary hemispherical, glabrous, 6-celled; ovules solitary. Fruit cylindric-oblong, supported at the base by the accrescent truncate calyx, dry, hard, 1 in. long, 6-celled; seeds solitary.

In the Syhadra hills, on the Southern Ghauts, Bombay. Flowers in February and March, *Dalzell.*

40. MABA LAMPONGA, Miq. Fl. Ind. Bat. Suppl. I. n. 1179. p. 584 (1860).

M. foliis obovato-oblongis vel ellipticis, apice rotundatis retusis, basi angustatis, glabrescentibus, coriaceis, breviter petiolatis, venis inconspicuis; floribus masculis (monstrosis in speciminibus?) axillaribus supra-axillaribus et lateralibus, paniculatis fasciculatis umbellatis vel interdum solitariis; floribus femineis solitariis, subsessilibus vel breviter pedunculatis, axillaribus, calyce tridentato, corollâ subcampanulatâ, lobis elliptico-oblongis, acuminatis, ovario ovoideo, glabro, stylo brevissimo, stigmatibus 3, patentibus.

Buds velutinous. Leaves minutely and appressedly downy when young, quickly glabrescent, obovate-oblong or elliptical, acute or sub-cuneate at the base, in most cases widely rounded and retuse at the apex, coriaceous, rather shining, griseo-pallid when dry, nearly veinless, but with slender net-veins when old, $3\frac{1}{2}$—2 in. long; petioles short, sub-trigonous. Flowers diœcious?

♂. Flowers (all monstrous?) axillary supra-axillary and lateral, sometimes in short panicles, at other times fascicled or umbelled, occasionally solitary, pedicelled; corolla represented by 3 ovate scales united at the base, alternating with the calyx-teeth, pubescent; moreover there are placed inside numerous narrower scales (monstrous stamens?) in several series, free or united in pairs, more or less hairy at the back, plainly imbricated.

♀. Flowers solitary, subsessile or shortly pedunculate, axillary; calyx coriaceous, cupuliform-globose, tridentate; teeth triangular acute, appressedly downy outside; tube of the corolla short, subcampanulate, glabrous as high as the calyx; lobes elliptic-oblong, acuminate, densely hirsute along the middle of the back. Ovary ovoid, glabrous; style very short, thick, stigmas 3, spreading, canaliculate in front.

South Sumatra in prov. Lampong; on sea coast. *Teysmann.*

41. MABA MERGUENSIS, sp. nov.

M. foliis ovalibus oblongis vel ovato-oblongis, apice acuminatis, basi subrotundis vel parum angustatis, glabris, submembranaceis vel tenuiter coriaceis, petiolatis; floribus masculis 8-nis, paniculatis, parvis, axillaribus, 3—4-meris, pedicellis brevissimis; calyce aperte campanulato, minute puberulo, ciliato; corollâ subglabrâ; staminibus 14—16, glabris; ovarii rudimento subtus glabro; floribus femineis densè cymosis, 3—9-nis, plerumque trimeris; staminodiis 3 vel 6, glabris, basi corollæ insertis; ovario glabro, 6-loculari, loculis uni-ovulatis; fructibus globosis, glabris; seminibus oblongis, albumine non ruminato.

Cfr. *Diospyros frutescens*, var., Blume, Bijdr. fl. ned. Ind. p. 668 (1825); *β. Tallak*, Alph. DC. Prodr. VIII. p. 230. n. 38 (1844).

A small tree, glabrous, with brown or dark-ashy branches, spreading at about 50°. Leaves oblong or ovate-oblong, glabrous or sometimes with puberulous midrib beneath, sub-membranous, dark above, paler and brownish beneath, rounded or slightly narrowed at base, acuminate at apex, not black-punctate, nearly flat, veins delicately raised on both sides, or sometimes slightly depressed on the upper surface, $2\frac{1}{2}$—$6\frac{3}{4}$ in. in length by 1—3 in. in width; petioles $\frac{1}{8}$—$\frac{1}{2}$ in. in length.

♂. Cymes panicled, bearing numerous flowers, pubescent with short lightish brown hairs, $\frac{1}{4}$ to 1 in. in length (excluding the flowers); pedicels very short; flowers small; calyx $\frac{1}{16}$ in. in height, openly campanulate, with slight short pubescence outside, and 3 or 4 widely deltoid lobes about half the depth of the calyx, ciliate; corolla nearly glabrous, about $\frac{1}{10}$ in. long, 3—4-lobed, lobes short; stamens 14—16, mostly or all in pairs, inserted at base of interior of corolla or hypogynous, glabrous, the interior ones the smaller, anthers about equalling the longer filaments; ovary rudimentary.

♀. Cymes dense, short, many-flowered, with thick pedicels, pubescent in flower, glabrescent in fruit; bracteoles ovate, pubescent, caducous, $\frac{1}{25}$ in. in length; flower usually trimerous, occasionally tetramerous, $\frac{1}{4}$ in. in length; calyx pubescent $\frac{1}{8}$ in. in length, glabrescent, spreading, lobes $\frac{1}{16}$ in. long, diverging, ovate, sides reflexed; corolla pubescent, lobes $\frac{1}{16}$ in. in length, oval, somewhat spreading; staminodes 3 or 6, inserted at base of the tube of the corolla, $\frac{1}{8}$ in. in length, linear, glabrous; styles 3, distant, glabrous, $\frac{1}{30}$ in. in length; ovary semi-ellipsoidal, glabrous except at base where there is a band of hairs, 6-celled, cells 1-ovuled. Fruit glabrous, smooth and shining, globular, about $\frac{1}{3}$—$\frac{3}{4}$ in. in diameter, sometimes 4-celled when young; albumen of seeds not ruminated.

Flowers in January, fruits in February.

Mergui Archipelago, *Griffith!* (in fruit), *Helfer!* 3618; Sumatra, *Korthals!*; Borneo, *O. Beccari!* n. 1670; ? Java, *Blume!*, *Kurl and Hasselt!*

42. MABA FASCICULOSA, F. Muell. Fragm. v. p. 163 (1866).

M. foliis ovato-lanceolatis vel oblongis, apice angustatis vel acuminatis, obtusis, basi angustatis, glabris, coriaceis, petiolatis; floribus masculis numerosis, dichotome cymosis, 3—4-*meris, staminibus* 8—18, *antheris glabris, filamentis sæpe minute ciliatis; floribus femineis* 3—∞-*nis,* 3—4-*meris, staminodiis* 0—4, *ovario glabro,* 6-*loculari, loculis uni-ovulatis; fructibus subglobosis, glabris.*

Benth. Fl. Austral. IV. p. 290. n. 5 (1869).

Diospyros fasciculosa, F. Muell. Austral. Veg. in Intercol. Exh. Ess. 1866—67, p. 35 (1867).

M. laxiflora, Benth. *l. c.* n. 4.

A tall shrub or lofty tree, glabrous with terete branches spreading at about 45°. Leaves ovate-lanceolate oval or oblong, narrowed or acuminate at apex, obtuse, more or less narrowed at base, minutely black-punctate beneath, 2—4$\frac{1}{2}$ in. long by $\frac{2}{3}$—1$\frac{3}{4}$ in. wide; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long; midrib and veins more or less raised on both surfaces; margins somewhat recurved.

♂. Flowers numerous, 3—4-merous, $\frac{1}{8}$—$\frac{3}{20}$ in. long, in fascicled axillary cymes $\frac{1}{3}$—$\frac{1}{2}$ in. long exclusive of the flowers; pedicels slender, subglabrous; bracteoles small, ovate, slightly pubescent, caducous. Calyx $\frac{1}{25}$—$\frac{1}{8}$ in. long, with short lobes, somewhat pubescent outside, glabrous inside. Corolla campanulate, 3—4-fid, glabrous or obsoletely pubescent, lobes rounded; stamens 8—18, anthers glabrous, filaments often minutely ciliate; ovary rudimentary, glabrous.

♀. Flowers 3 or many together, clustered, 3—4-merous; cymes axillary short; staminodes 0—4, glabrous; ovary shortly conical, glabrous, 6-celled with cells 1-ovuled, or 3-celled with 2 ovules in each cell separated by an incomplete dissepiment; style very short, 3-lobed at apex. Fruit subglobose or shortly ellipsoidal, shining, of pale colour, $\frac{3}{10}$—$\frac{3}{5}$ in. long; fruiting calyx 3—4-fid with spreading and reflexed deltoid lobes, tube cupuliform; seeds 1—4, albumen not ruminated.

Called in New Caledonia *Mêdeso.*

Australia, Rockingham, *Dallachy!*; Queensland Woods, *Hill!* 100; Brisbane River, *F. Mueller!*; Rockhampton, *O'Shanesy!*, *Thozet.*

New Caledonia, *Deplanche!* 48, 206; *Vieillard!* 899.

43. MABA RUMINATA, sp. nov.

M. foliis anguste ellipticis, utrinque angustatis, glabris, coriaceis, petiolatis; floribus femineis ∞-*nis, trimeris; fructibus subglobosis, glabris; calyce fructifero* 3-*fido, tubo hemisphærico, lobis late ovatis, patentibus; albumine seminum ruminato.*

Young parts and inflorescence puberulous; branches somewhat cinereous. Leaves narrowly elliptical, narrowed at both ends, glabrous, alternate, coriaceous, 3—5 in. long by $\frac{4}{5}$—$1\frac{3}{5}$ in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long; margins recurved; midrib slightly dilatato-depressed above; veins not conspicuous above, of same colour as lamina beneath.

♀. Cymes axillary, many-flowered, $\frac{1}{8}$—$\frac{1}{4}$ in. long, spreading; pedicels $\frac{1}{12}$—$\frac{1}{6}$ in. long, puberulous; fruiting calyx 3-fid, nearly $\frac{1}{2}$ in. across, glabrous or very nearly so, tube hemispherical; lobes widely ovate, convex from above, spreading; fruit about $\frac{1}{2}$ in. long, subglobose, glabrous, pale and shining; seeds with ruminated albumen.

New Caledonia, *Deplanche!* 311.

44. MABA CONFERTIFLORA, sp. nov.

M. foliis ovali-oblongis, apice obtuse vel emarginate acuminatis, glabris, coriaceis; floribus masculis aggregatis, brevissime cymosis, subsessilibus, trimeris, corollâ urceolato-tubulosâ; staminibus 12, *geminatis, glabris; floribus femineis axillaribus, subsessilibus, aggregatis, trimeris; staminodiis* 2—3, *stylo apice 3-lobo, ovario glabro,* 6 ?*-loculari.*

A small tree; shoots subglabrous with scattered short appressed hairs, cinereous. Leaves crowded, oval-oblong, obtusely or emarginately acuminate, coriaceous, glabrous except the midrib beneath, shining, with depressed veins on the upper surface, somewhat paler beneath; lateral veins inconspicuous and weak; leaves $1\frac{1}{2}$—$3\frac{1}{2}$ in. long (including subglabrescent petiole $\frac{1}{5}$ in. long) by $\frac{6}{7}$—$1\frac{1}{2}$ in. wide. Bracts shortly ovate.

♂. Flowers many, crowded, on short slightly hairy cymes, subsessile, rufous; calyx small, trifid, with scattered short hairs, spreading; corolla urceolate-tubular, shortly trifid, with 3 hairy lines down the middle lines of the lobes; stamens 12, united by their filaments in 6 pairs, the inner ones being the shorter, glabrous; anthers dehiscing widely from apex downwards; ovary rudimentary, glabrous.

♀. Flowers subsessile, crowded in axils of leaves, several abortive; calyx sub-glabrescent, coriaceous, spreading, 3-lobed; corolla 3-fid; staminodes 2—3; style 3-lobed at apex; ovary glabrous, 6 ?-celled, with a few hairs at base.

Labuan, *Lobb!*, *Motley!* 205.

45. MABA PUNCTATA, sp. nov.

M. foliis oblongis, apice breviter acuminatis vel apiculatis, basi subcordatis, coriaceis vel submembranaceis, minute pellucido-punctatis, supra glabris nitidis, subtus secus nervos puberulis, breviter petiolatis; floribus masculis dense cymosis, pubescentibus, trimeris, staminibus 9, *glabris; floribus femineis* 3—∞*-nis, breviter cymosis, supra-axillaribus, trimeris; fructibus sub-globosis, glabris,* 6*-locularibus.*

Diospyros punctata, Korthals, MSS. in Hb. Lugd. Batav. Ebenac. n. 15.

A small tree; young parts, inflorescence, &c. softly ferruginous-pubescent; shoots terete, pubescent. Leaves oblong, alternate, coriaceous or submembranous, minutely pellucid-punctate, subcordate at base, suddenly and sharply acuminate apiculate or mucronate at apex, glabrous and shining above with depressed midrib and lateral veins; midrib and about 9 or 10 lateral veins on each side, puberulous, distinct and in relief beneath; lower surface slightly and appressedly puberulous, with evanescent reddish pulverulence; $3\frac{1}{2}$—$10\frac{1}{2}$ in. long by $1\frac{1}{4}$—$3\frac{1}{2}$ in. wide; petioles $\frac{1}{6}$ in. long, thick, terete, pubescent; usually some depressed glands are visible on the lower surface of the leaves especially at the base.

♂. Inflorescence axillary, dense, many-flowered, short, $\frac{3}{10}$—$\frac{2}{5}$ in. long (exclusive of the flowers); bracts acute, numerous, hairy; pedicels varying in length up to $\frac{1}{4}$ in.; flowers $\frac{1}{3}$ in. long, white; calyx $\frac{1}{6}$ in. long, campanulate, shortly 3-fid, shortly pubescent outside, glabrous inside, lobes deltoid; corolla tubular, hypocrateriform, $\frac{1}{10}$ in. wide, shortly 3-lobed, sericeous outside, glabrous inside; lobes acute, spreading, $\frac{1}{10}$ in. long; stamens glabrous, 9, hypogynous, equal, 6 united by their filaments in 3 pairs of which the inner ones are the shorter, and 3 distinct; anthers longer than the filaments, linear, acute; ovary 0; receptacle glabrous.

♀. Inflorescence supra-axillary; cymes 3—many-flowered, $\frac{1}{5}$—$\frac{1}{4}$ in. long (exclusive of the flowers); bracts small acute caducous; fruiting pedicels thickened upwards, $\frac{1}{10}$—$\frac{1}{5}$ in. long; pericarp rather thick; fruit ovoid $\frac{3}{4}$ in. long by $\frac{5}{8}$ in. thick, terminated by style $\frac{1}{20}$ in. long 3-lobed at apex, glabrous (in dry state), subverrucose, shining, 6-celled, 6-seeded; fruiting calyx appressed to young fruit and sericeous, quite patent and nearly glabrate in older fruit, not accrescent, $\frac{1}{4}$ in. across, 3-fid with ovate or deltoid lobes.

Borneo, Foot of Gunong Pautie on Serpentine Rocks. *Mr Motley!* n. 766; *Korthals!*, *Beccari!* n. 1423.

PLATE IV. A fruiting branch, *natural size.* *a.* A piece of a male branch in flower, *natural size.* *b.* A piece of a female branch, *natural size.* *c.* A fruit, *natural size.*

46. MABA TEIJSMANNI.

M. arborea, hermaphrodita (?), *foliis subdistichis, ovalibus vel oblongis, apice breviter acuminatis, basi angustatis vel obtusis, glabris, breviter petiolatis; floribus cymosis,* 5—9-*nis,* 3—4-*meris, urceolatis, staminibus* 3—4, *basi corollæ insertis, ovario ovoideo glabro* 6-*loculari; fructibus coriaceis, calyce aucto reflexo.*

Rhipidostigma Teijsmanni, Hassk. Retz. I. p. 106 (1855).

A tree with erect-patent cinereous asperulous branches and terete glabrous or minutely glandular-ciliate shoots whose bark is often somewhat chartaceous; hermaphrodite (?). Leaves subdistichous, oval or oblong, cuspidate or shortly acuminate at apex, narrowed or obtuse at base, 2—$4\frac{1}{2}$ in. long by 1—$2\frac{1}{8}$ in. wide, glabrous; midrib and veins slenderly depressed on the upper surface; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long. Cymes axillary, $\frac{1}{4}$—$\frac{1}{2}$ in. long, 5—9-flowered, puberulous; calyx 3—4 lobed, glabrous, lobes $\frac{1}{5}$—$\frac{1}{4}$ in. long, ovate, with reflexed margins, spreading; corolla urceolate, shortly lobed, $\frac{1}{6}$—$\frac{1}{4}$ in. high, $\frac{1}{4}$ in. wide, slightly hairy outside; stamens 3—4, inserted at base of the corolla; ovary ovoid, 6-celled, glabrous, surrounded at base by a ring of hairs. Fruit smooth, $\frac{3}{5}$ in. long by $\frac{1}{2}$ in. thick, pericarp coriaceous, flesh slightly glutinous; seeds $\frac{1}{2}$ in. long or more, $\frac{1}{4}$ in. thick, albumen cartilaginous.

Java, *Teijsmann!*, flowers in April; Borneo, *O. Beccari!* n. 1822.

47. MABA HERMAPHRODITICA, Zoll. Syst. Verz. Ind. Archip. p. 135 (1854).

M. arborea, hermaphrodita (?), *foliis alternis, sparsis, oblongo-lanceolatis, utrinque attenuatis, subcoriaceis, glabris, breviter petiolatis; floribus breviter cymosis,* 3—6-*nis,* 3—4-*meris,* *urceolatis, staminibus* 3—4, *basi corollæ insertis, ovario glabro,* 6-*loculari; fructibus obovato-ellipsoideis vel oblongo-ovoideis.*

Rhipidostigma Zollingeri, Hassk. Retz. I. p. 104 (1855).

Hermaphrodite ⚥; glabrous except the inflorescence; a moderate-sized tree with erect-patent, terete, cinereous-white, punctately asperulous branches and green glabrescent shoots; bark of shoots papery. Petioles $\frac{1}{4}$ in. long; leaves subcoriaceous, oblong-lanceolate, attenuate at both ends, sometimes acuminate at apex, $1\frac{1}{2}$—3 in. long, $\frac{2}{3}$—$1\frac{1}{2}$ in. wide, with subreflexed margins. Cymes short, $\frac{1}{4}$—$\frac{1}{3}$ in. long, pedicelled, usually 3-flowered, sometimes 4—6-flowered; bracts very short, ovate, acute, at the base of the pedicels, green, deciduous; pedicels terete, $\frac{1}{12}$—$\frac{1}{8}$ in. long, green, glabrous. Calyx fleshy, green, 3—4-partite; tube very short, cupuliform, scarcely $\frac{1}{12}$ in. high; lobes ovate, acute, $\frac{5}{24}$ in. long, $\frac{1}{6}$ in. wide at the base, with reflexed margins. Corolla urceolate, white, $\frac{1}{6}$ in. high; tube subglobose; lobes 3—4, patent, ovate, shorter than the tube. Stamens 3—4, inserted at base of corolla and included, abortive (?). Ovary depresso-globose, glabrous but with a ring of hairs at base, 6-celled, cells 1-ovuled; styles 2 or 3, bifid at apex, glabrous. Fruit obovate-ellipsoidal or oblong-ovoid, $\frac{5}{12}$ in. long, $\frac{1}{3}$ in. thick, 1-celled, 1-seeded. Flowers in April.

Java, *Zollinger!* 3467. Local name *Ki Kòning Kajoe.*

The next following species is very imperfectly known; it was described by Zollinger from a drawing of Kuhl and Van Hasselt in the botanical garden at Buitenzorg.

48. MABA JAVANICA, Zoll. obs. bot. nov. p. 14 in Natuurk. tydschr. Neerl. Ind. Vol. XIV. (1857).

M. foliis ellipticis utrinque breviter acuminatis glabris vix lucidis, floribus subsessilibus, confertis, calycis lobis margine revolutis acutis, baccis breviter pedunculatis oblongis, stylorum vestigiis mucronatis.

Java, *Zollinger.*

49. MABA MAINGAYI, sp. nov.

M. monoeca (?), foliis obovato-ovalibus, apice rotundis emarginatis vel brevissime acuminatis, obtusis, basi cuneatis, coriaceis, subglabrescentibus, petiolatis; cymis 3—5-*floris, breviter pubescentibus, floribus* 4—3-*meris, calyce campanulato, lobis rotundatis valde contorte imbricatis, staminibus* 20—22, *in fl. fem.* 8 ?; *ovario glabro, fusiformi,* 3-*locularibus; loculis* 2-*ovulatis; fructibus oblongis, seminibus solitariis, albumine ruminato.*

A tree, monœcious according to Dr Maingay; young branches rufous-puberulous, afterwards softly subglabrescent, blackish, terete. Leaves obovate-oval, rounded emarginate or very shortly and obtusely acuminate at apex, cuneate at base, coriaceous, alternate, puberulous when young, subglabrescent, 2—4 in. long by 1—$2\frac{3}{5}$ in. wide; petioles $\frac{4}{10}$—$\frac{7}{10}$ in. long, dark, puberulous when young, subglabrescent.

♂. Stamens 20—22; filaments very short or wanting, arising from the receptacle; anthers very long, linear-lanceolate; calyx deeply 4—3-lobed; limb of corolla 4—3-lobed; ovary 0.

♀. Cymes 3—5-flowered, ferruginously and shortly pubescent, about $\frac{1}{2}$ in. long; flowers mostly on long stout pedicels nearly $\frac{1}{2}$ in. long, suberect, $\frac{1}{2}$ in. long or more. Calyx $\frac{1}{5}$—$\frac{1}{4}$ in.

long by $\frac{1}{3}$ in. wide, openly campanulate, ferruginously and shortly pubescent outside, glabrous inside, 3-fid but apparently subtruncate by the close contorted imbrication of the rounded lobes. Corolla somewhat salver-shaped, with inflated tube, glabrous, deeply 4—3-fid, with spreading oblong obtuse lobes, much imbricated in bud. Staminodes 8 (?), glabrous, small (in the bud), equal, inserted at base of corolla, in one row. Ovary glabrous, spindleshaped, about $\frac{3}{10}$ in. long by scarcely $\frac{1}{10}$ in. thick (in the bud); style very short; stigma 3-lobed; ovary 3-celled with 2 pendulous ovules in each cell at lower end of the ovary. Fruit 1-seeded, $1\frac{1}{2}$—2 in. in length by about 1 in. thick, oblong, narrowed at base; testa crustaceous, intruded deeply in about 15 planes into the albumen which is therefore ruminated; pericarp about $\frac{1}{10}$ in. thick, dark and somewhat rugose outside. Fruiting calyx not accrescent; tube stuffed; lobes patent.

Malacca, *Maingay!* 976, Oct. 25, 1867; Borneo, *O. Beccari!* n. 1550.

50. MABA MOTLEYI, sp. nov.

M. foliis ellipticis vel oblongis, apice rotundatis, basi angustatis, glabris, coriaceis, breviter petiolatis; floribus subsolitariis, tetrameris, pedunculis glabris; calycis lobis rotundatis, emarginatis, valde imbricatis; staminibus 25, *glabris, receptaculo insertis; in flore femineo staminodiis* 16, *ovario ovoideo-conico, glabro,* 3-*loculari, loculis biovulatis; fructibus oblongis, glabris,* 1-*spermis, calyce fructifero parvo, patente; seminibus teretibus, albumine ruminato.*

A tall slender tree, glabrous except the extremities and inflorescence where there is a weak ferruginous or whitish pubescence; branches spreading at 45°—55°. Leaves alternate, elliptical or oblong, coriaceous, glabrous, rounded at apex, slightly narrowed at base, 2—$3\frac{1}{3}$ in. long by $\frac{3}{5}$—2 in. wide, petiole $\frac{1}{10}$—$\frac{2}{5}$ in. long; midrib depressed above, lateral veins rather close, inconspicuous. Flowers yellow, subsolitary on short glabrous peduncles $\frac{1}{10}$—$\frac{1}{5}$ in. long, tetramerous.

♂. Corolla $\frac{1}{2}$ in. long, funnel-shaped, deeply 4-fid with spreading lobes, glabrous; stamens 25, mostly equal, inserted on the receptacle, free, with very short filaments; anthers linear, glabrous.

♀. Flowers $\frac{3}{5}$ in. long. Calyx $\frac{1}{5}$—$\frac{9}{40}$ in. long, with scattered short weak hairs, 4-fid; with rounded emarginate much imbricated lobes, giving the calyx the appearance of being truncate. Staminodes 16 or more, glabrous; style 4?-lobed at apex; ovary ovoid-conical, glabrous, 3-celled, cells bi-ovuled. Fruit oblong, glabrous, $\frac{2}{3}$ in. long by $\frac{1}{3}$ in. thick, 1-seeded. Fruit-calyx small, 4-fid, $\frac{1}{4}$ in. across, loose or spreading; seed $\frac{2}{5}$ in. long, nearly terete, $\frac{1}{5}$ in. thick; albumen ruminated.

Borneo, Bangarmassing, *Mr J. Motley!* n. 721; Labuan, *Mr Barber!* n. 167, common on sandy soil near the barracks.

51. MABA MYRMECOCALYX, sp. nov.

M. glabra, foliis ellipticis, alternis, apice breviter et obtuse acuminatis, basi obtusis vel subrotundis, coriaceis, nitentibus, petiolatis; fructibus solitariis, axillaribus vel lateralibus, breviter pedunculatis, ellipsoideis, nitentibus, apice obsolete pubescentibus, calyce fructifero hemisphærico, breviter 4-*lobo, coriaceo, verrucoso, glabro.*

Glabrous. Leaves alternate, elliptical or sometimes somewhat ovate, shortly and obtusely acuminate, obtusely narrowed or nearly rounded at base, coriaceous, shining and brown (in dried state) on both sides, 2—4¼ in. long by 1⅓—2⅔ in. wide; petioles ⅕—⅖ in. long; veins inconspicuous. Fruit solitary, on the young branches, axillary or lateral, ellipsoidal, shining, smooth, obsoletely pubescent at apex, 1—1½ in. long by ¾—1 in. thick, 5-seeded, cells imperfect; peduncle thickened upwards, ⅓ in. long, somewhat verrucose, glabrous; calyx cup-shaped, verrucose, about 1 in. across, shallowly 4-lobed, half as high as the fruit; seeds oblong, ½ in. long.

Borneo, *O. Beccari!* n. 3568.

52. MABA BECCARII, sp. nov.

M. foliis elliptico-oblongis vel ovalibus, apice breviter et obtuse acuminatis, basi rotundatis, coriaceis, breviter petiolatis; floribus breviter pedunculatis, solitariis vel aliquando binis; calyce 4-fido, truncato, lobis rotundatis, arcte imbricatis, sinistrorse contortis; fructu immaturo pubescente, 3-loculari (?), *apice conico; stylo brevi, apice 3-lobo.*

Young shoots ferruginous, shortly tomentose; branches puberulous, spreading at about 55°. Leaves alternate, elliptic-oblong or oval, shortly and obtusely acuminate at apex, rounded at base, coriaceous, quickly glabrate except the shortly tomentose and strong midrib beneath; veins indistinct; ¾—2 in. long by ⅗—1 in. wide; petioles ⅛ in. long, rather stout, shortly pubescent.

Known only after the fall of the corolla. Flowers shortly pedunculate, solitary or occasionally 2 together; peduncles $\frac{1}{12}$—⅛ in. long. Calyx campanulate, ¼ in. long by ⅓ in. thick, shortly pubescent with rufous hairs, 4-fid, apparently truncate by closely contorted imbrication of the rounded lobes; ovary (or young fruit) rufous with short hairs, conical above, exceeding the calyx, 3-celled (?); style short, 3-lobed at apex.

Borneo, *O. Beccari!* n. 1948.

53. MABA SERICEA.

M. foliis lanceolatis, distichis, apice acuminatis, basi angustatis, coriaceis, supra nervo excepto glaberrimis, subtus dense pubescentibus, pilis appressis flavis sericeis, breviter petiolatis; floribus polygamis (?), *subsessilibus, plerumque pentameris, calyce campanulato, corolla 5-fida, staminibus* 50—60 *vel in fl. hermaphrod. circiter* 32, *hispidis; ovario* 6-*loculari, sericeo, loculis* 1-*ovulatis; fructibus flavo-pubescentibus,* 6-*locularibus.*

Diospyros sericea, Alph. DC. Prodr. VIII. p. 236. n. 67 (1844), Miq. in Mart. Fl. Bras. VII. Eben. p. 3. n. 1. t. 1. f. 2 (1856).

A polygamous (?) small leafy tree, 15 feet high, with patulous horizontal ramification and the leaves arranged in a pinnate manner; shoots flavo-pubescent. Leaves alternate, lanceolate, somewhat narrowed at base, distichous, patent, coriaceous, dark shining and glabrous except along depressed midrib above, densely flavo-sericeous beneath; with lateral veins not conspicuous; 1—3 in. long by ¼—1 in. wide; petioles $\frac{1}{16}$—⅐ in. long, pubescent. Bracts lanceolate, fulvo-pubescent; flowers usually pentamerous.

♂. Flowers 1—3 or more together, subsessile, axillary, tawny-hairy, $\frac{5}{12}$ in. long. Calyx campanulate, $\frac{3}{10}$ in. long, hairy on both sides, with 5 deltoid lobes $\frac{1}{3}$ length of calyx. Corolla 5-fid, with ovate-oblong lobes, glabrous within. Stamens 50—60, with linear slender hispid-pilose anthers and short filaments combined at the base, inserted at base of corolla; ovary rudimentary.

Hermaphrodite flowers solitary, on peduncles $\frac{1}{10}$—$\frac{1}{5}$ in. long; calyx 4—5-fid, with ovate lobes; corolla 4—5-fid; stamens about 32, perfect (?), not all in one row, somewhat silky, linear; inserted at base of corolla. Ovary 6-celled, 6-ovuled, silky, ovate-conical. Styles 3, emarginate, imbricated. Fruit subsessile, ellipsoidal, suddenly conical at apex, flavo-pubescent with longer hairs interspersed, about 1 in. long by $\frac{1}{2}$ in. thick, fleshy, 6-celled; fruiting peduncles thickened upwards with wide articulation at apex; fruiting calyx $\frac{2}{5}$ in. long, with 4 or 5 lanceolate lobes $\frac{1}{3}$ in. long, lying close to fruit or spreading, tomentose inside; seeds solitary, elongated, fusiform; embryo twice the length of the albumen.

Brazil, Minas Geraes, *Claussen!* 67, 464, 1062 (fruit glabrate, globular $\frac{1}{12}$ in. in diameter, drupaceous, 1-celled); *Martius!*; Goyáz, along the Caminho da Carreira, *Burchell!* 6970, 6986—2; called in Brazil "Culhõens de Macāto," *Burchell*; near Bahia, *Blanchet* 3358 ex Miq. in Mart. *l. c.*

54. MABA (?) CORDATA, sp. nov.

M. ramulis piloso-hispidis; foliis oblongis, apice acuminatis, basi supra-cordatis, submembranaceis, supra subglabris, subtus secus costam nervos et margines piloso-hispidis, breviter petiolatis; floribus femineis subsessilibus, 1—2-nis, fructibus immaturis basi globosis, apice abrupte et longe conicis, dense pilosis, 6-locularibus; loculis 1-spermis (-ovulatis); stylo 3-lobo; calyce 4—5-partito, extus pubescente, lobis lanceolatis patentibus.

Shoots, midrib, lateral veins below, and margins of leaves pilose-hispid with tawny hairs; branches terete; leaves alternate, oblong, acuminate, deeply cordate, submembranous, nearly glabrous and yellowish green above with inconspicuous slightly-depressed veins, paler beneath with clearly marked tawny veins, 4—8 in. long by $1\frac{1}{10}$—$2\frac{1}{10}$ in. wide; petioles $\frac{1}{6}$—$\frac{1}{5}$ in. long, ♀. Flowers subsessile, axillary, 1 or 2 together; young fruit with globular base and long suddenly conical apex terminating in 3-lobed style, together $\frac{1}{2}$ in. long by $\frac{1}{5}$ in. thick, densely fulvo-pilose, 6-celled, cells 1-ovuled (-seeded); fruiting calyx pubescent outside, glabrous inside, 4—5-partite; lobes lanceolate, acute, $\frac{2}{5}$ in. long, spreading, subcoriaceous.

Borneo, *O. Beccari!* n. 1429.

55. MABA MYRMECOCARPA.

M. foliis oblongis, apice acutis vel breviter acuminatis, basi angustatis, nervo excepto glabrescentibus, submembranaceis, breviter petiolatis; fructibus axillaribus, solitariis, subsessilibus, ovoideis, dense ferrugineo-hispidis, 6-locularibus; calyce fructifero 3-partito, utrinque fusco-sericeo, lobis acutis.

Diospyros (?) *myrmecocarpus*, Mart. in Fl. Bras. VII. Eben. p. 7 (1856).

Tree; shoots with spreading ferruginous hispid hairs, dull, glabrescent, terete. Leaves rather thickly membranous, narrowly oblong, 4—7 in. long by $1\frac{1}{3}$—2 in. wide, contracted at the base, acute or shortly acuminate at the apex (covered in the dried state, especially beneath, with small warts); midrib on both sides fusculo-strigillose; lateral veins 8—15, uniting within the margin, and somewhat sunk above, with scattered hairs; leaves at length glabrescent except the midrib; petioles about $\frac{1}{10}$ in. long; flowers unknown.

Young fruit axillary, solitary, subsessile, ovoid, somewhat pointed at apex, scarcely 1 in. long by $\frac{3}{5}$ in. thick; calyx 3-partite, fusco-sericeous on both sides, $\frac{3}{5}$ in. high, 1 in. across; lobes lanceolate acute; young fruit 6-celled, with dense ferruginous hairs arising from small rough warts, containing 6 young seeds.

Equatorial Brazil, Province Rio Negro, found in thick damp woods by river Japurá, near Manacarú, in January, *Martius!*

56. MABA MYRISTICOIDES.

M. foliis ovali-oblongis, apice acutis acuminatis, basi angustatis, tenuiter coriaceis, subglabrescentibus, breviter petiolatis; floribus masculis aggregatis, sessilibus, axillaribus, pubescentibus, sæpissime trimeris rarius pentameris, staminibus 12 *in floribus trimeris, circiter* 30 *in floribus pentameris, antheris linearibus sericeis, filamentis brevibus glabris.*

Macreightia myristicoides, Spruce MSS.

A small tree 3—7 ft. high; young parts subferruginous-hairy; branches 3 together, long, subsimple, brown, with appressed hairs. Leaves oval-oblong, somewhat narrowed at base, acuminate and acute at apex, alternate, thinly coriaceous, midrib depressed above, 3—7$\frac{1}{2}$ in. long by 1—2$\frac{2}{3}$ in. wide, with appressed scattered brown hairs especially beneath, subglabrescent; petioles $\frac{1}{7}$—$\frac{1}{5}$ in. long, with scattered spreading hairs.

♂. Flowers numerous, crowded in sessile axillary clusters, about $\frac{1}{4}$ in. long or pentamerous flowers $\frac{1}{3}$ in. long, covered with ferruginous straight hair, mostly trimerous, very rarely pentamerous; calyx campanulate, 3-fid with deltoid acute lobes (very rarely 5-fid); corolla white with ferruginous hairs; stamens 12 in trimerous flowers (about 30 in pentamerous flowers), anthers linear, with long straight silky hairs, filaments combined, short, glabrous; ovary 0.

N.W. Brazil, Near Panurè by Rio Uaupés, *Spruce!* 2542, October.

57. MABA (?) CAULIFLORA.

M. foliis ovali-oblongis, apice acuminatis vel breviter cuspidatis, basi rotundatis vel parum angustatis, submembranaceis, subglabrescentibus, breviter petiolatis; floribus femineis sessilibus, lateralibus, secus caules aggregatis, pentameris, pubescentibus, calyce 5-*partito, corollâ* 5-*partitâ, lobis oblongo-lanceolatis, staminodiis* 10—14, *geminatis, pilosis, stylo trifido, ovario rigide piloso,* 6 (?)-*loculari.*

Diospyros cauliflora, Mart. in Fl. Bras. VII. Eben. p. 7 (1856), non Blume.

A small tree; trunk sub-simple; branches crowded at the extremity, slender, terete, pubescent with brown appressed hairs, glabrescent. Leaves submembranous, oval-oblong, acuminate or shortly cuspidate, rounded or slightly narrowed at base, shining, with a few scattered appressed hairs and prominent veins beneath, subglabrescent, glabrous above with depressed veins, alternate, 3—9 in. long by 1$\frac{3}{5}$—3$\frac{1}{4}$ in. wide; petioles $\frac{1}{5}$—$\frac{3}{10}$ in. long, pubescent.

♀. Flowers sessile, in groups on the stem, $\frac{1}{3}$—$\frac{5}{12}$ in. high; calyx 5-partite, coriaceous, cup-shaped, appressedly hairy outside, lobes obtuse; corolla 5-partite, strigillose outside, coriaceous, lobes oblong-lanceolate, somewhat spreading; staminodes 10—14, inserted at base of corolla, united by their filaments in 5—7 pairs, the inner ones being the shorter; barren anthers covered with long rigid pilose hairs; style 3-fid, half concealed by hairs of ovary, hairy; stigmas bifid at apex. Ovary, with long rigid pilose hairs, (8-celled ex Fl. Brasil. *l. c.*) 6 ?-celled.

Surinam, by river Marowyne, *Wullschlägel;* Cayenne, *Martin!*

58. MABA HILAIREI, sp. nov.

M. foliis oblongis, utrinque angustatis, apice acuminatis, costâ et margine exceptis glabrescentibus, coriaceis, petiolatis, floribus femineis 3—7-*nis, breviter pedicellatis, fulvo-sericeis, pentameris, calyce campanulato profunde* 5-*fido, corollâ profunde* 5-*lobâ, lobis patentibus, staminodiis* 11—13, *leviter pilosis, corollæ basi insertis, stylis* 3, *ovario dense ferrugineo-piloso,* 6-*loculari.*

Young parts pubescent; branches terete. Leaves oblong, narrowed at both ends, acuminate at apex, glabrescent except midrib and margins beneath which are fulvo-puberulous; coriaceous, alternate, of dark slate colour above, reddish dull-brown beneath, with two rows of depressed glands not far from the midrib on each side, veins rather distant not conspicuous, midrib depressed above, 3—4 in. long by 1—1$\frac{1}{3}$ in. wide; petioles $\frac{1}{6}$ in. long, somewhat twisted.

♀. Inflorescence fulvo-sericeous, 3—7-flowered, $\frac{1}{4}$—$\frac{1}{3}$ in. long, bracts, ovate, small; peduncle almost obsolete, pedicels $\frac{1}{12}$—$\frac{1}{4}$ in. long; flowers pentamerous, $\frac{3}{8}$ in. long; calyx $\frac{1}{6}$ in. long, campanulate, deeply 5-fid, lobes lanceolate or ovate pubescent on both sides; corolla deeply 5-lobed, stellate, lobes spreading, fulvo-pubescent along middle outside, glabrous inside, contorted in æstivation; staminodes 11—13, somewhat pilose, inserted at the base of the corolla. Ovary densely ferruginous-pilose, globose, flattened at top; terminated by 3 distinct styles pubescent below, $\frac{1}{6}$ in. long, diverging in full flower, glabrous and lobed at apex; ovary 6-celled, cells 1-ovuled.

Brazil, Province of Espiritu Santo, *A. St Hilaire!* 1816—1821, n. 375.

59. MABA MELLINONI, sp. nov.

*M. foliis ellipticis, apice acuminatis, basi angustatis, supra nervo excepto glabris, subtus pallidioribus sparsis pilis appressis, petiolatis; floribus femineis sub-*3-*nis, breviter cymosis,* 5—6-*meris, pedicellis brevissimis, calyce campanulato* 5—6-*fido, utrinque pubescente, lobis deltoideis, corollâ* 5—6-*partitâ, staminodiis* 25—30, *minutis, uniserialibus, corollæ basi insertis, subglabris, ovario dense rufo-piloso,* 6-*loculari, stylis* 3.

Young parts pubescent. Leaves alternate, elliptical, acuminate at apex, narrowed at base, about 2$\frac{1}{4}$ in. long by $\frac{7}{8}$ in. wide, of dark slate green colour above and glabrous except depressed midrib, paler beneath and appressedly pubescent on midrib and with scattered appressed hairs on the lamina; margins recurved; petioles $\frac{1}{8}$ in. long.

♀. Flowers about 3 together, in short axillary cymes, 5—6-merous, $\frac{1}{3}$ in. long by $\frac{3}{16}$ in.

wide; pedicels very short; calyx $\frac{1}{5}$ in. long, campanulate, pubescent on both sides, especially inside, 5—6-fid, lobes deltoid; corolla (not expanded) glabrous except 5—6 pubescent lines in lower part, 5—6-partite; lobes elliptical, probably spreading in full flower; staminodes 25—30, very small, subequal, inserted in one row at base of corolla, nearly glabrous; ovary densely rufous-pilose, subglobose, flattish at top, 6-celled, 6-ovuled, surmounted by 3 erect contiguous styles which are pubescent at base and acutely bilobed at apex.

S. America, French Guiana, Maroni, *Mellinon!*

IV. DIOSPYROS, Dalech. Hist. Lib. III. cap. XXI. p. 349 (1587).

Flores diœci, rarius monœci vel polygami, 3—7-, *sæpius* 4—5-*meri, cymosi. Calyx lobatus vel rarius truncatus, in fructu sæpe accrescens. Corolla lobata; lobis obtusis vel rarius acutis, in præfloratione sinistrorse contortis.*

FLOS MASCULUS: *stamina* 4—∞, *sæpius circiter* 16 *et biserialibus; ovarium plerumque abortivum.*

FLOS FEMINEUS: *staminodia sæpius* 4—8, *interdum* 0; *ovarium* 4—16-, *rarissime* 6-, *nunquam 3-loculare.*

Arbores vel frutices, foliis alternis vel rarius suboppositis nunquam verticillatis, cymis axillaribus vel rarius secus ramos vetustiores lateralibus, interdum unifloris.

Linn. Gen. Plant. pp. 143, 383. n. 403 (1737), Alph. DC. Prodr. p. 222. n. VI. (1844), Agardh. Theor. Syst. Plant. t. X. figs. 11, 12 (1858). Cfr. Cesalp. De Plantis, lib. II. cap. LII. p. 86 (1583).

Ermellinus, Cesalp. De Plantis, lib. III. cap. XXI. p. 104 (1583).
Pseudolotus, Camer. Epit. p. 156 (1586).
Lotus (sp.), Camer. Epit. p. 157 (1586), non Linn.
LIGNUM VITÆ, Gerarde Herball, p. 1309 (1597), non auct. al.
Guaiacum (sp.), Gerarde Herball, p. 1310 (1597).
Pishamin, Parkins. Theatr. Bot. p. 1523. f. 4 (1640).
Guaiacana, Tourn. Inst. rei Herb. p. 600. t. 371 (1700).
Ficus (sp.), Kæmpf. Amœnit. Exot. p. 805 (1712).
Hebenaster, Rumph. vol. III. (lib. iv.) p. 15. t. 6 (1750).
Paralea, Aubl. Plant. Guin. vol. I. p. 576. t. 231 (1775).
Dactylus, Forsk. Fl. Ægypt. Arab. p. XXXVI. n. 481 (1775).
Embryopteris, Gaertn. De Fructibus et Seminibus Plantarum, vol. I. p. 145. t. 29 (1788).
Ebenus, Commers. ex Jussieu Gen. Pl. p. 156 (1789), non Linn.
Cavanillea, Desrouss. in Encycl. Méth. III. p. 663 (1789), non auct. al.
Garcinia (sp.), Desrouss. in Encycl. Méth. III. p. 701 (1789), non Linn.
(?) *Euclea* (sp.), Lour. Fl. Cochinch. p. 629 (1790).
Annona (sp.), Jacq. Fragm. Bot. p. 40. t. 44. f. 7 (1800—9).
Cargillia, R. Br. Prodr. Fl. Nov. Holl. p. 526 (1810).
Monodora (sp.), Dunal, Monogr. Anon. p. 80 (1817).
Leucoxylum, Blume, Bijdr. Fl. Ned. Ind. p. 1169 (1826).
Noltia, Schum. Plant. Guin. p. 189 (1827).

Guatteria (sp.), Wall. List. n. 7295 (1828—32).

Sapota (sp.), Blanco, Fl. Filip. p. 409 (1837).

Patonia (sp.), Wight, Illustr. I. p. 19 (1840).

Gunisanthus, Alph. DC. Prodr. VIII. p. 219. n. 3. (1844).

Rospidios, Alph. DC. *l. c.* p. 220. n. 4.

Danzleria, Bert. ex Alph. DC. *l. c.* p. 224. n. 8.

Vaccinium (sp.), Wall. ex Voigt, Hort. Suburb. Calcutt. p. 345 (1845).

Flowers diœcious or rarely polygamous, very rarely monœcious and then casually so; inflorescence usually short, cymose, axillary on the young branches, or occasionally arising from the old wood and lateral, more or less pubescent or tomentose, bracteate, 1- few- or many-flowered. Calyx 3—7-lobed, usually 4—5-fid, rarely truncate or obscurely lobed at apex, usually pubescent outside, often accrescent in fruit; in *D. Toposia*, Hamilt., closed in bud and afterwards irregularly broken. Corolla urceolate campanulate tubular or salver-shaped, usually pubescent outside at least along the middle line of the lobes and with 3—7 usually 4—5 spreading or recurved, rarely erect, obtuse or occasionally acute lobes; usually contracted at the throat, that is, at the top of the tube; lobes sinistrorsely (as regarded from inside) contorted in bud; in *D. oocarpa*, Thw., irregularly imbricated; never valvate. Stamens in ♂ flowers 4—∞ usually about 16 and more or less united by their filaments in pairs or otherwise, sometimes altogether separate, glabrous or hairy, inserted at the base of the corolla or hypogynous or very rarely about middle of corolla; when in pairs one stamen is placed in front of the other, the interior one being usually shorter than the exterior, sometimes equal; anthers oblong linear or lanceolate, never globose nor squarish, often apiculate at the apex by projecting connective, 2-celled; dehiscing laterally by longitudinal slits or rarely by apical pores; pollen widely ellipsoidal or globose; filaments usually shorter than the anthers and slender, sometimes almost obsolete, occasionally geniculate. In ☿ flowers staminodes usually present and fewer than the stamens in the ♂. Styles 1—4, or obsolete; stigma emarginate or punctiform. Ovary in ♂ flowers rudimentary or absent; in ☿ hairy or glabrous, ovoid conical or globose (in *D. Diepenhorstii* "stipitato—constricted at base"), 4—16-celled, usually 8-celled; ovules solitary in the cells, or in the section *Cargillia* 2 together. Fruit usually globose oblong or conical, glabrous glabrate pubescent hispid or tomentose, often about 1 in. in diameter, but varying from ½ in. to about 4 inches, often pulpy and edible, with thin or thick skin, containing 1—10 seeds; seeds usually oblong with dark more or less shining testa. Albumen cartilaginous, white and equable or in some species more or less ruminated by sinuous intrusion of the testa; embryo as in the family. Fruiting calyx often accrescent with the lobes erect spreading or reflexed and frequently dilated at the base, sometimes plicate, coriaceous or foliaceous.

The name is derived from ζεύς, διός Jupiter, and πυρός grain, with reference to the presumed life-giving properties of the fruit; but the allusion is by no means obvious.

DIOSPYROS.

KEY TO THE SECTIONS AND SPECIES.

- Seeds with ruminated albumen; leaves in some species opposite. § I. MELONIA.
- Albumen of seeds not ruminated; leaves always alternate.
  - Calyx truncate and entire or very shortly lobed; stamens glabrous. § II. EBENUS.
  - Calyx distinctly lobed, or stamens more or less hairy.
    - Fruit conical; ovary usually 4-celled, cells 1-ovuled. § III. NOLTIA.
    - Fruit globular, ovoid, obovoid or oblong; ovary 4—16-celled, cells 1—2-ovuled.
      - Pedicels long or cymes lax; stamens 8—21.
        - ♀ flowers solitary or on distinct peduncles. § IV. GUNISANTHUS.
        - ♀ flowers cymose. § V. GUAIACANA.
      - Pedicels short or cymes dense, or stamens very numerous.
        - Stamens all or half of them inserted about middle of corolla. § VI. CUNALONIA.
        - Stamens inserted on the receptacle or at base of corolla.
          - Corolla tubular, often salver-shaped; stamens 4—32, when numerous usually unequal.
            - Stamens quite glabrous. § VII. ERMELLINUS.
            - Stamens more or less hairy. § VIII. PATONIA.
          - Corolla urceolate or campanulate; stamens 8—22, usually unequal.
            - Ovary 4—16-celled, cells 1-ovuled.
              - Anthers dehiscing laterally by apical pores. § IX. LEUCOXYLUM.
              - Anthers dehiscing laterally by longitudinal slits.
                - Ovary glabrous (except apex). § X. DANZLERIA.
                - Ovary hairy. § XI. PARALEA.
            - Ovary 4-celled, cells 2-ovuled. § XII. CARGILLIA.
          - Corolla deeply lobed, subrotate; stamens 15—50, subequal. § XIII. ROSPIDIOS.
          - Corolla egg-shaped or oblong, shortly lobed; stamens numerous, subequal. § XIV. CAVANILLEA.
    - Calyx closed with connate lobes in bud, afterwards bursting irregularly. § XV. AMUXIS.

§ I. Melonia.

Leaves opposite or subopposite, or frequently so.

- Leaves deeply cordate at base; fruit pyriform. — 1. *D. calophylla.*
- Leaves rounded or narrowed not cordate at base; fruit globular or ellipsoidal.
  - ♂ flowers subsessile; ♀ flowers sessile.
    - Stamens 14—20; petioles $\frac{1}{4}$—$\frac{1}{2}$ in. long. — 2. *D. insignis.*
    - Stamens about 8; petioles $\frac{1}{10}$—$\frac{1}{6}$ in. long. — 3. *D. oppositifolia.*
  - ♂ flowers in short cymes; ♀ flowers subsessile or pedunculate.
    - Leaves rounded or obtuse at base; staminodes 0—6. — 4. *D. Tupru.*
    - Leaves narrowed at base; staminodes 8 or 10. — 5. *D. melanoxylon.*

Leaves all alternate.

- Fruiting calyx not much plicate; tube without internal elevated rim at top.
  - Fruit terete, smooth.
    - Flowers shortly urceolate or subglobose.
      - Leaves of nearly the same colour on both surfaces.
        - Leaves pubescent; staminodes 10. — 6. *D. decandra.*
        - Leaves glabrescent; staminodes 4.
          - ♀ flowers solitary; ovary 8—6-celled. — 7. *D. sylvatica.*
          - ♀ flowers about 3 together; ovary 4-celled. — 8. *D. Kurzii.*
      - Leaves of different colours on the two surfaces.
        - Stamens about 22; leaves elliptical. — 9. *D. ehretioides.*
        - Stamens 14—16; leaves oval-oblong. — 10. *D. rotundiflora.*
    - Flowers tubular or urceolate-oblong.
      - Leaves without conspicuous lateral veins; flowers axillary.
        - Leaves glabrous or quickly glabrescent.
          - Leaves usually elliptical; flowers subsessile. — 11. *D. hirsuta.*
          - Leaves usually oval-oblong; flowers in short cymes. — 12. *D. mespiliformis.*
        - Leaves more or less tomentose, especially beneath. — 13. *D. burmanica.*
      - Leaves with clearly marked lateral veins; flowers lateral and axillary. — 14. *D. lateralis.*
  - Fruit somewhat tetragonal, verrucose. — 15. *D. verrucosa.*
- Fruiting calyx not plicate; tube with internal elevated rim at top. — 16. *D. Korthalsiana.*
- Fruiting calyx much plicate; tube without internal elevated rim at top.
  - ♀ flowers solitary.
    - ♀ calyx-lobes acuminate; ovary 4—6-celled. — 17. *D. affinis.*
    - ♀ calyx-lobes obtuse or rounded and apiculate; ovary 8-celled. — 18. *D. crumenata.*
  - ♀ cymes many-flowered.
    - Leaves submembranous, $2\frac{1}{2}$—5 in. long. — 19. *D. frutescens.*
    - Leaves coriaceous, 4—8 in. long. — 20. *D. densiflora.*

§ II. Ebenus.

Fruit sub-verrucose; æstivation of corolla irregular. 21. *D. oocarpa.*
Fruit not verrucose; æstivation of corolla regularly contorted.
  Corolla-lobes acuminate.
    Flowers glabrous. 22. *D. truncata.*
    Flowers fulvo-velutinous. 23. *D. halesioides.*
  Corolla-lobes not acuminate.
    Ovary hairy.
      Corolla nearly glabrous, but margins ciliate; pedicels manifest. 24. *D. borneensis.*
      Corolla hairy; flowers sessile or subsessile.
        Leaves whitish beneath; flowers fuliginous-hispid. 25. *D. batocana.*
        Leaves not white beneath; flowers not fuliginous.
          Stamens 16. 26. *D. quæsita.*
          Stamens 10—13.
            Styles short.
              Leaves acuminate. 27. *D. toxicaria.*
              Leaves obtuse, not acuminate. 28. *D. tessellaria.*
            Style manifest, 4-lobed at apex. 29. *D. haplostylis.*
          Stamens 20—32.
            ♂ cymes few-flowered; leaves coriaceous.
              Fruiting calyx-lobes reflexed. 30. *D. melanida.*
              Fruiting calyx-lobes erect. 31. *D. nodosa.*
            ♂ cymes 5—15-flowered; leaves submembranous. 32. *D. anonæfolia.*
    Ovary glabrous.
      Flowers 1—5 together, subsessile, pubescent; fruiting calyx not plicate.
        Stamens 30—40; leaves subsessile, cordate, coriaceous; branches usually straight. 33. *D. leucomelas.*
        Stamens 11—15; leaves petiolate, not cordate, coriaceous; branches flexuous. 34. *D. chrysophyllos.*
        Stamens 16; leaves petiolate, not cordate, submembranous; branches straight. 35. *D. senensis.*
      Flowers solitary, shortly pedunculate, glabrous; fruiting calyx plicate. 36. *D. rotundifolia.*

§ III. NOLTIA.

- Ovary hairy.
  - Leaves not cordate at base.
    - Ovary (or young fruit) 4- (or 6-) celled, not fuliginous.
      - Stamens 4—5; leaves acuminate.
        - Leaves thin, 2—4 in. long. 37. *D. attenuata.*
        - Leaves coriaceous, 5—12 in. long. 38. *D. acuta.*
      - Stamens 6—16; leaves obtuse. 39. *D. tricolor.*
    - Ovary (or young fruit) 8-celled, fuliginous-hispid. 40. *D. fuliginea.*
    - Ovary 10-celled, fulvous-pubescent. 41. *D. Brandisiana.*
  - Leaves more or less cordate at base.
    - Calyx 4-fid; fruit appressedly pubescent. 42. *D. subacuta.*
    - Calyx 4—5-partite; fruit covered with patent hairs.
      - Corolla appressedly sericeous. 43. *D. pruriens.*
      - Corolla glabrous. 44. *D. apiculata.*
- Ovary glabrous except apex. 45. *D. Barteri.*

§ IV. Gunisanthus.

- Flowers urceolate or not very slender.
  - Calyx quite glabrous.
    - Ovary 8-celled; staminodes 4. — 46. *D. microrhombus.*
    - Ovary 4-celled; staminodes 0. — 47. *D. foliolosa.*
  - Calyx more or less hairy.
    - Peduncles ebracteate, articulated at apex. — 48. *D. pilosula.*
    - Peduncles bracteate.
      - Bracts densely imbricated at base of peduncles. — 49. *D. suberifolia.*
      - Bracts not densely imbricated.
        - Ovary glabrous. — 50. *D. squarrosa.*
        - Ovary hairy.
          - Peduncles stout; flowers pentamerous. — 51. *D. paniculata.*
          - Peduncles slender; flowers tetramerous. — 52. *D. gracilipes.*
- Male flowers narrowly tubular, very slender. — 53. *D. graciliflora.*

§ V. Guaiacana.

- Fruiting calyx tough; leaves coriaceous or subcoriaceous; stamens 12—20; diœcious.
  - ♀ Cymes 3-flowered, stiff. — 54. *D. Pervillei.*
  - Cymes many-flowered.
    - Flowers pentamerous; stamens 20. — 55. *D. dictyoneura.*
    - Flowers tetramerous; stamens 12—16.
      - Leaves not cordate at base.
        - ♀ Cymes racemose; veins of leaves in relief on both sides. — 56. *D. asterocalyx.*
        - ♀ Cymes panicled; veins of leaves markedly depressed on upper surface. — 57. *D. Horsfieldii.*
      - Leaves cordate at base. — 58. *D. Boivini.*
- Fruiting calyx usually foliaceous; leaves membranous; stamens 8; polygamous. — 59. *D. Loureiriana.*

§ VI. Cunalonia.

- Stamens 20 or 24, all inserted in pairs about middle of corolla; corolla-lobes obtuse. — 60. *D. Dendo.*
- Stamens 8, half inserted at base and half at middle of corolla; corolla-lobes acute. — 61. *D. Cunalon.*

§ VII. Ermellinus.

Fruiting calyx-tube without internal elevated rim at top.
  Stamens 8; leaves not velutinous.
    Leaves glabrous; branches glabrous except extremities.
      Leaves cuneate at base, $1\frac{1}{2}$—3 in. long. 62. *D. tetrasperma.*
      Leaves obtuse at both ends, 4—5 in. long. 63. *D. Carthei.*
    Leaves appressedly pubescent, at least on veins beneath; shoots pubescent. 64. *D. polyalthioides.*
  Stamens 9—10; leaves velutinous-pubescent; flowers 4—5-merous. 65. *D. Kirkii.*
  Stamens 12; leaves velutinous beneath; lateral flowers 3-merous. 66. *D. velutina.*
  Stamens 12; leaves glabrous, not velutinous; flowers 5-merous. 67. *D. plectosepala.*
  Stamens 14—20.
    Fruit obconical. 68. *D. stricta.*
    Fruit rounded at base.
      Corolla-lobes acute or acuminate; ovary hairy.
        Calyx-lobes lanceolate. 69. *D. eriantha.*
        Calyx-lobes rounded.
          Leaves and petioles glabrous. 70. *D. variegata.*
          Leaves and petioles more or less pubescent. 71. *D. dasyphylla.*
      Corolla-lobes obtuse; ovary glabrous.
        Leaves ferruginous-pubescent beneath, rounded at base. 72. *D. Beccarii.*
        Leaves glabrescent, narrowed at base.
          ♂ Calyx glabrous outside, tomentose inside; peduncles $\frac{1}{4}$—$\frac{1}{2}$ in. long 73. *D. oleifolia.*
          Calyx more or less hairy outside; peduncles very short.
            ♂ Calyx deeply 4-lobed; ovary 4-celled.
              Leaves somewhat ovate. 74. *D. flavicans.*
              Leaves somewhat obovate. 75. *D. sapotoides.*
            ♂ Calyx 4—5-fid; ovary 10-celled. 76. *D. aurea.*
  Stamens 32. 77. *D. nigricans.*
Fruiting calyx with internal elevated rim at top of its tube.
  Flowers diœcious; stamens 16—32; leaves opaque. 78 *D. Ebenum.*
  Flowers polygamous; stamens 8; leaves minutely pellucid-punctate. 79. *D. pellucida.*

§ VIII. PATONIA.

- Stamens 4. — 80. *D. tetrandra.*
- Stamens 8—30.
  - Corolla 4—5-fid, or lobed much beyond the apex.
    - Leaves subcaudate at apex. — 81. *D. Sprucei.*
    - Leaves obtuse, not acuminate. — 82. *D. maritima.*
    - Leaves more or less acuminate, not caudate.
      - Ovary 4-celled. — 83. *D. philippinensis.*
      - Ovary 8-celled.
        - ♀ Flowers 6 or more together. — 84. *D. pilosanthera.*
        - ♀ Flowers solitary.
          - Filaments glabrous except apex. — 85. *D. lanceæfolia.*
          - Filaments pilose. — 86. *D. Gardneri.*
  - Corolla 4—5-lobed only at apex; filaments hairy.
    - Leaves ovate, paler beneath. — 87. *D. Heudelotii.*
    - Leaves oblong, of same colour on both surfaces.
      - Stamens 11—14. — 88. *D. undulata.*
      - Stamens 15—30.
        - ♂ Flowers clustered, several together. — 89. *D. multiflora.*
        - ♂ Flowers 2 together. — 90. *D. biflora.*

§ IX. LEUCOXYLUM.

- Corolla glabrous inside; stamens glabrous; leaves less than 1 in. in width.
  - ♂ Flowers solitary; calyx 3-lobed. — 91. *D. parvifolia.*
  - ♂ Flowers 3—4 together; calyx 4-lobed. — 92. *D. buxifolia.*
- Corolla tomentose on both sides; filaments tomentose; leaves more than 1 in. in width. — 93. *D. Vescoi.*

§ X. DANZLERIA.

- Ovary 4-celled. 94. *D. Morrisiana.*
- Ovary 8-celled.
  - ♂ Flowers 1—4 together (or rarely panicled), or stamens not glabrous.
    - ♂ Flowers surrounded at base by 5—6 imbricated bracts nearly as long as the calyx. 95. *D. squamosa.*
    - Bracts not enveloping the calyx.
      - Corolla fleshy. 96. *D. comorensis.*
      - Corolla not fleshy.
        - Calyx-lobes deltoid or rounded.
          - Corolla glabrous or nearly so. 97. *D. montana.*
          - Corolla hairy outside. 98. *D. Zollingeri.*
        - Calyx-lobes ovate or lanceolate.
          - Stamens hairy.
            - Flowers pubescent outside.
              - Leaves ciliate; calyx-lobes obtuse. 99. *D. ciliata.*
              - Leaves not ciliate; calyx-lobes lanceolate.
                - Flowers sessile or subsessile.
                  - Fruit $\frac{1}{2}$—$\frac{2}{3}$ in. in diameter. 100. *D. Lotus.*
                  - Fruit 1—$1\frac{1}{3}$ in. in diameter. 101. *D. virginiana.*
                - Flowers shortly pedunculate. 102. *D. Kaki.*
            - Flowers glabrous outside, but calyx-lobes ciliate. 103. *D. chartacea.*
          - Stamens glabrous.
            - Corolla-lobes acute. 104. *D. vaccinioides.*
            - Corolla-lobes obtuse.
              - Stamens 10—12. 105. *D. cayennensis.*
              - Stamens 16. 106. *D. lævis.*
  - ♂ Flowers in dense axillary clusters; stamens glabrous.
    - Glabrous, not spinous; stamens 12. 107. *D. Thouarsii.*
    - More or less tomentose, often spinous; stamens 16. 108. *D. chloroxylon.*

§ XI. Paralea.

- ♀ Flowers arising from the old wood.
  - Albumen radiately striate.
    - ♂ Flowers pentamerous; stamens 20. — 109. *D. pergamena.*
    - ♂ Flowers tetramerous; stamens 16. — 110. *D. cauliflora.*
  - Albumen equable, not radiately striate.
    - Ovary ovoid-conical, 10—12-celled. — 111. *D. ramiflora.*
    - Ovary stipitato-constricted at base, 14—16-celled. — 112. *D. Diepenhorstii.*
- Flowers arising from the young branches.
  - Fruiting calyx foliaceous; leaves with long narrow acumen at apex. — 113. *D. sumatrana.*
  - Fruiting calyx not foliaceous; leaves without long narrow acumen at apex.
    - ♂ Flowers solitary. — 114. *D. pendula.*
    - ♂ Flowers cymose, not solitary.
      - ♂ Flowers very hard and crass (in the dry state). — 115. *D. macrophylla.*
      - ♂ Flowers not very hard.
        - ♀ Fruiting calyx-tube without elevated internal rim at top.
          - Flowers diœcious; fruit ½—1½ in. in diameter.
            - Ovary 4—8-celled.
              - Stamens glabrous.
                - Corolla nearly glabrous. — 116. *D. ovalifolia.*
                - Corolla hairy outside.
                  - ♂ Flowers 1—3 together. — 117. *D. texana*
                  - ♂ Flowers 5—7 together. — 118. *D. mabacea.*
              - Stamens not glabrous.
                - Filaments glabrous, anthers silky.
                  - Leaves glabrous, yellowish. — 119. *D. pentamera.*
                  - Leaves margined with tomentum, at least when young. — 120. *D. Paralea.*
                - Filaments hairy.
                  - Corolla glabrous (except 4 lines). — 121. *D. rhodocalyx.*
                  - Corolla pubescent outside.
                    - Leaves coriaceous, 2—5 in. long. — 122. *D. macrocarpa.*
                    - Leaves firmly membranous, 6—7½ in. long. — 123. *D. perforata.*
            - Ovary 10-celled. — 124. *D. oblonga.*
          - Flowers polygamous; fruit 1½—4 in. in diameter, edible. — 125. *D. Ebenaster.*
        - ♀ Fruiting calyx-tube with internal elevated rim at top.
          - Flowers tetramerous; corolla-lobes obtuse. — 126. *D. samoënsis.*
          - Flowers 3—4-merous; corolla-lobes acute. — 127. *D. Olen.*

§ XII. CARGILLIA.

Leaves oblong or oval, without evident glands. Australian.	128. *D. Cargillia.*
Leaves oval, scattered with glands. Philippine Islands.	129. *D. Malacapai.*

§ XIII. ROSPIDIOS.

Stamens glabrous.	
Spinous; flowers very small; stamens 16.	130. *D. spinosa.*
Not spinous; flowers of moderate size; stamens 18—24.	
Leaves about 1 in. long.	131. *D. ovalis.*
Leaves 2—5 in. long.	132. *D. hispida.*
Stamens more or less pilose.	
Leaves not markedly paler beneath.	
Leaves cordate at base.	
Calyx deeply 5-fid.	
Leaves with margins but slightly revolute, and veins clearly depressed on the upper surface.	133. *D. Goudotii.*
Leaves with margins much revolute, and tertiary veins not clearly depressed on the upper surface.	134. *D. gaultheriæfolia.*
Calyx shortly 4—7-fid.	
Stamens 20.	135. *D. subrotata.*
Stamens 40—50.	136. *D. polyandra.*
Leaves not cordate at base.	
Leaves pubescent beneath.	
♂ flowers 3 together; stamens 18—24.	137. *D. coccolobæfolia.*
♂ flowers several together; stamens 30—45.	
♂. Cymes very dense; stamens 30.	138. *D. Pearcei.*
♂. Cymes less dense; stamens 36—45.	139. *D. peruviana.*
Leaves glabrous.	140. *D. Weddellii.*
Leaves markedly paler beneath.	
Stamens 26—45.	
Leaves 6—9 in. long; corolla very silky outside.	141. *D. glomerata.*
Leaves 2—3 in. long; corolla glabrous except hairy lines outside.	142. *D. capreæfolia.*
Stamens 15—17.	143. *D. Mannii.*
[Cfr.	144. *D. artanthæfolia.*]

§. XIV. CAVANILLEA.

- Fruit not densely hairy; corolla 4—6-lobed at apex.
  - ♂ flowers $\frac{1}{3}$—$\frac{2}{5}$ in. thick.
    - Leaves cuneate at base.
      - Stamens 12—20; leaves oval or oblong-lanceolate. — 145. *D. Poeppigiana.*
      - Stamens 25—32; leaves obovate. — 146. *D. emarginata.*
    - Leaves rounded or obtusely narrowed at base.
      - Leaves subvelutinous beneath. — 147. *D. rigida.*
      - Leaves glabrous on both sides.
        - Styles 4, hairy at base. — 148. *D. Embryopteris.*
        - Style glabrous, lobed at apex. — 149. *D. coriacea.*
  - ♂ flowers $\frac{1}{2}$—$\frac{2}{3}$ in. thick. — 150. *D. crassiflora.*
- Fruit densely hairy; corolla 4—5-fid.
  - Stamens glabrous. — 151. *D. discolor.*
  - Stamens densely pilose. — 152. *D. argentea.*

§ XV. AMUXIS. — 153. *D. Toposia.*

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1. DIOSPYROS CALOPHYLLA, sp. nov.

D. foliis suboppositis, oblongis, apice acuminatis, basi profunde cordatis, glabrescentibus, subcoriaceis, nitidis, uni-coloribus, petiolatis; floribus femineis aggregatis, breviter pedunculatis, axillaribus; fructibus pyriformi-obconicis, rufo-tomentosis, pluri locularibus; calyce fructifero apice 4-lobo, rufo-tomentoso, appresso, fructum æquante.

A small tree, of rich brown colour in the dry state, glabrous except the inflorescence and young parts which are rufous-puberulous. Leaves oblong, subcoriaceous, deeply cordate at base, acuminate at apex, subopposite, shining, of same colour on both sides, 6—12 in. long or more by 1$\frac{1}{4}$—3$\frac{1}{2}$ in. wide; petioles $\frac{1}{8}$—$\frac{1}{2}$ in. long, rufous-puberulous, glabrescent. Fruits clustered, axillary, several together (5—10 in flower) on rufous-puberulous peduncles $\frac{1}{8}$—$\frac{1}{4}$ in. long; bracts small; fruit pyriform-obconic, rufous-tomentose, 1—1$\frac{1}{4}$ in. long × $\frac{7}{8}$—1 in. thick, several-celled; enclosed to full height by appressed thick calyx which is shortly and roundedly 4-lobed at apex and rufous-tomentose on both sides; fruit tipped with remains of rufous-tomentose style.

Madagascar, not far from the sea at Angonsti, *Richard!* 36.

2. DIOSPYROS INSIGNIS, Thw. En. Ceyl. Pl. p. 180. n. 10. (1860).

D. foliis suboppositis et alternis, firmiter membranaceis, ovatis vel oblongis, acuminatis, basi rotundatis vel angustatis, glabris; floribus masculis lateralibus, glomeratis, subsessilibus, tetrameris, staminibus 14—20; *floribus femineis* 1—3-*nis, axillaribus, sessilibus, ovario* 8-*loculari, fructibus subglobosis lævibus, calyce aucto crasso lignoso, margine reflexo; albumine ruminato.*

Bedd. Ic. Pl. Ind. Or. (Pt. VII.) p. 26. t. 130 (1871).

A very large tree, with young shoots somewhat pubescent. Leaves subopposite and alternate, toughly membranous, glabrous, ovate or oblong, acuminate, rounded or somewhat narrowed at base, midrib and lateral veins clearly marked on the under surface, tertiary veins nearly parallel and transverse to the midrib, 4—14 in. long by $1\frac{1}{2}$—6 in. wide; petioles $\frac{1}{4}$—$\frac{5}{12}$ in. long.

♂ Flowers crowded many together in subsessile lateral clusters. Calyx $\frac{1}{8}$—$\frac{1}{5}$ in long, campanulate, shortly 4-fid, lobes ovate, acute, shortly pubescent outside, glabrous inside; corolla $\frac{1}{3}$ in. long, shortly tubular, 4-lobed, pale-tomentose outside, lobes short; stamens 14—20, nearly glabrous, many in pairs, the longer filaments geniculate and with light-coloured hairs.

♀ Flowers 1—3 together, axillary, sessile. Ovary 8-celled; fruit subglobular, smooth, subglabrous, inconspicuously depresso-areolated, $1\frac{1}{2}$ in. in diameter, supported on a thick accrescent woody calyx whose tube forms a cup with elevated rim, margin reflexed; seeds $\frac{6}{7}$—$\frac{11}{12}$ in. long by $\frac{3}{10}$—$\frac{1}{2}$ in. wide; albumen ruminated. A valuable timber tree.

Ceylon,—*Thwaites!* 2000 ft. alt., C.P. 2730, 3477; S. India, Anamallays, *Beddome*, 2000—3000 ft. alt. Called "Gona-gass" in Ceylon.

3. DIOSPYROS OPPOSITIFOLIA, Thw. En. Ceyl. Pl. p. 181. n. 11. (1860).

D. foliis oppositis, obtusis, breviter acuminatis, basi rotundatis, coriaceis, glabratis, breviter petiolatis; floribus masculis anguste tubulosis, subsessilibus, tetrameris, calyce 4-*fido, campanulato, corollâ breviter* 4-*fidâ, staminibus circiter* 8, *inæqualibus.*

Bedd. Ic. Pl. Ind. Or. (Pt. VII.) p. 27. t. 131 (1871).

A moderate-sized tree; branches glabrous terete. Leaves oval, firmly coriaceous, glabrous (or the younger ones slightly pubescent), rounded at base, obtuse or shortly acuminate at apex, opposite or subopposite, 2—6 in. long by $1\frac{1}{2}$—3 in. wide; net-veins inconspicuous, nearly transverse and feebly depressed on the lower surface; petioles $\frac{1}{10}$—$\frac{1}{6}$ in. long, tumid-crass, dark, glabrous.

♂ Flowers sessile or subsessile, few together, $\frac{3}{8}$ in. long. Calyx $\frac{1}{12}$ in. long, not quite glabrous, 4-lobed nearly to the middle, with acute lobes. Corolla slender, hispid, $\frac{1}{3}$ in. long, lobes about $\frac{1}{3}$ the depth of the corolla. Stamens about 8, very unequal; (the filaments and connectives are figured as having short hairs). The timber of this tree resembles that of *D. quæsita*, Thw., Calamander.

Ceylon, *Thwaites!*; C.P. 3011; Hinidoon Corle, up to elevation of 1000 feet; local name "Kaloomidereya-gass."

4. DIOSPYROS TUPRU, Buch. Journey vol. I. p. 183 (1807).

D. foliis alternis et suboppositis, ellipticis ovatis vel subrotundis, apice obtusis rarius acuminatis, basi rotundatis vel rarius angustatis, coriaceis, subtus tomentosis, petiolatis; pedunculis florum masculorum longitudine petioli, apice 3—4-floris, calyce campanulato apice 4—6-lobo, staminibus 12—18; floribus femineis solitariis brevissime pedunculatis, staminodiis 0—6, fructibus subglobosis vel ovoideis glabris, calyce profunde 4—6-fido, lobis ovatis margine extus reflexis; albumine ruminato.

Hamilt. (olim Buch.) in Trans. Linn. Soc. xv. p. 111 (1827).

Diospyros exculpta, Hamilt. *l.c.* p. 110, *D. exsculpta*, Alph. D.C. Prodr. VIII. p. 223. n. 3 (1844), Bedd. Fl. Sylv. t. 66 (1870) ? fr.

Diospyros insculpta, Hamilt. *l.c.* p. 112, Alph. DC. *l.c.* n. 6.

Diospyros tomentosa, Roxb. Hort. Beng. p. 40 (1814), Fl. Ind. edit. 1832 vol. 2. p. 532, Roxb. draw. n. 1728 in Hb. Kew, R. Wight Ic. tt. 182, 183 (1840), non Poir.

? *D. speciosa*, Wood, Rep. For. Oudh 1867—68, p. 33 (*name only*, 1869).

Called *Tupru* (Carnatic), *Kend* (Hindoo), *Kendu* (Bengal) according to Hamilton, and *Kallindoo* (Sanscrit), *Kyou* and *Tumala* (Bengal), according to Roxburgh; *Tunki* in the Cuddapah district, and *Tumboornee* in the Bombay presidency, according to Beddome.

A tree either of small moderate or large size up to 60—80 ft. high; diœcious or polygamous; the heart-wood is black in some trees and of a hard and heavy substance called at Munghur *Batti* and at Saseram *Abnus*. The latter word is said to be of Persian origin and a source from which our word Ebony is derived. Trunk grey-black, bark very closely cracked both transversely and longitudinally. Branches cinereous, alternate or opposite, ramified as in the oak; young shoots ferruginous-pubescent. Leaves opposite subopposite and alternate, elliptical ovate or subrotund, bright green, more or less coriaceous, usually almost glabrous on the upper side and tomentose beneath, sometimes glabrous on both sides; obtuse or rounded at the base; emarginate rounded or obtusely narrowed or sometimes apiculate at apex; 3—14 in. long by $1\frac{1}{2}$—$7\frac{1}{2}$ in. wide; petioles $\frac{1}{8}$—$\frac{3}{4}$ in. long; lateral veins usually prominent beneath; deciduous.

♂ Flowers 3 or 4, on recurved thickened tomentose peduncles equalling or rather longer than the petioles, 4—5-merous, white, $\frac{1}{3}$—$\frac{5}{12}$ in. long; bracts small; pedicels very short; calyx tomentose, campanulate; corolla much longer than the calyx, with short lobes, hairy outside; stamens 12—18, inserted on the receptacle, glabrous (?); ovary rudimentary, hairy.

♀ Flowers solitary, subsessile or shortly stalked, 4—6-merous; peduncles about $\frac{1}{16}$ in. long; bracts 3—4, scale-like, caducous; calyx campanulate, 4—6-fid; corolla shortly 4—6-lobed; staminodes 0—6; ovary 4 (-6 ?) -celled, somewhat hairy; styles 2 (—3). Fruit egg-shaped or globose, glabrescent, about 1 in. long by $\frac{3}{4}$ in. thick, usually 4-celled and 3-seeded; seeds $\frac{1}{2}$ in. long by $\frac{1}{3}$ in. wide and $\frac{1}{4}$ in. thick; fruiting calyx surrounding the base of the fruit or spreading, pubescent on both sides, $\frac{2}{3}$—$\frac{3}{4}$ in. across, not or scarcely accrescent; testa shining, marked with reticulated depressions; albumen cartilaginous, ruminated, grey. "The fruit when ripe is sweet and not very bad tasted;" according to Hamilton the cotyledons in *D. insculpta* are conduplicate. This valuable tree sheds all its leaves in the cold season, and they appear again in the beginning of the hot weather (Beddome); not uncommon in the Cuddapah, Salem and Kurnool forests in Madras. Difficult to distinguish from *D. melanoxylon*, Roxb., to which species it ought perhaps to be united.

N.W. India. *Hb. Royle!*; Brundekund, *Edgeworth!* 6004; *Hb. Stocks!*; plains of Behar, &c., *Dr Hooker!* 440, 441; Magadi (used for small beams and posts), Hejuru, S.W. Mysore (a large tree; timber good), *Buch. Ham.* Journey vol. I. p. 183. vol. II. p. 125. W. Himalaya, *Dr Stewart!*

5. DIOSPYROS MELANOXYLON, Roxb. Coromand. p. 36. t. 46 (1795).

D. foliis oppositis suboppositis vel alternis, ovalibus vel oblongis, apice rotundatis vel leviter angustatis, basi cuneatis vel rarius rotundatis, pubescentibus tenuiter coriaceis, petiolatis; pedunculis florum masculorum longitudine petioli plurifloris, calyce campanulato, tomentoso, breviter 4—5-lobo, staminibus 12—16, rarius 8, glabris vel antheris leviter hirsutis; floribus femineis solitariis brevissime pedunculatis 5—4-meris, staminodiis 8 vel 10, ovario 4—(8-) loculari, fructibus globosis vel ovoideis, pubescentibus vel glabratis, plerumque 4-spermis; albumine ruminato.

Alph. DC. Prodr. VIII. p. 224. n. 7 (1844), non Blum. nec Hassk.

D. Wightiana, Wall.! List n. 4406 (1828—32), Alph. DC. Prodr. VIII. p. 223. n. 2, Beddom. Fl. Sylv. Madras, t. 67 (1870).

D. Roylii, Wall. List n. 4134 (1828—32), *D. Roylei*, Alph. DC. *l. c.* p. 239. n. 89.

D. dubia, Wall. n. 4407, Alph. DC. *l. c.* p. 223. n. 4, non Goepp.

Cfr. *D. rubiginosa*, Roth, Nov. Pl. Sp. p. 385 (1821), et *D. montana*, Heyne ex Roth *l. c.*, non Roxb.

Tunki Tumi and *Tumbi* in Tamil and Telugu, ex Beddome, Fl. Sylv. Madr. t. 67 (1870); *Tumida* of the Telingas, ex Roxb. *l. c.*; *Tumballi* of the Tamuls, ex Roxb. Fl. Ind. edit. 1832, vol. II. p. 531; *Tindoo* of the Hindoos ex Roxb. *l. c.*; (*Tendoo*, Beddome); Coromandel ebony tree; *Thomboorah Marum* in Hb. Wight; *Toomrie*, Dr Ritchie (Belgaum).

A large tree with a trunk 8—10 feet in circumference, sometimes only a small shrub; diœcious; young shoots very downy, pale-ferruginous. Leaves opposite subopposite and alternate, pubescent especially beneath, thinly coriaceous, oval or oblong, cuneate or rarely rounded at base, rounded or somewhat narrowed at apex, 2—6 in. long by 1—2$\frac{1}{2}$ wide; petioles $\frac{1}{6}$—$\frac{1}{3}$ in. long; veins less conspicuous than in *D. Tupru*; deciduous.

♂. Flowers $\frac{1}{6}$—$\frac{1}{3}$ in. long, in panicled tomentose-ferruginous drooping cymes $\frac{1}{4}$—$\frac{1}{2}$ in. long, longer than the petioles, several or many together, with small bracts at base and apex of short pedicels. Calyx shortly 4—6-lobed, campanulate, tomentose on both sides. Corolla 4—6-lobed at apex, glabrous inside, densely silky outside, 1$\frac{1}{2}$—3 times the length of the calyx. Stamens 12—16 (rarely 8 only), in pairs when 16, glabrous or with lines of short hairs back and front on the anthers which are longer than the filaments; ovary wanting or rudimentary and hairy.

♀. Flowers rather larger than the male, solitary, subsessile, pentamerous or tetramerous; calyx hairy on both sides, 5-winged (in *D. Wightiana*, Wall.!) by the patent projection of the margins of the lobes; staminodes 8 or 10; styles 2 bifid somewhat hairy; ovary 4! (—8) -celled, densely hairy; cells 1-ovuled. Fruit globular or ovoid, somewhat hairy or glabrescent, usually 4-celled and 4-seeded, about 1 in. long; albumen of seed somewhat ruminated; according to Roxburgh 2—8 seeds ripen; fruiting calyx nearly flat about $\frac{2}{3}$ in. across.

Neilgherries and Serramallee Hills, India, *R. Wight!* (*D. dubia*); Adjeeghur, and Bisrumgunge ghaut, *Royle* (*D. Roylii*); Belgaum, *Dr Ritchie!* 1108; Calicut!, *Hb. Wight!* 1723, Subbulpore, 1727, 1721, 1725; *Hb. Griffith!*, 3630, 3626 (1); Bababoodun Hills, Mysore, *Mr Law!*; common in dry forests in Madras, according to *Major Beddome.* The ebony tree of Malabar and Coromandel. Mysore, a small shrub, common, *Dr Brandis!*, May 1868.

It is only the centre of large trees that is black and valuable, and the quantity found varies with the age of the tree. The outer portion of the wood is white and soft, and either decays soon or is destroyed by insects which leave the black part untouched. The ripe fruit is eaten by the natives in the Circars, but is astringent and not very palatable. The bark of the tree possesses tonic and astringent properties, and in decoction proves useful in atonic diarrhœa, dyspepsia and diseases of debility. [See E. J. Waring, *Pharmacopœia of India*, p. 132 (1868).]

Cfr. *D. decandra*, Lour.

6. DIOSPYROS DECANDRA, Loureiro, Fl. Cochinch. p. 227 (1790).

D. foliis ovato-lanceolatis vel ellipticis, apice obtuse acuminatis, basi plus minus angustatis, alternis, tenuiter coriaceis, leviter pubescentibus, petiolatis; floribus femineis sub- 3-*nis, cymosis,* 4—5-*meris; corollâ urceolatâ; staminodiis* 10, *glabris; ovario* 6—8-*loculari; fructibus subglobosis edulibus.*

Alph. DC. Prodr. VIII. p. 238. n. 85 (1844), non Boj.

A large tree with rather patent branches, producing excellent heavy timber, white but marked with many black veins and sometimes with black heart-wood. Leaves thinly coriaceous, slightly pubescent, especially on the midrib, which is somewhat depressed on the upper surface, of nearly the same brown colour (in the dry state) on both sides, alternate, elliptical or ovate-lanceolate, shortly and obtusely acuminate at apex, more or less narrowed at base, 2—3 in. long (besides petiole $\frac{5}{16}$—$\frac{3}{8}$ in. long) by $\frac{7}{16}$—$1\frac{1}{2}$ in. wide; venation as in *D. melanoxylon.*

♀. Inflorescence rufous-hairy, more or less glabrescent; peduncles axillary, ranging up to $\frac{1}{2}$ in. long or rather more, bearing 3 or more flowers on short pedicels. Flowers whitish. Flower-bud depresso-ovato, $\frac{1}{6}$ in. long by $\frac{3}{16}$ in. thick; calyx deeply 4—5-fid, enclosing the young corolla, with valvate (?) deltoid lobes whose sides are somewhat revolute; corolla shortly lobed, glabrous inside, tube urceolate, lobes obtuse, reflexed in full flower; staminodes glabrous, 10 according to *Loureiro,* short, inserted at the base of the corolla. Ovary 6- or 8-celled and -ovuled; ovules pendulous. Style short, lobed at apex. Fruit compresso-rotund or subglobose, subglabrate at least in part, 6—8-celled in the cases examined, about 1 in. in diameter or perhaps larger, yellow, edible, pulpy, sweet but astringently so, 6—8-seeded, strongly scented, not very pleasant to the taste. Fruiting calyx spreading, nearly as wide as the fruit when young. Seeds bony, "compresso-ovate."

The fruit according to *Loureiro* is brought to market for sale.

N. Cochinchina. *Loureiro!* A.D. 1774 (seen in Hb. Mus. Brit.). Local name *Cay Thi.*

Very possibly *D. melanoxylon*, Roxb. ought to be united with this species; but the leaves in the latter are all alternate, so far as the specimen seen by me shews.

7. DIOSPYROS SYLVATICA, Roxb. Coromand. p. 37. t. 47 (1795).

D. foliis alternis, ovalibus, sæpe acuminatis, basi angustatis, vix coriaceis, glabris vel subglabris, breviter petiolatis; floribus masculis cymosis, ∞-nis, globosis, parvis, sæpius 4-meris interdum 3- vel 5-meris; staminibus 13—22, glabris; floribus femineis solitariis pedunculatis, sæpe in ramulis junioribus racemose dispositis, 4—3-meris, globosis; staminodiis 4, glabris; ovario 8- vel 6-loculari; fructibus globosis; albumine ruminato.

Alph. DC. Prodr. VIII. p. 231. n. 41 excl. var. β velutina (1844).

Thw. Enum. Ceyl. Pl. p. 178. n. 3 (1860).

Bedd. Ic. Pl. Ind. Or. (Part VII.) p. 25. t. 121 (1871).

D. orixensis, Klein ex Willd. Sp. Pl. IV. p. 110 (1805), Alph. DC. *l. c.* p. 230. n. 35, non Wight.

Native names: *Tella-gada* of the Telingas; *Nella-gada* (Hb. Roxb.); *Soodoo-Kadoombaireya-gass* in Ceylon.

A pretty large tree; foliage turning black when dry; branches spreading at 60°—75°, glabrous or the young shoots pubescent. Leaves alternate, oval, pointed or acuminate, thin, usually somewhat narrowed at base; nearly or quite glabrous, 2—6 in. long by $\frac{3}{4}$—3 in. wide; petioles $\frac{1}{6}$—$\frac{5}{12}$ in. long, often puberulous; midrib and veins depressed on upper side, but not conspicuous; lateral veins not very close.

♂. Cymes axillary, several- or many-flowered, $\frac{1}{5}$—$\frac{9}{20}$ in. long (excluding the flowers), more or less shortly-pubescent; ultimate pedicels short; flowers small, $\frac{1}{10}$—$\frac{1}{7}$ in. long, white and fragrant when growing, 3—5-merous, usually 4-merous; calyx very short $\frac{1}{30}$—$\frac{1}{25}$ in. high by $\frac{1}{10}$ in. wide, 3—5-fid, pubescent or ciliate, glabrous inside; corolla obconic-subglobular, lobed at apex, nearly glabrous; stamens 13—22, mostly in pairs and inserted at base of corolla, glabrous (or rarely with a few short hairs); anthers about the length of the filaments, dehiscing laterally from apex; ovary rudimentary, somewhat hairy at apex or glabrous.

♀. Flowers solitary, on peduncles $\frac{1}{5}$—$\frac{1}{2}$ in. long, larger than the ♂, 3—4- usually 4-merous; staminodes 4, glabrous, alternating with the corolla-lobes; ovary 6- or 8-celled, glabrous or hairy at apex; cells 1-ovuled; styles 3 or 4; fruit globose, glabrous or with a few appressed hairs around apex, $\frac{1}{2}$ in. or more in diameter; fruiting calyx spreading, accrescent, $\frac{3}{4}$ in. in diameter. Seeds 2—8; albumen somewhat ruminated. Wood very hard, used for fancy work.

India, Circars, *Roxburgh !*; Concan, *Law !*; Bombay, *Law !*, 3000 ft. alt. Ceylon, *Thwaites !* C. P. 2729, damp forests up to 4000 ft. alt.

8. DIOSPYROS KURZII, sp. nov.

D. foliis alternis, ovato-ovalibus, mox glabratis, apice acuminatis, basi cuneatis, breviter petiolatis, nitentibus, nervis lateralibus crebris tenuibus; floribus femineis sub-3-nis, breviter cymosis, tetrameris, urceolatis; staminodiis 4, glabris; ovario 4-loculari, 4-ovulato, stylis 2, basi connatis.

Young branches pubescent with short appressed silky fulvous or brown hairs; branches at about 40°. Leaves ovate-oval, quickly glabrescent, alternate, dark, very dark and shining above with crowded delicate lateral veins which are also in relief beneath where the leaf is slightly paler, acuminate at apex, more or less narrowed at base, thinly coriaceous; $2\frac{1}{2}$—$3\frac{1}{2}$ in. long by 1—$1\frac{2}{5}$ in. wide; petioles $\frac{1}{8}$—$\frac{1}{5}$ in. long; midrib depressed above.

♀ Cymes axillary, $\frac{1}{8}$—$\frac{1}{4}$ in. long (excluding flowers), about 3-flowered, with very short pedicels, pubescent, and with small caducous bracts at base of calyx. Flowers $\frac{1}{5}$ in. long. Calyx $\frac{1}{15}$ in. long, puberulous outside, glabrous inside, shortly 4-fid, bigger in young fruit. Corolla 4-lobed at apex, with rounded lobes pubescent on both sides, urceolate; staminodes 4, glabrous, alternating with corolla-lobes; ovary glabrous except apex, 4-celled; cells 1-ovuled. Styles 2, straight, erect, slender, hairy, long, connate at base.

South Andaman, *S. Kurz !*

9. DIOSPYROS EHRETIOIDES, Wall. List, n. 4137 (1828—32).

D. foliis alternis, ellipticis, vix coriaceis, discoloribus; floribus masculis ∞-nis, cymosis, subglobosis; staminibus 22—29, glabris, ovarii rudimento hirsuto; floribus femineis solitariis, breviter pedunculatis; fructibus globosis, glabratis; albumine ruminato.

Alph. D.C. Prodr. VIII. p. 231. n. 42 (1844).

D. mollis, Wall. ex Steud. Nomencl. bot. edit. ii. I. p. 514 (1840).

Young shoots and inflorescence ferruginous-pubescent; branches spreading, alternate, terete. Leaves elliptical, rounded or somewhat narrowed at base, rounded obtusely pointed or apiculate at apex, alternate, thinly coriaceous or submembranous, glabrous except the veins, ferruginous or reddish-brown beneath, greener or slaty-brown above, 3—9 in. long by $2\frac{1}{3}$—5 in. wide; petioles $\frac{1}{5}$—$\frac{1}{2}$ in. long.

♂ Cymes compound, trichotomous, 4 times the length of the petioles, patent, abundant on the young shoots; bracts hooked at the apex; flowers $\frac{1}{8}$ in. wide, pubescent, globose, reflexed on very short pedicels. Calyx 4-fid, pubescent, with obtuse lobes. Corolla campanulate, twice the length of the calyx, with ciliated much contorted lobes. Stamens 22—29, glabrous, subequal, crowded on the receptacle, mostly distinct; filaments short. Ovary rudimentary, represented by a few hairs.

♀ Flowers solitary; peduncles $\frac{1}{5}$—$\frac{3}{10}$ in. long, on the young shoots. Fruiting calyx with recurved lobes, somewhat pubescent or nearly glabrate, about $\frac{3}{4}$ in. broad (when expanded); fruit glabrous, globular, $1\frac{1}{3}$ in. in diameter; albumen ruminated.

Tavoy and Moolmyne, *Wallich !*; Pegu, *McLelland !* (Fruits in January); Pegu and Ta Oo, Tenasserim, local name *Aekchinza*, *Dr Brandis !*

10. Diospyros rotundiflora, sp. nov.

D. foliis ovali-oblongis, alternis, apice acuminatis, basi rotundatis, tenuiter coriaceis, supra nitentibus glabris, subtus subglabris, breviter petiolatis; floribus masculis paniculatis, subglobosis vel ovoideis, 4—3-meris, pubescentibus; staminibus 14—16, biserialibus, glabris, ovario rudimentario hirsuto.

Young parts and inflorescence subtomentose; branches cinereous, terete. Leaves oval-oblong, alternate, acuminate at apex, rounded at base, thinly coriaceous, of rich brown colour when dry, shining and glabrous above with veins inconspicuous, nearly glabrous beneath, 3—7 in. long by $1\frac{1}{2}$—$2\frac{1}{4}$ in. wide; petioles shortly pubescent, $\frac{1}{5}$ in. long; lateral veins about 10 on each side the midrib.

♂. Cymes axillary and lateral, many-flowered, less than 1 in. long; ultimate pedicels very short or obsolete; bracts ovate, sometimes larger than the flowers; flowers subglobose or ovoid, $\frac{1}{8}$—$\frac{1}{6}$ in. in diameter, pubescent; calyx subhemispherical or widely campanulate, 4- occasionally 3-fid; lobes deltoid; corolla shortly 4—3-fid, lobes rounded; stamens 14—16, biseriate, subequal, glabrous; ovary rudimentary, hairy.

Borneo, *O. Beccari!* n. 3567.

Near *D. ehretioides*, Wall.

11. Diospyros hirsuta, Linn. fil. Suppl. p. 440 (1781).

D. foliis alternis, ellipticis oblongis vel ovatis, tenuiter coriaceis, breviter petiolatis, nervis lateralibus sæpius inconspicuis; floribus masculis dense cymosis, oblongis, pubescentibus, 4—5-meris; staminibus 5—16, subglabris; floribus femineis 1—6-nis; staminodiis 5—10; ovario 4—10-loculari, loculis 1-ovulatis; stylis 2—5, brevibus; fructibus globosis vel ellipsoideis, tomentosis vel glabratis, calyce fructifero, stellato- vel depresso-cupuliformi; seminibus oblongis, albumine ruminato.

Alph. DC. Prodr. VIII. p. 223. n. 5 (1844), Thw. Enum. Ceyl. Pl. p. 181. n. 15 (1860), Bedd. Icon. Pl. Ind. Orient. (Part VII.) p. 28. t. 137 (1871), non Desf.

A tree of moderate size, diœcious or occasionally monœcious; produces an ebony. Buds inflorescence and in some cases the young branches and underside of leaves pubescent. Leaves alternate, more or less elliptical oblong or ovate, thinly coriaceous, obtusely or acutely acuminate at apex, narrowed or rounded at base, glabrous and shining above, sometimes pubescent beneath, 2—12 in. long by 1—4 in. wide; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long; midrib depressed on upper side; lateral veins usually inconspicuous beneath. Flowers subsessile, axillary, 4—5-merous; bracts rounded, caducous. According to Dr Thwaites female flowers are occasionally intermixed in the male cymes and in that case are much smaller than when occurring alone.

♂. Flowers $\frac{1}{3}$—$\frac{1}{2}$ in. long, oblong, in dense cymes. Calyx hairy on both sides, 4—5-fid; lobes acute. Corolla tubular, at least double the length of the calyx, 4—5-fid, glabrous inside. Stamens 5—16, glabrous or mainly so, when numerous often united by the filaments in pairs. Ovary rudimentary, hairy.

♀. Flowers $\frac{1}{3}$ in. long, thicker than in the ♂, 1—6 together. Margins of calyx-lobes wavy and reflexed. Corolla-lobes reflexed, rounded, mucronate. Staminodes 5—10; barren anthers, glabrous or with setose tips, filaments glabrous or hairy. Ovary ovoid, covered with ferruginous or rufous hairs, 4—10-celled. Styles 2—5, short. Fruit globose or ellipsoidal, pale-glabrate or rarely tomentose and ferruginous or rufous, $\frac{1}{2}$—$1\frac{1}{4}$ in. long, 1—10-seeded. Fruiting calyx stellate-flat or shallow-cupuliform, $\frac{1}{2}$—1 in. in diameter, 4—5-fid; lobes with reflexed margins. Seeds oblong, usually compressed, transversely scored outside; albumen ruminated.

The following forms seem to me difficult to separate from *D. hirsuta*, L. f., but the combination of them all into a single species makes it a very variable and widely spread one.

D. lucida, Wall. List, n. 4127 (1828—32), non Hort., Alph. DC. Prodr. VIII. p. 233. n. 52 (1844). = (?) *D. nilagirica*, Bedd. Icon. pl. Ind. Or. (Pt. vii.) p. 27. t. 136 (1871).

D. Candolleana, Wight, Icon. tt. 1221, 1222 (1850), non Thw.

D. Moonii, Thw. Enum. Ceyl. Pl. p. 182. n. 16 (1860), Bedd. *l.c.* p. 28. t. 138 (1871). Perhaps a distinct species.

D. canarica, Bedd. *l.c.* p. 27. t. 134. = *D. oligandra*, Bedd. Rep. Forests Madras 1867—68 p. 25 (1868) name only.

D. Thwaitesii, Bedd. *l.c.* p. 27. t. 135. = *D. Candolleana*, Thw. *l.c.* p. 181. n. 14, non Wight.

The following key serves to contrast the typical characters of these forms, but intermediate states exist.

Ovary (6-) 8—10-celled; filaments of stamens often hairy.
- Ovary usually 10-celled. Stamens 5.
  - Leaves elliptical, narrowed at base. *hirsuta* proper.
  - Leaves oblong, wide near base. *Moonii.*
- Ovary 8-celled. Stamens 16. *nilagirica.*

Ovary 4-celled; filaments of stamens 10—12, glabrous.
- Staminodes 4—5.
  - Staminodes quite glabrous. Fruit pale, glabrate. *Candolleana.*
  - Staminodes setose at tip. Fruit rufous-hairy, at length glabrate. *Thwaitesii.*
- Staminodes 8—10, glabrous. *canarica.*

Hirsuta proper. Ceylon, *Thwaites!* 382.

Moonii. Ceylon, *Thwaites!* 2833; *Moon!*; *Walker!*; (?) Tennaserim and Andaman, *Hb. Helfer!* 3632.

Nilagirica. Sispara ghat. Nilgiris, India. *Major Beddome*; (*lucida*) Singapore, *Wallich!* 4127; Malacca, *Maingay!* 970, 973; *Griffith!* 3637.

Candolleana. Courtallum and Quilon, *Wight!* 1715, Canara, Mangalor, *Wight!* 1728; *Hohenacker!* 591 (Native name *Karmarn*); Concan, *Dr Gibson!* 128; Goa, *Dalzell!*; Moodlis, *Dr Ritchie!* 96, 3 (tree 24 ft. high); Phoondu Ghaut, *Dr Ritchie!* 96/2 (tree 36 ft.); Ram Ghaut, *Dr Ritchie!* 96 (Native name *Kalevin*).

Thwaitesii. Ceylon, Local name *Homedereya-gass*, *Thwaites!* 3394.

Canarica. S. Canara, *Major Beddome!* "yields an ebony," native name *Kara mara.*

A specimen from Malacca (*Maingay!* 969), with less dense cymes and 14 stamens with somewhat pilose anthers, may belong to this species.

I cannot discover any authentic and satisfactory specimen of *D. hirsuta*, Linn. fil.; there is not a specimen so named by the younger Linnæus in the elder Linnæus' herbarium, and one in Sir J. E. Smith's herbarium sent from Ceylon by *Burmann* and labelled "Diospyros hirsuta, H. L. fil." is not a *Diospyros* nor even a member of the family.

12. DIOSPYROS MESPILIFORMIS, Hochst. in Pl. Schimp. Abyss. Exsicc. sect. ii. nn. 655, 1243 (1842).

D. foliis ellipticis vel oblongis, alternis, tenuiter coriaceis, glabrescentibus vel leviter pubescentibus, breviter petiolatis, nervis inconspicuis; floribus masculis axillaribus, 5—4-meris, ferrugineo-tomentosis. breviter cymosis, urceolato-oblongis, staminibus 10—16, *subglabris; floribus femineis* 1—3-*nis. axillaribus,* 5—4- *rarius* 3-*meris, staminodiis* 6—8, *uniserialibus, glabris, ovario ovoideo vel conico, sericeo,* 4- *vel* 8-*loculari, loculis* 1-*ovulatis; fructibus subglobosis, glabratis, edulibus, calyce fructifero margine undulato; albumine ruminato.*

Alph. DC. Prodr. VIII. p. 672 (1844).

D. senegalensis, Perrott. ex Alph. DC. *l. c.* p. 234. n. 59.

D. bicolor, Klotzsch in Peters Mossamb. I. p. 184 (1862).

A shrub or tree from 6 to 40 feet high or more. Wood much thought of by the natives, white, compact, and useful for many purposes, or black in the centre like ebony. Branches terete, brown-cinereous, glabrescent, more or less patent; the young shoots and inflorescence ferruginous-tomentose. Leaves alternate, oblong or elliptical, somewhat narrowed or rounded at either end, thinly coriaceous (the younger ones very softly membranous) glabrescent and shining, or with scattered appressed pubescence beneath, often rubescent, especially on the midrib beneath, 2—6 in. long by $\frac{2}{3}$—$2\frac{1}{2}$ in. wide; petiole $\frac{1}{6}$—$\frac{1}{3}$ in. long; midrib depressed above, lateral and net-veins delicate; margins just recurved. Flowers white, diœcious.

♂. Inflorescence axillary, cymose, bearing few to many flowers, $\frac{1}{6}$—$\frac{2}{3}$ in. long exclusive of the flowers. Flowers ferruginous-tomentose, about $\frac{1}{3}$ in. long, pentamerous, occasionally tetramerous; bracteoles lanceolate. Calyx about $\frac{1}{6}$ in. long, 5- occasionally 4-fid, campanulate or campanulate-oblong, hairy on both sides; lobes ovate or lanceolate. Corolla in general shortly 5-fid, urceolate-oblong, twice the length of the calyx or more, sericeous outside, glabrous inside; lobes spreading, pointed. Stamens 10—16, often in pairs, nearly glabrous but with a narrow band of light-coloured hairs on the back of the anthers, inserted at the base of the corolla; filaments short; connective produced at apex; pollen widely ellipsoidal, smooth. Ovary rudimentary, hairy, or 0.

♀. Flowers pentamerous or tetramerous or rarely trimerous, solitary or in very short 1—3-flowered axillary cymes; peduncles $\frac{1}{8}$—$\frac{2}{3}$ in. long; bracts narrow, caducous. Calyx hairy on both sides, campanulate, deeply lobed; lobes ovate acuminate with undulated margins.

Corolla pubescent outside, glabrous inside, exceeding the calyx, shortly lobed, lobes pointed. Staminodes 6—8, in one row, inserted at base of the corolla, glabrous. Ovary ovoid or conical, sericeous, terminated by 2 short hirsute bilobed styles, 4- or 8-celled and -ovuled. Fruit glabrate subglobose, $\frac{2}{3}$—1 in. in diameter, edible, often slightly wrinkled, 4—5-seeded. Fruiting calyx somewhat or but little increased, with undulated margins, appressed to fruit or spreading. Seeds shining, $\frac{1}{2}$—$\frac{2}{3}$ in. long; albumen cartilaginous, somewhat ruminated; embryo straight; cotyledons linear-lanceolate; radicle shorter than the cotyledons.

Difficult to distinguish by technical characters from the Indian species *Diospyros hirsuta*, Linn. fil., of which it may be taken as the African representative; its forms also are subject to considerable variation.

Tropical Africa. Abyssinia; native name *Ajé* or *Ajejeh*, near Docheladscheranne, *Schimper!* sect. ii. nn. 655, 1243, in ♂ flower, June; *Petit* in Hb. Franq. iv. Coll. n. 434, in fruit; *Schimper* (1862)! n. 155, September, 4400 ft. alt.; Nubia, Fayohel, *Kotschy!* n. 470, in fruit; *Dr Martin St Ange!* in young fruit; *Tinné* Expedition, nn. 170, 394, fruit in flavour like that of Theobroma Cacao; Benischangul, *Cienkowski*, n. 96 *b*; Gallabat, Matamma, *Schweinfurth!* n. 973, 974; Mozambique, between Tette and the sea coast, *Dr Kirk!* in fruit; near Lupata, *Dr Kirk!* in fruit, January; native names, Sechuana dialect, *Makudima;* Tette dialect, *Kasinjamtolmera;* 50 miles above Tette, *Kaurabassa;* Sena, *Dr Peters!;* Niger, *Barter!* 1208, 1334; Senegambia, *Leprieur!*, *Whitfield!*, *Lelierre!*, *Perrottet!*, *Dr Daniell!* ("*Monkey Guava*"); Livingstone Expedition, ("*Mocheka*") *Dr Kirk!;* Angola, Distr. Golungo Alto, *Dr Welwitsch!* 2529, frequent, "*Musolveira;*" Benguela, in woods from Serra da Xella towards Mumpulla, *Dr Welwitsch!* 2530; Congo, in rocky and sandy woods near Ambriz, *Dr Welwitsch!* 2528; Cape Coast, *Brass!*

13. DIOSPYROS BURMANICA, Kurz in Journ. Asiat. Soc. Bengal, Vol. XL. Pt. ii. p. 73. n. 96 (1871).

D. foliis alternis, ovalibus, apice obtusis vel brevissime acuminatis, basi rotundis vel subcuneatis, breviter petiolatis, tenuiter coriaceis, junioribus supra tomento tenui fugaci adspersis subtus appresse tomentosis; floribus masculis 4—6-meris, breviter cymosis, tomentosis, urceolato-oblongis, staminibus sæpius 8, rarius 14—16, glabris; floribus femineis solitariis, fructibus globosis, glabris, nitentibus, vulgo 4-spermis, albumine seminum pulcherrime ruminato, calyce fructifero margine undulato.

A small tree with young parts appressedly fulvo-pubescent. Branches cinereous. Leaves alternate, oval, obtuse or slightly acuminate at apex, rounded or wedge-shaped at base, thinly coriaceous, covered when young especially beneath with appressed pubescence, at length more or less glabrate, 1—6 in. long by $\frac{1}{3}$—2 in wide; veins not prominent; petioles $\frac{1}{24}$—$\frac{1}{4}$ in. long.

♂. Flowers cream-coloured, urceolate-oblong, 4—6- usually 5-merous, $\frac{1}{3}$—$\frac{1}{2}$ in. long, fulvo-tomentose, in short 3—5-flowered axillary cymes, on young branches. Calyx $\frac{1}{6}$—$\frac{1}{4}$ in. long, hemispherical, tomentose on both sides, shortly 4—6-fid; lobes deltoid or cordate-ovate. Corolla shortly 4—5-lobed, glabrous inside; lobes rounded, reflexed. Stamens glabrous.

usually 8, occasionally 14—16, hypogynous or inserted at base of corolla, about $\frac{1}{5}$ in. long; anthers linear, acute, longer than the filaments. Ovary rudimentary, fulvo-pubescent.

♀. Flowers solitary, pentamerous. Fruit globose, shining, 1—$1\frac{1}{2}$ in. in diameter, commonly 4-seeded. Seeds about $\frac{3}{4}$ in. long, reticularly wrinkled outside, shining; albumen grey, beautifully ruminated. Fruiting calyx about $\frac{7}{8}$ in. in diameter, at base of fruit, tomentose; margins undulated.

Burma, in sandy and hilly woods, 3rd Kioudweng, 13 May, 1837, *Griffith!* 3638 [see Journal of Travels, p. 104 (1847)]; Pegu, Zeebenthlah, October, 1861, *Dr Brandis!* 952 (vernacular name *Tayben*); *M^cLelland!*; *Kurz!* 3010.

Authentic specimens of this species seen during the printing of this paper prove that it embraces the specimens of Griffith and McLelland, which I had previously named *D. octandra* and so printed on pages 33 and 41.

14. DIOSPYROS LATERALIS, sp. nov.

D. ramis teretibus cinereis glabris, gemmis et inflorescentiâ ferrugineo-pubescentibus; foliis alternis, ovalibus, apice acuminatis, basi angustatis, tenuiter coriaceis vel submembranaceis, glabris, petiolatis, nervis tenuibus manifestis; floribus masculis tubulosis breviter cymosis; cymis lateralibus secus ramos et axillaribus, 3—9-*floris; calyce oblongo apice* 5—6-*lobo, corollâ tubulosâ plerumque* 5-*fidâ, staminibus* 10—14, *inaequalibus, fere glabris, antheris apice minute setulosis, filamentis glabris; ovario hirsuto rudimentario.*

Buds and inflorescence ferruginous-pubescent; branches terete, glabrate, cinereous. Leaves oval, alternate, narrowly acuminate, cuneate at base, glabrous, submembranous or thinly coriaceous, 2—$4\frac{1}{4}$ in. long by 1—$2\frac{1}{5}$ in. wide; lateral veins slender, manifest, about 6 on each side the midrib, margins slightly recurved; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long, glabrous.

♂. Cymes lateral on the older branches and axillary on the younger ones, $\frac{1}{5}$—$\frac{1}{3}$ in. long, 3—9-flowered; flowers tubular; calyx oblong, $\frac{3}{10}$ in. long, 5—6-lobed at apex, glabrous inside, lobes somewhat spreading, often unequal; corolla tubular, exceeding the calyx, usually 5-fid, lobes obtuse; stamens 10—14, very nearly glabrous, unequal, anthers minutely setulose at apex, filaments glabrous; ovary rudimentary, hairy.

Borneo, *O. Beccari!* n. 1600.

15. DIOSPYROS VERRUCOSA, sp. nov.

D. fructicosa, foliis ovato-oblongis, alternis, apice angustatis, obtusis, mucronulatis, basi rotundatis, tenuiter coriaceis, supra subglabris, subtus appresse pubescentibus; fructibus solitariis, pedunculatis, subtetragono-ellipsoideis vel -globosis, verrucosis, pubescentibus, 4-*locularibus,* 4-*spermis; pedunculis robustis, patentibus; calyce fructifero parvo, patente,* 4-*lobo, pubescente; seminibus oblongis, sulcatis, albumine ruminato.*

A shrub; branches numerous, at length glabrescent, terete; young shoots densely and shortly pubescent, subferruginous. Leaves ovate-oblong, alternate, thinly coriaceous, somewhat narrowed and mucronulate at apex, rounded at base, nearly glabrous above except that the depressed midrib is sometimes puberulous, paler with soft appressed pubescence beneath and rufous-pubescent on the raised midrib and lateral veins; $1\frac{1}{2}$—$3\frac{1}{2}$ in. long by $\frac{5}{6}$—$1\frac{4}{5}$ in. wide, including petiole $\frac{1}{10}$—$\frac{1}{5}$ in. long, rufous, densely puberulous; lateral veins about 6 on each side the midrib.

♀. Flowers solitary, on distinct densely puberulous rather slender peduncles, axillary; bracts small, rufous-hairy, caducous, near base of peduncle; fruiting peduncles stout, thickened upwards with wide articulation at apex, nearly $\frac{1}{3}$ in. across, puberulous or subglabrate, $\frac{1}{3}$—$\frac{2}{5}$ in. long, patent; fruiting calyx subtomentose-pubescent on both sides, spreading, 4-fid, $\frac{1}{2}$ in. across; with depresso-deltoid lobes slightly recurved at apex. Fruit oblong or globose, pulpy, roundedly 4-sided, verrucose, at length smoother with pale ferruginous short pubescence between the raised warty prominences, obtusely umbonate at apex, 1—$1\frac{1}{4}$ in. long by $\frac{5}{6}$—1 in. across from one side to the opposite side, 4-celled; cells 1-seeded; seeds $\frac{2}{3}$—$\frac{3}{4}$ in. long, enclosed in a thin chartaceous envelope, transverse section a quadrant of a circle of radius $\frac{3}{16}$ in., shewing several intrusions of the testa into the albumen corresponding with depressed lines on the exterior of the seed; embryo nearly straight, nearly the length of the seed; radicle superior, much shorter than the compressed 1—3-nerved cotyledons. In one case the calyx is triangular and flat. The pulp of the fruit is eaten.

E. Africa, Prov. Zanguebar, *Dr Kirk!*; Zambesia, Rovuma R., 20 miles above the mouth. *Dr Kirk!*, August 1862.

16. DIOSPYROS KORTHALSIANA.

D. glabra, foliis alternis ellipticis, apice obtuse acuminatis, basi cuneatis, coriaceis, costis inconspicuis; fructibus solitariis, axillaribus, pedunculatis, glabris, apice cum stylorum reliquiis appresse ferrugineo-hirsutis, ellipsoideis, 8-locularibus, 8-spermis; seminibus oblongis, albumine ruminato.

Diospyros macrocarpa, Korthals MSS. in Hb. Lugd. Batav. Ebenaceæ No 2, non mihi.

Glabrous; branches (in dry state) drab. Leaves elliptical, cuneate at base, obtusely acuminate at apex, alternate, coriaceous, 2—4 in. long (besides petiole $\frac{1}{4}$—$\frac{3}{8}$ in. long) by $\frac{3}{4}$—$1\frac{1}{2}$ in. wide, palish brown (in dry state) on both sides, shining above, midrib depressed above; veins inconspicuous.

Fruit solitary, on axillary peduncles which are $\frac{3}{8}$ in. long, suberect, thickened upwards and thicker than the extension of the young branches from which they grow; fruit ellipsoidal, glabrous except at apex, dark and shining, about $1\frac{1}{2}$ in. across, by scarcely 2 in. (?) long, 8-celled, 8-seeded, tipped by appressedly ferruginous-hairy remains of style. Pericarp in the dry state $\frac{1}{16}$ in. long; seeds $\frac{3}{4}$ in. long by $\frac{1}{4}$ in. wide and $\frac{1}{8}$ in. thick in the dry state, pendulous from inner side; albumen somewhat ruminated.

Fruiting calyx nearly glabrate outside, appressedly hairy and smooth inside, very crass.

shallowly cup-shaped, 1 in. across, 4-cornered and shortly 4-lobed; about $\frac{1}{4}$ in. high; tube with elevated rim; lobes much thinner, reflexed; verrucose-rugose outside.

Borneo, *Korthals!*

17. DIOSPYROS AFFINIS, Thw. Enum. Ceyl. Pl. p. 179. n. 6 (1860).

D. foliis alternis, ovalibus vel oblongis, apice obtusissimis, basi angustatis vel subrotundis, tenuiter coriaceis, glabris, petiolatis; floribus masculis 3—7-*nis, cymosis, pubescentibus,* 4-*meris, calyce apice lobato, corollâ urceolato-oblongâ; staminibus* 6—16 *glabris vel leviter hirsutis; floribus femineis solitariis, pedunculatis; staminodiis* 6—8, *glabris, ovario* 6 (?) -*loculari; calyce inter lobos marsupio-dilatato, lobis acuminatis; albumine ruminato.*

Bedd. Ic. Pl. Ind. Or. (Part VII.) p. 26 t. 127 (1871).

A moderate-sized tree; buds ferruginous hairy; branches quickly glabrescent. Leaves oval or oblong, alternate, quite obtusely narrowed at apex, narrowed at base, thinly coriaceous, glabrous or puberulous below, $1\frac{1}{4}$—$4\frac{1}{2}$ in. long by $\frac{1}{2}$—$1\frac{1}{3}$ in. wide; petioles $\frac{1}{6}$—$\frac{1}{2}$ in. long; midrib canaliculate above, net-veins numerous, raised on both sides.

♂. Flowers $\frac{1}{3}$ in. long, cymose, 3—7 together; cymes $\frac{1}{2}$—$\frac{2}{3}$ in. long excluding the flowers; ultimate pedicels short, not exceeding $\frac{1}{10}$ in. long. Calyx semi-ellipsoidal, with short hairs on both sides, 4-toothed at apex, $\frac{1}{8}$ in. long. Corolla shortly salver-shaped, tawny-hairy outside; tube inflated below, constricted at top; lobes 4, spreading, oval, somewhat pointed at apex, about $\frac{1}{3}$ the length of the tube. Stamens 6—16, usually about 9 and some or all in pairs, short, usually hypogynous and unequal; filaments glabrous, shorter than the anthers which are glabrous or somewhat hairy. Ovary rudimentary, represented by a bunch of hairs.

♀. Flowers solitary, $\frac{1}{2}$ in. long and wide; peduncles axillary, $\frac{1}{4}$—$\frac{9}{20}$ in. long, equalling or rather shorter than the flower, puberulous. Calyx $\frac{5}{12}$ in. long, hairy inside, subglabrate outside, 4-fid, plicate; lobes acuminate with very wide sinuses; somewhat enlarged in fruit. Corolla shortly 4-fid with 6—8 glabrous staminodes at base inside. Ovary 6-celled (4-celled?, conical), hairy; styles 2, bifid at apex. Fruit globular, apiculate, usually 4-seeded, 1 in. long, finally glabrous; seeds $\frac{1}{2}$ in. long or more by $\frac{1}{3}$ in. wide, with ruminated albumen.

Ceylon, *Thwaites!* C.P. 2924.

18. DIOSPYROS CRUMENATA, Thw. Enum. Ceyl. Pl. p. 179. n. 5 (1860).

D. foliis alternis, ovalibus vel oblongis, obtuse et breviter acuminatis, coriaceis, glabris, petiolatis; floribus masculis breviter cymosis, 3—5-*nis, tetrameris, pubescentibus, staminibus circiter* 12, *glabris; floribus femineis solitariis, breviter pedunculatis, tetrameris, calyce inter lobos marsupio-dilatato, staminodiis* 8, *glabris; ovario* 8-*loculari, hirsuto; fructibus subglobosis glabrescentibus, albumine ruminato.*

Bedd. Ic. Pl. Ind. Or. (Part VII.) p. 26. t. 126 (1871).

A large tree; branches glabrous. Leaves oval or oblong, alternate, obtusely shortly and abruptly acuminate at apex, rounded or narrowed at base, coriaceous, glabrous, with midrib channelled above and net-veins numerous and raised on both sides, 2—5 in. long by 1—2 in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long, canaliculate above.

♂. Cymes 3—5-flowered, near together, about $\frac{1}{2}$ in. long, hairy. Calyx obscurely 4-dentate, glabrous inside, $\frac{1}{5}$—$\frac{1}{4}$ in. long. Corolla $\frac{5}{12}$ in. long, shortly 4-lobed; lobes recurved. Stamens about 12, glabrous, hypogynous.

♀. Flowers solitary, rather more than $\frac{1}{2}$ in. long; peduncles hairy, $\frac{1}{4}$ in. long, thickened upwards. Calyx $\frac{1}{3}$ in. high by $\frac{2}{3}$ in. wide, shallowly 4-lobed, plainly plicate, coriaceous, puberulous outside, hairy inside; lobes obtuse or apiculate and rounded; between the lobes marsupio-dilated. Corolla slightly exceeding the calyx ferruginous-tomentose, shortly 4-lobed; lobes with undulated margins and tomentose on both sides. Staminodes 8, in one row, glabrous, inserted at base of interior of corolla-tube. Ovary 8-celled, hairy. Fruit subglobose, 6—8-seeded, 1$\frac{1}{2}$—2 in. in diameter, at length glabrous, resting at base on tetragonal thickened spreading calyx, 1$\frac{1}{4}$ in. in diameter; seeds black, shining, 1 in. long, $\frac{1}{2}$ in. wide, with ruminated albumen.

Ceylon, 2000—4000 ft. alt. *Thwaites!* C.P. 2438.

19. DIOSPYROS FRUTESCENS, Blume, Bijdr. Fl. Ned. Ind. p. 668 excl. *var.* (1825).

D. foliis alternis, ellipticis, apice acuminatis, basi angustatis, firmiter submembranaceis, glabris, nitentibus; floribus femineis axillaribus vel lateralibus, tetrameris, ∞-nis; calycis lobis margine revolutis; corollâ suburceolatâ, 4-fidâ; staminodiis 8, æqualibus, uniserialibus; ovario hispido; fructibus globosis, subglabris, succulentis.

Alph. DC. Prodr. VIII. p. 230. n. 38 excl. *var.* (1844), non Hassk. Plant. Javan. rar. p. 467 (1848).

Young shoots puberulous; branches dark, terete, smooth. Leaves alternate, elliptical, firmly submembranous, somewhat narrowed at base, acuminate at apex, glabrous and shining, 2$\frac{1}{2}$—5 in. long by 1$\frac{1}{6}$—2$\frac{1}{2}$ in. wide, petioles $\frac{1}{6}$—$\frac{1}{4}$ in. long; veins inconspicuous above; midrib depressed above; margins neatly recurved.

♀. Cymes axillary or lateral, fasciculate, many-flowered, $\frac{1}{8}$—$\frac{3}{4}$ in. long (excluding the flowers), shortly pubescent, ferruginous (or fuliginous); common peduncle obsolete; pedicels $\frac{1}{8}$—$\frac{1}{3}$ in. long; bracts ovate acuminate, near base of pedicels. Flowers about $\frac{1}{3}$ in. long; calyx about $\frac{1}{6}$ in. long, puberulous outside, deeply 4-lobed; lobes with sides reflexed from a longitudinal internal edge; corolla glabrous except 4 longitudinal puberulous lines outside, 4-fid, suburceolate; tube $\frac{1}{8}$ in. long and thick, lobes $\frac{1}{8}$ in. long spreading, ovate, subciliate and pointed at apex by inflexion of sides near apex, contorted in æstivation; staminodes 8, equal, inserted in one row near base of corolla, $\frac{3}{8}$ in. long, appearing at mouth of corolla-tube, filaments longer than the barren anthers, hairy near top; ovary globose below, ovoid above, terminated by 2 hairy styles, ferrugineo- or (nigro-) hispid, 4- (10-) celled, 4- (10-) ovuled; stigmas emarginate. Fruit globose, subglabrate, 1$\frac{1}{2}$ in. in diameter, succulent; (seed scarcely 1 in. long by $\frac{1}{2}$ in. wide, transversely sulcate outside; albumen ruminated). Fruiting calyx $\frac{2}{3}$ in. across, spreading, puberulous outside, with raised 4-sided thickened stellate border inside; lobes wide, short, undulated.

Java, *Blume!*, *Horsfield* drawings n. 128 (part) in Hb. Kew.

20. DIOSPYROS DENSIFLORA, Wall. List, n. 4140 (1828—32).

D. foliis alternis, anguste ovalibus, utrinque obtusis, interdum subacutis, glabris, coriaceis, supra nitidis venis inconspicuis, petiolatis; floribus cymosis, tetrameris, pubescentibus, calyce profunde lobato, lobis margine reflexis, corollâ tubulosâ, staminibus 15—16, *antheris glabris, filamentis brevissimis hirsutis; fructibus globosis, ferrugineo-pilosis, calyce fructifero plicato, seminibus oblongis, transverse notatis, albumine ruminato* (?).

Alph. DC. Prod. VIII. p. 233. n. 56. (1844).

Branches glabrous terete. Leaves alternate, coriaceous, oval-oblong, glabrous, obtuse or obtusely acuminate at apex, slightly narrowed at base, 4—8 in. long by $1\frac{1}{2}$—$3\frac{2}{3}$ in. wide; petioles about $\frac{1}{3}$ in. long, glabrous, strong, wrinkled; midrib depressed and lateral veins slightly raised on upper face.

♂. Cymes panicled, about 1 in. long, hairy, many-flowered with hairy bracts and short pedicels; flowers about $\frac{1}{2}$ in. long, tetramerous; calyx $\frac{1}{8}$ in. long, hairy on both sides, 4-partite, lobes ovate with reflexed sides; corolla cylindrical, hairy outside, glabrous inside, 4 times the length of the calyx; stamens 15—16, anthers glabrous, on very short hairy filaments; ovary rudimentary, hairy.

♀. Cymes $\frac{2}{3}$ in. long, puberulous, about 12-flowered, trichotomous; pedicels longer than the peduncle; bracts lanceolate, at base of pedicels; flowers 1 in. long. Fruit globose, $\frac{2}{3}$—$\frac{4}{5}$ in. long, ferruginous-pilose; seeds oblong, transversely scored, albumen ruminated (?). Fruiting calyx 4-partite, $\frac{3}{4}$ the length of the fruit, puberulous, lobes much widened at base with auricled imbricated bases forming 4 dependent processes, plicate; pedicels $\frac{1}{3}$—$\frac{1}{2}$ in. long.

Moolmyne and Amherst, *Wallich!*; Martaban, Burma, below 500 feet alt., a small tree, *Dr Brandis!*

21. DIOSPYROS OOCARPA, Thw. Enum. Ceyl. Pl. p. 180. n. 9 (1860).

D. foliis alternis, glabris, ovatis vel ovalibus, apice obtuse acuminatis, basi rotandatis vel parum angustatis, tenuiter coriaceis, nervis inconspicuis; floribus masculis, 3—7-*nis, brevissime cymosis,* 3—4-*meris, calyce subintegro, vel dentato, corollæ præfloratione irregulari, staminibus* 9—12, *glabris; floribus femineis* 1—3-*nis, subsessilibus; ovario* 6—8-*loculari; fructibus subglobosis vel oblongis, puberulis vel glabratis, rugoso-areolatis, albumine non ruminato; calyce fructifero vix aucto.*

D. Arnottiana, Miq. (in Pl. Ind. Or. Hohenacker, n. 562!) ex Thw. *l. c.* p. 423 (1864).

Ceylon name *Kaloo-kadoombaireya-gass.*

A moderate-sized tree; young shoots pubescent or puberulous, quickly glabrescent. Leaves alternate, glabrous, ovate or oval, obtusely acuminate at apex, more or less rounded towards base, inconspicuously veined with midrib canaliculate above, thinly coriaceous 2—4½ in. long by 1—2 in. wide; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long.

♂. Flowers 3—7 together, arranged in dense axillary fulvous-silky cymes equalling or falling short of the petioles, with very short pedicels and rounded concave bracts. Calyx $\frac{1}{6}$ in. long, tubular, subentire or 3—4 dentate (or even 3—4-fid), glabrous inside. Corolla $\frac{5}{12}$ in. long, 3—4-fid, with obtuse lobes, one of which is completely enclosed by the others in bud, the other lobes imbricating sometimes dextrorsely and at other times sinistrorsely. Stamens 9—12, alternately in pairs and single, glabrous, inserted at the base of corolla-tube or some hypogynous; the outer ones of the pairs the longer; anthers shorter than the longer filaments (at least in bud); ovary rudimentary, hairy.

♀. Flowers 1—3 together, scarcely longer than the male; ovary 6—8-celled. Fruit egg-shaped when ripe, cylindrical when young, scattered with short appressed ferruginous hairs, glabrescent, 1⅓ in. long by $\frac{2}{3}$ in. thick when ripe, rugoso- or sub-verrucose, resting at base on scarcely increased calyx, solitary, 2-or more-celled, "usually 6-seeded." Seeds with albumen not ruminated.

Ceylon, *Thwaites!* C.P. 1914; Concan, *Dalzell!*; Bababoodun Hills, Mysore, *Mr Law!*

22. DIOSPYROS TRUNCATA, Zoll. et Mor. in Moritzi, Systemat. Verzeichn. Javan. Pflnzn. p. 43. n. 1156 (1846).

D. foliis alternis, ovali-oblongis, apice obtuse acuminatis, basi cuneatis, glabris, tenuiter coriaceis, breviter petiolatis, marginibus revolutis; floribus masculis 2—8-nis in alis subsessilibus vel brevissime cymosis, glabris; calyce tubuloso subintegro, corollæ lobis acuminatis, staminibus 11—14, glabris; floribus femineis 1—2-nis, brevissime pedunculatis, tetrameris; calycis lobis latissimis retusis reflexis; corollæ lobis acutis, patentibus; staminodiis 8—10; stylis 4; fructibus 8-locularibus, glabris.

D. laxa, Teijsmann et Binnendijk, Pl. n. h. Bogor. in Kruidk. Arch. III. p. 406 (1855).

D. melanoxylon, Blume! Bijdr. Fl. Ned. Ind. p. 669 (1825), non Roxb.

A tree with terminal buds slightly hairy; branches glabrous, terete, lax, widely spreading and forming a beautiful crown or top. Leaves oval-oblong, obtusely acuminate at apex, attenuate or narrowed at base, thinly coriaceous, with margins more or less reflexed, midrib depressed above, and delicate not contiguous lateral veins inconspicuously raised on both sides; of a yellowish green colour, alternate, glabrous, 3—6 in. long by 1—2 in. wide; petioles $\frac{1}{6}$—$\frac{1}{4}$ in. long.

♂. Flowers 2—4—8 together on very short axillary somewhat pubescent cymes, glabrous, yellowish green, slender, $\frac{7}{10}$ in. long; bracts small, pubescent; pedicels very short. Calyx tubular, somewhat inflated in middle, 4-toothed at apex, $\frac{7}{20}$ in. long by $\frac{1}{8}$ in. thick; corolla tubular, narrowly conical in bud, 4-fid (?), with acuminate lobes; stamens 11—12—14, glabrous, at base of corolla or on disk; filaments short; ovary obsolete.

♀. Flowers 1—2 together, $\frac{1}{6}$—$\frac{1}{4}$ in. long, on peduncles about $\frac{1}{10}$ in. long, axillary, as long as the petioles. Bracts caducous. Calyx glabrous with 4 very wide retuse reflexed (short ?) lobes; corolla twice the length of the calyx, 4-fid, with acute, patent, pale-yellow, lobes white at the base; staminodes 8—10; styles 4 connate at the base. Fruit glabrous, 8-celled, $\frac{1}{2}$ in. thick, globose, shining; fruiting calyx forming a shallow 4-cornered cup for base of fruit, with 4 reflexed undulate-plicate retuse lobes; $\frac{1}{2}$—$\frac{3}{4}$ in. in diameter.

According to Moritzi this tree resembles *D. Ebenum*, Linn. fil., and has even in the young twigs indications of black wood which becomes quite black in the older branches. The male flowers open in March.

In woods. Java. *De Vriese!* ♂ fl.; *Zollinger!* n. 1156; *Dr Horsfield!* Ebenaceæ, n. 4, in fruit; *Binnendijk!* ♂ fl.; *Hasskarl!*; *Blume!*

23. DIOSPYROS HALESIOIDES, Grisebach Cat. Pl. Cubensium, p. 168. n. 2 (1866).

D. foliis alternis, obovato-ovalibus, apice acutatis, basi obtuse cuneatis, subcoriaceis, pellucido-punctatis, subtus fulvo-velutinis, robustè venosis, suprà pubescentibus; floribus masculis in cymis brevissimis axillaribus dispositis, 1—3-nis, velutinis, calyce breviter 4-fido campanulato, corollâ urceolato-oblongâ, breviter 4-fidâ, lobis ovatis acuminatis, staminibus 12, glabris; fructibus solitariis, subsessilibus, depresso-globosis, ferrugineo-sericeis, 8-locularibus; calyce fructifero ampliato, lobis erecto-patentibus.

Terminal buds ferruginous-hairy; young shoots pubescent at apex, glabrescent shining and terete below. Leaves alternate, somewhat obovate, acute and apiculate at apex, cuneate to an obtuse base, subcoriaceous, fulvo-velutinous and conspicuously rich-veined beneath, darker nearly glabrescent and nitescent above, except veins; midrib depressed above; 1—$2\frac{1}{2}$ in. long by $\frac{1}{2}$—$1\frac{1}{4}$ in. wide, pellucid-punctate; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. long, ferruginous-pubescent; bracts ovate, small, fulvo-pubescent.

♂. Flowers $\frac{1}{2}$ in. long, in (1—) 3-flowered sessile or subsessile fulvous-hairy short cymes, on short pedicels; calyx campanulate, fulvo-velutinous, unequally 4-lobed, $\frac{1}{5}$—$\frac{1}{4}$ in. long; lobes deltoid, acute, less than half the length of the tube, unequal. Corolla hairy outside, glabrous inside, campanulate-oblong, $\frac{3}{8}$—$\frac{1}{2}$ in. long, 4-fid; lobes ovate-lanceolate. Stamens (11—) 12, glabrous, unequal, 8 in 2 rows opposite lobes of corolla, the inner 4 of which are on shorter filaments than the outer 4 and inserted above them (or united with them in 4 pairs) and 4 alternate with the corolla-lobes, and inserted on corolla near its base; filaments slender; ovary rudimentary, fulvous-hairy.

♀. Flowers unknown. Fruit solitary, sessile, depresso-globose, 1 in. thick by $\frac{3}{4}$ in. high, ferruginous-silky, 8-celled; fruiting calyx accrescent, deeply 4-fid, somewhat spreading, $1\frac{2}{5}$ in. across the top; lobes $\frac{1}{2}$-ellipsoidal, with margins somewhat recurved; albumen not ruminated.

Eastern Cuba, *Wright!* 2936 ♂ fl., 2937 in fruit.

24. DIOSPYROS BORNEENSIS, sp. nov.

D. foliis alternis, oblongis, apice breviter acuminatis, basi angustatis, subglabris, tenuiter coriaceis, breviter petiolatis; floribus femineis secus ramos vetustos glomeratis, pedunculatis, pubescentibus, calyce tubuloso subintegro, corollâ 5-fidâ, lobis ovalibus reflexis obtusis; staminodiis 10, uniserialibus, glabris, basi corollæ insertis; stylo 4-lobo; ovario conico, ferrugineo-tomentoso, 8-loculari, loculis 1-ovulatis; fructibus magnis, globosis, tomentosis.

A tree; wood yellow, tough and stringy with black streaks. Terminal buds ferruginous-tomentose; young shoots puberulous; branches dark, glabrescent, terete. Leaves oblong, alternate, thinly coriaceous, shortly acuminate at apex, somewhat narrowed at base, puberulous when young, glabrescent, 5—6½ in. long by 1½—2 in. wide, including petiole $\frac{1}{5}$—$\frac{1}{4}$ in. long; canaliculate on upper side, midrib depressed above, lateral veins about 12 on each side, distinct on under side, indistinct above, forming (by anastomosing) a marginal vein clearly marked beneath; tertiary veins oblique.

♀. Flowers clustered, greenish white, large, on distinct fulvo-tomentose peduncles $\frac{3}{10}$—$\frac{2}{5}$ in. long inserted on tubercles on older branches. Bracts small, obtuse, $\frac{1}{12}$—$\frac{1}{10}$ in. long, at base of peduncles. Calyx about $\frac{2}{5}$ in. long by $\frac{1}{4}$ in. thick, finely fulvo-tomentose outside, appressedly silky inside, tubular, irregularly and shallowly toothed at apex. Corolla about 8 in. long when straightened, nearly glabrous but with ciliate margins to the lobes, 5-fid; lobes oval, imbricated, reflexed. Staminodes 10, glabrous, in one row, inserted at base of corolla. Style 4-lobed; ovary conical, ferruginous-tomentose, 8-celled; cells 1-ovuled; dissepiments thin. Fruit large, with a sweet black pulp, globose, fulvo-tomentose, crowned with short style, surrounded half-way up by burst calyx.

Native name *malam kuning*. Tamgong Vinbong, Labuan, rather uncommon, *Mr Motley!* 7.

25. DIOSPYROS BATOCANA, sp. nov.

D. foliis alternis, ovalibus, utrinque rotundatis, coriaceis, glabris, supra nitentibus, subtus pallidis albidis, margine reflexo, nervis tenuibus, petiolis rugosis; floribus masculis sessilibus glomeratis secus ramos annotinos in nodulis dispositis, pentameris, fuligineo-hispidis; calyce apice lobato, corollâ crassâ, ovoideâ; staminibus circiter 12, *glabris, inæqualibus, ovarii rudimento hispido.*

A large bush, quite or nearly glabrous except the buds and inflorescence; branches nigro-cinereous, spreading at about 45°. Leaves alternate, oblong-elliptical, shining above, whitish beneath, firmly coriaceous, glabrous or with a few minute black setæ beneath, more or less rounded at both ends, with reflexed margins and veins in delicate relief on both sides, 2—2½ in. long by $\frac{3}{4}$—1 in. wide; petioles $\frac{1}{6}$—$\frac{1}{4}$ in. long, thick, angular, obliquely and in other directions wrinkled, often twisted or recurved.

♂. Inflorescence arranged on nodules, covered with fuliginous and ferruginous hispid hairs, on the branches of previous season; flowers sessile, several together, clustered; bracts at base of calyx; calyx fuliginous and ferruginous-hispid on both sides, 5-lobed at apex; corolla fuliginous-hispid outside, 5-fid, pale and glabrous inside, crass, ovoid; lobes imbricated sinistrorsely, obtuse; stamens 12—16 (?), glabrous, unequal, on short filaments; ovary wanting, represented by ferruginous hairs.

Tropical Africa, Setoka, "Mikumbo," Batoka country, *Dr Kirk!*, fruit eaten, ♂ fl. July.

26. DIOSPYROS QUÆSITA, Thw. Enum. Ceyl. Pl. p. 179. n. 7 (1860).

D. foliis alternis, ellipticis vel oblongis, apice breviter et abrupte acuminatis, obtusis, basi parum angustatis, glabris, coriaceis, petiolatis, nervis reticulatis gracilibus; floribus masculis 3—9-*nis, breviter cymosis,* 4—5-*meris, hirsutis, calyce tubuloso apice dentato; staminibus* 16, *glabris, ovarii rudimento hirsuto; floribus femineis solitariis, pedunculatis, pentameris, calyce inter lobos marsupio-dilatato, lobis acutiusculis; fructibus subglobosis, externe rugosis, subglabris, seminibus compressis, albumine non ruminato.*

Bedd. Ic. Pl. Ind. Or. (Pt. VII.) p. 26. t. 128 (1871).

A large tree, nearly glabrous except the buds and flowers; branches terete, dark, spreading at about 40°. Leaves alternate, coriaceous, elliptical or oblong, abruptly and shortly acuminate, somewhat narrowed at base, glabrous except a few scattered weak appressed hairs beneath; 3—7 in. long by $1\frac{1}{2}$—3 in. wide, turning fuscous in drying, with petioles canaliculate and about $\frac{1}{2}$ in. long, midrib depressed on the upper surface, lateral veins numerous, not conspicuous.

♂. Cymes 3—9-flowered, pilose, about equalling the petiole; flowers (closed) $\frac{1}{2}$—$\frac{3}{5}$ in. long. Calyx $\frac{5}{12}$ in. long by $\frac{3}{13}$ in. wide, obscurely 4—5-lobed at apex, pubescent, tubular, somewhat inflated about middle; lobes depresso-deltoid; corolla about $\frac{1}{2}$ in. long (closed), oblong, clothed outside with subferruginous felt, 4—5-lobed; lobes ovate, about $\frac{2}{5}$ths depth of corolla-tube; stamens 16, hypogynous, glabrous, not united in pairs, $\frac{1}{6}$ in. long; anthers longer than the filaments; ovary rudimentary, hairy.

♀. Flowers solitary, pentamerous; corolla shortly 5-lobed; fruiting calyx 5-fid, marsupio-dilatated with ovate cordate lobes having reflexed sides and base, hairy inside, $1\frac{1}{3}$ in. across; fruiting peduncle stout, patent, $\frac{2}{5}$ in. long; fruit subglobose, 2 in. in diameter (immature), rugose, nearly glabrous, 8-celled (?); seeds 1 in. long, shining, compressed; albumen not ruminated.

According to Dr Thwaites this tree is the true Calamander of the Cinghalese; in Ceylon it is called *Kaloomidereya-gass.*

Ceylon, *Thwaites!* C. P. 3010.

27. DIOSPYROS TOXICARIA, sp. nov.

D. foliis alternis, elongato-ovatis, apice acuminatis, basi rotundis, glabris, nitentibus, coriaceis, subtus reticulatis; petiolis robustis; floribus masculis ferrugineo-tomentosis, sessilibus, aggregatis, e pulvino convexo surgentibus, bracteis basi obtectis, calyce apice lobato, corollâ breviter 4-fidâ, staminibus 11—13, *glabris, ovarii rudimento dense piloso; fructibus solitariis, subsessilibus, ferrugineo-tomentosis, subglobosis,* 8-*locularibus; calyce fructifero, cyathiformi, breviter* 4-*lobo, aucto.*

A tree, 20—30 feet high, glabrous except the fruit and inflorescence, bark rather rough, gum sometimes exudes from it. Leaves elongate-ovate, rounded at the base, acuminate at the apex, alternate, coriaceous, shining, $2\frac{1}{2}$—5 in. long (besides robust petiole about $\frac{1}{6}$ in. long) by 1—2 in. wide, midrib depressed above, finely reticulated as in *D. tessellaria.*

♂. Flowers 5—12 together sessile or axillary, very short ferruginous hairy dense nodular cymes; bracts imbricated, unequal, ferrugineo-tomentose outside, nearly glabrous inside, oval, some (outer ones) $\frac{1}{3}$ in. long; buds ovoid, ferrugineo-tomentose, $\frac{1}{3}$ in. long; calyx lobed at apex, hairy outside, glabrous inside. Corolla shortly 4-fid, hairy outside, glabrous inside. Stamens 11—13, glabrous; ovary wanting; receptacle hairy. Native names

Sifotatu, Alacainisi. Madagascar, Tranomaro, sands near the sea, July 1862, *Dr Meller!* Natives say that birds die soon after eating the fruit.

♀. Bracts caducous, imbricated. Fruit solitary, ferruginous-tomentose, subsessile; young fruit scarcely the length of the accrescent calyx, ovoid or subglobose, $\frac{1}{2}$ in. high, umbilicate-depressed at apex, 8-celled, 8-ovuled; pericarp thick; fruiting calyx ferruginous-tomentose on both sides, cup-shaped, shortly 4-lobed. Native name *Vorongi.* Madagascar, Tranomaro, July 1862, *Dr Meller!*

The following may belong to this species:

(1) Specimen with ovoid fruit ferruginous-tomentose 2 in. high, fruiting calyx $1\frac{1}{2}$ in. across appressed to base of fruit, leaves $4\frac{1}{2}$—$6\frac{1}{2}$ in. long by $1\frac{5}{8}$—$2\frac{1}{4}$ in. wide; Madagascar, *Lastelle!* 1841, seen in the Paris Museum.

(2) Fruit nearly 1 in. long, rufous-tomentose. Fruiting calyx spreading $\frac{3}{4}$ in. across; Madagascar, *Chapelier!* Seen in the Paris Museum.

(3) Fruit $1\frac{1}{3}$ in. long by $1\frac{1}{6}$ in. thick, ferruginous-tomentose, 8-celled; fruiting calyx spreading. Without leaves. Said to come from S. Africa, but probably this is a mistake; *Gerard!* n. 190, seen at the Kew Museum.

28. DIOSPYROS TESSELLARIA, Poir. in Enclyc. Méth. V. p. 430. n. 5 (1804).

D. ramulis fusco-cinereis; foliis alternis, ovalibus vel ovatis, apice rotundatis, basi rotundis, glabris, nitentibus, coriaceis, tenuiter reticulatis, petiolatis; floribus masculis sessilibus, sæpius aggregatis, e pulvino convexo ferrugineo piloso surgentibus, tetrameris, tomentosis, bracteis imbricatis ovato-rotundatis extus sericeis intus glabris; calyce apice lobato, staminibus 12—13, *glabris; floribus femineis aggregatis, tetrameris; fructibus subglobosis, ferrugineo-sericeis vel subglabratis, 8-locularibus, edulibus.*

Alph. DC. Prodr. VIII. p. 225. n. 12 (1844).

Ebenus tessellaria, Commers. ex Poir. *l. c.*

D. Ebenum, Poir. *l. c.* p. 429. n. 4, non Koenig.

D. reticulata, Willd. sp. pl. IV. p. 1109, n. 6 (1805), Alph. DC. *l. c.* n. 11 excl. *β timoriana*, non Decaisne.

A tree or shrub with dark-cinereous glabrous branches. Leaves alternate, oval or ovate, rounded at both ends, especially at base, where they are sometimes slightly cordate, glabrous, coriaceous, shining, finely reticulated, 3—6 in. long by $1\frac{1}{2}$—$3\frac{1}{2}$ in. wide; margins slightly reflexed: petioles stout, $\frac{1}{4}$—$\frac{1}{3}$ in. long. Flowers densely clustered, sessile, arising from lateral nodules on the young branches, fulvo-sericeous, tetramerous; calyx tubular, lobed at the apex, $\frac{1}{8}$—$\frac{1}{4}$ in. long; ♂. stamens 12—13, glabrous, mostly in pairs, inserted on the receptacle. Fruit globular or ellipsoidal, 8-celled, nearly glabrate or sericeous, edible, 8-celled; fruiting calyx hemispherical or rarely flat, thickly coriaceous, sericeous, lobes short, rounded. Wood valuable; this species probably yields the ebony of Mauritius.

Mauritius, in the forests of the highest parts of the island, *Bouton!; Ayres!, Telfair!;* shrub 6—8 feet high, fruit good to eat, sweet, fruiting calyx flat, *Bouton!*

D. rubra, Gaertn. fil. Fruct. et Sem. Pl. III. p. 138. t. 208 (1805), differs by a flat 5-lobed fruiting calyx and 10-celled fruit; it may however belong to *D. tessellaria*, Poir., or if not to *D. chrysophyllos*, Poir.

29. Diospyros haplostylis, Boiv. MSS.

D. foliis alternis, ovalibus, apice anguste acuminatis, basi angustatis, coriaceis, nitidis, subglabris, subtus tenuiter reticulatis, breviter petiolatis; floribus masculis 3—6-*nis, brevissime cymosis vel aggregatis, ferrugineo-sericeis,* 4—5-*meris, calyce breviter lobato, staminibus* 10—12, *glabris, biserialibus; floribus femineis solitariis, brevissime pedunculatis, ferrugineo-sericeis,* 4—5-*meris, staminodiis* 4, *glabris, ovario sericeo, depresso-globoso,* 8-*loculari, stylo apice* 4-*lobo; fructibus subglobosis, glabrescentibus,* 8-*locularibus.*

A shrub of 12 feet or an erect tree 22 feet high or more; heart-wood black, very hard; young parts puberulous; branches glabrescent, terete, subcinereous, smooth. Leaves alternate, oval, narrowly acuminate at apex, somewhat narrowed at base, coriaceous, glabrous with depressed midrib above, highly and minutely reticular beneath with scattered appressed inconspicuous hairs, somewhat undulated, 2—$3\frac{1}{2}$ in. long by 1—$1\frac{3}{4}$ in. wide; petioles $\frac{1}{4}$ in. long.

♂. Flowers clustered, 3—6 together, subsessile on young branches, ferruginous-pubescent, $\frac{3}{8}$ in. long by $\frac{1}{8}$ in. thick; bracts caducous, smaller than the flowers; calyx hairy on both sides, $\frac{1}{4}$ in. long by $\frac{1}{8}$ in. thick, lobes 4, erect, deltoid, one-third the depth of the calyx; corolla hairy outside, glabrous inside, lobes one-third the length of the corolla; stamens 10, 12, glabrous, hypogynous, biseriate, nearly equal, $\frac{1}{8}$—$\frac{1}{7}$ in. long, anthers $\frac{1}{10}$ in. long, linear; ovary rudimentary, ferruginous-sericeous.

♀. Flowers solitary, ferruginous-sericeous, nearly $\frac{1}{2}$ in. long, very shortly pedunculate; calyx $\frac{5}{16}$ in. long by $\frac{1}{4}$ in. thick, campanulate, lobes 4—5, ovate deltoid, one-third depth of calyx; corolla 4-fid, lobes somewhat spreading; staminodes 4, glabrous, short, alternate with corolla-lobes; ovary densely sericeous, fleshy, depresso-globose, $\frac{1}{10}$ in. high by $\frac{1}{6}$ in. thick, 8-celled, terminated at apex by style; style $\frac{1}{10}$ in. long, 4-lobed at apex. Fruit ferruginous-sericeous when young, glabrescent, subglobose, $1\frac{1}{4}$ in. long by 1 in. thick, 8-celled; cells 1-seeded; fruiting peduncle $\frac{1}{8}$ in. long, sericeous; fruiting calyx $\frac{3}{8}$ in. long and wide, hairy on both sides, campanulate or nearly flat, spreading.

Madagascar, Nossi Be, *Boivin!* 2108 bis, *Pervillé!* 439, 505; mountains at Diego, Suares, *Bernier!* 259 (excl. fr.).

30. Diospyros melanida, Poir. in Encycl. Méth. v. p. 431. n. 7 (1804).

D. foliis alternis, ovalibus, utrinque rotundatis vel obtuse angustatis, glabris, coriaceis, petiolatis, mediocriter reticulatis; floribus masculis 1—3-*nis, aggregatis, sessilibus,* 5—6-*meris, calyce subglabro apice lobato, corollâ* 5—6-*fidâ, extus sericeâ, staminibus* 22—24, *glabris, basi corollæ insertis; floribus femineis solitariis, fructibus subglobosis, sessilibus,* 10-*locularibus, calyce fructifero aucto,* 5—6-*lobo, tubo concavo, lobis recurvis sæpe undulatis; seminibus oblongis, albumine cartilagineo, non ruminato.*

Alph. DC. Prodr. VIII. p. 227. n. 22 (1844).

Ebenus melanida, Commers. ex Poir. *l. c.*

(?) *D. pterocalyx*, Boj. Hort. Maurit. p. 200. n. 7 (1837), Alph. DC. *l. c.* p. 225. n. 14?

A tree with glabrous stem and branches. Leaves alternate, oval, rounded or obtusely narrowed at either end, glabrous, coriaceous, 1—8 in. long by $\frac{1}{2}$—3 in. wide, petioles $\frac{1}{6}$—$\frac{1}{2}$ in. long; margins slightly recurved, net-veins delicate, often coloured beneath.

♂. Flowers 1—3 together, sessile, 5—6-merous, $\frac{1}{2}$ in. long; calyx tubular, cup-shaped, $\frac{1}{3}$ in. long, subentire or 5—6-lobed at apex, subglabrous; corolla 5—6-fid, silky outside, glabrous inside, lobes oval, rounded, spreading and reflexed; stamens 22—24, glabrous, inserted at the base of the corolla; ovary rudimentary, hairy.

♀. Flowers solitary. Fruit sessile, subglobose, as large as a moderate-sized apple, glabrous, shining, 10-celled, surrounded one-third way up by tube of calyx which has 5—6 wide reflexed and often undulated lobes. Seeds oblong, albumen not ruminated.

Mauritius, *Bouton!* The following localities are less certain; Bourbon, *Richard!*, *Boivin!;* Round Island, Mauritius, *Sir H. Barkly!;* Rodriguez, *Bouton!*

31. DIOSPYROS NODOSA, Poir. Encycl. Méth. V. p. 432 n. 9. (1804).

D. foliis ovalibus vel oblongis, alternis, utrinque rotundatis vel obtuse angustatis, glabris, coriaceis, petiolatis, mediocriter reticulatis; floribus masculis 1—3-*nis, subsessilibus, sæpius* 5-*meris, calyce glabro, apice lobato, staminibus* 20—32, *glabris; floribus femineis solitariis, subsessilibus, staminodiis* 12, *ovario hirsuto, stylo* 5-*lobo; fructibus subglobosis, glabratis, calyce fructifero aucto, tubo cyathiformi, lobis erectis.*

Alph. DC. Prodr. VIII. p. 226. n. 18 (1844).

D. angulata, Poir. *l. c.* p. 434. n. 16, Alph. DC. *l. c.* p. 226. n. 16.

D. mauritiana, Alph. DC. *l. c.* p. 226. n. 15 (1844).

D. macrocalyx, Alph. DC. *l. c.* p. 226. n. 17, non Kl.

(?) *D. capensis*, Alph. DC.! *l. c.* p. 226. n. 19.

(?) *D. Neraudii*, Alph. DC. *l. c.* p. 227. n. 23.

(?) *D. Boutoniana*, Alph. DC. *l. c.* p. 236. n. 72.

A glabrous shrub or tree; branches especially of the male plants often nodose at the inflorescence. Leaves oval or oblong, alternate, more or less rounded at both ends or occasionally cuneate at base, coriaceous, $1\frac{1}{2}$—6 in. long by 1—3 in. wide; petioles $\frac{1}{4}$—$\frac{2}{3}$ in. long.

♂. Flowers axillary, subsessile, about 1—3 together; calyx glabrous or glabrescent, subtruncate or shortly 4—6- usually 5-lobed at apex, cup-shaped, about $\frac{1}{3}$ in. long; corolla about $\frac{1}{2}$ in. long, sericeous outside, glabrous inside, deeply 4—6- usually 5-lobed; lobes oval, spreading; stamens 20—32, glabrous, hypogynous or inserted at the base of the corolla, somewhat combined at base; filaments short; ovary rudimentary, hairy.

♀. Flowers solitary, axillary, subsessile; bracts imbricated, caducous; calyx shortly 5-lobed, nearly glabrous, cup-shaped; corolla short; staminodes 12, separate, inserted at base of corolla; ovary hairy; style 5-lobed. Fruit globular or ovoid, glabrate, 1¼—2 in. high, resting at base on cup-shaped nearly glabrous calyx which in some cases reaches half-way up the fruit and has erect lobes.

Mauritius, *Boivin!*, *Gardner!*, *Duport!*, *Commerson!*, 299, 301. Madagascar, *Boivin!*

D. capensis, Alph. DC. is reported from the Cape of Good Hope, probably by mistake.

Perhaps ought to be united to *D. melanida*, Poir.

32. DIOSPYROS ANONÆFOLIA, Alph. DC. Prodr. VIII. p. 227. n. 21 (1844).

D. foliis elliptico-oblongis, alternis, obtusis, basi subacutis, glabris, submembranaceis, petiolatis; floribus masculis aggregatis, subsessilibus, calyce elongato-cyathiformi, basi acuto, glabro, obscure 5-lobo; corollâ profunde 5-fidâ, extus sericeâ; staminibus 20—24, geminatis, glabris, corollæ basi insertis.

Branches and buds glabrous. Leaves alternate, elliptic-oblong, obtuse, glabrous, submembranous, subacute at the base, 5—7 in. long by 2—3 in. wide, paler beneath; petioles ½ in. long.

♂. Flowers fascicled, 5—15 together, subsessile; bracts ovate, glabrous, caducous; calyx elongate, cup-shaped, acute at the base, smooth, glabrous, obscurely 5-lobed at the apex, ¼ in. long; corolla deeply 5-lobed, silky outside, rather longer than the calyx. Stamens 20—24, united in pairs at base, glabrous, inserted at base of corolla.

Mauritius (or Bourbon?) ex Alph. DC. *l. c.*

Perhaps ought to be united to *D. nodosa* or to *D. melanida* or to both.

33. DIOSPYROS LEUCOMELAS, Poir. Encycl. Méth. V. p. 432. n. 8 (1804).

D. foliis alternis, ovalibus vel orbicularibus, apice rotundis, basi cordatis, subamplexicaulibus, coriaceis, glabris, nitidis, subsessilibus; floribus diœcis, 1—3-nis, axillaribus, sessilibus, 6—5-meris; calyce tubuloso-campanulato, apice lobato, extus sericeo; corollâ profunde lobatâ; staminibus 30—40, glabris, receptaculo insertis; fructibus solitariis, glabris, calyce cyathiformi duplo et ultra longioribus, 8—12-locularibus.

Alph. DC. Prodr. VIII. p. 236. n. 70 (1844).

Ebenus leucomelas, Commers. MSS. n. 149 Ic. ex Poir. *l. c.*

Diospyros reticulata, Sieb.! Pl. Maurit. n. 114. non Willd. nec Decaisne.

Diospyros amplexicaulis, Lindl. et Paxt.! Flower Garden, II. p. 11. n. 271. fig. 139 (1851).

Diospyros Commersoni, Gaertn. fil. Carp. III. p. 136. t. 208 (1805).

D. melanida, Neraud ex Alph. DC. Prodr. VIII. p. 236. n. 70 (1844), non Poir.

Cfr. *D. Hebenaster*, Gaertn. Fruct. et Sem. Plant. II. p. 478. t. 179. f. 9 (1791), non *D. Ebenaster*, Retz.

A lofty tree with white wood but with black lines in the heart; trunk with a dark bark, much branched; branches glabrous, pale-cinereous, spreading at about 40°. Leaves oval or orbicular, alternate, subsessile, cordate at base, rounded at apex, subamplexicaul, coriaceous, quite glabrous and shining; often marked by coloured net-veins and occasionally by black blotches; 2—$5\frac{1}{2}$ in. long by $1\frac{1}{2}$—$3\frac{3}{4}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long. Bracts imbricated, subtomentose with grey hair, rounded, $\frac{1}{10}$ in. high by $\frac{1}{4}$ in. wide, surrounding the base of the calyx.

♂. Flowers 1 or few together, axillary on young shoots or clustered on the shoots of previous season, sessile; calyx tubular, somewhat campanulate, with usually 6 short teeth at apex, covered outside with short brown or cinereous tomentum, $\frac{1}{8}$—$\frac{1}{2}$ in. high; corolla campanulate, 6—5-lobed, shortly ferruginous-sericeous outside, glabrous inside; tube $\frac{1}{4}$—$\frac{3}{10}$ in. long: lobes $\frac{1}{5}$ in. long, spreading and recurved at extremities. Stamens 30—40, glabrous, inserted on the receptacle, nearly equal; anthers linear $\frac{1}{7}$ in. long: filaments about $\frac{1}{10}$ in. long, often somewhat combined at base; ovary only represented by a trace of hair.

♀. Flowers arranged as in ♂. Fruit sessile, solitary, on branches deprived of their leaves (in the dry state), very glutinous, *ex* Poiret, glabrous *ex* Alph. DC., depresso-globose, umbilicate at the apex, about 1 in. high by $1\frac{1}{4}$ in. thick, 8—12-celled; fruit-calyx cup-shaped, about $\frac{1}{2}$ the height of the fruit which it receives, 6-lobed at apex, coriaceous; seeds cinereous, $\frac{3}{4}$ in. long; albumen not ruminated, white.

Mauritius, *Commerson!*; *Sieber!* 114; on the crest of the mountain to the left of the second Fenetre, above the French fort, Sept. ♂ fl., *Ayres!*; in forests on mountains at Savane and at Trois Ilots, Fl. May, Dec. *Ayres* MSS.; a specimen from Round Island near Mauritius without flower or fruit by *Sir H. Barkly!* probably belongs to this species; Madagascar, *Gaertner*, *Chapelier!*

34. DIOSPYROS CHRYSOPHYLLOS, Poir. Encycl. Méth. v. p. 433, n. 12 (1804).

D. ramulis flexuosis, foliis lanceolato-oblongis, apice utrinque obtuse angustatis, glabris, coriaceis, petiolatis; floribus diœcis, 1—3-nis, subsessilibus, 5—4-meris, axillaribus; calyce cyathiformi, extus pubescente, apice lobato, corollâ profunde lobatâ, staminibus 11—15, glabris; ovario in floribus femineis glabro, 10-loculari, stylis 5; fructibus solitariis, globosis, nitidis, 7—10-locularibus, calyce fructifero subtruncato, pateriformi.

Alph. DC. Prodr. VIII. p. 225. n. 13 (1844).

A shrub or tree with glabrous flexuous branches, subscandent (?). Leaves lanceolate-oblong, somewhat narrowed at base, usually more or less narrowed towards apex, obtuse, $2\frac{1}{4}$—$5\frac{1}{2}$ in. long by $\frac{3}{4}$—$1\frac{1}{4}$ in. wide, besides petiole $\frac{1}{5}$—$\frac{3}{10}$ in. long; somewhat paler and brilliant beneath (golden-coloured); coriaceous, margins reflexed. Flowers subsessile, axillary, pentamerous or tetramerous. Calyx ferruginous-pubescent outside, cup-shaped, dentate at apex. Corolla ferruginous-sericeous outside, deeply lobed.

♂. Flowers 1—3 together. Stamens 11—15.

♀. Flowers solitary, more than $\frac{5}{8}$ in. long. Calyx $\frac{3}{10}$—$\frac{3}{8}$ in. long; lobes $\frac{3}{40}$ in. deep, widely ovate, wavy at margins, obtuse; tube very crass especially at base, felted outside, glabrous and shining inside. Corolla with 5 short imbricated rounded lobes, constricted at top of calyx, hairy outside in upper part, often remaining at top of young fruit. Staminodes 9, glabrous. Ovary glabrous but surrounded at base with a ring of hairs, 10-celled; styles 5, sericeous at base; stigmas lobed at apex. Fruit glabrous, globose about 1 in. in diameter, shining, green, 7—10-celled. Fruiting calyx subtruncate, $\frac{7}{8}$ in. across at top, glabrescent.

Mauritius, *Bojer!*, *Gardner!*, *Bouton!*, *Commerson!*

35. DIOSPYROS SENENSIS, Klotzsch in Peters Mossamb. I. p. 183 (1862).

D. foliis alternis, obovato-oblongis, apice breviter acuminatis vel rotundatis, basi cuneatis vel subrotundatis, submembranaceis, subtus flavido-pubescentibus, breviter petiolatis; floribus 1—5-nis, breviter cymosis, axillaribus, tetrameris, pedicellis brevissimis; calyce anguste tubuloso, apice lobato, corollâ 4-fidâ, staminibus 16, *geminatis, glabris; in flore femineo staminodiis* 0, *ovario glabro* (?), 8-*loculari; fructibus solitariis, glabris,* 2—8-*locularibus.*

A shrub from 10 feet high to a tree 30 feet; occasionally subhermaphrodite or polygamous; branches terete, pale-cinereous or smooth and reddish; young shoots softly flavido-pubescent. Leaves membranous, alternate, obovate-oblong, suddenly narrowed or acuminate or occasionally rounded at apex, cuneate or nearly rounded at base, subglabrescent and deep green above with depressed midrib, somewhat flavido-pubescent or subglabrescent beneath, 2—7¼ in. long × 1—3½ in. wide, besides hairy petiole $\frac{1}{10}$—½ in. long. Inflorescence axillary, in short 1—5-flowered cymes, flavido-pubescent, with small caducous bracts at base of very short pedicels; flowers greenish-yellow, fragrant; ♂ peduncles not exceeding ⅓ in. long.

♂. Calyx ¼—⅖ in. long by $\frac{1}{10}$ in. thick, tubular, subtruncate or with 4 short rounded lobes at apex, flavido-pubescent outside and hairy inside. Flowers greenish-yellow, fragrant. Corolla tubular, about twice the length of the calyx, 4-fid, glabrous, except 4 hairy lines down the middle of the lobes; lobes oblong, obtuse; stamens 16, glabrous, in pairs, partly inserted at base of corolla and partly hypogynous; ovary usually rudimentary or wanting, occasionally 5-celled in subhermaphrodite flowers.

♀. Flowers shorter than in ♂; staminodes 0; ovary glabrous (?), 8-celled; calyx hairy on both sides. Fruit solitary, glabrous, but hairy around base of style, acorn-shaped, 1 in. long by ⅔ in. thick; half inclosed in subtruncate calyx, 2—4—8-celled, not eaten (*Dr. Kirk*); style making a short conical projection; fruiting calyx shortly pubescent especially inside; albumen horny; seeds with green vittæ on surface (*Kirk*); cotyledons cordate, acute, foliaceous.

Tropical Africa, Mozambique, Rios de Sena, *Dr. Peters!;* Lower Shire Valley, *Dr. Kirk!* ♂ fl. January; Lupata, *Dr. Kirk!*, young fruit, January; Forest below Strigogo, left bank of Zambezi, *Dr. Kirk!*, fruit, April; North of Shire, Banks of Zambezi, *Dr. Meller!* ♂, fl. January; Abbeokuta, *Dr. Irving!* 141; Abbeokuta &c., Niger expedition, *Barter!* 290, 3251, 3390; Eppah, *Barter!* 3250 (peduncles 1-flowered, $\frac{1}{10}$—$\frac{3}{10}$ in. long, in bud).

36. DIOSPYROS ROTUNDIFOLIA, sp. nov.

D. foliis alternis, obovato-rotundis, utrinque rotundatis, glabris, coriaceis, breviter petiolatis, margine revolutis, nervis subtus inconspicuis; floribus solitariis, axillaribus, glabris, diœcis, breviter pedunculatis; calyce apice 5-*lobo; corollâ* 5-*fidâ; staminibus* 30, *glabris, receptaculo insertis; fructibus globosis, apice umbilicatis, nitidis, uncialibus,* 8-*locularibus* (?); *calyce fructifero aucto, apice* 5-*lobo, plicato, pateriformi.*

Young parts puberulous; branches pale-cinereous. Leaves obovate-rotund, alternate, coriaceous, with recurved margins in the dry state, rounded at both ends, ¾—1½ in. long by $\frac{9}{10}$—1¼ in. wide, besides petioles $\frac{1}{10}$—⅙ in. long, glabrous; veins inconspicuous beneath.

Peduncles axillary, solitary, crowded in upper axils, puberulous, recurved, $\frac{1}{10}$—$\frac{1}{6}$ in. long, 1-flowered; bracts caducous.

♂. Flowers glabrous, about $\frac{3}{8}$ in. long; calyx $\frac{3}{16}$ in. long, hemispherical-campanulate, with 5 shallow apiculate lobes; corolla 5-fid, with oval spreading lobes; stamens in one case 30 !, glabrous, nearly equal, inserted on the glabrous receptacle; filaments short, straight; anthers about $\frac{1}{8}$ in. long, 2-celled, dehiscing laterally from apex; ovary 0.

♀. Fruiting calyx accrescent, exceeding the young fruit, $\frac{1}{2}$ in. high and broad, really 5-lobed at apex but apparently 5-fid by calyx being plicate and reflexed downwards and outwards into 5 sides; quasi-lobes broadly ovate dilated at base and plicate so as to make the calyx 5-winged; styles 5, connate at base, $\frac{1}{15}$ in. long, glabrous; stigmas bifid; young fruit depresso-globose, $\frac{1}{8}$ in. high, glabrous, 8 ?-celled; ripe fruit globose, umbilicate at apex, shining, $\frac{3}{4}$—1 in. in diameter; fruiting calyx pateriform, $\frac{3}{4}$—1 in. across, $\frac{1}{13}$ in. high, with raised border above, plicate; seeds compressed, $\frac{1}{2}$ in. long by $\frac{5}{16}$ in. wide.

S. Africa, Delagoa Bay, *Forbes!* 34.

37. Diospyros attenuata, Thw. Enum. Ceyl. Pl. p. 182. n. 18 (1860).

D. foliis alternis, anguste ovatis vel oblongis, apice acuminatis, basi cuneatis, tenuiter coriaceis, glabratis, breviter petiolatis, creberrime venulosis; floribus masculis 3—10-*nis, subsessilibus, oblongis,* 4—5-*meris, staminibus* 4—5; *floribus femineis solitariis, ovario* 4-*loculari, fructibus ovoideo-conicis, acuminatis, subglabrescentibus,* 2—3-*spermis, albumine non ruminato.*

Bedd. Ic. Pl. Ind. Or. (Part VII.) p. 28. t. 139 (1871).

A moderate-sized tree; young shoots appressedly puberulous, quickly glabrescent; leaves alternate, narrowly ovate or oblong, acuminate at apex, more or less narrowed at base, quickly glabrescent, thinly subcoriaceous, 2—4 in. long by $\frac{2}{3}$—$1\frac{1}{3}$ in. wide; petioles $\frac{1}{12}$—$\frac{1}{6}$ in. long; midrib depressed above; net-veins very close together, in relief on both sides, delicate.

♂. Flowers clustered, 3—10 together, sessile or subsessile on $\frac{1}{8}$ in. long axillary nodules, strigose with black and subferruginous mixed hairs, 4—5-merous; calyx $\frac{1}{12}$ in. long, 4—5-fid, hairy on both sides, lobes narrowly deltoid. Corolla slender in bud, much exceeding the calyx, 4—5-lobed, $1\frac{1}{6}$ in. long, lobes rather shorter than the tube; stamens 4—5, in one row, anthers glabrous, connective prolonged at apex, filaments short, without or with light brown hairs; ovary 0 or rudimentary, conical, with light ferruginous hairs.

♀. Flowers solitary, sessile; calyx $\frac{1}{4}$—$\frac{1}{3}$ in. long, lobes more or less reflexed at the margin; corolla but little exceeding the calyx; staminodes 4—5; stigmas 2, short; ovary hairy, 4-celled, ovoid; cells 1-ovuled. Fruit conical, with an ovoid base and acuminate apex, pale, softly hairy or nearly glabrescent, 2—3-seeded; fruiting calyx loose, deeply 4—5-lobed, not accrescent; seeds oblong, shining, acuminate; albumen not ruminated.

Ceylon, Pasdoon Corle, *Thwaites!* C. P. 3478.

38. Diospyros acuta, Thw. Enum. Ceyl. Pl. p. 182. n. 17 (1860).

D. foliis alternis, lanceolato-oblongis, apice acuminatis, basi subrotundatis, coriaceis, glabris, robuste petiolatis, nervis inconspicuis; floribus masculis aggregatis, sessilibus, 4—5-

meris, calycis lobis lanceolatis, staminibus 4—5; *floribus femineis* 1—4-*nis; fructibus ovoideis acuminatis,* 2—3-*spermis, seminibus acuminatis, albumine non ruminato.*

A moderate-sized tree, glabrous except the buds and inflorescence; branches terete. Leaves alternate, lanceolate-oblong, acuminate at apex, more or less rounded towards base, coriaceous, 5—12 in. long by $1\frac{1}{2}$—4 in. wide, turning reddish beneath (when dry); petioles $\frac{1}{2}$—1 in. long, stout, channelled above; midrib depressed above; lateral veins inconspicuous. Inflorescence appressedly fulvous-hairy, diœcious or sometimes monœcious, in which case the female capitula are towards the top of the branches, and the male ones beneath.

♂. Inflorescence dense, many-flowered, axillary, sessile; calyx $\frac{1}{6}$ in. long, 4—5-lobed beyond the middle, lobes lanceolate, acute, hairy on both sides; corolla $\frac{1}{5}$ in. long, 4—5-fid; stamens 4—5, short, glabrous; ovary rudimentary, very small.

♀. Flowers 1—4 together; calyx $\frac{1}{3}$—$\frac{5}{12}$ in. long, lobes more or less reflexed at the margin; corolla about as long as the calyx; stigmas 2—3, $\frac{1}{12}$ in. long, spathulate; ovary 4- or 6-celled; fruit acuminate, $1\frac{1}{2}$ in. long, resting on a scarcely increased calyx, usually 2—3-seeded; seeds shining, oblong, acuminate, 1 in. long; albumen not ruminated.

Ceylon, Pasdoon Corle, *Thwaites!* C. P. 3476.

39. DIOSPYROS TRICOLOR.

D. fruticosa, foliis alternis, ellipticis, utrinque obtusis, supra subglabris viridibus, subtus albido-sericeis, costâ ferrugineâ; floribus axillaribus, sessilibus, 1—4-*nis, tetrameris, pubescentibus, calyce quadrifido, corollâ tubulosâ, staminibus* 6—8 *vel pluribus, inæqualibus; floribus femineis solitariis, staminodiis* 7—8, *ovario ovoideo, sericeo, in stylum subulatum attenuato; fructibus subpyramidatis, glabris, junioribus* 4-*locularibus; seminibus* 2—4.

Noltia tricolor, Schum. et Thonn. Plant. Guin. p. 189 (1827), in Kong. Danske Vidensk. Sel. Phys. og Mathem. Skr. III. p. 209 (1828).

A much-branched shrub, 2—4 feet high; branches terete, ferruginous-tomentose, diverging, sometimes flexuous, procumbent. Leaves alternate, distichous, elliptical, obtuse, nearly rounded at base, with few lateral veins, green and glabrescent above, white-silky with the midrib and margin often ferruginous beneath, 1—3 in. long by $\frac{2}{3}$—2 in. wide, the young ones silvery-silky on both sides; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long, tomentose. Flowers solitary or 3—4 together, axillary, sessile.

♂. Calyx 4-fid, lobes acute, silky-tomentose, ferruginous; corolla tubular, 3 times the length of the calyx, scarcely dilated below, sub-4-lobed, subcoriaceous, "red," silky outside, $\frac{1}{6}$ in. long; lobes acute, erect, inflexed at the margin; filaments 6—"8 or more, unequal, 4 often double the length of the rest, half the length of the corolla, pubescent below, inserted on the receptacle, either distinct or 2—3 together at the base," anthers subulate, erect; ovary rudimentary.

♀. Flowers solitary; corolla rather inflated at the base; staminodes 7—8, distinct; ovary ovoid, silky, attenuated into a subulate style; stigma acute; fruit conical-oblong and ferruginous-silky when young, afterwards conical, obsoletely tetragonal, yellow, quite glabrous, 1 in. long by $\frac{1}{2}$ in. wide, 1-celled, 4-seeded; seeds oblong; pulp sweetish; calyx of young fruit $\frac{1}{5}$ in. high, 4-fid with acutely deltoid lobes, erect; young fruit 4-celled, 2 cells of which are each 1-seeded.

Local name, *Aumbæ*. West tropical Africa, Guinea, *Thonning!*, common in the vicinity of the shore; Cape Coast, *Brass!*

I have without doubt referred this plant to *Diospyros*, thus following the suggestions of Messrs Bentham and Planchon. See Niger Flora p. 442 (1849) and Annal. Sc. Nat. ser. IV. vol. 3. p. 293 (1855).

Plate V. fig. 1. A fruiting branch, from Brass' specimen in Hb. Mus. Brit. *natural size.*

40. DIOSPYROS FULIGINEA, sp. nov.

D. foliis ovali-oblongis, apice anguste et valde acuminatis, basi sæpius rotundatis, glabris vel subtus subglabris, coriaceis, costâ superne depressâ, venis inconspicuis, margine tenuiter revoluto, petiolo tereti, robusto, fusco; fructibus ternis, 8-locularibus, 8-spermis, in cymis distinctis axillaribus fuligineo-hispidis dispositis; calyce fructifero aperte campanulato, 4-fido, fuligineo-hispido, lobis deltoideis, erecto-patentibus, non plicatis.

Branches cinereous, scattered more or less with small fuliginous spots, glabrescent; young shoots fuliginous-hispidulous; leaves alternate, oval-oblong, narrowly and usually suddenly acuminate at apex, usually rounded at base, glabrous or scattered with short appressed inconspicuous hairs beneath, coriaceous, $4\frac{1}{2}$—7 in. long by $1\frac{1}{3}$—$2\frac{1}{4}$ in. wide, midrib depressed above, veins inconspicuous, margin finely revolute; petioles stout, terete, fuscous, with short dark hairs, $\frac{2}{5}$ in. long.

♀. Cymes many-flowered (?), fuliginous-hispid, $\frac{2}{3}$ in. long exclusive of the flowers, bearing in one case 3 fuliginous-hispid fruits with firm pedicels $\frac{1}{4}$—$\frac{1}{3}$ in. long; young fruit globose with conical apex, exceeding the calyx, 8-celled, 8-seeded; fruiting calyx widely campanulate, about $\frac{1}{2}$ in. in diameter, 4-fid, thickly coriaceous, not plicate; lobes deltoid, spreading.

Borneo, *O. Beccari!* n. 2486.

41. DIOSPYROS BRANDISIANA, Kurz in Journ. Asiat. Soc. Beng. vol. XL. Pt. II. p. 72. n. 93 (1871).

D. foliis ovalibus, alternis, apice acuminatis, basi rotundatis vel acutis, chartaceis, adultis glabris vel secus costas sparse appresse hirsutis, breviter petiolatis; floribus cymosis e ramis ortis vel axillaribus, 5—4-meris, calyce 5-fido, lobis lineari-lanceolatis, acutis, corollâ 5-fidâ, lobis obtusis, staminibus circiter 16, filamentis brevissimis, pubescentibus, antheris glabris; in floribus femineis staminodiis 5, ovario dense fulvo-pubescente, 10-loculari, fructu immaturo ovoideo, acuminato.

Flora, 1871, p. 342.

A tree with young parts shortly pubescent. Leaves alternate, oblong to elliptic-oblong and oblong-lanceolate, acuminate, rounded or acute at base, entire, chartaceous, 4—6—8 in. long, the adult ones glabrous or usually sparsely and appressedly hirsute on the midrib; petioles $\frac{1}{12}$—$\frac{1}{6}$ in. long, puberulous, somewhat depressed above. Flowers $\frac{1}{3}$—$\frac{5}{12}$ in. long in the bud, pentamerous or tetramerous, in rather dense much-branched minutely-bracteated black-brown cymes springing from the branches or axillary; pedicels $\frac{1}{12}$—$\frac{1}{6}$ in. long, afterwards elongated, tomentose; bracts minute, oblong-lanceolate, tomentose; calyx covered with slight black or dark brown tomentum, $\frac{1}{12}$—$\frac{1}{8}$ in. long, deeply lobed, lobes linear-lanceolate, acute; corolla-tube appressedly pubescent, $\frac{5}{24}$ in. long, rather widened towards the base and commonly 5-sided, lobes equalling the tube, oblong, obtuse.

♂. Stamens 14—16; filaments very short, pubescent; anthers linear, mucronulate, glabrous. Receptacle hairy.

♀. Staminodes 5; ovary densely fulvo-pubescent, 10-celled, terminated by the rather long simple crass style. Very young fruit, ovoid-conical, acuminate, shortly pubescent.

Burma, Martaban, Dombamee forests, *Dr Brandis!*

42. DIOSPYROS SUBACUTA, sp. nov.

D. fruticosa, foliis ovato-oblongis, distichis, apice acuminatis, basi rotundatis vel subcordatis, sub-glabris, margine subciliatis, nitidis, subsessilibus, nervis inconspicuis; fructibus solitariis, axillaribus, oblongis, apice conicis, appresse pubescentibus, pedunculatis; calyce fructifero 4-fido, pateriformi, pubescente.

Shrub; young parts rufous-pilose-hispid with scattered hairs; branches dark, terete, glabrescent. Leaves ovate-oblong, distichous, subcoriaceous, glabrous or nearly so and shining, acuminate at apex, rounded or subcordate at base, subsessile, rich brown beneath in the dry state, darker above with elevated midrib; veins inconspicuous; 1$\frac{1}{2}$—3 in. long by $\frac{5}{8}$—1 in. wide, spreading; petioles $\frac{1}{16}$—$\frac{1}{12}$ in. long, thick, dark, subpilose; margins of leaves subciliate with pilose long hairs.

♀. Fruit solitary, axillary, $\frac{3}{4}$ in. long by $\frac{1}{4}$ in. thick, oblong, conical at apex, covered with short appressed brown pubescence, with several (?) cells; flowering peduncles $\frac{1}{4}$ in. (or more?) erect-patent, rough; bracts caducous; fruiting calyx 4-fid, pubescent, $\frac{1}{3}$ in. across by $\frac{1}{6}$ in. high, shallowly cup-shaped.

Madagascar, Ste Marie, *Boivin!*

43. DIOSPYROS PRURIENS, Dalz. in Kew Journ. Bot. vol. IV. p. 110. n. 2 (1852).

D. foliis alternis, ovali-oblongis, apice breviter acuminatis, basi rotundatis vel subcordatis, tenuiter coriaceis, supra nervo excepto subglabrescentibus, subtus præsertim secus nervos piloso-hirsutis, breviter petiolatis; floribus masculis axillaribus, pedunculis confertis, 1—2-floris, calyce 4-partito, utrinque piloso, corollâ profunde 4-fidâ, extus sericeâ, lobis obtusis, staminibus 13—14, glabris, hypogynis; floribus femineis 4—5-meris, staminodiis 4—5, glabris, ovario ferrugineo-hispido, 4-loculari, loculis 1-ovulatis; fructibus piloso-prurientibus, ovoideo-conicis, 4-locularibus, loculis 1-spermis.

Bedd. Ic. Pl. Ind. Or. (Pt. VII.) p. 26. t. 129 (1871); (?) Thw. Enum. Ceyl. Pl. p. 423, (1864).

Young shoots, peduncles, petioles and underside of leaves, especially on the veins, softly pilose-hirsute, fulvous; branches terete, dark, glabrescent. Leaves oval-oblong; shortly and usually obtusely acuminate at apex, rounded or subcordate at base, thinly subcoriaceous, alternate, 2—4 in. long by $\frac{4}{5}$—1$\frac{2}{3}$ in. wide, with petiole $\frac{1}{10}$—$\frac{1}{5}$ in. long, subglabrescent above, except the depressed midrib; lateral veins not strong.

♂. Peduncles near together in the upper axils, $\frac{1}{5}$—$\frac{3}{10}$ in. long, 1- or 2-flowered; flowers $\frac{1}{2}$ in. long or more; bracts rounded, caducous, glabrous inside; calyx $\frac{1}{4}$ in. long, 4-partite, fulvo-pilose on both sides, lobes linear-oblong, lax; corolla appressedly sericeous outside, glabrous inside, $\frac{1}{2}$—$\frac{3}{4}$ in. long, deeply 4-fid, constricted at top of tube, lobes ovate-oblong, obtuse, imbricated sinistrorsely. Stamens 13—14, glabrous, unequal, hypogynous, connate at base, shorter than the corolla-tube, surrounding the hairy rudiment of the ovary; filaments about as long as the anthers.

♀. Flowers solitary, crowded in the upper axils on peduncles $\frac{1}{10}$—$\frac{1}{5}$ in. long; calyx $\frac{1}{4}$ in. long with oblong spreading lobes, hairy on both sides, 4—5-partite; corolla $\frac{2}{5}$ in. long, 4-fid, constricted about middle; staminodes 5 (in one case), inserted at base of corolla, glabrous, linear; ovary ferruginous-hispid, 4-celled, cells 1-ovuled; styles 2, short, almost concealed by the long hairs on the ovary, glabrous, bifid at apex. Fruit ovoid-conical, $\frac{2}{3}$—1 in. long, 4-celled, 4-seeded, densely clothed with fulvous stinging hairs. Fruiting calyx spreading or reflexed, not accrescent.

Bombay, Chorla Ghaut, *Dr Ritchie!* 1833; *Dalzell!;* Bababoodun hills, Mysore, *Mr Law!;* (?) Ceylon, Saffragam district, 2000 ft. alt., *Dr Thwaites,* C.P. 2836.

44. DIOSPYROS APICULATA, sp. nov.

D. foliis alternis, oblongis, apice acute acuminatis, basi cordatis, tenuiter coriaceis, supra glabrescentibus, subtus præsertim secus nervos hispidis, breviter petiolatis; floribus masculis sub-3-nis, subsessilibus, axillaribus; calyce 4—5-partito, piloso-pubescente, corollâ tubulosâ, glabrâ, 4-lobâ, staminibus 6—7 vel 12, inæqualibus, glabris, ovarii rudimento hirsuto; floribus femineis 1—3-nis, brevissime cymosis; fructibus solitariis, subsessilibus, ferrugineo-setosis, ovoideo-conicis, apice apiculatis, 4-locularibus; albumine seminum non ruminato.

A tree with slender stem, about 4 feet high in the specimen seen; young parts ferruginous-hispid. Leaves oblong, alternate, thinly subcoriaceous, much acuminate at apex, cordate at base, hispid beneath, especially on the clearly marked veins, glabrescent above, with depressed midrib, of the same colour on both sides except the hairs, 4—$7\frac{1}{2}$ in. long by $1\frac{1}{2}$—$2\frac{2}{3}$ in. wide, margins just reflexed; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. long, hispid. Bracts finely hispid.

♂. Flowers about 3 together, subsessile, axillary, $\frac{5}{12}$ in. long; calyx 4—5-partite, about $\frac{1}{4}$ in. long, pilose-pubescent on both sides except near the base inside, lobes lanceolate-linear; corolla glabrous, $\frac{2}{5}$ in. long, tubular, 4-lobed, lobes spreading, oval, obtusely pointed at apex, contorted sinistrorsely in bud, $\frac{1}{16}$ in. long; stamens glabrous, 6—7 or 12, unequal, anthers linear-oblong, pointed at apex; filaments often geniculate, dilated and connate at base, inserted in a very short tube at the very base of the corolla; ovary rudimentary, small, hairy.

♀. Flowers 1—3 together, on very short axillary finely hispid cymes. Fruit solitary, subsessile, finely ferruginous-setose especially upwards but not densely so and subglabrescent in lower part, ovoid-conical, about 1 in. long by $\frac{1}{2}$—$\frac{2}{3}$ in. wide, apiculate at apex, ovoid at base, with indications inside of 4 cells, terminated by 2 (?) adjacent styles; seeds 4 (?), $\frac{5}{8}$ in. long; albumen somewhat farinaceous (in dry state), not ruminated.

Penang, Goot hill, *Dr Maingay!* no. 1514.

45. Diospyros Barteri, sp. nov.

D. fruticosa, foliis alternis, ovali-ovatis, apice apiculatis acuminatis, basi cordatis, firmiter membranaceis, supra nervo excepto glabris, subtus pallidis hispido-sericeis præsertim secus nervos, breviter petiolatis; floribus femineis solitariis, subsessilibus, hispidis, calyce 4—5-*partito, lobis lineari-lanceolatis, corollâ extus hispidâ,* 5-*fidâ, lobis acutis, staminodiis* 11, *brevibus, uniserialibus, pilosis, ovario glabro (apice excepto),* 4-*loculari, loculis* 1-*ovulatis; fructibus conicis, acuminatis, glabris sed apice hirsutis, seminibus oblongis, albumine non ruminato.*

A shrub with young shoots rufous-hispid or afterwards fuscous-hispid; older branches dark, terete, glabrate, spreading at about 50°. Leaves alternate, oval-ovate, acuminate, apiculate, at base cordate, firmly membranous, dark green and glabrous except the depressed midrib and with depressed veins above; paler with hispid-pilose ferruginous hairs, especially on the veins beneath, 2—3 in. long by 1—$1\frac{1}{2}$ in. wide; petioles hispid, $\frac{1}{10}$—$\frac{1}{7}$ in. long.

♀. Flowers solitary, subsessile, axillary, with narrow rufous-hispid-pilose caducous bracts. Calyx $\frac{1}{4}$ in. long, rufous-hispid-pilose, 4—5-partite with linear-lanceolate lobes somewhat spreading in flower and sub-horizontal not accrescent in fruit; hispid inside. Corolla conical in bud, as long as the calyx, ferruginous-hispid outside, glabrous inside, 5-fid, lobes acute, imbricated. Staminodes 11, short, in one row, distinct (except 1 pair), pilose. Pistil conical; ovary glabrous except apex, 4-celled, cells 1-ovuled; styles 2, bilobed at apex, pilose below, as long as the young ovary. Fruit oblong-conical, $1\frac{1}{6}$ in. long, glabrous (except apex), shining, with shortly ferruginous-pubescent remains of styles, 2-celled; cells 1-seeded; seeds $\frac{2}{3}$ in. long; albumen not ruminated.

W. Africa, Guinea, Lagos. Niger Expedition. *Barter!* 20194.

46. Diospyros microrhombus, sp. nov.

D. foliis distichis, rhomboideo-ovalibus, ad apicem emarginatum angustatis, basi cuneatis, interdum sub-obliquis, subglabris, coriaceis, subsessilibus; floribus femineis solitariis, graciliter pedunculatis, glabris, calyce profunde 4-*lobo, lobis rotundatis, erecto-patentibus, corollâ breviter* 4-*fidâ, staminodiis* 4, *glabris, corollæ basi insertis, ovario glabro, ovoideo-conico,* 8-*loculari.*

Of a dark colour when dry; branches covered with short patent pale pubescence, terete; wood very good. Leaves subsessile, distichous, rhomboid-oval, narrowed to an emarginate apex, cuneate at base and sometimes slightly oblique, glabrous or very nearly so, coriaceous, $\frac{1}{2}$ in. long by $\frac{1}{4}$ in. wide, dark slatish green above, brownish beneath; veins indistinct.

♀. Flowers solitary, on long slender glabrous peduncles which measure $\frac{1}{3}$—$\frac{3}{4}$ in. long and bear appressed oblong glabrous bracts about middle and near base; flowers $\frac{1}{3}$ in. long, glabrous; calyx $\frac{1}{5}$ in. long, deeply 4-lobed; lobe $\frac{1}{2}$-oval, $\frac{1}{8}$ in. wide, rounded, erect-patent; corolla erect, $\frac{1}{6}$ in. high, glabrous, 4-sided and shortly 4-fid; staminodes 4, glabrous, alternate with the lobes of the corolla and inserted at its base; ovary glabrous, 8-celled, ovoid-conical, terminated at apex by a 4-lobed conical style; divisions of the style emarginate at apex.

"Ebenier de Madagascar, son bois est superbe; Iles de France et Bourbon," Hb. Mus. Paris.!

47. DIOSPYROS FOLIOLOSA, Wall. List n. 4143 (1828—32).

D. glabra, foliis alternis, oblongo-lanceolatis, apice attenuato-acuminatis, basi obtusis, nitidis, tenuiter coriaceis, reticulatis, petiolatis; floribus masculis laxe cymosis, ovoideis, tetrameris, calyce parvo, corollâ ovoideo-urceolatâ, breviter lobatâ, staminibus 12—16, *geminatis, connectivo et filamentis leviter pubescentibus, ovarii rudimento acuminato; floribus femineis solitariis, axillaribus, pedunculatis,* 4- *rarius* 3-*meris, staminodiis nullis, stigmatibus* 4—3, *sessilibus, ovario* 4-*loculari, loculis* 1-*ovulatis; fructibus globosis, junioribus pubescenti-squamosis, senioribus glabratis; calyce fructifero fructum æquantibus vel excedentibus, lobis cordato-ovatis, foliaceis, nervosis.*

Alph. DC. Prodr. VIII. p. 234. n. 58 (1844).

Diospyros calycina, Bedd., Ann. Rep. Forests, Madras Pres. for 1867—68, p. 26 (1868), Flora Sylvatica, Madras, t. 68 (1870), Ic. Pl. Ind. Or. (Part vii.) p. 25. t. 123 (1871), non Audib.

D. auriculata, Wight! (MS. in Hb. Kew), Hb. Wight!, Kew List n. 1716, non Stiehler.

A good sized tree, glabrous in all parts except the stamens ovary and young fruit. Leaves alternate, oblong-lanceolate, thinly coriaceous, attenuate-acuminate at apex, narrowed or rounded at base, shining, green on both sides, 2—$4\frac{1}{2}$ in. long by $\frac{1}{2}$—$1\frac{1}{4}$ in. wide, midrib depressed on upper side; net-veins delicate in relief on both sides; petioles $\frac{1}{8}$—$\frac{1}{4}$ in. long.

♂. Cymes axillary, lax, about half the length of the leaves, 3—9-flowered; flowers $\frac{1}{4}$ in. long, ovoid; calyx small, about $\frac{1}{12}$ in. high by $\frac{1}{6}$ in. across, 4-fid, with deltoid or ovate lobes; corolla urceolate, often gibbous at base, 4-fid, bright yellow in colour, much contracted at the top of the tube, lobes short, pointed, spreading; stamens 12—16, inserted on the receptacle and united in pairs by their short compressed more or less hairy filaments; anthers equal, lanceolate, dehiscing from the base, converging at the apex above the rudimentary 5-lobed ovary which terminates with a long acumen; connective somewhat hairy.

♀. Flowers solitary, axillary, on peduncles $\frac{2}{3}$—1 in. long; calyx with 4 or rarely 3 cordate imbricated veined accrescent partitions; corolla urceolate, gibbous; tube nearly globose, lobes 4 or rarely 3, short, reflexed; staminodes 0; stigmas 4 or 3, sessile; ovary 4-celled; cells 1-ovuled. Fruit globose, covered when young with hairlike scales, glabrescent, $\frac{2}{3}$ in. in diameter; fruiting calyx about as long as the fruit or longer, sometimes 1 in. long, somewhat glandular at base within around base of fruit; lobes cordate-ovate, foliaceous.

Very abundant in the ghat forests from bottom to 3000 ft. alt. in the Tinnevelly district and southern portions of Madura; it is called *Vellay Toveray*, and yields a valuable light-coloured wood, *Beddome;* Courtallum, *Wallich!*

48. DIOSPYROS PILOSULA, Wall. List. n. 4132 (1828—32).

D. foliis alternis, obovato-oblongis vel anguste ellipticis, apice acuminatis, basi obtusis, tenuiter coriaceis, supra glabris, nitidis, subtus secus nervos pubescentibus, petiolatis; floribus masculis pedunculatis, staminibus 12, *glabris, inæqualibus; floribus femineis solitariis, pedun-*

culatis, calyce 4-partito, lobis lanceolatis acutis, staminodiis 0, *ovario rufo-hispido, 4-loculari, loculis* 1-*ovulatis.*

Gunisanthus pilosulus, Alph. DC. Prodr. VIII. p. 220 (1844).

A tree or shrub; branches terete, fulvo-pubescent when young, afterwards glabrescent and cinereous. Leaves narrowly elliptical or obovate-oblong, acuminate at apex, somewhat narrowed at base, alternate, thinly coriaceous, glabrous and shining above with depressed midrib, appressedly pubescent beneath and ciliate when young, glabrescent except the veins beneath, 3—4¼ in. long by 1—1½ in. wide; petioles about ⅙ in. long, pubescent when young; lateral veins not conspicuous.

♂. Flowers on the young shoots, tetramerous, pilose, about ¾ in. long, on slender peduncles about ⅓ in. long; calyx ¼ in. long, lobes deep lanceolate acute lax; corolla rather slender, tube tapering upwards ¼—⅓ in. long, lobes lanceolate acute rather longer than the tube, at length spreading; stamens 12, glabrous, very unequal, $\frac{1}{10}$—⅕ in. high, inserted on the receptacle, filaments often geniculate, anthers about $\frac{1}{20}$ in. long.

♀. Flowers solitary, on the young shoots, rather slender; peduncles ⅓—½ in. long, pubescent, articulated at the apex to the flower, without bracts, tapering downwards in fruit; calyx of the young fruit 4-partite, pubescent outside, glabrous inside, lobes lanceolate, ¼—⅓ in. long, spreading, acute; corolla deeply 4-fid, silky outside, tube cylindrical, narrowed upwards, shorter than the calyx, lobes acute; young fruit rufous-hispid, 4-celled; staminodes 0; style very short, covered by the hairs of the ovary; stigmas 2, glabrous; ovary 4-celled, cells 1-ovuled.

Among the mountains of Silhet, *Wallich!*; Pegu, *Dr Brandis!*, local name *Gjut.*

49. DIOSPYROS SUBERIFOLIA, Decaisne MSS. in Hb. Mus. Paris.

D. foliis alternis, ovalibus vel obovato-oblongis, apice rotundatis emarginatis vel apiculatis, basi obtusis, subtus subtomentosis, margine minute repando-crenulatis, subsessilibus; floribus masculis pubescentibus, pedunculis axillaribus, 1—2-*nis,* 1-*floris, basi e bractearum nidulo exorientibus, calyce* 5-*partito, corollâ urceolatâ, breviter* 5—6-*dentatâ, staminibus circiter* 20, *antheris hispidulis, filamentis glabris, ovarii rudimento hirsuto.*

Stems dark-cinereous, rough, glabrescent, softly subtomentose when young. Leaves oval or obovate-oblong, alternate, coriaceous, subsessile, softly sub-tomentose at least beneath, slightly convex from above, rounded emarginate or apiculate at apex, rounded or somewhat narrowed at base, margins minutely repand-crenulate, 1—3 in. long by ½—1½ in. wide, net-veins not very conspicuous; petioles very short.

♂. Flowers pedunculate, axillary; peduncles solitary or 2 together, arising from a nest of bracts at base, pubescent, ¼ in. long or more; calyx 5-partite, pubescent outside, glabrous inside, $\frac{1}{15}$ in. long, lobes ovate; corolla urceolate, shortly or irregularly 5—6-lobed, puberulous outside, glabrous inside, $\frac{3}{16}$ in. long; stamens 21 (one of which is very thin) in one case, inserted on the receptacle or some at the very base of the corolla, some in pairs; anthers hispidulous upwards, lanceolate-linear, apiculate; filaments very short, slender, glabrous; ovary rudimentary, hairy.

Cultivated in hort. Paris.!; supposed to have been brought from Chili.

50. DIOSPYROS SQUARROSA, Klotzsch in Peters Mossamb. I. p. 184 (1862).

D. foliis alternis, ovalibus, utrinque rotundatis vel obtusis, tenuiter coriaceis, breviter pubescentibus præsertim secus nervos vel subglabrescentibus, petiolatis; floribus femineis axillaribus, solitariis, pedunculatis, tetrameris; calyce profunde 4-fido, corollâ 4-partitâ, partitionibus obtusis, patentibus, subglabris, staminodiis 0, ovario subgloboso, glabro, 8-loculari; stylis 4, bifidis; fructibus subglobosis nitidis, calycis fructiferi lobis dependentibus, seminibus compressis.

A tree, or much branched shrub, with young shoots delicately hispid, virgate; branches glabrescent, terete, spreading at about 80°. Leaves elliptical or somewhat obovate, alternate, thinly coriaceous, rounded at both ends or sometimes narrowed; with scattered patent pubescence or subglabrescent, subnitescent above; paler, with patent pubescence, rufous and denser on midrib and lateral veins beneath; patent, delicately reticulated, $1\frac{1}{2}$—$3\frac{1}{2}$ in. long by $\frac{5}{6}$—2 in. wide; petiole $\frac{3}{20}$—$\frac{1}{4}$ in long, pubescent.

♀. Flowers axillary, solitary, drooping, tetramerous; peduncles recurved, $\frac{1}{4}$—$\frac{1}{3}$ in. long, patently pubescent; bracts caducous, at about middle of peduncle, lanceolate, $\frac{3}{10}$ in. long; calyx covered with short appressed tawny hairs on both sides, loosely hemispherical, $\frac{1}{4}$ in. long, with 4 deep oval or ovate lobes; corolla 4-partite, openly cup-shaped or rotate nearly glabrous, but with scattered pale appressed hairs along middle of lobes; lobes reflexed, about $\frac{1}{3}$ in. long, obtuse; stamens 0; ovary glabrous, somewhat 4-sided, $\frac{1}{12}$ in. high, 8-celled, cells 1-ovuled; styles 4, glabrous, bifid to about middle, not persistent on fruit; fruit glabrous, somewhat 4-sidedly globular, about $\frac{2}{5}$ in. high; fruiting calyx with pendent lobes, not accrescent.

Africa, R. Zambezi at Senna (left bank), and Rivoque near Tette, January, in ♀ flower and fruit, local name "Mutshenje tuna tuna," Sechuana dialect, *Dr Kirk!*; Sena, *Dr Peters!*, in hedges near water-courses.

51. DIOSPYROS PANICULATA, Dalz. in Kew Journ. Bot. IV. p. 109. n. 1 (1852), Bedd. Ic. Pl. Ind. Or. (Pt. VII.) p. 25. t. 125 (1871).

D. foliis oblongis, alternis, utrinque obtusis, glabris, subcoriaceis vel submembranaceis, reticulatis, petiolatis; floribus masculis numerosis paniculatis pentameris fuligineo-pubescentibus, calycis lobis foliaceis reticulato-venosis, staminibus 20 geminatis glabris; floribus femineis solitariis pedunculatis pentameris; fructibus ovoideis glanduloso-hirsutis, 4-locularibus, calyce aucto plicato.

A middle-sized or large tree with glabrous somewhat angular branches. Leaves oblong, alternate, thinly subcoriaceous or submembranous, narrowed rounded or obtusely acuminate at apex, but little narrowed at base, highly reticulated, with veins, except the midrib, in relief on both sides, 4—9 in. long by $1\frac{1}{2}$—$3\frac{1}{4}$ in. wide; petioles $\frac{1}{6}$—$\frac{1}{2}$ in long; net-veins pellucid when young.

♂. Cymes paniculate, many-flowered, in axils of fallen leaves, pubescent with fuliginous hairs, 1—$1\frac{1}{2}$ in. long; flowers $\frac{2}{5}$—$\frac{1}{2}$ in long. Calyx 5-partite shortly nigro-puberulous on both sides, $\frac{1}{5}$ in. long, lobes foliaceous, widely oval, obtuse, net-veined, with a callous internal keel and margins widely reflexed. Corolla pentagonal, fuliginous-hairy outside, glabrous inside, 5-fid, constricted in the middle; lobes oval, spreading in flower or reflexed. Stamens 20, glabrous, in pairs, the inner ones rather shorter, inserted on the disk or on the corolla; filaments short; ovary 0.

♀. Fruit solitary, axillary, on strong peduncles $\frac{1}{2}$—$\frac{2}{3}$ in. long, erect-patent; bracts caducous, large, ovate, about middle of peduncle; calyx glabrescent; fruit ovoid, $\frac{3}{4}$—$1\frac{1}{4}$ in. long, rounded at apex, tipped with remains of style, with mixed fuliginous and ferruginous hairs and glands, 3—4-celled; fruiting calyx 5-lobed, accrescent, 5-partite, $\frac{1}{2}$—$\frac{2}{3}$ in. high, more or less plicate, umbilicate below, lobes much widened auricled and imbricated at base, forming 5 dependent processes.

Tallewarru, Canara Ghauts, *Dr Ritchie!* 1884, a large tree in fruit, May; Syhadree mountains, near Chorla Ghât, Bombay, *Dalzell!* 2—3000 ft. alt.; Anamallays, *Major Beddome!* 285 (young fruit 4-celled, cells 1-ovuled or -seeded, style $\frac{1}{8}$ in. long, glabrous above, lobed at apex).

52. DIOSPYROS GRACILIPES, sp. nov.

D. foliis alternis, ovalibus vel ovatis, apice sæpius acuminatis, obtusis, basi angustatis, glabris, coriaceis, reticulatis, breviter petiolatis; floribus femineis lateralibus, secus ramos vetustiores vel ramulos dispositis, tetrameris, pedunculis gracilibus, aggregatis, 1-floris, calyce 4-fido, pubescente, ovario breviter pubescente, 8-loculari; fructibus oblongis obtusis, calyce fructifero aucto patente coriaceo.

From a shrub 10 feet high to a large tree, glabrous except the extremities and inflorescence; branches at 25°—35°. Leaves alternate, oval ovate or nearly oblong, obtuse, usually acuminate at the apex, more or less narrowed at base, glabrous, coriaceous, of the same (metallic) colour on both sides, reticulated, shining, 2—5 in. long $\frac{3}{4}$—3 in. wide; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long.

♀. Peduncles slender, on young branches or clustered on the old wood, 1—$1\frac{3}{4}$ in. long, puberulous or glabrescent, 1-flowered, with small deciduous bracts below the middle; calyx $\frac{1}{3}$ in. long, coriaceous, covered on both sides with close pale tawny pubescence, deeply 4-fid, with ovate-deltoid lobes dilated towards the base undulating at the margin and shortly acuminate; ovary shortly hairy, ovoid-tetragonal, 8-celled, cells 1-ovuled; styles 4, short; fruiting calyx spreading, with very short pubescence on both sides, whitened within, $1\frac{1}{4}$—$1\frac{1}{2}$ in. across, 4-fid; fruit oblong, rounded at apex, $\frac{7}{10}$ in. long, $\frac{1}{3}$ in. thick, nearly glabrous, whitened in parts, 8-celled.

Madagascar, *Bojer!*; Forest Lomoumé, Nossi Be, *Pervillé!* 275; East side, *Chapelier!* 82; native name *Ozou-matana.*

53. DIOSPYROS GRACILIFLORA, sp. nov.

D. foliis ovalibus, alternis, apice anguste acuminatis, subcaudatis, basi cuneatis, firmiter submembranaceis, costâ utrinque puberulâ, ceterum glabris, breviter petiolatis; floribus masculis solitariis, gracillimis, gracillime pedunculatis, tetrameris; staminibus 8, glabris, ovarii rudimento glabro.

Branches slender, terete, puberulous, leafy; leaves oval or somewhat obovate, alternate, narrowly acuminate or subcaudate at apex, cuneate at base, shining, firmly submembranous, glabrous except the midrib which is puberulous on both sides and depressed above, $1\frac{1}{2}$—$4\frac{1}{2}$ in. long by $\frac{2}{5}$—$1\frac{1}{3}$ in. wide; veins inconspicuous above, lateral veins few; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. long, puberulous.

♂. Flowers solitary, very slender, $\frac{2}{3}$ in. long, on very slender remotely setulose peduncles $\frac{1}{3}$—$\frac{2}{3}$ in. long, which arise from small bracts on the young branches; calyx $\frac{1}{12}$ in. long, campanulate, 4-fid, puberulous outside, glabrous inside, ciliate, lobes rounded; corolla narrowly tubular (in bud), $\frac{3}{5}$ in. long by $\frac{1}{16}$ in. thick, deeply 4-fid, glabrous, somewhat constricted below lobes; lobes obtuse, much contorted; stamens 8, biseriate, glabrous, unequal, anthers oblong, filaments more or less connate at base into hypogynous ring; ovary rudimentary, glabrous.

Borneo, *O. Beccari!* n. 1560.

54. DIOSPYROS PERVILLEI, sp. nov.

D. foliis anguste ovalibus, alternis, apice acuminatis, basi cuneatis, glabris, coriaceis, unicoloribus, petiolatis, nervis gracillimis; fructibus 1—3-nis, rigide cymosis, subglobosis, subglabris, nitidis, plurilocularibus, calyce aucto, reflexo, coriaceo, 4-partito, nervoso.

A tree 40 feet high, very nearly glabrous in all its parts; branches at about 50°. Leaves alternate, narrowly elliptical, acuminate at apex, narrowed at base, glabrous, coriaceous, of same (metallic) colour on both sides, shining, about 6 in. by 2—$2\frac{3}{4}$ in. wide; petioles about $\frac{1}{2}$ in. long, strong; veins numerous, slender.

♀. Cymes about 3-flowered, rigid in fruit, common peduncle $\frac{1}{2}$—$\frac{3}{4}$ in. long, fruiting pedicels $\frac{1}{3}$—$\frac{1}{2}$ in. long; fruiting calyx 4-partite, coriaceous, veined, lobes reflexed, oblong, rounded at apex, $\frac{3}{4}$—1 in. long by $\frac{1}{2}$—$\frac{3}{5}$ in. wide. Fruit subglobose, 1 in. long by $\frac{3}{4}$ in. thick, nearly glabrate but with a few scattered short appressed weak hairs, shining, with remains of 4 styles at apex, 8-celled and -seeded (?); seeds $\frac{2}{3}$ in. long, albumen not ruminated (?).

Madagascar, Nossi Be, *Pervillè!* 525.

55. DIOSPYROS DICTYONEURA, sp. nov.

D. foliis ovali-oblongis, alternis, glabris, apice, acuminatis, basi parum angustatis vel subrotundatis, coriaceis, utrinque reticulatis, nitentibus, petiolatis; floribus masculis pentameris, cymosis; cymis unciallibus, multifloris, axillaribus; calyce partito, basi plicato; corollâ tubulosâ, carnosâ; staminibus 20, plerisque binis, antheris linearibus glabris, filamentis brevibus hispidis.

Shoots terete, softly puberulous. Leaves alternate, oval-oblong, acuminate at apex, slightly narrowed or slightly subrotundate at base, glabrous, coriaceous, shining, with raised well-marked net-veins on both sides, 6—7 in. long by $2\frac{1}{4}$—$2\frac{3}{4}$ in. wide; midrib depressed above; margins recurved; petioles stout, wrinkled, $\frac{2}{5}$—$\frac{1}{2}$ in. long.

♂. Cymes axillary, 1 in. long exclusive of the flowers, many-flowered, pubescent; flowers pubescent, pentamerous; calyx about $\frac{1}{4}$ in. broad and high, partite, plicate at base, lobes ovate-deltoid, sides sometimes plicate towards base, subobtuse, shortly pubescent on both sides;

corolla glabrous inside, $\frac{1}{2}$ in. long, shortly 5-fid, lobes rounded; stamens 20, mostly in pairs, subequal; anthers linear glabrous, filaments very short, hispid, more or less combined at base. Ovary rudimentary, represented by a bunch of hairs.

Borneo, *O. Beccari!* 2542, 2615.

56. DIOSPYROS ASTEROCALYX, sp. nov.

D. foliis alternis, ovalibus, apice breviter acuminatis, basi obtusis, glabris, coriaceis, subtus conspicue reticulatis, petiolatis; floribus femineis racemosis tetrameris, racemis 5—7-*floris, basi bracteatis; calyce profunde* 4-*lobo, ferrugineo-velutino, stellato, lobis margine revolutis, corollâ urceolatâ* 4-*fidâ; staminodiis* 3—4*; ovario velutino,* 8-*loculari, loculis* 1-*ovulatis.*

Buds and inflorescence ferruginous-velutinous, in other parts glabrate; leaves alternate, oval, shortly acuminate at apex, obtuse at base, coriaceous, conspicuously net-veined beneath, $2\frac{1}{2}$—$7\frac{1}{2}$ in. long by $1\frac{1}{5}$—$3\frac{1}{8}$ in. wide; margins recurved; petioles $\frac{1}{3}$—$\frac{2}{3}$ in. long.

♀. Flowers racemose; racemes 1—$2\frac{1}{4}$ in. long, pedicels patent, unequal, ranging up to $\frac{1}{2}$ in. long, the lower ones the longer. Calyx thickly coriaceous, deeply 4-lobed, stellate, $\frac{2}{3}$—$\frac{3}{4}$ in. in diameter; lobes widely ovate but much revolute. Corolla widely urceolate, under $\frac{1}{2}$ in. high, 4-fid, lobes hairy on both sides, obtuse; staminodes 3 (in one case), glabrous, inserted at base of corolla, alternate with its lobes. Ovary velutinous, ovoid, conical at apex, 8-celled, cells 1-ovuled; style very short, lobed at apex, velutinous; stigmas glabrous.

Borneo, *O. Beccari!* n. 2612.

57. DIOSPYROS HORSFIELDII, sp. nov.

D. foliis alternis, ovalibus vel oblongis, apice acuminatis, basi subrotundatis vel obtusis, glabrescentibus, tenuiter coriaceis, supra nitentibus depresso-venosis, subtus reticulatis, breviter petiolatis; cymis lateralibus vel axillaribus, fuligineo-hispidis, calyce plicato, 4-*lobo, corollâ urceolatâ* 4-*lobâ, staminibus* 14—16 (*in fl. fem.* 12, *sterilibus*), *antheris glabris, filamentis hispidis, ovario in floribus femineis dense hispido,* 8-*loculari; fructibus globosis.*

Diospyros frutescens, Hasskarl, Plant. Javan. Rar. p. 467 (1848), non Blume.

Branches numerous, terete and glabrous, spreading at about 70°, green when young, afterwards turning black. Leaves oblong or elliptical, alternate, soon quite glabrous, acuminate at apex, somewhat narrowed or nearly rounded at base, with veins plainly depressed on upper surface and in conspicuous relief beneath, shining above, thinly coriaceous, 4—$9\frac{1}{2}$ in. long by $1\frac{3}{4}$—$4\frac{1}{2}$ in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long.

♂. Cymes chiefly in the upper axils, fuliginous-hispid, bearing 3—5 flowers, drooping; peduncles $\frac{3}{10}$—$\frac{4}{5}$ in. long; pedicels $\frac{1}{10}$—$\frac{1}{5}$ in. long; bracts oval, leaf-like; flowers $\frac{1}{3}$—$\frac{1}{2}$ in. long; calyx $\frac{1}{5}$—$\frac{1}{3}$ in. long, 4-lobed, lobes ovate, plicate-connivent, thickened and fuliginous-hispid on both sides over a lanceolate area proceeding from base to above the middle and with broad membranous everted glabrous and green margins; corolla urceolate, tetragonal, fuliginous-hispid outside, straw-coloured and glabrous inside, 4-lobed, lobes ovate, rather obtuse, reflexed; stamens 14—16, inserted at the base of the corolla or on the disk, often in pairs united by their short hairy filaments; anthers glabrous; ovary rudimentary, minute.

♀. Cymes corymbose, many-flowered, 1—3 in. long, frequently on older branches, bracteate, fuliginous; flowers $\frac{2}{5}$—$\frac{1}{2}$ in. long; calyx $\frac{1}{5}$—$\frac{1}{4}$ in. long, like ♂ but occasionally 5-partite; corolla tetragonal, 4-partite; staminodes 12, in one row, attached by their hairy filaments to base of corolla, anthers glabrous; styles 4, short, spreading; ovary densely hispid, with black and rufous mixed hairs, 8-celled; cells 1-ovuled. Fruit globose, with a central pit at apex around remains of styles, about $\frac{1}{2}$—$\frac{3}{4}$ in. in diameter, black-hairy or nearly glabrescent; fruiting calyx reaching about $\frac{1}{2}$ in. up fruit, lobes auricled at base.

Malacca, *Griffith!* 3620; Java, *Dr Horsfield!* Eben. 1 (1182) drawings n. 128 (pt.) in Hb. Kew.; *Leschenault!* 1669; *Perrottet!*

58. DIOSPYROS BOIVINI, sp. nov.

D. foliis alternis, ovato-lanceolatis vel-oblongis, apice obtuse acuminatis, basi cordatis, subglabris, subcoriaceis, breviter petiolatis; floribus masculis laxe cymosis, tomentoso-pubescentibus, tetrameris, calyce campanulato, corollâ 4-fidâ, lobis late rotundatis, staminibus 12—14, *glabris, plerisque geminatis, ovarii rudimento pubescente.*

Young branches and inflorescence ferruginous-pubescent; shoots terete, shining, rather dark. Leaves alternate, ovate-lanceolate or -oblong, rather obtusely acuminate at apex, cordate at base, subcoriaceous, shining brown and nearly glabrous above with somewhat sunken veins, rather paler and nearly glabrous beneath with somewhat ruddy raised midrib and clear but not close net-veins, $2\frac{1}{2}$—$6\frac{1}{4}$ in. long by 1—$2\frac{3}{8}$ in. wide; petiole $\frac{1}{12}$—$\frac{1}{6}$ in. long, thick, pubescent.

♂. Cymes lax, many-flowered, near ends of branches, $\frac{1}{2}$—2 in. long, shortly hispid-pubescent, ferruginous; bracts lanceolate; flowers campanulate, $\frac{3}{8}$ in. long, $\frac{1}{3}$ in. wide, tetramerous, tomentose-pubescent; calyx nearly $\frac{3}{8}$ in. long, campanulate, shortly 4-lobed or occasionally deeper, lobes depresso-deltoid, somewhat wavy; pubescent on both sides; corolla just exceeding the calyx, 4-fid, ferruginous-velutinous outside, glabrous within, lobes widely rounded, contorted sinistrorsely; stamens (12 ex Baillon in note) 14! (in 2 flowers), glabrous, mostly in pairs, nearly equal, inner ones rather shorter, $\frac{1}{6}$ in. long, anthers oblong-linear, $\frac{1}{8}$ in. long, dehiscing laterally; ovary rudimentary, pubescent.

Madagascar, Voyage of *M. Boivin!* 1847—1852.

59. DIOSPYROS LOUREIRIANA, G. Don, Gen. Syst. Gard. and Bot. iv. p. 39. n. 22 (1837).

D. foliis alternis, oblongis vel obovato-oblongis, apice plus minus acuminatis, basi rotundatis vel subcordatis, glabrescentibus, ciliatis, submembranaceis, supra saturate- subtus flavescenti-viridibus, petiolatis; pedunculis axillaribus sub-3-floris, glanduloso-pubescentibus, pedicellis basi bracteis foliaceis ovatis glandulosis deciduis suffultis; calyce 4-fido in fructu aucto, corollâ urceolatâ 4-lobâ, staminibus 8 *uniserialibus pilosis in fl. fem. effatis, ovario in fl. fem.* 8-

loculari, tomentello, stylis 4; fructibus globosis uncialibus, seminibus oblongis, albumine non ruminato.

Alph. DC. Prodr. VIII. p. 239. n. 95 (1844).

Diospyros Lotus, Lour. Fl. Cochin. p. 226. n. 1 (1790), non Linn. nec Blanco.

Diospyros macrocalyx, Klotzsch in Peters Mossamb. p. 182 (1862), non Alph. DC.

A shrub 2—8 ft. high or small tree with young parts and inflorescence glandular-puberulous and with a few scattered pilose hairs. Leaves alternate, oblong or obovate-oblong, submembranous, weakly pubescent on the veins and ciliate on the margins when young, glabrescent, obtuse rounded or subcordate at base, more or less acuminate at apex, $1\frac{1}{2}$ to 4 in. long by $\frac{3}{5}$—$2\frac{3}{10}$ in. wide, besides petiole $\frac{1}{7}$—$\frac{7}{10}$ in. long; flowers subhermaphrodite or polygamous, drooping; calyx foliaceous.

♂. Cymes 3- or few-flowered, glandular-hairy; peduncles $\frac{1}{5}$—$\frac{3}{10}$ in. long, twice the length of the pedicels, bearing ovate cordate sessile bracts at apex; flowers about $\frac{1}{5}$ in. long; calyx green, about $\frac{1}{7}$ in. long with 4 deltoid lobes about $\frac{1}{15}$ in. deep, glandular-pubescent (closed in specimen), valvate in æstivation; corolla deeply 4-lobed, somewhat pubescent outside, urceolate, white; lobes contorted in æstivation; stamens 8, in one row, inserted at base of corolla, subsessile, pilose, lanceolate; ovary ovoid-conical or subglobose, puberulous, abortive or 8? -celled, surmounted by a 4-lobed style.

♀ Cymes about 3- or many-flowered, about $\frac{2}{5}$—1 in. long, glandular-hairy; peduncle about $\frac{1}{4}$ in. long; flowers like the ♂; staminodes 8, puberulous; ovary globose, shortly tomentose, 8-celled, cells 1-ovuled; styles 4, included in the corolla; fruit globose, about 1 in. in diameter, puberulous or glabrate, 4-celled, 4-seeded. Fruiting calyx accrescent, deeply 4-lobed, more or less covering the fruit, about 1 in. long; lobes ovate, subglabrate, dilated and widely subcordate at base. Fruiting peduncle strong, $\frac{1}{4}$—$\frac{1}{2}$ in. long; pedicels about $\frac{1}{3}$ in. long; seeds $\frac{1}{3}$ in. long, oblong, embryo $\frac{1}{6}$ in. long; cotyledons narrow, rather longer than the radicle; albumen cartilaginous, not ruminated.

Local name in Sena (Mozambique) *nhamodéma,* according to Dr Klotzsch. The natives use the roots to clean and dye their teeth red; fruits in January and February; grows in the neighbourhood of Sena, *Dr Peters!;* Senna, *Kirk!;* Rovuma River, Shiramba, *Kirk!;* between Lupata and Tette, *Kirk!;* Quiloa, *Kirk!;* Congo, *Burton!;* Angola, district Golungo Alto, *Welwitsch!* No. 2535, frequent in thickets throughout the whole district, especially in mountainous woods, fruit said to be edible; var. *vernalis,* leaves $\frac{3}{4}$—2 in. long by $\frac{1}{3}$—$\frac{2}{3}$ in. wide, flowers solitary on shorter peduncles, fruiting calyx smaller, less foliaceous, a shrub 2—6 ft. high, Angola, district Golungo Alto, *Welwitsch* 2535 *b.* The characters approach those of the genus *Royena.* A specimen in the herbarium of the British Museum without flowers from Sierra Leone gathered by *Afzelius!* may possibly belong to this species.

60. DIOSPYROS DENDO, Welw. MSS.

D. foliis alternis, ovali-oblongis, apice acuminatis, basi leviter angustatis, tenuiter coriaceis, glabrescentibus, nitido-virentibus, persistentibus; floribus brevissime cymosis, axillaribus, 5—6-

meris, diœcis, calyce campanulato, utrinque pubescente, ♂ 5—6-fido, ♀ profunde lobato; corollâ aperte campanulatâ, glabrâ, ♂ 5—6-fidâ, lobis reflexis, ♀ profunde 5—6-fidâ; ♂ staminibus 20 vel 24, exsertis, subæqualibus, geminatis, corollæ medio insertis, pubescentibus; ♀ staminodiis 0, ovario ovoideo, glabro, 4-locularibus, loculis 1-ovulatis; fructibus subglobosis, glabris, 2-spermis; seminibus sub-hemisphericis, albumine non ruminato; calyce fructifero aucto, patente.

Plate X. *a.* a male flowering branch, *natural* size. *b.* a male flower, *magnified* 3 diameters. *c.* a male corolla laid open, shewing the stamens, *magnified* 3 diameters. *d.* a pair of stamens, *magnified* 6 diameters. *e.* a female flowering branch, *natural* size. *f.* a female flower, *magnified* 3 diameters. *g.* the same after the removal of the corolla, *magnified* 3 diameters. *h.* a vertical section of the last, shewing ovules inside the ovary, *magnified* 4 diameters. *i.* a fruiting branch, *natural* size. *k.* a fruit, *natural* size. *l. m.* a seed, *natural* size. *n.* transverse section of a seed, natural size. *o.* embryo, *magnified* 6 diameters.

A tree 25—35 feet high, valuable as timber. Wood very black and hard in the centre. Trunk 1—2 ft. in diameter. Branches terete, smooth, of dark brown colour, glabrescent; young parts shortly and closely fulvo-pubescent. Leaves alternate, elliptic-oblong, shortly and obtusely acuminate at apex, slightly or scarcely narrowed at base, thinly coriaceous or submembranous, darker above, shining, glabrescent or midrib and sometimes principal veins puberulous on both sides, midrib depressed above; evergreen, 2—5$\frac{1}{2}$ in. long by 1—2$\frac{2}{3}$ in. wide; petioles $\frac{1}{8}$—$\frac{1}{4}$ in. long, puberulous; principal lateral veins distant, clear and slender beneath, inconspicuous above, arching; tertiary veins transverse, slender. Internodes much shorter than the leaves. Inflorescence axillary or slightly supra-axillary, shortly and closely fulvo-pubescent, in short clustered several-flowered cymes. Flowers 5—6-merous, diœcious; pedicels short.

♂. Flowers $\frac{1}{5}$ in. long; calyx $\frac{1}{10}$ in. long, campanulate, 5—6-fid, shortly pubescent on both sides, lobes ovate; corolla glabrous, 5—6-fid; tube campanulate; lobes $\frac{1}{8}$ in. long, elliptical, wholly reflexed, rounded at apex, contorted sinistrorsely in æstivation. Stamens 20, 24, appearing at the mouth of the open corolla, equal or subequal, biseriate, distinct, one pair inserted alternate and another pair opposite to each corolla-lobe; inner series inserted slightly below the outer about the middle of the corolla, that is, about the top of its tube; anthers linear, erect, hairy, sessile or subsessile; pollen globular, smooth. Ovary rudimentary, glabrous.

♀. Flowers $\frac{1}{4}$ in. long. Calyx campanulate, deeply 5—6-lobed, shortly pubescent on both sides; lobes ovate-lanceolate; accrescent in fruit. Corolla openly campanulate, glabrous or nearly so, deeply 5—6-fid; lobes oblong, erect or spreading, obtuse. Staminodes 0. Ovary glabrous, obtusely conical, 4-celled, bilobed at apex; cells 1-ovuled. Style 0; stigmas 2, compressed, with thin margins. Fruit subglobose, glabrous, about $\frac{1}{2}$ in. in diameter, 2-seeded. Seeds sub-hemispherical, $\frac{1}{4}$ in. in diameter; albumen white, not ruminated, cartilaginous; embryo axile, $\frac{1}{7}$ in. long, nearly straight; radicle $\frac{1}{12}$ in. long, bent near upper end; cotyledons ovate, equal, thin, not veined. Fruiting calyx spreading, 1—1$\frac{1}{2}$ in. across, puberulous; lobes ovate or lanceolate, subobtuse.

See Welwitsch, Synopse das Amostras de Madeiras &c. p. 10 (1862).

W. Tropical Africa, Angola, Distr. Golungo Alto, frequent in dense primitive woods, flowers from December to February, fruits in March, *Dr Welwitsch!* nos. 2537, 2538. Native name *Dendo* or *N-Dendo.*

61. Diospyros (?) Cunalon, Alph. DC. Prodr. viii. p. 237 n. 79 (1844).

D. foliis alternis, late lanceolatis, apice obtusis, glabris, brevissime petiolatis, margine revolutis; floribus breviter racemoso-cymosis, calyce campanulato, lobis 4 rarius 5 rotundatis, corollæ lobis 4 profundis acutis, staminibus 8, corollæ adnatis, 4 basi, 4 medio loborum; ovario globoso, stylis 2; baccis globosis, 4-locularibus, loculis monospermis.

(Cunalon), Blanco, Flora de Filipinas pp. 304, 305 (1837).

A tree with erect and branching trunk. Leaves alternate, broadly lanceolate, obtuse at apex, glabrous; the margins entire and reflexed; petioles very short. Flowers in small racemose panicles. Calyx free, persistent, campanulate, with 4 or rarely 5 rounded lobes. Corolla longer than the calyx, with 4 deep acute lobes. Stamens 8, inserted on the corolla, 4 at the base and the other 4 at the middle of the lobes; filaments shorter than the corolla, compressed; anthers erect, acute. Ovary globose, enclosed within the flower; styles 2, linear, compressed; stigmas simple. Fruit baccate, globose, juicy, 4-celled; cells 1-seeded; seeds oblong, convex and canaliculate outside, angular inside, very hard and horny, and "covered with a thin aril."

Cebu, Philippine Islands, *Blanco, loc. cit.*

The leaves and fruit turn very black at maturity and are used by the islanders to dye cloth. The black colour produced is good and fast and without notable smell. Flowers in October. Called *Cunalon* in Bisayas, Philippine Islands.

62. Diospyros tetrasperma, Sw. Prodr. p. 62 (1788).

D. foliis alternis, anguste obovatis, apice obtusis, basi cuneatis, glabris, subcoriaceis, breviter petiolatis; floribus masculis 3—4-nis, breviter cymosis, calyce campanulato, subglabrescente, 4- rarius 5-fido, corollâ tubulosâ, extus sericeâ, breviter 4-fidâ, staminibus 8, glabris, geminatis; floribus femineis solitariis, staminodiis 4, ovario conico, pubescente, 4-loculari, loculis 1-ovulatis, fructibus globosis, glabris, seminum albumine "radiato-striato quasi fibroso, carnoso, albo."

Fl. Ind. Occ. p. 678 (1800), Gaertn. f. Carp. iii. p. 138. t. 208 (1805), Alph. DC. Prodr. viii. p. 222. n. 1 (1844).

D. obovata, Jacq. Hort. Schœnbr. iii. p. 34. t. 312 (1798), non Wight.

A shrub glabrous except the inflorescence and young parts; stem 1 in. thick; branches pale, at about 40°; shoots slender, subvelutinous. Leaves alternate, oblanceolate-oblong or obovate, subcoriaceous, the younger ones sometimes pellucid-punctate, cuneate at base into short petiole, rounded or obtuse at apex, deep green above, paler beneath; veins raised on both sides; $1\frac{1}{2}$—3 in. long by $\frac{1}{2}$—1 in. wide; petioles $\frac{1}{10}$—$\frac{1}{7}$ in. long.

♂ flowers in 3—4-flowered cymes; cymes recurved, $\frac{1}{4}$ in. long, with short appressed hairs. Flowers about $\frac{1}{4}$ in. long. Pedicels very short. Bracts small, caducous. Calyx about

$\frac{1}{8}$ in. long, green, nearly glabrescent, campanulate, 4—5- usually 4-fid; lobes deltoid or rounded. Corolla tubular, pale with appressed short hair outside, with 4 spreading obtuse lobes half the length of the tube. Stamens 8, distinct, 2 alternating with each corolla-lobe, the inner ones being shorter and inserted at very base of corolla-tube, or hypogynous, the outer ones longer with filament and anther about equal and inserted rather above base of corolla tube, or hypogynous; all glabrous. Ovary rudimentary, with short hairs.

♀ flowers solitary, on erect peduncles about $\frac{1}{10}$ in. long; calyx and corolla as in ♂; staminodes 4, alternating with corolla-lobes and inserted at base of its tube; ovary conical, hairy, 4-celled, 4-ovuled, continuous with hairy style which is 4-lobed and glabrous at apex. Fruit globose, about $\frac{1}{2}$ in. thick, pale, glabrous, 4-celled, 4-seeded. Fruiting calyx 4—5-fid, not or scarcely accrescent, concave or somewhat spreading, glabrous. Fruiting peduncle $\frac{1}{10}$—$\frac{1}{5}$ in. long, patent; seeds $\frac{1}{3}$ in. long; testa rather rough; albumen not ruminated, but somewhat striated in a radiated manner.

Jamaica, *Mr March!* No. 1190; *Purdie!* (♂ and ♀ fl. and fr., October); *Swartz*, ♂ fl. July; St Domingo, *Jacquin*, ♂ fl. May; Cuba, teste *Grisebach* (the specimen Pl. Cub. Wright, n. 348, has a somewhat different foliage and fruit-calyx).

63. DIOSPYROS CARTHEI, sp. nov.

D. foliis alternis, elliptico-oblongis, utrinque obtusis, glabris, coriaceis, petiolatis; floribus masculis sub-5-nis, subsessilibus, confertis, axillaribus, tubulosis, ferrugineo-pubescentibus, 4—6-fidis, calyce campanulato, corollâ gracili; lobis obtusis, staminibus 8, *inæqualibus, ovarii rudimento piloso.*

Glabrous and dark except inflorescence and buds; branches terete. Leaves elliptic-oblong, alternate, coriaceous, not pellucid-punctate, of same colour on both sides, 4—5 in. long by $1\frac{5}{8}$—$1\frac{7}{8}$ in. wide; petioles $\frac{3}{8}$ in. long, spreading.

♂. Flowers about 5 together, subsessile, crowded, axillary, tubular, slender, $\frac{1}{4}$—$\frac{3}{8}$ in. long, ferruginous-pubescent, the colour greenish beneath the hairs; calyx $\frac{1}{5}$ in. long, campanulate, 4—6- (5—6!) -fid; lobes lanceolate. Corolla 4-fid, slender, $\frac{1}{5}$—$\frac{3}{8}$ in. long, ferruginous-hairy outside, constricted in midrib; lobes imbricated, obtuse. Stamens 8, unequal by shorter or longer filaments, glabrous, $\frac{1}{10}$—$\frac{1}{5}$ in. long, anthers dehiscing longitudinally along their sides; pollen ellipsoidal. Ovary rudimentary, represented by hairs.

Manila, Philippine Islands, *Carthe!*

64. DIOSPYROS POLYALTHIOIDES, Korthals MSS. in Hb. Ludg. Batav. Eben. nn. 5—9, 12—14.

D. foliis alternis, oblongis, apice acutè acuminatis, basi obtusis, tenuiter coriaceis, suprà glabris, subtùs subglabris; floribus masculis, aggregatis, breviter cymosis, axillaribus, oblongis, sericeis, calyce campanulato, 4—5-*fido, corollâ tubulosâ, breviter* 4-*fidâ, lobis obtusis patentibus, staminibus* 8, *glabris, receptaculo insertis, inæqualibus; floribus femineis axillaribus, breviter cymosis; fructibus subsolitariis, breviter pedunculatis, globosis, pubescentibus,* 8-*locularibus; calyce fructifero aucto, profunde* 4-*lobo, ampliato, lobis undulatis, latis, erectis.*

Diœcious. Shoots ferruginous-pubescent, terete. Leaves oblong, alternate, obtusely narrowed or nearly rounded at base, acutely acuminate at apex, thinly coriaceous, glabrous and rather shining above except the depressed midrib, nearly glabrous beneath except the midrib and weak slender lateral veins, 6—8 in. long (besides hairy petiole $\frac{5}{16}$—$\frac{3}{8}$ in. long) by $1\frac{1}{2}$—$2\frac{1}{2}$ in. wide; margins just recurved; lower surface somewhat red; not pellucid-punctate; a few dark depressed glands usually exist on the lower surface, especially near the base and in the fruiting specimens.

♂. Cymes axillary, many-flowered, sericeous-ferruginous, $\frac{3}{16}$—$\frac{5}{16}$ in. long (excluding the flowers; pedicels about $\frac{1}{12}$ in. long; bracts small. Flowers sericeous, about $\frac{1}{2}$ in. long in bud, crowded. Calyx nearly $\frac{1}{4}$ in. long, campanulate, 4—5-fid; lobes deltoid or oval, hairy on both sides; corolla shortly 4-fid, tubular; lobes obtuse, much imbricated in bud, oval, $\frac{1}{6}$ in. long; glabrous inside, spreading; tube constricted at the top. Stamens 8, glabrous, inserted on the receptacle, unequal, combined more or less by their filaments at base; anthers linear, acute (when young), longer than their filaments. Ovary 0. Rarely a flower is trimerous.

♀. Cymes axillary, about $\frac{1}{4}$ in. long, sericeous-ferruginous, bearing 3—many flowers; bracts caducous; pedicels $\frac{1}{8}$ in. long. Calyx plicate, $\frac{2}{5}$ in. high, longer than the corolla. Flowers 4—5-merous. Fruit subsolitary, on peduncles $\frac{1}{8}$—$\frac{1}{4}$ in. long, enclosed when young by accrescent deeply 4-lobed calyx: fruit globose, ferruginous-hairy, about $\frac{1}{3}$ in. in diameter (perhaps not mature), 8-celled, (8-ovuled), 8-seeded. Pericarp rather thick; dissepiments thin. Fruiting calyx $\frac{3}{4}$ in. high, deeply 4-lobed, hairy on both sides; ample at the sinuses; lobes widely ovate with margins wavy, wide at base.

Borneo, *Korthals!*

Plate VII. A branch in male flower, *natural size.* *a.* Calyx laid open and stamens, the corolla having been removed, *magnified* 3 diameters. *b.* A branch in young fruit, *natural size.*

65. DIOSPYROS KIRKII, sp. nov.

D. foliis ovalibus, alternis, utrinque rotundatis, coriaceis, velutinis, petiolatis; floribus masculis axillaribus, breviter cymosis, 4- rarius 5-meris, calyce campanulato, sæpius 4-fido, corollâ tubulosâ, breviter 4-lobâ, staminibus 9—10, glabris, inæqualibus; floribus femineis solitariis, breviter pedunculatis, staminodiis 8, ovario globoso, 4-loculari, fulvo-tomentoso, loculis 1-ovulatis; fructibus edulibus.

A fruit-tree with young shoots ferruginous-tomentose-puberulous; branches cinereous glabrescent, terete. Leaves elliptical or oval-oblong, alternate, coriaceous, rounded at both ends; velutinous-puberulous, sub-nitescent above with delicate slightly raised veins; velutinous-pubescent fulvous beneath with raised rufous midrib and lateral veins; $1\frac{1}{2}$—4 in. long by $\frac{3}{4}$—$2\frac{1}{4}$ in. wide; petioles hairy $\frac{1}{5}$—$\frac{1}{3}$ in. long.

♂. Inflorescence axillary, in several-flowered cymes, rufous-tomentose, raised on peduncles about $\frac{1}{5}$ in. long, with short pedicels, bracteate; flowers $\frac{3}{10}$ in. long, tetramerous or rarely pentamerous; calyx $\frac{3}{50}$ in. high ferruginous-velutinous outside, appressedly hairy inside, 4-fid, campanulate, rarely with 5 unequal lobes; corolla inflated-tubular, with 4 short ovate patent lobes, glabrous inside; stamens 9, 10, glabrous, inserted at base of corolla or on receptacle, unequal, on short filaments; ovary 0.

♀. Flowers solitary, on short peduncles, $\frac{9}{20}$ in. high; fulvo-velutinous; calyx 4—5-lobed, $\frac{2}{5}$ in. long; with lanceolate erect lobes $\frac{3}{10}$—$\frac{7}{20}$ in. deep, hairy on both sides; corolla truncately conical, with 5 (or 4?) very short spreading obtuse lobes, glabrous inside; staminodes 8, inserted at base of corolla and 1 on receptacle (in flower examined), glabrous; ovary fulvous-velutinous, globular, 4-celled with 2 styles hairy at base; cells 1-ovuled; stigmas glabrous, lobed; young fruit fulvo-velutinous, with calyx-lobes appressed or erect; pulp of fruit good when made into a cake.

Africa, Zambesia, above Tette, common. *Dr Kirk!*

66. DIOSPYROS VELUTINA, sp. nov.

D. foliis alternis, ovalibus vel oblongis, coriaceis, subtus fulvo-velutinis interdum pubescentibus et pellucido-punctatis, petiolatis; floribus masculis ternis, breviter cymosis, 3—4-meris, ferrugineo-hirsutis; calyce campanulato, 3—4-fido, lobis obtusis, corollâ tubulosâ, 3—4-lobâ, staminibus 12, *glabris, inæqualibus; floribus femineis solitariis breviter pedunculatis, calyce 3—5-lobo, corollâ 4-lobâ, ovario dense fulvo-sericeo, subgloboso, 8-loculari; stylis* 4; *fructibus globosis, albumine non ruminato.*

A diœcious shrub about 6 feet high or small tree; shoots, leaves especially on the under-side, and inflorescence ferruginous-velutinous; branches glabrescent, terete, shining, spreading at about 60°. Leaves oval or oblong, somewhat narrowed (sometimes acutely), obtuse, rounded or even cordate at either or both ends, coriaceous, shining and comparatively glabrescent above with (in some specimens) more or less depressed veins, densely ferruginous-velutinous beneath, or in some specimens becoming less hairy and then with small pellucid dots, alternate, $1\frac{1}{2}$—6 in. long by $\frac{7}{8}$—$2\frac{1}{3}$ in. wide; petioles $\frac{1}{5}$—$\frac{1}{3}$ in. long, ferruginous-velutinous. Inflorescence short, axillary, ferruginous-velutinous; bracts narrow, caducous.

♂. Flowers usually 3 together, on peduncles $\frac{3}{40}$—$\frac{7}{40}$ in. long, trimerous or the central ones tetramerous, about $\frac{1}{2}$ in. long; calyx $\frac{1}{5}$ in. long, ferruginous-velutinous outside, glabrous inside, 3- or 4-fid, with obtuse lobes; corolla ferruginous-sericeous, 3- or 4- lobed, tubular, nearly $\frac{1}{2}$ in. long, lobes $\frac{1}{6}$—$\frac{1}{5}$ in. deep, oval, glabrous inside, spreading; stamens 12, glabrous, some in pairs, unequal in the pairs, the inner ones the shorter; filaments short, anthers linear-oblong; ovary ferrugineous-hairy, rudimentary.

♀. Flowers and peduncles solitary, ferruginous, hairy; peduncles $\frac{1}{5}$—$\frac{3}{10}$ in. long; flowers $\frac{2}{5}$ in. long; calyx $\frac{3}{10}$ in. long by $\frac{1}{3}$ in. wide, 3—5-lobed; lobes $\frac{1}{4}$ in. deep by $\frac{1}{3}$ in. wide, ferruginous-tomentose on both sides, rounded or deltoid, cordate at base, with undulating sides, often emarginate at apex, with central boss inside near base; corolla shortly tubular with 4 short acute spreading lobes; staminodes 2 (in one case), glabrous; ovary densely fulvo-sericeous, subglobose, 8-celled, with a short neck terminated by 4 styles; cells 1-ovuled; stigmas emarginate. Fruit globose, shining, pale, glabrate, except at the apex, about 5-celled and 5-seeded, pulpy, $\frac{1}{2}$—$\frac{3}{4}$ in. thick; seeds about $\frac{3}{8}$ in. long, enveloped in pulp considered by *Mr Miers*, in Ann. Mag. Nat. Hist. ser. ii. vol. VIII. p. 164 (1851), to be of the nature of an aril, not however in the dried state suggesting such an origin; fruiting calyx 3—4-lobed, spreading, tomentose, $\frac{1}{2}$—$\frac{3}{4}$ in. across; lobes more or less emarginate, especially in the trimerous ones; albumen horny, not ruminated, but (in some specimens) obscurely striate in a radiating manner.

Brazil, Rio de Janeiro, Jurujuba Bay, *Mr Miers!* 3709; Serra de Araripe, *Gardner!* 1512 (♂ fl. Sept.); between Franqueira and Canariera, *Gardner!* 2284 (albumen radiately striate, fruit in March); New-Granada, Prov. Mariquita, Piedros, banks of Magdalena, 1300 ft. alt., *Triana!* 2612; Mexico, Carmen and neighbourhood, *Dr Warra!* 226 (plant in young fruit with acute leaves and calyx 3-fid having pointed lobes). Possibly 2 or 3 different species are here described together. Cfr. *Maba inconstans*, Griseb. which is like this plant in some states.

67. Diospyros plectosepala, sp. nov.

D. foliis alternis, ovalibus, apice acuminatis, basi angustatis, subglabris, tenuiter coriaceis, breviter petiolatis; floribus masculis brevissime cymosis, axillaribus, pentameris, hirsutis, bracteatis, campanulato-oblongis, calyce profunde lobato, lobis rotundis valde contortis, corollæ lobis ovalibus obtusis, staminibus 12 *glabris inæqualibus, ovarii rudimento hirsuto.*

Branches terete, sparsely hispid with mixed brown and black short hairs. Leaves alternate, oval, acuminate at apex, somewhat narrowed at base, thinly coriaceous, scattered especially beneath with a few inconspicuous appressed short stiff hairs, $1\frac{1}{2}$—$4\frac{1}{2}$ in. long by $\frac{1}{2}$—$1\frac{3}{5}$ in. wide, dark green above; lateral veins few, delicate; petioles $\frac{1}{8}$—$\frac{1}{4}$ in. long, hispid.

♂. Flowers few or several together, in very abbreviated hispid axillary cymes, pentamerous, $\frac{1}{3}$ in. long, campanulate-oblong; bracts small. Calyx deeply 5-lobed, scarcely half the length of the flower, hirsute outside, glabrous inside, lobes round, much imbricated, cordate at base. Corolla densely hirsute outside with pale appressed hairs, glabrous inside, 5-fid; lobes oval, obtuse. Stamens 12, glabrous, unequal, hypogynous or inserted at very base of corolla. Ovary minute, rudimentary, hairy.

Borneo, *O. Beccari!* n. 3225.

68. Diospyros stricta, Roxb. Cat. Pl. Fl. Ind. p. 93 (1813).

D. trunco stricto, apice tantum ramoso; foliis alternis, ovato-oblongis, apice valde acuminatis, basi subrotundis, submembranaceis, ciliatis, subtus sparse pubescentibus, breviter petiolatis; cymis masculis brevissimis, 3—6-*floris, bracteatis, floribus subsessilibus,* 4-*meris, hirsutis, calyce parvo, profunde lobato, corollâ urceolato-oblongâ, staminibus* 14—16, *glabris; fructibus solitariis, breviter pedunculatis, obovoideis, basi conicis, glabris; seminibus oblongis, albumine non ruminato.*

Roxb. Hort. Beng. p. 40 (1814); Fl. Ind., edit. 1832, II. p. 539. n. 14; Drawings no. 2507 in Hb. Kew; Wall. List n. 4121 (1828—32); Alph. DC. Prodr. VIII. p. 232. n. 47 (1844).

A tall slender conical tree with a trunk perfectly straight, as in firs, to the very top; branches spreading at 40°, terete; young shoots subtomentose, covered with dull tawny patent short hairs, glabrescent. Leaves ovate-oblong, much acuminate at apex, obtuse at base, submembranous, alternate, erect-patent, pubescent beneath, ciliate, glabrous above except on the midrib, 2—$3\frac{1}{2}$ in. long by about 1 in. wide; petioles about $\frac{1}{7}$ in. long, pubescent; veins inconspicuous especially on upper face.

♂. Flowers $\frac{1}{3}$ in. long, 3—6 together, crowded and subsessile on short pubescent cymes about the length of the petioles, tetramerous. Bracts numerous, hairy, at base of very short pedicels. Calyx tawny-hirsute outside, small, $\frac{1}{10}$ in. long, with 4 deep ovate apiculate lobes, glabrous inside. Corolla salver-shaped, $\frac{3}{10}$ in. long, tawny-hirsute, much contracted towards top of tube; tube inflated below, $\frac{2}{11}$ in. long; lobes oval, patent or reflexed, shorter than the tube. Stamens 14—16, glabrous, single, about half the length of the corolla-tube, most inserted in one row at base of corolla and nearly equal, some inserted on the disk; filaments about as long as the anthers. Receptacle convex.

♀. Fruit solitary, on patent peduncles which are about $\frac{1}{4}$ in. long and thicker towards the apex and continuous with small tawny-hairy shortly 4-lobed calyx. Fruit egg-shaped but somewhat conical towards base, $1\frac{1}{4}$ in. long by $\frac{9}{10}$ in. thick, unequally 4?-celled, glabrous. Seeds oblong, albumen not ruminated.

East Bengal, Tipperah, *Roxburgh* (♂ fl. March); *Griffith!* 3624 (in fruit); Chittagong, *Drs J. D. Hooker and T. Thomson!;* Silhet, &c. *Roxburgh,* Hort. Beng. p. 40.

69. DIOSPYROS ERIANTHA, Champ. in Kew Journ. Bot. IV. p. 302 (1852).

D. foliis distichis, oblongo-lanceolatis, apice acuminatis, basi obtusis, tenuiter coriaceis, suprà nitidis, subtus secus venas pilosis, breviter petiolatis; floribus masculis 1—3-nis, axillaribus, subsessilibus, basi bracteatis, tetrameris, hirsutis, calyce profunde lobato, corollâ hypocrateriformi, lobis lanceolatis, acuminatis, patentibus, staminibus 14—16, glabris; floribus femineis solitariis, staminodiis 8, uniserialibus, glabris, ovario villoso, 4-loculari, loculis 1-ovulatis; fructibus oblongis, subglabratis, monospermis, albumine non ruminato.

Benth. Fl. Hongkongens. p. 210. n. 2 (1861).

A small tree, with young shoots; margins, mid-rib and lateral veins of underside of leaves and inflorescence covered with stiff appressed rusty pubescence; branches spreading at about 35°, glabrescent, terete. Leaves oblong-lanceolate, much acuminate at apex, obtuse or nearly rounded at base, distichous, thinly coriaceous, shining and with slight depressed inconspicuous midrib and lateral veins above; ruddier and with raised and rather conspicuous midrib and lateral veins beneath; $2\frac{1}{2}$—$4\frac{1}{2}$ in. long by $\frac{3}{5}$—$1\frac{1}{4}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{6}$ in. long, pubescent when young. Bracts much imbricated, numerous, especially in ♀, concealing the very short peduncle and young flowers, pubescent when young, wide, rounded or obtusely narrowed.

♂. Flowers subsolitary, 1—3 together, axillary, not nodding, subsessile, tetramerous, $\frac{1}{2}$ in. long. Calyx deeply 4-fid, $\frac{1}{6}$ in. long, with lanceolate hirsute lobes. Corolla tubular, salver-shaped, hirsute outside, glabrous inside, 4-lobed, white; tube $\frac{3}{10}$ in. long; lobes $\frac{3}{16}$ in. long, spreading, acuminate, lanceolate, imbricated sinistrorsely. Stamens 14—16, inserted in pairs at base of corolla, glabrous; anthers acuminate; the interior filaments shorter, the outer ones longer. Ovary rudimentary, small.

♀. Flowers solitary, subsessile, tetramerous; calyx $\frac{3}{16}$ in. long, like ♂. Corolla equalling the calyx; lobes acute. Staminodes 8, glabrous, in one row. Ovary hairy, 4-celled; cells 1-ovuled; style bifid to the middle with contiguous emarginate lobes, glabrous except at base. Fruit glabrate or nearly so, oblong, about $\frac{1}{2}$ in. long, shining, 1-seeded. Fruiting calyx $\frac{2}{5}$ in. long, with apiculate lobes, somewhat spreading. Albumen not ruminated; embryo straight.

Hong Kong, *C. Wright!* 64; in the Happy Valley woods, *Champion!* 133, 147; Borneo, *Korthals!*

D. Teysmanni, Miq. in Fl. Ind. Bat. Suppl. I. pp. 250, 583 (1860), belongs to the above species; it however differs by rather smaller leaves with nearly or quite glabrous lateral veins and with the upper surface paler than in the above species. Local name *Kajoe-ngingeh.* Near Kabagoesan on the coast in Lampong, S. Sumatra, *Teijsmann!*

70. DIOSPYROS VARIEGATA, Kurz in Journ. Asiat. Soc. Beng. vol. XL. pt. ii. p. 73. n. 95 (1871).

D. foliis oblongis, acutis vel acuminatis, tenuiter coriaceis, glabris, petiolatis; floribus masculis tetrameris, ternis vel paucis, in cymis axillaribus breviter pedicellatis, calyce puberulo, lobis late oblongis obtusis, corollæ tubo quam calyce paulum longiore, lobis ovatis acutis tubi longitudine, staminibus circiter 16 *inæqualibus, antheris glabris.*

Flora, 1871, p. 342.

A moderate-sized tree, quite glabrous except the buds. Leaves varying from elliptic-oblong to oblong, usually rather unequal and but little narrowed at base, acute or acuminate, entire, 5—10 in. long, thinly coriaceous, glabrous; petioles $\frac{1}{6}$—$\frac{1}{2}$ in. long, crass; lateral veins prominent below; net-veins rather distant and conspicuous beneath.

♂. Flowers yellow, tetramerous, in bud $\frac{1}{3}$—$\frac{5}{12}$ in. long, elongated, very shortly pedicelled, 3 or few together, in axillary shortly-stalked minutely puberulous bracteated cymes, on young usually leafless shoots, simulating racemes; bracts wide, rather acute, puberulous. Calyx puberulous; lobes widely-oblong, obtuse, about $\frac{1}{6}$ in. long. Corolla urceolate (-oblong?); tube a little longer than the calyx; lobes ovate, acute, equalling the tube. Stamens about 16, unequal, inserted at the base of the corolla; filaments short; anthers linear, cordate at the base, acuminate, glabrous.

Pegu, *Dr Brandis!*

71. DIOSPYROS DASYPHYLLA, Kurz in Journ. Asiat. Soc. Beng. vol. XL. pt. ii. p. 71. n. 92. (1871).

D. foliis oblongis vel ovali-oblongis, apice acutis vel breviter acuminatis, basi rotundatis vel subcordatis, chartaceis, secus nervos puberulis, breviter petiolatis; floribus masculis tetrameris, in cymis brevibus fulvo-pubescentibus axillaribus vel supra foliorum delapsorum cicatrices erumpentibus dispositis, calyce partito, lobis rotundatis, corollâ tubulosâ, paulum ampliatâ, staminibus circiter 16, *filamentis valde inæqualibus, ovarii rudimento fulvo-hirsuto.*

Flora, 1871, p. 333.

A tree (?) with branchlets densely tawny-pubescent. Leaves varying from oblong to oval-oblong, on petioles $\frac{1}{12}$—$\frac{1}{6}$ in. long, densely tawny-pubescent, rounded or subcordate at base, acute or shortly acuminate, 4—6 in. long by $1\frac{1}{2}$—3 in. wide, chartaceous, with long cilia when young, afterwards softly puberulous on the veins above and below.

♂. Flowers in bud nearly $\frac{4}{5}$ in. long, tetramerous, shortly pedicelled, arranged in short tawny-pubescent cymes, axillary or above the scars of fallen leaves; bracts suborbicular, puberulous, ciliated, about $\frac{1}{12}$ in. long. Calyx ferruginous-pubescent, lobed almost to the base; lobes rounded, ciliated. Corolla-tube appressedly tawny- or ferruginous-pubescent, $\frac{1}{4}$ in. long, widely tubular; corolla-lobes equalling the tube, acute, oblong, canescent-velutinous outside. Stamens about 16, inserted at the base of the corolla; filaments very unequal, some $\frac{1}{12}$—$\frac{1}{6}$ in. long, but mostly very short; anthers oblong, acute. Ovary rudimentary, with tawny hairs.

Karen hills, Taipo mountains, Burmah (between Sitang Hills and Salween River), at 4000 ft. alt., *Dr Brandis!*

72. DIOSPYROS BECCARII, sp. nov.

D. ramulis petiolis et inflorescentiâ ferrugineo-pubescentibus; foliis alternis, ovali-oblongis, apice acuminatis, basi rotundatis vel rarius parum angustatis, tenuiter coriaceis, superne glabris, subtus ferrugineo-pubescentibus; floribus femineis solitariis, subsessilibus, basi pluribracteatis, axillaribus; calyce 4-partito, lobis margine revolutis vel undulatis; corollâ 4-fidâ, lobis obtusis; staminodiis 8, *glabris; ovario glabro, 4-loculari, loculis 1-ovulatis.*

Young parts, petioles, underside of leaves and inflorescence ferruginous-pubescent; shoots longitudinally wrinkled. Leaves oval-oblong, narrowly acuminate, obtuse at apex, rounded or rarely slightly narrowed at base, thinly coriaceous, glabrous above with indistinct veins, flat, 2—6 in. long by 1—2$\frac{1}{3}$ in. wide; petioles stout, terete, $\frac{1}{6}$—$\frac{1}{4}$ in. long.

♀. Flowers solitary, axillary, subsessile, with several caducous ovate bracts at base; bracts unequal, shorter than the calyx; calyx campanulate, $\frac{1}{3}$—$\frac{2}{5}$ in. long, hairy on both sides, 4-partite; lobes ovate, with reflexed or undulated margins; corolla (immature) 4-fid, glabrous inside; lobes obtuse; staminodes 8, glabrous, equal, in one row; ovary glabrous, ovoid, 4-celled, cells 1-ovuled.

Borneo, *O. Beccari!* nn. 2492, 2591.

73. DIOSPYROS OLEIFOLIA, Wall. List n. 4128 (1828—32).

D. foliis alternis, ovalibus vel oblongis apice obtuse acuminatis, basi angustatis, subcoriaceis, glabrescentibus, utrinque lævibus nitidisque, nervis subtilissimis impressis inconspicuis, petiolatis; floribus masculis ternis, breviter cymosis, tetrameris; calyce extus glabro, intus tomentoso, lobis latis acutis, corollâ urceolato-oblongâ, lobis brevibus rotundatis, staminibus circiter 20, *ovarii rudimento pubescente; fructibus solitariis, subglobosis.*

DC. Prodr. VIII. p. 239. n. 88 (1844); Kurz in Journ. Asiat. Soc. Beng. vol. XL. Pt. II. p. 72. n. 94 (1871); Flora, 1871, p. 342.

A moderate-sized tree with dark bark, glabrous except young parts, which are ferruginous-tomentose. Leaves alternate, oblong-elliptical or oblong-lanceolate, narrowed at both ends, 2$\frac{3}{4}$—6—9 in. long by 1—2$\frac{1}{2}$—2$\frac{3}{4}$ in. wide, subcoriaceous, pale, smooth and shining on both sides, the yellowish midrib and inconspicuous veins all slightly depressed on the upper surface; petioles $\frac{1}{6}$—$\frac{1}{3}$—$\frac{3}{8}$ in. long; margins just recurved.

♂. Cymes drooping, $\frac{1}{3}$—1 in. long, axillary, slightly pubescent, usually 3-flowered; common peduncle $\frac{1}{4}$—$\frac{5}{8}$ in. long; pedicels $\frac{1}{8}$—$\frac{1}{6}$ in. long, hispidulous; flowers tetramerous, white. Calyx nearly $\frac{1}{4}$ in. long, glabrous outside, densely fulvo-tomentose inside; lobes wide, acute. Corolla more than twice the length of the calyx, fulvo-tomentose outside; tube wide and inflated, about $\frac{1}{4}$—$\frac{7}{24}$ in. long; lobes short, rounded; stamens about 20, inserted at the base of the corolla and on the receptacle; filaments very short; anthers linear, acuminate, about $\frac{1}{6}$ in. long. Ovary rudimentary, minute, fulvo-pubescent.

♀. Fruit solitary, on young branches, very shortly pedunculate, sub-globose, $\frac{2}{3}$—$\frac{3}{4}$ in. in diameter, more or less rufous-pubescent, yellowish, in one case 3-celled and 3-seeded. Fruiting calyx $\frac{1}{4}$ in. long, 4-fid (in one case 3-fid), tomentose inside, pubescent outside; lobes ovate-deltoid.

Pegu, *Dr Brandis, Kurz!* no. 3012. Java, Wynkoopers Bay, *Teijsmann* (Malay name *Kayu arang*); Amherst, *Wallich!* 4128, *Anderson!*, *H. Falconer!*, Herb. Hort. Bot. Calc. No. 242.

74. DIOSPYROS FLAVICANS.

D. foliis alternis, ovali-oblongis, apice acuminatis, basi obtusis, tenuiter coriaceis, glabris, breviter petiolatis; inflorescentiâ axillari, brevissime cymosâ, pauciflorâ, bracteis longis imbricatis, floribus 4—5-*meris, calyce partito, corollâ hypocrateriformi tetragonâ, lobis obtusis, staminibus in flore masculo geminatis,* 14—20, *corollæ basi insertis, glabris; ovario in flore femineo glabro, tetragono-pyramidali,* 4-*loculari, loculis* 1-*ovulatis; fructibus oblongis, glabris.*

Guatteria? flavicans, Wall. List, n. 7295 (1828—32).

A diœcious shrub 8—10 feet high or small tree, with virgate terete and somewhat flexuous branches, appressedly ferruginous-pubescent as well as the leaves when young, glabrescent, spreading at about 50°. Leaves alternate, oval-oblong, usually much acuminate at apex into a long obtuse point, somewhat narrowed at base, thinly coriaceous, 2—5$\frac{1}{4}$ in. long by $\frac{7}{8}$—2 in. wide, besides petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long; quickly glabrescent, somewhat shining on both sides; midrib somewhat depressed and lateral veins not conspicuous on upper surface, the latter clear and slender and anastomosing near margin beneath. Inflorescence axillary, shortly cymose, ferruginous-pubescent, with long bracts, 1—several-flowered; flowers white.

♂. Cymes very short; flowers clustered (or solitary); with short pedicels bearing long lanceolate foliaceous bracts at base sometimes $\frac{1}{3}$ in. long. Calyx $\frac{1}{4}$—$\frac{1}{3}$ in. long, pilose on both sides, 4-partite or deeply lobed rarely 5-lobed, lobes ovate acute foliaceous, with plicate-valvate sides, lax. Corolla salver-shaped, about double the length of the calyx, pubescent outside, glabrous inside; tube tetragonal, 4—5-fid or partite. Stamens 14—16—18—20, inserted at or near base of tube of corolla, in pairs, the inner shorter on bent filaments, glabrous; anthers apiculate, equalling or shorter than the filaments; ovary 0.

♀. Cymes 1—few-flowered, $\frac{1}{4}$—$\frac{1}{3}$ in. long; bracts pubescent outside, glabrous inside, varying in size, leaf-like, at base of pedicels, $\frac{1}{6}$—$\frac{1}{3}$ in. long. Calyx $\frac{3}{10}$—$\frac{2}{5}$ in. long, pubescent on both sides, 4-partite; lobes widely ovate, cordate, with undulated and recurved sides and base, plicate, foliaceous. Corolla caducous. Ovary glabrous, tetragonally pyramidal, 4-celled terminated at apex by an erect glabrous bilobed style $\frac{1}{12}$ in. long or shorter; cells 1-ovuled

Fruit glabrous, oblong, $\frac{3}{4}$—1 in. long by $\frac{3}{10}$—$\frac{2}{5}$ in. thick, obtusely tetragonal, rounded at apex and terminated by remains of style, 4-celled. Fruiting calyx loosely embracing base of fruit, $\frac{3}{10}$ in. high, deeply 4-fid; margins wavy-reflexed.

Mergui, Tenasserim, *Griffith!* (Cfr. Notulæ, vol. IV. p. 291. n. 2. 1854) n. 3639; Malacca, *Griffith!* Kew List 454, 3623; Penang, *G. Porter!* from the hills (Wall. List 7295!); (?) Tenasserim and Andamans, *Herb Helfer!* 3640; Malacca *Maingay!* 972, "♂ Feb. 19, 1868, stamens 17—18, ♀ testa subosseous."

An instance of phyllomania occurs in a specimen probably of this species collected by *Helfer!* n. 423, Tenasserim or Andamans.

75. Diospyros sapotoides, Kurz MSS.

D. foliis alternis, obovato-ovalibus, apice breviter acuminatis, basi cuneatis, mox glabrescentibus, tenuiter coriaceis, breviter petiolatis; floribus masculis aggregatis, subsessilibus, tetrameris, urceolato-oblongis, calyce profunde lobato, utrinque pubescente; corollâ 4-fidâ, lobis obtusis, staminibus circiter 16, glabris, biserialibus, inæqualibus, ovario rudimentario.

Branches terete, smooth. Leaves alternate, obovate-oval, shortly acuminate at apex, cuneate at base, quickly glabrescent, thinly coriaceous, glaucescent (bluish green in dry state) above, 3—10 in. long by $1\frac{1}{8}$—$3\frac{1}{2}$ in. wide; lateral veins 12—15 on each side the midrib, arching and anastomosing near the margin; petioles $\frac{1}{5}$—$\frac{1}{2}$ in. long.

♂. Flowers $\frac{1}{2}$ in. long, urceolate-oblong, tetramerous, clustered, several together, subsessile, in axillary nodose dense abbreviated cymes. Calyx about $\frac{1}{4}$ in. long, openly campanulate, hairy on both sides, deeply lobed; lobes cordate-ovate. Corolla 4-fid, hirsute outside at least along 4 hairy lines on tube; lobes oval, rounded. Stamens 15—16, in two rows, glabrous; inner row shorter. Ovary wanting.

Pegu; flowers in April, *S. Kurz!* n. 3013.

76. Diospyros aurea, (?) Teijsmann et Binnendijk Pl. Nov. Hort. Bogor. in Nederl. Kruidk. Arch. III. p. 405 (1855).

D. ramis fastigiatis; foliis bifariis, elliptico-oblongis, breviter acuminatis, basi acute angustatis, glabris, nitidis, tenuissime coriaceis, petiolis crassiusculis; floribus masculis aggregatis subsessilibus tetrameris, calycis lobis deltoideis acutis, corollâ tubulosâ, lobis ovali-oblongis patentibus, staminibus 16, glabris, antheris apiculatis; floribus femineis solitariis 4—5-meris, staminodiis 10—11, "stigmate profunde 3-fido"; baccâ globosâ, aurantiacâ.

Walp. Ann. v. p. 478 (1858).

A small tree; trunk 4 feet high with fastigiate terete contiguous leafy branches which form a dense head; young shoots petioles and pedicels ferruginous-puberulous as well as the midrib of the leaves beneath. Leaves alternate, distichous, glabrescent, oval-oblong, acuminate at apex, narrowed at base into petiole, very thinly coriaceous, shining, with midrib depressed and lateral veins slightly raised above, 4—$8\frac{1}{2}$ in. long by $1\frac{1}{2}$—$2\frac{1}{4}$ in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long, rather thick.

♂ Flowers in very short many-flowered dense nodular cymes with very short pedicels, in the axils of fallen leaves, $\frac{1}{2}$—$\frac{2}{3}$ in. long, slender. Calyx $\frac{1}{8}$—$\frac{1}{7}$ in. long, scattered with few inconspicuous short ferruginous hairs, 4-fid; glabrous inside; lobes narrowly deltoid, acute, spreading. Corolla tubular, 4-fid, glabrous except 4 lines of short hairs outside; tube $\frac{1}{8}$ in. thick in middle where it is slightly inflated; lobes oval-oblong, spreading. Stamens 16, glabrous, unequal, inserted on the tube of the corolla a little above its base, $\frac{1}{15}$—$\frac{1}{5}$ in. long; anthers ovate, apiculate, $\frac{1}{20}$—$\frac{1}{14}$ in. long; the longer filaments exceeding the anthers in length. Ovary rudimentary, glabrous.

♀. Flowers axillary, glabrous, subsessile, of a golden colour, solitary; calyx 4—5-lobed, with shallow rounded wide plicate lobes, glabrous. Corolla 4—5-fid, constricted at the apex, scarcely twice the length of the calyx. Ovary 10-celled, glabrous. Staminodes 10—11. Stigma deeply 3-fid (?). Fruit globose, $\frac{1}{2}$—$\frac{5}{6}$ in. in diameter, of orange colour, tipped by style, subsessile, with flat or reflexed calyx. Gum sometimes exudes from the young branches.

Java, *Dr Horsfield!* Ebenaceæ nos. 3, 6; Bantam, *Teijsmann and Binnendijk.*

77. DIOSPYROS NIGRICANS, Wall. List n. 6351 (1828—32).

D. foliis alternis ovali-oblongis, apice valde acuminatis, basi obtuse angustatis, firmiter membranaceis, glabris, nitidis, breviter petiolatis; floribus masculis 3—6-nis, axillaribus, brevissime cymosis, subsessilibus, tetrameris, corollà gracili, profunde lobatâ, staminibus 32, inæqualibus, antheris minutis, glabris; fructibus solitariis, breviter pedunculatis, glabris, 4-locularibus, sub-globosis, loculis monospermis, albumine non ruminato, calyce fructifero 4-partito patente vel reflexo.

Alph. DC. Prodr. VIII. p. 239. n. 87 (1844), non Dalz.

A tree 50 feet high, with many lax cinereous, glabrescent branches; young shoots and petioles minutely puberulous. Leaves oval-oblong, much acuminate at apex, somewhat narrowed at base, alternate, turning black when dry, firmly membranous, glabrous except on midrib which is puberulous and depressed on the upper surface; lateral veins and net-veins delicate, not conspicuous above; 3—5 in. long by 1—1$\frac{3}{4}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{7}$ in. long.

♂. Flowers in few (3—6)-flowered short axillary puberulous cymes, subsessile, $\frac{1}{4}$—$\frac{1}{3}$ in. long; bracts small, imbricated. Calyx with scattered short ferruginous hairs outside shortly 4-lobed. Corolla with few scattered short hairs outside deeply ($\frac{2}{3}$rds) lobed, slender; lobes reflexed at apex. Stamens 32 in one case, very unequal, many minute, glabrous.

♀. Fruit glabrous, ovoid or globose, pointed at apex, about $\frac{2}{3}$ in. long, 4-celled, 4-seeded, solitary. Fruiting calyx 4-partite, with scattered ferruginous hairs outside, nearly glabrous inside; with oval, flat, spreading or reflexed lobes, $\frac{1}{3}$ in. long. Seeds oblong, $\frac{2}{5}$ in. long; albumen not ruminated, embryo nearly as long as the albumen. Fruiting peduncles shortly hispid, $\frac{1}{5}$ in. long, patent, unilateral, bearing 2 small bracts.

Khasia, Churra, 2000 ft. alt.; *Drs J. D. Hooker and T. Thomson!* 842, June, in fruit; East Bengal, *Griffith!* 3628; (Silhet), *Wallich!* 6351.

78. DIOSPYROS EBENUM, Koenig in Physiogr. Sålsk. Handl. I. p. 176 (1776).

D. ligno duro in centro nigro, foliis alternis, ovalibus vel oblongis, apice obtuse acuminatis, basi obtuse angustatis, tenuiter coriaceis, reticulatis, glabris, breviter petiolatis; floribus masculis subsessilibus, breviter cymosis, sæpius 3-5-nis, tetrameris, calyce campanulato, ciliato, breviter 4-lobo, corollâ tubulosâ, medio constrictâ, glabrâ, 4-fidâ, staminibus 16—32, filamentis 8; floribus femineis solitariis, staminodiis 16 geminatis vel paucioribus, ovario 8-loculari, glabro vel appresse pubescente, calyce fructifero aucto, tubo campanulato margine intus elevato, lobis patentibus vel reflexis, fructibus subglobosis, glabris vel appresse pubescentibus, seminum albumine non ruminato.

Alph. DC. Prodr. VIII. p. 234. n. 63 (1844); Ettingsh. Blatt-skel. Dikot. p. 89. t. 37. f. 13 (1861); Linn. fil. Suppl. Pl. p. 440 (1781); Roxb. drawings; Beddome, Fl. Sylvat. Madr. t. 65 (1870); Wight. Ic. t. 188. (1840).

D. glaberrima, Rottb. in Act. Hafn. 1783. vol. II. p. 540. t. 5.

D. melanoxylon, Willd. Hb. n. 19243; Sp. pl. IV. p. 1109. n. 8 (1805); non Roxb.

D. reticulata, Wall.! List, p. 159. n. 4120 E. (1828—32), non Willd.

D. Ebenaster, Spach, Hist. Végét. IX. p. 407 (1840), t. 135 (1846), non Retz.

D. nigricans, Dalz. in Kew Journ. Bot. IV. p. 110 (1852); Bedd. Ic. Pl. Ind. Or. (VII.) p. 25, excl. t. 124 (1871); non Wall.

D. assimilis, Bedd. Report Forests of Madras for 1866—67, p. 20. t. 1 (1867).

A large tree with glabrous branches. Leaves glabrous, alternate, oblong or oval, obtusely communicate or retuse at apex, somewhat narrowed at base, thinly coriacious 2—7 in. long by $\frac{3}{4}$—$2\frac{3}{4}$ in. wide, with petioles $\frac{1}{6}$—$\frac{1}{3}$ in. long; net-veined, of same colour on both sides.

♂. Flowers 3—15 together, subsessile, on short pubescent cymes which about equal the petioles, about $\frac{5}{12}$ in. long in bud; Bracts small, caducous. Calyx funnel-shaped, about $\frac{1}{4}$ in. long, shortly 4-lobed, nearly or quite glabrous outside with ciliated margins, hairy inside; lobes rounded. Corolla tubular, constricted at middle, glabrous, 4-fid, with imbricated lobes. Stamens 16, unequal, more or less in pairs, glabrous, inserted at base of corolla, or ranging up to 32 on 8 filaments; ovary rudimentary or wanting.

♀. Flowers solitary, with 2 bracts at base, shortly stalked. Calyx much longer than in the ♂, deeply 4-fid with an elevated callous marginal ring round its mouth. Staminodes 16, in pairs, or fewer. Style 1; stigmas 4; ovary 8-celled, glabrous or appressedly pubescent. Albumen of seeds not ruminated. Fruit depresso-globose or subglobose, $\frac{1}{2}$ in. long, or globose and $\frac{1}{2}$—1 in. in diameter, glabrous or appressedly pubescent. Fruit-calyx about $\frac{3}{4}$—1 in. across, with spreading or reflexed lobes, receiving the base of the fruit by the cup-shaped tube which has an elevated circular margin felted inside.

East India, *Koenig!*; Chorla Ghaut, *Dalzell* (called *Kárá mará* in S. Canara); Assam, *Griffith!*; Ceylon, Columbo, *Ferguson!*, *Thwaites!* 1912, 1913, 2437, 2439; East Bengal, *Griffith!* 3621; Malacca, *Griffith!* 3635 "Cayoo Arang, Ebony Wood," *Maingay!* 971. "Flowers 4—5 merous; Satiny-black. Leaves shining above. Flower yellowish;" *Wight* 1714; *Wallich!* List n. 4120; Sumatra and Molucca Isl. ex Miq. Fl. Ind. Bat. II. p. 1048 (1856); New Caledonia, *Vieillard!* 898, *Thiebault!* 344.

This valuable tree is not uncommon in the mountain forests on both sides of the Presidency of Madras and in Ceylon; it yields the best kind of Ebony, generally jet-black but sometimes slightly streaked with yellow or brown; it is very heavy, close and even-grained, and stands a high polish; unseasoned it weighs 90 to 100 lbs. the cubic foot, and 81 lbs. when seasoned; it is used for inlaying and ornamental turnery and sometimes for furniture, but there is not much demand for it in Madras. The sap-wood is white, hard, close-grained, and strong, but not durable; it is however used by the natives for various purposes; it is called *Nalluti* in the Cuppapah and Kurnool hill-forests where the tree is very common and well known. Beddome *l. c.*

D. reticulata, Decaisne, Herb. Timor. in Nouv. Ann. Mus. III. p. 406 (1834), non Willd.; *D. reticulata, β. timoriana*, Alph. DC. Prodr. VIII. p. 225. n. 11 *var.* (1844); *D. timoriana*, Miq. Fl. Ind. Bat. II. p. 1045 (1856), ought probably to be referred to *D. Ebenum*, Koen., but I have not seen an authentic specimen.

D. hebecarpa, A. Cunn. ex Benth. Fl. Austr. IV. p. 286 (1869) is probably the same species; the fruit is $\frac{3}{4}$—1 in. in diameter, covered with short hairs or glabrescent. Australia, Queensland, Cape York, *W. Hill!*; Endeavour River, *A. Cunningham!*; New Caledonia, Wagap, *Vieillard!* 2869.

A specimen in Hb. Mus. Paris collected by *Pancher!* in New Caledonia may be the same species (D. Ebenum, Koen.) but the leaves are more coarsely reticulated and the fruiting peduncles are longer ($\frac{1}{4}$ in.). Cfr. *D. samoensis*, A. Gray.

79. DIOSPYROS PELLUCIDA, sp. nov.

D. foliis alternis, ovali-oblongis, apice acuminatis, basi angustatis, firmiter membranaceis, minute pellucido-punctatis, utrinque nitidis, glabris, breviter petiolatis; floribus solitariis, axillaribus, subsessilibus, polygamis, tetrameris, calyce profunde lobato, lobis acuminatis, leviter plicatis, corollæ lobis profundis acutis, staminibus in fl. masc. 8, uniserialibus, glabris, fructibus globosis subglabratis, 8-locularibus.

Branches spreading at about 45°, terete, dark, glabrous, or minutely puberulous at the extremities. Leaves oval-oblong, alternate, firmly membranous, glabrous, of nearly same dark colour and shining on both sides, minutely pellucid-punctate, acuminate at apex, somewhat narrowed at base, $4\frac{1}{2}$—$6\frac{1}{2}$ in. long by $1\frac{4}{5}$—$2\frac{1}{5}$ in. wide, including petiole $\frac{1}{6}$ in. long; midrib depressed and veins inconspicuously reticulated above, lateral veins anastomosing within the margin beneath. Flowers solitary, axillary, very nearly sessile; polygamous (a male flower and a young fruit growing on the same specimen), tetramerous. Calyx $\frac{3}{10}$ in. long, spreading, puberulous, but glabrescent outside, deeply 4-lobed, lobes $\frac{1}{4}$ in. long, ovate, cordate and dilated at base, acuminate at apex, spreading, with margins reflexed outwards, especially near base, somewhat plicate; tube thickened and hairy inside, cup-shaped, the thickened portion extending upwards a short distance up the middle of the lobes.

♂. Corolla conical in bud, $\frac{1}{3}$ in. high, glabrous above, puberulous below outside, deeply lobed; lobes acute. Stamens 8, equal, in one row, glabrous, $\frac{1}{8}$ in. long; anthers compressed, $\frac{1}{12}$ in. long. Style $\frac{3}{20}$ in. long, straight, erect, slightly puberulous below the lobed apex, receptacle (rudimentary ovary) puberulous.

♀. Young fruit $\frac{1}{3}$ in. high by $\frac{3}{10}$ in. thick, bluntly pointed at apex, pubescent;

fruit globose, subglabrate, $\frac{1}{2}$ in. in diameter, unequally 8-celled. Fruiting calyx not lengthened, spreading, about $\frac{1}{6}$ in. high, supporting base of fruit; tube with raised rim within.

Philippine Islands, *Cuming!* 1496, 1506.

80. DIOSPYROS TETRANDRA, sp. nov., non Span.

D. foliis alternis, elliptico-oblongis, acuminatis, basi angustatis, tenuiter coriaceis, glabris, graciliter reticulatis, petiolatis; floribus masculis 3-nis, brevissime cymosis, tetrameris, tubulosis, extus hispidis, calyce late campanulato, 4-fido, corollâ breviter 4-fidâ, staminibus 4, æqualibus, antheris hispidis, ovarii rudimento hirsuto; floribus femineis, 1—3-nis, subsessilibus, stylis 4; fructibus solitariis, subsessilibus, globosis, nitidis; calyce fructifero aucto, concavo-plicato.

A tree (?), shining and quite glabrous except buds, inflorescence, &c.; young branches terete, with smooth bark. Leaves alternate, elliptic-oblong, acuminate, somewhat narrowed at base, thinly coriaceous, 4—8 in. long by $1\frac{1}{2}$—3 in. wide; midrib narrowly depressed above; lateral veins clear and slender beneath, arching and anastomosing within the margin, inconspicuous and very delicate as well as the net-veins above; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long, with bladdery tumours on the under-side (especially on the younger ones of the male plants) extending from the top downwards and disappearing from the older petioles.

♂. Inflorescence axillary, very short, 3-flowered, with short rufous setæ; flowers subsessile, $\frac{3}{8}$ in. long, slender, with short rufous hairs. Calyx $\frac{1}{8}$ in. long, 4-fid; lobes acute, somewhat spreading. Corolla tubular, shortly 4-fid; lobes spreading, rounded, $\frac{1}{16}$ in. long. Stamens 4, inserted on the receptacle or at very base of corolla, equal, distinct; anthers linear, with reddish short hairs, apiculate, as long as the glabrous filaments. Ovary rudimentary, rufous-hairy.

♀. Inflorescence axillary, 1—3-flowered, shortly pubescent, without the flowers about equalling the petiole; bracts ovate, shortly pubescent; pedicels $\frac{1}{10}$ in. long; flowers nearly $\frac{1}{2}$ in. long, 4—5-, usually 4-, merous, with short appressed hairs. Calyx $\frac{1}{4}$ in. high by $\frac{1}{2}$ in. wide, rather larger in fruit, 4-lobed, lobes cordate, acuminate or emarginate, roundly plicate. Corolla elongate-urceolate, with reflexed ovate lobes. Staminodes... Ovary... Styles 4, hairy. Fruit solitary, globose, $\frac{3}{4}$ in. in diameter, shining, with short inconspicuous appressed hairs, or subglabrate; fruiting calyx $\frac{1}{2}$—$\frac{3}{4}$ in. wide, $\frac{1}{4}$—$\frac{3}{8}$ in. high; lobes forming below dependent hollows, ascending above.

Guiana, *Martin!*, *Rudge!* A. D. 1806, *Poiteau!*

Plate VI. A branch in male flower-bud, *natural size*. *a.* A piece of a male branch with more advanced flowers, *natural size*. *b.* A male flower on branch, *magnified* 3 diameters. *c.* A male calyx, *magnified* 6 diameters. *d.* The andrœcium with rudimentary ovary in centre, *magnified* 6 diameters. *e.* A female branch with empty calyx, *natural size*. *f.* A piece of a fruiting branch, the fruit fractured, *natural size*.

81. DIOSPYROS SPRUCEI, sp. nov.

D. foliis alternis oblongis, apice valde acuminatis, basi subrotundis, coriaceis, supra glabris nitidis, subtus ferrugineo-tomentosis, nervis manifestis, petiolatis; floribus masculis aggregatis, *dense cymosis, ferrugineo-tomentosis, tetrameris, calyce campanulato, lobis deltoideis, corollâ tubulosâ, lobis rotundatis patentibus, staminibus* 16, *glabris, geminatis, inæqualibus, corollæ tubo brevioribus, ovarii rudimento rufo-tomentoso.*

A slender straight tree, 60 feet high, with ferruginous-pubescent branches. Leaves oblong, nearly rounded at base, much acuminate and sub-caudate at apex, coriaceous, glabrous and with depressed veins on the upper side, ferruginous-tomentose with strong veins beneath, alternate, about 1 ft. long by 3—3¾ in. wide, edges recurved; petioles ½—¾ in. long, thick, "recurved" (Spruce).

♂. Flowers ferruginous-tomentose outside, in many-flowered ferruginous cymes; cymes about ½ in. long (excluding the flowers); pedicels about $\frac{1}{20}$ in. long, stout. Calyx ⅕ in. long, campanulate, shortly tomentose on both sides, 4-fid with deltoid lobes. Corolla about ½ in. long, tubular, with 4 patent lobes, glabrous inside, tube ⅖ in. long; lobes ⅕ in. long, rounded, pale green. Stamens 16, nearly or quite glabrous, in 8 pairs, sub-equal in those pairs which are opposite the corolla lobes and unequal in the alternate pairs; the longer ones ⅕ in. long with the anthers about equalling the filaments; inserted at base of corolla; anthers with very few hairs on the back or glabrous; filaments glabrous. Ovary rudimentary, rufous-tomentose.

South America, Columbia, San Carlos, frequent in the woods near river Guasié, ♂ fl. October. *Spruce!* 3138.

Plate VIII. A branch in male flower, *natural size*. *a.* A male flower-bud, *magnified* 2 diameters. *b.* A male flower expanded, *magnified* 2 diameters. *c.* A male corolla laid open, shewing the stamens, *magnified* 3½ diameters. *d, e.* Contiguous pairs of stamens, *magnified* 3½ diameters.

82. DIOSPYROS MARITIMA, Blume, Bijdr. Fl. Ned. Ind. p. 669 (1825).

D. foliis alternis, ovalibus vel oblongis, utrinque obtusis, coriaceis, glabris, petiolatis, floribus masculis aggregatis, 3—7-*nis, subsessilibus, elongato-campanulatis, pubescentibus, calyce campanulato, apice* 4—5- *rarius* 3-*dentato, corollâ tubulosâ,* 4-*fidâ, staminibus* 15—18, *inæqualibus, plerisque geminatis, antheris glabris, filamentis basi hirsutis brevissimis; floribus femineis solitariis vel binis, staminodiis* 4—10, *glabris, ovario* 8-*loculari, ferrugineo-pubescente, fructibus subglobosis, glabrescentibus, seminum albumine non ruminato.*

Alph. DC. Prodr. VIII. p. 234. n. 62 (1844), Decaisne in Nouv. Ann. Mus. III. p. 406 (1834).

Cargillia laxa, R. Br. Prodr. p. 526. n. 1 (1810), Alph. DC. Prodr. VIII. p. 243. n. 2 (1844), Benth. Fl. Austr. IV. p. 287 (1869).

Cargilia maritima, Hassk. Cat. Pl. Hort. Bot. Bogor. II. p. 159 (1844).

Cargillia megalocarpa, F. Muell. Fragm. V. p. 163 (1866).

Maba megalocarpa, F. Muell. *l. c.*

Diospyros tetrandra, Spanoghe! in Linnæa XV. p. 336 (1841), non mihi.

Diospyros megalocarpa, F. Muell. Austral. Veg. in Intercolonial Essays, 1866—67, p. 35 (1867).

A small tree 8—10 feet high with moderately thick trunk, dense head and drooping branches, or a handsome tree attaining 50 feet, glabrous except the buds and inflorescence;

branches and shoots terete, rather slender. Leaves oblong or oval, coriaceous or thinly so, of nearly same colour on both sides, shining above, alternate, usually rounded or obtuse near base, obtuse at apex, $2—10\frac{1}{2}$ in. long by $1\frac{1}{2}—3\frac{1}{4}$ in. wide, often with 2 glands at base near the petiole; petioles $\frac{1}{3}—\frac{1}{2}$ in. long; midrib depressed above; lateral veins rather clear beneath, raised and not conspicuous above. Bracts several, rather small, on very short stalks.

♂. Flowers 3—7 together, crowded, subsessile, $\frac{2}{5}$ in. long in bud, elongate-campanulate. Calyx campanulate, 4—5- rarely 3-toothed at the apex, silky-puberulous on both sides, $\frac{1}{8}$ in. long, coriaceous; lobes $\frac{1}{4}$ depth of calyx, depresso-deltoid. Corolla 4-fid, silky outside, 2—4 times the length of the calyx, tubular, $\frac{3}{8}$ in. long. Stamens 15—18, inserted at base of corolla, mostly in pairs, unequal; filaments very short, hirsute at base; anthers lanceolate-subulate or oblong, glabrous; pollen white, globose. Ovary rudimentary, hairy.

♀. Flowers 1—2 together, subsessile, about $\frac{1}{4}$ in. long in bud. Calyx like ♂ but thicker especially in fruit. Corolla $\frac{2}{3}$ rds 4-fid. Staminodes 4—10, glabrous. Styles 4, short. Ovary ferruginous-pubescent, 8-celled; cells 1-ovuled (4-celled, cells 2-ovuled according to R. Brown). Fruiting calyx broadly cup-shaped or flatly appressed to base of fruit, 4—5-lobed, coriaceous, about $\frac{3}{4}$ in. across, often $\frac{1}{2}$ in. high. Fruiting peduncle very short and much thickened and continuous with calyx. Fruit depresso-globular, glabrescent, $\frac{2}{3}$—1 in. high by $\frac{5}{6}$—1 in. thick, 4 (?) -celled and seeded, marked at the apex by remains of short style. Seeds nearly $\frac{1}{2}$ in. long, somewhat compressed, brown and shining; albumen white, not ruminated. Radicle longer than the ovate cotyledons.

N. Australia, Gulf of Carpentaria, opposite Groote Island, *R. Brown!*; Escape Cliffs, *Hulls*; Queensland, Cape York, *W. Hill!*; Timor, *Zippelius*, *Decaisne!*, *Gaichenol!*, *Spanoghe!*; S. Java, *Blume!*, *Zollinger!* n. 1833; Java, *Leschenault!*; Straits of Sunda, *Ld. Macartney!* Java, *Hasskarl!*; *De Vriese and Teijsmann!* 1859—60. Menado, Celebes, poisonous tree. *Teijsmann and De Vriese!*.

83. DIOSPYROS PHILIPPINENSIS, Alph. DC. Prodr. VIII. p. 231. n. 43 (1844).

D. foliis alternis, ovalibus, apice obtuse acuminatis, basi angustatis, tenuiter coriaceis, glabrescentibus, breviter petiolatis; floribus femineis 1—3-*nis, breviter cymosis, bracteatis, tetrameris, pubescentibus, calyce profunde lobato, corollâ tubulosâ, 4-fidâ, staminodiis* 6, *leviter pubescentibus, ovario ovoideo-conico, fulvo-pubescente, 4-loculari, loculis* 1-*ovulatis.*

Young shoots buds inflorescence and underside midrib and margin of young leaves covered with short tawny tomentum; branches glabrescent. Leaves oval, rather shortly and obtusely acuminate at apex, obtusely narrowed at base, thinly coriaceous, alternate, glabrescent, shining above, $2\frac{1}{2}—5\frac{1}{4}$ in. long by $1\frac{1}{3}—2\frac{1}{2}$ in. wide; petioles $\frac{1}{5}—\frac{1}{4}$ in. long; midrib depressed above; lateral veins distant, slender, inconspicuous especially above.

♀. Flowers in axillary 1—3-flowered bracteated cymes with several imbricated scales at the base, or solitary near the base of the young shoots of the year; peduncles or pedicels $\frac{1}{10}—\frac{1}{5}$ in. long; bracts rounded, tawny-pubescent; scales at the base of the young shoots several, much imbricated; flowers $\frac{3}{8}$ in. long, tawny-pubescent outside, erect. Calyx $\frac{1}{6}$ in. long, loose, glabrous and shining inside, deeply 4-fid with rounded or sometimes apiculate imbricated lobes. Corolla glabrous inside, 4-fid; tube $\frac{1}{4}$ in. long by $\frac{1}{8}$ in. thick; lobes oval, spreading and recurved, somewhat cordate at base, round at apex, imbricated. Staminodes 6 (in one flower), equal, somewhat tawny-hairy. Style very short, cut at apex. Ovary ovoid-conical, tawny hairy! 4-celled!; cells 1-ovuled; according to Alph. DC. *l. c.* the ovary is glabrous and 6- (or 6—8-) celled.

Manila, Philippine Islands, *Cuming!* 1142.

84. DIOSPYROS PILOSANTHERA, Blanco, Fl. Filipin. p. 304 (1837).

D. caule arboreo, foliis alternis, lanceolatis, coriaceis, glabris, basi 2—3-glandulosis, brevissime petiolatis; floribus [femineis?] axillaribus sessilibus, 6-nis vel ultra, calyce 4—5-lobo, lobis revolutis, corollâ calyce longiore pilosâ, 5-lobâ, staminibus 5—6, antheris medio pilosis (sterilibus?), stylis 4, baccâ 10-spermâ.

Alph. DC. Prodr. VIII. p. 237. n. 77 (1844).

A tree with hard wood. Leaves alternate, lanceolate, glabrous, coriaceous, with 2 or 3 glandular depressions at the base beneath; petioles very short.

♀ (?) Flowers axillary, sessile, 6 or more together; calyx with 4 or 5 large teeth recurved and bordered at maturity; corolla longer than the calyx, covered with hair outside, naked at the throat, 5-lobed; stamens 5—6; filaments short; anthers with a line of hairs along the middle; stigmas 4; fruit baccate, 10-seeded, edible; like a small guava; seeds horny, semicircular and thin at the two sides, and convex on the exterior.

Philippine Islands, *Blanco*.

85. DIOSPYROS LANCEÆFOLIA, Roxb. Cat. Pl. Fl. Ind. (1813).

D. foliis alternis, oblongis vel lanceolatis, apice acuminatis, basi angustatis, coriaceis, glabris; floribus masculis fasciculatis, dense cymosis, 3—5-nis, pubescentibus, tetrameris, calyce campanulato, corollâ tubulosâ, staminibus 14—16, geminatis, inæqualibus, subglabris; floribus femineis solitariis, subsessilibus, axillaribus, 4—5-meris, staminodiis 8—10, ovario pubescente, 8-loculari, fructibus subglobosis, tomentosis, seminum albumine non ruminato.

Fl. Ind., Edit. 1832, vol. II. p. 537; Roxb. drawings no. 2508; Alph. DC. Prodr. VIII. p. 232. n. 46 (1844).

D. multiflora, Wall. List, n. 4144 (1828—1832), Alph. DC. Prodr. VIII. p. 231. n. 45, non Blanco.

(?) *D. amœna*, Wall. List. n. 4139 (1828—32), Alph. DC. *l. c.* p. 231. n. 44 (1844), Ettingsh. Blatt-skel. Dikot. t. 41. f. 11 (1861).

Goolal or *Goolul* is the vernacular name in Sillet, ex Roxb. *l. c.*

A pretty large tree, furnishing hard durable timber suitable for the construction of houses; glabrous except the buds under side of young leaves inflorescence and fruit. Leaves oblong oblong-lanceolate or -ovate, more or less narrowed at base, acuminate at apex, with midrib depressed on upper side, coriaceous, alternate, rather pale on both sides, with veins not conspicuous above, 2—3—6—9$\frac{1}{2}$ in. long by $\frac{1}{2}$—$\frac{3}{4}$—2 in. wide, besides petioles $\frac{1}{3}$—$\frac{1}{2}$ in. long.

♂. Flowers fascicled on very short dense cymes, 3—5 together, densely ferruginous-pubescent, $\frac{2}{5}$—$\frac{1}{2}$ in. long, tetramerous. Calyx campanulate, $\frac{1}{10}$—$\frac{1}{5}$ in. long, hairy on both sides, 4-fid or more shortly 4-lobed, with deltoid lobes. Corolla tubular with inflated tube, glabrous inside; lobes spreading, shorter than the tube. Stamens 14, 16, united in pairs by thin filaments and inserted at base of corolla, or hypogynous; inner ones shorter, $\frac{1}{5}$—$\frac{2}{9}$ in. long, glabrous except base of anthers or apex of filaments; common filaments $\frac{1}{12}$ in. long; connective apiculate. Ovary 0; receptacle hairy.

♀. Flowers solitary, subsessile, axillary, near together, 4—5-merous, $\frac{1}{2}$ in. long, densely tawny-pubescent; bracts short, pubescent, imbricated. Calyx $\frac{1}{3}$ in. high, 4—5-lobed; lobes with sides sometimes reflexed. Corolla-lobes cordate (ex Roxb.), imbricated. Staminodes 8—10, short, inserted at base of corolla. Style very short, about 8-lobed; ovary 8-celled, hairy. Fruit ovoid or globose, usually pointed at the apex, tawny-tomentose or appressedly silky, 1 in. or more long. Fruiting calyx pubescent on both sides, 1 in. across, with crass somewhat concave tube and 4 or 5 lobes spreading or recurved and much thinner towards the margins. Albumen not ruminated.

East Bengal, *Griffith!* 3631, 3634; Sillet, *Wallich!* 4144, 4139 (?); Khasia, Churra, foot of hills; *Drs J. D. Hooker and T. Thomson!* 20 June 1850, in young fruit. In Khasia? or Cachar? it is called *Soi-lo* and is a poison for fish, *Drs J. D. Hooker and T. Thomson!*

86. DIOSPYROS GARDNERI, Thw. Enum. Ceyl. Pl. p. 181. n. 12 (1860).

D. foliis alternis, oblongis, apice acuminatis, basi leviter angustatis, tenuiter coriaceis, glabris, breviter petiolatis; floribus masculis 1—4-*nis, subsessilibus, tetrameris, pubescentibus, calyce campanulato, corollâ hypocrateriformi, staminibus* 16, *pubescentibus; floribus femineis solitariis, ovario* 8-*loculari, fructibus depresso-globosis, subglabratis.*

Beddome, Ic. Pl. Ind. Or. (Pt. VII.) p. 27. t. 132 (1871).

Patonia Walkerii, Wight, Ill. I. p. 19 (1840).

A moderate-sized tree; young shoots puberulous, quickly glabrescent. Leaves alternate, thinly coriaceous or submembranous, glabrous, shining above with inconspicuous veins and channelled midrib, oblong, acuminate at apex, somewhat narrowed at base, 3—7 in. long by 1—2$\frac{1}{2}$ in. wide; petioles $\frac{1}{4}$—$\frac{3}{8}$ in. long; lateral veins depressed on the upper surface in the thinner-leaved specimens.

♂. Flowers pubescent, 1—4 together, subsessile, on very short axillary pubescent cymes. Bracts small. Calyx $\frac{1}{6}$ in., campanulate, 4-fid, covered with short hairs on both sides; lobes deltoid. Corolla about $\frac{1}{4}$—$\frac{1}{2}$ in. long, conical in bud, salver-shaped in full flower, covered outside with appressed ferruginous silky shining hairs, glabrous inside, tube somewhat inflated below, with 4 spreading lobes about half the length of the tube. Stamens 16 (or about 12 according to Dr Thwaites), in pairs; filaments short, pilose; anthers linear, glabrous or somewhat hairy. Ovary 0 or represented by a bunch of hairs.

♀. Flowers solitary, erect, axillary, $\frac{1}{2}$ in. long; peduncles $\frac{1}{8}$ in. long. Calyx $\frac{1}{4}$ in. long, covered with short tawny pubescence, openly campanulate, 4-fid; lobes with undulated and recurved margins. Corolla-lobes lanceolate, about $\frac{1}{2}$ the length of the tube. Ovary 8-celled. Fruit depresso-globose, about 1 in. long (unripe), glabrate or with remains of ferruginous pubescence. Fruiting calyx accrescent, about $\frac{1}{2}$ in. high by $1\frac{1}{4}$ in. across at top; lobes pointed and patent at apex; tube hemispherical. The timber of this tree is valuable for building and for cabinet-work, *Dr Thwaites* loc. cit.

Ceylon, *Thwaites!* C. P. 1908, *Macrae!* 30, *Walker!*, *Gardner!* 532, up to 2000 ft. alt., called *Kadoombaireya-gass.*

87. Diospyros Heudelotii, sp. nov

D. foliis alternis, ovato-ovalibus, apice breviter acuminatis, basi obtusis, tenuiter coriaceis, subglabratis, breviter petiolatis; floribus masculis aggregatis, 4—6-nis, subsessilibus, pubescentibus, calyce breviter 4—5-fido, corollâ tubulosâ, lobis obtusis, staminibus 13—15, filamentis brevibus hirsutis.

Bushy tree 3—4 metres high; young parts puberulous; branches terete, dark, at about 35°, quickly glabrescent. Leaves ovate-oval, alternate, obtusely narrowed at base, shortly acuminate at apex, thinly coriaceous; dark green, glabrous and with depressed veins above; paler with few weak scattered appressed whitish hairs and with raised veins beneath; 2—3 in. long by 1—$1\frac{1}{4}$ in. wide; petioles $\frac{1}{8}$—$\frac{1}{3}$ in. long, wrinkled, glabrous; margins of leaves just recurved.

♂. Cymes very short, 4—6-flowered, ferruginous-hairy; bracts short, hairy. Flowers (closed in specimen) shortly and appressedly pubescent, whitish, sweet-scented, subsessile. Calyx $\frac{1}{12}$ in. high, campanulate, 4—5-fid, with ovate lobes. Corolla oblong, inflated in middle, 4—5-lobed at apex, glabrous inside, $\frac{1}{4}$ in. long; lobes emarginate. Stamens 15, or in a tetramerous flower 13, inserted at very base of corolla or on receptacle, nearly equal, $\frac{2}{11}$ in. long; filaments pubescent, very short, more or less connate at base; anthers linear, narrower towards apex, with a few hairs on back; dehiscing laterally by slits. Ovary rudimentary, hairy.

Africa, Senegambia, *Heudelot!* 638, October, January.

Plate V. fig. 2. A male flowering branch, *natural size.* *a.* Male flower-bud, *magnified* 4 diameters. *b.* Half the corolla laid open, shewing some of the stamens, *magnified* 4 diameters. *c.* A pair of stamens, *magnified* 4 diameters.

88. Diospyros undulata, Wall. List, n. 4136 (1828—32).

D. foliis lanceolato-oblongis, alternis, apice acuminatis, basi angustatis vel subrotundatis, glabris, nitidis, firmiter membranaceis, petiolatis; floribus masculis breviter cymosis, 3—9-nis, ferrugineo-pubescentibus, tetrameris, calyce 4-fido, corollâ tubulosâ, breviter 4-lobâ, lobis obtusis, staminibus 11—14, pubescentibus; floribus femineis 1—3-nis, breviter pedunculatis vel subsessilibus, fructibus subglobosis, appresse pilosis, plurilocularibus, seminibus compressis, albumine non ruminato, calyce fructifero aucto, crasso, fructus basim amplectente.

Alph. DC. Prodr. VIII. p. 233. n. 55 (1844); G. Don, Gen. Syst. Gard. and Bot. IV. p. 40. n. 38 (1837).

Var. β ♂. D. macrophylla, Wall n. 4141 (1828—32), non Blume; *foliis fructuque majoribus.*

A tree; branches glabrous or young shoots puberulous. Leaves oblong or lanceolate-oblong, more or less acuminate at apex, acute or more or less rounded but not subcordate at base, firmly membranous, glabrous, shining, alternate, 3—15 in. long by 1—5 in. wide, besides petioles ¼—½ in. long, thinly coriaceous; margins reflexed; midrib depressed above; lateral veins inconspicuous or depressed above. Inflorescence axillary, ferruginous-hairy.

♂. Flowers ⅓—½ in. long (in bud), conic-oblong, ferruginous-hairy, sessile on 3—9-flowered cymes not exceeding them in length, except in *var. β*; bracts ovate. Calyx short, 4-fid, with deltoid acute lobes, less hairy inside except near the margins. Corolla tubular, glabrous inside, shortly 4-lobed, with obtuse spreading lobes. Stamens 11—14, inserted on the receptacle or at base of corolla, some in pairs, unequal except *var. β*; anthers linear, hairy, subsessile, filaments short, hairy. Ovary rudimentary, hairy.

♀. Flowers solitary or 3 together; peduncles or cymes short, not exceeding ⅕ in. long. Fruit subglobose, about 1 in. long by nearly the same width, flat at the top and slightly umbilicate at base of style, appressedly brown-hairy, about 6-celled and 6-seeded; pericarp thick; pulp mucilaginous; seeds compressed, about ⅔ in. long; albumen not ruminated; embryo ⅓ in. long; cotyledons foliaceous, lanceolate, about as long as the radicle; fruiting calyx erect, embracing about half the fruit, very crass, hairy inside; 4-fid, with the sinuses nearly filled on the inner side; lobes deltoid, occasionally spreading at the tips.

Amherst, *Wallich!* 4136; Moulmein, *Parish!*; Malacca, *Griffith!* 3619, 3636, *Maingay!* 977. *Var. β.* Tavoy, *Wallich!* 4141; Mergui, *Griffith!*; Malacca, *Maingay!* 974.

89. DIOSPYROS MULTIFLORA, Blanco, Fl. Filipin. p. 303 (1837), non Wall.

D. foliis alternis, lanceolato-oblongis, apice obtusis, basi cuneatis, coriaceis, subtus puberulis, petiolatis; floribus masculis 8-nis, aggregatis, brevissime cymosis, pubescentibus, calyce 4—5-fido, corollâ tubulosâ, apice lobatâ, staminibus 15—18, filamentis hirsutis, antheris glabris; fructibus venenosis.

Diospyros Canomoi, Alph. DC. Prodr. VIII. p. 237. n. 78 (1844).

D. Lotus, Blanco, Fl. Filipin. edit. II. p. 210 (1845), non Linn.

A tree, glabrous except the buds, inflorescence and underside of leaves; branches terete, dark; leaves lanceolate-oblong, alternate, coriaceous, obtusely lanceolate or rounded at apex, cuneate at base and often with 2 glands on the upper side, glabrous with depressed and not conspicuous veins above, tomentose-puberulous, subglabrescent beneath, 6—8 in. long by 1½—2½ in. wide; besides petioles ⅓—½ in. long; margins revolute.

♂. Flowers ferruginous-pubescent, ⅜ in. long, axillary in clusters of about 8 each, sessile, in very short ferruginous-pubescent cymes, tetramerous or pentamerous. Calyx $\frac{1}{10}$ in. long, 4—5-fid, ferruginous-tomentose on both sides; lobes deltoid, spreading in flower. Corolla glabrous inside, lobed at apex, rather fleshy, ⅓ in. long, tubular. Stamens 18 (in one case), 15 or more, hypogynous or at base of corolla, more or less combined at base by their hairy filaments; anthers linear, apiculate, glabrous. Ovary 0.

♀. Fruit poisonous; reported to intoxicate fish; "even the crocodile it causes to rush from the water hurriedly." Flowers sweet-scented. By rubbing the bark and leaves on eruptions, it is said that the latter disappear.

Local names *Canomoi, Canomai.*

Philippine Islands, *Cuming!* 1829, *Blanco.*

90. DIOSPYROS BIFLORA, Blanco, Fl. Filipin. p. 303 (1837).

D. foliis alternis, lanceolatis, glabris, subcoriaceis, breviter petiolatis; floribus masculis axillaribus, binis, calyce campanulato, 3—4-*lobo, corollâ carnosâ campanulato-oblongâ,* 4-*lobâ, staminibus* 17—30, *corollæ basi insertis, filamentis brevissimis lanuginosis, ovarii rudimento pubescente.*

Alph. DC. Prodr. VIII. p. 237. n. 76 (1844).

A tree of 30 feet high. Leaves alternate, lanceolate, quite glabrous, entire, subcoriaceous, with only 2 glands at the base below; petioles very short and without glands.

♂. Flowers axillary, 2 together, with a strong smell. Calyx campanulate, 3—4-lobed. Corolla fleshy, double the length of the calyx, inflated in the middle and narrowed above, forming a throat, with 4 reflexed lobes. Stamens 17—30, inserted on the corolla and not reaching the throat; filaments very short, woolly; anthers very long. Ovary hairy; style very short; stigma and fruit wanting.

Philippine Islands, *Blanco,* Tágatog name *Talang;* flowers in June.

91. DIOSPYROS (?) PARVIFOLIA, sp. nov.

D. foliis alternis, obovatis, apice rotundatis, basi cuneatis, coriaceis, glabrescentibus, nitidis, parvis, breviter petiolatis, venis inconspicuis; floribus masculis solitariis, subsessilibus, axillaribus, pubescentibus, calyce campanulato, trilobo, corollâ 4-*fidâ, staminibus* 12, *glabris, corollæ basi insertis, biserialibus, antheris apice dehiscentibus, ovarii rudimento ferrugineo-hirsuto.*

Branches cinereous, at about 30', the younger ones rufous-hispid at first, subsequently whitish-hairy, ultimately glabrate. Leaves alternate, obovate or obovate-oblong, rounded at apex, cuneate at base, hairy beneath when quite young, quickly glabrescent, coriaceous, with margins just reflexed, without conspicuous veins, shining, $\frac{1}{3}$—$\frac{2}{3}$ in. long by $\frac{1}{8}$—$\frac{1}{4}$ in. wide, including petiole $\frac{1}{20}$—$\frac{1}{12}$ in. long.

♂. Bracts rufous-hairy, ovate or lanceolate; flowers solitary, subsessile, rufous-hairy, axillary; calyx $\frac{1}{10}$—$\frac{1}{7}$ in. long, campanulate, 3-lobed, rufous-hairy on both sides, lobes $\frac{1}{3}$ depth of calyx, rounded; corolla openly campanulate, covered with silky ferruginous hairs outside, glabrous within, $\frac{1}{5}$ in. long (when straightened), 4-fid with reflexed and somewhat emarginate lobes; stamens 12, glabrous, inserted at or near base of tube of corolla, in 2 rows, distinct, the inner ones at a lower level, filaments $\frac{1}{40}$—$\frac{3}{50}$ in. long; anthers $\frac{3}{50}$ in. long, dehiscing laterally by apical pores; ovary rudimentary, represented by a bunch of ferruginous hairs.

Madagascar!

92. DIOSPYROS BUXIFOLIA.

D. foliis alternis, ovato-ellipticis, utrinque angustatis, coriaceis, supra lucidis, subtus sericeo-pubescentibus, subsessilibus, confertis, nervis inconspicuis; floribus axillaribus, subsessilibus, masculis 3—4-nis, confertis, femineis solitariis, calyce 4-fido, corollâ 4-fidâ, breviter et late campanulatâ, intus glabrâ, staminibus 10—16, *geminatis, glabris, in flore femineo* 0; *antheris apice rimosis; ovario femineo 4-loculari superne pubescente inferne glabro, loculis* 1-*ovulatis; fructibus oblongis,* 1—2-*spermis, albumine non ruminato.*

Leucoxylum buxifolium, Bl. Bijdr. Fl. Ned. Ind. p. 1169 (1826); Choisy, Mém. Ternstr. p. 43. t. 2 (1855); Miq. Fl. Ind. Bat. p. 1049 (1856).

Diospyros microphylla, Bedd. Ic. Pl. Ind. Or. (VII) p. 27. t. 133 (1871).

A large tree with glabrescent terete branches and straight trunk. Young shoots and inflorescence covered with pale ferruginous pubescence. Leaves distichous, close together, easily falling (in dried state), firm, occasionally minutely pellucid-punctate, the younger ones silky beneath, without conspicuous veins, ovate-oval, narrowed at both ends, subsessile, $\frac{3}{4}$—$2\frac{1}{4}$ in. long by $\frac{9}{20}$—$\frac{13}{20}$ in. wide; midrib depressed and often puberulous above. Flowers diœcious.

♂. Flowers 3 or 4 together, subsessile, very short axillary cymes; flower $\frac{1}{8}$ in. long, tetramerous. Calyx $\frac{1}{16}$ in. high, covered with short hairs, having 4 rounded imbricated lobes $\frac{3}{40}$ in. deep. Corolla $\frac{1}{10}$ in. high, with 4 rounded apiculate reflexed lobes $\frac{1}{20}$ in. deep, hairy along middle lines outside. Stamens 10—16 (16! in all the flowers examined), glabrous, united by their filaments in pairs, the inner ones the shorter; anthers ovate or oblong, dehiscing at apex; filaments slender, equalling or exceeding the anthers, inserted at base of corolla. Ovary rudimentary, hairy.

♀. Flowers solitary, subsessile. Calyx 4-fid, with rounded lobes much imbricated in bud, pubescent outside; corolla $\frac{1}{9}$ in. long, 4-fid, hairy outside; staminodes 0. Ovary 4-celled, ellipsoidal and glabrous below, conical and pubescent above, cells 1-ovuled; style bipartite, short. Fruit cylindrical or oblong, conical at apex, dry, 1-celled, 1- rarely 2-seeded, $\frac{1}{3}$—$\frac{7}{8}$ in. long by $\frac{1}{6}$—$\frac{1}{3}$ in. wide, pointed, glabrous and shining or subglabrous or fulvous pubescent at apex resting at base on small spreading pubescent or ciliate calyx; albumen cartilaginous not ruminated; cotyledons about equalling the radicle.

Malacca, *Maingay!* 966 "ovary rudimentary 4-lobed;" Java, *Blume!*, *Zollinger!* 3247, 3438; India, S. Canara, &c., *Major Beddome!*; Borneo, *O. Beccari!* n. 1973.

Major Beddome *l.c.* states that the S. Canara plant has the habit of *Leucoxylum buxifolium*, Miq., but he does not regard his plant as the same species with it. According to Zollinger in the Obs. Bot. Nov. p. 18 (1857), the flowers in both sexes are usually pentamerous, the stamens usually 10, free, and the ovary apparently 2-celled.

93. DIOSPYROS VESCOI, sp. nov.

D. foliis alternis, obovatis, apice rotundatis, basi angustatis, coriaceis, subtus puberulis, inconspicue reticulatis, confertis, petiolatis, margine revolutis; floribus masculis axillaribus, breviter cymosis, calyce laxe hemisphærico, 4-fido, extus tomentoso, corollâ campanulatâ utrinque *tomentosâ, breviter 4- rarius 3- vel 5-fidâ, lobis obtusis, staminibus 13—16, plerisque geminatis, corollæ basi insertis, antheris glabris, apice rimosis, filamentis tomentosis, ovario rudimentario.*

Young parts ferruginous, shortly pubescent; branches pale, cinereous, terete. Leaves alternate, obovate, rounded or emarginate at apex, narrowed or nearly rounded at base, coriaceous, puberulous with curved hairs on both sides especially beneath, crowded, 1—3 in. long by $\frac{5}{8}$—$1\frac{3}{8}$ in. wide; petioles $\frac{1}{3}$—$\frac{1}{2}$ in. long, puberulous, margins revolute, reticulated with delicate inconspicuous veins in faint relief on both sides, midrib slightly depressed above.

♂. Inflorescence axillary on young shoots, $\frac{1}{2}$—$\frac{3}{4}$ in. long, ferruginous, pubescent with short hairs; peduncle $\frac{5}{16}$—$\frac{5}{8}$ in. long; pedicels $\frac{1}{10}$—$\frac{1}{5}$ in. long; flowers $\frac{1}{5}$—$\frac{1}{4}$ in., openly campanulate; calyx $\frac{1}{8}$—$\frac{1}{6}$ in. long, hemispherical, tomentose outside, 4-fid, sometimes unequally so, lobes widely ovate-deltoid; corolla campanulate, shortly 4-fid, occasionally 3- or 5-lobed, tomentose on both sides, lobes ovate-oval, obtuse; stamens 13—16, all or mostly in pairs, inserted near base of corolla, inner ones shorter, anthers glabrous, equal, lanceolate, acuminate, filaments tomentose; ovary rudimentary, receptacle tomentose.

Madagascar, Port Leven, *Vesco!*, St Marie, *Boivin!* 2539 *b*.

94. DIOSPYROS MORRISIANA, Hance ex Walp. Ann. iii. p. 14 (1852).

D. foliis ovalibus, alternis, apice acuminatis, basi angustatis, tenuiter coriaceis, glabris, petiolatis; floribus masculis 3-nis, breviter cymosis, tetrameris, calyce utrinque pubescente, 4-fido, corollâ urceolatâ, breviter 4-lobâ, lobis obtusis, staminibus 16—25, sæpius 20, plerisque geminatis, corollæ basi insertis, antheris linearibus, pubescentibus, ovarii rudimento glabro; fructibus glabris, nitidis, subglobosis, 4-locularibus, loculis monospermis, seminum albumine non ruminato, calyce fructifero patente, subglabro.

A shrub (or tree?) quite glabrous except the buds inflorescence and extremities; branches dark, terete, spreading at about 30°—35°. Leaves oval, acuminate at apex, more or less narrowed at base, glabrous, alternate, thinly and firmly coriaceous, with recurved margins, 2—$3\frac{1}{2}$ in. long by 1—$1\frac{1}{4}$ in. wide, besides petiole $\frac{1}{3}$—$\frac{1}{2}$ in. long; shining above; veins few and slight

♂. Flowers whitish, $\frac{1}{4}$—$\frac{1}{3}$ in. long, tetramerous, 3 together on short drooping ferruginous-hairy axillary cymes; peduncles and pedicels each about $\frac{1}{12}$ in. long; calyx ferruginous-hairy on both sides, 4-fid, $\frac{1}{10}$ in. long, erect-patent, with deltoid lobes; corolla about $\frac{3}{10}$ in. long, tubuloso-urceolate in flower, ovate-conical in bud, lobes $\frac{1}{10}$ in. long recurved, obtuse; stamens numerous, 16—25, usually about 20, mostly united in pairs, outer ones the longer, inserted at base of corolla, about $\frac{1}{5}$ in. long; anthers linear, apiculate, hairy, dehiscing from apex; filaments short, glabrous; ovary rudimentary, glabrous.

♀. Flowers unknown. Fruit glabrous and shining, yellow, nearly globular, $\frac{1}{2}$—$\frac{2}{3}$ in. in diameter, 4-celled; cells 1-seeded. Fruiting calyx nearly flat, nearly glabrate, $\frac{7}{20}$ in. across; seeds $\frac{9}{20}$ in. long, compressed, chestnut-coloured; albumen cartilaginous, not ruminated. The male flowers appear in May; the fruit gathered in December is edible.

Hong Kong, *Hance!* no. 460, *C. Wright!* 313.

95. DIOSPYROS SQUAMOSA, Boj. ex Alph. DC. Prodr. VIII. p. 232. n. 49 (1844).

D. foliis alternis, ovalibus, utrinque obtusis, glaberrimis, coriaceis, petiolatis; floribus masculis, 1—3- *sæpius* 3-*nis, sessilibus, bracteis amplis ovato-rotundatis imbricatis calyce vix brevioribus, calyce campanulato* 4—5-*fido, corollâ breviter* 4-*fidâ infundibuliformi, staminibus* 22, *corollæ basi insertis, filamentis pubescentibus; fructibus cubico-globosis glabris, apice excepto calyce* 4-*fido aucto ferrugineo-sericeo occultis, stylis* 4 *brevibus glabris.*

Branches glabrous. Leaves alternate, oval, rather obtuse at both ends especially at base, coriaceous, quite glabrous, flat, $3\frac{1}{2}$—5 in. long by $1\frac{1}{4}$—$1\frac{3}{4}$ in. wide; petioles $\frac{1}{4}$ in. long; venation delicate, in relief on upper surface.

♂. Flowers 1—3 together, sessile, rather more than $\frac{1}{2}$ in. long, arising from points on the branchlets rather above the (scars of the fallen) leaves; bracts 5—6, imbricated, $\frac{1}{12}$—$\frac{1}{3}$ in. long, the outer ones the shorter, roundly ovate, scarcely falling short of the calyx, ferruginous-tomentose at the margins. Calyx campanulate 4—5-fid or shortly lobed, $\frac{5}{12}$—$\frac{7}{16}$ in. long, ferruginous-pilose outside, lobes widely ovate, erect-patent. Corolla funnel-shaped, shortly 4—5-fid (?), subglabrous, exceeding the calyx, lobes obtuse. Stamens 22, inserted at the base of the corolla; filaments short, pubescent, frequently united in pairs.

♀. Fruit cubic-globose, glabrous, $\frac{3}{8}$ in. high, concealed except at apex by accrescent calyx; styles 4, short, glabrous. Fruiting calyx crass, ferruginous-sericeous, 4-fid, $\frac{3}{4}$ in. across, tube tetragonal $\frac{3}{4}$ in. high, lobes shortly ovate spreading.

Madagascar, near Foul-pointe, *Helsonberg!*; *Chapelier!* Local name, *Valanguiran.*

96. DIOSPYROS COMORENSIS, sp. nov.

D. foliis alternis, ellipticis, apice sæpius acuminatis, basi angustatis, coriaceis, glabrescentibus; floribus masculis 3—4-*nis, breviter cymosis, tetrameris, calyce laxe cyathiformi,* 4-*fido, corollâ urceolatâ glabrâ carnosâ breviter* 4-*lobâ, staminibus* 16 *geminatis glabris corollæ basi insertis, ovarii rudimento glabro.*

Young parts pilosely pubescent; branches brown, scarcely terete. Leaves alternate, elliptical, coriaceous, narrowed at base and usually acuminate at apex, bluish brown above with cleanly depressed midrib and inconspicuous lateral veins, brown beneath with inconspicuous veins, nearly or quite glabrous, 2—$2\frac{1}{2}$ in. long by $\frac{3}{4}$—$1\frac{1}{5}$ in. wide including petiole $\frac{1}{3}$ in., often conduplicate in specimen.

♂. Cymes axillary 3—4-flowered, about $\frac{1}{3}$ in. long, pilose, subferruginous, recurved, pedicels $\frac{1}{7}$—$\frac{1}{5}$ in. long; flowers $\frac{3}{10}$—$\frac{2}{5}$ in. long, tetramerous, ovoid in buds; calyx $\frac{1}{5}$—$\frac{1}{4}$ in. long, pubescent on both sides, 4-fid, lobes erect-patent, deltoid; corolla $\frac{3}{10}$—$\frac{3}{8}$ in. long, narrowly ovoid in bud, glabrous, fleshy, lobes much imbricated; stamens 16, placed in pairs in two rows at base of corolla, glabrous, $\frac{1}{7}$—$\frac{7}{32}$ in. long; anthers linear longer than the filaments, pollen somewhat 4- (?) sidedly ellipsoidal. Ovary rudimentary, glabrous. Female plant at present unknown.

Comoro Islands, Mayotte, *Boivin!*

97. DIOSPYROS MONTANA, Roxb. Coromand. p. 37. t. 48 (1795).

D. trunco ramisque interdum spinosis, foliis alternis, ovalibus vel ovatis, apice obtusis vel acutis, basi interdum cordatis, tenuiter coriaceis, pubescentibus vel glabrescentibus, *deciduis, petiolatis; floribus masculis breviter cymosis, tetrameris, glabriusculis, calyce late campanulato, profunde 4-fido, lobis ovatis, ciliatis, corollâ urceolatâ, breviter 4-lobâ, staminibus 16, geminatis, glabris vel subglabris, corollæ basi insertis; floribus femineis solitariis, breviter pedunculatis, staminodiis 4—12, glabris, ovario glabro globoso, 8-loculari, loculis 1-ovulatis, stylis 4, glabris, seminibus 2—8, albumine non ruminato; calyce fructifero paulum aucto, plus minus reflexo.*

Wall. List n. 4115, Alph. DC. Prodr. VIII. p. 230. n. 34 (1844), Wight Ic. t. 1225 (1850).

D. cordifolia, Roxb. *l. c.* p. 38. t. 50 (1795); Wall. List n. 4116; Alph. DC. *l. c.* n. 36; Wight, Illustr. Ind. Bot. Vol. II. t. 148 (1850).

D. rugosula, R. Br. Prodr. p. 526 (1810).

D. bracteata, Roxb. Cat. Pl. Fl. Ind. (1813); Fl. Ind. edit. 1832, Vol. II. p. 539 ex specimine in Hb. Mart.!; Alph. DC. *l. c.* p. 239. n. 93.

D. heterophylla, Wall. Cat. Burm. 599, List n. 4138 (1828—32), Alph. DC. *l. c.* p. 230. n. 39.

D. sylvatica, Wall. List n. 4117! (1828—32), *β velutina*, Alph. DC. *l. c.* p. 231. n. 41 *var.*, non Roxb.

D. punctata, Decaisne, in N. Ann. Mus. Hist. Nat. III. p. 407 (1834); Herb. Timor. Discr. p. 79 (1835); Alph. DC. *l. c.* p. 230. n. 37.

D. rugulosa, Alph. DC. Prodr. VIII. p. 229. n. 32 (1844).

D. Goindu, Dalz. in Kew Journ. IV. p. 111 (1852).

D. Waldemarii, Klotzsch in Waldemar Reise, p. 101. t. 55 (1862).

Yerra-gada of the Telingas (R. montana, Roxb.) ex Roxb. *l. c.*; *Kak-woolymera* of the Telingas (R. cordifolia, Roxb.) ex Roxb. *l. c.*; *Vakanoi*, Neilygerry Mts., base, Leschanault! 198 (large tree), seen in Hb. Mus. Paris; *Tumala*, the Sanscrit name, *Bun-Gaub*, in Bengal, ex Roxb. Fl. Ind. (edit. 1832) vol. II. p. 538; *Kala Goindu* in Canara, *Kala Nuddi*, *teste* Dr Ritchie; *Makar Kend*, Hindwi dialect of Behar, ex Hamilt. in Tran. Linn. Soc. XV. p. 113 (1827); *Gavindu* or *Goindu*, ex Graham, Cat. Pl. Bomb. p. 108 (1839); *Jugalagunti* (signifies scolding wife), ex Buchanan, Journey, vol. I. p. 183 (1807).

A tree often with spines scattered over the trunk and larger branches; young branches softly pubescent, of a pale colour. Leaves oval, oblong, obovate, or ovate-oblong, alternate, sometimes cordate at base, thinly coriaceous, of nearly the same yellowish-green colour (in the dry state) on both sides, softly pubescent or glabrescent beneath, softly puberulous or glabrous above, with depressed midrib and weak veins, deciduous, 1—4—5 in. long by $\frac{1}{2}$—2—$2\frac{1}{2}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{2}$ in. long, pubescent or glabrescent. Flowers white, scentless.

♂. Cymes 3-flowered or panicled, $\frac{1}{4}$—$\frac{3}{4}$ in. long, patent or recurved; bracts ovate, ciliate, $\frac{1}{20}$ in. long; at base of the pedicels; flowers $\frac{1}{6}$—$\frac{3}{10}$ in. long. Calyx $\frac{1}{3}$ or $\frac{1}{4}$ length of flower, deeply 4-fid, on both sides pubescent or nearly glabrous, with deltoid or rounded ciliated lobes. Corolla urceolate, shortly 4-lobed; lobes rounded, recurved; glabrous or nearly so. Stamens 16, united at base in 8 pairs and inserted at base of corolla, glabrous or very nearly so, with very short hairs, appearing at mouth of open corolla. Ovary rudimentary, glabrous except apex.

♀. Flowers solitary, $\frac{1}{3}$—$\frac{2}{5}$ in. long, on recurved peduncles $\frac{1}{10}$—$\frac{1}{3}$ in. long, which bear small caducous bracts. Calyx puberulous or nearly glabrous, deeply 4-fid, $\frac{1}{6}$—$\frac{1}{3}$ in. long, with imbricated often ciliated lobes. Corolla rather exceeding the calyx, glabrous, 4-fid. Staminodes 4, 8, 12, glabrous. Ovary glabrous, globular, 8-celled, cells 1-ovuled; styles 4, glabrous, bifid at apex. Fruit globose, $\frac{1}{2}$—$1\frac{1}{2}$ in. in diameter, glabrous and shining; fruiting calyx more or less reflexed, somewhat accrescent; seeds 2—6—8, albumen not ruminated (in *D. rugosula*, R. Br., there are two contiguous slight intrusions of the testa along the outer side of the seed). The wood is dark-coloured or variegated, hard and durable. Dr Dalzell states that bees are very fond of the flowers.

There are two principal forms:

α. montana proper. *D. montana*, Roxb., *D. Goindu*, Dalz., *D. heterophylla*, Wall. Leaves oval, 3—4 in. long. ♂ flowers panicled, with calyx glabrous except ciliate margin. ♀ flowers with 4 staminodes.

β. cordifolia. D. cordifolia, Roxb., *D. punctata*, Decaisne, *D. rugosula*, R. Br., *D. Waldemarii*, Kl. Leaves oblong, often cordate at base, 1—$2\frac{1}{2}$ in. long. ♂ flowers 3 together with hairy calyx. ♀ flowers with 8 (*D. Waldemarii*) or 12 staminodes.

India, Madras, *Shuter!*; Othacalmundapum, Kew list 1724!; Patna, *Dr Ritchie!* 1240; Moradabad, *Dr T. Thomson!* 985; *Rottler!* 361; Sirhind, *Dr T. Thomson!*; Bengal, *Edgeworth!* 6006; Ambala, *Edgeworth!*; Pinjor Valley, *Edgeworth!*; Ceylon, *Thwaites!* C. P. 1909; sea coast, Tinnevelly district, Kew list 1717!; Pondichéry, *Perrottet!* Sikkim, *Dr Hooker!*; Courtallum, Kew list 1713!, 1726!; Bombay, *Dalzell!*; Concan, *Dr Stocks!*; Canara, *Dr Ritchie!* 970; Belgaum, *Dr Ritchie* 972; Himalaya, *Hoffmeister*, teste Kl. *l. c.*; Ava, *Wallich!*; India, Magadi, Hejuru, S. W. Mysore. Timor, *Decaisne!* N. Australia, Victoria River, *F. Mueller!*, Carpentaria, *R. Brown!*; Australia, Port Darwin, *Schultz!* n. 607, 608. The natives are prejudiced against this tree. Buch. Ham. Journey, vol. I. p. 183, vol. II. 125.

Cfr. *Diospyros* sp. Bedd. in Clegh. For. 259 (1861), Muchi tanki; a very hard light-coloured wood, Godavari forests, Madras.

98. DIOSPYROS ZOLLINGERI, sp. nov.

D. foliis alternis, obovato-oblongis, apice acuminatis, basi plerisque rotundatis, coriaceis, glabrescentibus, petiolatis; floribus masculis axillaribus cymosis tetrameris, fulvo-pubescentibus, calyce campanulato, lobis deltoideis, corollâ breviter 4-fidâ, campanulatâ, staminibus 16, *geminatis, glabriusculis, ovarii rudimento glabro.*

Young parts and inflorescence puberulous or pubescent. Leaves alternate, obovate-oblong, acuminate at apex, usually rounded at base, glabrescent, 4—8 in. long, by 1—$2\frac{1}{2}$ in. wide; midrib and lateral veins depressed above and in clear relief beneath; petioles about $\frac{1}{4}$ in. long.

♂. Inflorescence in short cymes axillary or in the axils of fallen leaves. Abortive buds in some cases are arranged in a panicle. Flowers tetramerous, tawny-pubescent; calyx campanulate, $\frac{1}{10}$ in. high by $\frac{1}{8}$ in. wide, 4-fid, nearly glabrous inside, lobes deltoid; corolla (in bud) $\frac{1}{6}$ in. long, ovoid, shortly 4-lobed, appressedly pubescent; stamens 16, united in 8 pairs at the top of the filaments, nearly equal, inserted at the base of the corolla, not quite glabrous, but with short hairs on the back of the anthers and on the upper part of the filaments; anthers dehiscing widely on both sides downwards from apex, pollen subglobose, smooth; ovary rudimentary, glabrous.

Java, *Zollinger!* n. 2651. A specimen from Assam, collected by Col. Jenkins, has also abortive buds arranged in a considerable panicle; it does not however appear to belong to this species, having a somewhat different foliage, resembling in this respect *D. variegata*, Kurz.

99. DIOSPYROS CILIATA, Alph. DC. Prodr. VIII. p. 229. n. 31 (1844), non Rafin.

D. foliis alternis, ovato-ellipticis, basi obtusis, apice acuminatis acutisve, ciliatis, membranaceis; floribus femineis axillaribus, breviter pedicillatis, tetrameris, calyce partito, lobis ovatis obtusis, corollâ campanulatâ.

Branches glabrous. Leaves alternate, ovate-elliptical, obtuse at base, acuminate or acute at apex, ciliate, 2—3 in. long (including petiole $\frac{5}{12}$ in. long) by 1—$1\frac{1}{2}$ in. wide, membranous, with the nervation of the leaves as in *D. virginiana* except that the margin is ciliate and the acumen is usually acute.

♀. Flowers axillary, on glabrous pedicels much shorter than the petiole or flower, tetramerous or sometimes pentamerous, $\frac{1}{3}$ in. long. Calyx 4-partite, silky inside at base, with ovate obtuse reflexed lobes; corolla glabrous, campanulate, narrower above, 4-fid, with obtuse lobes. Styles 4, united to the middle, glabrous, longer than the calycine lobes. Fruit edible.

S. Mexico, *Pavon!*

100. DIOSPYROS LOTUS, Linn. Sp. Pl. p. 1057 (1753).

D. foliis alternis, ovalibus, utrinque sæpius obtusis, submembranaceis, subtus sæpe pallidioribus et pubescentibus, petiolatis; floribus masculis 2—3-nis brevissime cymosis, subsessilibus, urceolatis, 4- rarissime 5-meris, axillaribus, calyce campanulato, lobis acutis, corollâ breviter lobatâ, staminibus 16, geminatis, antheris glabriusculis, filamentis glabris; floribus femineis solitariis, staminodiis 8, ovario apice excepto glabro, 8-locularibus, fructibus subglobosis, edulibus.

Pallas, Fl. Ross. t. 58 et t. 59 fig. inferior, tom. I. pars. II. p. 20 (1788).
Poir. in Lam. Encycl. Méth. vol. V. p. 428 (1804), t. 858 fig. inf. (1823).
Nouveau Duhamel, vol. VI. p. 83. t. 26 (1801—19).
Turpin, Dict. Sc. Nat. Planch. vol. III. t. 65 (1816—29).
Alph. DC. Prodr. VIII. p. 228. n. 28 (1844).
Reichenb. Pl. Ic. Fl. Germ. et Helv. (XVII) t. 1079 (1855), non Lour. Fl. Cochinch. p. 226 (1790).
Ermellinus, Cesalp. De Plantis, lib. III. cap. XXI. p. 104 (1583).
Pseudolotus, Camer. Epit. p. 156 (1586).
Lotus africana altera, Camer. Epit. p. 157 (1586).
Lignum Vitæ, Gerarde Herball, p. 1309 (1597).
Guaiacum patavinum, Gerarde Herball, p. 1310 (1597).

D. Kaki, var. β. Thunb. Fl. Japon. p. 158 (1784), var. γ. glabra, Alph. DC. Prodr. VIII. p. 229. n. 30 (1844); non Linn.

D. microcarpa, Sieb. in Ann. Soc. Hortic. Pays Bas 1844, p. 28.

D. japonica, Sieb. et Zucc. in Abh. Bayer. Acad. IV. 3. p. 136 (1846).

D. Umlovok, Griff. Itin. Not. p. 355 n. 137. (1848).

Dactylus trapezuntinus, Forskål Fl. Ægypt.—Arab. p. XXXVI. (1775).

A diœcious moderate-sized tree or shrub, from 15 ft. high upwards; bark dark, rough, scored, but less so than in D. *virginiana*, L.; young parts pubescent. Leaves alternate, submembranous, more or less elliptical, usually paler beneath and often pubescent, 2—6 in. long by 1—$2\frac{1}{4}$ in. wide; petioles $\frac{1}{3}$—$\frac{3}{4}$ in. long. Flowers tetramerous, or by exception pentamerous, axillary.

♂. Flowers subsessile, 2—3 together, about $\frac{1}{3}$ in. long, urceolate. Calyx campanulate, about $\frac{1}{8}$ in. long, shortly 4-fid; lobes ovate, acute. Corolla urceolate-oblong, nearly or quite glabrous, $\frac{1}{3}$rd way 4-lobed; lobes ciliate, obtuse, recurved. Stamens 16, combined by their glabrous filaments in 8 pairs; two pairs opposite each corolla-lobe; each pair consists of a shorter inner and longer outer stamen; filaments inserted at base of corolla-tube; anthers not quite glabrous. Ovary rudimentary.

♀. Flowers solitary, subsessile, wider than in the ♂. Calyx ultimately spreading. Corolla often remaining at apex of the young fruit, urceolate, yellowish white. Staminodes 8, in one row inserted at the base of the corolla, hairy. Ovary glabrous, except at apex from which 4 hairy lines often descend down the fruit, 8-celled, cells 1-ovuled; styles 4, somewhat pubescent below. Fruit subsessile or apparently sessile, often with a glaucous tinge, subglobose, $\frac{2}{5}$—$\frac{7}{10}$ in. thick; fruiting calyx spreading, $\frac{1}{2}$—$\frac{3}{5}$ in. across, with a ring of short dense appressed silky hairs on the inside below the fruit. Flesh of the fruit astringent.

Naturalized in the countries on the shores of the Mediterranean Sea. Russia in Asia, *Pallas*, called *Kurma* by the Persians, *Churma* or *Karà-churma* in Tartary, *Dikoi Phenik* in Astracan; Asia Minor, *Zohrab!*; Turkey in Asia, Lazistan, near Rhize, spontaneous, *Boissier!* n. 1464; Caucasus; China, *Hance!* n. 13753, Canton; Pekin Mountains, *Bunge;* Zsing Yune Pass, along North river, about 120 miles from Canton, *Hance;* Affghanistan, *Griffith!* n. 1289; N. W. India, common on the Huzārā from 3000—6000 ft. alt., male plant called *Gwaladar*, female *Amlōk*, *Dr Stewart!* n. 424; Tsu-sima Island, Straits of Corea, *Wilford!*; Japan, Nagasaki, *Oldham!* n. 529, called *Sinanokaki*.

101. DIOSPYROS VIRGINIANA, Linn. Sp. Pl. p. 1057 (1753).

D. foliis alternis, ovalibus, utrinque obtusis, submembranaceis, pubescentibus vel glabrescentibus, petiolatis, floribus masculis 1—3-nis, breviter cymosis, axillaribus, 4- rarius 5-meris, urceolatis, calyce campanulato, lobis lanceolatis, corollâ breviter lobatâ, staminibus 16, geminatis, paulum pubescentibus; floribus femineis solitariis breviter pedunculatis, staminodiis 8, ovario apice excepto glabro, 8-locularibus, fructibus subglobosis, edulibus.

Gaertn. fil. Carp. (III) p. 138. t. 207 (1805).

Michaux, Arb. Amer. Septr. II. p. 195. t. 12 (1812).

Collin, Förslag af några Nord-americas Träd, p. 23 (1823).

Watson, Dendr. Brit. II. t. 146 (1825).
Rafinesque, Medic. Fl. N. Amer. i. p. 153. t. 32 (1828).
Alph. DC. Prodr. VIII. p. 228. n. 29 (1844).
Belgique Horticole, IV. p. 118. *tab.* (1854).
Ettingsh. Blatt-skel. Dikot. p. 89. t. 38. f. 12 (1861).
Pishamin, Parkinson, Paradis. p. 570 (1629), Theatr. p. 1523. f. 4 (1640).
D. concolor, Moench, Meth. p. 470 (1794).
D. guaiacana, Robin, Voyages, vol. III. p. 417 (1807).
D. pubescens, Pursh, Fl. N. Amer. p. 265 (1814), non Pers.
D. caroliniana, Muhlenb. ex Rafin. Florul. Ludovic. p. 139 (1817).
D. Persimon, Wikstr. Jahr. Schwed. 1830, p. 92 (1834).
D. ciliata, Rafin. New Flora and Bot. N. Amer. part III. p. 25 (1836), non Alph. DC.
D. fertilis, Lodd. Cat. ex Loud. Arb. et Frut. Brit. II. 1197 (1838).
D. calycina, Audib. Cat. Hort. Tonn. ♀. ex Spach, Hist. Végét. IX. p. 405 (1840), non Wall. &c.
D. angustifolia, Audib. ex Spach, Hist. Végét. IX. p. 405 (1840).
D. lucida, Hort. ex Loud. Gard. Mag. 1841, p. 394, non Wall.
D. intermedia, Hort. ex Loud. Encycl. Trees and Shrubs, p. 627 (1842).

A tree attaining in favourable places 60 feet in height and 20 in. in diameter in the trunk, according to Michaux, from whom other details are taken. The trunk of full-grown trees is covered with much and deeply-cracked blackish bark; the sap-wood after drying keeps a clear greenish colour, and the heart is brown. The wood is hard, compact and tough, and is used for several mechanical purposes. The inner bark is said to be useful in intermittent fevers. Young parts pubescent. Branches spreading at 50°—60°. Leaves alternate, submembranous, more or less oval, slightly narrowed, rounded or even slightly cordate at base, usually shortly acuminate at apex, paler beneath and often pubescent; 2—7 in. long (besides pubescent petiole $\frac{1}{3}$—$\frac{7}{8}$ in. long) by 1—3½ in. wide. Flowers tetramerous or occasionally pentamerous, greenish.

♂. Flowers in short 1—3-flowered pubescent cymes which measure (excluding the flowers) about $\frac{1}{8}$ in. long. Calyx small, about $\frac{1}{10}$ in. high, partite, hairy, with lanceolate lobes. Corolla tubular-urceolate, $\frac{1}{4}$ in. long, or in subhermaphrodite flowers $\frac{1}{2}$ in. long, lobes one-third the length of the corolla. Stamens 16, in pairs, somewhat hairy. Ovary glabrous, rudimentary.

♀. Flower solitary, $\frac{1}{2}$ in. long and wide, on peduncles $\frac{1}{10}$ in. long; ovary glabrous, pilose at apex, 8-celled, cells 1-ovuled; styles 4, pilose at base. Fruit solitary, on peduncles $\frac{1}{8}$ in. long, subglobose, 1—1⅓ in. in diameter, glabrous, edible, tipped at apex with remains of style; skin thin, of a pale orange-colour when ripe, often marked externally with 4 depressed lines running down from the apex, and with a slight pruinose bloom; pulp with a sweetish apricot-like taste when ripe but somewhat astringent; seeds 6—8, sometimes 3—5, about $\frac{2}{3}$ in. long, $\frac{3}{8}$ in. broad and $\frac{1}{4}$ in. thick. Fruiting calyx spreading, 4-fid, occasionally 5-fid (in one case small and trifid in a cultivated specimen), $\frac{3}{4}$—1½ in. across, subglabrous; lobes broadly ovate, $\frac{1}{3}$—$\frac{3}{4}$ in. broad, usually somewhat concave from below and not appressed to the fruit, with recurved margins; tube convex from above with a circular depression at its outer margin.

The fruit, which is locally known by the name of *Persimon*, does not fully ripen north of New Jersey; it is said to become better fit to eat after it has suffered frost, and then it becomes very sweet but mawkish. Though eaten by the negroes, and often brought to market, it is not a table-fruit. There is however a sweet variety (*D. virginiana*, L. var. *dulcis*), which is said to yield a good table-fruit. "For an interesting account of the properties of the tree and its fruit, see the inaugural thesis of the late Professor Woodhouse, of the University of Pennsylvania." *Darlington*, Florula Cestrica, p. 47 (1826).

Two other inaugural essays have been devoted to the study of the fruit of this tree; one by Benj. R. Smith, printed in the *American Journal of Pharmacy*, October, 1846, pp. 161—167, and the other by John E. Bryan, in the same journal, May, 1860, pp. 215—217. From these essays the following results are taken. The fruit contains tannin, pectin (or perhaps malic acid), sugar, lignin and colouring matter and neither vegetable albumen starch nor resin. Of 600 grains of green persimon there were found to be 119 grs. of insoluble resinous matter, 64 grs. of saccharine matter slightly acid, 22 grs. of ligneous matter, 1 of green colouring matter, and the remaining 394 grs. were supposed to be water. It is further supposed that in the young fruit lignin serves as a sort of frame-work and as a means of circulation for the juices of the plant; but as the fruit ripens the lignin is converted into sugar, 20 parts of lignin producing 21 parts of sugar. The astringent principle is tannin analogous to that of Cinchona, Catechu, &c., and different from that of galls and oak-bark; and the fruit retains its astringency when dried carefully.

An astringent and styptic. The inner bark is used in intermittent fever, in diarrhœa, and with alum as a gargle in ulcerated sore throat. An indelible ink can also be made from the fruit. (See *Resources of the Southern Fields and Forests*, by Dr Porcher, pp. 423—427, Charleston, 1869.)

In the southern United States of N. America the fruit hangs during part of the winter on the tree a long time after the fall of the leaves; and when at length it too falls, it is eagerly eaten by both wild and domestic animals. In Virginia, Carolina and the western States, the fruit is gathered, kneaded with bran, made into cakes and baked. These cakes mixed with tepid water serve to make beer with the addition of hops and yeast to cause fermentation. Spirit is also distilled by further fermentation. Neither the beer nor the spirit is made for the purposes of commerce.

This species with its varieties has a foliage exceedingly like *D. Lotus*, L.; it differs from the latter by the male cymes and female peduncles being rather longer and by the larger flowers and fruit. Some specimens with regard to which the native country is unknown, though clearly belonging to one of these species, are extremely difficult to assign to either one of them with certainty.

Michaux also speaks of a variety with smaller fruit, compressed seeds, and leaves pubescent underneath: this is *D. pubescens*, Pursh. Fl. N. Amer. p. 265 (1814) and the var. *β. microcarpa*, Rafin. Med. Fl. II. p. 153. t. 32. A variety is occasionally met with in Sumter district, S. Carolina, with fruit about twice the ordinary size (Dr Porcher). *D. intermedia*, Hort. is a variety with more numerous (about 20) hairy stamens. Polygamous flowers occur in cultivated specimens of this species.

North America, United States, St Louis, *Drummond!*, *Riehl!* n. 178; New Orleans, *T. Drummond!* n. 204 bis; "Woods and old fields, Rhode Island and New York to Illinois, and southward," *Asa Gray*; "Florida! to Mississippi and northward," *Chapman*; Cumberland, *Olney!*; Missouri, *Buckley!*; Virginia, Portsmouth, *Rugel!*, *A. Gray!*; Kansas, *Engelmann*.

It is cultivated in British Guiana, and has long been introduced into Europe.

102. DIOSPYROS KAKI, Linn. fil. Suppl. p. 439 (1781).

D. foliis alternis, ovalibus, utrinque obtusis vel angustatis, submembranaceis, subtus pubescentibus, petiolatis; floribus masculis axillaribus, ternis, cymosis, tetrameris, urceolatis, calyce campanulato, lobis ovatis vel lanceolatis, corollâ extus pubescente, staminibus sæpius 16, *geminatis, leviter pubescentibus; floribus femineis sæpius solitariis, pedunculatis, staminodiis sæpius* 8, *ovario sæpius* 8-*loculari, fructibus globosis edulibus sæpe magnis.*

Wight Ic. t. 415 (1840); Alph. DC. Prodr. VIII. p. 229. n. 30 excl. *var.* γ. *glabra* (1844); Thunb. Fl. Jap. p. 157 excl. *var.* β. (1784); non Blanco.

Ficus hortensis, fructu ossiculato eduli, folio Pyri, Kæmpf. Amœnit. exotic. p. 805 (1712).

(?) *D. lobata*, Lour. Flor. Cochinch. p. 227 (1790); Alph. DC. *l. c.* p. 233. n. 53.

D. chinensis, Blume, Catal. Buitenz. p. 110 (1823); Flora, 1825, p. 254; Bijdr. Fl. Ned. Ind. p. 670 (1825).

D. Schi-Tse, Bunge, Enum. Pl. Chin. Bor. n. 237. p. 42 (1832).

Embryopteris Kaki, G. Don, Gen. Dict. Gard. and Bot. IV. p. 41 (1837).

D. costata, Carr. in Rev. Hortic. 1870, p. 134 (fig. p. 133).

D. Kaki var. *costata*, André in L'Illustration Hortic. vol. XVIII. p. 176. t. 78 (1871).

D. Roxburghi, Carrière in Revue horticole, 1872, p. 253. fig. 28, 29.

Local names; *Ono Kaki*, Kæmpfer Amœnit. pp. 805, 807. fig. p. 806 (1712); *Kakwe*, Javan name ex Bl. Bijdr. *l. c.*; *Khi*, Rumph. Herb. Amboin. vol. I. p. 137 (1750).

A small tree; young branches inflorescence and underside of leaves pubescent or subtomentose. Leaves alternate, submembranous, more or less oval and acuminate at apex, paler beneath, 2—7 in. long by 1—3½ in. wide; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long. Flowers pedunculate, diœcious or polygamous, tetramerous.

♂. Cymes axillary $\frac{1}{5}$—$\frac{1}{3}$ in. long, 3-flowered; pedicels about $\frac{1}{10}$ in. long; flowers usually drooping, variable in size, $\frac{1}{3}$—$\frac{3}{4}$ in. long; calyx slightly hairy, with 4 deep ovate or lanceolate lax lobes, shorter than the corolla; corolla hairy outside, urceolate, yellowish-white; stamens 16 (14—24), in pairs, more or less hairy, filaments short.

♀. Flowers usually solitary, or pubescent, axillary; 2-bracteate peduncles $\frac{3}{10}$—$\frac{3}{5}$ in. long, dilated and articulated to fruit at apex; calyx large, hairy on both sides, deeply 4-fid, about 1 in. or more wide, with widely ovate spreading lobes, cordate at the base, much accrescent in fruit at least in most cases, with a thickened and hairy shallow tube in fruit; corolla puberulous, about $\frac{1}{3}$—$\frac{1}{2}$ in. high and wide, 4-fid, with $\frac{1}{2}$-oval recurved lobes; staminodes 8 (or 16?); ovary 8—10-celled, glabrous or nearly so; style hairy, 4-fid; fruit glabrous or nearly so, globular, sometimes as big as an orange, reddish or yellow, 8—10-celled [?in *D. lobata*, Lour. 1 in. in diameter and lobed], in *D. costata*, Carr. 2 in. in diameter and more or less deeply ribbed or lobed. The Chinese preserve the fruits with sugar.

This species has been for a long time under cultivation in China, Japan, &c. and presents much variety in the size and shape of its fruit. By the kindness of M. Carrière I have been supplied with specimens of his *D. costata*, and I am also indebted to M. Decaisne for the inspection of original drawings of other forms of this species. On the whole view of the case I prefer to consider all as belonging to one species, which has under cultivation assumed much perplexing variation. Some varieties are considerably more hardy than others; the foliage also in some forms is fine and shining, in others smaller and more pubescent.

D. lobata, Lour. may very possibly belong to this species, but the fruit is described as only 1 in. in diameter.

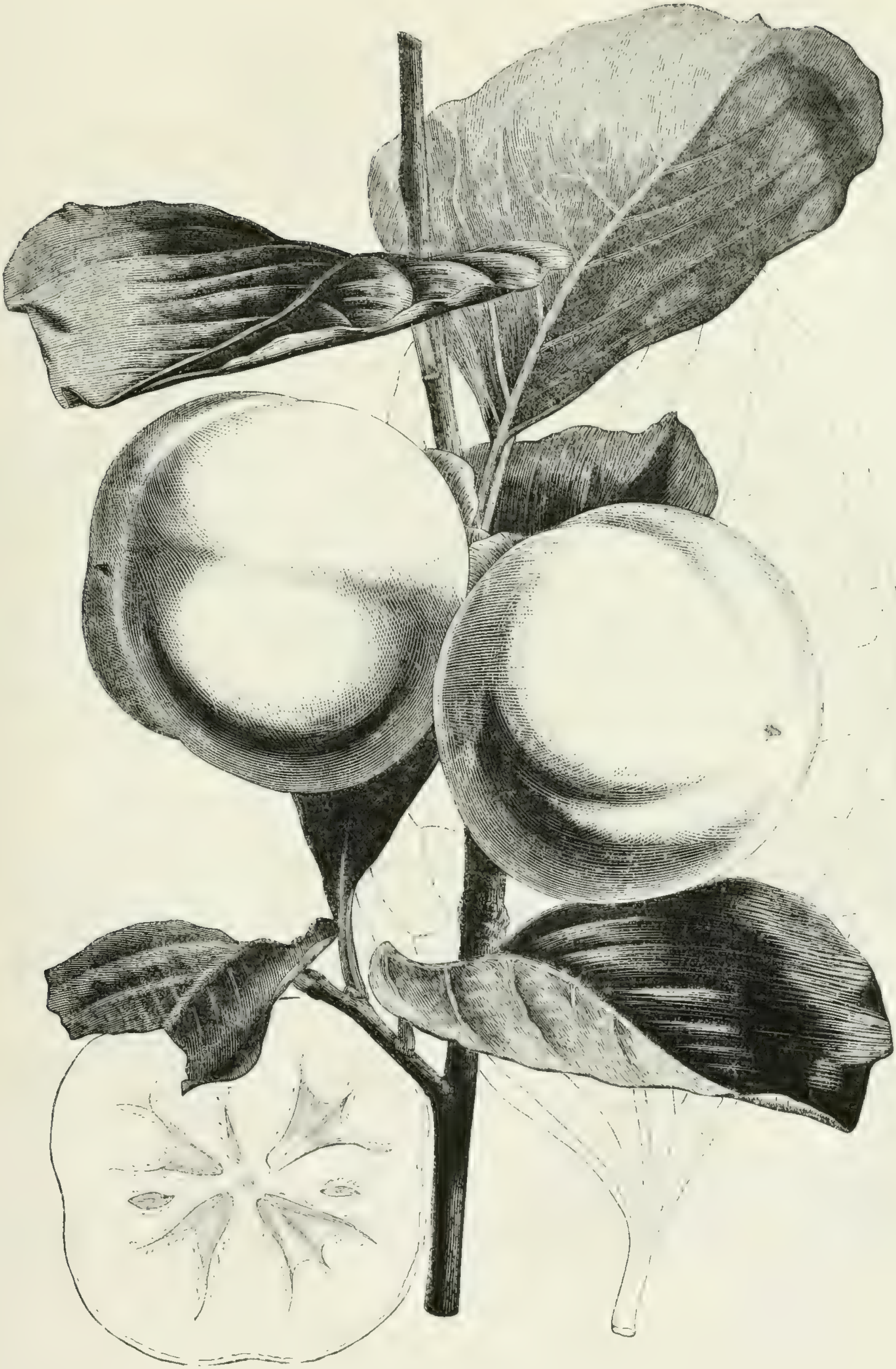
D. Kaki, L. f. is closely allied to *D. Lotus*, L. and *D. virginiana*, L., all of which species are very variable and indeed are likely to be confused amongst each other; *D. virginiana*, L. holds a middle position in respect of the size of the fruit and the length of the inflorescence.

Fruit [see Hasskarl in Bonplandia, VII. p. 255 (1859)] globose, with a diameter of 2 in. or very depressedly globose, $2\frac{1}{2}$ in. wide and $1\frac{4}{5}$ in. high, glabrous and shining, scarlet; skin thin, membranous; flesh of an orange-scarlet colour, edible, sweet, with yellow fibres joined at the base, and then forked and longitudinally dispersed towards the surface; fruiting calyx with a brick-coloured tube and green reflexed lobes. Seeds laterally compressed; *in the globose fruits* 6, oblong, one face nearly straight the other convex, blunt at the apex, acute at the base, a little produced laterally, 1 in. long, $\frac{1}{2}$ in. wide; *in the depresso-globose fruits* 8, one face rather straight the other more than semi-orbicular, widely rounded at the apex, rather acute at the base, scarcely produced; all dark, smooth, well wrapt in the fleshy pulp of the mesocarp, and when carefully removed from the flesh rather shining, marked on the convex face along its whole length with an acute yellowish raphe $\frac{1}{5}$ in. thick. Testa thin, coriaceous; albumen milk-white, cartilaginous; embryo small in proportion to the albumen, straight; radicle terete, very slightly curved or usually straight, white, $\frac{1}{6}$ in. long; cotyledons thin, whitish, lying parallel side by side, in the globose fruits ovate acute $\frac{1}{5}$ in. long by $\frac{1}{12}$ in. wide, in the depresso-globose fruits subrotund, $\frac{1}{8}$ in. in diameter.

Japan, Nagasaki, fr. ripe in Oct., *Oldham!* 528, *C. Wright!;* Tsu-sima Island, Str. of Corea, *C. Wilford!* 756, ♀ fl. May; Formosa, *Oldham!* n. 299; Khasia, *Dr Hooker!*, fruit, Sept.

For a discussion of *D. costata*, Carr., in addition to the above reference, see a note by M. Carrière on *D. Kaki* in Rev. Hortic. 1869, p. 284; a letter of M. Decaisne in the Gardener's Chronicle for 1870, p. 39; a letter of M. Carrière in Gard. Chr. 1870, p. 312; and a paper with woodcuts and coloured plate by M. Carrière in Rev. Hortic. 1871, p. 410; also André in L'Illustr. Hortic. *loc. cit.* where the same coloured plate is given, and the Gardener's Chronicle for 1872, p. 576, whence the accompanying figure has been obtained.

M. Carrière describes his *D. Roxburghi* as a monœcious shrub, at times subdiœcious by abortion; the male flowers very numerous in comparison with the female and bearing 15—20 stamens; the fruit $1\frac{3}{4}$ in. (2 in. in figure) in diameter, with numerous brown prominences especially towards the apex. He thinks it identical with the *D. Kaki*, Roxb. from India, but quite distinct from the *D. Kaki*, Thunb. from Japan; he finds it considerably more tender and sensitive to cold and much less productive of fruit. It must however be borne in mind that M. Carrière has described his species from a cultivated specimen, and also that the whole group of *Kaki* has been long cultivated in Japan, China and elsewhere, and thus it may be expected



that many differences have been acquired under cultivation extended over a wide geographical area which, while worthy of distinct notice in a horticultural point of view, yet do not properly amount to specific importance. The allied species *D. virginiana*, L. and *D. Lotus*, L. are in a similar manner subject to great variation and from similar causes.

I have examined the herbaria both of Linnæus and of Sir J. E. Smith (including that of Linnæus the younger) without finding an authentic specimen of *D. Kaki;* but I think there is no doubt but the *D. Kaki* of Thunberg is identical with that of Linn. fil.

103. DIOSPYROS CHARTACEA, Wall. List n. 4135 (1828—32).

D. glabra, foliis alternis, elongato-lanceolatis, apice acuminatis, basi obtusis, submembranaceis, minute pellucido-punctatis, breviter petiolatis; floribus masculis ternis, subsessilibus, tetrameris, calyce elongato-cylindrico, 4-fido, lobis ovatis, ciliatis; corollâ brevi, 4-fidâ, lobis obtusis, staminibus 16—20, *geminatis, antheris pilosis, ovarii rudimento glabro.*

Alph. DC. Prodr. VIII. p. 232. n. 51 (1844).

Glabrous or very nearly so. Leaves elongate-lanceolate, acuminate at apex, rounded or somewhat narrowed at base, submembranous, minutely pellucid-punctate, alternate, 2—9½ in. long, by ½—3 in. wide; petioles $\frac{1}{10}$—¼ in. long; lateral veins prominent below.

♂. Flowers subsessile, subglobose (in bud), $\frac{1}{16}$ in. in diameter, ternate, crowded, in small very short cymes; bracts 2 or 3 times shorter than the flowers, ovate, acute, subciliate. Flowers ⅙ in. long, quite glabrous except the margins of the ciliated calyx-lobes. Calyx shortly cylindrical, 4-fid, with ovate rounded much imbricated lobes. Corolla (in bud) scarcely longer than the calyx, 4-fid; lobes ovate, obtuse. Stamens 16—20, in pairs; anthers shortly pilose, filaments short, $\frac{1}{50}$ in. long (in bud). Ovary rudimentary, glabrous.

Burma, Trogla hills, Bank of Sallun, *Wallich!* 4135.

104. DIOSPYROS VACCINIOIDES, Lindl. in Hook. Exot. Fl. t. 139 (1825).

D. fruticosa, foliis alternis, ovalibus, apice apiculatis sæpe acutis, basi angustatis, supra glabris nitidis, coriaceis, subsessilibus; floribus masculis 1—3-nis subsessilibus axillaribus 4-meris, bracteis ovatis ciliatis imbricatis, calyce 4-partito, corollâ lobatâ, lobis lanceolatis acutis patentibus, staminibus 16, *geminatis, glabris, corollæ basi insertis; floribus femineis solitariis, subsessilibus, staminodiis* 4—8, *glabris, uniserialibus, ovario inferne glabro,* 8-*loculari; fructibus globosis vel ellipsoideis, albumine non ruminato, calyce non accrescente.*

Loddiges, Cab. t. 1549 (1829); Wall. List n. 4130 (1828—32).

Rospidios vaccinioides, Alph. DC. Prodr. VIII. p. 220 (1844), Benth. Fl. Hongk. p. 210 (1861), Hance in Ann. Sc. Nat. ser. V. vol. v. p. 227 (1866).

Non *Vaccinium Sprengelii*, Wall. List 6296! Vide Voigt, Hort. Suburb. Calcutt. p. 333 (1845).

Vaccinium fragrans, Wall. ex Voigt, *l. c.* p. 345.

D. vacciniifolia, Ettingsh. Beitr. Kenntniss. Foss. Fl. v. Sotzka in Untersteiermark, in Sitzungsberichte der Math.-naturw. Cl. Kais. Akad. Wissensch. XXVIII. p. 494. t. v. fig. 5—6 (1858).

A small, erect, twiggy, leafy and evergreen shrub, resembling *Buxus sempervirens*. Branches covered with shaggy rufous silky hairs when young, then puberulous and finally glabrate, spreading at about 30°. Leaves oval, apiculate and often acute at apex, more or less narrowed at base, alternate, subsessile, coriaceous, appressedly pubescent beneath when young, shining and glabrous above; $\frac{2}{3}$—$1\frac{1}{5}$ in. long by $\frac{1}{3}$—$\frac{5}{6}$ in. wide; midrib depressed above; without conspicuous veins.

♂. Flowers $\frac{9}{40}$ in. long, tetramerous, drooping, subsessile, in very short 1—3-flowered axillary rufous-hairy cymes; bracts ovate, ciliated, imbricated, caducous. Calyx 4-partite, with 4 lanceolate-subulate erect-patent lobes $\frac{1}{5}$ in. deep pubescent on both sides. Corolla scarcely longer than the calyx, 4-fid, with lanceolate spreading lobes, with 4 hairy lines outside. Stamens 16 (12 ex Alph. DC.), glabrous, in pairs, the inner ones shorter, inserted at the base of the corolla-tube; anthers rather shorter than the filaments, dehiscing laterally from apex. Ovary rudimentary, hairy.

♀. Flowers solitary, subsessile. Bracts caducous. Calyx $\frac{1}{4}$—$\frac{2}{5}$ in. long, hairy, 4-partite, with linear-lanceolate lobes. Corolla shorter than the calyx, 4-fid, with 4 hairy lines outside and lanceolate acute recurved lobes. Staminodes 4—8, glabrous, in one row, inserted at the base of the corolla-tube. Ovary ovoid, prolonged at apex into a pubescent 4-lobed style, glabrous below; (3-celled according to Lindley) 8-celled; cells 1-ovuled. Fruit globose or ellipsoidal, shining, usually glabrous except at apex, "3-celled," about $\frac{1}{2}$ in. high; cells 1-seeded; albumen not ruminated, cartilaginous; embryo axile, straight, cotyledons large foliaceous. Fruiting calyx not accrescent.

China; Hong Kong, *Major Champion!; C. Wilford!; Millett!; Hance!* 604; *C. Wright!* 312; S. China, *Seemann!* 2454; Malacca, *Griffith!* 3643; Singapore, Penang, &c. *Walker!*, *Wallich!*

Var. *pellucido-punctata*. Leaves pellucid-punctate, thinly coriaceous. Fruit with scattered hairs. South Andaman, *S. Kurz!*

105. DIOSPYROS CAYENNENSIS, Alph. DC. Prodr. VIII. p. 224. n. 8 (1844).

D. foliis alternis, oblongis, obtusis, basi angustatis, supra nitidis glaberrimis, subtus glabrescentibus, coriaceis, petiolatis; floribus 1—3-*nis brevissime cymosis subsessilibus ferrugineo-velutinis, calyce turbinato profunde* 4-*lobo, corollâ calyce sublongiore, staminibus* 10—12, *glabris, ovario in flore femineo ovoideo glabro* 8-*loculari.*

Danzleria axillaris, Bert. ex Alph. DC. *l. c.*

Young shoots and flowers ferruginous-velutinous. Leaves alternate, oblong, obtuse or obtusely acuminate, abruptly narrowed at base, coriaceous, shining and quite glabrous above, glabrescent beneath, green on both sides in the dry state, 3—5 in. long by 1—$1\frac{2}{3}$ in. wide, reticulated; more or less revolute at the margin, petioles $\frac{1}{2}$ in. long or shorter. Flowers drooping, puberulous, tetramerous; peduncles much shorter than the calyx.

♂. Flowers solitary or 3 together. Calyx $\frac{1}{3}$ in. long; lobes sub-erect, widely ovate, undulated, silky on both sides, thickened within over a triangular space at base. Corolla ovoid, sub-tetragonal, contorted in bud, fleshy, somewhat hairy outside. Stamens 10—12, glabrous, distinct or in pairs; anthers subulate.

♀. Flowers axillary, 1—3 together, $\frac{1}{2}$ in. long; pedicels equalling the petioles. Calyx turbinate at base, deeply 4-fid; lobes wide, cordate. Corolla silky outside, rather longer than the calyx. Ovary ovoid, glabrous, 8-celled, fruiting calyx nearly $\frac{1}{4}$ in. high by more than $\frac{1}{2}$ in. wide, 4-fid, with reflexed undulated lobes and shallowly cup-shaped crass tube having internal elevated rim, appressedly and shortly hairy inside.

Cayenne, French Guiana!; cultivated in Jamaica, *Berter!*, but not mentioned in Grisebach's Flora of the British West Indies.

106. DIOSPYROS LÆVIS, Boj. ex Alph. DC. Prodr. VIII. p. 232. n. 50 (1844).

D. glabra, foliis alternis, ellipticis, utrinque angustatis, coriaceis, breviter petiolatis; floribus masculis, 1—3-nis, subsessilibus, tetrameris, calyce campanulato, corollæ lobis obtusis, staminibus 16, *geminatis, glabris.*

Glabrous. Branches slender, black in dried state. Leaves alternate, elliptical, attenuate at both ends, rather obtuse at very apex, coriaceous, revolute at the margins, 3 in. long by $1\frac{1}{4}$—$1\frac{1}{2}$ in. wide; lateral veins scarcely conspicuous; midrib depressed above; petioles $\frac{1}{6}$ in. long.

♂. Flowers solitary or 3 together, subsessile, $\frac{1}{3}$ in. long, glabrous; calyx campanulate, shortly 4-fid, with acute ciliated deltoid lobes; corolla shortly 4-fid, double the length of the calyx or less, lobes widely ovate, obtuse; stamens 16, in pairs, unequal, apiculate, glabrous.

Madagascar, East coast, *Bojer!*, *Helsonberg!*

107. DIOSPYROS THOUARSII, sp. nov.

D. glabra, foliis alternis, ellipticis, utrinque paulum angustatis, coriaceis, subsessilibus; floribus masculis, aggregatis, brevissime cymosis, tetrameris, urceolatis, calyce parvo, 4-*lobo, staminibus* 12, *glabris, in flore femineo paucis, ovario ovoideo,* 8-*loculari.*

Dark, glabrous except the bracts. Leaves alternate, elliptical, subsessile, coriaceous, somewhat narrowed at both ends, obtuse; veins reticulated, inconspicuous; midrib somewhat depressed above; $1\frac{1}{2}$—3 in. long by $\frac{1}{2}$—$1\frac{1}{3}$ in. wide, rich dark brown beneath; margins just thickened beneath. Cymes axillary, many-flowered, short; bracts small, ciliated.

♂. Flowers $\frac{1}{6}$ in. long by $\frac{1}{10}$ in. wide, urceolate. Calyx $\frac{1}{20}$ in. long by $\frac{1}{10}$ in. wide, shortly 4-fid, lobes depresso-deltoid, apiculate; corolla $\frac{1}{7}$ in. long, $\frac{1}{12}$ in. wide, barrel-shaped, $\frac{1}{3}$ way 4-lobed; lobes imbricated sinistrorsely, depresso-ovate; stamens 12, mostly or all inserted at base of corolla (some in pairs?), glabrous; anthers much exceeding the filaments, dehiscing laterally from apex; pollen ellipsoidal, smooth. Ovary 0.

♀. Calyx $\frac{3}{10}$ in. long by $\frac{3}{10}$—$\frac{2}{5}$ in. wide, deeply 4-fid; lobes widely ovate, sub-cordate at base, apiculate at apex, suberect, accrescent. Corolla equalling the calyx, deeply 4-fid, not spreading. Staminodes 2 or more, 4 (?), small, inserted at base of corolla. Ovary ovoid, terminated by short 4-lobed style, 8-celled, cells 1-ovuled.

Madagascar, *Hb. Petit-Thouars!* in Mus. Paris.

108. DIOSPYROS CHLOROXYLON, Roxb. Coromand. I. p. 38. t. 49 (1795).

D. foliis alternis, ovalibus, basi sæpius rotundatis apice mucronatis, tenuiter coriaceis, subtus tomentosis, breviter petiolatis; floribus masculis aggregatis, subsessilibus, 4—10-nis, tetrameris, calyce campanulato, dense pubescente, profunde 4-fido, corollâ 4-fidâ, staminibus 16, *biserialibus, glabris; floribus femineis solitariis, sessilibus, staminodiis circiter* 8, *glabris, ovario glabro,* 8-*loculari; fructibus globosis, glabris, edulibus, seminum albumine non ruminato.*

Wall. List n. 4118 (1828—32), Alph. DC. Prodr. VIII. p. 230. n. 40 (1844).

D tomentosa, Poir. Encycl. V. p. 436. n. 22 (1804), non Roxb.

D. capitulata, R. Wight Ic. tt. 1224, 1588 bis (1850).

Cfr. *D. glauca,* Rottler in Gesellschaft Naturforschender freunde zu Berlin, Neue Schrift. IV. p. 221 (1803); Alph. DC. Prodr. VIII. p. 238. n. 84 (1844).

Nella-woolymera of the Telingas, Roxb. Corom. *l. c.*; *Neenye* or *Ninei* in Surat, Dr Gibson.

A tree of middle size with irregular trunk, or in low lands only a large bush; bark scabrous, dark rust-coloured; branches spreading, sometimes spinous; young shoots pubescent-tomentose, subfulvous. Leaves oval or oval-oblong, usually rounded at base and mucronate at apex, thinly coriaceous; pubescent or subglabrescent and dark green on upper side; more or less tomentose and sub-tawny beneath, alternate, $\frac{3}{5}$—3 in. long by $\frac{2}{3}$—$1\frac{3}{10}$ in. wide; petioles $\frac{1}{12}$—$\frac{1}{4}$ in. long; midrib depressed above; lateral veins not conspicuous. Inflorescence tawny densely pubescent; flowers white.

♂. Flowers clustered sessile or subsessile, 4—10 together, about $\frac{1}{6}$ in. long, tetramerous; on peduncles $\frac{1}{12}$ in. long, with very short pedicels; bracts oval, small, glabrous inside. Calyx about $\frac{1}{10}$ in. high, campanulate, densely tawny-pubescent, deeply 4-fid, with apiculate lobes, glabrous inside. Corolla 4-fid, glabrous except 4 lines of hairs outside. Stamens 16, in 2 rows, inserted more or less in pairs, receptacle or at base of corolla, glabrous, the inner ones shorter; longer filaments as long as anthers; anthers dehiscing laterally from apex; connective apiculate or prolonged. Receptacle glabrous; ovary rudimentary glabrous.

♀. Flowers solitary, sessile (or subsessile in Wight Ic. t. 1588 bis), about $\frac{1}{4}$ in. long, tetramerous; bracts longer than in ♂, shorter than the calyx. Calyx $\frac{1}{7}$ in. long, campanulate, pale tawny, densely pubescent; lobes $\frac{2}{3}$ rds depth of calyx, apiculate. Corolla erect, glabrous, except 4 hairy lines down middle of lobes; lobes $\frac{1}{2}$—$\frac{2}{3}$ rds depth of corolla, ovate-lanceolate. Staminodes 7—9, glabrous, in one row, hypogynous or inserted at base of corolla. Ovary glabrous, surmounted by 4 erect glabrous styles, 8-celled; cells 1-ovuled, often approximated in pairs. Fruit globose, glabrous, $\frac{1}{3}$ in. or rather more in diameter, on nearly flat calyx $\frac{1}{3}$ in. in diameter. When ripe it is eaten raw among the Orixa mountains and is very palatable. Seeds 2—3; albumen not ruminated, testa thick slightly irregular inside. The wood of the larger trees is yellowish, very hard and durable, and is used by the natives for various economical purposes.

Tranquebar, *Vahl;* Madras!; Bombay; Surat and Nassick, *Dr Gibson!;* Canara and Mysore, *Mr Law!;* Kew List, n. 1712, 1719, 3617; *Wallich!;* East Bengal!

109. DIOSPYROS (?) PERGAMENA, sp. nov.

D. glabra, foliis obovato-oblongis, alternis, basi leviter angustatis, apice anguste et abrupte acuminatis, firmiter pergamenis, petiolatis, floribus masculis in ramis vetustis dense aggregatis sessilibus et breve-cymosis, pentameris pubescentibus parvis, staminibus 20 *binis, ovarii rudimento hirsuto; fructibus pedunculatis globosis uncialibus glabratis* 3-*locularibus* 3-*spermis; albumine radiatim striato.*

Glabrous, young shoots terete; leaves alternate, obovate-oblong, narrowly and suddenly acuminate at apex, slightly narrowed at base, of the texture of firm parchment, dark brown above with depressed veins, paler with raised veins loosely reticulated beneath, 8—9 in. long by $2\frac{3}{10}$—3 in. wide; petioles $\frac{1}{2}$ in. long.

♂. Flowers densely clustered on the older branches, sessile and in short cymes, pentamerous, hirsute, subglobose (?), $\frac{1}{10}$ in. in diameter (immature); calyx 5-fid, glabrous inside, lobes ovate; corolla 5-fid, lobes obtuse; stamens 20 in 10 pairs hispid, ovary rudimentary hairy.

♀. Fruit solitary (?), glabrate, subglobose, about 1 in. in diameter, 3-celled, 3-seeded (in one case); peduncle nearly $\frac{1}{2}$ in. long; calyx 5-partite, $\frac{1}{2}$ in. in diameter, closely hairy on both sides, reflexed; lobes involute; seeds $\frac{3}{4}$ in. long by $\frac{2}{5}$ in. thick; albumen radiately striate, not ruminated,

Borneo, *O Beccari!* n. 1787.

110. DIOSPYROS CAULIFLORA, Blume, Bijdr. Fl. Ned. Ind. p. 668 (1825).

D. foliis alternis, ovalibus, utrinque attenuatis, nitidis, glabris, breviter petiolatis; floribus masculis axillaribus tetrameris, staminibus 16, *geminatis, inæqualibus; floribus femineis lateralibus secus ramos vetustiores paniculatis, calyce profunde* 4-5-*lobo, lobis basi margine sinuato reflexis nigrescentibus, corollâ urceolatâ,* 4-*fidâ, fauce constrictâ, fructibus globosis, glabris, edulibus,* 4—8-*locularibus, albumine radiatim striato.*

Alph. DC. Prodr. VIII. p. 238. n. 81 (1844); Hasselt in Hasskarl Pl. Javan. p. 767. n. 351 (1848); non Mart.

A lofty diœcious tree. Leaves alternate, elliptical or oblong, attenuate at both ends, 4—9 in. long by 1—$3\frac{7}{10}$ in. wide, shining, glabrous; midrib and lateral veins depressed above; petioles $\frac{1}{6}$—$\frac{1}{4}$ in. long.

♂. Flowers axillary; calyx 4-fid; corolla 4-fid; stamens 16, in pairs, unequal in the pairs.

♀. Flowers crowded in dense lateral panicles on the older wood, racemose; racemes 3—5 flowered, with bracteoles at the ramifications; peduncles nearly 1 in. long, thickened at the apex, turning black. Calyx deeply 4—5-lobed; lobes turning black, narrow, reflexed, at the base with wavy margin. Corolla urceolate, 4-fid, much constricted at the top of the tube; tube tetragonal, covered with black hairs especially at the angles; lobes pale yellow, horizontal.

Fruit fleshy, globose, glabrous, edible, 1 in. in diameter, 4—8-celled, green; seeds solitary in the cells, some imperfect; albumen radiately striate; embryo turning yellow.

Java, Bantam, 500 ft. alt. *Hasselt, Reinwardt!, Blume!*

111. Diospyros ramiflora, Roxb. Hort. Beng. p. 40 (1814).

D. foliis alternis, ovalibus vel oblongis, apice acuminatis, basi angustatis, coriaceis, glabris; floribus femineis dense fasciculatis secus ramos vetustiores, 4—6-meris, tomentosis, calyce campanulato irregulariter lobato accrescente, corollâ urceolato-oblongâ obtuse lobatâ, staminodiis 10—12, *glabris, ovario ovoideo-conico, fulvo-tomentoso,* 10- *vel* 12-*loculari, fructibus globosis, magnis, subscabris, edulibus,* 10—12-*spermis.*

Roxb. Fl. Ind., edit. 1832, Vol. ii. p. 535. n. 7; Drawings in Herb. Kew; Wall. List n. 4119 (1828—32); Wight Ic. t. 189 (1840); Alph. DC. Prodr. viii. p. 233. n. 57 (1844).

A large diœcious tree with glabrous leaves and branches and straight trunk. Leaves oval or oblong, acuminate at apex, somewhat narrowed at base, coriaceous, alternate, shining and of same colour on both sides, margins slightly undulated and recurved, 4—10 in. long by 1—3 in. wide, veins not conspicuous above, petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long; midrib wide and channelled above.

♀. Flowers urceolate-oblong, collected in small short fascicles on the thick woody branches, tetramerous pentamerous or hexamerous; the inflorescence is however sometimes on young shoots or in racemes or panicles. Pedicels and calyces clothed with olive-coloured down; calyx $\frac{2}{7}$ in. long, urceolate, with inflated tube and deltoid lobes $\frac{1}{10}$—$\frac{1}{6}$ in. deep; corolla $\frac{1}{3}$—$\frac{1}{2}$ in. long, white, covered with short felt outside, glabrous inside except on the obtuse imbricated lobes which are about $\frac{1}{4}$ the depth of the corolla, at first spreading and then revolute; tube somewhat inflated. Staminodes 10 or 12, double the number of the parts of the flower, glabrous, in one row, shorter than the corolla-tube. Ovary about the length of the calyx, ferruginous-hairy, ovoid-conical, 10 or 12-celled; cells 1-ovuled; style short; stigmas 5 or 6. Fruit globular, $2\frac{1}{2}$—3 in. in diameter, slightly scabrous, resting on the very thick enlarged calyx which is about $1\frac{1}{2}$ in. in diameter, 10—12-celled; cells 1-seeded; seeds transversely lined outside; albumen somewhat ruminated (?)

Native name *Oori-gaub* or *Goolul* on eastern frontier of Bengal, where, according to Dr Roxburgh, the tree grows wild to a great size, and supplies the natives with very strong hard wood. Silhet, *Wallich!* 4119; Tipperah, *Roxburgh.*

The position of this species in the genus is uncertain in consequence of the want of knowledge of the male plant and of the nature of the albumen in the seed; thus when more intimately known, the species may require to be removed to Sect. i. Melonia or elsewhere.

112. Diospyros Diepenhorstii, Miq. Fl. Ind. Bat. Suppl. i. pp. 250, 583 (1860).

D. foliis oblongis, apice breviter acuminatis, basi obtusis, firmiter coriaceis, glabris, breviter petiolatis; floribus femineis secus ramos vetustiores aggregatis, ovario ovoideo, heptagono, glabro, 14-*loculari, basi abrupte stipitato-constricto, calyce coriaceo cupulato grossificante, extus parce appresso-pubescente.*

Buds somewhat hirsute; branchlets glabrous. Leaves oblong or obovate-oblong, the upper ones sublanceolate, rounded or obtuse at the base, shortly acuminate, glabrous, of firm parchment-like texture, smooth above with the lateral veins usually depressed, pale beneath with the lateral veins prominent mostly patent united within the margin and loosely reticulated, 9—10 in. long.

♀. Flowers densely crowded on the old branches, with short pedicels and hirtellous bracts; ovary rather thickened, resting on a coriaceous cup-shaped patent obtusely 5-lobed calyx sparsely covered outside with appressed puberulence, abruptly stipitate-constricted at the base, ovoid-heptagonal, glabrous, marked at the apex with 7 pits and teeth, 14— (? 16)-celled.

According to Miquel it is clearly related to *D. ramiflora*, Roxb., but quite a distinct species. Malay name *Djantoe-dipo.* I have not seen a specimen.

West Sumatra in Province Priaman, *Diepenhorst.*

113. DIOSPYROS SUMATRANA, Miq. Plantæ Junghuhnianæ, p. 203 (1851—55).

D. foliis distichis, oblongis, apice anguste acuminatis, basi cuneatis, submembranaceis, breviter petiolatis; floribus femineis laxiuscule cymosis, fructibus immaturis ovoideo-oblongis, subglabris, 4-locularibus, loculis monospermis, albumine non ruminato, calyce fructifero profunde 4-lobo appresse pubescente, aucto, foliaceo, lobis suberectis, ovatis, acuminatis, basi cordatis.

Young parts inflorescence petioles and midrib of leaves beneath covered with short stiff puberulence. Leaves firmly submembranous, oblong, alternate, with a long narrow acumen at apex, cuneate at base, distichous, $2\frac{1}{2}$—5 in. long by $\frac{5}{8}$—2 in. wide; petioles $\frac{1}{8}$—$\frac{1}{5}$ in. long; veins slightly depressed above, in relief beneath; lateral veins 5—6; very minutely and vaguely pellucid-punctate.

♀. Cymes rather lax, about $\frac{1}{4}$ in. long or shorter (excluding the flowers); peduncles $\frac{1}{8}$ in. long, about 3-flowered; pedicels with yellowish hairs, thickened upwards; bracts foliaceous, caducous. Fruit (unripe?) oblong, $\frac{5}{8}$—$\frac{7}{8}$ in. long, glabrous; style, erect, glabrous (broken), distinct. Fruiting calyx deeply 4-lobed, appressedly pubescent, erect or sub-erect, nearly as long as the young fruit; lobes ovate, acutely acuminate, wide and cordate at base, foliaceous, wavy; seed $\frac{1}{2}$ in. long; albumen not ruminated; young fruit glabrous, 4-celled; cells 1-seeded.

Sumatra, *Korthals!*, distr. Angkola, *Junghuhn!*; Borneo, *Korthals!*

114. DIOSPYROS PENDULA, Hasselt ex Hasskarl Pl. Javan. p. 468. n. 352 (1848).

D. foliis oblongo-lanceolatis, utrinque acuminatis, glabris, breviter petiolatis, floribus masculis solitariis, femineis 1—2-nis vel breve-racemosis, calyce 4-fido, nigro-piloso, corollæ lobis revolutis, filamentis 8, villosis, antheras 2—3 gerentibus, ovario ovoideo, 4—8-loculari, fructibus carnosis.

A lofty diœcious tree. Leaves oblong-lanceolate, acuminate at both ends, $2\frac{1}{2}$—$3\frac{1}{2}$ in. long by 1—$2\frac{1}{2}$ wide, shortly petiolate, glabrous.

♂. Flowers solitary. ♀. Flowers solitary 2 together or collected in small racemes, pendulous. Calyx 4-fid, bright green, nigro-pilose. Corolla pale yellow, with reflexed lobes. Filaments 8, short, hairy, bearing 2—3 anthers. Ovary ovoid, 4—8-celled; style thick, attenuate at the apex (or 2—4 connate styles); stigmas 4, emarginate. Fruit fleshy, pilose when young, 4—8-celled; cells 1-seeded.

Java, Bantam Province, Mt. Pulassarie, flowers in June, between 4000 ft. alt. and the crater, *Hasselt.*

115. DIOSPYROS MACROPHYLLA, Blume Bijdr. Fl. Ned. Ind. p. 670 (1825).

D. foliis alternis, ovalibus vel ovali-oblongis, apice acuminatis, basi rotundatis vel interdum subcordatis, tenuiter coriaceis, supra glabris, subtus glabriusculis, nervis gracilibus, breviter petiolatis; floribus masculis paniculatis, pedicellis brevibus, calyce breviter 3—5-fido, urceolato, corollâ breviter 5-lobâ, crasso-ligneâ, staminibus 12, geminatis, glabris, cymis femineis paucifloris brevibus, fructibus tomentosis subglobosis, calyce fructifero aucto.

Alph. DC. Prodr. VIII. p. 228. n. 27 (1844), non Wall.

D. phyllomegas, Steud. Nomencl. Bot. edit. ii. vol. I. p. 514 (1840).

A tree 60 feet high, with dark terete branches. Leaves alternate, oval or oval-oblong, acuminate at apex, rounded or sub-cordate at base, thinly coriaceous, nearly glabrescent above with clear slender arching lateral veins, glabrous above, 3—10 in. long by $1\frac{1}{2}$—$4\frac{3}{4}$ in. wide; petioles $\frac{1}{6}$—$\frac{1}{4}$ in. long.

♂. Flowers axillary, paniculate, $\frac{1}{4}$ in. long, pubescent; panicles many-flowered; 1—$1\frac{1}{2}$ in. long, ultimate pedicels mostly short. Calyx shortly 3—5-fid, globose-urceolate, $\frac{3}{16}$ in. long, lobes deltoid; corolla silky outside, ovoid in bud, shortly 5-lobed, tube very crass and hard; stamens 12, unequal, in pairs, glabrous.

♀. Cymes few-flowered, short, calyx 4—5-fid, hairy on both sides, accrescent in fruit; fruit tomentose, sub-globose, 1 in. or more in diameter.

Java, in mountainous places, *Blume!* Local name *Kitjallung.*

116. DIOSPYROS OVALIFOLIA, R. Wight, Ic. t. 1227 (1850).

D. foliis alternis, ovalibus, apice obtusis, basi angustatis, tenuiter coriaceis, glabris, petiolatis, floribus aggregatis, 3—6-nis, brevissime cymosis, 4—5-meris, urceolatis, calyce brevi, pubescente, corollâ subglabrâ, lobis obtusis, staminibus 13—20, glabris, subæqualibus, plerisque geminatis; in flore femineo staminodiis 0—7, ovario hirsuto 4- vel 6-loculari, loculis 1-ovulatis; fructibus solitariis, globosis, glabratis, seminum albumine non ruminato.

Thw. Enum. Ceyl. Pl. p. 181. n. 13 (1864).

A moderate-sized tree, glabrous except the inflorescence. Leaves oval- or obovate-oblong, rounded or obtusely pointed at apex, more or less narrowed at base, thinly coriaceous, alternate, midrib depressed above, $1\frac{1}{2}$—6 in. long by $\frac{1}{2}$—$2\frac{1}{2}$ in. wide, with petiole $\frac{1}{6}$—$\frac{1}{3}$ in. long, turning yellowish when dry, paler beneath with reddish midrib.

♂. Flowers clustered, 3—6 together, on very short hairy cymes, in the axils of fallen or present leaves, 4—5-merous, $\frac{1}{5}$—$\frac{1}{4}$ in. long. Calyx $\frac{1}{10}$ in. long, tawny-hairy on both sides, openly campanulate, with rounded or somewhat deltoid lobes about half the length of the calyx. Corolla twice the length of the calyx or more, urceolate, glabrous or nearly so, 4—5-fid or less deeply lobed, with obtuse spreading or recurved lobes. Stamens 13—15—20, glabrous, mostly inserted on the receptacle and in pairs, nearly equal, $\frac{1}{8}$ in. long, filaments $\frac{1}{20}$ in. long. Ovary rudimentary, hairy.

♀. Flowers clustered, 3—6 together, on very short cymes, 4—5-merous, thicker than in ♂. Staminodes 0—7, glabrous, hypogynous or at base of corolla. Ovary conical, tawny hairy, 4- or 6-celled (2-celled ex Wight *l. c.*): cells 1-ovuled; stigma 2—3-lobed. Fruit solitary, subsessile, glabrate, globose, ⅝ in. in diameter, usually 1-seeded. Fruiting calyx reflexed, tomentose, thickened but not dilated or but slightly so. Seeds with albumen not ruminated.

Ceylon, 2000—4000 ft. alt., *Thwaites!* 1815, 1816, 2533, Trincomalee, *Moon!*; Madras, Coimbatore, *Wight!* n. 1720; Anamalay hills, *Beddome!*

117. DIOSPYROS TEXANA, Scheele in Linnæa XXII. p. 145 (1849).

D. foliis alternis, obovatis, apice rotundis, basi cuneatis, submembranaceis, subtus pubescentibus, subsessilibus; floribus masculis 1—3-*nis, breviter pedunculatis, pubescentibus, calyce* 5—6-*partito, corollâ urceolatâ,* 5—6-*fidâ, lobis obovatis, staminibus* 16—20, *biserialibus, glabris; floribus femineis solitariis, staminodiis* 0, *ovario sub*-8-*loculari, ovoideo, dense sericeo, fructibus globosis.*

A tall much-branched shrub, 12—15 feet high, with fastigiate branches spreading at 60°—70°, cinereous, verrucose, leafy, softly pubescent and pale at the apex; warts subrotund, of dark reddish colour. Leaves alternate, oblong-obovate, wedge-shaped at base, rounded or emarginate at apex, submembranous, softly pubescent, pale, glabrescent on upper side; veins inconspicuous above; nearly flat or with revolute margins, ½—2 in. long by ¼—1 in. wide; petioles $\frac{1}{100}$—$\frac{1}{20}$ in. long, hairy. Flowers with scent of vanilla.

♂. Flowers 1—3 together, in axils of present or fallen leaves, drooping, ¼—⅓ in. long, softly pubescent, pale, crowded on young shoots; peduncles ⅙—⅓-⅝ in. long, pubescent; bracts caducous. Calyx with 5 or rarely 6 deep ovate or lanceolate lobes, shorter than the tube of the corolla, ⅛ in. long, pubescent on both sides. Corolla urceolate, with 5 or perhaps rarely 6 recurved lobes about half the length of the corolla-tube, glabrous inside. Stamens 16—20, distinct, in 2 rows, glabrous; anthers longer than the filaments, dehiscing from the apex. (In one case a stamen is abnormal, an anther having two filaments.) Ovary rudimentary, with grey hairs.

♀. Flowers solitary, pentamerous, ⅓ in. high by ½ in. wide. Calyx large but not accrescent; peduncles ⅕—⅓ in. long, recurved, bearing caducous small bracts. Corolla with spreading lobes. Staminodes 0. Style 4-fid; stigmas dilated. Ovary 8 (?)-celled, densely pilose. Fruit globose, ½ in. in diameter, dark, covered with scattered hairs, fleshy, sweet-tasted, ultimately shining. Albumen not ruminated. Fruiting calyx spreading or reflexed, 5-partite; lobes ⅜ in. long, oblong, pubescent on both sides.

North America, Texas, Galveston Bay, *Drummond!*, Fasc. III. n. 329. (♂ fl.); *Drummond!*, Fasc. II. n. 201. (Fr.); *Lindheimer!*, Flora Texana exsiccata, Fasc. III. n. 451 (♂), 452 (♀ fl.), 453 (Fruit); Mexico, between Laredo and Bejem, Feb. 1828, *Berlandier!* (♂ fl.); collected in expedition from Western Texas to El Paso, New Mexico, May—Oct. 1849, by *Charles Wright!* n. 423; Texas, *Trécul*, Oct. 1849, n. 1249, in woods by the sides of streams; Herb. Berlandierianum Texano-Mexicanum, n. 3030! (*D. mexicana*, Scheele MSS.), Ann. 1828; "Hill sides, Fort Inge to Escondido Creek, and near Eagle Pass, Western Texas, flowers in March, fruit ripe in August about 1 in. in diameter," *Torrey*.

118. DIOSPYROS MABACEA, F. Muell. Austral. Veg. in Intercolonial Exhibition Essays, 1866—67, p. 35 (1867).

D. foliis alternis, ovalibus, apice breviter acuminatis, basi cuneatis, chartaceis, costis exceptis subglabris, breviter petiolatis; floribus masculis 5—7-nis, dense cymosis, tetrameris, calyce campanulato, 4-fido, lobis deltoideis acutis, corollâ extus sericeâ, campanulatâ, profunde lobatâ, lobis ovalibus, staminibus 15—16, *glabris.*

Cargillia mabacea, F. Muell. Fragm. Phytogr. Austr. v. p. 162 (1866), Benth. Fl. Austral. IV. p. 287. n. 2 (1869).

Maba quadridentata, F. Muell. Fragm. *l. c.*

A tree, 20 feet high; young branches strigose-pubescent. Leaves oval or oblong, chartaceous, alternate, cuneate at base, narrowed or shortly and obtusely acuminate at apex, pubescent on midrib and principal veins beneath and on the depression of the midrib above, nearly glabrous on the rest of the leaf, somewhat shining beneath, of same dark green colour on both sides, 3—4 in. long by $1\frac{1}{8}$—$1\frac{2}{3}$ in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long, hispid-pubescent; lateral veins slightly depressed on upper surface of leaves.

♂. Flowers 5—7 together, in short axillary hairy cymes; peduncles very short; pedicels $\frac{1}{20}$—$\frac{1}{10}$ in. long; bracts ovate; flowers tetramerous. Calyx campanulate, dark green, puberulous outside, glabrous inside, 4-fid; lobes deltoid acute. Corolla pale silky outside, about twice the length of the calyx, deeply 4-lobed; lobes oval, emarginate, imbricated sinistrorsely in bud. Stamens 15!, 16, many in pairs and inserted at base of corolla, a few hypogynous, all glabrous, $\frac{1}{8}$ in. long; anthers narrowly lanceolate; filaments short. Ovary rudimentary, represented by a bunch of hairs.

♀. "Fruit a scarlet berry."

Australia, Tweed River, *C. Moore!*

119. DIOSPYROS PENTAMERA, Woolls and F. Muell. ex F. Muell. Austral. Veg. in Intercolonial Exhibition Essays, 1866—67, p. 35 (1867).

D. foliis alternis, ovali-lanceolatis, apice obtuse acuminatis, basi attenuatis, coriaceis, glabris, breviter petiolatis; floribus masculis 3—5-*nis, pubescentibus, brevissime cymosis, sæpius pentameris, calyce hemisphærico, lobis deltoideis, corollâ brevi, profunde* 5-*lobâ, staminibus* 15—20, *filamentis brevissimis glabris, antheris villosis, ovarii rudimento pubescente; floribus femineis* 1—3-*nis, fructibus solitariis, subsessilibus, globosis, apice excepto glabratis,* 1—4-*locularibus, seminum albumine non ruminato.*

Cargillia pentamera, Woolls et F. Muell. in F. Muell. Fragm. Phytogr. Austr. IV. p. 82 (1864), Benth. Fl. Austr. IV. p. 288. n. 4 (1869).

Maba pentamera, F. Muell. *l. c.* v. 163 (1866).

C. arborea, A. Cunn. MSS.

A large tree attaining 80—100 feet in height and 2—3 feet in diameter, glabrous except the young shoots and inflorescence, somewhat rigid in habit. Leaves alternate, lanceolate or oval, cuneate or attenuate at base and usually as much so at the apex, coriaceous, shining above, pale and yellowish at least beneath, $1\frac{1}{2}$—3 in. long by $\frac{1}{3}$—1 in. wide; petioles $\frac{1}{12}$—$\frac{1}{6}$ in. long; midrib slightly either raised or depressed above; net-veins numerous rather prominent above, rather inconspicuous beneath. Leaves usually marked beneath by some small dark glands arranged along 2 straight lines equally distant from the midrib.

♂. Flowers 3—5 together, $\frac{1}{7}$ in. long, on short silky drooping cymes which measure without the flowers about $\frac{3}{20}$—$\frac{1}{5}$ in. long; pedicels very short. Calyx hemispherical, pubescent outside, glabrous or shining inside, half the length of the flower, 5-fid, with deltoid lobes; rarely 4—6-fid. Corolla sub-globose, open at the mouth, pale and shortly pubescent outside, glabrous inside, deeply 5-lobed, imbricated sinistrorsely in bud; lobes oval. Stamens 15—20 ex Bentham *l. c.*, usually 20 in pairs or in groups; anthers "tetragono-linear, rostellate, dehiscing laterally from apex downwards," silky; filaments very short, glabrous. Ovary rudimentary, hairy.

♀. Flowers subsessile, 1—3 together, usually solitary (?). Fruit solitary, subsessile, globose or spheroidal, about $\frac{1}{2}$ in. long, glabrate except the apex, tipped by the remains of the hairy style, 2—4-usually 4- or rarely 1-celled; cells 1-seeded; pericarp thin, crustaceous; dissepiments membranous; seeds $\frac{1}{3}$ in. long or rather more; albumen white, cartilaginous; embryo $\frac{1}{6}$ in. long; radicle clavate-cylindrical, slender, equalling the flat narrowly or linear-lanceolate cotyledons.

Fruiting calyx $\frac{1}{5}$ in. long, receiving the base of the fruit, puberulous or pubescent on both sides, 5- rarely 4-fid; lobes often somewhat spreading.

Australia, Moreton Bay, *Leichhardt;* Brisbane river, *A. Cunningham!;* New South Wales, *C. Moore!*, Paris Exhibition, Sydney woods 30, *W. Macarthur!* 49. Plentiful on the mountain brushes of the Hastings River, *C. Moore;* Clarence River, *C. Moore!*, *Beckler*, *J. Wilcox;* Queensland, *Hill!;* Ash Island, Hunter's River, *Mrs Forde* and *Miss Scott* ex W. Woolls, Contr. Fl. Austr. p. 192 (1867). Fruit eaten by the *Carpophaga magnifica*, Selby. Called *Black Myrtle* by the colonists. Timber soft when fresh, but exceedingly tough.

120. DIOSPYROS PARALEA, Steud. Nomencl. Bot. edit. ii. vol. I. p. 514 (1840).

D. foliis ovalibus, alternis, apice acuminatis, basi obtusis supra nitidis glabrescentibus, subtus costâ margineque tomentosis vel subglabrescentibus, coriaceis, breviter petiolatis; floribus masculis aggregatis, subsessilibus, ferrugineo-tomentosis, tetrameris, calyce 4-fido, corollâ campanulatâ, 4-fidâ, staminibus circiter 16, *antheris lineari-lanceolatis, hirsutis, filamentis brevibus glabris; floribus femineis subsolitariis vel aggregatis, subsessilibus, staminodiis* 8, *ovario tomentoso*, 8-*loculari; fructibus globosis, subglabratis, seminum albumine non ruminato.*

Alph. DC. Prodr. VIII. p. 224. n. 10 (1844), Miq. in Mart. Fl. Bras. VII. p. 6. t. 3 (1856).

Paralea guianensis, Aubl. Pl. Guyan. I. p. 576 (1775). *P. guyannensis*, Aubl. *l. c.* t. 231 (stam. char. et fig. excl.); *Paralia guianensis*, Desv. ex Hamilt. Prodr. Pl. Ind. Occ. p. 45. n. 89 (1825).

D. ferruginea, Spltgbr. in Vriese Ned. Kruidk. Arch. p. 327 (1848).

D. longifolia, Spruce in Journ. Proceed. Linn. Soc. Lond. V. p. 7 (1861), Pl. Bras. exsicc. n. 1516 (1851).

A small moderate-sized or lofty tree of hard white wood; young shoots buds and inflorescence ferruginous-tomentose. Leaves oval-oblong or ovate-oblong, alternate, more or less rounded occasionally somewhat narrowed at base, acuminate at apex, coriaceous, shining and glabrous or nearly so with depressed midrib above, pubescent puberulous or nearly glabrescent beneath especially on the marked midrib and recurved margins (hairs rufous in dried state; leaves bordered when young with white hairs which fall off, according to Aublet); lateral veins several, slender; leaves 3—8 in. long by 1—2½ in. wide; petioles ¼—⅗ in. long, glabrescent. Bracts rufous-hairy.

♂. Flowers in axillary subsessile clusters, $\frac{3}{10}$ in. long, rufous-hairy. Calyx $\frac{3}{20}$ in. long, acutely 4-fid; lobes deltoid. Corolla fleshy, campanulate or campanulate-oblong, sweet-scented, quadrangular, with a short inflated tube and 4 short lobes, glabrous inside (ferruginous on both sides ex Vriese), 4-fid. Stamens 16!, united by their filaments in 8 pairs (18 ex Aublet, about 13 ex Alph. DC., 8 ex Vriese), the inner ones the shorter; anthers linear-lanceolate, hairy at the back of the outer ones and at the front of the inner ones; filaments short, glabrous. Ovary rudimentary, ferruginous-hairy.

♀. Flowers (subsolitary ex DC.) few together in axillary subsessile clusters. Staminodes 8. Ovary 8-celled; cells 1-ovuled. Fruit solitary or 2 together, subsessile, shining and glabrate or with some persistent ferruginous hairs, globose, about 1 in. in diameter, pericarp at length splitting from apex, 3—4-seeded. Fruiting calyx with 4 lobes cordate at base, rufous-hairy especially on the undulating margins, on the centre of the back and inside, suberect or spreading, ⅔—1 in. across. Seeds oblong, ½ in. long; albumen not ruminated.

A decoction of the bark is said to be useful in case of fever in Guiana, where the plant is known by the name of *Parala.*

French Guiana, Cayenne, *Sagot!* n. 1253, *Martin!;* Guiana, *Mrs Parker!;* British Guiana, *Schomburgk!* 1492; Surinam, *Hostmann!* 547, *Splitgerber,* 541; S. Venezuela, near the rivers Casiquiari, Vasiva and Pacimoni, *Spruce!* 3159 ♂ flower (arbor gracilis 18-pedalis, ramulis longis pendulis. Flores flavo-virides. In ripis inundatis per totum Casiquiarem, necnon in Orinoco superiore, Nov. 1853); Brazil, by the south bank of the Rio Negro close to its junction with the Solimoes, *Spruce!* 1516, in fruit (small tree with subverticillate subsimple branches, fruit green, seeds immersed in flesh, *D. longifolia,* Spruce).

According to Mr Spruce, his *D. longifolia* has the branches arranged in whorls of five (very rarely three or four), while in *D. Paralea* the branches are alternate. The branches however in *D. Paralea* are sometimes verticillate.

121. Diospyros rhodocalyx, Kurz in Journ. Asiat. Bengal, Vol. xl. Part ii. p. 71. n. 91 (1871).

D. foliis oblongis vel ovali-oblongis, apice obtusis, basi angustatis, chartaceis, suprà glabris, lucidis, subtùs secus costam pubescentibus, breviter petiolatis; floribus masculis tetrameris, brevissime cymosis, calyce dense fulvo-pubescente, lobis oblongo-lanceolatis obtusiusculis, corollâ glabrâ, urceolatâ, staminibus circiter 16 *corollæ basi insertis; floribus femineis solitariis, staminodiis* 8—10, *ovario dense fulvo-tomentoso, 4-loculari* (?).

Flora, 1871, p. 332.

A small tree with young parts appressedly pubescent. Leaves oblong or oval-oblong, rarely obovate-oblong, retuse or rarely (on the same stock) obtusely apiculate, on slender and short petioles, acute or obtuse at base, chartaceous, of variable size 1—2 or 3—4 in. long, glabrous and shining above, for the most part slightly pubescent beneath on the midrib; veins conspicuous, net-veins lax. Flowers tetramerous, small, sessile or subsessile, axillary; bracts linear, densely fulvo-tomentose, short.

♂. Cymes very short, tomentose, calyx densely tawny-pubescent; lobes oblong, lanceolate, rather obtuse, corolla glabrous, scarcely $\frac{1}{6}$ in. long; tube inflated; lobes short, oblong. Stamens about 16, inserted at the base of the corolla; filaments short, bearded; anthers linear, acuminate. Ovary rudimentary.

♀. Flowers solitary, sessile or subsessile. Calyx larger than in the ♂; lobes widely oblong, obtuse, at the base with margin plicate-dilatated and tinged with red. Corolla $\frac{1}{4}$ in. long. Staminodes 8—10. Ovary oblong densely fulvo-tomentose, 4-celled (?).

Siam, Rádbúrí and Kánbúrí, *Teijsmann* 6000, 6007 in Herb. Bogor.

According to Kurz, somewhat resembling in general habit "*D. heterophylla*, Wall., and best placed near *D. tomentosa*," Poir. I have not seen a specimen.

122. DIOSPYROS MACROCARPA, sp. nov.

D. foliis alternis, oblongis, apice acutis vel subacuminatis, basi cuneatis, coriaceis, subtus pubescentibus, subglabrescentibus, breviter petiolatis; floribus masculis axillaribus, breviter cymosis, subsessilibus, pubescentibus, tetrameris, calyce campanulato, 4-fido, corollâ 4-fidâ campanulatâ, staminibus 16, *geminatis, filamentis dense pilosis; fructibus solitariis, subsessilibus, ovoideis, glabratis, seminum albumine non ruminato.*

Cargillia macrocarpa, Vieill. Hb.

Young parts shortly densely and softly pubescent. Branches dark, glabrescent. Leaves oblong, acute or slightly acuminate at apex, somewhat narrowed at base into petiole, alternate, coriaceous, dark-cinereous glabrous and shining above with slightly depressed midrib, paler appressedly pubescent and sub-glabrescent beneath with duller veins; margins more or less undulated; 2—5 in. long (besides petiole $\frac{1}{10}$—$\frac{1}{4}$ in. long) by $\frac{2}{3}$—$1\frac{3}{4}$ in. wide.

♂. Cymes axillary, short, bearing about 3—5 subsessile flowers, about equalling the petiole, softly pubescent, subferruginous; common peduncle in bud about $\frac{1}{10}$ in. long. Bracts minute. Flower-bud ovoid, about $\frac{1}{8}$ in. long, scarcely exceeding the calyx. Calyx campanulate, erect, 4-fid, with deltoid acute lobes; glabrous except near margin and shining inside. Corolla 4-fid, with rounded or mucronate sinistrorsely imbricated lobes; hairy outside, glabrous inside. Stamens 16 in 8 pairs, the pairs arranged in one row; filaments short, dilated and united in pairs at the base and (in young state) almost forming a short tube at base of corolla, densely setose-pilose especially the outer ones; anthers lanceolate-linear apiculate comparatively glabrous, but the outer ones surrounded by the dense long hairs of the filaments. Ovary rudimentary, small, pubescent, surmounted by 2 styles.

♀. Fruit solitary, subsessile, ovoid, glabrate, about $1\frac{1}{2}$ in. high by 1 in. thick or more, apparently 4-celled, fleshy, with rather thin pericarp. Bracts caducous. Fruiting calyx flat, 4-fid, $\frac{3}{8}$—$\frac{5}{8}$ in. across, shortly sub-pubescent outside, pubescent inside; lobes widely ovate, obtuse or mucronate. Seed rather more than 1 in. long, oblong, albumen not ruminated.

New Caledonia, Balade, Wagap, *Vieillard!* n. 890; *Pancher!* n. 251.

123. DIOSPYROS PERFORATA, sp. nov.

D. glabra, foliis ovali-oblongis, alternis, apice acuminatis, basi angustatis, firmiter membranaceis, perforato-punctatis, breviter petiolatis, nervis patentibus; floribus masculis aggregatis, subsessilibus, pubescentibus, campanulatis, calyce profunde 4-fido, corollâ urceolatâ (?), 4-fidâ, lobis latis, staminibus 16, *geminatis, receptaculo insertis, interioribus brevioribus, antheris hispidis, filamentis superne hispidis inferne glabris, ovario* 0, *receptaculo leviter hispido.*

Glabrous except the inflorescence and buds; branches cinereous, longitudinally wrinkled. Leaves alternate, oval-oblong, acuminate, narrowed at base, firmly membranous, scattered with small dark glands especially alongside the midrib beneath, in places perforated, 6—$7\frac{1}{2}$ in. long by $1\frac{3}{4}$—$2\frac{1}{4}$ in. wide, dark and shining above with depressed patent lateral veins, pale brown beneath with rather distinctly marked lateral veins; petioles channelled above, $\frac{1}{4}$ in. long.

♂. Flowers clustered, subsessile, few or several together on axillary or lateral nodules, tawny-pubescent, in bud about $\frac{1}{8}$ in. long; bracts small, imbricated, on very short pedicels, dark-cinereous; calyx deeply 4-fid, lobes deltoid, glabrous inside; corolla urceolate (?), 4-fid, glabrous inside, appressedly silky outside, contorted sinistrorsely in æstivation, 4-fid, slightly exceeding the calyx, lobes rounded; stamens 16, united by their filaments in 8 pairs, inner ones rather shorter, anthers longer than the filaments, with hairs on the back and front especially on the back of the outer ones and on the front of the inner ones, connective apiculate, filaments with spreading hairs above, glabrous beneath; ovary 0 or minute, represented by a few short hairs on the receptacle.

Ceram Island, Moluccas, *De Vriese!* 1857—61.

124. DIOSPYROS OBLONGA, Wall. List n. 4124 (1828—32).

D. foliis oblongis, alternis, apice breviter acuminatis, basi obtusis, subcoriaceis, glabris, petiolatis; floribus femineis 1—3-*nis, brevissime cymosis, confertis, pentameris, calyce profunde lobato, hispidis, lobis undulatis basi auriculatis, corollâ calycem æquante, carnosâ, ovario ferrugineo-pubescente,* 10-*loculari; fructibus subglobosis, subglabratis, seminum albumine non ruminato.*

Alph. DC. Prodr. VIII. p. 228. n. 26 (1844); G. Don, Gen. Syst. Gard. and Bot. IV. p. 40 (1837) excl. synon.

A tree or shrub; branches terete or warty, glabrous, puberulous at the extremities, dark. Leaves oblong, subcoriaceous, alternate, rounded near base, glabrous, shortly acuminate at apex, 5—$9\frac{1}{2}$ in. long by 2—4 in. wide, with petioles $\frac{1}{3}$—$\frac{2}{3}$ in. long; midrib strong and lateral veins numerous clear parallel and spreading, both depressed on upper surface.

♀. Flowers crowded, subsessile or in very short 1—3-flowered cymes, on short young shoots or axillary, pentamerous. Calyx covered on both sides with a mixture of black and ferruginous short hairs, $\frac{3}{10}$ in. long, deeply 5-lobed, rather crass; lobes erect-patent with wavy margins auricled at base. Corolla 5-lobed, ferruginous-hairy outside, glabrous inside, fleshy, not exceeding the calyx (in bud). Staminodes 5, glabrous. Ovary 10-celled, ferruginous-hairy; cells 1-ovuled. Fruit subglobose, nearly glabrate, $\frac{5}{8}$ in. long; surrounded at base by blackish hispid calyx $\frac{3}{4}$ in. across with appressed or somewhat spreading lobes having wavy margins and auricled pouting bases; albumen horny, not ruminated.

Penang, *Wallich!* n. 4124; Singapore, *Maingay!* n. 967.

125. DIOSPYROS EBENASTER, Retz. Obs. Bot., fasc. v., p. 31 (1789).

D. glabra, foliis alternis, ellipticis vel oblongis, apice plerisque obtusis, basi angustatis, firmiter membranaceis, nitidis, petiolatis; floribus 4—6-meris, pubescentibus, axillaribus, polygamis, pedunculis brevibus unifloris vel masculis plurifloris, calyce 4—6-fido, lobis ovatis margine revolutis, corollâ urceolatâ, apice lobatâ, staminibus 8—20, leviter pubescentibus, ovario pubescente, 4—10-loculari; fructibus magnis edulibus, 4—10-spermis, albumine non ruminato.

Alph. DC. Prodr. VIII. p. 235. n. 64 (1844); non Spach; nec *D. Hebenaster*, Gaertn. Fruct. et Sem. Pl. II. p. 478. t. 179. f. 9 (1791).

D. digyna, Jacq. Hort. Schœnbr. vol. III. p. 35. t. 313 (1798); Alph. DC. *l.c.* p. 238. n. 80; non Hort.

D. revoluta, Poir. in Encycl. Méth. v. p. 435. n. 18 (1804); Alph. DC. *l.c.* p. 234. n. 60.

D. obtusifolia, Humb. et Bonpl. ex Willd. Sp. Pl. IV. p. 1112 (1805); Humb. Bonpl. et Kunth, Nov. Gen. et Sp. Pl. III. p. 253. t. 247 (1818); Alph. DC. *l.c.* p. 227. n. 24; non Bert.

D. Sapota, Roxb. Hort. Beng. p. 40 (1814), Fl. Ind. edit. 1832, vol. II. p. 535; Bot. Mag. LXIX. t. 3988 (1843); Alph. DC. *l.c.* p. 228. n. 25.

D. sapotanigra, DC. Ess. Prop. Med. Pl. p. 200 (1816).

D. edulis, Lodd. ex Sweet, Hort. Brit. p. 270 (1827); Alph. DC. *l.c.* p. 239. n. 90.

D. decandra, Boj. Hort. Maurit. p. 200 (1837), non Lour.

Sapota nigra, Blanco, Fl. Filip. p. 409 (1837).

D. membranacea, Alph. DC.! *l.c.* p. 227. n. 20 (1844).

D. nigra, Blanco, Fl. Filip. edit. ii. p. 211 (1845).

D. laurifolia, A. Rich. Fl. Cub. in Ramon de la Sagra, Hist. de Cuba, vol. XI. p. 86. t. 55 (1845—55), ex Walp. Ann. Bot. v. p. 480 (1858).

D. brasiliensis, Mart. Fl. Bras. VII. p. 5. t. 2. f. 2 (1856).

Hebenaster, Rumph. Amb. III. lib. IV. p. 13. t. 6 (1750).

Sapotte Negro, Sonnerat, Voy. à la Nouv. Guin. p. 45. tt. 14—16 (1776).

A tall shrub or even lofty tree, quite glabrous except the inflorescence; branches dark. Leaves alternate, elliptical or oblong, usually obtuse at apex, somewhat narrowed at base, firmly membranous, shining, evergreen, 3—12 in. long by $1\frac{1}{2}$—$3\frac{1}{2}$ in. wide; midrib depressed above; net-veins not conspicuous; petioles ranging up to $\frac{3}{4}$ in. long. Flowers polygamous, 4—6-merous, $\frac{1}{2}$—1 in. long, pubescent; peduncles axillary, pubescent, solitary, those producing male flowers with several flowers, those with hermaphrodite or female flowers 1-flowered, $\frac{1}{5}$—$\frac{1}{3}$ in. long, bracteate. Calyx ample, $\frac{1}{4}$—$\frac{1}{2}$ in. long, somewhat hairy on both sides, 4—6-fid, lobes ovate, with revolute margins and sinuses. Corolla urceolate, twice the length of the calyx, yellowish white or greenish, thick and fleshy, 4—6-lobed at apex, silky or nearly glabrescent. Stamens 8—20, slightly hairy, often some or all in pairs; filaments somewhat pilose. Styles 2—5; ovary pubescent, 4—10-celled. Fruit globose, $1\frac{1}{2}$—4 in. in diameter, glabrous, shining, of olive yellowish-green colour when ripe, filled with a dark soft and paste-like pulp, edible; towards the centre of the pulp are 4—10 cells, each containing a large oval compressed seed; albumen cartilaginous, not ruminated. Fruiting calyx spreading, much thickened in middle, 5—6-fid, 1—$1\frac{1}{4}$ in. in diameter or less, puberulous on both sides, lobes undulated. Fruiting peduncles about $\frac{1}{3}$ in. long.

Local names, *Faux Magostan* in Mauritius, *Lolin* in Amboina, *Sapotte negro,* &c.

Philippine Islands, *Sonnerat, Blanco,* flowers in July; Celebes, *Jacquin*; Amboina, *Rumf.* Cultivated in Mauritius, at Calcutta, and Malacca, *Maingay!* 975; introduced into England and France &c., where it requires a hot-house for protection. Occurs also in cultivated places in tropical America, perhaps introduced; Mexico, Orizaba, *Botteri!* 909; Vera Cruz, *Galeotti!* 4609 (2000 ft. alt.); Cuernavaca, *Humboldt* and *Bonpland!* 3984 (5000 ft. alt.); Lizardo, *Wawra!* 249; Miradon, *Wawra!* 1029; Brazil, Rio Janeiro, *Schott and Pohl!* 4568; Cuba, *Richard;* Montserrat, *Ryan!* ex Hb. Vahl.

Blanco *loc. cit.* states that the tree in the Philippine Islands grows to a height of 24—30 feet and is carefully cultivated as well as indigenous. He says that the flesh of the fruit is blackish, and although it is eaten the taste is not well flavoured, that the leaves have caustic properties, and that the unripe fruit is reported to poison fish. An evergreen tree 30—50 ft. high with light even-grained wood grown at Cordova, Mexico, and called *Zapotillo,* probably belongs to this species; a specimen exists in the Kew Museum. The type of this species cannot be found in Retz' herbarium at Lund in Sweden.

126. DIOSPYROS SAMOËNSIS, A. Gray in Amer. Acad. v. p. 326 (1862).

D. foliis alternis, ovali- vel ovato-oblongis, apice obtuse angustatis, basi angustatis, coriaceis, glabris, petiolatis; floribus masculis 3—9-nis, tetrameris, pubescentibus, calyce campanulato, 4-fido, lobis obtusis, corollâ campanulatâ brevi, 4-fidâ, lobis obtusis; staminibus 8—10, glabris; floribus fœmineis solitariis, petiolatis, ovario hirsuto, 8-loculari; fructibus globosis glabris, calycis fructiferi aucti tubo concavo depresso-cupuliformi, intus margine elevato; seminum albumine non ruminato.

Branches glabrous or young ones scarcely puberulous. Leaves alternate, glabrous, oval or ovate-oblong, coriaceous, obtusely narrowed at apex, somewhat narrowed at base, 3—6 in. long by $1\frac{1}{2}$—3 in. wide; midrib depressed above; lateral and net-veins raised, slender; petioles $\frac{1}{4}$—$\frac{2}{5}$ in. long.

♂. Peduncles 3—9-flowered; flowers tetramerous, $\frac{1}{5}$—$\frac{1}{4}$ in. long, ovoid in bud. Calyx campanulate, $\frac{1}{6}$ in. long, shortly puberulous, 4-fid; corolla silky outside, 4-fid; stamens 8—10, glabrous, unequal, some in pairs.

♀. Calyx-lobes rounded; calyx about equalling the corolla; peduncles solitary, $\frac{1}{4}$—$\frac{1}{2}$ in. long, puberulous, 1-flowered, equalling the flower; ovary hairy, 8-celled; fruit globose, $\frac{3}{4}$—$1\frac{1}{2}$ in. in diameter, glabrous; fruiting calyx-tube flat or cupuliform with a raised border receiving the base of the fruit, and with 4 obtuse spreading or recurved lobes, glabrous, about $\frac{3}{4}$ in. wide; seeds $\frac{2}{5}$—$\frac{3}{5}$ in. long, closely packed together; albumen not ruminated, white.

Navigators' Islands, South Pacific Exploring Expedition !; Friendly Islands, *W. H. Harvey !,* (caustic berry for burning ringworms, &c.) "Tutuna." The foliage and fruiting calyx resemble *D. Ebenum,* Kœnig, but the plant is of a paler green colour and the flowers shorter.

According to the Rev. Thomas Powell in Seemann's Journal of Botany, vol. VI. p. 281 (1868), the wood of this large tree is hard and used for axe-handles and spear-points; the fruit is used for poisoning fish; and the secretion of the fruit is a vesicatory and turns the human skin black. Also the Samoan children are said to insert the midrib of the cocoa-nut leaflet into the fruit and apply the liquid thus obtained to their arms to produce blisters and eventually permanent prominences which they consider an ornament. Mr Powell describes the flowers as hermaphrodite.

127. DIOSPYROS OLEN, sp. nov.

D. glabra, foliis alternis, ovalibus, apice breviter obtuseque acuminatis, basi cuneatis, coriaceis, nitidis, utrinque delicate reticulatis, breviter petiolatis; floribus femineis solitariis, breviter pedunculatis, axillaribus, bracteatis, 4- rarius 3-meris, calyce subglabro profunde lobato, lobis late ovatis acuminatis basi cordatis, tubo intus margine elevato, corollâ 4—3-fidâ, lobis acutis, staminodiis 0—6, glabris, ovario superne pubescente, inferne glabro, 8-loculari.

Dark-cinereous, and except the inflorescence glabrous. Leaves oval, alternate, coriaceous, narrowed at base and usually with a short obtuse acumen at apex, of the same cinereous colour on both sides, somewhat shining on upper surface, with midrib depressed on upper side and net-veins delicately raised on both sides, $2\frac{1}{2}$—4 in. long by 1—$1\frac{1}{3}$ in. wide; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long.

♀. Peduncles solitary, 1-flowered, axillary, $\frac{1}{6}$—$\frac{1}{4}$ in. long, patent, pubescent, bearing caducous patent lanceolate alternate bracts. Flowers usually tetramerous, rarely trimerous. Calyx about 1 in. across when spreading, deeply lobed, very nearly glabrous; lobes widely ovate acuminate, widened and cordate at base, $\frac{1}{4}$—$\frac{1}{2}$ in. long, spreading or reflexed, with numerous parallel slight longitudinal veins; tube with raised internal hairy border at top. Corolla about $\frac{1}{3}$ in. high, glabrous in upper part, puberulous at least in places beneath, 3—4-fid, conical above in bud; lobes acute arching in flower outwards; tube urceolate in flower. Staminodes inserted on corolla, glabrous, about 6 (in a trimerous flower) or wanting (in a tetramerous one). Ovary glabrous beneath, pubescent and suddenly narrowed towards apex; terminating in a short 4-fid style glabrous at apex; ovary 8-celled; cells 1-ovuled.

Indigenous name *Olèn.* I. Lifu, *Deplanche!* No. 31.

128. DIOSPYROS CARGILLIA, F. Muell. Austral. Veg. in Intercolonial Exhibition Essays, 1866—67, p. 35 (1867).

D. foliis alternis, ovalibus vel oblongis, apice obtusis, basi cuneatis, coriaceis, glabris, pallidis, breviter petiolatis, nervis subtus inconspicuis; floribus masculis breviter cymosis, pubescentibus, tetrameris, axillaribus, campanulatis; calyce 4-fido, lobis deltoideis, corollæ lobis obtusis, staminibus sæpius 16, glabris, geminatis, corollæ basi insertis; floribus femineis 1—3-nis, brevissime cymosis, axillaribus tetrameris; staminodiis 8, uni-serialibus, glabris, corollæ basi insertis; ovario ovoideo, pubescente, 4-loculari, loculis bi-ovulatis; fructibus glabratis, ovoideis vel globosis, 1-spermis, albumine non ruminato.

Annona microcarpa, Jacq. Fragm. Bot. p. 40. t. 44. f. 7 (1800—1809).

Monodora microcarpa, Dunal, Monogr. Anon. p. 80 (1817). Cfr. Brown in Tuckey, Congo, p. 475 (1818).

Cargillia australis, R. Br. Prodr. Fl. Nov. Holl. et Ins. Van-Diem. p. 527. n. 2 (1810); Alph. DC. Prodr. VIII. p. 243. n. 1 (1844); Bot. Mag. t. 3274 (1833); Ettingsh. Blatt-skel. Dikot. p. 90. t. 35. f. 6 (1861); Benth. Fl. Austral. IV. p. 288. n. 3 (1869); F. Muell. Fragm. IV. p. 82 (1864).

Maba Cargillia, F. Muell. Fragm. V. p. 162 (1866).

This species is cited on pages 30, 31, 36, 46, 54 by the name of *Diospyros australis.*

A large shrub or tree 20—40 or even 100 feet high, glabrous, or the young parts and inflorescence with short hairs; trunk sometimes 2 feet in diameter. Leaves oblong or oval, alternate, coriaceous, obtuse at apex, more or less narrowed at base, palish green especially beneath, $1\frac{1}{2}$—$4\frac{1}{2}$ in. long by $\frac{2}{3}$—$1\frac{1}{2}$ in. wide including petiole $\frac{1}{6}$—$\frac{1}{4}$ in. long; midrib flattish depressed above; lateral and net-veins in relief and not conspicuous; frequently with small black spots arranged in a row on each side of midrib beneath. Flowers diœcious, tetramerous (or rarely trimerous?).

♂. Flowers several together arranged on short axillary pubescent drooping cymes which without the flowers measure $\frac{1}{5}$—$\frac{1}{3}$ in. long; calyx $\frac{1}{10}$ in. high, covered with pale appressed short hispid hairs, shortly lobed, campanulate, with deltoid lobes; corolla $\frac{1}{4}$ in. long, deeply lobed, covered outside with pale short hairs, glabrous inside, ovoid in bud, campanulate in open flower; lobes erect or recurved at apex, obtuse; stamens 12—16, usually 16, in pairs, glabrous, inserted at base of corolla; anthers longer than the filaments, lanceolate linear, dehiscing by lateral slits near apex; ovary rudimentary, hairy.

♀. Flowers 1—3 together, $\frac{1}{3}$ in. long, on very short cymes, campanulate, pubescent; calyx $\frac{2}{11}$ in. high by $\frac{1}{3}$ in. thick, 4-fid; corolla deeply 4-lobed, lobes obtuse; staminodes 8, in one row, inserted at base of corolla, glabrous, with lateral slits; ovary ovoid, hairy, 4-celled; cells often with 2 ovules, without any trace of a dissepiment between them, alternate with the calycine lobes; style hairy, 2-lobed at apex; stigma 2-lobed and glabrous. Fruit globular or ovoid, $\frac{1}{2}$—$\frac{3}{5}$ in. thick, fuscous and glabrescent when ripe, edible, ultimately 1-celled and 1-seeded; albumen of seed not ruminated; fruiting calyx about $\frac{1}{3}$ in. high, cup-shaped, shortly puberulous or nearly glabrous. Fruit called *Grey plums.* Slender-growing tree, with elongated trunk and elegant rigid foliage. Wood close, very tough and firm.

In the forest regions towards the coast through New South Wales and Queensland. Australia, *Hügel!;* Queensland, Brisbane River, Moreton Bay, *F. Mueller;* Rockhampton, *Dallachy!;* Crocodile Creek, *Bowman;* New South Wales, Port Jackson to the Blue Mountains, *R. Brown!, F. Mueller!;* Berrima and Richmond River, *C. Moore;* Hastings and Mackay Rivers, *Beckler!;* Illawarra, *A. Cunningham!;* Sydney, *Bynoe!;* Sydney woods, Paris Exhibition No. 20, *M. Macarthur!;* New South Wales, Kiama, *W. H. Harvey!;* Cabramatta River, *W. Woolls.*

129. Diospyros Malacapai, Alph. DC. Prodr. viii. p. 237. n. 75 (1844).

D. foliis alternis, ovalibus, glandulis sparsis; floribus axillaribus, 1—3-nis, calyce 4-lobo, baccâ globosâ, 4-loculari, loculis 2-spermis.

A small tree having yellow wood, with some black spots; said to keep off bugs when fresh. Leaves alternate, oval, with some scattered glands especially at the end. Flowers axillary, 1—3 together; calyx 4-lobed; fruit baccate, globose, 4-celled; cells 2-ovuled.

Local name *Malacapai* (Blanco, Fl. Filipin. p. 302, 1837); Tagatog, Philippine Islands, *Blanco.*

130. Diospyros spinosa, sp. nov.

D. spinosa, foliis alternis, ovalibus, apice acuminatis vel obtusiusculis, basi rotundatis vel subcordatis, junioribus subtus pubescentibus; margine revolutis, breviter petiolatis; floribus *masculis brevissime cymosis, parvis, tetrameris, calyce hemisphærico, corollâ profunde 4-lobâ, staminibus 16, glabris, ovarii rudimento glabro.*

Dull, spinous; young parts and inflorescence ferruginously tomentose-pubescent; branches terete. Leaves alternate, coriaceous, oval, acuminate or pointed at apex, rounded or subcordate at base, with whitish loose hairs beneath when young, subglabrescent, dark green above, browner beneath, $1\frac{1}{2}$—3 in. long by $\frac{3}{4}$—$1\frac{1}{2}$ in. wide; margins recurved; petioles $\frac{1}{8}$ in. long, pubescent, terete.

♂. Inflorescence arranged in very short axillary cymes on the young branches; flowers $\frac{1}{12}$ in. long (in bud), subglobose, tetramerous. Calyx about half the length of the flower, hemispherical, appressedly hairy outside, glabrous inside, deeply 4-fid; lobes rounded, sinistrorsely contorted in bud. Corolla subglobose, glabrous except 4 hairy lines outside along middle of lobes, deeply 4-lobed; lobes rounded, sinistrorsely contorted in æstivation. Stamens 16, glabrous, subequal (?), in two rows (?), distinct, inserted at or near base of corolla; anthers lanceolate, acute, longer than the filaments, dehiscing laterally from the apex; ovary rudimentary, glabrous.

Brazil, *Martius!* Herb. Reg. Monac., Ebenaceæ n. 144.

131. Diospyros ovalis, sp. nov.

D. fruticosa, foliis alternis, ovalibus, utrinque rotundatis, apice mucronatis, basi subcordatis, suprà glabris nitidis, subtus villosis, subcoriaceis, breviter petiolatis; floribus masculis breviter cymosis, tetrameris, profunde lobatis, corollæ lobis obovatis patentibus, staminibus circiter 20, glabris.

A shrub, 2—3 feet high. Young parts underside of leaves and inflorescence and especially the buds subferruginous-pubescent. Branches terete, glabrescent and nitescent, numerous. Leaves (of the shoots of the current season) oval, alternate, subcoriaceous, mucronate at apex, subcordate at base, dark shining and glabrous above except depression of midrib, without conspicuous veins, shaggy underneath with ciliated margins, about 1 in. long by $\frac{1}{3}$—$\frac{2}{3}$ in. wide; petioles about $\frac{1}{20}$ in. long, pubescent.

♂. Inflorescence at the base of the shoots of the current season, cymose, with few or several flowers rather loosely arranged; cymes (excluding the flowers) $\frac{1}{2}$—$\frac{2}{3}$ in. long; pedicels about $\frac{1}{4}$ in. long; bracts oval, densely pubescent. Flowers $\frac{9}{20}$ in. long, tetramerous, green. Calyx $\frac{1}{5}$ in. long, partite with lanceolate erect-patent lobes, pubescent on both sides. Corolla $\frac{2}{5}$ in. high ($\frac{1}{2}$ in. long when straightened), glabrous except 4 lines of hairs outside; lobes $\frac{3}{10}$ in. deep, obovate, erect-patent and recurved at apex. Stamens 20 (18—20 ex Benth. MS. in Hb. Cantab.), equal, glabrous; anthers linear; filaments short, united almost in a short tube. Ovary rudimentary, glabrous (?), minute.

Brazil, Pernambuco, sandy open places, Rio Preto, September. *Gardner!* 2813.

132. DIOSPYROS HISPIDA, Alph. DC. Prodr. VIII. p. 236. n. 68 (1844).

D. foliis alternis, ovalibus vel ovali-oblongis, apice cuspidatis vel acuminatis, basi sæpius obtusis, subcoriaceis, subtus ferrugineo-hispidis, breviter petiolatis; floribus masculis 2—4-nis, breviter cymosis, 4-meris, calyce hispido, 4-partito, lobis lanceolatis, corollâ profunde lobatâ, lobis oblongis, staminibus 18—24, subæqualibus, glabris, ovarii rudimento pubescente; floribus femineis 4—5-meris; fructibus solitariis, globosis, dense ferrugineo-hispidis, carnosis, 8-locularibus; calyce fructifero 4—5-partito, patente, lobis lanceolatis.

Miq. in Mart. Fl. Bras. VII. (Eben.) p. 4. n. 2 (1856).

An arborescent shrub or tree, 10—30 feet high, with shoots underside of leaves and inflorescence ferruginous-hispid; branches spreading. Leaves oval or oval-oblong, cuspidate or acuminate, usually obtuse at base sometimes narrowed or in ♂ subcordate, subcoriaceous, alternate, 2—5 in. long by 1—2¼ in. wide, darker and pubescent-velutinous above; petioles $\frac{1}{5}$—$\frac{2}{5}$ in. long, hairy.

♂. Flowers $\frac{3}{5}$ in. long, in 2—4-flowered distant or usually contiguous cymes ($\frac{1}{4}$—$\frac{3}{4}$ in. long); pedicels $\frac{1}{10}$—$\frac{1}{5}$ in. long. Calyx $\frac{1}{5}$—$\frac{1}{2}$ in. long, ferruginous-hispid on both sides, 4-partite, with lanceolate lobes. Corolla green, deeply 4-lobed, pubescent along longitudinal stripes; lobes oblong, somewhat narrowed at apex. Stamens 18—24, subequal, some or all in pairs, glabrous; anthers linear; filaments short. Ovary rudimentary, globose, hairy.

♀. Flowers few together, in short cymes, tetramerous or pentamerous. Fruit solitary, on pedicels $\frac{1}{10}$—$\frac{1}{5}$ in. long, densely ferruginous-hispid, globose, pointed at apex, about 1 in. in diameter, fleshy, 8-celled, 8-seeded. Fruiting calyx 4—5-partite, spreading, 1½ in. across; lobes lanceolate. Seeds 8, oblong, compressed, ¼ in. long.

Brazil, between Goiavéira and Corrego de Jeraguá, *Burchell!* 7437, ♂ fl. Aug., tree 20—30 ft. high, corolla green; between Córrego-fúndo and Pórto-Reál, *Burchell!* 8396, in young fruit, November, tree 20 ft. high; Gozáz, 10 ft. high, *Burchell!* 6994; Minas Geraes, *Claussen!* 478.

133. DIOSPYROS GOUDOTII, sp. nov.

D. foliis alternis, ovato-oblongis, apice acuminatis, basi subcordatis, subsessilibus, submembranaceis, suprà glabrescentibus, subtus puberulis; fructibus globosis, solitariis, axillaribus, pedunculatis, papilloso-verrucosis, pilis aspersis, calyce patente, 5-lobo, non aucto.

Young parts tawny- or ferruginous-pubescent; shoots terete, puberulous, glabrescent. Leaves alternate, ovate-oblong, widest near the middle, submembranous, acuminate at apex, subcordate at base, subsessile, glabrescent and dark green above with conspicuously depressed veins, puberulous and reddish brown below at least on veins, 6—10 in. long by 2—4 in. wide; petioles $\frac{1}{8}$—$\frac{1}{5}$ in. long, ferruginous-pubescent. Fruit globose about 1 in. in diameter, scattered with pilose hairs, ferruginous-pilose at apex where is base of broken style; papillose-verrucose. Fruiting calyx not accrescent, hairy on both sides, spreading, $\frac{2}{3}$ in. across, with 5 ovate or lanceolate lobes $\frac{1}{6}$—¼ in. long. Fruiting peduncle $\frac{3}{8}$—½ in. long, ferruginous hispid-pubescent, thick, erect-patent, axillary, solitary, 1-fruited; bracts at base of peduncle, ovate, imbricated, caducous, ranging up to ¼ in. long,

New Granada, Muzo, *Goudot!* No. 3.

134. DIOSPYROS GAULTHERIÆFOLIA, Mart. Fl. Bras. VII. (Eben.) p. 5. n. 5. t. 2. f. 1 (1856).

D. foliis distichis, oblongis, apice obtusis, basi subcordatis, tenuiter coriaceis, subtus præsertim secus nervos ferrugineo-hispidis, breviter petiolatis, marginibus in sicco late reflexis; floribus masculis aggregatis, brevissime cymosis, 5-meris, calyce campanulato, 5-fido, corollâ profundè lobatâ, staminibus ∞ ∞ pilosis, floribus femineis subsessilibus aggregatis; fructibus solitariis vel binis, globosis, apice abrupte conicis, setosis, papilloso-verrucosis, albumine non ruminato.

A shrub or small tree 12—14 feet high; with rufous-hairy terete branches, spreading at 60°, glabrescent. Leaves oblong, distichous, obtusely lanceolate at apex, subcordate at base, thinly coriaceous, margins widely reflexed in the dry state; dark shining and glabrous except the midrib, with depressed veins above; ferruginous-hispid especially on the veins beneath; 2—5½ in. long by 1—2 in. wide; petioles $\frac{1}{10}$ in. long.

♂. Flowers clustered in axils of leaves; cymes short, with oblong bracts glabrous inside, $\frac{3}{11}$ in. long, pentamerous; calyx campanulate, ferruginous-hairy on both sides, $\frac{9}{20}$ in. long, lobes ovate-oblong, $\frac{1}{4}$ in. long; corolla glabrous outside except a few pilose hairs along 5 longitudinal lines outside, 5-sided in bud, deeply 5-lobed; stamens 45—75, anthers linear, slender, with long scattered ferruginous hairs, filaments short, combined at base and inserted at base of corolla or on the receptacle, nearly glabrous; ovary 0 or minute.

♀. Flowers in subsessile clusters. Fruit solitary or 2 together, globose but abruptly pointed at apex, with long ferruginous stiff hairs that easily rub off, papillose-verrucose, scarcely 1 in. long, pulpy. Fruiting peduncle hairy, $\frac{1}{5}$ in. long; testa thick; albumen not ruminated; fruiting calyx with (4 or) 5 deep lanceolate lobes hairy inside, spreading, nearly 1 in. across.

Brazil, Bahia, *Blanchet* 1886; common in sandy shrubby places near Maçeio, Alagoas, February, 1838, *Gardner!* 1412, in ♀ fl. and fr. The anthers in the figure quoted above are incorrectly drawn as glabrous except the apex.

135. DIOSPYROS SUBROTATA, sp. nov.

D. foliis distichis, ovalibus, apice sæpe acuminatis, basi subcordatis, tenuiter coriaceis, breviter petiolatis, costâ exceptâ glabrescentibus; floribus masculis axillaribus, cymosis, 5—6-meris, calyce aperte campanulato, corollâ partitâ, subrotatâ, lobis obtusis patentibus, staminibus circiter 20, antheris pilosis, linearibus; floribus femineis sub-6-nis, fructibus pubescentibus.

A shrub of 8 feet high, or a small tree of 18—30 feet; young parts with pale appressed pubescence, glabrescent except the midrib of leaves and inflorescence. Leaves oval- or ovate-oblong, subcordate at base, more or less acuminate at apex, thinly coriaceous, with midrib depressed on upper side, distichous, with margins slightly reflexed, 3—7 in. long by 1½—3 in. wide; petioles $\frac{1}{10}$—$\frac{1}{5}$ in. long.

♂. Inflorescence axillary, cymose, with several or numerous flowers and spreading pedicels, pubescent with short appressed hairs; cymes $\frac{1}{5}$—$\frac{2}{5}$ in. long; pedicels $\frac{1}{20}$—$\frac{3}{20}$ in. long; flowers pentamerous or hexamerous; calyx openly campanulate, with short deltoid lobes, with short appressed inconspicuous pubescence, $\frac{1}{10}$—$\frac{1}{5}$ in. long; corolla subrotate, nearly $\frac{1}{2}$ in. in diameter, with deep oval spreading convex lobes, $\frac{1}{6}$ in. long, with longitudinal stripes of appressed hair outside, glabrous inside, rather thick; stamens about 20; anthers pilose, linear; filaments consolidated, short, pistil 0.

♀. Flowers about 6 together in axillary cymes. Fruiting pedicels $\frac{1}{4}$—$\frac{1}{2}$ in. long or very short (sessile ex Burchell MSS.); fruit depresso-subrotund, 4-5-sided, yellow, shining, with scattered appressed short hairs, and nearly smooth skin, probably about 1 in. in diameter; fruiting calyx $\frac{3}{4}$ in. across with acute deltoid spreading lobes and short appressed hairs inside.

Brazil, at Pará, *Burchell!* 9923, 9952, ♂ fl. December; at Baião, *Burchell!* 9275. Fruit in June.

136. DIOSPYROS POLYANDRA, Spruce in Journ. Proc. Linn. Soc. Lond. v. p. 7 (1861).

D. foliis distichis, ovato-oblongis, apice acuminatis, basi subcordatis, tenuiter coriaceis, subtus pubescentibus, breviter petiolatis; floribus masculis axillaribus, cymosis, 4—7-*sæpius* 6-*meris, calyce hemisphærico extus fulvo-pubescente, lobis acutis, corollæ lobis profundis, patentibus, staminibus* 40—50; *antheris linearibus, pilosis, filamentis brevissimis, basi connatis.*

A tree 18—30 ft. high, with a trunk 9 in. in diameter, and branches arranged in subterminal whorls, long, subsimple, leafy throughout, tawny-hairy at the extremities. Leaves ovate-oblong, acuminate at apex, subcordate at base, $3\frac{1}{2}$—6 in. long by $1\frac{1}{2}$—3 in. wide, with petioles $\frac{1}{10}$—$\frac{1}{4}$ in. long, distichous, thinly coriaceous, with recurved edges, with scattered appressed pubescence, glabrescent above; veins depressed on upper surface of leaf.

♂. Inflorescence in axillary not very crowded cymes which without the flowers measure about $\frac{1}{2}$ in. long, densely hispid-pubescent, tawny; pedicels $\frac{1}{10}$—$\frac{3}{10}$ in. long; bracteoles deciduous; flowers 4—7- usually 6-merous, white, sweet-scented, about $\frac{1}{3}$ in. long and cylindric-conical in bud; calyx hemispherical, about $\frac{1}{7}$ in. long, with short acute lobes, glabrous inside, tawny-hairy outside; corolla with oval-oblong deep lobes spreading in flower, glabrous inside, with longitudinal stripes of hair outside; stamens 40—50; anthers linear, pilose; filaments very short, connate at base; pistil 0.

Brazil, south bank of Rio Negro at confluence with river Solimoes, *Spruce!* 1528; frequent on the banks of the Casiquiare, *Spruce!* 3166. ♂ flowers in May and sparingly in November. According to Mr Spruce, the branches are arranged in whorls of five (very rarely three or four).

137. DIOSPYROS COCCOLOBÆFOLIA, Mart. Fl. Bras. VII. (Eben.) p. 6. n. 7. tab. 1. fig 1 (1856).

D. foliis alternis, ovalibus, utrinque obtusis, discoloribus, tenuiter coriaceis, subglabris, petiolatis; floribus masculis breviter cymosis, axillaribus, calyce sæpius 4-*partito, lobis ovatis vel lanceolatis, patentibus, ciliatis, corollâ* 4—6-*partitâ, lobis oblongis patentibus, staminibus* 18—24, *plerisque geminatis, hirsutis; floribus femineis* 1—4-*nis,* 4-*meris, staminodiis* 4, *ovario ovoideo-conico, piloso,* 4-*loculari, loculis* 1-*ovulatis.*

A small or moderate-sized diœcious tree, glabrescent in most parts. Shoots and lower surface of leaves pubescent especially on veins and margins, sometimes glabrous. Leaves oval, thinly coriaceous, or thickly membranous, somewhat or scarcely contracted and sometimes oblique at base, rounded obtuse or emarginate at apex, with about 8 lateral veins on each side at about 50°—60° with midrib, alternate, 2—$4\frac{1}{2}$ in. long by 1—$3\frac{1}{4}$ in. wide; angular divergence $\frac{2}{5}$; net-veins pellucid in Gardner's specimen, not so in Martin's nor in Pohl's; bluish green above, browner beneath; hairs ferruginous; petioles $\frac{1}{5}$—$\frac{3}{10}$ in. long, somewhat decurrent, leaving large scars on the branch; bracts transversely oblong, glabrous inside.

♂. Inflorescence in axillary, tawny-pubescent, usually 3-flowered drooping cymes $\frac{1}{10}$—$\frac{1}{3}$ in. long. Flowers $\frac{1}{2}$ in. long, green. Calyx $\frac{1}{4}$—$\frac{2}{5}$ in. high with (3 or) 4 lobes, tawny-hairy outside; lobes ovate or lanceolate, ciliated, almost as deep as the calyx, erect-patent. Corolla glabrous, or with hairy lines on back, with 4—6 very deep oblong lobes much imbricated in the bud, erect-patent. Stamens 18—24, many or all united by their filaments in pairs, $\frac{1}{5}$—$\frac{1}{4}$ in. long, nearly equal, inserted at very base of corolla (hairy either on the anthers or filaments), contiguous; filaments short and with spreading hairs (not so in Gardner's specimen), anthers linear-oblong, glabrous (pilose at base in Gardner's specimen), $\frac{1}{7}$ in. long; pollen widely ellipsoidal. Ovary rudimentary, fulvo-sericeous, hemispherical, small; style 0.

♀. Inflorescence and outside of calyx fulvo-sericeous. Flowers axillary, solitary or 2—4 together; peduncles $\frac{1}{6}$—$\frac{1}{4}$ in. long, thick, solitary or 2 together, articulated to the branches. Calyx $\frac{1}{6}$ in. high, with 4 ovate-acute lobes. Corolla tubular, 4-fid, twice the height of the calyx, white, glabrous. Staminodes 4, inserted at the base of the corolla and alternate with its lobes, filiform, included, with rigid hairs at base, glabrous above. Ovary ovoid-conical, covered with shining erect hairs, continuous with 4 linear oblong truncate-obtuse stigmas, "apparently 4-celled" with 1 ovule in each cell.

A fruit, collected by Gardner from Brazil, where it is called *Marmaleiro*, and is said to be good to eat, probably belongs to this species; it is subglobose, rugose in the dry state, and nearly glabrous, but pointed and tawny-pubescent at apex, $\frac{4}{7}$ in. thick; the calyx is spreading, slightly pubescent, with 4 deep ovate-oblong lobes, about $\frac{1}{2}$ in. across.

Brazil, Serra de Araripe, *Gardner!* 1511 (♂ fl. October); in hot dry places near the river S. Francis in prov. Minas, *e.g.* near Salgado and in the desert towards Vâo do Paranau, ♀ flowers in August and September, *Martius!*; near Oliveira, *Pohl!* 455.

138. DIOSPYROS PEARCEI, sp. nov.

D. foliis alternis, ovato-oblongis, apice acuminatis, basi obtusis vel rotundatis, tenuiter coriaceis, subtus appresse pubescentibus, petiolatis; floribus masculis aggregatis, subsessilibus, sæpius pentameris, calyce campanulato, extus pubescente, 5-fido, lobis deltoideo-acutis, corollâ subrotatâ, lobis patentibus, staminibus circiter 30, receptaculo insertis, antheris linearibus, pilosis, filamentis brevibus basi connatis.

Young parts densely tawny-pubescent; an evergreen (?) tree, 15 ft. high. Leaves ovate-oblong, rounded or slightly narrowed at base, alternate, acuminate at apex, thinly coriaceous, dark green and glabrous above except the depressed midrib and veins, with scattered appressed pubescence beneath, 6—$7\frac{1}{2}$ in. long by $1\frac{2}{3}$—$2\frac{1}{4}$ in. wide: petiole $\frac{1}{4}$—$\frac{1}{2}$ in. long, pubescent.

♂. Flowers very numerous and crowded, subsessile, $\frac{1}{5}$—$\frac{1}{3}$ in. long, conical in bud, white, pentamerous or occasionally hexamerous. Calyx campanulate, $\frac{1}{4}$ in. long, pubescent, 5-fid; lobes deltoid-acute, glabrous inside. Corolla with hairy lines outside, twice the length of the calyx, deeply 5-lobed, subrotate, lobes spreading. Stamens about 30; anthers linear, pilose, with long terminal apiculus; filaments short, combined at base, inserted on receptacle; ovary 0.

S. America, Peru (?), Monterico, 3000—4000 ft. alt., rare, *Pearce!*

139. DIOSPYROS PERUVIANA, sp. nov.

D. foliis alternis, oblongis, apice acuminatis, basi subrotundis vel angustatis, coriaceis, subtus pubescentibus, petiolatis; floribus masculis aggregatis, cymosis, 5—6-meris, calyce campanulato, extus pubescente, 5—6-fido, lobis lanceolatis vel ovatis, corollâ profunde lobatâ, lobis rotundatis patentibus, staminibus 36—45, *pilosis; floribus femineis aggregatis, subsessilibus, fructibus subglobosis, papilloso-rugosis, setosis, calyce fructifero patente, non aucto.*

Young parts underside of leaves and inflorescence ferruginous-pubescent. Leaves alternate, more or less oblong, acuminate at apex, coriaceous, deep green, shining and glabrescent except the depressed veins above, pubescent beneath especially on the veins and recurved margins, 3—6 in. long by $1\frac{1}{5}$—2 in. wide; petioles $\frac{1}{7}$—$\frac{1}{3}$ in. long.

♂. Flowers cymose, several together, $\frac{1}{3}$—$\frac{2}{5}$ in. long, crowded; cymes (excluding the flowers) $\frac{1}{4}$—$\frac{2}{5}$ in. long; pedicels $\frac{1}{10}$—$\frac{1}{4}$ in. long; calyx campanulate $\frac{1}{6}$—$\frac{1}{4}$ in. long, densely pubescent outside slightly so inside, 5—6-fid, lobes ovate or lanceolate, acute; corolla deeply 5—6-lobed, $\frac{1}{3}$—$\frac{2}{5}$ in. long, lobes rounded, spreading widely in full flower, much imbricated sinistrorsely in bud, each with a longitudinal stripe of dense ferruginous silky hairs outside; stamens 36—45, appearing at the mouth of the open corolla, anthers linear, pilose, filaments glabrous or nearly so, combined at the base; ovary wanting.

Var. α. *Sprucei.* "A small tree, 15 feet high, not rarely pendulous at apex, with long subpinnate branches, 5 or occasionally 3 or 4 together." Leaves ovate-oblong, nearly rounded at base. ♂ flowers white, scentless, about $\frac{2}{5}$ in. long. Stamens about 45. ♀ flowers in subsessile clusters. Fruit sub-spheroidal, $\frac{3}{4}$ in. thick, $\frac{6}{7}$ in. long, papillose-rugose, covered with ferruginous setæ, with remains of 4 styles at apex, yellow, rather fleshy, fruiting calyx not accrescent, 7-fid, spreading, about $\frac{1}{2}$ in. across, bearing remains of calyx at base of fruit. Tarapoto, E. Peru, in young woods, ♂ fl. in January, 1856, fruit in October, 1855, *Spruce!* n. 4411.

Var. β. *ocanensis.* Leaves oblong-lanceolate, narrowed at base. ♂ flowers greenish, dashed with rose-colour, about $\frac{1}{3}$ in. long. Stamens 36. New Granada, Ocaña, 3500 ft. alt., flowers in June, *Schlim!* n. 698. Perhaps a distinct species.

140. DIOSPYROS WEDDELII, sp. nov.

D. foliis alternis, oblongis, apice obtusè acuminatis, basi cuneatis, coriaceis, glabris, breviter petiolatis, nervis inconspicuis; cymis femineis puberulis, paucifloris; fructibus globosis, verrucosis, breviter pubescentibus; seminum albumine non ruminato; calyce fructifero parvo, patente, 5-*fido, utrinque puberulo, lobis ovato-deltoideis.*

Branches terete, young shoots puberulous, quickly glabrescent; bark of older branches pale. Leaves alternate, oblong, obtusely acuminate at apex, alternate at base, coriaceous, glabrous, undulated in the dry state, $1\frac{1}{2}$—5 in. long by $\frac{3}{4}$—$1\frac{1}{2}$ in. wide; petioles $\frac{1}{8}$—$\frac{1}{4}$ in. long, veins inconspicuous.

♀. Cymes axillary, puberulous, few-flowered, $\frac{1}{4}$ in. long. Fruit globular, verrucose, shortly pubescent between the rough points, $1\frac{1}{4}$ in. in diameter, tipped with remains of ferruginous-silky style. Albumen of seeds not ruminated. Fruiting calyx small, flat, 5-fid, puberulous on both sides, $\frac{1}{2}$ in. in diameter, lobes ovate-deltoid.

Brazil, near Rio de Janeiro, *Weddell!* 577.

141. DIOSPYROS GLOMERATA, Spruce in Journ. Proc. Linn. Soc. Lond. v. p. 7 (1861).

D. foliis alternis, ovato-oblongis, apice acutis acuminatis, basi rotundatis vel subcordatis, firmiter membranaceis, subtus pallidis apresse pubescentibus, breviter petiolatis; floribus masculis aggregatis, axillaribus, sessilibus, sericeis, 5—6-meris, calyce campanulato, corollâ profunde lobatâ, lobis patentibus, staminibus 26—33, *sericeis; fructibus immaturis subglobosis, sub*-10-*locularibus.*

A slender tree 20—30 feet high; branches 5 together arranged in 3 subterminal whorls, very long (12 feet), simple or rarely forked, leafy and flowering to the base; terminal buds narrowly conical, covered with dense short yellowish hair; young shoots puberulous with short brown curly-patent hairs, terete, glabrescent, dark, smooth. Leaves alternate, ovate-oblong, firmly membranous, usually rounded or subcordate at base, acuminate and acute at apex, 6—12 in. long by 2—$4\frac{1}{2}$ in. wide; dark green with few scattered weak pale hairs, glabrescent, and with depressed midrib above; pale and covered with appressed hairs and with raised and darker veins beneath; petioles $\frac{1}{5}$—$\frac{1}{4}$ in. long, patent, slightly bent upwards at point of attachment of leaf. Flowers sub-polygamous, pentamerous or hexamerous.

♂. Flowers numerous in crowded axillary sessile clusters, pale, silky, "white," scentless, about $\frac{1}{4}$ in. long, pentamerous or occasionally hexamerous; bracts rounded, imbricated, hairy. Calyx campanulate, 5—6-fid, with acute deltoid or ovate lobes, glabrous or nearly so inside. Corolla deeply 5—6-lobed; lobes oblong-obovate, glabrous inside, incurved near apex, erect-patent, distant upwards when in full flower, imbricated sinistrorsely in bud; stamens nearly equal, 26—33, clustered and more or less united at base, inserted at base of corolla or on receptacle; anthers linear, with long straight silky hairs on back and front; filaments short, glabrous. Ovary 0 or in subhermaphrodite flowers ovoid pubescent 10(?)-celled terminated at apex by 5-lobed style. Young fruit subglobose, about 10-celled.

N. W. Brazil, near Panurè by shady banks of Rio Uaupés, *Spruce!* 2701, November; *Martius!;* French Guiana, *Martin!*

142. DIOSPYROS CAPREÆFOLIA, Mart. MSS. in Herb.

D. foliis alternis, ovali- vel ovato-oblongis, apice acuminatis, basi angustatis, tenuiter coriaceis, subtus pallidis, subglabris, breviter petiolatis; floribus masculis subsessilibus, 4—5-*meris, calyce campanulato,* 4—5-*fido, corollâ subrotatâ, staminibus circiter* 45, *pilosis, corollæ basi insertis; floribus femineis solitariis, sessilibus,* 5-*meris, ovario dense hirsuto, stylis* 4 (?)

A tree 40 feet high; terminal buds small, rufous-hairy, lateral, often hard; young shoots with scattered rufous hairs, glabrescent; branches spreading at 40°—70°, terete, with a rather pale cuticle. Leaves oval or ovate-oblong, somewhat narrowed at base, acuminate at apex, thinly coriaceous, dark green shining and glabrous except depressed midrib and with depressed veins above, pale, subglabrous except the veins beneath, alternate, 2—3 in. long by $\frac{7}{10}$—$1\frac{3}{10}$ in. wide: petioles $\frac{1}{10}$—$\frac{1}{8}$ in. long.

♂. Flowers few together, in subsessile clusters, tetramerous or pentamerous; calyx $\frac{1}{5}$ in. long, with scattered appressed hairs, campanulate, felted within, 4—5-fid, lobes deltoid acute $\frac{1}{10}$ in. long; corolla $\frac{7}{10}$ in. long, glabrous except longitudinal stripes of brown hairs outside, subrotate, lobes oval, spreading, $\frac{1}{4}$ in. long; stamens 45 (in one pentamerous flower), inserted at base of corolla, anthers linear, with a few pilose erect hairs; filaments glabrous, combined at base; ovary rudimentary.

♀. Flowers solitary, sessile, pentamerous, bracteate at base. Calyx 5-fid, with deltoid lobes, hairy on both sides; corolla spreading, $1\frac{1}{2}$ in. across or more, glabrous outside; ovary densely hairy, subrufous. Styles 4 (?), glabrous, erect, exceeding the ovary.

Brazil, Cape Frio, Rio de Janeiro, *Sello* 1011*!*; Maranhão, *Don!*; Guinea, Surinam, *Martius!* 1678.

143. DIOSPYROS MANNII, sp. nov.

D. foliis alternis, ovali-oblongis, apice acuminatis, basi angustatis, firmiter membranaceis, subtus pallidis, subglabris nervis exceptis, breviter petiolatis; floribus masculis dense cymosis, axillaribus et secus ramos vetustos lateralibus, 5—6-meris, calyce profunde lobato, corollâ subrotatâ, staminibus 15—17, *subæqualibus, hispido-pilosis.*

A tree, with young shoots rufous-hispid or afterwards fuscous-hispid; older branches dark, glabrate, spreading at about 50°. Leaves oval-oblong, narrowed at base, acuminate at apex, alternate, firmly membranous, glabrous and with depressed veins above, glabrous (except a few isolated erect hairs) and paler on the lamina and with rufous hispid hairs on the raised midrib and lateral veins beneath, flat, 5—$7\frac{1}{2}$ in. long by $1\frac{1}{2}$—$2\frac{1}{3}$ in. wide; petioles fuscous, hispid, $\frac{1}{8}$—$\frac{3}{10}$ in. long.

♂. Inflorescence often on older branches, in several- or many-flowered dense short rufous-hispid cymes in the axils of present or fallen leaves; pedicels short; flowers $\frac{3}{5}$ in. long, pentamerous or hexamerous. Calyx ferruginous-hairy on both sides, $\frac{3}{10}$—$\frac{2}{5}$ in. long, deeply 5—6-fid, with lanceolate somewhat spreading lobes. Corolla subrotate in full flower, ovoid-conical in bud, $\frac{1}{2}$ in. high, 5—6-partite, glabrous except patches of short pale hairs along exterior of lanceolate-oblong spreading lobes. Stamens 15—17, nearly equal, about $\frac{1}{3}$ in. long, appearing at open mouth of corolla, hispid-pilose, with pale ferruginous hairs, on short filaments, not in pairs. Ovary wanting, represented by a few hispid hairs.

West Equinoctial Africa, Gaboon River, ♂ fl. July, *Mann!* 924.

144. DIOSPYROS ARTANTHÆFOLIA, Mart. Fl. Bras. VII. (Eben.) p. 7 (1856).

D. foliis alternis, oblongis, apice cuspidato-acuminatis, basi rotundatis vel angustatis, crassiuscule membranaceis, subtus fusco-hirtis, pallentibus, petiolatis; floribus femineis axillaribus, solitariis vel binis, calyce 5-partito, hirtulo, baccis depresso-globosis, 8-locularibus, dense rufo-setosis; calycis fructiferi lobis obtusis deltoideis.

Sinuous branches petioles and underside of leaves especially on the midrib and rather prominent veins villous with brown hairs. Leaves rather thickly membranous, oblong or ovate-oblong, 4—7 in. long by 2—4 in. wide, cuspidate-acuminate, rounded or contracted at the base, dark green, rather paler beneath, with 8—13 lateral veins on each side, alternate; petioles $\frac{1}{3}$ in. long; veins depressed above.

♀. Flowers axillary, solitary or 2 together, subsessile in fruit; calyx 5-partite, somewhat hairy; fruiting calyx divided beyond the middle; lobes triangular, rather obtuse, tawny-setulose especially in middle. Berry densely rufous-setose, 8-celled, depresso-globose, setæ shining.

S. America, N. Peru, Maynas, in woods, *Pöppig!* 2266.

145. DIOSPYROS PŒPPIGIANA, Alph. DC. Prodr. VIII. p. 224. n. 9 (1844).

D. foliis alternis, ovali-lanceolatis, apice obtuse acuminatis, basi cuneatis, tenuiter coriaceis, subtus appresse-pubescentibus, breviter petiolatis; floribus masculis breviter cymosis, fulvo-pubescentibus, calyce aperte campanulato, breviter 4—5-fido, corollâ tubulosâ, apice obtuse lobatâ, staminibus 12—20, subæqualibus, filamentis brevibus glabris, antheris hispidis; fructibus globosis, appresse papilloso-pubescentibus, calyce fructifero non aucto patente.

Miq. in Mart. Fl. Bras. VII. (Eben.) p. 4. n. 4 (1856).

A small bushy tree, rarely erect, 15—25 feet high; alternate branches and underside of leaves with scattered appressed hairs. Leaves oval- or oblong-lanceolate, obtusely acuminate or narrowed at apex, cuneate or abruptly narrowed at base, alternate, thinly coriaceous with very slightly reflexed margins, glabrescent above except the depression of midrib, 2—4 in. long by $\frac{5}{8}$—$1\frac{2}{3}$ in. wide; petioles $\frac{1}{7}$—$\frac{1}{4}$ in. long; lateral veins inconspicuous.

♂. Inflorescence tawny-pubescent, cymose, bearing few or several flowers, in short cymes which measure about $\frac{1}{4}$ in. long exclusive of the flowers; pedicels short, reflexed; flowers $\frac{2}{5}$ in. long, tetramerous or pentamerous; bracts ovate, acute, caducous; calyx $\frac{1}{10}$ in. high, openly and shortly campanulate, shortly 4—5-fid with acute lobes, dark, with pale pubescence outside, glabrous inside; corolla $\frac{7}{20}$ in. long, tubular, bright tawny-hairy outside, glabrous inside, shortly 4—5-lobed at apex, lobes obtuse; stamens 12—15 or 18—20, nearly equal; anthers hispid, linear, hypogynous; filaments short, glabrous, combined at base more or less in pairs; ovary small, rudimentary, with short inconspicuous hairs. Fruit globular, nearly 1 in. in diameter, shining but with scattered appressed short brown hairs especially at apex arising from papillose bases, 6—8-celled. Fruiting calyx $\frac{1}{2}$ in. in diameter, spreading but appressed to base of fruit, 4—5-lobed, not accrescent.

Brazil, Amazon, *Pöppig!* 2639; Povoaçaô dos Juris, *Martius!* n. 3053; Rio Negro, frequent on margin of Gapó from Barcellos upwards, Nov., *Spruce!* 1938; *St Hilaire!;* Rio Uaupés, Gapó, October, *Spruce!* 2635.

146. DIOSPYROS EMARGINATA, sp. nov. Plate IX.

D. foliis alternis, obovatis, apice retusis vel emarginatis, basi cuneatis, coriaceis, costâ exceptâ glabrescentibus, inconspicue reticulatis, breviter petiolatis; floribus masculis axillaribus, *conferto-cymosis, fulvo-hirsutis, calyce 4—5-fido, corollâ tubulosâ, apice 4—5-lobâ, staminibus 25—32, subæqualibus, filamentis brevibus, antheris hispidis; fructibus globosis, subglabris, calyce fructifero vix aucto.*

A tall straight tree, 90 feet high, with a trunk 2 feet thick; shoots with a few inconspicuous appressed hairs. Leaves obovate, alternate, retuse or emarginate at apex, cuneate at base, coriaceous, quite glabrescent except the midrib beneath and its depression above, with highly reticulated but inconspicuous veins; $1\frac{1}{2}$—3 in. long by $\frac{9}{10}$—$1\frac{1}{5}$ in. wide; petioles about $\frac{1}{8}$ in. long.

♂. Inflorescence axillary, tawny-hairy; cymes $\frac{1}{3}$—$\frac{1}{2}$ in. long, bearing several flowers on short pedicels; flowers $\frac{2}{5}$ in. long, tetramerous or pentamerous, drooping, tawny; calyx $\frac{1}{10}$ in. high, shortly and openly campanulate, 4—5-fid with sub-acute lobes, dark, with short scattered appressed hairs outside, glabrous inside; corolla tubular, with tawny-silky hairs outside, glabrous inside, 4—5-lobed at apex; stamens 25—32, nearly equal; anthers hispid, linear, filaments glabrous towards base, more or less combined at base in pairs or otherwise; ovary rudimentary, hairy.

♀. Fruit globular, about 1 in. in diameter, subglabrous but with a few scattered appressed short hairs. Fruiting calyx about $\frac{1}{2}$ in. in diameter, flat and appressed to base of fruit.

Brazil, Rio Negro, Gapó below Barcellos, November. Always within (and not on) the skirts of inundated forests, nearly related to *D. Pöppigiana*, Alph. DC. but less common, *Spruce!* 1913.

Plate IX. A branch in male flower, *natural size*. *a.* a piece of a branch with male flower abnormally thickened by an insect, *not magnified*. *b.* interior of male flower cut open, *magnified* 3 diameters. *c.* a stamen, *magnified* 10 diameters. *d.* a fruit, *natural size*.

147. DIOSPYROS RIGIDA, sp. nov.

D. foliis alternis, oblongis, basi rotundis, rigide coriaceis, supra glabris, subtus pallide subvelutinis, costâ robustâ, nervis inconspicuis, petiolatis; fructibus cymosis, depresso-globosis glanduloso-pulverulentis, ceterum glabris; calyce fructifero cyathiformi, fructum æquante, coriaceo, puberulo, profunde 4-lobo, lobis late ovatis erectis.

Shoots shortly fuscous-hispid, terete; leaves alternate, oblong or oval-oblong, rounded at base, rigidly coriaceous, glabrous above, pale beneath and covered with thin velutinous tomentum, 5—14 in. long by $1\frac{1}{2}$—$3\frac{5}{6}$ wide, midrib stout, slightly depressed on the upper side, net-veins not conspicuous; petioles stout, wrinkled, puberulous, $\frac{1}{2}$—$\frac{3}{4}$ in. long.

♀. Fruit about 3 together on the young branches, depresso-globose, 1 in. long, covered with reddish glandular pulverulence (as in *D. Embryopteris*), otherwise glabrous; peduncles $\frac{3}{4}$—$1\frac{1}{3}$ in. long, nigro-hispidulous, rigid; fruiting calyx cup-shaped, as high as the fruit, $1\frac{1}{3}$ in. in diameter, coriaceous, puberulous, deeply 4-lobed; lobes widely ovate, erect.

Borneo, *O. Beccari!* n. 2285.

148. DIOSPYROS EMBRYOPTERIS, Pers. Synops. II. p. 624. n. 6 (1807).

D. foliis alternis, oblongis vel anguste ovalibus, apice sæpius acuminatis, basi obtusis, coriaceis vel submembranaceis, glabris, petiolatis, reticulatis; floribus masculis axillaribus, racemose cymosis, 3—7-nis, 4- rarius 5-meris, pubescentibus, flavescentibus, calyce patente, 4—5-fido, corollâ campanulatâ, lobis obtusis, staminibus 24—∞, pubescentibus, antheris linearibus, fila-

mentis brevissimis; floribus femineis 1—5-nis, subsessilibus vel cymosis, 4-meris, staminodiis 1—12, pubescentibus, ovario farinaceo-glanduloso, sæpius 8-loculari, stylis 4; fructibus globosis vel ellipsoideis.

Excl. syn. Lam., Bot. Reg. t. 499 (1820), Alph. DC. Prodr. VIII. p. 235. n. 65 (1844), Griff. Notulæ IV. p. 289 (1854), Thw. En. Ceyl. Pl. p. 178. n. 1 (1860), Bedd. Fl. Sylv. Madras t. 69 (1870), non Boj.

Embryopteris peregrina, Gaertn. Fruct. I. p. 145. t. 29. f. 2 (1788).

Garcinia malabarica, Desrouss. in Encycl. Méth. III. p. 701 (1789).

Embryopteris glutinifera, Roxb. Coromand. I. p. 49. t. 70 (1795); *E. globularia*, ex Miq. Fl. Ind. Bat. II. p. 1048. n. 16 (1856).

Diospyros glutinosa, Roxb. Hort. Bengal. p. 40 (1814); König ex Roxb. Fl. Ind., edit. 1832, II. p. 533.

Diospyros glutinifera, Wall. List n. 4123 B (1828—32).

Diospyros malabarica, Kosteletsky, Med. Pharmac. Flora (III.) p. 1099 (1834).

Embryopteris gelatinifera, G. Don, Gen. Syst. Gard. and Bot. IV. p. 41 (1837).

Diospyros citrifolia, Wall. ex. Alph. DC. *l. c.*

Embryopteris glutenifera, Wight, Ic. Pl. Ind. Or. Vol. III. pt. 2, p. 4, tt. 843, 844 (1843—47).

Diospyros melanoxylon, Hassk. Cat. Pl. Hort. Bot. Bogor. II. p. 159 (1844), Ettingsh. Blatt-Skel. Dikot. t. 41. f. 9 (1861), non Roxb.

A middle-sized or large evergreen tree, glabrous and shining except the buds inflorescence and fruit; there is however occasionally a slight puberulence upon the petioles, &c. Branches straight, spreading. Bark scaly. Leaves oblong or narrowly oval, alternate, usually rounded at base, sometimes subcordate or slightly narrowed, acute lanceolate acuminate or obtuse at apex, highly reticulated with veins in relief on both sides with the exception of the midrib which is depressed on the upper side, coriaceous, of a pale green colour, persistent, 3—12 in. long by $\frac{3}{4}$—$3\frac{1}{4}$ in. wide; petioles $\frac{1}{5}$—$\frac{4}{5}$ in. long, usually channelled above. Flowers yellowish-white, diœcious or polygamous.

♂. Cymes about 3—7-flowered, tawny- or fuliginous-pubescent or puberulous, $\frac{1}{4}$—$\frac{3}{4}$ in. long (excluding the flowers); flowers ovoid, $\frac{1}{5}$ in. long in bud, $\frac{2}{5}$ in. long when open, tetramerous or occasionally pentamerous; calyx $\frac{1}{4}$ in. long by $\frac{2}{5}$ in. wide, 4-fid, pubescent, lobes pubescent inside; corolla $\frac{1}{3}$ in. long, with pubescent patches of hair outside, glabrous inside, shortly cylindrical, lobes about $\frac{1}{10}$ in. long, spreading, imbricated sinistrorsely in bud; stamens indefinite, 24—64 or more, nearly equal, inserted on the receptacle or at base of corolla, anthers linear, more or less hairy on back and front, filaments very short, hairy; ovary 0 or rudimentary; receptacle hairy.

♀. Flowers 1—5 together, subsessile or cymose, tetramerous, larger than in the male plant, cymes ranging up to $\frac{3}{5}$ in. long, glabrescent or pubescent; bracts caducous; calyx deeply lobed, pubescent or glabrescent, lobes dilatate-subcordate at base, erect-patent, ovate, $\frac{1}{5}$—$\frac{3}{5}$ in. long; corolla about $\frac{1}{2}$ in. long, with short nearly erect lobes; staminodes 1—12, hairy (sometimes perhaps perfect stamens), inserted at base of corolla or partly hypogynous; ovary glabrous (normally), reddish-glandular, or with a basal ring of hairs (rarely hairy?), 8 (—10) -celled; styles 4, hairy at base, dilated and lobed at apex, spreading; fruit usually solitary, subsessile or pedunculate, globular or ovoid, often large ($1\frac{1}{2}$—2 in. long), glandular or glabrate, 6—8—10-celled and

-seeded, of a yellowish rusty colour, covered with a rubiginous mealiness; fruiting calyx deeply 4-lobed, puberulous or glabrate, as wide as or wider than the fruit, spreading more or less or erect, with lobes dilatate-subcordate at base, imbricated sinistrorsely.

An officinal preparation (*Extractum Diospyri* of the Pharmacopœia of India) is a valuable astringent obtained from the fruit of this species, and is useful in diarrhœa chronic dysentery and leucorrhœa and as a local application to bruises and sprains.

Of this variable species the following varieties may be noticed:—

β. atrata, Thw. *l. c.* Leaves thinly coriaceous; buds, peduncles and calyx fuliginous-pilose.

γ. nervosa, Thw. *l. c.* Veins on both sides of the coriaceous leaves very prominent; leaves rounded at the base. Buds, peduncles and calyx nigro-pilose. Fruiting calyx-lobes erect.

Local names. *Panitsjika-maram,* Reede, Hort. Malabar. pt. III. p. 45. t. 41 (1782). Malabarensibus; *Tembiri,* Brachmanis; *Fruita da Grude,* Lusitanis; *Lym-appel,* Belgis. *Tumika* of the Telingas, ex Roxb. Corom. *l. c.* *Mangostan-utan* of the Malays. *Tindooka,* the Sanscrit name, ex Roxb. Fl. Ind. *Gaub* in Bengal. *Kibaragma* or *Kledong* in Java. *Timberee-gass* in Ceylon. *Kūsi* in Banda, India. *Gusvakendhu* in Goomsur forests, Madras.

The fruit when unripe contains a large quantity of tannin, and when ripe is eaten but is not very palatable. The astringent viscid mucus of the fruit is used in Bengal for paying the bottom of boats, and an infusion is employed to steep fishing-nets in to make them more durable. It is also used for book-binding since it preserves the books from insects. Masts and yards of country vessels are made from this tree in Ceylon.

India, Silhet, *Wallich!* 4123; Quilon, Hurdwar, Amherst, Tavoy, *Wallich;* N. W. India, *Hb. Royle, M. P. Edgeworth!;* Bengal, Behar, *Hooker fil. and T. Thomson!* (Cult.?); Assam plains!; Upper Assam, *Jenkins!* 277; Ceylon, *Thwaites!* C. P. 1915, *Walker!, Hb. Wight!* 1711 *bis, Gardner!* 531 (β or γ); Canara, Mangalor, *Hohenacker!* 869; Siam, *Sir R. Schomburgk!* 115; Java, *Dr Horsfield!* Eben. 2, 7, 8; *Zollinger!* 3565; E. Doon, *Dr Brandis!*

Var. β. Ceylon, *Thwaites!* C. P. 2731; Mergui, *Griffith!* 3626, 3627; Tenasserim, *Packmann!*

Var. γ. Ceylon, *Thwaites!* C. P. 1910.

149. DIOSPYROS CORIACEA, sp. nov.

D. tota coriacea, glabrata; foliis alternis, oblongo-lanceolatis vel ovalibus, apice acuminatis, basi fere rotundatis vel breviter angustatis, petiolatis; floribus femineis solitariis vel raro binis breviter pedunculatis axillaribus, calyce lato, plicato, 4—3-fido, lobis obtusis, corollâ breviter semi-ellipsoideâ 4—3-fidâ, lobis rotundatis valde contortis, staminodiis 5 glabris, ovario minute granuloso-glanduloso, subgloboso, 8-loculari, stylo apice lobato, fructibus subglobosis lævibus, calyce fructifero ampliato longitudine fructus.

Shoots dark-cinereous, glabrate, terete; leaves alternate, oblong-lanceolate or oval, acuminate at apex, nearly rounded or somewhat narrowed at base, coriaceous, glabrate, moderately reticulated, 2—4 in. long by $\frac{1}{2}$—$1\frac{1}{2}$ in. wide; petioles $\frac{1}{6}$—$\frac{1}{3}$ in long.

♀. Flowers solitary or rarely 2 together, in upper axils, glabrous, coriaceous; peduncles $\frac{1}{8}$—$\frac{1}{6}$ in. long; calyx wide, plicate, $\frac{1}{2}$ in. wide, $\frac{1}{3}$ in. high, 4—3-fid, lobes obtuse; corolla shortly ovoid, as high as the calyx, $\frac{1}{5}$ in. wide, 4—3-fid, lobes rounded, much contorted;

staminodes 5 (in one case), glabrous; ovary subglobose, glabrous, covered with minute glandular pulverulence, 8-celled, cells 1-ovuled; style lobed at apex, glabrous; fruit subglobose, $\frac{2}{5}$ in. high, glabrous, smooth; fruiting calyx $\frac{2}{3}$ in. in diameter, widely plicate, about as high as the fruit.

Borneo, *O. Beccari!* n. 1422, 3455.

150. DIOSPYROS CRASSIFLORA, sp. nov.

D. foliis alternis, oblongis, apice anguste acuminatis, basi angustatis, glabris, unicoloribus, tenuiter coriaceis, patentibus, petiolatis, nervis inconspicuis; floribus masculis crassis, 1—3-*nis, brevissime cymosis, axillaribus, calyce depresso-hemisphærico,* 4—5-*fido, utrinque puberulo, lobis rotundatis, corollâ ellipsoideâ, carnosâ, apice* 4—6-*lobâ, staminibus* ∞ ∞, *subæqualibus, pluriserialibus, dorso pubescentibus, hypogynis, ovario minuto, hirsuto.*

A tall tree, nearly glabrous except the inflorescence; branches dark, terete. Leaves alternate, spreading, oblong, narrowly acuminate at apex, narrowed more or less at base, of same green colour on both sides, very thinly coriaceous, shining above with depressed midrib and inconspicuous veins, with clear lateral and delicate tertiary veins beneath, 7—8 in. long by 2—2$\frac{3}{8}$ in. wide; petioles $\frac{1}{3}$—$\frac{1}{2}$ in. long.

♂. Flowers $\frac{1}{2}$—$\frac{2}{3}$ in. long, 1—3 together, on very short, shortly pubescent axillary peduncles or cymes. Calyx depresso-hemispherical, toughly coriaceous, $\frac{1}{2}$ in. in diameter, shortly puberulous on both sides, 4—5-fid; lobes rounded. Corolla "fleshy, of a light pink colour and of the size and form of a pigeon's egg," shortly tomentose outside, nearly glabrous inside, 4—6-toothed at apex; teeth contorted sinistrorsely as regarded from within, $\frac{1}{12}$—$\frac{1}{8}$ in. deep, obtuse. Stamens very numerous, about $\frac{3}{8}$ in. long, inserted on the receptacle, subequal, in several rows; anthers linear, acute, 2-celled, somewhat hairy on the back; filaments very short. Ovary minute, hairy.

Female flower and fruit unknown.

West Tropical Africa, Old Calabar, *Rev. W. C. Thomson!*, 12 March, 1863.

151. DIOSPYROS DISCOLOR, Willd. Sp. Pl. IV. p. 1108 (1805).

D. foliis alternis, oblongis, apice acuminatis, basi rotundatis, coriaceis, supra nitidis glabris, subtus pallidis appresse pilosis vel glabrescentibus, petiolatis, nervis inconspicuis; floribus masculis in cymis brevibus trifloris secus ramulos juniores terminaliter confertis, sæpius tetrameris, sericeis, calycis lobis ovalibus, rotundatis, corollâ infundibuliformi profunde 4-*fidâ, staminibus* 24—28, *subæqualibus, glabris, geminatis; floribus femineis solitariis, axillaribus, sessilibus, staminodiis* 4, 5, 10, *glabris, corollæ basi insertis, ovario dense piloso,* 8-*loculari; fructibus subglobosis, carnosis, pilosis,* 4—6-*spermis, albumine non ruminato, calyce fructifero fructus basi appresso.*

Alph. DC. Prodr. VIII. p. 235. n. 66 (1844).

Cavanillea philippensis, Desrouss. in Lam. Encycl. III. p. 663 (1789).

C. Mabolo, Lam. Encycl. tab. 454 (1823).

D. Mabola, Roxb. Hort. Beng. p. 40 (1814), Lindl. Bot. Reg. t. 1139 (1828).

D. Embryopteris, Boj. Hort. Maurit. p. 200 (1837), non Pers.
Embryopteris discolor, G. Don. Gen. Syst. Gard. and Bot. IV. p. 41 (1837).
Diospyros Kaki, Blanco, Fl. Filip. edit. i. p. 302 (1837), non Linn. f.
D. Blancoi, Alph. DC. Prodr. VIII. p. 237. n. 74 (1844).
D. embriopteris, Blanco, Fl. Filip. edit. ii. p. 209 (1845).
D. melanida, Sieber!, Fl. Maurit. Suppl. n. 29; non Poir.

A tree of moderate size, 40 feet or more high; the trunk furnishes a hard compact ebony of an exceedingly deep black colour. Young shoots and inflorescence fulvo-sericeous. Leaves oblong, alternate, coriaceous, rounded at base, acuminate at apex, brown glabrous and shining above, pale and appressedly pilose beneath, with shining silvery hairs that penetrate the skin and cause it to itch, ultimately glabrescent, 5—8—12 in. long (including petiole $\frac{1}{4}$—$\frac{3}{4}$ in. long) by 2—3—4 in. wide, rigid; lateral veins delicate, inconspicuous; midrib depressed above, stout beneath, wrinkled when dry as well as the petioles and young shoots. Sometimes small glands are found on the under side of the leaves.

♂. Flowers about $\frac{7}{10}$ in. long, subsessile on short contiguous 3-flowered cymes, which are arranged in terminal or axillary racemes, sweet-scented, tetramerous or occasionally pentamerous. Bracteoles shortly deltoid, acute. Calyx turbinate-campanulate, coriaceous, wider than the corolla-tube, $\frac{2}{5}$ long, deeply lobed, lobes oval, rounded or mucronate; silky outside, glabrous inside. Corolla silky outside, glabrous inside, coriaceous, funnel-shaped; lobes rather longer than the tube, spreading, oval. Stamens glabrous, 24—28, in pairs, nearly equal, hypogynous or inserted at the base of the corolla-tube, erect, more or less united at their base; filaments shorter than the linear laterally dehiscing anthers; ovary hairy, rudimentary.

♀. Flowers solitary, axillary, bracteate at base, about $\frac{3}{4}$ in. long, subterminal-spicate, tetramerous or pentamerous, sessile. Calyx open, about $\frac{1}{2}$ in. high; lobes nearly $\frac{1}{2}$ in. long and wide, $\frac{1}{2}$-oval, coriaceous, cordate at base, appressedly silky outside, glabrous and shining inside, imbricated in various ways. Corolla $\frac{3}{5}$ in. long, shortly tubular, contracted about middle, silky outside except near base, glabrous inside; tube $\frac{2}{7}$ in. long, truncate-ovate, lobes about as long as the tube, spreading, $\frac{1}{2}$-oval, obtuse, margins incurved, imbricated sinistrorsely. Staminodes usually 4, occasionally 5 or even 10, much shorter than the corolla; filaments about as long as the barren (?) anthers; all glabrous, alternate with corolla-lobes; ovary very densely pilose, large, 8- or more-celled, fleshy, 8!-celled in specimen of Dr Maingay, depresso-conical, cells 1-ovuled; styles 4, distinct, hairy outside or glabrous, arched, converging at apex. Fruit thick, fleshy, globose or subglobose, densely hairy, reddish, like a quince, 4—6-seeded, with flesh rose-coloured, 3—4 in. in diameter, pulp white; hairs ferruginous; albumen cartilaginous, not ruminated; fruiting calyx flattish, appressed, rather more than 1 in. in diameter.

The wood is very hard, of a dark flesh colour, which in time becomes black like ebony. The fruit has an agreeable smell like a quince (but sometimes not so), and is edible after removing the hairs and skin. Local names *Mabolo* in Tâgalog, *Amaga* in Bisaya, *Talang* in Pampango, according to *Blanco, l. c.*

Philippine Islands, Manila, *Gaudichaud!; Blanco.* Cultivated in Mauritius (Hb. Kunth!) and in the Calcutta and Paris Gardens; also introduced at Mahé I. Seychelles, *Horne!* 345;

Guadalope, *Perottet!* cultivated (?); Malaya, Pulo Ticus, "Stem thin," *Dr Maingay!* 970/2; Borneo, *O. Beccari!* n. 1892, *Wallich!* 4131.

A form with leaves pale and having numerous inconspicuous veins on both sides, probably introduced, is found at Rio de Janeiro, Brazil, *Glaziou!* 1560, 1561.

152. DIOSPYROS ARGENTEA (*D. argenteus*), Griff. Not. IV. p. 288 (1854).

D. foliis alternis, oblongis, apice acuminatis, basi rotundatis vel cordatis, coriaceis, supra glabris, subtus dense argenteo-pilosis, breviter petiolatis; floribus masculis breviter cymosis, sæpius tetrameris, sericeis; calyce 4-fido, campanulato-cylindrico, lobis ovalibus; corollâ breviter tubulosâ, lobis ovalibus; staminibus 22—24, subæqualibus, hirsutis, geminatis, ovarii rudimento pubescente; floribus femineis solitariis, breviter pedunculatis, staminodiis 4—5, ovario dense hirsuto 4-loculari, loculis imperfecte divisis; fructibus ellipsoideis, strigoso-pilosis, 8-locularibus, seminibus 6—8, albumine non ruminato; calyce fructifero 4-partito, aucto; lobis oblongis.

Buds lanceolate-acuminate, with silvery silky hairs; branchlets somewhat compressed, covered as well as inflorescence and petioles with very brilliant silvery and silky hairs which at length become ferruginous-silvery. Leaves alternate, oblong, coriaceous, cordate or rounded at base, sharply acuminate at apex, glabrous above, densely velutinous-pilose beneath with silky shining silvery hairs, which afterwards become ferruginous-silvery and at length mostly fall off, leaving an appressed pubescence and the under surface of the leaf pale, 7—11 in. long by 2—3¼ in. wide; petioles ¼—⅓ in. long; margins reflexed; midrib stout, depressed above; lateral veins inconspicuous.

♂. Cymes axillary, spreading, near ends of branchlets, ⅓—⅔ in. long (exclusive of the flowers), bearing 3—∞ flowers; common peduncle ⅕—⅖ in. long; ultimate pedicels short; bracts ovate, glabrous inside. Flowers (closed) nearly ½ in. long, silky outside, usually tetramerous. Calyx $\frac{3}{10}$ in. long, campanulate-cylindrical, silky on both sides, 4- (in one case 3-) fid, lobes oval. Corolla ⅖ in. long, shortly tubular, 4-lobed, silky on both sides especially outside, lobes ⅛ in. deep, oval. Stamens 22—24, in pairs, nearly equal, very hairy, filaments much shorter than the anthers; ovary rudimentary, hairy.

♀. Flowers solitary, in axils of upper leaves; peduncles ⅕—¼ in. long. Calyx about ½ in. long, 4-fid, densely furred on both sides, campanulate; calyx-lobes ovate, apiculate. Corolla $\frac{7}{10}$ in. long, 4-fid, tomentose; lobes oval, apiculate, imbricated, hairy inside. Staminodes 4—5, alternate with corolla-lobes, hairy above; ovary globose, densely hairy, 4-celled; cells imperfectly divided; ovules 8; styles 4, hairy, erect, ⅕—⅓ in. Fruit with 1 oval bract at the base ⅖ in. long, $\frac{3}{10}$ in. wide, glabrous inside, egg-shaped, 2½—3 in. long by 1½—2 in. thick, very strigosely pilose, greenish-white or yellowish, shortly cuspidate at the apex, 8-celled. Fruiting calyx 4-partite, sometimes 3 in. wide; lobes very large, oblong, concave, obtuse, with metallic lustre, very silvery-silky outside, veined inside, 1½—2 in. long by ⅝ in. wide; seeds 6—8, subcylindrical, slightly attenuated at both ends; ranging up to 2 in. long by ⅝ in. wide, imbedded in pulp; albumen cartilaginous or horny, white; embryo $\frac{7}{12}$—¾ in. long; radicle thick, clavate, about equalling or shorter than the cotyledons.

Malacca, *Griffith!* 3625; *Maingay!* n. 968.

153. DIOSPYROS TOPOSIA, Hamilt. in Trans. Linn. Soc. Vol. XV. p. 115 (1827).

D. foliis alternis, oblongis ovatis vel lanceolatis, apice acuminatis, basi obtusis, coriaceis, glaberrimis, crebre reticulatis, petiolatis; floribus masculis axillaribus, cymosis, calyce initio clauso lobis connatis demum irregulariter apice rupto, corollâ urceolatâ, apice 4—5-lobâ, staminibus ∞, glabris; floribus femineis solitariis, staminodiis 12—16, ovario 4- (rarius 6-) loculari, fructibus subglobosis vel ellipsoideis, glanduloso-pubescentibus vel glabrescentibus, seminibus 1—4; calyce fructifero 3—4-lobo, pubescente.

Ettingsh. Blat.-Skel. Dikot. t. 42. f. 7 (1861); Bedd. Ic. Pl. Ind. Or. (Part. VII.) p. 25, t. 122 (1871); Alph. DC. Prodr. VIII. p. 237. n. 73 (1844).

D. racemosa, Roxb. Hort. Beng. p. 40 (1814); Fl. Ind., edit. 1832, vol. II. p. 536; Wight, Ic. t. 416.

D. lanceolata, Wall. List n. 4122 (1828—32), non Poir.

D. incisa, Hamilt. ex. Wall. *l. c.*

Embryopteris racemosa, G. Don, Gen. Syst. Gard. and Bot. IV. p. 41 (1837).

Called *Toposi* in Bengal, where it is cultivated on account of the fragrancy of the flowers; *Kahakaala-gass* in Ceylon, see Thw. Enum. Ceyl. Pl. p. 179. n. 4 (1860); *Goolul* in Silhet and Tipperah, see Roxb. Hort. Beng. p. 40 (1814).

A large or middle-sized tree with glabrous terete branches. Leaves alternate, oblong ovate or oval, acuminate at apex, obtusely narrowed or rounded at base, coriaceous, closely and clearly net-veined, with midrib depressed on upper surface, shining above, quite glabrous, 3—8 in. long by 1—3¾ in. wide; petioles ⅓—½ in. long. Foliage like *D. paniculata*, Dalz.

♂. Cymes axillary ¼—1 in. long, slightly hairy or glabrescent, usually 3-flowered, in cultivated specimens 3—12-flowered; flowers ⅓ in. long, yellow, pedicels shorter than the calyx; bracts caducous, at the top of peduncle: calyx at first closed in bud with connate lobes, afterwards irregularly broken from apex in unequal acute lobes, scattered with inconspicuous short setæ, about ⅕ in. high. Corolla urceolate, 4-lobed at apex, glabrous except a few short hairs outside along the middle lines of the lobes; Dr Hamilton states that the corolla is 5-lobed. Stamens numerous, indefinite, in one case 33, glabrous, mostly hypogynous; filaments very short; ovary rudimentary.

♀. Flowers solitary; fruiting peduncle ⅛—⅓ in. long, sometimes at base shortly adnate to the branch so as to become supra-axillary; bracts at top of peduncle, caducous. Calyx as in ♂. Corolla tubular-urceolate, 4-lobed at apex. Staminodes 12—16. Ovary 4- rarely 6-celled. Style 0, stigma 4-lobed. Fruit oblong or subglobose, ⅝—1 in. long, glandular and covered with short weak close tawny hairs or glabrescent. Fruiting calyx hairy, with 3—4 oblong or rounded lobes, ½—⅞ in. across, spreading; seeds 1—4, albumen cartilaginous, not ruminated but with very faint radiating striæ near the circumference.

East Bengal, *Griffith!* 3622; Ceylon, not uncommon in damp forests up to an elevation of 4000 feet, *Thwaites!* C.P. 1911, 2514, *Gardner!* 533; Silhet, *Roxburgh*, *Wallich!* 4122; ? Khasia, *Dr Hooker!* (part).

A specimen from Borneo, collected by *O. Beccari!* n. 3052, with leaves 5—11 in. long by 1¾—4 in. wide, and subglobose 4-celled 4-seeded fruit with deeply trifid calyx nearly 1 in. in diameter, probably belongs to this species.

THE FOLLOWING SPECIES OF DIOSPYROS ARE TOO IMPERFECTLY KNOWN TO BE PLACED IN THEIR POSITIONS IN THE SECTIONS.

154. DIOSPYROS GRATA, Wallich, List n. 4142 (1828—32).

D. foliis alternis, oblongis, utrinque angustatis, obtusis, glabris, floribus femineis solitariis, subsessilibus, ovario fulvo-hispido; fructibus globosis, subglabratis, calyce fructifero 5-fido, pentagono, utrinque pubescente.

Alph. DC. Prodr. VIII. p. 232. n. 48 (1844).

Branches nearly glabrous, pubescent at the extremities. Leaves alternate, glabrous, oblong, narrowed at both ends, obtusely acuminate at apex, 3—6 in. long by 1—2 in. wide; midrib depressed above; veins slender, crowded, not conspicuous; petioles $\frac{1}{5}$—$\frac{1}{6}$ in. long, glabrous.

♀. Fruit solitary, subsessile, globose, about 1 in. in diameter, glabrate or with remains of ferruginous hairs; fruiting calyx stellate, 5-fid and 5-cornered, hairy on both sides, tawny, $\frac{3}{4}$ in. across; peduncles very short, hairy.

Nepal, *Wallich!* Cfr. *D. lanceæfolia*, Roxb.

155. DIOSPYROS ORIXENSIS, Wight Hb.!, non Klein.

D. foliis alternis, ellipticis, apice obtuse angustatis vel breviter acuminatis, basi obtusis, glabrescentibus, tenuiter coriaceis, breviter petiolatis; fructibus solitariis axillaribus subglobosis, breviter pedunculatis; calyce fructifero profunde 4-fido, appresso vel leviter patente, extus piloso, lobis obtusis.

Young shoots petioles and peduncles hirsute, afterward puberulous, ultimately glabrous, terete; leaves alternate, thinly coriaceous, elliptical, obtuse at base, obtusely narrowed or shortly acuminate at apex, glabrescent, brown on both sides, midrib slightly depressed above and veins inconspicuously raised above, more manifest beneath, subnitescent, $1\frac{1}{2}$—$3\frac{1}{4}$ in. long by $\frac{2}{3}$—$1\frac{1}{2}$ in. wide; petioles $\frac{1}{8}$ in. long, strong.

♀. Fruit solitary, axillary, dark, subglobose, about $\frac{2}{3}$ in. in diameter, on peduncle about equalling the petiole; bracts caducous; fruiting calyx deeply 4-fid, appressed to base of fruit or somewhat spreading, $\frac{1}{3}$ in. across, subpilose outside; lobes obtuse; seeds 2—3, oblong, $\frac{1}{4}$ in. long.

Courtallum, *Hb. Wight!*

156. DIOSPYROS DODECANDRA, Loureiro Fl. Cochinch. p. 228. n. 5 (1790).

D. foliis alternis, late-lanceolatis; floribus axillaribus; corollæ tubo subgloboso, lobis 4, brevibus; staminibus 18, corollæ basi insertis; baccis compressis, lentiformibus, 8-spermis.

Alph. DC. Prodr. VIII. p. 238. n. 86 (1844).

Embryopteris Loureiriana, G. Don, Gen. Syst. Gard. and Bot. IV. p. 41 (1837).

A large tree with sub-patent branches. Leaves widely lanceolate, quite entire, alternate. Flowers hermaphrodite according to Loureiro, axillary, white; corolla 4-lobed, tube subglobose, large, lobes short; stamens 18, inserted at the base of the corolla. Fruit pallid, compressed, lentiform, 1-celled [?], 8-seeded, pulpy; pulp moderate, somewhat sweet, astringent, edible, not good-tasted; seeds compresso-ovate, bony, large.

Spontaneous and cultivated in Cochinchina, *Loureiro*. Local name, *Càу Thī trâm.* Wood like that of *D. decandra*, Lour., but without the very black veins in the heart; white and smooth and with dense fibres. Used in gardens to support black pepper plants.

157. DIOSPYROS (?) PILOSA, Alph. DC. Prodr. VIII. p. 219 (1844).

D. caule arboreo, foliis alternis, ovato-lanceolatis, subtus tomentosis, breviter petiolatis; floribus masculis racemosis, rubro-fuscis, calyce 5-lobo, lobis ovatis, corollâ 5-lobâ, tubo brevi, laciniis ovato-oblongis, crassis, patentibus, calyce sublongioribus, filamentis 15 *brevibus, antheris oblongis.*

Euclea pilosa, Loureiro, Fl. Cochinch. p. 629 (1790).

A large tree with ascending branches; diœcious. Wood fit for house-building. Leaves alternate, ovate-lanceolate, quite entire, tomentose beneath; petioles short. Flowers reddish-brown, "in terminal racemes."

♂. Calyx 5-lobed, lobes ovate, pilose on both sides; corolla 5-lobed, tube short, lobes ovate-oblong, crass, pilose, patent, rather longer than the calyx; filaments 15, short, anthers oblong, erect.

Cochinchina, *Loureiro*. Vernacular name *Cây Nhaoc.*

158. DIOSPYROS HASSELTII, Zoll. Obs. Bot. Nov. p. 15. n. 3 in Natuurk. Tydschr. Neerl. Ind. Vol. XIV. (1857).

D. foliis ovalibus, utrinque attenuatis, nitidis, glabris; floribus axillaribus, racemosis, racemis suberectis, calycis marginibus in axillis loborum deflexis, laciniis acutis, pedicellis subclavatis pilosis, corollæ (fem. ?) tubo 4-gono, pilis nigris præsertim ad angulos tecto, staminibus 8 [12], *iisdem quæ lobis corollæ alternant simplicibus longioribus, aliis brevioribus bicruris, stylis* 4, *bifidis; baccâ glabrâ,* 8-*loculari.*

Java. Described by Zollinger from a drawing of Kuhl and van Hasselt No. 2 b in the Buitenzorg botanical garden.

159. DIOSPYROS KUHLII, Zoll. Obs. Bot. Nov. p. 15. n. 1 in Natuurk. Tydschr. Neerl. Ind. Vol. XIV. (1857).

D. foliis oblongis, utrinque acuminatis, integris; floribus lateralibus axillaribus, pedicellis calycem æquantibus, staminibus 8 [12] *alternatim bicruris (antheris interioribus brevioribus) aliis simplicibus, stylis* 2 *bifidis, baccâ pilosâ.*

Java. Described by Zollinger from a drawing of Kuhl and van Hasselt No. 3 in the Buitenzorg botanical garden.

160. DIOSPYROS PENDULIFLORA, Zoll. Obs. Bot. Nov. p. 15. n. 2 in Natuurk. Tydschr. Neerl. Ind. Vol. XIV. (1857).

D. foliis oblongis, utrinque acutis, acuminatis; floribus masculis lateralibus pendulis, pedunculo bifido, pedicellis flores æquantibus, calyce nigro-piloso 4-lobo, corollâ apertâ, staminibus 8 [circiter 20 ?], filamentis brevibus pilosis, alternatim 2- vel 3-cruris; floribus femineis solitariis pendulis, corollæ lobis erectis, staminibus 12 sterilibus, baccâ pilosâ 5—8-loculari.

Java. Described by Zollinger from a drawing of Kuhl and van Hasselt No. 2 a in the Buitenzorg botanical garden.

161. DIOSPYROS (?) CYSTOPUS, Miq. Fl. Ind. Bat. Suppl. I. pp. 250, 584 (1860).

D. ramulis teretibus præsertim superne cum petiolis foliisque subtus maxime secus nervos rufo-pubescentibus, glabrescentibus, foliis alternis, oblongis, apice caudato-acuminatis, basi rotundatis, tenuiter subcoriaceis, supra glabris, subtus costulis patentibus utrinque 18—12 tenuibus venulosis pertensis, in sicco glauco-fuscescentibus.

Young parts rufous-hispid; branches terete. Leaves alternate, oblong, caudate-acuminate at apex, rounded at base, thinly sub-coriaceous, glabrous above, rufous-hispidulous beneath especially on the raised midrib and lateral veins; about 9 in. long by $2\frac{1}{2}$—3 in. wide; petioles $\frac{1}{4}$ in. long, channelled; lateral veins about 15 on each side, inconspicuous above, slender and more conspicuous beneath; midrib much raised beneath, tapering towards the apex. Flowers and fruit unknown, and therefore the plant is of uncertain position.

Sumatra; Lampong, near Kebang, *Teijsmann!* Local name *Daréhan-darehan.*

162. DIOSPYROS PYRRHOCARPA, Miq. Fl. Ind. Bat. Suppl. I. pp. 250, 583 (1860).

D. ramulis novellis cum petiolis costâque subtus pubescentibus glabrescentibus; foliis e basi rotundatâ usque acutiusculâ elliptico-oblongis plerisque breviter obtuso-acuminatis, coriaceis, glabris, supra secus costam canaliculatis, subtus pallidis costulis 9—7 tenuibus arcuato-patulis a margine leviter incurvo distanter unitis, dense tenereque reticulatis; floribus secus ramulos inferne lateralibus solitariis brevi-pedunculatis, cum calyce 4—5-partito (lobis acuminatis coriaceis) utrinque rufo-tomentosis; baccis cerasi majoris mole depresso-globosis, calyce adaucto reflexo (lobis antice convexis) suffultis rufo-ochrascenti-tomentosis.

West Sumatra, in province Priaman, *Diepenhorst;* Malay name *Hampadoe-Kajoe.*

163. DIOSPYROS PLATYPHYLLA, Welw. MSS.

D. arborea, laxe ramosa apice foliosa, foliis alternis, ellipticis rotundis vel obovatis, apice obtusis, basi rotundatis sæpe inæqualibus, valde coriaceis, supra glabrescentibus nitidis, subtus tomentosis reticulatis, breviter petiolatis; fructibus edulibus.

A moderate-sized tree, with lax tortuous dark-cinereous branches leafy and angular at the apex. Leaves alternate, elliptical rotund or obovate, rounded or obtuse at apex, rounded and often unequal at base, very coriaceous, glabrescent and shining above, more or less tomentose beneath, reticulated but inconspicuously so above, 3—6 in. long by $1\frac{1}{4}$—$3\frac{1}{2}$ in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long. Flowers monstrous in the specimen, the inflorescence consisting entirely of densely imbricated ferruginous-tomentose foliaceous scales.

W. Tropical Africa, Angola, Pungo Andongo, in sandy woods from Calunda to Condo, fruit said to be edible, *Dr Welwitsch!* no. 2531; native name *Musolveira*, the same as that of *Diospyros mespiliformis*, Hochst, of which it may very possibly prove to be an aberrant form.

164. DIOSPYROS PLATYCALYX, sp. nov.

D. foliis alternis, obovato-oblongis, apice rotundatis, basi cuneatis, coriaceis, utrinque puberulis vel glabrescentibus, petiolatis; fructibus solitariis, subglobosis, glaberrimis, apice umbilicatis, nitidis, 10 (?)*-locularibus, breviter pedunculatis; calyce fructifero profunde* 5—6*-lobo, plicato, aucto, lobis late ovatis, cordatis, auriculatis; seminibus compressis, albumine non ruminato.*

Tree of 20 feet; young shoots with short patent whitish tomentum; branches glabrescent, terete, palish. Leaves obovate-oblong, alternate, coriaceous, undulated, rounded at apex, cuneate at base, brown, shining, with slight veins, puberulous or glabrescent on both sides, of nearly the same colour on both sides, 2—3 in. long by $\frac{3}{4}$—1 in. wide; petioles puberulous, $\frac{1}{4}$—$\frac{1}{3}$ in. long; midrib slightly depressed on upper surface. Fruit solitary, in axils of fallen leaves, on shoots of previous season, $\frac{7}{10}$ in. long (besides the calyx) by $\frac{5}{8}$ in. thick, subglobose, quite glabrous, umbilicate at apex, 10 (?)-celled, with cells 1-seeded, shining, paler than leaves and calyx. Fruiting peduncle stout, with wide convex articulation, $\frac{1}{10}$—$\frac{1}{7}$ in. long, glabrate. Fruiting calyx $\frac{5}{8}$ in. deep by $1\frac{1}{2}$ in. wide, concealing half the fruit, nearly glabrous, deeply 5—6-lobed; lobes widely ovate, acute, cordate, much auricled at base, firmly membranous, with sides of lobes reflexed, folding with contiguous lobes and forming 5 dependent spurs the points of which are $\frac{1}{4}$ in. below the level of the articulation of the fruit; seeds compressed, $\frac{1}{2}$ in. long or more, albumen not ruminated.

Seychelles Islands, *Pervillé!* 640.

165. DIOSPYROS LEUCOCALYX, sp. nov.

D. fruticosa, glabra, foliis alternis, oblongis, apice obtuse acuminatis, basi rotundatis vel subcordatis, costâ et nervis lateralibus subtus validis, petiolis validis tumido-crassis; calyce fructifero 4-partito, intus albido-pruinoso, lato, lobis late cordatis, acuminatis, foliaceis.

A small shrub, glabrous, dark green but shining. Leaves alternate, subcoriaceous, oblong, obtusely acuminate at apex, rounded or subcordate at base, 1 foot long by 5 inches wide; midrib and lateral veins strong beneath; petioles more than $\frac{1}{2}$ in. long, strong, dark, tumid-crass. Fruiting calyx 4-partite, white-pruinose within, 2 in. high by 3 in. or more wide, erect-patent; lobes widely cordate, ovate, acuminate at apex, foliaceous.

Madagascar, Ambanivoule, *Goudot!* A. D. 1833.

166. DIOSPYROS BERNIERI, sp. nov.

D. foliis alternis, ovali-lanceolatis, apice subacuminatis, basi angustatis, coriaceis, glabris, breviter petiolatis, nervis inconspicuis; fructibus solitariis, appresse hirsutis, breviter pedunculatis; calyce fructifero utrinque pubescente, 4-fido, tubo concavo, tetragono, incrassato, lobis reflexis, undulatis, late ovatis.

Glabrous except the inflorescence; branches pale, terete. Leaves alternate, dark above, oval-lanceolate, somewhat narrowed at base, obtusely or sometimes acutely sub-acuminate at apex, coriaceous, veins indistinct, reddish brown beneath; midrib depressed above, blackish beneath; 2—$3\frac{1}{2}$ in. long by $\frac{3}{4}$—1 in. wide; petiole $\frac{1}{5}$ in. long, black in the dry state.

Fruiting peduncles very thick, $\frac{1}{4}$ in. long and as thick, pubescent, solitary; fruiting calyx $\frac{1}{2}$ in. high by nearly 2 in. across, pubescent on both sides, 4-fid; tube concave, 4-sided, thickened; lobes reflexed, wavy, widely ovate. Fruit ferruginous, shortly and appressedly hairy.

Madagascar, common in the forests of Tintingue; vernacular name *Voane Silac*, *Bernier!* 113.

Foliage of *D. lævis*, Bojer.

167. DIOSPYROS PRUINOSA, sp. nov.

D. bracteis exceptis glaberrima, foliis alternis, ovato-ovalibus, utrinque obtusis, vix coriaceis, brevissime petiolatis, nervis inconspicuis; floribus masculis axillaribus, brevissime cymosis; fructibus solitariis, axillaribus, 8-locularibus, breviter pedunculatis, bracteatis, subglobosis, cum calyce 4—5-fido plicato patente aucto violaceo-pruinosis.

Quite glabrous except the small shortly and slightly ciliated bracts; branches pale brown, terete. Leaves alternate, ovate-oval, more or less obtuse at both ends, submembranous or subcoriaceous, of a rich brown colour when dry, rather paler beneath, 1—$2\frac{1}{5}$ in. long by $\frac{1}{2}$—$1\frac{1}{6}$ in. wide; petioles $\frac{1}{16}$—$\frac{1}{12}$ in. long; veins indistinct, spreading; midrib flat above, darker beneath.

♂. Cymes axillary, 3—8-flowered, $\frac{1}{6}$ in. long, dark.

♀. Fruit solitary, axillary on the young branches, shortly globose, $\frac{1}{2}$ in. thick by $\frac{3}{8}$ in. high, 8-celled, several-seeded, as well as the calyx violaceo-pruinose; peduncles dark $\frac{1}{6}$—$\frac{1}{4}$ in. long; bracts several, ovate, about $\frac{1}{16}$ in. long; calyx plicate-patent, 1 in. in diameter, 4—5-(usually 4-)fid, undulated; lobes widely ovate cordate, apiculate or mucronate at apex.

Madagascar, Ste Marie, *Boivin!* 2538; Port Leven, *Vesco!* 1850.

168. DIOSPYROS CUNEIFOLIA, Hb. Delessert.

D. foliis alternis, obovatis, apice rotundatis, basi cuneatis, breviter hispidis, subsessilibus, confertis; fructibus solitariis, subglobosis, pilis brevibus hispidis aspersis, pedunculatis; calyce fructifero pubescente, 5(—6?)-partito, lobis oblongis, patentibus.

Shoots puberulous, glabrescent. Leaves alternate, obovate, subsessile, crowded, rounded at apex, cuneate at base, shortly hispid, $\frac{3}{4}$—$1\frac{1}{4}$ in. long by about $\frac{1}{2}$ in. wide.

♀. Fruiting peduncle axillary, solitary, ferruginous-hispid, $\frac{1}{3}$ in. long; fruiting calyx 5-(or 6-?)partite, pubescent; lobes oblong, spreading, $\frac{3}{8}$ in. long by $\frac{1}{8}$ in. wide; fruit solitary, somewhat depressedly globose, dark, covered with scattered pale short hispid hairs, about $\frac{1}{3}$ in. long.

Mexico, *Pavon* in Hb. Delessert!

169. DIOSPYROS APEIBACARPOS, Raddi, Quarante nuove del Brasile, in Atti Soc. Modena, Vol. XVIII. p. 12. n. 10 (1820).

D. foliis alternis, lanceolatis, acutis, supra glabris, subtus villoso-sericeis, brevissime petiolatis; baccis depressis, papillis adspersis et setis crebris, subdecaspermis; calyce 5-lobo.

Alph. DC. Prodr. VIII. p. 239. n. 96 (1844), Mart. Fl. Bras. VII. p. 8 (1856) excl. syn.

A tree of about 30 feet high, with not very thick trunk, very slightly branched; the young branches rather setose at the extremity. Leaves alternate, lanceolate, elongated at the apex, entire, smooth above, scattered with yellowish hairs beneath which are closer along the midrib and round the margin. Calyx 5-lobed. Fruit depressed, scattered with papillæ and short setulæ almost like the hairs with which the petioles of the leaves are covered, size and shape of the *Apeiba* of Aublet, 1—2 in. thick, 8—10-seeded.

Brazil, Estrella Mountains, *Raddi*, fruits in April; Minas, San Francisco River, in woods, *Martius;* near Borba by River Madeira, *Riedel.*

Martius in Fl. Bras. VII. (Eben.) p. 8 states that this plant is the same as *D. sericea*, Alph. DC.; but the fruit appears to be different.

170. DIOSPYROS (?) XYLOPIOIDES, Mart. in Fl. Bras. VII. p. 8. n. 4 (1856).

D. ramulis subdistichis fulvo- et apices versus albido-sericeis; foliis subcoriaceis lanceolatis acuminatis basi acutis (20—36‴ *long.*, 3—5‴ *lat.*) *supra glabris, subtus sericeis pilis appressis flexuosis albis, in nervo margine petioloque fulvulis; floribus axillaribus geminis ternisve bracteisque fulvo-sericeis.*

Arborea. Rami cortice tenui deductili, qualis in multis Diospyris obtinet. Ramuli præsertim in extremitatibus dense albo- aut fulvulo-sericei. Folia, tam figura quam dispositione et compage ea *Xylopiae frutescentis* et nonnullarum affinium assimilantia, supra saturate viridia nervo impresso, subtus pilis mollibus appressis in margine et nervo frequentioribus, venis vix conspicuis.

Flower-bud sessile, narrowly campanulate, $\frac{1}{5}$ in. long; calyx trifid or tripartite, glabrous inside, lobes ovate-lanceolate; corolla not exceeding the calyx, silky outside, puberulous inside, tripartite or tripetalous, valvate; stamens 3, erect, with hairy lines, filaments short; ovary rudimentary.

S. America. Guiana, in woods (*Martius!*).

Scarcely a true *Diospyros* and nearer to *Maba*, but probably neither, and perhaps the type of a new genus. The foliage is exceedingly like that of *Maba sericea*.

EXCLUDED AND NOMINAL SPECIES OF DIOSPYROS.

Diospyros acapulcensis, Kunth = *Maba acapulcensis*.

Diospyros acuminata, Wall. List, n. 4129 (1828—32). Cfr. Laurineæ.

Diospyros albens, Presl = *Maba albens*.

Diospyros ambigua, Vent. non Sap., = *Royena ambigua*, Vent.

Diospyros Berterii, Alph. DC. = *Maba inconstans*, Griseb.

Diospyros cauliflora, Mart. in Fl. Bras. VII. p. 7 (1856), non Blume, = *Maba cauliflora*.

Diospyros cerasifolia, D. Don, Prodr. Fl. Nep. p. 144 (1825) = *Eurya symplocina*, Blume.

Diospyros conduplicata, Kunth = *Maba inconstans*, Griseb.

Diospyros cupulosa, F. Muell. = *Maba rufa*, Labill.

Diospyros fasciculosa, F. Muell. = *Maba fasciculosa*, F. Muell.

Diospyros feminina, Hamilt. ex Alph. DC. Prodr. VIII. p. 238. n. 83 (1844), = *Eurya symplocina*, Blume.

Diospyros frondosa, Wall. List, n. 4125 (1828—32) = *Bocagea elliptica*, Hook. fil. et Thoms.

Diospyros geminata, F. Muell. = *Maba geminata*, R. Br.

Diospyros grandifolia, Wall. ex Voigt, Hort. Suburb. Calcutt. p. 345 (1845). *Name only.*

Diospyros hexasperma, Hasselt = *Maba elliptica*, Forst.

Diospyros hirsuta, Desf. non Linn. fil., = *Royena hirsuta*, Linn.

Diospyros humilis, F. Muell. = *Maba humilis*, R. Br.

Diospyros inconstans, Jacq. = *Maba inconstans*, Griseb.

Diospyros lanceolata, Poir., non Wall., = *Maba lanceolata*.

Diospyros lycioides, Desf. = *Royena pallens*, Thunb.

Diospyros microcarpa, Span. in Hook. Comp. Bot. Mag. I. p. 348 (*name only*, 1835), non Sieb.

Diospyros myrmecocarpus, Mart. = *Maba myrmecocarpa*.

Diospyros oblonga, G. Don, Gen. Syst. Gard. IV. p. 40 (1837) = ? *D. venosa*, Wall.

Diospyros obovata, Wight, Icon. t. 1226 (1850), non Jacq., = Sapotacea.

Diospyros obtusifolia, Bert., non Humb. et Bonpl., = *Maba inconstans*, Griseb.

Diospyros psidioides, Kunth = *Maba inconstans*, Griseb.

Diospyros pubescens, Pers., non Pursh, = *Royena hirsuta*, Linn.

Diospyros punctata, Korth., non Decaisne, = *Maba punctata.*

Diospyros salicifolia, Humb. et Bonpl. = *Maba salicifolia.*

Diospyros sericea, Alph. DC. = *Maba sericea.*

Diospyros sericocarpa, F. Muell. = *Maba rufa*, Labill.

Diospyros serrata, Hamilt. ex D. Don, Prodr. Fl. Nep. p. 143 (1825) = *Eurya acuminata*, DC.

Diospyros venosa, Wall. List, n. 4126 (1828—32) = ? Anonacea.

Diospyros virginica dulcis = *Carpodinus edulis*, Don. Cfr. Alph. DC. Prodr. VIII. p. 329 (1844).

Diospyros (sp.), Salt! Voyage to Abyssinia, p. 14 (1814) = *Euclea multiflora*, var.

V. TETRACLIS, gen. nov.

♂. *Flores dioeci. Flores cymosi, tetrameri, subglobosi. Calyx depresso-globosus; lobis brevibus depresso-deltoideis, præfloratione valvatis. Corolla carnosa, 4-fida, extus puberula, intus hirsuta; lobis præfloratione valvatis. Stamina circiter 30, pleraque geminata, prope corollæ basim inserta; filamentis brevibus compressis pubescentibus; antheris hispidulis oblongis liberis, lateraliter bilocularibus; pollen globosum, læve. Ovarii rudimentum nullum.*

♀. *Bracteæ caducæ. Fructus superus, solitarius, pedunculatus, subglobosus, subtomentosus, ferrugineus, carnosus, 8 (?)-locularis et -spermus; pericarpio crasso. Calyx profunde 4-lobus, accrescens, appressus, semina pendula oblonga, testâ non nitidâ.*

Arbor madagascariensis; foliis coriaceis alternis simplicibus integerrimis exstipulatis; floribus axillaribus, apice 4 lineis cruciatis præfloratione notatis.

1. TETRACLIS CLUSIÆFOLIA, sp. nov.

T. foliis oblongis vel obovato-oblongis, apice rotundatis, emarginatis vel breviter acuminatis, obtusis, basi cuneatis, subglabris, coriaceis, petiolatis, nervis tenuibus, crebris.

PLATE XI. A fruiting branch, *natural* size. *a.* A male inflorescence, *natural* size. *b.* A male flower, *magnified* 4 diameters. *c.* The same after the removal of the calyx, *magnified* 4 diameters. *d.* A vertical section of a male flower, *magnified* 6 diameters. *e.* A transverse section of male flower, *magnified* 5 diameters.

A large tree with young parts puberulous; branches dark, somewhat angular, glabrescent, remotely rugose-verrucose. Leaves oblong or obovate-oblong, rounded emarginate or shortly acuminate at apex, cuneate at base, with recurved margins, subglabrous, yellowish green and shining on both sides; midrib depressed above; lateral veins close, widely spreading, very feeble, and in relief on both sides of the leaf; [the lateral veins on the lower side of the leaves are too distinctly rendered by the lithographer in Plate XI.]; 5—10 in. long (besides channelled petioles $\frac{5}{8}$—1 in. long) by $1\frac{7}{8}$—3 in. wide.

♂. Cymes on young branches, shortly subferruginous-pubescent, bearing 3—10 flowers, $\frac{1}{2}$—$\frac{7}{8}$ in. long (not including the flowers); common peduncle $\frac{1}{4}$—$\frac{1}{2}$ in. long; pedicels $\frac{1}{16}$—$\frac{1}{8}$ in. long; flowers $\frac{1}{5}$—$\frac{1}{4}$ in. in diameter, pubescent. Calyx nearly as high as the corolla, 4-lobed at apex, at base somewhat 4-sided outside. Corolla puberulous outside, hirsute or hispidulous inside. Stamens 30 (in one flower), mostly united by their filaments in pairs; anthers hispidulous, filaments hairy inserted near the base of the corolla; pollen globular, smooth, about $\frac{1}{790}$ in. in diameter. Ovary wanting.

♀. Fruit (unripe?) solitary, $\frac{3}{4}$—$\frac{7}{8}$ in. high, by 1—$1\frac{1}{4}$ in. thick, crowned at apex by remains of 4-partite style; fruiting peduncles $\frac{1}{4}$—$\frac{1}{2}$ in. long, thickened upwards, puberulous; fruiting calyx 4-sided, softly hairy on both sides; lobes widely ovate, acute, somewhat cordate and pouting at base, reaching half the height of the fruit, thickly coriaceous.

Madagascar, *Richard!* 388, Nossi-bé; *Pervillé!* 6.

FOSSIL EBENACEÆ.

About 60 specific names of this family relating to fossils have been published; the first was published by Dr Alexander Braun, about 25 years ago, and the last by Prof. W. Ph. Schimper, in the present year (1872). All these fossils occur in Tertiary strata, with the exception of one, namely *Diospyros primæva* Heer from the beds of Nebraska in North America, which beds have been recently referred to the Cretaceous period, though they were formerly supposed from the facies of the contained flora to be Tertiary. The majority of the species have been founded on leaves alone; and the venation of these no doubt accords more or less closely with that of those species of Ebenaceæ, such as *Diospyros Lotus*, *Royena hirsuta*, *Euclea lanceolata*, &c., which fossil botanists seem to regard as the types of their respective genera. There is in fact much variety of venation amongst the recent species of the family; and with respect to recent plants it is quite impossible to assign to the family, with even a moderate amount of certainty, a given leaf of an unknown genus. A few of the fossil species have been described from the calyx fruit or seed, with or without leaves; and the best of these specimens, such as those which have been named *Diospyros brachysepala*, and *Euclea relicta*, present fair evidence of belonging to the structure of Ebenaceæ, while even in these instances the genus cannot be properly fixed, and other families are not absolutely excluded. With regard to many of the fossil species, the utmost inference founded on reasonable grounds which can be deduced, is a favourable suggestion of Ebenaceæ for the family to which the specimens may probably belong; and with regard to other specimens of the published species, it appears to me that Ebenaceæ is not a probable family for them. It would be much the better plan to refer all fossils, which have nearest affinities to Ebenaceæ, to a fossil genus *Ebenacites*, as was done in the first instance by Saporta, but subsequently relinquished by him in favour of *Diospyros*. On the whole then, as I place but little confidence in the determination of the fossils, I wish in no way to confirm them in their present places; but since they have been published as Ebenaceous, I quote them as they stand, with the addition, in some cases, of additional particulars and remarks; I have added the synonymy in accordance with the views of the principal authorities in fossil botany, and have drawn up artificial keys for the genera, and also for the species in each genus, in order to set forth the distinctive characters of the genera and species, so far as their published descriptions allow, and to found a basis for their systematic arrangement.

Key to the Fossil Genera.

Leaves small, not exceeding 1 in. long, midrib alone robust. I. Royena.

Leaves exceeding 1 in. long; lateral veins more or less clearly marked.

- Calyx 4—5-merous.
  - Leaves narrowly elliptical, 3—4 in. long, narrowed at both ends. II. Euclea.
  - Leaves ovate lanceolate oval or oblong or exceeding 4 in. long. III. Diospyros.
- Calyx 3-merous. IV. Macreightia.

Diospyros hæringiana, Ettingsh. has narrowly elliptical leaves 2—3½ in. long, but it was published previously to the reference of any fossil to the genus *Euclea*. Fossils with a trimerous calyx, especially if the foliage is Ebenaceous, have been referred by authors to *Macreightia*, a genus which I have merged in the older genus *Maba*; if then they still merit reference to a recent Ebenaceous genus, they must all be included under the genus *Maba*.

I. Key to the Fossil Species of Royena.

Leaves linear, ⅛ in. broad. 1. *R. Myosotis.*

Leaves oblanceolate or wider, ¼—½ in. broad.

- Leaves oblanceolate or oblong, 1 in. long.
  - Leaves oblanceolate. 2. *R. græca.*
  - Leaves oblong. 3. *R. Amaltheæ.*
- Leaves oval or round, ⅙—⅗ in. long.
  - Leaves cuneate-orbicular. 4. *R. euboea.*
  - Leaves oval. 5. *R. Pentelici.*

1. ROYENA MYOSOTIS, Ung. Foss. Fl. Eub. in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXVII. p. 69. t. XIV. fig. 5—8 (1867).

R. foliis lineari-lanceolatis minimis in petiolum brevem attenuatis integris coriaceis, nervo medio solo distincto; calyce quinquelobo quatuor lineas lato, laciniis inæqualibus rotundatis.

Diospyros Myosotis, Ung. Gen. et Sp. Pl. Foss. p. 436 (1850), Syll. Pl. Foss. III. p. 28. t. IX. f. 13, 15, (1866); Schimp. Pal. Vég. II. p. 952 (1872); Web. Palæontogr. II. p. 190. t. XIV. f. 5 *b*, non f. 5 *a*.

In Miocene formations, Kumi, Negropont; Eocene, in marly schist, Radoboj, Croatia.

Leaves $\frac{1}{2}$—1 in. long by $\frac{1}{8}$ in. wide. Calyx $\frac{1}{3}$ in. in diameter. Cfr. *Porana*. According to Prof. Schimper, Unger has comprised several different plants under this species, and Ettingshausen in Sitzungsber. Math.—Naturw. Akad. Wissensch. XXXVIII. p. 492 (1858), has shewn that the leaf which Unger described and figured for this species in Foss. Fl. v. Sotzka, p. 172. t. 43. f. 15 (1851), is a leaflet of *Cassia phaseolites*, while he thinks that Unger's fig. 16 may be a calyx of *Celastrus*.

2. ROYENA GRÆCA, Ung. Foss. Fl. Eub. in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXVII. p. 68. t. XI. fig. 40—51 (1867).

R. foliis lanceolato-lingulatis breviter petiolatis integerrimis coriaceis, nervo primario valido, nervis secundariis tenuissimis transversissimis ramosissimis; calyce firmo patente semiquinquefido deciduo, laciniis inæqualibus ovato-acuminatis extus striatis 8-millim. longis, margine parum involutis; drupâ siccâ quadriloculari.

Diospyros græca, Saporta in Bull. Soc. Géol. France, XXV. p. 321 (1868).

Schimp. Pal. Vég. II. p. 954. n. 1 (1872).

In Miocene formations at Kumi, Negropont.

Leaves 1 in. long by $\frac{27}{100}$ in. wide, oblong, obtuse, cuneate at base; petiole very short.

3. ROYENA AMALTHEÆ, Ung. Foss. Fl. Eub. in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXVII. p. 69. t. XIV. f. 1 (1867).

R. foliis ovato-lanceolatis minimis obtusis in petiolum attenuatis integerrimis coriaceis, nervis secundariis crebris tenuibus ramosis reticulatim conjunctis.

Schimp. Pal. Vég. II. p. 955. n. 2 (1872).

In Miocene formations at Kumi, Negropont.

Leaves oblanceolate about 1 in. long by $\frac{1}{4}$ in. wide. Cfr. *R. græca*, Ung.

4. ROYENA EUBOEA, Ung. Foss. Eub. in Denkschr. Kais. Akad. Wissensch. Math.-Naturw. Vol. XXVII. p. 69. tab. XIV. fig. 2—4 (1867).

R. foliis minimis petiolatis cuneato-orbicularibus coriaceis integerrimis nervo primario valido, nervis secundariis inconspicuis.

Schimp. Pal. Vég. II. p. 955. n. 3 (1872).

In Miocene formations at Kumi in Negropont.

Leaves $\frac{1}{5}$—$\frac{1}{2}$ in. long by $\frac{1}{4}$—$\frac{1}{2}$ in. wide; petioles $\frac{1}{20}$—$\frac{1}{10}$ in. long. Of quite uncertain family.

5. ROYENA PENTELICI, Ung. Foss. Fl. Eub. in Denkschr. Kais. Akad. Wissensch. Math.-Naturw. Vol. XXVII. p. 70. t. XIV. f. 9 (1867).

R. foliis minimis ovato-ellipticis petiolatis integerrimis coriaceis, nervis secundariis subsimplicibus fere inconspicuis.

Schimp. Pal. Vég. II. p. 955. n. 4 (1872).

In Miocene formations at Kumi in Negropont.

Leaf $\frac{3}{5}$ in. long by $\frac{1}{3}$ in. wide; petiole $\frac{1}{16}$ in. long. Not unlike a short leaf of *R. glabra* L., but placed in this genus on insufficient evidence.

II. KEY TO THE FOSSIL SPECIES OF EUCLEA.

- Leaves petiolate.
  - Leaf 3 in. long, acuminate at both ends. — 1. *E. miocenica.*
  - Leaf 4 in. long, narrowed at both ends, not acuminate. — 2. *E. Apollinis.*
- Leaf sessile. — 3. *E. relicta.*

1. EUCLEA MIOCENICA, Ung. Syll. Pl. Foss. III. p. 25. t. VIII. fig. 8, 8* (1866).

E. foliis lanceolatis utrinque acuminatis petiolatis integerrimis coriaceis, nervo primario valido, nervis secundariis flexuosis ramosis rete nervorum tertiariorum laxo inter se conjunctis.

Heer Mioc. Balt. Fl. p. 84. t. XXVIII. fig. 3—8 (1869); Schimp. Pal. Vég. II. p. 956. n. 1 (1872).

In marly schist, Croatia, *Unger;* Rixhöft, Samland, W. Prussia.

Leaf 3 in. long by $\frac{5}{8}$ in. wide; petiole $\frac{3}{16}$ in. long. Genus and family quite uncertain.

2. EUCLEA APOLLINIS, Ung. Syll. Pl. Foss. III. p. 26. t. VIII. fig. 10, 10* (1866).

E. foliis lanceolatis breviter petiolatis integerrimis coriaceis, nervo primario valido, nervis secundariis crebris flexuosis ramosisque rete nervorum tertiariorum laxo inter se conjunctis.

Rhododendron Apollinis, Ettingsh. ex Ung. *l. c.*

In marly schist, Eocene; Radoboj, Croatia, *Unger.*

Leaf 4 in. long by $\frac{5}{6}$ in. wide; petiole scarcely $\frac{1}{4}$ in. long, narrowed at both ends. Leaf very like *E. miocenica* Ung. but rather larger. Prof. Schimper unites this with *E. miocenica.*

3. EUCLEA RELICTA, Ung. Foss. Fl. Eub. in Denkschr. Kais. Wissensch. Math.-Naturw. Vol. XXVII. p. 68. t. XI. f. 39 (1867).

E. foliis lanceolatis utrinque attenuatis sessilibus integerrimis coriaceis, nervo primario valido, nervis secundariis angulo subrecto exorientibus flexuosis ramosissimis in retem nervorum tertiariorum laxum divisis.

Schimp. Pal. Vég. II. p. 956. n. 2 (1872).

In Miocene formations at Kumi in Negropont.

Leaf $3\frac{1}{4}$ in. long by $\frac{9}{16}$ in. wide, narrowed at both ends; of quite uncertain family.

III. KEY TO THE FOSSIL SPECIES OF DIOSPYROS.

The species whose leaves have been described may be arranged as follows.

Leaves more or less narrowed at base, or if sometimes rounded at base then acuminate at apex.
- Leaves acute or subacute at apex (sometimes obtuse in *D. brachysepala*).
  - Petioles short, not exceeding $\frac{1}{2}$ in. in length.
    - Leaves reticulated, secondary veins manifest, branched.
      - Leaves membranous, often unequal at the base. — 1. *D. anceps.*
      - Leaves coriaceous, equal at the base.
        - Leaves $2\frac{1}{3}$—3 in. long, ovate. — 2. *D. vetusta.*
        - Leaves about 6 in. long, oval-oblong. — 3. *D. Loveni.*
    - Secondary veins simple or subsimple, manifest.
      - Leaves about 6 in. long. — 4. *D. Wodani.*
      - Leaves about 7 in. long. — 5. *D. Lignitum.*
    - Secondary veins obsolete. — 6. *D. Weberii.*
  - Petioles long.
    - Lateral veins rather distant.
      - Tertiary veins transverse. — 7. *D. alaskana.*
      - Tertiary oblique or in various directions.
        - Midrib very stout; net-veins subobsolete. — 8. *D. incerta.*
        - Midrib moderate; leaves reticulated.
          - Calyx 4-fid, with widely ovate or rounded lobes. — 9. *D. brachysepala.*
          - Calyx 5-partite, with linear lobes. — 10. *D. paradisiaca.*
    - Lateral veins crowded, — 11. *D. lotoides.*
- Leaves obtuse or subobtuse (sometimes acute in *D. varians*).
  - Leaves lanceolate or ovate.
    - Secondary veins serpentine. — 12. *D. primæva.*
    - Secondary veins not serpentine, subremote.
      - Leaves about $3\frac{1}{2}$—$4\frac{1}{2}$ in. long, membranous. — 13. *D. Auricula.*
      - Leaves about 2 in. long, subcoriaceous. — 14. *D. dubia.*
    - Secondary veins numerous, not serpentine.
      - Leaves $\frac{1}{3}$—$1\frac{1}{5}$ in. wide.
        - Tertiary veins reticulated. — 15. *D. varians.*
        - Tertiary veins less ramified. — 16. *D. obscura.*
      - Leaves $1\frac{7}{8}$ in. wide. — 17. *D. palæogæa.*
  - Leaves oval or elliptical.
    - Leaves $2\frac{1}{8}$—$3\frac{1}{2}$ in. long.
      - Leaves $\frac{9}{10}$—$\frac{7}{10}$ in. wide. — 18. *D. hæringiana.*
      - Leaves $1\frac{1}{3}$ in. wide. — 19. *D. pannonica.*
    - Leaves $1\frac{1}{4}$ in. long by $\frac{5}{8}$ in. wide — 20. *D. Royena.*

Leaves rounded, not cordate at base, subobtuse, not rounded at apex.
- Calyx 4-partite with oblong lobes; leaves oval, subcoriaceous. — 21. *D. stenosepala.*
- Calyx deeply 4-fid, with widely elliptical lobes; leaves oval-oblong, coriaceous — 22. *D. bilinica.*

Leaves rounded at both ends, not cordate — 23. *D. oblongifolia.*

Leaves cordate at base. — 24. *D. Parthenon.*

25. *D. obliqua* with linear lobes, and 26. *Ebenacites rugosus* with wider lobes, are known only from the calyx; and 27. *D. Zollikoferi* is described from a cluster of seeds only.

1. DIOSPYROS ANCEPS, Heer, Fl. Tert. Helvet. III. p. 12. t. CII. fig. 15—18 (1859).

D. foliis ovato-ellipticis, apice acuminatis, basi obtusis, membranaceis, hic illic inæquilateris, integerrimis, petiolatis, nervis secundariis remotiusculis, sub angulo sat aperto egredientibus, curvatis, ramosis, ipsis et ramis arcuato-conjunctis, rete laxo.

Gaudin et Strozzi Mém. Foss. Tosc. II. p. 51. n. 48. t. VII. f. 6 (1859); Heer, Mioc. Balt. Flora, p. 84. t. XXVII. fig. 7—9 (1869); Schimper, Pal. Vég. II. p. 948. n. 12 (1872).

Miocene; Germany and Tuscany.

Leaves acute or subacute, $1\frac{1}{2}$—$3\frac{1}{2}$ in. long by $\frac{5}{8}$—$1\frac{2}{3}$ in. wide; petioles $\frac{1}{4}$—$\frac{1}{3}$ in. long.

2. DIOSPYROS VETUSTA, Giebel, Fl. Sächs. Thüring. Braunkohl. in Zeitschr. XVI. p. 57 (1860).

D. foliis alternis, ovato-ellipticis, apice acutis vel acuminatis, basi angustatis, coriaceis, nervis secundariis subtilissimis, areis reticulatis; fructu globoso, 5-angulato, 5-spermo; calyce fructifero patente, 5-fido, lobis rotundatis.

Heer, Sächs.—Thüring. Braunk. p. 10 [416]. n. 24. t. VII. f. 1—6 (1861). Schimp. Pal. Vég. II. p. 946. n. 6 (1872).

Eocene; Lignites of the Ligurian strata of Skopau in Thüringe, Saxony.

Leaves $2\frac{1}{3}$—3 in. long by $\frac{9}{10}$—$1\frac{1}{4}$ in. wide; petiole about $\frac{3}{10}$ in. long; calyx $\frac{3}{4}$ in. in diameter; fruit $\frac{3}{5}$ in. in diameter.

3. DIOSPYROS LOVENI, Heer, Fl. Foss. Arct. p. 118. t. VII. fig. 7 *b*, *c*, 8, t. XLVII. fig. 8 (1868).

D. foliis firmis, coriaceis, integerrimis, nervis secundariis remotis, sub angulo acuto egredientibus, valde camptodromis, ramosis, areis argute reticulatis.

Schimp. Pal. Vég. II. p. 949. n. 15 (1872).

Miocene; Atanekerdluk, North Greenland.

Leaves perhaps 6 in. long by $1\frac{5}{8}$ in. wide, areolate, elliptic-oblong.

4. DIOSPYROS WODANI, Ung. Gen. et Sp. Pl. Foss. p. 435. n. 2 (1850).

D. foliis ovato-oblongis, apice acuminatis basi attenuatis petiolatis integerrimis membranaceis, nervo medio valido, nervis secundariis remotis subsimplicibus sursum arcuatis tenuibus; baccâ globosâ exsuccâ semipollicari, calyce quinquelobo deciduo patente, laciniis lanceolatis obtusis striatis pollicaribus.

Ung. Syll. Pl. Foss., pug. iii., in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXV. p. 27. t. IX. fig. 10—12 (1866), Ettingsh. Beitr. z. Foss. Fl. v. Radoboj, p. 55, Schimp. Pal. Vég. II. p. 951. n. 22 (1872).

Plumeria Flos-saturni, Ung. Gen. et Sp. Pl. Foss. p. 433 (1850); Syll. Pl. Foss. III. p. 27. t. IX. fig. 10—12 (1866).

Anona macrophylla, Ung. Gen. et Sp. Pl. Foss. p. 442. n. 3 (1850).

Eocene; in marly schist, Radoboj, Croatia.

Fruit $\frac{3}{5}$ in. in diameter; calyx deeply 5-lobed, $1\frac{5}{8}$ in. in diameter, patent; lobes narrowly elliptical, obtuse, striate, $\frac{3}{4}$—$\frac{7}{8}$ in. long by $\frac{1}{4}$—$\frac{1}{3}$ in. wide. Leaves about 6 in. long. Of uncertain family.

5. DIOSPYROS LIGNITUM, Ung. Syll. Pl. Foss., pug. iii., in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXV. p. 30. t. IX. f. 9 (1866).

D. foliis ovato-oblongis utrinque attenuatis petiolatis integerrimis membranaceis, nervo primario valido, nervis secundariis distantibus simplicibus ramosisque; seminibus suborbiculari-oblongis obtusis lævibus compressis, chalazâ parvâ immersâ.

Anona Lignitum, Ung. *l.c.*, pug. i. p. 25. t. X. fig. 1—7 (1861), Gen. et Sp. Pl. Foss. p. 441 (1850).

Miocene; lignite of Salzhausen in Wetterau and Frofajach in Styria.

Leaves 7 in. long by $1\frac{3}{4}$ in. wide; seeds $\frac{3}{8}$ in. long by $\frac{5}{16}$ in. broad. The seed is not typical of the family. Not given by Schimper in his *Traité de Paléontologie végétale* among the Ebenaceæ.

6. DIOSPYROS WEBERII, Massal. Syll. Pl. Foss. Tert. Venit. p. 77 (1859).

D. foliis (?) ovatis acutis subpetiolatis integerrimis, nervo primario valido, nervis secundariis nullis; calyce quinquelobo deciduo minimo patente, laciniis apiculatis.

D. Myosotis, Web. Tert. Fl. Niederrhein. Braunkohl. in Dunker et Meyer, Beitr. Naturgesch. Vorwelt, Vol. II. p. 19. t. XIV. fig. 5 *a* (1852), non Ung.

In Tertiary formations, Italy, &c.

Calyx $\frac{1}{3}$ in. in diameter. The leaves (at least) probably belong to *Royena Myosotis,* Ung.

7. DIOSPYROS ALASKANA, Schimp. Pal. Vég. II. p. 949. n. 17 (1872).

D. foliis ellipticis utrinque acutis, subcoriaceis, nitidis, integerrimis, longe petiolatis, nervo medio valido, nervis secundariis subtilibus (validis ex Lesq.), superioribus alternantibus, inferioribus oppositis, omnibus valde curvatis camptodromis sub angulo acuto egredientibus, areis nervulis transversis obsoletis ramosis reticulatis.

D. lancifolia, Lesquer. in Sill. Amer. Journ. ser. ii. vol. XXVII. p. 361. n. 13 (Maio, 1859); Heer, Foss. Pfl. v. Van Couver u. Brit. Columb. p. 8. t. I. f. 10—12, II. f. 1—3; Fl. Foss. Alaskana, p. 35. t. III. f. 12 (1869); non Al. Br.

D. lanceolata, "Lesq." ex Schimp. *l.c.*, non Poir.

British Columbia and Neniltschik (Alaska); Billingham Bay, Washington Territory, N. America.

Leaf 4 in. long by $1\frac{1}{3}$ in. wide; petiole $\frac{1}{2}$ in. long.

8. DIOSPYROS INCERTA, Massalongo, Synops. Fl. Foss. Senigall. p. 76. n. 187 (1858).

D. foliis ovato-lanceolatis utrinque attenuatis acuminatis petiolatis penninerviis integerrimis, costâ validissimâ, nervis secundariis rectiusculis, sub angulo 45—50° exorientibus subæquidistantibus, subramosis, apice bifurcatis, arcuatim conjunctis, nervulis venisque subobsoletis.

Massal. Fl. Foss. Senigall. p. 295. t. XXVI—XXVII. fig. 6, 29 (1859).

Miocene; Senigallia, Italy.

Leaves 4—$4\frac{1}{2}$ in. long (including petiole $\frac{3}{8}$—$\frac{7}{16}$ in. long) by $1\frac{1}{2}$ in. wide. The specific name is more suitable than the generic.

9. DIOSPYROS BRACHYSEPALA, Al. Br. in Leonh. et Bronn, Neues Jahrb. für Mineral. 1845, p. 170.

D. foliis elliptico-oblongis, apice obtuse vel acute angustatis vel acuminatis, basi cuneatis vel obtusis, subcoriaceis vel submembranaceis, integerrimis, petiolatis, penninerviis, nervis secundariis alternantibus, remotiusculis, sub angulo acuto vel recto egredientibus, curvatis ramosis, ipsis et ramis dorsalibus marginem versus arcuato-conjunctis, brochiodromis; calyce quadrifido, medio cicatrice annulari impressâ notato, lobis brevibus, late ovatis vel rotundis, apiculatis; baccâ exsuccâ, semipollicari.

Ung. Blätt. Swoszowice in Nat. Abh. Gesamm. t. XIV. f. 15 (1849); Heer, Miocene Baltische Flora, p. 84. n. 65. t. XXVII. fig. 1—6, t. XXVIII. f. 1. (1869), Fl. Tert. Helvet. III. t. CII. fig. 1—14 (1859), Fl. Foss. Arct. p. 117. t. XV. f. 10—12, t. XVII. f. 5 *h, i*, t. XLVII. fig. 4 *b, c, d*, 5—7 (1868), Braunk. Bornstädt in Abh. Nat. Halle, t. III. fig. 7, 8 (1869), Phil. Trans. vol. 159. pt. II. p. 475. n. 48. t. L. f. 13, t. LV. f. 8 (1870); Sismonda, Pal. Piém. p. 55. t. XI. f. 6, t. XVI. f. 5, t. XIX. f. 3 (1865); Ettingsh. Foss. Fl. Bilin, in Denkschrift. Akad. Kais. Wissensch. Math.-Naturw. Bd. XXVIII. p. 232. t. XXXVIII. f. 28, t. XXXIX. f. 1 (1868); Schimp. Pal. Vég. II. p. 949. n. 18 (1872).

Tetrapteris Harpyiarum, Ung. Foss. Fl. v. Sotzka, p. 46, t. XXIX. fig. 9 (1850), in Denkschr. Bd. II. t. L (1851).

Getonia petrææformis, Ung. Foss. Fl. v. Sotzka, t. XXXIII. f. 2—4, in Denkschr. Bd. II. t. LIV. (G. *petrææfolia* ex Schimp. *l. c.*)

G. macroptera, Ung. Foss. Fl. v. Sotzka, t. XXXIII. fig. 8, in Denkschr. Bd. II. t. LIV.

G. truncata, Goepp. Tert. Fl. v. Schossnitz, p. 37. t. XXV. fig. 11 (1855).

D. lancifolia, A. Br. ex Bruckm. Fl. Oening. Foss. in Jahr. Ver. Nat. Würtemb. p. 232 (1850), non Lesq.

D. langifolia, "Al. Braun" ex Stizenb. Verst. Baden, p. 83 (1851).

Arbutus diospyrifolius, Massal. Lett. Scarab. p. 29. n. 203 in Ann. Sc. Nat. Bologn. (1854); Fl. Foss. Senigall. p. 296. t. XXVI—XXVII. f. 3, t. XLV. f. 7 (1859).

D. longifolia, "Stiz." ex Heer, Fl. Tert. Helvet. III. p. 11 (1859), non Spruce.

D. latifolia, "Al. Br." ex Schimp. Pal. Vég. II. p. 949 (1872).

Upper middle and lower Miocene formations; North Greenland, W. Prussia, France, Switzerland, Italy.

Leaves $1\frac{1}{2}$—5 in. long by $\frac{5}{8}$—$2\frac{1}{3}$ in. wide; petioles ranging up to $\frac{3}{4}$ in. long. Calyx $\frac{1}{5}$—$\frac{1}{2}$ in. in diameter.

10. DIOSPYROS PARADISIACA, Ettingsh. Foss. Fl. Bilin in Denkschrift. Akad. Wissensch. Math.-Naturw. XXVIII. p. 234. t. XXXVIII. fig. 29—31, 34 (1868).

D. foliis lanceolatis, utrinque angustatis, basi acutis, integerrimis membranaceis, petiolatis, nervo primario distincto recto, nervis secundariis tenuibus, inferioribus sub angulo 45°, mediis et superioribus sub angulis obtusioribus, arcubus laqueorum maculis externis instructis, nervis tertiariis tenuissime dictyodromis; baccâ ovideâ, exsuccâ; calyce 5-partito, patente, deciduo, laciniis linearibus obtusis, nervoso-striatis, vix semipollicaribus.

Schimp. Pal. Vég. II. p. 946. n. 5 (1872).

Miocene; Tripoli de Kutschlin, Bohemia.

Leaf $3\frac{1}{2}$ in. long or more by $\frac{9}{10}$ in. wide; petiole $\frac{1}{2}$ in. long; calyx-lobes $\frac{1}{5}$—$\frac{2}{5}$ in. long by $\frac{1}{16}$—$\frac{1}{8}$ in. wide; fruit (?) $\frac{2}{5}$ in. long.

11. DIOSPYROS LOTOIDES, Ung. Syll. Pl. Foss., pug. III., in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXV. p. 30. t. X. fig. 1—12 (1866).

D. foliis lanceolato-oblongis utrinque attenuatis longe-petiolatis, margine plus minus undulato, integerrimis plurinerviis, nervo primario valido, nervis secundariis crebris, sub angulo plus minus acuto emissis, marginem versus arcu brevi conjunctis, apice se conjunctis, nervis tertiariis transversalibus utplurimum obsoletis vel parum conspicuis, calyce minimo, 5-fido, patente, laciniis rotundatis.

Ung. Foss. Fl. d. ält. Braunk. d. Wetterau, p. 59; Schimp. Pal. Vég. II. p. 951. n. 20 (1872).

Borraginites myosotiflorus, Ludw. Palæontogr. in Mey. Beitr. Naturg. Vorw. VIII. p. 116 (1860).

Miocene; lignite at Wetterau, Germany.

Leaves 3—$5\frac{1}{2}$ in. long by $\frac{7}{8}$—$1\frac{3}{8}$ in. wide; petiole $\frac{3}{16}$—$1\frac{1}{4}$ in. long. Apparently not Ebenaceous. Cfr. *Juglans acuminata*, Ludw. and J. *ventricosa*, Ludw.

12. DIOSPYROS PRIMÆVA, Heer, Phyll. Crét. der Nebraska, p. 19. t. I. fig. 6, 7.

D. foliis oblongo-ovalibus, apice obtusiusculis, integerrimis; nervis secundariis serpentinis, ramosis, camptodromis.

Schimp. Pal. Vég. II. p. 948. n. 14 (1872).

Upper cretaceous deposits (?) in Nebraska, N. America.

D. anceps of the Miocene of Europe and *D. alaskana* of the molasse of North America greatly approach this species from Nebraska.

13. DIOSPYROS AURICULA, Ung. Gen. et Sp. Pl. Foss. p. 436 (1850), Syll. Pl. Foss. III. p. 26. t. IX. f. 1—4.

D. foliis ovatis utrinque attenuatis integerrimis membranaceis, nervo primario valido, nervis secundariis subremotis, sub angulo acuto egredientibus, sursum repetito-arcuato-anastomosatis, subarcuatis, apice ramosis; calyce quadrifido vel quinquefido deciduo patente, laciniis subquadratis emarginatis basi callosis striatisque semipollicaribus.

Schimp. Pal. Vég. II. p. 947. n. 10 (1872).

D. auriculata, Stichler, Synops. Pflanz. Vorw. I. 147 (1861), non Wight.

Eocene; Croatia, in marly schist at Radoboj.

Leaves somewhat narrowed at both ends, obtuse at apex, $3\frac{5}{8}$—$4\frac{3}{8}$ in. long by $1\frac{1}{8}$—$1\frac{11}{16}$ in. wide; petiole $\frac{6}{16}$—$\frac{7}{16}$ in. long; calyx-lobes $\frac{1}{2}$ in. long. The characters given agree with *Diospyros*.

14. DIOSPYROS DUBIA, Goeppert, Tert. Fl. Java, p. 47. t. XII. f. 72 (1854).

D. foliis ovatis subobtusis subcoriaceis integerrimis penninerviis, nervis secundariis alternantibus subremotis sub angulo acuto 60° circa exorientibus adscendentibus curvatis ramosis, ramulis ante marginem in maculas transeuntibus in rete solutis.

Schimp. Pal. Vég. II. p. 947. n. 9 (1872), non Wight.

Pliocene?; Pesawahan, Java, *Junghuhn* 353.

Leaf (estimated from a fragment) probably about 2 in. long by 1 in. wide. Family quite conjectural.

15. DIOSPYROS VARIANS, Saporta in Ann. Sc. Nat. ser. V. vol. III. p. 111. t. IV. fig. 14, t. VI. fig. 4 (1865), vol. VIII. p. 91. t. X. fig. 7, 8 (1867).

D. foliis lanceolatis ellipticis oblongo-lanceolatis vel ovatis, apice obtusis quandoque breviter attenuatis, basi parum inæqualibus, subcoriaceis, breviter petiolatis, integerrimis; petiolo transverse rugoso; nervo primario valde expresso; nervis secundariis tenuibus numerosis reticulatis; nervis tertiariis in rete flexuoso subtiliter venuloso coeuntibus.

Schimp. Pal. Vég. II. p. 944. n. 1 (1872).

Tertiary; S.E. France, frequent.

Leaves $2\frac{1}{6}$ in. long by $\frac{1}{3}$ in. wide, $3\frac{1}{4}$ in. by $1\frac{1}{5}$ in., $3\frac{1}{3}$ in. by $\frac{3}{4}$ in.

16. DIOSPYROS OBSCURA, Sap. Etud. II. p. 283 in Ann. Sc. Nat. ser. V. vol. IV. p. 138 (1865).

D. foliis lanceolatis, coriaceis, breviter lateque petiolatis; nervo primario valido, secundariis obliquis, secus marginem areolatis, inconspicuis.

Schimp. Pal. Vég. II. p. 947. n. 8 (1872).

Upper Tertiary; S.E. France, Armissan; rare.

Only differs from *D. varians* by thicker and little longer petiole, by the more regularly lanceolate leaves, and by the less ramified secondary ascending veins, which are united near the margin by very obtuse curves.

17. DIOSPYROS PALÆOGÆA, Ettingsh. Foss. Fl. Bilin in Denkschrift. Akad. Wissensch. Math.-Naturw. XXVIII. p. 233. t. XXXVIII. f. 24—26, 32 (1868).

D. foliis ovalibus, obtuse acuminatis, basi angustatis, integerrimis, coriaceis, petiolatis, 4—5 pollices longis, nervo primario distincto, nervis secundariis crebris, tenuibus, flexuosis, ramosis; baccâ globosâ, exsuccâ, fere pollicari; calyce firmo, quinque-partito, patente, deciduo, semipollicari, laciniis ovato-lanceolatis, acuminatis.

Schimper, Pal. Vég. vol. II. p. 945. n. 4 (1872).

Miocene; Tripoli de Kutschlin, Bohemia.

Leaf $4\frac{3}{4}$ in. by $1\frac{7}{8}$ in. wide; petiole $\frac{1}{8}$ in. long; calyx nearly $\frac{3}{4}$ in. in diameter.

18. DIOSPYROS HÆRINGIANA, Ettingsh. Tert. Fl. Häring, p. 61. t. XXI. f. 26, t. XXII. f. 11 (1851).

D. foliis lanceolatis vel elongato-lanceolatis, petiolatis, integerrimis, sub-coriaceis, utrinque angustatis, petiolo rugoso; nervatione dictyodroma, nervo primario valido, nervis secundariis tenuibus, sub angulo 60—80° orientibus, arcuatis, ramosis; calyce 4-fido, segmentis parum productis, acutis.

Saporta in Ann. Sc. Nat. ser. iv. vol. XIX. p. 72. t. IX. f. 1 (1863); Schimper, Pal. Vég. II. p. 945. n. 2 (1872).

Tertiary; in calcareous bituminous schist at Häring, Tyrol; marly beds, S.E. France.

Leaves $1\frac{4}{5}$—$3\frac{1}{2}$ in. long by $\frac{1}{2}$—$\frac{7}{10}$ in. wide, narrowly elliptical, obtuse at apex; calyx $\frac{1}{4}$ in. in diameter.

19. DIOSPYROS PANNONICA, Ettingsh. Foss. Fl. Wien, p. 19. t. III. f. 8 (1851).

D. foliis ellipticis, basi angustioribus, integerrimis, petiolatis, nervis secundariis undulatis, sub angulo 50—60° orientibus, apice ramosis et in rete abeuntibus, nervis reticulatis e nervo primario sub angulo recto, e nervis secundariis sub angulo acuto egredientibus, ramosis.

Schimper doubtfully unites this to *D. anceps*, Heer.

Vienna, in marly schist.

Leaf $2\frac{2}{3}$ in. long by $1\frac{1}{5}$ in. wide. Very like leaf of *D. brachysepala*, but apparently more obtuse.

20. DIOSPYROS ROYENA, Ung. Syll. Pl. Foss., pag. iii., in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXV. p. 29. t. IX. fig. 18, 19 (1866).

D. foliis ovalibus breviter petiolatis integerrimis sesquipollicem longis, nervo primario distincto, nervis secundariis crebris tenuibus ramosis; calyce firmo quinquelobo patente deciduo semipollicari, laciniis acuminatis.

Schimp. Pal. Vég. II. p. 952 (1872).

In marly schist; Radoboj, Croatia.

Leaves $1\frac{1}{4}$ in. long by $\frac{5}{8}$ in. wide; petiole $\frac{3}{16}$ in. long. Calyx-lobes $\frac{3}{16}$ in. long by $\frac{1}{8}$ in. wide. Family quite uncertain.

21. DIOSPYROS STENOSEPALA, Heer, Fl. Foss. Alaskana, p. 35. t. VIII. fig. 7, 8 (1869).

D. foliis ovalibus, basi rotundatis, integerrimis, subcoriaceis; calyce fructifero quadripartito, lobis oblongis, apice rotundatis.

Schimp. Pal. Vég. II. p. 949. n. 16 (1872).

Miocene; English Bay, Alaska, N. America, *Furuhjelm.*

The shape and size of the calyx correspond with those in *D. brachysepala*, but the lobes are longer and narrower. Calyx $\frac{3}{4}$ in. in diameter; leaves $1\frac{3}{4}$ in. wide.

22. Diospyros bilinica, Ettingsh. Foss. Fl. Bilin, II. p. 45, in Denkschrift. Akad. Wissensch. Math.-Naturw. XXVIII. p. 233. tab. XXXIX. fig. 17, 18 (1868).

D. foliis coriaceis, oblongo-ellipticis, crassiuscule petiolatis, basi rotundatis, apice subobtusis, integerrimis, nervo primario basi valido, apicem versus sensim angustato, nervis secundariis sub angulis acutis orientibus, tenuissimis, subremotis, nervis tertiariis obsoletis; calyce profunde quadrifido, deciduo, patente, minimo, laciniis ovalibus, longitudinaliter nervoso-striatis, basi coarctatis.

Schimp. Pal. Vég. II. p. 947. n. 10 (1872).

D. bohemica, Schimp. *l. c.* p. 945. n. 3.

Miocene; menilite-opal of the valley of Schichow near Bilin, Bohemia.

Leaf $4\frac{1}{2}$ in. long by $1\frac{5}{8}$ in. wide; petiole $\frac{3}{8}$ in. long. Calyx $\frac{1}{4}$—$\frac{3}{8}$ in. in diameter, lobes rounded. The leaf much resembles that of *D. Auricula,* Ung., but differs by the more considerable thickness of the petiole and midrib.

23. Diospyros oblongifolia, Heer, Braunk. von Bornstädt in Abhandl. d. Nat. Gessellsch. zu Halle. XI. Bd. p. 17. t. III. f. 9 (1869).

D. foliis oblongis, utrinque obtusis, integerrimis; nervis suprabasilaribus ultra medium productis, cæteris utrinque 4 remotis, patentioribus, apice cum nervis tertiariis transversis arcuato-conjunctis, nervulis e nervo medio et e nervis secundariis sub angulo recto emissis, inter se parallelis, reticulo minuto.

Schimp. Pal. Vég. II. p. 950. n. 19 (1872).

Eocene; Bornstädt near Eisleben, Saxony, about N. Lat. $51\frac{1}{2}°$.

Leaf nearly 3 in. long by rather more than 1 in. wide, rounded at both ends.

24. Diospyros Parthenon, Ung. Syll. Pl. Foss., pug. iii., in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXV. p. 29. t. IX. f. 8 (1866).

D. foliis ovato-acuminatis basi subcordatis integerrimis membranaceis longe petiolatis, nervo primario valido, nervis secundariis crebris tenuibus subpatentibus apice diviso anastomosatis.

Schimper, Pal. Vég. II. p. 948. n. 13 (1872).

Miocene; lignite at Wetterau, Germany.

Leaf 4 in. long by $1\frac{9}{16}$ in. wide; petiole $\frac{13}{16}$ in. long. The long petiole associated with a subcordate leaf-base is not suggestive of Ebenaceæ.

25. Diospyros obliqua, Ung. Syll. Pl. Foss., pug. iii. in Denkschrift. Kais. Akad. Wissensch. Math.-Naturw. XXV. p. 29. t. IX. fig. 17, 17* (1866).

D. calyce quinquelobo deciduo minimo patente, laciniis e basi lata angustatis linearibus obtusis.

Schimper, Pal. Vég. II. p. 951. n. 23 (1872).

In marly schist, Radoboj, Croatia.

Calyx about $\frac{1}{2}$ in. in diameter; lobes about $\frac{1}{8}$ in. long. Like *Royena Myosotis*, Ung., but calyx-lobes narrower. Cf. *Porana*.

26. EBENACITES RUGOSUS, Sap. in Sap. et Math. Exam. Anal. Fl. Tert. Provence, p. 31 (1861).

E. foliis (?) ovatis, petiolatis, integerrimis; nervis secundariis curvatis, nervis tertiariis sinuosis transversim reticulatis; floribus unisexualibus; calyce 5-lobo lobis inæqualibus, extus rugoso-sulcatis, intus lævibus, æstivatione imbricatis; masculorum corollâ erectâ breviter urceolatâ calycibus breviore; feminorum segmentis calycinis primùm erectis, ovarium 2—3-stylum foventibus, demùm patentibus, indurato-persistentibus, baccam globosam ipsis breviorem stipantibus.

Diospyros rugosa, Sap. in Ann. Sc. Nat. ser. iv. vol. xvii. p. 264. t. xi. f. 3. A, B, C, D, E, F. (1862); Schimp. Pal. Vég. II. p. 946. n. 7. (1872).

Tertiary; beds of gypsum at Aix, S.E. France, common.

The styles are not Ebenaceous in character. Male flower $\frac{3}{8}$ in. diameter, calyx and corolla lobed half way, lobes ovate; female flower $\frac{1}{4}$ in. thick, calyx deeply lobed, lobes ovate acuminate, styles $\frac{1}{5}$ in. long.

27. DIOSPYROS ZOLLIKOFERI, Ung. Syll. Pl. Foss., pug. iii., in Denkschrift. Kais. Akad. xxv. p. 27. t. ix. f. 6 (1866).

D. seminibus ovoideis, compressis, distinctis, numero octo in orbem dispositis—residuis fructus baccati globularis.

Schimp. Pal. Vég. II. p. 951. n. 21. (1872).

Miocene; Hengsberg, Styria.

Seeds $\frac{5}{26}$—$\frac{3}{8}$ in. long by $\frac{1}{8}$—$\frac{3}{16}$ in. wide.

IV. KEY TO THE FOSSIL SPECIES OF MACREIGHTIA.

Peduncles not thickened upwards
- Leaves $1\frac{2}{3}$—$2\frac{1}{3}$ in. long by $\frac{3}{5}$—$\frac{1}{10}$ in. wide. — 1. *M. germanica.*
- Leaves more than 3 in. long by $1\frac{1}{3}$ in. wide. — 2. *M. microcalyx.*

Peduncles thickened upwards.
- Calyx-lobes ovate-acuminate. — 3. *M. longipes.*
- Calyx-lobes ovate or cuneiform, obtuse — 4. *M. münzenbergensis.*

1. MACREIGHTIA GERMANICA, Heer, Fl. Tert. Helvet. III. p. 13. t. ciii. fig. 1, 2 (1859).

M. foliis late-lanceolatis acuminatis in petiolum mediocrem attenuatis integerrimis vel margine inæquali passim denticulatis coriaceeis, nervo medio robusto, nervis secundariis e *nervo primario sub angulo acuto egredientibus subsimplicibus rectis parallelis; calyce firmo pedunculato tripartito, lobis basi ovato-acuminatis nervosis; baccâ rotundâ calyce basi cinctâ.*

Ettingsh. Foss. Fl. Bilin, p. 234. (1868); Schimp. Pal. Vég. II. 953. n. 1 (1872); Ung. Syll. Pl. Foss., pug. iii. p. 26. t. viii. fig. 12, 13 (1866).

Celastrus europæus, Ung. Gen. et Sp. Pl. Foss. p. 459 (1850), Syll. Pl. Foss., pug. ii. p. 10. t. ii. fig. 10—15 (1864).

Tertiary; Parschlug, Styria; Croatia; Oeningen, &c.

Calyx only like Macreightia (Maba) in being trimerous; if the leaves are sometimes denticulate as stated, the plant cannot belong to Ebenaceæ. In *Celastrus europæus* the leaves measure $1\frac{2}{3}$—$2\frac{1}{3}$ in. long by $\frac{3}{5}$—$\frac{1}{10}$ in. wide; petioles about $\frac{1}{3}$ in. long.

2. MACREIGHTIA MICROCALYX. Ettingsh. Foss. Fl. v. Bilin in Denkschrift. Akad. Wissensch. Math.-Naturw. xxviii. p. 234. t. xxxix. f. 2—5. n. 2 (1868).

M. foliis lanceolo-oblongis, basi angustatis, obtusis, apicem versus angustatis, margine integerrimis, nervis secundariis camptodromis, nervo primario valido, nervis tertiariis obsoletis; calyce submembranaceo, pedunculato, tripartito, extus piloso, lobis ovato-acutis, basi latis, apice breviter cuspidatis, nervoso-striatis; baccâ rotundâ, calycis basi cinctâ.

Schimper, Pal. Vég. II. p. 953. n. 2 (1872).

Miocene; Kutschlin, Bohemia.

Leaf $1\frac{1}{3}$ in. wide by more than 3 in. long. Calyx $\frac{1}{3}$—$\frac{1}{2}$ in. long; perhaps a fourth lobe at the back of the impression is concealed by the front ones.

3. MACREIGHTIA LONGIPES, Ettingsh. Beitr. z. Kenntn. d. Tertfl. Steierm. p. 58. t. IV. f. 10, 11.

M. calyce longe pedunculato, pedunculo sursum sensim incrassato, lobis ovato-acuminatis. acutis.

Schimp. Pal. Vég. II. p. 954. n. 3 (1872).

Lignite at Leoben, Styria, Austria.

4. MACREIGHTIA MÜNZENBERGENSIS, Ettingsh. Foss. Fl. d. ält. Braunk. d. Wetterau, p. 59.

M. calyce tripartito lobis ovatis vel cuneiformibus, obtusis, nervosis.

Schimp. Pal. Vég. II. p. 954. n. 4 (1872).

Hydrocharis ovata, Ludw. Palæontogr. in Mey. Beitr. Naturg. Vorw. VIII. t. XXIV. f. 6 (1860).

Tertiary; sandstone at Münzenberg, Darmstadt, S.W. Germany.

Calyx (?)-lobes $\frac{2}{7}$—$\frac{3}{8}$ in. long. As much like a calyx of Barringtonia or a split fruit of Viola as Maba. Peduncles thickened upwards.

The following names of fossil species have been published apparently without descriptions or with extremely meagre ones.

Diospyros laurina, Massal. Syllab. Pl. Foss. Tert. Venet. p. 77 (1859).

Macreightia italica, Massal. *l. c.*

Macreightia? umbellata, Massal. *l. c.*

Diospyros discreta, Saporta, Vég. Sud-Est France, Ép. Tert. in Ann. Sc. Nat., ser. v. vol. xv. p. 321 (1872).

Diospyros ambigua, Saporta, *l. c.*, non Vent.

Diospyros rhodendrifolia, Saporta, *l. c.*

Diospyros corrugata, Saporta, *l. c.*

Diospyros styracifolia, Saporta, *l. c.* p. 333; *D. tyracifolia,* Sap. in. Bull. Soc. Géol. France, xxv. p. 895 (1868).

Diospyros raminervis, Saporta, *l. c.* p. 333; in Bull. Soc. Géol. France, xxv. p. 895 (1868).

Diospyros Scheuzeri, A. Br. ex Ung. Pflanzenwelt, p. 233 (1851), is *Labatia Scheuzeri,* A. Br. ex Stiehler, Synops. Pflanz. Vorwelt, I. p. 147 (1861).

ADDITIONS AND CORRECTIONS.

During the time that the previous sheets have taken in passing through the press numerous new specimens of Ebenaceæ have reached this country and been presented to my notice, containing indeed several new species and affording additional matter for the more complete knowledge of old ones. So far as circumstances allowed, I have incorporated the additional matter in its proper place, and such information as was not sufficiently early for that purpose, I now add at the end: I also take the same opportunity of correcting the misprints, mistakes, and omissions that have been noticed, and of making any slight additions that further research has rendered desirable. The estimate given on page 61 for the number of recent species in the family and in the genera *Maba* and *Diospyros* will require a slight increase in each case; thus the whole Natural Order contains more than 260 recent species, of which about 100 are new or not previously described; and if the fossils are included the whole number will be at least 300.

P. 27, l. 10. *For* ILICINIÆ *read* ILICINEÆ.
P. 28, l. 5 from bottom. For *Bertolini* read *Bertoloni.*
P. 30, l. 11. *For* Paralia *read* Paralea.
„ l. 27. *For* Blum. *read* Blanco.
P. 33, l. 4. *Add*, Java, Sumatra and Borneo.
„ l. 18. *Strike out* Java (?).
P. 37, l. 10. *Strike out* Mart.
„ l. 18, 24, 33. *For* capræfolia *read* capreæfolia.
P. 40, l. 19. *For* 4117. Diospyros sylvatica, Roxb. *read* 4117. Diospyros sylvatica, Wall.
„ l. 24. *For* Roxb. *read* Wall.
P. 41. *Insert*, among the numbers of HB. GRIFFITH AND HELFER, 3620. Diospyros Horsfieldii.
P. 44, l. 7. *For* Hendelotii *read* Heudelotii.

P. 44. *Insert*

BECCARI. 1865—1868.

Borneo, District Sarawak.

No. 1399. Maba punctata.
1422. Diospyros coriacea.
1423. Maba punctata.
1429. Maba (?) cordata.
1550. Maba Maingayi.
1560. Diospyros graciliflora.
1600. Diospyros lateralis.
1670. Maba merguensis.
1787. Diospyros pergamena.
1822. Maba Teijsmanni.
1837. Maba (?) cordata.
1892. Diospyros discolor, Willd.
1948. Maba Beccarii.
1949. Diospyros (sp.).
1973. Diospyros buxifolia.

No. 2285. Diospyros rigida.
2486. Diospyros fuliginea.
2492. Diospyros Beccarii.
2542. Diospyros dictyoneura.
2591. Diospyros Beccarii.
2612. Diospyros asterocalyx.
2615. Diospyros dictyoneura.
3052. Cfr. Diospyros Toposia, Ham.
3101. Cfr. Diospyros.
3120. Diospyros (sp.).
3225. Diospyros plectosepala.
3455. Diospyros coriacea.
3567. Diospyros rotundiflora.
3568. Maba myrmecocalyx.

P. 44, l. 20. *For* 1854 *read* 1852.
„ Under GERRARD AND McKEN *insert* 18. Royena cordata, E. Mey.
P. 45. „ „ „ 1606. Euclea daphnoides.
„ l. 10. *For* cordata, E. Mey. *read* glandulosa, Harv.
„ At bottom, *add*, 1740. Cape of Good Hope. Euclea lanceolata, E. Mey.
P. 46. Under BOLUS *add*, 128. Graaf Reinet. Royena pallens, Thunb.
527. „ Royena cordata, E. Mey.
572. „ Euclea ovata, Burch.
655. „ Euclea undulata, Thunb.
P. 50. Under BERNIER *add*, 259 (excl. fr.). Madagascar. Diospyros haplostylis, Boiv.
P. 51, l. 9. *For* Maba capreæfolia *read* Diospyros capreæfolia, *and prefix* 1011.
P. 57, l. 4 from bottom. After *lobis* insert *integris*.
P. 60, l. 2. *After* equal *insert* entire.
P. 67, l. 32. *For* p. 38 *read* p. 37.
P. 69, *Insert* 1827. Ferreola guineensis, Schum. and Thonn. Plant. Guin. p. 448. Guinea, Africa.
„ l. 4 from bottom. *For* Sweet. *read* Sweet,.
P. 72. *Insert* 1843. Rymia polyandra, Endl. Cat. Hort. Acad. Vindob. II. p. 123. n. 4583. Cape of Good Hope.
P. 73. *Insert* 1850. Anona macrophylla, Ung. Gen. et Sp. Pl. Foss. p. 442. Croatia.
„ last two lines *for* 1850 *read* 1851.
P. 75. *Insert* 1861. Diospyros longifolia, Spruce in Journ. Proc. Linn. Soc. Lond. v. p. 7. South America.
1861. Diospyros glomerata, Spruce in Journ. Proc. Linn. Soc. Lond. v. p. 7. South America.

1861. Diospyros polyandra, Spruce in Journ. Proc. Linn. Soc. Lond. V. p. 7. S. America.

1861. Macreightia myristicoides, Spruce in Journ. Proc. Linn. Soc. Lond. V. p. 8. S. America.

„ l. 6 from bottom. *For* Island *read* Islands.

P. 76, *insert* 1865. Diospyros obscura, Sap. Étud. II. p. 283 in Ann. Sc. Nat. ser. V. vol. IV. p. 138. S.E. France.

P. 77, *insert* 1868. Diospyros palæogæa, Ettingsh. Foss. Fl. Bilin in Denkschrift. Akad. Kais. Wissensch. Math.—Naturw. XXVIII. p. 233. t. XXXVIII. fig. 24—26, 32. Bohemia.

1868. Diospyros bilinica, Ettingsh. Foss. Fl. Bilin in Denkschrift. Akad. Kais. Wissensch. Math.—Naturw. XXVIII. p. 233. t. XXXIX. fig. 17, 18. Bohemia.

1868. Diospyros paradisiaca, Ettingsh. Foss. Fl. Bilin in Denkschrift. Akad. Kais. Wissensch. Math. Naturw. XXVIII. p. 234. t. XXXVIII. fig. 29—31, 34. Bohemia.

1868. Macreightia microcalyx, Ettingsh. Foss. Fl. Bilin in Denkschrift. Akad. Kais. Wissensch. Math.—Naturw. XXVIII. p. 234. t. XXXIX. f. 2—5.

1868. Diospyros græca, Saporta in Bull. Soc. Géol. France XXV. p. 321. S. E. France.

1868. Diospyros styracifolia, Saporta in Bull. Soc. Géol. France XXV. p. 895. S. E. France.

1868. Diospyros raminervis, Saporta in Bull. Soc. Géol. France XXV. p. 895. S. E. France.

1869. Diospyros oblongifolia, Heer Braunk. v. Bornstädt in Abh. Nat. Gesell. Halle, XI. Bd. p. 17. t. III. fig. 9. Saxony.

1869. Diospyros stenosepala, Heer Fl. Foss. Alaskana in Kongl. Svenska Vetenskaps—Akad. Handl., Ny Följd, VIII. p. 35. t. VIII. fig. 7, 8. N. America.

Macreightia longipes, Ettingsh. Beitr. z. Kenntn. d. Steierm. p. 58. t. IV. fig. 10, 11, ex Schimp. Pal. Vég. II. p. 954 (1872). Austria.

Macreightia münzenbergensis, Ettingsh. Foss. Fl. d. ält. Braunk. d. Wetterau, p. 59, ex Schimp. Pal. Vég. II. p. 954 (1872). Germany.

Diospyros primæva, Heer Phyll. Crét du Nebraska, p. 19. t. I. fig. 6, 7. N. America.

1872. Diospyros bohemica, Schimp. Tr. Pal. Vég. II. p. 945. n. 3. Bohemia.

1872. Diospyros alaskana, Schimp. Tr. Pal. Vég. II. p. 949. n. 17. N. America.

1872. Diospyros Roxburghi, Carrière in Revue Horticole, p. 253. N.E. India.

1872. Diospyros discreta, Saporta in Ann. Sc. Nat., ser. V. vol. XV. p. 321 (*sine descriptione*). S.E. France.

1872. Diospyros rhododendrifolia, Saporta in Ann. Sc. Nat., ser. V. vol. XV. p. 321 (*sine descriptione*). S.E. France.

1872. Diospyros corrugata, Saporta in Ann. Sc. Nat., ser. V. vol. XV. p. 321 (*sine descriptione*). S.E. France.

P. 92, l. 7. *For* Stamens 16—17 *read* Stamens 16—22.

P. 94, l. 25. *For* Stamens 16—17 *read* 16—22.

PP. 95, 96. *Euclea rigida*, E. Mey. must be removed from *E. pseudebenus*, E. Mey. to *E. lancea*, Thunb.

P. 99, l. 3 from bottom. *For* Stamens 16 *read* Stamens 12—16.

P. 100. To the localities for EUCLEA DIVINORUM *add* Basuta country, *T. Baines!*

„ *Add* as a synonym of EUCLEA MULTIFLORA

Diospyros (Sp.), Salt, Voyage Abyss. p. 14 (1814);

And among the localities for this species, *add* Sofala Bay, Mozambique, 19 August, 1809, *Salt!*

P. 103, l. 10 from bottom. *For* (1847) *read* (1851).

P. 104, l. 13. *After* Fruit *insert* edible.

P. 106. *To* EUCLEA UNDULATA, Thunb. *add* Wooded chasms, Swellendam, a tree, *Lichtenstein;* Basuta country, "Tolangoola," *T. Baines!*. According to Thunberg, the berries when bruised and fermented yield a vinegar.

P. 107, last line. *After* pilose *insert* except *M.* (?) *cordata.*

P. 141. In the character of MABA CORDATA *insert*

floribus masculis tubulosis, 5-meris, lobis lanceolatis, calyce partito, staminibus 12, *glabris, inæqualibus.*

And in the description *insert*

♂. Flowers tubular, $\frac{3}{4}$ in. long, pentamerous. Calyx $\frac{1}{3}$ in. long, partite; lobes narrowly lanceolate, glabrous inside. Corolla-tube equalling the calyx, pubescent outside above; lobes $\frac{3}{8}$ in. long, oval-lanceolate, subacute, puberulous outside, glabrous inside. Stamens 12, unequal, glabrous, inserted on the receptacle; anthers apiculate; filaments unequal, more or less combined at base. Ovary 0. Pedicels $\frac{1}{8}$ in. long; bracts caducous, unequal. Borneo, *O. Beccari!* n. 1837.

P. 142, l. 16. *For* MSS. *read* in Journ. Proc. Linn. Soc. Lond. v. p. 8 (1861).

P. 144. At the end of the descriptions of the species of MABA *insert*

EXCLUDED SPECIES OF MABA.

Maba Cargillia, F. Muell. Fragm. v. p. 162 (1866) = *Diospyros Cargillia,* F. Muell.

Maba pentamera, F. Muell. Fragm. v. p. 163 (1866) = *Diospyros pentamera,* Woolls et F. Muell.

Maba quadridentata, F. Muell. Fragm. v. p. 162 (1866) = *Diospyros mabacea,* F. Muell.

P. 144, l. 2 from bottom. *Add* non Solander ex Lowe Man. Fl. Madeira vol. II. p. 34 (1872).

P. 151, l. 11. *For* glabrous *read* subglabrous.

P. 159, l. 19. *Prefix the reference* Kern. Hort. Semperv. t. 64.

P. 161, l. 16. *Add* ; non Wall.

P. 197, l. 1. *For* Golunto *read* Golungo.

P. 201, l. 7 from bottom. *Add* ; non Hort.

P. 218, l. 9. *After* Bat. *insert* II.

P. 221, l. 17. Add *D. dioica,* Spanoghe in Hook. Comp. Bot. Mag. I. p. 348 (1835).

P. 222. Among the localities for DIOSPYROS MONTANA, Roxb., *insert*

Pegu, *Kurz!* 3008, 3009.

P. 225. At end of synonymy of DIOSPYROS VIRGINIANA, Linn., *add*

D. stricta, Hort. ex Loud. Encycl. *l.c.*, non Roxb.

D. digyna, Hort. ex Loud. Encycl. *l.c.*, non Jacq.

P. 240, l. 4 from bottom. *Add* non Heer.

P. 244. To the synonymy of DIOSPYROS EBENASTER Retz. *add* Cfr. *Lolin,* Valentyn, Oost-Ind. Deel III. Stuk i. p. 223. t. LXIV (1726).

P. 286, l. 7. *For* rhodendrifolia *read* rhododendrifolia.

ALPHABETICAL INDEX OF THE LATIN NAMES.

The Numbers on the right-hand of the names denote the pages on which the names severally occur; the Numbers on the left-hand, in the case of adopted specific names of recent species, denote the number of the species under its genus.

INDEX OF SUBJECTS.

III. *On the equation which determines the form of the strata in Legendre's and Laplace's theory of the Figure of the Earth.*

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1. The equation to which the present memoir refers is the following:

$$\frac{Y_n}{a}\int_0^a \rho a^2 da - \frac{1}{(2n+1)\,a^{n+1}}\int_0^a \rho \frac{d}{da}\left(Y_n a^{n+3}\right) da - \frac{a^n}{2n+1}\int_a^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = 0 \;\ldots\ldots\ldots\ldots(1).$$

This equation will be found in the *Mécanique Céleste*, Livre III. § 30. Laplace uses substantially this notation; but he denotes our a_1 by unity. English writers generally use accented letters for the variables under the integral sign; thus instead of $\int_0^a \rho a^2 da$ they use $\int_0^{a'} \rho' a'^2 da'$. This is more explicit than Laplace's notation; but with a little care no error will arise from the omission of the accents. In Laplace's notation symbols which are not under the integral sign refer to the value a; when they refer to the special value a_1 we shall use the suffix 1.

The above equation, as is well known, presents itself in the theory of the Figure of the Earth considered as a heterogeneous fluid; and in that theory the equation has to hold for all positive integral values of n except $n = 2$. In the case of $n = 2$ the right-hand member is not zero.

2. Clairaut first obtained the equation for the case of $n = 2$: see his *Figure de la Terre*, page 273. D'Alembert afterwards arrived at results equivalent to the cases of $n = 1$ and $n = 3$, besides the case of $n = 2$: see his *Recherches...du Systême du Monde*, Vol. III. pages 226...228, and his *Opuscules Mathématiques*, Vol. V. page 5.

The general equation for all positive integral values of n appears in two memoirs, one by Legendre and one by Laplace, both published in the memoirs of the French Academy for 1789. Legendre claims for himself the priority of date; see page 372 of the volume.

3. It has been the object of mathematicians to demonstrate that except $Y_n = 0$ there is no value of Y_n which will satisfy (1); or at least no value which is admissible

in the theory of the Figure of the Earth. I propose to examine these demonstrations, to shew distinctly on what assumptions they rest, and what objections may, in my opinion, be brought against some of them.

4. From the primary equation (1) a certain differential equation has been derived; this was effected by Clairaut and D'Alembert for their particular cases, and by Legendre and Laplace generally. It will be necessary for our purpose to give the steps of the process separately.

Multiply (1) by a^{n+1}; thus

$$Y_n a^n \int_0^a \rho a^2 da - \frac{1}{2n+1}\int_0^a \rho \frac{d}{da}(Y_n a^{n+3})\, da - \frac{a^{2n+1}}{2n+1}\int_a^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = 0 \text{ ....(2)}.$$

Differentiate with respect to a; thus we obtain

$$\frac{dY_n}{da} a^n \int_0^a \rho a^2 da + na^{n-1} Y_n \int_0^a \rho a^2 da - a^{2n}\int_a^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = 0 \text{ ....(3)}.$$

Divide by a^{2n}; thus

$$\frac{dY_n}{da} a^{-n} \int_0^a \rho a^2 da + na^{-n-1} Y_n \int_0^a \rho a^2 da - \int_a^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = 0 \text{ ....(4)}.$$

Differentiate with respect to a; thus we obtain

$$\frac{d^2 Y_n}{da^2}\frac{1}{a^n}\int_0^a \rho a^2 da + 2\frac{dY_n}{da}\frac{\rho}{a^{n-2}} - Y_n\left\{\frac{n(n+1)}{a^{n+2}}\int_0^a \rho a^2 da - \frac{2\rho}{a^{n-1}}\right\} = 0 \text{ ....(5)}.$$

This is the differential equation deduced from the primary equation (1); any value of Y_n which satisfies (1) must satisfy (5): and as (5) may be considered an equation of a more usual kind than (1) the labours of mathematicians have naturally been much directed to the discussion of (5).

5. But of course the primary equation (1) is really that which is to be solved, if possible; so that supposing we have a value of Y_n which satisfies (5) we have to examine conversely if it will also satisfy (1), or to investigate the conditions under which it will satisfy (1). This is a very simple matter, but does not appear to me to have ever been distinctly considered.

We may represent (1) symbolically thus, $I(Y_n) = 0$, where I denotes a certain complex operation chiefly of the nature of integration. Now suppose we have obtained a solution of (5), say $Y_n = \phi(a)$, we have then to determine the value of $I\{\phi(a)\}$ so as to ascertain whether it vanishes, or under what conditions it will vanish.

We start then with the value $\phi(a)$ for Y_n, which by hypothesis satisfies (5). Integrate with respect to a: thus we arrive at an equation which can only differ from (4) by the presence of some arbitrary constant, that is a constant with respect to a. Thus we have

$$\frac{d\phi(a)}{da} a^{-n} \int_0^a \rho a^2 da + na^{-n-1}\phi(a) \int_0^a \rho a^2 da - \int_a^{a_1} \rho \frac{d}{da}\left\{\frac{\phi(a)}{a^{n-2}}\right\} da = H \text{ ....(6)},$$

where H denotes this arbitrary constant. Multiply by a^{2n}; thus

$$\frac{d\phi(a)}{da}\, a^n \int_0^a \rho a^2 da + n a^{n-1}\phi(a) \int_0^a \rho a^2 da - a^{2n}\int_a^{a_1} \rho \frac{d}{da}\left\{\frac{\phi(a)}{a^{n-2}}\right\} da = Ha^{2n} \quad \text{.............(7)}.$$

Integrate with respect to a. Thus

$$\phi(a)\, a^n \int_0^a \rho a^2 da - \frac{1}{2n+1}\int_0^a \rho \frac{d}{da}\{\phi(a)\, a^{n+3}\}\, da - \frac{a^{2n+1}}{2n+1}\int_a^{a_1} \rho \frac{d}{da}\left\{\frac{\phi(a)}{a^{n-2}}\right\} da = \frac{Ha^{2n+1}}{2n+1} + K \text{.....(8)},$$

where K is another arbitrary constant.

Divide by a^{n+1}. Thus

$$\frac{\phi(a)}{a}\int_0^a \rho a^2 da - \frac{1}{(2n+1)\, a^{n+1}}\int_0^a \rho \frac{d}{da}\{\phi(a)\, a^{n+3}\}\, da - \frac{a^n}{2n+1}\int_a^{a_1} \rho \frac{d}{da}\left\{\frac{\phi(a)}{a^{n-2}}\right\} da = \frac{H a^n}{2n+1} + \frac{K}{a^{n+1}} \text{.(9)}.$$

Thus we see that

$$I\{\phi(a)\} = \frac{Ha^n}{2n+1} + \frac{K}{a^{n+1}} \quad \text{..(10)}.$$

Moreover the values of H and K can now be definitely expressed.

For since (6) is true for all values of a, we may put $a = a_1$; thus we see that

$$H = \frac{\int_0^{a_1} \rho a^2 da}{a_1^{2n}} \left\{\frac{d}{da}\, a^n \phi(a)\right\}_1 \quad \text{..(11)}.$$

Again since (8) is true for all values of a, we may put $a = a_1$; thus we see that

$$\frac{Ha_1^{2n+1}}{2n+1} + K = \phi(a_1)\, a_1^n \int_0^{a_1} \rho a^2 da - \frac{1}{2n+1}\int_0^{a_1} \rho \frac{d}{da}\{\phi(a)\, a^{n+3}\}\, da \quad \text{.............(12)}.$$

Thus the values of H and K are assigned by (11) and (12).

It is obvious from (10) that $I\{\phi(a)\}$ will not vanish for all values of a unless we have both

$$H = 0 \text{ and } K = 0.$$

6. Let us now examine Legendre's process for shewing that there is no admissible solution of the primary equation (1), except $Y_n = 0$.

Legendre first considers what we can infer from supposing a very small.

Whatever be the law of density, as we shall assume that the density decreases from the centre to the surface, we may put $\rho = ga^{-m}$, where g and m are constants, m being positive or zero. If m is not zero then the density is infinite at the centre; but this is admissible provided $\int_0^a \rho a^2 da$ vanishes with a, so that the mass may be infinitesimal when the volume is such. Now $\int a^{2-m} da = \frac{a^{3-m}}{3-m} +$ constant; so that provided m is less than 3 the mass does vanish with a.

Legendre denotes $\int_0^a \rho a^2 da$ by σ; and he transforms the differential equation (5) by putting

$$Q = \sigma Y_n.$$

Thus we shall obtain from (5)

$$\frac{d^2Q}{da^2} - n(n+1)\frac{Q}{a^2} - \frac{a^2}{\sigma}\frac{d\rho}{da}Q = 0 \text{ ....(13).}$$

With the assumed value of ρ we have

$$\frac{a^2}{\sigma}\frac{d\rho}{da} = -\frac{m(3-m)}{a^2},$$

so that when a is very small (13) becomes

$$\frac{d^2Q}{da^2} = \{n(n+1) - m(3-m)\}\frac{Q}{a^2}.$$

The solution of this differential equation is

$$Q = Aa^c + Ba^{1-c},$$

where A and B are arbitrary constants, and

$$c = \frac{1}{2} + \sqrt{\left\{\left(n+\frac{1}{2}\right)^2 - m(3-m)\right\}}.$$

The term Ba^{1-c} must be rejected as inadmissible in our case; for if this were retained we should have Q infinite when a is zero, and therefore also Y_n infinite. Thus the only admissible value of the arbitrary constant B is zero. Therefore when a is very small we have

$$Y_n = \frac{Aa^c}{\sigma}.$$

7. Legendre next shews that numerically Y_n increases from the centre to the surface. Divide both sides of (1) by a^n; and then differentiate with respect to a: thus we obtain

$$\frac{dY_n}{da}\frac{\sigma}{a^{n+1}} - \frac{(n+1)\sigma}{a^{n+2}}Y_n + \frac{1}{a^{2n+2}}\int_0^a \rho\frac{d}{da}(Y_n a^{n+3})\,da = 0,$$

therefore

$$\sigma a^{n+1}\frac{dY_n}{da} = (n+1)\sigma a^n Y_n - \int_0^a \rho\frac{d}{da}(Y_n a^{n+3})\,da \text{ ....(14).}$$

Let $$\omega = \int_0^a a^3\frac{d\rho}{da}da, \text{ so that } \sigma = \frac{\rho a^3}{3} - \frac{\omega}{3}.$$

Then, by integration by parts,

$$\int_0^a \rho\frac{d}{da}(Y_n a^{n+3})\,da = \rho a^{n+3}Y_n - \int_0^a a^{n+3}Y_n\frac{d\rho}{da}da$$

$$= (3\sigma + \omega)a^n Y_n - \int_0^a a^n Y_n\frac{d\omega}{da}da$$

$$= 3\sigma a^n Y_n + \int_0^a \omega\frac{d}{da}(a^n Y_n)\,da.$$

Hence (14) may be expressed thus:

$$\sigma a^{n+1}\frac{dY_n}{da} = (n-2)\,\sigma a^n Y_n - \int_0^a \omega \frac{d}{da}(a^n Y_n)\,da \quad\text{.......................(15)}.$$

Now since we assume that ρ decreases from the centre to the surface ω is always negative.

And n may be supposed greater than 2, for the case of $n=1$ is treated by Legendre separately.

Equation (15) then shews that if $a^n Y_n$ and $\frac{d}{da}(a^n Y_n)$ have the same sign at the origin, that is when a is zero, they always have the same sign.

For suppose this sign at the origin to be positive. If the sign of $\frac{dY_n}{da}$ can ever become negative suppose that it first becomes negative when a has the value a_0. Then for this value the right-hand side of (15) would be positive and the left-hand side negative; which is absurd. [Legendre puts the argument differently; but not as I think so decisively.] Now from the value of Y_n in Art. 6, combined with the value of σ, it is obvious that at the origin Y_n and $\frac{dY_n}{da}$ have the same sign; therefore they always have.

8. Then for the final step. We have seen that $I\{\phi(a)\}$ will not vanish unless both H and K vanish. The simplest process would be to remark that if $\phi(a)$ is an admissible solution H will not vanish: see equation (11). For we have just shewn that $\frac{d}{da}\{a^n\phi(a)\}$ cannot vanish for any value of a.

But Legendre did not investigate the value of $I\{\phi(a)\}$, and so his process is less simple. His process is in effect equivalent to shewing that the right-hand member of (12) will not vanish. He introduces ω as in Art. 7, and then it follows that to make the right-hand side of (12) vanish we must have

$$(2n-2)\,\sigma_1 a_1^n \phi(a_1) = \int_0^{a_1} \omega \frac{d}{da}\{a^n\phi(a)\}\,da;$$

and this is impossible, for on the right-hand side we have a quantity of one sign, and on the left-hand side a quantity of the contrary sign.

9. Such then is Legendre's treatment of the problem. His main assumption is that ρ decreases from the centre to the surface. Also he assumes that near the origin the law of density is determined by $\rho = ga^{-m}$, where m is zero or positive and less than 3. It is of course easy to suggest forms for the density which do not admit of being represented by ga^{-m} when a is very small. For instance such a term as $a^2 \log a$ might occur in the expression of the density; and this cannot be expanded in powers of a. The assumption which Legendre makes of the law of density near the origin is required by him for the

purpose of shewing that $a^n Y_n$ and $\frac{d}{da}(a^n Y_n)$ have the same sign at the origin; so that if it should appear a more obvious assumption we may at once assume this instead of assuming the law of density.

We see that Legendre does not undertake to shew that the primary equation $I(Y_n)=0$ has *no* solution, but only that it has *no admissible* solution with respect to the Figure of the Earth; that is he shews that there is no solution in which Y_n is restricted to be always finite.

10. Let us now illustrate the process by examples. We begin with one which has been partly considered by Legendre.

Suppose $n=1$ in (3); then it becomes

$$\sigma \frac{d}{da}(aY) = a^2 \int_a^{a_1} \rho \frac{d}{da}(aY)\, da,$$

where Y is used instead of Y_1 to represent the general value of the unknown quantity.

Multiply by ρ: thus

$$-\rho\sigma \frac{d}{da}(aY) + \rho a^2 \int_a^{a_1} \rho \frac{d}{da}(aY)\, da = 0.$$

Hence by integration

$$\sigma \int_a^{a_1} \rho \frac{d}{da}(aY)\, da = C, \text{ a constant,} \qquad (16);$$

therefore $$\int_a^{a_1} \rho \frac{d}{da}(aY)\, da = \frac{C}{\sigma};$$

therefore $$-\rho \frac{d}{da}(aY) = -\frac{C}{\sigma^2}\frac{d\sigma}{da} = -\frac{C}{\sigma^2}\rho a^2;$$

therefore $$\frac{d}{da}(aY) = \frac{C}{\sigma^2}a^2 \qquad (17);$$

therefore $$aY = C\int \frac{a^2}{\sigma^2}\, da + E,$$

and $$Y = \frac{C}{a}\int \frac{a^2}{\sigma^2}\, da + \frac{E}{a};$$

where E is a constant.

We see from (17) that $\frac{C}{\sigma_1}$ is equivalent to the H of equation (11). Hence $I(Y)=0$ will not be satisfied unless $C=0$.

And we see that then by (12)

$$K = E\sigma_1 - \frac{E}{3}\int_0^{a_1} \rho\, 3a^2 da$$
$$= E\sigma_1 - E\sigma_1 = 0.$$

Hence when $n=1$ if $Y_n = \frac{E}{a}$ we do have $I(Y_n)=0$. This result is of no importance in Legendre's theory because he expressly limits Y_n to be finite; and moreover by properly choosing the origin the term Y_n for the case of $n=1$ is made to disappear. But the result will be of service to us hereafter.

11. Now leaving n unrestricted let us take a special law of density, namely one given by Legendre. Let

$$\rho = ga^{-m} + ha^{m-3};$$

as we have seen in Art. 6 this is admissible provided m is less than 3. With this value of ρ we find that

$$\sigma = \frac{g}{3-m} a^{3-m} + \frac{h}{m} a^m,$$

and

$$\frac{a^2}{\sigma}\frac{d\rho}{da} = -\frac{m(3-m)}{a^2} \quad \text{.......................................} (18).$$

Hence, proceeding as in Art. 6, we find that the differential equation gives

$$Y_n = \frac{Aa^c}{\sigma} + \frac{Ba^{1-c}}{\sigma},$$

where c has the value there assigned.

We will now with Legendre omit the term which involves the negative power of a, as inadmissible in our theory: thus we have

$$Y_n = \frac{Aa^c}{\sigma}.$$

Hence from (11) we have

$$\begin{aligned} H &= \frac{\sigma_1 A}{a_1^{2n}} \left\{\frac{d}{da}\frac{a^{n+c}}{\sigma}\right\}_1 \\ &= \frac{A}{a_1^{2n}\sigma_1}\left\{(n+c)\,a^{n+c-1}\sigma - a^{n+c}\frac{d\sigma}{da}\right\}_1 \\ &= \frac{A}{a_1^{2n}\sigma_1}\{(n+c)\,a^{n+c-1}\sigma - a^{n+c+2}\rho\}_1 \\ &= \frac{A}{\sigma_1} a_1^{c-1-n}\left\{(n+c)\left(\frac{g}{3-m}a_1^{3-m} + \frac{h}{m}a_1^m\right) - (ga_1^{3-m} + ha_1^m)\right\}. \end{aligned}$$

This will vanish of course if $A=0$, which makes Y_n also vanish.

It will also vanish if

$$(n+c)\left(\frac{g}{3-m}a_1^{3-m} + \frac{h}{m}a_1^m\right) = ga_1^{3-m} + ha_1^m \text{...........................} (19).$$

But Legendre says that if (19) be supposed to hold we shall find that $\frac{d\rho}{da}$ is not always negative; this is quite true and is easily verified.

For (19) may be thus expressed

$$\sigma_1 = \frac{\rho_1 a_1^3}{n+c}.$$

But
$$\sigma_1 = \frac{\rho_1 a_1^3}{3} - \frac{1}{3}\int_0^{a_1} a^3 \frac{d\rho}{da}\, da,$$

so that σ_1 is greater than $\frac{\rho_1 a_1^3}{3}$ if $\frac{d\rho}{da}$ is always negative.

And as $n+c$ is greater than 3 we have *a fortiori* σ_1 greater than $\frac{\rho_1 a_1^3}{n+c}$ if $\frac{d\rho}{da}$ is always negative.

It is obvious that (19) might always be satisfied as it determines only the ratio of g to h; and Legendre implies that if we had not assumed ρ to decrease from the centre to the surface we should thus have found a value of Y_n to satisfy $I(Y_n) = 0$. Legendre however seems to overlook the equation (18); in virtue of this if $\frac{d\rho}{da}$ changes sign it must be when σ vanishes, and in his theory it is of course impossible that σ, which is proportional to the mass, should vanish when a is not zero.

Thus it will be seen that equation (19) is quite inadmissible in Legendre's theory, in which ρ and σ denote positive quantities, and σ must increase with a. Therefore with this particular law of density we have consistently with the general proposition no solution admissible of $I(Y_n) = 0$, except $Y_n = 0$.

12. Nevertheless it will be convenient for our purpose to examine the consequences which follow from (19). Assuming that we make H vanish by means of this relation we proceed to consider whether K will also vanish. Return to equation (12).

We have
$$\int_0^{a_1} \rho \frac{d}{da}\{\phi(a)\, a^{n+3}\}\, da$$

$$= \rho_1 a_1^{n+3}\phi(a_1) - \int_0^{a_1} a^{n+3}\phi(a)\frac{d\rho}{da}\, da$$

$$= \rho_1 a_1^{n+3}\phi(a_1) - \int_0^{a_1} a^{n+3} A\frac{a^c}{\sigma}\frac{d\rho}{da}\, da$$

$$= \rho_1 a_1^{n+3}\phi(a_1) + Am(3-m)\int_0^{a_1} a^{n+c-1}\, da$$

$$= \rho_1 a_1^{n+3}\phi(a_1) + \frac{Am(3-m)}{n+c} a_1^{n+c}$$

$$= \rho_1 a_1^{n+3}\phi(a_1) + \frac{\phi(a_1)\, a_1^n \sigma_1}{n+c} m(3-m).$$

Hence
$$K = \phi(a_1)\, a_1^n\left\{\sigma_1 - \frac{1}{2n+1}\left[\rho_1 a_1^3 + \frac{m(3-m)\,\sigma_1}{n+c}\right]\right\}.$$

To make this vanish we must have

$$\{(2n+1)(n+c)-m(3-m)\}\,\sigma_1=(n+c)\,\rho_1 a_1^{\,3} \quad \text{.......................(20).}$$

But
$$n^2+n+\frac{1}{4}-m(3-m)=\left(c-\frac{1}{2}\right)^2;$$

thus
$$2n^2+2nc+n+c-m(3-m)=(n+c)^2,$$

so that (20) becomes

$$(n+c)^2\,\sigma_1=(n+c)\,\rho_1 a_1^{\,3},$$

that is
$$(n+c)\,\sigma_1=\rho_1 a_1^{\,3},$$

and this is in fact equivalent to (19), so that K does vanish.

We have therefore the following result: if $\rho=ga^{-m}+ha^{m-3}$, and $Y_n=\dfrac{Aa^c}{\sigma}$, then $I(Y_n)=0$, provided h and g be taken in the proportion assigned by (19).

13. It may however be conjectured that since Y_n becomes infinite when σ vanishes in the preceding Article we shall find our primary equation (1) is really not satisfied.

But on examination this apparent difficulty will disappear. The first term in (1) is $\dfrac{Y_n\sigma}{a}$; so that provided the product of Y_n into σ remains finite there is nothing inadmissible. Then by integration by parts the other terms of (1) may be transformed into

$$-\frac{1}{2n+1}\rho a^2 Y_n+\frac{1}{(2n+1)\,a^{n+1}}\int^a Y_n a^{n+3}\frac{d\rho}{da}\,da$$

$$+\frac{1}{2n+1}\rho a^2 Y_n-\frac{a^n}{2n+1}\left\{\frac{\rho Y_n}{a^{n-2}}\right\}_1+\frac{a^n}{2n+1}\int_a^{a_1}\frac{Y_n}{a^{n-2}}\frac{d\rho}{da}\,da,$$

that is into

$$-\frac{a^n}{2n+1}\left\{\frac{\rho Y_n}{a^{n-2}}\right\}_1+\frac{1}{(2n+1)\,a^{n+1}}\int_0^a Y_n a^{n+3}\frac{d\rho}{da}\,da+\frac{a^n}{2n+1}\int_a^{a_1}\frac{Y_n}{a^{n-2}}\frac{d\rho}{da}\,da.$$

Now $\dfrac{d\rho}{da}=-\dfrac{m(3-m)\,\sigma}{a^4}$, so that the product $Y_n\dfrac{d\rho}{da}$ remains finite when σ vanishes. Hence the conclusion arrived at in Art. 12 remains undisturbed by the vanishing of σ.

14. In the example of Art. 12 it will be seen that when by a certain relation between constants, we had secured the vanishing of H, then K also vanished. And this in fact might have been anticipated. For since (8) is true for all values of a, we may put $a=0$; then since we assume that Y_n is finite or zero when $a=0$, we see that K must be zero.

Thus any solution of (5) *which is never infinite* necessarily makes $K=0$, and reduces $I(Y_n)$ to $\dfrac{Ha^n}{2n+1}$.

15. Let us now advert to Laplace's discussion of the primary equation. It will be sufficient to touch briefly on this as the *Mécanique Céleste* is readily accessible, and Laplace's method has been reproduced by others. Laplace's discussion is essentially of the same kind as Legendre's, the differences being not important.

Laplace, however, considers that the density must be *finite* at the origin. This amounts to saying that the m in Art. 6 must be zero; and therefore the c of that Article becomes $n+1$. It seems to me that there is no reason for Laplace's assertion, and that Legendre is correct in saying that all that is essential is that the mass should be infinitesimal when the volume is such. If we take the ordinary law of gravity we might at first consider it to be a serious difficulty that the force becomes infinite for particles in contact; but really there is no difficulty inasmuch as the attraction of a sphere on a particle at its surface is infinitesimal when the sphere is such.

By assuming that the density must be finite at the origin, Laplace's discussion is rendered somewhat less general than Legendre's. Laplace gives a proposition like that of Legendre's in Art. 7.

Laplace's final step, like Legendre's, consists in shewing that the right-hand member of (12) does not vanish, except when $\phi(a)$ is always zero.

For by integration by parts we have

$$(2n+1)\,\phi(a_1)\,a_1^{\,n}\int_0^{a_1}\rho a^2 da - \int_0^{a_1}\rho\,\frac{d}{da}\left\{\phi(a)\,a^{n+3}\right\}da$$

$$=\frac{1}{3}(2n-2)\,\rho_1 a_1^{\,n+3}\phi(a_1) - \int_0^{a_1}\left\{\frac{1}{3}(2n+1)\,\phi(a_1)\,a_1^{\,n}a^3 - \phi(a)\,a^{n+3}\right\}\frac{d\rho}{da}\,da.$$

This is necessarily positive. For $\dfrac{d\rho}{da}$ is negative by supposition; and by Art. 7, or by Laplace's equivalent demonstration, $\phi(a)$ increases with a, so that the expression $\dfrac{1}{3}(2n+1)\,\phi(a_1)\,a_1^{\,n} - \phi(a)\,a^n$ is necessarily positive.

16. Laplace's main assumption is that ρ decreases from the centre to the surface. Also he assumes that at the origin the density is finite. It might seem on glancing at his process that he also assumes Y_n to be *always* capable of expansion in powers of a; but in reality it is sufficient for him to assume that such is the case when a is very small. The assumption is made to enable him to shew that Y_n and $\dfrac{dY_n}{da}$ have the same sign at the origin; and therefore we may if we please assume this if it appear more obvious instead of the assumption which Laplace makes.

On the whole although the difference is slight between the processes of Legendre and Laplace, the former appears to me rather the better. Laplace like Legendre confines himself to shewing that the equation $I(Y_n)=0$ has no *admissible* solution except $Y_n=0$.

17. I now proceed to the treatment of the equation which we find in O'Brien's *Mathematical Tracts* published in 1840. As I am persuaded that this treatment is altogether illusory, I must present it fairly to the reader. I will therefore use very nearly the author's words, but change the notation slightly to conform to that already adopted.

Let v and v' be two quantities which satisfy the equations

$$I(v) = a^n, \qquad I(v') = \frac{1}{a^{n+1}} \quad \text{.......................................(21)},$$

and let C and C' be two constants: then

$$Y_n = Cv + C'v'$$

will be a solution of (5); for performing the operation I on both sides of the equation

$$Y_n = Cv + C'v',$$

we have

$$I(Y_n) = I(Cv + C'v') = CI(v) + C'I(v')$$

(evidently from the nature of the operation I),

or

$$I(Y_n) = Ca^n + \frac{C'}{a^{n+1}}$$

by (21).

Now if we get rid of the integral signs in this equation by the same process we have applied to the equation (1), the second side will evidently be made zero by the differentiations, and thus we shall arrive at the same differential equation as before; hence

$$Y_n = Cv + C'v'$$

is a solution of (5). Now this is evidently true whatever be the values of C and C'; hence this solution contains two arbitrary constants, and is therefore the *most general* solution that (5) admits of. Hence all values of Y_n, which satisfy (1), since they also satisfy (5), must be values of $Cv + C'v'$. To determine what values of $Cv + C'v'$ satisfy (1) substitute $Cv + C'v'$ for Y_n in (1), and we find

$$CI(v) + C'I(v') = 0,$$

or

$$Ca^n + \frac{C'}{a^{n+1}} = 0$$

by (21).

Now this equation ought to be true for all values of a; hence $C = 0$, and $C' = 0$; hence it is evident that only one value of $Cv + C'v'$, namely zero, satisfies (1); and hence it follows from the equation (1) that $Y_n = 0$.

This is the demonstration I propose to examine.

18. The first point to which I would call attention is the excessive generality of the proposition which is thus supposed to be established. It will be observed that no restriction is put on Y_n or on ρ. Thus we are not compelled to have Y_n finite when a is zero. And ρ is not such a quantity as would necessarily correspond to *density* in the physical problem, for here ρ may be positive or negative. Also n is not restricted to be a positive integer, but may be any number positive or negative, whole or fractional. In fact in this process

the equation $I(Y_n)=0$ is quite liberated from all its physical connexion; and it is maintained that whatever ρ and n and Y_n may be, the equation has no solution except $Y_n=0$.

19. My next remark is that the highly general proposition thus maintained is certainly untrue. The assertion is that Y_n must be zero. Now to overthrow this assertion it is sufficient to point to two examples which have already been discussed. We have shewn in Art. 10 that whatever ρ may be, when $n=1$ the value $\frac{E}{a}$ satisfies the equation $I(Y_n)=0$. And we have shewn in Art. 12 that whatever n may be, with the values of ρ and Y_n there assigned we have $I(Y_n)=0$. These two examples as we have already shewn do not bear against Legendre's demonstration of his duly restricted proposition; but they are decisive against the highly general proposition which is now under discussion.

20. The *form* in which the argument is put is certainly strange. In order to shew that the equation $I(Y_n)=0$ has no solution, it is *assumed* that the equations $I(Y_n)=a^n$ and $I(Y_n)=\frac{1}{a^{n+1}}$ can both be solved. But the points assumed appear as difficult as that which is to be demonstrated. For in fact out of the three equations here brought before us if we assume that any two can be solved the same kind of argument might be employed to shew that the third equation could not be solved.

21. The best method of shewing where the argument fails is to employ the reverse process of Art. 5. We may admit that the general solution of (5) will be of the form

$$G\psi(a)+E\chi(a),$$

where G and E are arbitrary constants, and $\psi(a)$ and $\chi(a)$ functions of a; these functions could not be specifically determined unless ρ were given explicitly. If we perform the operation I on this expression the result will be $\frac{Ha^n}{2n+1}+\frac{K}{a^{n+1}}$.

The value of H will be given by

$$H=\frac{G\sigma_1}{a_1^{2n}}\left\{\frac{d}{da}a^n\psi(a)\right\}_1+\frac{E\sigma_1}{a_1^{2n}}\left\{\frac{d}{da}a^n\chi(a)\right\}_1.$$

The value of K will be of the same form and we may denote it thus

$$K=G\lambda(a_1)+E\mu(a_1),$$

where $\lambda(a_1)$ and $\mu(a_1)$ are certain functions of a_1 which we need not state explicitly.

Now O'Brien supposes, and quite correctly as we see, that G and E may be so taken as to make H vanish. When G and E are so taken he denotes $G\psi(a)+E\chi(a)$ by v', *assuming in fact that K cannot also vanish.* But of course it is theoretically quite possible that the constants which occur may be so related as to make K vanish when H vanishes. In fact in the two examples we have brought forward we have made one of the two G and E zero, say G; and then we found that the coefficients of E in H and K vanish.

22. It might perhaps be suggested that the argument intends to shew that no general form can be found which will satisfy $I(Y_n) = 0$ *whatever may be the value of* ρ. The answer is twofold. In the first place this is really not the problem to be solved; what we wish to shew is that whatever value may be assigned to ρ there is no admissible value of Y_n. In the second place the example of Art. 10 shews that the argument is not valid even when the problem is changed in the manner suggested.

23. I shall now consider a proof given by Mr Pratt. In the first edition of Pratt's *Mechanical Philosophy* Laplace's process was adopted. In the second edition a new process was given to which attention is invited in the preface, where this passage occurs, "Mr O'Brien in his *Mathematical Tracts, Part I.*, has given an excellent demonstration, and shorter than Laplace's, but the one now given is shorter and simpler even than his." Mr Pratt's proof is reproduced in his separate publication on *Attractions...and the Figure of the Earth.*

We may observe here that if O'Brien's proof had been sound its excessive generality might have compensated for the want of almost any other qualities.

24. The proof given by Mr Pratt is interesting; it is founded solely on the primary equation without any use of the differential equation (5). But the assumptions on which it rests are rather large.

It assumes that Y_n and ρ can each be expanded in a series of ascending positive powers of a for all the values of a between 0 and a_1. It does not assume that ρ decreases from the centre to the surface; but it does assume that ρ does not vanish at the origin. Both Legendre and Laplace suppose that ρ has a value at the origin which is not zero; but with them this is no new assumption, being in fact involved in their main assumption that ρ decreases from the centre to the surface.

The assumed forms for Y_n and ρ are used in the primary equation which is developed in powers of a, and it is found impossible to make the terms separately vanish.

25. The assumptions respecting the possibility of expanding Y_n and ρ are certainly considerable assumptions. The assumption respecting Y_n shortens the demonstration as it enables us to omit the process by which Legendre and Laplace shew that Y_n and $\frac{dY_n}{da}$ have the same sign at the origin. The assumption that ρ can be expanded in a series of ascending powers of a, is of course a large assumption analytically, for it excludes many conceivable forms for ρ. And in a physical point of view the problem is very much restricted; for it is assumed that ρ is the *same* function of a throughout the investigation; whereas it is obviously conceivable that ρ may be discontinuous in form and even in value. To take a simple example. Let us examine *if by the method now under discussion* we can shew that there is no value of Y_n except zero, which will satisfy the primary equation under the following suppositions as to ρ; from $a = 0$ to $a = b$ let $\rho = D + Ea^m + \ldots$, and from $a = b$ to $a = a_1$ let $\rho = Ma^\mu + Na^\nu + \ldots$. Here we suppose m and the other exponents which occur in the first expression for ρ to be positive; in the second expression

for ρ we will suppose the exponents μ, ν, ... to be arranged in ascending algebraical order of magnitude, but they need not be restricted to be positive.

Let $$Y_n = Wa^s + \dots$$

First suppose a to be less than b. Then

$$\int_0^a \rho a^2 da = \frac{Da^3}{3} + \frac{Ea^{m+3}}{m+3} + \dots$$

$$\int_0^a \rho \frac{d}{da}(Y_n a^{n+3})\, da = WDa^{s+n+3} + \dots$$

$$\int_a^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = \int_a^b \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da + \int_b^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = WD(b^{s-n+2} - a^{s-n+2}) + \dots$$

$$+ \frac{WM(s+2-n)}{\mu+s+2-n}(a_1^{\mu+s+2-n} - b^{\mu+s+2-n}) + \dots$$

Substitute in the primary equation. Thus we get

$$\frac{WD}{3} a^{s+2} - \frac{WD}{2n+1} b^{s-n+2} a^n - \frac{WM(s+3-n)}{(2n+1)(\mu+s+2-n)}(a_1^{\mu+s+2-n} - b^{\mu+s+2-n}) a^n + \dots = 0.$$

We have then to see if the coefficient of each power of a vanishes.

If we suppose $s+2$ less than n we should have to suppose $WD=0$; which is inadmissible.

If we suppose $s+2=n$ we are led to $WD\left(\frac{1}{3} - \frac{1}{2n+1}\right) = 0$; this also is inadmissible.

But if we suppose $s+2$ greater than n we have as our first condition

$$Db^{s-n+2} + \frac{M(s+2-n)}{\mu+s+2-n}(a_1^{\mu+s+2-n} - b^{\mu+s+2-n}) = 0.$$

This condition presents nothing which is obviously impossible. We should then have to continue the investigation by examining the coefficients of the various powers of a.

Next suppose a to be greater than b. Then

$$\int_0^a \rho a^2 da = \int_0^b \rho a^2 da + \int_b^a \rho a^2 da = \frac{Db^3}{3} + \dots + \frac{M}{\mu+3}(a^{\mu+3} - b^{\mu+3}) + \dots$$

$$\int_0^a \rho \frac{d}{da}(Y_n a^{n+3})\, da = \int_0^b \rho \frac{d}{da}(Y_n a^{n+3})\, da + \int_b^a \rho \frac{d}{da}(Y_n a^{n+3})\, da$$

$$= WDb^{s+n+3} + \dots + \frac{s+n+3}{\mu+s+n+3} MW(a^{\mu+s+n+3} - b^{\mu+s+n+3}) + \dots$$

$$\int_a^{a_1} \rho \frac{d}{da}\left(\frac{Y_n}{a^{n-2}}\right) da = \frac{WM(s-n+2)}{\mu+s-n+2}(a_1^{\mu+s-n+2} - a^{\mu+s-n+2}) + \dots$$

If we substitute in the primary equation we shall find that the algebraically lowest power of a is a^{-n-1}; to make the coefficient of this vanish we must have

$$Db^{s+n+3} + \ldots - \frac{(s+n+3)\,M}{\mu+s+n+3}\,b^{\mu+s+n+3} + \ldots = 0.$$

And so on.

I do not mean to assert that all the conditions thus obtained can be satisfied. It may very possibly happen that by discussing the conditions it can be shewn that they are incompatible, and thus that the original proposition remains true. But it is obvious that the demonstration given does not in itself apply to the case of discontinuity in ρ; and that if we are to include this case the process must be much developed, so as to lose all the advantage which it may now claim as to brevity.

26. There is still another consideration. We have in the preceding Article allowed the form of Y_n to remain the same throughout. But it is conceivable that corresponding to a change in ρ there may be also a change in Y_n.

Hence we might suppose that from $a=0$ to $a=b$, we have $Y_n = Wa^s + \ldots$; and then from $a=b$ to $a=a_1$ we have $Y_n = Ua^t + \ldots$

To shew that this is an impossible solution will involve a more troublesome investigation than that of the preceding Article.

We may observe that the physical problem we are considering would require a certain continuity of *value*, though not necessarily of *form*. We should in fact require that the successive strata should be in contact throughout. This will be secured if we make Y_n continuous *in value;* that is if

$$Wb^s + \ldots\ldots\ldots = Ub^t + \ldots$$

It is not necessary that ρ should be continuous in value; if we please to impose this condition in the preceding Article we have

$$D + Eb^m + \ldots = Mb^\mu + Nb^\nu + \ldots$$

It will be easily seen on examination that the investigations given by Legendre and Laplace are not restricted to the case in which ρ is continuous in form or value.

27. On the whole then although the demonstration which we are discussing depends very simply and clearly on the assumptions made, yet as the assumptions are very large and considerably restrict the physical range of the problem, it seems to me less satisfactory than the demonstrations of Legendre and Laplace.

28. The most recent investigation which I have seen is that given in the treatise on *Natural Philosophy* by Thomson and Tait. This though slightly different in form is in substance the same as O'Brien's. But as I have abundantly shewn the extreme generality which such a process contemplates cannot be attained, because it does not exist. No further

discussion is required as to the principle of this demonstration, but I may just make one remark on a point of detail, which is not important but might lead to brief embarrassment. Suppose that instead of zero on the right-hand side of the primary equation we had a given function of a, say A_n so that

$$I(Y_n) = A_n \text{ ....(22).}$$

Then instead of (5) we should obtain

$$\frac{d^2Y_n}{da^2}\frac{1}{a^n}\int_0^a \rho a^2 da + 2\frac{dY_n}{da}\frac{\rho}{a^{n-2}} - Y_n\left\{\frac{n(n+1)}{a^{n+2}}\int_0^a \rho a^2 da - \frac{2\rho}{a^{n-1}}\right\}$$

$$= \frac{d}{da}\left\{a^{-2n}\frac{d}{da}(a^{n+1}A_n)\right\} \text{ ....(23).}$$

Now we may say that the solution of (23) is $G\psi(a) + E\chi(a) + z$ where G and E are arbitrary constants and z is any particular solution of (23). This is an elementary fact in the theory of differential equations; $G\psi(a) + E\chi(a)$ being the solution of (23) without its second term, that is the solution of (5). But we should not say that z is such as to make $I(z) = A_n$; that is we must not assume that any solution of (23) will satisfy (22); we do not know whether (22) admits of any solution at all prior to special investigation.

29. I am unwilling to leave the equation after criticising the treatment it has hitherto received without an attempt at a new investigation.

I continue to mean by an admissible value of Y_n a value which is never infinite; and also I will assume that $a\frac{dY_n}{da}$ must never be infinite: this is an essential condition in the physical problem with which our equation (1) is connected. I shall then shew that equation (1) has no admissible solution provided ρ be such that σ never vanishes except when a vanishes.

The equation (5) may be written

$$\frac{\rho a^2}{\sigma} = \frac{n(n+1)Y_n - a^2\dfrac{d^2Y_n}{da^2}}{2aY_n + 2a^2\dfrac{dY_n}{da}},$$

that is

$$\frac{\rho a^2}{\sigma} = \frac{1}{2a}\,\frac{n(n+1)Y_n - a^2\dfrac{d^2Y_n}{da^2}}{\dfrac{d}{da}(aY_n)} \text{ ....(24).}$$

Since σ is never to vanish except with a it follows from (24) that $\frac{d}{da}(aY_n)$ must be some function of a which never vanishes; let it be denoted by $\phi(a)$. We may then suppose $\phi(a)$ to be positive, for if it were not positive we could make it so by changing the sign of every term in the numerator and denominator of the right-hand member of (24).

Thus

$$\frac{d}{da}(aY_n) = \phi(a),$$

where $\phi(a)$ is always positive; therefore

$$aY_n = \int \phi(a)\, da + \text{constant}.$$

Since aY_n vanishes with a we may write the last result thus

$$aY_n = \int_0^a \phi(a)\, da.$$

Hence

$$a^n Y_n = a^{n-1} \int_0^a \phi(a)\, da,$$

$$\frac{d}{da}(a^n Y_n) = (n-1)\, a^{n-2} \int_0^a \phi(a)\, da + a^{n-1}\phi(a);$$

the last expression is necessarily positive, and will not vanish when $a = a_1$.

Thus the condition $H = 0$ of Art. 5 cannot be satisfied; and therefore the equation (1) is not satisfied.

The only exception which presents itself to the foregoing argument is this: it may be said that it is not absolutely necessary that $\frac{d}{da}(aY_n)$ should be always of the same sign, for $\frac{d}{da}(aY_n)$ may perhaps vanish provided $n(n+1)\, Y_n - a^2 \frac{d^2 Y_n}{da^2}$ vanishes simultaneously. But this may be shewn to be impossible.

As before we have

$$Y_n = \frac{1}{a} \int_0^a \phi(a)\, da;$$

thus

$$\frac{d^2 Y_n}{da^2} = \frac{2}{a^3} \int_0^a \phi(a)\, da - \frac{2}{a^2}\phi(a) + \frac{\phi'(a)}{a},$$

so that

$$n(n+1)\, Y_n - a^2 \frac{d^2 Y_n}{da^2} = \frac{n^2+n-2}{a} \int_0^a \phi(a)\, da + 2\phi(a) - a\phi'(a) \qquad \text{(25)}.$$

The last expression cannot vanish simultaneously with $\phi(a)$. For suppose that as a increases from zero $\phi(a)$ *first* vanishes when a has a certain value. Since $\phi(a)$ begins by being positive $\phi'(a)$ must be negative or zero when $\phi(a)$ first vanishes; hence the expression on the right-hand side of (25) is then necessarily positive and cannot vanish. Thus from (24) we should have $\sigma = 0$, which is contrary to the supposition.

30. Thus I do not assume that ρ continually decreases from the centre to the surface, nor even that ρ is always positive; but only that σ never vanishes except at the origin: so that if this demonstration be accepted it will form the most general which the proposition has yet received.

It will be observed that the conditions which we have imposed are not satisfied by the two examples which are discussed in Arts. 10 and 12. In Art. 10 we see that aY_n does not vanish with a; and in Art. 11 we have σ vanishing for a certain value of a. Thus these two examples do not supply any objection to the new demonstration.

31. I may observe that from (24) if we supposed Y_n to be given we may by integration theoretically determine the value of ρ. For the right-hand member of (24) would be thus a given function of a, say $\psi(a)$; and by integration

$$\log \sigma = \int \psi(a)\, da;$$

this gives σ, and then by differentiation we find ρ. In this way we might establish relations between Y_n, ρ, and a, which would satisfy (5); but by virtue of the demonstration in Art. (29) we cannot obtain *admissible* solutions of (1).

I. TODHUNTER.

October, 1871.

IV. *On the Centro-surface of an Ellipsoid.* *By* Professor CAYLEY.

[Read *March* 7, 1870.]

THE Centro-surface of any given surface is the locus of the centres of curvature of the given surface, or say it is the locus of the intersections of consecutive normals, (the normals which intersect the normal at any particular point of the surface being those at the consecutive points along the two curves of curvature respectively which pass through the point on the surface). The terms, *normal, centre of curvature, curve of curvature,* may be understood in their ordinary sense, or in the generalised sense referring to the case where the Absolute (instead of being the imaginary circle at infinity) is any quadric surface whatever; viz. the normal at any point of a surface is here the line joining that point with the pole of the tangent plane in respect of the quadric surface called the Absolute: and of course the centre of curvature and curve of curvature refer to the normal as just defined.

The question of the centro-surface of a quadric surface has been considered in the two points of view, viz. 1°, when the terms "normal," &c. are used in the ordinary sense, and the equation of the quadric surface (assumed to be an ellipsoid) is taken to be $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$; 2°, when the Absolute is the surface $X^2+Y^2+Z^2+W^2=0$, and the equation of the quadric surface is taken to be $aX^2+\beta Y^2+\gamma Z^2+\delta W^2=0$:—in the first of them by Salmon, *Quart. Math. Jour.* t. II. pp. 217—222 (1858), and in the second by Clebsch, *Crelle,* t. 62. pp. 64—107 (1863): see also Salmon's *Solid Geometry,* 2nd Ed. 1865, pp. 143, 402, &c. In the present Memoir, as shewn by the title, the quadric surface is taken to be an Ellipsoid; and the question is considered exclusively from the first point of view: the theory is further developed in various respects, and in particular as regards the nodal curve upon the centro-surface: the distinction of real and imaginary is of course attended to. The new results suitably modified would be applicable to the theory treated from the second point of view; but I do not on the present occasion attempt so to present them.

The Ellipsoid; Parameters ξ, η, &c. Art. Nos. 1—6.

1. The position of a point (X, Y, Z) on the ellipsoid

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$$

may be determined by means of the parameters, or elliptic co-ordinates, ξ, η; viz. these are such that we have

$$\frac{X^2}{a^2+\xi}+\frac{Y^2}{b^2+\xi}+\frac{Z^2}{c^2+\xi}=1,$$

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1;$$

or, what is the same thing, ξ, η are the roots of the quadric equation

$$\frac{X^2}{a^2+v}+\frac{Y^2}{b^2+v}+\frac{Z^2}{c^2+v}=1.$$

(In its actual form this is a cubic equation, but there is a root $v=0$, which is to be thrown out, and the quadric equation is thus,

$$\begin{aligned}&v^2\\&+v\,(a^2+b^2+c^2-X^2-Y^2-Z^2)\\&+\{b^2c^2+c^2a^2+a^2b^2-(b^2+c^2)\,X^2-(c^2+a^2)\,Y-(a^2+b^2)\,Z^2\}=0,\end{aligned}$$

or putting

$$\begin{aligned}P&=a^2+b^2+c^2,\\Q&=b^2c^2+c^2a^2+a^2b^2,\\R&=a^2b^2c^2,\end{aligned}$$

the equation is

$$v^2+v\,(P-X^2-Y^2-Z^2)+Q-(b^2+c^2)\,X^2-(c^2+a^2)\,Y^2-(a^2+b^2)\,Z^2=0).$$

2. It is convenient to write throughout

$$\begin{aligned}b^2-c^2&=\alpha,\\c^2-a^2&=\beta,\\a^2-b^2&=\gamma,\end{aligned}$$

(whence $\alpha+\beta+\gamma=0$).

As usual, a is taken to be the greatest and c the least of the semi-axes, we have thus α, γ each of them positive, and β negative, $=-\beta'$ where β' is a positive quantity $=\alpha+\gamma$. A distinction arises in the sequel between the two cases $a^2+c^2>2b^2$ and $a^2+c^2<2b^2$, but the two cases are not essentially different, and it is convenient to assume $a^2+c^2>2b^2$, that is, $a^2-b^2>b^2-c^2$ or $\gamma>\alpha$, say $\gamma-\alpha$ positive. The limiting case $a^2+c^2=2b^2$ or $\gamma=\alpha$ requires special consideration.

3. We have

$$\begin{aligned}-\beta\gamma\,X^2&=a^2\,(a^2+\xi)\,(a^2+\eta),\\-\gamma\alpha\,Y^2&=b^2\,(b^2+\xi)\,(b^2+\eta),\\-\alpha\beta\,Z^2&=c^2\,(c^2+\xi)\,(c^2+\eta).\end{aligned}$$

It is in fact easy to verify that these values satisfy as well the equation of the ellipsoid as the assumed equations defining the elliptic co-ordinates ξ, η. We may also obtain the relations

$$X^2 + Y^2 + Z^2 = a^2 + b^2 + c^2 + \xi + \eta,$$

$$a^2X^2 + b^2Y^2 + c^2Z^2 = a^4 + b^4 + c^4 + b^2c^2 + c^2a^2 + a^2b^2 + (a^2 + b^2 + c^2)(\xi + \eta) + \xi\eta.$$

These, however, are obtained more readily from the equation in v, viz. the roots thereof being ξ, η, we have

$$-\xi - \eta = a^2 + b^2 + c^2 - X^2 - Y^2 - Z^2,$$

$$\xi\eta = b^2c^2 + c^2a^2 + a^2b^2 - (b^2 + c^2) X^2 - (c^2 + a^2) Y^2 - (a^2 + b^2) Z^2,$$

which lead at once to the relations in question.

4. Considering ξ as constant, the locus of the point (X, Y, Z) is the intersection of the ellipsoid with the confocal ellipsoid

$$\frac{X^2}{a^2 + \xi} + \frac{Y^2}{b^2 + \xi} + \frac{Z^2}{c^2 + \xi} = 1;$$

viz. this is one of the curves of curvature through the point; and similarly considering η as constant, the locus of the point is the intersection of the ellipsoid with the confocal ellipsoid

$$\frac{X^2}{a^2 + \eta} + \frac{Y^2}{b^2 + \eta} + \frac{Z^2}{c^2 + \eta} = 1;$$

viz. this is the other of the curves of curvature through the point.

5. If instead of ξ and η we write h and k, we may consider h as extending between the values $-a^2$, $-b^2$, and k as extending between the values $-b^2$, $-c^2$.

$h = \text{const.}$ will thus give the series of curves of curvature one of which is the section by the plane $X = 0$, or ellipse semi-axes b, c; say this is the *minor-mean* series. In particular $h = -a^2$ gives the ellipse just referred to; and $h = -b^2$, or say $h = -b^2 - \epsilon$ gives two detached portions of the ellipse semi-axes a, c; viz. each of these portions extends from an umbilicus above the plane of xy, through the extremity of the semi-axis a to an umbilicus below the plane of xy.

And in like manner $k = \text{const.}$ gives the series of curves of curvature one of which is the section by the plane $Z = 0$, or ellipse semi-axes a, b; say this is the *major-mean* series. In particular $k = -c^2$ gives the ellipse just referred to; and $k = -b^2$, or say $k = -b^2 + \epsilon$ gives the remaining portions of the ellipse semi-axes a, c; viz. these are two portions each extending from an umbilicus above the plane of xy, through the extremity of the semi-axis c, to an umbilicus above the plane of xy.

The ellipse last referred to may be called the umbilicar section, the other two principal sections being the major-mean section and the minor-mean section respectively.

In the limiting case $h = k = -b^2$, we have the umbilici, viz. these are given by

$$\frac{X^2}{a^2} = -\frac{\gamma}{\beta}, \quad Y = 0, \quad \frac{Z^2}{c^2} = -\frac{\alpha}{\beta}.$$

The two series of curves of curvature cover the whole real surface of the ellipsoid; so that at any real point thereof we have $\xi = h$, $\eta = k$, or else $\xi = k$, $\eta = h$, where h, k are negative real values lying within the foregoing limits $-a^2$, $-b^2$ and $-b^2$, $-c^2$ respectively. But observe that ξ, η taken separately may each extend between the limits $-a^2$, $-c^2$.

6. Suppose $\xi=\eta$, the equation in v will have equal roots, or the condition is

$$(P-X^2-Y^2-Z^2)^2=4\{Q-(b^2+c^2)X^2-(c^2+a^2)Y^2-(a^2+b^2)Z^2\},$$

viz. this surface by its intersection with the ellipsoid determines the envelope of the curves of curvature. This envelope is in fact a system of eight imaginary lines, four of them belonging to one of the systems of right lines on the ellipsoid, the other four to the other of the systems of right lines. For in the values of X^2, Y^2, Z^2 writing $\eta=\xi$, we find

$$\pm\sqrt{-\beta\gamma}\,\frac{X}{a}=a^2+\xi,$$

$$\pm\sqrt{-\gamma\alpha}\,\frac{Y}{b}=b^2+\xi,$$

$$\pm\sqrt{-\alpha\beta}\,\frac{Z}{c}=c^2+\xi,$$

or representing for shortness the left-hand functions by $\pm X'$, $\pm Y'$, $\pm Z'$, the eight lines are

$$\begin{array}{l|l|l|l}
a^2+\xi=\ \ X' & =\ \ X' & =-X' & =-X' \\
b^2+\xi=\ \ Y' & =-Y' & =\ \ Y' & =-Y' \\
c^2+\xi=\ \ Z' & =-Z' & =-Z' & =\ \ Z' \\
\hline
a^2+\xi=-X' & =\ \ X' & =\ \ X' & =-X' \\
b^2+\xi=\ \ Y' & =-Y' & =\ \ Y' & =-Y' \\
c^2+\xi=\ \ Z' & =\ \ Z' & =-Z' & =-Z'
\end{array}$$

so that in the two tetrads each line intersects the four lines of the other tetrad, but it does not intersect the remaining three lines of its own tetrad. The intersections are four points corresponding to $\xi=-a^2$, being the imaginary umbilici in the plane $X=0$, four to $\xi=-b^2$ being the real umbilici in the plane $Y=0$, four to $\xi=-c^2$ being the imaginary umbilici in the plane $Z=0$, and four corresponding to $\xi=\infty$, which may be called the umbilici at infinity*.

Sequential and Concomitant Centro-curves. Art. No. 7.

7. Consider any particular curve of curvature; the normals at the several points thereof successively intersect each other in a series of points forming a curve; and we have thus, corresponding to the particular curve of curvature, a curve on the centro-surface, which

* According to Salmon, *Solid Geometry*, p. 229, the number of umbilici for a surface of the n^{th} order is $=n(10n^2-25n+16)$; viz. for $n=2$, this is $=12$, as in the ordinary theory, not recognizing the umbilici at infinity. But whether properly umbilici or not, the 4 points which I call the umbilici at infinity do in the present theory present themselves in like manner with the 12 umbilici.

curve may be called the *sequential centro-curve.* Again the same normals, viz. those at the several points of the particular curve of curvature, are intersected, the normal at each point by the consecutive normal belonging to the other curve of curvature through that point; and we have thus corresponding to the particular curve of curvature, a curve on the centro-surface, which curve may be called the *concomitant centro-curve.* If instead of a single curve of curvature we consider the whole series, say of the major-mean curves of curvature, we have a series of major-mean sequential centro-curves, and also a series of major-mean concomitant centro-curves; and similarly considering the series of the minor-mean curves of curvature we have a series of minor-mean sequential centro-curves and also a series of minor-mean concomitant curves; the configuration of the several curves will be discussed further on, but it may be convenient to remark here that the centro-surface may be considered as consisting of two portions, say,

(A) locus of the major-mean sequential centro-curves; and also of the minor-mean concomitant centro-curves;

(B) locus of the minor-mean concomitant centro-curves, and also of the major-mean sequential centro-curves.

Investigation of expressions for the Co-ordinates of a point on the Centro-surface.
Art. Nos. 8 to 13.

8. Consider the normal at the point (X, Y, Z). Taking in the first instance (x, y, z) as current co-ordinates, the equations are

$$\frac{x-X}{\frac{X}{a^2}}=\frac{y-Y}{\frac{Y}{b^2}}=\frac{z-Z}{\frac{Z}{c^2}}, =\lambda \text{ suppose},$$

or, what is the same thing,

$$x=X\left(1+\frac{\lambda}{a^2}\right),\ \ y=Y\left(1+\frac{\lambda}{b^2}\right),\ \ z=Z\left(1+\frac{\lambda}{c^2}\right).$$

Suppose now that the normal meets the consecutive normal, or normal at the point $X+dX,\ Y+dY,\ Z+dZ$; and let x, y, z belong to the point of intersection of the two normals; we must have

$$0=dX\left(1+\frac{\lambda}{a^2}\right)+\frac{X}{a^2}d\lambda,$$

$$0=dY\left(1+\frac{\lambda}{b^2}\right)+\frac{Y}{b^2}d\lambda,$$

$$0=dZ\left(1+\frac{\lambda}{c^2}\right)+\frac{Z}{c^2}d\lambda,$$

which determine the direction of the consecutive point; the equations in fact give

$$0=\begin{vmatrix} dX, & \frac{dX}{a^2}, & \frac{X}{a^2} \\ dY, & \frac{dY}{b^2}, & \frac{Y}{b^2} \\ dZ, & \frac{dZ}{c^2}, & \frac{Z}{c^2} \end{vmatrix},$$

or, what is the same thing,

$$0 = \begin{vmatrix} a^2dX, & dX, & X \\ b^2dY, & dY, & Y \\ c^2dZ, & dZ, & Z \end{vmatrix},$$

which is the differential equation of the curve of curvature. This equation must therefore be satisfied by taking for $X+dX$, $Y+dY$, $Z+dZ$, the co-ordinates of the consecutive point along either of the curves of curvature,—say along that which is the intersection with the surface,

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1.$$

9. To verify this, observe that we then have

$$\frac{XdX}{a^2}+\frac{YdY}{b^2}+\frac{ZdZ}{c^2}=0,$$

$$\frac{XdX}{a^2+\eta}+\frac{YdY}{b^2+\eta}+\frac{ZdZ}{c^2+\eta}=0;$$

or, what is the same thing,

$$XdX : YdY : ZdZ = a^2(a^2+\eta)\,\alpha : b^2(b^2+\eta)\,\beta : c^2(c^2+\eta)\,\gamma.$$

But from the equations $-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta)$ &c., these become

$$XdX : YdY : ZdZ = \frac{X^2}{a^2+\xi} : \frac{Y^2}{b^2+\xi} : \frac{Z^2}{c^2+\xi};$$

or, what is the same thing,

$$dX : dY : dZ = \frac{X}{a^2+\xi} : \frac{Y}{b^2+\xi} : \frac{Z}{c^2+\xi},$$

and substituting these values in the determinant equation, it becomes

$$0 = \frac{XYZ}{(a^2+\xi)(b^2+\xi)(c^2+\xi)} \begin{vmatrix} a^2, & 1, & a^2+\xi \\ b^2, & 1, & b^2+\xi \\ c^2, & 1, & c^2+\xi \end{vmatrix},$$

which is identically true, since evidently the determinant vanishes.

10. Proceeding with the solution, we have from the three equations

$$XdX+YdY+ZdZ+\lambda\left(\frac{XdX}{a^2}+\frac{YdY}{b^2}+\frac{ZdZ}{c^2}\right)+d\lambda\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)=0,$$

and observing that from the equation

$$X^2+Y^2+Z^2=a^2+b^2+c^2+\xi+\eta.$$

considering therein η as constant, we have

$$XdX+YdY+ZdZ=\tfrac{1}{2}\,d\xi,$$

the equation becomes

$$\tfrac{1}{2}d\xi + d\lambda = 0\,;$$

and the three equations then are

$$0 = dX\left(1+\frac{\lambda}{a^2}\right) - \tfrac{1}{2}\frac{X}{a^2}d\xi, \text{ \&c.},$$

or say

$$0 = dX\,(a^2+\lambda) - \tfrac{1}{2}\,Xd\xi, \text{ \&c.}$$

But from the equation $-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta)$, considering therein η as a constant, we have

$$\frac{dX}{X} = \frac{\tfrac{1}{2}\,d\xi}{a^2+\xi},$$

and the equations thus become

$$0 = \frac{a^2+\lambda}{a^2+\xi} - 1, \text{ \&c.},$$

viz. these are all satisfied if only $\lambda = \xi$.

11. The co-ordinates of the point of intersection of the two normals thus are

$$x = X\left(1+\frac{\xi}{a^2}\right),\quad y = Y\left(1+\frac{\xi}{b^2}\right),\quad z = Z\left(1+\frac{\xi}{c^2}\right),$$

or squaring, and substituting for X^2, &c., their values as given by

$$-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta), \text{ \&c.},$$

the equations become

$$\begin{aligned}-\beta\gamma\, a^2x^2 &= (a^2+\xi)^3(a^2+\eta),\\ -\gamma\alpha\, b^2y^2 &= (b^2+\xi)^3(b^2+\eta),\\ -\alpha\beta\, c^2z^2 &= (c^2+\xi)^3(c^2+\eta),\end{aligned}$$

viz. these equations give (x, y, z) the co-ordinates of a point on the centro-surface, the intersection of the normal at the point (X, Y, Z) of the ellipsoid, (determined by the parameters ξ, η) by the normal at the consecutive point along the curve of curvature

$$\frac{X^2}{a^2+\eta} + \frac{Y^2}{b^2+\eta} + \frac{Z^2}{c^2+\eta} = 1,$$

or say η is the sequential parameter*.

Of course by interchanging ξ and η we should obtain the co-ordinates of the point of intersection of the normal at the same point (X, Y, Z) by the normal at the consecutive point along the other curve of curvature: ξ being in this case the sequential parameter.

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* The expressions are given in effect, but not explicitly, Salmon, p. 143.

12. I stop for a moment to consider the foregoing two equations

$$\lambda = \xi, \quad d\lambda = -\tfrac{1}{2}\, d\xi,$$

which at first sight appear inconsistent. But observe that in the foregoing solution λ is the parameter of the point (x, y, z) of the centro-surface considered as a point on the normal at (X, Y, Z); $\lambda + d\lambda$ is the parameter of the *same point* considered as a point on the normal at the consecutive point $(X+dX,\ Y+dY,\ Z+dZ)$: the value $\lambda + d\lambda = \xi + d\xi$ would belong to a different point, viz. the consecutive point of the centro-surface considered as a point on the consecutive normal—wherefore the $d\lambda$ of the solution ought not to be $= d\xi$. In further explanation, observe that the equations

$$x = X\left(1 + \frac{\lambda}{a^2}\right), \text{ \&c. where } \lambda = \xi,$$

if we pass from (x, y, z) to the consecutive point on the centro-surface, give

$$dx = dX\left(1 + \frac{\lambda}{a^2}\right) + \frac{X}{a^2}\, d\xi\,;$$

but since by what precedes,

$$0 = dX\left(1 + \frac{\lambda}{a^2}\right) - \tfrac{1}{2}\frac{X}{a^2}\, d\xi,$$

this is

$$dx = \tfrac{3}{2}\frac{X}{a^2}\, d\xi.$$

Or since

$$a^2 x = X(a^2 + \xi),$$

this is

$$\frac{dx}{x} = \tfrac{3}{2}\frac{d\xi}{a^2 + \xi};$$

and similarly

$$\frac{dy}{y} = \tfrac{3}{2}\frac{d\xi}{b^2 + \xi},$$

$$\frac{dz}{z} = \tfrac{3}{2}\frac{d\xi}{c^2 + \xi},$$

which are the correct values of dx, dy, dz as derived from the equations

$$-\beta\gamma a^2 x^2 = (a^2 + \xi)^3 (a^2 + \eta), \text{ \&c.}$$

13. The equations $-\beta\gamma a^2 x^2 = (a^2+\xi)^3 (a^2+\eta)$, &c. give expressions for the co-ordinates (x, y, z) of a point on the centro-surface in terms of the two parameters (ξ, η): the elimination of (ξ, η) from these equations will therefore lead to the equation of the surface: but the discussion of the surface may also be effected by means of these expressions for the co-ordinates in terms of the two parameters.

Discussion by means of the equations $-\beta\gamma a^2 x^2 = (a^2+\xi)^3 (a^2+\eta)$, &c.; *Principal Sections, &c.* Art. Nos. 14 to 24 (*several subheadings*).

14. To fix the ideas consider the section of the surface by the plane $z = 0$; we have in the surface $z = 0$, that is $\xi = -c^2$, or else $\eta = -c^2$, values which give respectively

$$\begin{array}{ll} -\beta\gamma a^2x^2 = -\beta^3(a^2+\eta), & -\beta\gamma a^2x^2 = -\beta(a^2+\xi)^3, \\ -\gamma\alpha b^2y^2 = \alpha^3(b^2+\eta). & -\gamma\alpha b^2y^2 = \alpha(b^2+\xi)^3. \end{array}$$

Or, what is the same thing,

$$\begin{array}{ll} \frac{\gamma}{\beta^2}a^2x^2 = a^2+\eta, & \gamma a^2x^2 = (a^2+\xi)^3, \\ \frac{\gamma}{\alpha^2}b^2y^2 = -b^2-\eta. & \gamma b^2y^2 = -(b^2+\xi)^3. \end{array}$$

The first set of equations gives

$$\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,$$

which is the equation of an ellipse.

The second set gives

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=\gamma^{\frac{2}{3}},$$

or in a rationalised form

$$(a^2x^2+b^2y^2-\gamma^2)^3+27a^2b^2\gamma^2x^2y^2=0,$$

which is the equation of an evolute of an ellipse.

15. The ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$ is a cuspidal curve on the surface, and the section by the plane $z=0$ is consequently made up of this ellipse counting three times, and of the evolute; it is therefore of the twelfth order; and the order of the surface is in fact $=12$.

It is clear that the section of the centro-surface arises from the section $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, viz. the normal at any point of this ellipse lies in the plane $Z=0$, and its intersection by a normal at the consecutive point of the ellipse gives a point of the evolute; the evolute being thus the sequential centro-curve of this section: the intersection by the normal at the consecutive point on the other curve of curvature gives a point on the ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$, which ellipse is therefore the concomitant centro-curve. Observe that this other curve of curvature cuts the ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$ at right angles, and that the normals at the consecutive points above and below the point on the ellipse will meet each other and also the normal at the point of the same ellipse at the point on the ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$: this shows that the last-mentioned ellipse is a cuspidal curve on the centro-surface.

16. The three principal sections of the centro-surface are consequently

$$x=0,\quad \frac{b^2y^2}{\gamma^2}+\frac{c^2z^2}{\beta^2}=1,\quad \text{and}\quad (by)^{\frac{2}{3}}+(cz)^{\frac{2}{3}}=\alpha^{\frac{2}{3}};$$

$$y=0,\quad \frac{c^2z^2}{\alpha^2}+\frac{a^2x^2}{\gamma^2}=1,\quad \text{and}\quad (cz)^{\frac{2}{3}}+(ax)^{\frac{2}{3}}=\beta^{\frac{2}{3}};$$

$$z=0,\quad \frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,\quad \text{and}\quad (ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=\gamma^{\frac{2}{3}};$$

viz. each section is made up of an ellipse counting three times and of an evolute (of an ellipse). I have for shortness represented the three evolutes by their irrational equations. It will presently appear that the section (imaginary) by the plane infinity is of the like character.

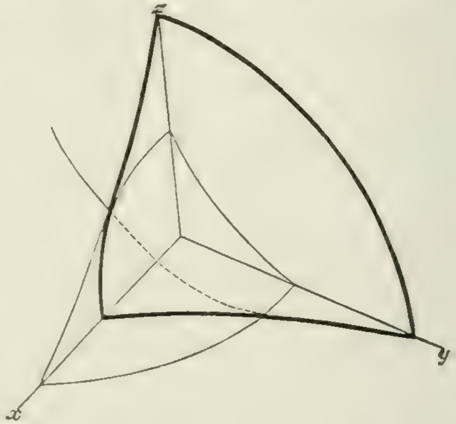
17. Considering only the positive directions of the axes, we have on each axis two points, viz.

$$\text{axis of } x, \quad x = \frac{\gamma}{a}, \quad x = \frac{-\beta}{a};$$

$$\text{axis of } y, \quad y = \frac{\alpha}{b}, \quad y = \frac{\gamma}{b};$$

$$\text{axis of } z, \quad z = \frac{-\beta}{c}, \quad z = \frac{\alpha}{c};$$

through each of which, in the two different planes through the axis respectively, there passes an ellipse and an evolute. In the assumed case $a^2 + c^2 > 2b^2$, the disposition of the points is as shown in the figure.



Plane of xz, evolute is outside ellipse,

yz, „ inside „

xy, „ cuts „

but in the contrary case $a^2 + c^2 < 2b^2$, the disposition is

Plane of xz, evolute is outside ellipse,

yz, „ cuts „

xy, „ is inside „

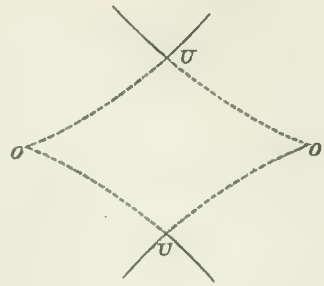
there is no real difference, and to fix the ideas I attend exclusively to the first-mentioned case

$$a^2 + c^2 > 2b^2.$$

18. In each of the principal planes, the evolute and ellipse, quà curves of the orders 6 and 2, respectively, intersect in 12 points, 3 in each quadrant; viz. of the 3 points two unite together into a twofold point or point of contact, and the third is a point of simple intersection; assuming for the moment that this is so, the figure at once shows that in the plane of xz or umbilicar plane the contact is real, the intersection imaginary; in the plane of xy, or major-mean plane, the contact is imaginary, the intersection real; but in the plane of yz or minor-mean plane the contact and intersection are each imaginary. The contacts arise, as will appear, from the umbilici of the ellipsoid, and may be termed "umbilicar centres," or "omphaloi;" the simple intersections "points of outcrop," or simply "outcrops." By what precedes there are in the umbilicar plane, four real umbilicar centres (in each quadrant one); and in the major-mean plane four real outcrops (in each quadrant one); the other umbilicar centres and outcrops are respectively imaginary.

19. The surface consists of two sheets intersecting in a *nodal curve* connecting the outcrop with the umbilicar centre. As to the form of this curve there is a cusp at the

outcrop, and the curve does not terminate at the umbilicar centre, but on passing it, from crunodal becomes acnodal (viz. there is no longer through the curve any real sheet of the surface): moreover the curve is not at the umbilicar centre perpendicular to the plane of xz, and there is consequently on the opposite side of the plane a symmetrically situate branch of the curve, viz. the umbilicar centre is a node on the nodal curve. Completing the curve, the nodal curve consists of two distinct portions, one on the positive side of the plane of yz or minor-mean plane consisting of two cuspidal branches as shown in the figure; the other a symmetrically situate portion on the negative side of the minor-mean plane.



Intersections of Evolute and Ellipse.

20. Consider in the plane of xy the ellipse and evolute,

$$\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,\quad (a^2x^2+b^2y^2-\gamma^2)^3+27\,\gamma^2a^2b^2x^2y^2=0.$$

First, these are satisfied by

$$\left.\begin{array}{l} a^2x^2=-\dfrac{\beta^3}{\gamma}, \\ b^2y^2=-\dfrac{\alpha^3}{\gamma}, \end{array}\right\} \text{Co-ordinates of Umbilicar centres in plane of } xy \text{ (imaginary)},$$

viz. the equations respectively become

$$-\frac{\beta}{\gamma}-\frac{\alpha}{\gamma}=1,\quad (-\frac{\beta^3+\alpha^3}{\gamma}-\gamma^2)^3+27\alpha^3\beta^3=0,$$

the first of which is $\alpha+\beta+\gamma=0$, and the second is $(\alpha^3+\beta^3+\gamma^3)^3-27\alpha^3\beta^3\gamma^3=0$. But the equation $\alpha+\beta+\gamma=0$ gives $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$, and the two equations are thus identically satisfied. Moreover the condition for a contact is at once found to be

$$\beta^2\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2b^2y^2]=\alpha^2\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2a^2x^2],$$

or, what is the same thing,

$$(\alpha^2-\beta^2)\,(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2\,(\alpha^2a^2x^2-\beta^2b^2y^2)=0;$$

and substituting the foregoing values, this is

$$(\alpha^2-\beta^2)\,(-\frac{\alpha^3+\beta^3}{\gamma}-\gamma^2)^2+9\gamma^2\,\frac{-\alpha^2\beta^3+\alpha^3\beta^2}{\gamma}=0,$$

that is,

$$\frac{\alpha^2-\beta^2}{\gamma^2}\,(\alpha^3+\beta^3+\gamma^3)^2+9\gamma\alpha^2\beta^2\,(\alpha-\beta)=0;$$

which, putting therein $\alpha+\beta=-\gamma$, and $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$, is also satisfied; that is, the points in question are points of contact of the ellipse and evolute.

21. Secondly, consider the values

$$\left.\begin{aligned} a^2x^2 &= -\frac{\beta^3}{\gamma}\left(\frac{\gamma-\alpha}{\alpha-\beta}\right)^3, \\ b^2y^2 &= -\frac{\alpha^3}{\gamma}\left(\frac{\beta-\gamma}{\alpha-\beta}\right)^3, \end{aligned}\right\}\ \text{Co-ordinates of outcrops in plane of } xy \text{ (real).}$$

Substituting in the equation of the ellipse, we have

$$\alpha(\beta-\gamma)^3+\beta(\gamma-\alpha)^3+\gamma(\alpha-\beta)^3=0,$$

which is

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)=0,$$

or the equation is satisfied identically: and substituting in the equation of the evolute, we have first

$$a^2x^2+b^2y^2-\gamma^2=-\frac{\alpha^3(\beta-\gamma)^3+\beta^3(\gamma-\alpha)^3+\gamma^3(\alpha-\beta)^3}{\gamma(\alpha-\beta)^3};$$

which in virtue of $\alpha(\beta-\gamma)+\beta(\gamma-\alpha)+\gamma(\alpha-\beta)=0$ becomes

$$a^2x^2+b^2y^2-\gamma^2=-\frac{3\,\alpha\beta\gamma\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{\gamma(\alpha-\beta)^3},$$

$$=-\frac{3\alpha\beta(\beta-\gamma)(\gamma-\alpha)}{(\alpha-\beta)^2},$$

and then, completing the substitution, it is seen that the equation of the evolute is also satisfied. The points last considered are simple intersections, and we have thus the complete number $8+4=12$ of the intersections of the evolute and ellipse.

22. We have α, γ positive, β negative; whence $\alpha-\beta$ is positive, $\beta-\gamma$ negative; $\gamma-\alpha\,(=a^2+c^2-2b^2)$ is positive, and hence, the outcrops in the plane of xy are real; the umbilicar centres are imaginary for this plane, but real for the plane of zx, the co-ordinates being

$$\left.\begin{aligned} c^2z^2 &= -\frac{\alpha^3}{\beta}, \\ a^2x^2 &= -\frac{\gamma^3}{\beta}, \end{aligned}\right\}\ \text{Co-ordinates of Umbilicar centres in plane of } xz \text{ (real).}$$

Nodes of the Evolute.

23. The Evolute is a curve with four nodes, all of them imaginary; viz. for the evolute in the plane of xy, the equation of which is

$$(a^2x^2+b^2y^2-\gamma^2)^3+27\,\gamma^2a^2b^2x^2y^2=0,$$

these are

$$\left.\begin{aligned} a^2x^2 &= -\gamma^2, \\ b^2y^2 &= -\gamma^2, \end{aligned}\right\}\ \text{Co-ordinates of Nodes of evolute in plane of } xy \text{ (imaginary).}$$

in fact these values satisfy as well the equation of the evolute, as the two derived equations

$$6a^2x\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2b^2y^2]=0,$$

$$6b^2y\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2a^2x^2]=0,$$

or the points in question are nodes of the evolute.

The evolute has the four cusps on the axes and two cusps at infinity, in all 6 cusps as just mentioned; it has 4 nodes: and the order being 6, the class is

$$30-2.4-3.6,=4.$$

Section by the plane infinity.

24. The surface itself is finite, and the section by the plane infinity is therefore imaginary, but by what precedes the nodal curve must have real points at infinity, viz., there must be real acnodal points on this imaginary section. The section by the plane infinity resembles in fact the principal sections; viz., writing successively $\xi=\infty$, and $\eta=\infty$, we have

$$-\beta\gamma a^2x^2:-\gamma\alpha b^2y^2:-\alpha\beta a^2z^2=a^2+\eta:b^2+\eta:c^2+\eta \quad\text{or}$$
$$=(a^2+\xi)^3:(b^2+\xi)^3:(c^2+\xi)^3,$$

giving respectively

$$a^2x^2+b^2y^2+c^2z^2=0,\quad\text{and}\quad(a\alpha x)^{\frac{2}{3}}+(b\beta y)^{\frac{2}{3}}+(c\gamma z)^{\frac{2}{3}}=0,$$

where the first equation represents an imaginary conic which counts three times; and the second equation, the rationalised form of which is

$$(a^2\alpha^2x^2+b^2\beta^2y^2+c^2\gamma^2z^2)^3-27\,a^2b^2c^2\alpha^2\beta^2\gamma^2x^2y^2z^2=0,$$

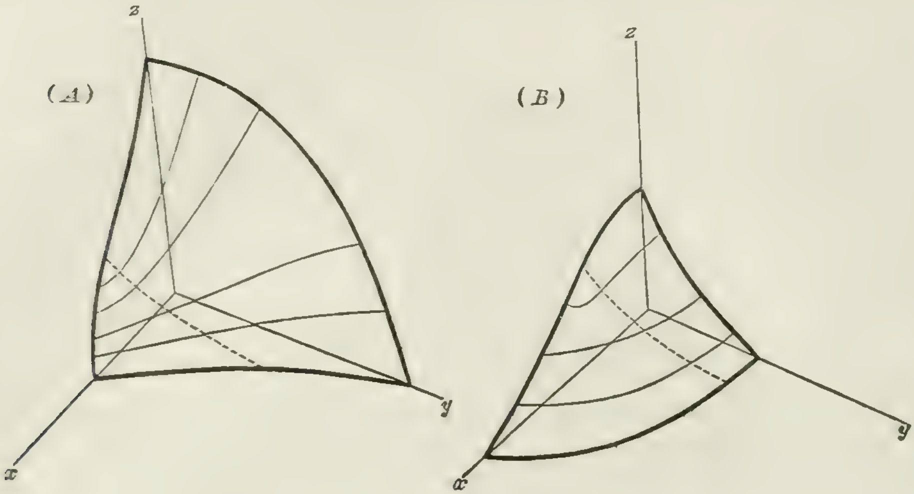
an imaginary evolute. The conic and evolute have four contacts and four simple intersections (in all $4.2+4=12$ intersections) which are all of them imaginary. But the evolute has four real nodes (acnodes) $a^2\alpha^2x^2=b^2\beta^2y^2=c^2\gamma^2z^2$; or, what is the same thing, there are four real lines $a^2\alpha^2x^2=b^2\beta^2y^2=c^2\gamma^2z^2$, which are respectively asymptotes of the nodal curve: viz., inasmuch as the equation of the surface contains only the squares x^2, y^2, z^2, the lines in question will be not merely parallel to, but will be, the asymptotes of the nodal curve.

The plane infinity may be reckoned as a principal plane, and we may say that in each of the four principal planes there are four umbilicar centres, four outcrops, and four evolute-nodes.

The generation of the surface considered geometrically. Arts. Nos. 25 to 28.

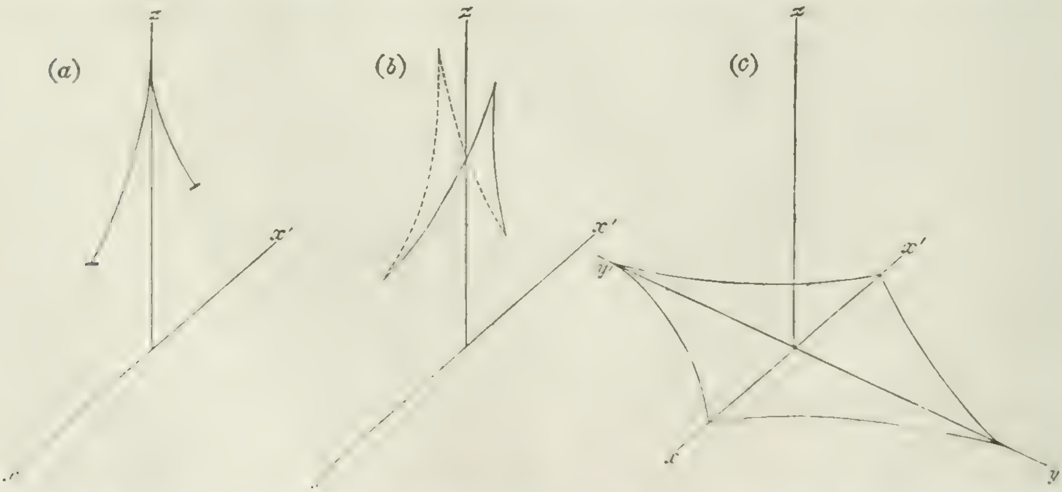
25. I have deferred until this point the discussion of the generation of the centro-surface by means of the centro-curves, for the reason that it can be carried on more precisely

now that we know the forms of the principal sections and of the nodal curve. The two



figures exhibit (as regards one octant of the surface) the portions already distinguished as (A), and (B): they intersect each other in the nodal curve, shown in each of the figures.

26. Consider first the generation of the portion (A) by means of the major-mean sequential centro-curves. The major-mean curves of curvature (attending to those below



the plane of xy) commence with a portion (extending from umbilicus to umbilicus) of the ellipse $\frac{X^2}{a^2}+\frac{Z^2}{c^2}=1$, this may be termed the vertical curve, and they end with the whole ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, which may be termed the horizontal curve. The normals at the several points of the vertical curve successively intersect along a portion (terminated each way at an umbilicar centre) of the evolute in the plane of xz or umbilicar plane; viz. this portion of the evolute, shown fig. (a), is the sequential centro-curve belonging to the vertical curve of curvature. The curve of curvature is at first a narrow oval surrounding the vertical curve; the corresponding form of the sequential centro-curve is at once seen to be a four-cusped curve as in fig. (b), and which we may imagine as derived from the curve (a) by first doubling this curve and then opening out the two component parts thereof: the two upper cusps of the curve (b) are situate on the yz-ellipse of the centro-surface, and the two lower cusps upon two detached portions respectively of the xz-ellipse of the centro-surface. And as the curve of curvature gradually broadens out and ultimately coincides with the XY-section of the ellipsoid, the four-cusped curve continues to open itself out, and ultimately coincides as shown figure (c) with the xy-evolute of the centro-surface, viz. this evolute is the sequential centro-curve belonging to the horizontal curve of curvature or XY-section of the ellipsoid. The successive sequential curves are also shown (so far as regards an octant of the surface) in the figure (A).

27. We consider next the generation of the portion (B) by means of the major-mean concomitant centro-curves. Starting as before with the vertical curve of curvature, the concomitant centro-curve is a finite portion (terminated each way at an umbilicar centre) of the xz-ellipse of the centro-surface. As the curve of curvature opens itself out into an oval, the concomitant centro-curve in like manner opens itself out into an oval, the two further vertices thereof situate on two detached portions of the xz-evolute of the centro-surface, and the two nearer vertices on the yz-evolute of the central surface. And as the curve of curvature continues to open itself out, and ultimately coincides with the horizontal curve or XY-section of the ellipsoid, so the concomitant centro-curve continues to open itself out and ultimately coincides with the xy-ellipse of the centro-surface. The successive forms (so far as relates to an octant of the surface) are shown in the figure (B). We have in each case attended only to the curves of curvature below the plane of xy, and the corresponding centro-curves above the plane of xy, but of course every thing is symmetrical as regards the two sides of the plane.

28. There is a precisely similar generation of the portion (A) by the minor-mean concomitant centro-curves, and of the portion (B) by means of the major-mean sequential centro-curves.

The Nodal Curve. Art. Nos. 29 to 60.

29. If two different points on the ellipsoid correspond to the same point on the centro-surface, this will be a point on the Nodal Curve: the conditions for this if (ξ, η), (ξ_1, η_1) are the parameters for the two points on the ellipsoid, are obviously

$$(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1),$$

$$(b^2+\xi)^3(b^2+\eta) = (b^2+\xi_1)^3(b^2+\eta_1),$$

$$(c^2+\xi)^3(c^2+\eta) = (c^2+\xi_1)^3(c^2+\eta_1);$$

these equations in effect determine η as a function of ξ, so that the equations

$$-\beta\gamma a^2x^2 = (a^2+\xi)^3\ (a^2+\eta),\ \&\text{c.}$$

then determine the co-ordinates (x, y, z) of a point on the Nodal Curve in terms of the single parameter ξ.

The relation between ξ and η would be obtained by eliminating ξ_1, η_1 from the foregoing equation: but it is easier to eliminate η and η_1, thus obtaining between ξ_1 and ξ a relation in virtue of which ξ_1 may be regarded as a known function of ξ; η and η_1 can then be expressed in terms of ξ, ξ_1, so that each of these quantities will be in effect a known function of ξ*.

30. The relation between ξ, ξ_1 is in the first instance given in the form

$$\begin{vmatrix} a^2\,[(a^2+\xi)^3-(a^2+\xi_1)^3], & (a^2+\xi)^3, & (a^2+\xi_1)^3 \\ b^2\,[(b^2+\xi)^3-(b^2+\xi_1)^3], & (b^2+\xi)^3, & (b^2+\xi_1)^3 \\ c^2\,[(c^2+\xi)^3-(c^2+\xi_1)^3], & (c^2+\xi)^3, & (c^2+\xi_1)^3 \end{vmatrix} = 0.$$

Throwing out a factor $(\xi-\xi_1)^2$, this becomes

$$\begin{aligned}\Sigma\,[a^2\,\{3a^4+3a^2\,(\xi+\xi_1)+\xi^2+\xi\xi_1+\xi_1^2\}\qquad&\\ \times(b^2-c^2)\,.\,(1,\ 1,\ 1)\between\big((b^2+\xi)\,(c^2+\xi_1),\ (b^2+\xi_1)\,(c^2+\xi)\big)^2] &= 0,\end{aligned}$$

where the left-hand side is a symmetrical function of ξ, ξ_1 vanishing for $\xi=\xi_1$, and therefore divisible by $(\xi-\xi_1)^2$; it is also divisible by Δ, $=(b^2-c^2)\,(c^2-a^2)\,(a^2-b^2)$ $(=\alpha\beta\gamma)$. To work this out, write $\xi+\xi_1=p$, $\xi\xi_1=q$, the equation may be written

$$\Sigma\,\{(b^2-c^2)\,a^2 \left|\begin{array}{l} 3a^4 \\ +\,3a^2p \\ +\,p^2-q \end{array}\right| \left|\begin{array}{l} 3b^4c^4 \\ +\,3b^2c^2\,(b^2+c^2)\,p \\ +\,(b^4+c^4)\,(p^2-q) \\ +\,b^2c^2\,(p^2+8q) \\ +\,3\,(b^2+c^2)\,pq \\ +\,3q^2 \end{array}\right| \} = 0,$$

where the left-hand side divides by $\Delta\,(p^2-4q)$.

31. Developing and reducing, and omitting this factor, the final result is

$$6R+3Qp+P\,(p^2+2q)+3pq=0,$$

where as before P, Q, R denote $a^2+b^2+c^2$, $b^2c^2+c^2a^2+a^2b^2$, $a^2b^2c^2$, respectively; that is

$$6R+3Q\,(\xi+\xi_1)+P\,(\xi^2+4\xi\xi_1+\xi_1^2)+3\,(\xi+\xi_1)\,\xi\xi_1=0.$$

or as this may be written

$$\begin{aligned} &6R+3Q\xi+P\xi^2 \\ &+\xi_1\,(3Q+4P\xi+3\xi^2) \\ &+\xi_1^2\,(P+\quad 3\xi\qquad)=0, \end{aligned}$$

viz. the parameters ξ, ξ_1 have a symmetrical (2, 2) correspondence.

* This was my first method of solution; and I have thought the results quite interesting enough to retain them —but it will appear in the sequel that I have succeeded in expressing ξ, η, ξ_1, η_1, in terms of a single parameter σ.

32. From the equations $(a^2+\xi)^3(a^2+\eta)=(a^2+\xi_1)^3(a^2+\eta_1)$, &c., we have

$$\Sigma(b^2-c^2)\{(a^2+\xi)^3(a^2+\eta)-(a^2+\xi_1)^3(a^2+\eta_1)\}=0,$$
$$\Sigma b^2c^2(b^2-c^2)\{(a^2+\xi)^3(a^2+\eta)-(a^2+\xi_1)^3(a^2+\eta)\}=0;$$

and observing that the term in { } is

$$\begin{aligned} &a^6(3\xi+\eta-3\xi_1-\eta_1)\\ &+a^4(3\xi^2+3\xi\eta-3\xi_1^2-3\xi_1\eta_1)\\ &+a^2(\xi^3+3\xi^2\eta-\xi_1^3-3\xi_1^2\eta_1)\\ &+\quad(\xi^3\eta \qquad -\xi_1^3\eta_1),\end{aligned}$$

these are readily reduced to

$$\begin{aligned}(3\xi+\eta-3\xi_1-\eta_1)P+(3\xi^2+3\xi\eta-3\xi_1^2-3\xi_1\eta_1)&=0,\\ (3\xi+\eta-3\xi_1-\eta_1)R+\xi^3\eta-\xi_1^3\eta_1 &=0,\end{aligned}$$

or what is the same thing

$$\begin{aligned}3(\xi-\xi_1)(P+\xi+\xi_1)+\eta(P+3\xi)-\eta_1(P+3\xi_1)&=0,\\ 3(\xi-\xi_1)R \qquad +\eta(R+\xi^3)-\eta_1(R+\xi_1^3)&=0,\end{aligned}$$

and if we hence determine the ratios $3(\xi-\xi_1):\eta:\eta_1$, the first of the resulting terms divides by $\xi-\xi_1$, and we have

$$\begin{aligned}3:\eta:\eta_1=&-P(\xi^2+\xi\xi_1+\xi_1^2)+3R-3\xi\xi_1(\xi+\xi_1)\\ &:R(2\xi_1-\xi)-\xi_1^3(P+\xi+\xi_1)\\ &:R(2\xi-\xi_1)-\xi^3(P+\xi+\xi_1).\end{aligned}$$

Hence observing that by the relation between ξ, ξ_1 the first term is

$$=3\{P\xi\xi_1+Q(\xi+\xi_1)+3R\},$$

the equations become

$$\begin{aligned}1:\eta:\eta_1=&P\xi\xi_1+Q(\xi+\xi_1)+3R\\ &:R(2\xi_1-\xi)-\xi_1^3(P+\xi+\xi_1)\\ &:R(2\xi-\xi_1)-\xi^3(P+\xi+\xi_1);\end{aligned}$$

and we thus have

$$\eta=\frac{R(2\xi_1-\xi)-\xi_1^3(P+\xi+\xi_1)}{P\xi\xi_1+Q(\xi+\xi_1)+3R},$$

which, considering ξ_1 as a given function of ξ, gives η as a function of ξ.

33. I write $\xi+\xi_1=2\mathrm{x}$, $\xi-\xi_1=2\mathrm{y}$, so that $p=2\mathrm{x}$, $q=\mathrm{x}^2-\mathrm{y}^2$, the relation between ξ, ξ_1 takes the form

$$6(R+Q\mathrm{x}+P\mathrm{x}^2-\mathrm{x}^3)-(6\mathrm{x}+2P)\mathrm{y}^2=0,$$

or, what is the same thing,

$$\mathrm{y}^2=\frac{(\mathrm{x}+a^2)(\mathrm{x}+b^2)(\mathrm{x}+c^2)}{\mathrm{x}+\frac{1}{3}(a^2+b^2+c^2)};$$

so that taking x at pleasure and considering y as denoting this function of x, the values of ξ, ξ_1 belonging to a point on the nodal curve are $\xi=(x+y)$, $\xi_1=(x-y)$; and the value of η is then given as before.

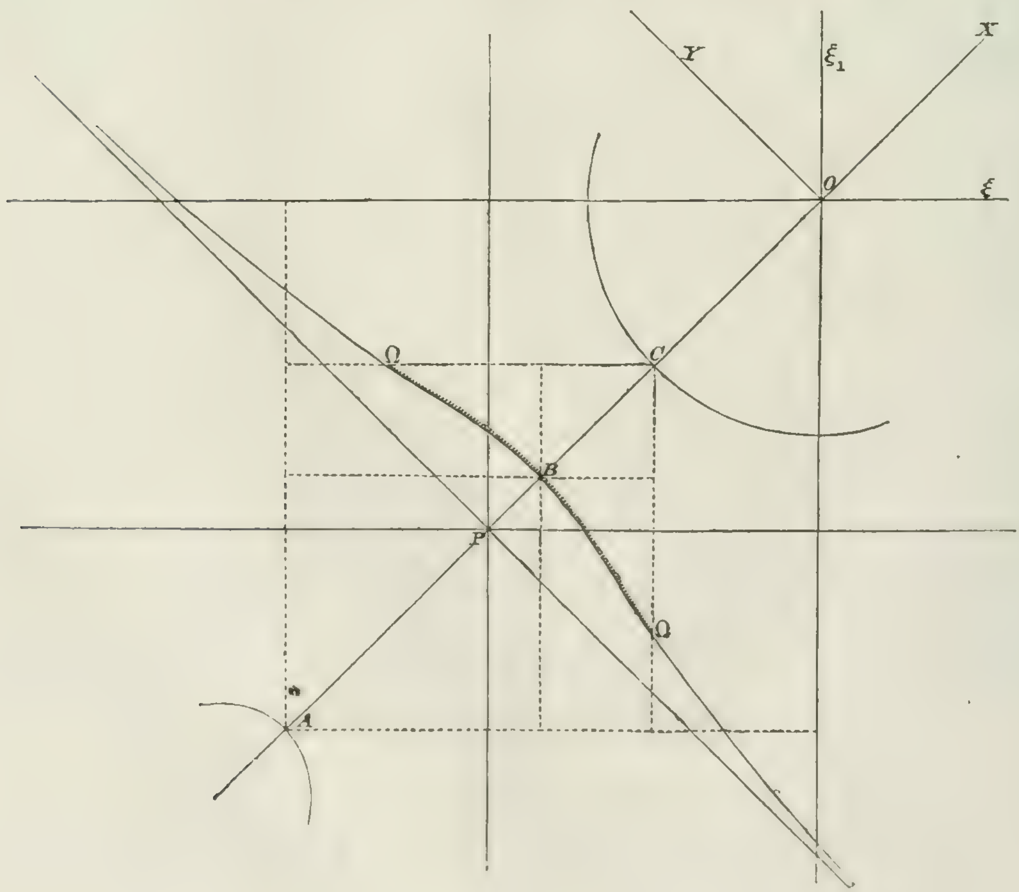
34. The form just given is analytically the most convenient, but there is some advantage in writing $\frac{1}{\sqrt{2}}x$, $\frac{1}{\sqrt{2}}y$, in the place of x, y respectively; viz. we then have

$$y^2=\frac{(x+a^2\sqrt{2})(x+b^2\sqrt{2})(x+c^2\sqrt{2})}{x+\frac{1}{3}\sqrt{2}(a^2+b^2+c^2)},$$

where $\xi=\frac{1}{\sqrt{2}}(x+y)$, $\xi_1=\frac{1}{\sqrt{2}}(x-y)$, so that if (ξ, ξ_1) be taken as rectangular co-ordinates of a point in a plane, (x, y) will be the rectangular co-ordinates of the same point referred to axes inclined at angles of 45° to the first-mentioned axes respectively.

35. The curve is a cubic curve symmetrical in regard to the axis of x, and having the three asymptotes,

$$x=-\tfrac{1}{3}(a^2+b^2+c^2)\sqrt{2},\quad y=\pm\{x+\tfrac{1}{3}(a^2+b^2+c^2)\sqrt{2}\},$$



viz. these all meet in the point P the co-ordinates of which are

$$\mathrm{x} = -\tfrac{1}{3}(a^2+b^2+c^2)\sqrt{2}, \quad \mathrm{y}=0:$$

moreover we have $\mathrm{y}=0$ for the values $\mathrm{x} = -a^2\sqrt{2},\ -b^2\sqrt{2},\ -c^2\sqrt{2}$, that is, the curve meets the axis of x in the points A, B, C; the order in the direction of $-\mathrm{x}$ being C, B, P, A as shown in the figure: and with these data it is easy to draw the curve: the portion which gives the crunodal part of the nodal curve is that extending from B to the points Ω; viz. at B we have $\xi=\xi_1=-b^2$ corresponding to the umbilicar centre; and at Ω, Ω we have ξ or $\xi_1=-c^2$, ξ_1 or $\xi=-c^2+\dfrac{3\alpha\beta}{\alpha-\beta}$ corresponding to the outcrop.

36. The nodal curve passes through (I) the umbilicar centres, (II) the outcrops, (III) the nodes of the evolute. The geometrical construction led to the conclusion that the real umbilicar centre was a node on the nodal curve, and that the real outcrop was a cusp (the tangent lying in the principal plane). It will presently appear generally, as regards the several points real or imaginary, that the umbilicar centre is a node on the nodal curve, and the outcrop a cusp—the tangent at the outcrop being in the principal plane: as regards the node on the evolute this is a simple point on the nodal curve, and by reason of the symmetry in regard to the principal plane, the nodal curve will at this (imaginary) point cut the principal plane at right angles. Hence considering the intersections of the nodal curve by a principal plane, the umbilicar centre, outcrop and node of the evolute count respectively as 2 points, 3 points and 1 point, and as for each kind the number is 4, the whole number of intersections is $4(2+3+1), =24$. It may be shown that these are the only intersections of the nodal curve with the principal plane; and this being so, it follows that the order of the nodal curve is $=24$; which agrees with the result of a subsequent analytical investigation.

37. The umbilicar centres or points (I) belong to values such as $\xi=\xi_1=-a^2$ which are the *united values* in the equation between (ξ, ξ_1), viz. writing herein $\xi_1=\xi$ the equation becomes

$$(\xi+a^2)(\xi+b^2)(\xi+c^2)=0,$$

so that the united values are $\xi=\xi_1=-a^2$, $-b^2$ or $-c^2$. (It may be remarked, that treating this cubic as a degenerate quartic, a united value would be $\xi=\xi_1=\infty$, corresponding to the umbilicar centres at infinity.)

To a value such as $\xi=-a^2$ there corresponds (not only the value $\xi_1=-a^2$, but also) a value $\xi_1=-a^2+\dfrac{3\beta\gamma}{\beta-\gamma}$, as it is easy to verify. And the outcrops or points (II) belong to such values $\xi=-a^2$, $\xi_1=-a^2+\dfrac{3\beta\gamma}{\beta-\gamma}$.

And the nodes of the evolute or points (III) belong to values such as $\xi=\omega b^2+\omega^2c^2$, $\xi_1=\omega^2b^2+\omega c^2$ (ω an imaginary cube root of unity) which, as it is easy to see, satisfy the relation between (ξ, ξ_1). But to complete the theory we require to have the values of η, η_1 and also the co-ordinates of the points on the centro-surface, and of the two points on the ellipsoid.

38. I exhibit the results first for the umbilicar centres (imaginary), outcrops (imaginary), and nodes of the evolute (imaginary), in the plane $x=0$; secondly for the real umbilicar centres in the plane $y=0$ and for the real outcrops in the plane $z=0$. The formulæ contain an expression Ω which is a symmetrical function of α, β, γ (or a, b, c), viz. it is

$$\Omega = \alpha^2 - \beta\gamma = \beta^2 - \gamma\alpha = \gamma^2 - \alpha\beta = \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2) = -(\beta\gamma+\gamma\alpha+\alpha\beta).$$

We have

I. $$\xi = -a^2,\ \eta = -a^2;\ \xi_1 = -a^2,\ \eta_2 = -a^2.$$

$$\left.\begin{array}{ll} X=0, & X_1=0, \\ Y^2 = -b^2\dfrac{\gamma}{\alpha}, & Y_1^2 = -b^2\dfrac{\gamma}{\alpha}, \\ Z^2 = -c^2\dfrac{\beta}{\alpha}, & Z_1^2 = -c^2\dfrac{\beta}{\alpha}, \end{array}\right\} \text{(Umbilicus).}$$

$$\left.\begin{array}{l} x = 0, \\ b^2y^2 = -\dfrac{\gamma^3}{\alpha}, \\ c^2z^2 = -\dfrac{\beta^3}{\alpha}, \end{array}\right\} \text{(Umbilicar centre).}$$

II. $$\xi = -a^2, \qquad \eta = -a^2 + \frac{9\beta\gamma\Omega}{(\beta-\gamma)^3}$$

$$\left(\text{or } \eta + b^2 = -\gamma\frac{(\alpha-\beta)^3}{(\beta-\gamma)^3},\ \eta + c^2 = \frac{\beta(\gamma-\alpha)^3}{(\beta-\gamma)^3}\right).$$

$$\xi_1 = -a^2 + \frac{3\beta\gamma}{\beta-\gamma},\ \eta_1 = -a^2.$$

$$\begin{array}{ll} X=0, & X_1 = 0, \\ Y^2 = -b^2\dfrac{\gamma}{\alpha}\dfrac{(\alpha-\beta)^3}{(\beta-\gamma)^3}, & Y_1^2 = -b^2\dfrac{\gamma}{\alpha}\dfrac{\alpha-\beta}{\beta-\gamma}, \\ Z^2 = -c^2\dfrac{\beta}{\alpha}\dfrac{(\gamma-\alpha)^3}{(\beta-\gamma)^3}, & Z_1^2 = -c^2\dfrac{\beta}{\alpha}\dfrac{\gamma-\alpha}{\beta-\gamma}, \end{array}$$

Ellipse, concomitant. Ellipse, sequential.

$$\left.\begin{array}{l} x=0, \\ b^2y^2 = -\dfrac{\gamma^3(\alpha-\beta)^3}{\alpha(\beta-\gamma)^3}, \\ c^2z^2 = -\dfrac{\beta^3(\gamma-\alpha)^3}{\alpha(\beta-\gamma)^3}, \end{array}\right\} \text{(Outcrop.)}$$

(Observe that at point Y_1, Z_1 of ellipse $\dfrac{Y^2}{b^2}+\dfrac{Z^2}{c^2}=1$, the co-ordinates of the centre of curvature are $y = \dfrac{\alpha Y_1^3}{b^4}$, $z = -\dfrac{\alpha Z_1^3}{c^4}$, and it thence appears that this is the point in regard to which the ellipse is sequential.)

III. $$\xi = \omega b^2 + \omega^2 c^2,\quad \eta = -a^2; \qquad \xi_1 = \omega^2 b^2 + \omega c^2,\quad \eta_1 = -a^2,$$

$$\begin{array}{ll} X = 0, & X_1 = 0, \\ Y^2 = -b^2\omega^2, & Y_1^2 = -b^2\omega, \\ Z^2 = -c^2\omega, & Z_1^2 = -c^2\omega^2, \end{array}$$

$$\left.\begin{array}{l} x = 0, \\ b^2y^2 = -a^2, \\ c^2z^2 = -a^2 \end{array}\right\} \quad \text{(Node of evolute.)}$$

39. Observe that these are the only ways in which it is possible to satisfy the equations

$$0 = (a^2 + \xi)^3 (a^2 + \eta) = (a^2 + \xi_1)^3 (a^2 + \eta_1),$$

viz., starting from this equation we have

I. $$a^2 + \xi = 0, \; a^2 + \xi_1 = 0,$$

whence in the equations for η, η_1, substituting the values $\xi = \xi_1 = -a^2$, we have

$$\begin{aligned} 1 : \eta : \eta_1 = & \; Pa^4 - 2Qa^2 + 3R, \\ & : -a^2R + a^6(P - 2a^2), \\ & : -a^2R + a^6(P - 2a^2), \end{aligned}$$

that is

$$1 : \eta : \eta_1 = -a^2\beta\gamma : a^4\beta\gamma : a^4\beta\gamma,$$

or, $$\eta = \eta_1 = -a^2.$$

40. II. $a^2 + \xi = 0$ without $a^2 + \xi_1 = 0$, consequently $a^2 + \eta_1 = 0$; writing $\xi = -a^2$, in the relation between (ξ, ξ_1) this is

$$6R + 3Q(\xi_1 - a^2) + P(\xi_1^2 - 4a^2\xi_1 + a^4) - 3a^2\xi_1(\xi_1 + a^2) = 0,$$

viz., this is

$$\xi_1^2(b^2 + c^2 - 2a^2) + \xi_1(-a^4 - a^2b^2 - a^2c^2 + 3b^2c^2) + a^2(a^4 - 2a^2b^2 - 2a^2c^2 + 3b^2c^2) = 0,$$

where the left-hand side should divide by $\xi_1 + a^2$; the equation in fact is

$$(\xi_1 + a^2)\{\xi_1(b^2 + c^2 - 2a^2) + a^4 - 2a^2b^2 - 2a^2c^2 + 3b^2c^2\} = 0\,;$$

or, what is the same thing,

$$(\xi_1 + a^2)\{(\xi_1 + a^2)(\beta - \gamma) - 3\beta\gamma\} = 0,$$

$$\text{whence } \xi_1 = -a^2 + \frac{3\beta\gamma}{\beta - \gamma}.$$

41. Considering these values of ξ, ξ_1 as given, the verification of the value $\eta_1 = -a^2$, and determination of $\eta = -a^2 + \dfrac{9\beta\gamma\Omega}{(\beta - \gamma)^3}$ is somewhat complex.

Writing for a moment $\Lambda = -\dfrac{3\beta\gamma}{\beta - \gamma}$, we have

$$\begin{aligned} 1 : \eta : \eta_1 = & \; P(a^4 + a^2\Lambda) - Q(2a^2 + \Lambda) + 3R \\ & : -R(a^2 + 2\Lambda) - (a^2 + \Lambda)^3(2a^2 - P + \Lambda) \\ & : -R(a^2 - \Lambda) - a^6(2a^2 - P + \Lambda). \end{aligned}$$

The first term is

$$a^4P - 2a^2Q + 3R + \Lambda(a^2P - Q),$$

which is

$$= - a^2\beta\gamma + \Lambda(a^4 - b^2c^2),$$

and for the value of η, proceeding to the third term, this is

$$= - a^2R - a^6(2a^2 - P) + \Lambda(R - a^6),$$

which is

$$= a^4\beta\gamma - a^2\Lambda(a^4 - b^2c^2),$$

so that without any further reduction $\eta_1 = - a^2$.

42. We have then

$$\eta = \frac{- R(a^2 + 2\Lambda) - (a^2 + \Lambda)^3(a^2 - b^2 - c^2 + \Lambda)}{- a^2\beta\gamma + \Lambda(a^4 - b^2c^2)},$$

and I assume

$$\eta = - a^2 + \frac{9\beta\gamma}{(\beta - \gamma)^3}\Omega,$$

and investigate the value of Ω.

We have

$$- R(a^2 + 2\Lambda) - (a^2 + \Lambda)^3(a^2 - b^2 - c^2 + \Lambda), = a^4\beta\gamma + \Lambda\odot, \text{ suppose.}$$

The equation therefore is

$$\frac{a^4\beta\gamma + \Lambda\odot}{- a^2\beta\gamma + \Lambda(a^4 - b^2c^2)} = - a^2 + \frac{9\beta\gamma}{(\beta - \gamma)^3}\Omega,$$

that is

$$\Lambda\odot = - a^2\Lambda(a^4 - b^2c^2) + \frac{9\beta\gamma}{(\beta - \gamma)^3}\Omega\{- a^2\beta\gamma + \Lambda(a^4 - b^2c^2)\} = 0$$

or writing $\frac{9\beta\gamma}{(\beta - \gamma)^3} = - \frac{3\Lambda}{(\beta - \gamma)^2}$, omitting the factor Λ, and multiplying by $(\beta - \gamma)^2$, this is

$$(\beta - \gamma)^2\{\odot + a^2(a^4 - b^2c^2)\} + 3\Omega\{- a^2\beta\gamma + \Lambda(a^4\, b^2c^2)\} = 0,$$

in which equation

$$\odot = - 2R - a^6 - (3a^4 + 3a^2\Lambda + \Lambda^2)(a^2 - b^2 - c^2 + \Lambda),$$

and thence $\quad \odot + a^2(a^4 - b^2c^2) = \text{same} + a^2(a^4 - b^2c^2),$

$$\begin{aligned} &= - 3a^6 + 3a^4(b^2 + c^2) - 3a^2b^2c^2 \\ &\quad + \Lambda\{- 6a^4 + 3a^2(b^2 + c^2)\} \\ &\quad + \Lambda^2(- 4a^2 + \quad b^2 + c^2) \\ &\quad - \Lambda^3 \\ &= 3a^2\beta\gamma + 3a^2\Lambda(\beta - \gamma) + \Lambda^2(\beta - \gamma - 2a^2) - \Lambda^3. \end{aligned}$$

43. Hence, substituting for Λ its value and multiplying by $(\beta - \gamma)^3$, we have

$$(\beta - \gamma)^3\{\odot + a^2(a^4 - b^2c^2)\}$$

$$= 3a^2\beta\gamma(\beta - \gamma)^3 - 9a^2\beta\gamma(\beta - \gamma)^3 + 9\beta^2\gamma^2(\beta - \gamma - 2a^2)(\beta - \gamma) + 27\beta^3\gamma^3,$$

which is

$$= -6a^2\beta\gamma(\beta-\gamma)^3 + 9\beta^2\gamma^2(\beta-\gamma)^2 - 18\alpha^2\beta^2\gamma^2(\beta-\gamma) + 27\beta^3\gamma^3;$$

viz. this is

$$= \{-6a^2(\beta-\gamma)+9\beta\gamma\}\{(\beta-\gamma)^2+3\beta\gamma\}\beta\gamma,$$

$$= \{-6a^2(\beta-\gamma)+9\beta\gamma\}(\beta^2+\beta\gamma+\gamma^2)\beta\gamma,$$

and the equation thus is

$$\{-2a^2(\beta-\gamma)+3\beta\gamma\}(\beta^2+\beta\gamma+\gamma^2)\beta\gamma + \Omega\left\{-a^2\beta\gamma - \frac{3\beta\gamma}{\beta-\gamma}(a^4-b^2c^2)\right\}(\beta-\gamma)=0,$$

or finally

$$\Omega\{a^2(\beta-\gamma)+3(a^4-b^2c^2)\} = (-2a^2(\beta-\gamma)+3\beta\gamma)(\beta^2+\beta\gamma+\gamma^2).$$

But $c^2=a^2+\beta$, $b^2=a^2-\gamma$, and hence $\alpha^4-b^2c^2=-\alpha^2(\beta-\gamma)+\beta\gamma$, and therefore

$$a^2(\beta-\gamma)+3(a^4-b^2c^2) = -2a^2(\beta-\gamma)+3\beta\gamma;$$

the equation thus divides by $-2a^2(\beta-\gamma)+3\beta\gamma$ and we have

$$\Omega = \beta^2+\beta\gamma+\gamma^2,$$

or as this may also be written $\Omega=\alpha^2-\beta\gamma$, $=\beta^2-\gamma\alpha$, $=\gamma^2-\alpha\beta$. So that Ω has the value originally so denoted, and we have then

$$\eta = -a^2 + \frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega.$$

44. III. Lastly the equation $0=(a^2+\xi)^3(a^2+\eta)=(a^2+\xi_1)^3(a^2+\eta_1)$ is satisfied if $\alpha^2+\eta=0$, $\alpha^2+\eta_1=0$: the equations

$$(b^2+\xi)^3(b^2+\eta) = (b^2+\xi_1)^3(b^2+\eta_1),$$

$$(c^2+\xi)^3(c^2+\eta) = (c^2+\xi_1)^3(c^2+\eta_1),$$

then give

$$(b^2+\xi)^3 = (b^2+\xi_1)^3,$$

$$(c^2+\xi)^3 = (c^2+\xi_1)^3,$$

which can be satisfied by $\xi=\xi_1$, leading to $\xi=\xi_1=-a^2$, which is the case I., or else by

$$b^2+\xi = \omega(b^2+\xi_1),$$

$$c^2+\xi = \omega^2(b^2+\xi_1),$$

that is

$$\xi = \omega b^2+\omega^2c^2,\quad \xi_1 = \omega^2b^2+\omega c^2.$$

To show that these values satisfy the relation between ξ, ξ_1, observe that they give

$$\xi+\xi_1 = -b^2-c^2,\quad \xi\xi_1 = b^4-b^2c^2+c^4,$$

whence also

$$\xi^2+4\xi\xi_1+\xi_1{}^2 = 3(b^4+c^4),$$

and the relation becomes

$$6a^2b^2c^2 - 3\,[a^2(b^2+c^2) + b^2c^2]\,(b^2+c^2)$$
$$+ [a^2 + (b^2+c^2)]\,.\,3\,(b^4+c^4) - 3\,(b^2+c^2)\,(b^4 - b^2c^2 + c^4) = 0,$$

which is an identity.

45. I will show that these values of ξ, ξ_1 give the foregoing values $\eta = \eta_1 = -a^2$. We have

$$1 : \eta - \eta_1 : \eta + \eta_1 = P\xi\xi_1 + Q\,(\xi + \xi_1) + 3R$$
$$: (\xi_1 - \xi)\,\{3R - (\xi^2 + \xi\xi_1 + \xi_1^2)\,(P + \xi + \xi_1)\}$$
$$: (\xi_1 + \xi)\,\{R - (\xi^2 - \xi\xi_1 + \xi_1^2)\,(P + \xi + \xi_1)\},$$

this is

$$1 : \eta - \eta_1 : \eta + \eta_1 = \alpha^2\,(b^2+c^2) : 0\,(\xi_1 - \xi) : -2a^2\alpha^2\,(b^2+c^2),$$

or

$$\eta - \eta_1 = 0,\ \ \eta + \eta_1 = -2a^2;\ \text{ that is }\ \eta = \eta_1 = -a^2.$$

46. For the real umbilicar centres and outcrops we have

I. $\qquad \xi = -b^2,\ \eta = -b^2,\ \xi_1 = -b^2,\ \eta_2 = -b^2.$

$$X^2 = -a^2\frac{\gamma}{\beta}, \qquad X_1^2 = -a^2\frac{\gamma}{\beta},$$
$$Y = 0, \qquad Y_1 = 0,$$
$$Z^2 = -c^2\frac{\alpha}{\beta}, \qquad Z_1^2 = -c^2\frac{\alpha}{\beta},$$

$$\left.\begin{aligned} a^2x^2 &= -\frac{\gamma^3}{\beta},\\ y &= 0,\\ c^2z^2 &= -\frac{\alpha^3}{\beta}, \end{aligned}\right\}\ \text{(real umbilicar centre.)}$$

II. $\quad \xi = -c^2, \qquad \eta = -c^2 + \dfrac{9\alpha\beta}{(\alpha-\beta)^3}$

$$\left(\text{or }\ \eta + a^2 = -\beta\,\frac{(\gamma-\alpha)^3}{(\alpha-\beta)^3},\ \ \eta + b^2 = \frac{\alpha\,(\beta-\gamma)^3}{(\alpha-\beta)^3}\right).$$

$$\xi_1 = -c^2 + \frac{3\alpha\beta}{\alpha-\beta},\ \ \eta_1 = -c^2.$$

$$\left.\begin{aligned} X^2 &= -a^2\frac{\beta}{\gamma}\frac{(\gamma-\alpha)^3}{(\alpha-\beta)^3}, & X_1^2 &= -a^2\frac{\beta}{\gamma}\frac{\gamma-\alpha}{\alpha-\beta},\\ Y^2 &= -b^2\frac{\alpha}{\gamma}\frac{(\beta-\gamma)^3}{(\alpha-\beta)^3}, & Y_1^2 &= -b^2\frac{\alpha}{\gamma}\frac{\beta-\gamma}{\alpha-\beta},\\ Z &= 0, & Z_1 &= 0, \end{aligned}\right\}$$

ellipse concomitant. $\qquad$ ellipse sequential.

$$\left.\begin{aligned} a^2x^2 &= -\frac{\beta^3}{\gamma}\frac{(\gamma-\alpha)^3}{(\alpha-\beta)^3} \\ b^2y^2 &= -\frac{\alpha^3}{\gamma}\frac{(\beta-\gamma)^3}{(\alpha-\beta)^3} \end{aligned}\right\} \text{(real outcrop).}$$

$$z = 0.$$

Nodal curve in vicinity of umbilicar centre, $a^2x^2 = -\dfrac{\gamma^3}{\beta}$, $y=0$, $c^2z^2 = -\dfrac{\alpha^3}{\beta}$. Art. Nos. 47 to 49.

47. Write
$$\xi = -b^2 + q, \qquad \eta = -b^2 + r,$$
$$\xi_1 = -b^2 + q_1, \qquad \eta_1 = -b^2 + r_1,$$

we have to find the relation between q, q_1, r, r_1; first for q, q_1, the equation of correspondence gives

$$\begin{aligned} &6R \\ &+ 3Q(-2b^2 + q + q_1) \\ &+ P(6b^4 - 6b^2\overline{q+q_1} + q^2 + 4qq_1 + q_1^2) \\ &+ 3(-2b^6 + 3b^4\overline{q+q_1} - b^2(q^2 + 4qq_1 + q_1^2) + qq_1\overline{q+q_1}) = 0, \end{aligned}$$

that is
$$\begin{aligned} &3(q+q_1)(3b^4 - 2b^2P + Q) \\ &+ (q^2 + qq_1 + q_1^2)(-3b^2 + P) \\ &+ 3qq_1(q+q_1) = 0, \end{aligned}$$

viz. this is
$$\begin{aligned} &-3(q+q_1)\alpha\gamma \\ &+ (q^2 + 4qq_1 + q_1^2)(\gamma - \alpha) \\ &+ 3qq_1(q+q_1) = 0, \end{aligned}$$

whence approximately $q + q_1 = 0$; but it will appear that the value is required to the second order; we have therefore

$$\begin{aligned} q + q_1 &= \tfrac{1}{3}\frac{\gamma-\alpha}{\gamma\alpha}(q^2 + 4qq_1 + q_1^2) \\ &= -\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}q^2. \end{aligned}$$

48. Now the equations
$$(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1), \text{ and } (c^2+\xi)^3(c^2+\eta) = (c^2+\xi_1)^3(c^2+\eta_1),$$

putting therein for ξ, η, ξ_1, η_1, their values, give the first of them

$$\log\left(1+\frac{r}{\gamma}\right) + 3\log\left(1+\frac{q}{\gamma}\right) = \log\left(1+\frac{r_1}{\gamma}\right) + 3\log\left(1+\frac{q_1}{\gamma}\right),$$

that is
$$r + 3q - \frac{1}{2\gamma}(r^2+3q^2) + \frac{1}{3\gamma^2}(r^3+3q^3) = r_1 + 3q_1 - \frac{1}{2\gamma}(r_1^2+3q_1^2) + \frac{1}{3\gamma^2}(r_1^3+3q_1^3),$$

and similarly the second equation

$$r + 3q + \frac{1}{2\alpha}(r^2 + 3q^2) + \frac{1}{3\alpha^2}(r^3 + 3q^3) = r_1 + 3q_1 + \frac{1}{2\alpha}(r_1^2 + 3q_1^2) + \frac{1}{3\alpha^2}(r_1^3 + 3q_1^3);$$

whence multiplying by γ, α, and adding,

$$(\gamma + \alpha)\left\{r + 3q + \frac{1}{3\alpha\gamma}(r^3 + 3q^3)\right\} = (\gamma + \alpha)\left\{r_1 + 3q_1 + \frac{1}{3\alpha\gamma}(r_1^3 + 3q_1^3)\right\},$$

which, neglecting terms of the third order is

$$r + 3q = r_1 + 3q_1.$$

Subtracting the two equations we have

$$\tfrac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right)(r^2 + 3q^2) + \tfrac{1}{3}\left(\frac{1}{\alpha^2} - \frac{1}{\gamma^2}\right)(r^3 + 3q^3) = \tfrac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right)(r_1^2 + 3q_1^2) + \tfrac{1}{3}\left(\frac{1}{\alpha^2} - \frac{1}{\gamma^2}\right)(r_1^3 + 3q_1^3),$$

viz. this is

$$r^2 + 3q^2 + \tfrac{2}{3}\frac{\gamma - \alpha}{\gamma\alpha}(r^3 + 3q^3) = r_1^2 + 3q_1^2 + \tfrac{2}{3}\frac{\gamma - \alpha}{\gamma\alpha}(r_1^3 + 3q_1^3),$$

or, what is the same thing,

$$r^2 - r_1^2 + 3(q^2 - q_1^2)) + \tfrac{2}{3}\frac{\gamma - \alpha}{\gamma\alpha}\{r^3 - r_1^3 + 3(q^3 - q_1^3)\} = 0,$$

which, putting therein $r - r_1 = -3(q - q_1)$, is

$$-r - r_1 + q + q_1 + \tfrac{2}{3}\frac{\gamma - \alpha}{\gamma\alpha}(-r^2 - rr_1 - r_1^2 + q^2 + qq_1 + q^2) = 0,$$

say this is

$$-r - r_1 + q + q_1 + 2\Delta = 0;$$

combining herewith

$$r - r_1 + 3q - 3q_1 \quad = 0,$$

we have

$$r + q - 2q_1 - \Delta = 0,$$

and

$$r_1 - 2q + q_1 - \Delta = 0,$$

where

$$\Delta = \tfrac{1}{3}\frac{\gamma - \alpha}{\gamma\alpha}(-r^2 - rr_1 - r_1^2 + q^2 + qq_1 + q_1^2),$$

but substituting herein the values $r = -q + 2q_1$, $r_1 = 2q - q_1$, this becomes

$$\Delta = \tfrac{1}{3}\frac{\gamma - \alpha}{\gamma\alpha}(-2q^2 + 4qq_1 - 2q_1^2), = -\tfrac{8}{3}\frac{\gamma - \alpha}{\gamma\alpha}q^2,$$

and then

$$r = -q + 2q_1 + \Delta,$$

that is

$$r + 3q = 2(q + q_1) + \Delta, = -\frac{4(\gamma - \alpha)}{\gamma\alpha}q^2.$$

49. We have then

$$a^2x^2 = -\frac{\gamma^3}{\beta}\left(1 + \frac{r}{\gamma}\right)\left(1 + \frac{q}{\gamma}\right)^3,$$

$$= -\frac{\gamma^3}{\beta}\left(1 + \frac{r + 3q}{\gamma} + \frac{3q(r + q)}{\gamma^2}\right).$$

$$= -\frac{\gamma^3}{\beta}\left(1 + \frac{r+3q}{\gamma^2} - \frac{6}{\gamma^2}q^2\right)$$

$$= -\frac{\gamma^3}{\beta}\left\{1 + q^2\left(-\frac{4(\gamma-\alpha)}{\gamma^2\alpha} - \frac{6\alpha}{\gamma^2\alpha}\right)\right\}$$

$$= -\frac{\gamma^3}{\beta}\left\{1 + q^2\frac{2(\beta-\gamma)}{\gamma^2\alpha}\right\};$$

and in the same way from $c^2z^2 = -\frac{\alpha^3}{\beta}\left(1-\frac{r}{\alpha}\right)\left(1-\frac{r}{\alpha}\right)^3$, we have

$$c^2z^2 = -\frac{\alpha^3}{\beta}\left\{1 - q^2\,\frac{2(\alpha-\beta)}{\gamma\alpha^2}\right\}:$$

moreover we have at once

$$b^2y^2 = -\frac{q^3r}{\gamma\alpha} = \frac{3q^4}{\gamma\alpha}.$$

Hence, writing $x+\delta x$, $0+\delta y$, $z+\delta z$ for x, y, z, we find

$$\delta x = \tfrac{1}{2}x.\frac{2(\beta-\gamma)}{\gamma^2\alpha}.q^2,$$

$$\delta y = \pm\frac{1}{b}\sqrt{\frac{3}{\gamma\alpha}}.q^2,$$

$$\delta z = \tfrac{1}{2}z.\frac{-2(\alpha-\beta)}{\gamma\alpha^2}.q^2,$$

or, what is the same thing,

$$dx : dy : dz = x\frac{2(\beta-\gamma)}{\gamma^2\alpha} : \pm\frac{2}{b}\sqrt{\frac{3}{\gamma\alpha}} : z\frac{-2(\alpha-\beta)}{\gamma\alpha^2},$$

where x, z denote the values at the umbilicar centre.

Nodal curve in vicinity of real outcrop, viz.

$$a^2x^2 = -\frac{\beta^3}{\gamma}\frac{(\gamma-\alpha)^3}{(\alpha-\beta)^3},\quad b^2y^2 = -\frac{\alpha^3}{\gamma}\frac{(\alpha-\gamma)^3}{(\alpha-\beta)^3},\quad z=0.\qquad \text{Art. Nos. 50 to 52.}$$

50. Write

$$\xi = -c^2 + q,\qquad \eta = -c^2 + \frac{q\alpha\beta\Omega}{(\alpha-\beta)^3} + \theta,$$

$$\xi_1 = -c^2 + \frac{3\alpha\beta}{\alpha-\beta} + q_1,\qquad \eta_1 = -c^2 + \theta_1;$$

and first for the relation between q and q_1, writing for a moment $\frac{3\alpha\beta}{\alpha-\beta} + q_1 = Q_1$, and therefore $\xi_1 = -c^2 + Q_1$, the equation of correspondence gives

$$-3\alpha\beta(q+Q_1) + (q^2 + 4qQ_1 + Q_1^2)(\alpha-\beta) + 3qQ_1(q+Q_1) = 0,$$

which putting for Q_1 its value is

$$-3\alpha\beta\left(q+q_1+\frac{3\alpha\beta}{\alpha-\beta}\right)$$

$$+(\alpha-\beta)\left(q^2+4qq_1+q_1^2+(4q+2q_1)\frac{3\alpha\beta}{\alpha-\beta}+\frac{9\alpha^2\beta^2}{(\alpha-\beta)^2}\right)$$

$$+3q\left\{q(q+q_1)+(q+2q_1)\frac{3\alpha\beta}{\alpha-\beta}+\frac{9\alpha^2\beta^2}{(\alpha-\beta)^2}\right\}=0;$$

that is

$$-3\alpha\beta(q+q_1)$$

$$+3\alpha\beta(4q+2q_1)+(\alpha-\beta)(q^2+4qq_1+q_1^2)$$

$$+\frac{27\alpha^2\beta^2}{(\alpha-\beta)^2}q+\frac{9\alpha\beta}{\alpha-\beta}(q^2+2qq_1)+3qq_1(q+q_1)=0,$$

or, what is the same thing,

$$\left(9\alpha\beta+\frac{27\alpha^2\beta^2}{(\alpha-\beta)^2}\right)q+3\alpha\beta q_1$$

$$+q^2\left\{\frac{\alpha^2+7\alpha\beta+\beta^2}{\alpha-\beta}+qq_1\frac{\alpha^2+16\alpha\beta+\beta^2}{\alpha-\beta}+q_1^2(\alpha-\beta)\right\}$$

$$+3qq_1(q+q_1)=0,$$

or for small values

$$\left(3+\frac{9\alpha\beta}{(\alpha-\beta)^2}\right)q+q_1=0,\ \text{ that is }\frac{3\Omega}{(\alpha-\beta)^2}q+q_1=0.$$

51. Moreover, from equation $(c^2+\xi)^3(c^2+\eta)=(c^2+\xi_1)^3(c^2+\eta_1)$, we have

$$q^3\frac{9\alpha\beta\Omega}{(\alpha-\beta)^3}=\left(\frac{3\alpha\beta}{\alpha-\beta}\right)^3.\theta_1,\ \text{ that is, }\theta_1=\tfrac{1}{3}\frac{\Omega}{\alpha^2\beta^2}\cdot q^3,$$

or, since q and q_1 are of the same order, θ_1 is of the order q_1^3. Hence, starting from the equations $-\beta\gamma a^2x^2=(a^2+\xi_1)^3(a^2+\eta_1)$ &c., the terms of x, y arising from the variation of η_1 are indefinitely small in regard to those arising from the variation of ξ_1; and we have

$$\frac{2\delta x}{x}=\frac{3q_1}{-\beta+\dfrac{3\alpha\beta}{\alpha-\beta}},=-3q_1\frac{(\alpha-\beta)}{\beta(\gamma-\alpha)},$$

$$\frac{2\delta y}{y}=\frac{3q_1}{\alpha+\dfrac{3\alpha\beta}{\alpha-\beta}},=\ 3q_1\frac{\alpha-\beta}{\alpha(\beta-\gamma)},$$

and for $\delta z\ (=z)$ we have

$$c^2(\delta z)^2=-\frac{1}{\alpha\beta}\left(\frac{3\alpha\beta}{\alpha-\beta}\right)^3\theta_1,=\frac{-9\Omega q^3}{(\alpha-\beta)^3},=\frac{(\alpha-\beta)^3}{3\Omega^4}q_1^3,$$

so that writing for greater simplicity, $(\alpha-\beta)\,q_1=-\alpha\beta\varpi$, the formulæ become

$$\frac{2\delta x}{x}=\frac{3\alpha}{\gamma-\alpha}\,\varpi,$$

$$\frac{2\delta\eta}{\eta}=-\frac{3\beta}{\beta-\gamma}\,\varpi,$$

$$c\delta z=\frac{(-\alpha\beta\varpi)^{\frac{3}{2}}}{\Omega\sqrt{3}}.$$

52. This shows that there is at the outcrop a cusp, the cuspidal tangent being in the plane of xy. It appears moreover that this tangent coincides with the tangent of the evolute. In fact from the equation $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}-\gamma^2=0$ of the evolute we have

$$\frac{(ax)^{\frac{2}{3}}.\,dx}{x}+\frac{(by)^{\frac{2}{3}}.\,dy}{y}=0,$$

or substituting for $(x,\ y)$ their values at the outcrop,

$$\frac{\beta\,(\gamma-\alpha)}{\gamma^{\frac{1}{3}}\,(\alpha-\beta)}\,\frac{dx}{x}+\frac{\alpha\,(\beta-\gamma)}{\gamma^{\frac{1}{3}}\,(\alpha-\beta)}\,\frac{dy}{y}=0\,;$$

that is,

$$\beta\,(\gamma-\alpha)\frac{dx}{x}+\alpha\,(\beta-\gamma)\frac{dy}{y}=0,$$

which is satisfied by the foregoing values of $\dfrac{\delta x}{x}$, and $\dfrac{\delta y}{y}$, and the two tangents therefore coincide.

We have

$$4\,\{(\delta x)^2+(\delta y)^2\}=\frac{-9\varpi^2\alpha^2\beta^2}{\gamma\,(\alpha-\beta)^3}\cdot\left\{\frac{\beta\,(\gamma-\alpha)}{a^2}+\frac{\alpha\,(\beta-\gamma)}{b^2}\right\},$$

which in virtue of

$$a^2\alpha\,(\beta-\gamma)+b^2\beta\,(\gamma-\alpha)+c^2\gamma\,(\alpha-\beta)=3\alpha\beta\gamma,$$

is
$$4\,\{(\delta x)^2+(\delta y)^2\}=\frac{-9\varpi^2\alpha^2\beta^2}{a^2b^2\,(\alpha-\beta)^3}\,\{3\alpha\beta-c^2\,(\alpha-\beta)\}$$

(observe $3\alpha\beta-c^2\,(\alpha-\beta),\ =-c^2\,(\gamma-\alpha)-3a^2\alpha$, is negative)

$$=\frac{-9\varpi^2\alpha^2\beta^2}{a^2b^2\,(\alpha-\beta)^2}\,\xi_1,$$

if ξ_1 be the value at the outcrop. Writing δs for the element of the arc we have

$$\delta s=-\tfrac{3}{2}\,\frac{\alpha\beta}{ab\,(\alpha-\beta)}\,\sqrt{-\xi_1}\,.\,\varpi,$$

$$\delta z=\frac{(-\alpha\beta\varpi)^{\frac{3}{2}}}{\Omega\sqrt{3}},$$

which exhibit the form at the outcrop.

The Nodal Curve; expressions for the co-ordinates in terms of a single parameter σ. Art. Nos. 53 to 60.

53. After the foregoing investigation of the nodal curve, I was led to perceive that it is possible to express ξ, η, ξ_1, η_1 in terms of a single variable σ, and thus to obtain expressions for the co-ordinates of a point of the nodal curve in terms of the single variable σ. The result was obtained by the consideration that the acnodal portion of the nodal curve could only arise from imaginary values of ξ, η; the question thus was, what imaginary values of these quantities give real values for the co-ordinates x, y, z. To make y real we may assume

$$\xi = -b^2 - p(\theta - \phi i),$$
$$\eta = -b^2 + p(\theta + \phi i)^3,$$

($i = \sqrt{-1}$ as usual): this being so, if Δ denote one or other of the quantities

$$\gamma,\ -\alpha\ (= a^2 - b^2,\ c^2 - b^2),$$

the expressions for $-\beta\gamma a^2x^2$, $-\gamma\alpha b^2y^2$ will be

$$= \{\Delta - p(\theta - \phi i)\}^3 \{\Delta + p(\theta + \phi i)^3\},$$

and we have therefore the condition that this shall be real (for the two values $\Delta = \gamma$, $\Delta = -\alpha$): being real, it will in certain cases be positive, and we shall then have real values for the remaining co-ordinates x, z.

54. The condition of reality is easily found to be

$$\Delta^2(3\theta^2 - \phi^2 + 3) - 6\theta p\Delta(\theta^2 + \phi^2 + 1) + p^2\{3(\theta^2 + \phi^2)^2 + 3\theta^2 - \phi^2\} = 0,$$

viz. this equation in Δ must have the roots γ, $-\alpha$, or the expression on the left hand must be

$$= (3\theta^2 - \phi^2 + 3)\{\Delta^2 - (\gamma - \alpha)\Delta - \alpha\gamma\}:$$

we have therefore

$$\frac{(\gamma - \alpha)^2}{-\gamma\alpha} = \frac{36(\theta^2 + \phi^2 + 1)^2}{(3\theta^2 - \phi^2 + 3)\{3(\theta^2 + \phi^2)^2 + 3\theta^2 - \phi^2\}},$$

$$\gamma - \alpha = \frac{6\theta p(\theta^2 + \phi^2 + 1)}{3\theta^2 - \phi^2 + 3};$$

and writing $\theta^2 + \phi^2 = X$, $3\theta^2 - \phi^2 = Y$, the first of these is

$$\frac{(\gamma - \alpha)^2}{-\gamma\alpha} = \frac{9(X + Y)(X + 1)^2}{(Y + 3)\{3(X^2 - 1) + Y + 3\}},$$

which regarding X, Y as the co-ordinates of a point in a plane is a cubic curve having the point $X + 1 = 0$, $Y + 3 = 0$ as a node: hence writing $Y + 3 = 3\sigma(X + 1)$, X and Y will be each of them a rational function of σ. The second equation is

$$\frac{6\theta p(X + 1)}{Y + 3} = \gamma - \alpha, \text{ that is } p = \frac{(\gamma - \alpha)\sigma}{2\theta}, = \frac{(\gamma - \alpha)\sigma}{\sqrt{X + Y}};$$

and we have also

$$2\theta = \sqrt{X+Y}, \quad 2\phi = \sqrt{3X-Y};$$

the equations thus become

$$\xi = -b^2 - \frac{(\gamma-\alpha)\sigma}{2}\left\{1 - i\sqrt{\frac{3X-Y}{X+Y}}\right\},$$

$$\eta = -b^2 + \frac{(\gamma-\alpha)\sigma(X+Y)}{8}\left\{1 + i\sqrt{\frac{3X-Y}{X+Y}}\right\}^3,$$

which are better written in the form

$$\xi = -b^2 - \tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1 - \sqrt{\frac{-3X+Y}{X+Y}}\right\},$$

$$\eta = -b^2 + \tfrac{1}{8}(\gamma-\alpha)\sigma(X+Y)\left\{1 + \sqrt{\frac{-3X+Y}{X+Y}}\right\}^3,$$

where X, Y are given functions of σ. We in fact thus obtain an analytical expression of the nodal curve, quite independent of the considerations as to real and imaginary which suggested the process: the foregoing values substituted for ξ, η will give $-\beta\gamma a^2x^2$, &c. equal to rational functions of σ, so that taking for ξ_1, η_1 the same expressions, only changing therein the sign of the radical $\sqrt{\frac{-3X+Y}{X+Y}}$, these values of ξ_1, η_1 give the very same values of $-\beta\gamma a^2x^2$, &c., or the values of ξ, η, ξ_1, η_1 satisfy the conditions

$$(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1), \text{ \&c.}$$

for a point on the nodal curve.

55. To complete the investigation, writing as above $Y+3 = 3\sigma(X+1)$, we obtain

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha} = \frac{(3\sigma+1)X + 3\sigma - 3}{\sigma(X+\sigma-1)};$$

or putting for a moment

$$\frac{(\gamma-\alpha)^2\sigma}{-\gamma\alpha} = K,$$

we have

$$X = \frac{(K-3)(\sigma-1)}{3\sigma+1-K}, \quad X+1 = \frac{K(\sigma-2)+4}{3\sigma+1-K};$$

$$Y+3 = \frac{3K\sigma(\sigma-2)+12\sigma}{3\sigma+1-K}, \quad Y = \frac{3(\sigma-1)\{K(\sigma-1)+1\}}{3\sigma+1-K};$$

$$X+Y = \frac{(\sigma-1)(3\sigma-2)K}{3\sigma+1-K}, \quad -3X+Y = \frac{3(\sigma-1)\{K(\sigma-2)+4\}}{3\sigma+1-K};$$

or substituting for K its value we have

$$K(\sigma-2)+4=-\frac{(\gamma-\alpha)^2}{\gamma\alpha}\left\{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}$$

$$=-\frac{(\gamma-\alpha)^2}{\gamma\alpha}\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right),$$

$$3\sigma+1-K=\frac{1}{\gamma\alpha}\{(3\sigma+1)\gamma\alpha+(\gamma-\alpha)^2\sigma\},\quad =\frac{1}{\gamma\alpha}(\Omega\sigma+\gamma\alpha),$$

if as before $\Omega=\beta^2-\gamma\alpha$; and the result is

$$\xi=-b^2-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\},$$

$$\eta=-b^2-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}\left\{1+\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}^3,$$

and changing the sign of the radical we have the values of ξ_1, η_1.

56. Write for moment

$$\left[\Delta-\tfrac{1}{2}(\gamma-a)\sigma\left\{1-\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}\right]=(\Delta-\mathrm{a}+\mathrm{a}\sqrt{S})^3=A+B\sqrt{S},$$

$$\Delta-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}\left\{1+\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}^3=A'+B'\sqrt{S};$$

then in the product of these two expressions the rational part is $=AA'+BB'S$; but from the manner in which they were arrived at we have $0=AB'+A'B$, and the rational part is thus

$$=-\frac{B'}{B}(A^2-B^2S).$$

We have

$$A^2-B^2S=\{(\Delta-\mathrm{a})^2-\mathrm{a}^2S\}^3,$$

$$B'=-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}(3+S),$$

$$B=\tfrac{1}{2}(\gamma-\alpha)\{3(\Delta-\mathrm{a})^2+\mathrm{a}^2S\};$$

hence the rational part in question is

$$=\tfrac{1}{4}\frac{(\gamma-\alpha)^2}{\Omega\sigma+\gamma\alpha}\frac{\sigma(\sigma-1)(3\sigma-2)(3+S)}{3(\Delta-\mathrm{a})^2+\mathrm{a}^2S}\{(\Delta-\mathrm{a})^2+\mathrm{a}^2S\}^3,$$

which putting therein $\Delta=0$ gives the value of $-\gamma\alpha b^2y^2$; and putting $\Delta=\gamma$, or $\Delta=-\alpha$, gives the value of $-\beta\gamma a^2x^2$ or $-\alpha\beta c^2z^2$.

57. We have

$$1 - S = \frac{1}{\sigma(3\sigma - 2)}\left[3\sigma^2 - 2\sigma - 3\left\{\sigma^2 - 2\sigma - \frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}\right]$$

$$= \frac{4\left\{\sigma + \dfrac{3\alpha\gamma}{(\gamma-\alpha)^2}\right\}}{\sigma(3\sigma-2)},$$

$$3 + S = \frac{3}{\sigma(3\sigma-2)}\left[3\sigma^2 - 2\sigma + \left\{\sigma^2 - 2\sigma - \frac{4\alpha\gamma}{(\gamma-\alpha)^2}\right\}\right]$$

$$= \frac{12}{\sigma(3\sigma-2)}\left(\sigma + \frac{\alpha}{\gamma-\alpha}\right)\left(\sigma - \frac{\gamma}{\gamma-\alpha}\right).$$

Hence we have at once the value of

$$-\gamma\alpha b^2y^2, \ = \tfrac{1}{4}\,\frac{(\gamma-\alpha)^2}{\Omega\sigma + \gamma\alpha}\cdot\frac{\sigma(\sigma-1)(3\sigma-2)}{\mathrm{a}^2}\,\mathrm{a}^6(1-S)^3,$$

$$\text{where } \mathrm{a} = \tfrac{1}{2}(\beta - \gamma)\sigma.$$

58. Moreover

$$(\Delta - \mathrm{a})^2 - \mathrm{a}^2S = \Delta^2 - 2\mathrm{a}\Delta + \mathrm{a}^2(1-S)$$

$$= \frac{1}{3\sigma-2}\left[(3\sigma-2)\{-(\gamma-\alpha)\Delta\sigma + \Delta^2\} + (\gamma-\alpha)^2\sigma^2 + 3\alpha\gamma\right],$$

where the term in [ ] is

$$\sigma^2(\gamma-\alpha)(\gamma-\alpha-3\Delta) + \sigma\{3\Delta^2 + 2(\gamma-\alpha)\Delta + 3\alpha\gamma\} - 2\Delta^2,$$

and since $\Delta = \gamma$ or $-\alpha$, that is $\Delta^2 - (\gamma-\alpha)\Delta - \alpha\gamma = 0$, the coefficient of σ is

$$= \Delta\{6\Delta - (\gamma-\alpha)\},$$

or the term is the product of two linear functions of σ; and we have

$$(\Delta - \mathrm{a})^2 - \mathrm{a}^2S = \frac{1}{3\sigma-2}\{\sigma(\gamma-\alpha) - 2\Delta\}\{\sigma(\gamma-\alpha-3\Delta) + \Delta\}.$$

Similarly

$$3(\Delta - \mathrm{a})^2 + \mathrm{a}^2S = 3(\Delta^2 - 2\mathrm{a}\Delta) + \mathrm{a}^2(3+S)$$

$$= \frac{3}{3\sigma-2}\left[(3\sigma-2)\{-(\gamma-\alpha)\sigma\Delta + \Delta^2\} + \sigma\{(\gamma-\alpha)\sigma + \alpha\}\{(\gamma-\alpha)\sigma - \gamma\}\right],$$

where the term in [ ] is

$$(\gamma-\alpha)^2\sigma^3 - (\gamma-\alpha)(\gamma-\alpha+3\Delta)\sigma^2 + \{3\Delta^2 + 2(\gamma-\alpha)\Delta - \alpha\gamma\}\sigma - 2\Delta^2,$$

in which the coefficient of σ is $= \Delta\{2\Delta + 3(\gamma-\alpha)\}$, and the term is a product of three linear functions: hence

$$3(\Delta - \mathrm{a})^2 + \mathrm{a}^2S = \frac{3(\sigma-1)}{3\sigma-2}(\overline{\gamma-\alpha}\,\sigma - \Delta)(\overline{\gamma-\alpha}\,\sigma - 2\Delta).$$

59. Substituting these values we have the expression

$$\frac{1}{\Omega\sigma+\gamma\alpha}\,\frac{\{(\gamma-\alpha)\,\sigma+\alpha\}\,\{(\gamma-\alpha)\,\sigma-\gamma\}\,\{(\gamma-\alpha)\,\sigma-2\Delta\}^2\,\{(\gamma-\alpha-3\Delta)\,\sigma+\Delta\}^3}{\{(\gamma-\alpha)\,\sigma-\Delta\}\,(3\sigma-2)^2};$$

which writing therein $\Delta=\gamma$ gives $-\beta\gamma a^2x^2$, and writing $\Delta=-\alpha$ gives $-\alpha\beta c^2z^2$; we have above an expression for $-\gamma\alpha b^2y^2$ requiring only a simple reduction, and the final results are

$$-\beta\gamma a^2x^2=\frac{\{(\gamma-\alpha)\,\sigma+\alpha\}\,\{(\gamma-\alpha)\,\sigma-2\gamma\}^2\,\{(\beta-\gamma)\,\sigma+\gamma\}^3}{(\Omega\sigma+\gamma\alpha)\,(3\sigma-2)^2}.$$

$$-\gamma\alpha b^2y^2=\frac{(\sigma-1)\,\sigma^2\,\{(\gamma-\alpha)^2\,\sigma+3\alpha\gamma\}^3}{(\Omega\sigma+\gamma\alpha)\,(3\sigma-2)^2}.$$

$$-\alpha\beta c^2z^2=\frac{\{(\gamma-\alpha)\,\sigma-\gamma\}\,\{(\gamma-\alpha)\,\sigma+2\alpha\}^2\,\{(\alpha-\beta)\,\sigma-\alpha\}^3}{(\Omega\sigma+\gamma\alpha)\,(3\sigma-2)^2}.$$

where it is to be observed that, equating the denominator to 0, we have a triple root $\sigma=\infty$; to indicate this, we may insert in the denominator the factor $(1-0\sigma)^3$.

60. We see here the meaning of all the factors, viz.

Planes.

	$x=0$	$y=0$	$z=0$	∞
Evolute nodes	$\sigma=-\dfrac{\alpha}{\gamma-\alpha}$	$\sigma=1$	$\sigma=\dfrac{\gamma}{\gamma-\alpha}$	$\sigma=\dfrac{-\gamma\alpha}{\Omega}$
Umbilicar centres	$\sigma=\dfrac{2\gamma}{\gamma-\alpha}$	$\sigma=0$	$\sigma=\dfrac{-2\alpha}{\gamma-\alpha}$	$\sigma=\frac{2}{3}$
Outcrops	$\sigma=\dfrac{-\gamma}{\beta-\gamma}$	$\sigma=\dfrac{-3\gamma\alpha}{(\gamma-\alpha)^2}$	$\sigma=\dfrac{\alpha}{\alpha-\beta}$	$\sigma=\infty$

For the real curve σ extends from $\dfrac{\alpha}{\alpha-\beta}$ through 0 to $-\dfrac{\gamma\alpha}{\Omega}$, viz.

$\sigma=\dfrac{\alpha}{\alpha-\beta}$ gives outcrop in plane $z=0$,

$\sigma=\quad 0\quad$ " umbilicar centre in plane $y=0$,

$\sigma=\dfrac{-\gamma\alpha}{\Omega}\quad$ " evolute-node in plane ∞.

It is to be noticed that the order of magnitude of the terms in the table is

$$\infty,\ \frac{2\gamma}{\gamma-\alpha},\ \frac{\gamma}{\gamma-\alpha},\ 1,\ \tfrac{2}{3},\ \frac{-\gamma}{\beta-\gamma},\ \frac{\alpha}{\alpha-\beta},\ 0,\ \frac{-\gamma\alpha}{\Omega},\ \frac{-\alpha}{\gamma-\alpha},\ \frac{-2\alpha}{\gamma-\alpha},\ \frac{-3\gamma\alpha}{(\gamma-\alpha)^2},\ -\infty.$$

so that the values $\frac{\alpha}{\alpha-\beta}$, 0, $\frac{-\gamma\alpha}{\Omega}$ which belong to the real curve are contiguous; this is as it should be. Several of the preceding investigations conducted by means of the quantities ξ, η, ξ_1, η_1 might have been conducted more simply by means of the formulæ involving σ.

The Eight Cuspidal Conics. Art. Nos. 61 to 71.

61. The centro-surface is the envelope of the quadric

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}-1=0.$$

Hence it has a cuspidal curve given by means of this equation and the first and second derived equations

$$\frac{a^2x^2}{(a^2+\xi)^3}+\frac{b^2y^2}{(b^2+\xi)^3}+\frac{c^2z^2}{(c^2+\xi)^3}=0,$$

$$\frac{a^2x^2}{(a^2+\xi)^4}+\frac{b^2y^2}{(b^2+\xi)^4}+\frac{c^2z^2}{(c^2+\xi)^4}=0,$$

which equations determine a^2x^2, b^2y^2, c^2z^2 in terms of ξ, viz. we have

$$-\beta\gamma\, a^2x^2=(a^2+\xi)^4,$$

$$-\gamma\alpha\, b^2y^2=(b^2+\xi)^4,$$

$$-\alpha\beta\, c^2z^2=(c^2+\xi)^4;$$

so that, comparing with the equations $-\beta\gamma\, a^2x^2=(a^2+\xi)^3\,(a^2+\eta)$ &c. which give the centro-surface, we see that for the cuspidal curve $\xi=\eta$; or the cuspidal curve now in question arises from the eight lines on the ellipsoid, which lines are the envelope of the curves of curvature: it is clear that the curve is imaginary.

62. From the foregoing equations we have:

$$\sqrt{\alpha}\, ax+\sqrt{\beta}\, by+\sqrt{\gamma}\, cz=\sqrt{-\alpha\beta\gamma},$$

$$\alpha^{\frac{3}{4}}\sqrt{ax}+\beta^{\frac{3}{4}}\sqrt{by}+\gamma^{\frac{3}{4}}\sqrt{cz}=0,$$

the second of which is best written in the rationalised form

$$(1,\ 1,\ 1,-1,-1,-1)\,(\alpha\sqrt{\alpha}\, ax,\ \beta\sqrt{\beta}\ by,\ \gamma\sqrt{\gamma}\, cz)^2=0,$$

and combining herewith the equation

$$\sqrt{\alpha}\, ax+\sqrt{\beta}\ by\ +\sqrt{\gamma}\, cz=\sqrt{-\alpha\beta\gamma},$$

then for any given signs of $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{-\alpha\beta\gamma}$ the first of these equations represents a quadric surface, the second a plane, or the two equations together represent a conic.

By changing the signs of the radicals (observing that when all the signs are changed simultaneously the curve is unaltered) we obtain in all 8 conics, but only four quadric surfaces; viz. the two conics

$$\sqrt{\alpha}\,ax + \sqrt{\beta}\ by\ \sqrt{\gamma}\,cz = \pm\sqrt{-\alpha\beta\gamma}$$

lie on the same quadric surface.

63. The conics form two sets of four, corresponding to the two sets of four lines on the ellipsoid. The analysis seems to establish a correspondence of each conic of the one set to a single conic of the other set; viz. the conics have been obtained in pairs as the intersections of the same quadric surface by a pair of planes: there is a like correspondence of each line of the one set to a single line of the other set, viz. the lines meet in pairs on the umbilici at infinity, but this correspondence is included in a more general property: in fact each line of the one set meets each line of the other set in an umbilicus; and the corresponding conics (not only meet but) touch at the corresponding umbilicar centre; and *quà* touching conics they have two points and intersection, and consequently lie on the same quadric surface. It is to be added that the two conics touch also at the umbilicar centre the cuspidal conic of the principal section.

64. The 8 conics form two tetrads, and the principal conics (reckoning as one of them the conic at infinity) another tetrad: the complete cuspidal curve consists therefore of three tetrads of conics: with these we may form (one conic out of each tetrad) 16 triads; viz. each conic of one tetrad is combined with each conic of either of the other tetrads, and with a determinate conic of the third tetrad, to form a triad. And the conics of each triad, not only meet but touch at an umbilicar centre, the common tangent being also by what precedes, the tangent of the evolute at that point, which point is also a node of the nodal curve.

65. In fact consider the two conics

$$\sqrt{\alpha}\,ax \pm \sqrt{\beta}\ by\ + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$(1,\ 1,\ 1, -1, -1, -1)\,(\alpha\sqrt{\alpha}\,ax,\ \pm\beta\sqrt{\beta}\,by,\ \gamma\sqrt{\gamma}\,cz)^2 = 0\,;$$

for the intersections with the plane $y=0$ we have

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$(\alpha\sqrt{\alpha}\,ax - \gamma\sqrt{\gamma}\,cz)^2 = 0\,;$$

so that the two conics each meet the plane in question in the same two coincident points, that is they each touch the plane $y=0$ at the same point, viz. the point given by the equations

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$\alpha\sqrt{\alpha}\,ax - \gamma\sqrt{\gamma}\,cz = 0\,;$$

viz. this is the point, $ax - \dfrac{\gamma\sqrt{\gamma}}{\sqrt{-\beta}}$, $cz = \dfrac{\alpha\sqrt{\alpha}}{\sqrt{-\beta}}$, which is one of the umbilicar centres

$$\left(a^2x^2 = -\frac{\gamma^3}{\beta},\ c^2z^2 = -\frac{\alpha^3}{\beta}\right);$$

and the common tangent at this point is

$$\sqrt{\alpha}\, ax + \sqrt{\gamma}\, cz = \sqrt{-\alpha\beta\gamma},$$

which is also the common tangent of the ellipse and evolute in the plane $y = 0$.

66. It has been seen that the nodal curve meets each principal conic at four outcrops, which points are cusps of the nodal curve: it is to be further shown that the nodal curve meets each of the 8 cuspidal conics in four points (giving 32 new points, which may be called 'outcrops,' the 16 points heretofore so called being distinguished as the principal outcrops or 16 outcrops, and the points now in question as the 32 outcrops), which points are cusps of the nodal curve.

In fact to obtain the intersections of the nodal curve with the 8 cuspidal conics, we must in the equation of the nodal curve, or (what is the same thing) in the expressions of ξ, η in terms of σ, write $\eta = \xi$.

67. Putting for shortness,

$$\Theta = \tfrac{1}{4}\,\frac{(\gamma-\alpha)^2\,\sigma\,(\sigma-1)\,(3\sigma-2)}{\Omega\sigma+\gamma\alpha},$$

and as before

$$S = \frac{3\left(\sigma+\dfrac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\dfrac{2\gamma}{\gamma-\alpha}\right)}{\sigma\,(3\sigma-2)},$$

we have thus

$$\Theta\,(1+\sqrt{S})^3 = 1-\sqrt{S},$$

or, what is the same thing,

$$\Theta\,(1+3S) - 1 + \sqrt{S}\,\{\Theta\,(3+S)+1\} = 0:$$

we have without difficulty

$$\Theta\,(3+S)+1 = \frac{(\gamma-\alpha)^2}{\Omega\sigma+\gamma\alpha}\left\{3\sigma^3-6\sigma^2+4\sigma+\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\},$$

$$\Theta\,(1+3\sqrt{S})-1 = \frac{(\gamma-\alpha)^2\,(3\sigma-2)\left(\sigma+\dfrac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\dfrac{2\gamma}{\gamma-\alpha}\right)}{\Omega\sigma+\gamma\alpha}$$

so that the resulting equation contains the factor

$$\sqrt{\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)},\ = \sqrt{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}};$$

omitting it, the equation becomes

$$\sqrt{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}}\,(3\sigma-2)^{\frac{3}{2}}\sqrt{\sigma}+\sqrt{3}\left\{3\sigma^3-6\sigma^2+4\sigma+\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}^2 = 0,$$

or putting for shortness $\dfrac{4\gamma\alpha}{(\gamma-\alpha)^2} = M$, and rationalising, this is

$$-(\sigma^2-2\sigma-M)\,(3\sigma-2)^3\,\sigma + 3\,(3\sigma^3-6\sigma^2+4\sigma+M)^2 = 0,$$

and, working this out, the terms in σ^6, σ^5 disappear, and the result is

$$(36+27M)\,\sigma^4-(64+36M)\,\sigma^3+32\sigma^2+16M\sigma+3M^2 = 0,$$

or, as this may also be written,

$$3M^2 + M(27\sigma^4 - 36\sigma^3 + 16\sigma) + 4(9\sigma^4 - 16\sigma^3 + 8\sigma^2) = 0,$$

a quartic equation in σ: to each of the 4 roots there correspond 8 intersections, viz. there will be in all 32 intersections, lying in 4's upon the 8 cuspidal conics.

68. To show that these points are cusps, or stationary points on the nodal curve, starting from the expressions of $-\beta\gamma\alpha^2x^2$ &c. in terms of σ we have, first for dy,

$$-\frac{2dy}{y} = d\sigma\left\{\frac{1}{\sigma-1} + \frac{2}{\sigma} + \frac{3(\gamma-\alpha)^2}{(\gamma-\alpha)^2\sigma + 3\alpha\gamma} - \frac{\Omega}{\Omega\sigma+\gamma\alpha} - \frac{6}{3\sigma-2}\right\},$$

or, as this may be written,

$$-\frac{2dy}{y} = d\sigma\left\{\frac{1}{\sigma-1} + \frac{2}{\sigma} + \frac{12}{4\sigma+3M} - \frac{4+3M}{(4+3M)\sigma+M} - \frac{6}{3\sigma-2}\right\},$$

$$= d\sigma\left\{\frac{3\sigma^2-6\sigma+4}{3\sigma^3-5\sigma^2+2\sigma} + \frac{(32+24M)\sigma - 9M^2}{(16+12M)\sigma^2 + (16M+9M^2)\sigma + 3M^2}\right\},$$

$$= \frac{d\sigma\,.\,4\{(36+27M)\sigma^4 - (64+36M)\sigma^3 + 32\sigma^2 + 16M\sigma + 3M^2\}}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}},$$

viz. the numerator vanishes when σ is a root of the quartic equation.

69. We have next

$$-\frac{2\,dx}{x} = d\sigma\left\{\frac{\gamma-\alpha}{(\gamma-\alpha)\sigma+\alpha} + \frac{2(\gamma-\alpha)}{(\gamma-\alpha)\sigma-2\gamma} + \frac{3(\beta-\gamma)}{(\beta-\gamma)\sigma+\gamma} - \frac{\Omega}{\Omega\sigma+\gamma\alpha} - \frac{6}{3\sigma-2}\right\},$$

which, putting $\dfrac{\gamma}{\gamma-\alpha} = B$, and therefore $\dfrac{\alpha}{\gamma-\alpha} = B-1$, and $\dfrac{\gamma}{\beta-\gamma} = C$, is

$$= d\sigma\left\{\frac{1}{\sigma+B-1} + \frac{2}{\sigma-2B} + \frac{3}{\sigma+C} - \frac{4+3M}{(4+3M)\sigma+M} - \frac{6}{3\sigma-2}\right\},$$

and adding the fractions except $\dfrac{3}{\sigma+C}$, the numerator is

$$\begin{aligned}&\sigma^2(27MB+36B-4)\\ &+\sigma\{54(B^2-B)M + 72B^2 - 80B + 8\}\\ &+\quad 4M - 16B^2 + 16B,\end{aligned}$$

which observing that $B^2 - B = \frac{1}{4}M$ is

$$\begin{aligned}=\quad &\sigma^2(27MB+36B-4)\\ &+\sigma\left(\frac{27}{2}M^2 + 18M - 8B + 8\right),\end{aligned}$$

and, substituting for M and B their values, this is found to be

$$= \frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma^2 + \frac{8(2\gamma+\alpha)^3\alpha}{(\gamma-\alpha)^4}\sigma,$$

$$= \frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma\left(\sigma + \frac{2\alpha}{\gamma-\alpha}\right).$$

70. Hence observing that $C=\frac{\gamma}{\beta-\gamma}=\frac{-\gamma}{2\gamma+\alpha}$, the whole coefficient of $d\sigma$ is

$$=\frac{\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\left(\sigma^2+\frac{2\alpha}{\gamma-\alpha}\sigma\right)}{(3\sigma-2)(\sigma+B-1)(\sigma-2B)[(3M+4)\sigma+M]}+\frac{3}{\sigma+C},$$

and the numerator of this is

$$=\qquad\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^3}\sigma\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)[(2\gamma+\alpha)\sigma-\gamma],$$

$$+3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M-B\sigma)\{(3M+4)\sigma+M\},$$

which is

$$=3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M)\{(3M+4)\sigma+M\},$$

$$+\sigma\Bigg[-3B(3\sigma-2)\{(3M+4)\sigma+M\},$$

$$+\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^2}\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)(\overline{2\gamma+\alpha}\,\sigma-\gamma)\Bigg]:$$

the term in { } is

$$=\qquad\sigma^2\left[-(27M+36)B+\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\right]$$

$$+\sigma\left[3B(3M+8)+\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^3}\left(-\gamma+\frac{2\alpha(2\gamma+\alpha)}{\gamma-\alpha}\right)\right]$$

$$+\quad\left[6BM-\frac{8(2\gamma+\alpha)^2\gamma\alpha}{(\gamma-\alpha)^4}\right],$$

which is found to be

$$=-4\sigma^2+\sigma(8+15M+\tfrac{27}{2}M^2)-2M-\tfrac{9}{2}M^2,$$

and the whole numerator is thus

$$3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M)[(3M+4)\sigma+M]$$

$$-4\sigma^3+\sigma^2(8+15M+\tfrac{27}{2}M^2)+\sigma(-2M-\tfrac{9}{2}M^2),$$

which is

$$=(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2.$$

71. We have thus

$$-\frac{2dx}{x}=d\sigma\frac{(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}\left(\sigma+\frac{\gamma}{\beta-\gamma}\right)},$$

and thence also

$$-\frac{2dz}{z}=d\sigma\frac{(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}\left(\sigma-\frac{\alpha}{\alpha-\beta}\right)},$$

so that dx and dz also vanish when σ is a root of the quartic equation: the points in question are therefore cusps of the nodal curve.

Centro-surface as the envelope of the quadric $\Sigma a^2x^2(a^2+\xi)^{-2}=1$. Art. Nos. 72 to 76.

72. The equations $-\beta\gamma a^2x^2=(a^2+\xi)^3(a^2+\eta)$, &c. considering therein ξ, η as variable give the centro-surface: considering η as a given constant but ξ as variable they give the sequential centro-curve; and considering ξ as a given constant but η as variable they give the concomitant centro-curve.

73. Suppose first that η is a given constant; to eliminate ξ we may write the equations in the form

$$-(\beta\gamma)^{\frac{1}{3}}(ax)^{\frac{2}{3}}(a^2+\eta)^{-\frac{1}{3}}=(a^2+\xi),\ \&c.,$$

and then multiplying first by $\alpha(a^2+\eta)$, &c. and adding, and secondly by α, &c., and adding (observing that $\Sigma\alpha(a^2+\xi)(a^2+\eta)=-\alpha\beta\gamma$, $\Sigma\alpha(a^2+\xi)=0$); we have

$$\Sigma(\alpha ax)^{\frac{2}{3}}(a^2+\eta)^{\frac{2}{3}}=(\alpha\beta\gamma)^{\frac{2}{3}},$$

$$\Sigma(\alpha ax)^{\frac{2}{3}}(a^2+1)^{-\frac{1}{3}}=0,$$

which equations, considering therein η as a given constant, are the equations of a sequential centro-curve.

If from the two equations we eliminate η we should have the equation of the centro-surface; the second equation is the derivative of the first in regard to η; and it thus appears that the equation of the centro-surface might be obtained by equating to zero the discriminant of the rationalised function

$$\text{norm.}\ [\{\Sigma(\alpha ax)^{\frac{2}{3}}(a^2+\eta)^{\frac{2}{3}}\}-(\alpha\beta\gamma)^{\frac{2}{3}}];$$

but the form is too inconvenient to be of any use.

74. Taking next ξ as a given constant; and writing the equations in the form

$$-\beta\gamma a^2x^2(a^2+\xi)^{-3}=(a^2+\eta),\ \&c.,$$

then multiplying by $\alpha(a^2+\xi)$, &c. and adding; and again multiplying by α, &c. and adding, we have

$$\Sigma a^2x^2(a^2+\xi)^{-2}=1,$$

$$\Sigma a^2x^2(a^2+\xi)^{-3}=0;$$

or writing these equations at full length,

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}-1=0,$$

$$\frac{a^2x^2}{(a^2+\xi)^3}+\frac{b^2y^2}{(b^2+\xi)^3}+\frac{c^2z^2}{(c^2+\xi)^3}\qquad=0,$$

which equations, considering therein ξ as a constant, are the equations of any concomitant centro-curve: since the equations are each of the second order it thus appears that the concomitant centro-curves are quadri-quadrics.

75. If from the two equations we eliminate ξ, we have the equation of the centro-surface; the second equation is the derivative of the first in regard to ξ; and it thus appears that the equation of the centro-surface is obtained by equating to zero the discriminant in regard to ξ of the integralised function

$$(a^2+\xi)^2(b^2+\xi)^2(c^2+\xi)^2\{(\Sigma a^2x^2(a^2+\xi)^{-2}-1\},$$

or, what is the same thing, the discriminant of the sextic function

$$(a^2+\xi)^2(b^2+\xi)^2(c^2+\xi)^2-\Sigma a^2x^2(b^2+\xi)^2(c^2+\xi)^2.$$

76. If instead hereof we consider the homogeneous function

$$w^2(a^2+\xi)^2(b^2+\xi)^2(c^2+\xi)^2-\Sigma a^2x^2(b^2+\xi)^2(c^2+\xi)^2,$$

then the coefficients are of the second order in (x, y, z, w), and the discriminant, being of the tenth order in the coefficients, is of the order 20 in (x, y, z, w). But the sextic function has a twofold factor $\left(1+\frac{\xi}{\infty}\right)^2$ if $w^2=0$, and it has evidently a twofold factor if $x^2=0$ or $y^2=0$ or $z^2=0$, that is, the discriminant contains the factor $x^2y^2z^2w^2$; or, omitting this factor, it will be of the order 12 in (x, y, z, w); whence writing $w=1$, the centro-surface is of the order 12. I have in this manner actually obtained the equation of the centro-surface: see the memoir "On a certain Sextic Torse," *Camb. Phil. Trans.* t. XI. (1871), pp. 507—523.

Another generation of the Centro-surface. Art. Nos. 77 to 83.

77. By what precedes the equation of the centro-surface is obtained as the condition in order that the equation

$$\{\Sigma a^2x^2(a^2+\xi)^{-2}\}-1=0$$

may have two equal roots. But taking m an arbitrary constant, this is the derived equation of

$$\{\Sigma a^2x^2(a^2+\xi)^{-1}\}+\xi+m=0,$$

and as such it will have two equal roots if the last-mentioned equation has three equal roots; and conversely, we have thus the equation of the centro-surface by expressing that the last-mentioned equation, or, what is the same thing, the quartic equation

$$(\xi+m)(\xi+a^2)(\xi+b^2)(\xi+c^2)-\Sigma a^2x^2(\xi+b^2)(\xi+c^2)=0$$

has three equal roots. The condition for this is that the quadrinvariant and the cubinvariant shall each of them vanish; the two invariants are respectively a quadric and a cubic function of m; viz. the equations are

$$(a,\ b,\ c)(m,\ 1)^2=0,\quad (a',\ b',\ c',\ d')(m,\ 1)^3=0;$$

where the degrees in (x, y, z) of a, b, c are 0, 2, 4 and those of a', b', c', d' are 0, 2, 4, 6 respectively: the equation of the centro-surface then is

$$\begin{vmatrix} & & a, & b, & c \\ & a, & b, & c & \\ a, & b, & c & & \\ & a', & b', & c', & d' \\ a', & b', & c', & d' & \end{vmatrix}=0,$$

which is of the right order 12; but it would be difficult to obtain thereby the developed equation.

78. For the nodal curve the cubic equation must be satisfied by each root of the quadric equation, or, what is the same thing, the quadric function must completely divide the cubic function; the conditions are

$$\left\| \begin{array}{cccc} & a, & b, & c \\ a, & b, & c, & \\ a', & b', & c', & d' \end{array} \right\| = 0,$$

where the degrees may be taken to be

$$\left| \begin{array}{cccc} -2, & 0, & 2, & 4 \\ 0, & 2, & 4, & 6 \\ 0, & 2, & 4, & 6 \end{array} \right|$$

and the order of the nodal curve is thus $=24$: two of the equations in fact are

$$\left| \begin{array}{ccc} & a, & b \\ a, & b, & c \\ a', & b', & c' \end{array} \right| = 0, \qquad \left| \begin{array}{ccc} & b, & c \\ a, & c, & \\ a\,, & c', & d' \end{array} \right| = 0,$$

which are surfaces of the orders 4, 6; or the nodal curve is a complete intersection 4×6. By the results above obtained as to the nodal curve, it appears that the two surfaces must have an ordinary contact at each of the 16 umbilicar centres, and a stationary or singular contact at each of the 48 outcrops.

79. The derivation of the centro-surface from the surface

$$\frac{a^2x^2}{a^2+\xi} + \frac{b^2y^2}{b^2+\xi} + \frac{c^2z^2}{c^2+\xi} + \xi - m = 0$$

requires to be further explained. The surface, say $V=0$, is a quadric surface depending on the two parameters ξ, m; the axes coincide in direction with those of the ellipsoid, and their relative magnitudes are as

$$\frac{1}{a}\sqrt{a^2+\xi} \, : \, \frac{1}{b}\sqrt{b^2+\xi} \, : \, \frac{1}{c}\sqrt{c^2+\xi},$$

viz. these are as the axes of the confocal surface

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} - 1 = 0$$

divided by a, b, c respectively; to fix the absolute magnitudes observe that the equation may be written

$$x^2+y^2+z^2-m-\xi\left(\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} - 1\right) = 0,$$

viz. the surface $V=0$ is a surface through the spheroconic which is the intersection of the confocal surface by the arbitrary sphere $x^2+y^2+z^2-m=0$; but, while the surface is hereby and by the preceding condition as to the axes completely determined, the geometrical significance is anything but clear.

80. Considering then the quadric surface $V=0$, depending on the parameters ξ, m; suppose that m remains constant while ξ alone varies; we have thus three consecutive surfaces $V=0$, $V'=0$, $V''=0$; and these I say intersect in a point of the centro-surface; the point in question will depend on the two parameters (ξ, m), and if these vary simultaneously we have the whole system of points on the centro-surface; but if only one of them varies, the other being constant, we have a curve on the centro-surface.

The three equations may be replaced by $V=0$, $\delta_\xi V=0$, $\delta_\xi^2 V=0$; of which the first alone contains m; and it thus appears that if m be the variable parameter, the equations of the curve are $\delta_\xi V=0$, $\delta_\xi^2 V=0$, viz. the curve is then the quadriquadric curve which is the concomitant centro-curve of the curve of curvature for the parameter ξ. But if the variable parameter be ξ, then this is a curve on the 12-thic surface $\Omega=0$ obtained by the elimination of ξ from the equations $V=0$, $\delta_\xi V=0$; viz. we have $\Omega = S^3 - T^2 = 0$, where $S=(a, b, c)(m, 1)^2$, $T=(a', b', c', d')(m, 1)^3$, and the curve in question is the curve $S=0$, $T=0$, which is the cuspidal curve on the surface $\Omega=0$; the elimination of m from the two equations $S=0$, $T=0$ gives as above the equation of the centro-surface.

81. The surface $\Omega = S^3 - T^2 = 0$ obtained as above by the elimination of ξ from the equations $V=0$, $\delta_\xi V=0$, (or, what is the same thing, by equating to zero the discriminant of V in regard to ξ,) may be termed the sociate-surface: we have then the quartic and sextic surfaces $S=0$, $T=0$ intersecting in the before-mentioned curve, which may be called the sociate-edge; and the locus of these sociate-edges is the centro-surface.

82. We may if we please, changing the parameter in one of the functions, consider the two series of surfaces $S=0$, $T=0$ depending on the parameters m, m' respectively; a surface of the first series will correspond to one of the second series when the parameters are equal, $m=m'$, and we have then a sociate-edge. Taking a point anywhere in space, through this point there pass two surfaces $S=0$, and three surfaces $T=0$; but there is no pair of corresponding surfaces, or sociate-edge. If however the point be taken anywhere on the centro-surface, then there is a pair of corresponding surfaces $S=0$, $T=0$; that is, through each point of the centro-surface there passes a single sociate-edge; and if the point be taken anywhere on the nodal curve of the centro-surface, then there are two pairs of corresponding surfaces; that is, through each point of the nodal curve there are two sociate-edges: this explains the method above made use of for finding the equations of the nodal curve, by giving to the equations $S=0$, $T=0$, considered as equations in m, two equal roots.

83. The *à-posteriori* verification that the surfaces $V=0$, $V'=0$, $V''=0$ intersect in a point of the centro-surface, is not without interest; the parameters ξ_1, η_1 of the point of intersection are found to be $\xi_1=\xi$, $\eta_1 = m - a^2 - b^2 - c^2 - 3\xi$; whence in the equation $V=0$, writing $-\beta\gamma a^2 x^2 = (a^2+\xi_1)^3 (a^2+\eta_1)$ and $m = a^2+b^2+c^2+3\xi_1+\eta_1$, the resulting equation considered as an equation in ξ should have three roots $\xi=\xi_1$: the fourth root is at once seen to be $\xi=\eta_1$, and we ought therefore to have identically

$$-\frac{\alpha(a^2+\xi_1)^3(a^2+\eta_1)}{a^2+\xi}-\&\text{c.}+\alpha\beta\gamma(\xi-3\xi_1-\eta_1-a^2-b^2-c^2)$$
$$=\frac{\alpha\beta\gamma(\xi-\xi_1)^3(\xi-\eta_1)}{(a^3-\xi)(b^2+\xi)(c^2+\xi)};$$

and by decomposing the right-hand side into its component fractions this is at once seen to be true.

Third generation of the Centro-surface. Art. Nos. 84 and 85.

84. Instead of the foregoing equation $V=0$, consider the equation

$$W=\left(\frac{a^2x^2}{a^2+\xi}+\frac{b^2y^2}{b^2+\xi}+\frac{c^2z^2}{c^2+\xi}+\xi\right)-\left(\frac{a^2x^2}{a^2+\eta}+\frac{b^2y^2}{b^2+\eta}+\frac{c^2z^2}{c^2+\eta}+\eta\right)=0.$$

The equations $d_\xi W=0$, $d_\xi^2 W=0$ contain only ξ, and are in fact identically the same as the equations $d V=0$, $d_\xi^2 V=0$; the elimination of ξ from the equations $d_\xi V=0$, $d_\xi^2 W=0$ would therefore lead to the equation of the centro-surface: and the centro-surface is connected with the surfaces $W=0$, $\delta_\xi W=0$, $d_\xi^2 W=0$ and the parameters ξ, η in the same way as it is with the surfaces $V=0$, $d_\xi V=0$, $d_\xi^2 V=0$ and the parameters ξ, m. That is, if from the equations $W=0$, $d_\xi W=0$ we eliminate ξ we have a surface $\Omega=0$ (depending upon η) and having a cuspidal curve; and the locus of the cuspidal curve (as η varies) is the centro-surface. But the equation $W=0$ divides by $\xi-\eta$, and throwing out this factor it becomes

$$\frac{a^2x^2}{(a^2+\xi)(a^2+\eta)}+\frac{b^2y^2}{(b^2+\xi)(b^2+\eta)}+\frac{c^2z^2}{(c^2+\xi)(c^2+\eta)}-1=0,$$

so that the surface $\Omega=0$ is obtained by eliminating ξ from this equation and the derived equation in regard to ξ; or, what is the same thing, by equating to zero the discriminant in regard to ξ of the cubic function

$$(a^2+\xi)(b^2+\xi)(c^2+\xi)-\Sigma\frac{a^2x^2}{a^2+\eta}(b^2+\xi)(c^2+\xi);$$

this surface is in fact the torse generated by the normals at the several points of the curve of curvature belonging to the parameter η; the cuspidal curve is the edge of regression of this torse, that is, it is the sequential centro-curve of the curve of curvature; and we thus fall back upon the original investigation for the centro-surface.

85. In verification I remark that if X, Y, Z be the coordinates of a point on the curve of curvature in question, and (x, y, z) current coordinates, then the tangent plane of the torse, or plane through the normal and the tangent of the curve of curvature, has for its equation

$$\frac{Xx}{a^2+\eta}+\frac{Yy}{b^2+\eta}+\frac{Zz}{c^2+\eta}-1=0,$$

and if in this equation we consider the point (X, Y, Z) to be the point belonging to the parameters (η, ξ), viz. if we have $-\beta\gamma X^2=a^2(a^2+\xi)(a^2+\eta)$, &c., then this plane will be always touched by the before-mentioned ellipsoid,

$$\frac{a^2x^2}{(a^2+\xi)(a^2+\eta)}+\frac{b^2y^2}{(b^2+\xi)(b^2+\eta)}+\frac{c^2z^2}{(c^2+\xi)(c^2+\eta)}=1;$$

the condition for the contact in fact is

$$\Sigma \frac{X^2}{(a^2+\eta)^2} \frac{(a^2+\xi)(a^2+\eta)}{a^2} = 1,$$

viz. substituting for (X, Y, Z) their values, this is

$$-\frac{1}{\alpha\beta\gamma} \Sigma \alpha (a^2+\xi)^2 = 1,$$

which is true. And this being so the ellipsoid and the plane have each the same envelope, viz. this is the torse in question.

Reciprocal Surface. Art. No. 86.

86. The centro-surface is the envelope of

$$\frac{a^2x^2}{(a^2+\xi)^2} + \frac{b^2y^2}{(b^2+\xi)^2} + \frac{c^2z^2}{(c^2+\xi)^2} - 1 = 0\,;$$

hence the reciprocal surface in regard to the sphere $x^2+y^2+z^2-k^2=0$, is the envelope of

$$\frac{(a^2+\xi)^2}{a^2} X^2 + \frac{(b^2+\xi)^2}{b^2} Y^2 + \frac{(c^2+\xi)^2}{c^2} Z^2 - k^2 = 0,$$

that is

$$a^2X^2+b^2Y^2+c^2Z^2-k^2+2\xi(X^2+Y^2+Z^2)+\xi^2\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)=0,$$

viz. the envelope is

$$(a^2X^2+b^2Y^2+c^2Z^2-k^4)\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-(X^2+Y^2+Z^2)^2=0,$$

or, expanding and multiplying by $a^2b^2c^2$, this is

$$a^2(b^2-c^2)^2\,Y^2Z^2+b^2(c^2-a^2)\,Z^2X^2+c^2(a^2-b^2)^2\,X^2Y^2$$
$$-k^4(b^2c^2\,X^2+c^2a^2\,Y^2+a^2b^2\,Z^2)=0,$$

or, what is the same thing,

$$a^2\alpha^2\,Y^2Z^2+b^2\beta^2\,Z^2X^2+c^2\gamma^2\,X^2Y^2-k^4(b^2c^2\,X^2+c^2a^2\,Y^2+a^2b^2\,Z^2)=0,$$

which may be written

$$\mathrm{a}^2Y^2Z^2+\mathrm{b}^2Z^2X^2+\mathrm{c}^2X^2Y^2+\mathrm{f}^2X^2+\mathrm{g}^2Y^2+\mathrm{h}^2Z^2=0,$$

where $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}) = (a\alpha,\ b\beta,\ c\gamma,\ 2k^2bc,\ 2k^2ca,\ 2k^2ab)$,

and consequently, $\mathrm{af}+\mathrm{bg}+\mathrm{ch} = 2k^2abc\,(\alpha+\beta+\gamma)=0.$

It would doubtless be interesting to discuss this surface as it here presents itself, and with reference to its geometrical signification as the locus of the pole, in regard to the sphere, of the plane through two intersecting consecutive normals of the ellipsoid: but I abstain from any consideration of the question.

Delineation of the centro-surface for given numerical values of the semiaxes. Art. Nos. 87 and 88.

87. I constructed on a large scale a drawing of the centro-surface for the values

$$a^2 = 50,\ \ b^2 = 25,\ \ c^2 = 15.$$

(These were chosen so that a, b, c should have approximately the integer values 7, 5, 4, and that $a^2 + c^2$ should be well greater than $2b^2$; they give a good form of surface, though perhaps a better selection might have been made; there is a slight objection to the existence of the relation $a^2 = 2b^2$, as in the xy-section it brings a cusp of the evolute on the ellipse). We have therefore

$$\alpha = 10,\ \ \beta = -35,\ \ \gamma = 25\,;$$

the ellipses in the principal planes of the centro-surface are

$$\frac{y^2}{(5)^2} + \frac{z^2}{(8{\cdot}937)^2} = 1,$$

$$\frac{z^2}{(2.582)^2} + \frac{x^2}{(3{\cdot}535)^2} = 1,$$

$$\frac{x^2}{(4{\cdot}950)^2} + \frac{y^2}{(2)^2} = 1,$$

and these determine on each axis the two points which are the cusps of the evolutes. We have moreover for the umbilicar centre $x = 2{\cdot}988$, $y = 0$, $z = 1{\cdot}380$, and for the outcrop $x = 1{\cdot}127$, $y = 1{\cdot}947$, $z = 0$.

88. For the delineation of the nodal curve (crunodal portion) we have first to find the values of ξ, ξ_1; these are given in terms of x, y *ante* No. 33, where y is a given function of x, and x extends between the values $\{-b^2$ and $-\tfrac{1}{3}(a^2+b^2+c^2)\}$ -25 and $-26\tfrac{2}{3}$. It was thought sufficient to divide the interval into 6 equal parts, that is, the values of x were taken to be -25, $-25{\cdot}\dot{3}$, ... $-26{\cdot}\dot{6}$. The values of ξ, ξ_1 being found, those of η, η_1 were obtained from them by means of the original equations $(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)(a^2+\eta)$ &c. viz. we have thus for the determination of η, η_1 three simple equations, affording a verification of each other.

For the performance of these calculations (viz. of the values of y, ξ, ξ_1, η, η_1) I am indebted to the kindness of Mr J. W. L. Glaisher, of Trinity College. The results being obtained it is then easy to calculate as well the co-ordinates (x, y, z) of the point on the nodal curve as also the co-ordinates (X, Y, Z) and (X_1, Y_1, Z_1) of the corresponding two points on the ellipsoid (these last are of course not required for the delineation of the nodal curve, but it was interesting to obtain them). The whole series of the results is given in the annexed Table, and from them the drawing was constructed.

I find also in the neighbourhood of the umbilicar centre (if $\xi = -25 + q$),

$$\delta x = \quad \cdot 02868\, q^2,$$
$$\delta y = \pm \cdot 02484\, q^2,$$
$$\delta z = \quad \cdot 02191\, q^2.$$

and in the neighbourhood of the outcrop if $\xi_1 = -38 \cdot 333 + \frac{70}{9}\varpi$,

$$\delta x = \quad 1 \cdot 127\, \varpi,$$
$$\delta y = -1 \cdot 704\, \varpi,$$
$$\delta z = \pm 4 \cdot 582\, \varpi^{\frac{3}{2}}.$$

x	y	ξ	ξ_1	η	η_1	x	y	z	X	Y	Z	X_1	Y_1	Z_1
−25	0·	−25·	−25·	−25·	−25·	2·988	0·	1·380	5·326	0·	2·070	5·326	0·	2·070
$-25\cdot\dot{3}$	4·2669	−21·0664	−29·6002	−38·3193	−16·6728	2·543	0·360	0·996	4·394	2·289	2·491	6·233	1·957	1·023
$-25\cdot\dot{6}$	6·3191	−19·3475	−31·9858	−42·9911	−15·4693	2·148	0·721	0·662	3·504	3·189	2·283	5·956	2·580	0·584
−26·	8·1240	−17·8760	−34·1240	−45·7879	−15·1047	1·786	1·175	0·373	2·780	3·847	1·948	5·626	3·005	0·293
$-26\cdot\dot{3}$	9·8760	−16·4573	−36·2094	−47·5684	−15·0106	1·448	1·500	0·139	2·159	4·391	1·426	5·251	3.346	0·098
$-26\cdot\dot{6}$	11·6667	−15·0000	−38·3333	−48·7037	−15·0000	1·127	1·947	0·	1·610	4·869	0·	4·829	3·651	0·

The calculations were performed before I had obtained the formulæ in σ, which would have given the results more easily.

V. *On* DR. WIENER'S *Model of a cubic surface with 27 real lines; and on the construction of a double-sixer.* By PROF. CAYLEY.

[Read *May* 15, 1871.]

I.

I CALL to mind that a cubic surface has upon it in general 27 lines which may be all of them real. We may out of the 27 lines (and that in 36 different ways) select 12 lines forming a "double-sixer," viz. denoting such a system of lines by

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,$$

$$b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6;$$

then no two lines a meet each other, nor any two lines b, but each line a meets each line b, except that the two lines of a pair (a_1, b_1), (a_2, b_2), ... (a_6, b_6) do *not* meet each other. And such a system of twelve lines leads at once to the remaining fifteen lines; viz. we have a line c_{12}, the intersection of the planes which contain the pairs of lines (a_1, b_2) and (a_2, b_1) respectively.

The model is formed of plaster, and is contained within a cube, the edge of which is $= 18{\cdot}2$ inches: the lines a, b, c are colored blue, yellow, and red respectively; the lines a_1, b_2, b_5 being at right angles to each other, in such wise that taking the origin at the centre of the cube, the axes parallel to the edges, and the unit of length $= 1{\cdot}6$ inches, the equations of these three lines are

$$\begin{array}{lll} a_1, & x = 0,\ y = & 0, \\ b_2, & x = 0,\ z = & 1, \\ b_5, & y = 0,\ z = & -1. \end{array}$$

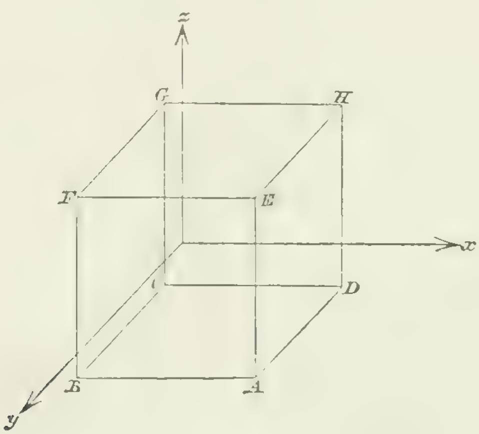
The model is a solid figure bounded by portions of the faces of the cube, and by a portion of the cubic surface, being a surface with three apertures, the collocation of which is not easily explained.

To determine the construction I measured on the faces of the cube, the coordinates of the two extremities of each of the twelve lines; these were measured in tenths of an inch (taking account of the half division, or twentieth of an inch), and the resulting numbers divided by 16 to reduce them to the before-mentioned unit of 1·6 inches. These reduced values are shewn in the table: knowing then the coordinates of two points on each line, the equations of the several lines became calculable; the true theoretical form of these results—(viz. the form which, but for errors of the model, or of the measurement, they would have assumed)—is

b_1,	$x = B_1 z + D$,	$y = B_1' z + D'$,
b_2,	$x = 0$,	$z = 1$,
b_3,	$x = B_3(z + \beta_3)$,	$y = B_3'(z + \beta_3)$,
b_4,	$x = B_4(z + \beta_4)$,	$y = B_4'(z + \beta_4)$,
b_5,	$y = 0$,	$z = -1$,
b_6,	$x = B_6(z + \beta_6)$,	$y = B_6'(z + \beta_6)$.
a_1,	$x = 0$,	$y = 0$,
a_2,	$x = A_2 z + C_2$,	$y = A_2'(z - 1)$,
a_3,	$x = A_3(z + 1)$,	$y = A_3'(z - 1)$,
a_4,	$x = A_4(z + 1)$,	$y = A_4'(z - 1)$,
a_5,	$x = A_5(z + 1)$,	$y = A_5' z + C_5'$,
a_6,	$x = A_6(z + 1)$,	$y = A_6'(z - 1)$;

but in consequence of such errors, the results are not accurately of the form in question.

The faces of the cube being as in the diagram



the Table is

Equations calculated from the measurements of the model.		$ABCD$ $z=-5{\cdot}688$	$EFGH$ $z=+5{\cdot}688$	$AEBF$ $y=+5{\cdot}688$	$BCFG$ $x=-5{\cdot}688$	$CDGH$ $y=-5{\cdot}688$	$AEDH$ $x=+5.688$
$x=0$ $y=0$	a_1	$x=0$ $y=0$	$x=0$ $y=0$				
$x=-{\cdot}780z-{\cdot}187$ $y=-{\cdot}423z+{\cdot}406$	a_2	$x=4{\cdot}250$ $y=2{\cdot}812$	$x=-4{\cdot}625$ $y=-2{\cdot}000$				
$x=-{\cdot}654z-{\cdot}656$ $y=-{\cdot}588z+{\cdot}531$	a_3	$x=3{\cdot}062$ $y=3{\cdot}875$	$x=-4{\cdot}375$ $y=-2{\cdot}812$				
$x=-2{\cdot}912z-2{\cdot}959$ $y=-{\cdot}736z+{\cdot}752$	a_4	....	....	....	$y={\cdot}0625$ $z={\cdot}9375$	....	$y=2{\cdot}937$ $z=-2{\cdot}969$
$x=1{\cdot}024z+1{\cdot}014$ $y=-1{\cdot}049z-{\cdot}277$	a_5	$x=-4{\cdot}812$ $y=5{\cdot}688$	....	....	....	....	$y=-5{\cdot}063$ $z=4{\cdot}562$
$x={\cdot}264z+{\cdot}187$ $y=-{\cdot}104z+{\cdot}219$	a_6	$x=-1{\cdot}313$ $y={\cdot}8125$	$x=1{\cdot}687$ $y=-{\cdot}375$				
$x=-1{\cdot}611z+{\cdot}151$ $y=-1{\cdot}438z+{\cdot}288$	b_1	....	....	....	$y=5{\cdot}500$ $z=-3{\cdot}625$	....	$y=-4{\cdot}656$ $z=3{\cdot}437$
$x=0$ $z=-1$	b_2	....	....	$x=0$ $z=-1$	....	$x=0$ $z=-1$	
$x=-1{\cdot}352z-{\cdot}685$ $y=-2{\cdot}034z-{\cdot}984$	b_3	....	....	$x=3{\cdot}750$ $z=-3{\cdot}281$	....	$x=-3{\cdot}812$ $z=2{\cdot}313$	
$x=-{\cdot}753z-{\cdot}0315$ $y=-{\cdot}500z-{\cdot}0315$	b_4	$x=4{\cdot}250$ $y=2{\cdot}812$	$x=-4{\cdot}313$ $y=-2{\cdot}875$				
$y=0$ $z=+1$	b_5	....	....	....	$y=0$ $z=1$	....	$y=0$ $z=1$
$x=1{\cdot}123z-{\cdot}702$ $y=-1{\cdot}123z+{\cdot}702$	b_6	....	....	$x=-5{\cdot}688$ $z=-4{\cdot}438$	....	....	$y=-5{\cdot}688$ $z=5{\cdot}688$

I hence calculate the intersections: considering any two lines which ought to intersect, then projecting on the horizontal plane and calculating x, y the coordinates of the point of intersection of the two projections, these values of x, y substituted in the equations should give the same value of z; but if the lines do not accurately intersect, then the values of z will be different.

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	*	0 0 − 1	0 0 − ·495 ± ·011	0 0 − ·052 ∓ ·010	0 0 + 1	0 0 + ·625
a_2	− ·077 + ·455 − ·129 ∓ ·013	*	+ ·381 + ·771 − ·796 ± ·067	+ 4·292 + 2·803 − 5·704 ± ·036	− ·967 − ·008 ∓ ·008 + 1	− ·398 + ·227 + ·346 ± ·076
a_3	− ·423 + ·699 − ·264 ± ·021	− ·001 ∓ ·001 + 1·119 − 1	*	− 4·782 − 3·227 + 6·350 ± ·042	− 1·310 − ·028 ∓ ·028 + 1	− ·673 + ·344 + ·162 ± ·146
a_4	− ·957 + 1·238 − ·674 ± ·014	− ·023 ∓ ·023 + 1·488 − 1	+ 1·286 + 1·736 − 1·398 ± ·060	*	− 5·871 + ·008 ± ·008 + 1	− 1·330 + ·847 − ·344 ± ·215
a_5	+ 2·519 − 1·801 + 1·462 ± ·008	− ·005 ∓ ·005 + ·772 − 1	+ ·282 + ·476 − ·717 ± ·001	+ ·412 + ·194 − ·518 ± ·070	*	+ 18·764 − 14·155 + 15·282 ± 4·052
a_6	+ ·194 + ·214 + ·040 ± ·022	− ·038 ∓ ·038 + ·323 − 1	+ ·045 + ·283 − ·582 ± ·042	− ·131 + ·284 − ·423 ± ·208	+ ·451 + ·057 ± 057 + 1	*

Starting from the assumed equations of b_2, b_3, b_4, b_5, b_6, a_1, and calculating by the theory the remaining lines, the equations of the b lines (those of b_1 being calculated) are

$$b_1, \qquad x = 1{\cdot}321\, z - {\cdot}310, \quad y = -1{\cdot}295\, z + {\cdot}581,$$

$$b_2, \qquad x = 0, \quad z = -1,$$

$$b_3, \qquad x = -1{\cdot}352\,(z + {\cdot}510), \quad y = -2{\cdot}034\,(z + {\cdot}510),$$

$$b_4, \qquad x = -{\cdot}753\,(z + {\cdot}052), \quad y = -{\cdot}500\,(z + {\cdot}052),$$

$$b_5, \qquad y = 0, \quad z = +1,$$

$$b_6, \qquad x = 1{\cdot}123\,(z - {\cdot}624), \quad y = -1{\cdot}123\,(z - {\cdot}624);$$

and the equations of the a-lines (those of all but a_1 being calculated) are

$$a_1, \qquad x = 0, \quad y = 0,$$

$$a_2, \qquad x = -{\cdot}753\, z - {\cdot}091, \quad y = -{\cdot}498\,(z - 1),$$

$$a_3, \qquad x = -{\cdot}609\,(z + 1), \quad y = -{\cdot}677\,(z - 1),$$

$$
\begin{array}{ll}
a_4, & x = -2\cdot506\,(z+1), \\
 & y = -\ \cdot841\,(z-1), \\
a_5, & x = \cdot874\,(z+1), \\
 & y = -\cdot967\,z - \cdot288, \\
a_6, & x = \cdot170\,(z+1), \\
 & y = -\cdot071\,(z-1);
\end{array}
$$

and thence for the points of intersection the coordinates are

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	*	0 0 − 1	0 0 − ·510	0 0 − ·052	0 0 + 1	0 0 + ·624
a_2	− ·170 + ·446 + ·105	*	+ ·662 + ·996 − 1	+ 197· + 131· − 262· } *i.e.* lines a_2, b_4 nearly parallel.	− ·844 0 + 1	− ·336 + ·336 + ·325
a_3	− ·515 + ·782 − ·155	0 + 1·354 − 1	*	− 1·805 − 2·007 + 3·964	− 1·218 0 + 1	− ·641 − ·641 + ·053
a_4	− 1·071 + 1·323 − ·573	0 + 1·682 − 1	+ 1·438 + 2·164 − 1·574	*	− 5·012 0 + 1	− 1·259 + 1·259 − ·497
a_5	+ 3·189 − 2·849 + 2·649	0 + ·679 − 1	+ ·259 + ·291 − ·702	+ ·383 + ·255 − ·561	*	+ 6·410 − 6·410 + 6·333
a_6	+ ·241 + ·041 + ·417	0 + ·142 − 1	− ·074 + ·112 − ·565	+ ·131 + ·087 − ·226	+ ·340 0 + 1	*

II.

I have in a paper "On the double-sixers of a cubic surface," *Quart. Math. Journal*, t. X. (1868), pp. 58–71, obtained analytical expressions for the twelve lines of a double-sixer, and also calculated numerical values, which however (as there remarked) did not come out convenient ones for the construction of a figure. A different mode of treatment since occurred to me, by means of the following equation of the cubic surface

$$
\left(\frac{x}{\alpha'} - \frac{y}{\beta'} + \frac{z}{\gamma'} - \frac{w}{\delta'}\right)\left(\frac{xz}{\alpha\gamma} - \frac{yw}{\beta\delta}\right) - k\left(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} - \frac{w}{\delta}\right)\left(\frac{xz}{\alpha'\gamma'} - \frac{yw}{\beta'\delta'}\right) = 0,
$$

which as will appear is a very convenient one for the purpose; we in fact obtain at once eight lines of the double-sixer; viz. these are

1. $x=0,\ w=0,$	$2'$. $x=0,\ y=0,$
3. $y=0,\ z=0,$	$4'$. $z=0,\ w=0,$
5. $\dfrac{x}{\alpha}-\dfrac{y}{\beta}=0,\ \dfrac{z}{\gamma}-\dfrac{w}{\delta}=0,$	$5'$. $\dfrac{x}{\alpha'}-\dfrac{w}{\delta'}=0,\ \dfrac{y}{\beta'}-\dfrac{z}{\gamma'}=0,$
6. $\dfrac{x}{\alpha'}-\dfrac{y}{\beta'}=0,\ \dfrac{z}{\gamma'}-\dfrac{w}{\delta'}=0,$	$6'$. $\dfrac{x}{\alpha}-\dfrac{w}{\delta}=0,\ \dfrac{y}{\beta}-\dfrac{z}{\gamma}=0,$

and also five lines not belonging to the double-sixer, viz.

$$12.\qquad x=0,\quad \left(-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

$$23.\qquad y=0,\quad \left(\frac{x}{\alpha'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\alpha'\gamma'}=0,$$

$$34.\qquad z=0,\quad \left(\frac{x}{\alpha'}-\frac{y}{\beta'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

$$41.\qquad w=0,\quad \left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}\right)\frac{1}{\alpha'\gamma'}=0,$$

$$56.\qquad \frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}=0,\quad \frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}=0.$$

The remaining lines of the double-sixer are then easily determined; viz. the lines 3, 5, 6, and 12 are met by the line $2'$, and by a second line $1'$; this, as a line meeting 3, 5, 6, will be given by equations of the form

$$x-\frac{\alpha}{\beta}y=\phi\left(\frac{\delta}{\gamma}z-w\right),\quad x-\frac{\alpha'}{\beta'}y=\phi\left(\frac{\delta'}{\gamma'}z-w\right),$$

and observing that these equations, writing therein $x=0$, give

$$\frac{z}{\gamma}-\frac{w}{\delta}=-\frac{\alpha}{\beta\delta\phi}y,\quad \frac{z}{\gamma'}-\frac{w}{\delta'}=-\frac{\alpha'}{\beta'\delta'\phi}y,$$

the condition of intersection with the line 12 gives

$$\phi=-\frac{\alpha'-k\alpha}{\delta'-k\delta},$$

which is the value of ϕ in the foregoing equations: and to these we may join the resulting equation

$$y\gamma\gamma'(\alpha\beta'-\alpha'\beta)=z\phi\beta\beta'(\gamma\delta'-\gamma'\delta).$$

Proceeding in like manner for the lines 3′, 2, 4, the equations for the remaining four lines of the double-sixer are

2.
$$\phi = \frac{\alpha' - k\alpha}{\beta' - k\beta},$$
$$x - w\frac{\alpha'}{\delta'} = \phi\left(y - z\frac{\beta'}{\gamma'}\right),$$
$$x - w\frac{\alpha}{\delta} = \phi\left(y - z\frac{\beta}{\gamma}\right),$$
$$w\gamma\gamma'(\alpha\delta' - \alpha'\delta) = z\phi\delta\delta'(\beta\gamma' - \beta'\gamma).$$

4.
$$\phi = \frac{\gamma' - k\gamma}{\delta' - k\delta},$$
$$\phi\left(x\frac{\delta'}{\alpha'} - w\right) = y\frac{\gamma'}{\beta'} - z,$$
$$\phi\left(x\frac{\delta}{\alpha} - w\right) = y\frac{\gamma}{\beta} - z,$$
$$x\phi\beta\beta'(\alpha\delta' - \alpha'\delta) = y\alpha\alpha'(\beta\gamma' - \beta'\gamma).$$

1′.
$$\phi = \frac{\alpha' - k\alpha}{\delta' - k\delta},$$
$$x - y\frac{\alpha'}{\beta'} = \phi\left(z\frac{\delta'}{\gamma'} - w\right),$$
$$x - y\frac{\alpha}{\beta} = \phi\left(z\frac{\delta}{\gamma} - w\right),$$
$$y\gamma\gamma'(\alpha\beta' - \alpha'\beta) = z\phi\beta\beta'(\gamma\delta' - \gamma'\delta).$$

3′.
$$\phi = -\frac{\gamma' - k\gamma}{\beta' - k\beta},$$
$$\phi\left(x\frac{\beta'}{\alpha'} - y\right) = z - w\frac{\gamma'}{\delta'},$$
$$\phi\left(x\frac{\beta}{\alpha} - y\right) = z - w\frac{\gamma}{\delta},$$
$$x\phi\delta\delta'(\alpha\beta' - \alpha'\beta) = w\alpha\alpha'(\gamma\delta' - \gamma'\delta).$$

It may be added that

In plane $x = 0$,

intersection of 1′ lies on line $z : w = (\alpha\beta' - \alpha'\beta)\gamma\gamma' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta$,

" 2 " $y : z = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\alpha\delta' - \alpha'\delta)\gamma\gamma'$,

and that line joining these intersections is line 12.

In plane $y = 0$,

intersection of 2 lies on line $x : w = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : -(\beta\gamma' - \beta'\gamma)\delta\delta'$,

" 3′ " $z : w = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\alpha\beta' - \alpha'\beta)\delta\delta'$,

and that line joining these intersections is line 23.

In plane $z = 0$,

intersection of 3′ lies on line $x : y = (\gamma\delta' - \gamma'\delta)\alpha\alpha' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta$,

" 4 " $x : w = -(\beta\gamma' - \beta'\gamma)\alpha\alpha' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta$,

and that line joining these intersections is line 34.

And in plane $w = 0$,

intersection of 4 is on line $y : z = (\alpha\delta' - \alpha'\delta)\beta\beta' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta$,

" 1′ " $x : y - \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\gamma\delta' - \gamma'\delta)\beta\beta'$

and that line joining these intersections is line 14.

The equations of the remaining ten lines of the surface may be obtained without difficulty, and also the forty-five triple planes, but I do not stop to effect this; the planes $x = 0$, $y = 0$, $z = 0$, $w = 0$, are, it is clear, triple planes, containing the lines 1, 2′, 12; 2′, 3, 23; 3, 4′, 34; and 4′, 1, 41 respectively.

If, to fix the ideas, the planes $x = 0$, $y = 0$, $z = 0$, $w = 0$ are taken to be those of the tetrahedron $ABCD$ ($x = BCD$ &c., as usual), then the edges AB, BC, CD, DA (but not the remaining opposite edges AC, BD) will be lines on the surface. Each plane of the tetrahedron, for instance ABC ($w = 0$), is met by the ten lines not contained therein in two vertices A, C, three points on the edge BA, three points on the edge BC, and two other points, viz., these are the intersections of the plane ABC by the lines 4 and 1′. For the construction of a model it is sufficient to determine the three points on each edge, and the two points say in the plane ABC, and in the plane DBC ($x = 0$) respectively, for then each of the remaining eight lines will be determined as a line joining two points in these two planes respectively. If in the first instance k is considered as a variable parameter, then the two points in the plane $w = 0$ are given as the intersections of two fixed lines by a variable line (14) rotating round the fixed point $\frac{x}{a} - \frac{y}{\beta} + \frac{z}{\gamma} = 0$, $\frac{x}{a} - \frac{y}{\beta} + \frac{z}{\gamma} = 0$; and the like as regards the two points in the plane $x = 0$. By making with assumed values of the other parameters the proper drawings for the two planes $w = 0$, $x = 0$, it is easy to fix upon a convenient value of the parameter k; and I have in this manner succeeded in making a string model of the double-sixer; viz., the coordinates x, y, z, w were taken to be as the perpendicular distances of the current point from the faces of a regular tetrahedron (the coordinates being positive for an interior point); the values of a, β, γ, δ were put $= 3, 4, 5, 6$ and those of $a', \beta', \gamma', \delta' = 1, 1, 1, 1$; the value of k fixed upon as above was $k = -\frac{1}{8}$; this however brings the lines 2 and 4 too close together (viz., the shortest distance between them is not great enough), and also their apparent intersection too close to their intersections with the line 6′; and it is probable that a slightly different value of k would be better.

The results just obtained may be exhibited in a compendious form as follows:

	x	$:y$	$:z$	$:w$
1 is line BC	0			0
2′ ,, CD	0	0		
3 ,, DA		0	0	
4′ ,, AB			0	0
5 meets CD			γ	δ
,, AB	α	β		
6 ,, CD			γ'	δ'
,, AB	α'	β'		
6′ ,, BC		β	γ	
,, AD	α			δ
5′ ,, BC		β'	γ'	
,, AD	α'			δ'
1′ ,, AD	$\alpha'-k\alpha$			$\delta'-k\delta$
,, BCD		$-(\alpha'-k\alpha)(\gamma\delta'-\gamma'\delta)\beta\beta'$	$(\delta'-k\delta)(\alpha\beta'-\alpha'\beta)\gamma\gamma'$	$(\delta'-k\delta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$
,, ABC	$(\alpha'-k\alpha)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\alpha'-k\alpha)(\gamma\delta'-\gamma'\delta)\beta\beta'$	$-(\delta'-k\delta)(\alpha\beta'-\alpha'\beta)\gamma\gamma'$	
3′ ,, BC		$\beta'-k\beta$	$\gamma'-k\gamma$	
,, ACD	$-(\beta'-k\beta)(\gamma\delta'-\gamma'\delta)\alpha\alpha'$		$(\gamma'-k\gamma)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\gamma'-k\gamma)(\alpha\beta'-\alpha'\beta)\delta\delta'$
,, ABD	$(\beta'-k\beta)(\gamma\delta'-\gamma'\delta)\alpha\alpha'$	$(\beta'-k\beta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$		$-(\gamma'-k\gamma)(\alpha\beta'-\alpha'\beta)\delta\delta$
2 ,, AB	$\alpha'-k\alpha$	$\beta'-k\beta$		
,, BCD		$(\beta'-k\beta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\beta'-k\beta)(\alpha\delta'-\alpha'\delta)\gamma\gamma'$	$(\alpha'-k\alpha)(\beta\gamma'-\beta'\gamma)\delta\delta'$
,, ACD	$(\alpha'-k\alpha)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$		$-(\beta'-k\beta)(\alpha\delta'-\alpha'\delta)\gamma\gamma'$	$-(\alpha'-k\alpha)(\beta\gamma'-\beta'\gamma)\delta\delta'$
4 ,, CD			$\gamma'-k\gamma$	$\delta'-k\delta$
,, ABD	$-(\delta'-k\delta)(\beta\gamma'-\beta'\gamma)\alpha\alpha'$	$-(\gamma'-k\gamma)(\alpha\delta'-\alpha'\delta)\beta\beta'$		$(\delta'-k\delta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$
,, ABC	$(\delta'-k\delta)(\beta\gamma'-\beta'\gamma)\alpha\alpha'$	$(\gamma'-k\gamma)(\alpha\delta'-\alpha'\delta)\beta\beta'$	$(\gamma'-k\gamma)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	

1′ and 2 meet BCD on line $\left(\ .\ -\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\ .\ -\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$

2 and 3′ ,, CDA ,, $\left(\frac{x}{\alpha'}\ \ .\ +\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}\ \ .\ +\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\alpha'\gamma'}=0,$

3′ and 4 ,, DAB ,, $\left(\frac{x}{\alpha'}-\frac{y}{\beta'}\ \ .\ -\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}\ \ .\ -\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$

4 and 1′ ,, ABC ,, $\left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}\ \ .\ \right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}\ \ .\ \right)\frac{1}{\alpha'\gamma'}=0,$

or calculating the numerical values from the foregoing assumed data,

	x	$:y$	$:z$	$:w$	say
1 is line BC	0			0	
2′ ... CD	0	0			
3 ... DA		0	0		
4′ ... AB			0	0	
5 meets CD			5	6	$z = 45{\cdot}5,\ w = 54{\cdot}5.$
... AB	3	4			$x = 42{\cdot}9,\ y = 57{\cdot}1.$
6 meets CD			1	1	$z = 50,\ w = 50.$
... AB	1	1			$x = 50,\ y = 50.$
6′ meets BC		4	5		$y = 44{\cdot}4,\ z = 55{\cdot}6.$
... AD	3			6	$x = 33{\cdot}3,\ w = 66{\cdot}7.$
5′ meets BC		1	1		$y = 50,\ z = 50.$
... AD	1			1	$x = 50,\ w = 50.$
1′ meets AD	11			14	$x = 44,\ z = 56.$
... BCD		−44	70	126	$y = -25{\cdot}1,\ z = 39{\cdot}9,\ w = 71{\cdot}8.$
... ABC	99	44	−70		Not required.
3′ meets BC		12	13		$y = 48,\ z = 52.$
... ACD	−36		117	78	Not required.
... ABD	36	108		−78	$x = 47{\cdot}2,\ y = 141{\cdot}7,\ w = -102{\cdot}3.$
2 meets AB	11	12			$x = 47{\cdot}8,\ y = 52{\cdot}2.$
... BCD		100	180	66	$y = 26{\cdot}4,\ z = 44,\ w = 16{\cdot}2.$
... ACD	−99		189	66	Not required.
4 meets CD			13	14	$z = 48{\cdot}1,\ w = 51{\cdot}9.$
... ABD	42	156		−126	$x = 50{\cdot}5,\ y = 187{\cdot}6,\ w = -151{\cdot}5.$
... ABC	42	156	117		Not required.

1′ and 2 meet BCD on line $35y - 32z + 30w = 0,$

2 and 3′ ... CDA ...... $26x + 22z - 21w = 0,$

3′ and 4 ... DAB ...... $8x - 7y - 6w = 0,$

4′ and 1 ... ABC ...... $52x - 47y - 44z = 0,$

which last four equations serve as a verification.

The outside numerical values are given in the manner most convenient for the construction of a drawing; viz. when the coordinates refer to a point on an edge of the tetrahedron, or say on the side of an equilateral triangle, then taking the length of this edge (or side) to be = 100, the numerical values are fixed so that the sum of the two coordinates may be = 100, and the two co-ordinates thus denote the distances from the extremities of the edge or side: but when the three co-ordinates belong to a point in the face of the tetrahedron, or say in the plane of an equilateral triangle, then the sum of the coordinates is made = 86·6, and the three coordinates thus denote the perpendicular distances from the sides of the triangle.

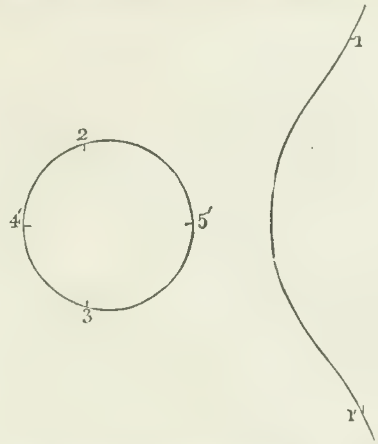
III.

It is possible to find on a cubic curve a double-sixer of points 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′ such that any six points such as 1, 2, 3, 4′, 5′, 6′ lie in a conic. In fact considering a cubic surface having upon it the double-sixer of lines 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′, the section by any plane is a cubic curve meeting the lines, say in the points 1, 2, 3, 4, 5, 6, 1′, 2′, 3′, 4′, 5′, 6′: each of the lines 1, 2, 3 meets each of the lines 4′, 5′, 6′, and consequently the six lines lie in a quadric surface: therefore the points 1, 2, 3, 4′, 5′, 6′ lie in a conic: and so in the other cases; the number of the conics is of course = 60.

The cubic curve may be a given curve, and six of the points upon it (not being points on a conic) may also be taken to be given; for instance the points 1, 2, 3, 1′ 4′, 5′. For take through the points 2, 3 respectively any two lines 1, 2; through 1′, 4′, 5′ respectively the lines 1′, 4′, 5′ each meeting each of the lines 2, 3: and through 1 a line meeting each of the lines 4′, 5′. It is easy to see that a cubic surface may be drawn through the cubic curve and the lines 1, 2, 3, 1′, 4′, 5′: for the passage through the cubic curve is 9 conditions; the surface then passes through the point 2 and to make it pass through the line 2 is 3 conditions; similarly the surface passes through the point 3, and to make it pass through the line 3 is 3 conditions. The surface now passes through 1′ and through the points of intersection of the line 1′ with the lines 2, 3: to make it pass through the line 1′ is 1 condition; similarly to make it pass through the lines 4′, 5′, 1 is in each case 1 condition; or there are in all 19 conditions, so that the cubic surface is completely determined. Take now through the points 1, 2, 3, 4′, 5′, a conic meeting the cubic in the point 6′: then through the lines 1, 2, 3, 4′, 5′ we have a quadric surface passing through this conic, and therefore through 6′: hence through 6′ we may draw a line 6′ meeting each of the lines 1, 2, 3; and since the cubic surface passes through the point 6′ and also through the intersections of the line 6′ with the lines 1, 2, 3, it passes through the line 6′. We complete in this manner by constructions in the plane of the cubic the system of the twelve points, viz., each new point is given as the intersection of the cubic curve by a conic drawn through five points of the cubic curve; and it is then shown as for the point 6′ and the line 6′ through it, that through each new point there can be drawn a line denoted by the same number and meeting each of the lines which it ought to meet, and hence lying on the cubic surface: the twelve points are thus the intersections of the plane of the cubic curve by the twelve lines of the double-sixer; and it follows that the six points which ought to lie in a conic (in every case where such conic has not been used in the plane construction) do actually lie in a conic.

I was anxious to construct such a double-sixer of pounds on a cubic curve; for this purpose I take the equation of the curve to be $y^2 = \left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{b}\right) \left(1 - \frac{x}{c}\right)$, or say for shortness $y^2 = X$; where, to fix the ideas, a, b are supposed to be positive, a greater than b; and c to be negative.

The cubic curve is thus a parabola symmetrical in regard to the axis of x, and consisting of a loop and infinite branch; and I take upon it the points 1, 2, 3, 1′, 4′, 5′ as shown in the



figure, viz., the coordinates of these points are as stated in the Table, where m is the x coordinate, and $\sqrt{M} = \sqrt{\left(1-\frac{m}{a}\right)\left(1-\frac{m}{b}\right)\left(1-\frac{m}{c}\right)}$ and so in other cases, $\sqrt{14} = 3{\cdot}74165$.

	x	y	x	y
1	m	$\sqrt{M}$	6	$\sqrt{14} = 3{\cdot}742$
2	0	1	0	1
3	0	-1	0	-1
4	θ	$\sqrt{\Theta}$	$2 - \frac{75}{1369} = 1{\cdot}945$	$-\frac{2280}{(37)^3}\sqrt{14} = -{\cdot}168$
5	ϕ	$\sqrt{\Phi}$	$-1 + \frac{75}{361} = -0{\cdot}792$	$\frac{1110}{(19)^3}\sqrt{14} = {\cdot}606$
6	m_1	$-\sqrt{M_1}$	$\frac{13}{3} = 4{\cdot}333$	$-\frac{4}{9}\sqrt{14} = -1{\cdot}641$
1′	m	$-\sqrt{M}$	6	$-\sqrt{14} = -3{\cdot}742$
2′	σ	$\sqrt{\Sigma}$	$-\frac{1560\,(14-\sqrt{14})}{(31\sqrt{14}-5)^2} = 1{\cdot}299$	.. $= +{\cdot}676$
3′	τ	$\sqrt{\mathrm{T}}$	$-\frac{1560\,(14+\sqrt{14})}{(31\sqrt{14}+5)^2} = 1{\cdot}887$	.. $= +{\cdot}247$
4′	c	0	-1	0
5′	b	0	2	0
6′	m_1	$\sqrt{M_1}$	$\frac{13}{3} = 4{\cdot}333$	$\frac{4}{9}\sqrt{14} = 1{\cdot}641.$

The numerical values belong to the curve $y^2 = \left(1-\frac{x}{3}\right)\left(1-\frac{x}{2}\right)\left(1+x\right)$ and to $m = 6$.

Starting with the points 1, 2, 3, 1′, 4′, 5′ we have to find the remaining points 6′, 6, 4, 5, 2′, 3′.

Point 6′ by means of the conic 1234′5′6.

The equation of the conic is

$$(x-b)(x-c)-bc\,y^2+k\,xy=0,\quad (2, 3, 4', 5'),$$

and making this pass through the point 1 ($x=m$, $y=\sqrt{M}$) we find

$$(m-b)(m-c)+ka\sqrt{M}=0.\quad (1).$$

Hence taking the coordinates of 6′ to be m_1, $\sqrt{M_1}$ we have

$$(m_1-b)(m_1-c)+ka\sqrt{M_1}=0,\quad (6'),$$

and thence

$$\frac{\sqrt{M_1}}{\sqrt{M}}=\frac{(m_1-b)(m_1-c)}{(m-b)(m-c)}=\frac{M_1}{M}\frac{(m-a)}{(m_1-a)},$$

that is

$$\frac{\sqrt{M_1}}{\sqrt{M}}=\frac{(m_1-b)(m_1-c)}{(m-b)(m-c)}=\frac{m_1-a}{m-a},$$

we have thus for m, a quadric equation satisfied by $m=m_1$, so that throwing out the factor $m-m_1$, the equation is a linear one, viz., we find

$$m_1=\frac{ma-ab-ac+bc}{m-a},$$

or what is the same thing

$$m_1-a=\frac{(a-b)(a-c)}{m-a},$$

and thence also

$$\sqrt{M_1}=\frac{(a-b)(a-c)}{(m-a)^2}\sqrt{M},$$

viz., $\sqrt{M_1}$ is determined rationally in terms of m, $\sqrt{M}$; this is of course as it should be, since the point 6′ is uniquely determinate.

Point 6 by means of the conic 2361′4′5′.

In precisely the same manner the coordinates are $m_1, -\sqrt{M_1}$, where m_1, $\sqrt{M_1}$ denote as before.

Point 4 by means of the conic 2341′5′6′.

The equation of the conic is

$$Fx+Gy+H=\frac{1-y^2}{x},\qquad (2, 3),$$

where

$$Fb \qquad + H = \frac{1}{b}, \tag{5'}$$

$$Fm_1 + G\sqrt{M_1} + \cdot H = \frac{1-M_1}{m_1}, \tag{6'}$$

$$Fm - G\sqrt{M} + H = \frac{1-M}{m}, \tag{1'}$$

which give without difficulty

$$abc\, F = -a - c + P,$$

$$\sqrt{M}\, abc\, G = (m-b)\ (-m+P),$$

$$abc\, H = ab + ac + bc - bP,$$

where $P = 2a - c - \dfrac{2\,(a-c)\,(b-c)}{m+m_1-2c}$, a quantity which will presently be expressed in terms of m only.

And then

$$F\theta + G\sqrt{\Theta} + H = \frac{1-\Theta}{\theta},$$

or say

$$F\,(\theta - b) + G\sqrt{\Theta} = \frac{1-\Theta}{\theta} - \frac{1}{b}$$

$$= -\,(\theta - b)\ \left(\frac{1}{ab} + \frac{1}{bc} - \frac{\theta}{abc}\right),$$

that is

$$(\theta - b)\ (abc\ F + a + c - \theta) + G\ abc\sqrt{\Theta} = 0,$$

viz., that is

$$(\theta - b)\ (P - \theta) + (m-b)\,\frac{\sqrt{\Theta}}{\sqrt{M}}\,(P - m) = 0$$

or, rationalising and throwing out the factor $\theta - b$,

this is

$$(\theta - b)\ (\theta - P)^2 - (m-b)\ (m-P)^2 \cdot \frac{(\theta - a)\ (\theta - b)}{(m-a)\ (m-b)} = 0,$$

which is a cubic equation satisfied by $\theta = m$ and $\theta = m_1$; so that throwing out the factors $\theta - m$, $\theta - m_1$ we have for θ a linear equation.

Putting for shortness

$$\begin{cases} A = (m-a)^2 - (a-b)\ (a-c), \\ B = (m-b)^2 - (b-c)\ (b-a), \\ C = (m-c)^2 - (c-a)\ (c-b), \end{cases}$$

the value of θ may be expressed in the forms

$$\theta - a = \frac{B^2}{C^2}(c-a),\ \theta - b = \frac{A^2}{C^2}(c-b),\ \theta - c = \frac{4\,(m-a)\,(m-b)\,(m-c)\,(b-c)\,(a-c)}{C^2}.$$

We have moreover

$$P - c = \frac{2\,(a-c)\,(m-b)\,(m-c)}{C},\quad P - m = -\frac{(m-c)\,A}{C},$$

equations which express P in terms of m only; also

$$\theta - P = \frac{-2\,(a-c)\,(m-b)\,(m-c)\,B}{C^2},$$

and then

$$\sqrt{\Theta} = -\sqrt{M}\,\frac{\theta-b}{m-b}\cdot\frac{P-\theta}{P-m},$$

whence

$$\sqrt{\Theta} = 2\sqrt{M}\,(b-c)\,(c-a)\,\frac{AB}{C^3},$$

so that θ, $\sqrt{\Theta}$ are now determined.

Point 5 by means of the conic $2351'4'6'$.

The conic is

$$Fx + Gy + H = \frac{1-y^2}{x}, \qquad (2,\ 3)$$

where

$$Fc \,.\qquad\qquad + H = \frac{1}{c}, \qquad (4')$$

$$Fm_1 + G\sqrt{M_1} + H = \frac{1-M_1}{m_1}, \qquad (6')$$

$$Fm - G\sqrt{M} + H = \frac{1-M}{m}, \qquad (1').$$

Every thing is the same as for the point 4 except that b, c are interchanged: hence writing Q instead of P, and using A, B, C to denote as before, we have

$$abc\ F = -a - b + Q,$$

$$\sqrt{M}\,abc\ G = (m-c)\,(-m+Q),$$

$$abc\ H = ab + ac + bc - cQ,$$

and

$$\phi - a = \frac{C^2\,(b-a)}{B^2},$$

$$\phi - b = \frac{4\,(m-a)\,(m-b)\,(m-c)\,(c-b)\,(a-b)}{B^2},$$

$$\phi - c = \frac{A^2\,(b-c)}{B^2},$$

$$Q-b=\frac{2\,(m-b)\,(m-c)\,(a-b)}{B},$$

$$Q-m=-\frac{A\,(m-b)}{B},$$

$$\phi-Q=-\frac{2\,(m-b)\,(m-c)\,C\,(a-b)}{B^2},$$

and

$$\sqrt{\Phi}=\quad 2\sqrt{M}\,(c-b)\,(b-a)\,\frac{AC}{B^3},$$

which determine ϕ, $\sqrt{\Phi}$.

Point $3'$ by means of conic $1263'4'5'$.

Equation of conic is

$$Fx+Gy+H=\frac{(x-b)\,(x-c)}{y},\qquad (4', 5')$$

and we have

$$G+H=bc,\qquad (2)$$

$$Fm+G\sqrt{M}+H=\frac{(m-b)\,(m-c)}{\sqrt{M}},\qquad (1)$$

$$Fm_1-G\sqrt{M_1}+H=\frac{(m_1-b)\,(m_1-c)}{-\sqrt{M_1}},\qquad (6).$$

Eliminating F, we have

$$G\,(m_1\sqrt{M}+m\sqrt{M_1})+H\,(m-m_1)=\frac{m_1\,(m-b)\,(m-c)}{\sqrt{M}}+\frac{m\,(m_1-b)\,(m_1-c)}{\sqrt{M_1}}$$

which is easily reduced first to

$$G\,\frac{2mm_1-a(m+m_1)}{(m-a)\sqrt{M}}+H\,(m-m_1)=(m+m_1)\frac{(m-a)\,(m-b)}{\sqrt{M}}$$

and then to

$$G\,\{aA+2m\,(a-b)\,(a-c)\}-H\,\frac{(m-a)\,A}{\sqrt{M}}+abc\,\{-A+2m\,(m-a)\}=0$$

and combining herewith $G+H=bc$, we have

$$H=\frac{2bc\,m\left[a\,(m-a)+(a-b)\,(a-c)\right]}{aA+2m\,(a-b)\,(a-c)+\dfrac{(m-a)\,A}{\sqrt{M}}};$$

$$G=bc-H;$$

and we have then

$$F(m+m_1)+G(\sqrt{M}-\sqrt{M_1})+2H=0,$$

that is

$$F\{2m(m-a)-A\}+G\frac{A\sqrt{M}}{m-a}+2H(m-a)=0,$$

or what is the same thing

$$F\{2m(m-a)-A\}=-bc\frac{A\sqrt{M}}{m-a}-H.\left\{2(m-a)-\frac{A\sqrt{M}}{m-a}\right\}.$$

We then have

$$Fx+H=y\left(-G+\frac{(x-b)(x-c)}{y^2}\right)$$

$$=y\left(-G-\frac{abc}{x-a}\right)=-\frac{y(Ha+Gx)}{x-a},$$

that is

$$(Fx+H)^2=-\frac{(x-b)(x-c)}{abc(x-a)}(Ha+Gx)^2, \text{ or}$$

$$abc(x-a)(Fx+H)^2+(x-b)(x-c)(Gx+Ha)^2=0,$$

or, developing and throwing out the factor x, this is

$$\begin{aligned}&G^2x^3\\ +&+\{2a\,GH-(b+c)G^2+abc\,F^2\}x^2\\ &+\{a^2H^2-2a(b+c)GH+bc\,G^2+abc(2FH-aF^2)\}x\\ &+\{-(b+c)a^2H^2+2abc\,GH+abc(H^2-2aFH)\}=0.\end{aligned}$$

This must be satisfied by $x=m$, $x=m_1$; hence the left hand must be $=G^2(x-m)(x-m_1)(x-\sigma)$, or equating the constant terms we have

$$G^2mm_1\sigma=aH\{-2abc\,F+2bc\,G+(bc-ab-ac)H\},$$

which gives σ; and we then have

$$\sqrt{\Sigma}=-\frac{\sigma-a}{G\sigma+Ha}(F\sigma+H),$$

but I have not attempted the further reduction of these expressions.

The numerical values for the example are

$$3F=\frac{-140+62\sqrt{14}}{5+21\sqrt{14}},\quad G=\frac{-10+62\sqrt{14}}{5+21\sqrt{14}},\quad H=\frac{-104\sqrt{14}}{5+\sqrt{14}},$$

whence σ as in the Table.

Point $2'$ by means of conic $1362'4'5'$.

The equation of the conic is

$$Fx + Gy + H = \frac{(x-b)\,(x-c)}{y}, \qquad (4',\ 5')$$

where

$$-G + H = -bc, \qquad (3)$$

$$Fm + G\sqrt{M} + H = \frac{(m-b)\,(m-c)}{\sqrt{M}}, \qquad (1)$$

$$Fm_1 - G\sqrt{m_1} + H = \frac{(m_1-b)\,(m_1-c)}{-\sqrt{M_1}}, \qquad (6)$$

which are the same as for point $3'$, if only we reverse the signs of F, H and $\sqrt{M}$, $\sqrt{M_1}$.

Hence the formulæ are

$$H = -\frac{2bcm\,[a\,(m-a) + (a-b)\,(a-c)]}{aA + 2m\,(a-b)\,(a-c) - \dfrac{(m-a)A}{\sqrt{M}}},$$

$$G = \quad bc + H,$$

$$F\,\{2m\,(m-a) - A\} = -bc\,\frac{A\sqrt{M}}{m-a} - H\left\{2\,(m-a) + \frac{A\sqrt{M}}{m-a}\right\},$$

$$G^2\,mm_1\,\tau = aH\,\{-2abc\,F - 2bc\,G + (bc - ab - ac)\,H\},$$

which gives τ, and then

$$\sqrt{T} = \frac{(\tau - a)}{G\tau - Ha}\,(F\tau + H),$$

which are also unreduced.

The numerical values are

$$3F = \frac{140 + 62\sqrt{14}}{5 - 21\sqrt{14}},\quad G = \frac{-10 - 62\sqrt{14}}{5 - 21\sqrt{14}},\quad H = \frac{-104\sqrt{14}}{5 - 21\sqrt{14}},$$

whence τ as in the Table.

VI. *Tables of the first* 250 *Bernoulli's Numbers* (*to nine figures*) *and their logarithms* (*to ten figures*). By J. W. L. GLAISHER, B.A., F.R.A.S. *Fellow of Trinity College, Cambridge.*

[Read *May* 29, 1871.]

THE only table of the logarithms of Bernoulli's Numbers that has hitherto been calculated with which the author is acquainted, is that given in Grunert's supplement to Klügel's Wörterbuch (Article, Bernoullische Zahlen); it contains the logarithms of the first eighteen numbers. This table, which is stated to be copied from Eytelwein's Higher Analysis, is also reprinted in the Penny (and English) Cyclopædia.

The present table which contains the logarithms of the first 250 Bernoulli's Numbers to ten decimal places was constructed as follows:

The first seven logarithms were found by merely taking out the logarithms of the corresponding numbers; the rest were calculated by means of the formula*

$$B_n = \frac{2\,(1\,.\,2\ldots 2n)}{(2\pi)^{2n}}\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots\right),$$

from which we obtain

$$\log B_n = \log 2 + \log 1 + \log 2 \ldots + \log (2n) - 2n \log (2\pi) + \mu\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots\right),$$

(where μ is the modulus $\cdot 43429448\ldots$), the square and higher powers of

$$\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots$$

being insensible to 10 places, when n is greater than 7.

Several of the logarithms were calculated by taking out the logarithms of the number as well as by this method.

A table of the values of $\log (1\,.\,2\ldots x)$ as far as $x = 1200$ to 18 decimal places, was published by C. F. Degen, at Copenhagen, in 1824; this has been used in the construction

* There is some confusion in the notation for Bernoulli's Numbers; they are sometimes written B_1, B_2, B_3, &c., and sometimes B_1, B_3, B_5, &c. B_n in the one system corresponds to B_{2n-1} in the other. The former system is here adopted as being obviously the better.

of the accompanying table, but every number quoted has been verified, to guard against any misprint of Degen's.

The whole work has been performed in duplicate, and as the calculation was not of a difficult nature, there is a high probability that the table is free from error.

It would have been easy to give more decimal places, but as there exists no printed table of logarithms of natural numbers to more than ten figures, it seemed useless to calculate the present logarithms farther; the work was however extended to fifteen places, so that the last figure (which has been corrected) is in all cases accurate.

The table in the Wörterbuch was found to be very inaccurate; the following is the list of errors*.

In B_6	$\bar{1}.4033154004$	should be	$\bar{1}.4033154003$,
,, B_8	0.8507783387	,,	0.8507783327,
,, B_{12}	4.9374188514	,,	4.9374188511,
,, B_{14}	7.4361345055	,,	7.4361345056,
,, B_{15}	8.7792940212	,,	8.7792940203,
,, B_{17}	11.6330790754	,,	11.6330790755,
,, B_{18}	13.1370898829	,,	13.1370898839.

For values of n greater than 250, we can replace

$$1.2...2n \text{ by } 2^{2n+1}\sqrt{(\pi n)}\, n^{2n} e^{-2n}\left(1+\frac{1}{24n}+\frac{1}{1152n^2}\right),$$

so that

$$\log B_n = \log 4 + \left(2n+\frac{1}{2}\right)\log n - \left(2n-\frac{1}{2}\right)\log \pi - 2\mu n + \frac{\mu}{24n},$$

the term $\frac{\mu}{1152n^2}$ being cancelled by an equal term of opposite sign in the expansion.

Bernoulli's Numbers have been calculated as far as B_{31}, the first 15 by Euler and the rest by Rothe; they are all given in Crelle's *Journal*, t. xx. p. 11. These as far as B_{25}, have been verified by two calculations of Euler's constant by means of them†. An error should be mentioned in Euler's original calculation of B_{13}, which is given (*Acta Petropol.* pt. II. for 1781, p. 46) as $\frac{8553103}{2}$ instead of $\frac{8553103}{6}$. This error is reproduced in the *Penny Cyclopædia* (Article, Numbers of Bernoulli) and probably elsewhere.

* All these errors occur also in Eytelwein's *Grundlehren der höhern Analysis*, Berlin, 1824 (t. I. p. 488); so that they were not introduced by inaccurate copying on Grunert's part. Euler's error in B_{13}, alluded to further on, is not reproduced by Eytelwein.

† *Proceedings of the Royal Society*, No. 129, 1871.

TABLE OF THE LOGARITHMS OF BERNOULLI'S NUMBERS.

n	$\log B_n$	n	$\log B_n$	n	$\log B_n$
1	$\bar{1}$·22184 87496	46	68·96324 71164	91	188·85301 53421
2	$\bar{2}$·52287 87453	47	71·30849 81818	92	191·78392 45182
3	$\bar{2}$·37675 07096	48	73·67213 32834	93	194·72424 94541
4	$\bar{2}$·52287 87453	49	76·05377 13567	94	197·67388 91731
5	$\bar{2}$·87942 60688	50	78·45304 68146	95	200·63274 48416
6	$\bar{1}$·40331 54003	51	80·86960 86234	96	203·60071 97008
7	0·06694 67896	52	83·30311 94507	97	206·57771 90030
8	0·85077 83327	53	85·75325 48783	98	209·56364 99490
9	1·74013 50433	54	88·21970 26748	99	212·55842 16287
10	2·72355 76597	55	90·70216 21211	100	215·56194 49641
11	3·79183 95878	56	93·20034 33859	101	218·57413 26542
12	4·93741 88511	57	95·71396 69440	102	221·59489 91229
13	6·15397 24516	58	98·24276 30368	103	224·62416 04676
14	7·43613 45056	59	100·78647 11692	104	227·66183 44113
15	8·77929 40203	60	103·34483 96399	105	230·70784 02554
16	10·17944 59554	61	105·91762 51042	106	233·76209 88349
17	11·63307 90755	62	108·50459 21641	107	236·82453 24750
18	13·13708 98839	63	111·10551 29855	108	239·89506 49493
19	14·68871 54679	64	113·72016 69394	109	242·97362 14401
20	16·28548 03295	65	116·34834 02653	110	246·06012 84990
21	17·92515 37399	66	118·98982 57554	111	249·15451 40104
22	19·60571 51352	67	121·64442 24580	112	252·25670 71551
23	21·32532 57440	68	124·31193 53982	113	255·36663 83756
24	23·08230 51026	69	126·99217 53150	114	258·48423 93431
25	24·87511 14502	70	129·68495 84142	115	261·60944 29248
26	26·70232 52332	71	132·39010 61345	116	264·74218 31528
27	28·56263 51260	72	135·10744 49274	117	267·88239 51945
28	30·45482 61057	73	137·83680 60487	118	271·03001 53231
29	32·37776 92183	74	140·57802 53621	119	274·18498 08894
30	34·33041 27436	75	143·33094 31529	120	277·34723 02954
31	36·31177 45314	76	146·09540 39514	121	280·51670 29672
32	38·32093 53181	77	148·87125 63663	122	283·69333 93304
33	40·35703 28735	78	151·65835 29261	123	286·87708 07852
34	42·41925 68522	79	154·45654 99288	124	290·06786 96825
35	44·50684 42463	80	157·26570 72990	125	293·26564 93016
36	46·61907 53547	81	160·08568 84529	126	296·47036 38271
37	48·75527 01978	82	162·91636 01686	127	299·68195 83282
38	50·91478 53168	83	165·75759 24642	128	302·90037 87373
39	53·09701 09079	84	168·60925 84803	129	306·12557 18298
40	55·30136 82495	85	171·47123 43696	130	309·35748 52052
41	57·52730 73841	86	174·34339 91902	131	312·59606 72671
42	59·77430 50258	87	177·22563 48049	132	315·84126 72058
43	62·04186 26660	88	180·11782 57847	133	319·09303 49796
44	64·32950 48541	89	183·01985 93166	134	322·35132 12983
45	66·63677 76334	90	185·93162 51160	135	325·61607 76057

n	$\log B_n$		n	$\log B_n$		n	$\log B_n$	
136	328·88725	60639	176	464·52554	03298	216	608·11800	48903
137	332·16480	95371	177	468·02595	85605	217	611·79562	27795
138	335·44869	15762	178	471·53127	71748	218	615·47723	87890
139	338·73885	64045	179	475·04146	86808	219	619·16283	45997
140	342·03525	89024	180	478·55650	58935	220	622·85239	20597
141	345·33785	45939	181	482·07636	19292	221	626·54589	31818
142	348·64659	96328	182	485·60101	02012	222	630·24332	01415
143	351·96145	07892	183	489·13042	44143	223	633·94465	52744
144	355·28236	54370	184	492·66457	85605	224	637·64988	10748
145	358·60930	15409	185	496·20344	69140	225	641·35898	01929
146	361·94221	76446	186	499·74700	40268	226	645·07193	54329
147	365·28107	28587	187	503·29522	47241	227	648·78872	97510
148	368·62582	68490	188	506·84808	41000	228	652·50934	62536
149	371·97643	98257	189	510·40555	75133	229	656·23376	81950
150	375·33287	25320	190	513·96762	05832	230	659·96197	89755
151	378·69508	62338	191	517·53424	91851	231	663·69396	21397
152	382·06304	27092	192	521·10541	94467	232	667·42970	13746
153	385·43670	42383	193	524·68110	77442	233	671·16918	05075
154	388·81603	35936	194	528·26129	06981	234	674·91238	35044
155	392·20099	40301	195	531·84594	51697	235	678·65929	44683
156	395·59154	92765	196	535·43504	82574	236	682·40989	76374
157	398·98766	35254	197	539·02857	72929	237	686·56417	73831
158	402·38930	14251	198	542·62650	98377	238	689·92211	82087
159	405·79642	80706	199	546·22882	36798	239	693·68370	47476
160	409·20900	89952	200	549·83549	68301	240	697·44892	17617
161	412·62701	01626	201	553·44650	75191	241	701·21775	44396
162	416·05039	79584	202	557·06183	41937	242	704·99018	68953
163	419·47913	91828	203	560·68145	55137	243	708·76620	51664
164	422·91320	10424	204	564·30535	03493	244	712·54579	42129
165	426·35255	11435	205	567·93349	77774	245	716·32893	94154
166	429·79715	74843	206	571·56587	70785	246	720·11562	62735
167	433·24698	84479	207	575·20246	77346	247	723·90584	04050
168	436·70201	27956	208	578·84324	94252	248	727·69956	75437
169	440·16219	96600	209	582·48820	20253	249	731·49679	35385
170	443·62751	85386	210	586·13730	56019	250	735·29750	43517
171	447·09793	92869	211	589·79054	04120			
172	450·57343	21128	212	593·44788	68992			
173	454·05396	75699	213	597·10932	56917			
174	457·53951	65520	214	600·77483	75990			
175	461·03005	02866	215	604·44440	36100			

[*Tables of the first* 250 *Bernoulli's Numbers* (*to nine figures*) *and their logarithms* (*to ten figures*)].

SUPPLEMENT. Added *February* 29, 1872.

[READ *March* 11, 1872.]

AFTER the reading of this paper, which originally contained only the logarithms of the first 250 Bernoulli's Numbers, it occurred to me that it would greatly increase the value of the Table if the first nine or ten figures of the numbers themselves were also tabulated.

If only seven figures are required, the number is taken out from the logarithm so easily that it is of no great consequence whether the numbers are or are not tabulated; but if more than seven figures are required, the case is very different. In the first place the operation of taking out the number is far more laborious, and secondly, ten-figure logarithmic tables are very difficult to procure. The only complete table is Vlacq's (often erroneously called Briggs's), published at Gouda, 1628, and at London, with an English introduction, 1631. This was reprinted by Vega in his *Thesaurus Logarithmorum Completus*, Leipsic, 1794, but the form is not quite so convenient as in the original. Both Vlacq and Vega are now very scarce. The former, further, contains over 400 errors which were found by M. Lefort, by comparison with the great French Manuscript Tables, and published by him in tome IV. of the *Annales de l'Observatoire Impérial de Paris* (1858), pp. [148]...[150], and very many of these occur also in Vega. The accompanying table of Bernoulli's Numbers was calculated from B_{19} to B_{250} by taking out the numbers answering to the logarithms by Vlacq's table, the logarithms corresponding to these numbers were then taken out from the same table and compared with the originals, so that the verification was perfect.

The first five figures of the numbers were then read with Lefort's table of errata previously referred to and one error* found thereby was corrected.

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* It corresponded to B_{224}; log 44656 should be 64987 98191 not 64987 48191. Since this paper was read I have formed a list of errata in Vlacq, supplementary to that given by Lefort: see *Monthly Notices of the Royal Astronomical Society*, for May and June, 1872.

As the tenth figure, obtained by ten-figure logarithms, is not to be depended on as accurate, it seemed best to reject it and only tabulate nine figures of the numbers.

The first eighteen Bernoulli's Numbers in the table were formed by division from the exact numbers (as vulgar fractions) given by Ohm in Crelle's *Journal*, Vol. xx. p. 11. It should be mentioned that Ohm has given the first thirty-one numbers, the values of the first twenty-five of which have been verified rigorously, and the rest partially by a calculation made by means of them for the determination of Euler's constant (*Proc. Roy. Soc.* 1871, p. 514). The numerals in square brackets denote the number of decimal places that follow the figures tabulated before the decimal point; for example B_{160} is 161811355 followed by 401 figures before the decimal point is reached, so that the integral portion of B_{160} consists of 410 figures.

If a_n be the characteristic of $\log B_n$, then the quantity in square brackets corresponding to B_n is $a_n - 8$; as there are altogether $a_n + 1$ figures before the decimal point, of which the first nine are tabulated.

It may be remarked that the value of a table often consists as much in its insuring accuracy, as in its saving the user the trouble of calculating any of the results tabulated. Even if the formula from which a table is calculated is very simple and admits of ready computation, it by no means follows that on that account the table is not worth constructing, as although it may not save a great deal of labour, it gives an amount of confidence to the consulter that he might not feel in his own calculations. This remark does not apply in full force to the present case, as the calculation of the results was quite as laborious as in the average of tables, but it affords one of the chief reasons that seemed to render it desirable to supplement the logarithms by the numbers.

J. W. L. GLAISHER.

TABLE OF BERNOULLI'S NUMBERS.

n	B_n	n	B_n	n	B_n
1	·16666 6667	46	91885 5282[60]	91	71287 8213[180]
2	·03333 3333	47	20346 8968[63]	92	60802 9315[183]
3	·02380 9524	48	47003 8340[65]	93	52996 7764[186]
4	·03333 3333	49	11318 0434[68]	94	47194 2592[189]
5	·07575 7576	50	28382 2496[70]	95	42928 4138[192]
6	·25311 3553	51	74064 2490[72]	96	39876 7450[195]
7	1·1666 6667	52	20096 4548[75]	97	37819 7804[198]
8	7·0921 5686	53	56657 1701[77]	98	36614 2337[201]
9	54·971 1779	54	16584 5112[80]	99	36176 0903[204]
10	529·12 4242	55	50368 8599[82]	100	36470 7727[207]
11	6192·1 2319	56	15861 4682[85]	101	37508 7554[210]
12	86580· 2531	57	51756 7436[87]	102	39345 8673[213]
13	14255 17·17	58	17488 9218[90]	103	42088 2112[216]
14	27298 231·1	59	61160 5200[92]	104	45902 2962[219]
15	60158 0874[0]	60	22122 7769[95]	105	51031 7258[222]
16	15116 3158[2]	61	82722 7768[97]	106	57822 7623[225]
17	42961 4643[3]	62	31958 9251[100]	107	66762 4822[228]
18	13711 6552[5]	63	12750 0822[103]	108	78535 3076[231]
19	48833 2319[6]	64	52500 9231[105]	109	94106 8941[234]
20	19296 5793[8]	65	22301 8179[108]	110	11484 9339[238]
21	84169 3048[9]	66	97684 5219[110]	111	14272 9587[241]
22	40338 0719[11]	67	44098 3620[113]	112	18059 5596[244]
23	21150 7486[13]	68	20508 5709[116]	113	23261 5353[247]
24	12086 6265[15]	69	98214 4333[118]	114	30495 7517[250]
25	75008 6675[16]	70	48412 6008[121]	115	40685 8061[253]
26	50387 7810[18]	71	24553 0888[124]	116	55231 0313[256]
27	36528 7765[20]	72	12806 9268[127]	117	76277 2794[259]
28	28498 7693[22]	73	68676 1671[129]	118	10715 5711[263]
29	23865 4275[24]	74	37846 4686[132]	119	15310 2009[266]
30	21399 9493[26]	75	21426 1013[135]	120	22244 8917[269]
31	20500 9757[28]	76	12456 7271[138]	121	32862 6792[272]
32	20938 0059[30]	77	74345 7875[140]	122	49355 9289[275]
33	22752 6965[32]	78	45535 7953[143]	123	75349 5712[278]
34	26257 7103[34]	79	28612 1128[146]	124	11691 4852[282]
35	32125 0821[36]	80	18437 7236[149]	125	18435 2615[285]
36	41598 2782[38]	81	12181 1545[152]	126	29536 8262[288]
37	56920 6955[40]	82	82482 1872[154]	127	48079 3213[291]
38	82183 6294[42]	83	57225 8779[157]	128	79502 1251[294]
39	12502 9043[45]	84	40668 5305[160]	129	13352 7842[298]
40	20015 5832[47]	85	29596 0921[163]	130	22776 4065[301]
41	33674 9829[49]	86	22049 5226[166]	131	39451 8404[304]
42	59470 9705[51]	87	16812 5971[169]	132	69385 2577[307]
43	11011 9103[54]	88	13116 7362[172]	133	12388 9637[311]
44	21355 2595[56]	89	10467 8940[175]	134	22455 4260[314]
45	43328 8970[58]	90	85432 8936[177]	135	41312 1318[317]

The numerals in square brackets denote the number of additional figures before the decimal point, thus B_{25}, to nine figures, is 750086675 followed by 16 ciphers before the decimal point.

n	B_n	n	B_n	n	B_n
136	77135 8135[320]	176	33538 2448[456]	216	13122 1468[600]
137	14615 3607[324]	177	10615 9426[460]	217	62462 9915[603]
138	28099 0461[327]	178	33984 2097[463]	218	30008 1201[607]
139	54809 5712[330]	179	11001 9250[467]	219	14549 0488[611]
140	10845 7328[334]	180	36016 8638[470]	220	71185 5852[614]
141	21769 8078[337]	181	11922 3517[474]	221	35147 3982[618]
142	44319 9879[340]	182	39903 4275[477]	222	17511 3707[622]
143	91506 2566[343]	183	13502 8180[481]	223	88034 9809[625]
144	19158 6735[347]	184	46193 2544[484]	224	44656 1291[629]
145	40672 5630[350]	185	15975 2224[488]	225	22854 9457[633]
146	87542 2379[353]	186	55847 5373[491]	226	11801 4517[637]
147	19101 7369[357]	187	19734 4362[495]	227	61479 4185[640]
148	42250 0132[360]	188	70482 9544[498]	228	32310 6916[644]
149	94719 5935[363]	189	25442 3670[502]	229	17130 4273[648]
150	21521 4997[367]	190	92815 5160[505]	230	91617 6136[651]
151	49554 8577[370]	191	34217 5716[509]	231	49426 7597[655]
152	11562 2594[374]	192	12747 3364[513]	232	26896 8471[659]
153	27334 0660[377]	193	47985 2481[516]	233	14763 1899[663]
154	65468 6814[380]	194	18251 1695[520]	234	81730 3774[666]
155	15885 2491[384]	195	70136 6744[523]	235	45634 6231[670]
156	39043 5480[387]	196	27230 0386[527]	236	25697 9002[674]
157	97199 3869[390]	197	10680 1485[531]	237	14594 1022[678]
158	24507 6362[394]	198	42316 5095[534]	238	83583 0488[681]
159	62578 9210[397]	199	16936 5005[538]	239	48273 0509[685]
160	16181 1355[401]	200	68469 4485[541]	240	28113 9431[689]
161	42365 2880[404]	201	27958 0913[545]	241	16510 2686[693]
162	11230 4707[408]	202	11530 1297[549]	242	97765 7858[696]
163	30139 7179[411]	203	48023 6885[552]	243	58372 0796[700]
164	81884 3757[414]	204	20199 9526[556]	244	35139 3896[704]
165	22519 1059[418]	205	85802 0724[559]	245	21327 4737[708]
166	62684 1129[421]	206	36802 4794[563]	246	13050 4736[712]
167	17659 9085[425]	207	15939 2446[567]	247	80508 2534[715]
168	50351 5444[428]	208	69702 6718[570]	248	50068 8416[719]
169	14527 8103[432]	209	30775 2809[574]	249	31390 1607[723]
170	42414 9089[435]	210	13718 4676[578]	250	19838 2954[727]
171	12529 6600[439]	211	61736 2736[581]		
172	37448 3005[442]	212	28047 0313[585]		
173	11323 1581[446]	213	12862 5090[589]		
174	34635 1085[449]	214	59543 9442[592]		
175	10716 4338[453]	215	27822 9779[596]		

The numerals in square brackets denote the number of additional figures before the decimal point, thus B_{175}, to nine figures, is 107164338 followed by 453 ciphers before the decimal point.

VII. *Further Observations on the state of an Eye affected with a peculiar malformation.* By GEORGE BIDDELL AIRY, M.A., LL.D., D.C.L., *Honorary Fellow of Trinity College; formerly Lucasian Professor, late Plumian Professor, in the University of Cambridge; Astronomer Royal.*

[Read *Feb.* 12, 1872.]

FOR the method which I have employed now for the fourth time in examining the state of the eye, I refer generally to my communication to the Cambridge Philosophical Society dated 1825, February 5. A very minute hole is made, by the point of a fine needle, in a blackened card, which is so pierced in another part that it can be slid upon a graduated scale, of which one end abuts against the orbital bone of the eye; the graduated scale thus giving a very approximate measure of the distance of the minute hole from the cornea of the eye in every experiment. With a properly-formed eye, and with the card raised between the eye and the bright sky, the minute hole is seen, at the distance of distinct vision, as a brilliant point. With the anomalous eye, the hole is seen at one distance as a nearly horizontal line pretty sharply defined, and at a greater distance as a line at right angles to the former line (and thus approaching to a vertical direction) pretty sharply defined. At no distance is it seen as a point. To this form of refraction Dr Whewell gave the name of *astigmatism*, which it has since retained.

Without further explanation, I will give the results of a late examination, in combination with those of previous examinations, in the same form as in my paper of 1866, November 19 (*Proceedings of the Camb. Phil. Soc.* Pt. IV.).

I. Distance from the cornea of the left eye at which the luminous point presents the appearance of a nearly horizontal line.

In 1825, 3·5 inches;	Reciprocal	·286	
			Difference − ·073.
In 1846, 4·7 ...... ;	......	·213	
			...... − ·028.
In 1866, 5·4 ...... ;	......	·185	
			...... − ·006.
In 1871, 5·6 ...... ;	......	·179	

II. Distance from the cornea of the left eye at which the luminous point presents the appearance of a nearly vertical line.

In 1825, 6·0 inches;	Reciprocal =	·166	
			Difference − ·054.
In 1846, 8·9 ...... ;	......	·112	
			...... − ·018.
In 1866, 10·6 ...... ;	......	·094	
			...... + ·006.
In 1871, 10·0 ...... ;	......	·100	

III. Measure of the astigmatic power of the left eye at different epochs; estimated in each case by the differences of the reciprocals for the same date in the two preceding tables.

In 1825, astigmatism =	·120	
		Difference = ·019.
In 1846, ...	·101	
		...... ·010.
In 1866, ...	·091	
		...... ·012.
In 1871, ...	·079	

IV. Distance from the cornea of the right eye at which the luminous point is seen distinctly.

In 1846, 4·7 inches; Reciprocal =	·213	
		Difference = − ·031.
In 1866, 5·5 ...... ;	·182	
		...... + ·003.
In 1871, 5·4 ...... ;	·185	

The changes in the last period of five years are small. The element which appears to have undergone the greatest change is the astigmatism; this result of observation is opposed to that of earlier years.

I am inclined to think that the self-adjusting power of the eyes for different distances is sensibly less than it was in 1866; and that the stigmatic refraction of the right eye is less perfect than it was formerly.

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,
1871, *December* 27.

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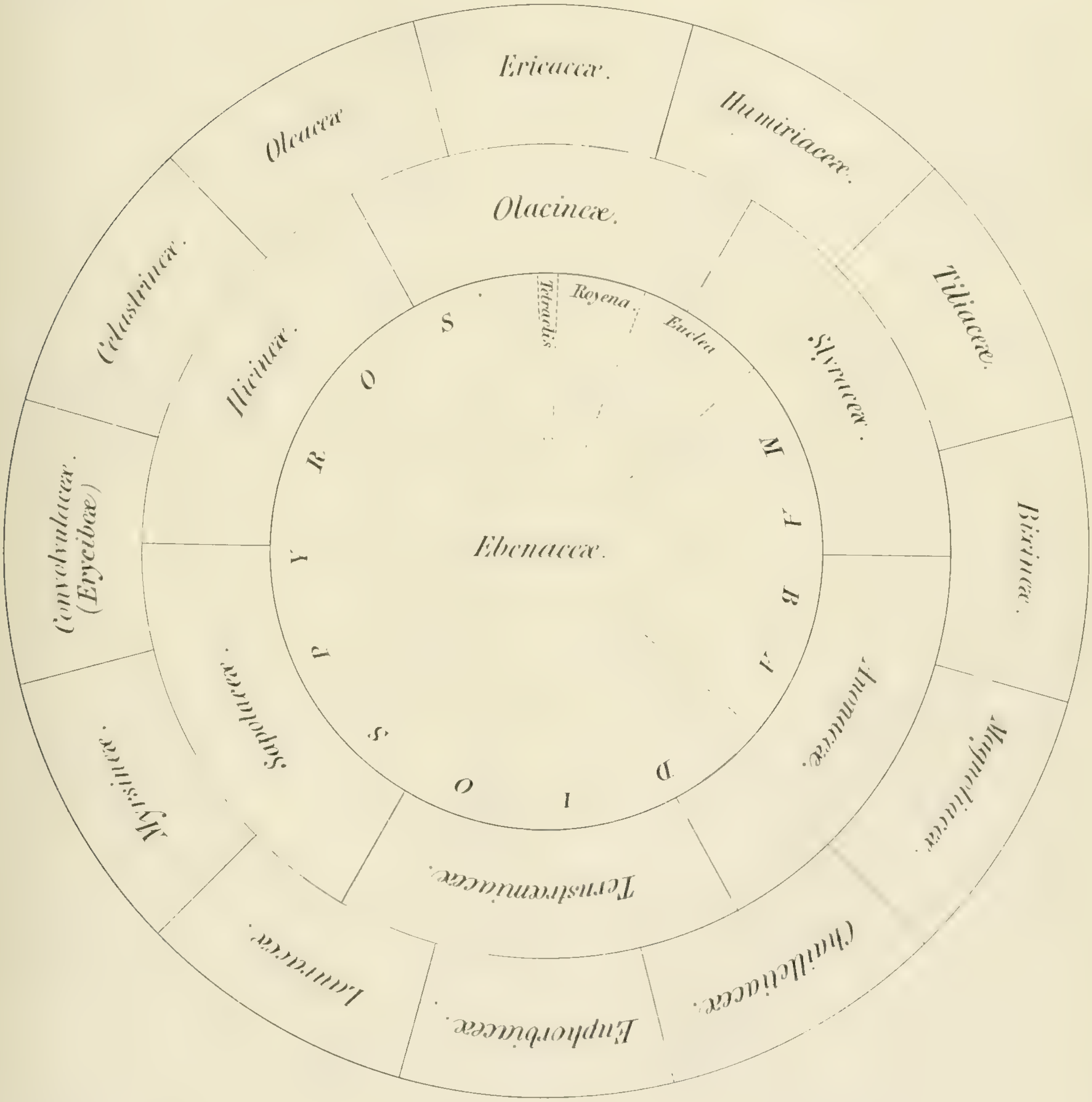
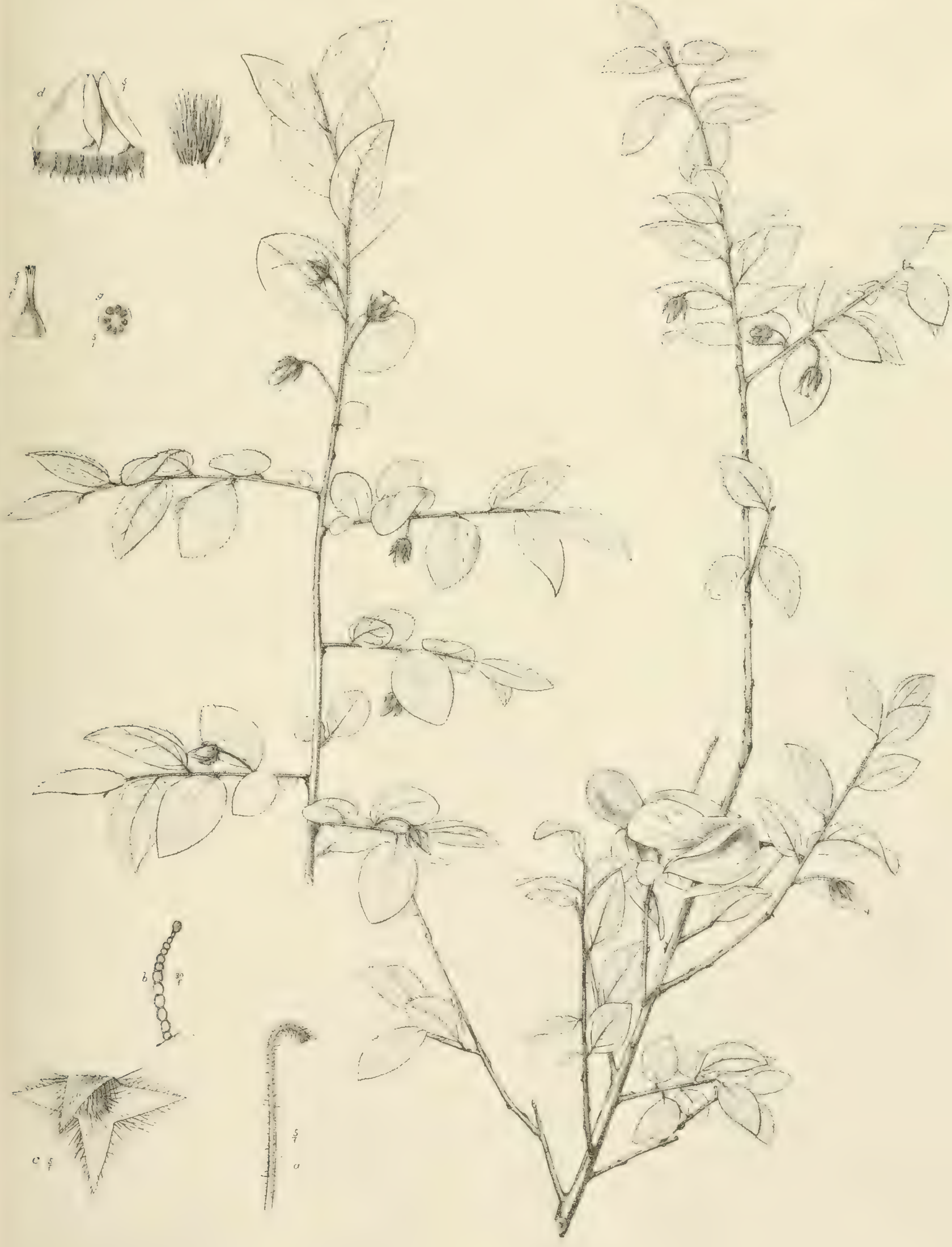
PLAN EXHIBITING AFFINITIES OF EBENACEÆ.





A.T Hallick del et lith.


Fig 1
Fig 2
a
b
d
e
f


b
a
c


Fig 1.
a
b
c
Fig 2


a
6/1
d
3/1
b
6/1
c
f
e

A. T. Hollick del. et lith.

Mintern Bros. imp.

Mintern Bros imp

I. *On the geometrical representation of Cauchy's theorems of Root-limitation.* By Professor CAYLEY.

[Read *Feb.* 16, 1874.]

THERE is contained in Cauchy's Memoir "Calcul des Indices des Fonctions," *Journ. de l'Ecole Polytech.* t. XV. (1837) a general theorem, which, though including a well-known theorem in regard to the imaginary roots of a numerical equation, seems itself to have been almost lost sight of. In the general theorem (say Cauchy's two-curve theorem) we have in a plane two curves $P=0$, $Q=0$, and the real intersections of these two curves, or say the "roots," are divided into two sets according as the Jacobian

$$d_xP \,.\, d_yQ - d_xQ \,.\, d_yP$$

is positive or negative, say these are the Jacobian-positive and the Jacobian-negative roots: and the question is to determine for the roots within a given contour or circuit, the difference of the numbers of the roots belonging to the two sets respectively.

In the particular theorem (say Cauchy's rhizic theorem) P and Q are the real part and the coefficient of i in the imaginary part of a function of $x+iy$ with, in general, imaginary coefficients (or, what is the same thing, we have $P+iQ=f(x+iy)+i\phi(x+iy)$, where f, ϕ are real functions of $x+iy$): the roots of necessity are of the same set: and the question is to determine the number of roots within a given circuit.

In each case the required number is theoretically given by the same rule, viz., considering the fraction $\frac{P}{Q}$, it is the excess of the number of times that the fraction changes from + to − over the number of times that it changes from − to +, as the point (x, y) travels round the circuit, attending only to the changes which take place on a passage through a point for which P is $=0$.

In the case where the circuit is a polygon, and most easily when it is a rectangle the sides of which are parallel to the two axes respectively, the excess in question can be actually determined by means of an application of Sturm's theorem successively to each side of the polygon, or rectangle.

In the present memoir I reproduce the whole theory, presenting it under a completely geometrical form, viz. I establish between the two sets of roots the distinction of *right-* and *left-handed:* and (availing myself of a notion due to Prof. Sylvester*) I give a geometrical form to the theoretic rule, making it depend on the "intercalation" of the intersections of the two curves with the circuit: I also complete the Sturmian process in regard to the sides of the rectangle: the memoir contains further researches in regard to the curves in the case of the particular theorem, or say as to the rhizic curves $P=0$, $Q=0$.

The General Theory. Articles Nos. 1 to 19.

1. Consider in a plane two curves $P=0$, $Q=0$ (P and Q each a rational and integral function of x, y), which to fix the ideas I call the *red* curve and the *blue* curve respectively†: the curve $P=0$ divides the plane into two sets of regions, say a positive set for each of which P is positive, and a negative set for each of which P is negative: it is of course immaterial which set is positive and which negative, since writing $-P$ for P the two sets would be interchanged: but taking P to be given, the two sets are distinguished as above. And we may imagine the negative regions to be coloured red, the positive ones being left uncoloured, or say they are white. Similarly the curve $Q=0$ divides the plane into two sets of regions, the negative regions being coloured blue, and the positive ones being left uncoloured, or say they are white. Taking account of the twofold division, and considering the coincidence of red and blue as producing black, there will be four sets of regions, which for convenience may be spoken of as *sable, gules, argent, azure:* viz. in the figures we have

$\overbrace{P \quad Q}$	
$-\ \ -$	sable, shown by cross lines,
$-\ \ +$	gules, „ „ vertical lines,
$+\ \ +$	argent, left white,
$+\ \ -$	azure, shown by horizontal lines,

sable and argent ($-\,-$ and $+\,+$) being thus positive colours, and gules and azure ($-\,+$ and $+\,-$) negative colours. See figures towards end of Memoir.

2. Consider any point of intersection of the two curves. There will be about this point four regions, sable and argent being opposite to each other, as also gules and azure: whence selecting an order

sable, gules, argent, azure;

if to have the colours in this order we have to go about the point, or root, right-handedly, the root is right-handed: but if left-handedly, then the root is left-handed: or, what is more

* See his memoir, *A theory of the Syzygetic relations &c. Phil. Trans.* 1853. The Sturmian process is by Sturm and Cauchy applied to two independent functions ϕx, fx of a variable x; but the notion of an intercalation as applied to the order of succession of the roots of the equations $\phi(x)=0$, $f(x)=0$ is due to Sylvester, and it was he who showed that what the Sturmian process determined was in fact the intercalation of these roots: but, not being concerned with circuits, he was not led to consider the intercalation of a circuit.

† It is assumed throughout that the two curves have no points (or at least no real points) of multiple intersection; i.e. they nowhere touch each other, and neither curve passes through a multiple point of the other curve.

convenient, going always right-handedly, then, if the order of the colours is

sable, gules, argent, azure,

the root is right-handed: but if the order is

sable, azure, argent, gules,

the root is left-handed.

3. The distinction of right- and left-handed corresponds to the sign of the Jacobian

$$\frac{d(P,\ Q)}{d(x,\ y)}\ (= d_xP\,.\,d_yQ - d_xQ\,.\,d_yP),$$

and we may (reversing if necessary the original sign of one of the functions) assume that for a right-handed root the Jacobian is positive, for a left-handed one, negative.

4. I consider a trajectory which may be either an unclosed curve not cutting itself, or else a circuit, viz. this is a closed curve not cutting itself. A circuit is considered as described right-handedly: an unclosed trajectory is considered as described according to a currency always determinate *pro hâc vice:* viz. one extremity is selected as the beginning and the other as the end of the trajectory: but the currency may if necessary or convenient be reversed: thus if an unclosed trajectory forms part of a circuit the currency is thereby determined: but the same unclosed trajectory may form part of two opposite circuits, and as such may have to be taken with opposite currencies. It is assumed that a trajectory does not pass through any intersection of the P and Q curves.

5. A trajectory has its P- and Q-sequence, viz. considering in order its intersections with the two curves, we write down a P for each intersection with the red curve and a Q for each intersection with the blue curve, thus obtaining an intermingled series of P's and Q's, which is the sequence in question. In the case of a circuit, the sequence is considered as a circuit, viz. the first and last terms are considered as contiguous, and it is immaterial at what point the sequence commences. The sequence will of course vanish if the trajectory does not meet either of the curves.

6. A P- and Q- sequence gives rise to an "intercalation," viz. if in the sequence there occur together any even number of the same letter these are omitted (whence also any odd number of the same letter is reduced to the letter taken once): and if by reason of an omission there again occur an even number of the same letter these are omitted: and so on. The intercalation contains therefore only the letters P and Q alternately: viz. in the case of an unclosed trajectory the intercalation may contain an even number of letters, beginning with the one and ending with the other letter, and so containing the same number of each letter—or it may contain an odd number of letters, beginning and ending with the same letter, and so containing one more of this than of the other letter; say the intercalation is PQ or QP, or else PQP or QPQ. The intercalation may vanish altogether, thus if the sequence were $QPPQ$ this would be the case.

7. In the case of a circuit the intercalation cannot begin and end with the same letter, for these, as contiguous letters, would be omitted; and since any letter thereof may

be regarded as the commencement it is PQ or QP indifferently. A little consideration will show that the whole number of letters must be evenly even, or, what is the same thing, the number of each letter must be even. Thus imagine the circuit beginning in sable, and let the intercalation begin with PQ; viz. P we pass from sable to azure, and Q we pass from azure to argent: in order to get back into sable we must either return the same way (Q argent to azure, P azure to sable), but then the sequence is $PQQP$, and the intercalation vanishes: here the number of letters is 0, an evenly even number: or else we must complete the cycle of colours P argent to gules, Q gules to sable: and the sequence and therefore also the intercalation then is $PQPQ$, where the number of letters is 4, an evenly even number.

8. In the case of any trajectory whatever, the half number of letters in the intercalation is termed the "index," viz. this is either an integer or an integer $+ \frac{1}{2}$. But in the case of a circuit the index is an even integer, and the half-index is therefore an integer. The index may of course be $= 0$.

9. But we require a further distinction: instead of a P- and Q- sequence we have to consider a $\pm P$- and Q- sequence. To explain this observe that a passage over the red curve may be from a negative to a positive colour (azure to sable or gules to argent), this is $+P$, or from a positive to a negative colour (sable to azure or argent to gules), this is $-P$. And so the passage over the blue curve may be from a negative to a positive colour (gules to sable or azure to argent), this is $+Q$, or else from a positive to a negative colour (sable to gules or argent to azure), this is $-Q$. The sequence will contain the P and Q intermingled in any manner, but the signs will always be $+ -$ *alternately;* for $+(P$ or $Q)$, denoting the passage into a positive colour, must always be immediately succeeded by $-(P$ or $Q)$, denoting the passage into a negative colour. Whence, knowing the sequence independently of the signs, we have only to prefix to the first letter the sign $+$ or $-$ as the case may be, and the sequence is then completely determined.

10. Passing to a $\pm$ intercalation, observe that in omitting any even number of P's or Q's, the omitted signs are always $+ - + -$ &c. or else $- + - +$ &c., viz. the omitted signs begin with one sign and end with the opposite sign. Hence the signs being in the first instance alternate, they will after any omission remain alternate: and the letters being also alternate, the intercalation can contain only $+P$ and $-Q$ or else $-P$ and $+Q$. Hence in the case of a circuit the intercalation is either $(+P-Q)$, say this is a *positive* circuit, or else $(-P+Q)$, say this is a *negative* circuit. There is of course the *neutral* circuit $(PQ)_0$ for which the intercalation vanishes.

11. Consider a circuit not containing within it any root; as a simple example let the circuit lie wholly in one colour, or wholly in two adjacent colours, say sable and gules: in the former case the sequence, and therefore also the intercalation, vanishes: in the latter case the sequence is $+Q-Q$, and therefore the intercalation vanishes: viz. in either case the intercalation is $(PQ)_0$.

12. Consider next a circuit containing within it one right-handed root; for instance let the circuit lie wholly in the four regions adjacent to this root, cutting the two curves each twice; the sequence and therefore also the intercalation is $+P-Q+P-Q$; viz. this is a positive circuit $(+P-Q)_1$, where the subscript number is the half-index, or half of the number of P's or of Q's. Similarly if a circuit contains within it one left-handed root, for instance if the circuit lies wholly in the four regions adjacent to this root, cutting the two curves each twice, the sequence and therefore also the intercalation is $-P+Q-P+Q$, viz. this is a negative circuit $(-P+Q)_1$: and the consideration of a few more particular cases leads easily to the general and fundamental theorem:

13. *A circuit is positive* $(+\mathrm{P}-\mathrm{Q})_\delta$ *or negative* $(-\mathrm{P}+\mathrm{Q})_\delta$ *according as it contains within it more right-handed or more left-handed roots; and in either case the half-index* δ *is equal to the excess of the number of one over that of the other set of roots. If the circuit is neutral* $(\mathrm{PQ})_0$, *then there are within it as many left-handed as right-handed roots.*

14. The proof depends on a composition of circuits, but for this some preliminary considerations are necessary.

Imagine two unclosed trajectories forming a circuit, and write down in order the intercalation of each. The whole number of letters must be even: viz. the numbers for the two intercalations respectively must be both even or both odd. I say that if the terminal letter of the first intercalation and the initial letter of the second intercalation are different, then also the initial letter of the first intercalation and the terminal letter of the second intercalation will be different: if the same, then the same. In fact the intercalations may be each PQ or each QP, or one PQ and the other QP: or each PQP, or each QPQ, or one PQP and the other QPQ. Supposing the letters in question are different, then the intercalations may be termed similar; but if the same, then the intercalations may be termed contrary.

15. In the first case, that is when the intercalations are similar, the two together form the intercalation of the circuit; the sum of their numbers of letters (that is twice the sum of their indices) will be evenly even, and the half of this, or sum of the indices, will be the index of the circuit; each intercalation will be $(+P-Q)$ or else each will be $(-P+Q)$; and the circuit will be $(+P-Q)$ or $(-P+Q)$ accordingly.

In the second case, that is when the intercalations are contrary, they counteract each other in forming the intercalation of the circuit: it is the *difference* of the numbers of letters, or twice the difference of the indices, which is evenly even, and the half of this, or difference of the indices, which is the index of the circuit: one intercalation is $(+P-Q)$, and the other is $(-P+Q)$: and the circuit will agree with that which has the larger index.

In particular if the circuit consist of a single unclosed trajectory, taken forwards and backwards; then the trajectory taken one way is $(+P-Q)$, taken the other way it is $(-P+Q)$; the number of terms is of course equal, and the circuit is $(PQ)_0$.

16. Consider now two circuits $ABCA$ and $ACDA$, having a common portion CA, or, more accurately, the common portions AC and CA: write down in order the intercalations of

$$ABC,\quad CA,\quad AC,\quad CDA:$$

the two mean terms destroy each other, and we can hence deduce the intercalation of the entire circuit $ABCDA$.

Suppose *first*, that ABC and CDA are similar; then if CA is similar to ABC it is also similar to CDA, that is AC is contrary to CDA: and so if CA is contrary to ABC, then AC is similar to CDA.

To fix the ideas suppose CA similar to ABC, but AC contrary to CDA, then $ABCA$ is similar to CA; but $ACDA$ will be similar or contrary to AC, i.e. contrary or similar to CA, that is to $ABCA$, according as index of $AC >$ or $<$ index of CDA.

Suppose Ind. $AC <$ Ind. CDA, then $ACDA$ is similar to $ABCA$.

$$\begin{aligned}
\text{Ind. } ABCDA &= \text{Ind. } ABC + \text{Ind. } CDA,\\
\text{Ind. } ABCA &= \text{Ind. } ABC + \text{Ind. } AC,\\
\text{Ind. } ACDA &= \text{Ind. } CDA - \text{Ind. } AC,
\end{aligned}$$

and thence
$$\text{Ind. } ABCDA = \text{Ind. } ABCA + \text{Ind. } ACDA,$$

the whole circuit being in this case similar to each of the component ones.

But if Ind. $AC >$ Ind. CDA, then $ACDA$ is contrary to $ABCA$.

$$\begin{aligned}
\text{Ind. } ABCDA &= \text{Ind. } ABC + \text{Ind. } CDA,\\
\text{Ind. } ABCA &= \text{Ind. } ABC + \text{Ind. } CA,\\
\text{Ind. } ACDA &= -\text{Ind. } CDA + \text{Ind. } AC,
\end{aligned}$$

and thence
$$\text{Ind. } ABCDA = \text{Ind. } ABCA - \text{Ind. } ACDA,$$

and the investigation is like hereto if CA is contrary to ABC but AC similar to CDA.

17. *Secondly*, if ABC and CDA are contrary, then if CA is similar to ABC it is contrary to CDA, that is AC is similar to CDA; and so if CA is contrary to ABC it is similar to CDA, that is AC is contrary to CDA.

Suppose CA similar to ABC, and AC similar to CDA; then $ABCA$ is also similar to ABC, and $ACDA$ similar to CDA; viz. ABC, CA and $ABCA$ are similar to each other, and contrary to AC, CDA, $ACDA$ which are also similar to each other.

$$\begin{aligned}
\text{Ind. } ABCDA &= \text{Ind. } ABC \sim \text{Ind. } CDA,\\
\text{Ind. } ABCA &= \text{Ind. } ABC + \text{Ind. } CA,\\
\text{Ind. } ACDA &= \text{Ind. } CDA + \text{Ind. } AC,
\end{aligned}$$

and thence
$$\text{Ind. } ABCDA = \text{Ind. } ABCA \sim \text{Ind. } ACDA,$$

and the investigation is like hereto if CA is contrary to ABC and AC contrary to CDA.

18. It thus appears that in every case

$$\text{Ind. } ABCDA = \text{Ind. } ABCA + \text{Ind. } ACDA,$$
$$\text{or} = \text{Ind. } ABCA \sim \text{Ind. } ACDA,$$

according as the component circuits are similar or contrary, and in the latter case the entire circuit is similar to that which has the largest index.

Moreover, any circuit whatever can be broken up into two smaller circuits, and these again continually into smaller circuits until we arrive at the before-mentioned elementary circuits, and the theorem as to the number of roots within a circuit is true as regards these elementary circuits; wherefore the theorem is true as regards any circuit whatever.

19. In the case where a trajectory is a finite right line, y is a given linear function of x, or the coordinates x, y can if we please be expressed as linear functions of a parameter u, so that as the describing point passes along the line, u varies between given limits, say from $u=0$ to $u=1$. The functions P, Q thus become given rational and integral functions of a single variable u (or it may be x or y), and the question of the P- and Q- sequence and intercalation relates merely to the order of succession of the roots of the equations $P=0$, $Q=0$, where P and Q denote functions of a single variable as above. To fix the ideas let the trajectory be a line parallel to the axis of x; and in this case taking x as the parameter, and supposing that y_0 is the given value of y, P and Q are the functions of x obtained by writing y_0 for y in the original expressions of these functions. Of course the theory will be precisely the same for a line parallel to the axis of y: and by combining two lines parallel to each axis we have the case of a rectangular circuit. We require, for each side of the rectangle considered according to its proper currency, the intercalation PQ, QP, PQP or QPQ as the case may be, and also the sign $+$ or $-$ of the initial letter of the first intercalation; for then writing down the intercalations in order, with the signs for the several letters, $+$ and $-$ alternately (the first sign being $+$ or $-$ as the case may be), we have or deduce the intercalation of the circuit, and thus obtain the value of the difference of the numbers of the included right- and left-handed roots. We thus see how the whole theory depends on the case where the trajectory is a right line.

Intercalation-theory for a right line. Articles Nos. 20 to 31.

20. Considering then the case where the trajectory is a line parallel to the axis of x, P and Q will denote given rational functions of x; the curves $P=0$, $Q=0$ being of course each of them a set of right lines parallel to the axis of y: the regions will be bands each of them included between two such lines; and colouring them as explained in the general case, the colours will be as before, sable, gules, argent, azure, each region having in the neighbourhood of the trajectory (what we are alone concerned with) the same colour that it had in the original case where P and Q were functions of (x, y). We may regard the trajectory as described according to the currency $x=-\infty$ to $x=+\infty$: we have in regard to the trajectory a P- and Q- sequence, and intercalation, a $\pm$ P- and Q- sequence, &c., as in

the original case. The intercalation may be as before PQ, QP, PQP or QPQ, and in each of these cases it may be positive, that is $(+P-Q)$, or else negative, that is $(-P+Q)$.

21. The question of sign may in the present case be disposed of without difficulty. For the initial point of the trajectory, we know the signs of P, Q, that is the colour of the region: suppose for example that we have $P=-$, $Q=+$, or that the region is gules: then if the intercalation begin with P, this means that we either first pass a red line, or before doing so we pass an even number of blue lines: but in the last case the colours are sable gules sable gules,... always ending in gules; and the passage over the red line is gules to argent, viz. this is $+P$; and so in general the initial P or Q of the intercalation has the sign opposite to that of the P or Q belonging to the commencement of the trajectory.

22. For the solution of the problem we connect with P, Q a set of functions R, S, T, &c.: the intercalation is in fact given by means of the gain or loss of changes of sign in these functions on substituting therein the initial and final values of the variable x. It is convenient to consider the functions as arranged in a column

$$\begin{matrix} P \\ Q \\ R \\ S \\ \vdots \end{matrix}$$

say this is the column $PQRS...$, and to connect therewith a signaletic bicolumn: viz. the left-hand column is here the series of signs of these functions for the initial value of x, and the right-hand column is the series of signs for the terminal value of x: the bicolumn thus consisting of as many rows each of two signs, as there are functions. But such a bicolumn may be considered apart from any series of functions, as a set of rows each of two signs taken at pleasure.

We say that the "gain" of a bicolumn is

$$= -\text{(No. of changes of sign in left-hand column)} + \text{(No. in right-hand ditto)},$$

the gain being of course positive or negative; and a negative gain being regarded as a loss. Also if a positive gain be converted into an equal negative gain or *vice versâ*, we may speak of the gain as *reversed*.

23. A bicolumn may be divided in any manner into parts, taking always the last row of any part as being also the first row of the next succeeding part. This being so, the gain of the whole bicolumn is equal to the sum of the gains of its parts.

In a bicolumn of two rows, if we reverse either row (that is write therein $-$ for $+$ and $+$ for $-$), we reverse the gain: and hence dividing a bicolumn into bicolumns each of two rows, viz. first and second rows, second and third rows, and so on, it at once appears that if we reverse alternate rows (viz. either the first, third, fifth, &c., rows, or the second, fourth, sixth, &c., rows) we reverse the gain. It of course follows that reversing all the rows, we leave the gain unaltered.

24. If to any bicolumn we prefix at the top thereof the second row reversed, we either leave the gain unaltered or we alter it by ± 1. In fact, as regards either column, if this originally begin with a change, the process introduces no change therein; but if it begins with a continuation, then the process introduces a change. Hence if the columns begin each with a change or each with a continuation, the gain is unaltered: but if one begins with a change, and the other with a continuation, then the gain is altered by ± 1; viz. the left-hand column beginning with a continuation the gain is altered by -1, and the right-hand column beginning with a continuation the gain is altered by $+1$.

The column $PQRST\ldots$ is taken to satisfy the following conditions: two consecutive terms never vanish together (that is, for the same value of the variable): if for a given value of the variable, any term vanishes, the preceding and succeeding terms have then opposite signs; the last term, say V, is of constant sign.

25. Considering P, Q as given functions without a common measure, such a column of functions is obtained by the well-known process of seeking for the greatest common measure, reversing at each step the sign of the remainder: viz. we thus derive a set of functions R, S, $T\ldots$ where

$$\begin{aligned} P &= \lambda Q - R, \\ Q &= \mu R - S, \\ R &= \nu S - T, \\ S &= \rho T - U, \\ &\vdots \end{aligned}$$

the degrees of the successive functions R, S, T, ..., being successively less and less, so that the last of them, say V, is an absolute constant: or we may stop the process as soon as we arrive at a function V, the sign of which remains unaltered for all values between the initial and final values of the variable. It may be observed that the process may be regarded as applicable in the case where the degree of Q exceeds that of P: viz. we then have $\lambda = 0$, $R = -P$, and the column begins $(P, Q, -P, S, \ldots)$, the subsequent terms being, except as to sign, the same as if P, Q had been interchanged.

Reversing the sign of P or Q, we reverse in the bicolumn a set of alternate rows, and thus reverse the gain: and reversing both signs we reverse all the rows, and leave the gain unaltered—of course the intercalation (considered irrespectively of sign) is in each case unaltered. It is convenient to take the signs in such manner that for the initial value of x, the signs of P, Q shall be each positive: or, what is the same thing, taking P, Q with their proper signs, we may in the bicolumn, by reversing if necessary each or either set of alternate rows, make the left-hand column to begin with the signs $++$.

26. The complete rule now is—for a given trajectory form the bicolumn for $PQRS\ldots$, and if necessary, by reversing each or either set of alternate rows, make the left-hand column to begin with $++$: then if there is a gain the intercalation begins with P, if a loss with Q, the gain or loss showing the number of P's. To find the number of Q's prefix at the top of the bicolumn the second row reversed—then the gain or loss (equal to or differing by unity from the original value) shows the number of Q's. It may happen that for P the gain is $=0$; then for Q the gain is 0 or ± 1, and the intercalation vanishes or is Q.

27. I give some simple examples.

	0	2	4
$P=x-1$	$-$	$+$	$+$
$Q=x-3$	$-$	$-$	$+$
$R=\quad -1$	$-$	$-$	$-$

0 P 2 Q 4

	0	2	4
$P=x-3$	$-$	$-$	$+$
$Q=x-1$	$-$	$+$	$+$
$R=\quad +1$	$+$	$+$	$+$

0 Q 2 P 4

In the left-hand example taking the intervals to be successively $0-2$, $0-4$, $2-4$, the bicolumns modified as above are

$0-2$		$0-4$		$2-4$	
$-$	$-$	$-$	$+$	$-$	$+$
$+$	$-$	$+$	$-$	$+$	$+$
$+$	$+$	$+$	$-$	$+$	$-$
$+$	$+$	$+$	$+$	$-$	$-$

viz. Interval $0-2$;	for P gain $=1$,	P first;	for Q gain	$=0$;	Intercalation is	P;
„ $0-4$	„ „ $=1$,		„ „	$=1$;	„ „	PQ;
„ $2-4$	„ „ $=0$	...	„ loss	$=1$;	„ „	Q.

And similarly in the right-hand example we have

$0-2$		$0-4$		$2-4$	
$-$	$+$	$-$	$+$	$-$	$-$
$+$	$+$	$+$	$-$	$+$	$-$
$+$	$-$	$+$	$-$	$+$	$+$
$-$	$-$	$-$	$-$	$-$	$-$

Interval $0-2$	for P gain $=\ 0$,	......	for Q gain	$=-1$;	Intercalation is	Q;
„ $0-4$	„ „ $=-1$,	Q first,	„ „	$=-1$;	„ „	QP;
„ $2-4$	„ „ $=+1$,	P first,	„ „	$=\ 0$;	„ „	P.

28. Or to take a slightly more complicated example,

	1	3	$5\pm\epsilon$	7	9
$P=x^2-\ 8x+12$	$+$	$-$	$-$	$+$	$+$
$Q=x^2-12x+32$	$+$	$+$	$-$	$-$	$+$
$R=-\quad x+\ 5$	$+$	$+$	$\mp$	$-$	$-$
$S=\qquad +\ 1$	$+$	$+$	$+$	$+$	$+$

P Q P Q

0 1 2 3 4 5 6 7 8 9

And hence for the several intervals,

	$1-3$	$1-5$	$1-7$	$1-9$	$3-5$	$3-7$	$3-9$	$5-7$	$5-9$	$7-9$
	$-\ -$	$-\ +$	$-\ +$	$-\ -$	$-\ +$	$-\ +$	$-\ -$	$-\ -$	$-\ +$	$-\ +$
	$+\ -$	$+\ -$	$+\ +$	$+\ +$	$+\ +$	$+\ -$	$+\ -$	$+\ -$	$+\ -$	$+\ +$
	$+\ +$	$+\ -$	$+\ -$	$+\ +$	$+\ -$	$+\ -$	$+\ +$	$+\ +$	$+\ -$	$+\ -$
	$+\ +$	$+\ \mp$	$+\ -$	$+\ -$	$-\ \pm$	$-\ +$	$-\ +$	$\pm\ +$	$\pm\ +$	$-\ -$
	$+\ +$	$+\ +$	$+\ +$	$+\ +$	$+\ +$	$+\ +$	$+\ +$	$-\ -$	$-\ -$	$-\ -$
Showing	P	PQ	PQP	$PQPQ$	Q	QP	QPQ	P	PQ	Q

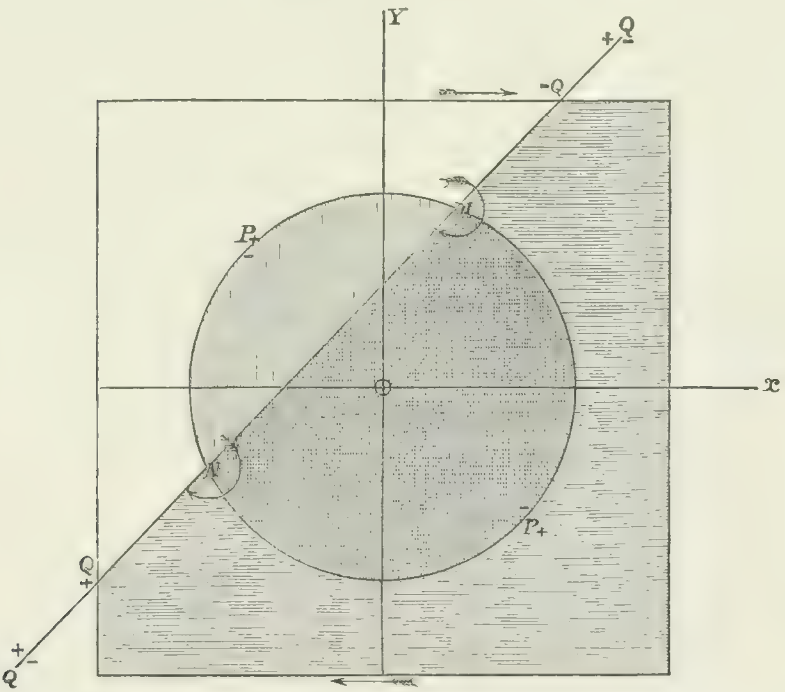
For instance

Interval $1-9$ for P gain $=2$, P first, for Q gain $=2$: Intercalation is $PQPQ$.

It may be added that P being $+$ for $x=1$, the $\pm$ intercalation is $+PQPQ$.

29. As an example of circuits take the following: curves are $P=0$, $Q=0$, where

$$P=x^2+y^2-4,$$
$$Q=y-x-1\,;$$



viz. $P=0$ (see figure) is a circle, centre the origin, radius $=2$: the inside here of $(P=-)$ being coloured red: and $Q=0$ is a right line cutting the axes of x, y at the points $(-1, 0)$ and $(0, 1)$ respectively, or say running N.E. and S.W., the lower region $(Q=-)$ being coloured blue: the square is an arbitrary circuit ($x=\pm 3$, $y=\pm 3$) surrounding the circle, and the regions within the square are coloured by what precedes sable, gules, azure, argent, as shown in the figure: the line and circle intersect in two points M, N, and going right-handedly round these respectively, for M the order is sable, gules, argent, azure, viz. M is a right-handed root; while for N the order is sable, azure, argent, gules, viz. N is a left-handed root: the two points are accordingly in the figure denoted by $+M$ and $-N$ respectively.

30. Now considering successively the four smaller squares of the figure, say these are the squares N.E., S.E., S.W., N.W.: and going right-handedly round each of these:

In the square N.E., the sequence and therefore also the intercalation is $+P-Q+P-Q$, viz. this is an intercalation $(+P-Q)$, showing an excess 1 of right-handed roots, and of course consisting with the single right-handed root M.

In the square S.E., the sequence is $-P+P$, viz. this is an intercalation $(PQ)_0$, showing an equality of right and left-handed roots, and consisting with no root.

In the square S.W., the sequence and therefore also the intercalation is $-P+Q-P+Q$: viz. this is an intercalation $(-P+Q)_1$, showing an excess 1 of left-handed roots, and consisting with the single left-handed root N.

And in the square N.W., the sequence is $-Q+P-P+Q$, viz. this is an intercalation $(PQ)_0$, showing an equality of right and left-handed roots, and consisting with no root.

Again take the whole large square: the sequence is $-Q+Q$: viz. the intercalation is $(PQ)_0$, showing an equality of right and left-handed roots, and consisting with there being one of each.

So taking the squares N.E. and N.W. conjointly, the sequence and therefore also the intercalation is $-Q+P-Q+P$, viz. this is an intercalation $(+P-Q)_1$, as for the single square N.E.

31. As regards the analytical determination it will be sufficient to consider a single square, say N.E.: going round right-handedly, the trajectories will be

(1) $x=0$, $y=0$ to 3.

(2) $y=3$, $x=0$ to 3.

(3) $x=3$, $y=3$ to 0; or if $y'=-y$, then $y'=-3$ to 0.

(4) $y=0$, $x=3$ to 0; or if $x'=-x$, then $x'=-3$ to 0.

And we thus have

$$(1)\quad \begin{array}{r|c|c|} & 0 & 3 \\ P= y^2-4 & - & + \\ Q= y-1 & - & + \\ R= -1 & + & - \end{array}\ \text{, that is }\ \begin{array}{|c|c|} 0 & 3 \\ \hline - & + \\ \hline + & - \\ + & - \\ - & - \end{array}$$

, for P gain $=-1$, Q begins, $1P$;
„ Q „ $=-1$, „ $1Q$;
wherefore intercalation is QP,
or since at origin $P=-$, $R=-$,
or region is sable, it is $-Q+P$.

$$(2)\quad \begin{array}{r|c|c|} & 0 & 3 \\ P= x^2+5 & + & + \\ Q= -x+2 & + & - \\ R= -1 & - & - \end{array}\ \text{, that is }\ \begin{array}{|c|c|} 0 & 3 \\ \hline - & + \\ \hline + & + \\ + & - \\ - & - \end{array}$$

, for P gain $=\ 0$,
„ Q „ $=-1$.
Intercalation is $-Q$.

$$(3)\quad \begin{array}{r} \\ P=\ y'^2+5 \\ Q=-y'+4 \\ R=\quad -1 \end{array} \begin{array}{|c|c|} \multicolumn{1}{c}{-3} & \multicolumn{1}{c}{0} \\ \hline + & + \\ - & - \\ - & - \end{array}\,,\ \text{that is}\ \begin{array}{|c|c|} \multicolumn{1}{c}{-3} & \multicolumn{1}{c}{0} \\ \hline - & - \\ \hline + & + \\ + & + \\ - & - \end{array}\,,$$

for P gain $=0$,
„ Q „ $=0$.
Intercalation vanishes.

$$(4)\quad \begin{array}{r} \\ P=\ x'^2-4 \\ Q=\ x'-1 \\ R=\quad +1 \end{array} \begin{array}{|c|c|} \multicolumn{1}{c}{-3} & \multicolumn{1}{c}{0} \\ \hline + & - \\ - & - \\ + & + \end{array}\,,\ \text{that is}\ \begin{array}{|c|c|} \multicolumn{1}{c}{-3} & \multicolumn{1}{c}{0} \\ \hline - & - \\ \hline + & - \\ + & + \\ + & + \end{array}\,,$$

for P, gain $=+1$, P first,
„ Q „ $=$ 0.
Intercalation is $+P$.

Hence for the four sides, combining the intercalations, we have $-Q+P-Q+P$, and since there are no terms to be omitted, this is the intercalation of the N.E. square: which is right.

The Rhizic Theory. Articles, Nos. 32 to 38.

32. Consider now $F(z)=(\ast)(z, 1)^n$ a rational and integral function of z, of the order n with in general imaginary (complex) coefficients, or, what is the same thing, let $F(z)=f(z)+i\phi(z)$, where the functions f, ϕ are real*. Writing herein $z=x+iy$, let P, Q be the real part and the coefficient of the imaginary part in the function $F(x+iy)$: or, what is the same thing, assume

$$P+iQ=f(x+iy)+i\phi(x+iy),$$

then it is clear that to any root $\alpha+i\beta$ (real or imaginary) of the equation $F(z)=0$, there corresponds a real intersection, or root, $x=\alpha$, $y=\beta$, of the curves $P=0$, $Q=0$. The functions P, Q as thus serving for the determination of the roots of the equation $F(z)=0$, are termed "rhizic functions," and similarly the curves $P=0$, $Q=0$ are "rhizic curves." The assumed equation shows at once that we have

$$d_y(P+iQ)=i\,d_x(P+iQ),$$

or, what is the same thing,

$$d_yP=-d_xQ\,,\ \ d_xP=d_yQ.$$

And we hence see that

$$\frac{d(P, Q)}{d(x, y)}\,, = (d_xP)^2+(d_yP)^2, \text{ or } (d_xQ)^2+(d_yQ)^2,$$

is positive: viz. that the roots $P=0$, $Q=0$ are all of them right-handed (the essential thing is that they are same-handed; for by reversing the signs of P and Q they might be made left-handed: but it is convenient to take them as right-handed): hence the theorem—which in the general case, P and Q arbitrary functions, serves to determine

* It is assumed that the equation $F(z)=0$ has no equal roots: this being so, the curves $P=0$, $Q=0$, will have no point of multiple intersection; which accords with the assumption made in the general case of two arbitrary curves.

the difference of the numbers of the right and left-handed roots—in the particular case where P and Q are rhizic functions serves to determine the number of intersections of the curves $P=0$, $Q=0$: or, what is the same thing, the number of the (real or imaginary) roots of the equation $F(z)=0$: viz. we thus determine the number of roots within a given circuit.

33. The rhizic curves $P=0$, $Q=0$ have various properties. 1°. Each curve has n real points at infinity, or, what is the same thing, n real asymptotes: and the P and Q points at infinity succeed each other, a P-point and then a Q-point, and so on alternately.

In fact from the equation

$$P+iQ=(a'+ia'')(x+iy)^n\ldots+(k'+k''i),$$

writing herein $a'+ia''=a(\cos\alpha+i\sin\alpha)$, and $x+iy=\rho(\cos\theta+i\sin\theta)$, we have

$$P+iQ=a\rho^n[\cos(n\theta+\alpha)+i\sin(n\theta+\alpha)]\ldots+k'+k''i;$$

and it thus appears that for the curve $P=0$, the points at infinity are given by the equation $\cos(n\theta+\alpha)=0$, while for the curve $Q=0$ they are given by the equation $\sin(n\theta+\alpha)=0$: which proves the theorem.

Representing infinity as a closed curve or circuit, each point at infinity must be represented by two opposite points on the circuit; so that writing down P for each P-point and Q for each Q-point we have $2n$ P's and $2n$ Q's succeeding each other, a P-point and then a Q-point, and so on alternately.

It may be assumed that taking the circuit right-handedly, the P's are + and the Q's −, (this depends only on the colouring, but it corresponds with the foregoing assumption that the roots $P=0$, $Q=0$ are right-handed): the theorem just obtained then really is that for the circuit infinity, the intercalation is $(+P-Q)_n$: and we have herein a proof of the theorem that a numerical equation of the order n with real or imaginary coefficients has precisely n real or imaginary roots. But the force of this will more distinctly appear presently.

34. 2°. Neither of the curves $P=0$, $Q=0$ can include as part of itself a closed curve or circuit.

The foregoing relations between the differential coefficients give

$$d_x^2P+d_y^2P=0,\quad d_x^2Q+d_y^2Q=0,$$

which equations for the two curves respectively lead to the theorem in question. For as regards the curve $P=0$, take z a co-ordinate perpendicular to the plane of xy, and consider the surface $z=P$: if the curve $P=0$ included as part of itself a closed curve, then corresponding to some point (x, y) within the curve we should have z a proper maximum or minimum, viz. there would be a summit or an imit; at the point in question we should have $d_xP=0$, $d_yQ=0$; and also (as the condition of a summit or imit) $d_x^2P.d_y^2P-(d_xd_yP)^2=+$, implying that d_x^2P and d_y^2P have at this point the same sign: but this is inconsistent with the foregoing relation $d_x^2P+d_y^2P=0$.

35. 3°. The curves $P=0$, $Q=0$ have not in general any double (or higher multiple) points. A point which is a double (or higher multiple) point on one of these curves is not of necessity a point on the other curve: but being a point on the other curve it is on that curve a point of the same multiplicity. For changing if necessary the co-ordinates, the point in question may be taken to be at the origin: forming the equation

$$P+iQ=(a'+a''i)(x+iy)^n \ldots +(k'+k''i)(x+iy)^2+(l'+l''i)(x+iy)+m'+m''i=0,$$

the point $x=0$, $y=0$ will not be a double point on the curve $P=0$, unless we have $m'=0$, $l'=0$, $l''=0$; these conditions being satisfied, it will not be a point on the curve $Q=0$ unless also $m''=0$; but this being so, it will be a double point on the curve $Q=0$: and the like for points of higher multiplicity. But a point which is a multiple point on each curve, represents four or more coincident intersections of the curves $P=0$, $Q=0$, that is four or more equal roots of the equation $F(z)=0$; so that assuming that the equation has no equal roots, the case does not arise: and we in fact exclude it from consideration.

To fix the ideas assume that the curves $P=0$, $Q=0$ are each of them without double points. As already seen, neither of them includes as part of itself a closed curve. Hence in the figure the curve $P=0$ must consist of n branches each drawn from a point P in the circuit (viz. the circuit infinity) to another point P in the circuit; and in such manner that no two branches intersect each other: this implies that the two points P of the same branch must include between them an even number (which may of course be $=0$) of points P. And the like as regards the curve $Q=0$.

36. 4°. No branch of the P-curve can meet a branch of the Q-curve more than once. In fact drawing the two branches to meet twice, the colouring would at once show that of the two intersections or roots, one must be right, the other left-handed: whence, the roots being all right-handed, the branches do not meet twice. And in exactly the same way it appears that no P-branch can meet two Q-branches, or any Q-branch meet two P-branches. And under these restrictions it requires only a consideration of a few successive cases to show that the n P-branches, and the n Q-branches can only be drawn on the condition that each P-branch shall intersect once and only once a single Q-branch; which of course implies that each Q-branch intersects once and once only a single P-branch: and further, that there shall be precisely n intersections: viz. the n P-branches and the n Q-branches must satisfy the conditions just stated. And the theorem of the n roots is thus obtained as a consequence of the impossibility (except under the same conditions) of drawing the n P-branches and the n Q-branches, so as to give rise to right-handed roots only. But the case of double or higher multiple points would need to be specially considered.

37. It is interesting for a given value of n to consider $\phi(n)$ the number of different ways in which the P-branches and the Q-branches can be drawn. We have $2n$ points P and $2n$ points Q, in all $4n$ points: starting from any point P, these may be numbered in order 1, 2, 3, ... $4n$, the points P bearing odd numbers and the points Q even numbers.

We may consider the P-branch which joins 1 with some P-point β, and (intersecting this) the Q-branch which joins some two Q-points α and γ: the numbers $1\alpha\beta\gamma$ are then in order of increasing magnitude: and excluding these four points there remain the points corresponding to numbers between 1 and α, between α and β, between β and γ, and between γ and 1. Now since the P-branch 1β meets the Q-branch $\alpha\gamma$, no branch from a point between 1 and α can meet either of these curves; hence these points form a system by themselves, capable of being connected together by P-branches and Q-branches: the number of them must therefore be a multiple of 4: and the like as to the points between α and β, between β and γ, and between γ and 1. Taking the number of the points in the four systems to be $4x$, $4y$, $4z$, and $4w$ respectively, we have $x+y+z+w=n-1$, and the first mentioned four points bear the numbers

$$\begin{aligned} &1 \\ \alpha &= 4x+2, \\ \beta &= 4x+4y+3, \\ \gamma &= 4x+4y+4z+4. \end{aligned}$$

For the four systems the number of ways of drawing the P- and Q-branches are ϕx, ϕy, ϕz, ϕw respectively: that is x, y, z, w being any partition whatever of $n-1$ (order attended to), and $\phi(0)$ being $=1$, we have

$$\phi(n) = \Sigma\phi(x)\,\phi(y)\,\phi(z)\,\phi(w),$$

which is the condition for the determination of ϕn.

Taking then θ for the value of the generating function

$$1 + t\phi(1) + t^2\phi(2) \ldots + t^n\phi(n) + \ldots,$$

it hereby appears that we have

$$\theta = 1 + t\theta^4;$$

or writing this for a moment $\theta = u + t\theta^4$, and expanding by Lagrange's theorem, but putting finally $u=1$, we have the value of θ, that is of the generating function,

$$\begin{aligned} &= 1 + [4]^0\frac{t}{1} + [8]^1\frac{t^2}{1.2} + [12]^2\frac{t^3}{1.2.3} \ldots\ldots + [4n]^{n-1}\frac{t^n}{1.2\ldots n} + \ldots \\ &= 1 + \quad t + \quad 4t^2 + \quad 22t^3 + \quad 140t^4 + \ldots, \end{aligned}$$

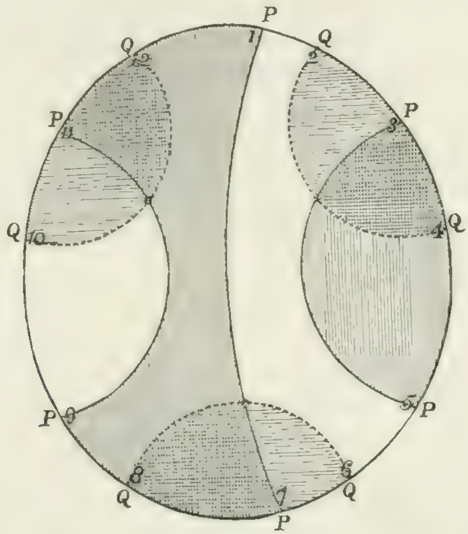
that is $\qquad \phi(1)=1, \quad \phi(2)=4, \quad \phi(3)=22, \quad \phi(4)=140, \ldots$

and generally $\qquad \phi(n) = \dfrac{[4n]^{n-1}}{[n]^n}, \quad = \dfrac{4n\,.\,4n-1\ldots 3n+2}{2.3\ldots n}.$

The results are easily verified for the successive particular cases; thus $n=1$, the points are 1, 2, 3, 4, and the P- and Q-branches respectively are 13, 24: $\phi(1)=1$. Again $n=2$, the points are 1, 2, 3, 4, 5, 6, 7, 8: we may join 13, 24 or 13, 28 or 17, 28 or 17, 68, leaving in each case four contiguous numbers which may be joined in a single manner: that is $\phi(2)=4$. Or, what is the same thing, the partitions of 1 are 0001, 0010, 0100, 1000, whence $\phi(2) = 4\{\phi(0)\}^3\phi(1) = 4$. Again $n=3$, the partitions of 2 are 0002, &c.

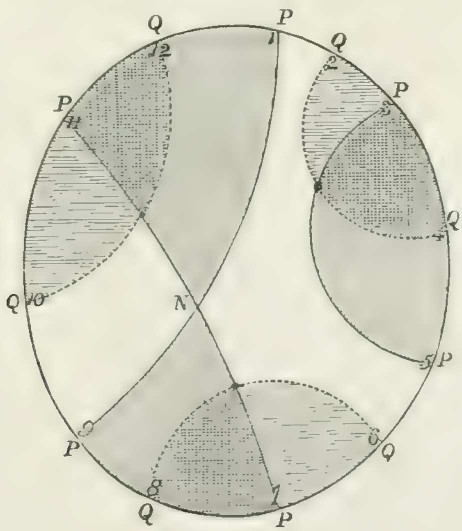
(4 of this form) and 1100 (6 of this form): that is $\phi(3) = 4\{\phi(0)\,\phi(2)\} + 6\{\phi(0)\}\{\phi(1)\}^2$, $= 4 \,.\, 4 + 6 \,.\, 1 = 22$, and so on.

38. Starting from the $4n$ points P and Q, and joining them in any manner subject to the foregoing conditions, we have a diagram representing two rhizic curves: and colour-



ing the regions we verify that the n roots are all of them right-handed. We have for instance the annexed figure $(n = 3)$.

Having drawn such a figure we may by a continuous variation of the several lines, in a variety of ways introduce a double point in the P-curve, or in the Q-curve: and by a



continued repetition of the process introduce double points in each or either curve: thus for instance we may from the last figure derive a new figure in which the P-curve has a node at N. It will be observed that here it is no longer the case that each P-branch intersects one and only one Q-branch: the P-branch $1-9$ does not meet any Q-branch, but the P-branch $7-11$ meets two Q-branches. But looking at the figure in a different manner, and considering the P-branches through N as being either $11-N-1$ and $7-N-9$, or $1-N-7$ and $9-N-11$, then in either case each P-branches intersects one and only one Q-branch: and in this way, in a diagram in which the two curves have each or either of them double points, but neither curve passes through a double point of the other curve, the theorem may be regarded as remaining true—we in fact consider the diagram as the limit of a diagram wherein the curves have no double points. It will be recollected that the equation $F(z)$ being without equal roots, we cannot have either curve passing through a multiple point of the other curve. And we thus see that the various figures drawn as above, without double points are, so to speak the types of all the different forms of a system of rhizic curves $P=0$, $Q=0$.

In connexion with the present paper I give the following list of Memoirs:—

CAUCHY. Calcul des Indices des fonctions. *Jour. de l'Ecole Polyt.* t. XV. (1837) pp. 176—229. First part seems to have been written in 1833: second part is dated 20 June, 1837. Refers to a memoir presented to the Academy of Turin the 17th Nov. 1831, wherein the principles of the "Calcul des Indices des fonctions" are deduced from the theory of definite integrals: I have not seen this.

STURM AND LIOUVILLE. Demonstration d'un theorème de M. Cauchy relative aux racines imaginaires des equations. *Liouv.* t. I. (1836) pp. 278—289.

STURM. Autres demonstrations du même theorème. Do. pp. 290—308.

These two papers contain proofs of the particular theorem relating to the roots of an equation $F(z)=0$, but do not refer to the general theorem relating to the intersection of the two curves $P=0$, $Q=0$: the special theorem of the existence of the n roots of the equation $F(z)=0$ is considered.

SYLVESTER. A theory of the Syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions and that of the greatest algebraical common measure. *Phil. Trans.* t. CXLIII. 1853, pp. 407—548.

DE MORGAN. A proof of the existence of a root in every algebraic equation, with an examination and extension of Cauchy's theorem on imaginary roots, and remarks on the proofs of the existence of roots given by Argand and Mourey. *Camb. Phil. Trans.* t. X. (1858).

Contains the important remark that the two curves $P=0$, $Q=0$ are such that two branches, one of each curve, cannot inclose a space; also that the two curves always [i.e. at a simple intersection] intersect orthogonally.

AIRY, G. B. Suggestion of a proof of the theorem that every algebraic equation has a root. *Camb. Phil. Trans.* t. X. (1859).

CAYLEY, A. On a proof of the theorem that every algebraic equation has a root. *Phil. Mag.* t. XVIII. (1859), pp. 436—439.

WALTON, W. On a theorem in maxima and minima. *Quart. Math. Jour.* t. X. (1869) pp. 253—262. CAYLEY, A. Addition thereto, pp. 262—263. (Relates to the curves $P=0$, $Q=0$.)

WALTON, W. Note on rhizic curves. *Quart. Math. Jour.* t. XI. (1870) pp. 91—98. First use of the term "rhizic curves:" relates chiefly to the configuration of each curve at a multiple point, and of the two at a common multiple point.

WALTON, W. On the spoke-asymptotes of rhizic curves. *Q. M. J.* t. XI. (1871) pp. 200—202.

WALTON, W. On a property of the curvature of rhizic curves at multiple points. Do. pp. 274—281.

BJÖRLING. Sur la séparation des racines d'equations algebriques. *Mem. d'Upsal* (1870) pp. 1—35. (Contains delineations of some rhizic curves.)

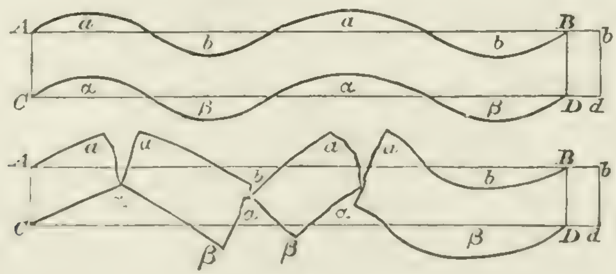
II. *On the Inequalities of the Earth's Surface viewed in connection with the secular cooling. By Rev.* Osmond Fisher, M.A., F.G.S., F.C.P.S., *late Fellow and Tutor of Jesus College.*

[Read *December* 1, 1873.]

In a paper which I read before this Society in 1868, I attributed the elevating force which has raised mountain-ranges to the contraction of the heated interior of the earth, and consequent wrinkling of the crust so as to accommodate itself to the diminished nucleus. This was an old hypothesis; but I believe the amount of horizontal pressure produced by that means had not been estimated before. I shewed that it is equal at the earth's surface to the weight of a piece of rock of the same section as the stratum, and 2000 miles long; enough to crumple up and distort any rocks*, and I also proved that a still greater horizontal pressure than this would be produced at any moderate depth. Towards the conclusion of the paper I made a rough estimate of the dimensions of the mountains which such a process might produce, upon certain hypotheses as to the amount of compression and thickness of the crust, which on a cursory view appeared to me probable. The object of the present paper is to attempt to arrive at a more definite conclusion upon this part of the subject.

In order to render clear what follows I am obliged to recapitulate the substance of a small part of a paper upon the formation of mountains which I have already published in the *Geological Magazine*†.

Let $ABCD$ be a layer of rock of unit of width, length l, and depth k. And suppose the abutments at AC and BD to approach each other through the space le, where e is a



* See Pratt's *Figure of the Earth*, fourth edition, p. 203, note.

† *Geol. Mag.* Vol. x. p. 218.

small fraction. Then the layer of rock in question would assume some new form, as one of those given in the figure, or any other whatsoever possible.

Let us now seek for some simple laws which must govern the disturbed strata in spite of the confusion which appears to reign among them. Let a, a, &c., be the areas formed by the upper curved line above AB, and b, b, &c., the areas formed by the same line below AB. It is not necessary that the a's should be equal to one another, nor yet the b's. They are used simply to designate the areas in respect of their *positions*. We will call AB "The datum level."

In like manner let α, β be similar areas for the lower datum level CD. Then the space included between the curved lines must be equal to

$$AbCd = kl\ (1 + e).$$

It is also evidently equal to

$$ABCD + a + a + \&\text{c.} + \beta + \beta + \&\text{c.}$$
$$- b - b - \&\text{c.} - \alpha - \alpha - \&\text{c.},$$

or, denoting the sums of the quantities of the same sort by the symbol Σ, we get

$$kl\ (1 + e) = kl + \Sigma\ (a) - \Sigma\ (b) + \Sigma\ (\beta) - \Sigma\ (\alpha).$$
$$\therefore kle = \Sigma\ (a) - \Sigma\ (b) + \Sigma\ (\beta) - \Sigma\ (\alpha). \qquad (1).$$

Since the pressure is supposed to take place in a horizontal direction, it will not have any direct effect to raise the centre of gravity of the portion of the crust under consideration; so that, if the layer in question rest upon a liquid substratum, we may expect some portions of the disturbed crust to dip into the superheated rocks. But in that case a corresponding volume of such subjacent rock must rise into the anticlinals.

Hence, $\qquad \Sigma\ (\alpha) = \Sigma\ (\beta).$

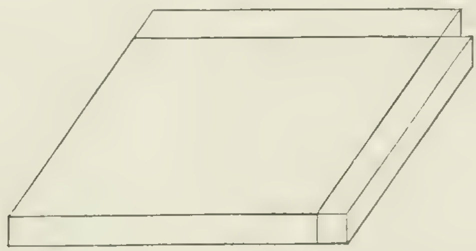
And the equation becomes $\qquad kle = \Sigma\ (a) - \Sigma\ (b).$

In order to render this reasoning applicable to the section of a surface of any form it is only necessary that the pressure, which causes the compression, should be everywhere tangential to the surface, and that gravity should be perpendicular to it. Hence it is applicable to the earth's surface, although that surface is not strictly regular, and may contain local elevations and depressions affecting the *mean* figure (that is the figure as unaffected by corrugation), which, though of small amount as compared to the dimensions of the earth, may be large as compared with the quantities of which we have to take cognizance in this investigation.

But the above suppositions cannot represent accurately what has occurred in nature. For they assume the upper and under parts of the crust of the earth to have been compressed horizontally by the same amount. In reality compression must have gone on gradually ever since a solid crust was first formed, and must be greatest at the upper datum level. We must therefore consider e to be the mean coefficient of compression. In the next place it will be necessary to extend our considerations from a section of unit of horizontal thickness to any proposed *area* of the earth's surface, and eventually to that of the whole globe.

A clear conception of what will then be the upper "datum level" is important. It will be an imaginary surface which occupies the position that the surface of the crust would occupy at the present time, had it been perfectly compressible in a horizontal direction; so that no corrugations would have been formed in it.

As soon as elevated tracts had once been formed by the corrugation of the cooled crust, all the material above the datum level, however distributed, must have been derived from matter originally beneath it, and subsequently raised above it. As often as further compression of the crust has taken place, every fresh addition to the sum total of the quantity of matter above the datum level must have accrued by the elevation of matter from below it. For the matter which was already above it, however freshly corrugated, or rearranged by water action, or otherwise, cannot have been thereby altered in quantity. Moreover each additional contraction will have acted upon a crust thicker than it was before on account of its having become in the meanwhile solid to a greater depth.



We have hitherto confined our symbols to a vertical section of the earth's crust of unit of width. We will now extend them to a portion of the crust whose length is l and width w, and the depth k; e and e' being the mean coefficients of compression in the directions of length and width. Then, if we neglect the product ee', our equation will become

$$klw\,(e+e') = \Sigma\,(A) - \Sigma\,(B),$$

where A and B are now the *volumes* of the elevations above and depressions below the datum level.

If in this equation we put $w=1$ and $e'=0$, it reduces to our former equation.

If we put $e=e'$ we get

$$2klwe = \Sigma\,(A) - \Sigma\,(B) \ldots\ldots (\text{A}),$$

which we may take as the general expression corresponding to any area of the surface.

The tendency of the corrugations over a given area will be to form two systems at right angles to one another, thus relieving the whole compression. Where one corrugation intersects another, we shall have a part of the volume common to both. But physically the same space cannot be occupied by two distinct volumes of rock. Hence at every such intersection there must be an increase of altitude in the ridges sufficient to contain within the contour an additional volume equal to the common portion. This appears to occur in nature where a peak often occupies the place of intersection of two ranges, as if the denudation which has shaped out the mountains has had a greater amount of matter to remove in such a situation.

The result expressed by the equation may be extended to the whole globe by considering its surface composed of elementary rectangular areas, and summing them; whence

$$8\pi r^2 ke = \Sigma(A) - \Sigma(B)\ldots\ldots(\mathrm{B}).$$

In which expression

r is the radius of the earth measured to the present position of the datum level.

k is the present thickness of the cooled crust.

e is the *mean* total linear compression of the crust.

$\Sigma(A)$ is the volume of the total elevations above the datum level.

$\Sigma(B)$ is the volume of the total depressions below it.

And $\pi = 3\cdot14159$.

It is worthy of remark, that the relation between lateral compression and elevation expressed by the above equation, (B), in no way depends upon the arrangement of the disturbed rocks, nor upon the time at which the successive movements have taken place, nor upon whether or not subsequent denudation, or any other mode of action, has redistributed the elevated matter. Nay, even if by human agency excavations have been made, their results will be included in our equation, which is perfectly general and true, so long as it is strictly interpreted. It requires also in particular to be noticed that it is not assumed that mountain-ranges need be anticlinals. All that is assumed is that they are carved out of tracts elevated as a whole by lateral pressure.

This appears to be the proper place for considering the manner in which the results of the deposition of sediment over a limited area would enter our equation. Geologists believe thick deposits to have been accumulated upon sinking areas, and attribute the subsidence of the area to the weight of the deposit itself. If rocks plastic from heat, or any other cause, come within a moderate distance of the surface this is possible. But the displacement of such plastic matter from beneath the new deposit must cause the rise of an equal volume somewhere else, so that if the terms of our equation were in any way affected (which would depend upon whether the effect extended below the datum level) it would be by an addition to the term $\Sigma(\beta)$, and a consequent equal addition to $\Sigma(\alpha)$, and their sum would be zero. So that, as already intimated, our equation will include any such effect of denudation and the resulting deposition.

Such an action as has been just referred to would produce a slight amount of crumpling in the strata, but only a very slight one, unless the new deposit *sunk through* some subjacent strata, and pressed them sideways out of its way; as I have shewn has happened in the contorted drift of Norfolk*. But it is only possible for this to have happened where the deposit produces a very much greater pressure upon the area which it occupies, than the rocks displaced by it did; that is when it is very much thicker than

* *Geol. Mag.* Vol. v. p. 550.

they were. And it is clearly impossible that it should elevate any displaced strata to a greater altitude than its own, which is necessarily limited to that of the sea-level where the deposit is going on. The test of such a mode of action would be to enquire whether any portion of the strata which ought to lie beneath the thick deposit can be discovered in a contorted condition abutting upon it.

Thus far, for the sake of perfect generality, it has been assumed that the disturbed upper strata rest upon a liquid, or at least a plastic, substratum. In what follows, the enquiry will be made whether this condition of the subjacent rocks is probable. And the method taken will be to consider whether the contrary supposition, that the subjacent rocks are *solid*, will account for the amount of corrugation which the earth actually exhibits.

If the earth had cooled as a solid body, the outer layers at any epoch having attained their complete amount of contraction sooner than the interior, would have been too large to fit the interior after the cooling had proceeded further. They would therefore become corrugated. But in this case that corrugation would have necessarily taken place wholly in an upward direction; and there could be no places where any portion of the surface could have become depressed below the datum level. Hence upon this hypothesis we may introduce into our datum-level equation the supposition that $\Sigma(B)=0$. And it becomes

$$8\pi r^2 ke = \Sigma(A).$$

A little consideration will give the following geometrical relation:

The volume of the Sea above the datum level = the area of the whole surface of the globe × the depth of the datum level below the sea-level − the volume of rock displacing water between those levels.

Let S = area of the Sea.

D = its mean depth.

L = area of the Land.

W = volume of the Land.

d = the depth of the datum level below the surface of the Sea, considering these as parallel.

Now the volume of rock displacing water between the datum level and the sea-level will be $\Sigma(A) - W$, and since in the case supposed $\Sigma(A) = 8\pi r^2 ke$, we shall obtain the following relation:

$$SD = (S+L)d - (8\pi r^2 ke - W).$$

But it is to be observed that this relation assumes the surface of the ocean to be everywhere parallel to the datum level, which is itself supposed to be parallel to the original surface of the globe when it was first covered with a solid crust. But the distribution of the great continents and oceans, as well as observations on the plumbline, render it probable that there is some cause which makes the density greater beneath the great oceans, and tends to accumulate the water upon them, so that their depth is greater than it would be were that solely due to a relative depression or locally diminished curvature of the sea-bed.

Let us then call X the accumulated volume of water in excess of that, which the oceans would contain, were their surface parallel everywhere to the datum level; retaining D as the observed mean depth of the sea, but using d as measured from the *supposed* surface as just defined.

In order to allow for the accumulated water, we must correct our equation by subtracting X from the actual volume of the ocean, so that

$$SD - X = (S + L)\, d - (8\pi r^2 ke - W).$$

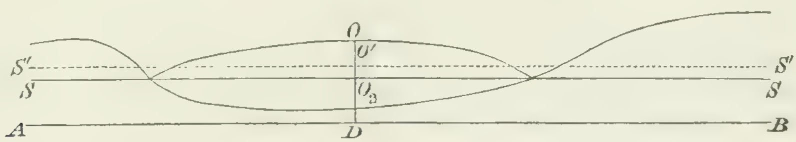
Observing that $S + L$ is the whole area of the globe, and therefore equals $4\pi r^2$, and that $\frac{X}{S+L}$ would be the depth of a layer of water of the volume of X if it were equally spread over the whole globe, which depth we will call δ. The above equation gives

$$2ke = d + \delta - \frac{\text{vol. of sea} - \text{vol. of land}}{\text{area of the globe}}.$$

The area of the ocean is estimated to be 146 millions of square miles, and that of the land 51 millions*.

Mr Carrick Moore† has shewn that Sir John Herschel was misled by an inaccurate expression in *The Cosmos* in his estimate of the mean height of the land being 1800 feet‡, and that it ought to be put at half that, or 900 feet. The average depth of the ocean is reckoned at three miles, and its volume at 438 millions of cubic miles. By substituting these numbers we obtain

$$2ke = d + \delta - 2{\cdot}2.$$



AB the datum level.
SS the sea level supposed parallel to it.
O the actual surface of the ocean.
$S'S'$ the surface of the accumulated water levelled down.
$O_2D = d$, $O'D = d + \delta$.

Our next step must be to fix upon a value for $d + \delta$. This is the depth of the datum level below the surface of an imaginary ocean, which is rather less deep than the actual one.

Now it is not probable that there is any place where the bottom of the ocean coincides with the datum level. Nevertheless there seems reason to think that this will approximately be the case in the deeper parts. If this be so, $d + \delta$ will not differ

* Herschel's *Physical Geography*, second edition, p. 19.

† *Nature*, Vol. v. p. 479.

‡ *Physical Geography*, p. 118.

greatly from the measure of the deepest parts of the ocean, and is not likely to exceed it by much, because, as already mentioned, the surface from which it is reckoned is somewhat lower than the actual surface of the ocean.

If the deepest parts of the ocean do not coincide with the datum level they must be one of two things. They must be either depressions below it, belonging to the series $\Sigma(B)$, or else they must be the places where the ocean bottom is least raised above the datum level. That they should be depressions beneath that level is scarcely possible under our present assumption that the earth has cooled as a solid; whence we have concluded that $\Sigma(B)=0$. But if we take the other alternative, and suppose the ocean-bottom to be raised above the datum level, even where the depths are very great, then we are taking $d+\delta$ too small, and our estimate for $2ke$ will be too small. And this appears by far the more probable case.

On the whole, then, we may feel pretty well assured that we are within the mark if we put for $d+\delta$ the measure of the more profound parts of the ocean. The *mean* depth being taken at three miles we shall not be much in excess if we put $d+\delta$ as equal to four miles. Our value for $2ke$ then becomes

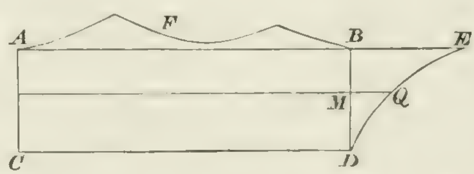
$$\begin{aligned} 2ke &= 4 - 2{\cdot}2 \text{ miles}, \\ &= 1{\cdot}8 \text{ miles}, \\ &= 9504 \text{ feet (about)}, \end{aligned}$$

a value which is more likely to be too small than too large.

We will now proceed to seek for a probable value for $2ke$ on physical grounds, in order to compare it with that just found.

For this purpose we must estimate the compression, kle, corresponding to the present thickness k of the crust and to a length l upon the present surface.

Let $ABCD$ be a section of a portion of the crust of length l and depth k. Then e is the mean coefficient of compression for the whole, which will depend upon the



depth to which the cooling down to the melting temperature under pressure has advanced. The area which by its having become compressed will have gone to form the corrugations AFB will be BDE, in which BE is the quantity by which AE has been compressed, while CD has not been compressed at all.

Then if $BM=x$, and $c=$ the compression at the depth x,

$$MQ = lc.$$

And the area in question will be $\int_0^k lc\,dx,$

which must be equal to the area AFB, or to kle.

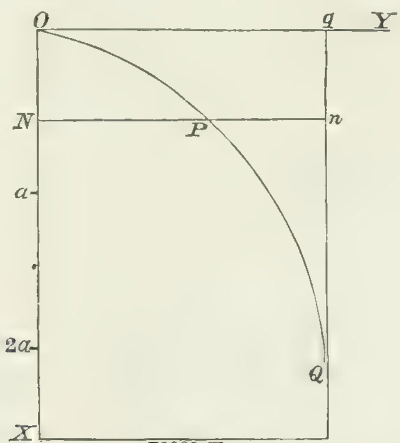
Hence $$ke = \int_0^k c\,dx.$$

We have now to express c in terms of x.

For this purpose I avail myself of Sir W. Thomson's paper "On the Secular Cooling of the Earth," using the symbols and a modification of the diagram there given, but not necessarily adhering to his values of the constants. In the event I think it will appear that we must adopt some modification of his views as to the condition of the globe at the time at which a rigid crust commenced to form, because we shall find that they lead us to a value of $2ke$ which is not in accordance with natural appearances. Sir William Thomson's views are thus expressed. "The earth although once all melted, or melted all round its surface*, did in all probability really become solid at its melting temperature all through, or all through the outer layer which had been melted, and not until the solidification was thus complete or nearly so did the surface begin to cool†."

The point which is material in the application of his investigation is, that cooling is to take place by conduction and not by convection.

Diagram modified from Sir W. Thomson's paper on the secular cooling of the Earth. ON the depth below the surface $=x$. NP the excess of temperature at that depth above the temperature of the surface $=v$. Oq the excess of the melting temperature above that of the surface $=V$.



V is the melting temperature of rock; which Sir William Thomson places provisionally at 7000° Far., a very high estimate. Mr Mallet has shewn that the temperature of slags run from an iron furnace is less than 4000° Far.‡

$$b = \frac{V}{\frac{1}{2}\sqrt{\pi}} = \frac{7000^\circ}{\cdot 88622} = 7900^\circ \text{ Far.}$$

* Dr Sterry Hunt supports the latter view. *American Journal of Science*, Vol. v. p. 264.

† *Secular Cooling of the Earth*, § (s). *Trans. of the Royal Society of Edinburgh*, 1862. Also Thomson and Tait's *Nat. Phil.*, p. 711.

‡ *Trans. Royal Soc.* 1873, p. 198.

The quantity a is thus defined: $a=2\sqrt{\kappa t}$, where κ is the conductivity of rock, found by Sir W. Thomson to be "400 for unit of length a British foot and unit of time a year." And t is the number of years since the Earth became solid, which must have elapsed in order that the rate of increase of temperature in descending into the Earth should be what it is observed to be. We learn (§ p.) that $NP=\frac{b}{a}\int_0^x \epsilon^{-\frac{x^2}{a^2}}dx$.

When then the rigid crust was commencing to form, the whole globe, or at any rate its outer portion to a considerable depth, was at the melting temperature V^0 Far. represented on the diagram by Oq or Nn. At a subsequent time the temperature at the depth ON or x was v^0, represented by the ordinate NP. Hence at that depth cooling has taken place through $(V-v)^0$, represented by Pn. Now the *compression* which each layer of the crust undergoes is caused by the *contraction* of all the matter beneath it.

Let E be the cubic contraction for 1° Far. Hence the layer of the sphere at the depth x whose area is $4\pi r^2$ and thickness δx will have contracted by $E4\pi r^2 Pn\,\delta x$, and consequently the whole contraction at that depth will be $E4\pi r^2\int Pn\,dx$, where the integral is to be taken from $x=x$ to that depth to which contraction has taken place. We may neglect the change in r for a moderate depth, since it is multiplied by the small quantity E. The diagram shews that the difference between the actual and the melting temperature at the depth $2a$ becomes small. It is $V\times \cdot 00468$.[1]

[1] Generally
$$Pn=V-\frac{b}{a}\int_0^x \epsilon^{-\frac{x^2}{a^2}}dx$$
$$=V-\frac{b}{a}\int_0^x\left\{1-\frac{x^2}{a^2}+\frac{1}{1\,.\,2}\left(\frac{x^2}{a^2}\right)^2-\frac{1}{\underline{|3}}\left(\frac{x^2}{a^2}\right)^3+\dots\right.$$
$$\left.+(-1)^n\frac{1}{\underline{|n}}\left(\frac{x^2}{a^2}\right)^n+\&c.\right\}$$
$$=V-\frac{b}{a}\left\{x-\frac{1}{3}\frac{x^3}{a^2}+\frac{1}{1\,.\,2}\frac{1}{5}\frac{x^5}{a^4}-\dots\right.$$
$$\left.+(-1)^n\frac{1}{\underline{|n}}\cdot\frac{1}{2n+1}\frac{x^{2n+1}}{a^{2n}}+\&c.\right\}$$
The general term of the series is
$$(-1)^n b\frac{1}{\underline{|n}}\frac{1}{2n+1}\left(\frac{x}{a}\right)^{2n+1}.$$
Suppose the superior limit of the integral to be $2a$. Then this becomes
$$(-1)^n b\frac{1}{\underline{|n}}\frac{2^{2n+1}}{2n+1}.$$
And the next term to this will be
$$-(-1)^n b\frac{1}{\underline{|n+1}}\frac{2^{2(n+1)+1}}{2(n+1)+1}$$
$$=-(-1)^n b\frac{1}{\underline{|n+1}}\frac{2^{2n+3}}{2n+3}.$$
And the sum of the two terms
$$=(-1)^n b\frac{2^{2n+1}}{\underline{|n+1}}\left(\frac{n+1}{2n+1}-\frac{2^2}{2n+3}\right)$$
$$-(-1)^n b\frac{2^{2n+1}}{\underline{|n+1}}\frac{2n^2-3n-1}{(2n+1)(2n+3)}.$$

In order to obtain a value of the contraction at the depth x, which we are assured is larger than the truth, we will take the integral from $x=x$ to $x=2a$ for the contraction to that depth, and add the contraction of all below on the assumption that it continues to the centre as great as it is at the depth $2a$.

We will therefore consider the whole cubic contraction at the depth x to be

$$E\,4\pi r^2 \int_x^{2a} Pn\,dx + \frac{E\,4\pi r'^3}{3} V \times \cdot 00468,$$

where $r' = r - 2a$.

But what we want is the linear compression of the shell in a horizontal direction at this depth, and that will be the same as the linear contraction of a spherical surface at the same depth. We will therefore find the mean contraction of the whole below that, as if the contraction was equally distributed throughout its volume.

Suppose E' to be the mean coefficient of contraction of the sphere below the given depth x.

Therefore $$E'\frac{4\pi r^3}{3} = E\,4\pi r^2 \int_x^{2a} Pn\,dx + E\frac{4\pi r'^3}{3} V \times \cdot 00468.$$

Hence $$E' = \frac{3E}{r}\int_x^{2a} Pn\,dx + EV \times \cdot 00468 \text{ nearly},$$

and the coefficient of *linear* contraction, which is that of compression for the next overlying shell, is one third of this,

$$\text{or } c = \frac{E}{r}\int_x^{2a} Pn\,dx + \frac{1}{3}EV \times \cdot 00468 \text{ nearly}.$$

The entire compression of the shell, or ke, which equals $\int_0^k c\,dx$, will therefore be less than

$$\frac{E}{r}\int_0^k \left(\int_x^{2a} Pn\,dx\right) dx + \frac{1}{3}EV \times \cdot 00468 \int_0^k 1\,dx.$$

And because the melting temperature is nearly reached at the depth $2a$, we may take that for the thickness of the corrugated crust. So that

$$ke < \frac{E}{r}\int_0^{2a} \left(\int_x^{2a} Pn\,dx\right) dx + \frac{2a}{3}EV \times \cdot 00468.$$

In this expression $n=2$ will give the third and fourth terms of the series; $n=4$, the fifth and sixth, &c., &c.

The sum of the two first terms of the series is $-0{\cdot}666666$. Making n successively 2, 4,... 18 (for which last value the first significant digit will not occur before the 7th place of decimals), we get for the sum of the first 20 terms

$$1{\cdot}548749 - 0{\cdot}666666 = 0{\cdot}882083,$$

whence at the depth $2a$, $Pn = V - b \times 0{\cdot}882083$, $\left(\text{where } b = \dfrac{V}{\frac{1}{2}\sqrt{\pi}}\right)$,

$$= V\left(1 - \frac{0{\cdot}882083}{\frac{1}{2}\sqrt{\pi}}\right), \quad (\sqrt{\pi} = 1{\cdot}77245),$$

$$= V \times 0{\cdot}00468.$$

In which expression

$$Pn = V - \frac{b}{a}\int_0^x \epsilon^{-\frac{x^2}{a^2}}\,dx.$$

After a somewhat tedious calculation the value of the factor under the sign of integration comes out

$$V2a^2 - a^2b \times 1{\cdot}549800.$$[1]

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[1] It has been already proved that

$$Pn = V - \frac{b}{a}\left\{x - \frac{1}{3}\frac{x^3}{a^2} + \frac{1}{1\,.\,2}\frac{x^5}{a^4} - \right.$$

$$\left. + (-1)^n \frac{1}{\underline{|n}}\,\frac{1}{2n+1}\,\frac{x^{2n+1}}{a^{2n}} + \right\};$$

$$\therefore \int Pn\,dx = Vx - \frac{b}{a}\left\{\frac{x^2}{2} - \frac{1}{3}\cdot\frac{1}{4}\frac{x^4}{a^2} + \frac{1}{1\,.\,2}\cdot\frac{1}{5}\cdot\frac{1}{6}\frac{x^6}{a^4} - \right.$$

$$\left. + (-1)^n \frac{1}{\underline{|n}}\,\frac{1}{2n+1}\,\frac{1}{2n+2}\,\frac{x^{2n+2}}{a^{2n}} + \right\}.$$

The general term of the series is

$$(-1)^n \frac{b}{a}\frac{1}{\underline{|n}}\,\frac{1}{(2n+1)(2n+2)}\,a^2\left(\frac{x}{a}\right)^{2n+2}$$

$$= (-1)^n ab\,\frac{1}{\underline{|n}}\,\frac{1}{(2n+1)(2n+2)}\left(\frac{x}{a}\right)^{2n+2}.$$

Taking $2a$ as the superior limit of the integral, the general term becomes

$$(-1)^n ab\,\frac{1}{\underline{|n}}\,\frac{2^{2n+2}}{(2n+1)(2n+2)}$$

$$= (-1)^n ab\,\frac{1}{\underline{|n}}\,\frac{2^{2n+1}}{(2n+1)(n+1)}$$

$$= (-1)^n ab\,\frac{1}{\underline{|n+1}}\,\frac{2^{2n+1}}{2n+1}.$$

And putting $n+1$ for n the next term to this will be

$$-(-1)^n ab\,\frac{1}{\underline{|n+2}}\,\frac{2^{2n+3}}{2n+3}.$$

And the sum of the two terms

$$(-1)^n ab\,\frac{2^{2n+1}}{\underline{|n+1}}\left\{\frac{1}{2n+1} - \frac{2^2}{(n+2)(2n+3)}\right\}$$

$$= (-1)^n ab\,\frac{2^{2n+1}}{\underline{|n+2}}\,\frac{2n^2-(n-2)}{(2n+1)(2n+3)} \quad\text{.................................(A).}$$

Giving to n the series of values 2, 4, 6, ..., 14, we obtain for the sum of the terms from the 3rd to the 16th, both inclusive,

$$ab \times 0{\cdot}0606654.$$

Hence we have for the value of $\int Pndx$, when $x = 2a$ (or at the superior limit),

$$V2a - ab\left(2 - \frac{4}{3} + 0{\cdot}606654\right) = V2a - ab \times 1{\cdot}273320;$$

Taking the observed rate of cooling at $\frac{1}{51}$th of a degree Far. per foot in descending, the time since the supposed solidification at 7000° took place will have been 100,000,000 years, and $a = 2\sqrt{\kappa t} = 400{,}000$ feet, or about 80 miles. And $b = \dfrac{V}{\frac{1}{2}\sqrt{\pi}} = 7900^{\circ}$.

$$\therefore \int_x^{2a} Pn\,dx = V(2a-x) - ab \times 1{\cdot}273320$$
$$+\frac{b}{a}\left\{\frac{x^2}{2} - \frac{1}{3}\cdot\frac{1}{4}\frac{x^4}{a^2} + \frac{1}{\lfloor 2}\cdot\frac{1}{5}\cdot\frac{1}{6}\frac{x^6}{a^4} - \right.$$
$$\left. + (-1)^n \frac{1}{\lfloor n}\frac{1}{2n+1}\frac{1}{2n+2}\frac{x^{2n+2}}{a^{2n}} + \right\};$$
$$\therefore \int\left(\int_x^{2a} Pn\,dx\right)dx = V\left(2ax - \frac{x^2}{2}\right) - 1{\cdot}273320abx$$
$$+\frac{b}{a}\left\{\frac{1}{2}\frac{1}{3}x^3 - \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}\frac{x^5}{a^2} + \frac{1}{\lfloor 2}\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}\frac{x^7}{a^4} - \right.$$
$$\left. + (-1)^n \frac{1}{\lfloor n}\frac{1}{2n+1}\frac{1}{2n+2}\frac{1}{2n+3}\frac{x^{2n+3}}{a^{2n}} + \right\}.$$

The integral has to be taken from $x=0$ at the surface to that value of x which expresses the thickness of the rigid crust beyond which corrugations will not have been formed.

Now the melting temperature V is reached within an extremely small error at the depth $2a$. We may therefore take $2a$ as the superior limit, and the above quantity becomes

$$\int_0^{2a}\left(\int_x^{2a} Pn\,dx\right)dx = V(4a^2 - 2a^2) - 1{\cdot}273320 \times 2a^2b$$
$$+\frac{b}{a}\left\{\frac{1}{2}\cdot\frac{1}{3}(2a)^3 - \frac{1}{3\,.\,4\,.\,5}\frac{(2a)^5}{a^2} + \frac{1}{\lfloor 2}\cdot\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}\frac{(2a)^7}{a^4} - \right.$$
$$\left. + (-1)^n \frac{1}{\lfloor n}\frac{1}{(2n+1)(2n+2)(2n+3)}\frac{(2a)^{2n+3}}{a^{2n}} + \right\};$$

of which series the general term after simplification becomes

$$(-1)^n a^2 b \frac{2^{2n+2}}{\lfloor n+1}\frac{1}{(2n+1)(2n+3)};$$

and the next to it
$$-(-1)^n a^2 b \frac{2^{2n+4}}{\lfloor n+2}\frac{1}{(2n+3)(2n+5)};$$

and their sum, after reduction,
$$(-1)^n a^2 b \frac{2^{2n+2}}{\lfloor n+2}\frac{2n^2+n+6}{(2n+1)(2n+3)(2n+5)}.$$

This is the sum of the two corresponding terms in the former series (A) multiplied by the factor

$$a\frac{2}{2n+5}\left\{1 + \frac{2(n+2)}{2n^2-(n-2)}\right\}.$$

Giving to n the values 2, 4, 6, ... 14, we obtain for the sum of the terms of the series from the 3rd to the 16th, both inclusive,

$$a^2b \times 0{\cdot}193840.$$

Hence
$$\int_0^{2a}\left(\int_x^{2a} Pn\,dx\right)dx = V2a^2 - a^2b \times 2{\cdot}543640 + a^2b\left(\frac{2^3}{2\times 3} - \frac{2^5}{3\times 4\times 5}\right) + a^2b \times 0{\cdot}193840$$
$$= V2a^2 - a^2b \times 1{\cdot}549800.$$

With these values of the constants we obtain for the value of our Integral,

$$281100000000000 \text{ sq. feet} = 2811 \times 10^{11} \text{ sq. feet.}$$

In order to fix upon a value of E, the coefficient of contraction for 1° Far., I refer to some experiments on a large scale conducted by Mr Mallet*. And upon reducing his results I find for slag from an iron furnace $E = \cdot 0000215$,[1] whence $\frac{E}{3}$ the linear contraction $= \cdot 000007$; which does not differ more than might be expected from the estimates made at much lower temperatures for rock by Mr Adie†. It appears that the integral

$$= 4\kappa t V\left(2 - \frac{1\cdot 54980}{\frac{1}{2}\sqrt{\pi}}\right) = 4\kappa t V \times 0\cdot 2533,$$

which varies as the time multiplied by the melting temperature, as the true value of ke would also do, if it were fully integrated. If we take the Earth's mean radius at 20,890,000 feet,[2] we obtain as the final result for the two terms that ke is less

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[1] To obtain the coefficient of contraction for 1° F. from Mr Mallet's experiments.

The cones of slag began to solidify when the containing iron moulds were at 450° F. At that time the mean contents of the moulds was 8231·9229 cubic inches, and the volume of the slag when measured at 53° F. was 7700·2303 cubic inches.

The slag entered the cones at a temperature of 3680°.

By Mr Mallet's estimate, founded on the *time* in which solidification commenced, the solidification commenced at about "3062°, or say 3000° F."

Hence the number of degrees through which the temperature fell was $3062° - 53° = 3009°$.

We have therefore to determine E, the coefficient of cubical contraction between incipient consolidation and 53° F.

$$7700 = 8232\,(1 - E\,3009),$$

$$E = \frac{352}{8232 \times 3000} = \frac{\cdot 0647}{3000}$$

$$= \frac{\cdot 0000647}{3} = \cdot 0000215;$$

$$\text{and } \frac{E}{3} = \cdot 0000071.$$

[2] The Earth's mean radius in feet = 20890000.

Hence the first term of the quantity to which ke is approximately equal will be

$$= \frac{2811 \times 10^{11} \times 215}{2089 \times 10^{4} \times 10^{7}}$$

$$= 289\cdot 30 \text{ feet.}$$

The second term in the value of ke is $\frac{2a}{3} EV \cdot 00468$ ($= mV$, suppose)

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* *Loc. cit.*, p. 201.

† *Trans. Roy. Soc. Edin.* Vol. XIII. p. 370.

than $289 + 183$ feet. Probably half the value of the latter term will be a full estimate. For among other reasons for its being over estimated, it will be perceived that the contraction for the decrease of temperature, $V \times 0{\cdot}00468$, has been attributed to the crust whose depth is $2a$ in addition to what is proper to it. We may say then that

$$ke = 380 \text{ feet},$$

and
$$2ke = 760 \text{ feet}.$$

In other words, if all the elevations which would be produced upon the above suppositions were levelled down, they would form a coating over the whole globe of less than 800 feet in thickness. This is an exceedingly small result, especially when we consider that our calculation has been conducted so as to make the result larger, rather than smaller, than the exact value would be, if the nature of the analysis admitted of that being found. But now if we introduce for V the temperature which Mr Mallet has determined for melting slag, viz. 4000° F., instead of 7000°, and the corresponding value for t of 33,000,000 years[1], the result becomes surprisingly small indeed. For then, by the rule of proportion proved on the last page, $2ke = 144$ feet only.

In this case $2a$ would be about 100 miles.

In what direction are we to look for an explanation to account for the discrepancy between a value for $2ke$ anywhere between these very wide limits, and the value 9504 feet, which we have previously obtained by estimating the existing irregularities of the earth's surface; for the latter is from 11·8 to 66 times as large as the former? It does not seem probable that any error in the values of the melting temperature between those wide limits, or in the value of the conductivity on the one hand; nor in estimating the magnitude of the irregularities of the earth's surface on the other; can explain the difference.

$$= 800000 \times {\cdot}000007 + 7000 \times {\cdot}00468$$
$$= 8 \times 10^5 \times 7 \times 10^3 \times \frac{7}{10^6} \times \frac{468}{10^5}$$
$$= \frac{8 \times 7 \times 7 \times 468}{10^3}$$
$$= 183 \text{ feet.}$$

Probably half this value will be a full estimate; for among other things in the calculation the contraction for mV is attributed to the depth $2a$ in addition to what is proper to it.

[1] To find the time since the Earth was all melted, on the supposition that it has cooled as a solid, and that the temperature was then 4000° Far. as determined for slag in Mr Mallet's experiments.

Referring to Thomson's paper *loc. cit.*,

$$\frac{dv}{dx} = \frac{V}{\sqrt{\pi\kappa}} \frac{1}{t^{\frac{1}{2}}} \epsilon^{-\frac{x^2}{4\kappa t}}$$
$$= \frac{V}{35{\cdot}4} \frac{1}{t^{\frac{1}{2}}} \epsilon^{-\frac{x^2}{1600t}}.$$

Taking $\frac{1}{51}$ of 1° as the rate of decrease at the surface where $x = 0$, we get from the above

$$t = 33 \text{ millions of years.}$$

Very likely it will be replied that the great basins, shallow as compared to their areas, which are occupied by the oceans, are not the troughs of great corrugations, but are depressions caused by vertical contraction in the direction of the radius; in which case they would not be contemplated by our datum-level equation. It is possible that this may be so to some extent, and I have accordingly made a correction which will help to meet the case. But if the elevated tracts which form continents are due to lateral compression of the crust, of which I think there can be no doubt, then the depressions which are their necessary correlatives must be sought beneath the oceans. Were the strata of the earth in the main horizontally disposed, vertical movements, however difficult to be accounted for, might nevertheless be appealed to. But the generally corrugated condition of the more ancient rocks forbids us to ignore the effect of lateral compression in elevating and relatively depressing the surface. It is thus that we believe the areas to have been raised, out of which denuding agencies have modelled the present land, while the depressions have served as the necessary receptacles for the waste materials so abundantly removed.

In passing I may observe, that I attribute the greater amount of corrugation observable in the older rocks simply to their exhibiting the accumulated effects of repeated compression, to which the newer deposits have not been so often subjected. But as regards the original crust of the earth, formed from the cooling of the melted surface, I have shewn that the compression, and therefore corrugation, if such has been produced, must diminish towards the lower parts.

The observed corrugations alone seem sufficient to shew that the amount of compression, to which the earth's surface has been subjected, must have been very much greater than the result of our calculation can account for. For instance, in Professor Ramsay's restored sections, figured in the first volume of the memoirs of the Geological Survey, the compression appears to lie between $\frac{1}{13}$ and $\frac{1}{21}$, while the country through which they are carried is by no means violently disturbed: the rocks consisting of the Old Red and Carboniferous systems. Were the subjacent strata exposed, they might be expected to exhibit greater flexure.

It has been shewn that the compression of a shell at the depth x is less than

$$\frac{E}{r}\int_x^{2a} Pn\,dx + \frac{1}{3}EV \times 0.00468.$$

The compression at the surface will be obtained by putting $x=0$. And then, with the high value of 7000° F. for the melting temperature, we get the compression less than 0·0058, which is nearly ten times as little as that exhibited in the sections.[1]

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[1] The compression of the shell at the depth x is less than

$$\frac{E}{r}\int_x^{2a} Pn\,dx + \frac{1}{3}EV \times \cdot 00468,$$

the first term of which $= \frac{E}{r}\left\{V(2a-x) - ab\,1\cdot 273320 + \frac{b}{a}\,(\text{series in } x)\right\}$;

where $ab = \dfrac{aV}{\frac{1}{2}\sqrt{\pi}}$.

It is easy to find a relation between $2ke$ and the compression at the surface, on the supposition of the law of cooling adopted above. It is,

$$\frac{\text{compression at the surface}}{2ke} = \frac{3.6371}{2\sqrt{\kappa t}}.[1]$$

For the compression at the surface put $x=0$, and this becomes

$$\frac{E}{r}(V2a - ab \times 1{\cdot}273320)$$

$$= \frac{E}{r} aV\left(2 - \frac{1{\cdot}273320}{\frac{1}{2}\sqrt{\pi}}\right)$$

$$= \frac{E}{r} aV\left(2 - \frac{2{\cdot}546640}{\sqrt{\pi}}\right)$$

$$= \frac{E}{r} aV(1{\cdot}8426073)$$

$$= 400000 \times 7000 \times \frac{18964}{10^{16}}$$

$$= \frac{4 \times 7 \times 18964}{10^{3}}$$

$$= \frac{53}{10^4} = {\cdot}0053\,;$$

and the second term $= {\cdot}00023$, the two together being less than $\frac{6}{1000}$.

[1] Compression at the surface $= \frac{E}{r} 2\sqrt{\kappa t}\, V 1{\cdot}8426073$ .... (1)

And $\qquad 2ke = 2\frac{E}{r} 4\kappa t\, V\left(2 - \frac{1{\cdot}54980}{\frac{1}{2}\sqrt{\pi}}\right)$ ....(2).

Substituting its value for $\sqrt{\pi}$, *viz.* 1·7467, and dividing (1) by (2), we obtain the ratio in the text.

Putting $\qquad 2ke = 9504$ feet,

$$t = 100{,}000{,}000 \text{ years},$$

$$a = 2\sqrt{\kappa t} = 400000\,;$$

we get the compression at the surface, which $= \dfrac{2ke \times 3{\cdot}6371}{2\sqrt{\kappa t}}$

$$= \frac{9504 \times 3{\cdot}6371}{400000}$$

$$= 0{\cdot}08$$

$$= \frac{1}{12} \text{ nearly.}$$

If, however, we take the lower value for V, *viz.* 4000° Far., then

$$t = 33{,}000{,}000 \text{ years},$$

$$\sqrt{\kappa t} = 114890\,;$$

But if we give $2ke$ the value of 9504 feet, as estimated, then, with the value 7000° Far. for V, the compression at the surface comes out $\frac{1}{12}$, and with the value 4000° Far. for V, it comes out nearly double that or $\frac{1}{7}$. These values are so large that although they agree well enough with Geological appearances, they would involve so great an accompanying diminution of the earth's radius, amounting to from 600 to 300 miles, and such a different former condition of the interior, represented by the proportionally increased value of E, as to render the steps of the above investigation manifestly inapplicable. Nevertheless from appearances, I think we must be prepared to give up our contraction theory altogether, or to admit a change of dimensions approaching to such an amount.

Provided the formulæ used were applicable, then if we could discover the mean compression at the surface by observation, knowing the value of $2ke$, we should have data sufficient for determining the melting temperature, the time, and the coefficient of contraction in the three equations:

$$\textit{compression at surface} = \frac{2\,ke \times 3{\cdot}6371}{2\sqrt{\kappa t}} \quad \text{.........................} \quad (1)$$

$$\frac{dv}{dx} = \frac{V}{\sqrt{\pi \kappa t}}\,\epsilon^{\frac{-x^2}{4\kappa t}}$$

in which at the surface $x = 0$ and

$$\frac{dv}{dx} = \frac{1}{51},$$

or

$$\frac{1}{51} = \frac{V}{\sqrt{\pi\kappa t}} \quad \text{..................................} \quad (2)$$

$$ke = \frac{E}{r}\,4\kappa t\,V \times 0{\cdot}2533 \quad \text{..................} \quad (3)$$

Can we then attribute the intense corrugations which meet our observation to any other source than that of lateral pressure caused by a shrinking globe? It appears to me difficult to conceive any other. No local change in the condition of the superficial strata seems competent to produce a sufficient amount of extension in the superficial strata. I cannot perceive, as I have already endeavoured to explain, how the deposition of thick beds upon the sea-bottom could effect it by their weight, and I have elsewhere, I think, proved that they could not do so by causing a new distribution of heat within the crust*. There is likewise much reason to believe that consolidation and metamorphism are accompanied by a contraction in volume.

$$\begin{aligned}
\text{and the compression} &= \frac{9504 \times 3{\cdot}6371}{229780} \\
&= \frac{9000 \times 4}{230000} \text{ nearly} \\
&= \frac{36}{230} \\
&= {\cdot}15 \text{ nearly} \\
&= \frac{1}{7} \text{ nearly.}
\end{aligned}$$

* *Geological Magazine*, Vol. x. p. 218.

The supposition that the earth became solid throughout, or nearly so, before a crust began to be formed, necessitates the consequence that the contraction, out of which the compression would arise, must be almost wholly confined to the cooled upper portion: and it is upon this supposition that we have calculated its amount, and found it so much smaller than is warranted by natural appearances. If, however, we could suppose that a solid crust was formed upon the surface, long before its interior parts had fallen to the melting temperature, it seems that a much greater amount of compression might result, through the contraction extending far below the cooled upper portion. The objection made to this view is, that the crust would break up as fast as it formed, and sink into the underlying fluid until the whole was brought to the melting temperature. But Mr Mallet has shewn, in the paper already more than once referred to*, that the difference between the specific gravities of solidifying and molten rock is so small, being scarcely 6 per cent., that when we consider the intermediate condition of viscosity, we need not assume this breaking up and sinking of the crust. And if in its early stages shrinkage cracks did form, it seems likely that the fluid which welled up into them would immediately solidify and seal them up. Scrope tells us that "the interior of a lava-stream often retains a very high temperature for a great length of time after its emission, continuing to send forth vapour from its crevices and fumeroles, and probably remaining liquid, and even more or less in motion, throughout its central and lower portion for years†."

Sir W. Thomson clearly contemplates the mode of solidification from without inwards as not impossible, for he says "If experimenters will find the latent heat of fusion and the variation of conductivity and specific heat of the earth's crust up to its melting point, it will be easy to modify the solution given above so as to make it applicable to the case of a liquid globe gradually solidifying from without inwards in consequence of heat conducted through the solid crust to a cold external medium."

If this supposition is admissible, there may have been a considerably larger nucleus enclosed within the crust in early times than we have at present, and a great portion of that nucleus may have consisted in superheated water, the rocks being in a state of igneo-aqueous solution[1]: and much of this water may have been blown off in steam during

[1] The following remarks upon the above passage were received from a quarter which disposes me to place great reliance on them:

"It is probable, or rather certain, that water substance, if it exists at great depths under great pressure and at high temperature, is neither a gas nor a liquid, being above its critical point.

"In this state substances are easily dissolved in it, not however so much on account of a greater tendency to combine with water, as on account of a greater tendency of their own to dissipation. At still higher temperature the water substance becomes itself dissociated into oxygen and hydrogen. But it does not follow that the dissolved substances will be precipitated. The magma may be all the more complete the higher the temperature, because, though the bonds of affinity have fallen away, the prison-walls prevent the elements from escaping. But of all the known regions of the Universe the most unsafe to reason about is that which is under our feet."

* *Phil. Trans.* 1873, p. 160, § 50.

† *Volcanos*, 2nd Edition, p. 84, § 8.

volcanic eruptions, by that means materially contributing to the diminution of the volume. I have suggested in my former paper read before this Society, that Mr Sorby's observations on the water enclosed in granitic crystals, along with crystals of chlorides, renders it probable that the steam emitted in eruptions may be a constituent part of the deep-seated rocks, for it is probable that but a small part of the water contained in any magma would become confined in the interior of the crystals.

Here, however, the question arises whether it would be possible for a crust to form over a layer of molten rock in a condition of igneo-aqueous fusion. Would not the escape of the water cause a state of constant ebullition which would prevent the formation of any crust until it had ceased through the escape of all the water?

The idea of igneo-aqueous solution involves a condition of chemical combination between the water and the elements of the rock such that while that condition lasted there would be no tendency to separation between the two, and evaporation would be in abeyance. We can therefore conceive that at depths where the heat and pressure were sufficient there might be no tendency to evaporation and consequent ebullition, so that after the water had escaped to a certain depth ebullition would cease, and a crust be formed; but that more water would be ready to separate to a greater depth when its affinity for the rock became lessened through the abstraction of heat, or diminution of pressure owing to the crust being partially supported by corrugation.

If such was the condition of the interior in the early stages of the cosmogony, a large portion of the oceans now above the crust may once have been beneath it, and thus we gain a novel conception of a sense in which the fountains of the abyss may once have been broken up.

A somewhat analogous escape of elastic vapour from beneath a denser envelope is I believe considered to be now taking place in the Sun.

The comparatively small specific gravity of the earth as a whole, considering the great pressure to which its interior parts must be subject, has been held to prove that it is even now in a state of expansion through intense heat*.

But the question of its true condition is surrounded with difficulties. The supposition made by Sir Wm. Thomson of a cool nucleus covered by a sufficiently deep layer of molten rock is adopted by Dr Sterry Hunt†, and would afford the conditions required by the argument of this paper. It would also afford the mean rigidity required to meet the objection to a fluid interior drawn from the absence of internal tides. But it would not account for the small specific gravity of the whole, nor yet would it meet the argument from precession in the form in which it was originally advanced by Mr Hopkins: for a layer of fluid beneath the crust would destroy a rigid connection between it and the interior. This form of the argument however has been attacked, and apparently with some success‡. The late Archdeacon Pratt, in defence of the general

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* Herschel's *Phys. Geography*, 2nd Edit. p. 7.

† *American Journal of Science*, Vol. v. p. 264.

‡ General Barnard on "Problems of Rotary Motion," *Smithsonian Contributions*, No. 240.

argument from precession, placed the matter in a simple light in a letter to *Nature* in 1871 *, and in that form I think it would be met by the supposition of a cool nucleus covered by a molten ocean with a solidified crust. But I would invite attention to the fact that the conclusions I have arrived at, concerning a highly fluid condition of the interior, have reference to an early period of its history; while the tests of the tides, and of precession, are confined in their application to the present. Nevertheless I am disposed to think that, at any rate, what may be termed a superheated condition of the mass still exists at no very great depth below the surface. By which I mean that if it be solid the solidity is due to pressure.

* *Nature*, Vol. IV. p. 344.

III. *On the Inequalities of the Earth's Surface as produced by Lateral Pressure, upon the hypothesis of a liquid substratum.* By OSMOND FISHER, Clk., M.A., F.G.S.

[Read *Feb.* 22, 1875.]

So long ago as in 1831, the late Professor Sedgwick wrote, "As the earth has apparently diminished in temperature, we have a right to look for some indication of a contraction of its dimensions. May not some of the great parallel corrugations of the older systems of strata have been produced by such a partial contraction *?" This theory is now commonly accepted, and there seems no necessity for limiting its application to the older systems of rocks.

The principal difficulties connected with the inequalities of the earth's surface appear to be with regard to the basins of the great oceans. Were the earth a perfectly smooth spheroid, without any inequalities in its surface, even in that case an excess of density in particular regions would determine a flow of water towards them, and it is conceivable that dry land and oceans might exist, even although the radial distances of the land-surfaces and of the sea-bottoms from the centre of figure might be perfectly equal. That the distribution of the oceans is to some extent actually due to such a cause appears certain; for otherwise a whole hemisphere could not be almost entirely covered with water. On this point Archdeacon Pratt remarks, "There is no doubt that the solid parts of the earth's crust beneath the Pacific Ocean, must be denser than in the corresponding parts on the opposite side, otherwise the ocean would flow away to the other parts of the earth." And after explaining the reason for this statement he adds, "There must therefore be some excess of matter in the solid parts of the earth between the Pacific Ocean and the earth's centre, which retains the water in its place. This effect may be produced in an infinite variety of ways; and therefore, without data, it is useless to speculate regarding the arrangement of matter which actually exists in the solid parts below†." And Herschel considers that the prevalence of land and water in two opposite hemispheres "proves the force by which the continents are sustained to be one of *tumefaction*, inasmuch as it indicates a situation of the centre of gravity of the total mass of the

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* 'On the general structure of the Cumbrian Mountains.' *Trans. Geol. Soc.*, Jan. 5, 1831. I am indebted to Mr Bonney for this reference.

† *Figure of the Earth*, 4th ed., p. 236. 1871. A great ice-cap would also have its effect. See Croll's *Climate and Time*, Chaps. XXIII, XXIV.

earth somewhat eccentric relatively to that of the general figure of the external surface—the eccentricity lying in the direction of our antipodes: and is therefore a proof of the comparative *lightness* of the materials of the terrestrial hemisphere *."

A like conclusion as to the greater comparative density of the bed of the Ocean, was arrived at by Archdeacon Pratt from the fact that at seven Coast Stations out of thirteen, six being in the Anglo-gallic, and one in the Russian arc, it has been found that a deflection of the plumb-line exists towards the sea †. "In fact," he remarks, "the density of the crust beneath the mountains must be less than that below the plains, and still less than that below the ocean-bed. If solidification from a fluid state commenced at the surface, the amount of radial contraction in the solid parts beneath the surface of the mountain-region has been less than in the parts beneath the sea-bed. In fact it is this unequal contraction which appears to have caused the hollows in the external surface, which have become the basins into which the waters have flowed to form the Ocean." Latterly, however, he attributed the formation of mountainous regions to horizontal compression ‡.

Mr Mallet, in his paper on Volcanic Energy, takes a similar view. He thinks that the land- and sea-boundaries were shaped out by radial contraction during the first great stage of the operation of refrigeration, while the crust was thin and flexible, owing to the rapid contraction of its viscous portion, which must then have been much thicker than the solid sheet above it §.

Mr Hopkins appears to have been opposed to such views as maintain a difference of radial contraction; and to have held that lateral compression was the cause of the formation of the greater inequalities as well as of the lesser ones, for we find him in discussing M. Elie de Beaumont's theories to have used these words: "The physical cause to which our Author refers the phenomena of elevation—the shrinking of the earth's crust—is that which appears to me most unlikely to produce that paroxysmal action which his theory so essentially requires; and most likely to produce those slow and gradual movements which it scarcely recognizes. The actual *depressions of the great oceanic basins*, and generally the more widely extended geological depressions of the present or former periods, may, I think, be referred with great probability to this cause ‖."

The theories respecting the formation of the larger features of the earth's surface are discussed by Professor Le Conte in so lucid and unprejudiced a style, that his papers are well worthy of study. The following passage conveys his conclusions. "Mountain chains and mountain ranges are therefore, I think, beyond question produced by horizontal thrust, crushing together the whole rock mass, and swelling it up vertically; the horizontal thrust being the necessary result of secular contraction of the interior of

* *Physical Geography*, § 13. 1862.

† *Figure of the Earth*, pp. 200 and 206. 4th Ed. 1871.

‡ In the third edition of the *Figure of the Earth* no mention was made of horizontal compression, but mountains were attributed to vertical expansion. In the fourth the author's estimate of the horizontal force of compression is referred to.

§ *Trans. Royal Society*, 1872, § 52 and § 60. Mr Mallet's theory has been discussed by Mr Scrope in the *Geological Mag.*, Dec. II. Vol. I. pp. 28, 127, and by the author, *Journal of Geol. Soc.*, Vol. XXXI. p. 469, and *Phil. Mag.*, Oct. 1875.

‖ Presidential Address to the Geol. Soc., 1853. *Geol. Journ.*, Vol. IX. p. lxxxix.

the earth. The smaller inequalities, such as ridges, peaks, gorges, and in fact nearly all that constitutes scenery, are produced by subsequent erosion. I feel considerable confidence in the substantial truth of the foregoing statement of the formation of *mountain-chains.* As to the mode of formation of continents and sea-bottoms, I feel less confidence. It is possible that even these may be formed by a similar unequal yielding to horizontal thrust, and a similar crushing together and upswelling. If so, it would be necessary to suppose the amount of horizontal yielding in this case much less, but the depth effected much greater than in the case of mountain-chains. But as we find no unmistakable structural evidences of such crushing, except in the case of mountain-chains, I have preferred to attribute the formation of continents and sea-bottoms to unequal *radial* contraction*."

The last sentence of this passage appears to invite the remark that we cannot expect in general to have evidence of crushing except in those areas which are open to investigation, viz. on dry land. But *there* it is not confined to mountain-chains. Contorted strata are to be also found in what would be termed level countries, often covered with horizontal deposits of later date†: and this fact in itself proves that these contorted strata have been once covered by an ocean, offering a strong presumption that there are contorted strata now at the bottoms of the oceans.

I have lately shown, in a paper read before this Society in December, 1873, that if the earth be supposed to have become solid throughout simultaneously, and if the surface of the ocean be now parallel to what would have been the surface of the solid globe if no inequalities of contour had resulted from its contraction (and I call such a surface the *datum level*), then the inequalities produced by the contraction through cooling would have been very much less than those which actually exist: for I have proved them to be from sixty-six to eleven and a half times as great as they would have been under the supposed circumstances, according to what assumption we make regarding the surface-temperature at which a permanent crust began to form, between the limits of 4000° Fahr. and 7000° Fahr. respectively. And I have suggested, as a mode of accounting for this discrepancy, that the earth need not have become solid throughout simultaneously, and consequently may have been considerably larger than it is now at the time when a solid crust was first formed.

But it is obvious that in strictness this reasoning fails if the surface of the water, to which the inequalities are referred, be not parallel to what would be the surface of the solid globe, if no inequalities of contour had resulted from its contraction, that is to the "datum level"; its increased depth in the great oceans being in that case supposed due to greater density beneath them. And therefore I have in my investigation allowed a considerable margin to meet this supposition. But I suppose that the soundings at the edges of the oceanic basins, and indeed over their entire areas, will be admitted to prove that they are in the main essentially depressions.

It also fails—and this case is more important—if, in accordance with some of the

* *American Journal of Science*, 3rd series, Vol. IV. p. 462. 1872.

† For example the highly contorted carboniferous strata of parts of Belgium.

views already referred to, the oceans occupy depressions in the earth's surface produced by unequal contraction in the radial direction. Upon such a supposition the view of Archdeacon Pratt and Professor Le Conte appears to receive support from my result; for if lateral compression from the cooling of a *solid* globe will not account for the inequalities of the surface, it seems natural to call in to our assistance difference of radial contraction.

But it appears to me that the explanation of the formation of oceanic areas by this means (if mere cooling be its sole cause) is encompassed with more difficulties than upon the other view of compression. And these appear to have struck Professor Le Conte himself: for later in his paper* he says, "I am fully aware that there are some phenomena of movement of the earth's crust, which are not explained by the foregoing theory. I refer especially to those great and wide-spread oscillations which have marked the great divisions of *time,* and have left their impress in the general unconformability of strata. The last of these great oscillations took place during the Post-tertiary period. I cannot explain these oscillations."

That there have been such oscillations of level seems incontestable. The same areas have risen and sunk, again and again, during the lapse of ages. The deposits of the Appalachian Chain we are told attain a thickness of eight miles †. Consequently the rocks now exposed at the localities where the lower beds of the series are at the surface, must at one time have been some miles at least below the sea-level. If that which is now dry land was once so far below the sea, it is not only probable from analogy that an area which is now sea was then correspondingly raised, but it is *necessary* that the area now occupied by sea must have presented land-surfaces, in order to have furnished the detritus out of which the rocks of the present dry land were formed. And in the case of the Appalachians, that land has been proved to have been situated to the Northward, and also the detritus to have come "especially from a continental mass to the Eastward ‡," where the Atlantic now rolls.

The Islands of New Zealand, occupying a central position in the aqueous hemisphere, contain a series of deposits analogous to those of the northern hemisphere§, whence it follows that they must have been often submerged during periods when land-surfaces existed in what is now an extended ocean, from which lands the materials of their rocks must have been derived.

In considering this great subject, we are at once brought face to face with the vexed question of the condition of the earth's interior, whether it be rigid or not. On the one hand we have Sir W. Thomson's arguments || for the rigidity, as deduced from the amount of precession. And although General Barnard has demonstrated that a fluid nucleus enclosed in a *rigid* crust would behave as a solid under the action of the precession producing forces¶, yet for practical purposes this will not assist the view which

* *Loc. cit.*, p. 472.

† Hall's *Palæontology of New York*, p. 67.

‡ Le Conte, *American Journal of Science*, Vol. IV. p. 464.

§ Capt. F. W. Hutton, *Geol. Mag.*, Decade II., Vol I., p. 27, note. See also Hochstetter's *Geology of New Zealand.* Auckland, 1864.

|| 'On the Rigidity of the Earth,' *Phil. Trans.*, 1864.

¶ *Smithsonian Contributions to Knowledge*, No. 240. Washington City, 1871.

requires a fluid nucleus, for no crust can be supposed so rigid as not to yield to the pressure of tides internal to it.

On the other hand, the close accordance of the figure of the earth with the present values of the radius, and of the angular velocity, neither of which can be considered to have been always exactly the same as they are at present, shows that there is probably some capability in the matter of which it is composed for readjusting itself when the values of these elements are changed. Combined with these considerations, we have those which point to a very high temperature in the interior, resting not only on the observed increase of heat in descending, but also, as Herschel has remarked, on the mean density, so much less than might be expected when the great interior pressure is taken into account. These facts taken together appear to give probability to Sir W. Thomson's suggestion, that the high rigidity of the interior may be due to pressure, which may be sufficiently great to enable it to resist the deforming influence of the disturbing forces of the Sun and Moon. But although we cannot conceive of unequal radial contraction from cooling merely apart from rigidity, yet, on the other hand, rigidity will not of itself render unequal radial contraction probable. If the interior of the earth consists of matter so highly heated that it owes its rigidity to pressure, this seems to require that if there be inequality of contraction, it must reside within the limits of the crust, cooled below the temperature of fusion for the pressure; that is, in that portion in which liquefaction would not result upon a reduction of the pressure.

If unequal radial contraction from cooling has taken place, it seems that it must have arisen from one of two causes, or from those two co-operating, viz. firstly, from a difference in the conditions of cooling under which any two areas in question have been placed, or secondly, from a difference in the constitution of the materials of the globe at those two places.

Now if we take 0·00002 as the coefficient of contraction of rock for 1° Fahr., which is what Mr Mallet informs me to be the mean, and if we take 4000° Fahr. on the same authority for the temperature of melting rock, then 0·08 is the coefficient for contraction from the melting point to 0° Fahr., and one-third of this, or 0·027, that of linear contraction. Hence it would require a thickness of 400 miles cooled from 4000° to zero to give rise to a contraction of 10 miles upon any given radius, and if upon some other radius we conceive the contraction to have been from some cause or another one-tenth more, then on these suppositions we might have a difference of contraction of one mile upon the two radii in question. The excessive extravagance of these assumptions, and the inadequacy of the result even if we ignore the phenomenon of oscillation of level, sufficiently prove that the existing inequalities of the surface cannot be explained by difference of radial contraction through cooling. Nor is this all: for in regard to the conditions of cooling, it seems likely that those areas which are sea bottoms would cool more rapidly, on account of the heat being conveyed away by the ice-cold water which is found at the bottom of deep seas. So that the tendency from mere cooling would be continuously to deepen the seas, without any corresponding cause to re-elevate their floors. The supposition that the thick deposits which go on at the bottom of some seas may act as a "jacket" and prevent the escape of heat, meets this objection only to a small extent; first, because these deposits appear to have taken place over sinking areas, and to be so taking place still (and we can hardly attribute the stoppage as well as the reversal of the movement to a cause of such slight potency as difference of cooling has been shown to be). And secondly, because the heat conducted into the new deposits, and by some relied upon to re-elevate them, must be abstracted from the couches beneath, so that there can be no absolute increase in the amount of heat beneath the area in question*.

If we now consider the second cause capable of producing a difference of radial contraction, viz. a diversity of the materials of the globe at the two places in question, it is palpable that this cannot explain *oscillations* of level. For that would require the materials to become changed in their properties from time to time, in a manner highly improbable. Humboldt's suggestion of secular currents in the interior to explain the oscillations of level† is directly opposed to the condition of rigidity of the nucleus. In short, it seems that no modification of the theory of difference of radial contraction, arising from cooling merely, can be relied upon; and it becomes important to enquire how far the theory of lateral compression, combined with the hypothesis of a liquid substratum, can account for the phenomena of elevation and depression of areas of the earth's surface.

And here it is proper to notice a paper of much interest by Professor Shaler‡, who does not appeal to radial contraction, but only to compression. He insists upon a distinction, which he thinks ought to be drawn between the corrugations exhibited in mountain-ridges, and the broader features of "continental fold and oceanic depression §." "We find in continental folds broad curves of the surface, which narrow without exception towards the south, and which exhibit in no part of their structure the evidence of powerful lateral thrust, which are the most conspicuous phenomena of mountain-chains." He seeks an explanation of this circumstance by attributing the larger folds of continental and oceanic areas to the adaptation of a solid crust to a diminished nucleus, and suggests that the cause of the corrugations, producing mountain-chains, must be sought in changes going on within the crust itself, and in no way connected with the regions below. If I understand his meaning, he considers that there has been formed a solid crust, of which the lower layers were formerly much hotter than they are at present, and that the tendency to equalization of temperature between the lower and upper portions of the crust, has produced corrugations most intense in its upper layers. But besides this he thinks that the nucleus within the crust has shrunk away from the crust as a whole, so that corrugations of a wider and less abrupt character have been produced in the crust as a whole; these wider folds forming the broader features of continental fold and oceanic depression.

In support of his view, as regards the contraction within the crust itself, he adduces Sir Wm. Thomson's estimate of the secular diminution of the rate of increase of heat in descending. But he does not fully explain why the nucleus should contract away from the crust as a whole. If the contraction be simply due to loss of heat from a solid

* See a paper by the author, *Geol. Mag.*, Vol. x. p. 251.

† *Cosmos*, Vol. IV. p. 19. Sabine's Ed.

‡ Reprinted from *Proc. Boston Soc. of Nat. Hist.*, June 6, 1866, in *Geol. Mag.*, Vol. v. p. 511.

§ *Loc. cit.*, p. 514.

earth, the temperature increasing continuously from that of the surface to that of solidity due to pressure, there ought not to be any abrupt passage from the rate of contraction of the crust itself to that of the matter inside it, so that the series ought to be maintained which this author denies to exist; "at one extremity of which could be placed the greatest relief of continental fold and oceanic depression, and passing gradually to the most inconsiderable flexures."

Now it is remarkable that while, upon calculating the amount of inequalities of the surface, large and small, which can be got out of the compression arising from the contraction due to cooling, taking the temperature in descending to increase according to Sir W. Thomson's law for a solid globe, I have found that it falls very far short indeed of that which exists; yet I have shown above, that we must, for all that, look to causes connected with lateral compression and not to difference of radial contraction to account for these inequalities.

I conclude, therefore, that Professor Shaler's idea is in the main correct, although he does not distinctly explain the reason of the crust being left behind by the contracting globe. He holds that the earth first solidified from the centre, and after a time from the surface also, so that there is probably at present a thin layer of molten material between the solidified crust and a solidified interior. According to this view all the contraction required to make up the deficiency between the inequalities as calculated for a solid globe, and the inequalities as they exist, must be obtained from the amount by which this intervening layer has contracted since the time when the crust first solidified. I do not see how this can suffice, unless we invoke some further cause of diminution of volume, besides the contraction which would be due to mere cooling: such as the escape of gases, and of water in the state of steam, from volcanic vents*. But in any case the conclusions of Professor Shaler and myself, formed on totally distinct grounds, lead to the same general result, viz. that there is a solid crust, resting in corrugations upon a liquid or quasi-liquid heated layer. The existence of such a layer of liquid or viscous matter seems to be rendered probable by the following consideration. The increase of temperature, though rapid near the surface, becomes less and less as we descend, so that, if the earth were once wholly melted, the temperature near the centre is not very greatly above what it is at a depth which, compared to the earth's radius, is small. Consequently, if it requires great pressure to solidify the materials at such a temperature, it is probable that the melting temperature may be reached before the pressure is sufficient to solidify. Of course this reasoning is worthless, unless we admit, as I hold we may, that the crust need not break up and sink.

A further argument of great weight in favour of the existence of such a layer is that the contraction of the sphere has been evidently relieved by corrugations, not uniformly distributed over the surface, but locally along lines of elevation, often separated by considerable intervals. There must therefore have been a lateral movement of the crust

* The eminent vulcanologist, the late Mr Poulett Scrope, wrote to the author in October, 1875; "There is one of the points you put forward which never struck me before, but which now appears to me most valuable, namely that the enormous amount of steam that has escaped from the interior in early times as well as down to the present, has been, and is, the cause of those *subsidences* of the crust, to which the basins of seas and oceans, and the crumplings of the terrestrial rocks are owing, far more than to any general contraction of the nucleus by cooling."

over the nucleus towards these lines of elevation, which can hardly have been the case unless there exists a more or less fluid layer upon which it rests*. Should such a liquid layer exist, it would be subject to tides. But they would be analogous to the oceanic tides rather than to tides in the solid earth, and their period would probably be greater than half the diurnal period, as is the case with the ocean-tides. Consequently the crest of the wave would not be opposite the disturbing body, and would not affect precession as much as if it were. But it might have an effect in causing disturbances in the crust, and inducing earthquake action, as the observations of M. Perry show to be probably the case.

Assuming then that a solid crust rests in corrugations upon a liquid or viscous layer, which is capable of yielding to such forces as the gravitation of the crust exerts, we have to consider the conditions of equilibrium of such a crust.

In the first place, the disturbances which we see that it has experienced, to the greatest depths which denudation exposes to our observation, prove that when once disturbed in lengths of any extent it may be considered flexible. Moreover, when it rests in corrugations upon the subjacent liquid, it must be in unstable equilibrium; that is, the corrugations can have no particular relation to the places where they occur, but might exist equally well at some other. Here we at once encounter a condition which renders oscillations of the surface possible, provided we can account for the disturbance of the position of equilibrium.

We are not able strictly to investigate by means of mathematical formulæ the form which such a crust as I have supposed would assume. The surface of equilibrium would not be developable into a spherical one. But, nevertheless, it appears to me that we may obtain some insight into the general character of the corrugations, by enquiring what form would be assumed by a heavy flexible crust, resting upon a liquid within a rectangular trough shorter than the crust, for this would give an approximate idea of the contour of the surface upon the course of a section of the sphere perpendicular to its surface and cutting a set of corrugations at right angles.

It may be assumed that the trough and crust were originally of the same length, and that the corrugations have been produced by the ends of the trough having been made to approach each other.

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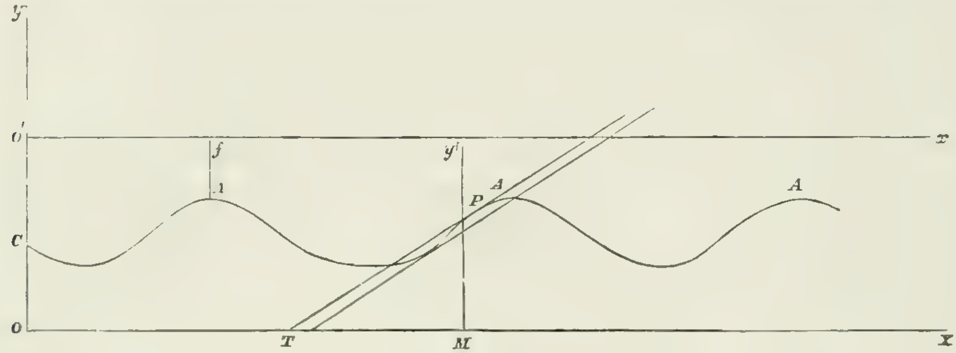
* Captain Dutton, U.S.A., in an interesting paper "On the Contractional Hypothesis" published in 1874, in which he uses a line of argument similar to that followed by the author in his paper "On the Inequalities of the Earth's Surface," of Dec. 1, 1873, and equally concludes that no contraction from cooling can adequately explain the plications of the Strata, has the following important remark: "The determination of plications to particular localities presents difficulties in the way of the contractional hypothesis which have been underrated. It has been assumed that if a contraction of the interior were to occur, the yielding of the outer crust would take place at localities of least resistance. But this could be true only on the assumption that the crust could have a horizontal movement in which the nucleus does not necessarily share. A vertical section through the Appalachian region and westward to the 100th meridian shows a surface highly disturbed for about two hundred and fifty miles, and comparatively undisturbed for more than a thousand. No one would seriously argue that the contraction of the nucleus had been confined to portions underlying the disturbed regions: yet if the contraction was general, there must have been a large amount of slip of some portion of the undisturbed segment over the nucleus." He does not consider such a "slip" possible, even with a substratum of "liquid lava." *American Journal of Science and Arts*, Jan. 1874, p. 113.

The author has however shown that the coefficient of friction required to allow of such a movement of the crust would not be smaller than might be thought admissible. See "Mr Mallet's theory of Volcanic Energy tested," *Phil. Mag.*, Oct. 1875, Vol. 50, p. 317.



Suppose COX to be a vertical section of the trough,

CPA the section of the crust whose thickness is c,

OX, horizontal, the axis of x,

OY, vertical, that of y,

$OM = x$, $MP = y$, $PTM = \theta$,

r the radius of curvature at P,

ρ the density of the crust,

σ that of the liquid,

p the pressure of the liquid at P,

t the force of compression in the crust in the direction of the tangent at P.

We may suppose for the present that the flexibility of the crust is not impaired by its thickness.

Then applying the conditions of equilibrium we obtain the following equations:

(1) $t = g\rho c\,(C - y)$, where C is an arbitrary constant.

(2) $p = g\rho c \cos\theta - \dfrac{t}{r}$.

Let h be the value of the ordinate at the highest point A, and f the depth of a layer of fluid of the density of the crust, which upon a unit of area would produce a pressure equal to the compression at A. Let δ be the depth of a layer of fluid of the density of the crust, which would produce a pressure equal to the fluid-pressure at A.

Then from (1) we have

$$\text{compression at } A = g\rho cf = g\rho c\,(C - h);$$

$$\therefore f = C - h.$$

Also $p =$ the pressure due to the depth below $A +$ the pressure at A;

$$\therefore\ p = g\sigma\,(h - y) + g\rho\delta.$$

Therefore, substituting in (2),

$$g\sigma(h-y)+g\rho\delta=g\rho c\left(\cos\theta+\frac{y-C}{r}\right).$$

Change the origin to O' by putting $y'=y-C$;

$$\therefore\ y=y'+C=y'+h+f.$$

Hence when $y=h$, $y'=-f$, so that f is the depth of A below the new origin, and

$$h-y=-(y'+f);$$

whence, by substituting for y,

$$-g\sigma(y'+f)+g\rho\delta=g\rho c\left(\frac{dx}{ds}+\frac{y'}{r}\right),$$

$$\text{or}\quad -\sigma y'-(\sigma f-\rho\delta)=\rho c\left(\frac{dx}{ds}+\frac{y'}{r}\right).$$

Integrated this gives,

$$-\frac{\sigma y'^2}{2}-(\sigma f-\rho\delta)\,y'=\rho c y'\frac{dx}{ds}+D,$$

$$\text{or,}\ \rho c y'\frac{dx}{ds}+(\sigma f-\rho\delta)\,y'+\frac{\sigma}{2}y'^2=-D\text{............(A)}.$$

If this equation be solved with regard to y' it will give two values of y' for one value of $\frac{dx}{ds}$ or $\cos\theta$.

And if it be solved with regard to $\frac{dx}{ds}$, or $\cos\theta$, it will give the same value for $\cos\theta$ whenever y' is the same.

The curve is therefore of an undulating form as in the figure.

If the two values of y' be made equal, they will give the points of contrary flexure.

If we put $\frac{dx}{ds}=1$, this will give the maximum and minimum ordinates.

Now we know that one of these is $-f$. Hence we may find D;

$$\therefore\ D=\sigma\frac{f^2}{2}+\rho(c-\delta)f.$$

Substituting this value we get for the said ordinates,

$$-f-\frac{\rho}{\sigma}(c-\delta)\pm\frac{\rho}{\sigma}(c-\delta),$$

and their difference,

$$2\frac{\rho}{\sigma}(c-\delta).$$

We have from equation (2), by making r infinite, since t is not infinite, at a point of contrary flexure,

$$p = g\rho c \cos\theta,$$

$$\text{or} \quad -g\sigma(y'+f) + g\rho\delta = g\rho c \cos\theta.$$

Substituting the corresponding value of $\cos\theta$ in the general equation, we get

$$y' = \pm\sqrt{f^2 + 2\frac{\rho}{\sigma}(c-\delta)f}$$

for the depth of the point of contrary flexure. The negative sign must be taken.

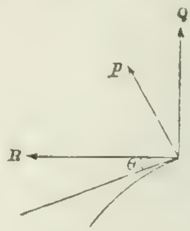
The radii of curvature are, at an anticlinal,

$$r = \frac{cf}{c-\delta},$$

and at a synclinal,

$$r = \frac{cf}{c-\delta} + 2\frac{\rho}{\sigma}c.$$

To determine the constant f we must discuss the conditions at an anticlinal. Consider any element ds of the curve, and let the horizontal and vertical components which act upon it from beyond at one extremity of it be



R and Q,

p the fluid pressure,

t the compression,

θ the inclination of the tangent to the horizon, as before.

Resolving vertically, we have for equilibrium,

$$p\cos\theta ds + t\sin\theta - g\rho c ds + Q = 0,$$

and horizontally,

$$p\sin\theta ds - t\cos\theta + R = 0.$$

Eliminating t,

$$pds - g\rho c\cos\theta ds + Q\cos\theta + R\sin\theta = 0 \text{..............................(1).}$$

Now this being true when ds is indefinitely diminished, since Q and R are not thereby altered, we have generally,

$$Q\cos\theta + R\sin\theta = 0.$$

But at an anticlinal Q and R must be the vertical and horizontal components of the compression at one side of the anticlinal. From the nature of the case the vertical components of the compression on either side of the anticlinal must act in the same direction; and therefore they cannot be in equilibrium unless they are separately zero. And Q is one of them;

$$\therefore\ Q = 0;$$

$$\therefore \text{ also } R\sin\theta = 0,$$

whence either $\sin\theta = 0$, or $R = 0$.

Let us consider these two cases in order. When the element is under the action of the forces Q and R at one extremity of it, and ds is indefinitely diminished, observing that $Q\cos\theta + R\sin\theta = 0$, we have from (1) in the limit

$$pds - g\rho c\cos\theta ds = 0,$$

$$\therefore \frac{pds}{g\rho c\cos\theta ds} = 1.$$

Now at an anticlinal we know that $p = g\rho\delta$, and (in the case we are now considering) $\cos\theta = 1$, and these are the values which these quantities assume there when ds is indefinitely diminished.

Hence at an anticlinal the above becomes in the limit

$$\frac{g\rho\delta}{g\rho c} = 1,$$

that is, when the curve is horizontal at an anticlinal,

$$\delta = c.$$

This condition evidently belongs to the case of a crust lying horizontally upon the liquid, which is a particular case of the general problem. In this instance the difference of the maximum and minimum ordinates $\frac{2\rho}{\sigma}(c-\delta)$ as found p. 443 vanishes, as it ought to do.

Next consider the second case, viz. when $R = 0$. In this case $f = 0$ since f measures R.

The relation $f = 0$ shows that there is no compression at the anticlinal.

In this case $\tan\theta = -\frac{Q}{R}$ takes the form of $\frac{0}{0}$.

The origin of co-ordinates is by the same relation for f brought to the level of the anticlinals.

Our equation (A) now becomes, suppressing the dash over the y,

$$\rho c y\frac{dx}{ds} - \rho\delta y + \frac{\sigma y^2}{2} = -D;$$

$$\therefore \frac{dx}{ds} = \frac{\rho\delta y - \frac{\sigma y^2}{2} - D}{\rho c y}.$$

This expression shows that $D = 0$, otherwise $\frac{dx}{ds}$ (or $\cos\theta$) would be infinite at an anticlinal where, reckoning from the newly determined origin, $y = -f = 0$.

Dividing the numerator and denominator by y,

$$\frac{dx}{ds} = \frac{\rho\delta - \sigma\frac{y}{2}}{\rho c}.$$

Integrating this equation, and taking the origin at an anticlinal, we obtain

$$x^2 + y^2 - \frac{4\rho}{\sigma} x \sqrt{c^2 - \delta^2} - \frac{4\rho}{\sigma} \delta y = 0.$$

This represents a circular arc.

If λ be the distance from one anticlinal to the next, we obtain for δ the value

$$\delta = \sqrt{c^2 - \frac{\sigma^2}{16\rho^2}\lambda^2},$$

and for the final equation

$$\left(x - \frac{\lambda}{2}\right)^2 + \left(y - \frac{1}{2}\sqrt{\frac{16\rho^2}{\sigma^2} c^2 - \lambda^2}\right)^2 = 4\frac{\rho^2}{\sigma^2} c^2;$$

which gives the co-ordinates of the centre

$$\frac{\lambda}{2}, \text{ and } \frac{1}{2}\sqrt{\frac{16\rho^2}{\sigma^2} c^2 - \lambda^2};$$

and the radius $\frac{2\rho}{\sigma} c$.

These conditions are possible so long as λ is less than $\frac{4\rho}{\sigma} c$; in which case the festoon between the anticlinals becomes a semicircle.

It can easily be shown, by introducing the supposition that the curve is a circle into the equations of equilibrium (1), (2), that the difference of pressure varies as the difference of the ordinate. This variation would of course be true for *any* curve, but it would not be consistent with giving equilibrium of form to the curve unless it satisfied the equations of equilibrium.

It appears therefore that, when no extraneous force acts upon the crust, it will assume the form of a series of equal circular arcs, the radii of which depend solely upon the mass (ρc) of a unit of length of the crust, and upon the density σ of the liquid on which it is supported. The degradation of the undulating curve into a series of circular arcs takes place through f becoming zero. The curvature (p. 444) at an anticlinal in this case becomes infinitely great.

The next step is to conceive a long trough, bent lengthwise into a circular form of large radius, and that the crust and liquid are acted upon by a force gravitating towards the centre, and so we approximate to the case of a section of the earth's surface taken along a great circle. In this case we must suppose the curve reentering. No extraneous force acting, the circular festooned form will approximately represent the curve of equilibrium.

But for simplicity of conception it will be better to return to the idea of the rectangular trough, and to suppose that the curve is in the phase of an anticlinal where it meets the trough at either end; which condition is necessary for equilibrium if there is no friction there: and that there are m festoons.

Having found the form of equilibrium we have now to enquire what relation it holds to the length of the trough.

We will consider that the trough was originally of the length L, and that it is compressed until its length becomes $L(1-\epsilon)$. Then since $\frac{2\rho}{\sigma}c$ is the radius of every circular arc which can admit of equilibrium, and that the chords of the m festoons, or arcs, must equal the reduced length of the trough, putting ϕ for the angle subtended by each festoon, we must have

$$m \,.\, 2 \,.\, \frac{2\rho}{\sigma} c \sin\frac{\phi}{2} = L(1-\epsilon),$$

and because the whole length of the crust is that of the trough before compression,

$$\therefore\ m\frac{2\rho}{\sigma}c\phi = L;$$

whence

$$\frac{\sin\frac{\phi}{2}}{\frac{\phi}{2}} = 1-\epsilon \quad\text{.......... (1)},$$

and

$$m = \frac{\sigma}{2\rho}\,\frac{L}{c\phi} \quad\text{.......... (2)}.$$

Any value of ϵ less than unity substituted in (1) will give a corresponding value of ϕ, and thence from (2) a value of m. But none but integral values of m will be compatible with equilibrium, because the curve must meet either end of the trough at an anticlinal.

ϕ diminishes as m increases, and when m is infinitely great and ϕ infinitely small,

$$\frac{\sin\frac{\phi}{2}}{\frac{\phi}{2}} = 1-\epsilon$$

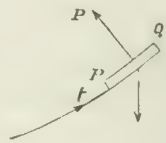
becomes unity, and therefore $\epsilon = 0$, or there is no compression.

m also increases as c diminishes. Hence, *cæteris paribus*, the festoons are more numerous with a thin crust than with a thick one. The geometrical relations show that the lengths of the festoons must increase, and their number diminish, as the compression is increased.

Consider now the crust to be in equilibrium with a certain pair of valves of m and ϵ, m being integral, and let the trough then be shortened, but not sufficiently so for the next integral value of m to be reached. What will happen? The festoons cannot alter their curvature so as to adapt themselves to their lessened chords. It seems then that the compression at the anticlinals must become finite, instead of being zero, and that the ends of the festoons will be pushed up against each other, by which means material will be accumulated there, and Q and R will become finite at the anticlinals.

This will accord with the undulating form of the curve, but it seems that the equilibrium must become unstable, on account of the weight resting on the anticlinal in the highest possible position. It will therefore be liable to slide down upon the surface of the liquid on one side or other of the anticlinal.

Let us suppose then that a portion of the crust PQ, whose weight is Q, has become thickened by the accumulation of material, and let P be the mean fluid-pressure upon this portion of crust. And let it be supposed that the curve is cut off above Q.



Then we have for the equilibrium of PQ,

$$Q \sin\theta = t$$

and

$$Q \cos\theta = P \times PQ.$$

The first of these equations combined with (1) p. 442, and with the equation to the circle, will give the position of the point P to which PQ will descend, balancing the curve below it in the same manner as it would be balanced if the curve were continuous above instead.

The fluid-pressure need not be altered by the above supposition, because the general pressure will keep the anticlinal filled, although some of it should tend to escape above PQ. But if the crust which was upon the other side of the anticlinal, surmounts it, and follows PQ down the incline, it will produce compression above Q, and it seems that Q must, in that case, descend to the synclinal, where if ρ be less than σ it will float and not disturb the general form of the curve beyond it. PQ, while thus descending, will force up the crust on the other side of the festoon, which in its turn surmounting the anticlinal, will descend the next incline. These changes, however, will take place rather by a relative movement of the liquid, than by an absolute movement of the crust, and will cause alternate elevation and depression of any given point in the crust.

It seems probable that the thickened portion of the crust may tend to descend more rapidly than it is followed by the crust surmounting the anticlinal, in which case the compression will be more or less relieved, and fissures formed part of the way down the side of the festoon, through which the liquid would escape.

This seems to offer a possible explanation of the situation of volcanic vents along lines parallel to mountain-chains.

The tendency to heap up matter about the anticlinals, would, in the case of the earth's crust, cause the strata to be accumulated in positions in which the tendency would be for them to slide among themselves and descend superficially to suit the angle of repose, thus causing the successive foldings and secondary corrugations, even on the smallest scale, such as occur on the flanks of the primary or central chains. And the folds might even tend to slope away from the crest, producing inverted bedding.

Some amount of secondary corrugation would also be produced by the bending of the crust into the festoon-like form. But for several reasons it is not possible to say how much. In the first place, this would increase with the thickness of the crust, and diminish with increase of radius of curvature. But in nature the curvature would have been greatest when the crust was thinnest, and would go on diminishing as the thickness increased. In short, if any point in the curve could be conceived to be at no time at an anticlinal, at such a point the curvature would go on perpetually decreasing as the crust grew thicker, and the compression of the general surface increased.

It will also be noticed that the under side of any festoon would be subject to tension from the bending, and disposed to open cracks; while the compression (t) to which it is exposed would tend to keep them closed. The upper side would hence be more corrugated than if there were no such compression as (t).

From the above remarks it does not appear that such an estimate as follows is of much service. Yet it may be as well to notice it.

Let $ECDF$ be a section of a portion of the crust originally a rectangle and bent into such a form that EF and CD are arcs of a circle whose centre is O. And suppose AB, an arc situated between EF and CD, to be the original length of this rectangle.

Let $EF=nAB$; $CD=mAB$; $OC=r$. Then we have

$$r-AC=\frac{r-c}{n}=\frac{r}{m}.$$

The greatest crumpling along EF would occur when $m=1$.

In this case
$$n=1-\frac{c}{r}$$
$$=1-\frac{\sigma}{2\rho}.$$

If $\sigma=\rho$ this would be $\frac{1}{2}$, which is a very large amount. But owing to the probable sliding of the parts over one another, the points E and F would not be upon the radii OA and OB, but fall beyond them, so that the crumpling would not be so great as that.

If the conditions of perfect fluidity in the liquid and perfect flexibility in the crust were fulfilled, and the crust were of insensible thickness, then our result would be true, and the festoons be all equal circular arcs. But under such circumstances as may be supposed to exist in nature, these conditions will be so imperfectly fulfilled, that all we can assert is that the above considerations will serve as a rough guide to what we may expect to occur. The conditions of equilibrium would be satisfied from point to point not very widely distant from each other, and we might expect the compression to be distributed unequally. Hence the festoons would be larger, and the anticlinals higher, in some regions than in others. The local thickenings of the crust would also render the forms of the curve locally to differ from the normal form. The effect of the hydrostatic pressure of the oceans must also be taken account of.

For suppose that any portion of the curve be covered by a heavy fluid (the ocean) of density μ. Let k be the height of this ocean above the level, upon which the origin was taken in the first instance (on p. 442). Then, in forming the equations of equilibrium, we shall have as before

$$t = g\rho c\,(C - y).$$

But for p we shall have,

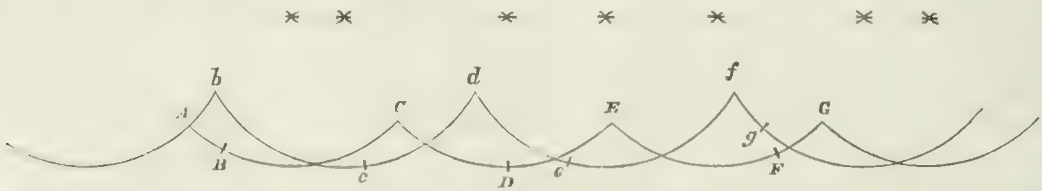
$$\begin{aligned} p &= \text{pressure at } A + \text{pressure due to the depth below } A - \text{pressure due to the ocean,}\\ &= g\sigma\,(h - y) + g\rho\delta - g\mu\,(k - y),\\ &= -g\,(\sigma - \mu)\,y + g\,(\sigma h - \mu k) + g\rho\delta. \end{aligned}$$

This is an equation of the same form as before. We cannot find the values of the constants exactly as we did before, but we can see that the curve must consist of two portions, one above the ocean and one below it.

At the point where these two portions meet one another, the direction of the curve must be continuous. The portion above this point will evidently follow the law already investigated in all respects. For the part below we have expressions for p and t of the same form as before. This part will, therefore, follow the law of a curve, which may be supposed to rest upon a liquid of the density $\sigma - \mu$ instead of σ. The law of form will, therefore, be the same as before, the constants being determined according to this suppositious case.

The *compression* (t) of the part of the curve beneath the ocean will, however, be greater than in the suppositious case, the crest being higher. But that will not alter its *form*, because (t) in any case varies according to the height of the anticlinal. The radius of the portion beneath the ocean will be $2\,\dfrac{\rho}{\sigma - \mu}\,c$.

To estimate how much the curvature would be affected by the ocean, we may consider the density of the subjacent liquid somewhat greater than that of granite (say) 3, and that of sea-water (say) 1. Then $\sigma - \mu = 2$, and the radius will be increased in the ratio of 3 : 2. This decrease of curvature will be independent of the depth of the ocean, and the change will occur at its margin. It may possibly have the effect of opening channels of communication between the surface and the subjacent liquid, and contribute somewhat to the formation of volcanic vents near the seaboard.



The diagram illustrates the effect of increased general compression in elevating and depressing points in the crust. The point A is supposed for simplicity to remain fixed. Then the points B, C, D, E, F, G, upon further compression assume the positions b, c, d, e, f, g.

One reason (p. 448) has already been suggested for alternate elevation and depression at any point. Another cause of the same thing may be seen in the fact that the length of the festoons, and height and distance of the anticlinals, will be increased, both by additional compression, and by a thickening of the crust. The first of these will result from secular cooling. The second from the same cause, and also from the distribution by denudations of the material accumulated, as shewn at p. 447, about the anticlinals. The latter would be more local than the former in its effects.

If we now pass to the case of any rectangular vessel, and suppose it to be compressed in two orthogonal directions, then the form of the corrugations in these two directions would be approximately governed by the laws we have investigated, provided the plane crust be supposed capable of adapting itself to a form not developable into a plane. And even in the case of a contracting sphere, the *character* of the corrugations may be assumed to be on the whole similar to that investigated, and they would be arranged about polygonal areas.

The consequence of the crust not being absolutely flexible would be, that it would rest within certain limits in forms either more or less curved than the proper surface of equilibrium, and it must be confessed that the great ratio, which the thickness of the crust bears to the radius of the festoon, renders the result of the investigation much more difficult of application to the case of nature than it would otherwise be. But we may accept it, and use it for what it is worth.

Still further, the general compression along a great circle will very inadequately represent what would occur when the compression over the whole surface has to be taken into account. In short, the case of nature is so extremely complicated, that it is very difficult to reason satisfactorily upon it. Nevertheless I think that the conclusions we have arrived at concerning the case of a thin heavy flexible crust resting on a perfect liquid, have some value as far as the results may be summed up in the following propositions:—

(1) A vertical section of the lower surface of the crust, where it meets the liquid, when carried across a series of the corrugations, would present a series of festoon-like arcs approximating to a circular form, concave upwards.

(2) The anticlinals would be of cusp-like form, resulting from the intersection of contiguous festoons. There would not be found anything of the character of rolling undulations with flat-crested anticlinals.

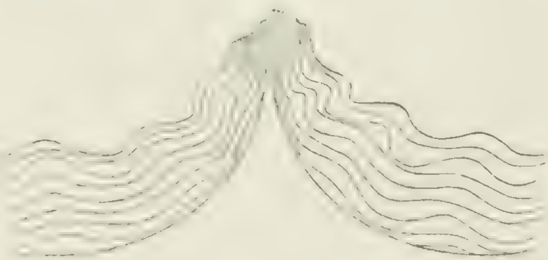
(3) The law of curvature of the festoons would not depend upon the amount of the general compression, though their amplitude, or the distance from crest to crest, would be increased, under circumstances of equilibrium (*i.e.* where m, p. 447, is integral), upon the compression being increased. The curvature would diminish upon the thickness being increased $\left(r = 2\dfrac{\rho}{\sigma}c\right)$.

(4) But the curve of equilibrium, even if it were once exactly established, could not be strictly maintained during successive contractions of the sphere, because we have

seen that, for a trough of given length, a flexible crust and a perfect liquid, equilibrium can only subsist for certain special amounts of contraction having relation to the length of the trough. No doubt the limits within which this would be possible would be enlarged by defects in flexibility and liquidity, and they would also be enlarged by a great increase in the length of the trough as compared to the thickness of the crust: for then m would always be nearly an integer.

This last seems to be the condition of things to which we may best compare the earth's crust, so long as it was thin, when there may also be supposed to have been a state of more complete liquidity in the subjacent matter. But as the crust became thicker, and the layer of subjacent matter probably more viscous, the diminished freedom of movement between the crust and the liquid would tend partially to assimilate the condition of things to the case of the trough of limited length. The behaviour of the crust might therefore be expected to partake of the two explained as belonging to these two cases. There would be a pushing up of the anticlinals against each other owing to the defect of freedom of movement which would assimilate the conditions to those of a trough of limited length, and there would be an alternate elevation and depression of different parts of the curve, the liquid flowing from one anticlinal into the adjoining one, in virtue of such limited freedom of motion as existed between it and the crust. And these conclusions seem to agree pretty well with the phenomena.

I have already remarked on the difficulty introduced by the large ratio which the thickness of the crust bears to the radius of the festoons. But, on the other hand, their curvature will have gone on diminishing as the crust has increased in thickness, to which thickness the material accumulated about the anticlinals will have contributed through the operation of denuding agency, still further diminishing the surface-curvature by its outspreading. The basset edges of the crust situated above the anticlinals of the liquid might occupy a considerable area on the upper surface, on account of its thickness. And throughout this area great disturbance of the strata might be expected to occur, owing to the approximate verticality which the beds would tend to assume, and from which they would settle into positions of repose.



The result noticed at (p. 447), and frequently since referred to, that there would be a tendency for material to accumulate about the anticlinals, which by its weight would tend to descend along the inclined side of the festoon, seems to agree very well with the phenomena. It would produce a fan-like arrangement of the strata about the crest itself, and a crumpling of those on its flanks more intense in its immediate neighbourhood, and diminishing in intensity as we recede from it. The effect would be local, and the crumplings might be on either on a large or a small scale, which would depend chiefly upon the nature of the rocks themselves. This is exactly what occurs, and it must be admitted that, however much it seems necessary to attribute the contortions of rocks to a general contraction of the earth's volume as a primary cause, yet the foldings which offer themselves to view in nature are of such a character, and often of such small dimensions, sometimes only a few feet from bend to bend, that it seems impossible to attribute them *directly*, to what must have operated on so grand a scale as the contraction of the globe itself. Causes have been pointed out why the crests might descend, and lose their pristine elevation. They would then become subject to be planed off by denudation. But if our theory is true, intensely crumpled rocks, wherever found, must have been at some time or another in an elevated position with reference to the neighbouring surface. And where an anticlinal with its accumulated load has sunk, additional compression and crumpling may have been produced by the alteration of the curvature there, but a ridge would nevertheless be preserved, if strong enough to resist denudation, more sharp in its outline perhaps than previously to sinking.

Owing to the anticlinals shifting their positions by a gradual motion parallel to themselves, elevations and depressions where they define a seaboard would be followed rather by the land encroaching upon the sea and the sea upon the land, than by new continents rising abruptly out of mid-ocean or old ones sinking bodily into it.

Not the least important result of the investigation is that, the more ready we conceive the crust of the earth to reply by flexure to the forces brought to bear upon it (and many facts seem to show that with time given it does so with considerable facility), the more does the compression (t) within it become limited to that due to the depth of any point beneath the crest of the anticlinal, whilst at the anticlinal itself the compression tends actually to vanish. This agrees perfectly with the investigations of the late Archdeacon Pratt upon the plumb-line in India, where he found the density of the mountainous tracts less, and of the ocean bed greater, than the average. But it does not support Mr Mallet's views regarding the source of volcanic energy being derived from the crushing of the rocks *. For although it is certain that the contraction of the earth might originate stresses within a *rigid* crust if unrelieved of practically infinite amount, yet if the crust is capable of readily yielding to them, the stresses become comparatively small.

Geologists have often observed that volcanic vents, and outpourings of basaltic rocks, have frequently occurred without much disturbance of the neighbouring strata; almost as if the crust had opened of its own accord and let them escape. A notable instance of such an outpouring of igneous rocks has lately been described by Professor Joseph Le Conte in the *American Journal of Science*, 3rd Series, Vol. VII. p. 167. 1874. He calls it "The great lava flood of the West;" and supposes it to have been poured forth from fissures. It covers nearly 300,000 square miles of the North-Western portion of the States, an area larger than the

* The author of this paper has, as he believes, shown the inadequacy of this cause in his article in the *Phil. Mag.* for October, 1875, entitled 'Mr Mallet's theory of Volcanic Energy tested.'

whole of France, and has an average depth of about 2000 feet. Such fissure eruptions require the compression of the crust locally to vanish in the manner which has been shown in this paper to be not improbable.

And what is the essential character of volcanic vents? They appear from description to be orifices from which superheated gases escape. Molten lava often fills the bottom of a crater for a long period. What keeps it hot? If it were a supply of lava from below, there need be a continuous escape above. But that does not appear to be requisite. It must then be the high temperature of the gases which pass through it, and in so doing support its temperature. If this be a correct explanation, then any place in the crust, which is not steam-tight, may originate a volcano. The gases would fuse their own way through, converting the rocks melted in their passage into lava. But some of the subjacent liquid layer would finally find an exit, on account of the pressure beneath being in general greater than that due to the thickness of the crust.

It must be admitted that the difficulty of accounting for the grand features of oceanic depressions and continental elevations has not been directly removed by the investigations above made. Our theory adapts itself more immediately to the features of smaller dimensions, such as are exhibited in a mountainous country. The remark however which has been already made, that the conditions of equilibrium would be satisfied from point to point, not very widely distant from one another, leads to the conclusion that the more elevated tracts, which form our present continents, may probably be those regions by whose elevation the contraction of the earth has been more recently relieved. The fluid pressure beneath the crust, which is what would tend to burst it upwards elsewhere, being due to the depth of the base of the crust below the highest point of the fluid, would be only slowly conveyed to great distances by a viscous couch; so that compression having been freshly relieved throughout a continental tract, the upward pressure beneath the far-off ocean-bed may be for a long while in abeyance. But after a long interval, during which the continental area has by the cooling of the crust become newly strengthened, the general lateral thrust may reassert its influence, and the ocean-floor, being now the weaker area, and exposed to the greater pressure from beneath, may in its turn become the scene of disturbance, and of elevation, the land in its turn sinking. That irregularities of contour, such as affect the land-surfaces, are continued under a modified form beneath the ocean, has been proved by recent soundings*; leading to the conclusion that its bed consists of old land-surfaces submerged. There is reason however to think that the escape of water from beneath the crust in volcanic eruptions through protracted ages has borne a share in the subsidence of the crust†. And it is a matter of common observation that volcanic areas where the energy is now well-nigh spent are almost always areas of depression.

* See 'A section of the bed of the Pacific from Yokohama to Cape Flattery,' *Nature*, Vol. x. p. 481.

† See the author's paper on the 'Inequalities of the Earth's Surface,' in these *Transactions*, Dec. 1873; also note, p. 440.

IV. *Exercises in Curvilinear and Normal Co-ordinates.* *By the* REV. J. W. WARREN, *Caius College.*

[Read *May* 22, 1876.]

EXERCISE THE FIRST.

CURVILINEAR CO-ORDINATES.

x, y, z representing the rectangular Cartesian co-ordinates of a point in space, and U being any function of these variables, suppose we write

$$u_1 = \phi_1(x, y, z), \quad u_2 = \phi_2(x, y, z), \quad u_3 = \phi_3(x, y, z).$$

U, by aid of these equations, may be expressed in terms of u_1, u_2, u_3.

Write

$$\frac{dx}{du_1} = a_1, \quad \frac{dy}{du_1} = a_2, \quad \frac{dz}{du_1} = a_3,$$

$$\frac{dx}{du_2} = b_1, \quad \frac{dy}{du_2} = b_2, \quad \frac{dz}{du_2} = b_3,$$

$$\frac{dx}{du_3} = c_1, \quad \frac{dy}{du_3} = c_2, \quad \frac{dz}{du_3} = c_3.$$

Hence

$$dx = a_1 du_1 + b_1 du_2 + c_1 du_3,$$

$$dy = a_2 du_1 + b_2 du_2 + c_2 du_3,$$

$$dz = a_3 du_1 + b_3 du_2 + c_3 du_3.$$

Write

$$B = \begin{vmatrix} a_1, & a_2, & a_3, \\ b_1, & b_2, & b_3, \\ c_1, & c_2, & c_3, \end{vmatrix}$$

Hence

$$Bdu_1 = \frac{dB}{da_1} dx + \frac{dB}{da_2} dy + \frac{dB}{da_3} dz,$$

$$Bdu_2 = \frac{dB}{db_1} dx + \frac{dB}{db_2} dy + \frac{dB}{db_3} dz,$$

$$Bdu_3 = \frac{dB}{dc_1} dx + \frac{dB}{dc_2} dy + \frac{dB}{dc_3} dz.$$

Hence
$$\frac{dU}{dx}=\frac{1}{B}\left\{\frac{dU}{du_1}\frac{dB}{da_1}+\frac{dU}{du_2}\frac{dB}{db_1}+\frac{dU}{du_3}\frac{dB}{dc_1}\right\},$$
$$\frac{dU}{dy}=\frac{1}{B}\left\{\frac{dU}{du_1}\frac{dB}{da_2}+\frac{dU}{du_2}\frac{dB}{db_2}+\frac{dU}{du_3}\frac{dB}{dc_2}\right\},$$
$$\frac{dU}{dz}=\frac{1}{B}\left\{\frac{dU}{du_1}\frac{dB}{da_3}+\frac{dU}{du_2}\frac{dB}{db_3}+\frac{dU}{du_3}\frac{dB}{dc_3}\right\}.$$

Write $dx^2+dy^2+dz^2$ equal to
$$C_{11}du_1^2+C_{22}du_2^2+C_{33}du_3^2+2C_{12}du_1du_2+2C_{23}du_2du_3+2C_{31}du_3du_1.$$

Write also $\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2+\left(\frac{dU}{dz}\right)^2$ equal to
$$A_{11}\left(\frac{dU}{du_1}\right)^2+A_{22}\left(\frac{dU}{du_2}\right)^2+A_{33}\left(\frac{dU}{du_3}\right)^2+2A_{12}\frac{dU}{du_1}\frac{dU}{du_2}+2A_{23}\frac{dU}{du_2}\frac{dU}{du_3}+2A_{31}\frac{dU}{du_3}\frac{dU}{du_1}.$$

It is sometimes convenient when we wish to compare our results with those arrived at by previous Authors to write
$$C_{11}=E,\quad C_{22}=G,\quad C_{33}=I,$$
$$C_{12}=F,\quad C_{23}=H,\quad C_{31}=J.$$

It is clear from the preceding equations that C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, C_{31} are functions of
$$A_{11},\ A_{22},\ A_{33},\ A_{12},\ A_{23},\ A_{31}.$$

The converse being also true, in fact we clearly have
$$C_{11}=E=a_1^2+a_2^2+a_3^2,$$
$$C_{22}=G=b_1^2+b_2^2+b_3^2,$$
$$C_{33}=I=c_1^2+c_2^2+c_3^2,$$
$$C_{12}=F=a_1b_1+a_2b_2+a_3b_3,$$
$$C_{23}=H=b_1c_1+b_2c_2+b_3c_3,$$
$$C_{31}=J=c_1a_1+c_2a_2+c_3a_3.$$

We also have
$$A_{11}=\frac{1}{B^2}\left\{\left(\frac{dB}{da_1}\right)^2+\left(\frac{dB}{da_2}\right)^2+\left(\frac{dB}{da_3}\right)^2\right\},$$
$$A_{22}=\frac{1}{B^2}\left\{\left(\frac{dB}{db_1}\right)^2+\left(\frac{dB}{db_2}\right)^2+\left(\frac{dB}{db_3}\right)^2\right\},$$
$$A_{33}=\frac{1}{B^2}\left\{\left(\frac{dB}{dc_1}\right)^2+\left(\frac{dB}{dc_2}\right)^2+\left(\frac{dB}{dc_3}\right)^2\right\},$$
$$A_{12}=\frac{1}{B^2}\left\{\frac{dB}{da_1}\frac{dB}{db_1}+\frac{dB}{da_2}\frac{dB}{db_2}+\frac{dB}{da_3}\frac{dB}{db_3}\right\},$$
$$A_{23}=\frac{1}{B^2}\left\{\frac{dB}{db_1}\frac{dB}{dc_1}+\frac{dB}{db_2}\frac{dB}{dc_2}+\frac{dB}{db_3}\frac{dB}{dc_3}\right\},$$
$$A_{31}=\frac{1}{B^2}\left\{\frac{dB}{dc_1}\frac{dB}{da_1}+\frac{dB}{dc_2}\frac{dB}{da_2}+\frac{dB}{dc_3}\frac{dB}{da_3}\right\}.$$

Hence clearly we can write

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}^2 = \begin{vmatrix} E, & F, & J \\ F, & G, & H \\ J, & H, & I \end{vmatrix} = \begin{vmatrix} C_{11}, & C_{12}, & C_{13} \\ C_{12}, & C_{22}, & C_{23} \\ C_{13}, & C_{23}, & C_{33} \end{vmatrix} = B^2,$$

$$\frac{1}{B^6}\begin{vmatrix} \frac{dB}{da_1}, & \frac{dB}{da_2}, & \frac{dB}{da_3} \\ \frac{dB}{db_1}, & \frac{dB}{db_2}, & \frac{dB}{db_3} \\ \frac{dB}{dc_1}, & \frac{dB}{dc_2}, & \frac{dB}{dc_3} \end{vmatrix}^2 = \begin{vmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \\ A_{13}, & A_{23}, & A_{33} \end{vmatrix} = \frac{1}{B^4};$$

$$C_{11}C_{22} - C_{12}{}^2 = B^2 \,.\, A_{33}; \quad B^2 (A_{11}A_{22} - A_{12}{}^2) = C_{33};$$

$$C_{22}C_{33} - C_{23}{}^2 = B^2 \,.\, A_{11}; \quad B^2 (A_{22}A_{33} - A_{23}{}^2) = C_{11};$$

$$C_{33}C_{11} - C_{13}{}^2 = B^2 \,.\, A_{22}; \quad B^2 (A_{33}A_{11} - A_{13}{}^2) = C_{22};$$

$$C_{31}C_{32} - C_{12}C_{33} = B^2A_{12}; \quad B^2 (A_{31}A_{32} - A_{12}A_{33}) = C_{12};$$

$$C_{12}C_{31} - C_{23}C_{11} = B^2A_{23}; \quad B^2 (A_{12}A_{31} - A_{23}A_{11}) = C_{23};$$

$$C_{12}C_{32} - C_{31}C_{22} = B^2A_{31}; \quad B^2 (A_{12}A_{32} - A_{31}A_{22}) = C_{31};$$

and since

$$A_{11} = \frac{1}{B^2}\left\{\left(\frac{dB}{da_1}\right)^2 + \left(\frac{dB}{da_2}\right)^2 + \left(\frac{dB}{da_3}\right)^2\right\},$$

$$A_{12} = \frac{1}{B^2}\left\{\frac{dB}{da_1}\,\frac{dB}{db_1} + \frac{dB}{da_2}\,\frac{dB}{db_2} + \frac{dB}{da_3}\,\frac{dB}{db_3}\right\},$$

$$A_{13} = \frac{1}{B^2}\left\{\frac{dB}{da_1}\,\frac{dB}{dc_1} + \frac{dB}{da_2}\,\frac{dB}{dc_2} + \frac{dB}{da_3}\,\frac{dB}{dc_3}\right\},$$

it clearly follows that

$$\frac{1}{B}\frac{dB}{da_1} = A_{11}a_1 + A_{12}b_1 + A_{13}c_1,$$

$$\frac{1}{B}\frac{dB}{da_2} = A_{11}a_2 + A_{12}b_2 + A_{13}c_2,$$

$$\frac{1}{B}\frac{dB}{da_3} = A_{11}a_3 + A_{12}b_3 + A_{13}c_3;$$

and symmetrically we also have

$$\frac{1}{B}\frac{dB}{db_1} = A_{22}b_1 + A_{23}c_1 + A_{12}a_1,$$

$$\frac{1}{B}\frac{dB}{db_2} = A_{22}b_2 + A_{23}c_2 + A_{12}a_2,$$

$$\frac{1}{B}\frac{dB}{db_3} = A_{22}b_3 + A_{23}c_3 + A_{12}a_3,$$

$$\frac{1}{B}\frac{dB}{dc_1} = A_{33}c_1 + A_{13}a_1 + A_{23}b_1,$$

$$\frac{1}{B}\frac{dB}{dc_2} = A_{33}c_2 + A_{13}a_2 + A_{23}b_2,$$

$$\frac{1}{B}\frac{dB}{dc_3} = A_{33}c_3 + A_{13}a_3 + A_{23}b_3.$$

It is clear that we can obtain from the previously written down values of C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, C_{31}, nine formulæ similar to the above nine expressing the values of a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, linearly in terms of

$$\frac{1}{B}\frac{dB}{da_1},\ \frac{1}{B}\frac{dB}{da_2},\ \frac{1}{B}\frac{dB}{da_3},\ \text{\&c., \&c.}$$

These formulæ clearly are

$$a_1 = C_{11}\cdot\frac{1}{B}\cdot\frac{dB}{da_1} + C_{12}\cdot\frac{1}{B}\cdot\frac{dB}{db_1} + C_{13}\cdot\frac{1}{B}\cdot\frac{dB}{dc_1},$$

$$a_2 = C_{11}\cdot\frac{1}{B}\cdot\frac{dB}{da_2} + C_{12}\cdot\frac{1}{B}\cdot\frac{dB}{db_2} + C_{13}\cdot\frac{1}{B}\cdot\frac{dB}{dc_2},$$

$$a_3 = C_{11}\cdot\frac{1}{B}\cdot\frac{dB}{da_3} + C_{12}\cdot\frac{1}{B}\cdot\frac{dB}{db_3} + C_{13}\cdot\frac{1}{B}\cdot\frac{dB}{dc_3};$$

$$b_1 = C_{22}\cdot\frac{1}{B}\cdot\frac{dB}{db_1} + C_{23}\cdot\frac{1}{B}\cdot\frac{dB}{dc_1} + C_{12}\cdot\frac{1}{B}\cdot\frac{dB}{da_1},$$

$$b_2 = C_{22}\cdot\frac{1}{B}\cdot\frac{dB}{db_2} + C_{23}\cdot\frac{1}{B}\cdot\frac{dB}{dc_2} + C_{12}\cdot\frac{1}{B}\cdot\frac{dB}{da_2},$$

$$b_3 = C_{22}\cdot\frac{1}{B}\cdot\frac{dB}{db_3} + C_{23}\cdot\frac{1}{B}\cdot\frac{dB}{dc_3} + C_{12}\cdot\frac{1}{B}\cdot\frac{dB}{da_3};$$

$$c_1 = C_{33}\cdot\frac{1}{B}\cdot\frac{dB}{dc_1} + C_{31}\cdot\frac{1}{B}\cdot\frac{dB}{da_1} + C_{32}\cdot\frac{1}{B}\cdot\frac{dB}{db_1},$$

$$c_2 = C_{33}\cdot\frac{1}{B}\cdot\frac{dB}{dc_2} + C_{31}\cdot\frac{1}{B}\cdot\frac{dB}{da_2} + C_{32}\cdot\frac{1}{B}\cdot\frac{dB}{db_2},$$

$$c_3 = C_{33}\cdot\frac{1}{B}\cdot\frac{dB}{dc_3} + C_{31}\cdot\frac{1}{B}\cdot\frac{dB}{da_3} + C_{32}\cdot\frac{1}{B}\cdot\frac{dB}{db_3}.$$

We also clearly have the nine formulæ

$$A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} = 1,$$
$$A_{22}C_{22} + A_{23}C_{23} + A_{21}C_{21} = 1,$$
$$A_{33}C_{33} + A_{31}C_{31} + A_{32}C_{32} = 1,$$

....

$$A_{11}C_{12} + A_{12}C_{22} + A_{13}C_{23} = 0,$$
$$A_{11}C_{13} + A_{12}C_{23} + A_{13}C_{33} = 0,$$

....

$$A_{22}C_{23} + A_{23}C_{33} + A_{21}C_{31} = 0,$$
$$A_{22}C_{12} + A_{23}C_{13} + A_{21}C_{11} = 0.$$

....

$$A_{33}C_{13} + A_{31}C_{11} + A_{32}C_{12} = 0,$$
$$A_{33}C_{23} + A_{31}C_{21} + A_{32}C_{22} = 0.$$

We have also the formulæ

$$-\frac{2}{B}dB = C_{11}\,.\,dA_{11} + C_{22}\,.\,dA_{22} + C_{33}\,.\,dA_{33}\,,$$

$$+2C_{23}\,.\,dA_{23}\,,$$

$$+2C_{13}\,.\,dA_{13}\,,$$

$$+2C_{12}\,.\,dA_{12}\,,$$

$$+\frac{2}{B}dB = A_{11}\,.\,dC_{11} + A_{22}\,.\,dC_{22} + A_{33}\,,\,dC_{33}\,,$$

$$+2A_{23}\,.\,dC_{23}\,,$$

$$+2A_{13}\,.\,dC_{13}\,,$$

$$+2A_{12}\,.\,dC_{12}\,.$$

All these formulæ as yet given express relations between the quantities U, u_1, u_2, u_3, x, y, z, and the differential coefficients of the *first* order of one or more of these quantities with respect to one of the quantities u_1, u_2, u_3, x, y, z; now, however, we proceed to determine the values of and relations between the differential coefficients of the *second* order.................. If we refer to the values of C_{11}, C_{22}, C_{33}, C_{23}, C_{31}, C_{12}, in terms of a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; it is clear that we may write

$$a_1\,.\,\frac{da_1}{du_1} + a_2\,.\,\frac{da_2}{du_1} + a_3\,.\,\frac{da_3}{du_1} = \frac{1}{2}\,\frac{dE}{du_1}\,,$$

$$b_1\,.\,\frac{da_1}{du_1} + b_2\,.\,\frac{da_2}{du_1} + b_3\,.\,\frac{da_3}{du_1} = \frac{dF}{du_1} - \frac{1}{2}\,\frac{dE}{du_2}\,,$$

$$c_1\,.\,\frac{da_1}{du_1} + c_2\,.\,\frac{da_2}{du_1} + c_3\,.\,\frac{da_3}{du_1} = \frac{dJ}{du_1} - \frac{1}{2}\,\frac{dE}{du_3}\,;$$

it is clear that we may also write

$$a_1\,.\,\frac{da_1}{du_2} + a_2\,.\,\frac{da_2}{du_2} + a_3\,.\,\frac{da_3}{du_2} = \frac{1}{2}\,\frac{dE}{du_2}\,,$$

$$b_1\,.\,\frac{da_1}{du_2} + b_2\,.\,\frac{da_2}{du_2} + b_3\,.\,\frac{da_3}{du_2} = \frac{1}{2}\,\frac{dG}{du_1}\,,$$

$$c_1\,.\,\frac{da_1}{du_2} + c_2\,.\,\frac{da_2}{du_2} + c_3\,.\,\frac{da_3}{du_2} = \frac{1}{2}\,\left\{\frac{dH}{du_1} + \frac{dJ}{du_2} - \frac{dF}{du_3}\right\}.$$

Hence by elimination we arrive at the equations

$$\frac{d^2x}{du_1^2} = \frac{da_1}{du_1} = \frac{1}{B}\,.\,\frac{dB}{da_1}\,.\,\frac{1}{2}\,\frac{dE}{du_1}\,,$$

$$+\frac{1}{B}\,.\,\frac{dB}{db_1}\,.\,\left\{\frac{dF}{du_1} - \frac{1}{2}\,\frac{dE}{du_2}\right\},$$

$$+\frac{1}{B}\,.\,\frac{dB}{dc_1}\,.\,\left\{\frac{dJ}{du_1} - \frac{1}{2}\,\frac{dE}{du_3}\right\};$$

$$\frac{d^2x}{du_1du_2}=\frac{da_1}{du_2}=\frac{db_1}{du_1} \text{ equals}$$

$$\frac{1}{B}\cdot\frac{dB}{da_1}\cdot\frac{1}{2}\frac{dE}{du_2}+\frac{1}{B}\cdot\frac{dB}{db_1}\cdot\frac{1}{2}\cdot\frac{dG}{du_1}$$

$$+\frac{1}{B}\cdot\frac{dB}{dc_1}\cdot\frac{1}{2}\left\{\frac{dH}{du_1}+\frac{dJ}{du_2}-\frac{dF}{du_3}\right\},$$

and clearly there are in all eighteen of such equations; viz. nine of the first species, and nine of the second.

By the previous written values of $\frac{1}{B}$, $\frac{dB}{da_1}$, &c., these two equations may clearly be written

$$\frac{d^2x}{du_1^2}=\frac{1}{2}\frac{dE}{du_1}\cdot\left\{A_{11}a_1+A_{12}b_1+A_{13}c_1\right\}$$

$$+\left(\frac{dF}{du_1}-\frac{1}{2}\frac{dE}{du_2}\right)\left\{A_{22}b_1+A_{23}c_1+A_{12}a_1\right\}$$

$$+\left(\frac{dJ}{du_1}-\frac{1}{2}\frac{dE}{du_3}\right)\left\{A_{33}c_1+A_{13}a_1+A_{23}b_1\right\}$$

....

$$\frac{d^2x}{du_1du_2}=\frac{1}{2}\frac{dE}{du_2}\left\{A_{11}a_1+A_{12}b_1+A_{13}c_1\right\}$$

$$+\frac{1}{2}\frac{dG}{du_1}\left\{A_{22}b_1+A_{23}c_1+A_{12}a_1\right\}$$

$$+\frac{1}{2}\left\{\frac{dH}{du_1}+\frac{dJ}{du_2}-\frac{dF}{du_3}\right\}\left\{A_{33}c_1+A_{13}a_1+A_{23}b_1\right\}.$$

We might now proceed to calculate

$$\frac{d^2U}{dx^2},\ \frac{d^2U}{dy^2},\ \frac{d^2U}{dz^2},\ \frac{d^2U}{dxdy},\ \&c.;$$

and the various geometrical quantities that are expressible in terms of these differential coefficients might thus be expressed in terms of Curvilinear co-ordinates. We might thus calculate a radius of normal curvature of any one of the three surfaces u_1, u_2, u_3; and thus in particular we might calculate the six radii of normal curvature due to the intersection of the surfaces u_1, u_2, u_3; such investigations, however, we shall find it unnecessary to undertake, inasmuch as they can be more simply arrived at by a change of the independent variables in the corresponding formulæ in Normal Co-ordinates.

EXERCISE THE SECOND.

Normal Co-ordinates.

The general system of curvilinear co-ordinates involves expressions which are functions of A_{11}, A_{22}, A_{33}, A_{12}, A_{23}, A_{13}, and therefore of C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, C_{13}, as well as of their various differential coefficients with respect to u_1, u_2, u_3; it is obvious that such a system is a little complicated, and therefore we may enquire whether any particular systems exist less cumbrous in form and capable of being applied to obtain useful results.

One such system of co-ordinates is obtained by writing A_{12}, A_{23}, A_{13}, each equal to zero. It is clear from the values found for du_1, du_2, du_3 in the previous chapter that such a system of co-ordinates are orthogonal, that is to say u_1, u_2, and u_3, mutually cut at right angles. I shall not pause at present to consider this particular system of co-ordinates, but I shall return to their consideration in a future chapter.

A second simplified case of the general system of co-ordinates is got by taking as our independent variables not the surfaces u_1, u_2, u_3, but the length of the intercepted normals between u_1, u_2, u_3, and three fixed surfaces parallel to u_1, u_2, u_3, respectively. Let us call the lengths of these three normals n_1, n_2, n_3; then x, y, z, representing the co-ordinates of a point common to u_1, u_2, u_3, and therefore to n_1, n_2, n_3. Let x_1, y_1, z_1 denote the co-ordinates of the remaining end of n_1; x_2, y_2, z_2 the co-ordinates of the remaining end of n_2; and x_3, y_3, z_3 the co-ordinates of the remaining end of n_3. We clearly have therefore

$$n_1^2=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2,$$
$$n_2^2=(x-x_2)^2+(y-y_2)^2+(z-z_2)^2,$$
$$n_3^2=(x-x_3)^2+(y-y_3)^2+(z-z_3)^2.$$

We clearly have therefore

$$\left(\frac{dn_1}{dx}\right)^2+\left(\frac{dn_1}{dy}\right)^2+\left(\frac{dn_1}{dz}\right)^2=1,$$
$$\left(\frac{dn_2}{dx}\right)^2+\left(\frac{dn_2}{dy}\right)^2+\left(\frac{dn_2}{dz}\right)^2=1,$$
$$\left(\frac{dn_3}{dx}\right)^2+\left(\frac{dn_3}{dy}\right)^2+\left(\frac{dn_3}{dz}\right)^2=1,$$
$$\frac{dn_2}{dx}\cdot\frac{dn_3}{dx}+\frac{dn_2}{dy}\cdot\frac{dn_3}{dy}+\frac{dn_2}{dz}\cdot\frac{dn_3}{dz}=\cos\theta,$$
$$\frac{dn_3}{dx}\cdot\frac{dn_1}{dx}+\frac{dn_3}{dy}\cdot\frac{dn_1}{dy}+\frac{dn_3}{dz}\cdot\frac{dn_1}{dz}=\cos\phi,$$
$$\frac{dn_1}{dx}\cdot\frac{dn_2}{dx}+\frac{dn_1}{dy}\cdot\frac{dn_2}{dy}+\frac{dn_1}{dz}\cdot\frac{dn_2}{dz}=\cos\psi,$$

where θ equals the angles between n_2 and n_3, ϕ equals the angle between n_1 and n_3, and ψ between n_1 and n_2.

The system of co-ordinates I have just mentioned I shall call *normal co-ordinates.* It is obvious that they introduce much simplicity into our general formulæ inasmuch as for such a system of co-ordinates we clearly have by the above equations

$$A_{11} = A_{22} = A_{33} = 1,$$

$$A_{23} = \cos\theta, \qquad A_{13} = \cos\phi, \qquad A_{12} = \cos\psi.$$

I shall for shortness in the case of normal co-ordinates write for A_{23}, A_{13}, A_{12}; A_1, A_2, A_3 respectively, retaining A_{12}, A_{23}, A_{31} for the general system alone. I shall also denote by $2S_1$, $2S_2$, $2S_3$, P_{12}, P_{23}, P_{31} what C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, C_{31} respectively become when we pass from general co-ordinates to the particular case of normal co-ordinates. It is obvious that S_1, S_2, S_3 are proportional to the squares of the sines of the angles between n_2 and n_3, n_1 and n_3, n_1 and n_2, whilst P_{12}, P_{23}, P_{13} are as the cosines of the angles between the three curves formed by the intersection of u_1, u_2, u_3.

It is hardly needful to write down all the formulæ of the general system again thus simplified by the use of normal co-ordinates; but the ten following formulæ I shall so frequently make use of that perhaps their exhibition will be useful:

$$P_{12} + 2A_3 \,.\, S_2 + A_2 \,.\, P_{23} = 0,$$
$$P_{13} + A_3 \,.\, P_{23} + 2A_2 \,.\, S_3 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P_{23} + 2A_1 \,.\, S_3 + A_3 \,.\, P_{13} = 0,$$
$$P_{21} + A_1 \,.\, P_{13} + 2A_3 \,.\, S_1 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P_{31} + 2A_2 \,.\, S_1 + A_1 \,.\, P_{12} = 0,$$
$$P_{32} + A_2 \,.\, P_{21} + 2A_1 \,.\, S_2 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$2S_1 + A_3 \,.\, P_{12} + A_2 \,.\, P_{13} = 1,$$
$$2S_2 + A_1 \,.\, P_{23} + A_3 \,.\, P_{12} = 1,$$
$$2S_3 + A_1 \,.\, P_{23} + A_2 \,.\, P_{13} = 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-\frac{dB}{B} = P_{23} \,.\, dA_1 + P_{13} \,.\, dA_2 + P_{12} \,.\, dA_3;$$

we have also obviously

$$\frac{1}{B^2} = 2A_1A_2A_3 + 1 - A_1^2 - A_2^2 - A_3^2.$$

I should here perhaps mention the theorem that if normals be drawn at every point of a given surface, and if equal lengths be taken along these normals, the surface locus of these points has also these normals of the first surface for its normals, this theorem which is easily demonstrated gives a clear conception of the manner in which the three normals become independent variables.

The three normal co-ordinates of a point in space having been supposed to be given, and the three fixed surfaces from which the normals start and to which u_1, u_2, u_3 move constantly parallel having been assigned, it is obvious that the angles between the normals, and therefore A_1, A_2, A_3, are fully determined; accordingly there must exist certain equations connecting the values of A_1, A_2, A_3 with those of n_1, n_2, n_3, and the object of the remainder of this Exercise will be to obtain six differential equations of the second order between A_1, A_2 and A_3, and n_1, n_2 and n_3.

I proceed to determine these six differential equations, first pointing out a few formulæ and geometric facts in relation to normal co-ordinates useful for our after calculations.

Taking unit lengths along the three normals that meet at any point, we clearly form a spherical triangle whose sides are θ, ϕ, ψ, and whose angles are $180-\Theta$, $180-\Phi$, $180-\Psi$, where Θ, Φ and Ψ are the angles between the three curves formed by the mutual intersection of the surfaces u_1, u_2, u_3 to which n_1, n_2, n_3 are normals. We write

$$\sin\Theta = k\,.\sin\theta\,; \quad \sin\Phi = k\,.\sin\phi\,; \quad \sin\Psi = k\,.\sin\psi\,;$$

by the ordinary formulæ of spherical trigonometry we therefore have

$$k^2 = \frac{\begin{vmatrix} 1, & \cos\psi, & \cos\phi \\ \cos\psi, & 1, & \cos\theta \\ \cos\phi, & \cos\theta, & 1 \end{vmatrix}}{\sin^2\theta\,.\sin^2\phi\,.\sin^2\psi}, \qquad k^4 = \frac{\begin{vmatrix} 1, & \cos\Psi, & \cos\Phi \\ \cos\Psi, & 1, & \cos\Theta \\ \cos\Phi, & \cos\Theta, & 1 \end{vmatrix}}{\sin^2\theta\,.\sin^2\phi\,.\sin^2\psi}\,;$$

$$\frac{1}{k^2} = B^2\,.\sin^2\theta\,.\sin^2\phi\,.\sin^2\psi.$$

If we now consider the parallelopiped formed by the three surfaces u_1, u_2, u_3 and three others very close to these three and parallel to them, and if we denote by ds_1 a side of this parallelopiped formed by the intersection of the surfaces u_2 and u_3, by ds_2 a side formed by the intersection of u_1 and u_3, and by ds_3 a side formed by the intersection of u_1 and u_2; we have then, by the known formulæ of solid geometry, for the area of our parallelopiped, the formula

$$\begin{vmatrix} 1, & \cos\Psi, & \cos\Phi \\ \cos\Psi, & 1, & \cos\Theta \\ \cos\Phi, & \cos\Theta, & 1 \end{vmatrix}^{\frac{1}{2}} \times ds_1 ds_2 ds_3 :$$

but this volume also equals dn_1 multiplied by area of face of parallelopiped to which dn_1 is perpendicular; i.e. the volume equals $dn_1\,.\,ds_2\,.\,ds_3\,.\sin\Theta$, whence

$$dn_1 \,.\, \sin\Theta = \begin{vmatrix} 1, & \cos\Psi, & \cos\Phi \\ \cos\Psi, & 1, & \cos\Theta \\ \cos\Phi, & \cos\Theta, & 1 \end{vmatrix}^{\frac{1}{2}} \times ds_1,$$

$$\therefore\; dn_1 \,.\, k \,.\, \sin\theta = k^2 \,.\, \sin\theta \,.\, \sin\phi \,.\, \sin\psi \,.\, ds_1;$$

whence we obtain $ds_1 = B\sin\theta\, dn_1$. Similarly we obtain

$$ds_2 = B \,.\, \sin\phi \,.\, dn_2; \quad ds_3 = B \,.\, \sin\psi \,.\, dn_3.$$

Now by our notation we have

$$a_1 = \frac{dx}{dn_1}, \quad b_1 = \frac{dx}{dn_2}, \quad c_1 = \frac{dx}{dn_3};$$

with similar formulæ for

$$a_2,\, b_2,\, c_2, \text{ and } a_3,\, b_3,\, c_3.$$

Now $\dfrac{dx}{dn_1} \,.\, dn_1$ is clearly the change of x caused by us going a distance ds_1 along the intersection of the two surfaces u_2 and u_3; calling this change Δx, and the cosine of the angle that ds_1 makes with the axis of x, $\cos\alpha_1'$, we clearly have

$$\cos\alpha_1' = \Delta x \div ds_1,$$

whence $$\cos\alpha_1' \,.\, ds_1 = \Delta x = \frac{dx}{dn_1}\, dn_1 = a_1 dn.$$

Whence we obtain $a_1 = B\sin\theta \,.\, \cos\alpha_1'$; and thus, if β_1', γ_1', &c. represent symmetric angles to α_1', we easily obtain

$$a_1 = B \,.\, \sin\theta \,.\, \cos\alpha_1'; \quad a_2 = B \,.\, \sin\theta \,.\, \cos\beta_1'; \quad a_3 = B \,.\, \sin\theta \,.\, \cos\gamma_1';$$

$$b_1 = B \,.\, \sin\phi \,.\, \cos\alpha_2'; \quad b_2 = B \,.\, \sin\phi \,.\, \cos\beta_2'; \quad b_3 = B \,.\, \sin\phi \,.\, \cos\gamma_2';$$

$$c_1 = B \,.\, \sin\psi \,.\, \cos\alpha_3'; \quad c_2 = B \,.\, \sin\psi \,.\, \cos\beta_3'; \quad c_3 = B \,.\, \sin\psi \,.\, \cos\gamma_3'.$$

Let now $\alpha_1\beta_1\gamma_1$ be the director angles of n_1, $\alpha_2\beta_2\gamma_2$ of n_2, and $\alpha_3\beta_3\gamma_3$ of n_3. We know that

$$\sin\Theta \,.\, \cos\alpha_1 = \cos\beta_2' \,.\, \cos\gamma_3' - \cos\beta_3' \,.\, \cos\gamma_2';$$

$$\therefore\; k \,.\, \sin\theta \,.\, \cos\alpha_1 = \cos\beta_2' \,.\, \cos\gamma_3' - \cos\beta_3' \,.\, \cos\gamma_2';$$

$$\therefore\; \frac{\cos\alpha_1}{B \,.\, \sin\phi \,.\, \sin\psi} = \cos\beta_2' \,.\, \cos\gamma_3' - \cos\beta_3' \,.\, \cos\gamma_2';$$

$$\therefore\; B\cos\alpha_1 = b_2c_3 - b_3c_2.$$

Hence by symmetry we clearly have

$$B \,.\, \cos\alpha_1 = b_2c_3 - b_3c_2; \quad B \,.\, \cos\alpha_2 = c_2a_3 - c_3a_2; \quad B \,.\, \cos\alpha_3 = a_2b_3 - a_3b_2;$$

$$B \,.\, \cos\beta_1 = b_3c_1 - b_1c_3; \quad B \,.\, \cos\beta_2 = c_3a_1 - c_1a_3; \quad B \,.\, \cos\beta_3 = a_3b_1 - a_1b_3;$$

$$B \,.\, \cos\gamma_1 = b_1c_2 - b_2c_1; \quad B \,.\, \cos\gamma_2 = c_1a_2 - c_2a_1; \quad B \,.\, \cos\gamma_3 = a_1b_2 - a_2b_1.$$

It served as an exercise to obtain these last nine formulæ thus; but we might also easily obtain them by simply solving the equations

$$dx = a_1 dn_1 + b_1 dn_2 + c_1 dn_3,$$
$$dy = a_2 dn_1 + b_2 dn_2 + c_2 dn_3,$$
$$dz = a_3 dn_1 + b_3 dn_2 + c_3 dn_3,$$

for dn_1, dn_2, dn_3, and then comparing the result with the formulæ

$$n_1 . dn_1 = (x - x_1)\, dx + (y - y_1)\, dy + (z - z_1)\, dz,$$
$$n_2 . dn_2 = (x - x_2)\, dx + (y - y_2)\, dy + (z - z_2)\, dz,$$
$$n_3 . dn_3 = (x - x_3)\, dx + (y - y_3)\, dy + (z - z_3)\, dz.$$

If we now consider such an expression as

$$\left\{\cos\alpha_3 . \frac{da_1}{dn_2} + \cos\beta_3 . \frac{da_2}{dn_2} + \cos\gamma_3 . \frac{da_3}{dn_2}\right\},$$

we clearly see by the preceding formulæ that it may also be written in the form

$$\frac{1}{B}\begin{vmatrix} \dfrac{da_1}{dn_2}, & \dfrac{da_2}{dn_2}, & \dfrac{da_3}{dn_2} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{vmatrix};$$

and hence obviously we have

$$\begin{aligned}
&\left\{\cos\alpha_3 . \frac{da_1}{dn_2} + \cos\beta_3 . \frac{da_2}{dn_2} + \cos\gamma_3 . \frac{da_3}{dn_2}\right\}^2 \\
&- \left\{\cos\alpha_3 . \frac{da_1}{dn_1} + \cos\beta_3 . \frac{da_2}{dn_1} + \cos\gamma_3 . \frac{da_3}{dn_1}\right\} \\
&\times \left\{\cos\alpha_3 . \frac{db_1}{dn_2} + \cos\beta_3 . \frac{db_2}{dn_2} + \cos\gamma_3 . \frac{db_3}{dn_2}\right\} \\
&- \frac{1}{B^2}\begin{vmatrix} \dfrac{da_1}{dn_2}, & \dfrac{da_2}{dn_2}, & \dfrac{da_3}{dn_2} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{vmatrix}^2 \\
&+ \frac{1}{B^2}\begin{vmatrix} \dfrac{da_1}{dn_1}, & \dfrac{da_2}{dn_1}, & \dfrac{da_3}{dn_1} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{vmatrix} \times \begin{vmatrix} \dfrac{db_1}{dn_2}, & \dfrac{db_2}{dn_2}, & \dfrac{db_3}{dn_2} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{vmatrix} = 0.
\end{aligned}$$

Now by means of the values of $\dfrac{da_1}{dn_2}$, &c., implicitly given at the close of Exercise the first on Curvilinear Co-ordinates, the first three lines of the preceding equation can obviously be expressed in terms of A_1, A_2, A_3 and the first differential coefficients of

S_1, S_2, S_3, P_{12}, P_{23}, P_{13} with respect to n_1, n_2, n_3; also the determinants when squared or multiplied clearly only contain the terms or constituents

$$-\left\{\left(\frac{da_1}{dn_2}\right)^2 + \left(\frac{da_2}{dn_2}\right)^2 + \left(\frac{da_3}{dn_2}\right)^2\right\}$$

$$+\left\{\left(\frac{da_1}{dn_1}\right)\left(\frac{db_1}{dn_2}\right) + \left(\frac{da_2}{dn_1}\right)\left(\frac{db_2}{dn_2}\right) + \left(\frac{da_3}{dn_1}\right)\left(\frac{db_3}{dn_2}\right)\right\};$$

$$a_1^2 + a_2^2 + a_3^2 = 2S_1,$$

$$b_1^2 + b_2^2 + b_3^2 = 2S_2.$$

$$a_1b_1 + a_2b_2 + a_3b_3 = P_{12},$$

$$a_1 \cdot \frac{da_1}{dn_2} + a_2 \cdot \frac{da_2}{dn_2} + a_3 \cdot \frac{da_3}{dn_2} = \frac{dS_1}{dn_2},$$

$$b_1 \cdot \frac{db_1}{dn_2} + b_2 \cdot \frac{db_2}{dn_2} + b_3 \cdot \frac{db_3}{dn_2} = \frac{dS_2}{dn_1},$$

$$a_1 \cdot \frac{da_1}{dn_1} + a_2 \cdot \frac{da_2}{dn_1} + a_3 \cdot \frac{da_3}{dn_1} = \frac{dS_1}{dn_1},$$

$$b_1 \cdot \frac{db_1}{dn_2} + b_2 \cdot \frac{db_2}{dn_2} + b_3 \cdot \frac{db_3}{dn_2} = \frac{dS_2}{dn_2},$$

$$b_1 \cdot \frac{da_1}{dn_1} + b_2 \cdot \frac{da_2}{dn_1} + b_3 \cdot \frac{da_3}{dn_1} = \frac{dP_{12}}{dn_1} - \frac{dS_1}{dn_2},$$

$$a_1 \cdot \frac{db_1}{dn_2} + a_2 \cdot \frac{db_2}{dn_2} + a_3 \cdot \frac{db_3}{dn_2} = \frac{dP_{12}}{dn_2} - \frac{dS_2}{dn_1}.$$

Of the constituents here written down the first two have the same coefficient and may therefore be united, and it is easy to see that when thus united these combined terms equal

$$-\frac{d^2S_1}{dn_2^{\,2}} + \frac{d^2P_{12}}{dn_1dn_2} - \frac{d^2S_2}{dn_1^{\,2}};$$

thus clearly the equation that we have indicated is a differential equation of the second order involving A_1, A_2, A_3, and the differential coefficients of S_1, S_2, S_3; P_{12}, P_{13}, P_{23} with respect to n_1, n_2, n_3. It only remains now actually to work this equation out; before I do so, however, I observe that although in the case of normal co-ordinates the adoption of the angles α_1, β_1, γ_1; α_1', β_1', γ_1'; α_2, β_2, γ_2 is very useful, and tends much to shorten our future calculations, yet most of the results so arrived at might be obtained by a purely analytical process without the introduction of any trigonometric conceptions whatever; for example, we might so obtain the previous equation, for clearly we may write it

$$\left\{\frac{dB}{dc_1}\cdot\frac{da_1}{dn_2}+\frac{dB}{dc_2}\cdot\frac{da_2}{dn_2}+\frac{dB}{dc_3}\cdot\frac{da_3}{dn_2}\right\}^2$$

$$-\left\{\frac{dB}{dc_1}\cdot\frac{da_1}{dn_1}+\frac{dB}{dc_2}\cdot\frac{da_2}{dn_1}+\frac{dB}{dc_3}\cdot\frac{da_3}{dn_1}\right\}$$

$$\times\left\{\frac{dB}{dc_1}\cdot\frac{db_1}{dn_2}+\frac{dB}{dc_2}\cdot\frac{db_2}{dn_2}+\frac{dB}{dc_3}\cdot\frac{db_3}{dn_2}\right\}$$

$$-\begin{vmatrix} \left(\frac{da_1}{dn_2}\right)^2+\left(\frac{da_2}{dn_2}\right)^2+\left(\frac{da_3}{dn_2}\right)^2; & a_1\cdot\frac{da_1}{dn_2}+a_2\cdot\frac{da_2}{dn_2}+a_3\cdot\frac{da_3}{dn_2}; & b_1\cdot\frac{da_1}{dn_2}+b_2\cdot\frac{da_2}{dn_2}+b_3\cdot\frac{da_3}{dn_2} \\ a_1\cdot\frac{da_1}{dn_2}+a_2\cdot\frac{da_2}{dn_2}+a_3\cdot\frac{da_3}{dn_2}; & a_1^2+a_2^2+a_3^2; & a_1b_1+a_2b_2+a_3b_3 \\ b_1\cdot\frac{da_1}{dn_2}+b_2\cdot\frac{da_2}{dn_2}+b_3\cdot\frac{da_3}{dn_2}; & a_1b_1+a_2b_2+a_3b_3; & b_1^2+b_2^2+b_3^2 \end{vmatrix}$$

$$+\begin{vmatrix} \frac{da_1}{dn_1}\cdot\frac{db_1}{dn_2}+\frac{da_2}{dn_1}\cdot\frac{db_2}{dn_2}+\frac{da_3}{dn_1}\cdot\frac{db_3}{dn_2}; & a_1\cdot\frac{da_1}{dn_1}+a_2\cdot\frac{da_2}{dn_1}+a_3\cdot\frac{da_3}{dn_1}; & b_1\cdot\frac{da_1}{dn_1}+b_2\cdot\frac{da_2}{dn_1}+b_3\cdot\frac{da_3}{dn_1} \\ a_1\cdot\frac{db_1}{dn_2}+a_2\cdot\frac{db_2}{dn_2}+a_3\cdot\frac{db_3}{dn}; & a_1^2+a_2^2+a_3^2; & a_1b_1+a_2b_2+a_3b_3 \\ b_1\cdot\frac{db_1}{dn_2}+b_2\cdot\frac{db_2}{dn_2}+b_3\cdot\frac{db_3}{dn_2}; & a_1b_1+a_2b_2+a_3b_3; & b_1^2+b_2^2+b_3^2 \end{vmatrix}.$$

If we look back now to the values of $\frac{da_1}{dn_1}$, $\frac{db_1}{dn_1}$, &c., implicitly given at the close of the previous Exercise, it is clear that the above in the case of general co-ordinates may be written

$$\left\{A_{31}\cdot\frac{1}{2}\frac{dE}{du_2}+A_{32}\frac{1}{2}\frac{dG}{du_1}+A_{33}\cdot\frac{1}{2}\left(\frac{dH}{du_1}+\frac{dJ}{du_2}-\frac{dF}{du_3}\right)\right\}^2$$

$$-\left\{A_{31}\cdot\frac{1}{2}\frac{dE}{du_1}+A_{32}\cdot\left(\frac{dF}{du_1}-\frac{1}{2}\frac{dE}{du_2}\right)+A_{33}\cdot\left(\frac{dJ}{du_1}-\frac{1}{2}\frac{dE}{du_3}\right)\right\}$$

$$\times\left\{A_{31}\cdot\left(\frac{dF}{du_2}-\frac{1}{2}\frac{dG}{du_1}\right)+A_{32}\cdot\frac{1}{2}\frac{dG}{du_2}+A_{33}\cdot\left(\frac{dH}{du_2}-\frac{1}{2}\frac{dG}{du_3}\right)\right\}$$

$$-\frac{1}{B^2}\begin{vmatrix} +\frac{1}{2}\frac{d^2E}{du_2^2}-\frac{d^2F}{du_1du_2}+\frac{1}{2}\frac{d^2G}{du_1^2}; & \frac{1}{2}\frac{dE}{du_2}; & \frac{1}{2}\frac{dG}{du_1} \\ \frac{1}{2}\frac{dE}{du_2} & ;\ E & ;\ F \\ \frac{1}{2}\frac{dG}{du_1} & ;\ F & ;\ G \end{vmatrix}$$

$$+\frac{1}{B^2}\begin{vmatrix} 0 & ;\ \frac{1}{2}\frac{dE}{du_1}; & \frac{dF}{du_1}-\frac{1}{2}\frac{dE}{du_2} \\ \frac{dF}{du_2}-\frac{1}{2}\frac{dG}{du_1}; & E\ ; & F \\ \frac{1}{2}\frac{dG}{du_2}; & F\ ; & G \end{vmatrix};$$

so that in the particular case of normal co-ordinates we clearly have

$$
\begin{aligned}
&\left\{A_2 \cdot \frac{dS_1}{dn_2} + A_1 \cdot \frac{dS_2}{dn_1} + \frac{1}{2}\left(\frac{dP_{23}}{dn_1} + \frac{dP_{31}}{dn_2} - \frac{dP_{12}}{dn_3}\right)\right\}^2 \\
&- \left\{A_2 \cdot \frac{dS_1}{dn_1} + A_1 \cdot \left(\frac{dP_{12}}{dn_1} - \frac{dS_1}{dn_2}\right) + \left(\frac{dP_{31}}{dn_1} - \frac{dS_1}{dn_3}\right)\right\} \\
&\times \left\{A_2 \cdot \left(\frac{dP_{12}}{dn_2} - \frac{dS_2}{dn_1}\right) + A_1 \cdot \frac{dS_2}{dn_3} + \left(\frac{dP_{23}}{dn_2} - \frac{dS_2}{dn_3}\right)\right\} \\
&- \frac{1}{B^2}\begin{vmatrix} \dfrac{d^2S_1}{dn_2^{\,2}} - \dfrac{d^2P_{12}}{dn_1 dn_2} + \dfrac{d^2S_2}{dn_1^{\,2}}; & \dfrac{dS_1}{dn_2}; & \dfrac{dS_2}{dn_1} \\ \dfrac{dS_1}{dn_2} & ;\ 2S_1; & P_{12} \\ \dfrac{dS_2}{dn_1} & ;\ P_{12}; & 2S_2 \end{vmatrix} \\
&+ \frac{1}{B^2}\begin{vmatrix} 0 & ;\ \dfrac{dS_1}{dn_1}; & \dfrac{dP_{12}}{dn_1} - \dfrac{dS_1}{dn_2} \\ \dfrac{dP_{12}}{dn_2} - \dfrac{dS_2}{dn_1}; & 2S_1; & P_{12} \\ \dfrac{dS_2}{dn_2} & ;\ P_{12}; & 2S_2 \end{vmatrix} = 0.
\end{aligned}
$$

Let us write the above equation in the form

$$(11) + (22) + (33) + (23) + (13) + (12) = 0,$$

where (11) equals the sum of all the terms that have two 1's in their suffix; (13) equals sum of all the terms that have a 1 and a 3 in suffix, and so on.

(11) equals

$$
\begin{aligned}
&\left\{\frac{1}{2}\frac{dP_{32}}{dn_1} + A_1 \cdot \frac{dS_2}{dn_1}\right\}^2 \\
&+ A_2 \cdot \frac{dS_2}{dn_1}\left\{\frac{dP_{13}}{dn_1} + A_1 \cdot \frac{dP_{12}}{dn_1} + A_2 \cdot \frac{dS_1}{dn_1}\right\} \\
&+ \frac{1}{B^2} \cdot \frac{dS_2}{dn_1}\left\{2S_1 \cdot \frac{dS_2}{dn_1} + 2S_2 \cdot \frac{dS_1}{dn_1} - P_{12} \cdot \frac{dP_{12}}{dn_1}\right\};
\end{aligned}
$$

but

$$2S_1 = B^2 . (1 - A_1^{\,2});\ 2S_2 = B^2 . (1 - A_2^{\,2}),$$

$$P_{12} = B^2 (A_1 A_2 - A_3);$$

therefore (11) equals

$$\frac{dS_2}{dn_1}\left\{\begin{aligned} &A_1 \cdot \frac{dP_{23}}{dn_1} + A_2 \cdot \frac{dP_{13}}{dn_1} + A_3 \cdot \frac{dP_{12}}{dn_1} \\ &+ \frac{dS_1}{dn_1} + \frac{dS_2}{dn_1} + \ldots \end{aligned}\right\} + \frac{1}{4}\left(\frac{dP_{23}}{dn_1}\right)^2;$$

but

$$S_1 + S_2 + S_3 + A_1 P_{23} + A_2 P_{13} + A_3 . P_{12} = \frac{3}{2},$$

$$P_{23} . dA_1 + P_{13} . dA_2 + P_{12} . dA_3 = -\frac{dB}{B},$$

so that clearly (11) equals

$$\frac{1}{4}\left(\frac{dP_{23}}{dn_1}\right)^2 - \frac{dS_2}{dn_1}\cdot\frac{dS_3}{dn_1} + \frac{1}{B}\cdot\frac{dS_2}{dn_1}\cdot\frac{dB}{dn_1}.$$

(22) is obviously symmetrical with (11), hence (22) equals

$$\frac{1}{4}\left(\frac{dP_{13}}{dn_2}\right)^2 - \frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_2} + \frac{1}{B}\cdot\frac{dS_1}{dn_2}\cdot\frac{dB}{dn_2}.$$

(33) clearly equals

$$\frac{1}{4}\left(\frac{dP_{12}}{dn_3}\right)^2 - \frac{dS_1}{dn_3}\cdot\frac{dS_2}{dn_3}.$$

(12) clearly equals

$$\begin{aligned}
&2\left\{A_2\cdot\frac{dS_1}{dn_2} + \frac{1}{2}\frac{dP_{13}}{dn_2}\right\}\left\{A_1\cdot\frac{dS_2}{dn_1} + \frac{1}{2}\frac{dP_{23}}{dn_1}\right\}\\
&-\left\{\frac{dP_{13}}{dn_1} + A_1\cdot\frac{dP_{12}}{dn_1} + A_2\cdot\frac{dS_1}{dn_1}\right\}\\
&\times\left\{\frac{dP_{23}}{dn_2} + A_2\cdot\frac{dP_{12}}{dn_2} + A_2\cdot\frac{dS_2}{dn_2}\right\}\\
&\qquad - A_1A_2\cdot\frac{dS_1}{dn_2}\cdot\frac{dS_2}{dn_1}\\
&+\frac{1}{B^2}\left\{\begin{array}{l} -2S_1\cdot\dfrac{dP_{12}}{dn_1}\cdot\dfrac{dS_2}{dn_2} - 2S_2\cdot\dfrac{dP_{12}}{dn_2}\cdot\dfrac{dS_1}{dn_1}\\ +P_{12}\cdot\dfrac{dS_1}{dn_1}\cdot\dfrac{dS_2}{dn_2} - P_{12}\cdot\dfrac{dS_1}{dn_2}\cdot\dfrac{dS_2}{dn_1} + P_{12}\cdot\dfrac{dP_{12}}{dn_1}\cdot\dfrac{dP_{12}}{dn_2}\end{array}\right\},
\end{aligned}$$

which equals

$$\begin{aligned}
&A_3\cdot\left\{\frac{dS_1}{dn_2}\cdot\frac{dS_2}{dn_1} - \frac{dS_1}{dn_1}\cdot\frac{dS_2}{dn_2}\right\}\\
&+A_1\cdot\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dS_2}{dn_1} - \frac{dP_{13}}{dn_1}\cdot\frac{dS_2}{dn_2}\right\}\\
&+A_2\cdot\left\{\frac{dP_{23}}{dn_1}\cdot\frac{dS_1}{dn_2} - \frac{dP_{23}}{dn_2}\cdot\frac{dS_1}{dn_1}\right\}\\
&-A_1\cdot\frac{dP_{12}}{dn_1}\cdot\frac{dP_{23}}{dn_2}\,\ldots\ldots(\alpha)\\
&-A_2\cdot\frac{dP_{12}}{dn_2}\cdot\frac{dP_{13}}{dn_1}\,\ldots\ldots(\alpha)\\
&-A_3\cdot\frac{dP_{12}}{dn_1}\cdot\frac{dP_{12}}{dn_2}\,\ldots\ldots(\alpha)\\
&-\frac{dP_{12}}{dn_1}\cdot\frac{dS_2}{dn_2}\,\ldots\ldots\ldots\ldots(\alpha)\\
&-\frac{dP_{12}}{dn_2}\cdot\frac{dS_1}{dn_1}\,\ldots\ldots\ldots\ldots(\alpha)\\
&+\frac{1}{2}\frac{dP_{13}}{dn_2}\cdot\frac{dP_{23}}{dn_1} - \frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}\ldots\ldots(\beta).
\end{aligned}$$

The five terms marked (α) equal

$$-\frac{1}{2}\frac{dP_{12}}{dn_1}\left\{A_1 \cdot \frac{dP_{23}}{dn_2} + A_3 \cdot \frac{dP_{12}}{dn_2} + 2\frac{dS_2}{dn_2}\right\}$$

$$-\frac{1}{2}\frac{dP_{12}}{dn_2}\left\{A_2 \cdot \frac{dP_{13}}{dn_1} + A_3 \cdot \frac{dP_{12}}{dn_1} + 2\frac{dS_1}{dn_1}\right\}$$

$$-\frac{1}{2}A_1 \cdot \frac{dP_{12}}{dn_1} \cdot \frac{dP_{23}}{dn_2} - \frac{1}{2}A_2 \cdot \frac{dP_{13}}{dn_1} \cdot \frac{dP_{12}}{dn_2};$$

and hence they equal

$$+\frac{1}{2}\frac{dP_{12}}{dn_1}\left\{P_{23} \cdot \frac{dA_1}{dn_2} + P_{12} \cdot \frac{dA_3}{dn_2} + P_{13} \cdot \frac{dA_2}{dn_2}\right\}$$

$$+\frac{1}{2}\frac{dP_{12}}{dn_2}\left\{P_{13} \cdot \frac{dA_2}{dn_1} + P_{12} \cdot \frac{dA_3}{dn_1} + P_{23} \cdot \frac{dA_1}{dn_1}\right\}$$

$$-\frac{1}{2}\frac{dP_{12}}{dn_1}\left\{A_1 \cdot \frac{dP_{23}}{dn_2} + P_{13} \cdot \frac{dA_2}{dn_2}\right\}$$

$$-\frac{1}{2}\frac{dP_{12}}{dn_2}\left\{A_2 \cdot \frac{dP_{13}}{dn_1} + P_{23} \cdot \frac{dA_1}{dn_1}\right\};$$

and hence these terms marked (α) equal

$$\frac{1}{2}\frac{dP_{12}}{dn_1}\left\{P_{23} \cdot \frac{dA_1}{dn_2} + P_{12} \cdot \frac{dA_3}{dn_2} + P_{13} \cdot \frac{dA_2}{dn_2}\right\}$$

$$+\frac{1}{2}\frac{dP_{12}}{dn_2}\left\{P_{13} \cdot \frac{dA_2}{dn_1} + P_{12} \cdot \frac{dA_3}{dn_1} + P_{23} \cdot \frac{dA_1}{dn_1}\right\}$$

$$+\frac{1}{4}\frac{dP_{12}}{dn_1}\left\{2 \cdot \frac{dS_3}{dn_2} + P_{23} \cdot \frac{dA_1}{dn_2} + A_2 \cdot \frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{1}{4}\frac{dP_{12}}{dn_2}\left\{2 \cdot \frac{dS_3}{dn_1} + P_{13} \cdot \frac{dA_2}{dn_1} + A_1 \cdot \frac{dP_{23}}{dn_1}\right\}$$

$$-\frac{1}{4}\frac{dP_{12}}{dn_1}\left\{A_1 \cdot \frac{dP_{23}}{dn_2} + P_{13} \cdot \frac{dA_2}{dn_2}\right\}$$

$$-\frac{1}{4}\frac{dP_{12}}{dn_2}\left\{A_2 \cdot \frac{dP_{13}}{dn_1} + P_{23} \cdot \frac{dA_1}{dn_1}\right\}.$$

The last line in (12), that is to say the terms marked β, equal

$$\frac{3}{4}\left\{\frac{dP_{31}}{dn_2} \cdot \frac{dP_{23}}{dn_1} - \frac{dP_{31}}{dn_1} \cdot \frac{dP_{23}}{dn_2}\right\}$$

$$-\frac{1}{4}\left\{\frac{dP_{31}}{dn_2} \cdot \frac{dP_{23}}{dn_1} + \frac{dP_{31}}{dn_1} \cdot \frac{dP_{23}}{dn_2}\right\};$$

(12) therefore may be written in the form

$$-\frac{1}{4}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}+\frac{dP_{23}}{dn_1}\cdot\frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{1}{2}\left\{\frac{dS_3}{dn_1}\cdot\frac{dP_{12}}{dn_2}+\frac{dS_3}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}$$

$$-\frac{1}{2B}\cdot\frac{dP_{12}}{dn_1}\cdot\frac{dB}{dn_2}$$

$$-\frac{1}{2B}\cdot\frac{dP_{12}}{dn_2}\cdot\frac{dB}{dn_1};$$

$$+A_3\cdot\left\{\frac{dS_1}{dn_2}\cdot\frac{dS_2}{dn_1}-\frac{dS_1}{dn_1}\cdot\frac{dS_2}{dn_2}\right\}\ \ldots\ldots\ldots\ldots\text{(i)};$$

$$+A_1\cdot\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dS_2}{dn_1}-\frac{dP_{13}}{dn_1}\cdot\frac{dS_2}{dn_2}\right\}\ \ldots\ldots\ldots\ldots\text{(ii)};$$

$$+A_2\cdot\left\{\frac{dP_{23}}{dn_1}\cdot\frac{dS_1}{dn_2}-\frac{dP_{23}}{dn_2}\cdot\frac{dS_1}{dn_1}\right\}\ \ldots\ldots\ldots\ldots\text{(iii)};$$

$$+\ \frac{3}{4}\cdot\left\{\frac{dP_{31}}{dn_2}\cdot\frac{dP_{23}}{dn_1}-\frac{dP_{31}}{dn_1}\cdot\frac{dP_{23}}{dn_2}\right\}\ \ldots\ldots\ldots\ldots\text{(iv)};$$

$$+\frac{1}{4}A_1\cdot\left\{\frac{dP_{23}}{dn_1}\cdot\frac{dP_{12}}{dn_2}-\frac{dP_{23}}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}\ \ldots\ldots\ldots\ldots\text{(v)};$$

$$+\frac{1}{4}A_2\cdot\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dP_{12}}{dn_1}-\frac{dP_{13}}{dn_1}\cdot\frac{dP_{12}}{dn_2}\right\}\ \ldots\ldots\ldots\ldots\text{(vi)};$$

$$+\frac{1}{4}P_{23}\cdot\left\{\frac{dP_{12}}{dn_1}\cdot\frac{dA_1}{dn_2}-\frac{dP_{12}}{dn_2}\cdot\frac{dA_1}{dn_1}\right\}\ \ldots\ldots\ldots\ldots\text{(vii)};$$

$$+\frac{1}{4}P_{13}\cdot\left\{\frac{dP_{12}}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dP_{12}}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}\ \ldots\ldots\ldots\ldots\text{(viii)}.$$

My object is now to express each of these last eight determinants in (12) in terms of the first differential coefficients of A_1, A_2, A_3 with respect to n_1, n_2, n_3.

$$A_3\left\{\frac{dS_1}{dn_2}\cdot\frac{dS_2}{dn_1}-\frac{dS_1}{dn_1}\cdot\frac{dS_2}{dn_2}\right\}$$

equals

$$B^4.A_1A_2A_3\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}$$

$$+2B\ .S_1A_2A_3\left\{\frac{dB}{dn_1}\cdot\frac{dA_2}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dA_2}{dn_1}\right\}$$

$$+2B\ .S_2A_1A_3\left\{\frac{dB}{dn_2}\cdot\frac{dA_1}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dA_1}{dn_2}\right\};$$

which equals

$$
\begin{aligned}
& B^4 A_1 A_2 A_3 \left\{\frac{dA_1}{dn_2} \cdot \frac{dA_2}{dn_1} - \frac{dA_1}{dn_1} \cdot \frac{dA_2}{dn_2}\right\} \\
&+ 2B^2 S_1 P_{23} A_2 A_3 \left\{\frac{dA_2}{dn_1} \cdot \frac{dA_1}{dn_2} - \frac{dA_2}{dn_2} \cdot \frac{dA_1}{dn_1}\right\} \\
&+ 2B^2 S_1 P_{12} A_2 A_3 \left\{\frac{dA_2}{dn_1} \cdot \frac{dA_3}{dn_2} - \frac{dA_2}{dn_2} \cdot \frac{dA_3}{dn_1}\right\} \\
&+ 2B^2 S_2 P_{12} A_1 A_3 \left\{\frac{dA_1}{dn_2} \cdot \frac{dA_3}{dn_1} - \frac{dA_1}{dn_1} \cdot \frac{dA_3}{dn_2}\right\} \\
&+ 2B^2 S_2 P_{13} A_1 A_3 \left\{\frac{dA_1}{dn_2} \cdot \frac{dA_2}{dn_1} - \frac{dA_1}{dn_1} \cdot \frac{dA_2}{dn_2}\right\}.
\end{aligned}
$$

The next term in (12) that we have to reduce is

$$A_1 \left\{\frac{dP_{13}}{dn_2} \cdot \frac{dS_2}{dn_1} - \frac{dP_{13}}{dn_1} \cdot \frac{dS_2}{dn_2}\right\}.$$

It equals

$$
\begin{aligned}
& B^4 A_1 A_2 A_3 \left\{\frac{dA_1}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dA_1}{dn_2} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ B^4 A_1^2 A_2 \cdot \left\{\frac{dA_3}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dA_3}{dn_2} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ 2BA_1^2 S_2 \cdot \left\{\frac{dB}{dn_1} \cdot \frac{dA_3}{dn_2} - \frac{dB}{dn_2} \cdot \frac{dA_3}{dn_1}\right\} \\
&+ 2BA_1 A_3 S_2 \left\{\frac{dB}{dn_1} \cdot \frac{dA_1}{dn_2} - \frac{dB}{dn_2} \cdot \frac{dA_1}{dn_1}\right\} \\
&+ 2BA_1 (A_2 P_{13} - S_2) \left\{\frac{dB}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dB}{dn_2} \cdot \frac{dA_2}{dn_1}\right\};
\end{aligned}
$$

or in this writing $$\frac{dB}{B} = -\{P_{23}\, dA_1 + P_{13}\, dA_2 + P_{12}\, dA_3\},$$

we clearly transform it into

$$
\begin{aligned}
& B^4 A_1 A_2 A_3 \left\{\frac{dA_1}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dA_1}{dn_2} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ B^4 A_1^2 A_2 \left\{\frac{dA_3}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dA_3}{dn_2} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ 2B^2 A_1^2 S_2 P_{23} \left\{\frac{dA_3}{dn_1} \cdot \frac{dA_1}{dn_2} - \frac{dA_3}{dn_2} \cdot \frac{dA_1}{dn_1}\right\} \\
&+ 2B^2 A_1^2 S_2 P_{13} \left\{\frac{dA_3}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dA_3}{dn_2} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ 2B^2 A_1 A_3 S_2 P_{13} \left\{\frac{dA_1}{dn_1} \cdot \frac{dA_2}{dn_2} - \frac{dA_1}{dn_2} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ 2B^2 A_1 A_3 S_2 P_{12} \left\{\frac{dA_1}{dn_1} \cdot \frac{dA_3}{dn_2} - \frac{dA_1}{dn_2} \cdot \frac{dA_3}{dn_1}\right\}
\end{aligned}
$$

$$+ 2B^2A_1P_{12} . (A_2P_{13} - S_2) \left\{\frac{dA_2}{dn_1} . \frac{dA_3}{dn_2} - \frac{dA_2}{dn_2} . \frac{dA_3}{dn_1}\right\}$$

$$+ 2B^2A_1P_{23} . (A_2P_{13} - S_2) \left\{\frac{dA_2}{dn_1} . \frac{dA_1}{dn_2} - \frac{dA_2}{dn_2} . \frac{dA_1}{dn_1}\right\}.$$

Thirdly, we reduce

$$A_2 \left\{\frac{dP_{23}}{dn_1} . \frac{dS_1}{dn_2} - \frac{dP_{23}}{dn_2} . \frac{dS_1}{dn_1}\right\}.$$

This clearly equals

$$\begin{aligned}
&B^4A_1A_2A_3 \left\{\frac{dA_1}{dn_1} . \frac{dA_2}{dn_2} - \frac{dA_1}{dn_2} . \frac{dA_2}{dn_1}\right\} \\
&+ B^4A_1A_2^2 \left\{\frac{dA_1}{dn_1} . \frac{dA_3}{dn_2} - \frac{dA_1}{dn_2} . \frac{dA_3}{dn_1}\right\} \\
&+ 2BA_2^2S_1 \left\{\frac{dB}{dn_2} . \frac{dA_3}{dn_1} - \frac{dB}{dn_1} . \frac{dA_3}{dn_2}\right\} \\
&+ 2BA_2A_3S_1 \left\{\frac{dB}{dn_2} . \frac{dA_2}{dn_1} - \frac{dB}{dn_1} . \frac{dA_2}{dn_2}\right\} \\
&+ 2BA_2S_1 \left\{\frac{dB}{dn_1} . \frac{dA_1}{dn_2} - \frac{dB}{dn_2} . \frac{dA_1}{dn_1}\right\} \\
&+ 2BA_1A_2P_{23} \left\{\frac{dB}{dn_2} . \frac{dA_1}{dn_1} - \frac{dB}{dn_1} . \frac{dA_1}{dn_2}\right\};
\end{aligned}$$

which clearly equals

$$\begin{aligned}
&B^4A_1A_2A_3 \left\{\frac{dA_1}{dn_1} . \frac{dA_2}{dn_2} - \frac{dA_1}{dn_2} . \frac{dA_2}{dn_1}\right\} \\
&+ B^4A_1A_2^2 \left\{\frac{dA_1}{dn_1} . \frac{dA_3}{dn_2} - \frac{dA_1}{dn_2} . \frac{dA_3}{dn_1}\right\} \\
&+ 2B^2A_2^2S_1P_{13} \left\{\frac{dA_3}{dn_2} . \frac{dA_2}{dn_1} - \frac{dA_3}{dn_1} . \frac{dA_2}{dn_2}\right\} \\
&+ 2B^2A_2^2S_1P_{23} \left\{\frac{dA_3}{dn_2} . \frac{dA_1}{dn_1} - \frac{dA_3}{dn_1} . \frac{dA_1}{dn_2}\right\} \\
&+ 2B^2A_2A_3S_1P_{12} \left\{\frac{dA_2}{dn_2} . \frac{dA_3}{dn_1} - \frac{dA_2}{dn_1} . \frac{dA_3}{dn_2}\right\} \\
&+ 2B^2A_2A_3S_1P_{23} \left\{\frac{dA_2}{dn_2} . \frac{dA_1}{dn_1} - \frac{dA_2}{dn_1} . \frac{dA_1}{dn_2}\right\} \\
&+ 2B^2A_2P_{12} (A_1P_{23} - S_1) \left\{\frac{dA_1}{dn_2} . \frac{dA_3}{dn_1} - \frac{dA_1}{dn_1} . \frac{dA_3}{dn_2}\right\} \\
&+ 2B^2A_2P_{13} (A_1P_{23} - S_1) \left\{\frac{dA_1}{dn_2} . \frac{dA_2}{dn_1} - \frac{dA_1}{dn_1} . \frac{dA_2}{dn_2}\right\}.
\end{aligned}$$

The four terms marked respectively (v), (vi), (vii), (viii), in the expression for (12) we must now reduce. It is clear that they may be written

$$+\frac{1}{2}BA_1(A_2A_3-A_1)\left\{\frac{dB}{dn_1}\cdot\frac{dP_{12}}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}$$

$$+\frac{1}{2}BA_2(A_1A_3-A_2)\left\{\frac{dB}{dn_2}\cdot\frac{dP_{12}}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dP_{12}}{dn_2}\right\}$$

$$+\frac{1}{4}B^2A_1A_3\left\{\frac{dA_2}{dn_1}\cdot\frac{dP_{12}}{dn_2}-\frac{dA_2}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}$$

$$+\frac{1}{4}B^2A_2A_3\left\{\frac{dA_1}{dn_2}\cdot\frac{dP_{12}}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dP_{12}}{dn_2}\right\}$$

$$+\frac{1}{4}B^2A_2A_3\left\{\frac{dA_1}{dn_2}\cdot\frac{dP_{12}}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dP_{12}}{dn_2}\right\}$$

$$+\frac{1}{4}B^2A_1A_3\left\{\frac{dA_2}{dn_1}\cdot\frac{dP_{12}}{dn_2}-\frac{dA_2}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\};$$

and the above clearly equals

$$\frac{1}{2}B(A_2^2-A_1^2)\left\{\frac{dB}{dn_1}\cdot\frac{dP_{12}}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}$$

$$+\frac{1}{2}B^2A_1A_3\left\{\frac{dA_2}{dn_1}\cdot\frac{dP_{12}}{dn_2}-\frac{dA_2}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}$$

$$+\frac{1}{2}B^2A_2A_3\left\{\frac{dA_1}{dn_2}\cdot\frac{dP_{12}}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dP_{12}}{dn_2}\right\}.$$

The first of these three lines equals

$$+BA_1(S_1-S_2)\left\{\frac{dB}{dn_1}\cdot\frac{dA_2}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dA_2}{dn_1}\right\}$$

$$+BA_2(S_1-S_2)\left\{\frac{dB}{dn_1}\cdot\frac{dA_1}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dA_1}{dn_1}\right\}$$

$$+B(S_1-S_2)\left\{\frac{dB}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dA_3}{dn_2}\right\},$$

whilst the last two develope into

$$BA_1A_3P_{12}\left\{\frac{dB}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}$$

$$+\quad BA_2A_3P_{12}\left\{\frac{dB}{dn_1}\cdot\frac{dA_1}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dA_1}{dn_1}\right\}$$

$$+\frac{1}{2}B^4A_1A_2A_3\left\{\frac{dA_2}{dn_1}\cdot\frac{dA_1}{dn_2}-\frac{dA_2}{dn_2}\cdot\frac{dA_1}{dn_1}\right\}$$

$$+\frac{1}{2}B^4A_1A_3\left\{\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA}{dn_2}\right\}$$

$$+\frac{1}{2}B^4A_1A_2A_3\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA}{dn_3}\right\}$$

$$+\frac{1}{2}B^4A_2A_3\left\{\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}.$$

From these nine lines it is now easy to pick out the coefficients of

$$\left\{\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_2}\right\},$$

$$\left\{\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}\right\},$$

$$\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_2}\right\};$$

the result is easily seen to be equal to

$$\left\{\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_2}\right\}\left[\begin{array}{l}\frac{1}{2}B^4A_1A_3,\\ +B^2A_1A_3P_{12}^2,\\ +\frac{1}{2}B^4A_1P_{12}\,.\,(A_1^2-A_2^2),\\ +\frac{1}{2}B^4P_{13}\,(A_1^2-A_2^2),\end{array}\right.$$

$$+\left\{\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}\left\{\begin{array}{l}\frac{1}{2}B^4A_2A_3,\\ +B^2A_2A_3\,.\,P_{12}^2,\\ +B^2A_2P_{12}\,.\,(S_1-S_2),\\ +B^2P_{23}\,.\,(S_1-S_2),\end{array}\right.$$

$$+\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}\left\{\begin{array}{l}B^4A_1A_2A_3,\\ -B^2A_1A_3\,.\,P_{12}\,.\,P_{23},\\ -B^2A_2A_3\,.\,P_{12}\,.\,P_{13},\\ +B^2A_2P_{13}\,(S_2-S_1),\\ +B^2A_1P_{23}\,(S_1-S_2).\end{array}\right.$$

It now only remains to reduce the term marked (iv) in 12, that is to say

$$+\frac{3}{4}\left\{\frac{dP_{31}}{dn_2}\cdot\frac{dP_{32}}{dn_1}-\frac{dP_{31}}{dn_1}\cdot\frac{dP_{32}}{dn_2}\right\};$$

on expansion this clearly becomes equal to

$$+\frac{3}{4}B^4\left\{\frac{dA_2}{dn_2}\cdot\frac{dA_1}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_1}{dn_2}\right\}$$

$$+\frac{3}{4}B^4A_3^2\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_2}{dn_2}\cdot\frac{dA_1}{dn}\right\}$$

$$+\frac{3}{4}B^4A_2A_3\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}\right\}$$

$$-\frac{3}{4}B^4A_1\quad\left\{\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_3}{dn_1}\cdot\frac{dA_1}{dn_2}\right\}$$

$$+\frac{3}{4}B^4A_1A_3\left\{\frac{dA_3}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_3}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}$$

$$-\frac{3}{4}B^4A_2\quad\left\{\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_2}\right\}$$

$$+\frac{3}{2}B\,(A_1A_3-A_2)\left\{\frac{dB}{dn_2}\cdot\frac{dP_{32}}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dP_{32}}{dn_2}\right\}\text{......}(\alpha),$$

$$+\frac{3}{2}B\,(A_2A_3-A_1)\left\{\frac{dB}{dn_1}\cdot\frac{dP_{31}}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dP_{31}}{dn_1}\right\}\text{......}(\alpha).$$

These two last lines marked (α) may be replaced by

$$+\frac{3}{2}B\,.\,\{P_{13}+A_3P_{23}\}\quad\left\{\frac{dB}{dn_1}\cdot\frac{dA_1}{dn_2}-\frac{dB}{dn_2}\cdot\frac{dA_1}{dn_1}\right\}$$

$$+\frac{3}{3}B\,.\,\{P_{23}+A_3P_{13}\}\quad\left\{\frac{dB}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}$$

$$+\frac{3}{2}B\,.\,\{A_2P_{13}+A_1P_{23}\}\left\{\frac{dB}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dB}{dn_1}\cdot\frac{dA_3}{dn_2}\right\}.$$

And hence by picking out coefficients we find that

$$\frac{3}{4}\left\{\frac{dP_{31}}{dn_2}\cdot\frac{dP_{32}}{dn_1}-\frac{dP_{31}}{dn_1}\cdot\frac{dP_{32}}{dn_2}\right\}$$

may be written

$$\left\{\frac{dA_3}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_3}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}\left[\begin{array}{l}+\frac{3}{4}B^4A_1A_3,\\[1ex] +\frac{3}{4}B^4A_2,\\[1ex] +\frac{3}{2}B^2\,.\,(A_2P_{13}-A_1P_{23})\,P_{12},\\[1ex] -\frac{3}{2}B^2\,.\,(P_{23}+A_3P_{13})\,P_{12},\end{array}\right.$$

$$+\left\{\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}\left[\begin{array}{l}-\frac{3}{4}B^4A_2A_3,\\[1ex] -\frac{3}{4}B^4A_1,\\[1ex] +\frac{3}{2}B^2\,.\,(A_2P_{13}-A_1P_{23})\,P_{23},\\[1ex] +\frac{3}{2}B^2\,.\,(P_{13}+A_3P_{23})\,P_{12},\end{array}\right.$$

$$+\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}\left\{\begin{array}{l}+\frac{3}{4}B^4\cdot(A_3{}^2-1),\\[1ex] -\frac{3}{2}B^2\cdot(P_{13}+A_3P_{23})\,P_{13},\\[1ex] -\frac{3}{2}B^2\cdot(P_{23}+A_3P_{13})\,P_{23}.\end{array}\right.$$

We are now clearly in a position to calculate what may be called the "Determinant" portion of (12), accordingly we first calculate the coefficient of

$$\left\{\frac{dA_3}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_3}{dn_1}\cdot\frac{dA_2}{dn_2}\right\};$$

it equals

$$\begin{aligned}&+\overset{(1)}{2B^2A_2A_3S_1P_{12}}\\ &-\overset{(2)}{B^4A_1{}^2A_2}-\overset{(3)}{2B^2S_2P_{13}A_1{}^2}+\overset{(4)}{2B^2A_1\{A_2P_{13}-S_2\}P_{12}}\\ &+\overset{(5)}{2B^2A_2{}^2S_1P_{13}}-\overset{(6)}{2B^2A_2A_3S_1P_{12}}\\ &-\overset{(7)}{\frac{1}{2}B^4A_1A_3}+\overset{(8)}{\frac{1}{2}B^4(A_2{}^2-A_1{}^2)(P_{13}+A_1P_{12})}-\overset{(9)}{B^2A_1A_3P_{12}{}^2},\\ &+\overset{(10)}{\frac{3}{4}B^4A_1A_3}+\overset{(11)}{\frac{3}{4}B^4A_2}+\overset{(12)}{\frac{3}{2}B^2\{A_2P_{13}-A_1P_{23}-A_3P_{12}\}P_{13}}-\overset{(13)}{\frac{3}{2}B^2P_{23}P_{12}}.\end{aligned}$$

We now must make use of the ten equations given at commencement of this Exercise: by their aid the four terms marked (3), (5), (8), (12) become

$$+\frac{3}{2}B^4(A_3{}^2-A_1A_2A_3)\,P_{13}+\frac{1}{2}B^4(A_2{}^2-A_1{}^2)\,A_1P_{12},$$

this equals

$$-\frac{3}{2}B^2A_3P_{12}P_{13}+B^2(S_1-S_2)\,A_1P_{12};$$

to this join on the three terms marked (4), (9) and (13), and we get

$$B^2P_{12}\left\{\begin{array}{l}A_1S_1-3A_1S_2-\frac{3}{2}A_3P_{13}+2A_1A_2P_{13}\\[1ex] -A_1A_3P_{12}-\frac{3}{2}P_{23}\end{array}\right\},$$

which clearly equals

$$B^2P_{12}\left\{\begin{array}{l}A_1S_1-3A_1S_2+3A_1S_3\\ +2A_1A_2P_{13}-A_1A_3P_{12}\end{array}\right\},$$

which clearly equals

$$B^2P_{12}\left\{\begin{array}{r}3A_1S_1-3A_1S_2+3A_1S_3+3A_1A_2P_{13}\\ -A_1\end{array}\right\},$$

which clearly equals

$$\frac{3}{2}B^4A_1P_{12}\{1-A_1^2+A_2^2-A_3^2+2A_1A_2A_3-2A_2^2\}-B^2A_1P_{12},$$

which finally equals $$+\frac{1}{2}B^2A_1P_{12}.$$

Now the terms marked (1) and (6) mutually cancel, there therefore remains only to consider the four terms marked (2), (7), (10), (11); these may be written in the form

$$B^4\left\{A_2\,.\,(1-A_1^2)+\frac{1}{4}(A_1A_3-A_2)\right\},$$

which clearly equals

$$B^2.\left(2A_2S_1+\frac{1}{4}P_{13}\right),$$

but $$2A_2S_1=-P_{13}-A_1P_{12}.$$

Whence joining on the previous found term $+\frac{1}{2}B^2A_1P_{12}$, we finally obtain for the coefficient of

$$\left(\frac{dA_3}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_3}{dn_1}\cdot\frac{dA_2}{dn_2}\right),$$

in (12), the term $$-\frac{B^2}{4}\{2A_1P_{12}+3P_{13}\},$$

and therefore by symmetry the coefficient of

$$\left\{\frac{dA_3}{dn_1}\cdot\frac{dA_1}{dn_2}-\frac{dA_3}{dn_2}\cdot\frac{dA_1}{dn_1}\right\}$$

equals $$-\frac{B^2}{4}\{2A_2P_{12}+3P_{23}\}.$$

The remainder of the "Determinant" portion of (12) is the coefficient of

$$\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_2}{dn_2}\right\},$$

which is, by picking out, found to equal

$$\begin{aligned}
&\overset{(1)}{B^4A_1A_2A_3}+\overset{(2)}{2B^2A_2A_3S_1P_{23}}+\overset{(3)}{2B^2A_1A_3S_2P_{13}}\\
&-\overset{(4)}{B^4A_1A_2A_3}-\overset{(5)}{2B^2A_1A_3S_2P_{13}}\\
&+\overset{(6)}{2B^2A_1\,.\,(A_2P_{13}-S_2)\,.\,P_{23}}\\
&-\overset{(7)}{B^4A_1A_2A_3}-\overset{(8)}{2B^2A_2A_3S_1P_{23}}+\overset{(9)}{2B^2A_2\,(A_1P_{23}-S_1)\,P_{13}}\\
&+\overset{(10)}{B^4A_1A_2A_3}-\overset{(11)}{B^2A_1A_3P_{12}P_{23}}-\overset{(12)}{B^2A_2A_3P_{12}P_{13}}\\
&+\overset{(13)}{\frac{1}{2}B^4A_2\,.\,(A_1^2-A_2^2)\,P_{13}}+\overset{(14)}{\frac{1}{2}B^4A_1\,.\,(A_2^2-A_1^2)\,P_{23}}\\
&+\overset{(15)}{\frac{3}{4}B^4\,.\,(A_3^2-1)}-\overset{(16)}{\frac{3}{2}B^2\,.\,(P_{13}+A_3P_{23})\,P_{13}}-\overset{(17)}{\frac{3}{2}B^2\,(P_{23}+A_3P_{13})\,P_{23}}.
\end{aligned}$$

The terms marked (6), (9), (11), (12), (16), and (17), since

$$P_{31} + 2A_2S_3 + A_3P_{23} = 0,$$

$$P_{32} + 2A_1S_3 + A_3P_{31} = 0;$$

may clearly be written

$$B^2A_1P_{23}\{2A_2P_{13} - 2S_2 - A_3P_{12} + 3S_3\} \ldots\ldots(\alpha),$$

$$+ B^2A_2P_{13}\{2A_1P_{23} - 2S_1 - A_3P_{12} + 3S_3\} \ldots\ldots(\alpha).$$

But $\quad 2S_3 - 2S_2 - A_3P_{12} = B^2\{A_2^2 - A_1A_2A_3\} = - B^2A_2P_{13},$

and $\quad 2S_3 - 2S_1 - A_3P_{12} = B^2\{A_1^2 - A_1A_2A_3\} = - B^2A_1P_{23}.$

Hence clearly the two lines (α), (α) equal

$$B^2A_1P_{23}\{A_2P_{13} + S_3\} + B^2A_2P_{13}\{A_1P_{23} + S_3\};$$

to this join the terms marked (13) and (14), and we obtain

$$B^2A_1P_{23}\{A_2P_{13} + S_3 + S_1 - S_2\}$$

$$+ B^2A_2P_{13}\{A_1P_{23} + S_3 + S_2 - S_1\},$$

which equals

$$+\frac{1}{2}B^2\{A_1P_{23} + A_2P_{13}\} = \frac{1}{2}B^2 \cdot (1 - 2S_3);$$

to this join the term marked (15), and we finally get for the coefficient of

$$\left\{\frac{dA_1}{dn_2}\frac{dA_2}{dn_1} - \frac{dA_1}{dn_1}\frac{dA_2}{dn_2}\right\},$$

the term $\quad +\frac{1}{2}B^2 - \frac{5}{2}B^2S_3.$

We can now write down the complete value of (12); referring to page 469, we find it equals

$$-\frac{1}{4}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dP_{23}}{dn_2} + \frac{dP_{23}}{dn_1} \cdot \frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{1}{2}\left\{\frac{dS_3}{dn_1} \cdot \frac{dP_{12}}{dn_2} + \frac{dS_3}{dn_2} \cdot \frac{dP_{12}}{dn_1}\right\}$$

$$-\frac{1}{2B}\frac{dP_{12}}{dn_1} \cdot \frac{dB}{dn_2} - \frac{1}{2B}\frac{dP_{12}}{dn_2} \cdot \frac{dB}{dn_1}$$

$$+\frac{B^2}{4}\{2A_1P_{12} + 3P_{13}\}\left\{\frac{dA_3}{dn_1}\frac{dA_2}{dn_2} - \frac{dA_3}{dn_2}\frac{dA_2}{dn_1}\right\}$$

$$+\frac{B^2}{4}\{2A_2P_{12} + 3P_{23}\}\left\{\frac{dA_3}{dn_2} \cdot \frac{dA_1}{dn_1} - \frac{dA_3}{dn_1}\frac{dA_1}{dn_2}\right\}$$

$$+\frac{B^2}{4}\{2 - 5S_3\}\left\{\frac{dA_1}{dn_2} \cdot \frac{dA_2}{dn_1} - \frac{dA_1}{dn_1} \cdot \frac{dA_2}{dn_2}\right\}.$$

We must now calculate (13); if we refer back to page 468, we easily find that it equals

$$
\begin{aligned}
&-\frac{dP_{12}}{dn_3}\left\{A_1 \cdot \frac{dS_2}{dn_1}+\frac{1}{2}\frac{dP_{23}}{dn_1}\right\}\\
&+\frac{dS_2}{dn_3}\left\{\frac{dP_{13}}{dn_1}+A_1 \cdot \frac{dP_{12}}{dn_1}+A_2 \cdot \frac{dS_1}{dn_1}\right\}\\
&\qquad -A_2\frac{dS_1}{dn_3} \cdot \frac{dS_2}{dn_1}.
\end{aligned}
$$

Clearly this may be written

$$
\begin{aligned}
&-\frac{1}{4}\left\{\frac{dP_{12}}{dn_3} \cdot \frac{dP_{23}}{dn_1}+\frac{dP_{12}}{dn_1} \cdot \frac{dP_{23}}{dn_3}\right\}\\
&+\frac{1}{2}\left\{\frac{dS_2}{dn_1} \cdot \frac{dP_{13}}{dn_3}+\frac{dS_2}{dn_3} \cdot \frac{dP_{13}}{dn_1}\right\}\\
&+\frac{1}{4}\left\{\frac{dP_{21}}{dn_1} \cdot \frac{dP_{23}}{dn_3}-\frac{dP_{21}}{dn_3} \cdot \frac{dP_{23}}{dn_1}\right\}\ldots\ldots(\alpha),\\
&+\frac{1}{2}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dS_2}{dn_3}-\frac{dP_{13}}{dn_3} \cdot \frac{dS_2}{dn_1}\right\}\ldots\ldots(\alpha),\\
&+A_1\left\{\frac{dP_{12}}{dn_1} \cdot \frac{dS_2}{dn_3}-\frac{dP_{12}}{dn_3} \cdot \frac{dS_2}{dn_1}\right\}\ldots\ldots(\alpha),\\
&+A_2\left\{\frac{dS_1}{dn_1} \cdot \frac{dS_2}{dn_3}-\frac{dS_1}{dn_3} \cdot \frac{dS_2}{dn_1}\right\}\ldots\ldots(\alpha).
\end{aligned}
$$

The last four lines marked (α) clearly may be written in the form

$$
\begin{aligned}
&\frac{dS_2}{dn_3}\left\{A_2 \cdot \frac{dS_1}{dn_1}+\frac{1}{2}A_1 \cdot \frac{dP_{12}}{dn_1}+\frac{1}{2}\frac{dP_{13}}{dn_1}\right\}\\
&+\frac{dS_2}{dn_3}\left\{S_1 \cdot \frac{dA_2}{dn_1}+\frac{1}{2}P_{12} \cdot \frac{dA_1}{dn_1}\right\}\\
&-\frac{dS_2}{dn_3}\left\{S_1 \cdot \frac{dA_2}{dn_1}+\frac{1}{2}P_{12} \cdot \frac{dA_1}{dn_1}\right\}\\
&-\frac{dS_2}{dn_1}\left\{A_2 \cdot \frac{dS_1}{dn_3}+\frac{1}{2}A_1 \cdot \frac{dP_{12}}{dn_3}+\frac{1}{2}\frac{dP_{13}}{dn_3}\right\}\\
&-\frac{dS_2}{dn_1}\left\{S_1\frac{dA_2}{dn_3}+\frac{1}{2}P_{12} \cdot \frac{dA_1}{dn_3}\right\}\\
&+\frac{dS_2}{dn_1}\left\{S_1\frac{dA_2}{dn_3}+\frac{1}{2}P_{12} \cdot \frac{dA_1}{dn_3}\right\}\\
+&\frac{1}{2}\frac{dP_{12}}{dn_1}\left\{A_1\frac{dS_2}{dn_3}+\frac{1}{2}A_2 \cdot \frac{dP_{12}}{dn_3}+\frac{1}{2}\frac{dP_{23}}{dn_3}\right\}\\
+&\frac{1}{2}\frac{dP_{12}}{dn_1}\left\{S_2 \cdot \frac{dA_1}{dn_3}+\frac{1}{2}P_{12}\frac{dA_2}{dn_3}-S_2\frac{dA_1}{dn_3}-\frac{1}{2}P_{12}\frac{dA_2}{dn_3}\right\}\\
-&\frac{1}{2}\frac{dP_{12}}{dn_3}\left\{A_1 \cdot \frac{dS_2}{dn_1}+\frac{1}{2}A_2 \cdot \frac{dP_{12}}{dn_1}+\frac{1}{2}\frac{dP_{23}}{dn_1}\right\}\\
-&\frac{1}{2}\frac{dP_{12}}{dn_3}\left\{S_2 \cdot \frac{dA_1}{dn_1}+\frac{1}{2}P_{12} \cdot \frac{dA_2}{dn_1}-S_2 \cdot \frac{dA_1}{dn_1}-\frac{1}{2}P_{12} \cdot \frac{dA_2}{dn_1}\right\}.
\end{aligned}
$$

But this is easily transformed into

$$
\begin{aligned}
&+ \ S_1 \left\{\frac{dS_2}{dn_1} \cdot \frac{dA_2}{dn_3} - \frac{dS_2}{dn_3} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ \frac{1}{2} S_2 \left\{\frac{dP_{12}}{dn_3} \cdot \frac{dA_1}{dn_1} - \frac{dP_{12}}{dn_1} \cdot \frac{dA_1}{dn_3}\right\} \\
&+ \frac{1}{2} P_{12} \left\{\frac{dS_2}{dn_1} \cdot \frac{dA_1}{dn_3} - \frac{dS_2}{dn_3} \cdot \frac{dA_1}{dn_1}\right\} \\
&+ \frac{1}{4} P_{12} \left\{\frac{dP_{12}}{dn_3} \cdot \frac{dA_2}{dn_1} - \frac{dP_{12}}{dn_1} \cdot \frac{dA_2}{dn_3}\right\}.
\end{aligned}
$$

And this clearly equals

$$
\begin{aligned}
&\frac{2}{B} S_1 S_2 \left\{\frac{dB}{dn_1} \cdot \frac{dA_2}{dn_3} - \frac{dB}{dn_3} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ \frac{1}{B} S_2 P_{12} \left\{\frac{dB}{dn_3} \cdot \frac{dA_1}{dn_1} - \frac{dB}{dn_1} \cdot \frac{dA_1}{dn_3}\right\} \\
&+ \frac{1}{B} S_2 P_{12} \left\{\frac{dB}{dn_1} \cdot \frac{dA_1}{dn_3} - \frac{dB}{dn_3} \cdot \frac{dA_1}{dn_1}\right\} \\
&+ \frac{1}{2B} P_{12}^2 \left\{\frac{dB}{dn_3} \cdot \frac{dA_2}{dn_1} - \frac{dB}{dn_1} \cdot \frac{dA_2}{dn_3}\right\} \\
&+ B^2 \cdot \frac{P_{12}}{4} \left\{\frac{dA_2}{dn_3} \cdot \frac{dA_3}{dn_1} - \frac{dA_2}{dn_1} \cdot \frac{dA_3}{dn_3}\right\} \\
&+ B^2 \cdot \frac{S_2}{2} \left\{\frac{dA_1}{dn_3} \cdot \frac{dA_3}{dn_1} - \frac{dA_1}{dn_1} \cdot \frac{dA_3}{dn_3}\right\} \\
&\qquad + B^2 \left\{\frac{dA_2}{dn_1} \cdot \frac{dA_1}{dn_3} - \frac{dA_2}{dn_3} \cdot \frac{dA_1}{dn_1}\right\} \left\{\frac{A_2 P_{12}}{4} - \frac{A_2 P_{12}}{2} - \frac{A_1 S_2}{2}\right\}.
\end{aligned}
$$

Hence remembering that $4S_1S_2 - P_{12}^2 = B^2$, and that $-\frac{A_1S_2}{2} - \frac{A_2P_{12}}{4} = +\frac{P_{23}}{4}$ above equals

$$
\begin{aligned}
&\frac{B}{2} \left\{\frac{dB}{dn_1} \cdot \frac{dA_2}{dn_3} - \frac{dB}{dn_3} \cdot \frac{dA_2}{dn_1}\right\} \\
&+ \frac{B^2 P_{12}}{4} \left\{\frac{dA_2}{dn_3} \cdot \frac{dA_3}{dn_1} - \frac{dA_2}{dn_1} \cdot \frac{dA_3}{dn_3}\right\} \\
&+ \frac{B^2 P_{23}}{4} \left\{\frac{dA_2}{dn_1} \cdot \frac{dA_1}{dn_3} - \frac{dA_2}{dn_3} \cdot \frac{dA_1}{dn_1}\right\} \\
&+ \frac{B^2 S_2}{2} \left\{\frac{dA_1}{dn_3} \cdot \frac{dA_3}{dn_1} - \frac{dA_1}{dn_1} \cdot \frac{dA_3}{dn_3}\right\}.
\end{aligned}
$$

Hence referring back to page 480, we clearly see that (13) equals

$$
\begin{aligned}
&- \frac{1}{4} \left\{\frac{dP_{21}}{dn_3} \cdot \frac{dP_{23}}{dn_1} + \frac{dP_{21}}{dn_1} \cdot \frac{dP_{23}}{dn_3}\right\} \\
&+ \frac{1}{2} \left\{\frac{dS_2}{dn_1} \cdot \frac{dP_{13}}{dn_3} + \frac{dS_2}{dn_3} \cdot \frac{dP_{13}}{dn_1}\right\}
\end{aligned}
$$

$$+\frac{B^2P_{12}}{4}\left\{\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_3}-\frac{dA_2}{dn_3}\cdot\frac{dA_3}{dn_1}\right\}$$

$$+\frac{3B^2P_{23}}{4}\left\{\frac{dA_2}{dn_1}\cdot\frac{dA_1}{dn_3}-\frac{dA_2}{dn_3}\cdot\frac{dA_1}{dn_1}\right\}$$

$$+\frac{2B^2S_2}{4}\left\{\frac{dA_1}{dn_3}\cdot\frac{dA_3}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_3}\right\};$$

(23) is clearly symmetrical with the above value of (13), and since we now know the value of (23), (13), (12), (11), (22), and (33), we can clearly write down the form of their sum, that is to say, the differential equation of second order indicated at commencement of this exercise, before we do so, however, it may conduce to clearness and conciseness to adopt the following *cyclic* notation: write

$$\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_3}-\frac{dA_2}{dn_3}\cdot\frac{dA_3}{dn_2}=\binom{23}{23},$$

$$\frac{dA_1}{dn_3}\cdot\frac{dA_3}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_3}=\binom{31}{23},$$

$$\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_3}-\frac{dA_1}{dn_3}\cdot\frac{dA_2}{dn_2}=\binom{12}{23}.$$

Similarly

$$\frac{dA_2}{dn_3}\cdot\frac{dA_3}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_3}=\binom{23}{31},$$

$$\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_3}-\frac{dA_1}{dn_3}\cdot\frac{dA_3}{dn_1}=\binom{31}{31},$$

$$\frac{dA_1}{dn_3}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_3}=\binom{12}{31}.$$

Similarly

$$\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_1}=\binom{23}{12},$$

$$\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}=\binom{31}{12},$$

$$\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}=\binom{12}{12}.$$

We also write or denote by Λ the expression

$$\frac{1}{4}\left(\frac{dP_{23}}{dn_1}\right)^2-\frac{dS_2}{dn_1}\cdot\frac{dS_3}{dn_1}$$

$$+\frac{1}{4}\left(\frac{dP_{13}}{dn_2}\right)^2-\frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_2}$$

$$+\frac{1}{4}\left(\frac{dP_{12}}{dn_3}\right)^2-\frac{dS_1}{dn_3}\cdot\frac{dS_2}{dn_3}$$

$$-\frac{1}{4}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}+\frac{dP_{23}}{dn_1}\cdot\frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{1}{2}\left\{\frac{dS_3}{dn_1}\cdot\frac{dP_{12}}{dn_2}+\frac{dS_3}{dn_2}\cdot\frac{dP_{12}}{dn_1}\right\}$$

$$-\frac{1}{4}\left\{\frac{dP_{12}}{dn_3}\cdot\frac{dP_{23}}{dn_1}+\frac{dP_{12}}{dn_1}\cdot\frac{dP_2}{dn_3}\right\}$$

$$+\frac{1}{2}\left\{\frac{dS_2}{dn_1}\cdot\frac{dP_{13}}{dn_3}+\frac{dS_2}{dn_3}\cdot\frac{dP_{13}}{dn_1}\right\}$$

$$-\frac{1}{4}\left\{\frac{dP_{12}}{dn_3}\cdot\frac{dP_{13}}{dn_2}+\frac{dP_{12}}{dn_2}\cdot\frac{dP_{13}}{dn_3}\right\}$$

$$+\frac{1}{2}\left\{\frac{dS_1}{dn_3}\cdot\frac{dP_{23}}{dn_2}+\frac{dS_1}{dn_2}\cdot\frac{dP_{23}}{dn_3}\right\}.$$

Adopting this expressive notation the differential equation of the second order indicated at page 468 may be written

$$-\frac{d^2S_1}{dn_2^{\,2}}+\frac{d^2P_{12}}{dn_1dn_2}-\frac{d^2S_2}{dn_1^{\,2}}$$

$$+\quad \Lambda$$

$$+\frac{1}{B}\left\{\begin{array}{l} +\dfrac{dS_1}{dn_2}\cdot\dfrac{dB}{dn_2}+\dfrac{dS_2}{dn_1}\cdot\dfrac{dB}{dn_1} \\ -\dfrac{1}{2}\dfrac{dP_{12}}{dn_2}\dfrac{dB}{dn_1}-\dfrac{1}{2}\dfrac{dP_{12}}{dn_1}\dfrac{dB}{dn_2}\end{array}\right\},$$

$$+\frac{B^2}{4}\left\{\begin{array}{lll} -(2A_1P_{12}+3P_{13})\cdot\binom{23}{12} & -(2A_2P_{12}+3P_{23})\cdot\binom{31}{12} & -(2-10S_3)\cdot\binom{12}{12} \\ -\;P_{12}\cdot\binom{23}{31} & -2S_2\cdot\binom{31}{31} & +3P_{23}\cdot\binom{12}{31} \\ -2S_1\cdot\binom{23}{23} & -\;P_{12}\cdot\binom{31}{23} & +3P_1\cdot\binom{12}{23}\end{array}\right\},$$

all equal zero.

Of course we have two more differential equations symmetrical with the above, we do not at present proceed to write these down, but go on to calculate the three remaining differential equations of the second order. For convenience we use n_1, n_2, n_3 in place of the correct symbols u_1, u_2, u_3. We have already seen that

$$\begin{vmatrix} \dfrac{da_1}{dn_1}, & \dfrac{da_2}{dn_1}, & \dfrac{da_3}{dn_1} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{vmatrix} \times \begin{vmatrix} \dfrac{db_1}{dn_1}, & \dfrac{db_2}{dn_1}, & \dfrac{db_3}{dn_1} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_2 \end{vmatrix}$$

Minus

$$\begin{vmatrix} \dfrac{da_1}{dn_2}, & \dfrac{da_2}{dn_2}, & \dfrac{da_3}{dn_2} \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{vmatrix}^2$$

admits of being written in two forms, and of course the difference of these two forms is zero, hence we obtain three differential equations of the second order.

Now the like remark applies to

$$\begin{vmatrix} \frac{da_1}{dn_3}, & \frac{da_2}{dn_3}, & \frac{da_3}{dn_3} \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \times \begin{vmatrix} \frac{db_1}{dn_1}, & \frac{db_2}{dn_1}, & \frac{db_3}{dn_1} \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$$

Minus

$$\begin{vmatrix} \frac{da_1}{dn_1}, & \frac{da_2}{dn_1}, & \frac{da_3}{dn_1} \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \times \begin{vmatrix} \frac{db_1}{dn_3}, & \frac{db_2}{dn_3}, & \frac{db_3}{dn_3} \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$$

Clearly we may either multiply together our determinants first and then interpret the symbols, *i.e.* substitute other values for them, or we may substitute the values for our symbols first and then multiply afterwards; subtracting these two equal results we get a differential equation of the second order, and two more can be formed in a similar manner: thus we have, multiplying our determinants,

$$\begin{vmatrix} \frac{da_1}{dn_3}\cdot\frac{db_1}{dn_1}+\frac{da_2}{dn_3}\cdot\frac{db_2}{dn_1}+\frac{da_3}{dn_3}\cdot\frac{db_3}{dn_1}; & b_1\frac{da_1}{dn_3}+b_2\frac{da_2}{dn_3}+b_3\frac{da_3}{dn_3}; & c_1\frac{da_1}{dn_3}+c_2\frac{da_2}{dn_3}+c_3\frac{da_3}{dn_3} \\ b_1\cdot\frac{db_1}{dn_1}+b_2\cdot\frac{db_2}{dn_1}+b_3\cdot\frac{db_3}{dn_1}; & b_1^2+b_2^2+b_3^2 & ;\ b_1c_1+b_2c_2+b_3c_3 \\ c_1\cdot\frac{db_1}{dn_1}+c_2\cdot\frac{db_2}{dn_1}+c_3\cdot\frac{db_3}{dn_1}; & b_1c_1+b_2c_2+b_3c_3 & ;\ c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

Minus

$$\begin{vmatrix} \frac{da_1}{dn_1}\cdot\frac{db_1}{dn_3}+\frac{da_2}{dn_1}\cdot\frac{db_2}{dn_3}+\frac{da_3}{dn_1}\cdot\frac{db_3}{dn_3}; & b_1\frac{da_1}{dn_1}+b_2\frac{da_2}{dn_1}+b_3\frac{da_3}{dn_1}; & c_1\frac{da_1}{dn_1}+c_2\frac{da_2}{dn_1}+c_3\frac{da_3}{dn_1} \\ b_1\cdot\frac{db_1}{dn_3}+b_2\cdot\frac{db_2}{dn_3}+b_3\cdot\frac{db_3}{dn_3}; & b_1^2+b_2^2+b_3^2 & ;\ b_1c_1+b_2c_2+b_3c_3 \\ c_1\cdot\frac{db_1}{dn_3}+c_2\cdot\frac{db_2}{dn_3}+c_3\cdot\frac{db_3}{dn_3}; & b_1c_1+b_2c_2+b_3c_3 & ;\ c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

but $$B = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}; \quad b_2c_3 - b_3c_2 = \frac{dB}{da_1}, \quad \text{\&c., \&c.}$$

and so this same result may be exhibited in the form

$$\left\{\frac{da_1}{dn_3}\cdot\frac{dB}{da_1}+\frac{da_2}{dn_3}\cdot\frac{dB}{da_2}+\frac{da_3}{dn_3}\cdot\frac{dB}{da_3}\right\}$$
$$\times\left\{\frac{db_1}{dn_1}\cdot\frac{dB}{da_1}+\frac{db_2}{dn_1}\cdot\frac{dB}{da_2}+\frac{db_3}{dn_1}\cdot\frac{dB}{da_3}\right\},$$

Minus

$$\left\{\frac{da_1}{dn_1}\cdot\frac{dB}{da_1}+\frac{da_2}{dn_1}\cdot\frac{dB}{da_2}+\frac{da_3}{dn_1}\cdot\frac{dB}{da_3}\right\}$$

$$\times\left\{\frac{db_1}{dn_3}\cdot\frac{dB}{da_1}+\frac{db_2}{dn_3}\cdot\frac{dB}{da_2}+\frac{db_3}{dn_3}\cdot\frac{dB}{da_3}\right\}.$$

Now to simplify all this observe that

$$\frac{da_1}{dn_3}\cdot\frac{db_1}{dn_1}+\frac{da_2}{dn_3}\cdot\frac{db_2}{dn_1}+\frac{da_3}{dn_3}\cdot\frac{db_3}{dn_1}$$

$$-\frac{da_1}{dn_1}\cdot\frac{db_1}{dn_3}-\frac{da_2}{dn_1}\cdot\frac{db_2}{dn_3}-\frac{da_3}{dn_1}\cdot\frac{db_3}{dn_3}$$

equals

$$-\frac{1}{2}\frac{d^2}{dn_1dn_2}\{a_1c_1+a_2c_2+a_3c_3\}$$

$$-\frac{1}{2}\frac{d^2}{dn_1dn_3}\{a_1b_1+a_2b_2+a_3b_3\}$$

$$+\frac{1}{2}\frac{d^2}{dn_1{}^2}\{b_1c_1+b_2c_2+b_3c_3\}$$

$$+\frac{1}{2}\frac{d^2}{dn_2dn_3}\{a_1{}^2+a_2{}^2+a_3{}^2\},$$

which equals

$$-\frac{1}{2}\left\{\frac{d^2J}{dn_1dn_2}+\frac{d^2F}{dn_1dn_3}+\frac{d^2H}{dn_1{}^2}-\frac{d^2E}{dn_2dn_3}\right\}.$$

We have also

$$b_1\cdot\frac{da_1}{dn_1}+b_2\cdot\frac{da_2}{dn_1}+b_3\cdot\frac{da_3}{dn_1}=\frac{dF}{dn_1}-\frac{1}{2}\frac{dE}{dn_2},$$

$$c_1\cdot\frac{da_1}{dn_1}+c_2\cdot\frac{da_2}{dn_1}+c_3\cdot\frac{da_3}{dn_1}=\frac{dJ}{dn_1}-\frac{1}{2}\frac{dE}{dn_3},$$

$$b_1\cdot\frac{da_1}{dn_3}+b_2\cdot\frac{da_2}{dn_3}+b_3\cdot\frac{da_3}{dn_3}=\frac{1}{2}\left\{\frac{dF}{dn_3}+\frac{dH}{dn_1}-\frac{dJ}{dn_2}\right\},$$

$$c_1\cdot\frac{da_1}{dn_3}+c_2\cdot\frac{da_2}{dn_3}+c_3\cdot\frac{da_3}{dn_3}=\frac{1}{2}\frac{dI}{dn_1},$$

$$b_1\cdot\frac{db_1}{dn_1}+b_2\cdot\frac{db_2}{dn_1}+b_3\cdot\frac{db_3}{dn_1}=\frac{1}{2}\frac{dG}{dn_1},$$

$$c_1\cdot\frac{db_1}{dn_1}+c_2\cdot\frac{db_2}{dn_1}+c_3\cdot\frac{db_3}{dn_1}=\frac{1}{2}\left\{\frac{dH}{dn_1}+\frac{dJ}{dn_2}-\frac{dF}{dn_3}\right\},$$

$$b_1\cdot\frac{db_1}{dn_3}+b_2\cdot\frac{db_2}{dn_3}+b_3\cdot\frac{db_3}{dn_3}=\frac{1}{2}\frac{dG}{dn_3},$$

$$c_1\cdot\frac{db_1}{dn_3}+c_2\cdot\frac{db_2}{dn_3}+c_3\cdot\frac{db_3}{dn_3}=\frac{1}{2}\frac{dI}{dn_2}.$$

We have also of course

$$b_1^2+b_2^2+b_3^2=G,$$
$$c_1^2+c_2^2+c_3^2=I,$$
$$b_1c_1+b_2c_2+b_3c_3=H.$$

If we refer back to "Exercise the first" we find that

$$B\,.\,\frac{da_1}{dn_1}=\frac{dB}{da_1}\,.\,\frac{1}{2}\frac{dE}{dn_1}+\frac{dB}{db_1}\,.\,\left\{\frac{dF}{dn_1}-\frac{1}{2}\frac{dE}{dn_2}\right\}$$
$$+\frac{dB}{dc_1}\,.\,\left\{\frac{dJ}{dn_1}-\frac{1}{2}\frac{dE}{dn_3}\right\};$$

and we may change a_1 into a_2 or a_3 in this, if we at the same time change b_1 into b_2 or b_3, and c_1 into c_2 or c_3. We have also

$$B\frac{db_1}{dn_1}=\frac{dB}{da_1}\frac{1}{2}\frac{dE}{dn_2}+\frac{dB}{db_1}\frac{1}{2}\frac{dG}{dn_1}$$
$$+\frac{dB}{dc_1}\frac{1}{2}\left\{\frac{dH}{dn_1}+\frac{dJ}{dn_2}-\frac{dF}{dn_3}\right\}.$$

And here also we may change a_1 into a_2 or a_3, b_1 into b_2 or b_3, c_1 into c_2 or c_3, and leave *all* else the same.

We can easily obtain in a similar manner

$$B\frac{da_1}{dn_3}=\frac{dB}{da_1}\,.\,\frac{1}{2}\frac{dE}{dn_3}+\frac{dB}{db_1}\frac{1}{2}\left\{\frac{dF}{dn_3}+\frac{dH}{dn_1}-\frac{dJ}{dn_2}\right\}$$
$$+\frac{dB}{dc_1}\,.\,\frac{1}{2}\frac{dI}{dn_1}.$$

And

$$B\frac{db_1}{dn_3}=\frac{dB}{da_1}\,.\,\frac{1}{2}\left\{\frac{dF}{dn_3}+\frac{dJ}{dn_2}-\frac{dH}{dn_1}\right\}$$
$$+\frac{dB}{db_1}\,.\,\frac{1}{2}\frac{dG}{dn_3}+\frac{dB}{dc_1}\,.\,\frac{1}{2}\frac{dI}{dn_2}.$$

Hence we can form the equation

$$\frac{1}{B^2}\begin{vmatrix} +\frac{1}{2}\frac{d^2J}{dn_1dn_2}+\frac{1}{2}\frac{d^2F}{dn_1dn_3}+\frac{1}{2}\frac{d^2H}{dn_1^2}-\frac{1}{2}\frac{d^2E}{dn_2dn_3}; & \frac{dF}{dn_1}-\frac{1}{2}\frac{dE}{dn_2}; & \frac{dJ}{dn_1}-\frac{1}{2}\frac{dE}{dn_3} \\ \frac{1}{2}\frac{dG}{dn_3} \qquad ; & G \quad ; & H \\ \frac{1}{2}\frac{dI}{dn_2} \qquad ; & H \quad ; & I \end{vmatrix}$$

$$-\frac{1}{B^2}\begin{vmatrix} 0 \quad ; & \frac{1}{2}\left\{\frac{dF}{dn_3}+\frac{dH}{dn_1}-\frac{dJ}{dn_2}\right\}; & \frac{1}{2}\frac{dI}{dn_1} \\ \frac{1}{2}\frac{dG}{dn_1} \quad ; & G \quad ; & H \\ \frac{1}{2}\left\{\frac{dH}{dn_1}+\frac{dJ}{dn_2}+\frac{dF}{dn_3}\right\}; & H \quad ; & I \end{vmatrix}$$

$$+\left\{A_{11}\cdot\frac{1}{2}\frac{dE}{dn_3}+A_{12}\cdot\frac{1}{2}\left(\frac{dF}{dn_3}+\frac{dH}{dn_1}-\frac{dJ}{dn_2}\right)+A_{13}\cdot\frac{1}{2}\frac{dI}{dn_1}\right\}\times$$

$$\times\left\{A_{11}\cdot\frac{1}{2}\frac{dE}{dn_2}+A_{12}\cdot\frac{1}{2}\frac{dG}{dn_1}+A_{13}\frac{1}{2}\left(\frac{dH}{dn_1}+\frac{dJ}{dn_2}-\frac{dF}{dn_3}\right)\right\}-$$

$$-\left\{A_{11}\cdot\frac{1}{2}\frac{dE}{dn_1}+A_{12}\cdot\left(\frac{dF}{dn_1}-\frac{1}{2}\frac{dE}{dn_2}\right)+A_{13}\cdot\left(\frac{dJ}{dn_1}-\frac{1}{2}\frac{dE}{dn_3}\right)\right\}\times$$

$$\times\left\{A_{11}\cdot\frac{1}{2}\left(\frac{dF}{dn_3}+\frac{dJ}{dn_2}-\frac{dH}{dn_1}\right)+A_{12}\cdot\frac{1}{2}\frac{dG}{dn_3}+A_{13}\cdot\frac{1}{2}\frac{dI}{dn_2}\right\},$$

all equal zero, so that in the particular case of normal co-ordinates we obtain

$$\frac{1}{2B^2}(4S_2S_3-P_{23}^2)\cdot\left\{\frac{d^2P_{31}}{dn_1dn_2}+\frac{d^2P_{12}}{dn_1dn_3}-\frac{d^2P_{23}}{dn_1^2}-2\frac{d^2S_1}{dn_2dn_3}\right\},$$

$$+\frac{1}{B^2}\begin{vmatrix} 0 \;; & \frac{dP_{12}}{dn_1}-\frac{dS_1}{dn_2}; & \frac{dP_{13}}{dn_1}-\frac{dS_1}{dn_3} \\ \frac{dS_2}{dn_3}; & 2S_2 \;; & P_{23} \\ \frac{dS_3}{dn_2}; & P_{23} \;; & 2S_3 \end{vmatrix}$$

$$-\frac{1}{B^2}\begin{vmatrix} 0 \;; & \frac{1}{2}\left(\frac{dP_{12}}{dn_3}+\frac{dP_{23}}{dn_1}-\frac{dP_{13}}{dn_2}\right); & \frac{dS_3}{dn_1} \\ \frac{dS_2}{dn_1} \;; & 2S_2 \;; & P_{23} \\ \frac{1}{2}\left(\frac{dP_{23}}{dn_1}+\frac{dP_{13}}{dn_2}-\frac{dP_{12}}{dn_3}\right); & P_{23} \;; & 2S_3 \end{vmatrix}$$

$$+\left\{\frac{dS_1}{dn_3}+A_3\frac{1}{2}\left(\frac{dP_{12}}{dn_3}+\frac{dP_{23}}{dn_1}-\frac{dP_{13}}{dn_2}\right)+A_2\frac{dS_3}{dn_1}\right\}\times$$

$$\times\left\{\frac{dS_1}{dn_2}+A_3\frac{dS_2}{dn_1}+A_2\frac{1}{2}\left(\frac{dP_{23}}{dn_1}+\frac{dP_{31}}{dn_2}-\frac{dP_{12}}{dn_3}\right)\right\}$$

$$-\left\{\frac{dS_1}{dn_1}+A_3\left(\frac{dP_{12}}{dn_1}-\frac{dS_1}{dn_2}\right)+A_2\left(\frac{dP_{13}}{dn_1}-\frac{dS_1}{dn_3}\right)\right\}\times$$

$$\times\left\{\frac{1}{2}\left(\frac{dP_{12}}{dn_3}+\frac{dP_{13}}{dn_2}-\frac{dP_{23}}{dn_1}\right)+A_3\frac{dS_2}{dn_3}+A_2\frac{dS_3}{dn_2}\right\};$$

all equal zero.

It is obvious that, as at page 468, we may write this last equation

$$(11)+(22)+(33)+(12)+(13)+(23)=0,$$

and we must now as there develope separately each of these terms.

Firstly, for (11) it equals

$$-\frac{1}{2}\frac{dP_{23}}{dn_1}\cdot\left\{\frac{dS_1}{dn_1}+A_3\cdot\frac{dP_{12}}{dn_1}+A_2\cdot\frac{dP_{13}}{dn_1}\right\}$$

$$-\left\{A_2\cdot\frac{dS_3}{dn_1}+\frac{1}{2}A_3\cdot\frac{dP_{23}}{dn_1}\right\}\left\{A_3\cdot\frac{dS_2}{dn_1}+\frac{1}{2}A_2\frac{dP_{23}}{dn_1}\right\}$$

$$-\frac{1}{B^2}\left\{\begin{array}{l}+S_2\dfrac{dS_3}{dn_1}\cdot\dfrac{dP_{23}}{dn_1}+S_3\dfrac{dS_2}{dn_1}\cdot\dfrac{dP_{23}}{dn_1}\\ -P_{23}\cdot\dfrac{dS_2}{dn_1}\cdot\dfrac{dS_3}{dn_1}-\dfrac{1}{4}P_{23}\cdot\left(\dfrac{dP_{23}}{dn_1}\right)^2\end{array}\right\}.$$

But this equals

$$-\frac{1}{2}\frac{dP_{23}}{dn_1}\left\{\frac{dS_1}{dn_1}+A_3\cdot\frac{dP_{12}}{dn_1}+A_2\cdot\frac{dP_{13}}{dn_1}\right\}$$

$$-\frac{1}{2}\frac{dS_3}{dn_1}\cdot\frac{dP_{23}}{dn_1}-\frac{1}{2}\frac{dS_2}{dn_1}\cdot\frac{dP_{23}}{dn_1}$$

$$-A_1\cdot\frac{dS_2}{dn_1}\cdot\frac{dS_3}{dn_1}-\frac{1}{4}A_1\left(\frac{dP_{23}}{dn_1}\right)^2.$$

But this equals

$$-\frac{1}{2}\frac{dP_{23}}{dn_1}\left\{\frac{dS_1}{dn_1}+\frac{dS_2}{dn_1}+\frac{dS_3}{dn_1}+A_1\frac{dP_{23}}{dn_1}+A_2\frac{dP_{13}}{dn_1}+A_3\frac{dP_{12}}{dn_1}\right\}$$

$$+A_1\left\{\frac{1}{4}\left(\frac{dP_{23}}{dn_1}\right)^2-\frac{dS_2}{dn_1}\cdot\frac{dS_3}{dn_1}\right\},$$

which finally equals

$$A_1\left\{\frac{1}{4}\left(\frac{dP_{23}}{dn_1}\right)^2-\frac{dS_2}{dn_1}\cdot\frac{dS_3}{dn_1}\right\}$$

$$-\frac{1}{2B}\frac{dB}{dn_1}\cdot\frac{dP_{23}}{dn_1}.$$

Next (22) equals

$$-A_3\frac{dS_1}{dn_2}\left\{A_2\cdot\frac{dS_3}{dn_2}+\frac{1}{2}\frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{1}{2}A_3\cdot\frac{dP_{13}}{dn_2}\left\{\frac{dS_1}{dn_2}+\frac{1}{2}A_2\frac{dP_{13}}{dn_2}\right\}$$

$$-\frac{1}{B^2}\left\{\frac{1}{4}P_{23}\left(\frac{dP_{13}}{dn_2}\right)^2-P_{23}\cdot\frac{dS_3}{dn_2}\cdot\frac{dS_1}{dn_2}\right\},$$

which clearly equals

$$A_1\left\{\frac{1}{4}\left(\frac{dP_{13}}{dn_2}\right)^2-\frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_2}\right\}.$$

(33) and (22) must be symmetrical, and hence (33) equals

$$A_1\left\{\frac{1}{4}\left(\frac{dP_{12}}{dn_3}\right)^2-\frac{dS_2}{dn_3}\cdot\frac{dS_1}{dn_3}\right\}.$$

Having thus calculated the values of (11), (22), and (33), we shall now proceed to calculate (23), it equals

$$-A_3 \cdot \frac{dS_1}{dn_2}\left\{A_3 \cdot \frac{dS_2}{dn_3} + \frac{1}{2}\frac{dP_{12}}{dn_3}\right\}$$

$$-A_2 \cdot \frac{dS_1}{dn_3}\left\{A_2 \cdot \frac{dS_3}{dn_2} + \frac{1}{2}\frac{dP_{13}}{dn_2}\right\}$$

$$-\frac{1}{4}A_2 A_3 \frac{dP_{12}}{dn_3} \cdot \frac{dP_{13}}{dn_2}$$

$$-\left\{\frac{dS_1}{dn_3} + \frac{1}{2}A_3 \cdot \frac{dP_{12}}{dn_3}\right\}\left\{\frac{dS_1}{dn_2} + \frac{1}{2}A_2 \frac{dP_{13}}{dn_2}\right\}$$

$$-\frac{1}{B^2}\left\{\begin{array}{c} +2S_2 \dfrac{dS_3}{dn_2}\dfrac{dS_1}{dn_3} + 2S_3 \dfrac{dS_2}{dn_3}\dfrac{dS_1}{dn_2} \\ -\dfrac{1}{2}P_{23}\dfrac{dP_{13}}{dn_2} \cdot \dfrac{dP_{12}}{dn_3} \end{array}\right\},$$

all which may be written

$$-\frac{1}{2}A_1 \frac{dP_{13}}{dn_2} \cdot \frac{dP_{12}}{dn_3}$$

$$-\frac{1}{2}\frac{dS_1}{dn_2}\left\{\frac{dS_1}{dn_3} + \frac{dS_2}{dn_3} + \frac{dS_3}{dn_3} + A_3 \cdot \frac{dP_{12}}{dn_3} + A_2 \cdot \frac{dP_{13}}{dn_3} + A_1 \frac{dP_{23}}{dn_3}\right\}$$

$$-\frac{1}{2}\frac{dS_1}{dn_3}\left\{\frac{dS_1}{dn_2} + \frac{dS_2}{dn_2} + \frac{dS_3}{dn_2} + A_3 \cdot \frac{dP_{12}}{dn_2} + A_2 \cdot \frac{dP_{13}}{dn_2} + A_1 \frac{dP_{23}}{dn_2}\right\}$$

$$-\frac{1}{2}\frac{dS_1}{dn_2}\left\{P_{12} \cdot \frac{dA_3}{dn_3} + P_{13} \cdot \frac{dA_2}{dn_3} + P_{23} \cdot \frac{dA_1}{dn_3}\right\}$$

$$-\frac{1}{2}\frac{dS_1}{dn_3}\left\{P_{12} \cdot \frac{dA_3}{dn_2} + P_{13} \cdot \frac{dA_2}{dn_2} + P_{23} \cdot \frac{dA_1}{dn_2}\right\}$$

$$+\frac{1}{2}\frac{dS_1}{dn_2}\left\{P_{12} \cdot \frac{dA_3}{dn_3} + P_{13} \cdot \frac{dA_2}{dn_3} + P_{23} \cdot \frac{dA_1}{dn_3}\right\}$$

$$+\frac{1}{2}\frac{dS_1}{dn_3}\left\{P_{12} \cdot \frac{dA_3}{dn_2} + P_{13} \cdot \frac{dA_2}{dn_2} + P_{23} \cdot \frac{dA_1}{dn_2}\right\}$$

$$+\frac{1}{2}\frac{dS_1}{dn_2} \cdot \frac{dS_3}{dn_3} + \frac{1}{2}A_2 \frac{dS_1}{dn_2} \cdot \frac{dP_{13}}{dn_3} + \frac{1}{2}A_1 \frac{dS_1}{dn_2} \cdot \frac{dP_{23}}{dn_3}$$

$$+\frac{1}{2}\frac{dS_1}{dn_3} \cdot \frac{dS_2}{dn_2} + \frac{1}{2}A_3 \frac{dS_1}{dn_3} \cdot \frac{dP_{12}}{dn_2} + \frac{1}{2}A_1 \frac{dS_1}{dn_3} \cdot \frac{dP_{23}}{dn_2}$$

$$-\frac{1}{2}\frac{dS_1}{dn_2} \cdot \frac{dS_2}{dn_2} - \frac{1}{2}\frac{dS_1}{dn_3} \cdot \frac{dS_3}{dn_2}$$

$$-\frac{1}{2}A_2 \frac{dS_1}{dn_3} \cdot \frac{dP_{13}}{dn_2} - \frac{1}{2}A_3 \cdot \frac{dS_1}{dn_2} \cdot \frac{dP_{12}}{dn_3}.$$

Hence (23) equals

$$+\frac{1}{2}A_1\left\{\frac{dS_1}{dn_2}\cdot\frac{dP_{23}}{dn_3}+\frac{dS_1}{dn_3}\cdot\frac{dP_{23}}{dn_2}\right\}$$

$$-\frac{1}{4}A_1\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dP_{12}}{dn_3}+\frac{dP_{13}}{dn_3}\cdot\frac{dP_{12}}{dn_2}\right\}$$

$$-\frac{1}{2B}\cdot\frac{dB}{dn_3}\cdot\frac{dS_1}{dn_2}-\frac{1}{2B}\cdot\frac{dB}{dn_2}\cdot\frac{dS_1}{dn_3}$$

$$+\frac{A_1}{4}\left\{\frac{d'P_{13}}{dn_3}\cdot\frac{d'P_{12}}{dn_2}-\frac{d''P_{13}}{dn_2}\cdot\frac{d''P_{12}}{dn_3}\right\}$$

$$+\frac{1}{2}\left\{\frac{d'S_1}{dn_2}\cdot\frac{d'S_3}{dn_3}-\frac{d''S_1}{dn_3}\cdot\frac{d''S_3}{dn_2}\right\}$$

$$+\frac{1}{2}\left\{\frac{d'S_1}{dn_3}\cdot\frac{d'S_2}{dn_2}-\frac{d''S_1}{dn_2}\cdot\frac{d''S_2}{dn_3}\right\}$$

$$+\frac{A_2}{2}\left\{\frac{d'S_1}{dn_2}\cdot\frac{d'P_{13}}{dn_3}-\frac{d''S_1}{dn_3}\cdot\frac{d''P_{13}}{dn_2}\right\}$$

$$+\frac{A_3}{2}\left\{\frac{d'S_1}{dn_3}\cdot\frac{d'P_{12}}{dn_2}-\frac{d''S_1}{dn_2}\cdot\frac{d''P_{12}}{dn_3}\right\}.$$

We are now going to reduce the ten terms marked $'$, and the ten marked $''$; firstly, we shall reduce the ten marked $'$, and then the ten marked $''$ can be deduced from these by changing dn_2 into dn_3, and dn_3 into dn_2.... To proceed, the ten terms marked $'$ equal

$$+\frac{1}{2}\frac{dS_1}{dn_2}\left\{\begin{array}{c}\dfrac{dS_3}{dn_3}+\dfrac{A_2}{2}\cdot\dfrac{dP_{13}}{dn_3}+\dfrac{A_1}{2}\cdot\dfrac{dP_{23}}{dn_3}\\ +\dfrac{P_{13}}{2}\cdot\dfrac{dA_2}{dn_3}+\dfrac{P_{23}}{2}\cdot\dfrac{dA_1}{dn_3}\end{array}\right\},$$

$$-\frac{1}{4}\frac{dS_1}{dn_2}\left\{P_{13}\cdot\frac{dA_2}{dn_3}+P_{23}\cdot\frac{dA_1}{dn_3}\right\};$$

$$-\frac{A_1}{4}\cdot\frac{dS_1}{dn_2}\cdot\frac{dP_{23}}{dn_3}.$$

$$+\frac{1}{2}\frac{dS_1}{dn_3}\left\{\begin{array}{c}\dfrac{dS_2}{dn_2}+\dfrac{A_3}{2}\cdot\dfrac{dP_{12}}{dn_2}+\dfrac{A_1}{2}\cdot\dfrac{dP_{23}}{dn_2}\\ +\dfrac{P_{12}}{2}\cdot\dfrac{dA}{dn_2}+\dfrac{P_{23}}{2}\cdot\dfrac{dA_1}{dn_2}\end{array}\right\},$$

$$-\frac{1}{4}\frac{dS_1}{dn_3}\left\{P_{12}\cdot\frac{dA_3}{dn_2}+P_{23}\cdot\frac{dA_1}{dn_2}\right\}$$

$$-\frac{A_1}{4}\cdot\frac{dS_1}{dn_3}\cdot\frac{dP_{23}}{dn_2}$$

$$+\frac{1}{8}\frac{dP_{13}}{dn_3}\left\{\begin{matrix} +2A_2\cdot\dfrac{dS_1}{dn_2}+A_1\cdot\dfrac{dP_{12}}{dn_2}+\dfrac{dP_{13}}{dn_2} \\ +2S_1\cdot\dfrac{dA_2}{dn_2}+P_{12}\cdot\dfrac{dA_1}{dn_2} \end{matrix}\right\}$$

$$-\frac{1}{8}\frac{dP_{13}}{dn_3}\left\{2S_1\cdot\frac{dA_2}{dn_2}+P_{12}\cdot\frac{dA_1}{dn_2}+\frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{1}{8}\frac{dP_{12}}{dn_2}\left\{\begin{matrix} +2A_3\cdot\dfrac{dS_1}{dn_3}+A_1\cdot\dfrac{dP_{13}}{dn_3}+\dfrac{dP_{12}}{dn_3} \\ +2S_1\cdot\dfrac{dA_3}{dn_3}+P_{13}\cdot\dfrac{dA_1}{dn_3} \end{matrix}\right\}$$

$$-\frac{1}{8}\frac{dP_{12}}{dn_2}\left\{2S_1\cdot\frac{dA_3}{dn_3}+P_{13}\cdot\frac{dA_1}{dn_3}+\frac{dP_{12}}{dn_3}\right\}.$$

It is easily seen therefore that if we omit all the terms that are destroyed either by equal terms in $''$ or else by virtue of the ten equations given at commencement of this Exercise that $'$ may be written

$$-\frac{1}{4}\frac{dS_1}{dn_2}\left\{P_{23}\cdot\frac{dA_1}{dn_3}+P_{13}\cdot\frac{dA_2}{dn_3}\right\}$$

$$-\frac{1}{4}\frac{dS_1}{dn_3}\left\{P_{23}\cdot\frac{dA_1}{dn_2}+P_{12}\cdot\frac{dA_3}{dn_2}\right\}$$

$$-\frac{1}{8}\frac{dP_{13}}{dn_3}\left\{2S_1\cdot\frac{dA_2}{dn_2}+P_{12}\cdot\frac{dA_1}{dn_2}\right\}$$

$$-\frac{1}{8}\frac{dP_{12}}{dn_2}\left\{2S_1\cdot\frac{dA_3}{dn_3}+P_{13}\cdot\frac{dA_1}{dn_3}\right\};$$

hence we easily see that if in $'-''$ we neglect all the terms that mutually destroy that, $'-''$ equals

$$+\frac{P_{12}}{4}\left\{\frac{dS_1}{dn_2}\cdot\frac{dA_3}{dn_3}-\frac{dS_1}{dn_3}\cdot\frac{dA_3}{dn_2}\right\}$$

$$+\frac{P_{13}}{4}\left\{\frac{dS_1}{dn_3}\cdot\frac{dA_2}{dn_2}-\frac{dS_1}{dn_2}\cdot\frac{dA_2}{dn_3}\right\}$$

$$+\frac{P_{13}}{8}\left\{\frac{dP_{12}}{dn_3}\cdot\frac{dA_1}{dn_2}-\frac{dP_{12}}{dn_2}\cdot\frac{dA_1}{dn_3}\right\}$$

$$+\frac{P_{12}}{8}\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dA_1}{dn_3}-\frac{dP_{13}}{dn_3}\cdot\frac{dA_1}{dn_2}\right\}$$

$$+\frac{S_1}{4}\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dA_2}{dn_3}-\frac{dP_{13}}{dn_3}\cdot\frac{dA_2}{dn_2}\right\}$$

$$+\frac{S_1}{4}\left\{\frac{dP_{12}}{dn_3}\cdot\frac{dA_3}{dn_2}-\frac{dP_{12}}{dn_2}\cdot\frac{dA_3}{dn_3}\right\}.$$

But this may be written

$$\frac{S_1P_{12} - S_1P_{12}}{2B}\left\{\frac{dB}{dn_2}\cdot\frac{dA_3}{dn_3} - \frac{dB}{dn_3}\cdot\frac{dA_3}{dn_2}\right\}$$

$$+ \frac{S_1P_{13} - S_1P_{13}}{2B}\left\{\frac{dB}{dn_3}\cdot\frac{dA_2}{dn_2} - \frac{dB}{dn_2}\cdot\frac{dA_2}{dn_3}\right\}$$

$$+ \frac{P_{13}P_{12} - P_{13}P_{12}}{4B}\left\{\frac{dB}{dn_3}\cdot\frac{dA_1}{dn_2} - \frac{dB}{dn_2}\cdot\frac{dA_1}{dn_3}\right\}$$

$$+ \left\{\frac{dA_1}{dn_3}\cdot\frac{dA_3}{dn_2} - \frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_3}\right\}\left[\begin{array}{l} +\frac{B^2}{4}A_1P_{12} + \frac{B^2}{8}A_1P_{12} \\ +\frac{B^2}{8}P_{13} + \frac{B^2}{4}A_2S_1 \end{array}\right]$$

$$+ \left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_3} - \frac{dA_1}{dn_3}\cdot\frac{dA_2}{dn_2}\right\}\left\{\begin{array}{l} +\frac{B^2}{4}A_1P_{13} \\ +\frac{B^2}{8}A_1P_{13} \\ +\frac{B^2}{8}P_{12} \\ +\frac{B^2}{4}A_3S_1 \end{array}\right.$$

$$+ \left\{\frac{dA_2}{dn_3}\cdot\frac{dA_3}{dn_2} - \frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_3}\right\}\left\{\begin{array}{l} +\frac{B^2}{4}A_1S_1 \\ +\frac{B^2}{4}A_1S_1\,. \end{array}\right.$$

And thus we obtain for the full developed form of (23) the expression

$$+ \frac{A_1}{2}\left\{\frac{dS_1}{dn_2}\cdot\frac{dP_{23}}{dn_3} + \frac{dS_1}{dn_3}\cdot\frac{dP_{23}}{dn_2}\right\}$$

$$- \frac{A_1}{4}\left\{\frac{dP_{13}}{dn_2}\cdot\frac{dP_{12}}{dn_3} + \frac{dP_{13}}{dn_3}\cdot\frac{dP_{12}}{dn_2}\right\}$$

$$- \frac{1}{2B}\cdot\frac{dB}{dn_3}\cdot\frac{dS_1}{dn_2} - \frac{1}{2B}\cdot\frac{dB}{dn_2}\cdot\frac{dS_1}{dn_3}$$

$$+ \frac{A_1B^2}{4}\left\{+P_{12}\cdot\binom{31}{23} + P_{13}\cdot\binom{12}{23} - 2S_1\cdot\binom{23}{23}\right\}.$$

We now proceed to calculate (12) which is clearly symmetrical with (13). (12) equals

$$+ \left\{A_2\cdot\frac{dS_3}{dn_2} + \frac{1}{2}\frac{dP_{13}}{dn_2}\right\}\left\{\frac{dS_1}{dn_1} + A_3\cdot\frac{dP_{12}}{dn_1} + A_2\cdot\frac{dP_{13}}{dn_1}\right\}$$

$$+ \frac{A_3}{2}\cdot\frac{dS_1}{dn_2}\cdot\frac{dP_{23}}{dn_1}$$

$$-\left\{A_2 \cdot \frac{dS_3}{dn_1} + \frac{A_3}{2} \cdot \frac{dP_{23}}{dn_1}\right\}\left\{\frac{dS_1}{dn_2} + \frac{A_2}{2} \cdot \frac{dP_{13}}{dn_2}\right\}$$

$$+\frac{A_3}{2} \cdot \frac{dP_{13}}{dn_2}\left\{A_3 \cdot \frac{dS_2}{dn_1} + \frac{A_2}{2} \cdot \frac{dP_{23}}{dn_1}\right\}$$

$$-\frac{1}{B^2}\left\{\begin{array}{l} + S_2 \cdot \dfrac{dS_3}{dn_1} \cdot \dfrac{dP_{13}}{dn_2} - 2S_2 \cdot \dfrac{dS_3}{dn_2} \cdot \dfrac{dP_{13}}{dn_1} \\ - S_3 \cdot \dfrac{dS_2}{dn_1} \cdot \dfrac{dP_{13}}{dn_2} + P_{23} \cdot \dfrac{dS_3}{dn_2} \cdot \dfrac{dP_{12}}{dn_1} \end{array}\right\}.$$

Which equals

$$+\frac{dS_3}{dn_2}\left\{A_2 \cdot \frac{dS_1}{dn_1} + A_1 \cdot \frac{dP_{12}}{dn_1} + \frac{dP_{13}}{dn_1}\right\}$$

$$+\frac{1}{2}\frac{dP_{13}}{dn_2}\left\{\frac{dS_1}{dn_1} + A_3 \cdot \frac{dP_{12}}{dn_1} + A_2 \cdot \frac{dP_{13}}{dn_1}\right\}$$

$$+\frac{1}{2}\frac{dP_{13}}{dn_2}\left\{\frac{dS_2}{dn_1} - \frac{dS_3}{dn_1}\right\}$$

$$-A_2 \cdot \frac{dS_3}{dn_1} \cdot \frac{dS_1}{dn_2}.$$

Which equals

$$+\frac{1}{2}\frac{dP_{13}}{dn_2}\left\{\begin{array}{l} + \dfrac{dS_1}{dn_1} + \dfrac{dS_2}{dn_1} + \dfrac{dS_3}{dn_1} + A_3 \cdot \dfrac{dP_{12}}{dn_1} + A_2 \cdot \dfrac{dP_{13}}{dn_1} + A_1 \cdot \dfrac{dP_{23}}{dn_1} \\ \qquad - 2\dfrac{dS_3}{dn_1} \qquad\qquad\qquad - A_1 \cdot \dfrac{dP_{23}}{dn_1} \\ \qquad\qquad + P_{12} \cdot \dfrac{dA_3}{dn_1} + P_{13} \cdot \dfrac{dA_2}{dn_1} + P_{23} \cdot \dfrac{dA_1}{dn_1} \\ \qquad\qquad - P_{12} \cdot \dfrac{dA_3}{dn_1} - P_{13} \cdot \dfrac{dA_2}{dn_1} - P_{23} \cdot \dfrac{dA_1}{dn_1} \end{array}\right\}$$

$$+\frac{dS_3}{dn_2}\left\{A_2 \cdot \frac{dS_1}{dn_1} + A_1 \cdot \frac{dP_{12}}{dn_1} + \frac{dP_{13}}{dn_1}\right\}$$

$$-A_2 \cdot \frac{dS_3}{dn_1} \cdot \frac{dS_1}{dn_2}.$$

Which equals

$$-\frac{1}{2}\frac{dP_{13}}{dn_2}\left\{P_{23} \cdot \frac{dA_1}{dn_1} + P_{13} \cdot \frac{dA_2}{dn_1} + P_{12} \cdot \frac{dA_3}{dn_1}\right\}$$

$$-\frac{dP_{13}}{dn_2}\left\{\frac{dS_3}{dn_1} + \frac{A_1}{2} \cdot \frac{dP_{23}}{dn_1}\right\}$$

$$+A_2\left\{\frac{dS_1}{dn_1} \cdot \frac{dS_3}{dn_2} - \frac{dS_1}{dn_2} \cdot \frac{dS_3}{dn_1}\right\}$$

$$+A_1 \cdot \frac{dP_{12}}{dn_1} \cdot \frac{dS_3}{dn_2}$$

$$+\frac{dP_{13}}{dn_1} \cdot \frac{dS_3}{dn_2}.$$

Which equals

$$
\begin{aligned}
&+\frac{1}{2B}\cdot\frac{dB}{dn_1}\cdot\frac{dP_{13}}{dn_2}\\
&-\frac{dP_{13}}{dn_2}\left\{\frac{dS_3}{dn_1}+\frac{A_1}{2}\cdot\frac{dP_{23}}{dn_1}+\frac{A_2}{2}\cdot\frac{dP_{13}}{dn_1}\right\}\\
&+\frac{dP_{13}}{dn_1}\left\{\frac{dS_3}{dn_2}+\frac{A_2}{2}\cdot\frac{dP_{13}}{dn_2}+\frac{A_1}{2}\cdot\frac{dP_{23}}{dn_2}\right\}\\
&-\frac{A_1}{2}\cdot\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}+A_1\cdot\frac{dP_{12}}{dn_1}\cdot\frac{dS_3}{dn_2}\\
&+A_2\left\{\frac{dS_1}{dn_1}\cdot\frac{dS_3}{dn_2}-\frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_1}\right\}.
\end{aligned}
$$

Which equals

$$
\begin{aligned}
&+\frac{1}{2B}\cdot\frac{dB}{dn_1}\cdot\frac{dP_{13}}{dn_2}-\frac{1}{2B}\cdot\frac{dB}{dn_1}\cdot\frac{dP_{13}}{dn_2}\\
&+\frac{1}{2B}\cdot\frac{dB}{dn_2}\cdot\frac{dP_{13}}{dn_1}\\
&+\frac{P_{12}}{2}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dP_{13}}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}\\
&-\frac{A_1}{2}\cdot\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}+A_1\cdot\frac{dP_{12}}{dn_1}\cdot\frac{dS_3}{dn_2}\\
&+A_2\left\{\frac{dS_1}{dn_1}\cdot\frac{dS_3}{dn_2}-\frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_1}\right\}.
\end{aligned}
$$

Which equals

$$
\begin{aligned}
&+\frac{A_1}{2}\left\{\frac{dP_{12}}{dn_1}\cdot\frac{dS_3}{dn_2}+\frac{dP_{12}}{dn_2}\cdot\frac{dS_3}{dn_1}\right\}\\
&-\frac{A_1}{4}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}+\frac{dP_{13}}{dn_2}\cdot\frac{dP_{23}}{dn_1}\right\}\\
&+\frac{1}{4B}\cdot\frac{dB}{dn_2}\cdot\frac{dP_{13}}{dn_1}+\frac{1}{4B}\cdot\frac{dB}{dn_1}\cdot\frac{dP_{13}}{dn_2}\\
&+\frac{A_1}{2}\left\{\frac{dP_{12}}{dn_1}\cdot\frac{dS_3}{dn_2}-\frac{dP_{12}}{dn_2}\cdot\frac{dS_3}{dn_1}\right\}\ldots\ldots.(\alpha)\\
&+A_2\left\{\frac{dS_1}{dn_1}\cdot\frac{dS_3}{dn_2}-\frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_1}\right\}\ldots\ldots\ldots(\alpha)\\
&+\frac{P_{12}}{2}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dP_{13}}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}\ldots(\beta)\\
&+\frac{1}{4B}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dB}{dn_2}-\frac{dP_{13}}{dn_2}\cdot\frac{dB}{dn_1}\right\}\ldots.(\beta)\\
&-\frac{A_1}{4}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}-\frac{dP_{13}}{dn_2}\cdot\frac{dP_{23}}{dn_1}\right\}\ldots(\gamma);
\end{aligned}
$$

the last five lines marked (α), (α), (β), (β), (γ) I now proceed to reduce.

The two lines marked (α), (α) equal

$$+\frac{dS_3}{dn_2}\left\{A_2 \cdot \frac{dS_1}{dn_1}+\frac{A_1}{2} \cdot \frac{dP_{12}}{dn_1}\right\}$$
$$-\frac{dS_3}{dn_1}\left\{A_2 \cdot \frac{dS_1}{dn_2}+\frac{A_1}{2} \cdot \frac{dP_{12}}{dn_2}\right\};$$

which equals

$$+\frac{1}{2}\left\{\frac{dS_3}{dn_1} \cdot \frac{dP_{13}}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dP_{13}}{dn_1}\right\}$$
$$+S_1\left\{\frac{dS_3}{dn_1} \cdot \frac{dA_2}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dA_2}{dn_1}\right\}$$
$$+\frac{P_{12}}{2}\left\{\frac{dS_3}{dn_1} \cdot \frac{dA_1}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dA_1}{dn_1}\right\}.$$

To the first of these last three lines add the line (γ), then $(\alpha)+(\alpha)+(\gamma)$ equals

$$+\frac{1}{2}\frac{dP_{13}}{dn_2}\left\{\frac{dS_3}{dn_1}+\frac{A_1}{2} \cdot \frac{dP_{23}}{dn_1}\right\}$$
$$-\frac{1}{2}\frac{dP_{13}}{dn_1}\left\{\frac{dS_3}{dn_2}+\frac{A_1}{2} \cdot \frac{dP_{23}}{dn_2}\right\}$$
$$+S_1\left\{\frac{dS_3}{dn_1} \cdot \frac{dA_2}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dA_2}{dn_1}\right\}$$
$$+\frac{P_{12}}{2}\left\{\frac{dS_3}{dn_1} \cdot \frac{dA_1}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dA_1}{dn_1}\right\};$$

or slightly modifying first two lines by aid of the ten formulæ given at commencement of this Exercise and adding on (β), (β), we obtain that $(\alpha)+(\alpha)+(\beta)+(\beta)+(\gamma)$ equals

$$+\frac{P_{23}}{4}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dA_1}{dn_2}-\frac{dP_{13}}{dn_2} \cdot \frac{dA_1}{dn_1}\right\}$$
$$+\frac{P_{13}}{4}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dA_2}{dn_2}-\frac{dP_{13}}{dn_2} \cdot \frac{dA_2}{dn_1}\right\}$$
$$+\frac{P_{12}}{2}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dA_3}{dn_2}-\frac{dP_{13}}{dn_2} \cdot \frac{dA_3}{dn_1}\right\}$$
$$+\frac{1}{4B}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dB}{dn_2}-\frac{dP_{13}}{dn_2} \cdot \frac{dB}{dn_1}\right\}$$
$$+S_1\left\{\frac{dS_3}{dn_1} \cdot \frac{dA_2}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dA_2}{dn_1}\right\}$$
$$+\frac{P_{12}}{2}\left\{\frac{dS_3}{dn_1} \cdot \frac{dA_1}{dn_2}-\frac{dS_3}{dn_2} \cdot \frac{dA_1}{dn_1}\right\}.$$

But the sum of these first four lines clearly equals

$$+\frac{P_{12}}{4}\left\{\frac{dP_{13}}{dn_1} \cdot \frac{dA_3}{dn_2}-\frac{dP_{13}}{dn_2} \cdot \frac{dA_3}{dn_1}\right\},$$

remembering that $-\frac{dB}{B}=P_{12} \cdot dA_3+P_{13} \cdot dA_2+P_{23} \cdot dA_1$; and by picking out coefficients it is thus easy to see that in $(\alpha)+(\alpha)+(\beta)+(\beta)+(\gamma)$ the coefficient of

$$\left\{\frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_1}-\frac{dA_2}{dn_1}\cdot\frac{dA_3}{dn_2}\right\}$$

equals
$$+\frac{B^2}{4}P_{12}-B^2A_3S_1+\frac{1}{2}P_{12}P_{13}{}^2-2P_{12}S_1S_3,$$

which equals
$$-B^2\left\{\frac{P_{12}}{4}+A_3S_1\right\}.$$

The coefficient of

$$\left\{\frac{dA_1}{dn_1}\cdot\frac{dA_3}{dn_2}-\frac{dA_1}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}$$

equals
$$+\frac{B^2}{4}A_3P_{12}+\frac{B^2}{2}A_3P_{12}-\frac{1}{2}P_{12}P_{13}P_{23}+P_{12}{}^2S_3,$$

which equals
$$+\frac{B^2}{4}A_3P_{12}.$$

The coefficient of

$$\left\{\frac{dA_1}{dn_2}\cdot\frac{dA_2}{dn_1}-\frac{dA_1}{dn_1}\cdot\frac{dA_2}{dn_2}\right\}$$

equals
$$-P_{12}P_{13}S_3+2S_1S_3P_{23},$$

which equals
$$-B^2A_1S_3.$$

Hence finally (12) equals

$$\begin{aligned}&+\frac{A_1}{2}\left\{\frac{dP_{12}}{dn_1}\cdot\frac{dS_3}{dn_2}+\frac{dP_{12}}{dn_2}\cdot\frac{dS_3}{dn_1}\right\}\\&-\frac{A_1}{4}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2}+\frac{dP_{13}}{dn_2}\cdot\frac{dP_{23}}{dn_1}\right\}\\&+\frac{1}{4B}\cdot\frac{dB}{dn_2}\cdot\frac{dP_{13}}{dn_1}+\frac{1}{4B}\cdot\frac{dB}{dn_1}\cdot\frac{dP_{13}}{dn_2}\\&+\frac{B^2}{4}\left\{+\left(P_{12}+4A_3S_1\right)\cdot\binom{23}{12}-A_3P_{12}\cdot\binom{31}{12}+4A_1S_3\binom{12}{12}\right\};\end{aligned}$$

(13) must be symmetrical with this, therefore it equals

$$\begin{aligned}&+\frac{A_1}{2}\left\{\frac{dP_{13}}{dn_1}\cdot\frac{dS_2}{dn_3}+\frac{dP_{13}}{dn_3}\cdot\frac{dS_2}{dn_1}\right\}\\&-\frac{A_1}{4}\left\{\frac{dP_{21}}{dn_1}\cdot\frac{dP_{23}}{dn_3}+\frac{dP_{12}}{dn_3}\cdot\frac{dP_{23}}{dn_1}\right\}\\&+\frac{1}{4B}\cdot\frac{dB}{dn_3}\cdot\frac{dP_{12}}{dn_1}+\frac{1}{4B}\cdot\frac{dB}{dn_1}\cdot\frac{dP_{12}}{dn_3}\Big\}\\&+\frac{B^2}{4}\left\{+(P_{13}+4A_2S_1)\binom{23}{31}-A_2P_{13}\binom{12}{31}+4A_1S_2\binom{31}{31}\right\}.\end{aligned}$$

Hence and by symmetry our second three differential equations are

$$
\begin{aligned}
&+\frac{1}{2}\left\{+2\frac{d^2S_1}{dn_2dn_3}+\frac{d^2P_{23}}{dn_1^{\,2}}-\frac{d^2P_{13}}{dn_1dn_2}-\frac{d^2P_{12}}{dn_1dn_3}\right\} \\
&+A_1\Lambda \\
&+\frac{1}{4B}\left\{-2\frac{dS_1}{dn_2}\cdot\frac{dB}{dn_3}-2\frac{dS_1}{dn_3}\cdot\frac{dB}{dn_2}-2\frac{dP_{23}}{dn_1}\cdot\frac{dB}{dn_1}+\frac{dP_{13}}{dn_1}\cdot\frac{dB}{dn_2}+\frac{dP_{13}}{dn_2}\cdot\frac{dB}{dn_1}+\frac{dP_{12}}{dn_1}\cdot\frac{dB}{dn_3}+\frac{dP_{12}}{dn_3}\cdot\frac{dB}{dn_1}\right\} \\
&+\frac{B^2}{4}\left\{\begin{array}{lll}
-2A_1S_1\cdot\binom{23}{23} & +A_1P_{12}\cdot\binom{31}{23} & +A_1P_{13}\cdot\binom{12}{23} \\
+(P_{13}+4A_2S_1)\cdot\binom{23}{31} & +4A_1S_2\cdot\binom{31}{31} & -A_2P_{13}\cdot\binom{12}{31} \\
+(P_{12}+4A_3S_1)\cdot\binom{23}{12} & -A_3P_{12}\cdot\binom{31}{12} & +4A_1S_3\cdot\binom{12}{12}
\end{array}\right\}=0.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\left\{+2\frac{d^2S_2}{dn_3dn_1}+\frac{d^2P_{31}}{dn_2^{\,2}}-\frac{d^2P_{12}}{dn_2dn_3}-\frac{d^2P_{23}}{dn_1dn_2}\right\} \\
&+A_2\Lambda \\
&+\frac{1}{4B}\left\{-2\frac{dS_2}{dn_1}\cdot\frac{dB}{dn_3}-2\frac{dS_2}{dn_3}\cdot\frac{dB}{dn_1}-2\frac{dP_{13}}{dn_2}\cdot\frac{dB}{dn_2}+\frac{dP_{12}}{dn_2}\cdot\frac{dB}{dn_3}+\frac{dP_{12}}{dn_3}\cdot\frac{dB}{dn_2}+\frac{dP_{23}}{dn_2}\cdot\frac{dB}{dn_1}+\frac{dP_{23}}{dn_1}\cdot\frac{dB}{dn_2}\right\} \\
&+\frac{B^2}{4}\left\{\begin{array}{lll}
-2A_2S_2\cdot\binom{31}{31} & +A_2P_{23}\cdot\binom{12}{31} & +A_2P_{12}\cdot\binom{23}{31} \\
+(P_{12}+4A_3S_2)\cdot\binom{31}{12} & +4A_2S_3\cdot\binom{12}{12} & -A_3P_{12}\cdot\binom{23}{12} \\
+(P_{23}+4A_1S_2)\cdot\binom{31}{23} & -A_1P_{23}\cdot\binom{12}{23} & +4A_2S_1\cdot\binom{23}{23}
\end{array}\right\}=0.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\left\{+2\frac{d^2S_3}{dn_1dn_2}+\frac{d^2P_{12}}{dn_3^{\,2}}-\frac{d^2P_{23}}{dn_3dn_1}-\frac{d^2P_{31}}{dn_3dn_2}\right\} \\
&+A_3\Lambda \\
&+\frac{1}{4B}\left\{-2\frac{dS_3}{dn_1}\cdot\frac{dB}{dn_2}-2\frac{dS_3}{dn_2}\cdot\frac{dB}{dn_1}-2\frac{dP_{12}}{dn_3}\cdot\frac{dB}{dn_3}+\frac{dP_{23}}{dn_3}\cdot\frac{dB}{dn_1}+\frac{dP_{23}}{dn_1}\cdot\frac{dB}{dn_3}+\frac{dP_{13}}{dn_2}\cdot\frac{dB}{dn_3}+\frac{dP_{13}}{dn_3}\cdot\frac{dB}{dn_2}\right\} \\
&+\frac{B^2}{4}\left\{\begin{array}{lll}
-2A_3S_3\cdot\binom{12}{12} & +A_3P_{31}\cdot\binom{23}{12} & +A_3P_{23}\cdot\binom{31}{12} \\
+(P_{23}+4A_1S_3)\cdot\binom{12}{23} & +4A_3S_1\cdot\binom{23}{23} & -A_1P_{23}\cdot\binom{31}{23} \\
+(P_{13}+4A_2S_3)\cdot\binom{12}{31} & -A_2P_{13}\cdot\binom{23}{31} & +4A_3S_2\cdot\binom{31}{31}
\end{array}\right\}=0.
\end{aligned}
$$

For the sake of convenience of reference I also write down the three remaining differential equations of the second order; one has already been given, and the other two are deducible by symmetry.

$$-\frac{d^2S_1}{dn_2^{\,2}}+\frac{d^2P_{12}}{dn_1dn_2}-\frac{d^2S_2}{dn_1^{\,2}}$$

$$+\Lambda$$

$$+\frac{1}{B}\left\{+\frac{dS_1}{dn_2}\cdot\frac{dB}{dn_2}+\frac{dS_2}{dn_1}\cdot\frac{dB}{dn_1}-\frac{1}{2}\frac{dP_{12}}{dn_2}\cdot\frac{dB}{dn_1}-\frac{1}{2}\frac{dP_{12}}{dn_1}\cdot\frac{dB}{dn_2}\right\}$$

$$+\frac{B^2}{4}\left\{\begin{array}{lll} -(2-10S_3)\cdot\binom{12}{12} & -(2A_1P_{12}+3P_{13})\cdot\binom{23}{12} & -(2A_2P_{12}+3P_{23})\cdot\binom{31}{12} \\ +3P_{13}\cdot\binom{12}{23} & -2S_1\cdot\binom{23}{23} & -P_{12}\cdot\binom{31}{23} \\ +3P_{23}\cdot\binom{12}{31} & -P_{12}\cdot\binom{23}{31} & -2S_2\cdot\binom{31}{31} \end{array}\right\}=0.$$

$$-\frac{d^2S_2}{dn_3^{\,2}}+\frac{d^2P_{23}}{dn_2dn_3}-\frac{d^2S_3}{dn_2^{\,2}}$$

$$+\Lambda$$

$$+\frac{1}{B}\left\{+\frac{dS_2}{dn_3}\cdot\frac{dB}{dn_3}+\frac{dS_3}{dn_2}\cdot\frac{dB}{dn_2}-\frac{1}{2}\frac{dP_{23}}{dn_2}\cdot\frac{dB}{dn_3}-\frac{1}{2}\frac{dP_{23}}{dn_3}\cdot\frac{dB}{dn_2}\right\}$$

$$+\frac{B^2}{4}\left\{\begin{array}{lll} -(2-10S_1)\cdot\binom{23}{23} & -(2A_2P_{23}+3P_{12})\cdot\binom{31}{23} & -(2A_3P_{23}+3P_{31})\cdot\binom{12}{23} \\ +3P_{12}\cdot\binom{23}{31} & -2S_2\cdot\binom{31}{31} & -P_{23}\cdot\binom{12}{31} \\ +3P_{31}\cdot\binom{23}{12} & -P_{23}\cdot\binom{31}{12} & -2S_3\cdot\binom{12}{12} \end{array}\right\}=0.$$

$$-\frac{d^2S_1}{dn_3^{\,2}}+\frac{d^2P_{13}}{dn_1dn_3}-\frac{d^2S_3}{dn_1^{\,2}}$$

$$+\Lambda$$

$$+\frac{1}{B}\left\{+\frac{dS_1}{dn_3}\cdot\frac{dB}{dn_3}+\frac{dS_3}{dn_1}\cdot\frac{dB}{dn_1}-\frac{1}{2}\frac{dP_{13}}{dn_1}\cdot\frac{dB}{dn_3}-\frac{1}{2}\frac{dP_{13}}{dn_3}\cdot\frac{dB}{dn_1}\right\}$$

$$+\frac{B^2}{4}\left\{\begin{array}{lll} -(2-10S_2)\cdot\binom{31}{31} & -(2A_3P_{31}+3P_{23})\cdot\binom{12}{31} & -(2A_1P_{31}+3P_{12})\cdot\binom{23}{31} \\ +3P_{23}\cdot\binom{31}{12} & -2S_3\cdot\binom{12}{12} & -P_{31}\cdot\binom{23}{12} \\ +3P_{12}\cdot\binom{31}{23} & -P_{31}\cdot\binom{12}{23} & -2S_1\cdot\binom{23}{23} \end{array}\right\}=0.$$

These six Equations for *Normal Co-ordinates* as well as the Determinant Equations for *Curvilinear Co-ordinates* on pages 467 and 486 are now given for (I believe) the first time. A slight examination of the Equations on pages 467 and 486 as well as of the calculations on immediately following pages proves that the six equations of *Curvilinear Co-ordinates* contain Second differential coefficients and Λ in the same manner as the six Equations of *Normal Co-ordinates*, but that the "determinant portion" (see page 482) consists of forty-five determinants in place of nine, inasmuch as the arrangements of three letters in pairs is three, but that of six letters is fifteen.

Writing in the Equations on pages 467 and 486, F, H, J, A_{12}, A_{13} and A_{23} all equal zero, and $A_{11}E = A_{22}G = A_{33}I =$ unity, we obtain

$$\frac{d^2E}{du_2^{\,2}} + \frac{d^2G}{du_1^{\,2}} - \frac{1}{2E}\left(\frac{dE}{du_2}\right)^2 - \frac{1}{2G}\left(\frac{dG}{du_1}\right)^2$$

$$-\frac{1}{2G}\frac{dE}{du_2}\frac{dG}{du_2} - \frac{1}{2E}\frac{dE}{du_1}\frac{dG}{du_1} + \frac{1}{2I}\frac{dE}{du_3}\frac{dG}{du_3} = 0,$$

and

$$\frac{d^2E}{du_2du_3} - \frac{1}{2E}\frac{dE}{du_2}\frac{dE}{du_3} - \frac{1}{2I}\frac{dI}{du_2}\frac{dE}{du_3} - \frac{1}{2G}\frac{dG}{du_3}\frac{dE}{du_3} = 0,$$

four more equations result from symmetry. These six equations are M. Lamé's Equations for *Orthogonal Curvilinear Co-ordinates.*

I now conclude this Chapter by deducing one *symmetrical* equation in $A_1A_2A_3$, $n_1n_2n_3$ by means of these six now given.

In order to do this multiply the first equation given on page 497 by P_{23},

.... second .... by P_{13},

.... third .... by P_{12},

.... first .... 498 by S_3,

.... second .... by S_1,

.... third .... by S_2,

and add these six results together; the result of course must be symmetrical in A_1, A_2, A_3, and must of course equal zero. We shall first seek the coefficients of $\binom{23}{23}, \binom{12}{31}, \binom{31}{12}$; $\binom{12}{23}, \binom{31}{31}, \binom{23}{12}$; $\binom{31}{23}, \binom{23}{31}, \binom{12}{12}$ in this equation, and then remaining portion............

The coefficient of $\binom{12}{23}$ equals

$$+\frac{B^2}{4}\left\{\begin{array}{l} + A_1P_{13}P_{22} - A_1P_{13}P_{23} \\ + P_{12}\,.\,(P_{23} + 4A_1S_3) + 3S_3\,.\,P_{13} \\ - S_1\,.\,(2A_3P_{23} + 3P_{13}) - S_2P_{13} \end{array}\right\};$$

which equals

$$+\frac{B^2}{4}\left\{\begin{array}{c} + P_{23}\,.\,(2A_3S_1 + P_{12}) + 3P_{13}\,(S_3 - S_1) - 4A_3S_1P_{23} \\ - S_2P_{13} + 4A_1S_3P_{12} \end{array}\right\};$$

which equals

$$+\frac{B^2}{4}\begin{Bmatrix} +P_{13}\,.\,(3S_3-3S_1-A_1P_{23}-S_2) \\ +4A_1P_{12}S_3-4A_3P_{23}S_1 \end{Bmatrix}.$$

Now from the ten Equations given at commencement of this Exercise, we get

$$S_2+S_3-S_1+A_1P_{23}=\frac{1}{2}\cdots\cdots\cdots\cdots$$

Hence above equals

$$+\frac{B^2}{4}\left\{-\frac{P_{13}}{2}+4\,(A_1S_3P_{12}-A_3S_1P_{23}+P_{13}S_3-P_{13}S_1)\right\},$$

but $(A_1P_{12}S_3+P_{13}S_3-A_3S_1P_{23}-S_1P_{13})$ is easily seen to equal zero, and hence the coefficient of $\binom{12}{23}$ equals

$$-\frac{B^2}{8}\,.\,P_{13},$$

and so by symmetry coefficient of $\binom{31}{23}$ equals

$$-\frac{B^2}{8}\,.\,P_{12}\,.$$

The coefficient of $\binom{23}{23}$ equals

$$+\frac{B^2}{4}\begin{Bmatrix} -2A_1S_1P_{23}+4A_2S_1P_{13} \\ +4A_3S_1P_{12}-2S_1S_3 \\ -2S_1S_2+S_1\,(10S_1-2) \end{Bmatrix};$$

which equals

$$+\frac{B^2S_1}{4}\begin{Bmatrix} -2\,(A_1P_{23}+S_2+S_3-S_1) \\ +4A_2P_{13}+4A_3P_{12}+8S_1-2 \end{Bmatrix};$$

which equals

$$+\frac{B^2S_1}{4}\,(+4-2-1)=+\frac{B^2S_1}{4}\,.$$

Hence so far as $\binom{23}{23},\binom{12}{31},\binom{31}{12};\binom{12}{23},\binom{31}{31},\binom{23}{12};\binom{31}{23},\binom{23}{31},\binom{12}{12}$ are concerned our *symmetrical* equation equals

$$+\frac{B^2}{8}\begin{Bmatrix} -P_{12}\,.\binom{31}{23}-P_{13}\,.\binom{12}{23}+2S_1\,.\binom{23}{23} \\ -P_{23}\,.\binom{12}{31}-P_{21}\,.\binom{23}{31}+2S_2\,.\binom{31}{31} \\ -P_{31}\,.\binom{23}{12}-P_{32}\,.\binom{31}{12}+2S_3\,.\binom{12}{12} \end{Bmatrix}.$$

Whilst other portion equals

$$
\begin{aligned}
&+ S_1 \left\{ + \frac{d^2P_{23}}{dn_2dn_3} - \frac{d^2S_2}{dn_3^2} - \frac{d^2S_3}{dn_2^2} \right\} \\
&+ S_2 \left\{ + \frac{d^2P_{13}}{dn_1dn_3} - \frac{d^2S_1}{dn_3^2} - \frac{d^2S_3}{dn_1^2} \right\} \\
&+ S_3 \left\{ + \frac{d^2P_{12}}{dn_1dn_2} - \frac{d^2S_1}{dn_2^2} - \frac{d^2S_2}{dn_1^2} \right\} \\
&+ \frac{P_{23}}{2} \left\{ - \frac{d^2P_{13}}{dn_1dn_2} - \frac{d^2P_{12}}{dn_1dn_3} + \frac{d^2P_{23}}{dn_1^2} + 2\frac{d^2S_1}{dn_2dn_3} \right\} \\
&+ \frac{P_{13}}{2} \left\{ - \frac{d^2P_{12}}{dn_2dn_3} - \frac{d^2P_{23}}{dn_1dn_2} + \frac{d^2P_{13}}{dn_2^2} + 2\frac{d^2S_2}{dn_1dn_3} \right\} \\
&+ \frac{P_{12}}{2} \left\{ - \frac{d^2P_{23}}{dn_1dn_3} - \frac{d^2P_{13}}{dn_2dn_3} + \frac{d^2P_{12}}{dn_3^2} + 2\frac{d^2S_3}{dn_1dn_2} \right\} .
\end{aligned}
$$

$$
+\frac{3}{8}\left|\begin{aligned}
&+\left(\frac{dP_{23}}{dn_1}\right)^2 + \left(\frac{dP_{13}}{dn_2}\right)^2 + \left(\frac{dP_{12}}{dn_3}\right)^2 \\
&-4\frac{dS_2}{dn_1}\cdot\frac{dS_3}{dn_1} - 4\frac{dS_1}{dn_2}\cdot\frac{dS_3}{dn_2} - 4\frac{dS_1}{dn_3}\cdot\frac{dS_2}{dn_3} \\
&-\frac{dP_{13}}{dn_1}\cdot\frac{dP_{23}}{dn_2} - \frac{dP_{23}}{dn_1}\cdot\frac{dP_{13}}{dn_2} + 2\frac{dS_3}{dn_1}\cdot\frac{dP_{12}}{dn_2} + 2\frac{dS_3}{dn_2}\cdot\frac{dP_{12}}{dn_1} \\
&-\frac{dP_{12}}{dn_3}\cdot\frac{dP_{23}}{dn_1} - \frac{dP_{12}}{dn_1}\cdot\frac{dP_{23}}{dn_3} + 2\frac{dS_2}{dn_1}\cdot\frac{dP_{13}}{dn_3} + 2\frac{dS_2}{dn_3}\cdot\frac{dP_{13}}{dn_1} \\
&-\frac{dP_{12}}{dn_3}\cdot\frac{dP_{13}}{dn_2} - \frac{dP_{12}}{dn_2}\cdot\frac{dP_{13}}{dn_3} + 2\frac{dS_1}{dn_3}\cdot\frac{dP_{23}}{dn_2} + 2\frac{dS_1}{dn_2}\cdot\frac{dP_{23}}{dn_3}
\end{aligned}\right|
$$

$$
+\frac{1}{B}\cdot\frac{dB}{dn_1}\left\{\begin{aligned}
&+\frac{P_{23}}{4}\cdot\frac{dP_{13}}{dn_2} + \frac{P_{23}}{4}\cdot\frac{dP_{12}}{dn_3} - \frac{P_{23}}{2}\cdot\frac{dP_{23}}{dn_1} \\
&+\frac{P_{13}}{4}\cdot\frac{dP_{23}}{dn_2} - \frac{P_{13}}{2}\cdot\frac{dS_2}{dn_3} + \frac{P_{12}}{4}\cdot\frac{dP_{23}}{dn_3} \\
&-\frac{P_{12}}{2}\cdot\frac{dS_3}{dn_2} + S_3\cdot\frac{dS_2}{dn_1} - \frac{S_3}{2}\cdot\frac{dP_{12}}{dn_2} \\
&+S_2\frac{dS_3}{dn_1} - \frac{S_2}{2}\cdot\frac{dP_{13}}{dn_3}
\end{aligned}\right\},
$$

$$+\frac{1}{B}\cdot\frac{dB}{dn_2} \text{ into a similar quantity;}$$

$$+\frac{1}{B}\cdot\frac{dB}{dn_3} \text{ into a similar quantity.}$$

But now obviously this may be written in the form

$$+\frac{d^2}{dn_1^2}\left(\frac{P_{23}^2}{4} - S_2S_3\right) + \frac{d^2}{dn_2^2}\left(\frac{P_{13}^2}{4} - S_1S_3\right)$$

$$+\frac{d^2}{dn_3^2}\left(\frac{P_{12}^2}{4}-S_1S_2\right)+\frac{d^2}{dn_2dn_3}\left(S_1P_{23}-\frac{1}{2}P_{13}P_{12}\right)$$

$$+\frac{d^2}{dn_1dn_3}\left(S_2P_{13}-\frac{1}{2}P_{23}P_{12}\right)+\frac{d^2}{dn_1dn_2}\left(S_3P_{12}-\frac{1}{2}P_{23}P_{13}\right)$$

$$-\frac{\Lambda}{2},$$

$$+\frac{1}{4B}\cdot\frac{dB}{dn_1}\left\{\frac{d}{dn_1}\left(4S_2S_3-P_{23}^2\right)+\frac{d}{dn_2}\left(P_{13}P_{23}-2P_{12}S_3\right)+\frac{d}{dn_3}\left(P_{12}P_{23}-2P_{13}S_2\right)\right\}$$

$$+\frac{1}{4B}\frac{dB}{dn_2}\text{ into a similar quantity }+\frac{1}{4B}\frac{dB}{dn_3}\text{ into a similar quantity;}$$

but it is clear this may be written

$$-\left(\frac{d^2}{dn_1^2}+\frac{d^2}{dn_2^2}+\frac{d^2}{dn_3^2}\right)\frac{B^2}{4}$$

$$-\frac{d^2}{dn_2dn_3}\left(\frac{A_1B^2}{2}\right)-\frac{d^2}{dn_1dn_3}\left(\frac{A_2B^2}{2}\right)-\frac{d^2}{dn_1dn_2}\left(\frac{A_3B^2}{2}\right)$$

$$-\frac{\Lambda}{2},$$

$$+\frac{1}{B}\left\{\frac{dB}{dn_1}\cdot\frac{d}{dn_1}\left(\frac{B^2}{4}\right)+\frac{dB}{dn_2}\cdot\frac{d}{dn_2}\left(\frac{B^2}{4}\right)+\frac{dB}{dn_3}\cdot\frac{d}{dn_3}\left(\frac{B^2}{4}\right)\right\}$$

$$+\frac{1}{B}\left\{\begin{array}{l}\dfrac{dB}{dn_1}\cdot\dfrac{d}{dn_2}\left(\dfrac{A_3B^2}{4}\right)+\dfrac{dB}{dn_1}\cdot\dfrac{d}{dn_3}\left(\dfrac{A_2B^2}{4}\right)\\ +\dfrac{dB}{dn_2}\cdot\dfrac{d}{dn_1}\left(\dfrac{A_3B^2}{4}\right)+\dfrac{dB}{dn_2}\cdot\dfrac{d}{dn_3}\left(\dfrac{A_1B^2}{4}\right)\\ +\dfrac{dB}{dn_3}\cdot\dfrac{d}{dn_1}\left(\dfrac{A_2B^2}{4}\right)+\dfrac{dB}{dn_3}\cdot\dfrac{d}{dn_2}\left(\dfrac{A_1B^2}{4}\right)\end{array}\right\};$$

if we examine above we find all such terms as $\left(\frac{dB}{dn_1}\right)^2$; $A_1\frac{dB}{dn_2}\cdot\frac{dB}{dn_3}$ mutually destroy, and thus we see our *symmetrical* differential equation of second order equals

$$\frac{d^2B}{dn_1^2}+\frac{d^2B}{dn_2^2}+\frac{d^2B}{dn_3^2}+2A_1\frac{d^2B}{dn_2dn_3}+2A_2\frac{d^2B}{dn_1dn_3}+2A_3\frac{d^2B}{dn_1dn_2}$$

$$+B\left\{\frac{d^2A_1}{dn_2dn_3}+\frac{d^2A_2}{dn_1dn_3}+\frac{d^2A_3}{dn_1dn_2}\right\}$$

$$+\frac{3}{2}\left\{\frac{dB}{dn_1}\left(\frac{dA_2}{dn_3}+\frac{dA_3}{dn_2}\right)+\frac{dB}{dn_2}\left(\frac{dA_3}{dn_1}+\frac{dA_1}{dn_3}\right)+\frac{dB}{dn_3}\left(\frac{dA_1}{dn_2}+\frac{dA_2}{dn_1}\right)\right\}$$

$$+\ \Lambda\div B$$

$$+\frac{B}{4}\left\{\begin{array}{l}-2S_1\cdot\binom{23}{23}+P_{12}\cdot\binom{31}{23}+P_{31}\cdot\binom{12}{23}\\ +P_{12}\cdot\binom{23}{31}-2S_2\cdot\binom{31}{31}+P_{23}\cdot\binom{12}{31}\\ +P_{31}\cdot\binom{23}{12}+P_{23}\cdot\binom{31}{12}-2S_3\cdot\binom{12}{12}\end{array}\right\}=0.$$

EXERCISE THE THIRD.

Normal Co-ordinates.

If we refer back to "Exercise the first," we find that

$$\frac{dx}{du_1} = a_1, \quad \frac{dx}{du_2} = b_1, \quad \frac{dx}{du_3} = c_1,$$

$$\frac{dy}{du_1} = a_2, \quad \frac{dy}{du_2} = b_2, \quad \frac{dy}{du_3} = c_2,$$

$$\frac{dz}{du_1} = a_3, \quad \frac{dz}{du_2} = b_3, \quad \frac{dz}{du_3} = c_3.$$

Hence it is easy to see

$$\frac{d}{du_1}(b_2c_3 - b_3c_2) + \frac{d}{du_2}(c_2a_3 - c_3a_2) + \frac{d}{du_3}(a_2b_3 - a_3b_2) = 0,$$

$$\frac{d}{du_1}(b_3c_1 - b_1c_3) + \frac{d}{du_2}(c_3a_1 - c_1a_3) + \frac{d}{du_3}(a_3b_1 - a_1b_3) = 0,$$

$$\frac{d}{du_1}(b_1c_2 - b_2c_1) + \frac{d}{du_2}(c_1a_2 - c_2a_1) + \frac{d}{du_3}(a_1b_2 - a_2b_1) = 0.$$

These equations are true both in Normal and Curvilinear Co-ordinates, and may be written as follows:

$$\frac{d^2B}{du_1da_1} + \frac{d^2B}{du_2db_1} + \frac{d^2B}{du_3dc_1} = 0,$$

$$\frac{d^2B}{du_1da_2} + \frac{d^2B}{du_2db_2} + \frac{d^2B}{du_3dc_2} = 0,$$

$$\frac{d^2B}{du_1da_3} + \frac{d^2B}{du_2db_3} + \frac{d^2B}{du_3dc_3} = 0.$$

Now multiply the first of these equations by $\frac{1}{B}\frac{dB}{da_1}$, the second by $\frac{1}{B}\frac{dB}{da_2}$, and the third by $\frac{1}{B}\frac{dB}{da_3}$; then add together, and using Normal Co-ordinates we obtain

$$\frac{dB}{dn_1} + A_2\frac{dB}{dn_3} + A_3\frac{dB}{dn_2} + B\left\{\frac{dA_2}{dn_3} + \frac{dA_3}{dn_2}\right\};$$

equal to

$$\left\{\frac{dB}{db_1}\frac{d}{dn_2}\left(\frac{1}{B}\frac{dB}{da_1}\right) + \frac{dB}{db_2}\frac{d}{dn_2}\left(\frac{1}{B}\frac{dB}{da_2}\right) + \frac{dB}{db_3}\frac{d}{dn_2}\left(\frac{1}{B}\frac{dB}{da_3}\right)\right\},$$
$$+\left\{\frac{dB}{dc_1}\frac{d}{dn_3}\left(\frac{1}{B}\frac{dB}{da_1}\right) + \frac{dB}{dc_2}\frac{d}{dn_3}\left(\frac{1}{B}\frac{dB}{da_2}\right) + \frac{dB}{dc_3}\frac{d}{dn_3}\left(\frac{1}{B}\frac{dB}{da_3}\right)\right\}.$$

This equation takes the simple form

$$\frac{dB}{dn_1} + A_2\frac{dB}{dn_3} + A_3\frac{dB}{dn_2} + B\left\{\frac{dA_2}{dn_3} + \frac{dA_3}{dn_2}\right\} =$$
$$= B.\left\{\begin{matrix} \cos\alpha_2\dfrac{d\cos\alpha_1}{dn_2} + \cos\beta_2\dfrac{d\cos\beta_1}{dn_2} + \cos\gamma_2\dfrac{d\cos\gamma_1}{dn_2} \\ \cos\alpha_3\dfrac{d\cos\alpha_1}{dn_3} + \cos\beta_3\dfrac{d\cos\beta_1}{dn_3} + \cos\gamma_3\dfrac{d\cos\gamma_1}{dn_3} \end{matrix}\right\}.$$

where $\alpha_1\ \beta_1\ \gamma_1$, $\alpha_2\ \beta_2\ \gamma_2$, $\alpha_3\ \beta_3\ \gamma_3$, have the same significance as in Second Exercise.

Hence, remembering that

$$\cos\alpha_1\frac{d\cos\alpha_1}{dn_1} + \cos\beta_1\frac{d\cos\beta_1}{dn_1} + \cos\gamma_1\frac{d\cos\gamma_1}{dn_1} = 0,$$
$$\cos\alpha_1 = \frac{dn_1}{dx},\quad \cos\beta_1 = \frac{dn_1}{dy},\quad \cos\gamma_1 = \frac{dn_1}{dz},$$
$$\cos\alpha_2 = \frac{dn_2}{dx},\quad \cos\beta_2 = \frac{dn_2}{dy},\quad \cos\gamma_2 = \frac{dn_2}{dz},$$
$$\cos\alpha_3 = \frac{dn_3}{dx},\quad \cos\beta_3 = \frac{dn_3}{dy},\quad \cos\gamma_3 = \frac{dn_3}{dz},$$

we easily see that the above equation may be written

$$\frac{1}{B}\left\{\frac{dB}{dn_1} + A_2\frac{dB}{dn_3} + A_3\frac{dB}{dn_2}\right\} + \frac{dA_2}{dn_3} + \frac{dA_3}{dn_2}$$
$$= \frac{d\cos\alpha_1}{dx} + \frac{d\cos\beta_1}{dy} + \frac{d\cos\gamma_1}{dz}.$$

But it is a well-known proposition in Solid Geometry that

$$\frac{d\cos\alpha_1}{dx} + \frac{d\cos\beta_1}{dy} + \frac{d\cos\gamma_1}{dz} = \frac{1}{R_1} + \frac{1}{R_2},$$

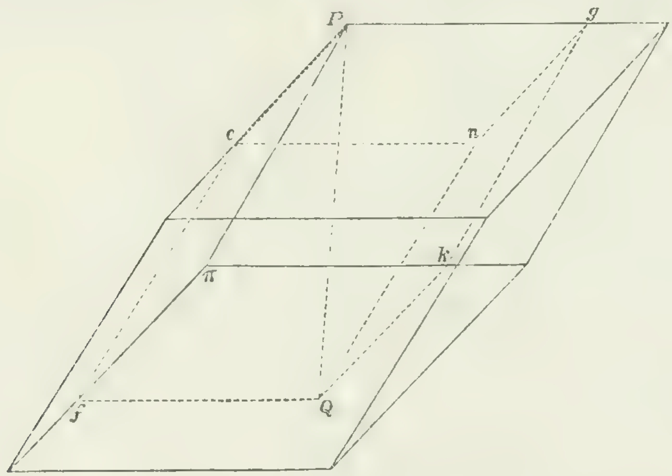
where R_1 and R_2 have their usual geometric significance; and hence we obtain the value of $\frac{1}{R_1}+\frac{1}{R_2}$ in Normal Co-ordinates,

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{B}\left\{\frac{dB}{dn_1}+A_2\frac{dB}{dn_3}+A_3\frac{dB}{dn_2}\right\}+\frac{dA_2}{dn_3}+\frac{dA_3}{dn_2}.$$

We may obtain this same result by a good many other methods.

I shall give a few for exercise.

If we pass from a point on the normal n_1 to a very near point on the same normal, it comes to the same in the end whether we do this directly by travelling a distance PQ along the normal n_1 to the surface u_1, or else first go a distance Pe along the intersection of the surfaces u_1 and u_2, next a distance ef along the intersection of the surfaces



u_2 and u_3, and then a distance fQ along the intersection of the surfaces u_1 and u_3. Hence clearly we have

$$PQ\,.\,\Delta_1=(ef)_1\frac{d}{dn_1}+(fQ)_2\frac{d}{dn_2}+(eP)_3\frac{d}{dn_3},$$

where $(ef)_1$, $(fQ)_2$, $(eP)_3$ denote the perpendiculars let fall from Q on the faces $ePgh$; $efP\Pi$; $P\Pi gk$; $\{(ef_1)=PQ;\ \ (fQ)_2=A_3\times PQ;\ \ (eP)_3=A_2\times PQ\}$. And Δ_1 denotes the operation of taking the difference of the values of a quantity at two very near positions P and Q on the normal n_1 to the surface u_1, and dividing this difference by the distance PQ; hence

$$\Delta_1=\frac{d}{dn_1}+A_3\frac{d}{dn_2}+A_2\frac{d}{dn_3};$$

$$\Delta_2=\frac{d}{dn_2}+A_1\frac{d}{dn_3}+A_3\frac{d}{dn_1};$$

$$\Delta_3=\frac{d}{dn_3}+A_2\frac{d}{dn_1}+A_1\frac{d}{dn_2}.$$

Whence we obtain

$$\frac{d}{dn_1} = 2S_1 \,.\, \Delta_1 + P_{13} \,.\, \Delta_3 + P_{12} \,.\, \Delta_2,$$

$$\frac{d}{dn_2} = 2S_2 \,.\, \Delta_2 + P_{23} \,.\, \Delta_3 + P_{12} \,.\, \Delta_1,$$

$$\frac{d}{dn_3} = 2S_3 \,.\, \Delta_3 + P_{23} \,.\, \Delta_2 + P_{13} \,.\, \Delta_1.$$

Now let dS_1 be an element of the surface u_1, R_1 and R_2 the chief radii of curvature of u_1 at the point where dS_1 is taken; also let λ and μ equal the elementary angles that R_1 and R_2 make with the consecutive normals along the lines of curvature; hence clearly

$$dS_1 = \lambda \,.\, R_1 \times \mu \,.\, R_2.$$

Now for the parallel surface neither λ nor μ receives any change; and hence we clearly see that

$$dS_1 - \text{value of } dS_1 \text{ for parallel surface}$$

$$\text{equals} \quad \lambda \,.\, \mu \,.\, R_1 \,.\, R_2 \left\{\frac{1}{R_1} + \frac{1}{R_2}\right\} \times PQ,$$

$$\text{which equals} \qquad dS_1 \times PQ \times \left\{\frac{1}{R_1} + \frac{1}{R_2}\right\}.$$

But we know that $dS_1 = ds_2 \,.\, ds_3 \times \sin\Theta$; therefore we have

$$dS_1 = B^2 \,.\, \sin\Theta \,.\, \sin\phi \,.\, \sin\psi \,.\, dn_2\, dn_3;$$

and hence

$$dS_1 = B \,.\, dn_2\, dn_3.$$

We clearly therefore have

$$dS_1 - \text{value of } dS_1 \text{ for parallel surface}$$

$$= (B - \text{value of } B \text{ for parallel surface}) \,.\, dn_2\, dn_3$$

$$+ (dn_2 \,.\, dn_3 - \text{value of } dn_2 \,.\, dn_3 \text{ for parallel surface}) \,.\, B,$$

and by above also equal to $B \,.\, dn_2\, dn_3 \,.\, PQ \,.\, \left\{\frac{1}{R_1} + \frac{1}{R_2}\right\}$.

Now $dn_2\, dn_3 -$ value of $dn_2\, dn_3$ for parallel surface is what we call in the calculus of variations $\delta \,.\, (dn_2\, dn_3)$, which it is well known equals

$$dn_2 \,.\, dn_3 \,.\, \left\{\frac{d\delta n_2}{dn_2} + \frac{d\delta n_3}{dn_3}\right\},$$

but $\delta n_2 = PQ \times A_3$ and $\delta n_3 = PQ \times A_2$, whence since PQ is of course supposed not to vary with n_2 or n_3 we have

$$\delta\,(dn_2 dn_3) = dn_2 \,.\, dn_3 \,.\, PQ \left\{\frac{dA_3}{dn_2} + \frac{dA_2}{dn_3}\right\};$$

hence we obtain as before

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{B}\left\{\frac{dB}{dn_1} + A_3\frac{dB}{dn_2} + A_2\frac{dB}{dn_3}\right\} + \frac{dA_3}{dn_2} + \frac{dA_2}{dn_3}.$$

I shall yet mention a third way to obtain above formula. Suppose that we write

$$\cos\alpha_1 \cdot \frac{d^2x}{dn_2^2} + \cos\beta_1 \cdot \frac{d^2y}{dn_2^2} + \cos\gamma_1 \cdot \frac{d^2z}{dn_2^2} = E',$$

$$\cos\alpha_1 \cdot \frac{d^2x}{dn_3^2} + \cos\beta_1 \cdot \frac{d^2y}{dn_3^2} + \cos\gamma_1 \cdot \frac{d^2z}{dn_3^2} = G',$$

$$\cos\alpha_1 \cdot \frac{d^2x}{dn_2 dn_3} + \cos\beta_1 \cdot \frac{d^2y}{dn_2 dn_3} + \cos\gamma_1 \cdot \frac{d^2z}{dn_2 dn_3} = F'.$$

It is a known result that $\frac{1}{R_1} + \frac{1}{R_2}$ is proportional to

$$EG' + E'G - 2FF'.$$

(See Traité du Calcul Différentiel et du Calcul Intégral, Par S. F. Lacroix. Tome second, Seconde Édition, Paris, 1814, 629, § 774)......But by previous Exercises we see that

$$A_3 \cdot \frac{dS_2}{dn_2} + A_2\left(\frac{dP_{23}}{dn_2} - \frac{dS_2}{dn_3}\right) + \left(\frac{dP_{12}}{dn_2} - \frac{dS_2}{dn_1}\right) = G',$$

$$A_3 \cdot \left(\frac{dP_{23}}{dn_3} - \frac{dS_3}{dn_2}\right) + A_2 \frac{dS_3}{dn_3} + \left(\frac{dP_{13}}{dn_3} - \frac{dS_3}{dn_1}\right) = E',$$

$$A_3 \cdot \frac{dS_2}{dn_3} + A_2 \frac{dS_3}{dn_2} + \frac{1}{2}\left(\frac{dP_{13}}{dn_2} + \frac{dP_{12}}{dn_3} - \frac{dP_{23}}{dn_1}\right) = F'.$$

Multiply first line by $2S_3$, and add it to second line multiplied by $2S_2$, and subtract from this third line multiplied by $2P_{23}$, result clearly is

$$\begin{aligned} &P_{23} \cdot \frac{dP_{23}}{dn_1} - 2S_3 \frac{dS_2}{dn_1} - 2S_2 \frac{dS_3}{dn_1} \\ &+ 2S_3\left(A_3 \frac{dS_2}{dn_2} + A_2 \frac{dP_{23}}{dn_2} + \frac{dP_{12}}{dn_2}\right) - 2S_2 A_3 \frac{dS_3}{dn_2} \\ &-2P_{23} A_2 \frac{dS_3}{dn_2} - P_{23} \frac{dP_{13}}{dn_2} \end{aligned}$$

+ symmetrical terms differentiated with regard to n_3.

But this clearly equals

$$\begin{aligned} &\frac{1}{2}\frac{d}{dn_1}(P_{23}^2 - 4S_2S_3) \\ &+ 2S_3\left(2A_3 \frac{dS_2}{dn_2} + A_2 \frac{dP_{23}}{dn_2} + \frac{dP_{12}}{dn_2}\right) \\ &- 2A_3S_3 \frac{dS_2}{dn_2} - 2A_3S_2 \frac{dS_3}{dn_2} \\ &- 2A_2P_{23} \frac{dS_3}{dn_2} - P_{23} \frac{dP_{13}}{dn_2} \end{aligned}$$

+ terms differentiated with regard to n_3.

But the above equals

$$-B\,.\frac{dB}{dn_1}-4S_2S_3\,.\frac{dA_3}{dn_2}-2S_3P_{23}\frac{dA_2}{dn_2}$$

$$-2A_3\frac{d}{dn_2}(S_2S_3)$$

$$-P_{23}\left(2A_2\frac{dS_3}{dn_2}+A_3\frac{dP_{23}}{dn_2}+\frac{dP_{13}}{dn_2}\right)+A_3P_{23}\frac{dP_{23}}{dn_2}$$

+a similar term differentiated with regard to n_3.

But this equals

$$-B\,.\frac{dB}{dn_1}+\frac{dA_3}{dn_2}(P_{23}{}^2-4S_2S_3)$$

$$+\frac{dA_2}{dn_2}(2S_3P_{23}-2S_3P_{23})+\frac{1}{2}A_3\frac{d}{dn_2}(P_{23}{}^2-4S_2S_3)$$

+a similar term differentiated with regard to n_3.

Hence once more, we find that

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{dA_2}{dn_3}+\frac{dA_3}{dn_2}+\frac{1}{B}\left\{\frac{dB}{dn_1}+A_3\frac{dB}{dn_2}+A_2\frac{dB}{dn_3}\right\}.$$

....

Having determined the value of $\frac{1}{R_1}+\frac{1}{R_2}$ in normal co-ordinates, I now proceed to find the value of $\frac{1}{R_1R}$ in terms of the same system.

By page 505 we have $-\Delta_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ equal to $\frac{1}{R_1{}^2}+\frac{1}{R_2{}^2}$, hence clearly we have

$$\frac{2}{R_1R_2}=\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2+\Delta_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

which equals

$$\frac{1}{B^2}\left(\frac{dB}{dn_1}+\frac{d}{dn_2}(A_3B)+\frac{d}{dn_3}(A_2B)\right)^2$$

$$+\left(\frac{d}{dn_1}+A_3\frac{d}{dn_2}+A_3\frac{d}{dn_3}\right)\frac{1}{B}\left(\frac{dB}{dn_1}+\frac{d}{dn_2}(A_3B)+\frac{d}{dn_3}(A_2B)\right),$$

which equals

$$\frac{1}{B}\left\{\begin{array}{l}\dfrac{d^2B}{dn_1{}^2}+A_3{}^2\dfrac{d^2B}{dn_2{}^2}+A_2{}^2\dfrac{d^2B}{dn_3{}^2}\\[2ex] +2A_3\dfrac{d^2B}{dn_1dn_2}+2A_2\dfrac{d^2B}{dn_1dn_3}+2A_2A_3\dfrac{d^2B}{dn_2dn_3}\end{array}\right\},$$

$$+\frac{d^2A_3}{dn_1dn_2}+A_3\frac{d^2A_3}{dn_2^2}+A_2\frac{d^2A_3}{dn_2dn_3}$$

$$+\frac{d^2A_2}{dn_1dn_3}+A_3\frac{d^2A_2}{dn_2dn_3}+A_2\frac{d^2A_2}{dn_3^2}$$

$$+\left(\frac{dA_2}{dn_3}+\frac{dA_3}{dn_2}\right)^2$$

$$+\frac{2}{B}\left(\frac{dA_2}{dn_3}+\frac{dA_3}{dn_2}\right)\left(\frac{dB}{dn_1}+A_3\frac{dB}{dn_2}+A_2\frac{dB}{dn_3}\right)$$

$$+\frac{1}{B}\frac{dB}{dn_2}\left\{\frac{dA_3}{dn_1}+A_3\frac{dA_3}{dn_2}+A_2\frac{dA_3}{dn_3}\right\}$$

$$+\frac{1}{B}\frac{dB}{dn_3}\left\{\frac{dA_2}{dn_1}+A_3\frac{dA_2}{dn_2}+A_2\frac{dA_2}{dn_3}\right\}.$$

Such is a formula for $\frac{2}{R_1R_2}$ in normal co-ordinates. I shall for conciseness refer to it thus, $\frac{2}{R_1R_2}=2M$. The following formulæ for $\frac{1}{R_1}+\frac{1}{R_2}$; $\frac{1}{R_1R_2}$ &c. are interesting.

Write $2V=f_1^2+f_2^2+f_3^2+2A_1f_2f_3+2A_2f_1f_3+2A_3f_1f_2$, then clearly, if *after* differentiation with regard to f_1, f_2, f_3 we write $f_1=1$, $f_2=0$, $f_3=0$, we shall have

$$\Delta_1=\frac{dV}{df_1}\frac{d}{dn_1}+\frac{dV}{df_2}\frac{d}{dn_2}+\frac{dV}{df_3}\frac{d}{dn_3},$$

$$B\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{d}{dn_1}\frac{dV}{df_1}B+\frac{d}{dn_2}\frac{dV}{df_2}B+\frac{d}{dn_3}\frac{dV}{df_3}B.$$

Let now r, s, t; a, b, c, be six such symbols, so that a is always the companion of r, b of s, and c of t. Moreover, let it be understood that each of r, s, and t are *independent* of one another, as well as a, b, and c. Finally, let it be understood that wherever we see an r, this r is in succession to be changed, first into f_1, next into f_2, thirdly into f_3. And wherever we see an a, let it be understood that *first* this a is to be changed into n_1, next into n_2, and lastly into n_3. Let the same hold for s, t, and for b and c. This being understood clearly, we may write

$$\Delta_1=\Sigma\frac{dV}{dr}\frac{d}{da}\text{ (consisting of three terms),} \qquad (\alpha)$$

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{B}\Sigma\frac{d}{da}\frac{dV}{dr}B\text{ (three terms);} \qquad (\beta)$$

but

$$\frac{2}{R_1R_2}=\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2+\Delta_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$=\frac{1}{B^2}\Sigma\left\{\frac{d}{da}\left(\frac{dV}{dr}B\right)\right\}\times\left\{\frac{d}{db}\left(\frac{dV}{ds}B\right)\right\}$$

$$+\Sigma\frac{dV}{ds}\frac{d}{db}\frac{1}{B}\frac{d}{da}\left(\frac{dV}{dr}B\right).$$

Hence we see that we may write

$$\frac{2}{R_1R_2}=\frac{1}{B}\Sigma\frac{d}{db}\frac{dV}{ds}\frac{d}{da}\frac{dV}{dr}B\ldots\ldots\ldots\ldots(\gamma).$$

because the right-hand side of this equation equals

$$\frac{1}{B}\Sigma\frac{d}{db}\frac{dV}{ds}B\frac{1}{B}\frac{d}{da}\frac{dV}{dr}B,$$

which equals

$$\frac{1}{B^2}\Sigma\left(\frac{d}{db}\frac{dV}{ds}B\right)\times\left(\frac{d}{da}\frac{dV}{dr}B\right)$$

$$+\Sigma\frac{dV}{ds}\frac{d}{db}\frac{1}{B}\frac{d}{da}\frac{dV}{dr}B,$$

which agrees with previous page.

In the same way as we obtained the equation

$$\frac{2}{R_1R_2}=\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2+\Delta\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

it is easy to obtain the equation

$$\frac{1}{R_1R_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\Delta\left(\frac{1}{R_1R_2}\right)=0.$$

Hence by formulæ (α), (β) and (γ),

$$\left(\frac{1}{B}\Sigma\frac{d}{dc}\frac{dV}{dt}B\right)\times\left(\frac{1}{B}\Sigma\frac{d}{db}\frac{dV}{ds}\frac{d}{da}\frac{dV}{dr}B\right)$$

$$+\Sigma\frac{dV}{dt}\frac{d}{dc}\frac{1}{B}\frac{d}{db}\frac{dV}{ds}\frac{d}{da}\frac{dV}{dr}B=0,$$

or

$$\Sigma\frac{d}{dc}\frac{dV}{dt}\frac{d}{db}\frac{dV}{ds}\frac{d}{da}\frac{dV}{dr}B=0\ldots\ldots\ldots\ldots(\delta).$$

Equation (δ) clearly consists of twenty-seven terms.

Another formula for $\frac{1}{R_1R_2}$ may be obtained thus:

Write

$$m=-\frac{1}{B^4}\cdot\begin{vmatrix} -\frac{d^2S_2}{dn_3^2}+\frac{d^2P_{23}}{dn_2dn_3}-\frac{d^2S_3}{dn_2^2}; & \frac{dP_{23}}{dn_3}-\frac{dS_3}{dn_2}; & \frac{dS_3}{dn_3} \\ \frac{dS_2}{dn_2} & ;\ 2S_2 & ;\ P_{23} \\ \frac{dP_{23}}{dn_2}-\frac{dS_2}{dn_3} & ;\ P_{23} & ;\ 2S_3 \end{vmatrix}$$

$$+\frac{1}{B^4}\begin{vmatrix} 0 & ; & \frac{dS_2}{dn_3} & ; & \frac{dS_3}{dn_2} \\ \frac{dS_2}{dn_3} & ; & 2S_2 & ; & P_{23} \\ \frac{dS_3}{dn_2} & ; & P_{23} & ; & 2S_3 \end{vmatrix}.$$

This clearly may be written $B^4 \times m =$

$$+B^2\left\{\frac{d^2S_2}{dn_3^{\,2}}-\frac{d^2P_{23}}{dn_2dn_3}+\frac{d^2S_3}{dn_2^{\,2}}\right\}$$
$$-2S_2\left\{\frac{dS_2}{dn_3}\,\frac{dS_3}{dn_3}-\frac{dP_{23}}{dn_2}\,\frac{dS_3}{dn_3}+\left(\frac{dS_3}{dn_2}\right)^2\right\}$$
$$-2S_3\left\{\frac{dS_2}{dn_2}\,\frac{dS_3}{dn_2}-\frac{dP_{23}}{dn_3}\,\frac{dS_2}{dn_2}+\left(\frac{dS_2}{dn_3}\right)^2\right\}$$
$$-P_{23}\left\{\frac{dS_2}{dn_2}\,\frac{dS_3}{dn_3}-\frac{dS_2}{dn_3}\,\frac{dS_3}{dn_3}-\frac{dS_2}{dn_3}\,\frac{dP_{23}}{dn_3}-\frac{dS_3}{dn_2}\,\frac{dP_{23}}{dn_2}+\frac{dP_{23}}{dn_2}\,\frac{dP_{23}}{dn_3}\right\}.$$

Hence B^4m equals

$$+B^2\left\{\frac{d^2S_2}{dn_3^{\,2}}-\frac{d^2P_{23}}{dn_2dn_3}+\frac{d^2S_3}{dn_2^{\,2}}\right\}$$
$$-\frac{dS_3}{dn_2}\left\{2S_2\frac{dS_3}{dn_2}+2S_3\frac{dS_2}{dn_2}-P_{23}\frac{dP_{23}}{dn_2}\right\}$$
$$-\frac{dS_2}{dn_3}\left\{2S_3\frac{dS_2}{dn_3}+2S_2\frac{dS_3}{dn_3}-P_{23}\frac{dP_{23}}{dn_3}\right\}$$
$$+\frac{1}{2}\frac{dP_{23}}{dn_3}\left\{2S_2\frac{dS_3}{dn_2}+2S_3\frac{dS_2}{dn_2}-P_{23}\frac{dP_{23}}{dn_2}\right\}$$
$$+\frac{1}{2}\frac{dP_{23}}{dn_2}\left\{2S_3\frac{dS_2}{dn_3}+2S_2\frac{dS_3}{dn_3}-P_{23}\frac{dP_{23}}{dn_3}\right\}$$
$$-\begin{vmatrix} S_2 & ; & S_3 & ; & P_{23} \\ \frac{dS_2}{dn_2} & ; & \frac{dS_3}{dn_2} & ; & \frac{dP_{23}}{dn} \\ \frac{dS_2}{dn_3} & ; & \frac{dS_3}{dn_3} & ; & \frac{dP_{23}}{dn_3} \end{vmatrix}.$$

Hence B^4m equals

$$+B^3\left\{\frac{d^2S_2}{dn_3^{\,2}}-\frac{d^2P_{23}}{dn_2dn_3}+\frac{d^2S_3}{dn_2^{\,2}}\right\}$$
$$-B\left\{\frac{dS_2}{dn_3}\,\frac{dB}{dn_3}+\frac{dS}{dn_2}\,\frac{dB}{dn_2}-\frac{1}{2}\frac{dP_{23}}{dn_2}\,\frac{dB}{dn_3}-\frac{1}{2}\frac{dP_{23}}{dn_3}\,\frac{dB}{dn_2}\right\}$$

$$- B^6 \begin{vmatrix} \frac{1}{2}(1-A_2^2)\,; & \frac{1}{2}(1-A_3^2)\,; & A_2A_3 - A_1 \\ -A_2 \,.\, \frac{dA_2}{dn_2}\,; & -A_3\frac{dA_3}{dn_2}\,; & A_2\frac{dA_3}{dn_2} + A_3\frac{dA_2}{dn_2} - \frac{dA_1}{dn_2} \\ -A_2 \,.\, \frac{dA_2}{dn_3}\,; & -A_3\frac{dA_3}{dn_3}\,; & A_2\frac{dA_3}{dn_3} + A_3\frac{dA_2}{dn_3} - \frac{dA_1}{dn_3} \end{vmatrix} .$$

Now first two lines equal

$$+ B^3 \left[\frac{d}{dn_3}\left\{\frac{1}{B}\left(\frac{dS_2}{dn_3} - \frac{1}{2}\,\frac{dP_{23}}{dn_2}\right)\right\} + \frac{d}{dn_2}\left\{\frac{1}{B}\left(\frac{dS_3}{dn_2} - \frac{1}{2}\,\frac{dP_{23}}{dn_3}\right)\right\}\right] .$$

But

$$\frac{1}{B}\left(\frac{dS_2}{dn_3} - \frac{1}{2}\frac{dP_{23}}{dn_2}\right) = (1-A_2^2)\frac{dB}{dn_3} + (A_1 - A_2A_3)\frac{dB}{dn_2} + \frac{B}{2}\left(\frac{dA_1}{dn_2} - A_2\frac{dA_3}{dn_2} - A_3\frac{dA_2}{dn_2} - 2A_2\frac{dA_2}{dn_3}\right) ,$$

and therefore $\frac{d}{dn_3}\left\{\frac{1}{B}\left(\frac{dS_2}{dn_3} - \frac{1}{2}\,\frac{dP_{23}}{dn_2}\right)\right\}$ equals

$$\begin{aligned} & \frac{d^2B}{dn_3^2}(1-A_2^2) + \frac{d^2B}{dn_2dn_3}(A_1 - A_2A_3) \\ & + \frac{1}{2}B\,\left\{\frac{d^2A_1}{dn_2dn_3} - A_3\frac{d^2A_2}{dn_2dn_3} - A_2\frac{d^2A_3}{dn_2dn_3} - 2A_2\frac{d^2A_2}{dn_3^2}\right\} \\ & + \frac{1}{2}B\,\left\{-\frac{dA_2}{dn_3}\,\frac{dA_3}{dn_2} - \frac{dA_2}{dn_2}\,\frac{dA_3}{dn_3} - 2\left(\frac{dA_2}{dn_3}\right)^2\right\} \\ & + \frac{dB}{dn_2}\left\{\frac{dA_1}{dn_3} - A_2\frac{dA_3}{dn_3} - A_3\frac{dA_2}{dn_3}\right\} \\ & + \frac{1}{2}\frac{dB}{dn_3}\left\{\frac{dA_1}{dn_2} - A_2\frac{dA_3}{dn_2} - A_3\frac{dA_2}{dn_2} - 6A_2\frac{dA_2}{dn_3}\right\} . \end{aligned}$$

Hence by symmetry we see that

$$+\left[\frac{d}{dn_3}\left\{\frac{1}{B}\left(\frac{dS_2}{dn_3} - \frac{1}{2}\,\frac{dP_{23}}{dn_2}\right)\right\} + \frac{d}{dn_2}\left\{\frac{1}{B}\left(\frac{dS_3}{dn_2} - \frac{1}{2}\,\frac{dP_{23}}{dn_3}\right)\right\}\right]$$

equals

$$\begin{aligned} & \frac{d^2B}{dn_3^2}(1-A_2^2) + 2\frac{d^2B}{dn_2dn_3}(A_1 - A_2A_3) + \frac{d^2B}{dn_2^2}(1-A_3^2) \\ & + B\left\{\frac{d^2A_1}{dn_2dn_3} - A_3\frac{d^2A_2}{dn_2dn_3} - A_2\frac{d^2A_3}{dn_2dn_3} - A_2\frac{d^2A_2}{dn_3^2} \,.\, A_3\frac{d^2A_3}{dn_2^2}\right\} \\ & + B\left\{-\frac{dA_2}{dn_3}\,\frac{dA_3}{dn_2} - \frac{dA_2}{dn_2}\,\frac{dA_3}{dn_3} - \left(\frac{dA_2}{dn_3}\right)^2 - \left(\frac{dA_3}{dn_2}\right)^2\right\} \end{aligned}$$

$$+\frac{3}{2}\frac{dB}{dn_2}\left\{\frac{dA_1}{dn_3}-A_2\frac{dA_3}{dn_3}-A_3\frac{dA_2}{dn_3}\right\}$$

$$+\frac{3}{2}\frac{dB}{dn_3}\left\{\frac{dA_1}{dn_2}-A_2\frac{dA_3}{dn_2}-A_3\frac{dA_2}{dn_2}\right\}$$

$$-3A_2\frac{dB}{dn_3}\frac{dA_2}{dn_3}-3A_3\frac{dB}{dn_2}\frac{dA_3}{dn_2}.$$

We reduce the determinant on page 512 thus; multiply third column by A_2, and add to it first column multiplied by A_3, result is

$$-B^6\begin{vmatrix} \frac{1}{2}(1-A_2^2); & \frac{1}{2}(1-A_3^2); & \frac{1}{2}A_2^2A_3-A_1A_2+\frac{1}{2}A_3 \\ -A_2\frac{dA_2}{dn_2}; & -A_3\,.\frac{dA_3}{dn_2}; & A_2^2\frac{dA_3}{dn_2}-A_2\frac{dA_1}{dn_2} \\ -A_2\,.\frac{dA_2}{dn_3}; & -A_3\,.\frac{dA_3}{dn_3}; & A_2^2\,.\frac{dA_3}{dn_3}-A_2\,.\frac{dA_1}{dn_3} \end{vmatrix};$$

(this is finally to be divided by A_2).

Multiply in this determinant third column by A_3, and add to it second multiplied by A_2^2, result is

$$-B^4\begin{vmatrix} S_2 & ; & S_3 & ; & A_2A_3P_{23}+A_3^2S_2+A_2^2S_3 \\ -A_2\frac{dA_2}{dn_2} & ; & -A_3\,.\frac{dA_3}{dn_2} & ; & -A_2A_3\frac{dA_1}{dn_2} \\ -A_2\frac{dA_2}{dn_3} & ; & -A_3\,.\frac{dA_3}{dn_3} & ; & -A_2A_3\frac{dA_1}{dn_3} \end{vmatrix};$$

but this determinant is to be divided by A_2A_3, therefore determinant on top of page 512 equals

$$-B^4\begin{vmatrix} +A_3S_2; & +A_2S_3; & \frac{1}{2}(2S_1-1) \\ \frac{dA_2}{dn_2}; & \frac{dA_3}{dn_2}; & \frac{dA_1}{dn_2} \\ \frac{dA_2}{dn_3}; & \frac{dA_3}{dn_3}; & \frac{dA_1}{dn_3} \end{vmatrix},$$

which, by page 482, equals

$$-B^4\left\{+\binom{31}{23}A_3S_2+\binom{12}{23}A_2S_3+\binom{23}{23}\frac{1}{2}(2S_1-1)\right\}.$$

We thus dividing all this by B^3 find $B\,.\,m$ equal to

$$\frac{d^2B}{dn_3^2}(1-A_2^2)+2\frac{d^2B}{dn_2dn_3}(A_1-A_2A_3)+\frac{d^2B}{dn_2^2}(1-A_3^2)$$

$$+ B\left\{\frac{d^2A_1}{dn_2dn_3} - A_3\frac{d^2A_2}{dn_2dn_3} - A_2\frac{d^2A_3}{dn_2dn_3} - A_2\frac{d^2A_2}{dn_3^2} - A_3\frac{d^2A_3}{dn_2^2}\right\}$$

$$+ B\left\{-\frac{dA_2}{dn_3}\cdot\frac{dA_3}{dn_2} - \frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_3} - \left(\frac{dA_2}{dn_3}\right)^2 - \left(\frac{dA_3}{dn_2}\right)^2\right\}$$

$$+ \frac{3}{2}\frac{dB}{dn_3}\left\{\frac{dA_1}{dn_3} - A_2\frac{dA_3}{dn_3} - A_3\frac{dA_2}{dn_3}\right\}$$

$$+ \frac{3}{2}\frac{dB}{dn_3}\left\{\frac{dA_1}{dn_2} - A_2\frac{dA_3}{dn_2} - A_3\frac{dA_2}{dn_2}\right\}$$

$$- 3A_2\frac{dB}{dn_3}\cdot\frac{dA_2}{dn_3} - 3A_3\frac{dB}{dn_2}\cdot\frac{dA_3}{dn_2}$$

$$- B\left\{+\binom{31}{23}A_3S_2 + \binom{12}{23}A_2S_3 + \binom{23}{23}\frac{1}{2}(2S_1 - 1)\right\}.$$

Referring back to page 508 and adding above to B multiplied by expression at foot of that page, we find that $B.(2M+m)$ equals

$$\frac{d^2B}{dn_1^2} + \frac{d^2B}{dn_2^2} + \frac{d^2B}{dn_3^2} + 2A_1\frac{d^2B}{dn_2dn_3} + 2A_2\frac{d^2B}{dn_1dn_3} + 2A_3\frac{d^2B}{dn_1dn_2}$$

$$+ B\left\{\frac{d^2A_1}{dn_2dn_3} + \frac{d^2A_2}{dn_1dn_3} + \frac{d^2A_3}{dn_1dn_2}\right\}$$

$$+ \frac{3}{2}\left\{\frac{dB}{dn_1}\left(\frac{dA_2}{dn_3} + \frac{dA_3}{dn_2}\right) + \frac{dB}{dn_2}\left(\frac{dA_3}{dn_1} + \frac{dA_1}{dn_3}\right) + \frac{dB}{dn_3}\left(\frac{dA_1}{dn_2} + \frac{dA_2}{dn_1}\right)\right\}.$$

$$+ B\left\{\frac{dA_2}{dn_3}\cdot\frac{dA_3}{dn_2} - \frac{dA_2}{dn_2}\cdot\frac{dA_3}{dn_3}\right\}\ldots\ldots\ldots\ldots(\alpha)$$

$$+ \frac{A_2}{2}\left\{\frac{dB}{dn_3}\cdot\frac{dA_3}{dn_2} - \frac{dB}{dn_2}\cdot\frac{dA_3}{dn_3}\right\}\ldots\ldots\ldots\ldots(\beta)$$

$$+ \frac{A_3}{2}\left\{\frac{dB}{dn_2}\cdot\frac{dA_2}{du_3} - \frac{dB}{dn_3}\cdot\frac{dA_2}{dn_2}\right\}\ldots\ldots\ldots\ldots(\gamma)$$

$$+ \frac{1}{2}\left\{\frac{dB}{dn_1}\cdot\frac{dA_2}{dn_3} - \frac{dB}{dn_3}\cdot\frac{dA_2}{dn_1}\right\}\ldots\ldots\ldots\ldots(\delta)$$

$$+ \frac{1}{2}\left\{\frac{dB}{dn_1}\cdot\frac{dA_3}{dn_2} - \frac{dB}{dn_2}\cdot\frac{dA_3}{dn_1}\right\}\ldots\ldots\ldots\ldots(\epsilon)$$

$$- B\left\{+\binom{31}{23}A_3S_2 + \binom{12}{23}A_2S_3 + \binom{23}{23}\frac{1}{2}(2S_1 - 1)\right\}\ldots\ldots(\zeta).$$

Now let us consider for a moment the five lines marked α, β, γ, δ and ϵ, we clearly have

$$\alpha = \quad - B\binom{23}{23}$$

$$\beta = + \frac{BA_2P_{13}}{2}\binom{23}{23} - \frac{BA_2P_{23}}{2}\binom{31}{23}$$

$$\gamma = + \frac{BA_3P_{12}}{2}\binom{23}{23} - \frac{BA_3P_{23}}{2}\binom{12}{23}$$

$$\delta = + \frac{BP_{23}}{2}\binom{12}{31} - \frac{BP_{12}}{2}\binom{23}{31}$$

$$\epsilon = + \frac{BP_{23}}{2}\binom{31}{12} - \frac{BP_{13}}{2}\binom{23}{12}.$$

Hence clearly we have $\alpha+\beta+\gamma+\delta+\epsilon$ equal to

$$+\frac{B}{4}\left\{\begin{array}{lll} -(2+4S_1)\binom{23}{23} & -2A_2P_{23}\binom{31}{23} & -2A_3P_{23}\binom{12}{23} \\ -2P_{12}\binom{23}{31} & * & +2P_{23}\binom{12}{31} \\ -2P_{13}\binom{23}{12} & +2P_{23}\binom{31}{12} & * \end{array}\right\}.$$

Hence adding the last line ζ, we find that $B.(2M+m)$ is equal to

$$\frac{d^2B}{dn_1^2}+\frac{d^2B}{dn_2^2}+\frac{d^2B}{dn_3^2}+2A_1\frac{d^2B}{dn_2dn_3}+2A_2\frac{d^2B}{dn_1dn_3}+2A_3\frac{d^2B}{dn_1dn_2}$$

$$+B\left\{\frac{d^2A_1}{dn_2dn_3}+\frac{d^2A_2}{dn_1dn_3}+\frac{d^2A_3}{dn_1dn_2}\right\}$$

$$+\frac{3}{2}\left\{\frac{dB}{dn_1}\left(\frac{dA_2}{dn_3}+\frac{dA_3}{dn_2}\right)+\frac{dB}{dn_2}\left(\frac{dA_3}{dn_1}+\frac{dA_1}{dn_3}\right)+\frac{dB}{dn_3}\left(\frac{dA_1}{dn_2}+\frac{dA_2}{dn_1}\right)\right\}$$

$$+\frac{B}{4}\left\{\begin{array}{lll} -8S_1\binom{23}{23} & +2P_{12}\binom{31}{23} & +2P_{13}\binom{12}{23} \\ -2P_{12}\binom{23}{31} & * & +2P_{23}\binom{12}{31} \\ -2P_{13}\binom{23}{12} & +2P_{23}\binom{31}{12} & * \end{array}\right\}.$$

Now from above subtract the symmetrical *zero* equation given at foot of page 502, we clearly obtain $B(2M+m)$ is equal to

$$-\frac{\Lambda}{B}$$

$$+\frac{B}{4}\left\{\begin{array}{lll} -6S_1\binom{23}{23} & +P_{12}\binom{31}{23} & +P_{31}\binom{12}{23} \\ -3P_{12}\binom{23}{31} & +2S_2\binom{31}{31} & +P_{23}\binom{12}{31} \\ -3P_{13}\binom{23}{12} & +P_{23}\binom{31}{12} & +2S_3\binom{12}{12} \end{array}\right\}.$$

But it was proved at foot of page 511 (see value of determinant given at foot of page 513) that B^2m is equal to

$$
\begin{aligned}
&+\frac{d^2S_2}{dn_3^2}-\frac{d^2P_{23}}{dn_2dn_3}+\frac{d^2S_3}{dn_2^2}\\
&-\frac{1}{B}\left\{+\frac{dS_2}{dn_3}\frac{dB}{dn_3}+\frac{dS_3}{dn_2}\frac{dB}{dn_2}-\frac{1}{2}\frac{dP_{23}}{dn_2}\frac{dB}{dn_3}-\frac{1}{2}\frac{dP_{23}}{dn_3}\frac{dB}{dn_2}\right\}\\
&-\frac{B^2}{4}\left\{+(4S_1-2)\binom{23}{23}+4A_3S_2\binom{31}{23}+4A_2S_3\binom{12}{23}\right\}.
\end{aligned}
$$

Add to this the second *zero* equation on page 498, and then divide the result by B, we clearly obtain Bm is equal to

$$
\begin{aligned}
&+\frac{\Lambda}{B}\\
&+\frac{B}{4}\left\{\begin{array}{l}
+6S_1\binom{23}{23}-P_{12}\binom{31}{23}-P_{13}\binom{12}{23}\\
+3P_{12}\binom{23}{31}-2S_2\binom{31}{31}-P_{23}\binom{12}{31}\\
+3P_{13}\binom{23}{12}-P_{23}\binom{31}{12}-2S_3\binom{12}{12}
\end{array}\right\}.
\end{aligned}
$$

Now comparing this Equation with the immediately previous found value of $B(2M+m)$ we clearly obtain $m=-M$, whence we have a second formula for $\frac{1}{R_1R_2}$, and then by the second equation on page 498 we obtain a third formula for $\frac{1}{R_1R_2}$. The formula $m=-\frac{1}{R_1R_2}$ is due to Gauss, who obtained it by a different process (see *Disquisitiones generales circa superficies curvas*, Göttingen, 1827).

I conclude this Exercise with a remarkable formula, the proof of which I leave to the reader as an Exercise in the preceding methods. I call the formula remarkable on account of the curious destruction of terms, the coefficients, for instance, of the nine determinants $\binom{12}{12}$ &c. completely vanishing. The reader should compare this following formula with pages 467 and 487.

$$
+A_{12}A_{13}\left|\begin{array}{ll}
\left\{A_{12}\frac{dG}{du_3}+A_{13}\frac{dI}{du_2}+A_{11}\left(\frac{dJ}{du_2}+\frac{dF}{du_3}-\frac{dH}{du_1}\right)\right\} & ;\ \left\{A_{12}\frac{dG}{du_2}+A_{13}\left(2\frac{dH}{du_2}-\frac{dG}{du_3}\right)+A_{11}\left(2\frac{dF}{du_2}-\frac{dG}{du_1}\right)\right.\\
\left\{A_{12}\left(2\frac{dH}{du_3}-\frac{dI}{du_2}\right)+A_{13}\frac{dI}{du_3}+A_{11}\left(2\frac{dJ}{du_3}-\frac{dI}{du_1}\right)\right\} & ;\ \left\{A_{12}\frac{dG}{du_3}+A_{13}\frac{dI}{du_2}+A_{11}\left(\frac{dJ}{du_2}+\frac{dF}{du_3}-\frac{dH}{du_1}\right)\right\}
\end{array}\right.
$$

$$+ A^2_{11} \left| \begin{matrix} \left\{A_{12}\left(\dfrac{dF}{du_3} + \dfrac{dH}{du_1} - \dfrac{dJ}{du_2}\right) + A_{13}\dfrac{dI}{du_1} + A_{11}\dfrac{dE}{du_3}\right\} & ; & \left\{A_{12}\dfrac{dG}{du_3} + A_{13}\dfrac{dI}{du_2} + A_{11}\left(\dfrac{dF}{du_3} + \dfrac{dJ}{du_2} - \dfrac{dH}{du_1}\right)\right\} \\ \left\{A_{12}\left(2\dfrac{dF}{du_1} - \dfrac{dE}{du_2}\right) + A_{13}\left(2\dfrac{dJ}{du_1} - \dfrac{dE}{du_3}\right) + A_{11}\dfrac{dE}{du_1}\right\} & ; & \left\{A_{12}\dfrac{dG}{du_1} + A_{13}\left(\dfrac{dH}{du_1} + \dfrac{dJ}{du_2} - \dfrac{dF}{du_1}\right) + A_{11}\dfrac{dE}{du_2}\right\} \end{matrix} \right|$$

is equal to

$$- A^2_{11}\frac{dA_{11}}{du_1}\frac{dH}{du_1} - 2A_{12}A_{13}\frac{dA_{11}}{du_2}\frac{dI}{du_2} - 2A_{12}A_{13}\frac{dA_{11}}{du_3}\frac{dG}{du_3}$$

$$+ A^2_{11}\frac{dA_{11}}{du_1}\frac{dJ}{du_2} + A_{11}A_{13}\left\{\frac{dA_{11}}{du_1}\frac{dI}{du_2} - \frac{dA_{11}}{du_2}\frac{dI}{du_1}\right\}$$

$$+ A^2_{11}\frac{dA_{11}}{du_1}\frac{dF}{du_3} + A_{11}A_{12}\left\{\frac{dA_{11}}{du_1}\frac{dG}{du_3} - \frac{dA_{11}}{du_3}\frac{dG}{du_1}\right\}$$

$$- \frac{dA_{11}}{du_2}\frac{dA_{11}}{du_3}$$

$$- \frac{dA_{11}}{du_2}\left(A^2_{11}\frac{dE}{du_3} + A^2_{12}\frac{dG}{du_3} + 2A_{11}A_{12}\frac{dF}{du_3}\right)$$

$$- \frac{dA_{11}}{du_3}\left(A^2_{11}\frac{dE}{du_2} + A^2_{13}\frac{dI}{du_2} + 2A_{11}A_{13}\frac{dJ}{du_2}\right).$$

February, 1875.

NOTE TO EXERCISE THE FIRST. (PAGE 460.)

I give here the values expressed in terms of general Curvilinear Co-ordinates of the radius of normal Curvature and the radius of geodesic Curvature of a Curve traced on a surface.

I represent by the notation r_{xy} the radius of normal Curvature with regard to the surface $x=0$ of the Curve formed by the intersection of the surfaces $x=0$, $y=0$, whilst γ_{xy} denotes the radius of geodesic Curvature with regard to the surface $x=0$ of the same Curve; $\{(u, x), (u, y)\}$ represents the value of the angle between two curves traced on the surface $u=0$, these curves being formed one by the intersection of the surfaces $u=0$, $x=0$, and the other by the intersection of the surfaces $u=0$, $y=0$. All the formulæ that follow are derived by simple transformation of co-ordinates from the corresponding *Homogeneous* formulæ expressed by means of Cartesian co-ordinates.

I also assume the known or else easily demonstrated formulæ

$$\frac{1}{\gamma_{xy}} = \frac{1}{\sin(x, y)}\left\{\frac{1}{r_{yx}} - \cos(x, y)\frac{1}{r_{xy}}\right\},$$

$$\frac{1}{\gamma_{yx}} = \frac{1}{\sin(x, y)}\left\{\frac{1}{r_{xy}} - \cos(x, y)\frac{1}{r_{yx}}\right\},$$

where (x, y) represents the angle between the surfaces $x=0$, $y=0$; finally r and γ will represent the radius of normal and geodesic curvatures of some curve traced on the surface $U(u_1, u_2, u_3)=0$, a direction of an element of this curve being defined by the differentials $du_1 : du_2 : du_3$. These preliminaries being thus explained my formulæ are:

$$\frac{1}{r} \cdot \left\{A_{11}\left(\frac{dU}{du_1}\right)^2 + A_{22}\left(\frac{dU}{du_2}\right)^2 + A_{33}\left(\frac{dU}{du_3}\right)^2 + 2A_{12}\frac{dU}{du_1}\frac{dU}{du_2} + 2A_{23}\frac{dU}{du_2}\frac{dU}{du_3} + 2A_{13}\frac{dU}{du_1}\frac{dU}{du_3}\right\}^{\frac{1}{2}}$$

$$\cdot \{C_{11} \cdot du_1^2 + C_{22}du_2^2 + C_{33}du_3^2 + 2C_{12}du_1du_2 + 2C_{23}du_2du_3 + 2C_{13}du_1du_3\}$$

equals

$$\left\{\frac{d^2U}{du_1^2}du_1^2 + \frac{d^2U}{du_2^2}du_2^2 + \frac{d^2U}{du_3^2}du_3^2 + 2\frac{d^2U}{du_1du_2}du_1du_2 + 2\frac{d^2U}{du_2du_3}du_2du_3 + 2\frac{d^2U}{du_1du_3}du_1du_3\right\}$$

$$+ \{K_{11} \cdot du_1^2 + K_{22} \cdot du_2^2 + K_{33}du_3^2 + 2K_{12}du_1du_2 + 2K_{23}du_2du_3 + 2K_{13}du_1du_3\},$$

where

$$K_{11} = \left\{\begin{array}{l} + \left\{A_{11}\left(\frac{1}{2}\frac{dC_{11}}{du_1}\right) + A_{12}\left(\frac{dC_{12}}{du_1} - \frac{1}{2}\frac{dC_{11}}{du_2}\right) + A_{13}\left(\frac{dC_{13}}{du_1} - \frac{1}{2}\frac{dC_{11}}{du_3}\right)\right\} \cdot \frac{dU}{du_1} \\ + \left\{A_{22}\left(\frac{dC_{12}}{du_1} - \frac{1}{2}\frac{dC_{11}}{du_2}\right) + A_{23}\left(\frac{dC_{13}}{du_1} - \frac{1}{2}\frac{dC_{11}}{du_3}\right) + A_{12}\left(\frac{1}{2}\frac{dC_{11}}{du_1}\right)\right\} \cdot \frac{dU}{du_2} \\ + \left\{A_{33}\left(\frac{dC_{13}}{du} - \frac{1}{2}\frac{dC_{11}}{du_3}\right) + A_{13}\left(\frac{1}{2}\frac{dC_{11}}{du_1}\right) + A_{23}\left(\frac{dC_{12}}{du_1} - \frac{1}{2}\frac{dC_{11}}{du_2}\right)\right\} \cdot \frac{dU}{du_3} \end{array}\right\}$$

and

$$-2K_{23}=\begin{bmatrix} +\left\{A_{11}\left(\dfrac{dC_{31}}{du_2}+\dfrac{dC_{12}}{du_3}-\dfrac{dC_{23}}{du_1}\right)+A_{12}\dfrac{dC_{22}}{du_3}+A_{13}\dfrac{dC_{33}}{du_2}\right\}\cdot\dfrac{dU}{du_1} \\ +\left\{A_{22}\dfrac{dC_{22}}{du_3}+A_{23}\dfrac{dC_{33}}{du_2}+A_{21}\left(\dfrac{dC_{31}}{du_2}+\dfrac{dC_{12}}{du_3}-\dfrac{dC_{23}}{du_1}\right)\right\}\cdot\dfrac{dU}{du_2} \\ +\left\{A_{33}\dfrac{dC_{33}}{du_2}+A_{31}\left(\dfrac{dC_{31}}{du_2}+\dfrac{dC_{12}}{du_3}-\dfrac{dC_{23}}{du_1}\right)+A_{32}\dfrac{dC_{22}}{du_3}\right\}\cdot\dfrac{dU}{du_3} \end{bmatrix},$$

K_{23}, K_{31}, K_{13} and K_{12} being obtained by a cyclic change of suffixes from above. There are *eighteen* coefficients of $\dfrac{dU}{du_1}$, &c. in all, compare these with the *eighteen* functions of first Exercise.

$$\operatorname{Cos}(Uu_1)=\frac{A_{11}\dfrac{dU}{du_1}+A_{12}\dfrac{dU}{du_2}+A_{13}\dfrac{dU}{du_3}}{A_{11}^{\frac{1}{2}}\cdot\left\{A_{11}\left(\dfrac{dU}{du_1}\right)^2+A_{22}\left(\dfrac{dU}{du_2}\right)^2+A_{33}\left(\dfrac{dU}{du_3}\right)^2+2A_{23}\dfrac{dU}{du_2}\dfrac{dU}{du_3}+2A_{13}\dfrac{dU}{du_1}\dfrac{dU}{du_3}+2A_{12}\dfrac{dU}{du_1}\dfrac{dU}{du_2}\right\}^{\frac{1}{2}}}.$$

$\operatorname{Cos}(Uu_2)$ and $\cos(Uu_3)$ are derived by a cyclic change of suffixes, by the aid of these cosines we can obviously eliminate $\dfrac{dU}{du_1}$, $\dfrac{dU}{du_2}$, $\dfrac{dU}{du_3}$; $\dfrac{d^2U}{du_1^2}$, &c. from the above value of $\dfrac{1}{r}$.

....

$$\frac{1}{\gamma_{u_3U}}\cdot\{C_{11}\cdot du_1^2+2C_{12}\cdot du_1du_2+C_{22}du_2^2\}\cdot\left\{C_{22}\left(\frac{dU}{du_1}\right)^2-2C_{12}\frac{dU}{du_1}\frac{dU}{du_2}+C_{11}\left(\frac{dU}{du_2}\right)^2\right\}^{\frac{1}{2}}\cdot\{C_{11}C_{22}-C_{12}\}^{\frac{1}{2}}.$$

equals

$$\left\{\begin{array}{l} \{C_{11}C_{22}-C_{12}^2\}\cdot\left\{\dfrac{d^2U}{du_1^2}du_1^2+2\dfrac{d^2U}{du_1du_2}du_1du_2+\dfrac{d^2U}{du_2^2}du_2^2\right\} \\ -du_1^2\begin{bmatrix} +\left\{\quad C_{22}\dfrac{1}{2}\dfrac{dC_{11}}{du_1}+C_{12}\left(\dfrac{1}{2}\dfrac{dC_{11}}{du_2}-\dfrac{dC_{12}}{du_1}\right)\right\}\dfrac{dU}{du_1} \\ +\left\{-C_{12}\dfrac{1}{2}\dfrac{dC_{11}}{du_1}-C_{11}\left(\dfrac{1}{2}\dfrac{dC_{11}}{du_2}-\dfrac{dC_{12}}{du_1}\right)\right\}\dfrac{dU}{du_2} \end{bmatrix} \\ -du_1du_2\begin{bmatrix} +\left\{\quad C_{22}\dfrac{dC_{11}}{du_2}-C_{12}\dfrac{dC_{22}}{du_1}\right\}\cdot\dfrac{dU}{du_1} \\ +\left\{\quad C_{11}\dfrac{dC_{22}}{du_1}-C_{12}\dfrac{dC_{11}}{du_2}\right\}\cdot\dfrac{dU}{du_2} \end{bmatrix} \\ -du_2^2\begin{bmatrix} +\left\{\quad C_{22}\left(\dfrac{dC_{12}}{du_2}-\dfrac{1}{2}\dfrac{dC_{22}}{du_1}\right)-C_{12}\dfrac{1}{2}\dfrac{dC_{22}}{du_2}\right\}\dfrac{dU}{du_1} \\ +\left\{\quad C_{11}\dfrac{1}{2}\dfrac{dC_{22}}{du_2}+C_{12}\left(\dfrac{1}{2}\dfrac{dC_{22}}{du_1}-\dfrac{dC_{12}}{du_2}\right)\right\}\dfrac{dU}{du_2} \end{bmatrix} \end{array}\right\};$$

from this we can eliminate du_1 and du_2, for since $du_3=0$, and $dU=0$, therefore

$$\frac{dU}{du_1}du_1+\frac{dU}{du_2}du_2=0,\ \text{ or }\ du_1=\lambda\frac{dU}{du_2}\ \text{ and }\ du_2=-\lambda\frac{dU}{du_1},$$

and if then we write $\dfrac{1}{\gamma_{u_3U}}$ equal to a constant or zero, we get a differential equation to determine the integral $U = 0$ of *Didonian* or *Geodesic* curves traced on the surface $u_3 = 0$. Write for shortness

$$\{(Uu_3)(u_3u_2)\} = \omega_2 \text{ and } \{(Uu_3)(u_3u_1)\} = \omega_1;$$

$$\cos\omega_1 = \frac{C_{22}\dfrac{dU}{du_1} - C_{12}\dfrac{dU}{du_2}}{C_{22}^{\frac{1}{2}}\left\{C_{22}\left(\dfrac{dU}{du_1}\right)^2 - 2C_{12}\dfrac{dU}{du_1}\dfrac{dU}{du_2} + C_{11}\left(\dfrac{dU}{du_2}\right)^2\right\}^{\frac{1}{2}}};$$

$$\cos\omega_2 = \frac{C_{11}\dfrac{dU}{du_2} - C_{12}\dfrac{dU}{du_1}}{C_{11}^{\frac{1}{2}}\left\{C_{22}\left(\dfrac{dU}{du_1}\right)^2 - 2C_{12}\dfrac{dU}{du_1}\dfrac{dU}{du_2} + C_{11}\left(\dfrac{dU}{du_2}\right)^2\right\}^{\frac{1}{2}}};$$

$$\sin\omega_1 = \frac{(C_{11}C_{22} - C_{12}^2)^{\frac{1}{2}}\dfrac{dU}{du_2}}{C_{22}^{\frac{1}{2}}\left\{C_{22}\left(\dfrac{dU}{du_1}\right)^2 - 2C_{12}\dfrac{dU}{du_1}\cdot\dfrac{dU}{du_2} + C_{11}\left(\dfrac{dU}{du_2}\right)^2\right\}^{\frac{1}{2}}};$$

$$\sin\omega_2 = \frac{(C_{11}C_{22} - C_{12}^2)^{\frac{1}{2}}\dfrac{dU}{du_1}}{C_{11}^{\frac{1}{2}}\left\{C_{22}\left(\dfrac{dU}{du_1}\right) - 2C_{12}\dfrac{dU}{du_1}\dfrac{dU}{du_2} + C_{11}\left(\dfrac{dU}{du_2}\right)^2\right\}^{\frac{1}{2}}};$$

and hence obviously

$$\left(\frac{C_{11}}{C_{22}}\right)^{\frac{1}{2}}\frac{\sin\omega_2}{\sin\omega_1} = \frac{\dfrac{dU}{du_1}}{\dfrac{dU}{du_2}},$$

so that if $\dfrac{d}{ds}$ denote a differentiation due to a passage along the junction line of the surfaces $U = 0$, $u_3 = 0$, we obviously have

$$\frac{1}{\gamma u_3 U} = \sin^2\omega_1\frac{C_{22}}{(C_{11}C_{22} - C_{12}^2)^{\frac{1}{2}}}\cdot\frac{d}{ds}\left\{\left(\frac{C_{11}}{C_{22}}\right)^{\frac{1}{2}}\frac{\sin\omega_2}{\sin\omega_1}\right\}$$

$$+\frac{\Gamma_1\,.\sin^3\omega_1 + \Gamma_2\,.\sin^2\omega_1\,.\sin\omega_2 + \Gamma_3\,.\sin\omega_1\,.\sin^2\omega_2 + \Gamma_4\,.\sin^3\omega_2}{\sin^3\Omega},$$

where $\Omega = \omega_1 + \omega_2$ and where Γ_1, Γ_2, Γ_3, and Γ_4 are functions merely of C_{11}, C_{22}, and C_{12} and their differential coefficients with regard to u_1 and u_2, and by supposing U alternately to coincide with u_1 and u_2, we clearly have

$$\Gamma_4 = \frac{1}{\gamma u_3 u_1}; \qquad \Gamma_1 = \frac{1}{\gamma u_3 u_2}.$$

These formulæ for $\dfrac{1}{r}$ and $\dfrac{1}{\gamma u_3 U}$ are the analytical foundation for most of the known Geometrical theorems regarding the Curvatures of Curves drawn on surfaces; it should also be observed how $\dfrac{1}{\gamma_{u_3 U}}$ is a function merely of C_{11}, C_{22}, C_{12}, ω_1 and ω_2, and their differential coefficients with regard to u_1 and u_2 as it should be.

If we eliminate $\sin^2\omega_1$ and $\sin^2\omega_2$ from the last written formula for $\dfrac{1}{\gamma_{u,U}}$ by aid of the equation

$$\sin^2\Omega = \sin^2\omega_1 + \sin^2\omega_2 - 2\cos\Omega \sin\omega_1 \sin\omega_2$$

we arrive after a few reductions at the following Geometrical interpretation of our formula. Conceive three curves s_1, s_2, s_3 drawn on the surface $u=0$ all to pass through the same point P on this surface. Let σ_1, σ_2, σ_3 equal the angles at P between the elements ds_2 and ds_3, &c., &c. so that $\sigma_1 + \sigma_2 + \sigma_3 = 0$, then we have

$$\frac{\sin\sigma_1}{\gamma_{s_1}} + \frac{\sin\sigma_2}{\gamma_{s_2}} + \frac{\sin\sigma_3}{\gamma_{s_3}},$$

equal to

$$\sin\sigma_1 \frac{d\sigma_2}{ds_1} - \sin\sigma_2 \frac{d\sigma_1}{ds_2} = \sin\sigma_2 \frac{d\sigma_3}{ds_2} - \sin\sigma_3 \frac{d\sigma_2}{ds_3} = \sin\sigma_3 \frac{d\sigma_1}{ds_3} - \sin\sigma_1 \frac{d\sigma_3}{ds_1}.$$

The reciprocals of the *six* radii of normal Curvature, the *three* sums (of the reciprocals of chief radii) of Curvature, and the *nine* first differential coefficients of the three angles at which the three surfaces cut, give us *eighteen* equations *linear* in the *eighteen* quantities $\dfrac{dC_{11}}{dv_1}$, &c., hence we can determine the Geometric meaning of these *eighteen* quantities and substitute these Geometric values if we wish.

NOTE TO EXERCISE THE THIRD. (PAGE 506).

I wish here to point out that the method of normal co-ordinates enjoys two advantages, so to speak, one of notation, the other as a deductive method; the advantages of notation are simplicity (with small Geometric sacrifice) caused by A_{11}, A_{22}, A_{33}, each being equal unity; but its advantage as a deductive method consists in the fact that there are certain quantities which experience either a known change or else no change, when we pass from a surface to a consecutive parallel surface; and if we express this fact analytically we arrive often at important results. In the third Exercise we by this deductive method arrived at the value of $\dfrac{1}{R_1 R_2}$ and $\dfrac{1}{R_1} + \dfrac{1}{R_2}$. I wish in this note to point out how the same deductive method would probably enable us to arrive at the six differential equations of the second Exercise.

I write for shortness Δ_3 in place of the symbol of passage along the normal n_3, i.e. in place of $\dfrac{d}{dn_3} + A_2 \dfrac{d}{dn_1} + A_1 \dfrac{d}{dn_2}$; we have likewise Δ_1 and Δ_2. When we are not concerned with Δ_1 or Δ_2, I omit the suffix and simply write Δ for Δ_3......... Now dS being an element of the surface u_3 and R, R', the corresponding chief radii of curvature, I have shown in this third Exercise that

$$(\alpha) \quad \frac{\Delta dS}{dS} = \frac{1}{R} + \frac{1}{R'},$$

$$(\beta) \quad \left(\frac{1}{R} + \frac{1}{R'}\right)^2 + \Delta\left(\frac{1}{R} + \frac{1}{R'}\right) = \frac{2}{RR'},$$

$$(\gamma) - \Delta\left(\frac{1}{RR'}\right) = \left(\frac{1}{RR'}\right)\left(\frac{1}{R} + \frac{1}{R'}\right).$$

Knowing as we do then the value of dS, by means of (α) in exercise the third, I obtained the value of $\frac{1}{R} + \frac{1}{R'}$, which I may call Σ, and thence by (β) I obtained a value for $\frac{1}{RR'}$ which I shall call Π, and this value observe contains differential coefficients of the *second* order. Lastly, by means of (γ) I thence obtained a differential equation of the *third* order equal zero; but it is not differential equations of the *third* order that we are in search of, but differential equations of the second order, and we thus obtain these. In addition to the formula Π which I have given for $\frac{1}{RR'}$ two others which I shall call Π_1 and Π_2 can be obtained. Π_2 contains differential coefficients of the second order, but Π_1 only contains differential coefficients of the *first* order, hence the formulæ (β) and (γ) lead to the following differential equations of the second order:

$$(\beta)' \qquad \Sigma^2 + \Delta\Sigma - \Pi_2 = 0,$$

$$(\beta)'' \qquad \Sigma^2 + \Delta\Sigma - \Pi_1 = 0,$$

$$(\gamma)' \quad - \Delta\Pi_1 = \Pi \,.\, \Sigma = \Pi_1 \,.\, \Sigma = \Pi_2 \,.\, \Sigma,$$

and to these of course must be added six similar series of differential equations of the second order, derived symmetrically from the surfaces u_1 and u_2.

Observe also that we may write the first (for example) of the Equations on page 497

$$\frac{1}{B}\left\{\Delta_2\Delta_3 + \Delta_3\Delta_2\right\} B + \Delta_2 \frac{dA_2}{dn_1} + \Delta_3 \frac{dA_3}{dn_1} - \Delta_1 \frac{dA_1}{dn_1} + \&c. = 0,$$

from which probably interesting results might follow, but I must now bear in mind the warning of our great Novelist, that "We can do nothing safely without some judgment as to where we are to stop."

V. *The Place of Music in Education as conceived by* ARISTOTLE (*Politics* V. [VIII.] cc. 3—7).

By Professor JEBB.

[Read *May* 17, 1875.]

THE object of education is to make the man a good citizen, and so to put him in the way of attaining happiness; that is, the conscious activity of the highest part of his nature in accordance with the law of his own excellence. Education should be the same for all the citizens; and in order of time physical training must come first, moral training second, intellectual training last. The State Education should aim *more* at the development of the contemplative than of the practical reason, since the legislator's object is to fit the citizen, above all things, for the wise and happy enjoyment of peace. The particular branches of Education, as ordinarily recognised, are, Aristotle says, four in number:—Grammar, Gymnastic, Music, and (as some reckon) Drawing. Grammar*, Gymnastic†, and Drawing‡ have evident practical utilities; but with what object is Music to be taught? This cannot be said to be either such a direct utility as is the end of Gymnastic, or such as is the end of Grammar and Drawing. Three objects, Aristotle says, might popularly be assigned: *παιδεία,* discipline; *παιδιά,* pastime; and *διαγωγή,* the rational employment of leisure. Classified more scientifically, the objects which are to be attained by the study of Music are, he concludes, these three:—*παιδεία,* discipline; *διαγωγή,* rational amusement; and *κάθαρσις,* the purification of the emotions.

It is of the third and last especially that I wish to say a few words, with a view to elucidating, if possible, the exact meaning which Aristotle attached to it, and which, as it seems to me, is as suggestive for our own day as it is significant of the Greek feeling towards art universally. But, before coming to *κάθαρσις,* it will be worth while to touch briefly on the two other objects—*παιδεία* and *διαγωγή.*

I. *παιδεία.* The *disciplinary* value of Music for youthful learners is twofold; *artistic* and *moral.* Artistic, as educing and training those perceptions which will make

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* 'For business—for economy—for learning—for political actions.'

† *πρὸς ὑγίειαν καὶ ἀλκήν.*

‡ As fitting to give accurate ideas of shape and of artistic finish, which guide one (for instance) in purchases, *ἐν τοῖς ἰδίοις ὠνίοις*: as making men connoisseurs of art; above all, as training the sense of beauty in the human form.

Music a delightful resource in mature life,—as, in short, preparing *διαγωγή*. Moral, since, as we listen to Music, we become *ποιοί τινες*: there is a definite affection of the soul; and, if the Music is rightly chosen, it disciplines the moral nature by establishing in us the faculty of rejoicing aright—*ἐθίζουσα δύνασθαι χαίρειν ὀρθῶς*. For Music can give us images, *ὁμοιώματα*, of certain feelings,—love, hatred, joy, sorrow; and pleasure in the imitations will create sympathy with the feelings represented. It is peculiar to the sense of hearing that it can thus be the channel of a moral imitation. The sense of touch and the sense of taste are not accessible to such suggestion. The sense of sight is so in only a slight degree. For though forms and colours are, in a way, *ethical*, or significant of character, they are so in a different manner from musical sounds or words. Musical sounds and words are imitative expressions of character and feeling. Forms and colours are not expressions, but only symbols; they are not *ὁμοιώματα* but *σημεῖα*.

Granting, however, that Music as a *discipline* has potentially this double value, the artistic and the moral, what kind of Music is to be chosen as especially useful for the discipline of the young? According to a division which, Aristotle says, had been used by some scientific writers (*τῶν ἐν φιλοσοφίᾳ τινάς*) of his day, *μέλη*, styles or genera of Music, were classified as

1. *ἠθικά*,
2. *πρακτικά*,
3. *ἐνθουσιαστικά*.

1. The meaning of *ἠθικά* is explained by the mention of the Dorian *μέλη* as being *ἠθικώτατα*. It is a grave and manly character in music, remote alike from excitement and from a voluptuous languor.

2. 'Practical' Music is that which accompanies and interprets action; — stirring, vigorous, animated, like martial music, but, on the other hand, steady and restrained. In the Problems Aristotle says that the Hypo-phrygian mode—in which the enthusiasm of the pure Phrygian was *tempered*—has an *ἦθος πρακτικόν* *: and so the iambic trimeter, as compared with the saltatory tetrameter, is said to be *πρακτικόν* †—as Horace expresses it, *natum rebus agendis*.

3. 'Enthusiastic' Music is such as the Phrygian—a wild, excited strain, fitted to stimulate the worshippers in the orgiastic rites of Dionysos or Cybele.

Now, for *παιδεία*, the Ethical Music is of course to be used,—the Dorian chiefly; though the Lydian Music may also, Aristotle thinks, satisfy the three conditions — absence of excess, the limit of what is practicable, and propriety.

But does Music, considered as a part of early training, imply the power of performing upon any instrument? Aristotle gives two reasons for answering *Yes*.—(i) A measure of practical knowledge is necessary to make a good judge of Music. (ii) A musical instrument may be for youths what the *πλαταγή* of Archytas is for children—

* Arist. *Problem.* xix. 41. And hence the reason, he adds, why *ἡ ὑποφρυγιστί* was never used in Tragedy by the chorus, which has no part in the action, but is merely a passive sympathiser—*κηδευτὴς ἄπρακτος*.

† Arist. *Poet.* c. 24, *τὸ δὲ ἰαμβικὸν καὶ τετράμετρον κινητικά, τὸ μὲν ὀρχηστικόν, τὸ δὲ πρακτικόν*.

a means of keeping them out of mischief. But here he states and answers an objection. May not the pursuit of executive skill in Music degrade the citizen into a *βάναυσος* or mechanic? Aristotle answers:—It *may* do so, if it is carried too far. We have to fix a *limit* up to which it may be studied by those who are being trained to the virtue of a citizen. This limit is determined by two things. First; the learning of Music must not interfere with other studies. Secondly, the *body* of the citizen must in no way be unfitted for war or those exercises which befit free men. No mechanic, any more than a slave, can do these actions which are according to virtue. Therefore youths must not enter upon such laborious musical training as is preparatory for the contests of artists (*τεχνικοὶ ἀγῶνες*). Nor must they attempt those brilliant pieces of an extraordinary difficulty (*τὰ θαυμάσια καὶ τὰ περιττά*) which have been brought into contests, and thence into education. In a word—the study of Music must stop short of what is *τεχνική*, professional. The feeling of the Greeks in and before Aristotle's time towards artistic specialists seems to have varied with the eminence of the artist a good deal more than it does among us. The artists of genius were recognised as great men. The ordinary artists were mechanics—men who had gone aside from the true political life, and whose moral natures were maimed.

II. *διαγωγή*. The distinction between *παιδιά* and *διαγωγή* must be clearly seen. *παιδιά* is *mere* recreation: it is for *the sake of rest* (*χάριν ἀναπαύσεως*), and fulfils its end if it is pleasant. *διαγωγή* is something more: it has two elements, corresponding to the two chief constituents of happiness itself—*τὸ καλόν* and *τὸ ἡδύ*. It is the employment of leisure in a manner befitting a citizen. Let it be remembered what is Aristotle's view of this *σχολή**. The soul is of two parts, Rational and Irrational; the Rational is divided into the Theoretic and the Practical Reason. As the Practical is subordinate to the Theoretic Reason, so useful or necessary actions are subordinate to noble actions. War leads up to peace. Work leads up to rest. Bravery and Patience are necessary for work, i.e. 'Philosophy,' intellectual culture, is necessary for the right use of rest. Temperance and Justice are necessary both for work and for rest. The aim of education is to teach men first how they shall *procure*, secondly how they shall *use*, leisure. Greek civilisation became more and more developed, the science of leisure—if one may use such a phrase—was more and more cultivated. Aristotle's word *σχολαστικός* means neither exactly 'leisurely,' nor, of course, 'scholastic,' but rather 'fitted for leisure,' i.e. qualified to use it intelligently: see *Polit.* VII. (VI.) 8 § 22, and VIII (V.) 11 § 5, *μήτε σχολάς ... μήτε συλλόγους σχολαστικούς*. 'As they became more fitted for leisure,' he says, 'through their material resources, and of a loftier spirit towards virtue,—having already, too, after the Persian wars, been lifted up in spirit by their achievements,—they began to lay hold on all learning, drawing no line—*οὐδὲν διακρίνοντες*—but pushing their search onward †.'

Here, then, is the reason of the place held by Music in the mature life of the normal citizen—it is one of the noblest and most elevating forms of *διαγωγή* or rational

* Arist. *Pol.* v. 14.

† v. (VIII.) 6, § 11.

recreation. And while it is thus a great general instrument of διαγωγή, it ministers, in that quality, to two special purposes—the culture of the intelligence, φρόνησις, and the purification of the emotions, *κάθαρσις*.

1. To the intelligence it renders, first of all, the service of relaxation, ἄνεσις: secondly, it affords a gentle exercise for the critical faculty in alliance with the imagination, thus aiding to render the perceptions subtle and exact. Athene's reason for throwing away the flute when she had found it, was not, Aristotle suggests, that it distorted the player's face, but rather that it contributed nothing to this essential object of the best Music—culture of the intelligence.

2. What, however, is to be understood by *κάθαρσις*, that purification of the emotions which is the highest and final moral function of Music?

The word *κάθαρσις*, as applied to Tragedy in Aristotle's Poetics, and here, in his Politics, to Music, has been variously explained.

In the Poetics, Tragedy is described as effecting, by means of pity and terror, the purification of such passions: *δι' ἐλέου καὶ φόβου περαίνουσα τὴν τῶν τοιούτων παθημάτων κάθαρσιν*.

The explanations which have been suggested are, so far as I know, four in number; for I set aside the notion, resting on verbal misconceptions, that *παθημάτων* and *κάθαρσις* could mean 'removal of calamities'—the prevention, that is, of such disasters as Tragedy represents.

1. *κάθαρσις* = that *moderation* of the emotions which results from familiarity with the objects that excite them: as the passions of pity or terror might be *moderated*, through habit, in the physician or the soldier. This explanation is manifestly not only inadequate, but not specially applicable to tragic fiction.

2. *κάθαρσις* = chastisement of the *bad* passions, effected by pity and terror at what Tragedy represents. When we see in Tragedy what the bad passions entail, we restrain them. This view appears untenable when we observe that it excludes pity and terror from the passions thus chastened; whereas Aristotle says, *τῶν τοιούτων παθημάτων*, *such* passions—such, namely, as pity and terror; e.g. love and hatred. Compare *Polit.* V. (VIII.) 7 § 5, *ταὐτὸ δὴ τοῦτο ἀναγκαῖον πάσχειν καὶ τοὺς ἐλεήμονας καὶ τοὺς φοβητικοὺς καὶ τοὺς ὅλους παθητικούς*: where *ταὐτὸ πάσχειν* = *καθάρσεως τυχεῖν*.

3. *κάθαρσις* = the separation from pity and terror of what is disagreeable or painful in such emotions when they are excited by real objects, and not, as in Tragedy, by fictions. Pleasure is doubtless attendant on *κάθαρσις παθημάτων*: but clearly *κάθαρσις* consists in something more than making an emotion pleasurable; and, moreover, the operation of *κάθαρσις* on the moral nature is manifestly something gradual, and when effected, lasting; it is not confined to a momentary impression; it is a process, and ultimately a healing of the soul.

4. *κάθαρσις* = 'the correction or refinement of the passions.' This is Twining's explanation; it is the nearest, I think, to the truth; but I do not think that he has found exactly the right point of view. I will quote his own words*:—

* Twining, *Poetics*, Vol. II. p. 17.

'The passions of savages, or of men in the first rude stages of civilisation, are ferocious and painful. They *pity*, or they *fear*, either violently or not at all. With them, there is hardly any medium between ungovernable agitation and absolute insensibility. Suppose such a people to have access, like the Athenians, to theatrical representations, and to have their emotions kept in frequent and pleasurable exercise by *fictitious* distress; the consequence, I think, would be that by degrees they would come to have more *feeling* and less *perturbation*. Instead of sympathetic emotions rarely excited, painfully felt, and soon extinguished, they would gradually acquire a calm, lasting, and useful habit of general tenderness and sensibility. The doctrine, therefore, of Aristotle, that tragedy purges the passions would perhaps only amount to this—that the habitual exercise of the passions by works of imagination in general, of the serious and pathetic kind (such as Tragedies, Novels, &c.) has a tendency to soften and refine those passions when excited by real objects in common life.'

This view appears essentially modern. The idea of *softening, refining, correcting,* was certainly not, I think, attached by Aristotle to *καθαίρειν, κάθαρσις* in this relation. In order to explain what, as I think, he did mean, a few prefatory remarks will be needful. Winckelmann observes that the two great characteristics of the Greek ideal, whether in art or in action, are what he calls *Heiterkeit* and *Allgemeinheit;* cheerful self-possession, and generality. The first explains itself; it is repose without sternness or sadness or monotony. The second, *generality*, is to be understood as the very opposite of laxity or vagueness; it means the concentration of impressions into types. This generality is best exemplified in sculpture. There, the character necessarily predominates over the situation. The sculptor has to choose a type which is intrinsically interesting, independently of a special situation or a critical moment: and then he has to present this type in its broad, central, incisive lines*. This he effects, not by accumulating details, but by abstracting from them. All that is accidental, that distracts the simple effect of the supreme types of humanity, all traces in them of the commonness of the world, he gradually purges away.

Now sculpture is not only the most Greek of Greek Arts†, it is the Greek soul and nature itself. 'In its poets and orators,' as Hegel says, 'in its historians and philosophers, Greece cannot be conceived from a central point, unless one brings, as a key to the understanding of it, an insight into the ideal forms of sculpture, and regards the images of statesmen and philosophers as well as epic and dramatic heroes from the artistic point of view; for those who act, as well as those who create and think, have in those beautiful days of Greece this *plastic* character.'

As Sculpture purges away the accidental, the common, the disturbing, so it is that Tragedy and Music *καθαίρουσι τὰ παθήματα*. Tragedy moves pity and terror; and Greek Tragedy moves them by the simple, massive presentation of great issues, from which the vulgar, the spurious, the petty, the maudlin are excluded; it is the conflict of the human will with necessity, it is the antithesis between written and unwritten law, it is the stedfast endurance of suffering incurred in a good cause, it is the duty to Apollo prevailing over the dread of the Furies. The objects or the issues, however grave, which in real

* See Pater, *Studies in the History of the Renaissance*, p. 188. † *ib.* 192.

life move such emotions, have seldom this *netteté*, this clearness of outline and freedom from alloy. What is the ultimate reason of that offence which persons of ordinary cultivation experience when some tragic or horrible occurrence is made the subject of florid comment? It is that such comment is doing deliberately and violently the very reverse of what, according to Aristotle, Tragedy has to do; it is using pity and terror not to clarify but to adulterate the feelings by confounding an essential pathos with its most trivial or repulsive accidents. A living poet and novelist, whose fictions have some moral as well as some artistic affinity with Greek Tragedy, was once asked how these works had been influenced by the study of Sophocles; and the answer was, 'in the delineation of the great primary emotions.'

Music also can move pity or terror or the feelings akin to them:

> And lo, with thee doth rise
> The lord of melodies,
> Sovereign of glorious sound, as thou of form:
> Love, hate, hope, fear, scorn, wrath, defiance, prayer,
> Each at Beethoven's mandate thrills the air,
> The low, sad night-wind or the rushing storm.

And Music, when rightly used, can effect the *katharsis* of these emotions. What Tragedy does in the sphere of action, πρᾶξις, so far idealised as to be represented by great and simple themes, this Music does in the sphere of moral imitation, ἠθικὴ ὁμοίωσις, which is peculiar to it. It gives a scope to the emotions from which everything foreign, turbid' vitiating or perplexing, is banished; it clarifies them; it presents them in a genuine simplicity, and at the same time with a majesty and a power from which even detail which was not impertinent would still necessarily make some detraction.

In order to see distinctly that the modern and perhaps peculiarly English idea of toning down, refining, is not necessarily, and if at all, only in a secondary sense, connected with katharsis, let us look at a striking and suggestive passage in the *Politics* V. (VIII.) 7, § 4. Aristotle has just been saying that Music is valuable for the three things, παιδεία, διαγωγή, κάθαρσις, and has drawn the inference that *all* the modes, ἁρμονίαι, may be advantageously applied for one or another of these purposes; the most ethical harmonies, for παιδεία; the practical and the enthusiastic for *akroasis*—for listening to while other people play *. But why is enthusiastic music thus universally available? Because, he says, the emotion which in some souls occurs vehemently, ἰσχυρῶς, exists originally, ὑπάρχει, in all human souls, in some degree: this is true of pity and terror; this is true also, he adds, of enthusiasm;—just as Wordsworth says, that the poet is a man who has the common sensibilities in a higher degree. 'Some persons,' Aristotle continues, 'are liable to the seizure by this tumult in the soul'—ὑπὸ ταύτης τῆς κινήσεως κατακώχιμοι εἰσίν. Then he gives an illustration. So much depends on the exact rendering of the words here, that I must give the Greek. ἐκ δὲ τῶν ἱερῶν μελῶν ὁρῶμεν τούτους, ὅταν χρήσωνται τοῖς ἐξοργιάζουσι τὴν ψυχὴν μέλεσι, καθισταμένους ὥσπερ ἰατρείας τυχόντας καὶ καθάρσεως. ταὐτὸ δὴ τοῦτο

* We must not alter πρὸς ἀκρόασιν into πρὸς κάθαρσιν as Twining does. Aristotle has used the conveniently general word ἀκρόασις precisely because it includes the two elements of the use πρὸς διαγωγήν—the cultivation of the intelligence, and κάθαρσις.

ἀναγκαῖον πάσχειν καὶ τοὺς ἐλεήμονας καὶ τοὺς φοβητικοὺς καὶ τοὺς ἄλλους παθητικούς, τοὺς δ' ἄλλους καθ' ὅσον ἐπιβάλλει ἕκαστον τούτων, καὶ πᾶσι γίγνεσθαί τινα κάθαρσιν καὶ κουφίζεσθαι μεθ' ἡδονῆς. The point is—Are we to identify *τὰ ἱερὰ μέλη* with *τὰ τὴν ψυχὴν ἐξοργιάζοντα μέλη*? I think it is clear that Aristotle meant to do so; and I render thus:—

'Now, as a result of the sacred chants, we see such persons (viz. those who are susceptible of enthusiasm in the higher degree), when they have experienced that music which brings the soul to a frenzy, becoming their true selves (*καθισταμένους**), having met, as it were, with that which can heal and purify them. The same thing must needs be felt by the compassionate, by the timid, in a word, by the emotional, and by the rest of the world, in such measure as each shares this or that susceptibility. All must experience a purification, a relief attended by pleasure.'

If this is a right version, then the *κάθαρσις*, the *ἰατρεία* of these naturally enthusiastic natures, is nothing else than enthusiasm itself: it consists in their liberation, by the inspiring and elevating power of the music, from all that restrains, obscures, or abuses that divine emotion. It is anything but a toning down or a refinement; it is an exalting, an intensifying influence.

But an objection might occur. Why should not the sentence be translated thus:—'We see such persons, after that they have experienced the orgiastic melodies, brought into their normal state by the sacred melodies'? The *ἱερὰ μέλη* would thus be tranquillizing melodies *following* the *ἐξοργιάζοντα*. Twining, in his introduction to the *Poetics*, is, however, clearly right in identifying the *ἱερά* with the *ἐξοργιάζοντα*. *χρήσωνται* is to be translated by the English perfect†,—'when they *have* used' or experienced: the *κατάστασις*, or pleasurable subsidence of the soul, is a consequence of the orgiastic music.

To sum up:—Where does Aristotle place Music in Education? He says that, for the youthful, it is a discipline both artistic and moral; but differs here from the modern view in requiring such a measure of executive skill as will make the connoisseur a *practical* judge; a view inseparable from the Greek conception of art and art-criticism as part of the complete civic life, but one which has been partly superseded in modern days by the creation of a scientific æsthetic, bringing *principles*, ascertained through practice, within the apprehension of those who have had no practical experience. For the mature, Music is a noble recreation, fitted to develope the intelligence, fitted also to purify the emotions; that is, to detach from the emotions whatever is accident or alloy, and to afford them a field for their clear and essential exercise. And this without loss of intensity—rather, to the heightening of true power. Compare two passages in which a Greek poet and an English poet, eminently *not* Greek, have made a husband and wife converse on a great distress. Compare the parting of Hektor and Andromache in the VIth Book of the *Iliad* with the dialogue between Adam and Eve in the xth Book of *Paradise Lost*. No one would say, I think, that the scene in the *Iliad* was less strenuous, less intense, less passionate, than

* For *καθισταμένους*, cf. Arist. *Rhet.* I. 11, where *ἡδονή* is defined as *κίνησίς τις τῆς ψυχῆς καὶ κατάστασις ἀθρόα καὶ αἰσθητὴ εἰς τὴν ὑπάρχουσαν φύσιν*—'a settling down, sudden and sensible, into our proper nature.'

† The aorist subjunctive, after *ὅταν, ἐπειδάν, ἐπάν*, is to be translated differently in two different cases:—(1) when the leading verb is *future*, by our future perfect: *ἐπειδὰν τοῦτο ἴδω, ἐλεύσομαι*: when I *shall have* seen this, I shall come: (2) when the leading verb, being in the present indicative, denotes a *general* truth, by our *perfect*: *ὅταν τοῦτο ἴδω, ἀπέρχομαι*: when I *have* seen this, I (always) depart. Cf. Goodwin, *Greek Moods and Tenses*, p. 26.

the other. But the *κάθαρσις παθημάτων* is effected by the Greek as it surely is not by the English, and why? Because, in the *Iliad*, the type is not merged in the person; because cheerful repose, and generality or typical concentration, are so perfectly preserved in the Greek*.

How is Aristotle's view of Music related to Plato's? Much has been said of the 'antagonism' between Aristotle and Plato as to music; but I cannot help thinking that the commentators have confused the thing by dwelling on trivial differences of detail. The most definite utterance in Plato as to the *general* power of music is in the *Republic* III. pp. 401, 402: where he says, that musical training is so powerful 'because rhythm and harmony find their way into the secret places of the soul, on which they mightily fasten, bearing grace in their movements, and making his soul graceful who is rightly educated, and his ungraceful who is ill-trained; and also, because he who has received this true education of the inner being will most shrewdly perceive omissions or faults in art or nature, and with a true taste, while he praises and rejoices over, and receives into his soul the good, and becomes noble and good, he will justly blame and hate the bad, now in the days of his youth, even before he is able to know the reason of the thing; and when reason comes he will recognise and salute her as a friend with whom his education has made him long familiar.'

That love for the beautiful which is engendered by Music is, for Plato, an introduction to the more divine Erôs for the Ideas. Here, then, we have a clue to the essential difference, so far as there is one, between the Platonic and the Aristotelian view of Music. Plato connects music rather with religion, Aristotle with art; Plato regards it as a means of evoking, Aristotle of defining enthusiasm. Aristotle's remark that Plato is wrong in allowing the Phrygian along with the Dorian mode, while he excludes that instrument, namely, the flute, which goes best with the enthusiastic music, seems to show a certain insensibility to Plato's spirit†. The flute is the instrument which, in Greek music, marked the divergence of *πάθος* from *μάθησις*: the use of Music, according to Plato, was to enlist a divine *πάθος* in the service of a divine *μάθησις*.

There are phenomena of the present day which forcibly suggest the importance of Aristotle's view regarding the *universal* moral importance of Music as an element in education,—I mean the ecstatic movements, whatever special form they may take, and the passion for excitement. What are these but instances of repressed, untrained, and therefore ungovernable sensibilities taking the first opening that is offered to them? Experiment has shown what Music, when it is good enough, can do in educating such sensibilities. There is one point, however, on which the modern world would differ from Aristotle—the only important one. The *principles* of music now rest on a really scientific basis, not on grounds fully possessed only by special experts, and merely touched by philosophers as part of a wider domain. Again, the generalisations of æsthetic give critics who are not even specially musicians a stand-point of their own. Aristotle's plea, therefore, that it is necessary *χειρουργεῖν* in order *κριτὰς τῶν ἔργων εἶναι σπουδαίους* has no longer its Greek validity. In all else, the modern world may still, perhaps, not scorn to hear the old.

* (i) *P. L.* x. 720—844, 914—936: (ii) *Iliad*, vi. 475—571.

† Arist. *Polit.* v. (viii.) vi. § 9.

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I. *Exercises in Curvilinear and Normal Co-ordinates. By the* REV. JAMES WILLIAM WARREN, *Caius College.*

[Read *May* 7, 1877.]

EXERCISE THE FOURTH.

NORMAL CO-ORDINATES.

LET us suppose that the equations of three surfaces each contain such suitable parameters that when these parameters change in value each of the three surfaces may be made continuously to pass through all space. Clearly the Cartesian co-ordinates of any point in space may be supposed to be expressed in terms of the equations of three such surfaces.

Now imagine two distinct systems of three such surfaces, call one the system u_1, u_2, u_3, and the second the system U_1, U_2, U_3, we shall denote functions of u_1, u_2, u_3 and similar functions of U_1, U_2, U_3 by the same letters, but we shall dash these letters in the case of U_1, U_2, U_3, thus we shall have such corresponding symbols as C_{11}, C_{11}', &c.

It is clear that *in general* a surface of the system, u_1 say, will not coincide with any surface of the system U_1, but we shall now suppose the surfaces u_1, u_2, u_3 and U_1, U_2, U_3 to be such that one surface of the system u_1 coincides with one surface of the system U_1, and this common surface we shall for the sake of conciseness call the surface (U_1 and u_1). We make the same hypothesis for the surfaces u_2 and U_2, u_3 and U_3, so that we have the surfaces (U_2 and u_2) and (U_3 and u_3).

It is clear that in general U_1 is a function of u_1, u_2, u_3, whilst u_1 is a function of U_1, U_2, and U_3: the same remark is of course true for U_2 and U_3 and u_2, u_3; this being the case we clearly have

$$du_1 = \frac{du_1}{dU_1} dU_1 + \frac{du_1}{dU_2} dU_2 + \frac{du_1}{dU_3} dU_3,$$

$$\frac{d}{dU_1} = \frac{du_1}{dU_1} \frac{d}{du_1} + \frac{du_2}{dU_1} \frac{d}{du_2} + \frac{du_3}{dU_1} \frac{d}{du_3}.$$

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Suppose now the surfaces u_1 and U_1 to take up the position (U_1 and u_1), then clearly we have all over the surface (U_1 and u_1) both $du_1 = 0$ and $dU_1 = 0$; therefore since the relative value of the differentials dU_2 and dU_3 are quite arbitrary, we clearly have all over (U_1 and u_1) both $\frac{du_1}{dU_2} = 0$ and $\frac{du_1}{dU_3} = 0$, and hence clearly we have at the point in space where the three surfaces (U_1 and u_1), (U_2 and u_2), (U_3 and u_3) intersect (which point for shortness we may call the "*common point*"),

$$\frac{du_1}{dU_2} = 0, \ \frac{du_1}{dU_3} = 0, \ \frac{du_2}{dU_3} = 0, \ \frac{du_2}{dU_1} = 0, \ \frac{du_3}{dU_1} = 0, \ \frac{dU_3}{dU_2} = 0,$$

$$\frac{d^2u_1}{dU_2^2} = 0, \ \frac{d^2u_1}{dU_2 dU_3} = 0, \ \&\text{c.},$$

$$\frac{d^3u_1}{dU_3^3} = 0, \ \&\text{c.},$$

$$\frac{d}{dU_1} = \frac{du_1}{dU_1}\frac{d}{du_1}; \ \frac{d}{dU_2} = \frac{du_2}{dU_2}\frac{d}{du_2}; \ \frac{d}{dU_3} = \frac{du_3}{dU_3}\frac{d}{du_3} \quad \ldots\ldots\ldots\ldots (1).$$

It is clear however that we have no reason for asserting that at the *common point* any one of the *nine* quantities $\frac{d^2u_1}{dU_1^2}$, $\frac{d^2u_1}{dU_1 dU_2}$, &c. are zero, in fact it is not true that this is the case, and one of the first things we shall do is to express these nine quantities as linear functions of the nine differential coefficients $\frac{dA_{11}}{du_1}$, $\frac{dA_{11}}{du_2}$, &c.

The notation of this Exercise is the same as that used in my first three Exercises; with the following additions, we write

$$\frac{A_{23}}{(A_{22}A_{33})^{\frac{1}{2}}} = \Lambda_1; \ \frac{A_{31}}{(A_{33}A_{11})^{\frac{1}{2}}} = \Lambda_2; \ \frac{A_{12}}{(A_{11}A_{22})^{\frac{1}{2}}} = \Lambda_3.$$

$$A_{11}C_{11} = 2\Sigma_1; \quad A_{22}C_{22} = 2\Sigma_2; \ A_{33}C_{33} = 2\Sigma_3.$$

$$(A_{22}A_{33})^{\frac{1}{2}} . C_{23} = \Pi_{23}; \ (A_{33}A_{11})^{\frac{1}{2}} . C_{31} = \Pi_{31}; \ (A_{11}A_{22})^{\frac{1}{2}} . C_{12} = \Pi_{12}.$$

$$\begin{vmatrix} 2\Sigma_1 & \Pi_{12} & \Pi_{13} \\ \Pi_{12} & 2\Sigma_2 & \Pi_{23} \\ \Pi_{13} & \Pi_{23} & 2\Sigma_3 \end{vmatrix} = A^2, \qquad \begin{vmatrix} 1 & \Lambda_3 & \Lambda_2 \\ \Lambda_3 & 1 & \Lambda_1 \\ \Lambda_2 & \Lambda_1 & 1 \end{vmatrix} = \frac{1}{A^2}.$$

The same symbols dashed will denote the same functions with regard to the dashed or U system of co-ordinates, and it is clear that at the "*common point*" we shall have $\Lambda_1 = \Lambda_1'$, &c., $\Sigma_1 = \Sigma_1'$, $\Pi_1 = \Pi_1'$, &c., $A = A'$. When the dashed or U system of co-ordinates are normal co-ordinates (as we shall generally suppose them to be), we write in place of Λ_1', Σ_1', Π_{23}', &c., the letters A_1, S_1, P_{23} of my former Exercises, we then have of course at the "*common point*" $A_1 = \Lambda_1$, &c.; where however it must always be carefully borne in mind that A_1, S_1, P_{23}, &c., and the *normal* B are supposed to be expressed in terms of v_1, v_2, v_3, whilst the ten quantities that are strictly equal to them at the common point, *i.e.* Λ_1, Σ_1, Π_{23}, &c., A are *always* supposed to be expressed in terms of u_1, u_2, and u_3.

We shall also represent the three operating symbols $\frac{du_1}{dn_1}\frac{d}{du_1}$, $\frac{du_2}{dn_2}\frac{d}{du_2}$, $\frac{du_3}{dn_3}\frac{d}{du_3}$, (supposed to operate always *on functions expressed in terms of* u_1, u_2, u_3, *never of* n_1, n_2, n_3) by the symbols of abbreviation $\frac{d}{dv_1}$, $\frac{d}{dv_2}$, $\frac{d}{dv_3}$; these symbols, mind, are mere symbols of abbreviation, and wherever we see one of them it is always supposed to be a mere representative symbol for $\frac{du_1}{dn_1}\frac{d}{du_1}$, &c. The object of this Exercise may be now clearly stated, it is to determine if possible the *thirty* quantities $\frac{dS_1}{dn_1}-\frac{d\Sigma_1}{dv_1}$, $\frac{dA_1}{dn_1}-\frac{d\Lambda_1}{dv_1}$, &c., $\frac{dB}{dn_1}-\frac{dA}{dv_1}$ as *linear* functions of the *nine* quantities $\frac{dA_{11}}{du_1}$, &c.; and if we can find *thirty* such equations true at the "*common point*" it is tolerably clear that the general equations of normal co-ordinates can be immediately transformed into the general equations of Curvilinear Co-ordinates, and that in both cases they are similar in form.

A few more matters with regard to notation only need to be explained; we write, for the sake of conciseness,

$$\left.\begin{aligned}
A_{11}\frac{dC_{11}}{du_1}+A_{12}\left(2\frac{dC_{12}}{du_1}-\frac{dC_{11}}{du_2}\right)+A_{13}\left(2\frac{dC_{31}}{du_1}-\frac{dC_{11}}{du_3}\right)&=2(K_{11})_1\\
A_{12}\frac{dC_{11}}{du_1}+A_{22}\left(2\frac{dC_{12}}{du_1}-\frac{dC_{11}}{du_2}\right)+A_{23}\left(2\frac{dC_{31}}{du_1}-\frac{dC_{11}}{du_3}\right)&=2(K_{11})_2\\
A_{31}\frac{dC_{11}}{du_1}+A_{32}\left(2\frac{dC_{12}}{du_1}-\frac{dC_{11}}{du_2}\right)+A_{33}\left(2\frac{dC_{31}}{du_1}-\frac{dC_{11}}{du_3}\right)&=2(K_{11})_3
\end{aligned}\right\}\dots\dots\dots(2),$$

$$\left.\begin{aligned}
A_{11}\left(\frac{dC_{31}}{du_2}+\frac{dC_{12}}{du_3}-\frac{dC_{23}}{du_1}\right)+A_{12}\frac{dC_{22}}{du_3}+A_{13}\frac{dC_{33}}{du_2}&=2(K_{23})_1\\
A_{21}\left(\frac{dC_{31}}{du_2}+\frac{dC_{12}}{du_3}-\frac{dC_{23}}{du_1}\right)+A_{22}\frac{dC_{22}}{du_3}+A_{23}\frac{dC_{33}}{du_2}&=2(K_{23})_2\\
A_{31}\left(\frac{dC_{31}}{du_2}+\frac{dC_{12}}{du_3}-\frac{dC_{23}}{du_1}\right)+A_{32}\frac{dC_{22}}{du_3}+A_{33}\frac{dC_{33}}{du_2}&=2(K_{23})_3
\end{aligned}\right\}\dots\dots\dots(3),$$

Twelve more equations exist similar to (2) and (3), and of course we have eighteen similar equations dashed. Lastly let us write

$$\left.\begin{aligned}
du_1&=P_{11}\,.\,dn_1+P_{12}\,.\,dn_2+P_{13}\,.\,dn_3\\
du_2&=P_{21}\,.\,dn_1+P_{22}\,.\,dn_2+P_{23}\,.\,dn_3\\
du_3&=P_{31}\,.\,dn_1+P_{32}\,.\,dn_2+P_{33}\,.\,dn_3
\end{aligned}\right\}\dots\dots\dots(4);$$

we have, by what was already stated, at the "*common point*" $P_{12}=0$, $P_{13}=0$, $P_{23}=0$, $P_{31}=0$, $P_{21}=0$, $P_{32}=0$, and it only requires a little consideration to see that we also have at the *common point*

$$\left.\begin{aligned} P_{11} &= A_{11}^{\frac{1}{2}} \\ P_{22} &= A_{22}^{\frac{1}{2}} \\ P_{33} &= A_{33}^{\frac{1}{2}} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Observe also that the ten quantities Λ_1, Σ_1, Π_{23}, &c., A are all functions of angles.

Our system of notation being thus all clearly explained, I now go on to the direct object of this exercise. I divide what follows into sections for the sake of clearness; and our first section will have for its object the problem to express the nine quantities $\frac{d^2u_1}{dn_1^2}$, $\frac{d^2u_1}{dn_1 dn_2}$, &c. linearly in terms of the nine quantities $\frac{dA_{11}}{dv_1}$, $\frac{dA_{11}}{dv_2}$, &c., where, as already explained, $\frac{d}{dv_1}$, &c. are mere abbreviations.

I.

$$\left\{\frac{d^2u_1}{dn_1^2},\ \frac{d^2u_1}{dn_1 dn_2}, \text{ \&c. and } \frac{dA_{11}}{dv_1},\ \frac{dA_{11}}{dv_2}, \text{ \&c.}\right\}.$$

V being any function of u_1, u_2, u_3, and therefore of n_1, n_2, n_3, we in "Exercise the first" have defined the letters A_{11}, A_{12}, &c., or A_{11}', A_{12}', &c., thus

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = A_{11}\left(\frac{dV}{du_1}\right) + 2A_{23}\frac{dV}{du_2}\frac{dV}{du_3} + \text{\&c.};$$

hence we clearly must have

$$A_{11}\left(\frac{dV}{du_1}\right)^2 + 2A_{23}\frac{dV}{du_2}\frac{dV}{du_3} + \text{\&c.} = \left(\frac{dV}{dn_1}\right)^2 + 2A_1\frac{dV}{dn_2}\frac{dV}{dn_3} + \text{\&c.} \ldots\ldots\ldots(6).$$

In Equation (6) write V equal to u_1, or u_2, or u_3, and we immediately obtain three equations, one of which is

$$A_{11} = \left(\frac{du_1}{dn_1}\right)^2 + \left(\frac{du_1}{dn_2}\right)^2 + \left(\frac{du_1}{dn_3}\right)^2 + 2A_1\frac{du_1}{dn_2}\frac{du_1}{dn_3} + 2A_2\frac{du_1}{dn_1}\frac{du_1}{dn_3} + 2A_3\frac{du_1}{dn_1}\frac{du_1}{dn_2} \ldots(7);$$

differentiate now both sides of the equation (7) in succession with regard to n_1, n_2, and n_3, we easily obtain

$$\frac{dA_{11}}{du_1} = 2\frac{d^2u_1}{dn_1^2} + 2A_2\frac{d^2u_1}{dn_1 dn_3} + 2A_3\frac{d^2u_1}{dn_1 dn_2},$$

$$\frac{dA_{11}}{du_2} = 2\left(\frac{A_{11}}{A_{22}}\right)^{\frac{1}{2}} \cdot \frac{d^2u_1}{dn_1 dn_2},$$

$$\frac{dA_{11}}{du_3} = 2\left(\frac{A_{11}}{A_{33}}\right)^{\frac{1}{2}} \cdot \frac{d^2u_1}{dn_1 dn_3}.$$

We can clearly therefore write down the following nine equations:

$$\left.\begin{aligned}
&\frac{d^2u_1}{dn_1^{\,2}}=\frac{1}{2}\left\{\frac{dA_{11}}{du_1}-\frac{A_{12}}{A_{11}}\frac{dA_{11}}{du_2}-\frac{A_{13}}{A_{11}}\frac{dA_{11}}{du_3}\right\},\\
&\frac{d^2u_2}{dn_2^{\,2}}=\frac{1}{2}\left\{\frac{dA_{22}}{du_2}-\frac{A_{23}}{A_{22}}\frac{dA_{22}}{du_3}-\frac{A_{12}}{A_{22}}\frac{dA_{22}}{du_1}\right\},\\
&\frac{d^2u_3}{dn_3^{\,2}}=\frac{1}{2}\left\{\frac{dA_{33}}{du_3}-\frac{A_{31}}{A_{33}}\frac{dA_{33}}{du_1}-\frac{A_{32}}{A_{33}}\frac{dA_{33}}{du_2}\right\};\\
&\frac{d^2u_1}{dn_1\,dn_2}=\frac{1}{2}\left(\frac{A_{22}}{A_{11}}\right)^{\frac{1}{2}}\cdot\frac{dA_{11}}{du_2};\qquad \frac{d^2u_1}{dn_1\,dn_3}=\frac{1}{2}\left(\frac{A_{33}}{A_{11}}\right)^{\frac{1}{2}}\cdot\frac{dA_{11}}{du_3},\\
&\frac{d^2u_2}{dn_2\,dn_3}=\frac{1}{2}\left(\frac{A_{33}}{A_{22}}\right)^{\frac{1}{2}}\cdot\frac{dA_{22}}{du_3};\qquad \frac{d^2u_2}{dn_1\,dn_2}=\frac{1}{2}\left(\frac{A_{11}}{A_{22}}\right)^{\frac{1}{2}}\cdot\frac{dA_{22}}{du_1},\\
&\frac{d^2u_3}{dn_3\,dn_1}=\frac{1}{2}\left(\frac{A_{11}}{A_{33}}\right)^{\frac{1}{2}}\cdot\frac{dA_{33}}{du_1};\qquad \frac{d^2u_3}{dn_3\,dn_2}=\frac{1}{2}\left(\frac{A_{22}}{A_{33}}\right)^{\frac{1}{2}}\cdot\frac{dA_{33}}{du_2}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(8).$$

Thus we have accomplished the object of this article.

I now proceed to article the second.

II.

$$\left\{\left(\frac{dA_1}{dn_1}-\frac{d\Lambda_1}{d\nu_1}\right);\quad\left(\frac{dA_1}{dn_2}-\frac{d\Lambda_1}{d\nu_2}\right),\ \text{\&c. linearly in terms of }\frac{dA_{11}}{d\nu_1},\ \text{\&c.}\right\}.$$

Let us in equation (6) write V equal to $u_1-\lambda\,.\,u_2$ where λ is any indeterminate, then equate the coefficients of λ on both sides, we instantly obtain

$$\begin{aligned}
A_{23}=&\frac{du_2}{dn_1}\frac{du_3}{dn_1}+\frac{du_2}{dn_2}\frac{du_3}{dn_2}+\frac{du_2}{dn_3}\frac{du_3}{dn_3}\\
&+A_1\left\{\frac{du_2}{dn_2}\frac{du_3}{dn_3}+\frac{du_2}{dn_3}\frac{du_3}{dn_2}\right\}\\
&+A_2\left\{\frac{du_2}{dn_1}\frac{du_3}{dn_3}+\frac{du_3}{dn_1}\frac{du_2}{dn_3}\right\}\\
&+A_3\left\{\frac{du_2}{dn_2}\frac{du_3}{dn_1}+\frac{du_2}{dn_1}\frac{du_3}{dn_2}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).
\end{aligned}$$

Now differentiate both sides of this equation (9) in succession with regard to n_1, n_2 and n_3, and then suppose that we are at the "common point," we clearly obtain

$$\frac{du_1}{dn_1}\frac{dA_{23}}{du_1}=\frac{du_2}{dn_2}\frac{du_3}{dn_3}\frac{dA_1}{dn_1}+A_1\frac{du_2}{dn_2}\frac{d^2u_3}{dn_1dn_3}+A_1\frac{du_3}{dn_3}\frac{d^2u_2}{dn_1dn_2},$$

or substituting from system (8),

$$\frac{dA_1}{dn_1}=\left(\frac{A_{11}}{A_{22}A_{33}}\right)^{\frac{1}{2}}\frac{dA_{23}}{du_1}-\frac{A_{23}}{A_{22}A_{33}}\left\{\frac{1}{2}\left(\frac{A_{11}A_{22}}{A_{33}}\right)^{\frac{1}{2}}\frac{dA_{33}}{du_1}+\frac{1}{2}\left(\frac{A_{11}A_{33}}{A_{22}}\right)^{\frac{1}{2}}\frac{dA_{22}}{du_1}\right\};$$

therefore clearly we have

$$\frac{dA_1}{dn_1}=A_{11}{}^{\frac{1}{2}}\frac{d}{du_1}\left\{\frac{A_{23}}{(A_{22}A_{33})^{\frac{1}{2}}}\right\},$$

or finally,

$$\frac{dA_1}{dn_1}=\frac{d\Lambda_1}{dv_1}.$$

....

We also clearly have

$$\frac{du_2}{dn_2}\frac{dA_{23}}{du_2}=\frac{du_2}{dn_2}\frac{du_3}{dn_3}\frac{dA_1}{dn_2}+\frac{du_3}{dn_3}\cdot\frac{d^2u_2}{dn_2dn_3}+A_2\frac{du_3}{dn_3}\frac{d^2u_2}{dn_1dn_2}$$

$$+A_1\left\{\frac{du_2}{dn_2}\frac{d^2u_3}{dn_2dn_3}+\frac{du_3}{dn_3}\frac{d^2u_2}{dn_2{}^2}\right\},$$

or substituting from system (8),

$$\frac{dA_1}{dn_2}=\frac{1}{A_{33}{}^{\frac{1}{2}}}\frac{dA_{23}}{du_2}-\frac{A_{33}{}^{\frac{1}{2}}}{2A_{22}}\frac{dA_{22}}{du_3}-\frac{A_{31}}{2A_{22}\,.\,A_{33}{}^{\frac{1}{2}}}\frac{dA_{22}}{du_1}$$

$$-\frac{A_{23}}{2A_{22}\,.\,A_{33}}\left\{\frac{A_{22}}{A_{33}{}^{\frac{1}{2}}}\frac{dA_{23}}{du_2}+A_{33}{}^{\frac{1}{2}}\left(\frac{dA_{22}}{du_2}-\frac{A_{23}}{A_{22}}\frac{dA_{22}}{du_3}-\frac{A_{12}}{A_{22}}\frac{dA_{22}}{du_1}\right)\right\},$$

and this clearly may be written in the form

$$\frac{dA_1}{dn_2}=A_{22}{}^{\frac{1}{2}}\frac{d}{du_2}\left\{\frac{A_{23}}{(A_{22}A_{33})^{\frac{1}{2}}}\right\}+\frac{1}{2A_{22}{}^2A_{33}{}^{\frac{1}{2}}}\left\{(A_{12}A_{23}-A_{31}A_{22})\frac{dA_{22}}{du_1}+(A_{23}{}^2-A_{22}A_{33})\frac{dA_{22}}{du_3}\right\};$$

but by the formulæ given in Exercise the first,

$$(A_{12}A_{23}-A_{31}A_{22})=\frac{C_{31}}{B^2}=\frac{A_{11}\,.\,A_{22}\,.\,A_{33}\,.\,C_{31}}{A^2}=\frac{A_{11}{}^{\frac{1}{2}}\,.\,A_{33}{}^{\frac{1}{2}}\,.\,A_{22}\,.\,\Pi_{31}}{A^2},$$

$$(A_{23}{}^2-A_{22}\,.\,A_{33})=\frac{-C_{11}}{B^2}=\frac{-A_{11}\,.\,A_{22}\,.\,A_{33}\,.\,C_{11}}{A^2}=\frac{-2\Sigma_1\,.\,A_{22}\,.\,A_{33}}{A^2};$$

so that clearly we may write

$$\frac{dA_1}{dn_2}=\frac{d\Lambda_1}{dv_2}+\frac{1}{2\,.\,A^2\,.\,A_{22}}\cdot\left\{\Pi_{31}\frac{dA_{22}}{dv_1}-2\Sigma_1\frac{dA_{22}}{dv_3}\right\}.$$

Hence by symmetry we may obviously write down the nine formulæ

$$\left.\begin{aligned}
\frac{dA_1}{dn_1} &= \frac{d\Lambda_1}{d\nu_1}, \\
\frac{dA_2}{dn_2} &= \frac{d\Lambda_2}{d\nu_2}, \\
\frac{dA_3}{dn_3} &= \frac{d\Lambda_3}{d\nu_3}, \\
\frac{dA_1}{dn_2} &= \frac{d\Lambda_1}{d\nu_2} + \frac{1}{2A^2 \,.\, A_{22}} \,.\, \left\{\Pi_{13}\frac{dA_{22}}{d\nu_1} - 2\Sigma_1\frac{dA_{22}}{d\nu_3}\right\}, \\
\frac{dA_1}{dn_3} &= \frac{d\Lambda_1}{d\nu_3} + \frac{1}{2A^2 \,.\, A_{33}} \,.\, \left\{\Pi_{12}\frac{dA_{33}}{d\nu_1} - 2\Sigma_1\frac{dA_{33}}{d\nu_2}\right\}, \\
\frac{dA_2}{dn_3} &= \frac{d\Lambda_2}{d\nu_3} + \frac{1}{2A^2 \,.\, A_{33}} \,.\, \left\{\Pi_{12}\frac{dA_{33}}{d\nu_2} - 2\Sigma_2\frac{dA_{33}}{d\nu_1}\right\}, \\
\frac{dA_2}{dn_1} &= \frac{d\Lambda_2}{d\nu_1} + \frac{1}{2A^2 \,.\, A_{11}} \,.\, \left\{\Pi_{23}\frac{dA_{11}}{d\nu_2} - 2\Sigma_2\frac{dA_{11}}{d\nu_3}\right\}, \\
\frac{dA_3}{dn_1} &= \frac{d\Lambda_3}{d\nu_1} + \frac{1}{2A^2 \,.\, A_{11}} \,.\, \left\{\Pi_{23}\frac{dA_{11}}{d\nu_3} - 2\Sigma_3\frac{dA_{11}}{d\nu_2}\right\}, \\
\frac{dA_3}{dn_2} &= \frac{d\Lambda_3}{d\nu_2} + \frac{1}{2A^2 \,.\, A_{22}} \,.\, \left\{\Pi_{13}\frac{dA_{22}}{d\nu_3} - 2\Sigma_3\frac{dA_{22}}{d\nu_1}\right\}
\end{aligned}\right\} \quad \text{..............(10)}.$$

We have thus accomplished the object proposed in the second section, and accordingly go on to third section.

III.

$$\left\{\left(\frac{dS_1}{dn_1} - \frac{d\Sigma_1}{d\nu_1}\right); \quad \left(\frac{dP_{23}}{dn_1} - \frac{d\Pi_{23}}{d\nu_1}\right), \text{\&c. linearly in terms of } \frac{dA_{11}}{d\nu_1}, \text{\&c.}\right\}.$$

The operating symbols $\dfrac{d^2}{dn_1\, dn_2}$ and

$$\left(P_{11}\frac{d}{du_1} + P_{21}\frac{d}{du_2} + P_{31}\frac{d}{du_3}\right)\left(P_{12}\frac{d}{du_1} + P_{22}\frac{d}{du_2} + P_{32}\frac{d}{du_3}\right)$$

are clearly equivalent. Hence it is easy to deduce that at the common point the following operational formulæ are true,

$$\frac{d^2}{dn_1^{\,2}} = A_{11}\frac{d^2}{du_1^{\,2}} + \frac{d^2u_1}{dn_1^{\,2}}\,\frac{d}{du_1} \quad \text{.....................................(11)},$$

$$\frac{d^2}{dn_1\, dn_2} = (A_{11}A_{22})^{\frac{1}{2}}\,\frac{d^2}{du_1\, du_2} + \frac{d^2u_1}{dn_1\, dn_2} \,.\, \frac{d}{du_1} + \frac{d^2u_2}{dn_1\, dn_2}\,\frac{d}{du_2} \quad \text{........................(12)}.$$

Operate with these symbols on the Cartesian co-ordinates of a point and substitute in such expressions as

$$\frac{dx}{dn_1}\frac{d^2x}{dn_1^2}+\frac{dy}{dn_1}\frac{d^2y}{dn_1^2}+\frac{dz}{dn_1}\frac{d^2z}{dn_1^2},$$

$$\frac{dx}{dn_1}\frac{d^2x}{dn_1\,dn_2}+\frac{dy}{dn_1}\frac{d^2y}{dn_1\,dn_2}+\frac{dz}{dn_1}\frac{d^2z}{dn_1\,dn_2},$$

we easily thus deduce the four following typical equations:

$$\frac{dS_1}{dn_1}=\frac{A_{11}^{\frac{3}{2}}}{2}\frac{dC_{11}}{du_1}+A_{11}^{\frac{1}{2}}C_{11}\frac{d^2u_1}{dn_1^2}\quad\text{.......(13)},$$

$$\frac{dP_{12}}{dn_1}-\frac{dS_1}{dn_2}=A_{11}\,.\,A_{22}^{\frac{1}{2}}\left(\frac{dC_{12}}{du_1}-\frac{1}{2}\frac{dC_{11}}{du_2}\right)+A_{22}^{\frac{1}{2}}\,.\,C_{12}\frac{d^2u_1}{dn_1^2}\quad\text{.......(14)},$$

$$\frac{dS_1}{dn_2}=A_{11}\,.\,A_{22}^{\frac{1}{2}}\,.\,\frac{1}{2}\frac{dC_{11}}{du_2}+A_{11}^{\frac{1}{2}}\left\{C_{11}\frac{d^2u_1}{dn_1\,dn_2}+C_{12}\frac{d^2u_2}{dn_1\,dn_2}\right\}\quad\text{.......(15)},$$

$$\frac{dP_{12}}{dn_3}+\frac{dP_{13}}{dn_2}-\frac{dP_{23}}{dn_1}=(A_{11}\,A_{22}A_{33})^{\frac{1}{2}}\,.\left(\frac{dC_{12}}{du_3}+\frac{dC_{13}}{du_2}-\frac{dC_{23}}{du_1}\right)+A_{11}^{\frac{1}{2}}\left\{C_{12}\frac{d^2u_2}{dn_2\,dn_3}+C_{13}\frac{d^2u_3}{dn_2\,dn_3}\right\}\text{.....(16)}.$$

Now first let us consider formulæ (13). Suppose that we substitute in it for $\frac{d^2u_1}{dn_1^2}$ from the group of formulæ (8), we clearly obtain

$$\frac{dS_1}{dn_1}=\frac{A_{11}^{\frac{1}{2}}}{2}\left\{A_{11}\frac{dC_{11}}{du_1}+C_{11}\frac{dA_{11}}{du_1}-\frac{C_{11}A_{12}}{A_{11}}\frac{dA_{11}}{du_2}-\frac{C_{11}A_{31}}{A_{11}}\frac{dA_{11}}{du_3}\right\},$$

which clearly may be written in the form

$$\frac{dS_1}{dn_1}=\frac{d\Sigma_1}{dv_1}-\frac{C_{11}}{2}\left\{\Lambda_3\frac{dA_{11}}{dv_2}+\Lambda_2\frac{dA_{11}}{dv_3}\right\}\quad\text{.......(17)}.$$

Again, let us consider the formula (15) and substitute from the group of formulæ (8), we clearly obtain

$$\frac{dS_1}{dn_2}=\frac{1}{2}A_{11}\frac{dC_{11}}{dv_2}+\frac{1}{2}C_{11}\frac{dA_{11}}{dv_2}+\frac{C_{12}\,.\,A_{11}^{\frac{1}{2}}}{2A_{22}^{\frac{1}{2}}}\frac{dA_{22}}{dv_1};$$

so that clearly we have

$$\frac{dS_1}{dn_2}=\frac{d\Sigma_1}{dv_2}+\frac{\Pi_{12}}{2A_{22}}\,.\,\frac{dA_{22}}{dv_1}\quad\text{.......(18)}.$$

Again, from the formula (16) and its analogues we can easily deduce the result

$$\frac{dP_{12}}{dn_3}=(A_{11}\,.\,A_{22}\,.\,A_{33})^{\frac{1}{2}}\frac{dC_{12}}{du_3}$$

$$+A_{11}^{\frac{1}{2}}\left\{C_{12}\frac{d^2u_2}{dn_2dn_3}+C_{13}\frac{d^2u_3}{dn_2dn_3}\right\}+A_{22}^{\frac{1}{2}}\left\{C_{12}\frac{d^2u_1}{dn_1dn_3}+C_{23}\frac{d^2u_3}{dn_1dn_3}\right\};$$

and making use of formula (8) as in previous two cases we easily obtain the result

$$\frac{dP_{12}}{dn_3}=\frac{d\Pi_{12}}{dv_3}+\frac{1}{2A_{33}}\left\{\Pi_{13}\frac{dA_{33}}{dv_2}+\Pi_{23}\frac{dA_{33}}{dv_1}\right\}\quad\text{.......(19)}.$$

Lastly, suppose that we take the sum of formulæ (14) and (15), we then clearly obtain

$$\frac{dP_{12}}{dn_1} = A_{11} \cdot A_{22}^{\frac{1}{2}} \cdot \frac{dC_{12}}{du_1} + A_{22}^{\frac{1}{2}} \cdot C_{12} \cdot \frac{d^2u_1}{dn_1^{\ 2}} + A_{11}^{\frac{1}{2}} C_{12} \frac{d^2u_2}{dn_1 dn_2} + A_{11}^{\frac{1}{2}} C_{11} \frac{d^2u_1}{dn_1 dn_2};$$

hence substituting from the group of formulæ (8) we clearly obtain that

$$\frac{dP_{12}}{dn_1} = (A_{11} \cdot A_{22})^{\frac{1}{2}} \cdot \frac{dC_{12}}{dv_1} + C_{12} \cdot \frac{1}{2} \left\{ A_{22}^{\frac{1}{2}} \frac{dA_{11}}{du_1} - \frac{A_{12}}{A_{11}} \frac{dA_{11}}{dv_2} - \frac{A_{22}^{\frac{1}{2}} A_{13}}{A_{11} \cdot A_{33}^{\frac{1}{2}}} \frac{dA_{11}}{dv_3} \right\}$$

$$+ C_{12} \cdot \frac{1}{2} \left\{ \left(\frac{A_{11}}{A_{22}} \right)^{\frac{1}{2}} \frac{dA_{22}}{dv_1} \right\} + C_{11} \cdot \frac{1}{2} \left\{ \frac{dA_{11}}{dv_2} \right\},$$

hence clearly we obtain that

$$\frac{dP_{12}}{dn_1} = \frac{d\Pi_{12}}{dv_1} + \frac{1}{2A_{11}} \left\{ (2\Sigma_1 - \Lambda_3 \Pi_{12}) \frac{dA_{11}}{dv_2} - \Lambda_2 \Pi_{12} \frac{dA_{11}}{dv_3} \right\} \dots\dots\dots\dots\dots (20).$$

It is clear that the group of formulæ of which (17)...(20) form a part are eighteen in number; it is perhaps needless to write them all down, but I shall give six for the sake of reference, and the others can be formed by a cyclic change of suffixes.

$$\left.\begin{aligned}
\frac{dS_1}{dn_1} &= \frac{d\Sigma_1}{dv_1} - \frac{\Sigma_1}{A_{11}} \cdot \left\{ \Lambda_2 \cdot \frac{dA_{11}}{dv_3} + \Lambda_3 \cdot \frac{dA_{11}}{dv_2} \right\} \\
\frac{dS_1}{dn_2} &= \frac{d\Sigma_1}{dv_2} + \frac{1}{2A_{22}} \cdot \left\{ \Pi_{12} \cdot \frac{dA_{22}}{dv_1} \right\} \\
\frac{dS_1}{dn_3} &= \frac{d\Sigma_1}{dv_3} + \frac{1}{2A_{33}} \cdot \left\{ \Pi_{13} \cdot \frac{dA_{33}}{dv_1} \right\} \\
\frac{dP_{23}}{dn_1} &= \frac{d\Pi_{23}}{dv_1} + \frac{1}{2A_{11}} \cdot \left\{ \Pi_{12} \cdot \frac{dA_{11}}{dv_3} + \Pi_{13} \cdot \frac{dA_{11}}{dv_2} \right\} \\
\frac{dP_{23}}{dn_2} &= \frac{d\Pi_{23}}{dv_2} + \frac{1}{2A_{22}} \cdot \left\{ (2\Sigma_2 - \Lambda_1 \Pi_{23}) \cdot \frac{dA_{22}}{dv_3} - \Lambda_3 \Pi_{23} \cdot \frac{dA_{22}}{dv_1} \right\} \\
\frac{dP_{23}}{dn_3} &= \frac{d\Pi_{23}}{dv_3} + \frac{1}{2A_{33}} \cdot \left\{ (2\Sigma_3 - \Lambda_1 \Pi_{23}) \cdot \frac{dA_{33}}{dv_2} - \Lambda_2 \Pi_{23} \cdot \frac{dA_{33}}{dv_1} \right\} \\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \dots\dots\dots\dots\dots (21).$$

There are six more formulæ similar to the first three, and six similar to the last three, which the reader can easily write down.

If we examine the groups of formulæ (10) and (21), we find that they are governed by a series of remarkable symmetrical laws; *firstly*, none of the three differential coefficients $\frac{dA_{11}}{du_1}$; $\frac{dA_{22}}{du_2}$; $\frac{dA_{33}}{du_3}$ enter our formulæ; *secondly*, whatever suffix appears in the n that we differentiate with, the same double suffix enters the A, *i.e.* n_1 and A_{11} are associated, but n_1 and A_{22} are never associated; *thirdly*, A_{11} A_{23} and A_{33} enter in a logarithmic form. Other laws may also strike the reader. I now proceed to the fourth section.

IV.

$$\left\{\left(\frac{dB'}{dn_1}-\frac{dA}{dv_1}\right) \text{ \&c. linearly in terms of } \frac{dA_{11}}{dv_2} \text{ \&c.}\right\}$$

B we have defined in "Exercise the first," A we have defined in this Exercise, the relation between A and B is shortly stated by the formula

$$A^2 = A_{11} \, . \, A_{22} \, . \, A_{33} \, . \, B^2 \qquad \text{(22)},$$

B' is the B of normal co-ordinates, at the "*common point*" we have of course $A = B'$.

Now it is clear from the definitions of A and B that we have

$$\left.\begin{aligned} -\frac{dB'}{B'} &= P_{23} \, . \, dA_1 + P_{13} \, . \, dA_2 + P_{12} \, . \, dA_3 \\ -\frac{dA}{A} &= \Pi_{23} \, . \, d\Lambda_1 + \Pi_{13} \, . \, d\Lambda_2 + \Pi_{12} \, . \, d\Lambda_3 \end{aligned}\right\} \qquad \text{(23)}.$$

Now the differentiation may be performed with regard to n_1, or n_2, or n_3; take for example n_1, then by substitution from the group of formulæ (10) we easily obtain that

$$\begin{aligned} -\frac{1}{B'}\frac{dB'}{dn_1} = -\frac{1}{A}\frac{dA}{dv_1} &+ \frac{\Pi_{13}}{2A^2 \, . \, A_{11}} \, . \left\{\Pi_{23}\frac{dA_{11}}{dv_2} - 2\Sigma_2\frac{dA_{11}}{dv_3}\right\}, \\ &+ \frac{\Pi_{12}}{2A^2 \, . \, A_{11}} \, . \left\{\Pi_{23}\frac{dA_{11}}{dv_3} - 2\Sigma_3\frac{dA_{11}}{dv_2}\right\}. \end{aligned}$$

But

$$\Pi_{13}\Pi_{23} - 2\Sigma_3 \, . \, \Pi_{12} = A^2 \, . \, \Lambda_3,$$

$$\Pi_{12}\Pi_{23} - 2\Sigma_2 \, . \, \Pi_{13} = A^2 \, . \, \Lambda_2;$$

hence clearly we may write down the three formulæ,

$$\left.\begin{aligned} \frac{dB'}{dn_1} &= \frac{dA}{dv_1} - \frac{A}{2A_{11}}\left\{\Lambda_3\frac{dA_{11}}{dv_2} + \Lambda_2\frac{dA_{11}}{dv_3}\right\} \\ \frac{dB'}{dn_2} &= \frac{dA}{dv_2} - \frac{A}{2A_{22}}\left\{\Lambda_3\frac{dA_{22}}{dv_1} + \Lambda_1\frac{dA_{22}}{dv_3}\right\} \\ \frac{dB}{dn_3} &= \frac{dA}{dv_3} - \frac{A}{2A_{33}}\left\{\Lambda_1\frac{dA_{33}}{dv_2} + \Lambda_2\frac{dA_{33}}{dv_1}\right\} \end{aligned}\right\} \qquad \text{(24)}.$$

The reader will observe that the three symmetrical laws mentioned at close of section III. hold also for these last three formulæ. I now proceed to the fifth section.

V.

{Connection between $(K_{11})'_1$ &c., $(K_{11})_1$ &c., $(\kappa_{11})_1$ &c.}

We have already defined $(K_{11})'_1$ &c. and $(K_{11})_1$, the only difference between them being that the former has reference to normal co-ordinates, the latter to general curvilinear co-ordinates. It remains to define $(\kappa_{11})_1$ &c. We do so thus: *let* $(\kappa_{11})_1$ &c. *be the same function of* $\Lambda_1\Lambda_2\Lambda_3$, $\Sigma_1\Sigma_2\Sigma_3$, $\Pi_{23}\Pi_{13}\Pi_{12}$, and A, *that* $(K'_{11})_1$ &c. *is of* $A_1A_2A_3$, $S_1S_2S_3$, $P_{23}P_{13}P_{12}$, and B. ....

Now let us make use first of the four typical formulæ (13) (14) (15) and (16), and next of the group of formulæ (21); substituting the values in formulæ (2) and (3) we easily obtain the following group of formulæ:

$$\left.\begin{aligned}
(K_{11})_1' &= A_{11}^{\frac{1}{2}} \cdot (K_{11})_1 + \frac{1}{A_{11}^{\frac{1}{2}}} \cdot \frac{d^2u_1}{dn_1^2} \\
(K_{11})_2' &= \frac{A_{11}}{A_{22}^{\frac{1}{2}}} \cdot (K_{11})_2 \\
(K_{11})_3' &= \frac{A_{11}}{A_{33}^{\frac{1}{2}}} \cdot (K_{11})_3 \\
(K_{23})_1' &= \left(\frac{A_{22}A_{33}}{A_{11}}\right)^{\frac{1}{2}} \cdot (K_{23})_1 \\
(K_{23})_2' &= A_{33}^{\frac{1}{2}} \cdot (K_{23})_2 + \frac{1}{A_{22}^{\frac{1}{2}}} \cdot \frac{d^2u_2}{dn_2dn_3} \\
(K_{23})_3' &= A_{22}^{\frac{1}{2}} \cdot (K_{23})_3 + \frac{1}{A_{33}^{\frac{1}{2}}} \cdot \frac{d^2u_3}{dn_2dn_3}
\end{aligned}\right\} \text{..........................} (25).$$

There are twelve more similar formulæ.

We also have eighteen formulæ connecting $(K_{11})_1'$ &c., and $(\kappa_{11})_1$ &c., but as we shall make in this exercise no use of these last formulæ it is hardly needful to write them down. Consider now for a moment the group of formulæ (25); we can clearly deduce from them the following two important equations:

$$\{(K'_{12})_3^2 - (K'_{11})_3 \cdot (K'_{22})_3\} = \left(\frac{A_{11}A_{22}}{A_{33}}\right)\{(K_{12})_3^2 - (K_{11})_3 \cdot (K_{22})_3\} \text{............} (26),$$

$$\begin{aligned}
\{(K_{12}')_1 \cdot (K_{13}')_1 - (K_{11}')_1 \cdot (K_{23}')_1\} &= (A_{22} \cdot A_{33})^{\frac{1}{2}}\{(K_{12})_1 (K_{13})_1 - (K_{11})_1 \cdot (K_{23})_1\}, \\
+ \left(\frac{A_{22}}{A_{11}}\right)^{\frac{1}{2}} \cdot (K_{12})_1 \frac{d^2u_1}{dn_1dn_3} &+ \left(\frac{A_{33}}{A_{11}}\right)^{\frac{1}{2}} \cdot (K_{13})_1 \frac{d^2u_1}{dn_1dn_2} - (A_{22} \cdot A_{33})^{\frac{1}{2}} \cdot (K_{23})_1 \cdot \frac{d^2u_1}{dn_1^2} \text{.....} (27).
\end{aligned}$$

Formula 26 shows us that the formula $m = -\frac{1}{R_1R_2}$ page 516, "Exercise the third", being proved to be true by normal co-ordinates must be true in general, and formula (27) enables us to derive last example given in "Exercise the third" from the corresponding equation proved for normal co-ordinates. I now proceed with the sixth section.

VI.

$$\left\{\begin{array}{ll}\left(-\frac{d^2S_1}{dn_2^2}+\frac{d^2P_{12}}{dn_1dn_2}-\frac{d^2S_2}{dn_1^2}\right)-A_{11}A_{22}\left(-\frac{1}{2}\frac{dC_{11}}{du_2^2}+\frac{d^2C_{12}}{du_1du_2}-\frac{1}{2}\frac{dC_{22}}{du_1^2}\right)\text{ \&c.} \\ \text{expressed in terms of} \qquad \frac{d^2u_1}{dn_1^2}\;\frac{d^2u_1}{dn_1dn_2}\text{ \&c.}\end{array}\right\}$$

In order to make our formulæ as concise as possible we use here $\frac{d^2u_1}{dn_1^2}$ &c., in place of $\frac{dA_{11}}{du_1}$ &c.; the group of formulæ (8) however give us the former nine quantities as linear functions of the last nine.

Write for conciseness the quantities in brackets in first line of this section equal to m_3' and m_3 respectively. It has been mentioned in the previous Exercises that

$$m'_3=\frac{d^2x}{dn_1^2}\cdot\frac{d^2x}{dn_2^2}+\frac{d^2y}{dn_1^2}\cdot\frac{d^2y}{dn_2^2}+\frac{d^2z}{dn_1^2}\,\frac{d^2z}{dn_2^2},$$

$$-\left(\frac{d^2x}{dn_1dn_2}\right)^2-\left(\frac{d^2y}{dn_1dn_2}\right)^2-\left(\frac{d^2z}{dn_1dn_2}\right)^2.$$

Hence substituting the values of $\frac{d^2x}{dn_1^2}$ &c., found by means of formulæ (11) and (12), we easily obtain that

$$\left.\begin{array}{l}-\frac{d^2S_1}{dn_2^2}+\frac{d^2P_{12}}{dn_1dn_2}-\frac{d^2S_2}{dn_1^2}=A_{11}A_{22}\left\{-\frac{1}{2}\frac{d^2C_{11}}{du_2^2}+\frac{d^2C_{12}}{du_1du_2}-\frac{1}{2}\frac{d^2C_{22}}{du_1^2}\right\} \\ +A_{11}\left(\frac{dC_{12}}{du_1}-\frac{1}{2}\frac{dC_{11}}{du_2}\right)\cdot\frac{d^2u_2}{dn_2^2}+A_{22}\left(\frac{dC_{12}}{du_2}-\frac{1}{2}\frac{dC_{22}}{du_1}\right)\cdot\frac{d^2u_1}{dn_1^2} \\ -(A_{11}A_{22})^{\frac{1}{2}}\frac{dC_{11}}{du_2}\cdot\frac{d^2u_1}{dn_1dn_2}-(A_{11}A_{22})^{\frac{1}{2}}\frac{dC_{22}}{du_1}\cdot\frac{d^2u_2}{dn_1dn_2} \\ -C_{11}\left(\frac{d^2u_1}{dn_1dn_2}\right)^2+C_{12}\left(\frac{d^2u_1}{dn_1^2}\cdot\frac{d^2u_2}{dn_2^2}-2\frac{d^2u_1}{dn_1dn_2}\cdot\frac{d^2u_2}{dn_1dn_2}\right)-C_{22}\left(\frac{d^2u_2}{dn_1dn_2}\right)^2\end{array}\right\}\text{........(28).}$$

We now use this same method to transform the quantity

$$\frac{1}{2}\cdot\left\{2\frac{d^2S_1}{dn_2dn_3}+\frac{d^2P_{23}}{dn_1^2}-\frac{d^2P_{13}}{dn_1dn_2}-\frac{d^2P_{12}}{dn_1dn_3}\right\}.$$

This quantity for conciseness we shall represent by the letter l'_1. We then have, as already stated in "Exercise the second,"

$$l_1'=\frac{d^2x}{dn_1dn_2}\cdot\frac{d^2x}{dn_1dn_3}+\frac{d^2y}{dn_1dn_2}\,\frac{d^2y}{dn_1dn_3}+\frac{d^2z}{dn_1dn_2}\cdot\frac{d^2z}{dn_1dn_3},$$

$$-\frac{d^2x}{dn_1^2}\,\frac{d^2x}{dn_2dn_3}-\frac{d^2y}{dn_1^2}\,\frac{d^2y}{dn_2dn_3}-\frac{d^2z}{dn_1^2}\,\frac{d^2z}{dn_2dn_3}.$$

Now make use of formulæ (11) and (12). We easily find by mere substitution that

$$\frac{1}{2}\left\{2\frac{d^2S_1}{dn_2dn_3}+\frac{d^2P_{23}}{dn_1^2}-\frac{d^2P_{31}}{dn_1dn_2}-\frac{d^2P_{12}}{dn_1dn_3}\right\}$$

is equal to

$$\frac{A_{11}\cdot(A_{22}A_{33})^{\frac{1}{2}}}{2}\left\{\frac{d^2C_{11}}{du_2du_3}+\frac{d^2C_{23}}{du_1^2}-\frac{d^2C_{13}}{du_1du_2}-\frac{d^2C_{12}}{du_1du_3}\right\}$$

$$+\frac{(A_{11}A_{22})^{\frac{1}{2}}}{2}\left\{\frac{d^2u_1}{dn_1dn_3}\cdot\frac{dC_{11}}{du_2}+\frac{d^2u_3}{dn_1dn_3}\left(\frac{dC_{23}}{du_1}+\frac{dC_{13}}{du_2}-\frac{dC_{12}}{du_3}\right)\right\}$$

$$+\frac{(A_{11}A_{33})^{\frac{1}{2}}}{2}\left\{\frac{d^2u_1}{dn_1dn_2}\cdot\frac{dC_{11}}{du_3}+\frac{d^2u_2}{dn_1dn_2}\left(\frac{dC_{12}}{du_3}+\frac{dC_{23}}{du_1}-\frac{dC_{13}}{du_2}\right)\right\}$$

$$-\frac{(A_{22}A_{33})^{\frac{1}{2}}}{2}\left\{\frac{d^2u_1}{dn_1^2}\left(\frac{dC_{12}}{du_3}+\frac{dC_{13}}{du_2}-\frac{dC_{23}}{du_1}\right)\right\}$$

$$-A_{11}\left\{\frac{d^2u_2}{dn_2dn_3}\left(\frac{dC_{12}}{du_1}-\frac{1}{2}\frac{dC_{11}}{du_2}\right)+\frac{d^2u_3}{dn_2dn_3}\left(\frac{dC_{13}}{du_1}-\frac{1}{2}\frac{dC_{11}}{du_3}\right)\right\}$$

$$+C_{11}\frac{d^2u_1}{dn_1dn_2}\frac{d^2u_1}{dn_1dn_3}+C_{23}\frac{d^2u_2}{dn_1dn_2}\frac{d^2u_3}{dn_1dn_3}$$

$$+C_{12}\cdot\left\{\frac{d^2u_1}{dn_1dn_3}\frac{d^2u_2}{dn_1dn_2}-\frac{d^2u_1}{dn_1^2}\cdot\frac{d^2u_2}{dn_2dn_3}\right\}$$

$$+C_{13}\cdot\left\{\frac{d^2u_1}{dn_1dn_2}\cdot\frac{d^2u_3}{dn_1dn_3}-\frac{d^2u_1}{dn_1^2}\cdot\frac{d^2u_3}{dn_2dn_3}\right\}\ldots\ldots\ldots\ldots(29).$$

We shall now conclude this Exercise with one more section; and as a partial verification of the formulæ we have arrived at, it shall be devoted to a deduction of the Equations of *Orthogonal Curvilinear Co-ordinates* from the corresponding formulæ of *Normal Co-ordinates.*

VII.

{Passage from Normal Co-ordinates to Orthogonal Curvilinear Co-ordinates.}

A system of Normal Co-ordinates it is clear cannot in general be also a system of Orthogonal Co-ordinates; that is to say, for every position they assume in space, yet one position of our normal co-ordinate surfaces can clearly be always made orthogonal. We assume such to be the case, and shall now substitute in the Equations given on pages 497 and 498 of Exercise the second from formulæ 10, 21, 24, 28 and 29 of this Exercise.

First, consider the formulæ on top of page 497, Exercise the second. It consists of *four* distinct portions, so to speak, and bearing in mind that for orthogonal co-ordinates we must have $2\Sigma_1=1$, $2\Sigma_2=1$, $2\Sigma_3=1$, $\Pi_{23}=0$, $\Pi_{13}=0$, $\Pi_{12}=0$, $\mathcal{A}=1$, a little consideration shows that the last three of these portions in the case of orthogonal co-ordinates are zero, there remains therefore only the first line, *i.e.* l_3 transformed, or in fact, our Equation 29. Let us write then in Equation 29,

$$C_{23}=0, \qquad C_{13}=0, \qquad C_{12}=0,$$
$$A_{11}C_{11}=1, \qquad A_{22}C_{22}=1, \qquad A_{33}C_{33}=1;$$
$$\frac{d^2u_1}{dn_1\,dn_3}=\frac{1}{2}\left(\frac{A_{33}}{A_{11}}\right)^{\frac{1}{2}}\cdot\frac{dA_{11}}{du_3}=-\frac{(A_{11}\,A_{33})^{\frac{1}{2}}}{2\,.\,C_{11}}\cdot\frac{dC_{11}}{du_3},$$
$$\frac{d^2u_1}{dn_1\,dn_2}=\ldots\ldots\ldots\ldots\ldots\ldots-\frac{(A_{11}\,A_{22})^{\frac{1}{2}}}{2\,.\,C_{11}}\cdot\frac{dC_{11}}{du_2},$$
$$\frac{d^2u_2}{dn_2\,dn_3}=\ldots\ldots\ldots\ldots\ldots\ldots-\frac{(A_{33}\,A_{22})^{\frac{1}{2}}}{2\,.\,C_{22}}\cdot\frac{dC_{22}}{du_3},$$
$$\frac{d^2u_3}{dn_2\,dn_3}=\ldots\ldots\ldots\ldots\ldots\ldots-\frac{(A_{22}\,A_{33})^{\frac{1}{2}}}{2\,.\,C_{33}}\cdot\frac{dC_{33}}{du_2}.$$

Now substitute these values in equation 29, multiply by 2, divide by $A_{11}\,.\,(A_{22}\,.\,A_{33})^{\frac{1}{2}}$; we then obtain

$$\frac{d^2C_{11}}{du_2\,du_3}-\frac{1}{2C_{11}}\,\frac{dC_{11}}{du_2}\,\frac{dC_{11}}{du_3}-\frac{1}{2C_{22}}\,\frac{dC_{22}}{du_3}\,\frac{dC_{11}}{du_2}-\frac{1}{2C_{33}}\,\frac{dC_{33}}{du_2}\,\frac{dC_{11}}{du_3}=0\;\ldots\ldots(30);$$

and this is the same as the second equation given on page 499 "Exercise the second."

Lastly, we give a short sketch of how we are to transform the equation on top of page 498, Exercise the second. It also consists of *four* portions or lines; the *first* line gives us, by aid of formula (28) after dividing by $A_{11}\,A_{22}$ and multiplying by 2,

$$-\frac{d^2C_{11}}{du_2^{\,2}}-\frac{d^2C_{22}}{du_1^{\,2}}+\frac{1}{2C_{22}}\,\frac{dC_{11}}{du_2}\,\frac{dC_{22}}{du_2}+\frac{1}{2C_{11}}\,\frac{dC_{11}}{du_1}\,\frac{dC_{22}}{du_1}$$
$$-\frac{1}{2C_{22}}\left(\frac{dC_{22}}{du_1}\right)^2-\frac{1}{2C_{11}}\left(\frac{dC_{11}}{du_2}\right)^2\;\ldots\ldots\ldots\ldots\ldots\ldots(\alpha);$$

the second line, that is to say Λ, contributes the terms

$$-\frac{1}{16C_{11}C_{22}C_{33}}\left\{\frac{dC_{11}}{du_3}\,\frac{dC_{22}}{du_3}+\frac{dC_{22}}{du_1}\,\frac{dC_{33}}{du_1}+\frac{dC_{33}}{du_2}\,\frac{dC_{11}}{du_2}\right\}\;\ldots\ldots\ldots\ldots(\beta);$$

the third line it is easily seen by aid of formula (24) can contribute no terms.

And lastly, by aid of formula (10) it is easy to see that the fourth or determinant portion contributes the terms

$$+\frac{1}{16C_{11}C_{22}C_{33}}\left\{\frac{dC_{11}}{du_2}\,\frac{dC_{33}}{du_2}+\frac{dC_{22}}{du_1}\,\frac{dC_{33}}{du_1}-3\,\frac{dC_{22}}{du_3}\,\frac{dC_{11}}{du_3}\right\}\;\ldots\ldots\ldots\ldots(\gamma).$$

Now divide (β) and (γ) by $A_{11}A_{22}$, multiply them by 2, add them to (α), and we obtain

$$-\frac{d^2C_{11}}{du_2^2}-\frac{d^2C_{22}}{du_1^2}+\frac{1}{2C_{11}}\frac{dC_{11}}{du_1}\cdot\frac{dC_{22}}{du_1}+\frac{1}{2C_{22}}\frac{dC_{11}}{du_2}\cdot\frac{dC_{22}}{du_2}$$

$$-\frac{1}{2C_{11}}\left(\frac{dC_{11}}{du_2}\right)^2-\frac{1}{2C_{22}}\left(\frac{dC_{22}}{du_1}\right)^2$$

$$-\frac{1}{2C_{33}}\frac{dC_{11}}{du_3}\cdot\frac{dC_{22}}{du_3}=0 \quad\ldots\ldots\ldots\ldots(31);$$

and this agrees with the first equation of page 499 "Exercise the second."

March, 1877.

II. *On Boltzmann's Theorem on the average distribution of energy in a system of material points. By* Professor J. Clerk Maxwell.

[Read *May* 6, 1878.]

Dr Ludwig Boltzmann, in his "Studien über das Gleichgewicht der lebendigen Kraft zwischen bewegten materiellen Punkten" [Sitzb. d. k. Akad. Wien, Bd. LVIII., 8 Oct. 1868], has devoted his third section to the general solution of the problem of the equilibrium of kinetic energy among a finite number of material points. His method of treatment is ingenious, and, as far as I can see, satisfactory, but I think that a problem of such primary importance in molecular science ought to be scrutinized and examined on every side, so that as many persons as possible may be enabled to follow the demonstration, and to know on what assumptions it rests. This is more especially necessary when the assumptions relate to the degree of irregularity to be expected in the motion of a system whose motion is not completely known.

Mr H. W. Watson, in his Treatise on the Kinetic Theory of Gases *, has developed with great clearness the steps of the investigation of the distribution of energy among a set of particles which are supposed to act on each other only at very small distances. The particles may be acted on by external forces such as gravity, but it is expressly stipulated that the time during which a particle is encountering other particles is very small compared with the time during which there is no sensible action between it and other particles; and also that the time during which a particle is simultaneously within the distance of molecular action of more than one other particle may be neglected.

Now this method of treating the question, however necessary it may be in the subsequent investigation of the processes of diffusion, &c. in gases, is inapplicable to the theory of the equilibrium of temperature in liquids and solids, for in these bodies the particles are never free from the action of neighbouring particles. It is true that in following the steps of the investigation, as given either by Boltzmann or by Watson, it is difficult, if not impossible, to see where the stipulation about the shortness and the isolation of the encounters is made use of. We may almost say that it is introduced rather for the sake of enabling the reader to form a more definite mental image

* *Clarendon Press Series*, 1876.

of the material system than as a condition of the demonstration. Be this as it may, the presence of such a stipulation in the enunciation of the problem cannot fail to leave in the mind of the reader the impression of a corresponding limitation in the generality of the solution.

In the theorem of Boltzmann which we have now to consider there is no such limitation. The material points may act on each other at all distances, and according to any law which is consistent with the conservation of energy, and they may also be acted on by any forces external to the system provided these also are consistent with that law.

The only assumption which is necessary for the direct proof is that the system, if left to itself in its actual state of motion, will, sooner or later, pass through every phase which is consistent with the equation of energy.

Now it is manifest that there are cases in which this does not take place. The motion of a system not acted on by external forces satisfies six equations besides the equation of energy, so that the system cannot pass through those phases, which, though they satisfy the equation of energy, do not also satisfy these six equations.

Again, there may be particular laws of force, as for instance that according to which the stress between two particles is proportional to the distance between them, for which the whole motion repeats itself after a finite time. In such cases a particular value of one variable corresponds to a particular value of each of the other variables, so that phases formed by sets of values of the variables which do not correspond cannot occur, though they may satisfy the seven general equations.

But if we suppose that the material particles, or some of them, occasionally encounter a fixed obstacle such as the sides of a vessel containing the particles, then, except for special forms of the surface of this obstacle, each encounter will introduce a disturbance into the motion of the system, so that it will pass from one undisturbed path into another. The two paths must both satisfy the equation of energy, and they must intersect each other in the phase for which the conditions of encounter with the fixed obstacle are satisfied, but they are not subject to the equations of momentum. It is difficult in a case of such extreme complexity to arrive at a thoroughly satisfactory conclusion, but we may with considerable confidence assert that except for particular forms of the surface of the fixed obstacle, the system will sooner or later, after a sufficient number of encounters, pass through every phase consistent with the equation of energy.

I shall begin with the case in which the system is supposed to be contained within a fixed vessel, and shall afterwards consider the case of a free system, or of a system contained in a vessel rotating uniformly about an axis which itself moves uniformly in a straight line.

I have found it convenient, instead of considering one system of material particles, to consider a large number of systems similar to each other in all respects except in the initial circumstances of the motion, which are supposed to vary from system to system, the total energy being the same in all. In the statistical investigation of the motion, we confine our attention to the *number* of these systems which at a given time are in a phase such that the variables which define it lie within given limits.

If the number of systems which are in a given phase (defined with respect to configuration and velocity) does not vary with the time, the distribution of the systems is said to be *steady*.

It is shown that if the distribution is steady, a certain function of the variables must be constant for all phases belonging to the same path. If the path passes through all phases consistent with the equation of energy, this function must be constant for all such phases. If however there are phases consistent with the equation of energy, but which do not belong to the same path, the value of the function may be different for such phases.

But whether we are able or not to prove that the constancy of this function is a necessary condition of a steady distribution, it is manifest that if the function is initially constant for all phases consistent with the equation of energy, it will remain so during the motion. This therefore is one solution, if not the only solution, of the problem of a steady distribution.

Now we know from the empirical laws of the diffusion of heat that the problem of the equilibrium of temperature in an isolated material system has one and only one solution. But we have found one solution of the problem of equilibrium of energy in a system of material points in motion. If, therefore, the real material system in which the equilibrium of temperature takes place is capable of being accurately represented by a system of material points (as defined in pure dynamics) acting on each other according to determinate, though unknown, laws, then the mathematical condition of the equilibrium of energy must be the dynamical representative of the physical condition of the equality of temperature.

It appears from the theorem that in the ultimate state of the system the average kinetic energy of two given portions of the system must be in the ratio of the number of degrees of freedom of those portions. This, therefore, must be the condition of the equality of temperature of the two portions of the system.

Hence at a given temperature the total kinetic energy of a material system must be the product of the number of degrees of freedom of that system into a constant which is the same for all substances at that temperature, being in fact the temperature on the thermodynamic scale multiplied by an absolute constant.

If the temperature, therefore, is raised by unity, the kinetic energy is increased by the product of the number of degrees of freedom into the absolute constant.

The observed specific heat of the body, expressed in dynamical measure, is the increment of the *total* energy when the temperature is increased by unity. The observed specific heat cannot therefore be less than the product of the number of degrees of freedom into the absolute constant, unless the potential energy diminishes as the temperature rises.

Dynamical Specification of the motion.

We shall begin by supposing the material system to be of the most general type, having its configuration determined by the n variables $q_1, q_2 \ldots q_n$, and its motion determined by the corresponding momenta $p_1, p_2 \ldots p_n$. The state of the system at any instant is completely defined if we know the values of these $2n$ variables for that instant.

We shall suppose the forces acting between the parts of the system to be of the most general kind consistent with the conservation of energy. This may be expressed by defining V, the potential energy of the system, as a function of $q_1 \ldots q_n$, the variables which define the configuration.

The kinetic energy of the system is denoted by T. We shall suppose it to be expressed in terms of the q's and p's as in Hamilton's method. The total energy is denoted by

$$E = V + T \quad \ldots\ldots (1),$$

and is a constant during the motion of the system.

Hamilton's equations of motion for this system are

$$\frac{\partial q_r}{\partial t} = \frac{dE}{dp_r} \quad \ldots\ldots (2),$$

$$\frac{\partial p_r}{\partial t} = -\frac{dE}{dq_r} \quad \ldots\ldots (3),$$

where q_r and p_r are the co-ordinate and the momentum corresponding to each other.

Let us now consider a finite motion of the system. Let the initial co-ordinates and momenta be distinguished by accented letters, and the final co-ordinates and momenta by the same letters unaccented.

To define completely such a motion requires $2n+1$ variables to be given. These may be the n initial co-ordinates, the n initial momenta, and the time occupied by the motion.

There is another method however in which the $2n+1$ variables are the n initial co-ordinates, the n final co-ordinates, and the total energy. When these quantities are given there are in general only a finite number of possible motions.

Definition of the "Action" of the system during the motion.

Twice the time integral of the kinetic energy, taken from the beginning to the end of the motion, and expressed in terms of the initial and final co-ordinates and of the total energy, is called the "Action" of the system during the motion. If we denote it by A,

$$A = \int 2T dt \qquad (4),$$

and is expressed as a function of $q_1' \ldots q_n'$, $q_1 \ldots q_n$, and E.

It is shewn in treatises on dynamics* that

$$\frac{dA}{dq_r'} = -p_r' \qquad (5),$$

and

$$\frac{dA}{dq_r} = p_r \qquad (6).$$

Hence

$$\frac{dp_r'}{dq_s} = -\frac{d^2A}{dq_r' dq_s} = -\frac{dp_s}{dq_r'} \qquad (7).$$

The indices r and s in this equation may be the same or different.

Also if t' and t are the values of the time at the beginning and at the end of the motion,

$$\frac{dA}{dE} = t - t' \qquad (8).$$

Hence

$$\frac{dp_r}{dE} = -\frac{dt'}{dq_r} \;(9) \text{ and } \frac{dp_s'}{dE} = -\frac{dt}{dq_s'} \qquad (10).$$

In the course of our investigation we shall have to compare the product of the differentials of the co-ordinates and momenta at the beginning of the motion with the corresponding product at the end of the motion. We shall write for brevity $ds = dq_1 \ldots dq_n$ for the product of the differentials of the co-ordinates, and $d\sigma = dp_1 \ldots dp_n$ for the product of the differentials of the momenta, and we shall use the product $ds' ds dE$ as a middle term in comparing $ds' d\sigma' dt'$ with $ds d\sigma dt$.

Now

$$ds' d\sigma' dt' = ds' ds dE \, \Sigma \pm \left(\frac{dp_1'}{dq_1} \cdots\cdots \frac{dp_n'}{dq_n} \frac{dt'}{dE} \right) \qquad (11),$$

where

$$\Sigma \pm \left(\frac{dp_1'}{dq_1} \cdots\cdots \frac{dp_n'}{dq_n} \frac{dt'}{dE} \right)$$

denotes the functional determinant

$$\begin{vmatrix} \frac{dp_1'}{dq_1}, & \cdots\cdots \frac{dp_1'}{dq_n}, & \frac{dp_1'}{dE} \\ \cdots\cdots & \cdots\cdots & \cdots \\ \frac{dp_n'}{dq_1}, & \cdots\cdots \frac{dp_n'}{dq_n}, & \frac{dp_n'}{dE} \\ \frac{dt'}{dq_1}, & \cdots\cdots \frac{dt'}{dq_n}, & \frac{dt'}{dE} \end{vmatrix} \qquad (12).$$

* Thomson and Tait's *Natural Philosophy*, § 330.

Substituting for the elements of this determinant their values as given by equations (7), (9), and (10) it becomes

$$\begin{vmatrix} -\frac{dp_1}{dq_1'}, \ldots\ldots -\frac{dp_n}{dq_1'}, & -\frac{dt}{dq_1'} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ -\frac{dp_1}{dq_n'}, \ldots\ldots -\frac{dp_n}{dq_n'}, & -\frac{dt}{dq_n'} \\ -\frac{dp_1}{dE}, \ldots\ldots -\frac{dp_n}{dE}, & -\frac{dt}{dE} \end{vmatrix} \quad \ldots\ldots\ldots\ldots(13).$$

Now the rows in this determinant are the same as the columns in the former one; the accented and unaccented letters being exchanged and the signs of all the elements changed. We may therefore express the relation between the two determinants in the abbreviated form

$$\Sigma \pm \left(\frac{dp_1'}{dq_1} \ldots\ldots \frac{dp_n'}{dq_n} \frac{dt'}{dE}\right) = (-)^{n+1} \Sigma \pm \left(\frac{dp_1}{dq_1'} \ldots\ldots \frac{dp_n}{dq_n'} \frac{dt}{dE}\right) \ldots\ldots\ldots\ldots (14).$$

Hence
$$ds'd\sigma'dt' = ds'dsdE\,\Sigma \pm \left(\frac{dp_1'}{dq_1} \ldots\ldots \frac{dp_n'}{dq_n} \frac{dt'}{dE}\right)$$
$$= (-)^{n+1}\, ds'dsdE\,\Sigma \pm \left(\frac{dp_1}{dq_1'} \ldots\ldots \frac{dp_n}{dq_n'} \frac{dt}{dE}\right)$$
$$= (-)^{n+1}\, d\sigma ds dt$$
$$= ds d\sigma dt \ldots\ldots\ldots\ldots(15).$$

If we suppose the time, $t-t'$, to be given, $dt = dt'$ and

$$ds'd\sigma = dsd\sigma \ldots\ldots\ldots\ldots(16),$$

or
$$dq_1' \ldots\ldots dq_n'\, dp_1' \ldots\ldots dp_n' = dq_1 \ldots\ldots dq_n\, dp_1 \ldots\ldots dp_n \ldots\ldots\ldots\ldots(17).$$

The initial state of the system is a function of $2n$ variables. We have hitherto supposed these to be the n co-ordinates and the n momenta, but since the total energy E is a function of these variables we may substitute for one of the momenta, say p_1', its value in terms of the n co-ordinates, the $n-1$ remaining momenta, and E, and thus express every quantity we have to deal with in terms of the latter set of variables. Then since by equation (2)

$$\frac{dE}{dp_1'} = \frac{\partial q_1'}{\partial t} = \dot{q}_1' \ldots\ldots\ldots\ldots(18),$$

$$dq_1' \ldots\ldots dq_n'\, dp_1' \ldots\ldots dp_n' = dq_1' \ldots\ldots dq_n'\, dp_2' \ldots\ldots dp_n'\, dE\, \frac{1}{\dot{q}_1'} \ldots\ldots\ldots\ldots(19).$$

Similarly we find for the final state of the system

$$dq_1 \ldots\ldots dq_n dp_1 \ldots\ldots dp_n = dq_1 \ldots\ldots dq_n dp_2 \ldots\ldots dp_n dE\, \frac{1}{\dot{q}_1} \ldots\ldots\ldots\ldots(20).$$

The left-hand members of these equations have been proved equal, and in the right-hand members dE is the same at the beginning and end of the motion. Dividing out dL we find

$$dq_1' \ldots\ldots dq_n' dp_2' \ldots\ldots dp_n' \frac{1}{q_1'} = dq_1 \ldots\ldots dq_n dp_2 \ldots\ldots dp_n \frac{1}{q_1} \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

This equation is applicable to the case in which the total energy is supposed not to vary from one particular instance of the motion to another, and in which, therefore, the $2n$ variables are no longer independent, but, being subject to the equation of energy, are reduced to $2n-1$.

Statistical Specification.

We have hitherto, in speaking of a phase of the motion of the system, supposed it to be defined by the values of the n co-ordinates and the n momenta. We shall call the phase so defined the phase (pq). We shall now adopt a wider definition by saying that the system is in the phase (a_1b) whenever the values of the co-ordinates are such that q_1 is between b_1 and b_1+db_1, q_2 between b_2 and b_2+db_2, and so on; also p_2 between a_2 and a_2+da_2, and so on. The limits of the first component of momentum, p_1, are not specified, because the value of p_1 is not independent of the other variables, being given in terms of E and the other $2n-1$ variables in virtue of the equation of energy.

The quantities a, b are of the same kind as p and q respectively, only they are not supposed to vary on account of the motion of the system. In the statistical method of investigation, we do not follow the system during its motion, but we fix our attention on a particular phase, and ascertain whether the system is in that phase or not, and also when it enters the phase and when it leaves it.

Boltzmann defines the probability of the system being in the phase (a_1b) as the ratio of the aggregate time during which it is in that phase to the whole time of the motion, the whole time being supposed to be very great. I prefer to suppose that there are a great many systems the properties of which are the same, and that each of these is set in motion with a different set of values for the n co-ordinates and the $n-1$ momenta, the value of the total energy E being the same in all, and to consider the number of these systems which, at a given instant, are in the phase (a_1b). The motion of each system is of course independent of the other systems.

Let N be the whole number of systems, and let the number of these which, at the time t, are in the phase (a_1b) be denoted by $N(a_1, b, t)$. The aim of the statistical method is to express $N(a_1, b, t)$ as a function of N, of the co-ordinates and momenta with their limits, and of t. It is manifest that N can only enter the function as a factor, for the different systems do not act on each other. Also any differential as da or db can only enter as a factor, for the number of systems within any phase must vary in the ratio of the interval between the limits of that phase. We may therefore write

$$N(a_1bt) = Nf(a_2, \ldots\ldots a_n,\ b_1, \ldots\ldots b_n,\ t)\, da_2 \ldots\ldots da_n db_1 \ldots\ldots db_n \qquad (22),$$

where we have to determine the form of the function f.

We shall now follow the motion of these systems from the time t', when we begin to watch the motion, to the time t when we cease to watch it.

Since the systems which at the time t form the group $N(a_1,\ b,\ t)$ are individually the same systems which at the time t' formed the group $N(a_1',\ b',\ t')$, we have

$$N(a_1,\ b,\ t) = N(a_1',\ b',\ t') \qquad (23),$$

or

$$Nf(a_2 \ldots\ldots t)\, da_2 \ldots\ldots db_n = Nf(a_2' \ldots\ldots t')\, da_2', \ldots\ldots db_n' \qquad (24).$$

But by equation (21)

$$da_2 \ldots\ldots db_n (\dot{b}_1)^{-1} = da_2' \ldots\ldots db_n' (\dot{b}_1')^{-1} \qquad (25).$$

Hence

$$f(a_2 \ldots\ldots t)\, \dot{b}_1 = f(a_2' \ldots\ldots t')\, \dot{b}_1' = C \qquad (26),$$

where C is a constant for all phases of the same motion, and we may write

$$f(a_2 \ldots\ldots t) = C(\dot{b}_1)^{-1} \qquad (27),$$

and

$$N(a_1,\ b,\ t) = NC(\dot{b}_1)^{-1} da_2 \ldots\ldots db_n \qquad (28).$$

If the distribution of the N systems in the different phases is such that the number in a given phase does not vary with the time, the distribution is said to be steady. The condition of this is that C must be constant for all phases belonging to the same path. It will require further investigation to determine whether or not this path necessarily includes all phases consistent with the equation of energy.

If, however, we assume that the original distribution of the systems according to the different phases is such that C is constant for all phases consistent with the equation of energy, and zero for all phases which that equation shows to be impossible, then the law of distribution will not change with the time, and C will be an absolute constant.

We have therefore found one solution of the problem of finding a steady distribution. Whether there may be other solutions remains to be investigated.

Let $N(b)$ denote the number of systems in which q_1 is between b_1 and b_1+db_1, q_2 between b_2 and b_2+db_2, and so on, and q_n between b_n and $b+db_n$, the momenta not being specified otherwise than by their being consistent with the equation of energy; then

$$N(b) = \int \ldots \int N(a_1,\ b)\, da_2 \ldots da_n \qquad (29),$$

the integration being extended to all values of the momenta consistent with the equation of energy.

To simplify the integration let us suppose the variables transformed so that the kinetic energy is expressed in terms of the squares of the component momenta,

$$T = \tfrac{1}{2}(\mu_1 a_1^2 + \mu_2 a_2^2 + \ldots + \mu_n a_n^2) \quad \text{.........(30)},$$

where $a_1 \ldots a_n$ are the transformed momenta, and $\mu_1 \ldots \mu_n$ are functions of the co-ordinates, which we may call moments of mobility, and which, in the case of material points, are the reciprocals of the masses.

Now let us assume

$$\tfrac{1}{2}\mu_n A_n^2 = T = E - V \quad \text{.........(31)},$$

$$\mu_{n-1} A_{n-1}^2 = \mu_n (A_n^2 - a_n^2) \quad \text{.........(32)},$$

$$\mu_{n-2} A_{n-2}^2 = \mu_{n-1} (A_{n-1}^2 - a_{n-1}^2) \quad \text{.........(33)},$$

....

$$\mu_2 A_2^2 = \mu_3 (A_3^2 - a_3^2) \quad \text{.........(34)}.$$

Then by the equation of energy

$$\mu_1 a_1^2 = \mu_2 (A_2^2 - a_2^2) \quad \text{.........(35)}.$$

Of these quantities, A_n is a function of the co-ordinates only, because E is given and V is a function of the co-ordinates, A_{n-1} is a function of the co-ordinates and a_n, A_{n-2} of the co-ordinates and of a_n and a_{n-1}, and so on.

Also by equation (2)

$$\dot{b}_1 = \frac{dT}{da_1} = \mu_1 a_1$$
$$= (\mu_1 \mu_2)^{\frac{1}{2}} (A_2^2 - a_2^2)^{\frac{1}{2}} \quad \text{.........(36)}.$$

To integrate the expression

$$\iiint \ldots \int C (\dot{b}_1)^{-1} \, da_2 \ldots da_n,$$

we begin by integrating with respect to a_2, thus

$$\int C (\dot{b}_1)^{-1} da_2 = \int C (\mu_1 \mu_2)^{-\frac{1}{2}} (A_2^2 - a_2^2)^{-\frac{1}{2}} da_2 \quad \text{.........(37)},$$

the limits of integration being $\pm A_2$. The result is

$$\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{2})} C (\mu_1 \mu_2)^{-\frac{1}{2}} A_2^0 \quad \text{.........(38)}.$$

For the next integration we have

$$\int (\mu_2 A_2)^0 da_3 = \int_{-A_3}^{A_3} \mu_3 (A_3^{\,2} - a_3^{\,2})^0 \, da_3 = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{2}{2}}{\Gamma\frac{3}{2}} A_3^{\frac{2}{2}} \quad \text{.........................(39).}$$

Hence after r integrations, r being any number less than n, the result is

$$NC\frac{(\Gamma(\frac{1}{2}))^{r+1}}{\Gamma\frac{r+1}{2}}(\mu_1\,\mu_2\ldots\mu_{r+1})^{-\frac{1}{2}}\,[\mu_{r+1}A_{r+1}^{\,2}]^{\frac{r-1}{2}}\,da_{r+2}\ldots da_r\,db_1\ldots db_n \quad \text{.................(40).}$$

Putting $r=n-1$ and remembering that $\mu_n A_n^{\,2} = 2E - 2V$, we find

$$N(b) = NC\frac{[\Gamma(\frac{1}{2})]^n}{\Gamma\left(\frac{n}{2}\right)}(\mu_1\,\mu_2\ldots\mu_n)^{-\frac{1}{2}}\,[2E-2V]^{\frac{n-2}{2}}\,db_1\ldots db_n \quad \text{.......................(41).}$$

This is the number of systems whose configuration is specified by the variables $b_1 \ldots b_n$, while the momenta may have any values consistent with the equation of energy.

The quantity $E-V$, which occurs in this equation, is, by equation (1), equal in magnitude to T, the kinetic energy of the system. The quantity T, however, is defined explicitly in terms of the velocities or the momenta of the system, whereas $E-V$ does not involve these quantities explicitly, but is expressed as a function of the configuration.

We shall find it convenient, however, especially in the study of more complicated problems, to remember that the number of systems in a given configuration is a function of the kinetic energy corresponding to that configuration.

If the kinetic energy is not expressed as a sum of squares, but in the more general form,

$$\begin{aligned} T = \tfrac{1}{2}[11]\,a_1^{\,2} + [12]\,a_1\,a_2 + \&c. \\ + \tfrac{1}{2}\,[22]\,a_2^{\,2} + [23]\,a_2\,a_3 + \&c. \end{aligned} \quad \text{..................................(42),}$$

where the quantities denoted by [11] &c. are functions of the co-ordinates, which we may call the moments and products of mobility of the system; then since the discriminant

$$\Delta = \begin{vmatrix} [11], & [12], \ldots\ldots & [1n] \\ [21], & [22], \ldots\ldots & [2n] \\ \ldots & \ldots & \ldots \\ [n1], & [n2], \ldots\ldots & [nn] \end{vmatrix} \quad \text{.....................................(43)}$$

is an invariant, its value is the same when T is reduced to a sum of squares, in which case all the elements except those in the principal diagonal of the determinant vanish, and we have

$$\Delta = \mu_1\mu_2\ldots\mu_n \quad \text{...(44).}$$

and we may write the value of $N(b)$,

$$N(b) = NC \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma\left(\frac{n}{2}\right)} \Delta^{-\frac{1}{2}} (2E - 2V)^{\frac{n-2}{2}} db_1 \dots db_n \dots\dots (45).$$

If the system consists of n' material particles, whose masses are $m_1 \dots m_{n'}$, then the number of degrees of freedom is $n = 3n'$ and

$$\mu_1 = \mu_2 = \mu_3 = m_1^{-1}, \quad \mu_4 = \mu_5 = \mu_6 = m_2^{-1} \text{ and so on} \dots\dots (46).$$

Hence in this case we may write

$$N(b) = NC' \frac{(\Gamma(\frac{1}{2}))^{3n'}}{\Gamma\left(\frac{3n'}{2}\right)} (m_1 \dots m_n)^{\frac{3}{2}} [2E - 2V]^{\frac{3n'-2}{2}} db_1 \dots db_n \dots\dots (47).$$

These expressions give the number of systems in a given configuration only when $E - V$ is positive for that configuration, for since the kinetic energy is necessarily positive, the potential energy cannot exceed the total energy. For configurations specified in such a way that if they existed V would be greater than E, the value of $N(b)$ is zero.

The value of $N(b)$ is also zero for configurations which, though they make V less than E, cannot be reached by a continuous path from the original configuration without passing through configurations which make V greater than E.

We shall return to this expression for the number of systems in a completely specified configuration, but in the mean time it will be useful to consider how many of these systems have one of their momenta, p_n, between given limits. In this way we shall be able to determine completely the average distribution of momentum among the variables without making any assumptions about the nature of the system which might limit the generality of our results.

In order to find the number of systems in the configuration (b) for which one of the momenta, say p_n, lies between a_n and $a_n + da_n$, we must stop before the last integration. Putting $r = n - 2$ in equation (40)

$$N(b_1 a_n) = NC \frac{\Gamma(\frac{1}{2})^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} (\mu_1 \dots \mu_{n-1})^{-\frac{1}{2}} (\mu_{n-1} A_{n-1}^2)^{\frac{n-3}{2}} da_n db_1 \dots db_n \dots\dots (48).$$

The whole number of systems in configuration (b) is given by (45). Hence the proportion of these systems for which a_n lies between a_n and $a_n + da_n$ is

$$\frac{2^{-\frac{1}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{1}{2}) \Gamma\left(\frac{n-1}{2}\right)} \frac{[E - V - \frac{1}{2}\mu_n a_n^2]^{\frac{n-3}{2}}}{[E - V]^{\frac{n-2}{2}}} \mu_n^{\frac{1}{2}} da_n \dots\dots (49).$$

If we write

$$\tfrac{1}{2}\mu_n a_n^2 = k_n \qquad (50),$$

then k_n denotes the part of the kinetic energy arising from the momentum a_n. The proportion of the systems in configuration (b) for which k_n is between k_n and $k_n + dk_n$ is

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{1}{2})\,\Gamma\left(\frac{n-1}{2}\right)} \frac{[E-V-k_n]^{\frac{n-3}{2}}}{[E-V]^{\frac{n-2}{2}}} k_n^{-\frac{1}{2}} dk_n \qquad (51).$$

Since any one of the variables may be taken for q_n, the law of distribution of values of the kinetic energy is the same for all the variables. The mean value of the kinetic energy corresponding to any variable is

$$K = \frac{1}{n}(E-V) = \frac{1}{n}T \qquad (52).$$

The maximum value is

$$T = nK \qquad (53).$$

The mean value of k^r is

$$\frac{1.3\ldots 2r-1}{n.n+2\ldots n+2r-2} n^r K^r \qquad (54).$$

When n is very large, the expression (51) approximates to

$$\frac{1}{\sqrt{2\pi}} \frac{1}{K} e^{-\frac{k}{2K}} dk \qquad (55).$$

Recapitulation.

The result of our investigation may therefore be stated as follows:

(α) We begin by considering a set of material systems which satisfy the general equations of dynamics (2) and (3), and the equation of energy (1). If in these systems the distribution of configurations satisfies equation (45), and the distribution of motion satisfies equation (51), these equations will continue to be satisfied during the subsequent motion of the system. One result of equation (51), to which we shall have to refer, is that the average kinetic energy corresponding to any one of the variables is the same for every one of the variables of the system.

(β) We now turn our attention to a system of real bodies enclosed in a rigid vessel impervious to matter and to heat. We know by experiment that in such a system the temperature cannot remain steady in every part unless the temperature of every part of the system is the same, and that this condition is necessary in whatever manner the configuration of the system may be varied by altering the position and mean density of the portions of sensible size into which we are able to divide it.

Now if the system of real bodies is a material system which satisfies the equations of dynamics, and if equations (45) and (51) are also satisfied, the condition of the system will, as we have shewn, (α), be steady in every respect, and therefore in respect of temperature. Hence by (β) the temperature of every part of the system must be the same.

Therefore if equations (45) and (51) are satisfied, the condition of equality of temperature is also satisfied.

But the condition of equality of temperature does not depend on the configuration of the system, for though we can alter the configuration by external constraint we cannot prevent the temperature from becoming equalized. It does not depend, therefore, on equation (45). We must therefore conclude, that if equation (51) is satisfied, the condition of equality of temperature is also satisfied, or, in other words, that equation (51) is the condition of equality of temperature.

Hence when two parts of a system have the same temperature, the average kinetic energy corresponding to any one of the variables belonging to these parts must be the same.

If the system is a gas or a mixture of gases not acted on by external forces, the theorem that the average kinetic energy of a single molecule is the same for molecules of different gases is not sufficient to establish the condition of equilibrium of temperature between gases of different kinds such as oxygen and nitrogen, because when the gases are mixed we have no means of ascertaining the temperature of the oxygen or of the nitrogen separately. We can only ascertain the temperature of the mixture by putting a thermometer into it.

We cannot legitimately assert that the temperatures of the oxygen and of the nitrogen must be equal because they are in contact with each other, for the only way in which we can conceive the oxygen or the nitrogen as existing in the mixture is by picturing the medium as a system of molecules, and as soon as we begin to see the molecules distinctly, heat becomes resolved into motion.

But since our investigation is equally applicable to a system of any kind, provided only it satisfies the equations of dynamics, we may suppose it to consist of pure oxygen and pure nitrogen separated by a solid diaphragm, the solid diaphragm consisting of molecules capable of motion, but acting on each other with forces which are sufficient to prevent any molecule from getting far apart from its neighbours except under the action of disturbing forces greater than any which would occur in a system at the given temperature. In this system, though the oxygen and the nitrogen cannot mix, each can make an exchange of molecular energy with the surface molecules of the diaphragm, and exchanges of energy can go on within the solid diaphragm itself without any exchange of molecules between distant parts of the diaphragm.

Hence, in this system, the average kinetic energy of a molecule of oxygen will become equal to that of a molecule of nitrogen in the final state of the system, that is to say, when the temperatures of all parts of the system have become equal, and since in that final state we have pure oxygen on one side and pure nitrogen on the other, we can verify the equality of temperature by means of a thermometer, and we can now assert that the temperatures, not only of oxygen and nitrogen, but of all bodies, are equal when the average kinetic energy of a single molecule of each of these substances is the same.

Approximate value of the probability when V is small compared with E.

To find the number of systems the configuration of which is specified as regards the limits of certain of the variables while the other variables are left undetermined, we should have to integrate the expressions in equations (41), (45), or (47) with respect to each of the undetermined variables in succession, the integrations being extended to all values of these variables which are consistent with the equation of energy.

These integrations cannot be performed unless the potential energy of the system is a known function of the variables which determine its configuration. We cannot therefore in general continue the integration so as to determine the number of systems in which the limits are specified for some, but not all, of the variables.

But when the number of variables is very great, and when the potential energy of the specified configuration is very small compared with the total energy of the system, we may obtain a useful approximation to the value of $[E-V]^{\frac{n-2}{2}}$ in an exponential form, for if we write, as in equation (53), $E=nK$,

$$[E-V]^{\frac{n-2}{2}} = E^{\frac{n-2}{2}} e^{\frac{n-2}{2}\log\left(1-\frac{V}{nK}\right)}$$

$$= E^{\frac{n-2}{2}} e^{-\frac{V}{2K}} \qquad \text{...(56)},$$

nearly, provided n is very great and V is small compared with E. The expression is no longer approximate when V is nearly as great as E, and it does not vanish, as it ought to do, when $V=E$.

Hence when the potential energy of the system in the given configuration is very small compared with its kinetic energy, we may use the approximately correct statement, that the number of systems in a given configuration is inversely proportional to the exponential function, the index of which is half the potential energy of the system in the given configuration divided by the average kinetic energy corresponding to each variable of the system.

If we divide the system into any two parts, A and B, we may consider V, the potential energy of the whole system, as made up of three parts, V_A and V_B, the potential energy of A and B, each on itself, and W, that of B with respect to A.

When, as in the case of a gas, the parts of the system are in a great degree independent of each other, the average values of V_A and V_B may be treated as constants, and the variations of V will be the same as those of W, so that the variable part of the exponential function will be reduced to

$$e^{-\frac{W}{2K}} \quad\quad\quad\quad (57).$$

If we suppose that A denotes a single molecule of a particular kind of gas, and that B denotes all the other molecules, of whatever kind, in the system, then, since there are many molecules similar to A, we may pass, from the number of systems in which A is within a given element of volume, to the average number of molecules similar to A which are within that element, or, in other words, the average density of the gas A within that element.

We may therefore interpret the expression (57) as asserting that the density of a particular kind of gas at a given point is inversely proportional to the exponential function whose index is half the potential energy of a single molecule of the gas at that point, divided by the average kinetic energy corresponding to a variable of the system.

We must remember that since the centre of mass of a molecule is determined by *three* variables, the mean kinetic energy of agitation of the centre of mass of a molecule is *three* times the quantity K which denotes the mean kinetic energy of a single variable.

PART II. *A Free system.*

In a material system not acted on by external forces the motion satisfies six equations besides the equation of energy, so that we must not include in our integration all the phases which satisfy the equation of energy, but only those of them which also satisfy these six equations.

In what follows, we shall suppose the system to consist of n particles, whose masses are $m_1 \dots m_n$, and whose co-ordinates x, y, z, and velocity-components u, v, w, are distinguished by the same suffix as the particle to which they belong.

Let us now consider a system consisting of s of these particles, and write

$$m_1 + m_2 + \&c. + m_s = M_s \quad\quad\quad\quad (58),$$

$$\left.\begin{aligned} m_1x_1 + m_2x_2 + \&c. + m_sx_s &= M_s X_s,\\ m_1y_1 + m_2y_2 + \&c. + m_sy_s &= M_s Y_s,\\ m_1z_1 + m_2z_2 + \&c. + m_sz_s &= M_s Z_s, \end{aligned}\right\} \quad\quad\quad\quad (59).$$

then M_s will be the mass of the minor system and X_s, Y_s, Z_s the co-ordinates of its centre of mass. If we also write

$$\left.\begin{aligned} m_1u_1 + \&c. + m_su_s &= M_s U_s,\\ m_1v_1 + \&c. + m_sv_s &= M_s V_s,\\ m_1w_1 + \&c. + m_sw_s &= M_s W_s, \end{aligned}\right\} \quad\quad\quad\quad (60),$$

$$\left.\begin{aligned} m_1(y_1w_1 - z_1v_1) + \&c. + m_s(y_sw_s - z_sv_s) &= F_s + M_s(Y_sW_s - Z_sV_s), \\ m_1(z_1u_1 - x_1w_1) + \&c. + m_s(z_su_s - x_sw_s) &= G_s + M_s(Z_sU_s - X_sW_s), \\ m_1(x_1v_1 - y_1u_1) + \&c. + m_s(x_sv_s - y_su_s) &= H_s + M_s(X_sV_s - Y_sU_s), \end{aligned}\right\} \quad \text{(61)},$$

then U_s, V_s, W_s will be the velocity-components of the centre of mass, and F_s, G_s, H_s the components of angular momentum round this point.

We shall also write

$$\tfrac{1}{2}m_1(u_1^2 + v_1^2 + w_1^2) + \&c. + \tfrac{1}{2}m_s(u_s^2 + v_s^2 + w_s^2) = T_s \quad \text{(62)}.$$

The seven conditions satisfied by the whole system are that the seven quantities U_n, V_n, W_n, F_n, G_n, H_n and E are constant during the motion.

Under these conditions the $3n$ momentum-components are not independent. We shall therefore transform equation (17) into one in which the differentials of the first seven velocity-components are replaced by the differentials of the seven constants.

The functional determinant is found by differentiating the seven quantities U_n, V_n, W_n, F_n, G_n, H_n and E with respect to the momenta m_1u_1, m_1v_1, m_1w_1; m_2u_2, m_2v_2, m_2w_2; and m_3u_3. We thus obtain

$$\begin{vmatrix} 1, & 0, & 0, & 0, & z_1, & -y_1, & u_1 \\ 0, & 1, & 0, & -z_1, & 0, & x_1, & v_1 \\ 0, & 0, & 1, & y_1, & -x_1, & 0, & w_1 \\ 1, & 0, & 0, & 0, & z_2, & -y_2, & u_2 \\ 0, & 1, & 0, & -z_2, & 0, & x_2, & v_2 \\ 0, & 0, & 1, & y_2, & -x_2, & 0, & w_2 \\ 1, & 0, & 0, & 0, & z_3, & -y_3, & u_3 \end{vmatrix} = \Delta \quad \text{(63)},$$

which we may write $$\Delta = \alpha\, r_{12}\, \dot{r}_{12} \quad \text{(64)},$$

where $$\alpha = (y_1 - y_2)(z_2 - z_3) - (y_2 - y_3)(z_1 - z_2) \quad \text{(65)},$$

or twice the projection on the plane of yz of the triangle whose vertices are m_1, m_2, and m_3, and

$$r_{12}\dot{r}_{12} = (u_1 - u_2)(x_1 - x_2) + (v_1 - v_2)(y_1 - y_2) + (w_1 - w_2)(z_1 - z_2) \quad \text{(66)},$$

or the rate of increase of the distance between m_1 and m_2 multiplied into that distance.

In a system composed of material particles, each component of momentum is equal to the corresponding velocity-component multiplied into the mass of the particle. We may therefore write $p_1 = m_1u_1$ and so on, and since the masses are invariable we may omit them from both members of equation (17), and write it

$$dx_1' \ldots dz_n'\, du_1' \ldots dw_n' = dx_1 \ldots dz_n du_1 \ldots dw_n \quad \text{(67)}.$$

But $$\begin{aligned} dU dV dW dF dG dH dE &= m_1^3 m_2^3 m_3 \alpha' r'_{12} \dot{r}'_{12} du_1' dv_1' dw_1' du_2' dv_2' dw_2' du_3' \\ &= m_1^3 m_2^3 m_3\, \alpha r_{12} \dot{r}_{12}\, du_1 \ldots\ldots du_3 \end{aligned} \quad \text{(68)}.$$

Hence $$\frac{dx_1' \dots dz_n' \, dv_3' \dots dw_n'}{m_1{}^3 m_2{}^3 m_3 \alpha' r'_{12} \dot{r}'_{12}} = \frac{dx_1 \dots dz_n \, dv_3 \dots dw_n}{m_1{}^3 m_2{}^3 m_3 \alpha r_{12} \dot{r}_{12}} = C \text{.....................(69)}.$$

and equation (29) becomes

$$N(b) = \int^{3n-7} C \,(m_1{}^3 m_2{}^3 m_3 \alpha r_{12} \dot{r}_{12})^{-1} \, dv_3 \dots dw_n \text{.............................(70)}.$$

We shall find it useful in what follows to define the energy of internal motion as the excess of the whole kinetic energy of the system over that which it would have if it were moving like a rigid body with the same configuration, and the same components of momentum and of angular momentum.

If we suppose the internal motion of the system to be destroyed in a very short time by internal forces, so that the configuration is not sensibly altered during the process, then the work done by the system against these forces is the measure of the energy of internal motion.

Writing T for the kinetic energy referred to the origin, K for that of the mass moving with the velocity of the centre of mass, J for the kinetic energy due to the rotation of the system as a rigid body, and I for the energy of internal motion, we have

$$I = T - K - J \text{ ....(71)},$$

where $$T = \Sigma \left[\tfrac{1}{2} m \,(u^2 + v^2 + w^2)\right] \text{ ....(72)},$$

$$K = \tfrac{1}{2} M \,(U^2 + V^2 + W^2) \text{ ....(73)},$$

$$J = \tfrac{1}{2} \,(Fp + Gq + Hr) \text{ ....(74)},$$

where p, q, r are the components of angular velocity with respect to the axes of x, y, z and are related to F, G, H by the equations

$$\left.\begin{aligned} Ap - Nq - Mr &= F, & aF - nG - mH &= p, \\ -Np + Bq - Lr &= G, & -nF + bG - lH &= q, \\ -Mp - Lq + Cr &= H, & -mF - lG + cH &= r, \end{aligned}\right\} \text{.....................(75)},$$

where

$$\left.\begin{aligned} A &= \Sigma m \left[(y - Y)^2 + (z - Z)^2\right] & L &= \Sigma m \,(y - Y)(z - Z) \\ B &= \Sigma m \left[(z - Z)^2 + (x - X)^2\right] & M &= \Sigma m \,(z - Z)(x - X) \\ C &= \Sigma m \left[(x - X)^2 + (y - Y)^2\right] & N &= \Sigma m \,(x - X)(y - Y) \end{aligned}\right\} \text{............(76)}.$$

Writing for the sake of brevity

$$D = \begin{vmatrix} A, & -N, & -M \\ -N, & B, & -L \\ -M, & -L, & C \end{vmatrix}, \qquad d = \begin{vmatrix} a, & -n, & -m \\ -n, & b, & -l \\ -m, & -l, & c \end{vmatrix} \text{ ....(77)},$$

the relations between the moments and products of mobility and those of inertia will be given by equations of the forms

$$\left.\begin{array}{ll} aD = BC - L^2 & Ad = bc - l^2 \\ lD = -MN - AL, & Ld = -mn - al, \\ & \hspace{-4em} Dd = 1. \end{array}\right\} \quad \text{.....................(78).}$$

If we write

$$\left.\begin{aligned} \xi &= u - U + qz - ry \\ \eta &= v - V + rx - pz \\ \zeta &= w - W + py - qx \end{aligned}\right\} \quad \text{.....................................(79),}$$

then ξ, η, ζ will be the velocity-components of a particle with respect to axes passing through the centre of mass of the system and rotating with the angular velocity whose components are p, q, r. We may therefore call ξ, η, ζ the velocity-components of the internal motion. If the system were to become rigid, the internal motion would become zero. The energy of internal motion may be expressed in terms of ξ, η, ζ, thus:—

$$\mathrm{I} = \Sigma\, \tfrac{1}{2} m\, (\xi^2 + \eta^2 + \zeta^2) \quad \text{.......................................(80).}$$

We have now to express the energy of internal motion of a system of $s-1$ particles in terms of the quantities U, V, W, F, G, H and T belonging to the system of s particles, together with the position and velocity of the s^{th} particle.

To avoid the repetition of suffixes we shall distinguish quantities belonging to the minor system of $s-1$ particles by accented letters, and quantities belonging to the complete system of s particles and the particle m_s by unaccented letters. We shall also write

We thus find

$$\left.\begin{aligned} \mu &= \frac{Mm}{M'}\,. \\ M' &= M - m \\ M'X' &= MX - mx \\ M'U' &= MU - mu, \\ F' &= F - \mu\,(y - Y)\,(w - W) + \mu\,(z - Z)\,(v - V) \\ A' &= A - \mu\,(y - Y)^2 - \mu\,(z - Z)^2 \\ L' &= L - \mu\,(y - Y)\,(z - Z) \\ T' &= T - \tfrac{1}{2} m\,(u^2 + v^2 + w^2) \\ K' &= K + \tfrac{1}{2}\mu\,[(U - u)^2 + (V - v)^2 + (W - w)^2] - \tfrac{1}{2} m\,(u^2 + v^2 + w^2). \end{aligned}\right\} \quad \text{......(81).}$$

Since the choice of the axes of reference is arbitrary, we may simplify the expressions by taking for origin the centre of mass of the system M, and for the axis of z the line passing through the particle m. We may also turn the axes of x and y about that of z till A becomes a maximum, the condition of which is

$$LM + CN = 0.$$

We shall also reckon velocities with reference to the centre of mass of the system M.

With these simplifications we find

$$\left.\begin{array}{lll} F' = F + \mu v z & G' = G - \mu u z & H' = H \\ A' = A - \mu z^2 & B' = B - \mu z^2 & C' = C \\ L' = L & M' = M & N' = N \\ a' = \dfrac{a}{1 - a\mu z^2}, & & l' = \dfrac{l'}{1 - b\mu z^2}, \\ b' = \dfrac{b}{1 - b\mu z^2}, & & m' = \dfrac{m}{1 - a\mu z^2}, \\ c' = c + \mu z^2 \left(\dfrac{l^2}{1 - b\mu z^2} + \dfrac{m^2}{1 - a\mu z^2}\right), & & n' = n = 0, \\ \multicolumn{3}{l}{D' = D(1 - a\mu z^2)(1 - b\mu z^2).} \end{array}\right\} \quad \text{.........(82).}$$

We are now able to calculate the energy of rotation, J', of the minor system:

$$2J' = a'F'^2 + b'G'^2 + c'H'^2 - 2l'G'H' - 2m'H'F' - 2n'F'G \quad \text{.....................(83)},$$

$$\left.\begin{aligned} &= 2J + \frac{1}{1 - a\mu z^2}[v^2 a\mu z^2 - 2v\mu z(Fa - Hm) + \mu z^2 (Fa - Hm)^2] \\ &\quad + \frac{1}{1 - b\mu z^2}[u^2 b\mu z^2 + 2u\mu z(Gb - Hl) + \mu z^2 (Gb - Hl)^2] \end{aligned}\right\} \quad \text{.........(84).}$$

Combining these results and reducing we find for the energy of internal motion of the system M'

$$I' = I - \tfrac{1}{2}\mu(1 - b\mu z^2)^{-1}(u - Gb + Hl)^2 - \tfrac{1}{2}\mu(1 - a\mu z^2)^{-1}(v - Fa + Hl)^2 - \tfrac{1}{2}\mu w^2 \text{.....(85).}$$

Hence $$\iiint I'^{\frac{q}{2}}\, du\,dv\,dw = \frac{(\Gamma(\frac{1}{2}))^3 \Gamma\left(\frac{q+2}{2}\right)}{\Gamma\left(\frac{q+5}{2}\right)} \left(\frac{2}{\mu}\right)^{\frac{3}{2}} (1 - a\mu z^2)^{\frac{1}{2}} (1 - b\mu z^2)^{\frac{1}{2}} I^{\frac{q+3}{2}} \quad \text{............(86),}$$

the integration being extended to all values of u, v, and w which make I' positive.

Now $(1 - a\mu z^2)(1 - b\mu z^2) = \dfrac{D'}{D}$, and this is an invariant.

Hence in general, whatever axes we choose,

$$\iiint [M_{s-1}^3 D_{s-1}]^{-\frac{1}{2}} I_{s-1}^{\frac{q}{2}}\, du_s\,dv_s\,dw_s = \frac{(\Gamma(\frac{1}{2}))^3 \Gamma\left(\frac{q+2}{2}\right)}{\Gamma\left(\frac{q+5}{2}\right)} [\tfrac{1}{2} m_s]^{-\frac{3}{2}} [M_s^3 D_s]^{-\frac{1}{2}} I_s^{\frac{q+3}{2}} \quad \text{.......(87).}$$

For the system consisting of the two particles m_1 and m_2 the energy of rotation is

$$J_2 = \frac{1}{2}\frac{M_2}{m_1 m_2 r_{12}^2}(F_2^2 + G_2^2 + H_2^2) \quad \text{...................................(88),}$$

and the energy of internal motion is

$$I_2 = \tfrac{1}{2}\frac{m_1 m_2}{M_2}\dot{r}_{12}^2 \qquad \text{(89)}.$$

Hence we may write equation (70)

$$N(b) = \int^{3n-7} C\,(m_1^3 m_2^3 m_3 \alpha_1 r_2)^{-1}\left(2\frac{M_2}{m_1 m_2}\right)^{-\frac{1}{2}} I_2^{-\frac{1}{2}}\,dv_3 \ldots dw_n \qquad \text{(90)}.$$

We have first to express I_2 in terms of quantities having the suffix $_3$.

If we make the plane of yz pass through the three particles m_1, m_2, m_3, so that the origin coincides with their centre of mass and has the same velocity, and the axis of z passes through m_3, then α is twice the area of the triangle whose vertices are m_1, m_2 and m_3,

$$F_2 = F_3 + \frac{M_3 m_3}{M_2} z_3 v_3,\quad G_2 = G_3 - \frac{M_3 m_3}{M_2} z_3 u_3,\quad H_2 = H_3 \qquad \text{(91)},$$

$$\alpha m_3 u_3 = G_3(y_1 - y_2) + H_3(z_1 - z_2) \qquad \text{(92)},$$

$$I_2 = I_3 - \tfrac{1}{2}\frac{M_3 m_3}{M_2}\left(1 + \frac{M_3 m_3}{m_1 m_2}\frac{z_3}{r_{12}^3}\right)\left[\left(u - \frac{M_2 G_3 z_3}{m_1 m_2 r_{12}^2 + M_3 m_3 z_3^2}\right)^2 + \left(v + \frac{M_2 F_3 z_3}{m_1 m_2 r_{12}^2 + M_3 m_3 z_3}\right)^2\right] - \tfrac{1}{2}\frac{M_3 m_3}{M_2} w^2 \qquad \text{(93)}.$$

We have now to integrate

$$\iint I_2^{-\frac{1}{2}}\,dv_3 dw_3,$$

extending the integration to all values of v_3 and w_3 which make I_2 positive, and remembering that equation (92) shows that u_3 is independent of v_3 and w_3. The result is

$$\iint I_2^{-\frac{1}{2}}\,dv_3 dw_3 = \frac{(\Gamma(\frac{1}{2}))^3}{\Gamma(\frac{3}{2})}\left[\frac{1}{4}\frac{M_3^2 m_3^2}{M_2^2}\cdot\frac{m_1 m_2 r_{12}^2 + M_3 m_3 z_3^2}{m_1 m_2 r_{12}^2}\right]^{-\frac{1}{2}} I_3^{\frac{1}{2}} \qquad \text{(94)}.$$

Now for the three particles m_1, m_2, m_3,

$$D_3 = \frac{m_1 m_2 m_3}{M_3^2}\left[r_{23}^2 m_2 m_3 + r_{31}^2 m_3 m_1 + r_{12}^2 m_1 m_2\right]\alpha^2 \qquad \text{(95)},$$

where r_{23}, r_{31} and r_{12} are the distances between the particles, and α is the area of the triangle $m_1 m_2 m_3$.

Also $$r_{23}^2 m_2 m_3 + r_{31}^2 m_3 m_1 + r_{12}^2 m_1 m_2 = \frac{M_3}{M_2}(m_1 m_2 r_{12}^2 + M_3 m_3 z_3) \qquad \text{(96)}.$$

We may now write equation (90) in the form

$$N(b) = \int^{3n-9} C\frac{(\Gamma\frac{1}{2})^3}{\Gamma(\frac{3}{2})}\left[\tfrac{1}{2} m_1^3 m_2^3 m_3^3 M_3^3 D_3\right]^{-\frac{1}{2}} I_3^{\frac{1}{2}}\,du_4 \ldots dw_n \qquad \text{(97)}.$$

Continuing the integration by equation (87) we find

$$N(b) = 2^{\frac{3n-8}{2}} C \frac{(\Gamma \frac{1}{2})^{3n-6}}{\Gamma\left(\frac{3n-6}{2}\right)} (m_1 \dots m_n)^{-\frac{3}{2}} M_n^{-\frac{3}{2}} D_n^{-\frac{1}{2}} I_n^{\frac{3n-8}{2}} \qquad \text{(98)},$$

where I_n is what we have defined as the energy of internal motion of the system, or the work which the system would do, in virtue of its motion, against the system of internal forces which would be called into play if the distances between the parts of the material system were in an insensibly small time to become invariable.

In order to determine the number of systems in a given configuration for which the velocity-components of the particle m_n lie between the limits $u \pm \frac{1}{2}du$, $v \pm \frac{1}{2}dv$, $w \pm \frac{1}{2}dw$, we must form the expression for $N(b, u_n, v_n, w_n)$ by stopping short before the last triple integration.

We thus find

$$N(b, u_n, v_n, w_n) = 2^{\frac{3n-11}{2}} C \frac{\{\Gamma(\frac{1}{2})\}^{3n-9}}{\Gamma\left(\frac{3n-9}{2}\right)} (m_1 \dots m_{n-1})^{\frac{3}{2}} M_{n-1}^{-\frac{3}{2}} D_{n-1}^{-\frac{1}{2}} I_{n-1}^{\frac{3n-11}{2}} du_n dv_n dw_n \qquad \text{(99)}.$$

If, as in equations (82) to (86), we suppose the origin of co-ordinates to be the centre of mass of the whole system, the axis of z to pass through the particle m_n, and the axes of x and y to be in the directions of the principal axes of the section of the momental ellipsoid normal to z, then writing

$$\xi = u - qz, \quad \eta = v + pz, \quad \zeta = w \qquad \text{(100)},$$

so that ξ, η, ζ are the velocity-components of m_n relative to axes moving as the system would do if it were then to become rigid, with the angular velocity whose components are p, q, r, we may write

$$I_{n-1} = I_n - \tfrac{1}{2}\mu (1 - b\mu z^2)^{-1} \xi^2 - \tfrac{1}{2}\mu (1 - a\mu z^2)^{-1} \eta^2 - \tfrac{1}{2}\mu z^2 \qquad \text{(101)}.$$

The sum of the last three terms of this expression, with its sign taken positive, represents the part of the internal motion of the system which is due to the fact that the particle m_n is moving with the relative velocity whose components are ξ, η, ζ.

We may also define it as the work which would be done by the particle m_n against the internal forces of the system, if these forces were suddenly to become such as to render the whole system rigid in an infinitely short time.

Comparing this result with that obtained in equation (48), we see that the law of distribution of the velocities of the particle m_n is the same as what it would be in a fixed vessel containing $n-2$ particles, provided that we substitute for u^2, v^2, w^2 the quantities $(1 - b\mu z^2)^{-1}\xi^2$, $(1 - a\mu z^2)^{-1}\eta^2$, ζ^2 respectively.

Hence the mean square of the velocity in the direction of the line joining the particle with the centre of mass is the same at all points of the system, but the mean square of the velocity in other directions is less than this in the ratio of $1 - a\mu z^2$ to 1, where z is the perpendicular from the centre of mass on the line of relative motion of the particle, and a is the moment of mobility of the system about an axis through the centre of mass and normal to the plane through that centre and the line of motion.

When the product of the mass of the particle into the square of its distance from the centre is so small that it may be neglected in comparison with the moments of inertia of the system, then quantities like $a\mu z^2$ and $b\mu z^2$ may be neglected in respect of unity, and we may assert that the mean square of the relative velocity, for a particle of given mass, is the same in all directions and at all points of the system; but that for different particles it varies inversely as their masses; so that the average energy of motion relative to the moving axes is the same for particles of all kinds throughout the system.

We have already learned from equation (98) that in a free system of n particles the number of cases in which the system is in a given configuration, or, in other words, the probability of that configuration, is proportional to the $\frac{3n-8}{2}$ power of the energy of internal motion corresponding to that configuration.

We have next to consider the manner in which this probability depends on the position of a particular particle, say of the last particle, m_n.

Let $I_n^{(0)}$ denote the energy of internal motion of the complete system when m_n is at the centre of mass of the system and is without any velocity relative to that centre. It is manifest that in this case m_n contributes nothing towards the energy of internal motion.

Now let m_n be carried from the centre of mass to the point $(0, 0, z)$ and left there without any velocity (that is, let $u = v = w = 0$).

Let W be the work which must be done against the forces of the system to effect this transference, then since the total energy of the system and the three angular momenta must be maintained constant, we shall have after this displacement, for the energy of internal motion of the remaining $n-1$ particles,

$$I_{n-1} = I_n^{(0)} - W \qquad (102).$$

But by equation (85)

$$I_n = I_{n-1} + \tfrac{1}{2}\mu\,(1 - b\mu z^2)^{-1}(u - qz)^2 + \tfrac{1}{2}\mu\,(1 - a\mu z^2)^{-1}(v + pz)^2 + \tfrac{1}{2}\mu w^2 \qquad (103).$$

Substituting the value of I_{n-1} from equation (102), and remembering that $u = v = w = 0$, we find for the energy of internal motion in the new configuration

$$I_n = I_n^{(0)} - W + \tfrac{1}{2}\mu\,(1 - b\mu z^2)^{-1} q^2 z^2 + \tfrac{1}{2}\mu\,(1 - a\mu z^2)\, p^2 z^2 \qquad (104).$$

The probability, therefore, of a configuration in which, the positions of all the other particles being given, that of m_n is varied, is proportional to $I_n^{\frac{3n-8}{2}}$, I_n being given by equation (104).

When, as in the case of a gas, there are a great many particles similar to m_n, we may speak of the density of the medium consisting of such particles in the element $dxdydz$. In this case, however, for reasons already given, neglect the quantities $a\mu z^4$ and $b\mu z^2$, and we may write m for μ. We may also choose our axes in the manner which is most convenient. We shall therefore make the axis of z that round which the system, if it were rendered rigid, would rotate with velocity ω, and we shall suppose this axis to be vertical, as otherwise a steady motion under the action of gravity could not exist, and we shall denote the horizontal distance from this axis by r.

We may now write for the density of the gas at the point (z, r)

$$\rho = \rho_0 \left[1 + (2I_n^{(0)})^{-1} (m\omega^2 r^2 - 2mgz)\right]^{\frac{3n-8}{2}} \quad \text{.......................(105)},$$

where ρ_0 is the density at the origin.

When n is a very large number and when the second term of the binomial is very small compared with unity, we may write for this the exponential expression

$$\rho = \rho_0 e^{\frac{3}{4}\frac{mn}{I}(\omega^2 r^2 - 2gz)} \quad \text{...................................(106)}.$$

If m_0 is the mass of a molecule of hydrogen, μm_0 will be the mass of a molecule of the kind of gas considered, where μ is the chemical equivalent of the gas.

Also if T is the temperature on the centigrade scale, and α the coefficient of dilatation of a perfect gas, then since the "velocity of mean square" of agitation of the molecules of hydrogen at 0° C. is $1 \cdot 844 \times 10^5$ centimetres per second, the kinetic energy of agitation of a system containing n molecules of any kind will be

$$\tfrac{3}{2} m_0 n \,(1 \cdot 844)^2\, 10^{10}\, (1 + \alpha T),$$

and the difference between this and the energy of internal motion may be neglected.

We thus find for the density at any point

$$\rho = \rho_0 e^{\frac{\mu}{2} \cdot \frac{\omega^2 r^2 - 2gz}{1 \cdot 844^2\, 10^{10} (1 + \alpha T)}} \quad \text{..................................(107)}.$$

Let us now consider a tube of uniform section placed on a whirling table so that one end, A, of the tube coincides with the axis while the other end, B, revolves about the axis with the angular velocity ω. The linear velocity of B is ωr, and we shall suppose, for the sake of easy calculation, that this velocity is one-tenth of the velocity of agitation of the molecules of hydrogen. The velocity of the end B would be $184 \cdot 4$ metres per second. If the tube contains hydrogen at 0° C., the ratio of the density of the gas at B to the density at A will be $e^{\frac{1}{200}}$, or approximately $1 + \frac{1}{200}$.

If it contains a gas whose chemical equivalent is μ, the ratio will be $1 + \frac{\mu}{200}$.

If the tube contains hydrogen and carbonic acid, and if a certain volume of the tube at A contains 200 parts of hydrogen and 200 of carbonic acid, then an equal volume of the tube at B will contain 201 parts of hydrogen and 222 parts of carbonic acid.

The time during which the experiment would require to be continued in order to obtain a given degree of approximation to the ultimate distribution of the mixed gases varies as the square of the length of the tube.

Thus in Loschmidt's experiments on the diffusion of gases he used a tube about a metre long, and continued his experiments from half an hour to an hour in order to obtain the results from which he could best deduce the coefficient of diffusion.

In these experiments the inequalities of distribution of hydrogen and carbonic acid were reduced to less than a third part of their original value in half an hour, and if the experiment had gone on for two hours the differences from the ultimate distribution would have been reduced to a hundredth part of their original value.

We may therefore consider two hours as ample time for an experiment on the ultimate distribution of these two gases in a tube one metre in length.

But if we make the whirling tube 20 centimetres long, the differences of distribution from the ultimate distribution would be reduced to a hundredth part of their original value in a twenty-fifth part of the time, that is to say in 4 minutes 48 seconds.

If it were found more convenient to have bulbs on the ends of the tubes, so as to be able to secure the gas at each end before it got mixed up by the violent commotion arising from the stopping of the whirling tube, we should have to allow a longer time for the whirling.

In order to obtain a similar distribution of the two gases in a vertical tube by the action of gravity the tube would require to be 1720 metres high, and in order to obtain the same degree of approximation to the ultimate distribution we should have to let the experiment go on for 675 years, carefully preserving the tube during that time from all inequalities of temperature, which, by causing convection-currents, would continually mix up the gases and prevent their partial separation.

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