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CONTENTS.

I. *A solution of the equations for the equilibrium of elastic solids having an axis of material symmetry, and its application to rotating spheroids.* By C. Chree, M.A., Fellow of King's College, Cambridge.

[Read Nov. 25, 1889.]

§ 1. If t_{xx}.. t_{yz}.. denote the stresses and u, v, w the displacements in an elastic solid of uniform density ρ, acted on by an external system of forces X, Y, Z, the three internal equations are of the form

$$\left.\begin{array}{l}\frac{dt_{xx}}{dx}+\frac{dt_{xy}}{dy}+\frac{dt_{xz}}{dz}+X-\rho\frac{d^2u}{dt^2}=0\\ \dots\dots\dots\dots\dots\dots\\ \dots\dots\dots\dots\dots\dots\end{array}\right\}\dots\dots\dots\dots(1).$$

If the axis of z be an axis of symmetry in the material, the stress-strain relations are*

$$\left.\begin{array}{ll} t_{xx}=(2\mathbf{f}+\mathbf{f}')\frac{du}{dx}+\mathbf{f}'\frac{dv}{dy}+\mathbf{d}'\frac{dw}{dz}, & t_{yz}=\mathbf{d}\left(\frac{dv}{dz}+\frac{dw}{dy}\right),\\ t_{yy}=\mathbf{f}'\frac{du}{dx}+(2\mathbf{f}+\mathbf{f}')\frac{dv}{dy}+\mathbf{d}'\frac{dw}{dz}, & t_{zx}=\mathbf{d}\left(\frac{dw}{dx}+\frac{du}{dz}\right),\\ t_{zz}=\mathbf{d}'\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\mathbf{c}\frac{dw}{dz}, & t_{xy}=\mathbf{f}\left(\frac{du}{dy}+\frac{dv}{dx}\right).\end{array}\right\}\dots\dots\dots(2).$$

When the solid is in equilibrium in the absence of the bodily forces X, Y, Z, substituting in (1) from (2) and arranging the terms we get

$$\mathbf{f}\nabla^2u+(\mathbf{d}-\mathbf{f})\frac{d^2u}{dz^2}+(\mathbf{f}+\mathbf{f}')\frac{d\delta}{dx}+(\mathbf{d}+\mathbf{d}'-\mathbf{f}-\mathbf{f}')\frac{d^2w}{dxdz}=0\dots\dots\dots(3),$$

$$\mathbf{f}\nabla^2v+(\mathbf{d}-\mathbf{f})\frac{d^2v}{dz^2}+(\mathbf{f}+\mathbf{f}')\frac{d\delta}{dy}+(\mathbf{d}+\mathbf{d}'-\mathbf{f}-\mathbf{f}')\frac{d^2w}{dydz}=0\dots\dots\dots(4),$$

$$\mathbf{d}\nabla^2w+(\mathbf{c}-2\mathbf{d}-\mathbf{d}')\frac{d^2w}{dz^2}+(\mathbf{d}+\mathbf{d}')\frac{d\delta}{dz}=0\dots\dots\dots(5);$$

* Saint-Venant's *Théorie de l'Élasticité des Corps Solides de Clebsch*, p. 77.

where as usual

$$\delta \equiv \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad\text{...(6)},$$

$$\nabla^2 \equiv \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \quad\text{...(7)}.$$

Differentiating (3) with respect to x and (4) with respect to y, then adding and arranging the terms, we get

$$\left[(2\mathbf{f}+\mathbf{f}')\nabla^2 + (\mathbf{d}-2\mathbf{f}-\mathbf{f}')\frac{d^2}{dz^2}\right]\delta = \left[(2\mathbf{f}+\mathbf{f}'-\mathbf{d}-\mathbf{d}')\nabla^2 + (2\mathbf{d}+\mathbf{d}'-2\mathbf{f}-\mathbf{f}')\frac{d^2}{dz^2}\right]\frac{dw}{dz} \quad\text{...(8)}.$$

Differentiating (5) with respect to z we get

$$(\mathbf{d}+\mathbf{d}')\frac{d^2\delta}{dz^2} + \left[\mathbf{d}\nabla^2 + (\mathbf{c}-2\mathbf{d}-\mathbf{d}')\frac{d^2}{dz^2}\right]\frac{dw}{dz} = 0 \quad\text{........................(9)}.$$

Combining (8) and (9) we find for the equation from which δ or $\frac{dw}{dz}$ must be derived

$$\left[\mathbf{d}(2\mathbf{f}+\mathbf{f}')\nabla^2 . \nabla^2 + \{(2\mathbf{f}+\mathbf{f}')(\mathbf{c}-2\mathbf{d}) - \mathbf{d}'(2\mathbf{d}+\mathbf{d}')\}\nabla^2\frac{d^2}{dz^2} + \{\mathbf{cd} - (2\mathbf{f}+\mathbf{f}')(\mathbf{c}-\mathbf{d}) + \mathbf{d}'(2\mathbf{d}+\mathbf{d}')\}\frac{d^4}{dz^4}\right]\begin{matrix}\delta \\ \frac{dw}{dz}\end{matrix} = 0 \quad\text{......(10)}.$$

In this equation it is obvious that δ may be replaced by $\frac{du}{dx} + \frac{dv}{dy}$.

§ 2. Confining our attention to solutions containing only integral powers of the variables, it is obvious that (10) is satisfied by any term the sum of whose indices is less than 4. For our immediate purpose we do not require to carry the expression for δ above the terms of the second degree of the variables, and so the equations we shall really have to do with at present are (8) and (9) not (10).

All possible terms not higher than the second degree are included in

$$\delta = A_{1.0} + A_{2.1}x + B_{2.1}y + A_{2.0}z + \tfrac{1}{2}A_{3.0}(2z^2 - x^2 - y^2) + 3A_{3.2}(x^2 - y^2) + F_3(x^2 + y^2) + 6B_{3.2}xy + 3A_{3.1}xz + 3B_{3.1}yz \quad\text{.........(11)},$$

$$\frac{dw}{dz} = \text{similar expression with dashed letters} \quad\text{..........................(12)},$$

where $A_{1.0}$, $A'_{1.0}$ etc. are constants.

The first of the two suffixes attached to a letter indicates the dimensions of the corresponding terms in the expressions for the displacements. A second suffix has not been attached to F_3 and F_3' because these constants in consequence of (8) and (9) are immediately connected with $A_{3.0}$ and $A'_{3.0}$ by the relations

$$(\mathbf{d}-2\mathbf{f}-\mathbf{f}')\,A_{3,0}+2\,(2\mathbf{f}+\mathbf{f}')\,F_3=(2\mathbf{d}+\mathbf{d}'-2\mathbf{f}-\mathbf{f}')\,A'_{3,0}+2\,(2\mathbf{f}+\mathbf{f}'-\mathbf{d}-\mathbf{d}')\,F_3' \quad\ldots(13),$$

$$(\mathbf{d}+\mathbf{d}')\,A_{3,0}+(\mathbf{c}-2\mathbf{d}-\mathbf{d}')\,A'_{3,0}+2\mathbf{d}F_3'=0 \quad\ldots(14).$$

From (13) and (14) we could substitute at once for F_3 and F_3', but it will be more convenient to retain them at present.

Integrating (12) we find

$$\begin{aligned} w &= A'_{1,0}z+A'_{2,1}\,xz+B'_{2,1}yz+\tfrac{1}{2}A'_{2,0}z^2+\tfrac{1}{2}A'_{3,0}\,(\tfrac{2}{3}z^3-x^2z-y^2z) \\ &\quad+3A'_{3,2}z\,(x^2-y^2)+6B'_{3,2}xyz+\tfrac{3}{2}A'_{3,1}xz^2+\tfrac{3}{2}B'_{3,1}yz^2+F_3'z\,(x^2+y^2) \\ &\quad+\phi\,(x,y) \quad\ldots(15); \end{aligned}$$

where

$$\begin{aligned} \phi\,(x,y) &\equiv \bar{\alpha}_1x+\bar{\beta}_1y+\bar{\gamma}_2xy+\bar{\epsilon}_2\,(x^2-y^2)+\bar{\zeta}_2\,(x^2+y^2) \\ &\quad+\bar{\eta}_3\,(x^3-3xy^2)+\bar{\theta}_3\,(y^3-3yx^2)+\bar{\lambda}_3\,(x^3+3xy^2)+\bar{\mu}_3\,(y^3+3yx^2) \quad\ldots(16). \end{aligned}$$

Here $\bar{\alpha}_1$, etc. are new constants; and all possible terms of less than the fourth degree which can appear in the value of w are included.

On account of (5) we have the following relations between the constants occurring in (11) and (15):

$$(\mathbf{d}+\mathbf{d}')\,A_{2,0}+(\mathbf{c}-\mathbf{d}-\mathbf{d}')\,A'_{2,0}+4\mathbf{d}\bar{\zeta}_2=0 \quad\ldots(17),$$

$$(\mathbf{d}+\mathbf{d}')\,A_{3,1}+(\mathbf{c}-\mathbf{d}-\mathbf{d}')\,A'_{3,1}+4\mathbf{d}\bar{\lambda}_3=0 \quad\ldots(18),$$

$$(\mathbf{d}+\mathbf{d}')\,B_{3,1}+(\mathbf{c}-\mathbf{d}-\mathbf{d}')\,B'_{3,1}+4\mathbf{d}\bar{\mu}_3=0 \quad\ldots(19).$$

If for shortness

$$\left.\begin{aligned} \bar{A}_0 &\equiv -(\mathbf{f}+\mathbf{f}')\,A_{2,1}+(\mathbf{f}+\mathbf{f}'-\mathbf{d}-\mathbf{d}')\,A'_{2,1}, \\ \bar{A}_1 &\equiv (\mathbf{f}+\mathbf{f}')(A_{3,0}-6A_{3,2}-2F_3)+(\mathbf{d}+\mathbf{d}'-\mathbf{f}-\mathbf{f}')\,(A'_{3,0}-6A'_{3,2}-2F_3'), \\ \bar{A}_2 &\equiv 6\,(\mathbf{f}+\mathbf{f}')\,(B'_{3,2}-B_{3,2})-6\,(\mathbf{d}+\mathbf{d}')\,B'_{3,2}, \\ \bar{A}_3 &\equiv 3\,(\mathbf{f}+\mathbf{f}')\,(A'_{3,1}-A_{3,1})-3\,(\mathbf{d}+\mathbf{d}')\,A'_{3,1}, \end{aligned}\right\} \quad\ldots(20),$$

then substituting from (11) and (12) in (3), we have to determine u from

$$\mathbf{f}\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right)+\mathbf{d}\frac{d^2u}{dz^2}=\bar{A}_0+\bar{A}_1x+\bar{A}_2y+\bar{A}_3z \quad\ldots(21).$$

A complete solution, so far as terms of not higher than the third degree are concerned, is

$$\begin{aligned} u &= (\bar{A}_0+\bar{A}_1x+\bar{A}_2y+\tfrac{1}{3}\bar{A}_3z)\,z^2/2\mathbf{d} \\ &\quad+\alpha_1x+\beta_1y+\gamma_1z+\alpha_2\,(x^2-y^2)+\beta_2xy+\gamma_2xz+\epsilon_2yz \\ &\quad+\eta_2x^2+\zeta_2y^2-\mathbf{fd}^{-1}\,(\eta_2+\zeta_2)\,z^2+\alpha_3\,(x^3-3xy^2)+\beta_3\,(y^3-3yx^2)+\gamma_3xyz \\ &\quad+\epsilon_3x^2y+\eta_3y^3-\mathbf{fd}^{-1}\,(\epsilon_3+3\eta_3)\,yz^2+\zeta_3y^2x+\theta_3x^3-\mathbf{fd}^{-1}\,(\zeta_3+3\theta_3)\,xz^2 \\ &\quad+\lambda_3x^2z+\mu_3y^2z-\tfrac{1}{3}\mathbf{fd}^{-1}\,(\lambda_3+\mu_3)\,z^3 \quad\ldots(22). \end{aligned}$$

Similarly if for shortness

$$\left.\begin{aligned}
&\overline{B}_0 \equiv (\mathbf{f}+\mathbf{f}')(B'_{2,1}-B_{2,1})-(\mathbf{d}+\mathbf{d}')B'_{2,1},\\
&\overline{B}_1 \equiv (\mathbf{f}+\mathbf{f}')\{A_{3,0}-A'_{3,0}+6(A_{3,2}-A'_{3,2})-2(F_3-F_3')\}+(\mathbf{d}+\mathbf{d}')(A'_{3,0}+6A'_{3,2}-2F_3'),\\
&\overline{B}_2 = 6\{(\mathbf{f}+\mathbf{f}')(B'_{3,2}-B_{3,2})-(\mathbf{d}+\mathbf{d}')B'_{3,2}\},\\
&\overline{B}_3 = 3(\mathbf{f}+\mathbf{f}')(B'_{3,1}-B_{3,1})-3(\mathbf{d}+\mathbf{d}')B'_{3,1},
\end{aligned}\right\}\ldots(23),$$

we find from (11), (12) and (4),

$$\begin{aligned}
v = {} & (\overline{B}_0+\overline{B}_1 y+\overline{B}_2 x+\tfrac{1}{3}\overline{B}_3 z)\, z^2 \div 2\mathbf{d}\\
&+\alpha_1'x+\beta_1'y+\gamma_1'z+\alpha_2'(x^2-y^2)+\beta_2'xy+\gamma_2'xz+\epsilon_2'yz\\
&+\eta_2'x^2+\zeta_2'y^2-\mathbf{fd}^{-1}(\eta_2'+\zeta_2')z^2+\alpha_3'(x^3-3xy^2)+\beta_3'(y^3-3yx^2)+\gamma_3'xyz\\
&+\epsilon_3'x^2y+\eta_3'y^3-\mathbf{fd}^{-1}(\epsilon_3'+3\eta_3')yz^2+\zeta_3'y^2x+\theta_3'x^3-\mathbf{fd}^{-1}(\zeta_3'+3\theta_3')xz^2\\
&+\lambda_3'x^2z+\mu_3'y^2z-\tfrac{1}{3}\mathbf{fd}^{-1}(\lambda_3'+\mu_3')z^3 \ldots\ldots(24).
\end{aligned}$$

In consequence of the identity (6) the following relations subsist between the constants in (15), (22) and (24):

$$\alpha_1+\beta_1'+A'_{1,0}-A_{1,0}=0 \ldots\ldots(25),$$

$$2\alpha_2+2\eta_2+\beta_2'+A'_{2,1}-A_{2,1}=0 \ldots\ldots(26),$$

$$\beta_2-2\alpha_2'+2\zeta_2'+B'_{2,1}-B_{2,1}=0 \ldots\ldots(27),$$

$$\gamma_2+\epsilon_2'+A'_{2,0}-A_{2,0}=0 \ldots\ldots(28),$$

$$3\alpha_3+3\theta_3-3\beta_3'+\epsilon_3'+\tfrac{1}{2}(A_{3,0}-A'_{3,0}-6A_{3,2}+6A'_{3,2}-2F_3+2F_3')=0 \ldots\ldots(29),$$

$$-3\alpha_3+3\eta_3'+3\beta_3'+\zeta_3+\tfrac{1}{2}(A_{3,0}-A'_{3,0}+6A_{3,2}-6A'_{3,2}-2F_3+2F_3')=0 \ldots\ldots(30),$$

$$\tfrac{1}{2}\mathbf{d}^{-1}(A_1+\overline{B}_1)-A_{3,0}+A'_{3,0}-\mathbf{fd}^{-1}(\zeta_3+3\theta_3+\epsilon_3'+3\eta_3')=0 \ldots\ldots(31),$$

$$\epsilon_3-3\beta_3-3\alpha_3'+\zeta_3'+3B'_{3,2}-3B_{3,2}=0 \ldots\ldots(32),$$

$$2\lambda_3+\gamma_3'+3A'_{3,1}-3A_{3,1}=0 \ldots\ldots(33),$$

$$\gamma_3+2\mu_3'+3B'_{3,1}-3B_{3,1}=0 \ldots\ldots(34).$$

§ 3. Multiplying (31) by $\mathbf{d}/\mathbf{f}$, and adding it to the sum of (29) and (30), we obtain an equation identical with (13). There thus exist between the constants of the solution only 14 independent relations, viz. (14), (17), (18), (19), and (25) to (34). Since 65 constants occur in the solution this leaves 51 of them arbitrary, to be determined by the surface conditions.

Certain of these constants fall into sets which seem fitted for application to different problems. The constants of any one set are *associated* with one or more of the constants occurring in the expression (11) for δ. The following table gives an analysis of the constants:—

TABLE I.

Degree of terms in displacements in which Constants occur	Associated Constants	Representative Constants	Connecting equations	Unassociated Constants
First	$A_{1,0}$ $A'_{1,0}$ α_1 β_1'	$A_{1,0}$	(25)	$\bar{\alpha}_1$ α_1' $\bar{\beta}_1$ β_1 γ_1 γ_1'
Second	$A_{2,0}$ $A'_{2,0}$ γ_2 ϵ_2' $\bar{\zeta}_2$	$A_{2,0}$	(17), (28)	
	$A_{2,1}$ $A'_{2,1}$ α_2 β_2' η_2	$A_{2,1}$	(26)	$\bar{\gamma}_2$ γ_2' $\bar{\epsilon}_2$ ϵ_2 η_2' ζ_2
	$B_{2,1}$ $B'_{2,1}$ α_2' β_2 ζ_2'	$B_{2,1}$	(27)	
Third	$A_{3,0}$ $A'_{3,0}$ $A_{3,2}$ $A'_{3,2}$ F_3 F_3' α_3 β_3' ϵ_3' η_3' ζ_3 θ_3	$A_{3,0}$ $A_{3,2}$	(14), (29), (30), (31)	
	$A_{3,1}$ $A'_{3,1}$ γ_3' $\bar{\lambda}_3$ λ_3	$A_{3,1}$	(18), (33)	$\bar{\eta}_3$ η_3 $\bar{\theta}_3$ θ_3' λ_3' μ_3
	$B_{3,1}$ $B'_{3,1}$ γ_3 $\bar{\mu}_3$ μ_3'	$B_{3,1}$	(19), (34)	
	$B_{3,2}$ $B'_{3,2}$ α_3' β_3 ϵ_3 ζ_3'	$B_{3,2}$	(32)	

There are thus 47 associated constants, of which however only 33 are independent, and 18 unassociated constants. The associated constants all occur in the expressions for strains causing a dilatation δ; while the unassociated constants answer to strains in which the dilatation is zero.

§ 4. By applying the solution consisting of (15), (22) and (24) to the problem of a straight cylinder of uniform elliptic section free from force on the curved surface, it may be *demonstrated* * that Saint-Venant's solution for an elliptic beam acted on only by terminal forces is the *only possible one* when terms of the fourth degree of the variables x and y, measured in the cross section, are neglected. The constants entering into the solution are those associated with $A_{1,0}$, $A_{2,1}$, $B_{2,1}$, $A_{3,1}$, and $B_{3,1}$, and in addition the unassociated constants ϵ_2, γ_2' and $\bar{\gamma}_2$. It can be shown *explicitly* that the conditions on the curved surface require every other constant to be zero except certain of the unassociated constants appearing in terms of the first degree in the displacements. The terms however in which they appear merely represent rotations of the solid as a rigid body about the rectangular axes, and so do not properly refer to the elastic problem.

For the same problem in the general case of any form of cross section the only constants left after satisfying the conditions on the sides are those associated with $A_{1,0}$, $A_{2,1}$ and $B_{2,1}$. The solution agrees with Saint-Venant's, which is thus *proved* to be complete so far as it goes.

§ 5. The proof of the completeness of Saint-Venant's solution is laborious, involving some heavy algebraic calculations. As it merely confirms results that meet with general acceptance,—based it is true on somewhat insufficient grounds,—it could hardly be

* The method of proof is the same as for an isotropic beam. Cf. *Quarterly Journal*, Vol. XXII., 1887, p. 89, et seq.

expected to be found interesting. Accordingly the first application I shall make of the previous solution is to the problem of a spheroid of uniform density rotating with uniform angular velocity about its axis of figure, which is also an axis of symmetry of the material. So far as I know, this problem has hitherto been solved only for the case of an isotropic * material, and in the paper referred to it was hardly attempted to deduce from the solution the true character of the phenomena. Thus the results obtained here may possess an interest even for those who are not professed mathematicians.

§ 6. If ω denote the angular velocity and ρ the density of the spheroid it may be regarded as at rest, but acted on by "centrifugal" forces whose components, per unit volume, are

$$X = \omega^2 \rho x, \qquad Y = \omega^2 \rho y, \qquad Z = 0.$$

In place of (3) and (4) we get, reintroducing X and Y and slightly altering the form

$$(2\mathbf{f} + \mathbf{f}')\frac{d^2u}{dx^2} + \mathbf{f}\frac{d^2u}{dy^2} + \mathbf{d}\frac{d^2u}{dz^2} + (\mathbf{f} + \mathbf{f}')\frac{d^2v}{dx\,dy} + (\mathbf{d} + \mathbf{d}')\frac{d^2w}{dx\,dz} + \omega^2\rho x = 0 \quad \text{(3 a)},$$

$$(\mathbf{f} + \mathbf{f}')\frac{d^2u}{dx\,dy} + \mathbf{f}\frac{d^2v}{dx^2} + (2\mathbf{f} + \mathbf{f}')\frac{d^2v}{dy^2} + \mathbf{d}\frac{d^2v}{dz^2} + (\mathbf{d} + \mathbf{d}')\frac{d^2w}{dy\,dz} + \omega^2\rho y = 0 \quad \text{(4 a)},$$

while (5) remains unchanged.

A particular solution of these equations is

$$\left.\begin{aligned} u &= -\frac{\omega^2\rho x(x^2+y^2)}{8(2\mathbf{f}+\mathbf{f}')}, \\ v &= -\frac{\omega^2\rho y(x^2+y^2)}{8(2\mathbf{f}+\mathbf{f}')}, \\ w &= 0 \end{aligned}\right\} \quad \text{(35)}.$$

The general solution is contained of course in (22), (24) and (15). It would however be a needlessly long process to substitute the whole of these terms in the surface conditions. A comparatively small number of terms suffice to give a complete solution. As by means of these the surface conditions are exactly satisfied, the solution is on an entirely different footing from Saint-Venant's solution for beams, and the neglecting of the remaining terms requires no justification. The only terms required are those of the first degree depending on $A_{1.0}$ and its associated constants, and those of the third degree depending on $A_{3.0}$, $A_{3.2}$ and their associated constants. Further from the symmetry around the axis of z we may at once assume

$$\left.\begin{aligned} &\beta_1' = \alpha_1, \\ &\eta_3' = \epsilon_3' = \zeta_3 = \theta_3, \\ &\alpha_3 = \beta_3' = A_{3.2} = A'_{3.2} = 0 \end{aligned}\right\} \quad \text{(36)}.$$

* *Quarterly Journal of Pure and Applied Mathematics*, Vol. XXII.I, 1889, p. 11.

Thus the solution we propose to use is in full, substituting r^2 for $x^2 + y^2$,

$$\delta = A_{1,0} + \tfrac{1}{2} A_{3,0} (2z^2 - r^2) + F_3 r^2 - \tfrac{1}{2}\omega^2 \rho r^2/(2\mathbf{f} + \mathbf{f}') \quad \text{.....(37)},$$

$$\frac{u}{x} = \frac{v}{y} = \frac{u_r}{r} = \alpha_1 + \theta_3 r^2 + \tfrac{1}{2} z^2 \mathbf{d}^{-1} \{(\mathbf{f} + \mathbf{f}')(A_{3,0} - A'_{3,0} - 2F_3 + 2F_3') + (\mathbf{d} + \mathbf{d}')(A'_{3,0} - 2F_3') - 8\mathbf{f}\theta_3\}$$
$$- \tfrac{1}{8}\omega^2 \rho r^2/(2\mathbf{f} + \mathbf{f}') \quad \text{.....(38)},$$

$$w = A'_{1,0} z + \tfrac{1}{2} A'_{3,0} z (\tfrac{2}{3} z^2 - r^2) + F_3' z r^2 \quad \text{.....(39)}.$$

The constants appearing in this solution are connected, as shown in the table of constants, by the relations (14), (13)—taken as more convenient than its equivalent (31),—(25), (29) and (30). Owing however to the relations (36) the relation (25) simplifies into

$$\alpha_1 = \tfrac{1}{2}(A_{1,0} - A'_{1,0}) \quad \text{.....(25 a)};$$

while (29) and (30) both transform into the single equation

$$8\theta_3 + A_{3,0} - A'_{3,0} - 2F_3 + 2F_3' = 0 \quad \text{.....(29 a)}.$$

§ 7. Let the equation to the spheroid, prolate or oblate, be

$$a^{-2}(x^2 + y^2) + c^{-2} z^2 = 1 \quad \text{.....(40)}.$$

The direction-cosines of the normal at the point x, y, z are in the ratio $a^{-2}x : a^{-2}y : c^{-2}z$. Thus the conditions for a free surface are

$$a^{-2}(x t_{xx} + y t_{xy}) + c^{-2} z t_{xz} = 0 \quad \text{.....(41)},$$

$$a^{-2}(x t_{xy} + y t_{yy}) + c^{-2} z t_{yz} = 0 \quad \text{.....(42)},$$

$$a^{-2}(x t_{xz} + y t_{yz}) + c^{-2} z t_{zz} = 0 \quad \text{.....(43)}.$$

The first two are however here identical as is obvious from the symmetry.

The relations between the strains and stresses are given in (2). Employing these it will be seen that in the surface conditions the terms containing ω or the constants associated with $A_{3,0}$ and $A_{3,2}$ are of the third degree in the variables x, y, z, while the terms containing the constants associated with $A_{1,0}$ are only of the first degree in the variables. At the surface however the relation (40) holds; thus the terms in the surface conditions containing the constants associated with $A_{1,0}$ can be made of the third degree by multiplying them by $a^{-2}r^2 + c^{-2}z^2$ which is there identical with unity. Doing this, and equating separately to zero the coefficients of xr^2 and xz^2 in (41), we find

$$2(\mathbf{f} + \mathbf{f}')\alpha_1 + \mathbf{d}' A'_{1,0} + a^2 (6\mathbf{f} + 4\mathbf{f}')\theta_3 + a^2 \mathbf{d}'(F_3' - \tfrac{1}{2} A'_{3,0}) = \frac{\omega^2 \rho a^2 (3\mathbf{f} + 2\mathbf{f}')}{4(2\mathbf{f} + \mathbf{f}')} \quad \text{.....(44)},$$

$$2(\mathbf{f} + \mathbf{f}')\alpha_1 + \mathbf{d}' A'_{1,0} + c^2 (\mathbf{f} + \mathbf{f}')\mathbf{d}^{-1} \{(\mathbf{f} + \mathbf{f}')(A_{3,0} - A'_{3,0} - 2F_3 + 2F_3') + (\mathbf{d} + \mathbf{d}')(A'_{3,0} - 2F_3') - 8\mathbf{f}\theta_3\}$$
$$+ c^2 \mathbf{d}' A'_{3,0} + a^2 \{(\mathbf{f} + \mathbf{f}')(A_{3,0} - A'_{3,0} - 2F_3 + 2F_3') + \mathbf{d}'(A'_{3,0} - 2F_3') - 8\mathbf{f}\theta_3\} = 0 \quad \text{.....(45)}.$$

Treating the surface condition (43) similarly, and equating separately to zero the coefficients of zr^2 and z^3, we find

$$2\mathbf{d}'\alpha_1 + \mathbf{c}A'_{1.0} + c^2\{(\mathbf{f}+\mathbf{f}')(A_{3.0} - A'_{3.0} - 2F_3 + 2F_3') + \mathbf{d}'(A'_{3.0} - 2F_3') - 8\mathbf{f}\theta_3\}$$
$$+ a^2\{\mathbf{c}(F_3' - \tfrac{1}{2}A'_{3.0}) + 4\mathbf{d}'\theta_3\} = \frac{\omega^2\rho a^2\mathbf{d}'}{2(2\mathbf{f}+\mathbf{f}')} \text{........(46)},$$

$$2\mathbf{d}'\alpha_1 + \mathbf{c}A'_{1.0} + c^2\mathbf{d}'\mathbf{d}^{-1}\{(\mathbf{f}+\mathbf{f}')(A_{3.0} - A'_{3.0} - 2F_3 + 2F_3') + (\mathbf{d}+\mathbf{d}')(A'_{3.0} - 2F_3') - 8\mathbf{f}\theta_3\}$$
$$+ c^2\mathbf{c}A'_{3.0} = 0 \text{......(47)}.$$

The equations (44)—(47) combined with (13), (14), (25 a) and (29 a) are obviously sufficient, and no more than sufficient, to determine without ambiguity the 8 constants of the solution, viz. $A_{1.0}$, $A'_{1.0}$, α_1, $A_{3.0}$, $A'_{3.0}$, F_3, F_3' and θ_3.

§ 8. The actual determination of these constants is a somewhat laborious process, and presents no novel features. Further a statement of the values of the individual constants seems hardly likely to be of service in the solution of any other problem. I shall thus not occupy space by recording here the values of the constants or the algebraic steps by which they were obtained, but shall proceed at once to give the values of the displacements. Their accuracy may be easily tested by reference to the equations which they require to satisfy, viz. (3 a) or (4 a), (5), (41) or (42), and (43).

For shortness let

$$D \equiv \tfrac{5}{2}\mathbf{c}^2\mathbf{f} + \{\mathbf{c}(\mathbf{f}+\mathbf{f}') - \mathbf{d}'^2\}[3\mathbf{c} + 2c^2a^{-2}\mathbf{d}^{-1}\{\mathbf{c}(2\mathbf{f}+\mathbf{f}') - \mathbf{d}'(2\mathbf{d}+\mathbf{d}')\} + 8c^4a^{-4}(2\mathbf{f}+\mathbf{f}')] \text{...(48)};$$

then the values of the displacements are as follows:

$$\frac{D}{\omega^2\rho}\frac{u}{x} = \frac{D}{\omega^2\rho}\frac{v}{y} = \frac{D}{\omega^2\rho}\frac{u_r}{r}$$
$$= \tfrac{1}{4}a^2\mathbf{c}^2\left\{\frac{\mathbf{c}\mathbf{f}}{\mathbf{c}(\mathbf{f}+\mathbf{f}') - \mathbf{d}'^2} + 2\right\} + \tfrac{1}{4}c^2\mathbf{c}\mathbf{d}^{-1}\{\mathbf{c}(3\mathbf{f}+2\mathbf{f}') - 2\mathbf{d}'^2\} + c^4a^{-2}\{\mathbf{c}(3\mathbf{f}+2\mathbf{f}') - \mathbf{d}'^2\}$$
$$- \tfrac{1}{4}r^2\left[\mathbf{c}^2 + c^2a^{-2}\mathbf{c}\left\{\mathbf{d}' + \frac{\mathbf{c}(\mathbf{f}+\mathbf{f}') - \mathbf{d}'^2}{\mathbf{d}}\right\} + 4c^4a^{-4}\{\mathbf{c}(\mathbf{f}+\mathbf{f}') - \mathbf{d}'^2\}\right]$$
$$- \tfrac{1}{2}z^2[\mathbf{c}\mathbf{d}'\mathbf{d}^{-1}(2\mathbf{d}-\mathbf{d}') + \tfrac{1}{2}\mathbf{c}^2\mathbf{d}^{-1}(3\mathbf{f}+2\mathbf{f}') + 2c^2a^{-2}\{\mathbf{c}(3\mathbf{f}+2\mathbf{f}') - \mathbf{d}'^2\}] \text{..................(49)},$$

$$\frac{D}{\omega^2\rho}w = -z\left[\tfrac{1}{2}a^2\mathbf{c}\mathbf{d}'\left\{\frac{\mathbf{c}\mathbf{f}}{\mathbf{c}(\mathbf{f}+\mathbf{f}') - \mathbf{d}'^2} + 2\right\} + \tfrac{1}{2}c^2\frac{\mathbf{d}+\mathbf{d}'}{\mathbf{d}}\{\mathbf{c}(3\mathbf{f}+2\mathbf{f}') - 2\mathbf{d}'^2\} + 2c^4a^{-2}\mathbf{d}'(2\mathbf{f}+\mathbf{f}')\right]$$
$$+ zr^2\left[\mathbf{c}\mathbf{d}' + c^2a^{-2}\left\{\left(\mathbf{c}(\mathbf{f}+\mathbf{f}') - \mathbf{d}'^2\right)\frac{\mathbf{d}+\mathbf{d}'}{\mathbf{d}} + \mathbf{c}(2\mathbf{f}+\mathbf{f}')\right\}\right]$$
$$+ \tfrac{1}{3}z^3[\tfrac{1}{2}\mathbf{d}^{-1}\{\mathbf{c}(3\mathbf{f}+2\mathbf{f}')(\mathbf{d}+\mathbf{d}') + 2\mathbf{d}'^2(\mathbf{d}-\mathbf{d}')\} + 2c^2a^{-2}\mathbf{d}'(2\mathbf{f}+\mathbf{f}')] \text{.....................(50)}.$$

§ 9. The elastic constants occurring in the preceding solution are not those which direct experiment would immediately lead to, and thus the application of the formulae to a solid whose elastic properties had been determined by the usual methods might be found laborious. It will thus be advantageous to transform the expressions into others in which the elastic constants occurring are such as practical men may be expected to become conversant with.

It is necessary of course to fix on five constants, and there is little doubt as to what three of these should be. Suppose two straight bars of uniform rectangular section cut out of the material, the axis of one of the bars coinciding with the axis of symmetry of the material, while in the other this axis of symmetry is perpendicular to one of the lateral faces. Let E and E' denote the values of Young's modulus for the respective bars under longitudinal tension, and η, η' the ratios of the lateral contraction to the longitudinal expansion in the experiments determining E and E', the direction in which η' is measured being perpendicular to the axis of symmetry; and finally let G denote the modulus of torsion for the first of the two bars twisted about its longitudinal axis. Then the constants it is proposed to use here are E, E', η, η' and G. The notation is Saint-Venant's, who has pointed out how the several constants may be found by experiment.

Experimental methods at present in use ought to supply trustworthy values of E, E', and G with comparative ease. The determination of η and η' is by no means so easy, and not improbably two more convenient constants might be selected. Still it must be remembered that the strictures that have been so frequently passed on the seemingly unsatisfactory determination of "Poisson's ratio" are really in the main directed against experiments in which all substances, even hard drawn wires, are regarded as isotropic bodies. There is no very obvious reason why satisfactory results should not be obtained when observers take the trouble to find out what exactly are the quantities whose magnitudes they determine with such extreme nicety.

§ 10. In Saint-Venant's *Clebsch*, pp. 83, 84, are given the relations between the several constants for the kind of material treated here. The following relations are in part directly taken from this source, and in part deduced algebraically:

$$\left.\begin{aligned} \mathbf{d} &= G, \\ \mathbf{f} &= \tfrac{1}{2}E'/(1+\eta'), \\ \mathbf{c} &= E^2(1-\eta')/\{E(1-\eta')-2E'\eta^2\}, \\ \mathbf{d}'/\mathbf{c} &= \eta E'/E(1-\eta'), \\ (\mathbf{f}+\mathbf{f}')/\mathbf{c} &= \tfrac{1}{2}E'/E(1-\eta'), \\ \{\mathbf{c}(\mathbf{f}+\mathbf{f}')-\mathbf{d}'^2\}/\mathbf{c} &= \tfrac{1}{2}E'/(1-\eta'), \\ \mathbf{cf}/\{\mathbf{c}(\mathbf{f}+\mathbf{f}')-\mathbf{d}'^2\} &= (1-\eta')/(1+\eta') \end{aligned}\right\} \quad\text{.....................(51).}$$

§ 11. If now $D' \equiv \dfrac{1-\eta'^2}{E'}\left\{\dfrac{E(1-\eta')-2E'\eta^2}{E^2(1-\eta')}\right\}^2 D$

$$= \tfrac{1}{4}(11+\eta') + c^2a^{-2}\frac{E'}{E(1-\eta')}\left\{\frac{E}{G} - 2\eta(1+\eta') + 4c^2a^{-2}\frac{E-E'\eta^2}{E}\right\} \text{....(52),}$$

the equations (49) and (50) transform into:—

$$\frac{D'}{\omega^2\rho}\frac{u_r}{r} = \tfrac{1}{4}(3+\eta')\left\{\frac{a^2(1-\eta')}{E'} + \tfrac{1}{2}\frac{c^2}{G}\right\} + c^4a^{-2}\frac{E(1-\eta')(3+\eta') - 4E'\eta^2}{2E^2(1-\eta')}$$
$$- \tfrac{1}{4}r^2(1+\eta')\left\{\frac{1-\eta'}{E'} + c^2a^{-2}\left(\frac{\eta}{E} + \frac{1}{2G}\right) + 2c^4a^{-4}\frac{E(1-\eta') - 2E'\eta^2}{E^2(1-\eta')}\right\}$$
$$- \tfrac{1}{2}z^2\left\{\frac{2\eta(1+\eta')}{E} + \frac{3+\eta'}{4G} + c^2a^{-2}\frac{E(1-\eta')(3+\eta') - 4E'\eta^2}{E^2(1-\eta')}\right\} \text{.........................(49 a)},$$

$$\frac{D'}{\omega^2\rho}w = -\tfrac{1}{2}z\frac{3+\eta'}{E}\left[a^2\eta + \frac{c^2}{2E(1-\eta')}\left\{E(1-\eta') - 2E'\eta^2 + \frac{\eta EE'}{G}\right\} + \frac{4c^4a^{-2}\eta E'(E-E'\eta^2)}{E^2(1-\eta')(3+\eta')}\right]$$
$$+ \frac{zr^2}{E}\left[\eta(1+\eta') + \frac{c^2a^{-2}}{2E(1-\eta')}\left\{E(1-\eta')(3+\eta') - 4E'\eta^2 + \frac{\eta EE'(1+\eta')}{G}\right\}\right]$$
$$+ \frac{z^3}{12E^2}\left[E(3+\eta') + 2\eta^2E'\frac{1+3\eta'}{1-\eta'} + \frac{\eta EE'(3+\eta')}{G(1-\eta')} + 8c^2a^{-2}\frac{\eta E'(E-E'\eta^2)}{E(1-\eta')}\right] \text{...... (50 a)}.$$

From physical considerations alone we are led to treat D' as essentially a positive quantity. From (52) it is obviously positive when c/a is small, and if in any kind of material it could change sign as c/a increased then a spheroid of this material could be constructed such that all the displacements would become infinite however slow the rotation.

These expressions it must be admitted appear somewhat formidable. It will be found however that their length does not present an insuperable barrier to the drawing of general conclusions. To permit the mind more easily to grasp these conclusions we shall consider first some special cases of comparative simplicity.

§ 12. When terms in c^2 and z^2 are neglected we get the following solution, applicable to a very flat oblate spheroid,

$$\left.\begin{aligned} u_r &= \frac{\omega^2\rho(1-\eta')r}{E'(11+\eta')}\{(3+\eta')a^2 - (1+\eta')r^2\}, \\ w &= \frac{-2\omega^2\rho\eta z}{E(11+\eta')}\{(3+\eta')a^2 - 2(1+\eta')r^2\} \end{aligned}\right\} \text{........................ (53)}.$$

This solution does not satisfy the equations (3 a), (4 a) and (5), and there is no reason to expect any approximate solution of the kind to do so; because while a term in u of the order xz^2 may be negligible when z is small, yet when operated on by $\frac{d^2}{dz^2}$ its contribution to the equation (3 a) is just as important as that of any other term in the expressions for the displacements. It is thus impossible to test the accuracy of such approximate solutions by means of the internal equations.

§ 13. It is well known that the distribution of electricity on a flat circular plate has been deduced by a mathematical treatment which regards the plate as the limiting form of a flat oblate spheroid. It would also appear that except near the rim there is a good agreement between theory and experiment. We are thus led to investigate whether (53) may not satisfactorily be applied to the case of a rotating circular plate.

The only way of testing the matter is by finding how exactly (53) may satisfy the surface conditions for a right circular cylinder of radius a and length $2c$. These conditions are the following:—

$$\left.\begin{array}{ll} xt_{xx}+yt_{xy}=0 & \text{when } r=a, \text{ for all values of } z \text{ between } -c \text{ and } +c, \\ xt_{xz}+yt_{yz}=0 & \quad ,, \qquad\qquad ,, \qquad , \\ t_{xz}=t_{yz}=0 & \text{when } z=\pm c, \text{ for all values of } r<a, \\ t_{zz}=0 & \quad ,, \qquad\qquad ,, \qquad , \end{array}\right\} \ldots\ldots\ldots (54).$$

Of these the first and the last, which answer to the vanishing of the normal stresses on the curved surface and on the flat ends, are identically satisfied. This is not however exactly the case with the other two, as the solution yields tangential forces of the order za on the curved surface, and of the order cr on the flat ends. Thus while the surface conditions are not all identically satisfied, they are approximately satisfied in a thin plate, and the approximation becomes closer the thinner the plate.

It will be noticed however that if each term of the solution (53) were multiplied by one and the same constant the resulting solution would satisfy the surface conditions (54) to the same degree of approximation that (53) itself does. Thus all we are safely entitled to assume is that (53), which gives very approximately the absolute magnitudes of the displacements in a flat oblate spheroid, gives to a somewhat less close degree of approximation the laws of variation of the several displacements and their relative magnitudes in a thin circular plate. Considering that the volume of a flat spheroid is less than that of the corresponding flat plate in the ratio 2 : 3, we should expect the absolute magnitudes of the displacements to be decidedly larger in the plate.

§ 14. To derive its full interpretation from the solution (53) we require to know something of the relative magnitudes of the elastic constants which appear in it. In all ordinary elastic solids the constants **c**, **f** etc. can hardly fail to be positive quantities, and the same is obviously true of E, E' and G. It is conceivable that in some exceptional substances η or η' might be negative, though it seems a somewhat remote possibility. If we assume here that all the constants are positive, then it follows from the expressions in (51) for **c** and **d**$'$/**c** that

$$\left.\begin{array}{r} 1>\eta', \\ E(1-\eta')>2E'\eta^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55).$$

Thus in (53), u_r must be everywhere positive and w everywhere negative. Consequently every element of the flat spheroid, or of the thin circular plate, increases its distance from the axis of rotation and approaches simultaneously the central or, as it may be termed, "equatorial" plane.

Confining our attention at first to the flat spheroid, we notice that the centre of an originally plane section perpendicular to the axis of rotation diminishes its distance z from the equatorial plane by the amount

$$2\omega^2\rho a^2 z\eta\,(3+\eta')\div E\,(11+\eta')\ldots\ldots\ldots\ldots\ldots\ldots(56);$$

and the section itself becomes very approximately a paraboloid of revolution, whose latus rectum is

$$\frac{a^2}{c} E(11+\eta') \div 4\omega^2\rho a^2\eta(1+\eta') \quad \text{.......(57).}$$

The axis of the paraboloid is the axis of rotation, and the concavity is directed away from the equatorial plane.

The curvature of the originally plane cross sections continually increases with their distance from the equatorial plane, and for a given material and given angular velocity is independent of the radius a—supposed of course great compared to the thickness $2c$.

The diminution of the *polar* axis $2c$ is

$$4\omega^2\rho a^2 c\eta(3+\eta') \div E(11+\eta') \quad \text{.......(58).}$$

It thus varies directly as the density, as the thickness and as the squares of the angular velocity and the radius. It also varies directly as η and inversely as E. On the other hand it is quite independent of E', and increases only about 20 per cent. as η' increases from 0 to 1.

The increase in the equatorial semi-axis, or *radius*, a is

$$2\omega^2\rho a^3(1-\eta') \div E'(11+\eta') \quad \text{.......(59).}$$

It thus varies directly as the density, as the square of the angular velocity, and as the cube of the radius. It varies inversely as E' and diminishes as η' decreases, but is entirely independent of E or of η.

In the circular plate, as in the flat spheroid, every originally plane section perpendicular to the axis of rotation becomes very approximately a paraboloid of revolution about that axis; and the latus rectum of the generating parabola varies inversely as the original distance of the section from the central section, as the density, and as the square of the angular velocity, while it is independent of the radius of the plate. Owing to this change in its originally plane surfaces the plate will present a biconcave appearance. As the actual measurements of the displacements might be easier for the plate than for the spheroid it may be as well to state explicitly the following relations, the diminution in thickness being measured along the axis of rotation:

$$\frac{\text{Increase in radius of plate}}{\text{Diminution in thickness } \quad ,,} = \frac{a}{2c}\,\frac{(1-\eta')E}{\eta(3+\eta')E'} \quad \text{.......(60),}$$

$$\frac{\text{Curvature at centre of face of plate}}{\text{Diminution in thickness } \quad ,,} = \frac{1}{a^2}\,\frac{2(1+\eta')}{3+\eta'} \quad \text{.......(61).}$$

If the ratios on the left-hand sides of these equations could be experimentally determined it is obvious that a great deal of light would be thrown on the nature of the material.

§ 15. To arrive at a more complete knowledge of the effects of rotation, an analysis of the strains is necessary. For our purpose the most convenient normal strain components are

$$\text{the } longitudinal \equiv \frac{dw}{dz},$$

$$\text{,, } radial \equiv \frac{du_r}{dr},$$

$$\text{,, } transverse \equiv u_r/r.$$

The first is directed parallel to the axis of rotation, the second along the perpendicular on the axis of rotation directed outwards, and the third is perpendicular to the other two.

Referring to (53) we see that in a flat spheroid, or a thin circular plate, the longitudinal strain is everywhere a compression, and the transverse everywhere an extension, and that the numerical measures of both these strains are greater the nearer the element considered to the axis of rotation. A cylinder whose axis is the axis of rotation, and whose radius is

$$a\left\{\frac{3+\eta'}{3(1+\eta')}\right\}^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (62),$$

divides the volume into two portions in the inner of which the radial strain is an extension while in the outer it is a compression. The expression (62) is necessarily less than a so long as η' does not vanish, so that except in this extreme case the radial strain actually is a compression near the rim of the circular plate and in the superficial equatorial regions of the flat spheroid.

§ 16. The next case that presents itself is that of a very elongated prolate spheroid in which c/a is very large. Near the centre of its length the surface of such a spheroid differs very little from that of a right circular cylinder of radius a. We are thus led to expect that a solution obtained from (49 a) and (50 a) by making c/a infinite while z/a remains finite, being strictly applicable to the central portions of an indefinitely long prolate spheroid, will apply very approximately to the case of a right circular cylinder, provided the length of the cylinder be great compared to its radius and its terminal portions be excluded from the solution. The solution in question is

$$\left.\begin{aligned} u_r &= \frac{\omega^2\rho r}{8E'(E-E'\eta^2)}\left[a^2\{E(1-\eta')(3+\eta')-4E'\eta^2\}-r^2(1+\eta')\{E(1-\eta')-2E'\eta^2\}\right],\\ w &= -\frac{\omega^2\rho a^2 z\eta}{2E}\end{aligned}\right\}\ldots(63).$$

Unlike (53) this solution, though deduced as an approximation from the general solution, itself satisfies the internal equations. There can thus be no doubt that it gives the *absolute* magnitudes of the displacements in any rotating solid whose boundary conditions it may happen to satisfy. It will be found to satisfy identically the first three surface conditions (54) for a right circular cylinder of finite length. The last of equations (54) is not exactly satisfied, as from (63) we get for all values of z

$$t_{zz} = \omega^2\rho(a^2-2r^2)E\eta(1+\eta') \div 4(E-E'\eta^2).$$

It will be noticed however that

$$\int_0^a 2\pi r t_{zz} dr = 0,$$

and thus the sum of the normal forces over a terminal cross-section vanishes. Now Saint-Venant's solution for beams acted on by terminal forces only secures that the integral of the stresses taken over the ends should have required values, and notwithstanding it is regarded by the highest authorities as perfectly satisfactory provided the length of the beam be great compared to its greatest transverse dimension. Thus (63), which satisfies exactly 3 out of 4 surface conditions, and is as regards the remaining condition in no respect less satisfactory than is Saint-Venant's solution as regards the terminal conditions in the ordinary beam problem, will doubtless be accepted by the majority of elasticians as a very approximate solution for the case of a rotating circular cylinder whose length is great compared to its diameter. The portions of the cylinder immediately adjacent to its ends ought however to be excluded.

§ 17. Assuming $\eta' < 1$, and noticing that in accordance with (55) $E - E'\eta^2$ must be positive, we see from (63) that each element of the long cylinder, as of the flat plate, increases its distance from the axis of rotation and approaches the central plane $z = 0$. In the long cylinder, however, the longitudinal displacement varies only as the distance from the central section, so that each cross-section remains plane.

The shortening in a length $2c$ of the cylinder amounts to

$$\omega^2 \rho a^2 c\eta / E \dots\dots\dots\dots (64).$$

It thus bears to the shortening in the polar axis $2c$ of a flat oblate spheroid of the same density and central section, rotating with the same angular velocity, the ratio $11 + \eta' : 4(3 + \eta')$, which for uniconstant* isotropy is $45 : 52$, and is for every material less than $11 : 12$.

The increase in the radius of the long cylinder is

$$\omega^2 \rho a^3 (1 - \eta')/4E' \dots\dots\dots\dots (65).$$

This bears to the increase in the equatorial semi-axis of the flat oblate spheroid of the same density and central section, rotating with the same angular velocity, the ratio $11 + \eta' : 8$, which is for every material a little less than the ratio, $3 : 2$, of the volumes of a cylinder and spheroid of the same axial thickness and central section.

We also see from (63) that throughout the long cylinder the longitudinal strain is everywhere a compression, and the transverse strain an extension. Also the radial strain is an extension inside and a compression outside of the coaxial cylinder

$$r = a \left[\frac{E(1 - \eta')(3 + \eta') - 4E'\eta^2}{3(1 + \eta')\{E(1 - \eta') - 2E'\eta^2\}} \right]^{\frac{1}{2}} \dots\dots\dots\dots (66).$$

* i.e. Isotropy in which Poisson's ratio is 1/4, or in Thomson and Tait's notation $m = 2n$.

In order to apply to our problem this radius must not exceed a, which is the case only when

$$E\eta'(1-\eta') > E'\eta^2(1+3\eta').$$

When this inequality becomes an equality the radial strain just vanishes over the surface of the rotating cylinder, and if the inequality be reversed then the radial strain is everywhere an extension. In the case of uniconstant isotropy the radial strain is a compression throughout one-fifteenth of the area of the cross-section.

§ 18. The next case we proceed to consider is that of uniconstant isotropy. In a material of this kind there is only one elastic constant. The one employed here is Young's modulus E, which is identical in Thomson and Tait's notation with $5n/2$ or $5m/4$. The expressions for the displacements in this case are:

$$u_r = \frac{\omega^2\rho r}{60E(9+8c^2a^{-2}+16c^4a^{-4})}\{117a^2+195c^2+280c^4a^{-2}-5r^2(9+18c^2a^{-2}+20c^4a^{-4}) - 5z^2(51+56c^2a^{-2})\} \quad\text{.........(67)},$$

$$w = \frac{-\omega^2\rho z}{30E(9+8c^2a^{-2}+16c^4a^{-4})}\{39a^2+130c^2+60c^4a^{-2}-10r^2(3+19c^2a^{-2})-10z^2(5+2c^2a^{-2})\} \quad\text{...(68)}.$$

In considering the strains we shall also want the following expressions:

$$\frac{du_r}{dr} = \frac{\omega^2\rho}{60E(9+8c^2a^{-2}+16c^4a^{-4})}\{117a^2+195c^2+280c^4a^{-2}-15r^2(9+18c^2a^{-2}+20c^4a^{-4}) - 5z^2(51+56c^2a^{-2})\} \quad\text{.........(69)},$$

$$\frac{dw}{dz} = \frac{-\omega^2\rho}{30E(9+8c^2a^{-2}+16c^4a^{-4})}\{39a^2+130c^2+60c^4a^{-2}-10r^2(3+19c^2a^{-2})-30z^2(5+2c^2a^{-2})\} \quad\text{...(70)},$$

$$\frac{du_r}{dz}+\frac{dw}{dr} = \frac{-\omega^2\rho rz(39-20c^2a^{-2})}{6E(9+8c^2a^{-2}+16c^4a^{-4})} \quad\text{.........(71)}.$$

§ 19. Writing

$$\nu \equiv \omega^2\rho a^2(117+195c^2a^{-2}+280c^4a^{-4})/60E(9+8c^2a^{-2}+16c^4a^{-4}) \quad\text{.........(72)},$$

$$\alpha_1^2 \equiv a^2(117+195c^2a^{-2}+280c^4a^{-4})/5(9+18c^2a^{-2}+20c^4a^{-4}) \quad\text{.........(73)},$$

$$\beta_1^2 \equiv a^2(117+195c^2a^{-2}+280c^4a^{-4})/5(51+56c^2a^{-2}) \quad\text{.........(74)},$$

we get

$$u_r/r = \nu(1-r^2/\alpha_1^2-z^2/\beta_1^2) \quad\text{.........(75)}.$$

Thus as ν, α_1^2 and β_1^2 are necessarily positive for all values of c/a, it follows that u_r and u_r/r are positive inside and negative outside the spheroid whose equatorial and polar semi-axes are respectively α_1 and β_1. Obviously α_1^2 is more than twice a^2, whatever c/a may be. Treating a as constant and varying c, it is easily seen that β_1 is greater than c so long as c/a is less than $\sqrt{39/20}$, but that for greater finite values of c/a the value of β_1 is less than c. The least value of β_1/c is very nearly ·989, occurring when c/a is approximately 2·08. Thus for all values of c/a exceeding $\sqrt{39/20}$ the difference between β_1 and c is extremely small. They become equal when c/a becomes infinite.

It follows that so long as c/a is less than $\sqrt{39/20}$ every element of the spheroid increases its distance from the axis of rotation, and the transverse strain is everywhere an extension. When c/a exceeds $\sqrt{39/20}$ there is an extremely limited superficial volume surrounding each extremity of the axis of rotation within which the elements diminish their distances from the axis of rotation, and where the transverse strain is a compression; elsewhere the distance of an element from the axis of rotation increases, and the transverse strain is an extension.

When c/a equals $\sqrt{39/20}$, or when it becomes infinite, the volumes within which the elements diminish in distance from the axis of rotation and the transverse strain is a compression, become reduced to the extremities of the axis of rotation.

§ 20. Similarly from (68)

$$w = -\tau z\,(1 - r^2/\alpha_2^{\,2} - z^2/\beta_2^{\,2}) \qquad \text{.........(76)};$$

where

$$\tau \equiv \omega^2\rho a^2\,(39 + 130c^2a^{-2} + 60c^4a^{-4})/30\,E\,(9 + 8c^2a^{-2} + 16c^4a^{-4}) \qquad \text{.........(77)},$$

$$\alpha_2^{\,2} \equiv a^2\,(39 + 130c^2a^{-2} + 60c^4a^{-4})/10\,(3 + 19c^2a^{-2}) \qquad \text{.........(78)},$$

$$\beta_2^{\,2} = c^2\,(39a^2c^{-2} + 130 + 60c^2a^{-2})/10\,(5 + 2c^2a^{-2}) \qquad \text{.........(79)}.$$

Thus τ, $\alpha_2^{\,2}$ and $\beta_2^{\,2}$ being essentially positive, w is of the opposite sign to z inside and of the same sign outside the spheroid whose equatorial and polar semi-axes are respectively α_2 and β_2. It is easily proved that α_2 equals a when c/a has approximately the values ·43 and ·90, and that it is only when c/a lies between these limits that α_2 is less than a. The least value of α_2/a is about ·97, answering to $c/a = {}$·65 approximately. It is obvious that β_2 considerably exceeds c for all values of c/a.

It follows that when c/a lies between ·43 and ·90 there is a very limited superficial volume close to the equator, the elements within which increase in distance from the equatorial plane, while elsewhere the elements approach this plane. When c/a lies outside these limits every element throughout the spheroid approaches the equatorial plane.

§ 21. From (69)

$$\frac{du_r}{dr} = \nu\,\{1 - r^2/\alpha_3^{\,2} - z^2/\beta_1^{\,2}\} \qquad \text{.........(80)},$$

where ν is given by (72) and $\beta_1^{\,2}$ by (74), while $\alpha_3^{\,2}$ equals $\alpha_1^{\,2}/3$ and so is known from (73).

It is obvious from (73) that α_3 is always less than a. It may also easily be found that as c/a increases from zero, α_3/a commencing with the value $\sqrt{13/15}$ diminishes at first, attaining a minimum value of about ·908 when c/a is ·65 approximately. It then increases continually as c/a increases further, passing through its initial value $\sqrt{13/15}$ when c/a equals $\sqrt{39/20}$, and finally reaches the value $\sqrt{14/15}$ when c/a becomes infinite. It may be remarked as a somewhat curious fact that α_2/a and α_3/a attain their minimum values for the identically same value of c/a. The variations in the value of β_1 have been already traced.

The conclusions from these data are as follows:—The radial strain is for all values of c/a an extension throughout all but a small portion of the spheroid. There is always however in the equator a superficial volume throughout which the radial strain is a compression. As c/a increases from zero this superficial volume extends towards the poles, and eventually reaches them when $c/a = \sqrt{39/20}$. For greater values of c/a this volume forms a layer completely enclosing the rest of the spheroid. The thickness of this layer in the equator continually diminishes from about $\cdot 069a$ when $c/a = \sqrt{39/20}$ to about $\cdot 034a$ when $c/a = \infty$. At the poles the ratio of the thickness to c attains a maximum of about $\cdot 01$ when $c/a = 2\cdot 08$ approximately, and then continually diminishes and vanishes in the limit when c/a becomes infinite.

§ 22. From (70)

$$\frac{dw}{dz} = -\tau(1 - r^2/\alpha_2^2 - z^2/\beta_3^2) \quad \text{.................................}(81),$$

where τ is given by (77) and α_2^2 by (78), while $\beta_3^2 = \beta_2^2/3$ and so is known from (79).

Thus $\frac{dw}{dz}$ is negative inside and positive outside the spheroid whose equatorial and polar semi-axes are respectively α_2 and β_3. The variation of α_2 with the value of c/a has been already traced in § 20. As c/a increases from zero β_3/c diminishes from infinity and becomes unity when $c/a = \sqrt{39/20}$. It attains a minimum value of about $\cdot 986$ when $c/a = 2\cdot 21$ approximately, and then continually but slowly increasing becomes unity when c/a becomes infinite.

The observed variations in the values of α_2 and β_3 lead us to the following results:— When c/a is less than $\cdot 43$, or when it lies between $\cdot 90$ and $\sqrt{39/20}$, the longitudinal strain is a compression throughout the entire spheroid. When c/a lies between $\cdot 43$ and $\cdot 90$ the longitudinal strain is an extension throughout a small superficial volume in the equator, elsewhere it is a compression. When c/a has any finite value exceeding $\sqrt{39/20}$ the longitudinal strain is an extension in a small superficial volume surrounding each pole, being elsewhere a compression. Lastly when c/a becomes infinite the longitudinal strain is everywhere a compression, except at the poles themselves where it vanishes.

§ 23. It will be observed that $\frac{dw}{dz}$, $\frac{du_r}{dr}$ and $\frac{u_r}{r}$ are the normal strains when for the coordinate axes at each point we take the parallel to the axis of rotation, the perpendicular on this axis produced outwards, and a third axis at right angles to the other two. The only remaining strain is the tangential or shearing strain $\frac{du_r}{dz} + \frac{dw}{dr}$ in the plane of zr.

From the expression (71) for the shearing strain it will be seen that it vanishes along the whole of the polar axis and everywhere in the equatorial plane. On the positive side of this plane it is everywhere of one sign, and this sign is negative or positive according as c/a is less or greater than $\sqrt{39/20}$. When c/a equals $\sqrt{39/20}$ the shearing strain everywhere vanishes. As will more fully appear presently, this particular value of c/a, whose recurrence will have been already noticed, answers to a sort of turning point in the character of the phenomena presented by a rotating spheroid. It will in future be referred to as the *critical value*, and the corresponding spheroid will be termed the *critical spheroid*.

The general character of the preceding results as to the nature of the strains will be found concisely stated in the following table.

TABLE II.

Limiting values of c/a	Radial strain $\frac{du_r}{dr}$ where			Transverse strain u_r/r where			Longitudinal strain $\frac{dw}{dz}$ where			Shearing strain $\frac{du_r}{dz}+\frac{dw}{dr}$, for z positive, where		
	extension	zero	compression	extension	zero	compression	extension	zero	compression	positive	zero	negative
$0 - \cdot 43$	all the central regions	surface of spheroid semi-axes a_3, β_1	superficial region in equator	everywhere	nowhere	nowhere	nowhere	nowhere	everywhere	nowhere	polar axis and equatorial plane	everywhere except where zero
$\cdot 43 - \cdot 90$	”	”	”	”	”	”	superficial region in equator	surface of spheroid semi-axes a_2, β_3	all the central regions	”	”	”
$\cdot 90 - \sqrt{39/20}$	”	”	”	”	”	”	nowhere	nowhere	everywhere	”	”	”
$\sqrt{39/20} - \infty$	”	”	superficial layer all round	all the central regions	surface of spheroid semi-axes a_1, β_1	superficial regions round poles	superficial regions round poles	surface of spheroid semi-axes a_2, β_3	all the central regions	everywhere except where zero	”	nowhere

A clearer idea possibly of the general character of the phenomena may be obtained from a study of the accompanying figures (see Plate I.). Each figure is intended to represent the state of some particular strain throughout a section of the spheroid by a plane through the axis of rotation. The strain represented is the radial $\frac{du_r}{dr}$ when the lines are straight and horizontal, the transverse $\frac{u_r}{r}$ when the lines are curved, the longitudinal $\frac{dw}{dz}$ when the lines are straight and vertical. When the lines are thin the strain is an extension, when thick a compression. The boundary line is drawn thin or thick according as the particular strain is an extension or compression in the surface at the point considered.

The surface volumes in which the sign of a strain differs from that at the centre are as a rule very considerably exaggerated in thickness. If drawn accurately to scale some of them could hardly be seen without a microscope.

§ 24. The displacements whose experimental determination appears most feasible are the increase u_a in the equatorial semi-axis, and the diminution $-w_c$ in the polar semi-axis. The amounts of these quantities per unit of original length, i.e. u_a/a and $-w_c/c$, are given in the second and third columns of the following Table III. The fourth column gives the common maximum value ν of u_r/r and $\frac{du_r}{dr}$. This is found at the centre and, as will presently appear, see § 31, is the absolutely greatest strain existing anywhere in the spheroid. According to Saint-Venant's theory of rupture if the angular velocity be increased until ν reaches a certain limit, determined by experiment, the spheroid will rupture—or more correctly the material will cease to obey the laws of perfect elasticity. The fifth column gives the maximum longitudinal compression, i.e. τ or the value of $-\frac{dw}{dz}$ at the centre. The last column gives the maximum stress-difference at the centre—i.e. the difference $4E(\nu+\tau)/5$ between the algebraically greatest and least of the principal stresses found there. On the maximum stress-difference theory of rupture the absolutely greatest maximum stress-difference found in the solid supplies the place taken on Saint-Venant's theory by the greatest strain. In certain special cases the absolutely greatest value of the maximum stress-difference unquestionably is found at the centre, but I have not *proved* this universally true, so in general we are only entitled to regard the value given in the last column of the table as an inferior limit to the value of the absolutely greatest maximum stress-difference existing in the spheroid.

As a basis of comparison a may be regarded as remaining constant while c/a passes through the values indicated in the first column. The displacements and strains are thus all expressed in terms of $\omega^2\rho a^2/E$. This represents a numerical quantity whose value can be easily calculated when the angular velocity, the equatorial diameter, the density, and Young's modulus for the material are known.

TABLE III.

Value of c/a	Increase of equatorial diameter per unit length $\frac{u_a}{a} \Big/ \frac{\omega^2\rho a^2}{E}$	Decrease of polar diameter per unit length $-\frac{w_c}{c} \Big/ \frac{\omega^2\rho a^2}{E}$	Greatest strain $\nu \Big/ \frac{\omega^2\rho a^2}{E}$	Greatest longitudinal compression $\tau \Big/ \frac{\omega^2\rho a^2}{E}$	Maximum stress-difference at centre $\frac{4}{5}E(\nu+\tau)/\omega^2\rho a^2$
infinitely small	$\cdot 1\dot{3}$	$\cdot 1\dot{4}$	$\cdot 21\dot{6}$	$\cdot 1\dot{4}$	$\cdot 2\dot{8}$
·2	·1364	·1507	·2234	·1580	·3051
·4	·1456	·1647	·2422	·1913	·3468
·6	·1590	·1743	·2669	·2235	·3924
·8	·1717	·1719	·2874	·2367	·4192
1·0	·1803	$\cdot 1\dot{6}\dot{0}$	$\cdot 2\dot{9}\dot{8}$	$\cdot 2\dot{3}\dot{1}$	$\cdot \dot{4}\dot{2}$
1·2	·1851	·1472	·3037	·2176	·4171
1·4	·1875	·1352	·3047	·2029	·4060
1·6	·1886	·1255	·3041	·1898	·3951
1·8	·1891	·1180	·3030	·1791	·3857
2	·1892	$\cdot \dot{1}\dot{1}\dot{2}$	·3017	·1705	·3778
3	·1888	·0968	·2972	·1469	·3553
4	·1883	·0910	·2950	·1376	·3461
infinitely great	·1875	$\cdot 08\dot{3}$	$\cdot 291\dot{6}$	·125	$\cdot \dot{3}$

It will be understood of course that in the preceding as in the succeeding table the entries do not as a rule give the exact values, but the last figure of each decimal is chosen so as to make the result as correct as the number of figures retained will permit.

§ 25. The approximate positions and values of the maxima of the several quantities, supposing ω, ρ, E and a to be constants, can be obtained from the preceding table. The following more exact results were obtained by direct calculation from the formulae:—

TABLE IV.

Quantity	Value of c/a supplying maximum	Maximum
$\frac{u_a}{a} \Big/ \frac{\omega^2\rho a^2}{E}$	2·06	·1892
$-\frac{w_c}{c} \Big/ \frac{\omega^2\rho a^2}{E}$	·658	·1749
$\nu \Big/ \frac{\omega^2\rho a^2}{E}$	$\sqrt{39/20} = 1\cdot 396$	·3047
$\tau \Big/ \frac{\omega^2\rho a^2}{E}$	·826	·2367
$\frac{4}{5}E(\nu+\tau) \Big/ \omega^2\rho a^2$	·956	·4246

§ 26. The most notable results in the two preceding tables are the extremely small change in the increase per unit length of the equatorial diameter or in the value of the greatest strain as c/a increases from 1 to ∞, and the fact that the absolutely largest value of the greatest strain—and so according to Saint-Venant the greatest tendency to rupture—occurs in the critical spheroid.

It is important to bear in mind that the above maxima are calculated on the hypothesis that the length of the equatorial diameter is the same in all the spheroids. If this be varied and some other quantity kept constant different results of course will be obtained. If for instance c and a both vary while the volume remains constant, a biquadratic equation in c^2/a^2 is obtained whose roots determine for what forms of spheroid the greatest strain ν—or Saint-Venant's tendency to rupture—has its greatest and least values. All the terms of this equation are however of the same sign, and so no true maximum or minimum can exist. The correct interpretation is that when the mass of the spheroid is constant Saint-Venant's tendency to rupture continually diminishes as the polar axis $2c$ increases from 0 to ∞. The same conclusion also follows if the constant quantity be the moment of inertia about the axis of rotation.

§ 27. Taking the axes specially for each point considered, as in the case of the strains, we get for the stresses in the case of uniconstant isotropy the following expressions:—

$$\left.\begin{aligned} Z &= \tfrac{2}{5}E\left(\frac{du_r}{dr} + \frac{u_r}{r} + 3\frac{dw}{dz}\right), \\ R &= \tfrac{2}{5}E\left(3\frac{du_r}{dr} + \frac{u_r}{r} + \frac{dw}{dz}\right), \\ \Phi &= \tfrac{2}{5}E\left(\frac{du_r}{dr} + 3\frac{u_r}{r} + \frac{dw}{dz}\right), \\ R_z &= \tfrac{2}{5}E\left(\frac{du_r}{dz} + \frac{dw}{dr}\right) \end{aligned}\right\} \quad \text{.......(82).}$$

The first three are normal stresses directed respectively parallel to the axis of rotation, along the perpendicular on this axis directed outwards, and along the perpendicular to these two directions. The last is a tangential or shearing stress in the *meridian* plane, or plane containing z and r.

From (67)—(71) we obtain the following convenient expressions for the stresses:—

$$Z = \frac{\omega^2\rho c^2(39 - 20c^2a^{-2})}{15(9 + 8c^2a^{-2} + 16c^4a^{-4})}\left\{\frac{r^2}{a^2} - \left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right)\right\} \quad \text{.......(83),}$$

$$R = \frac{\omega^2\rho a^2}{15(9 + 8c^2a^{-2} + 16c^4a^{-4})}\left\{(39 - 20c^2a^{-2})\left(1 - \frac{r^2}{a^2}\right) + 4\frac{c^2}{a^2}(18 + 25c^2a^{-2})\left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right)\right\} \quad \text{......(84),}$$

$$\Phi = \frac{\omega^2\rho a^2}{15(9 + 8c^2a^{-2} + 16c^4a^{-4})}\left\{(39 - 20c^2a^{-2})\left(1 - \frac{r^2}{a^2}\right) + 4\frac{c^2}{a^2}(18 + 25c^2a^{-2})\left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right) + (18 + 36c^2a^{-2} + 40c^4a^{-4})\frac{r^2}{a^2}\right\} \quad \text{.......(85),}$$

$$R_z = \frac{-\omega^2\rho(39 - 20c^2a^{-2})rz}{15(9 + 8c^2a^{-2} + 16c^4a^{-4})} \quad \text{.......(86).}$$

§ 28. There are at every point, as is well known, three principal stresses parallel to three rectangular axes, whose directions are such that the tangential stresses vanish over the elements whose normals are these axes. Φ is one of these principal stresses, and the corresponding strain u_r/r is everywhere one of the three principal strains. The two other principal stresses lie in the plane zr, but coincide with Z and R only when R_z vanishes, and so in general only along the polar axis and in the equatorial plane. These principal stresses are the two values of

$$\tfrac{1}{2}\left[R+Z\pm\{(R-Z)^2+4R_z^2\}^{\frac{1}{2}}\right] \quad\ldots\ldots(87).$$

If we suppose the square root always to represent a positive quantity, then the algebraically greatest principal stress in the meridian plane answers to the upper sign, and the angle α which its direction makes with the perpendicular on the axis of rotation directed outwards is given by

$$\alpha=\tan^{-1}\left[\frac{Z-R+\{(R-Z)^2+4R_z^2\}^{\frac{1}{2}}}{2R_z}\right] \quad\ldots\ldots(88).$$

As this expression concerns us practically only when R_z is not zero, we may say that $\tan\alpha$ is everywhere of the same sign as R_z. It is thus by (86) negative or positive for z positive according as c/a is less or greater than the critical value $\sqrt{39/20}$. It follows that the angle which the direction of the algebraically greater principal stress in the meridian plane makes with the perpendicular on the axis of rotation directed outwards is oblique or acute according as c/a is less or greater than the critical value.

§ 29. On the surface of the spheroid $1-r^2/a^2-z^2/c^2$ vanishes, and it is very simply proved from the expressions (83)—(86) that the two principal stresses in the meridian plane are there directed along the tangent and the normal. Also, from above, the principal stress along the tangent is the algebraically greater or the algebraically less according as c/a is less or greater than the critical value. Further the principal stress directed along the normal is zero, this being in fact a consequence of the surface conditions. Thus the tangential meridional stress is a tension or a pressure according as c/a is less or greater than the critical value. The algebraical expression for this stress may easily be found to be

$$\frac{\omega^2\rho\,(39-20c^2a^{-2})}{15\,(9+8c^2a^{-2}+16c^4a^{-4})}\cdot\frac{a^2c^2}{p^2} \quad\ldots\ldots(89),$$

where p is the perpendicular from the centre of the spheroid on the tangent plane at the point considered. Comment on the applications of this remarkably simple result seems unnecessary.

The complete change that takes place in the character of the meridional surface stress as c/a passes through the value $\sqrt{39/20}$ seems an ample justification of our designation of it as the *critical value*. There also appears for this value of c/a an important change in the character of the surface value of Φ the stress perpendicular to the meridian plane.

For from (85) we find for the surface value of Φ the expression

$$\Phi_s{}^* = \frac{\omega^2 \rho a^2}{15(9 + 8c^2a^{-2} + 16c^4a^{-4})}\left\{(39 - 20c^2a^{-2})\frac{z^2}{c^2} + (18 + 36c^2a^{-2} + 40c^4a^{-4})\frac{r^2}{a^2}\right\} \quad \text{......(90).}$$

So long as c/a is less than the critical value it is obvious that Φ_s is positive for all values of r/z and so all over the surface. When c/a attains the critical value Φ_s is still everywhere positive but just vanishes at the poles. For all greater values of c/a, Φ_s is negative within a small area surrounding each pole, being elsewhere positive. Thus for all values of c/a below the critical the surface stress perpendicular to the meridian plane is everywhere a tension. But for all values of c/a above the critical there is a small area round each pole within which this stress is a pressure.

It may also be easily proved that the surface tension at right angles to the meridian has its greatest value at the poles or on the equator according as c/a is less or greater than ·55 approximately.

§ 30. In the critical spheroid the state of stress is extremely simple as the only stresses which do not vanish are R and Φ, and these are everywhere principal stresses. Of these R vanishes all over the surface and elsewhere is positive, while Φ vanishes only at the poles being elsewhere positive. Excepting at the poles Φ is everywhere greater than R; and so, as both are positive and the third principal stress is zero, Φ is everywhere a correct measure of the maximum stress-difference. Its greatest value obviously occurs at the centre. Thus the critical spheroid is one of the special forms in which it is actually *proved* that the tendency to rupture on the maximum stress-difference theory, as well as on the greatest strain theory, occurs at the centre. It will be noticed that over the surface of the critical spheroid Φ varies as the square of the perpendicular on the axis of rotation.

§ 31. For values of c/a other than the critical the determination of the algebraically greatest principal stresses is a matter of some little difficulty. It is however worthy of notice as it leads at once to the greatest principal strain, which is required in applying Saint-Venant's theory of rupture.

Let P and Q denote the algebraically greater and less of the two principal stresses in the meridian plane. Then the algebraically greatest principal stress is either Φ or P. From the formulae for Φ and P we easily find

$$\Phi \gtreqless P \text{ according as}$$

$$\frac{2u_r}{r} - \frac{du_r}{dr} - \frac{dw}{dz} \gtreqless \left\{\left(\frac{du_r}{dr} - \frac{dw}{dz}\right)^2 + \left(\frac{du_r}{dz} + \frac{dw}{dr}\right)^2\right\}^{\frac{1}{2}} \quad \text{.........................(91).}$$

Thus Φ is the greatest principal stress when $\frac{2u_r}{r} - \frac{du_r}{dr} - \frac{dw}{dz}$ is positive, and when its square exceeds $\left(\frac{du_r}{dr} - \frac{dw}{dz}\right)^2 + \left(\frac{du_r}{dz} + \frac{dw}{dr}\right)^2$; otherwise P is the greatest principal stress.

* Here and in what follows surface values are distinguished by the suffix s.

Substituting the expressions for the strains from (67), (69), (70), (71), I find by a straightforward and not very laborious calculation on the above lines that so long as c/a is below the critical value, Φ is everywhere—excepting the axis of rotation where it equals R which is there a principal stress—the algebraically greatest principal stress. Thus for all values of c/a below the critical u_r/r is at every point in the spheroid the greatest strain, and so is the correct measure of Saint-Venant's tendency to rupture. A glance at (67) will show that its greatest value is found at the centre. This is given in Table III. under the heading ν.

When c/a exceeds the critical value there is a small superficial volume round each pole within which Φ is not the algebraically greatest stress, though elsewhere it continues to be so. Within these small volumes, however, the values of the maximum stress-difference and of the greatest strain are for finite values of c/a much less than are the corresponding values found at the centre of the spheroid. Thus so far as the question of rupture is concerned, the fact that when c/a exceeds the critical value small regions exist around the poles in which Φ is not the greatest principal stress nor u_r/r the greatest strain is of no material consequence, though of course a point well worthy of notice on its own account. This leaves the value of ν given in Table III. a correct measure of the tendency to rupture on Saint-Venant's theory even when c/a exceeds the critical value.

§ 32. The determination of the maximum stress-difference throughout the whole of the spheroid would be a laborious process which seems hardly worth the trouble. The value at the centre is given in the last column of Table III. In the critical spheroid it was shown above that this is the absolutely greatest value of the maximum stress-difference, and in a previous paper* it was proved that the same was true for a sphere of any isotropic material.

If the values m and n of the elastic constants in the general case of isotropy be substituted in the general expression (53) for a flat rotating spheroid, it can easily be proved that the stress Z everywhere vanishes, and that consequently, excluding the surface where all meridian stresses are of order z at least, the principal stresses in the meridian plane are respectively R and zero, when terms in z^2 are neglected. Further the value of R is nowhere negative. The third principal stress is Φ along the perpendicular to the meridian plane. Φ is everywhere not less than R—it is equal to R along the axis of rotation,—and its greatest value exists in the axis, where it is constant so long at least as terms in z^2 are neglected. Thus the greatest value of the maximum stress-difference is correctly given by the value of Φ at the centre of the flat spheroid.

The expressions obtained from (63) for a very elongated prolate spheroid of isotropic material, whether uniconstant or not, are even more simply treated. The stresses Z, R and Φ are everywhere the principal stresses, and $\Phi - Z$ is everywhere a correct measure of the maximum stress-difference. It is easily proved that its greatest value occurs in the axis of rotation, at every point of which the value is the same.

* See the Society's *Transactions*, Vol. XIV., pp. 292—294.

We are thus certain that in the cases of the flat oblate spheroid, the sphere, the critical spheroid, and the elongated prolate spheroid, the numbers in the last column of Table III. give the greatest value of the maximum stress-difference occurring anywhere, and there seems to me every probability that such is in general the case. I thus believe this column to give in each instance the true measure of the tendency to rupture on the stress-difference theory; but except in the four special cases just mentioned, we are strictly speaking only warranted in regarding the results as supplying minima for the correct measures of the tendency to rupture.

§ 33. After our examination of these special cases it will be unnecessary to enter into great detail in discussing the general case, for which the displacements are given by the expressions (49 a) and (50 a).

Assuming the original elastic constants **c**, **f** etc., as well as η, η' etc. all positive, we have as already explained the relations (55). From the latter of these it follows that

$$\left.\begin{aligned} E > E'\eta^2, \\ E(1-\eta')(3+\eta') > 4E'\eta^2 \end{aligned}\right\} \quad \text{.................................(92).}$$

Bearing in mind these relations, we see from (49 a) and (50 a) that:—

$$\left.\begin{aligned} u_r/r &= \nu'(1 - r^2/\alpha'^2_1 - z^2/\beta'^2_1), \\ w &= -\tau' z(1 - r^2/\alpha'^2_2 - z^2/\beta'^2_2), \\ \frac{du_r}{dr} &= \nu'(1 - r^2/\alpha'^2_3 - z^2/\beta'^2_1), \\ \frac{dw}{dz} &= -\tau'(1 - r^2/\alpha'^2_2 - z^2/\beta'^2_3) \end{aligned}\right\} \quad \text{.................................(93),}$$

where ν', τ', α'^2_1, β'^2_1, α'^2_2, β'^2_2, $\alpha'^2_3 \equiv \alpha'^2_1/3$, and $\beta'^2_3 = \beta'^2_2/3$ are all positive constants depending on the values of c/a and on the elastic constants. For the special case of uniconstant isotropy these reduce to the corresponding undashed constants ν, τ, etc.

There is thus for each displacement, or normal strain, a *determining* spheroidal surface over which the displacement, or strain, vanishes. Also u_r/r and $\frac{du_r}{dr}$ are positive inside and negative outside their determining spheroids, while the reverse is true of w and $\frac{dw}{dz}$. When a determining spheroidal surface lies wholly outside of the material rotating spheroid the corresponding displacement or strain is, if u_r, u_r/r, or $\frac{du_r}{dr}$, everywhere positive, but if w or $\frac{dw}{dz}$ everywhere negative throughout the solid.

The only remaining strain is the shearing strain in the meridian plane, whose value is given by the simple expression

$$\frac{D'}{\omega^2\rho}\left(\frac{du_r}{dz} + \frac{dw}{dr}\right) = -\frac{rz}{G}\left\{\frac{3+\eta'}{4} - \frac{E'\eta(1+\eta')}{E(1-\eta')}c^2a^{-2}\right\} \quad \text{......................(94).}$$

Thus it vanishes everywhere along the polar axis and in the equatorial plane, and throughout the rest of the spheroid changes sign only with z. The sign is $-$ or $+$ for z positive according as c/a is less or greater than the *critical value*

$$\left\{\frac{E(1-\eta')(3+\eta')}{4E'\eta(1+\eta')}\right\}^{\frac{1}{2}} \qquad (95).$$

In the critical spheroid whose axes possess this ratio the shearing strain is everywhere zero.

§ 34. The expressions for ν', $\alpha_1'^2$ etc. are somewhat complicated, and a consideration of the magnitudes of the semi-axes of the determining spheroids does not so easily lead to the desired results as does the following method.

The signs of the displacements and strains at the centre of the spheroid are already known. Thus if we determine their signs at the surface of the material spheroid we can tell whether any portion of the solid lies outside of the determining spheroids. To get the sign of any displacement or strain at the surface, it is simplest to make the expression for it homogeneous by substituting $a^{-2}r^2 + c^{-2}z^2$ for unity. There are then in each expression only two coefficients whose signs have to be considered. Employing this method we find over the surface

$$\frac{D'}{\omega^2\rho}\left(\frac{u_r}{r}\right)_s = \tfrac{1}{4}r^2\left[\frac{2(1-\eta')}{E'} + c^2a^{-2}\left\{\frac{1}{G} - \frac{\eta(1+\eta')}{E}\right\} + \frac{4c^4a^{-4}}{E^2}(E - E'\eta^2)\right]$$
$$+ \tfrac{1}{4}z^2\left\{a^2c^{-2}\frac{(1-\eta')(3+\eta')}{E'} - \frac{4\eta(1+\eta')}{E}\right\} \ldots (96).$$

Employing the last of equations (55) it is easily proved that for all values of c/a, however large G/E may be, the coefficient of r^2 is positive. The coefficient of z^2 is obviously positive or negative according as c/a is less or greater than the critical value.

It follows that for all materials of the class here considered, so long as c/a is less than the critical value, every element of the rotating spheroid increases its distance from the axis of rotation and the transverse strain is everywhere an extension. When, however, c/a exceeds the critical value there is in all such materials a superficial region surrounding each pole wherein the distance of each element from the axis of rotation is diminished and the transverse strain is a compression.

§ 35. The expression for the surface value of w is not quite so manageable. It is the following:—

$$\frac{D'}{\omega^2\rho}(w)_s = r^2z\left[-\frac{\eta(1-\eta')}{2E} + \frac{c^2a^{-2}}{4E^2}\left\{E(3+\eta') - 2E'\eta^2 - \frac{\eta EE'}{G}\right\} - \frac{2c^4a^{-4}E'\eta(E-E'\eta^2)}{E^3(1-\eta')}\right]$$
$$+ z^3\left[-a^2c^{-2}\frac{\eta(3+\eta')}{2E} - \frac{1}{6E(1-\eta')}\left\{(1-\eta')(3+\eta') - \frac{2E'\eta^2(5+3\eta')}{E} + \frac{E'\eta(3+\eta')}{G}\right\}\right.$$
$$\left. - \tfrac{4}{3}c^2a^{-2}\frac{E'\eta(E-E'\eta^2)}{E^3(1-\eta')}\right] \qquad (97).$$

By means of the second of equations (55) it is not very difficult to prove that, whatever be the value of G/E, the coefficient of z^3 is negative for all materials of the kind here considered. The coefficient of r^2z is certainly negative if c/a be either very small or very large, but in general it will be positive when c/a lies between certain limits depending on the material, the superior of which is decidedly less than the critical value. C.f. § 20.

It follows that if c/a be either very small or very large every element diminishes its distance from the equatorial plane. In most if not all materials, however, of the kind treated here,—certainly in all isotropic materials,—there is between certain limiting values of c/a depending on the material a superficial equatorial region within which the elements increase in distance from the equatorial plane.

§ 36. For the surface value of $\dfrac{du_r}{dr}$ we get

$$\frac{D'}{\omega^2\rho}\left(\frac{du_r}{dr}\right)_s = r^2\left[-\frac{\eta'(1-\eta')}{2E'} - \tfrac{1}{4}c^2a^{-2}\left\{\frac{\eta'}{G}+\frac{3\eta(1+\eta')}{E}\right\} - \frac{c^4a^{-4}}{E'^2(1-\eta')}\left\{E\eta'(1-\eta') - E'\eta^2(1+3\eta')\right\}\right]$$
$$+\tfrac{1}{4}z^2\left\{a^2c^{-2}\frac{(1-\eta')(3+\eta')}{E'} - \frac{4\eta(1+\eta')}{E}\right\} \quad\text{.........................(98).}$$

The coefficient of z^2 is positive or negative according as c/a is less or greater than the critical value. The coefficient of r^2 is negative for all values of c/a for all materials in which

$$E\eta'(1-\eta') > E'\eta^2(1+3\eta') \quad\text{..................................(99).}$$

This includes all isotropic materials in which $m < 3n$.

For other materials however, including isotropic materials in which $m > 3n$ if such exist, the coefficient of r^2 becomes positive when c/a is sufficiently increased above the critical value.

We conclude that while c/a is below the critical value the radial strain is everywhere an extension, except in a superficial volume about the equator where it is a compression. As c/a increases the superficial volume approaches the poles and eventually reaches them when c/a attains the critical value. In materials whose elastic constants satisfy the relation (99) there is for all values of c/a above the critical a superficial layer completely surrounding the spheroid wherein the radial strain is a compression, while elsewhere it is an extension. In materials whose elastic constants do not satisfy (99),—including isotropic materials for which $m > 3n$,—when c/a exceeds a certain value, greater considerably than the critical value, the superficial volume in which the radial strain is a compression splits up into two volumes one surrounding each pole, and as c/a further increases these polar volumes continually contract. The materials in which this splitting up of the superficial layer into two polar volumes may naturally be expected are those in which Young's modulus for the direction parallel to the axis of rotation is small compared to that for the perpendicular directions.

§ 37. The surface value of $\frac{dw}{dz}$ is given by

$$\frac{D'}{\omega^2\rho}\left(\frac{dw}{dz}\right)_s = r^2\left[-\frac{\eta(1-\eta')}{2E}+\frac{c^2a^{-2}}{4E^2}\left\{E(3+\eta')-2E'\eta^2-\frac{\eta EE'}{G}\right\}-2c^4a^{-4}\frac{E'\eta(E-E'\eta^2)}{E^3(1-\eta')}\right]$$
$$+z^2\left\{-a^2c^{-2}\frac{\eta(3+\eta')}{2E}+\frac{2E'\eta^2(1+\eta')}{E^2(1-\eta')}\right\} \quad\ldots\ldots\ldots\ldots (100).$$

The coefficient of r^2 is the same as that of r^2z in (97), and its sign has been already treated of in considering that expression. The coefficient of z^2 is negative or positive according as c/a is less or greater than the critical value. The conclusions these data lead to are as follows:—

For small values of c/a the longitudinal strain is in all materials everywhere a compression. In most if not in all materials,—certainly in all isotropic materials—there exists within certain limiting values of c/a, the superior of which is decidedly below the critical value, a superficial region about the equator wherein the longitudinal strain is an extension; elsewhere it remains a compression. Between this superior limit of c/a and the critical value the longitudinal strain is everywhere a compression. Finally when c/a exceeds the critical value there exists in all materials a superficial region round each pole wherein the longitudinal strain is an extension; elsewhere it is a compression.

§ 38. It will be observed that on the whole the variations of the strains and displacements in the general case follow very closely the variations which occur in the special case of uniconstant isotropy. In fact, with one exception presently to be noticed, when α_2 etc. are replaced by α'_2 etc., $\sqrt{39/20}$ by the "critical value" (95), and ·43 and ·90 by the two positive values of c/a obtained by equating the coefficient of r^2 in (100) to zero, Table II. in § 23 may be applied to all but certain exceptional materials whose existence is somewhat problematical.

The single exception is that of materials in which the relation (99) does not hold. In such materials, as already explained, the superficial volume wherein the radial strain is a compression becomes for large values of c/a limited to circumpolar regions. This is a rather noticeable departure from the phenomena described in uniconstant isotropy, and is worthy of special attention because the relation it requires between the values of the elastic constants seems likely to be by no means uncommon in materials in which Young's modulus in the direction of the axis of symmetry is small compared to that in the perpendicular directions.

§ 39. The expressions for the stresses in the general case are on the whole wonderfully simple. The tangential or shearing stress in the meridian plane $= G \times$ (corresponding shearing strain), and so is the product of the right-hand side of (94) into $\omega^2\rho G/D'$. Its fluctuations in sign have been already noticed in treating the shearing strain. It will be noticed that the surfaces over which this shearing strain and stress have constant values are generated by the revolution about the axis of rotation of rectangular hyperbolas whose asymptotes are the axis of rotation and an equatorial diameter.

The expressions for the normal stresses, referred as previously to the fundamental directions at each separate point for axes, are as follows:—

$$Z = \frac{\omega^2 \rho c^2}{4E(1-\eta')D'} \left\{ E(1-\eta')(3+\eta') - 4c^2a^{-2}E'\eta(1+\eta') \right\} \left\{ \frac{r^2}{a^2} - \left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right) \right\} \quad \text{..............(101)},$$

$$R = \frac{\omega^2 \rho a^2}{4E(1-\eta')D'} \left[\{E(1-\eta')(3+\eta') - 4c^2a^{-2}E'\eta(1+\eta')\} \left(1 - \frac{r^2}{a^2}\right) \right.$$
$$\left. + c^2a^{-2}E' \left\{ \frac{E}{2G}(3+\eta') + \eta(1+3\eta') + \frac{2c^2a^{-2}}{E}\left(E(3+\eta') - 2E'\eta^2\right) \right\} \left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right) \right] \quad \text{...(102)},$$

$$\Phi = R + \frac{\omega^2 \rho E' r^2}{2D'} \left\{ \frac{1-\eta'}{E'} + c^2a^{-2}\left(\frac{\eta}{E} + \frac{1}{2G}\right) + 2c^4a^{-4}\frac{E(1-\eta') - 2E'\eta^2}{E^2(1-\eta')} \right\} \quad \text{.........................(103)}.$$

§ 40. From (101) it appears that for all values of c/a, whatever be the character of the material, the longitudinal stress vanishes over the surface of the spheroid whose equatorial and polar semi-axes are respectively $a/\sqrt{2}$ and c. It is a pressure inside and a tension outside this surface when c/a is less than the critical value, a tension inside and a pressure outside when c/a is greater than the critical value. The volume throughout which it is a tension is thus under all circumstances equal to that throughout which it is a pressure. In the critical spheroid itself the longitudinal stress everywhere vanishes. Over the surface of the material spheroid for all values of c/a the longitudinal stress varies as the square of the perpendicular on the axis of rotation.

In (102) it will be noticed that the coefficient of $(1 - r^2/a^2 - z^2/c^2)$ is essentially positive for all materials of the kind considered here, and that the coefficient of $(1 - r^2/a^2)$ is positive or negative according as c/a is less or greater than the critical value.

Thus so long as c/a is less than the critical value the radial stress is everywhere a tension, but when c/a exceeds the critical value it becomes a pressure in a superficial volume, whose thickness is greatest at the poles and zero in the equator. Over the surface of the spheroid, whatever be the value of c/a or the character of the material, the radial stress varies as the square of the perpendicular on the equatorial plane. The radial stress thus vanishes where the equatorial plane cuts the surface and in general nowhere else. In the critical spheroid however it vanishes at every point of the surface.

The stress Φ at right angles to the meridian plane is equal to the radial stress at every point on the axis of rotation and everywhere else is algebraically greater than it. It is everywhere a tension so long as c/a is less than the critical value, but when c/a exceeds the critical value it becomes a pressure in a superficial volume around each pole.

The remarks made on the position of the principal axes in the case of uniconstant isotropy, cf. § 28, apply verbatim to the general case. The stress Φ perpendicular to the meridian plane is everywhere a principal stress. Along the polar axis and in the equatorial plane the longitudinal and radial stresses Z and R are principal stresses, and this is also the case at every point of the critical spheroid, which has thus one of its principal stresses everywhere zero. With these exceptions however the principal stresses in the meridian plane do not act along the fundamental directions, and the angle which the algebraically greater of them makes with the perpendicular on the axis of rotation produced outwards is everywhere obtuse or acute according as c/a is less or greater than the critical value.

On the surface the only stress in the meridian plane is along the tangent, and it is a tension or a pressure according as c/a is less or greater than the critical value. Over the surface of any given spheroid it varies inversely as the square of the perpendicular from the centre on the tangent plane.

§ 41. In the general case it seems scarcely worth while constructing tables for the values of the changes in the lengths of the equatorial and polar diameters and for the strains at the centre of the spheroid. To be practically useful such tables would have to assign numerical values to η, η', G/E and E'/E. It is doubtful if satisfactory experimental determinations of these quantities exist for materials of the class here considered, and a large amount of time would be required to make the arithmetical calculations necessary if all values theoretically possible were to be included.

Further, materials of this class can doubtless support a greater strain in some directions than in others, so that the value of the greatest positive strain, or the greatest value of the maximum stress-difference, cannot on any possible theory *immediately* determine the tendency of the body to pass beyond the limits of perfect elasticity or to approach rupture. Saint-Venant it is true has applied his theory of rupture in a generalized form to such materials, but it seems on the whole advisable to postpone consideration of the question until a reasonable expectation exists that the theory corresponds to the facts.

§ 42. In the case of uniconstant isotropy the variation of the more important strains and displacements with the value of c/a have been already shown in Table III. Since however in this country the biconstant theory of isotropy is almost universally accepted, I have calculated the values of the several quantities of that table for the values 0, ·2, ·4, ·6 and 1 of the ratio of the elastic constants $n : m$. These answer respectively to the values ·5, ·4, ·3, ·2 and 0 of Poisson's ratio. Every solid probably that has the least claim to be regarded as isotropic will be admitted to have positive values for Poisson's ratio and for the rigidity, so that 0 and 1 are respectively the least and greatest values which can be attached to n/m. The results are thus of the utmost generality so far as isotropic materials are concerned. They are given in the following tables, V.—IX. The corresponding results for intermediate values of n/m could in general be obtained to a close degree of approximation by interpolation from the tables.

§ 43. The quantity treated in Table V. is the total increase in the equatorial diameter divided by its whole length. It is for shortness spoken of as the increase *per unit length*, but it must be clearly understood that the radial strain varies from point to point of a diameter, so that the change in any particular unit of length varies with the distance from the centre. In this as in the following three tables the numbers in the table must be multiplied by $\omega^2\rho a^2/E$ to get the absolute values. This factor is an arithmetical quantity, and as such independent of the particular system of units employed. The value of E must of course be determined by experiment and expressed in terms of the same system of units as the other quantities.

In comparing the results answering to a given value of n/m the equatorial semi-diameter a must be regarded as constant, so that the variations in the value of c/a must be treated as proceeding from variations in c alone. Thus what Table V., for instance, immediately shows is how the increase in the equatorial diameter of a spheroid of given equatorial diameter, formed of given material and rotating with a given angular velocity, depends on the ratio of the polar to the equatorial diameter.

Table VI. gives the total diminution of the polar diameter divided by its whole length. The actual longitudinal strain of course along the polar diameter is not in general constant but varies with the distance from the centre.

Table VII. gives the algebraically greatest principal strain at the centre. It might equally correctly have been represented by $\left(\frac{du_r}{dr}\right)_0$, because the radial and transverse strains are there the same. In certain cases—e.g. for the values 0, 1, ∞ of c/a—this has already been proved to be the algebraically greatest strain occurring anywhere in the spheroid, and is then known to be the exact measure of the tendency to rupture on Saint-Venant's theory. It may further be shown, as in the corresponding case in uniconstant isotropy, that this quantity is in general the correct measure of Saint-Venant's tendency to rupture.

Table VIII. gives the numerical value of the third principal strain at the centre. It is a negative quantity and so is a compression, and its direction is the polar diameter. It does not in itself supply a measure of the tendency to rupture on any theory and so is of less importance than the greatest strain. Its variations have been deemed worthy of tabulation because the centre is in itself the most important point in the spheroid, and because the value of any given normal strain throughout the spheroid is as a rule small or great according as its value at the centre is small or great.

The quantity tabulated in Table IX. is the maximum stress-difference at the centre. For the values 0, 1, ∞ of c/a it measures exactly on the stress-difference theory the tendency of the spheroid to rupture. For other values of c/a it can be regarded only as an inferior limit to the true tendency to rupture, as the existence of greater values elsewhere has not been formally disproved.

Being of the nature of a stress it is measured in terms of $\omega^2\rho a^2$, and is thus given in absolute measure in terms of the system of units of length, time and mass which may have been adopted.

Table X. is of a totally different character from the previous five. It gives the value of c/a in the critical spheroid answering to the assigned values of n/m. The importance of the critical spheroid has been pointed out and most of its properties have been noticed in treating of the general case or of uniconstant isotropy. In the latter case it was stated in § 26 that the absolutely largest value of the greatest strain, for a given material and given equatorial diameter, occurs in the critical spheroid. This is not however a peculiarity of uniconstant isotropy but, as may easily be proved from the expression for the greatest strain, is true of the general case of biconstant isotropy. We can thus lay down as a general law that:—

In a rotating spheroid of given equatorial diameter formed of an isotropic medium, the absolutely largest "greatest strain" at the centre, and so the greatest "tendency to rupture" on Saint-Venant's theory, invariably occurs in the critical spheroid.

TABLE V.

Increase in equatorial diameter per unit length.

$$\frac{u_a}{a} \div \frac{\omega^2 \rho a^2}{E}.$$

Value of c/a	Value of n/m: 0	·2	·4	·6	1
0	·087	·105	·124	·143	·1$\dot{8}$
·2	·091	·109	·127	·146	·183
·4	·102	·119	·137	·154	·191
·8	·126	·144	·163	·181	·217
1·0	·132	·151	·171	·190	·229
2·0	·131	·154	·178	·2009	·2475
4·0	·127	·1514	·1760	·2006	·2498
∞	·125	·15	·175	·2	·25

TABLE VI.

Diminution of polar diameter per unit length.

$$\frac{-w_c}{c} \div \frac{\omega^2 \rho a^2}{E}.$$

Value of c/a	Value of n/m: 0	·2	·4	·6	1
0	·304	·239	·175	·114	0
·2	·308	·244	·181	·121	·007
·4	·314	·255	·194	·136	·025
·8	·287	·241	·195	·149	·0565
1·0	·263	·$\dot{2}$	·181	·140	·0571
2·0	·197	·163	·129	·095	·027
4·0	·174	·141	·108	·074	·0076
∞	·1$\dot{6}$	·1$\dot{3}$	·1	·0$\dot{6}$	0

TABLE VII.

Greatest strain at centre.

$$\left(\frac{u_r}{r}\right)_0 \div \frac{\omega^2\rho a^2}{E}.$$

Value of c/a	Value of n/m	0	·2	·4	·6	1
0		·152	·179	·204	·229	·27
·2		·164	·188	·212	·234	·276
·4		·190	·213	·233	·252	·286
·8		·218	·252	·277	·296	·325
1·0		·210	·255	·286	·31	·343
2·0		·160	·231	·282	·319	·371
4·0		·135	·215	·272	·3150	·3747
∞		·125	·208	·268	·3125	·375

TABLE VIII.

Longitudinal compression at centre.

$$-\left(\frac{dw}{dz}\right)_0 \div \frac{\omega^2\rho a^2}{E}.$$

Value of c/a	Value of n/m	0	·2	·4	·6	1
0		·304	·239	·175	·114	0
·2		·327	·256	·190	·127	·011
·4		·380	·298	·225	·158	·038
·8		·437	·347	·271	·204	·0847
1·0		·421	·337	·264	·2	·0857
2·0		·319	·256	·198	·143	·040
4·0		·269	·215	·163	·112	·011
∞		·25	·2	·15	·1	0

TABLE IX.

Maximum stress-difference at centre.

$$\frac{2E}{3-n/m}\left\{\left(\frac{u_r}{r}\right)_0-\left(\frac{dw}{dz}\right)_0\right\}\div\omega^2\rho a^2.$$

Value of c/a	Value of n/m 0	·2	·4	·6	1
0	·304	·298	·292	·286	·2$\dot{7}$
·2	·327	·318	·309	·301	·286
·4	·380	·365	·352	·342	·324
·8	·437	·428	·422	·417	·410
1·0	·421	·422	·424	·425	·429
2·0	·319	·348	·369	·386	·411
4·0	·269	·307	·335	·356	·386
∞	·25	·292	·321	·344	·375

TABLE X.

Value of c/a in the critical spheroid.

$n/m =$	0	·1	·2	·4	·6	·8	·9	1
critical value of c/a	·764	·853	·954	1·217	1·633	2·518	3·715	∞

§ 44. The calculations on which these tables are based proceeded to 4 places of decimals. The last of these however has been retained only in a few cases where the variation of the quantity considered with the value of c/a is exceptionally slow. When less than 3 places of decimals are shown the value given in the table is the exact value of the quantity.

The results of the Tables V.—IX. are also shown graphically in the accompanying figures 1—5, Plate II., as they seem peculiarly well adapted for this form of treatment. In all the figures the abscissae of the curves answer to the values of c/a, a special curve being drawn for each value of n/m. In the first four figures the curves for the value $n/m = ·5$, answering to uniconstant isotropy, are also drawn. In the last figure this curve is omitted as in its earlier portion it could hardly be shown distinctly between the curves answering to the values ·4 and ·6 of n/m.

In the first four figures the ordinates give the numerical value of the coefficient of $\omega^2\rho a^2/E$, which is thus treated as the unit quantity. In the last figure the unit quantity is $\omega^2\rho a^2$ simply. In the first four figures and the corresponding tables when a direct comparison is instituted, for a given value of c/a, between the values of the quantities which answer to the various values of n/m, the materials compared must be supposed to have the same Young's modulus and density.

§ 45. From fig. 1, or Table V., it is seen at a glance that the way in which the increase in the equatorial diameter varies with the value of c/a is very similar for all possible values of n/m. As c/a increases from 0 to 1 the increase in the equatorial diameter rises continually in every case, though somewhat slowly. As c/a increases further the variations in the quantity considered are remarkably small, so that the increase in the equatorial diameter is practically nearly independent of the eccentricity in all prolate spheroids. When $n/m=1$ the curve continually approaches an asymptotic value as a superior limit. In the other curves the ordinates show true maxima for finite values of c/a, all greater than unity and so denoting prolate spheroids, and the value of c/a answering to the maximum continually diminishes as n/m diminishes, i.e. as Poisson's ratio increases. Also it is obvious that for a given value of Young's modulus and a given density, the increase in the equatorial diameter invariably increases as Poisson's ratio diminishes, whatever be the value of c/a.

§ 46. The ordinates of all the curves of fig. 2 show distinct maxima which answer to values of c/a less than 1, so that for a given material and a given equatorial diameter the diminution per unit length in the polar diameter is greatest in some form of oblate spheroid. It is also obvious from the figures that the spheroid in which the quantity is a maximum becomes more and more oblate as n/m diminishes, i.e. as Poisson's ratio increases.

The dependence of the diminution of the polar diameter on the value of Poisson's ratio is very marked. When Poisson's ratio becomes zero, the diminution of the polar diameter totally disappears in the limiting forms of the oblate and prolate spheroids answering to the values 0 and ∞ of c/a, and is extremely small in all spheroids which differ much from the spherical form.

A comparison of figures 1 and 2 shows very strikingly how the class of isotropic materials in which the increase in the equatorial diameter is most marked is precisely the class in which the diminution in the polar diameter is least conspicuous.

§ 47. The curves of fig. 3 resemble pretty closely those of fig. 1. Except in the case of $n/m=1$, the ordinates show true maxima for finite values of c/a, and the value of c/a at which the maximum appears continually diminishes as Poisson's ratio increases. The exact positions of the maxima are, as already explained, given by Table X. Except in the case of $n/m=0$, the dependence of the greatest strain on the eccentricity is decidedly more conspicuous in oblate than in prolate spheroids.

§ 48. The curves of fig. 4 show a general resemblance to those of fig. 2. Their ordinates however exhibit much more pronounced maxima. The spheroids in which these maxima occur are all oblate, and the oblateness increases but only to a very small extent as Poisson's ratio increases. It will also be noticed that for a given magnitude of spheroid the longitudinal compression at the centre diminishes rapidly as Poisson's ratio diminishes, and absolutely vanishes along with Poisson's ratio in the limiting oblate and prolate spheroids answering to the values 0 and ∞ of c/a. In fact the isotropic materials in which the greatest strain at the centre is largest are precisely those in which the longitudinal compression is least and conversely.

§ 49. In fig. 5 the closeness of the curves for all values of c/a less than unity seems very remarkable. This would indicate that on the stress-difference theory the numerical measure of the tendency to rupture at the centre in all oblate spheroids of isotropic material is nearly independent of the values of the elastic constants. This would not of course imply that the angular velocities causing rupture in oblate spheroids of the same size and shape are nearly the same for all isotropic materials of the same density, because one such material might stand a very much greater stress-difference than another. There is also a critical value of c/a, lying in every case between ·9 and 1, at which the value of the maximum stress-difference regarded as a function only of n/m becomes stationary. In all oblate spheroids in which c/a is less than ·9 the maximum stress-difference continually increases, though only to a small extent, as Poisson's ratio increases; whereas in all prolate spheroids the maximum stress-difference continually diminishes as Poisson's ratio increases. In oblate spheroids in which c/a lies between ·9 and 1 the maximum stress-difference is practically independent of the values of the elastic constants.

In all the stress-difference curves the ordinates possess distinct maxima. When $n/m=1$ this maximum appears when c/a is nearly 1·2. In each of the other curves the maximum appears when c/a is less than unity, i.e. in an oblate spheroid, and the oblateness of this spheroid continually increases as n/m diminishes, i.e. as Poisson's ratio increases. In no case, however, does the spheroid in which the maximum occurs differ very much from the spherical form.

II. *Non-Euclidian Geometry.* By Professor Cayley.

[Read January 27, 1890.]

I consider ordinary three-dimensional space, and use the words point, line, plane, &c. in their ordinary acceptations; only the notion of distance is altered, viz. instead of taking the Absolute to be the circle at infinity, I take it to be a quadric surface: in the analytical developments this is taken to be the imaginary surface $x^2 + y^2 + z^2 + w^2 = 0$, and the formulæ arrived at are those belonging to the so-called Elliptic Space. The object of the Memoir is to set out, in a somewhat more systematic form than has been hitherto done, the general theory; and in particular to further develope the analytical formulæ in regard to the perpendiculars of two given lines. It is to be remarked that not only all purely descriptive theorems of Euclidian geometry hold good in the new theory; but that this is the case also (only we in nowise attend to them) with theorems relating to parallelism and perpendicularity, in the Euclidian sense of the words. In Euclidian geometry, infinity is a special plane, the plane of the circle at infinity, and we consider (for instance) parallel lines, that is lines which meet in a point of this plane: in the new theory infinity is a plane in nowise distinguishable from any other plane, and there is no occasion to consider (although they exist) lines meeting in a point of this plane, that is parallel lines in the Euclidian sense. So again, given any two lines, there exists always, in the Euclidian sense, a single line perpendicular to each of the given lines, but this is not in the new sense a perpendicular line; there is nothing to distinguish it from any other line cutting the two given lines, and consequently no occasion to consider it: we do consider the lines—there are in fact two such lines—which in the new sense of the word are perpendicular to each of the given lines.

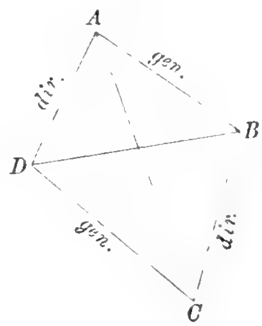
It should be observed that the term distance is used to include inclination: we have, say, a linear distance between two points; an angular distance between two lines which meet; and a dihedral distance between two planes. But all these are distances of the same kind, having a common unit, the quadrant, represented by $\frac{1}{2}\pi$; and in fact any distance may be considered indifferently as a linear, an angular, or a dihedral distance: the word, perpendicular, usually represented by $\perp$, refers of course to a distance $= \frac{1}{2}\pi$. We have moreover the distance of a point from a plane, that of a point from a line, and that of a plane from a line. Two lines which do not meet may be $\perp$, and in particular they may be reciprocal: in general they have two distances; and they have also a "moment" and "comoment", the values of which serve to express those of the two distances. Lines may be, in several distinct senses, as will be explained, parallel; and for this reason the word parallel is never used simpliciter; the notion of parallelism does not apply to planes, nor to points.

Elliptic space has been considered and the theory developed in connexion with the imaginaries called by Clifford biquaternions, and as applied to Mechanics: I refer to the names, Ball, Buchheim, Clifford, Cox, Gravelius, Heath, Klein, and Lindemann: in particular much of the purely geometrical theory is due to Clifford. Memoirs by Buchheim and Heath are referred to further on.

Geometrical Notions. Nos. 1 to 16.

1. The Absolute is a general quadric surface: it has therefore lines of two kinds, which it is convenient to distinguish as directrices and generatrices: through each point of the surface there is a directrix and a generatrix, and the plane through these two lines is the tangent plane at the point. A line meets the surface in two points, say A, C; the generatrix at A meets the directrix at C; and the directrix at A meets the

Fig. 1.



generatrix at C; and we have thus on the surface two new points B, D; joining these we have a line BD, which is the reciprocal of AC; viz. BD is the intersection of the planes BAD, BCD which are the tangent planes at A, C respectively, and similarly AC is the intersection of the planes ABC, ADC which are the tangent planes at B, D respectively.

According to what follows, reciprocal lines are $\perp$, but $\perp$ lines are not in general reciprocal; thus the two epithets are not convertible, and there will be occasion throughout to speak of reciprocal *lines.*

2. Two points may be harmonic; that is the two points and the intersections of their line of junction with the Absolute may form a harmonic range: the two points are in this case said to be $\perp$.

Two planes may be harmonic: that is the two planes and the tangent planes of the Absolute through their line of intersection may form a harmonic plane-pencil: the two planes are said to be $\perp$.

Two lines which meet may be harmonic: that is the two lines and the tangents from their point of intersection to the section of the Absolute by their common plane may form a harmonic pencil: the two lines are said to be $\perp$.

The locus of all the points $\perp$ to a given point is a plane, the reciprocal or polar plane of the given point; and similarly the envelope of all the planes $\perp$ to a given plane is a point, the pole of the given plane: a point and plane reciprocal to each other, or say a pole and polar plane, are said to be $\perp$.

3. If a point is situate anywhere in a given line, the $\perp$ plane passes always through the reciprocal line: each point of the reciprocal line is thus a point of the $\perp$ plane i.e. it is $\perp$ to the given point: that is, considering two reciprocal lines, any point on the one line and any point on the other line are $\perp$. Similarly any plane through the one line and any plane through the other line are $\perp$.

A line and plane may be harmonic; that is they may be reciprocal in regard to the cone, vertex their point of intersection, circumscribed to the Absolute; the line and plane are said to be $\perp$. The $\perp$ plane passes through the reciprocal line, and conversely every plane through the reciprocal line is a $\perp$ plane. It may be added that the line passes through the $\perp$ point of the plane; and conversely, that every line through the $\perp$ point of a plane is $\perp$ to the plane. Moreover if a line and plane be $\perp$, the line is $\perp$ to every line in the plane and through the point of intersection.

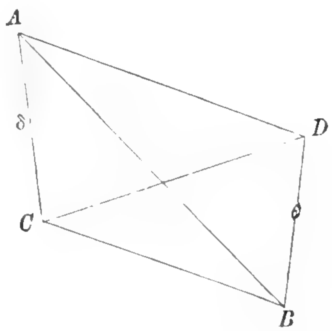
A line and point may be harmonic; that is they may be reciprocal in regard to the section of the Absolute by their common plane: the line and point are said to be $\perp$. The $\perp$ point lies in the reciprocal line, and conversely every point of the reciprocal line is a $\perp$ point. It may be added that the line lies in the $\perp$ plane of the point: and conversely that every line in the $\perp$ plane of a point is $\perp$ to the point. Moreover if a line and point be $\perp$, the line is $\perp$ to every line through the point and in the plane of junction.

4. We may have a triangle ABC composed of three lines BC, CA, AB in the same plane: the six parts hereof are the linear distances B, C; C, A; A, B of the angular points, and the angular distances of the sides CA, AB; AB, BC; BC, CA. Similarly we may have a trihedral composed of three lines meeting in a point, say the planes through the several pairs of lines are A, B, C respectively: the six parts hereof are the angular distances CA, AB; AB, BC; BC, CA of the three lines, and the dihedral distances B, C; C, A; A, B of the three planes. According to the definitions of distance hereinafter adopted, the relation of the six parts is that of the sides and angles of a spherical triangle: in particular, if two sides are each $=\frac{1}{2}\pi$, then the opposite angles are each $=\frac{1}{2}\pi$, and the included angle and the opposite side have a common value; and so also if two angles are each $=\frac{1}{2}\pi$, then the opposite sides are each $=\frac{1}{2}\pi$, and the included side and the opposite angle have a common value.

5. Let A, C be points on a line, and B, D points on the reciprocal line; by what precedes, each of the lines AB, AD, CB, CD is $=\frac{1}{2}\pi$: also each of the angles ACD, ACB,

CAB, CAD is $=\frac{1}{2}\pi$. The line AC is $\perp$ to the plane BCD and to the lines BC, CD, in that plane; it is also $\perp$ to the plane BAD and to the lines BA, AD in that plane; and similarly for the line BD. From the trihedral of the planes which meet in C, distance of planes ACB, ACD = distance of lines BC, CD, viz. the dihedral distance of two planes through the line AC is equal to the angular distance of their intersections with the $\perp$ plane BCD; and it is therefore equal also to the linear distance of their intersections with the

Fig. 2.



other $\perp$ plane BAD: and so from the triangle BCD, where BC, CD are each $=\frac{1}{2}\pi$, the angular distance BCD is equal to the linear distance BD; that is the distance of the planes ACB, ACD, that of the lines BC, CD that of the lines BA, AD and that of the points B, D are all of them equal; say the value of each of them is $=\theta$. And in like manner the distance of the planes ABD, CBD, that of the lines AB, BC, that of the lines AD, DC and that of the points A, C are all of them equal: say the value of each of them is $=\delta$.

The theorem may be stated as follows: all the planes $\perp$ to a given line intersect in the reciprocal line: and if we have through the given line any two planes, the distance of these two planes, the distance between their lines of intersection with any one of the $\perp$ planes, and the distance between their points of intersection with the reciprocal line are all of them equal.

And it thus appears also that a distance may be represented indifferently as a linear distance, an angular distance, or a dihedral distance.

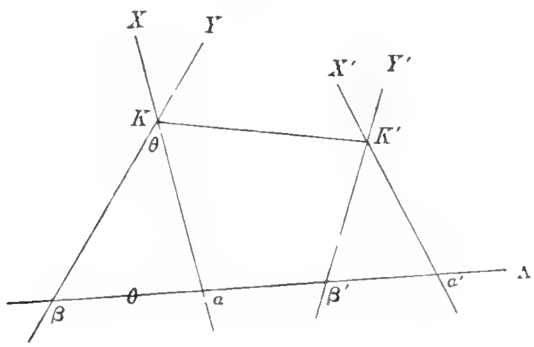
6. Consider a point and a plane: we may through the point draw a line $\perp$ to the plane, and intersecting it in a point called the 'foot': the distance of the point and plane is then (as a definition) taken to be equal to that of the point and foot. It may be added that the $\perp$ line is in fact the line joining the point with the $\perp$ point of the plane; and that the distance of the point and plane is equal to the complement of the distance of the point and the $\perp$ point. Or again, we may in the plane draw a line $\perp$ to the point, and determining with it a plane called the roof: and then (as an equivalent definition) the distance of the plane and point is equal to the distance of the plane and roof. It may be added that the $\perp$ line is in fact the intersection of the plane with the $\perp$ plane of the point, and that the distance of the point and plane is also equal to the complement of the distance of the plane and the $\perp$ plane of the point.

7. Consider a point and line: we have through the point a line ⊥ to the line and cutting it in a point called the foot; the distance of the point and line is then (as a definition) equal to the distance of the point and foot. It may be added that the foot is the intersection with the line of a plane ⊥ thereto through the point.

Again consider a plane and line: we have in the plane a line ⊥ to the line and determining with it a plane called the roof: the distance of the plane and line is then as a definition equal to the distance of the plane and roof. It may be added that the roof is the plane determined by the line and a point ⊥ thereto in the plane.

8. If two lines intersect, then their reciprocals also intersect. Say the intersecting lines are X, Y; and their reciprocals X', Y' respectively; then K, the point of intersection of X, Y, has for its reciprocal the plane of the lines X', Y'; and similarly K', the point of intersection of the lines X', Y', has for its reciprocal the plane of the lines X, Y: hence KK' has for its reciprocal the line of intersection of the planes XY and $X'Y'$; say this is the line Λ, meeting X, Y, X', Y', in the points α, β, α', β' respectively. Since

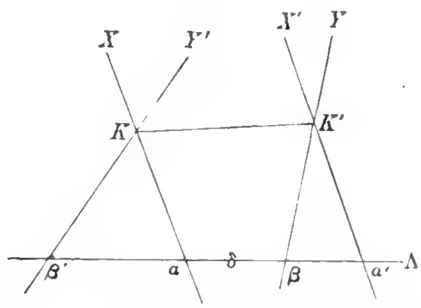
Fig. 3.



K, K' are points in the reciprocal lines X, X' (or in the reciprocal lines Y, Y') the distance KK' is $=\frac{1}{2}\pi$; and since the plane XY passes through the line Λ which is the reciprocal of KK', the line KK' is ⊥ to the plane XY and also to each of the lines X, Y (it is also ⊥ to the plane $X'Y'$ and to each of the lines X', Y'). Again since the lines KK' and Λ are reciprocal, each of the distances $K\alpha$, $K\beta$ is $=\frac{1}{2}\pi$; that is the line Λ is ⊥ to each of the lines X and Y (and similarly it is ⊥ to each of the lines X' and Y'). Moreover the angle at K or distance of the lines X and Y (which is equal to the distance of the planes $K'KX$ and $K'KY$) is equal to the distance $\alpha\beta$ of the intersections of Λ with the lines X and Y respectively. We have thus for the two intersecting lines X and Y, the two lines KK' and Λ each of them ⊥ to the two lines: where observe that KK' is the line of junction of the point of intersection of the two given lines with the point of intersection of the reciprocal lines; and that Λ is the line of intersection of the plane of the two given lines with the plane of the reciprocal lines. The linear distance along KK' between the two lines is $=0$; the dihedral distance between the planes which KK' determines with the two lines respectively is equal to the angular distance between the two lines. The linear distance along Λ is equal to the angular distance between the two lines; the dihedral distance between the two planes which Λ determines with the two lines respectively is $=0$.

9. If two lines are such that the first of them intersects the reciprocal of the second of them, then also the second will intersect the reciprocal of the first; the two lines are in this case said to be contrasecting lines; or more simply, to contrasect: and contrasecting lines are said to be $\perp$. Supposing that the two lines are X, Y and their reciprocals X', Y' respectively, we have here X, Y' intersecting in a point K, and X', Y intersecting in a point K': and the planes XY', $X'Y$ intersect in a line Λ which meets the lines X, Y, X', Y' in the points α, β, α', β' respectively. As before the lines KK' and Λ are reciprocal: the distance KK' is $=\frac{1}{2}\pi$; and KK' is $\perp$ to the plane XY', that is to each of the lines X, Y'; and also to the plane $X'Y$, that is to each of the lines X', Y; it is thus $\perp$ to each of the lines X and Y. Again each of the angles at α, β, α', β' is $=\frac{1}{2}\pi$; that is the line Λ is $\perp$ to each of the lines

Fig. 4.



X, Y', X', Y, or say to each of the lines X and Y. Moreover the angle at K or say the angular distance of the intersecting lines X and Y' is equal to the distance $\alpha\beta'$; and similarly the angle at K' or say the angular distance of the intersecting lines X' and Y is equal to the distance $\alpha'\beta$: but the distances $\alpha\alpha'$, $\beta\beta'$ are each equal to $\frac{1}{2}\pi$; and hence the distances $\alpha\beta'$, $\alpha'\beta$ are equal to each other and each of them is equal to the complement of the distance $\alpha\beta$. Thus in the case of two contrasecting lines we have the lines KK' and Λ each of them $\perp$ to the two given lines; where observe that KK' is the line joining the point of intersection of X with the reciprocal of Y and the point of intersection of Y with the reciprocal of X; and that Λ is the line of intersection of the plane through X and the reciprocal of Y with the plane through Y and the reciprocal of X. The linear distance KK' between the two lines along the first of these lines is thus $=\frac{1}{2}\pi$.

10. We have KK' and Λ reciprocal lines; on the first of these we have the points K, K' which are $\perp$ points: hence also the planes ΛK and $\Lambda K'$ are $\perp$; but the plane ΛK is the plane $\Lambda XY'$ or say the plane ΛX, and the plane $\Lambda K'$ is the plane $\Lambda X'Y$ or say the plane ΛY; hence the planes ΛX and ΛY are $\perp$. Similarly the line Λ cuts the two lines in the points α, β; and the line KK' determines with these two points

respectively the plane $KK'\alpha$, that is $KK'X$, and $KK'\beta$, that is $KK'Y$; and thus the linear distance between the two points α, β is equal to the dihedral distance between the two planes $KK'X$ and $KK'Y$. Thus the $\perp$ line Λ cuts the two lines in two points α, β the linear distance of which is, say δ: and it determines with them two planes the dihedral distance of which is $=\frac{1}{2}\pi$. And the other $\perp$ line KK' cuts the two lines in the points K, K' the linear distance of which is $=\frac{1}{2}\pi$, and it determines with them two planes the dihedral distance of which is $=\delta$.

11. Consider a line X and its reciprocal X': a line intersecting each of these also contrasects each of them and is thus $\perp$ to each of them: and similarly if Y be any other line and Y' its reciprocal, a line intersecting Y and Y' also contrasects each of them and is thus $\perp$ to each of them. Hence a line which meets each of the four lines X, X', Y, Y' is also $\perp$ to each of them, or attending only to the lines X, Y, say it is a $\perp$ of these lines: there are two $\perp$s; and clearly these are reciprocal to each other, for if a line meets X, Y, X', Y' then its reciprocal meets X', Y', X, Y, that is the same four lines. Looking back to figure 2 we may take AB, CD for the given lines, and AC, BD for the two $\perp$s; as just remarked these are reciprocal to each other. The $\perp$ AC cuts the two lines respectively in the two points A and C the linear distance of which is say $=\delta$; and it determines with them two planes ACB, ACD, the dihedral distance of which is say $=\theta$. Similarly the other $\perp$ BD meets the two lines respectively in the two points B and D the linear distance of which is $=\theta$, and it determines with them two planes BDA, BDC the dihedral distance of which is $=\delta$. In the plane triangles which are the faces of the tetrahedron $ABCD$, there is in each triangle an angle opposite to AC or BD and which, or say the angular distance of the two including sides, is thus $=\delta$ or θ. Except as aforesaid the sides, angles, and dihedral angles, or say the linear, angular, and dihedral distances of the tetrahedron are each of them $=\frac{1}{2}\pi$.

12. Considering the lines X and Y as given, the distances δ and θ depend upon two functions called the Moment and the Comoment: viz. moment $=0$ is the condition in order that the two lines may intersect (or, what is the same thing, in order that their reciprocals may intersect): comoment $=0$ is the condition in order that the two lines may contrasect, that is each line meet the reciprocal of the other one. It may be convenient to mention here that the actual relations are

$$\sin\delta\sin\theta = \text{Moment},\quad \cos\delta\cos\theta = \text{Comoment}.$$

In particular if moment $=0$, then the lines intersect; we have, say $\delta=0$, and therefore $\cos\theta =$ comoment; if comoment $=0$, then the lines contrasect, that is they are $\perp$: we have, say $\theta=\frac{1}{2}\pi$, that is $\sin\delta =$ moment. These are the two particular cases which have been considered above.

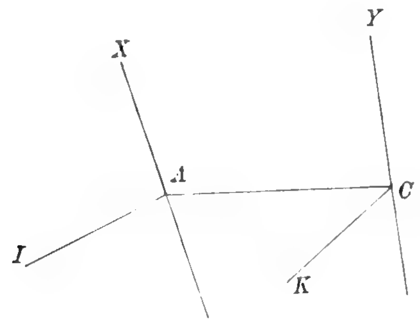
13. Consider as above the two lines, X, Y met by the $\perp$ δ in the two points A and C respectively. Consider at A a line I $\perp$ to the lines X, δ; and take Π the plane of the lines (X, δ) and Ω the plane of the lines (X, I). Similarly consider at C a line K $\perp$ to the lines Y, δ, and take Π_1 the plane of the lines (Y, δ) and Ω_1 the plane of the

lines (Y, K): we have thus through A two planes Π, Ω meeting in the line X; and through C two planes Π_1, Ω, meeting in the line Y. It requires only a little reflection to see that the distances of these planes are

$$(\Pi, \Pi_1) = \theta, \quad (\Omega, \Omega_1) = \delta.$$

$$(\Pi, \Omega) = \tfrac{1}{2}\pi, \ (\Pi_1, \Omega_1) = \tfrac{1}{2}\pi; \ (\Pi, \Omega_1) = \tfrac{1}{2}\pi, \ (\Pi_1, \Omega) = \tfrac{1}{2}\pi.$$

Fig. 5.



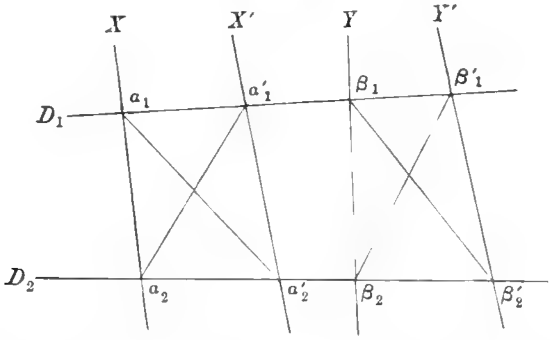
In fact Π, Π_1 are the before mentioned planes ACB, ACD the distance of which was $=\theta$: Ω, Ω_1 are planes having the common $\perp$ AC, which is the line through the poles of these planes, and such that the distance AC is equal to the distance of the two poles, that is the distance of the two planes. Moreover from the definitions the distances (Π, Ω) and (Π_1, Ω_1) are each $=\frac{1}{2}\pi$: the plane Π passes through the $\perp$ at C to the plane Ω_1 that is $(\Pi, \Omega_1) = \frac{1}{2}\pi$; and similarly the plane Π_1 passes through the $\perp$ at A to the plane Ω, that is $(\Pi_1, \Omega) = \frac{1}{2}\pi$; and we have thus the relations in question.

The consideration of these planes leads, (see *post* 31 and 32), to the before mentioned equation, $\cos\delta\cos\theta =$ comoment; if instead of one of the lines, say Y, we consider the reciprocal line Y', then the angles δ, θ are changed each of them into its complement, and we deduce immediately the other equation, $\sin\delta\sin\theta =$ Moment.

14. It may happen that instead of the determinate number 2, we have a singly infinite system of $\perp$s: viz. this will be so if the lines X, X', Y, Y', are generating lines (of the same kind) of a hyperboloid. They will be so if the lines X and Y each of them meet the same two lines (of the same kind) of the Absolute, say if X, Y each meet two directrices D_1, D_2, or two generatrices G_1, G_2; but it seems less easy to prove conversely that the lines X and Y must satisfy one of these two conditions. Suppose first that X, Y each meet the two directrices D_1, D_2; say X meets them in α_1, α_2 and Y in β_1, β_2 respectively. We have at α_1 a generatrix which meets D_2, suppose in α_2' and at α_2, a generatrix which meets D_1, suppose in α_1'; joining α_1', α_2', we have the line X' which is the reciprocal of A; viz. X' meets each of the lines D_1, D_2: similarly the generatrices at β_1, β_2 meet D_2, D_1 in the points β_2', β_1' respectively, and joining these we have the line Y' which is the reciprocal of Y: thus Y' meets each of the lines D_1 and D_2: the line D_1 meets the four generatrices in the points α_1, α_1', β_1, β_1' respectively, and the line D_2 meets the same four

generatrices in the points α_2', α_2, β_2', β_2: thus $AH(\alpha_1, \alpha_1', \beta_1, \beta_1') = AH(\alpha_2', \alpha_2, \beta_2', \beta_2)$, AH denoting anharmonic ratio as usual. But $AH(\alpha_2', \alpha_2, \beta_2', \beta_2) = AH(\alpha_2, \alpha_2', \beta_2, \beta_2')$ and thus the equation may be written $AH(\alpha_1, \alpha_1', \beta_1, \beta_1') = AH(\alpha_2, \alpha_2', \beta_2, \beta_2')$ viz. the lines X, X', Y, Y', cut D_1, D_2 homographically; and there is thus a singly infinite system of lines cutting D_1, D_2 homographically: that is X, X', Y, Y', are lines (of the same kind) of a hyperboloid. And similarly if X, Y each cut the same two generating lines G_1, G_2, then will X', Y' also cut these lines and X, X', Y, Y' will cut them homographically, that is X, X', Y, Y' will be lines (of the same kind) of a hyperboloid.

Fig. 6.



The condition may be otherwise stated; if the lines X, Y have for ⊥s any two directrices D_1, D_2 or any two generatrices G_1, G_2 of the Absolute, then in either case there will be a singly infinite series of ⊥s: the ⊥ distances are all of them equal; say we have $\theta = \delta$, and therefore $\sin^2\delta =$ moment, $\cos^2\delta =$ comoment; and therefore moment + comoment = 1; or as the equation is more properly written, ± moment ± comoment = 1.

15. Two lines X, Y each of them meeting the same two directrices D_1, D_2 are said to be "right parallels"; and similarly two lines X, Y each meeting the same two generatrices G_1, G_2 are said to be "left parallels": the selection as to which set of lines of the Absolute shall be called directrices and which shall be called generatrices will be made further on, (see *post* 35). We have just seen that if two lines are right parallels, or are left parallels, then in either case there is a singly infinite series of ⊥s. It may be remarked that reciprocal lines are at once right parallels and left parallels; and that in this case there is a doubly infinite series of ⊥s, viz. every line cutting the two lines is a ⊥.

Observe that right parallels do not meet, and left parallels do not meet: their doing so would imply in the one case the meeting of two directrices, and in the other case the meeting of two generatrices.

16. If instead of the foregoing definitions by means of two directrices or two generatrices, we consider a directrix and a generatrix of the Absolute, and define parallel lines by reference thereto, then it is at once seen that there are 3 chief forms, and several sub-forms; the directrix and generatrix meet in a point, or say an ineunt, of the Absolute, and lie in a plane which is a tangent plane of the Absolute: we may have two lines X, Y which

1°. Each pass through the ineunt, neither of them lying in the tangent plane.

2°. Each lie in the tangent plane, neither of them passing through the ineunt.

3°. One passes through the ineunt, but does not lie in the tangent plane: the other lies in the tangent plane, but does not pass through the ineunt.

Observe that in the cases 1° and 2° the lines X and Y intersect, but in the case 3° they do not intersect. The lines in the case 3° are I believe what Buchheim has termed β-parallels, his α-parallels being the foregoing right or left parallels*. The subforms arise by omitting in 1°, 2°, or 3°, as the case may be, the negative condition in regard to the two lines or to one of them; as the question is not here further pursued I do not attempt to give names to these several kinds of parallel lines.

Point-, line-, and plane- coordinates: General formulæ. Art. Nos. 17 to 20.

17. We consider point-coordinates (x, y, z, w): line-coordinates (a, b, c, f, g, h), where $af + bg + ch = 0$, and plane-coordinates $(\xi, \eta, \zeta, \omega)$; if we have a line which is at once through two points and in two planes, then the line-coordinates are given by

$$\begin{array}{lcccccccccc} & a & : & b & : & c & : & f & : & g & : & h \\ = & y_1z_2 - y_2z_1 & : & z_1x_2 - z_2x_1 & : & x_1y_2 - x_2y_1 & : & x_1w_2 - x_2w_1 & : & y_1w_2 - y_2w_1 & : & z_1w_2 - z_2w_1 \\ = & \xi_1\omega_2 - \xi_2\omega_1 & : & \eta_1\omega_2 - \eta_2\omega_1 & : & \zeta_1\omega_2 - \zeta_2\omega_1 & : & \eta_1\zeta_2 - \eta_2\zeta_1 & : & \zeta_1\xi_2 - \zeta_2\xi_1 & : & \xi_1\eta_2 - \xi_2\eta_1. \end{array}$$

Similarly if a plane be determined by three points thereof, then the coordinates of the plane are given by

$$\xi : \eta : \zeta : \omega = \begin{vmatrix} 1 & & & \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} : \begin{vmatrix} & 1 & & \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} : \begin{vmatrix} & & 1 & \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} : \begin{vmatrix} & & & 1 \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_8, & w_3 \end{vmatrix};$$

and if a point be given as the intersection of three planes, then the coordinates of the point are

$$x : y : z : w = \begin{vmatrix} 1 & & & \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix} : \begin{vmatrix} & 1 & & \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_1, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix} : \begin{vmatrix} & & 1 & \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix} : \begin{vmatrix} & & & 1 \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix}.$$

18. The conditions in order that a point (x, y, z, w) may be situate on a line (a, b, c, f, g, h) are

$$\begin{array}{rrrrl} & hy & -gz & +aw & = 0, \\ -hx & . & +fy & +bw & = 0, \\ gx & -fy & . & +cw & = 0, \\ -ax & -by & -cz & . & = 0, \end{array}$$

viz. these constitute a twofold relation.

* See Buchheim, A Memoir on Biquaternions. *Amer. Math. Jour.* t. 7 (1885), pp. 293—326.

Similarly the conditions in order that the plane $(\xi, \eta, \zeta, \omega)$ may contain the line (a, b, c, f, g, h) are

$$\begin{array}{rrrr} . & c\eta & -b\zeta & +f\omega = 0, \\ -c\xi & . & +a\zeta & +g\omega = 0, \\ b\xi & -a\eta & . & +h\omega = 0, \\ -f\xi & -g\eta & -h\zeta & . \quad = 0, \end{array}$$

viz. these constitute a twofold relation.

19. The condition in order that two lines (a, b, c, f, g, h), (A, B, C, F, G, H) may meet is

$$Af + Bg + Ch + Fa + Gb + Hc = 0.$$

Supposing that the two lines meet, we have at the point of intersection

$$\begin{array}{rrrrl} . & hy & -gz & +aw & = 0, \\ -hx & . & +fz & +bw & = 0, \\ gx & -fy & . & +cw & = 0, \\ -ax & -by & -cz & . & = 0, \end{array} \qquad \begin{array}{rrrrl} . & Hy & -Gz & +Aw & = 0, \\ -Hx & . & +Fz & +Bw & = 0, \\ Gx & -Fy & . & +Cw & = 0, \\ -Ax & -By & -Cz & . & = 0; \end{array}$$

and from these equations we can find the coordinates x, y, z, w of the point of intersection in a fourfold form, viz. we may write

$$\begin{array}{rccccccc} x : y : z : w = & fA + bG + cH & : & gA - aG & : & hA - aH & : & hG - gH \\ = & fB - bF & : & gB + cH + aF & : & hB - bH & : & fH - hF \\ = & fC - cF & : & gC - cG & : & hC + aF + bG & : & gF - fG \\ = & bC - cC & : & cA - aC & : & aB - bA & : & fA + gB + hC. \end{array}$$

There is no real advantage in any one over any other of these forms, but it is convenient to work with the last of them

$$x : y : z : w = bC - cB : cA - aC : aB - bA : fA + gB + hC.$$

20. In like manner if two lines intersect the plane which contains each of them is given by

$$\begin{array}{rccccccc} \xi : \eta : \zeta : \omega = & aF + gB + hC & : & bF - fB & : & cF - fC & : & cB - bC \\ = & aG - gA & : & bG + hC + fA & : & cG - gC & : & aC - cA \\ = & aH - hA & : & bH - hB & : & cH + fA + gB & : & bA - aB \\ = & gH - hG & : & hF - fH & : & fG - gF & : & aF + bG + cH; \end{array}$$

or say we have

$$\xi : \eta : \zeta : \omega = gH - hG : hF - fH : fG - gF : aF + bG + cH.$$

The Absolute. Nos. 21 to 27.

21. The equation is

in point coordinates $x^2+y^2+z^2+w^2=0$,

in plane coordinates $\xi^2+\eta^2+\zeta^2+\omega^2=0$,

in line coordinates $a^2+b^2+c^2+f^2+g^2+h^2=0$.

Hence $\perp$ of plane $(\xi, \eta, \zeta, \omega)$ is point $(\xi, \eta, \zeta, \omega)$,

$\perp$ of point (x, y, z, w) is plane (x, y, z, w).

Reciprocal of line (a, b, c, f, g, h) is line (f, g, h, a, b, c);

Points (x, y, z, w), (x', y', z', w') are $\perp$ if $xx'+yy'+zz'+ww'=0$;

Planes $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$ are $\perp$ if $\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega'=0$.

22. A line (a, b, c, f, g, h) and plane $(\xi, \eta, \zeta, \omega)$ are $\perp$ when the line passes through the $\perp$ point of the plane, that is the point $(\xi, \eta, \zeta, \omega)$: the conditions (equivalent to two equations) are

$$\begin{array}{rrrrl} . & h\eta & -g\zeta & +a\omega & =0, \\ -h\xi & . & +f\zeta & +b\omega & =0, \\ g\xi & -f\eta & . & +c\omega & =0, \\ -a\xi & -b\eta & -c\zeta & . & =0. \end{array}$$

A line (a, b, c, f, g, h) and point (x, y, z, w) are $\perp$ when the line lies in the $\perp$ plane of the point, that is in the plane (x, y, z, w): the conditions (equivalent to two equations) are

$$\begin{array}{rrrrl} . & cy & -bz & -fw & =0, \\ -cx & . & +az & +gw & =0, \\ bx & -ay & . & +hw & =0, \\ -fx & -gy & -hz & . & =0. \end{array}$$

Two lines (a, b, c, f, g, h), (a', b', c', f', g', h') which meet, that is for which $af'+bg'+ch'+a'f+b'g+c'h=0$, are $\perp$ if

$$aa'+bb'+cc'+ff'+gg'+hh'=0.$$

23. There will be occasion to consider the pair of tangent planes drawn through the line (a, b, c, f, g, h) to the Absolute. Writing for shortness

$$\begin{array}{rrrrrl} P= & . & hy & -gz & +aw, \\ Q= & -hx & . & +fz & +bw, \\ R= & gx & -fy & . & +cw, \\ S= & -ax & -by & -cz & . & , \end{array}$$

it may be shown that the equation of the pair of planes is

$$P^2 + Q^2 + R^2 + S^2 = 0.$$

In fact writing for a moment $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$ to denote the coefficients of (x, y, z, w) in P and Q respectively, so that $(\xi, \eta, \zeta, \omega) = (0, h, -g, a)$, $(\xi', \eta', \zeta', \omega') = (-h, 0, f, b)$, then equation of the planes is

$$(\xi' P - \xi Q)^2 + (\eta' P - \eta Q)^2 + (\zeta' P - \zeta Q)^2 + (\omega' P - \omega Q)^2 = 0,$$

that is $(\xi'^2 + \eta'^2 + \zeta'^2 + \omega'^2) P^2 - 2(\xi\xi' + \eta\eta' + \zeta\zeta' + \omega\omega') PQ + (\xi^2 + \eta^2 + \zeta^2 + \omega^2) Q^2 = 0,$

viz. this equation is

$$(b^2 + h^2 + f^2) P^2 + 2(fg - ab) PQ + (a^2 + g^2 + h^2) Q^2 = 0.$$

But P, Q, R, S are connected by the identical equations

$$\begin{aligned} . \quad cQ - bR + fS &= 0, \\ -cP \quad . \quad + aR + gS &= 0, \\ bP - aQ \quad . \quad + hS &= 0, \\ -fP - gQ - hR \quad . \quad &= 0, \end{aligned}$$

and using these equations to express R, S in terms of P, Q, viz. writing

$$R = -\frac{1}{h}(fP + gQ), \quad S = -\frac{1}{h}(bP - aQ),$$

we see that the last preceding equation is equivalent to $P^2 + Q^2 + R^2 + S^2 = 0$.

24. Similarly if

$$\begin{aligned} P_1 &= \quad . \quad cy - bz + fw, \\ Q_1 &= -cx \quad . \quad + az + gw, \\ R_1 &= bx - ay \quad . \quad + hw, \\ S_1 &= -fx - gy - hz \quad . \quad , \end{aligned}$$

functions which are connected by the identical relations

$$\begin{aligned} hQ_1 - gR_1 + aS_1 &= 0, \\ -hP_1 \quad . \quad + fR_1 + bS_1 &= 0, \\ gP_1 - fQ_1 \quad . \quad + cS_1 &= 0, \\ -aP_1 - bQ_1 - cR_1 \quad . \quad &= 0; \end{aligned}$$

then in like manner we have $P_1^2 + Q_1^2 + R_1^2 + S_1^2 = 0,$

for the equation of the pair of tangent planes from the reciprocal line (f, g, h, a, b, c) to the Absolute. And we may remark the identity

$$(P^2 + Q^2 + R^2 + S^2) + (P_1^2 + Q_1^2 + R_1^2 + S_1^2) = (a^2 + b^2 + c^2 + f^2 + g^2 + h^2)(x^2 + y^2 + z^2 + w^2).$$

We in fact have

$$P^2+Q^2+R^2+S^2=\begin{array}{c|cccc} & x & y & z & w \\ \hline x & a^2+g^2+h^2 & ab-fg & ac-hf & cg-bh \\ y & ab-fg & b^2+h^2+f^2 & bc-gh & ah-cf \\ z & ac-hf & bc-gh & c^2+f^2+g^2 & bf-ag \\ w & cg-bh & ah-cf & bf-ag & a^2+b^2+c^2 \end{array};$$

and in like manner

$$P_1^2+Q_1^2+R_1^2+S_1^2=\begin{array}{c|cccc} & x & y & z & w \\ \hline x & b^2+c^2+f^2 & -ab+fg & -ac+hf & -cg+bh \\ y & -ab+fg & c^2+a^2+g^2 & -bc+gh & -ah+cf \\ z & -ac+hf & -bc+gh & a^2+b^2+h^2 & -bf+ag \\ w & -cg+bh & -ah+cf & -bf+ag & f^2+g^2+h^2 \end{array}.$$

25. For the distance of two points (x, y, z, w) and (x', y', z', w') we have

$$\cos\delta=\frac{xx'+yy'+zz'+ww'}{\sqrt{x^2+y^2+z^2+w^2}\,\sqrt{x'^2+y'^2+z'^2+w'^2}},$$

whence also

$$\sin\delta=\frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{x^2+y^2+z^2+w^2}\,\sqrt{x'^2+y'^2+z'^2+w'^2}},$$

where in the numerator (a, b, c, f, g, h) stand for the coordinates of the line of junction of the two points, taken to be *equal* to $yz'-y'z$, $zx'-z'x$, $xy'-x'y$, $xw'-x'w$, $yw'-y'w$, $zw'-z'w$ respectively.

Similarly for the distance of two planes $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$ we have

$$\cos\delta=\frac{\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega'}{\sqrt{\xi^2+\eta^2+\zeta^2+\omega^2}\,\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

whence also

$$\sin\delta=\frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{\xi^2+\eta^2+\zeta^2+\omega^2}\,\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

where in the numerator (a, b, c, f, g, h) stand for the coordinates of the line of intersection of the two planes, taken to be equal to $\xi\omega'-\xi'\omega$, $\eta\omega'-\eta'\omega$, $\zeta\omega'-\zeta'\omega$, $\eta\zeta'-\eta'\zeta$, $\zeta\xi'-\zeta'\xi$, $\xi\eta'-\xi'\eta$ respectively.

The distance of a point (x, y, z, w) and plane $(\xi', \eta', \zeta', \omega')$ is the complement of the distance of the point (x, y, z, w) and the point $(\xi', \eta', \zeta', \omega')$ which is the $\perp$ point of the plane; viz. we have

$$\sin\delta = \frac{x\xi' + y\eta' + z\zeta' + w\omega'}{\sqrt{x^2+y^2+z^2+w^2}\,\sqrt{\xi'^2+\eta'^2+\xi'^2+\omega'^2}},$$

$$\cos\delta = \frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{x^2+y^2+z^2+w^2}\,\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

where in the numerator (a, b, c, f, g, h) stand for the coordinates of the line of junction of the two points. Of course the same result might have been equally well derived from the formula for the distance of two planes.

26. If we now consider a plane triangle ABC, and write

(x_1, y_1, z_1, w_1) for the coordinates of A,
(x_2, y_2, z_2, w_2) " " B,
(x_3, y_3, z_3, w_3) " " C,

then the coordinates

$$a, \quad b, \quad c, \quad f, \quad g, \quad h$$

of the line BC will be

$$y_2z_3 - y_3z_2, \quad z_3x_2 - z_2x_3, \quad x_2y_3 - x_3y_2, \quad x_2w_3 - x_3w_2, \quad y_2w_3 - y_3w_2, \quad z_2w_3 - z_3w_2,$$

and similarly for the coordinates of the lines BC, CA; the equations

$$a_1f_2 + b_1g_2 + c_1h_2 + a_2f_1 + b_2g_1 + c_2h_1 = 0, \ \&c.,$$

which express that these lines meet in pairs in the points A, B, C respectively are of course satisfied identically; and we then have for the sides and angles (linear and angular distances) of the triangle

$$\cos a = \frac{x_2x_3 + y_2y_3 + z_2z_3 + w_2w_3}{\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}\,\sqrt{x_3^2+y_3^2+z_3^2+w_3^2}},$$

$$\sin a = \frac{\sqrt{a_1^2+b_1^2+c_1^2+f_1^2+g_1^2+h_1^2}}{\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}\,\sqrt{x_3^2+y_3^2+z_3^2+w_3^2}},$$

$$\cos A = \frac{a_2a_3 + b_2b_3 + c_2c_3 + f_2f_3 + g_2g_3 + h_2h_3}{\sqrt{a_2^2+b_2^2+c_2^2+f_2^2+g_2^2+h_2^2}\,\sqrt{a_3^2+b_3^2+c_3^2+f_3^2+g_3^2+h_3^2}}; \quad \&c.$$

and this being so, if with the values of $\cos a$, $\cos b$, $\cos c$, we form the expression for $\cos a - \cos b \cos c$, then reducing to a common denominator, the expression for the numerator is at once found to be

$$= a_2a_3 + b_2b_3 + c_2c_3 + f_2f_3 + g_2g_3 + h_2h_3,$$

and thence easily

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

viz. the expressions for the angles in terms of the sides are those of ordinary spherical trigonometry.

27. Hence also

$$\sin A = \frac{\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}}{\sin b\sin c};$$

whence $$\sin A : \sin B : \sin C = \sin a : \sin b : \sin c,$$

and $$\cos A+\cos B\cos C = \frac{\cos a\,(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)}{\sin^2 a\sin b\sin c},$$

and consequently $$\cos a = \frac{\cos A+\cos B\cos C}{\sin B\sin C},$$

which completes the system of formulæ.

And similarly for a trihedral, that is if we have three planes A, B, C (meeting of course in a point, O) then the dihedral distances BC, CA, AB and the angular distances CA, AB; AB, BC; BC, CA are related to each other in the same way as the angles and sides of an ordinary spherical triangle.

Distance of a point and line. Nos. 28, 29.

28. The point is taken to be (x_1, y_1, z_1, w_1), the line (a, b, c, f, g, h). Drawing through the point a $\perp$ plane, say $(\xi, \eta, \zeta, \omega)$ meeting the line in the foot, and taking the coordinates hereof to be (x_2, y_2, z_2, w_2), then $\xi x_1+\eta y_1+\zeta z_1+w\omega_1=0$ and

$$\begin{array}{llll} \cdot\quad h\eta-g\zeta+a\omega=0, & \text{giving say,} & \xi= & \cdot\quad cy_1-bz_1+fw_1, \\ -h\xi\quad\cdot\quad+f\zeta+b\omega=0, & & \eta= & -cx_1\quad\cdot\quad+az_1+gw_1, \\ g\xi-f\eta\quad\cdot\quad+c\omega=0, & & \zeta= & bx_1-ay_1\quad\cdot\quad+hw_1, \\ -a\xi-b\eta-c\zeta\quad\cdot\quad=0, & & \omega= & -fx_1-gy_1-hz_1\quad\cdot\quad. \end{array}$$

We have here $$\xi^2+\eta^2+\zeta^2+\omega^2=(b^2+c^2+f^2)\,x_1^2+\&\text{c.},$$

where $(b^2+c^2+f^2)\,x_1^2+\&c.$ denotes the before mentioned quadric function of (x_1, y_1, z_1, w_1), which equated to zero, and regarding therein (x_1, y_1, z_1, w_1) as current coordinates represents the pair of tangent-planes from the reciprocal line (f, g, h, a, b, c) to the Absolute.

Resuming the question in hand we have then

$$\xi x_2+\eta y_2+\zeta z_2+\omega w_2=0,$$

which with

$$\begin{array}{llll} \cdot\quad hy_2-gz_2+aw_2=0, & \text{gives say} & -x_2= & \cdot\quad c\eta-b\zeta+f\omega, \\ -hx_2\quad\cdot\quad+fz_2+bw_2=0, & & -y_2= & -c\xi\quad\cdot\quad+a\zeta+g\omega, \\ gx_2-fy_2\quad\cdot\quad+cw_2=0, & & -z_2= & -b\xi-a\eta\quad\cdot\quad+h\omega, \\ -ax_2-by_2-cz_2\quad\cdot\quad=0, & & -w_2= & -f\xi-g\eta-h\zeta\quad\cdot\quad. \end{array}$$

that is
$$\begin{aligned}
x_2 &= (b^2+c^2+f^2)\,x_1 + (-ab+fg)\,y_1 + (-ac+hf)\,z_1 + (-cg+bh)\,w_1,\\
y_2 &= (-ab+fg)\,x_1 + (c^2+a^2+g^2)\,y_1 + (-bc+gh)\,z_1 + (-ah+cf)\,w_1,\\
z_2 &= (-ca+hf)\,x_1 + (-bc+gh)\,y_1 + (a^2+b^2+h^2)\,z_1 + (-bf+ag)\,w_1,\\
w_2 &= (-cg+bh)\,x_1 + (-ah+cf)\,y_1 + (-bf+ag)\,z_1 + (f^2+g^2+h^2)\,w_1.
\end{aligned}$$

We have therefore
$$x_1x_2+y_1y_2+z_1z_2+w_1w_2=(b^2+c^2+f^2)\,x_1^2+\&\text{c.},$$
and
$$x_2^2+y_2^2+z_2^2+w_2^2=(a^2+b^2+c^2+f^2+g^2+h^2)\,\{(b^2+c^2+f^2)\,x_1^2+\&\text{c.}\},$$
where $(b^2+c^2+f^2)\,x_1^2+\&c.$ denotes in each case the above-mentioned quadric function of $(x_1,\ y_1,\ z_1,\ w_1)$.

In verification of the expression for $x_2^2+y_2^2+z_2^2+w_2^2$ it is to be remarked that we have identically
$$\xi^2+\eta^2+\zeta^2+\omega^2+(af+bg+ch)^2\,(x_1^2+y_1^2+z_1^2+w_1^2)$$
$$=(a^2+b^2+c^2+f^2+g^2+h^2)\,\{(b^2+c^2+f^2)\,x_1^2+\&\text{c.}\}\,;$$
here on the left-hand side the whole coefficient of x_1^2 is
$$(b^2+c^2+f^2)^2+(ab-fg)^2+(ca-hf)^2+(cg-bh)^2+(af+bg+ch)^2,$$
where the last four terms are together $=(b^2+c^2+f^2)\,(a^2+g^2+h^2)$, and thus the whole coefficient is (as it should be) $=(b^2+c^2+f^2)\,(a^2+b^2+c^2+f^2+g^2+h^2)$: and similarly for the coefficients of the remaining terms.

29. Writing then δ for the required distance we have
$$\cos\delta=\frac{x_1x_2+y_1y_2+z_1z_2+w_1w_2}{\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}\,\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}},$$
that is
$$\cos\delta=\frac{\sqrt{(b^2+c^2+f^2)\,x_1^2+\&\text{c.}}}{\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}\,\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}},$$
where $(b^2+c^2+f^2)\,x_1^2+\&c.$ is the above-mentioned quadric function

	x_1	y_1	z_1	w_1
x_1	$b^2+c^2+f^2$	$-ab+fg$	$-ac+hf$	$-cg+bh$
y_1	$-ab+fg$	$c^2+a^2+g^2$	$-bc+gh$	$-ah+cf$
z_1	$-ac+hf$	$-bc+gh$	$a^2+b^2+h^2$	$-bf+ag$
w_1	$-cg+bh$	$-ah+cf$	$-bf+ag$	$f^2+g^2+h^2$

Distance of a plane and line. No. 30.

30. This may be deduced from the last preceding result: the formula as written down gives the distance of the $\perp$ plane $(x_1,\ y_1,\ z_1,\ w_1)$ from the reciprocal line $(f,\ g,\ h,\ a,\ b,\ c)$: hence writing $(\xi,\ \eta,\ \zeta,\ \omega)$ for $(x_1,\ y_1,\ z_1,\ w_1)$ and $(a,\ b,\ c,\ f,\ g,\ h)$ for

(f, g, h, a, b, c) we have for the distance of plane $(\xi, \eta, \zeta, \omega)$ and line (a, b, c, f, g, h) the expression

$$\cos\delta = \frac{\sqrt{(a^2+g^2+h^2)\,\xi^2 + \&c.}}{\sqrt{\xi^2+\eta^2+\zeta^2+\omega^2}\,\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}},$$

where $(a^2+g^2+h^2)\,\xi^2+\&c.$ denotes the quadric function

	ξ	η	ζ	ω
ξ	$a^2+g^2+h^2$	$ab-fg$	$ac-hf$	$cg-bh$
η	$ab-fg$	$b^2+h^2+f^2$	$bc-gh$	$ah-cf$
ζ	$ac-hf$	$bc-gh$	$c^2+f^2+g^2$	$bf-ag$
ω	$cg-bh$	$ah-cf$	$bf-ag$	$a^2+b^2+c^2$

The theory of two lines. Nos. 31 to 38.

31. Considering any two lines X, Y it has been seen that these have two $\perp$ s, viz. each $\perp$ is a line cutting as well the two lines X, Y as the reciprocal lines X', Y', say that one of them cuts the lines X, Y in the points A, C respectively, and the other of them cuts the two lines in the points B, D respectively: and take as before the distances AC and BD to be $=\delta$ and θ respectively.

The coordinates of the lines X, Y are

$$(a, b, c, f, g, h) \text{ and } (a_1, b_1, c_1, f_1, g_1, h_1) \text{ respectively;}$$

and if we consider as before the planes Π, Ω, Π_1, Ω_1 the coordinates of which are (l, m, n, p), $(\lambda, \mu, \nu, \varpi)$, (l_1, m_1, n_1, p_1), $(\lambda_1, \mu_1, \nu_1, \varpi_1)$ respectively, then X is the intersection of the planes Π, Ω, and we have

$$\begin{aligned} &a : b : c : f : g : h \\ &= l\varpi-\lambda p : m\varpi-\mu p : n\varpi-\nu p : m\nu-n\mu : n\lambda-l\nu : l\mu-m\lambda, \end{aligned}$$

and similarly Y is the intersection of the planes Π_1, Ω_1 and we have

$$\begin{aligned} &a_1 : b_1 : c_1 : f_1 : g_1 : h_1 \\ &= l_1\varpi_1-\lambda_1 p_1 : m_1\varpi_1-\mu_1 p_1 : n_1\varpi_1-\nu_1 p_1 : m_1\nu_1-n_1\mu_1 : n_1\lambda_1-l_1\nu_1 : l_1\mu_1-m_1\lambda_1. \end{aligned}$$

Also the planes (Π, Ω), (Π_1, Ω_1), (Π, Ω_1), (Π_1, Ω) being mutually $\perp$, we have

$$\begin{aligned} l\lambda + m\mu + n\nu + p\varpi &= 0, \\ l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 + p_1\varpi_1 &= 0, \\ l\lambda_1 + m\mu_1 + n\nu_1 + p\varpi_1 &= 0, \\ l_1\lambda + m_1\mu + n_1\nu + p_1\varpi &= 0; \end{aligned}$$

and for the inclinations to each other of the planes (Π, Π_1) and (Ω, Ω_1)

$$\cos\delta = \frac{\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1}{\sqrt{\lambda^2 + \&c.}\ \sqrt{\lambda_1^2 + \&c.}},$$

$$\cos\theta = \frac{ll_1 + mm_1 + nn_1 + pp_1}{\sqrt{l^2 + \&c.}\ \sqrt{l_1^2 + \&c.}}.$$

32. The expressions for the coordinates of the two lines give

$$\begin{aligned} aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1 = {} & (ll_1 + mm_1 + nn_1 + pp_1)(\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1) \\ & - (l\lambda_1 + m\mu_1 + n\nu_1 + p\varpi_1)(l_1\lambda + m_1\mu + n_1\nu + p_1\varpi) \\ = {} & (ll_1 + mm_1 + nn_1 + pp_1)(\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1) \\ = {} & \sqrt{l^2 + \&c.}\ \sqrt{l_1^2 + \&c.}\ \sqrt{\lambda^2 + \&c.}\ \sqrt{\lambda_1^2 + \&c.}\ \cos\delta\cos\theta. \end{aligned}$$

But we have

$$\begin{aligned} a^2 + b^2 + c^2 + f^2 + g^2 + h^2 &= (l^2 + m^2 + n^2 + p^2)(\lambda^2 + \mu^2 + \nu^2 + \varpi^2) - (l\lambda + m\mu + n\nu + p\varpi)^2 \\ &= (l^2 + \&c.)(\lambda^2 + \&c.); \end{aligned}$$

and similarly

$$\begin{aligned} a_1^2 + b_1^2 + c_1^2 + f_1^2 + g_1^2 + h_1^2 &= (l_1^2 + m_1^2 + n_1^2 + p_1^2)(\lambda_1^2 + \mu_1^2 + \nu_1^2 + \varpi_1^2) - (l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 + p_1\varpi_1)^2 \\ &= (l_1^2 + \&c.)(\lambda_1^2 + \&c.). \end{aligned}$$

Hence the last result gives

$$\frac{aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1}{\sqrt{a^2 + \&c.}\ \sqrt{a_1^2 + \&c.}} = \cos\delta\cos\theta;$$

or calling the expression on the left-hand side the comoment of the two lines, and denoting it by M_1, the equation just obtained is

$$\cos\delta\cos\theta = \text{comoment}, = M_1.$$

And if for either of the lines we substitute its reciprocal, then for δ, θ we have $\frac{1}{2}\pi - \delta$, $\frac{1}{2}\pi - \theta$ respectively, and consequently

$$\frac{af_1 + bg_1 + ch_1 + a_1f + b_1g + c_1h}{\sqrt{a^2 + \&c.}\ \sqrt{a_1^2 + \&c.}} = \sin\delta\sin\theta;$$

or calling the expression on the left-hand side the moment of the two lines and denoting it by M, the equation is

$$\sin\delta\sin\theta = \text{moment}, = M,$$

where observe that $M = 0$ is the condition for the intersection of the two lines, $M_1 = 0$ the condition for their contrasection*.

* The foregoing demonstration of the fundamental formulæ $\cos\delta\cos\theta = M_1$, $\sin\delta\sin\theta = M$, is in effect that given by Heath in his Memoir "On the Dynamics of a Rigid Body in Elliptic Space,' *Phil. Trans.* t. 175 (for 1884), pp. 281—324.

33. But to determine the coordinates (A, B, C, F, G, H) of the $\perp$ line AC or BD, and the coordinates of the points A and C or B and D of the points in which it meets the lines X and Y respectively, I employ a different method.

We consider the lines (a, b, c, f, g, h), $(a_1, b_1, c_1, f_1, g_1, h_1)$,
and their reciprocals (f, g, h, a, b, c), $(f_1, g_1, h_1, a_1, b_1, c_1)$.

A line (A, B, C, F, G, H) meeting each of these four lines is said to be a perpendicular. We have

$$\begin{aligned}(A, B, C, F, G, H)&(a,\ b,\ c,\ f,\ g,\ h) = 0,\\ \text{,,}\quad &(f,\ g,\ h,\ a,\ b,\ c) = 0,\\ \text{,,}\quad &(a_1, b_1, c_1, f_1, g_1, h_1) = 0,\\ \text{,,}\quad &(f_1, g_1, h_1, a_1, b_1, c_1) = 0,\end{aligned}$$

equations which determine say A, B, C, F in terms of G, H, and then substituting in $AF+BG+CH=0$ we have two values of $G : H$; i.e. there are two systems of values (A, B, C, F, G, H), that is two perpendiculars.

The equations may be written

$$\begin{aligned}(A+F)(a+f)+(B+G)(b+g)+(C+H)(c+h)&=0,\\ (A+F)(a_1+f_1)+(B+G)(b_1+g_1)+(C+H)(c_1+h_1)&=0,\\ (A-F)(a-f)+(B-G)(b-g)+(C-H)(c-h)&=0,\\ (A-F)(a_1-f_1)+(B-G)(b_1-g_1)+(C-H)(c_1-h_1)&=0.\end{aligned}$$

Hence we have

$$\begin{gathered}A+F \quad : \quad B+G \quad : \quad C+H, =\\ (b+g)(c_1+h_1)-(b_1+g_1)(c+h) : (c+h)(a_1+f_1)-(a+f)(c_1+h_1) : (a+f)(b_1+g_1)-(a_1+f_1)(b+g), =\\ \mathfrak{A}+\alpha \quad : \quad \mathfrak{B}+\beta \quad : \quad \mathfrak{C}+\gamma;\\ A-F \quad : \quad B-G \quad : \quad C-H, =\\ (b-g)(c_1-h_1)-(b_1-g_1)(c-h) : (c-h)(a_1-f_1)-(a-f)(c_1-h_1) : (a-f)(b_1-g_1)-(a_1-f_1)(b-g), =\\ \mathfrak{A}-\alpha \quad : \quad \mathfrak{B}-\beta \quad : \quad \mathfrak{C}-\gamma;\end{gathered}$$

equations which may be written

$$\begin{aligned}A+F,\ B+G,\ C+H &= 2\lambda(\mathfrak{A}+\alpha,\ \mathfrak{B}+\beta,\ \mathfrak{C}+\gamma),\\ A-F,\ B-G,\ C-H &= 2\mu(\mathfrak{A}-\alpha,\ \mathfrak{B}-\beta,\ \mathfrak{C}-\gamma),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{A}&=bc_1-b_1c+gh_1-g_1h, & \alpha&=bh_1-b_1h-(cg_1-c_1g),\\ \mathfrak{B}&=ca_1-c_1a+hf_1-h_1f, & \beta&=cf_1-c_1f-(ah_1-a_1h),\\ \mathfrak{C}&=ab_1-a_1b+fg_1-f_1g, & \gamma&=ag_1-a_1g-(bf_1-b_1f).\end{aligned}$$

34. We have

$$\begin{aligned}(\mathfrak{A}+\alpha)^2+(\mathfrak{B}+\beta)^2+(\mathfrak{C}+\gamma)^2 &= \{(a+f)^2+(b+g)^2+(c+h)^2\}\{(a_1+f_1)^2+(b_1+g_1)^2+(c_1+h_1)^2\}\\ &\quad -\{(a+f)(a_1+f_1)+(b+g)(b_1+g_1)+(c+h)(c_1+h_1)\}^2,\\ (\mathfrak{A}-\alpha)^2+(\mathfrak{B}-\beta)^2+(\mathfrak{C}-\gamma)^2 &= \{(a-f)^2+(b-g)^2+(c-h)^2\}\{(a_1-f_1)^2+(b_1-g_1)^2+(c_1-h_1)^2\}\\ &\quad -\{(a-f)(a_1-f_1)+(b-g)(b_1-g_1)+(c-h)(c_1-h_1)\}^2;\end{aligned}$$

or putting

$$\begin{aligned}\rho^2 &= a^2 + b^2 + c^2 + f^2 + g^2 + h^2,\\ \rho_1^{\,2} &= a_1^{\,2} + b_1^{\,2} + c_1^{\,2} + f_1^{\,2} + g_1^{\,2} + h_1^{\,2},\\ \sigma_1 &= aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1,\\ \sigma &= af_1 + bg_1 + ch_1 + a_1 f + b_1 g + c_1 h,\end{aligned}$$

the foregoing values are $= \rho^2\rho_1^{\,2} - (\sigma + \sigma_1)^2, \quad \rho^2\rho_1^{\,2} - (\sigma - \sigma_1)^2.$

Hence

$$A^2 + B^2 + C^2 + F^2 + G^2 + H^2 = 4\lambda^2\{\rho^2\rho_1^{\,2} - (\sigma + \sigma_1)^2\} = 4\mu^2\{\rho^2\rho_1^{\,2} - (\sigma - \sigma_1)^2\};$$

or we may write $\lambda^2 = \rho^2\rho_1^{\,2} - (\sigma - \sigma_1)^2$, or say $\lambda = \sqrt{\rho^2\rho_1^{\,2} - (\sigma - \sigma_1)^2}$,

$$\mu^2 = \rho^2\rho_1^{\,2} - (\sigma + \sigma_1)^2, \qquad \mu = -\sqrt{\rho^2\rho_1^{\,2} - (\sigma + \sigma_1)^2}.$$

Making a slight change of notation, if we put

$$M = \frac{af_1 + bg_1 + ch_1 + a_1 f + b_1 g + c_1 h}{\sqrt{a^2 + \&c.}\sqrt{a_1^{\,2} + \&c.}} = \frac{\sigma}{\rho\rho_1},$$

$$M_1 = \frac{aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1}{\sqrt{a^2 + \&c.}\sqrt{a_1^{\,2} + \&c.}} = \frac{\sigma_1}{\rho\rho_1},$$

then the values are

$$\lambda = rr_1\sqrt{1 - (M - M_1)^2}, \qquad \mu = -rr_1\sqrt{1 + (M + M_1)^2}.$$

And, this being so, the two systems of values of A, B, C, F, G, H, are

$$\begin{array}{l|l}\lambda(\mathfrak{A} + \alpha) + \mu(\mathfrak{A} - \alpha), & \lambda(\mathfrak{A} + \alpha) - \mu(\mathfrak{A} - \alpha),\\ \lambda(\mathfrak{B} + \beta) + \mu(\mathfrak{B} - \beta), & \lambda(\mathfrak{B} + \beta) - \mu(\mathfrak{B} - \beta),\\ \lambda(\mathfrak{C} + \gamma) + \mu(\mathfrak{C} - \gamma), & \lambda(\mathfrak{C} + \gamma) - \mu(\mathfrak{C} - \gamma),\\ \lambda(\mathfrak{A} + \alpha) - \mu(\mathfrak{A} - \alpha), & \lambda(\mathfrak{A} + \alpha) + \mu(\mathfrak{A} - \alpha),\\ \lambda(\mathfrak{B} + \beta) - \mu(\mathfrak{B} - \beta), & \lambda(\mathfrak{B} + \beta) + \mu(\mathfrak{B} - \beta),\\ \lambda(\mathfrak{C} + \gamma) - \mu(\mathfrak{C} - \gamma), & \lambda(\mathfrak{C} + \gamma) + \mu(\mathfrak{C} - \gamma);\end{array}$$

viz. the two perpendiculars are reciprocals each of the other.

35. Before going further I notice that if

$$\frac{a_1 + f_1}{a + f} = \frac{b_1 + g_1}{b + g} = \frac{c_1 + h_1}{c + h} \quad\text{or}\quad \frac{a_1 - f_1}{a - f} = \frac{b_1 - g_1}{b - g} = \frac{c_1 - h_1}{c - h},$$

then the four equations for (A, B, C, F, G, H) reduce themselves to three equations only: and thus instead of two perpendiculars we have a singly infinite series of perpendiculars, (see *ante* 15).

To explain the meaning of the equations, I observe that a line (a, b, c, f, g, h) will be a generating line of the one kind or say a "generatrix" of the Absolute if

$$a + f = 0, \quad b + g = 0, \quad c + h = 0:$$

and it will be a generating line of the other kind or say a "directrix" of the Absolute if $a-f=0,\ b-g=0,\ c-h=0$. Or what is the same thing, we have

$$(\text{a, b, c}, -\text{a}, -\text{b}, -\text{c}) \text{ where } \text{a}^2+\text{b}^2+\text{c}^2=0 \text{ for a generatrix,}$$

$$\text{and } (\text{a, b, c, a, b, c}) \text{ where } \text{a}^2+\text{b}^2+\text{c}^2=0 \text{ for a directrix of the Absolute.}$$

Consider now two directrices (a, b, c, a, b, c) and (a_1, b_1, c_1, a_1, b_1, c_1): if a line (a, b, c, f, g, h) meets each of these, then

$$(a+f)\,\text{a} + (b+g)\,\text{b} + (c+h)\,\text{c} = 0,$$

$$(a+f)\,\text{a}_1 + (b+g)\,\text{b}_1 + (c+h)\,\text{c}_1 = 0,$$

and consequently

$$a+f : b+g : c+h = \text{bc}_1-\text{b}_1\text{c} : \text{ca}_1-\text{c}_1\text{a} : \text{ab}_1-\text{a}_1\text{b},$$

and similarly if $(a_1, b_1, c_1, f_1, g_1, h_1)$ meets each of the two directrices then

$$a_1+f_1 : b_1+g_1 : c_1+h_1 = \text{bc}_1-\text{b}_1\text{c} : \text{ca}_1-\text{c}_1\text{a} : \text{ab}_1-\text{a}_1\text{b},$$

that is if the lines each of them meet the same two directrices of the Absolute, then

$$\frac{a_1+f_1}{a+f} = \frac{b_1+g_1}{b+g} = \frac{c_1+h_1}{c+h},$$

and conversely if these relations are satisfied then the lines each of them meet two directrices of the Absolute.

In like manner if the lines each meet two generatrices of the Absolute, then

$$\frac{a_1-f_1}{a-f} = \frac{b_1-g_1}{b-g} = \frac{c_1-h_1}{c-h},$$

and conversely if these relations are satisfied then the lines each of them meet the same two generatrices of the Absolute. In the former case the lines are said to be "right parallels" and in the latter case "left parallels."

A line (a, b, c, f, g, h) meets the Absolute in two points, and through each of these we have a directrix and a generatrix: that is, the line meets two directrices and two generatrices.

Through a given point we may draw, meeting the two directrices, or meeting the two generatrices, a line: that is, through a given point we may draw a line

$$(a_1,\ b_1,\ c_1,\ f_1,\ g_1,\ h_1)$$

which is a right parallel, and a line which is a left parallel to a given line. That is regarding as given the first line, and also a point of the second line, there are two positions of the second line such that for each of them, the ⊥'s of the pair of lines, instead of being two determinate lines, are a singly infinite series of lines.

36. Reverting to the general case we have found (A, B, C, F, G, H) the coordinates of either of the lines ⊥ to the given lines (a, b, c, f, g, h) and $(a_1, b_1, c_1, f_1, g_1, h_1)$: supposing that the ⊥ intersects the first of these lines in the point the coordinates of which are (x, y, z, w) and the second in the point the coordinates of which are

$$(x_1,\ y_1,\ z_1,\ w_1),$$

then we have for each set of coordinates a fourfold expression; the choice of the form is indifferent, and I write

$$x : y : z : w = cB - bC : aC - cA : bA - aB : fA + gB + hC,$$
$$x_1 : y_1 : z_1 : w_1 = c_1B - b_1C : a_1C - c_1A : b_1A - a_1B : f_1A + g_1B + h_1C,$$

and we have then for the distance of these two points,

$$\cos\phi = \frac{xx_1 + yy_1 + zz_1 + ww_1}{\sqrt{x^2 + y^2 + z^2 + w^2}\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}}, \quad \sin\phi = \frac{\sqrt{(yz_1 - y_1z)^2 + \&c.}}{\sqrt{x^2 + y^2 + z^2 + w^2}\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}};$$

where $\phi = \delta$ or θ, according to the sign of the radical $\lambda : \mu$ contained in the expressions for A, B, C, F, G, H.

I have not succeeded in obtaining in this manner the final formulæ for the determination of the distances: these in fact are, by what precedes, given by the equations

$$\sin\delta\sin\theta = M, \quad \cos\delta\cos\theta = M_1.$$

For then, writing ϕ to denote either of the distances δ, θ, at pleasure, we have

$$\frac{M^2}{\sin^2\phi} + \frac{M_1^2}{\cos^2\phi} = 1,$$

that is
$$\cos^4\phi + \cos^2\phi\,(M_1^2 - M^2 + 1) + M_1^2 = 0,$$
or
$$\cos^2\phi = \tfrac{1}{2}\{M_1^2 - M^2 + 1 \pm \sqrt{M_1^4 + M^4 - 2M_1^2M^2 - 2M_1^2 - 2M^2 + 1}\},$$

which is the expression for the cosine of the distance.

In the case where the two lines intersect $M = 0$, and if δ be the $\perp$ distance which vanishes, then $\delta = 0$, and consequently $\cos\theta = M_1$: the last-mentioned formula, putting therein $M = 0$ and taking the radical to be $= M_1^2 - 1$, gives $\cos^2\phi = M_1^2$, that is $\phi = \theta$, and $\cos^2\theta = M_1^2$ as it should be.

37. I verify as follows, in the case in question of two *intersecting* lines,

$$(af_1 + bg_1 + ch_1 + a_1f + b_1g + c_1h = 0),$$

the formula
$$\cos\theta = \frac{xx_1 + yy_1 + zz_1 + ww_1}{\sqrt{x^2 + y^2 + z^2 + w^2}\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}}.$$

We have here

$$\begin{aligned}
A &= \mathfrak{A} = bc_1 - b_1c + gh_1 - g_1h,\\
B &= \mathfrak{B} = ca_1 - c_1a + hf_1 - h_1f,\\
C &= \mathfrak{C} = ab_1 - a_1b + fg_1 - f_1g,\\
F &= \alpha = bh_1 - b_1h - cg_1 + c_1g,\\
G &= \beta = cf_1 - c_1f - ah_1 + a_1h,\\
H &= \gamma = ag_1 - a_1g - bf_1 + b_1h.
\end{aligned}$$

I stop to notice that these formulæ may be obtained in a different and somewhat more simple manner: the two lines (a, b, c, f, g, h) and $(a_1, b_1, c_1, f_1, g_1, h_1)$ intersect; hence their reciprocals also intersect: the equations of the plane through the two lines and that of the plane through the two reciprocal lines are respectively

$$(gh_1 - g_1h)\,x + (hf_1 - h_1f)\,y + (fg_1 - gf_1)\,z + (af_1 + bg_1 + ch_1)\,w = 0,$$
$$(bc_1 - b_1c)\,x + (ca_1 - c_1a)\,y + (ab_1 - a_1b)\,z + (fa_1 + gb_1 + hc_1)\,w = 0,$$

the line (A, B, C, F, G, H) is thus the line of intersection of these two planes, and it is thence easy to obtain the foregoing values.

From the values of A, B, C, F, G, H we have to find x, y, z, w and x_1, y_1, z_1, w_1 by the formulæ given above. We have

$$\begin{aligned} x = cB - bC = &\ c^2a_1 - cc_1a + chf_1 - ch_1f \\ & - abb_1 + a_1b^2 - bfg_1 + bgf_1 \\ = &\ (bg + ch)f_1 + a_1(b^2 + c^2) - b_1ab - c_1ac - bfg_1 - cfh_1 \\ = & -f(af_1 + bg_1 + ch_1) + a_1(b^2 + c^2) - b_1ab - c_1ac ; \end{aligned}$$

or writing here $a_1f + b_1g + c_1h$ in place of $-(af_1 + bg_1 + ch_1)$ this is a linear function of a_1, b_1, c_1, and similarly finding the values of y, z, w we have

$$\begin{aligned} x &= a_1(b^2+c^2+f^2) + b_1(fg - ab) + c_1(hf - ca), \\ y &= a_1(fg - ab) + b_1(c^2+a^2+g^2) + c_1(gh - bc), \\ z &= a_1(hf - ca) + b_1(gh - bc) + c_1(a^2 + b^2 + h^2), \\ w &= a_1(bh - cg) + b_1(cf - ah) + c_1(ag - bf). \end{aligned}$$

And in like manner (I introduce for convenience the sign $-$, as is allowable)

$$\begin{aligned} -x_1 &= a(b_1^2 + c_1^2 + f_1^2) + b(f_1g_1 - a_1b_1) + c(h_1f_1 - c_1a_1), \\ -y_1 &= a(f_1g_1 - a_1b_1) + b(c_1^2 + a_1^2 + g_1^2) + c(g_1h_1 - b_1c_1), \\ -z_1 &= a(h_1f_1 - c_1a_1) + b(g_1h_1 - b_1c_1) + c(a_1^2 + b_1^2 + h_1^2), \\ -w_1 &= a(b_1h_1 - c_1g_1) + b(c_1f_1 - a_1h_1) + c(a_1g_1 - b_1f_1). \end{aligned}$$

38. Write for shortness

$$\begin{array}{lll} p = a^2 + b^2 + c^2, & p_1 = f^2 + g^2 + h^2, & -\omega = a_1f + b_1g + c_1h, \text{ and therefore} \\ q = aa_1 + bb_1 + cc_1, & q_1 = ff_1 + gg_1 + hh_1, & -\omega = af_1 + bg_1 + ch_1. \\ r = a_1^2 + b_1^2 + c_1^2, & r_1 = f_1^2 + g_1^2 + h_1^2, & \end{array}$$

We have

$$\begin{array}{ll} x = a_1p - aq + f\omega, & x_1 = -ar + a_1q + f_1\omega, \\ y = b_1p - bq + g\omega, & y_1 = -br + b_1q + g_1\omega, \\ z = c_1p - cq + h\omega, & z_1 = -cr + c_1q + h_1\omega, \end{array}$$

$$w = -\begin{vmatrix} f, & g, & h \\ a, & b, & c \\ a_1, & b_1, & c_1 \end{vmatrix} \qquad w_1 = -\begin{vmatrix} f_1, & g_1, & h_1 \\ a, & b, & c \\ a_1, & b_1, & c_1 \end{vmatrix};$$

from which we easily obtain

$$x^2 + y^2 + z^2 = p(pr - q^2) + (p_1 + 2p)\omega^2,$$

and by expressing w^2 in the form of a determinant

$$w^2 = p_1(pr - q^2) - p\omega^2,$$

we obtain

$$x^2+y^2+z^2+w^2=(p+p_1)(pr-q^2+\omega^2),$$

and in like manner

$$x_1^{\,2}+y_1^{\,2}+z_1^{\,2}+w_1^{\,2}=(r+r_1)(pr-q^2+\omega^2).$$

And again

$$xx_1+yy_1+zz_1=q(pr-q^2)+(q_1+2q)\,\omega^2,$$

and by expressing ww_1 in the form of a determinant

$$ww_1=q_1(pr-q^2)-qw^2,$$

we find

$$xx_1+yy_1+zz_1+ww_1=(q+q_1)(pr-q^2+\omega^2).$$

Hence substituting in

$$\cos\theta=\frac{xx_1+yy_1+zz_1+ww_1}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^{\,2}+y_1^{\,2}+z_1^{\,2}+w_1^{\,2}}},$$

the factor $pr-q^2+\omega^2$ disappears, and we have

$$\cos\theta=\frac{q+q_1}{\sqrt{p+p_1}\sqrt{r+r_1}}=M_1,$$

the required result.

III. *On the full system of concomitants of three ternary quadrics.* By H. F. Baker, B.A., Fellow of S. John's College, Cambridge*.

§ I. *Summary.*

This Essay was undertaken to find the concomitants of three ternary quadrics. As such the net result is given in § III. For completeness I have given also a consecutive account of the present theory, § VII. It is possible that some of the concomitants given are themselves reducible, for some reductions effected have not been arrived at at all easily. With a view to rendering the process of obtaining them readable, I have studied extreme brevity—and it would seem quite practicable to apply the same abbreviated method to four conics. § II. is an explanation of the method; § IV. its application; § V. investigates a quasi-reducibility of 18 types of forms, reducibility on multiplication by u_x; § VI. gives some necessary identities; § VII. contains a connected account of the theory as given by Gundelfinger, Rosanes, and in Clebsch's lectures; and finally § VIII. gives some notes on the geometry of the forms—though apparently any competent expression thereof requires the establishment of new geometrical ideas. § IX. gives a list of memoirs on three conics.

§ II. *Explanation of the method.*

The method here followed for obtaining the system of concomitants of a system of ternary forms in terms of which all others can be expressed as rational integral algebraic functions is based on the remark, due to Gordan or Clebsch (Ueber ternäre Formen dritten Grades, *Math. Annal.* I. 90; Ueber biternäre Formen mit contragredienten Variabeln, *Math. Annal.* I. 359), that if, in the symbolic expression of any concomitant, containing one point variable x and one line variable u, wherein any letter a (symbol of a form a_x^n) which occurs, can occur only in the combinations a_x, (bca), (bau), we omit the power of a_x (which occurs, say, f times), change (bau), (cau)... into b_x, c_x... (say g such) and (bca), $(b'c'a)$... into (bcu), $(b'c'u)$... (say h such), we thereby both eliminate the letter a and also obtain a new invariantive combination; namely, we deduce a concomitant of one degree less than the original (and which in fact has its order lessened by $f-g$ and its class by $g-h$).

As then every concomitant of any degree r can be thus treated, it follows conversely that if we take every possible concomitant of the $(r-1)^{\text{th}}$ degree, then in any one such choose among the components of the type b_x, c_x... (any one of which may be repeated) a certain number g ($\ngtr n$) and replace them by (bau), (cau)... respectively, a being a symbol

* I am indebted to the great kindness of Professor Cayley for several suggestions tending to help the intelligibility of this essay.

(of the form a_x^n) which does not occur in the concomitant of the $(r-1)^{\text{th}}$ degree, and at the same time replace some (say h, $g+h \not> n$) of the type (bcu) by (bca), and then multiply the result by $a_x^{n-g-h} = a_x^f$ (thus obtaining a concomitant of the r^{th} degree) and do this for every value of f, g and h, and for every possible selection of the components acted on, and for the symbols a of every form of which the obtained combination is to be a concomitant, we shall obtain finally every possible concomitant of the r^{th} degree.

And under the title of "every possible concomitant of the $(r-1)^{\text{th}}$ degree," must of course be included all forms capable of arising by the process of the first paragraph from forms of the r^{th} degree, and therefore all products of the $(r-1)^{\text{th}}$ degree obtained by multiplying forms of lower degree. If however a form K of the $(r-1)^{\text{th}}$ degree can be written as the sum of products and powers of forms of lower degree, and of products of forms of the $(r-1)^{\text{th}}$ degree with powers of the identical covariant u_x (namely is, as we say, a rational integral algebraic function of other forms), then, as each constituent of the sum must necessarily be also of the $(r-1)^{\text{th}}$ degree, the process applied can only result in giving, from K, forms which are themselves sums of other forms of the r^{th} degree (some of these being, possibly, products). Thus, if in our enumeration of forms of the $(r-1)^{\text{th}}$ degree, we include simple products, we can exclude forms which are rational integral algebraic functions of other included forms, and we shall obtain a series of forms of the r^{th} degree, in terms of sums of multiples (by numbers or powers of u_x) of which, all forms of the r^{th} degree are expressible and which are therefore by the same reasoning competent to give the similar system of the $(r+1)^{\text{th}}$ degree. It is this sufficient system for the algebraic rational integral expression of all other concomitants which it is our aim to obtain for every degree.

Thus far with the general theory. For the case of three ternary quadrics, a_x^2, b_x^2, c_x^2, the method is considerably simplified. Here the derivatives from any concomitant of the $(r-1)^{\text{th}}$ degree are obtained by only five distinct operations. (1) (The x operation.) Leaving u untouched and replacing one x by the point $(vau)\,a_x = 0$ or $(vbu)\,b_x = 0$ or $(vcu)\,c_x = 0$ [i.e. replacing x_i by $(au)_i\,a_x = (a_j u_k - a_k u_j)\,a_x$, etc.]. (2) (The u operation.) Leaving x untouched and replacing one u by the line $a_y a_x = 0$ or $b_y b_x = 0$ or $c_y c_x = 0$ [i.e. putting for u_i, $a_i a_x$ or $b_i b_x$ or $c_i c_x$]. (3) (The xx operation.) Leaving u untouched and replacing two x's, that is, writing for $m_x n_x$, $(mau)\,(nau)$ or $(mbu)\,(nbu)$ or $(mcu)\,(ncu)$. (4) (The uu operation.) Leaving x untouched and replacing two u's, that is, putting for $u_p u_q$, $a_p a_q$ or $b_p b_q$ or $c_p c_q$. (5) (The xu operation.) Replacing one x and one u, that is, writing for $m_x n_p$, $(mau)\,a_p$ or $(mbu)\,b_p$ or $(mcu)\,c_p$; and upon any form each of these five operations, in their threefold method, must be applied in all possible ways. And *it is not necessary to consider products of the* $(r-1)^{\text{th}}$ *degree* in order to obtain all the requisite forms of the r^{th} degree. For first to clear the ideas it may be remarked that, since the number of places in which a letter a can be introduced, by changing either u_p into a_p or m_x into (mau), cannot be greater than two (for the second degrees of a are real coefficients), there is no utility in considering a product of more than two factors, for one, at least, of these factors will remain unchanged and be a factor in the result. Further there is no utility in either of the two first of the 'five distinct operations,' as applied to products, for either of these will only modify one of the factors of the product and not really bind the two together.

And finally any form obtainable from the product by any of the three remaining operations of the five *can and will arise among the derivatives of each of the factors* alone. This is best explained by example—the root of the matter lies in the remark that a simple invariantive product (of symbolical factors) involving a quantity r once, can be obtained by continued application of the two processes of changing u_i into a_i and x_i into $(au)_i$, (each time multiplying by a_x if requisite), *from the single term* r_x. So that the application of any one of the 'three distinct operations' spoken of to a product of two forms, which must, to bind them together, introduce a single letter, say a, into each, gives a result obtainable by taking one of them, introducing one a and multiplying by a_x and then operating continuously on this a_x, until the part of the result due to the other form is obtained. For example take the product $a_x^2 . b_x^2$ giving rise to $a_x b_x (acu)(bcu)$ and note that we can proceed thus: a_x^2, $a_x(acu)c_x$, $a_x(acu)(cbu)b_x$; or take $(abc)a_x b_x c_x . (b'c'u)(c'a'u)(a'b'u)$, giving rise to $(abc)(adu)b_x c_x (b'c'd)(c'a'u)(a'b'u)$, (where $d = a$ or b or c), and we can proceed thus: $(abc)a_x b_x c_x$, $(abc)(adu)b_x c_x d_x$, $(abc)(adu)b_x c_x (db'u)b_x'$, $(abc)(adu)b_x c_x (db'c')c_x' b_x'$, $(abc)(adu)b_x c_x (db'c')(c'a'u)(b'a'u)$, making the form arise from $(abc)a_x b_x c_x$: and it also arises from $(b'c'u)(c'a'u)(a'b'u)$. And this reasoning remains valid in case particular combinations of the letters are abbreviated by the use of other letters. To see this we may suppose the original letters explicitly reintroduced, in which case the form will generally be replaced by a sum of forms and a product of two forms replaced by a sum of products. But, for example, $(A+B+C)(D+E+F)$ gives for its derivative the sum of the derivatives of $(A+B+C)D$, $(A+B+C)E$, $(A+B+C)F$, which latter derivatives are proved to be also derivatives of $(A+B+C)$, as is also, therefore, effectively, the derivative of $(A+B+C)(D+E+F)$.

Passing now to the mode adopted of conducting the method thus justified—the three conics are written $a_x^2 = a_x'^2 = a_x''^2 = \ldots$, $b_x^2 = b_x'^2 = b_x''^2 = \ldots$, $c_x^2 = \ldots$, and their 'clusters' of tangents, namely $(aa'u)^2$, $(bb'u)^2$, $(cc'u)^2$ are written $u_\alpha^2 = u_{\alpha'}^2 = u_{\alpha''}^2 \ldots$, $u_\beta^2 = \ldots$, and $u_\gamma^2 = \ldots$; or say, we write $(aa')_i = a_j a_k' - a_k a_j' = \alpha_i$, etc. Then it is to be noticed that the factor a_α in a form involves always the real factor a_α^2—for $a_\alpha a_x u_\alpha = \frac{1}{3} a_\alpha^2 . u_x$; also a factor $(aa'u)$ [unless the form contain also $(aa'a'')$ in which case it would be written immediately $M u_\alpha a_\alpha''$ and not need the reductions in question] involves always the (real) second degrees of a_1, a_2, a_3. For $(aa'u) f(a) = -(aa'u) f(a') = \frac{1}{2}(aa'u)\{f(a) - f(a')\}$, and, in $f(a) - f(a')$, a, a' only occur in the combinations $(aa')_i$ and a, a' in the whole expression can be replaced by α_1, α_2, α_3, occurring to the second degree. So a factor $(\alpha\alpha' x)$ in an expression (wherein $(\alpha\alpha'\alpha'')$ is, possibly, not another factor) shews that the expression is reducible to a form containing α, α' only in the second-degree—combinations of the three $(\alpha\alpha')_1$, $(\alpha\alpha')_2$, $(\alpha\alpha')_3$. And these are reducible, for $(\alpha\alpha' x)^2 = \frac{4}{3} a_\alpha^2 . a_x^2$ and therefore $(\alpha\alpha' x)(\alpha\alpha' y) = \frac{4}{3} a_\alpha^2 . a_x a_y$. In fact $u_\alpha^2 = (aa'u)^2$, whence $(x\alpha'\alpha)^2 = (aa' . x\alpha')^2$ [where, *as always*, $(ab . xy)$ is used for

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = a_x b_y - a_y b_x = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ (xy)_1 & (xy)_2 & (xy)_3 \end{vmatrix} = \begin{vmatrix} (ab)_1 & (ab)_2 & (ab)_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}],$$

or $$(x\alpha'\alpha)^2 = (a_x a'_{\alpha'} - a_{\alpha'} a'_x)^2 = 2a_x^2 . a_{\alpha'}^2 - 2a_x a_{\alpha'} a'_x a'_{\alpha'} = 2a_x^2 a_\alpha^2 - \tfrac{2}{3} a_x^2 a_\alpha^2 = \tfrac{4}{3} a_\alpha^2 . a_x^2,$$

(and in case the expression does contain $(\alpha\alpha' x)(\alpha\alpha'\alpha'')$, this is $\frac{4}{3}a_\alpha^2 . a_x a_{\alpha''}$). So that in our investigations where we are seeking to retain only terms which do not contain real factors of lower degree, we can always omit terms containing a factor $(\alpha\alpha' x)$, since this involves the real factor a_α^2. Such terms in $(\alpha\alpha' x)$ are often, here, shortly written $\overline{\alpha\alpha'}$; and in fact in any expression containing α and α' we may interchange any α with any α', the result being only the neglect of reducible terms. For $Mu_\alpha v_{\alpha'} = M[u_{\alpha'} v_\alpha + (uv . \alpha\alpha')] = Mu_{\alpha'} v_\alpha + \overline{\alpha\alpha'}$, which is generally written here $Mu_\alpha v_{\alpha'} \equiv Mu_{\alpha'} v_{\alpha'}$, the sign $\equiv$ meaning, generally, "equal to except for terms containing real factors of lower degree," and always "may, in our tables, be replaced by." A particular case is when a form reduces entirely to products of forms of lower degree: this I write $\equiv 0$. A further aid to brevity consists in only writing down, when there are several forms similarly arising from the different conics, only a representative one, for example $u_\alpha b_\alpha b_x$ is used to represent the six forms $u_\alpha b_\alpha b_x$, $u_\alpha c_\alpha c_x$, $u_\beta c_\beta c_x$, $u_\beta a_\beta a_x$, $u_\gamma a_\gamma a_x$, $u_\gamma b_\gamma b_x$. The various forms of a fundamental identity, used, are

$$(abc)\,d_x = (bcd)\,a_x + (cad)\,b_x + (abd)\,c_x,$$
$$(abc)(def) = (bcd)(aef) + (cad)(bef) + (abd)(cef),$$
$$(abu)(bcv) - (abv)(bcu) = (abc)(buv),$$
$$(uab)(ucd) = (uac)(ubd) - (uad)(ubc),$$
$$(abu)(cdv) - (abv)(cdu) = (buv)(acd) + (uav)(bcd),$$
$$\{(cad)\,b_x + (abd)\,c_x\}^2 = \{(abc)\,d_x - (bcd)\,a_x\}^2;$$

which, since the squared terms, on expansion, are immediately interpretable as real terms (for the case when $a_x^2 \ldots$ are conics) gives

$$(cad)(abd)\,b_x c_x \equiv -(abc)(bcd)\,a_x d_x.$$

Further

$$b_x b_y' = b_y b_x' + (bb' . xy),$$
$$b_x b_y b_x' b_y' = \tfrac{1}{2}(b_x^2 . b_y'^2 + b_y^2 . b_x'^2) - (bb' . xy)^2.$$

And as typical, the following examples may be given,

1. $$\begin{aligned}(\alpha\beta\gamma)\,\underline{b_\gamma c_\beta b_\alpha' b_x'} b_x c_x &= (\alpha\beta\gamma)\,b_x c_\beta b_\alpha' b_\gamma' b_x c_x + (\alpha\beta\gamma)(bb' . \gamma x)\,c_\beta b_\alpha' b_x c_x \\ &= b_x^2 . (\alpha\beta\gamma)\,c_\beta b_\alpha' b_\gamma' c_x + (\alpha\beta\gamma)(bb' . \gamma x)\,c_\beta b_\alpha' b_x c_x && \equiv (\alpha\beta\gamma)(bb' . \gamma x)\,c_\beta b_\alpha' b_x c_x \\ &= \tfrac{1}{2}(\alpha\beta\gamma)(bb' . \gamma x)(b_x b_\alpha' - b_\alpha b_x')\,c_\beta c_x && \equiv \tfrac{1}{2}(\alpha\underline{\beta\gamma)(\beta'\gamma x)(\beta' x\alpha)\,c_\beta} c_x \\ &= \tfrac{1}{2}\{(\beta\gamma x)\,c_\alpha + (\gamma\alpha x)\,c_\beta + (\alpha\beta x)\,c_\gamma\}(\beta'\gamma x)(\beta' x\alpha)\,c_\beta \\ &= \tfrac{1}{2}(\beta\gamma x)(\beta'\gamma x)(\beta' x\alpha)\,c_\alpha c_\beta + \tfrac{1}{2}c_\beta^2 . (\gamma\alpha x)(\beta'\gamma x)(\beta' x\alpha) && \equiv \tfrac{1}{2}(\beta\gamma x)(\underline{\beta'\gamma x)(\beta' x\alpha)\,c_\alpha c_\beta} \\ &\qquad + \tfrac{1}{2}c_\gamma^2 . (\alpha\beta x)(\beta'\beta x)(\beta' x\alpha) \\ &= \tfrac{1}{2}(\beta\gamma x)(\beta' x\alpha)\,c_\alpha\{(\beta\gamma x)\,c_\beta' + (\beta'\beta x)\,c_\gamma + (\beta'\gamma\beta)\,c_x\} \\ &= \tfrac{1}{2}(\beta\gamma x)^2 . (\alpha\beta' x)\,c_\alpha c_\beta' + \overline{\beta\beta'} && \equiv \tfrac{1}{2}(\beta\gamma x)^2 . (\alpha\beta' x)\,c_\alpha c_\beta' \\ & && \equiv 0;\end{aligned}$$

and the second column will be, in the work, omitted. The thin lines └───┘, underneath, indicate the associations of the parts.

2. $(ubc)(ub'c')(abc')(ab'c)$

$= (bcu)(bc'a)(ub'c')(ab'c)$

$= \{(bcc')(bua) - (bca)(buc')\}(ub'c')(ab'c)$

$= (bcc')(bua)(ub'c')(ab'c) + (abc)(ub'c')(c'ub)(b'ca)$

$= \frac{1}{2} b_\gamma b_\gamma' (aub')(abu) + \underline{(abc)(ub'c')\{(ubb')(c'ca) + (bc'b')(uca) + (c'ub')(bca)\}}$

$= (abc)^2 . (ub'c')^2 - \frac{1}{2}(abu) b_\gamma' (ab'u) b_\gamma - \frac{1}{2}\{(uca) c_\beta' + (cc'a) u_\beta\}\{(cab)(c'ub') - (cab')(c'ub)\}$

$= (abc)^2 . (ub'c') - \frac{1}{2}(abu) b_\gamma' . \{(b'ub) a_\gamma + (uab) b_\gamma' + (ab'b) u_\gamma\} - \frac{1}{2}\{(uca) c_\beta' + (cc'a) u_\beta\}\{a_\beta (cc'u) - c_\beta (ac'u)\}$

$=$ etc.

$= (abc)^2 . (ub'c')^2 - \frac{1}{2}(abu)^2 . b_\gamma'^2 - \frac{1}{2}(cau)^2 . c_\beta'^2 - \frac{1}{2} a_\beta a_\gamma u_\beta u_\gamma + \frac{1}{4}(u_\beta a_\gamma - u_\gamma a_\beta)^2$

$= - a_\beta a_\gamma u_\beta u_\gamma .$

3. $(bcu)(abc')(ab'c)(b'a'u)(c'a'u)$

$= \{(ab'c)(c'a'u)\}\{(abc')(b'a'u)\}(bcu)$

$= \{(ab'c')(ca'u) + (b'cc')(aa'u) + (cac')(b'a'u)\}\{(ab'c')(ba'u) + (bc'b')(aa'u) + (abb')(c'a'u)\}(bcu)$

$=$ etc.

$= (ab'c')^2 . (ca'u)(ba'u)(bcu) - \frac{1}{2}(uab) a_\gamma b_\gamma . (ua'b')^2 - \frac{1}{2}(uc'a') c_\beta' a_\beta' . (uca)^2 + \frac{1}{4}(\alpha\beta\gamma) u_\alpha u_\beta u_\gamma$

$\equiv \frac{1}{4}(\alpha\beta\gamma) u_\alpha u_\beta u_\gamma .$

4. $(abc) a_\beta b_\gamma u_\beta u_\gamma c_x = \{(ubc) a_\gamma + (auc) b_\gamma + (abu) c_\gamma\} a_\beta b_\gamma u_\beta c_x$

$= (ubc) a_\gamma a_\beta b_\gamma u_\beta c_x + b_\gamma^2 . (auc) a_\beta u_\beta c_x + \frac{1}{3} c_\gamma^2 . (abu) a_\beta b_x u_\beta \equiv (ubc) a_\beta a_\gamma b_\gamma u_\beta c_x$

$= \{(uba) c_\beta + (uac) b_\beta + (abc) u_\beta\} a_\gamma b_\gamma u_\beta c_x$

$= (uba) a_\gamma b_\gamma . c_\beta c_x u_\beta + \frac{1}{3} b_\beta^2 . (uac) a_\gamma u_\gamma c_x + u_\beta^2 . (abc) a_\gamma b_\gamma c_x \equiv (uba) a_\gamma b_\gamma . c_\beta u_\beta c_x ;$

i.e. $$(abc) a_\beta b_\gamma u_\beta u_\gamma c_x \equiv 0,$$

and the second column would be omitted in the work.

Note. In verification of the theory given, it is worthy of remark that though Gordan (*Math. Annal.* I. 90, 'Ueber ternäre Formen dritten Grades') does not apparently recognise that it is not necessary to consider the derivatives of products of forms, yet this is really not so—the arrangement only is different. As a fact all his 34 concomitants do occur, independently of the products, in Tables I.—XXIX. (pp. 103—106), except the last one $u_s^2 u_t^2 u_p^5 (spt)$ (page 102, 12te Ordn.), which occurs on page 128 as equivalent to 7 of page 127, namely $u_s^2 u_t^2 c_s d_t (cdu)(bcu)(abu)^2 (adu)$. This last form would however, in accordance with our theory, arise also, independently of products, from u_t^3. For putting

$$u_s^2 v_s = (a'b'c')(b'c'v)(c'a'u)(a'b'u),$$

and then $v=c$, the form in question is

$$u_t^2 d_t\,(cdu)\,(a'b'c')\,(b'c'c)\,(c'a'u)\,(a'b'u)\,(bcu)\,(abu)^2\,(adu),$$

and we should have the following series of derivatives:

of degree 6. $u_t^2 d_t d_x^2$,

" " 7. $u_t^2 d_t d_x\,(dau)\,a_x^2$,

" " 8. $u_t^2 d_t d_x\,(dau)\,(abu)^2 . b_x$,

" " 9. $u_t^2 d_t\,(dcu)\,(dau)\,(abu)^2\,(bcu)\,c_x$,

" " 10. $u_t^2 d_t\,(dcu)\,(dau)\,(abu)^2\,(bcu)\,(cb'u)\,b_x'^2$,

" " 11. $u_t^2 d_t\,(dcu)\,(dau)\,(abu)^2\,(bcu)\,(cb'c')\,(b'c'u)\,b_x' c_x'$,

" " 12. $u_t^2 d_t\,(dcu)\,(dau)\,(abu)^2\,(bcu)\,(cb'c')\,(b'c'a')\,(b'a'u)\,(c'a'u)$.

which is the form in question.

The form here of seventh degree $u_t^2 d_t d_x\,(dau)\,a_x^2$ does occur in Gordan's work as the heading of Table XVII., page 111, under the form $u_t^2 a_t a_x\,(abu)\,b_x^2$: and in our arrangement there should occur under 3 of that table the form $u_t^2 a_t a_x\,(abu)\,(bcu)^2\,c_x$, which is the same as the form above of 8th degree. But this form it is unnecessary for Gordan to write down since it arises from the product $[u_t^2 a_t a_x^2 . c_x^3 = u_t^2 a_t a_x\,(a_x c_x^2)\,c_x]$ of two forms included in the table, § 4, page 101 (viz. under 1te Ordn. and 6te Ordn.), namely by changing x_i into $(bu)_i$ and getting $u_t^2 a_t a_x\,(abu)\,(cbu)^2\,c_x$. Our arrangement, if longer, possesses the advantage that all possibilities are exhausted in the course of the work—at any stage it is exhaustive so far as it has gone—while Gordan's arrangement is not trustworthy until the examination is completely finished.

§ III. *Statement of the system obtained.*

zero degree. $u_x = (011)$ (1 form)

degree 1. $a_x^2 = (102)$ (3 forms)

" 2. $(212) = (bcu)\,b_x c_x$ $u_a^2 = (220)_1$ $(bcu)^2 = (220)_2$ (9 forms)

degree 3.	$(300)_1 = a_a^2$	(3)	degree 4.	$(410) = (bcu)\,b_a c_a\;(=\frac{1}{4}u_\sigma a_\sigma^2)$	(3)
	$(300)_2 = b_a^2$	(6)		$(402)_1 = b_a c_a b_x c_x$	(3)
	$(300)_3 = (abc)^2$	(1)		$(402)_2 = (\beta\gamma x)^2$	(3)
	$(311)_1 = u_a b_a b_x$	(6)		$(421)_1 = (bcu)\,b_a c_x u_a$	(6)
	$(311)_2 = (abc)\,(bcu)\,a_x$	(3)		$(421)_2 = (bcu)\,b_\gamma c_x u_\gamma$	(6)
	†$(303) = (abc)\,a_x b_x c_x$	(1)		†$(421)_3 = (a'bc)\,(uca)\,(uab)\,a_x'$	(3)
	$(330) = (bcu)\,(cau)\,(abu)$	(1)		$(421)_4 = (\beta\gamma x)\,u_\beta u_\gamma$	(3)

degree 5.	$\dagger(501)_1 = (abc)\, a_x b_\alpha c_\alpha \; (= \frac{1}{12} a_x a_p)$	(3)	degree 6.	$(600) \;= (\alpha\beta\gamma)^2$	(1)
	$(501)_2 = (\beta\gamma x)\, a_\beta a_\gamma \; (= \frac{1}{6} b_x b_r = \frac{1}{6} c_x c_q)$	(3)		$(611)_1 = (\alpha\beta\gamma)\,(\beta\gamma x)\, u_\alpha$	(3)
	$(520) \;= u_\beta u_\gamma a_\beta a_\gamma$	(3)		$\dagger(611)_2 = a_\beta a_\gamma b_\gamma b_x u_\beta$	(6)
	$(512)_1 = (\beta\gamma x)\, a_\beta a_x u_\gamma$	(6)		$\dagger(630)_1 = (\alpha\beta\gamma)\, u_\alpha u_\beta u_\gamma$	(1)
	$\dagger(512)_2 = (abc)\, a_\beta u_\beta b_x c_x$	(6)		$\dagger(630)_2 = (abu)\, a_\beta b_\gamma u_\beta u_\gamma$	(6)
	$(512)_3 = (\beta\gamma x)\, c_x c_\beta u_\gamma$	(6)		$\dagger(630)_3 = (bcu)\, u_\beta u_\gamma b_\gamma c_\beta$	(3)
,, 7.	$\dagger(710)_1 = (\alpha\beta\gamma)\, a_\beta a_\gamma u_\alpha$	(3)		$(603)_1 = (\beta\gamma x)\,(\gamma\alpha x)\,(\alpha\beta x)$	(1)
	$\dagger(710)_2 = (bcu)\, a_\beta a_\gamma b_\gamma c_\beta$	(3)		$\dagger(603)_2 = (\beta\gamma x)\, a_x b_x a_\beta b_\gamma$	(6)
	$\dagger(721) \;= (\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma$	(6)		$\dagger(603)_3 = (\beta\gamma x)\, b_x c_x b_\gamma c_\beta$	(3)
,, 8.	$\dagger(801)_1 = (\beta\gamma x)\, b_\gamma c_\beta b_\alpha c_\alpha$	(3)	,, 9.	$\dagger(911) \;= a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x u_\beta$	(6)
	$\dagger(801)_2 = (a'bc)\, a_\beta a_\gamma b_\gamma c_\beta a_x'$	(3)	,, 10.	$\dagger(1010) = (\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'}$	(3)
	$\dagger(812) \;= (\alpha'\beta\gamma)\,(\gamma\alpha x)\,(\alpha\beta x)\, u_{\alpha'}$	(3)			

The 18 forms marked † are reducible when multiplied by u_x. Each form is either its own reciprocal or its reciprocal appears in the table. The number of forms corresponding to any type is given by the number in brackets which follows.

§ IV. *Establishment of the system. First and second degrees.*

The first degree forms from which we start are a_x^2, b_x^2, c_x^2.

From $a_x^2 = (102)$,

$$a_x\,(aa'u)\,a_x' = 0,$$
$$a_x\,(abu)\,b_x,$$
$$(aa'u)^2 = u_\alpha^2,$$
$$(abu)^2.$$

Thus the second degree forms are typified by

$$(212) \;= (bcu)\, b_x c_x,$$
$$(220)_1 = u_\alpha^2,$$
$$(220)_2 = (bcu)^2.$$

Third degree.

From $(212) \;= (bcu)\, b_x c_x$, $(220)_1 = u_\alpha^2$, $(220)_2 = (bcu)^2$, we proceed to shew that we get

$$(300)_1 = a_\alpha^2,$$
$$(300)_2 = b_\alpha^2,$$
$$(300)_3 = (abc)^2,$$
$$(311)_1 = u_\alpha b_\alpha b_x,$$
$$(311)_2 = (abc)\,(bcu)\,a_x,$$
$$(303) \;= (abc)\,a_x b_x c_x,$$
$$(330) \;= (bcu)\,(cau)\,(abu).$$

Derivatives from

$(212) = (bcu)\, b_x\, c_x$.	From $u_a = (220)$.	$(220)_2 = (bcu)^2$.
1. $(bcu)\, b_x\, (cau)\, a_x \equiv (311)_2$.	11. $u_a b_a b_x \equiv (311)_1$.	13. $(bcu)\,(bca)\, a_x \equiv (311)_2$.
2. $(bcu)\, b_x\, (cb'u)\, b_x' \equiv (311)_1$ and $(300)_2$.	12. $b_a^2 \equiv (300)_2$.	14. $(bcu)\,(bcc')\, c_x' \equiv (311)_1$ and $(300)_2$.
3. $(bcu)\, b_x\, (cc'u)\, c_x' \equiv (311)_1$.	17. $a_x^2 \equiv (300)_1$.	
4. $(bca)\, a_x b_x c_x \equiv (303)$.		15. $(bca)^2 \equiv (300)_3$.
5. $(bcc')\, c_x' b_x c_x = 0$.		16. $(bcc')^2 \equiv (300)_2$.
6. $(bcu)\,(bau)\,(cau) \equiv (330)$.		
7. $(bcu)\,(bb'u)\,(cb'u) = 0$.		
8. $(bca)\, b_x\, (cau) \equiv (311)_2$.		
9. $(bcb')\, b_x\, (cb'u) \equiv (311)_1$ and $(300)_2$.		
10. $(bcc')\, b_x\, (cc'u) \equiv (311)_1$.		

Of these 1. $\equiv -(abc)\,(abu)\, c_x \,.\, u_x \equiv (311)_2$.

2. $= -(bcu)^2 \,.\, b_x'^2 + (bcu)\, b'_x \,.\, \{(bb'u)\, c_x + (cb'b)\, u_x\}$

$$\equiv \tfrac{1}{2} c_x u_\beta\, (u_\beta c_x - u_x c_\beta) - \tfrac{1}{2} u_x c_\beta\, (u_\beta c_x - u_x c_\beta)$$

$\equiv \tfrac{1}{2} \{c_x^2 \,.\, u_\beta^2 - 2u_x\, c_x c_\beta u_\beta + u_x^2 \,.\, c_\beta^2\} \equiv (311)_1$ and $(300)_2$.

3. $= \tfrac{1}{2} u_\gamma b_x\, (b_\gamma u_x - b_x u_\gamma) \equiv (311)_1$.

7. $= \tfrac{1}{2} (bb'u)\, \{\quad\} = 0$.

9. $= -\tfrac{1}{2} c_\beta\, (u_\beta c_x - u_x c_\beta) \equiv (311)_1$ and $(300)_2$.

14. $= \tfrac{1}{2} b_\gamma u_x - b_x u_\gamma) \equiv (311)_1$ and $(300)_2$.

Fourth degree.

We proceed to shew that from

$(311)_1 = u_a b_a b_x$	we obtain $(410) = (bcu)\, b_a c_a$,
$(311)_2 = (abc)\,(bcu)\, a_x$	$(402)_1 = b_a c_a b_x c_x$,
$(303) = (abc)\, a_x b_x c_x$	$(402)_2 = (\beta\gamma x)^2$,
$(330) = (bcu)\,(cau)\,(abu)$	$(421)_1 = (bcu)\, b_a c_x u_a$,
	$(421)_2 = (bcu)\, b_\gamma c_x u_\gamma$,
	$(421)_3 = (a'bc)\,(uca)\,(uab)\, a_x'$,
	$(421)_4 = (\beta\gamma x)\, u_\beta u_\gamma$.

From $(311)_1 = u_a b_a b_x$.

1. $u_a b_a (bau)\, a_x \equiv (421)_2$.
2. $u_a b_a (bb'u)\, b'_x \equiv (421)_4$.
3. $u_a b_a (bcu)\, c_x \equiv (421)_1$.
4. $b'_a b'_x b_a b_x \equiv (402)_2$.
5. $c_a c_x b_a b_x \equiv (402)_1$.
6. $b'_a b_a (bb'u) = 0$.
7. $c_a b_a (bcu) \equiv (410)$.

From $(311)_2 = (abc)(bcu)\, a_x$.

8. $(abc)(bcu)(aa'u)\, a'_x \equiv (421)_1$.
9. $(abc)(bcu)(ab'u)\, b'_x \equiv \begin{cases}(410)\\(421)_1\\(421)_3\end{cases}$.
10. $(abc)(bca')\, a'_x a_x \equiv (402)_1$.
11. $(abc)(bcb')\, b'_x a_x \equiv (402)_1$.
12. $(abc)(bca')(aa'u) = 0$.
13. $(abc)(bcb')(ab'u) \equiv (410)$.

From $(abc)\, a_x b_x c_x$.

14. $(abc)(aa'u)\, b_x c_x a'_x \equiv 0$.
15. $(abc)(ab'u)\, b_x c_x b'_x \equiv (402)_1$.
16. $(abc)\, a_x (ba'u)(ca'u) \equiv (421)_3$.
17. $(abc)\, a_x (bc'u)(cc'u) \equiv (421)_1$.

From $(bcu)(cau)(abu)$.

18. $(bca')(cau)(abu)\, a'_x \equiv (421)_3$.
19. $(bcc')(cau)(abu)\, c'_x \equiv \begin{cases}(410)\\(421)_1\end{cases}$.
20. $(bcu)(caa')(aba') \equiv (410)$.
21. $(bcu)(cac')(abc') \equiv (410)$.

Of these

$2 = \frac{1}{2} u_\beta (\beta a x)\, u_a \equiv (421)_4$.

$4 = b_a^2 \,.\, b_x^2 - (bb'.\, x a)^2 \equiv (402)_2$.

$8 = \frac{1}{2}(bcu)\, u_a (c_a b_x - c_x b_a)$.

$9 = (abc)(b'cu)(ab'u)\, b_x + (abc)(ab'u)\{(bb'u)\, c_x - (cbb')\, u_x\}$

$\quad = (abc)(b'cu)(ab'u)\, b_x + \frac{1}{2} u_\beta c_x (acu)\, a_\beta - \frac{1}{2} c_\beta u_x (acu)\, a_\beta$

$\quad \equiv (410)$, $(421)_1$ and $(421)_3$.

$10 = (bca')^2 \,.\, a_x^2 + (bca')\, a_x \,.\, \{(aa'c)\, b_x - (baa')\, c_x\} \equiv \frac{1}{2} c_a b_x (b_a c_x - b_x c_a) - \frac{1}{2} b_a c_x (b_a c_x - b_x c_a)$

$\quad \equiv b_a c_a b_x c_x$.

$11 = -\frac{1}{2} c_\beta a_x (a_\beta c_x - a_x c_\beta)$.

$13 = +\frac{1}{2} c_\beta (cau)\, a_\beta$.

$14 = \frac{1}{2} u_a b_x c_x (c_a b_x - c_x b_a)$.

$15 \equiv (abc)\, c_x b'_x \{(bb'u)\, a_x - (abb')\, u_x\} = \frac{1}{2} u_\beta c_x a_x (a_\beta c_x - a_x c_\beta) - \frac{1}{2} u_x a_\beta c_x (a_\beta c_x - a_x c_\beta) \equiv +\frac{1}{2} u_x \,.\, a_\beta c_\beta a_x c_x$.

$17 = \frac{1}{2} u_\gamma a_x (aub)\, b_\gamma$.

$19 = \frac{1}{2} b_\gamma (abu)(u_\gamma a_x - u_x a_\gamma)$.

$21 = \frac{1}{2} a_\gamma (aub)\, b_\gamma$.

Fifth degree.

We proceed now to shew that from

$(410) = (bcu)\, b_\alpha c_\alpha,$	we obtain $(501)_1 = (abc)\, a_x b_\alpha c_\alpha.$
$(402)_1 = b_\alpha c_\alpha b_x c_x,$	$(501)_2 = (\beta\gamma x)\, a_\beta a_\gamma.$
$(402)_2 = (\beta\gamma x)^2,$	$(520) = u_\beta u_\gamma a_\beta a_\gamma.$
$(421)_1 = (bcu)\, b_\alpha c_x u_\alpha,$	$(512)_1 = (\beta\gamma x)\, a_\beta a_x u_\gamma.$
$(421)_2 = (bcu)\, b_\gamma c_x u_\gamma,$	$(512)_2 = (abc)\, a_\beta u_\beta b_x c_x.$
$(421)_3 = (a'bc)\,(uca)\,(uab)\, u'_x,$	$(512)_3 = (\beta\gamma x)\, c_x c_\beta u_\gamma.$
$(421)_4 = (\beta\gamma x)\, u_\beta u_\gamma,$	

From $(410) = (bcu)\, b_\alpha c_\alpha.$

1. $(bca)\, a_x b_\alpha c_\alpha \equiv (501)_1.$
2. $(bcc')\, c'_x b_\alpha c_\alpha \equiv (501)_2.$

From $(402)_1 = b_\alpha c_\alpha b_x c_x.$

3. $b_\alpha c_\alpha (bau)\, a_x c_x \equiv (512)_2.$
4. $b_\alpha c_\alpha (bb'u)\, b'_x c_x \equiv (512)_1.$
5. $b_\alpha c_\alpha (bc'u)\, c'_x c_x \equiv \begin{cases}(512)_1\\(501)_2.\end{cases}$
6. $b_\alpha c_\alpha (bau)(cau) \equiv 0.$
7. $b_\alpha c_\alpha (bc'u)(cc'u) \equiv (520).$

From $(\beta\gamma x)^2 = (402)_2.$

8. $(\beta\gamma x)(\beta\gamma \,.\, au)\, a_x \equiv (512)_1.$
9. $(\beta\gamma x)(\beta\gamma \,.\, cu)\, c_x \equiv (512)_3.$
10. $(\beta\gamma \,.\, au)^2 \equiv (520).$
11. $(\beta\gamma \,.\, cu)^2 \equiv 0.$

From $(421)_1 = (bcu)\, b_\alpha c_x u_\alpha.$

12. $(bcu)\ b_\alpha u_\alpha (cau)\, a_x \equiv 0.$
13. $(bcu)\ b_\alpha u_\alpha (cb'u)\, b'_x \equiv (520).$
14. $(bcu)\ b_\alpha u_\alpha (cc'u)\, c'_x \equiv (520).$
15. $(bcu)\ b_\alpha c_x b'_\alpha b'_x \equiv (512)_1.$
16. $(bcu)\ b_\alpha c_x c'_\alpha c'_x \equiv (501)_2$ and $(512)_1.$
17. $(bca)\ a_x b_\alpha c_x u_\alpha \equiv (512)_2.$
18. $(bcb')\ b'_x b_\alpha\, c_x u_\alpha \equiv (512)_1.$
19. $(bcc')\ c'_x b_\alpha\, c_x u_\alpha = 0.$
20. $(bcu)\, b_\alpha\, (cb'u)\, b'_\alpha \equiv (520).$
21. $(bcu)\, b_\alpha\, (cc'u)\, c'_\alpha \equiv (520).$
22. $(bca)\, b_\alpha u_\alpha\, (cau) \equiv 0.$
23. $(bcb')\, b_\alpha u_\alpha\, (cb'u) \equiv (520).$
24. $(bcc')\, b_\alpha u_\alpha\, (cc'u) \equiv (520).$
25. $(bcb')\, b_\alpha c_x\, b'_\alpha = 0.$
26. $(bcc')\, b_\alpha c_x c'_\alpha \equiv (501)_2.$

From $(421)_2 = (bcu)\, b_\gamma c_x u_\gamma.$

27. $(bcu)\ b_\gamma u_\gamma (cau)\, a_x \equiv 0.$
28. $(bcu)\ b_\gamma u_\gamma (cb'u)\, b'_x \equiv 0.$
29. $(bcu)\ b_\gamma u_\gamma (cc'u)\, c'_x \equiv 0.$
30. $(bcu)\ b_\gamma a_\gamma\, c_x a_x \equiv (512)_2.$
31. $(bcu)\ b_\gamma b'_\gamma c_x b'_x \equiv (512)_3.$
32. $(bca)\ b_\gamma u_\gamma\, c_x a_x \equiv (512)_2.$
33. $(bcb')\, b_\gamma u_\gamma\, c_x b'_x \equiv (512)_3.$
34. $(bcc')\, b_\gamma u_\gamma\, c_x c'_x = 0.$
35. $(bcu)\, b_\gamma a_\gamma\, (cau) \equiv 0.$
36. $(bcu)\, b_\gamma b'_\gamma\, (cb'u) \equiv 0.$
37. $(bca)\, b_\gamma u_\gamma\, (cau) \equiv 0.$
38. $(bcb')\, b_\gamma u_\gamma\, (cb'u) \equiv 0.$
39. $(bcc')\, b_\gamma u_\gamma\, (cc'u) \equiv 0.$
40. $(bca)\ b_\gamma a_\gamma c_x \equiv (501)_1.$
41. $(bcb')\, b_\gamma b'_\gamma c_x = 0.$

From $(421)_4 = (\beta\gamma x)\, u_\beta u_\gamma$.

42. $(\beta\gamma\,.\,au)\, u_\beta u_\gamma a_x \equiv 0$.

43. $(\beta\gamma\,.\,bu)\, u_\beta u_\gamma b_x \equiv 0$.

44. $(\beta\gamma x)\, a_\beta u_\gamma a_x \equiv (512)_1$.

45. $(\beta\gamma\,.\,au)\, a_\beta u_\gamma \equiv (520)$.

46. $(\beta\gamma\,.\,cu)\, c_\beta u_\gamma c_x \equiv 0$.

47. $(\beta\gamma x)\, a_\beta a_\gamma \equiv (501)_2$.

48. $(\beta\gamma x)\, c_\beta u_\gamma c_x \equiv (512)_3$.

From $(421)_3 = (a'bc)(uca)(uab)\, a'_x$.

49. $(a'bc)(uca)(uab)(a'a''u)\, a''_x \equiv 0$.

50. $(a'bc)(uca)(uab)(a'b'u)\, b'_x \equiv (520)$.

51. $(a'bc)(uca)(a''ab)\, a'_x a''_x \equiv (501)_1$ and $(512)_2$.

52. $(a'bc)(uca)(b'ab)\, a'_x b'_x \equiv (512)_1$.

53. $(a'bc)(uca)(c'ab)\, a'_x c'_x \equiv (501)_2$ and $(512)_1$.

54. $(a'bc)(uca)(a''ab)(a'a''u) \equiv 0$.

55. $(a'bc)(uca)(b'ab)(a'b'u) \equiv (520)$.

56. $(a'bc)(uca)(c'ab)(a'c'u) \equiv (520)$.

57. $(a'bc)(a''ca)(a''ab)\, a'_x \equiv (501)_1$.

58. $(a'bc)(b'ca)(b'ab)\, a'_x \equiv (501)_2$.

Of these

$2 = \frac{1}{2}(\gamma\alpha x)\, b_\gamma b_\alpha$.

$3 \equiv (bac)\, u_\alpha b_\alpha a_x c_x$ or say $(abc)\, u_\beta a_\beta b_x c_x$.

$4 = \frac{1}{2}(\beta\alpha x)\, u_\beta c_\alpha c_x$ or say $\equiv \frac{1}{2}(\beta\gamma x)\, u_\beta a_\gamma a_x$.

$5 \equiv b_\alpha c_\alpha c'_x \{(cc'u)\, b_x - (cc'b)\, u_x\} \equiv \frac{1}{2} u_\gamma b_\alpha b_x (\gamma\alpha x) - \frac{1}{2} u_x b_\gamma b_\alpha (\gamma\alpha x)$.

$6 \equiv (abc)(bcu)\, a_\alpha u_\alpha \equiv \frac{1}{3}(bcu)^2\,.\,a_\alpha^2$.

$7 = \frac{1}{2} u_\gamma b_\alpha (u_\gamma b_\alpha - u_\alpha b_\gamma)$.

$8 = (\beta\gamma x)\, a_\beta a_x u_\gamma - (\beta\gamma x)\, a_\gamma a_x u_\beta$.

$9 \equiv (\beta\gamma x)\, c_\beta c_x u_\gamma$.

$10 \equiv -2 a_\beta a_\gamma u_\beta u_\gamma$.

$11 \equiv 0$.

$12 = (cau)\, b_\alpha u_\alpha \{(bca)\, u_x + (cua)\, b_x + (uba)\, c_x\} \equiv u_x (cau)\, b_\alpha (abc)\, u_\alpha + c_x u_\alpha (uba)(cau)\, b_\alpha$

$\equiv -u_x (bcu)(abu)\, a_\alpha c_\alpha + c_x u_\alpha (uba)(bau)\, c_\alpha \equiv 0$.

$13\ [= -b_x b_\alpha u_\alpha\,.\,(b'cu)^2 -]$

$\equiv (bcu)\, u_\alpha b'_x (bb'u)\, c_\alpha \equiv \frac{1}{2} c_\alpha u_\beta u_\alpha (u_\beta c_x - u_x c_\beta)$.

$14 = \frac{1}{2} u_\gamma (b_\gamma u_x - b_x u_\gamma)\, b_\alpha u_\alpha$.

$15 \equiv (bcu)(bb'\,.\,\alpha x)\, c_x b'_\alpha = \frac{1}{2}(\beta\alpha x)(u_\beta c_\alpha - u_\alpha c_\beta)\, c_x$.

$16 \equiv (bcu)(cc'\,.\,x\alpha)\, b_\alpha c'_x = \frac{1}{2}(\gamma x\alpha)(b_\gamma u_x - b_x u_\gamma)\, b_\alpha$.

$18 = -\frac{1}{2} c_\beta (\beta\alpha x)\, c_x u_\alpha$.

$20 \equiv (bcu)\, b'_\alpha \{(bb'u)\, c_\alpha - (cbb')\, u_\alpha\} \equiv \frac{1}{2} u_\beta c_\alpha (u_\beta c_\alpha - u_\alpha c_\beta) - \frac{1}{2} c_\beta u_\alpha (u_\beta c_\alpha - u_\alpha c_\beta) \equiv -u_\alpha u_\beta c_\alpha c_\beta$.

$21 = \frac{1}{2} u_\gamma b_\alpha (b_\gamma u_\alpha - b_\alpha u_\gamma)$.

$22 \equiv -(abu)(bcu)\, c_\alpha a_\alpha \equiv 0$.

$23 = -\frac{1}{2} c_\beta u_\alpha (u_\beta c_\alpha - u_\alpha c_\beta)$.

$24 = u_\gamma b_\gamma b_\alpha u_\alpha$.

$26 = \frac{1}{2} b_\gamma b_\alpha (\gamma x \alpha)$.

$27 \equiv (bcu) a_\gamma u_\gamma (cbu) a_x = -(bcu)^2 . a_\gamma a_x u_\gamma \equiv 0$.

$28 \equiv (bcu) b'_\gamma u_\gamma (cbu) b'_x = -(bcu)^2 . b'_\gamma u_\gamma b'_x \equiv 0$.

$29 = (bcu) c_\gamma u_\gamma (bc'u) c'_x + (bcu) c'_\gamma u_\gamma (cbu) c'_x + (bcu) u_\gamma^2 (cc'b) c'_x \equiv 0$.

$30 \equiv (bca) b_\gamma u_\gamma c_x a_x$.

$31 \equiv (bcu)(bb' . \gamma x) b'_\gamma c_x \equiv \frac{1}{2} (\beta\gamma x)(u_\beta c_\gamma - u_\gamma c_\beta) c_x$.

$33 = -\frac{1}{2} c_\beta (\beta\gamma x) u_\gamma c_x$.

$35 \equiv -(abc)(abu) u_\gamma c_\gamma \equiv 0$.

$36 \equiv (bcu) u_\gamma b'_\gamma (cb'b) \equiv -\frac{1}{2} c_\beta u_\gamma (u_\beta c_\gamma - u_\gamma c_\beta)$.

$37 \equiv -(abu)(bcu) c_\gamma a_\gamma \equiv 0$.

$38 = -\frac{1}{2} c_\beta u_\gamma (u_\beta c_\gamma - u_\gamma c_\beta) \equiv 0$.

$39 = u_\gamma b_\gamma u_{\gamma'} b_{\gamma'} \equiv -(ub . \gamma\gamma')^2 \equiv -\frac{4}{3} c_\gamma^2 . (ubc)^2$.

$42 \equiv 0 \, (= u_\beta^2 \ldots - u_\gamma^2 \ldots)$.

$43 \equiv u_\beta^2 . b_\gamma u_\gamma b_x$.

$45 \equiv - a_\gamma u_\beta a_\beta u_\gamma$.

$46 = (c_\beta u_\gamma - c_\gamma u_\beta) c_\beta u_\gamma c_x \equiv 0$.

$49 = \frac{1}{2} u_\alpha (uca)(uab)(c_\alpha b_x - c_x b_\alpha) \equiv \frac{1}{2} u_\alpha (uca)(uac) b_\alpha b_x - \frac{1}{2} u_\alpha (uba)(uab) c_x c_\alpha \equiv 0$.

$$\begin{aligned}
50 &= (uca)(a'b'u) b'_x (bua)(bca') = (uca)(a'b'u) b'_x \{(buc)(baa') - (bua')(bac)\} \\
&= \tfrac{1}{2} b_\alpha (buc) b'_x (ub'c) u_\alpha - (uca)(bac)(a'b'u)\{(b'ua') b_x + (bb'a') u_x + (bub') a'_x\} \\
&\equiv \tfrac{1}{2} u_\alpha b_\alpha (ub'c)\{(bb'c) u_x + (bub') c_x\} - \tfrac{1}{2} (a'bb')(uca) u_x \{(acb)(ua'b') - (acb')(ua'b)\} \\
&\qquad + \tfrac{1}{2} (ubb')(uca) a'_x \{(acb)(ua'b') - (acb')(ua'b)\} \\
&\equiv \tfrac{1}{4} u_x u_\alpha c_\beta (c_\beta u_\alpha - c_\alpha u_\beta) - \tfrac{1}{4} u_\beta u_\alpha c_x (c_\beta u_\alpha - c_\alpha u_\beta) - \tfrac{1}{4} a'_\beta u_x (uca)\{c_\beta (aua') - a_\beta (cua')\} \\
&\qquad + \tfrac{1}{4} u_\beta a'_x (uca)\{c_\beta (aua') - a_\beta (cua')\} \\
&\equiv -\tfrac{1}{4} u_x . c_\alpha c_\beta u_\alpha u_\beta + \tfrac{1}{8} u_x u_\alpha c_\beta (c_\alpha u_\beta - c_\beta u_\alpha) + \tfrac{1}{4} u_x a'_\beta (uca)\{(caa') u_\beta - (uaa') c_\beta\} \\
&\qquad - \tfrac{1}{8} u_\alpha u_\beta c_\beta (c_\alpha u_x - c_x u_\alpha) - \tfrac{1}{4} u_\beta a'_x (uca) . \{(caa') u_\beta - (uaa') c_\beta\} \\
&\equiv -\tfrac{1}{4} u_x . c_\alpha c_\beta u_\alpha u_\beta + \tfrac{1}{8} u_x c_\alpha c_\beta u_\alpha u_\beta + \tfrac{1}{8} u_x c_\alpha u_\beta (c_\alpha u_\beta - c_\beta u_\alpha) - \tfrac{1}{8} u_x c_\beta u_\alpha (c_\alpha u_\beta - c_\beta u_\alpha) - \tfrac{1}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta \\
&\qquad + \tfrac{1}{8} u_\alpha u_\beta c_\beta (c_\alpha u_x - c_x u_\alpha) \\
&\equiv -\tfrac{1}{4} u_x . c_\alpha c_\beta u_\alpha u_\beta + \tfrac{1}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta - \tfrac{1}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta - \tfrac{1}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta - \tfrac{1}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta \\
&\qquad + \tfrac{1}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta \\
&\equiv -\tfrac{3}{8} u_x . c_\alpha c_\beta u_\alpha u_\beta.
\end{aligned}$$

$51 = \frac{1}{2} b_\alpha (a'bc) a'_x . (c_\alpha u_x - c_x u_\alpha) = \frac{1}{2} (abc) a_x b_\alpha c_\alpha . u_x - \frac{1}{2} (abc) c_x a_x u_\alpha b_\alpha$.

$52 = \frac{1}{2} a_\beta (uca) a'_x (a'_\beta c_x - a'_x c_\beta) \equiv \frac{1}{2} (aa' . \beta x)(uca) a'_\beta c_x \equiv \frac{1}{4} (\alpha\beta x) c_x (c_\alpha u_\beta - c_\beta u_\alpha)$.

$53 \equiv (a'bc)(uca)\,c'_x\,.\,\{(c'a'b)\,a_x + (c'aa')\,b_x\} \equiv (bc'a')(uca)\,a_x\,.\,\{(bc'a')\,c_x + (bcc')\,a'_x - (a'cc')\,b_x\}$
$\qquad + \tfrac{1}{2}\,c'_\alpha b_x c'_x (ubc)\,c_\alpha$

$\equiv \tfrac{1}{2}\,b_\gamma a_x a'_x\{(auc)(a'bc') - (auc')(a'bc)\} - \tfrac{1}{2}\,a'_\gamma a_x b_x\{(auc)(a'bc') - (auc')(a'bc)\} + \tfrac{1}{2}\,c'_\alpha b_x (ubc)(cc'\,.\,\alpha x)$

$\equiv \tfrac{1}{2}\,b_\gamma a_x a'_x\{u_\gamma (aa'b) - a_\gamma (ua'b)\} - \tfrac{1}{2}\,a'_\gamma a_x b_x\{u_\gamma (aa'b) - a_\gamma (ua'b)\} + \tfrac{1}{4}(\gamma\alpha x)\,b_x (b_\gamma u_\alpha - b_\alpha u_\gamma)$

$\equiv -\tfrac{1}{2}\,b_\gamma a_x (aa'\,.\,\gamma x)(ua'b) - \tfrac{1}{4}\,b_\alpha b_x u_\gamma (\alpha x\gamma) + \tfrac{1}{2}(aa'\,.\,x\gamma)\,b_x a_\gamma (ua'b) + \tfrac{1}{4}(\gamma\alpha x)\,b_\gamma b_x u_\alpha - \tfrac{1}{4}(\gamma\alpha x)\,b_\alpha b_x u_\gamma$

$\equiv -\tfrac{1}{4}\,b_\gamma (\alpha\gamma x)(b_\alpha u_x - b_x u_\alpha) - \tfrac{1}{4}(\gamma\alpha x)\,b_\alpha b_x u_\gamma + \tfrac{1}{4}(\gamma\alpha x)\,b_x (b_\alpha u_\gamma - b_\gamma u_\alpha)$
$\qquad + \tfrac{1}{4}(\gamma\alpha x)\,b_\gamma b_x u_\alpha - \tfrac{1}{4}(\gamma\alpha x)\,b_\alpha b_x u_\gamma$

$\equiv \tfrac{1}{4}(\gamma\alpha x)\,b_\gamma b_\alpha\,.\,u_x - \tfrac{1}{4}(\gamma\alpha x)\,b_\gamma b_x u_\alpha - \tfrac{1}{4}(\gamma\alpha x)\,b_\alpha b_x u_\gamma.$

$54 = -\tfrac{1}{2}(uca)\,u_\alpha (abc)\,b_\alpha \equiv \tfrac{1}{2}(bcu)(abu)\,a_\alpha c_\alpha \equiv -\tfrac{1}{6}\,a_\alpha^2\,.\,(bcu)^2.$

$55 = \tfrac{1}{2}(uca)\,a_\beta (cua')\,a'_\beta \equiv \tfrac{1}{2}(uca)\,a'_\beta\{(caa')\,u_\beta - (uaa')\,c_\beta\} \equiv \tfrac{1}{4}\,c_\alpha u_\beta (c_\alpha u_\beta - c_\beta u_\alpha)$
$\qquad - \tfrac{1}{4}\,u_\alpha c_\beta (c_\alpha u_\beta - c_\beta u_\alpha) \equiv -\tfrac{1}{2}\,c_\alpha c_\beta u_\alpha u_\beta.$

56. Consider it under the form $(ubc)(ub'c')(abc')(ab'c)$. This is given as example 2 of § II., where its value is written down. It is $\equiv u_\beta u_\gamma a_\beta a_\gamma$.

$57 = -(a'bc)\,a'_x b_\alpha c_\alpha.$

$58 = \tfrac{1}{2}\,a_\beta (aa'c)\,c_\beta a'_x = \tfrac{1}{4}\,c_\alpha (\alpha\beta x)\,c_\beta.$

This completes the establishment of the fifth degree. We have arrived at all the forms written down on page 71 and no others.

Sixth degree.

We proceed now to shew that from	we obtain
$(501)_1 = (abc)\,a_\alpha b_\alpha c_\alpha$,	$(600) = (\alpha\beta\gamma)^2$.
$(501)_2 = (\beta\gamma x)\,a_\beta a_\gamma$,	$(611)_1 = (\alpha\beta\gamma)(\beta\gamma x)\,u_\alpha$.
$(520) = u_\beta u_\gamma a_\beta a_\gamma$,	$(611)_2 = a_\beta a_\gamma b_\gamma b_x u_\beta$.
$(512)_1 = (\beta\gamma x)\,a_\beta a_x u_\gamma$,	$(630)_1 = (\alpha\beta\gamma)\,u_\alpha u_\beta u_\gamma$.
$(512)_2 = (abc)\,a_\beta u_\beta b_x c_x$,	$(630)_2 = (abu)\,a_\beta b_\gamma u_\beta u_\gamma$.
$(512)_3 = (\beta\gamma x)\,c_x c_\beta u_\gamma$,	$(630)_3 = (bcu)\,u_\beta u_\gamma b_\gamma c_\beta$.
	$(603)_1 = (\beta\gamma x)(\gamma\alpha x)(\alpha\beta x)$.
	$(603)_2 = (\beta\gamma x)\,a_x b_x a_\beta b_\gamma$.
	$(603)_3 = (\beta\gamma x)\,b_x c_x b_\gamma c_\beta$.

From $(501)_1 = (abc)\,a_x b_\alpha c_\alpha$.	From $(501)_2 = (\beta\gamma x)\,a_\beta a_\gamma$.	From $(520) = u_\beta u_\gamma a_\beta a_\gamma$.
1. $(abc)(aa'u)\,b_\alpha c_\alpha a'_x \equiv 0$.	3. $(\beta\gamma\,.\,a'u)\,a_\beta a_\gamma a'_x \equiv (611)_1$.	5. $a'_\beta a'_x u_\gamma a_\beta a_\gamma \equiv (611)_1$.
2. $(abc)(ab'u)\,b_\alpha c_\alpha b'_x \equiv (611)_2$.	4. $(\beta\gamma\,.\,bu)\,a_\beta a_\gamma b_x \equiv (611)_2$.	6. $c'_\beta c'_x u_\gamma a_\beta a_\gamma \equiv (611)_2$.
		7. $a'_\beta a'_\gamma a_\beta a_\gamma \equiv (600)$.

From $(512)_1 = (\beta\gamma x)\, a_\beta a_x u_\gamma$.

8. $(\beta\gamma \,.\, a'u)\, a_\beta a_x a'_x u_\gamma \equiv 0.$

9. $(\beta\gamma \,.\, bu)\, a_\beta a_x b_x u_\gamma \equiv 0.$

10. $(\beta\gamma \,.\, cu)\, a_\beta a_x c_x u_\gamma \equiv 0.$

11. $(\beta\gamma x)\, a_\beta (aa'u)\, a'_x u_\gamma \equiv (611)_1.$

12. $(\beta\gamma x)\, a_\beta (abu)\, b_x u_\gamma \equiv 0.$

13. $(\beta\gamma x)\, a_\beta (acu)\, c_x u_\gamma \equiv (611)_2.$

14. $(\beta\gamma x)\, a_\beta a_x a'_\gamma a'_x \equiv (603)_1.$

15. $(\beta\gamma x)\, a_\beta a_x b_\gamma b_x = (603)_2.$

16. $(\beta\gamma \,.\, a'u)\, a_\beta (aa'u)\, u_\gamma \equiv (630)_1.$

17. $(\beta\gamma \,.\, bu)\, a_\beta (abu)\, u_\gamma \equiv (630)_2.$

18. $(\beta\gamma \,.\, cu)\, a_\beta (acu)\, u_\gamma \equiv 0.$

19. $(\beta\gamma \,.\, a'u)\, a_\beta a_x a'_\gamma \equiv (611)_1.$

20. $(\beta\gamma \,.\, bu)\, a_\beta a_x b_\gamma \equiv 0.$

21. $(\beta\gamma x)\, a_\beta (aa'u)\, a'_\gamma \equiv (611)_1.$

22. $(\beta\gamma x)\, a_\beta (abu)\, b_\gamma \equiv (611)_2.$

From $(512)_2 = (abc)\, a_\beta u_\beta b_x c_x$.

23. $(abc)\, a_\beta u_\beta (ab'u)\, c_x a'_x \equiv 0.$

24. $(abc)\, a_\beta u_\beta (bb'u)\, c_x b'_x \equiv 0.$

25. $(abc)\, a_\beta u_\beta (bc'u)\, c_x c'_x \equiv 0.$

26. $(abc)\, a_\beta u_\beta b_x (ca'u)\, a'_x \equiv 0.$

27. $(abc)\, a_\beta u_\beta b_x (cb'u)\, b'_x \equiv 0.$

28. $(abc)\, a_\beta u_\beta b_x (cc'u)\, c'_x \equiv 0.$

29. $(abc)\, a_\beta a'_\beta a'_x b_x c_x \equiv (603)_2.$

30. $(abc)\, a_\beta c'_\beta b_x c_x \equiv (603)_2.$

31. $(abc)\, a_\beta a'_\beta (ba'u)\, c_x \equiv 0.$

32. $(abc)\, a_\beta c'_\beta (bc'u)\, c_x \equiv (611)_2.$

33. $(abc)\, a_\beta a'_\beta b_x (ca'u) \equiv (611)_2.$

34. $(abc)\, a_\beta c'_\beta b_x (cc'u) \equiv 0.$

35. $(abc)\, a_\beta u_\beta (ba'u)(ca'u) \equiv (630)_2.$

36. $(abc)\, a_\beta u_\beta (bb'u)(cb'u) \equiv 0.$

37. $(abc)\, a_\beta u_\beta (bc'u)(cc'u) \equiv (630)_2.$

From $(512)_3 = (\beta\gamma x)\, c_x c_\beta u_\gamma$.

38. $(\beta\gamma \,.\, a'u)\, c_x c_\beta u_\gamma a'_x \equiv 0.$

39. $(\beta\gamma \,.\, bu)\, c_x c_\beta u_\gamma b_x \equiv 0.$

40. $(\beta\gamma \,.\, c'u)\, c_x c_\beta u_\gamma c'_x \equiv 0.$

41. $(\beta\gamma x)(cau)\, c_\beta u_\gamma a_x \equiv 0.$

42. $(\beta\gamma x)(cbu)\, c_\beta u_\gamma b_x \equiv 0.$

43. $(\beta\gamma x)(cc'u)\, c_\beta u_\gamma c'_x \equiv 0.$

44. $(\beta\gamma x)\, c_x c_\beta a_\gamma a_x \equiv (603)_2.$

45. $(\beta\gamma x)\, c_x c_\beta b_\gamma b_x \equiv (603)_3.$

46. $(\beta\gamma \,.\, au)\, c_x c_\beta a_\gamma \equiv (611)_2.$

47. $(\beta\gamma \,.\, bu)\, c_x c_\beta b_\gamma \equiv 0.$

48. $(\beta\gamma x)(cau)\, c_\beta a_\gamma \equiv (611)_2.$

49. $(\beta\gamma x)(cbu)\, c_\beta b_\gamma \equiv 0.$

50. $(\beta\gamma \,.\, au)(cau)\, c_\beta u_\gamma \equiv (630)_2.$

51. $(\beta\gamma \,.\, bu)(cbu)\, c_\beta u_\gamma \equiv (630)_3.$

52. $(\beta\gamma \,.\, c'u)(cc'u)\, c_\beta u_\gamma \equiv 0.$

$1 = \tfrac{1}{2} u_{\alpha'} (c_{\alpha'} b_x - c_x b_{\alpha'})\, b_\alpha c_\alpha$ and $u_{\alpha'} c_{\alpha'} b_\alpha b_x c_\alpha \equiv u_{\alpha'} b_x c_\alpha (bc \,.\, \alpha\alpha') \equiv \tfrac{1}{2} b_x (cu \,.\, \alpha\alpha')(bc \,.\, \alpha\alpha').$

$$2 \equiv (abc)\, c_\alpha b'_x \{(uab)\, b'_\alpha + (ab'b)\, u_\alpha\} \equiv (abc)\, b'_x b'_\alpha \{(abc)\, u_\alpha + (uac)\, b_\alpha\} - \tfrac{1}{2} a_\beta c_\alpha u_\alpha (a_\beta c_x - a_x c_\beta)$$

$$\equiv (uac)(abc)\, b'_\alpha (bb' \,.\, \alpha x) + \tfrac{1}{2} c_\alpha c_\beta a_\beta a_x u_\alpha \equiv \tfrac{1}{2} (\beta\alpha x)(uac)(a_\beta c_\alpha - a_\alpha c_\beta) + \tfrac{1}{2} c_\alpha c_\beta a_\beta a_x u_\alpha$$

$$\equiv \tfrac{1}{2} \begin{vmatrix} u_\beta & u_\alpha & u_x \\ a_\beta & a_\alpha & a_x \\ c_\beta & c_\alpha & c_x \end{vmatrix} a_\beta c_\alpha + \tfrac{1}{2} c_\alpha c_\beta a_\beta a_x u_\alpha \equiv \tfrac{1}{2} u_\alpha c_\beta a_x a_\beta c_\alpha + \tfrac{1}{2} c_\alpha c_\beta a_\beta a_x u_\alpha \equiv c_\alpha c_\beta a_\beta a_x u_\alpha \equiv (611)_2.$$

$3 = (a'_\beta u_\gamma - a'_\gamma u_\beta)\, a_\beta a_\gamma a'_x$ and $a'_\beta u_\gamma a_\beta a_\gamma a'_x \equiv u_\gamma a_\beta a'_x (aa' \,.\, \gamma\beta) \equiv -\tfrac{1}{2} (\alpha\beta\gamma)(\alpha\beta x)\, u_\gamma.$

$4 \equiv b_\gamma u_\beta a_\beta a_\gamma b_x.$

$5 \equiv a'_x u_\gamma a_\beta (aa' \,.\, \gamma\beta) \equiv -\frac{1}{2} (a\beta\gamma)(a\beta x) u_\gamma.$

$7 \equiv (\alpha\beta\gamma)^2.$ For $= \frac{1}{2} (a_\beta{}^2 \,.\, a'_\gamma{}^2 + a_\gamma{}^2 a'_\beta{}^2)^2 - \frac{1}{2} (aa' \,.\, \beta\gamma)^2.$

$8 = (a'_\beta u_\gamma - a'_\gamma u_\beta) a_\beta a_x a'_x u_\gamma \equiv - a_x a_\beta u_\beta \,.\, a'_\gamma a'_x u_\gamma.$

$9 \equiv b_\gamma u_\beta u_\gamma a_\beta a_x b_x \equiv a_\beta a_x u_\beta \,.\, b_\gamma b_x u_\gamma.$

$10 \equiv c_\beta a_\beta c_x a_x \,.\, u_\gamma{}^2.$

$11 \equiv \frac{1}{2} (\beta\gamma x) u_\alpha (\alpha\beta x) u_\gamma \equiv -\frac{1}{2} u_x (\alpha\beta\gamma)(\gamma\alpha x) u_\beta.$

$$12 = \begin{vmatrix} a_\beta & a_\gamma & a_x \\ b_\beta & b_\gamma & b_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} a_\beta b_x u_\gamma \equiv u_\beta (a_\gamma b_x - a_x b_\gamma) a_\beta b_x u_\gamma \equiv b_x{}^2 \,.\, u_\beta u_\gamma a_\beta a_\gamma - a_x a_\beta u_\beta \,.\, b_\gamma b_x u_\gamma.$$

$$13 = \begin{vmatrix} a_\beta & a_\gamma & a_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} a_\beta c_x u_\gamma \equiv - c_\beta (a_\gamma u_x - a_x u_\gamma) a_\beta c_x u_\gamma \equiv - u_x \,.\, c_\beta a_\beta c_x a_\gamma u_\gamma + u_\gamma{}^2 \,.\, a_x c_x a_\beta c_\beta.$$

$14 \equiv (\beta\gamma x) a_\beta a'_x (aa' \,.\, x\gamma) \equiv \frac{1}{2} (\beta\gamma x)(\gamma\alpha x)(\alpha\beta x).$

$15 = (603)_2.$

$16 = (a'_\beta u_\gamma - a'_\gamma u_\beta)(aa'u) a_\beta u_\gamma \equiv -\frac{1}{2} u_\alpha u_\beta u_\gamma (\alpha\beta\gamma).$

$17 \equiv (abu) b_\gamma u_\beta a_\beta u_\gamma.$

$18 \equiv c_\beta a_\beta (acu) \,.\, u_\gamma{}^2.$

$19 \equiv a'_\beta u_\gamma a_\beta a_x a'_\gamma \equiv a'_\beta u_\gamma a_x (aa' \,.\, \beta\gamma) \equiv \frac{1}{2} (\alpha\beta\gamma)(\alpha x\beta) u_\gamma.$

$20 \equiv b_\gamma{}^2 \,.\, u_\beta a_\beta a_x.$

$21 = \frac{1}{2} (\beta\gamma x)(\alpha\beta\gamma) u_\alpha.$

$$22 = \begin{vmatrix} a_\beta & a_\gamma & a_x \\ b_\beta & b_\gamma & b_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} a_\beta b_\gamma \equiv u_\beta a_\gamma b_x a_\beta b_\gamma.$$

$23 \equiv (abc) a'_\beta u_\beta (bau) c_x a'_x \equiv (abc)(bau) c_x \,.\, a'_\beta u_\beta a'_x.$

$24 = (abc) c_x b'_x u_\beta \{(b'ua) b_\beta + (uba) b'_\beta + (bb'a) u_\beta\}.$

$25 \equiv (abc)(bau) c'_\beta u_\beta c_x c'_x \equiv (abc)(bau) c_x \,.\, c'_\beta u_\beta c'_x.$

$26 \equiv (abc) u_\beta a'_x \{(a'ua) c_\beta + (uca) a'_\beta\} \equiv \frac{1}{2} u_\alpha u_\beta b_x c_\beta (c_\alpha b_x - c_x b_\alpha) \equiv -\frac{1}{2} b_\alpha b_x u_\alpha \,.\, c_\beta c_x u_\beta.$

$27 \equiv (abu) a_\beta c_\beta b_x (cb'u) b'_x = \{(ab'u) b_x + \underbrace{(abb') u_x + (b'bu) a_x\} a_\beta c_\beta} b_x (cb'u) \equiv (b'bc) b_x \,.\, (cb'u) \,.\, a_x a_\beta u_\beta \equiv 0.$

$28 = \frac{1}{2} u_\gamma a_\beta u_\beta b_x (b_\gamma a_x - b_x a_\gamma) \equiv \frac{1}{2} u_\gamma b_\gamma b_x \,.\, u_\beta a_\beta a_x.$

$29 \equiv (abc) a'_\beta b_x c_x (aa' \,.\, \beta x) \equiv \frac{1}{2} (\alpha\beta x)(c_\alpha b_\beta - c_\beta b_\alpha) b_x c_x \equiv -\frac{1}{2} (\alpha\beta x) c_\beta b_\alpha b_x c_x.$

$30 \equiv (abc) a_\beta c'_x b_x (cc' \,.\, x\beta) \equiv \frac{1}{2} (\beta\gamma x) \alpha_\beta \beta_x (b_\gamma a_x - b_x a_\gamma) \equiv \frac{1}{2} (\beta\gamma x) a_\beta b_\gamma a_x b_x.$

$31 \equiv (abc) u_\beta a'_\beta (ba'a) c_\alpha \equiv -\frac{1}{2} b_\alpha u_\beta c_\alpha (c_\alpha b_\beta - c_\beta b_\alpha) \equiv 0.$

$32 \equiv (abc)(abc') c_x c'_\beta u_\beta \equiv (c'bc)(abc') c_x a_\beta u_\beta + (ab'c)^2 \,.\, c_x c_\beta u_\beta \equiv \frac{1}{2} b_\gamma a_\beta u_\beta (a_\gamma b_x - a_x b_\gamma) \equiv \frac{1}{2} a_\beta a_\gamma u_\beta b_\gamma b_x.$

$33 \equiv (aba') a_\beta c_\beta b_x (ca'u) \equiv -\frac{1}{2} b_\alpha c_\beta b_x (u_\alpha c_\beta - u_\beta c_\alpha) \equiv \frac{1}{2} c_\alpha c_\beta u_\beta b_x b_\alpha.$

$34 \equiv \frac{1}{2} b_x u_\gamma (b_\gamma a_\beta - b_\beta a_\gamma) a_\beta \equiv 0.$

$35 \equiv (abu)\, a_\beta c_\beta\, (ba'u)(ca'u) \equiv (abu)(ca'u)\, c_\beta \{(bau)\, a'_\beta + (ba'u)\, u_\beta\} \equiv -\tfrac{1}{2}\, c_\beta u_\beta b_\alpha\, (bcu)\, u_\alpha.$

$36 = \tfrac{1}{2}\, u_\beta a_\beta u_{\beta'} c_{\beta'}\, (uca) \equiv \tfrac{1}{2}\, (au\,.\,\beta\beta')\, u_\beta c_{\beta'}\, (uca) \equiv 0.$

$37 \equiv \tfrac{1}{2}\, u_\gamma a_\beta u_\beta b_\gamma\, (aub).$

$38 = (a'_\beta u_\gamma - a'_\gamma u_\beta)\, c_x c_\beta u_\gamma a'_x \equiv u_\gamma^2\,.\, a_\beta u_x c_x c_\beta - c_x c_\beta u_\beta\,.\, a_\gamma a_x u_\gamma.$

$39 \equiv b_x b_\gamma u_\gamma\,.\, c_\beta c_x u_\beta.$

$40 \equiv u_\gamma^2\,.\, c'_\beta c'_x c_x c_\beta.$

$$41 = \begin{vmatrix} c_\beta & c_\gamma & c_x \\ a_\beta & a_\gamma & a_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} c_\beta u_\gamma a_x \equiv c_x\,(a_\beta u_\gamma - a_\gamma u_\beta)\, c_\beta u_\gamma a_x \equiv -\, u_\beta c_\beta c_x\,.\, u_\gamma a_\gamma a_x.$$

$$42 = \begin{vmatrix} c_\beta & c_\gamma & c_x \\ b_\beta & b_\gamma & b_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} c_\beta u_\gamma b_x \equiv u_\beta\,(c_\gamma b_x - c_x b_\gamma)\, c_\beta u_\gamma b_x \equiv -\, c_x c_\beta u_\beta\,.\, b_x b_\gamma u_\gamma.$$

$$43 = \begin{vmatrix} c_\beta & c_\gamma & c_x \\ c'_\beta & c'_\gamma & c'_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} c_\beta c'_x u_\gamma \equiv c_x\,(c'_\beta u_\gamma - c'_\gamma u_\beta)\, c'_x u_\gamma \equiv 0.$$

$46 = (a_\beta u_\gamma - a_\gamma u_\beta)\, c_x c_\beta a_\gamma \equiv a_\beta a_\gamma u_\gamma c_x c_\beta.$

$48 \equiv b_\gamma u_\beta c_x c_\beta b_\gamma \equiv 0.$

$$48 = \begin{vmatrix} c_\beta & c_\gamma & c_x \\ a_\beta & a_\gamma & a_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} c_\beta a_\gamma \equiv c_x\,(a_\beta u_\gamma - a_\gamma u_\beta)\, c_\beta a_\gamma \equiv a_\beta a_\gamma c_\beta c_x u_\gamma.$$

$$49 = \begin{vmatrix} c_\beta & c_\gamma & c_x \\ b_\beta & b_\gamma & b_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} c_\beta b_\gamma \equiv u_\beta\,(c_\gamma b_x - c_x b_\gamma)\, c_\beta b_\gamma \equiv 0.$$

$50 = (a_\beta u_\gamma - a_\gamma u_\beta)\,(cau)\, c_\beta u_\gamma \equiv -\,(cau)\, u_\beta u_\gamma a_\gamma c_\beta.$

$51 \equiv b_\gamma u_\beta c_\beta u_\gamma\, (cbu).$

$52 \equiv c'_\beta c_\beta\,.\, u_\gamma^2\, (cc'u) \equiv 0.$

Thus justifying the system of the sixth degree.

Seventh degree.

We proceed now to shew that from

$(611)_1 = (\alpha\beta\gamma)\,(\beta\gamma x)\, u_\alpha,$

$(611)_2 = a_\beta a_\gamma b_\gamma b_x u_\beta,$

$(630)_1 = (\alpha\beta\gamma)\, u_\alpha u_\beta u_\gamma,$

$(630)_2 = (abu)\, a_\beta b_\gamma u_\beta u_\gamma,$

$(630)_3 = (bcu)\, u_\beta u_\gamma b_\gamma c_\beta,$

$(603)_1 = (\beta\gamma x)\,(\gamma\alpha x)\,(\alpha\beta x),$

$(603)_2 = (\beta\gamma x)\, a_x b_x a_\beta b_\gamma,$

$(603)_3 = (\beta\gamma x)\, b_x c_x b_\gamma c_\beta,$

we obtain

$(710)_1 = (\alpha\beta\gamma)\, a_\beta u_\gamma u_\alpha.$

$(710)_2 = (bcu)\, b_\beta a_\gamma b_\gamma c_\beta.$

$(721) \;= (\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma.$

From $(611)_1 = (\alpha\beta\gamma)(\beta\gamma x) u_\alpha$.

1. $(\alpha\beta\gamma)(\beta\gamma \cdot au) u_\alpha a_x \equiv (721)$.
2. $(\alpha\beta\gamma)(\beta\gamma \cdot bu) u_\alpha b_x \equiv (721)$.
3. $(\alpha\beta\gamma)(\beta\gamma x) b_\alpha b_x \equiv 0$.
4. $(\alpha\beta\gamma)(\beta\gamma \cdot bu) b_\alpha \equiv (710)_1$.

From $a_\beta a_\gamma b_\gamma b_x u_\beta = (611)_2$.

5. $a_\beta a_\gamma b_\gamma (ba'u) a_x' u_\beta \equiv 0$.
6. $a_\beta a_\gamma b_\gamma (bb'u) b'_x u_\beta \equiv 0$.
7. $a_\beta a_\gamma b_\gamma (bcu) c_x u_\beta \equiv 0$.
8. $a_\beta a_\gamma b_\gamma b_x a'_\beta a'_x \equiv 0$.
9. $a_\beta a_\gamma b_\gamma b_x c_\beta c_x \equiv 0$.
10. $a_\beta a_\gamma b_\gamma (ba'u) a'_\beta \equiv (710)_1$.
11. $a_\beta a_\gamma b_\gamma (bcu) c_\beta = (710)_2$.

From $(630)_1 = (\alpha\beta\gamma) u_\alpha u_\beta u_\gamma$.

12. $(\alpha\beta\gamma) b_\alpha b_x u_\beta u_\gamma = (721)$.
13. $(\alpha\beta\gamma) u_\alpha a_\beta a_\gamma = (710)_1$.

From $(abu) a_\beta b_\gamma u_\beta u_\gamma = (630)_2$.

14. $(aba') a_\beta b_\gamma u_\beta u_\gamma a_x' \equiv (721)$.
15. $(abb') a_\beta b_\gamma u_\beta u_\gamma b'_x \equiv 0$.
16. $(abc) a_\beta b_\gamma u_\beta u_\gamma c_x \equiv 0$.
17. $(abu) a_\beta b_\gamma a'_\beta u_\gamma a'_x \equiv (721)$.
18. $(abu) a_\beta b_\gamma c_\beta u_\gamma c_x \equiv 0$.
19. $(abu) a_\beta b_\gamma u_\beta a'_\gamma a'_x \equiv (710)_1$ and (721).
20. $(abu) a_\beta b_\gamma u_\beta b'_\gamma b'_x \equiv 0$.
21. $(aba') a_\beta b_\gamma a'_\beta u_\gamma$ vanishes.
22. $(abc) a_\beta b_\gamma c_\beta u_\gamma \equiv (710)_2$.
23. $(aba') a_\beta b_\gamma u_\beta a'_\gamma \equiv (710)_1$.
24. $(abb') a_\beta b_\gamma u_\beta b'_\gamma$ vanishes.
25. $(abu) a_\beta b_\gamma a'_\beta a'_\gamma \equiv (710)_1$.

From $(630)_3 = (bcu) u_\beta u_\gamma b_\gamma c_\beta$.

26. $(bca) a_x u_\beta u_\gamma b_\gamma c_\beta \equiv (710)_2$.
27. $(bcb') b'_x u_\beta u_\gamma b_\gamma c_\beta \equiv 0$.
28. $(bcu) a_\beta u_\gamma a_x b_\gamma c_\beta \equiv (710)_2$.
29. $(bcc') c'_\beta u_\gamma c'_x b_\gamma c_\beta \equiv 0$.
30. $(bca) a_\beta u_\gamma b_\gamma c_\beta \equiv (710)_2$.
31. $(bcc') c'_\beta u_\gamma b_\gamma c_\beta \equiv 0$.
32. $(bcu) a_\beta a_\gamma b_\gamma c_\beta = (710)_2$.

From $(\beta\gamma x)(\gamma\alpha x)(\alpha\beta x) = (603)_1$.

33. $(\beta\gamma \cdot au)(\gamma\alpha x)(\alpha\beta x) a_x \equiv 0$.
34. $(\beta\gamma \cdot bu)(\gamma\alpha x)(\alpha\beta x) b_x \equiv 0$.
35. $(\beta\gamma x)(\gamma\alpha \cdot au)(\alpha\beta \cdot au) \equiv 0$.
36. $(\beta\gamma x)(\gamma\alpha \cdot bu)(\alpha\beta \cdot bu) \equiv (721)$.

From $(603)_2 = (\beta\gamma x) a_x b_x a_\beta b_\gamma$.

37. $(\beta\gamma \cdot a'u) a_x' a_x b_x a_\beta b_\gamma \equiv 0$.
38. $(\beta\gamma \cdot b'u) b'_x a_x b_x a_\beta b_\gamma \equiv 0$.
39. $(\beta\gamma \cdot cu) c_x a_x b_x a_\beta b_\gamma \equiv 0$.
40. $(\beta\gamma x)(aa'u) b_x a'_x a_\beta b_\gamma \equiv 0$.
41. $(\beta\gamma x)(ab'u) b_x b'_x a_\beta b_\gamma \equiv 0$.
42. $(\beta\gamma x)(acu) b_x c_x a_\beta b_\gamma \equiv 0$.
43. $(\beta\gamma x) a_x (ba'u) a'_x a_\beta b_\gamma \equiv 0$.
44. $(\beta\gamma x) a_x (bb'u) b'_x a_\beta b_\gamma \equiv 0$.
45. $(\beta\gamma x) a_x (bcu) c_x a_\beta b_\gamma \equiv 0$.
46. $(\beta\gamma \cdot a'u)(aa'u) b_x a_\beta b_\gamma \equiv (710)_1$ and (721).
47. $(\beta\gamma \cdot b'u)(ab'u) b_x a_\beta b_\gamma \equiv 0$.
48. $(\beta\gamma \cdot cu)(acu) b_x a_\beta b_\gamma \equiv 0$.
49. $(\beta\gamma \cdot a'u) a_x (ba'u) a_\beta b_\gamma \equiv (721)$.
50. $(\beta\gamma \cdot b'u) a_x (bb'u) a_\beta b_\gamma \equiv 0$.
51. $(\beta\gamma \cdot cu) a_x (bcu) a_\beta b_\gamma \equiv (710)_2$.
52. $(\beta\gamma x)(aa'u)(ba'u) a_\beta b_\gamma \equiv (721)$.
53. $(\beta\gamma x)(ab'u)(bb'u) a_\beta b_\gamma \equiv 0$.
54. $(\beta\gamma x)(acu)(bcu) a_\beta b_\gamma \equiv 0$.

From $(\beta\gamma x)\, b_x c_x b_\gamma c_\beta = (603)_3$.

55. $(\beta\gamma \,.\, au)\, a_x b_x c_x b_\gamma c_\beta \equiv 0$.

56. $(\beta\gamma \,.\, b'u)\, b'_x b_x c_x b_\gamma c_\beta \equiv 0$.

57. $(\beta\gamma x)\,(bau)\, a_x c_x b_\gamma c_\beta \equiv 0$.

58. $(\beta\gamma x)\,(bb'u)\, c_x b'_x b_\gamma c_\beta \equiv 0$.

59. $(\beta\gamma x)\,(bc'u)\, c_x c'_x b_\gamma c_\beta \equiv 0$.

60. $(\beta\gamma \,.\, au)\,(bau)\, c_x b_\gamma c_\beta \equiv 0$.

61. $(\beta\gamma \,.\, b'u)\,(bb'u)\, c_x b_\gamma c_\beta \equiv 0$.

62. $(\beta\gamma \,.\, c'u)\,(bc'u)\, c_x b_\gamma c_\beta \equiv 0$.

63. $(\beta\gamma x)\,(bau)\,(cau)\, b_\gamma c_\beta \equiv 0$.

64. $(\beta\gamma x)\,(bb'u)\,(cb'u)\, b_\gamma c_\beta \equiv 0$.

Of these

$1 = (\alpha\beta\gamma)\,(a_\beta u_\gamma - a_\gamma u_\beta)\, u_\alpha a_x$.

$2 \equiv (\alpha\beta\gamma)\, u_\alpha u_\beta b_\gamma b_x$.

$3 \equiv -(\gamma\alpha x)\,(\alpha\beta x)\, b_\beta b_\gamma \equiv 0$.

$4 \equiv (\alpha\beta\gamma)\, u_\beta b_\gamma b_\alpha$.

$5 \equiv a_\beta' a_x' u_\beta \,.\, a_\gamma b_\gamma\, (bau)$.

$6 \equiv u_\beta{}^2 \,.\, a_\gamma b_\gamma b_x'\, (bb'a)$.

$7 \equiv c_\beta a_\gamma b_\gamma\, (bau)\, c_x u_\beta \equiv a_\gamma b_\gamma\, (bau) \,.\, c_x c_\beta u_\beta$.

$8 \equiv a_\beta\, (aa' \,.\, \gamma\beta)\, b_\gamma b_x a_x' \equiv -\frac{1}{2}\,(\alpha\beta\gamma)\,(\alpha\beta x)\, b_\gamma b_x \equiv \frac{1}{2}\,(\beta\gamma x)\,(\gamma\alpha x)\, b_\alpha b_\beta \equiv 0$.

9. For this consider $(abc)^2 \,.\, (\beta\gamma x)^2 = (a_\beta b_\gamma c_x - a_\beta b_x c_\gamma + a_\gamma b_x c_\beta - a_\gamma b_\beta c_x + a_x b_\beta c_\gamma - a_x b_\gamma c_\beta)^2$

$\equiv (a_\beta b_\gamma c_x + a_\gamma b_x c_\beta - a_x b_\gamma c_\beta)^2 \equiv 2 a_\beta a_\gamma b_\gamma c_\beta b_x c_x$.

$10 \equiv u_\beta a_\gamma b_\gamma\, (ba'a)\, a_\beta' \equiv -\frac{1}{2} u_\beta b_\gamma b_\alpha\, (\alpha\gamma\beta)$.

$14 = -\frac{1}{2}\,(\alpha\beta x)\, b_\gamma b_\alpha u_\beta u_\gamma \equiv -\frac{1}{2}\,(\alpha\beta\gamma)\, b_x b_\alpha u_\beta u_\gamma$.

$15 \equiv (ubb')\, a_\beta b_\gamma a_\beta u_\gamma b_x' \equiv 0$.

$16 \equiv (ubc)\, a_\beta b_\gamma u_\beta a_\gamma c_x \equiv (uba)\, c_\beta b_\gamma u_\beta a_\gamma c_x = (uba)\, b_\gamma a_\gamma \,.\, c_\beta u_\beta c_x$.

$17 \equiv (aba')\, a_\beta b_\gamma u_\beta u_\gamma a_x' \equiv -\frac{1}{2}\,(\alpha\beta x)\, b_\gamma b_\alpha u_\beta u_\gamma \equiv -\frac{1}{2}\,(\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma$.

$18 \equiv (abc)\, a_\beta b_\gamma u_\beta u_\gamma c_x = 16 \equiv 0$.

$19 = (abu)\, a_\gamma b_\gamma \,.\, u_\beta a_\beta' a_x' + \frac{1}{2}\,(\alpha\beta\gamma)\, b_\gamma u_\beta\, (u_\alpha b_x - u_x b_\alpha)$.

$20 \equiv (abu)\, a_\beta u_\beta b_\gamma'\, (bb' \,.\, \gamma x) = \frac{1}{2}\,(a_{\beta'} u_\gamma - a_\gamma u_{\beta'})\,(\beta'\gamma x)\, a_\beta u_\beta \equiv (\beta\gamma x)\, u_\beta u_\gamma \,.\, a_{\beta'}{}^2 + \overset{\frown}{\beta\beta'}$

$- (\beta\gamma x)\, a_\beta a_\gamma \,.\, u_{\beta'}{}^2 + \overset{\frown}{\beta\beta'} \equiv 0$.

$22 \equiv (bcu)\, a_\beta a_\gamma b_\gamma c_\beta$.

$23 = -\frac{1}{2}\,(\alpha\beta\gamma)\, b_\gamma b_\alpha u_\beta$.

$25 \equiv (aba')\, a_\beta b_\gamma u_\beta a_\gamma'$ or 23.

$26 = (abc)\, u_\beta u_\gamma a_x b_\gamma c_\beta \equiv (ubc)\, a_\beta u_\gamma a_x b_\gamma c_\beta \equiv u_x\, (abc)\, a_\beta u_\gamma b_\gamma c_\beta + (uca)\, c_\beta a_\beta \,.\, b_x b_\gamma u_\gamma + (uba)\, a_\beta u_\gamma c_x b_\gamma c_\beta$

$\equiv u_x \,.\, (ubc)\, \alpha_\beta a_\gamma b_\gamma c_\beta + (cba)\, u_\beta u_\gamma a_\beta c_x b_\gamma \equiv u_x \,.\, (ubc)\, a_\beta a_\gamma b_\gamma c_\beta + (cbu)\, u_\beta a_\gamma a_\beta c_x b_\gamma$

$\equiv u_x \,.\, (ubc)\, a_\beta a_\gamma b_\gamma c_\beta + (abu)\, a_\gamma b_\gamma \,.\, c_\beta u_\beta c_x \equiv u_x \,.\, (ubc)\, a_\beta a_\gamma b_\gamma c_\beta$,

a reduction not at all obvious.

$27 \equiv (bub')\, b_x' c_\beta u_\gamma b_\gamma c_\beta \equiv 0.$

$28 \equiv 26.$

$29 \equiv (ucc')\, c_\beta' \underbrace{b_\gamma c_x'} b_\gamma c_\beta.$

$30 \equiv (710)_2.$

$31 \equiv b_\gamma^2 \,.\, (ucc')\, c_\beta c_\beta'.$

$33 = (a_\beta u_\gamma - a_\gamma u_\beta)\, (\gamma\alpha x)\, (\alpha\beta x)\, a_x \equiv (\beta\alpha x)\, (\alpha\beta x)\, a_\gamma u_\gamma a_x - (\gamma\alpha x)\, (\alpha\gamma x)\, a_\beta u_\beta a_x \equiv 0.$

$$34 \equiv (\gamma\alpha x)\, (\alpha\beta x)\, b_x b_\gamma u_\beta \equiv (\gamma\alpha x)\, (\gamma\beta x)\, b_x b_\alpha u_\beta \equiv (\gamma\beta x)\, b_\alpha b_x \{(\beta\alpha x)\, u_\gamma + (\gamma\alpha\beta)\, u_x\}$$
$$\equiv -(\alpha\beta\gamma)\, (\beta\gamma\alpha)\, b_\alpha b_x \,.\, u_x + (\alpha\beta x)\, u_\gamma b_x\, (\beta\gamma\alpha)\, b_x \equiv (\gamma\alpha x)\, (\alpha\beta x)\, b_\beta b_\gamma \,.\, u_x.$$

$35 \equiv u_\alpha^2 \,.\, (\beta\gamma x)\, a_\beta a_\gamma.$

$36 \equiv (\beta\gamma x)\, (b_\gamma u_\alpha - b_\alpha u_\gamma)\, b_\alpha u_\beta \equiv (\beta\gamma x)\, b_\gamma b_\alpha u_\alpha u_\beta \equiv (\beta\gamma\alpha)\, b_\gamma b_x u_\alpha u_\beta.$

$37 = (a_\beta' u_\gamma - a_\gamma' u_\beta)\, a_x a_x' b_x a_\beta b_\gamma = u_\gamma b_\gamma b_x \,.\, a_\beta' a_x' a_\beta a_x - u_\beta a_\beta a_x \,.\, a_\gamma' a_x' b_\gamma b_x.$

$38 \equiv b_\gamma' u_\beta b_x' a_x b_x a_\beta b_\gamma$ or $b_x b_\gamma b_\gamma' b_x' \,.\, u_\beta a_\beta a_x.$

$39 \equiv c_\beta u_\gamma c_x a_x b_x a_\beta b_\gamma$ or $c_\beta c_x a_\beta a_x \,.\, u_\gamma b_\gamma b_x.$

$40 = \tfrac{1}{2}\, (\beta\gamma x)\, (\alpha\beta x)\, u_\alpha b_x b_\gamma \equiv \tfrac{1}{2}\, (\beta\gamma x)\, (\gamma\beta x) \,.\, u_\alpha b_\alpha b_x.$

$$41 = \begin{vmatrix} a_\beta & a_\gamma & a_x \\ b_\beta' & b_\gamma' & b_x' \\ u_\beta & u_\gamma & u_x \end{vmatrix} b_x b_x' a_\beta b_\gamma \equiv u_\beta\, (a_\gamma b_x' - a_x b_\gamma')\, b_x b_x' a_\beta b_\gamma \equiv -a_x a_\beta u_\beta \,.\, b_\gamma b_\gamma' b_x b_x'.$$

$$42 = \begin{vmatrix} a_\beta & a_\gamma & a_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} b_x c_x a_\beta b_\gamma \equiv -c_\beta\, (a_\gamma u_x - a_x u_\gamma)\, b_x c_x a_\beta b_\gamma \equiv -a_\beta a_\gamma b_\gamma c_\beta b_x c_x + a_\beta c_\beta a_x c_x \,.\, u_\gamma b_\gamma b_x \equiv 0$$
(see 9).

$$43 = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ a_\beta' & a_\gamma' & a_x' \\ u_\beta & u_\gamma & u_x \end{vmatrix} a_x a_x' a_\beta b_\gamma \equiv b_x\, (a_\beta' u_\gamma - a_\gamma' u_\beta)\, a_x a_x' a_\beta b_\gamma \equiv b_x b_\gamma u_\gamma \,.\, a_\beta' a_x' a_x a_\beta - u_\beta a_\beta a_x \,.\, a_\gamma' a_x' b_\gamma b_x.$$

$44 \equiv u_\beta\, (b_\gamma b_x' - b_\gamma' b_x)\, a_x b_x' a_\beta b_\gamma \equiv -u_\beta a_\beta a_x \,.\, (b_\gamma b_x' - b_\gamma' b_x)\, b_\gamma b_x'.$

$$45 = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} a_x c_x a_\beta b_\gamma \equiv b_x\, (c_\beta u_\gamma - c_\gamma u_\beta)\, a_x c_x a_\beta b_\gamma \equiv b_x b_\gamma u_\gamma \,.\, c_\beta a_\beta c_x a_x.$$

$$46 = (a_\beta' u_\gamma - a_\gamma' u_\beta)\, (aa'u)\, b_x a_\beta b_\gamma = -\tfrac{1}{2}\, (\alpha\beta\gamma)\, u_\alpha u_\beta b_x b_\gamma \equiv -\tfrac{1}{2}\, (x\beta\gamma)\, u_\alpha u_\beta b_\alpha b_\gamma$$
$$\equiv -\tfrac{1}{2}\, \{(\alpha\beta\gamma)\, u_\beta b_\gamma b_\alpha \,.\, u_x + (x\beta\alpha)\, u_\beta u_\gamma b_\alpha b_\gamma\}$$
$$\equiv -\tfrac{1}{2}\, (\alpha\beta\gamma)\, u_\beta b_\gamma b_\alpha \,.\, u_x + \tfrac{1}{2}\, (\alpha\beta\gamma)\, u_\beta u_\gamma b_\alpha b_x \equiv (710)_1 \text{ and } (721).$$

$47 \equiv b_\gamma' u_\beta\, (ab'u)\, b_x a_\beta b_\gamma \equiv (bb' \,.\, x\gamma)\, (ab'u)\, b_\gamma a_\beta u_\beta \equiv \tfrac{1}{2}\, (\beta x\gamma)\, (u_\beta a_\gamma - a_\beta u_\gamma)\, a_\beta u_\beta.$

$48 \equiv c_\beta u_\gamma\, (acu)\, b_x a_\beta b_\gamma$ or $(acu)\, a_\beta c_\beta \,.\, u_\gamma b_\gamma b_x.$

$$49 = (a_\beta' u_\gamma - a_\gamma' u_\beta)\, (ba'u)\, a_x a_\beta b_\gamma \equiv (ba'a)\, u_\beta a_\beta' u_\gamma a_x b_\gamma - u_\beta a_\beta a_x \,.\, a_\gamma' b_\gamma\, (ba'u)$$
$$\equiv -\tfrac{1}{2}\, b_\alpha\, (\alpha x\beta)\, u_\beta u_\gamma b_\gamma \equiv -\tfrac{1}{2}\, (\alpha\gamma\beta)\, b_\alpha b_x u_\beta u_\gamma.$$

$50 \equiv b_\gamma' u_\beta a_x\, (bb'u)\, a_\beta b_\gamma \equiv b_\gamma' u_\beta a_x\, (bb'a)\, u_\beta b_\gamma.$

$51 \equiv c_\beta u_\gamma\, (bcu)\, a_x a_\beta b_\gamma \equiv c_\beta u_\gamma\, (bca)\, a_x u_\beta b_\gamma$ or $26.$

$52 = \tfrac{1}{2} u_\alpha\, (\beta\gamma x)\, (u_\alpha b_\beta - u_\beta b_\alpha)\, b_\gamma \equiv -\tfrac{1}{2}\, (\beta\gamma x)\, u_\alpha u_\beta b_\gamma b_\alpha \equiv -\tfrac{1}{2}\, (\alpha\beta\gamma)\, u_\alpha u_\beta b_\gamma b_x.$

$53 = \tfrac{1}{2} u_\beta\, (\beta\gamma x)\, (u_\beta a_\gamma - u_\gamma a_\beta)\, a_\beta.$

$$54 = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} (acu)\, a_\beta b_\gamma \equiv b_x (c_\beta u_\gamma - c_\gamma u_\beta)(acu)\, a_\beta b_\gamma \equiv (acu)\, c_\beta a_\beta \,.\, b_x b_\gamma u_\gamma - (acu)\, a_\beta c_\gamma u_\beta b_x b_\gamma .$$

$$55 = (a_\beta u_\gamma - a_\gamma u_\beta)\, a_x b_x c_x b_\gamma c_\beta = c_\beta a_\beta c_x a_x \,.\, b_x b_\gamma u_\gamma - a_\gamma b_\gamma a_x b_x \,.\, u_\beta c_x c_\beta .$$

$$56 \equiv b_\gamma' u_\beta b_x' b_x c_x b_\gamma c_\beta = c_x c_\beta u_\beta \,.\, b_\gamma b_\gamma' b_x b_x' .$$

$$57 = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ a_\beta & a_\gamma & a_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} a_x c_x b_\gamma c_\beta \equiv b_x (a_\beta u_\gamma - a_\gamma u_\beta)\, a_x c_x b_\gamma c_\beta \equiv a_x a_\beta c_\beta c_x \,.\, b_x b_\gamma u_\gamma - a_\gamma b_\gamma a_x b_x \,.\, u_\beta c_\beta c_x .$$

$$58 = \tfrac{1}{2} u_{\beta'} (\beta\gamma x)(\beta'\underbrace{\gamma x)\, c_x}c_\beta \equiv 0.$$

$$59 = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta' & c_\gamma' & c_x' \\ u_\beta & u_\gamma & u_x \end{vmatrix} c_x c_x' b_\gamma c_\beta \equiv b_x (c_\beta' u_\gamma - c_\gamma' u_\beta)\, c_x c_x' b_\gamma c_\beta \equiv b_x b_\gamma u_\gamma \,.\, c_\beta' c_x' c_\beta c_x .$$

$$60 = (a_\beta u_\gamma - a_\gamma u_\beta)(bau)\, c_x b_\gamma c_\beta = (bau)\, c_x b_\gamma c_\beta a_\beta u_\gamma - u_\beta c_\beta c_x \,.\, (bau)\, b_\gamma a_\gamma \equiv (bac)\, c_x b_\gamma u_\beta a_\beta u_\gamma$$
$$\equiv (buc)\, c_x b_\gamma u_\beta a_\beta a_\gamma \equiv (bua)\, c_x b_\gamma u_\beta c_\beta a_\gamma \equiv (bua)\, b_\gamma a_\gamma \,.\, c_x u_\beta c_\beta .$$

$61 \equiv b_\gamma' u_\beta (bb'u)\, c_x b_\gamma c_\beta \equiv b_\gamma' u_\beta (bb'c)\, c_x b_\gamma u_\beta$, which vanishes.

$$62 \equiv c_\beta' u_\gamma (bc'u)\, c_x b_\gamma c_\beta \equiv c_\beta' u_\gamma (bc'c)\, c_x b_\gamma u_\beta \equiv b_\gamma^2 \,.\, (uc'c)\, c_\beta' c_x b_\gamma u_\beta .$$

$$63 = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ a_\beta & a_\gamma & a_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} (cau)\, b_\gamma c_\beta \equiv b_x (a_\beta u_\gamma - a_\gamma u_\beta)(cau)\, b_\gamma c_\beta \equiv b_\gamma b_x u_\gamma (cau)\, c_\beta a_\beta - (cau)\, c_\beta b_\gamma a_\gamma u_\beta b_x$$
$$\equiv -(abc)\, u_\beta u_\gamma a_\gamma b_x c_\beta \equiv -(cub)\, a_\beta u_\gamma a_\gamma b_x c_\beta \equiv -(cua)\, a_\beta u_\gamma b_\gamma b_x c_\beta$$
$$\equiv u_\gamma b_\gamma b_x \,.\, c_\beta a_\beta (cau) \equiv 0.$$

$$64 \equiv (\beta\gamma x)(bb'c)(cb'u)\, b_\gamma u_\beta \equiv \tfrac{1}{2} c_{\beta'} (\beta\gamma x)(u_{\beta'} c_\gamma - c_{\beta'} u_\gamma)\, u_\beta \equiv 0.$$

The seventh degree is therefore established.

Eighth, ninth and tenth degree. End of the system.

We proceed now to shew that from

$(710)_1 = (\alpha\beta\gamma)\, a_\beta a_\gamma u_\alpha$,

$(710)_2 = (bcu)\, a_\beta a_\gamma b_\gamma c_\beta$,

$(721) = (\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma$,

and thence

$(911) = a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x u_\beta$,

we obtain

$(801)_1 = (\beta\gamma x)\, b_\gamma c_\beta b_\alpha c_\alpha$,

$(801)_2 = (a'bc)\, a_\beta a_\gamma b_\gamma c_\beta a_x'$,

$(812) = (\alpha'\beta\gamma)(\gamma\alpha x)(\alpha\beta x)\, u_\alpha$

and thence

$(10\,.\,1\,.\,0) = b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} (\alpha'\beta\gamma)$,

and that this is the end of the system.

From $(710)_1 = (\alpha\beta\gamma)\, a_\beta a_\gamma u_\alpha$.

1. $(\alpha\beta\gamma)\, a_\beta a_\gamma b_\alpha b_x \equiv (801)_1$.

From $(710)_2 = (bcu)\, a_\beta a_\gamma b_\gamma c_\beta$.

2. $(a'bc)\, a_x' a_\beta a_\gamma b_\gamma c_\beta = (801)_2$.

3. $(bcb')\, b_x' a_\beta a_\gamma b_\gamma c_\beta \equiv 0$.

From $(721) = (\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma$.

4. $(\alpha\beta\gamma)\, b_\alpha (bau)\, a_x u_\beta u_\gamma \equiv 0$.
5. $(\alpha\beta\gamma)\, b_\alpha (bb'u)\, b_x' u_\beta u_\gamma \equiv 0$.
6. $(\alpha\beta\gamma)\, b_\alpha (bcu)\, c_x u_\beta u_\gamma \equiv 0$.
7. $(\alpha\beta\gamma)\, b_\alpha b_x a_\beta u_\gamma a_x \equiv (801)_1$.
8. $(\alpha\beta\gamma)\, b_\alpha b_x c_\beta u_\gamma c_x \equiv 0$.
9. $(\alpha\beta\gamma)\, b_\alpha b_x u_\beta a_\gamma a_x \equiv 0$.
10. $(\alpha\beta\gamma)\, b_\alpha b_x u_\beta b_\gamma' b_x' \equiv (812)$.
11. $(\alpha\beta\gamma)\, b_\alpha (bau)\, a_\beta u_\gamma \equiv 0$.
12. $(\alpha\beta\gamma)\, b_\alpha (bcu)\, c_\beta u_\gamma \equiv 0$.
13. $(\alpha\beta\gamma)\, b_\alpha (bau)\, u_\beta a_\gamma \equiv 0$.
14. $(\alpha\beta\gamma)\, b_\alpha (bb'u)\, u_\beta b_\gamma' \equiv 0$.
15. $(\alpha\beta\gamma)\, b_\alpha b_x a_\beta a_\gamma \equiv (801)_1$.

From $(801)_1 = (\beta\gamma x)\, b_\gamma c_\beta b_\alpha c_\alpha$.

16. $(\beta\gamma\, .\, au)\, a_x b_\gamma c_\beta b_\alpha c_\alpha \equiv (911)$.
17. $(\beta\gamma\, .\, b'u)\, b_x' b_\gamma c_\beta b_\alpha c_\alpha \equiv 0$.

From $(801)_2 = (a'bc)\, a_\beta a_\gamma b_\gamma c_\beta a_x'$.

18. $(a'bc)\, a_\beta a_\gamma b_\gamma c_\beta\, (a'a''u)\, a_x'' \equiv (911)$.
19. $(a'bc)\, a_\beta a_\gamma b_\gamma c_\beta\, (a'b'u)\, b_x' \equiv 0$.

From $(812) = (\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta x)\, u_{\alpha'}$.

20. $(\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta\, .\, au)\, a_x u_{\alpha'} \equiv 0$.
21. $(\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta\, .\, bu)\, b_x u_{\alpha'} \equiv 0$.
22. $(\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta\, .\, cu)\, c_x u_{\alpha'} \equiv 0$.
23. $(\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta x)\, b_{\alpha'} b_x \equiv 0$.
24. $(\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta\, .\, bu)\, b_{\alpha'} \equiv 0$.
25. $(\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta\, .\, cu)\, c_{\alpha'} \equiv 0$.
26. $(\alpha'\beta\gamma)\, (\gamma\alpha\, .\, au)\, (\alpha\beta\, .\, au)\, u_{\alpha'} \equiv 0$.
27. $(\alpha'\beta\gamma)\, (\gamma\alpha\, .\, bu)\, (\alpha\beta\, .\, bu)\, u_{\alpha'} \equiv 0$.

From $(911) = a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x u_\beta$.

28. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x a_\beta' a_x' \equiv 0$.
29. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x c_\beta' c_x' \equiv 0$.
30. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha\, (ca'u)\, a_x' u_\beta \equiv 0$.
31. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha\, (cb'u)\, b_x' u_\beta \equiv 0$.
32. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha\, (cc'u)\, c_x' u_\beta \equiv 0$.
33. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha\, (ca'u)\, a_\beta' \equiv 0$.
34. $a_\beta a_\gamma b_\gamma b_\alpha c_\alpha\, (cc'u)\, c_\beta' \equiv (10\, .\, 1\, .\, 0)$.

From $(10\, .\, 1\, .\, 0) = (\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'}$.

35. $(\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha b_{\alpha'}' b_x'$.

Of these

$1 \equiv (\alpha\beta x)\, a_\beta a_\gamma b_\alpha b_\gamma$ or say $(\beta\gamma x)\, b_\alpha c_\alpha b_\gamma c_\beta$.

$3 \equiv (bab')\, b_x' a_\gamma b_\gamma\, .\, c_\beta^2$.

$$4 = \begin{vmatrix} b_\alpha & b_\beta & b_\gamma \\ a_\alpha & a_\beta & a_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} b_\alpha a_x u_\beta u_\gamma \equiv - b_\gamma u_\alpha a_\beta b_\alpha a_x u_\beta u_\gamma \text{ or } - u_\beta a_\beta a_x\, .\, u_\gamma b_\gamma b_\alpha u_\alpha.$$

$5 = \frac{1}{2}\, (\alpha\beta\gamma)\, (\beta'\alpha x)\, u_{\beta'} u_\beta u_\gamma \equiv \frac{1}{2}\, u_{\beta'}^2\, .\, (\alpha\beta\gamma)\, (\beta\alpha x)\, u_\gamma + \widetilde{\beta\beta'}$.

$6 = \begin{vmatrix} b_\alpha & b_\beta & b_\gamma \\ c_\alpha & c_\beta & c_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} b_\alpha c_x u_\beta u_\gamma \equiv b_\gamma (c_\alpha u_\beta - c_\beta u_\alpha) b_\alpha c_x u_\beta u_\gamma \equiv - c_\beta u_\beta c_x \,.\, u_\alpha b_\alpha b_\gamma u_\gamma.$

7. Making one cyclical change forward this becomes

$$(\alpha\beta\gamma)\, b_\gamma c_\beta u_\alpha b_x c_x \equiv (x\beta\gamma)\, b_\gamma c_\beta u_\alpha b_\alpha c_x \equiv u_x \,.\, (\alpha\beta\gamma)\, b_\gamma c_\beta b_\alpha c_x + (x\alpha\gamma)\, b_\alpha b_\gamma \,.\, u_\beta c_\beta c_x + (x\beta\alpha)\, b_\gamma c_\beta u_\gamma b_\alpha c_x$$
$$\equiv u_x \,.\, (x\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha + (\gamma\beta\alpha)\, b_x c_\beta u_\gamma b_\alpha c_x \equiv u_x \,.\, (x\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha + (\gamma\beta x)\, b_x c_\beta u_\gamma b_\alpha c_\alpha$$
$$\equiv u_x \,.\, (x\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha + (\alpha\beta x)\, c_\alpha c_\beta \,.\, b_x b_\gamma u_\gamma \equiv u_x\, (x\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha \text{ (cf. 26, p. 54).}$$

$8 \equiv (x\beta\gamma)\, b_\alpha b_x c_\beta u_\gamma c_\alpha \equiv (x\beta\alpha)\, c_\beta c_\alpha \,.\, b_\gamma u_\gamma b_x.$

$9 \equiv (\alpha\beta x)\, b_\alpha b_\gamma u_\beta a_\gamma a_x \equiv (\alpha\gamma x)\, b_\alpha b_\gamma \,.\, u_\beta a_\beta a_x.$

$10 \equiv (\alpha\beta\gamma)\, (bb' \,.\, \alpha x)\, b_x u_\beta b_\gamma' \equiv \tfrac{1}{2} (\alpha\beta\gamma)\, (\beta'\alpha x)\, (\beta' x\gamma)\, u_\beta.$

$11 = \begin{vmatrix} b_\alpha & b_\beta & b_\gamma \\ a_\alpha & a_\beta & a_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} b_\alpha a_\beta u_\gamma \equiv u_\alpha (b_\beta a_\gamma - b_\gamma a_\beta)\, b_\alpha a_\beta u_\gamma \equiv b_\beta b_\alpha a_\beta u_\gamma u_\alpha a_\gamma \equiv 0.$

$12 = \begin{vmatrix} b_\alpha & b_\beta & b_\gamma \\ c_\alpha & c_\beta & c_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} b_\alpha c_\beta u_\gamma \equiv b_\gamma (c_\alpha u_\beta - c_\beta u_\alpha)\, b_\alpha c_\beta u_\gamma \equiv u_\beta u_\gamma b_\gamma b_\alpha c_\alpha c_\beta \equiv \tfrac{1}{2} (\alpha\beta\gamma)^2 \,.\, (ubc)^2.$

For $(ubc)^2 \,.\, (\alpha\beta\gamma)^2 = (u_\alpha b_\beta c_\gamma - u_\alpha b_\gamma c_\beta + u_\beta b_\gamma c_\alpha - u_\beta b_\alpha c_\gamma + u_\gamma b_\alpha c_\beta - u_\gamma b_\beta c_\alpha)^2$
$\equiv (- u_\alpha b_\gamma c_\beta + u_\beta b_\gamma c_\alpha + u_\gamma b_\alpha c_\beta)^2 \equiv 2 u_\beta u_\gamma b_\gamma b_\alpha c_\alpha c_\beta.$

$13 = \begin{vmatrix} b_\alpha & b_\beta & b_\gamma \\ a_\alpha & a_\beta & a_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} b_\alpha u_\beta a_\gamma \equiv b_\gamma (a_\alpha u_\beta - a_\beta u_\alpha)\, b_\alpha u_\beta a_\gamma \equiv - u_\alpha u_\beta a_\beta a_\gamma b_\gamma b_\alpha \equiv - \tfrac{1}{2} (uab)^2 \,.\, (\alpha\beta\gamma)^2.$

For $(uab)^2 (\alpha\beta\gamma)^2 = (u_\alpha a_\beta b_\gamma - u_\alpha a_\gamma b_\beta + u_\beta a_\gamma b_\alpha - u_\beta a_\alpha b_\gamma + u_\gamma a_\alpha b_\beta - u_\gamma a_\beta b_\alpha)^2$
$\equiv (u_\alpha a_\beta b_\gamma + u_\beta a_\gamma b_\alpha - u_\gamma a_\beta b_\alpha)^2 \equiv 2 u_\alpha u_\beta a_\beta a_\gamma b_\gamma b_\alpha.$

$14 = \tfrac{1}{2} (\alpha\beta\gamma)\, u_{\beta'}\, (\beta'\alpha\gamma)\, u_\beta = \tfrac{1}{2} u_{\beta'}^2 \,.\, (\alpha\beta\gamma)\, (\beta\alpha\gamma) + \overset{\frown}{\beta\beta'}.$

$15 \equiv (\alpha\beta x)\, b_\alpha b_\gamma a_\beta a_\gamma \equiv (801)_1.$

$16 = (a_\beta u_\gamma - a_\gamma u_\beta)\, a_x b_\gamma c_\beta b_\alpha c_\alpha = a_x a_\beta c_\beta c_\alpha b_\alpha b_\gamma u_\gamma - a_x a_\gamma b_\gamma b_\alpha c_\alpha c_\beta u_\beta,$

both represented by $c_x c_\alpha b_\alpha b_\gamma a_\gamma a_\beta u_\beta$ or (911).

$17 \equiv b_\gamma' u_\beta b_x' b_\gamma c_\beta b_\alpha c_\alpha \equiv b_\gamma' u_\beta c_\beta b_\alpha c_\alpha\, (bb' \,.\, \gamma x) \equiv (\beta'\alpha\gamma)\, (\beta'\gamma x)\, u_\beta c_\beta c_\alpha \equiv (\beta'\alpha\gamma)\, (\alpha\gamma x)\, u_\beta c_\beta c_{\beta'}$
$\equiv (\beta\alpha\gamma)\, (\alpha\gamma x)\, u_\beta \,.\, c_{\beta'}^2 + \overset{\frown}{\beta\beta'}.$

$18 = \tfrac{1}{2} u_\alpha a_\beta a_\gamma b_\gamma c_\beta\, (c_\alpha b_x - c_x b_\alpha) = \tfrac{1}{2} b_x b_\gamma a_\gamma a_\beta c_\beta c_\alpha u_\alpha - \tfrac{1}{2} c_x c_\beta a_\beta a_\gamma b_\gamma b_\alpha u_\alpha.$

$19 = \{(b'bc)\, a_x' + (a'b'c)\, b_x + (a'bb')\, c_x\}\, a_\beta a_\gamma b_\gamma c_\beta\, (a'b'u) \equiv \{(a'b'c)\, b_x + (a'bb')\, c_x\}\, a_\beta a_\gamma b_\gamma c_\beta\, (a'b'u)$

$\equiv (a'bb')\, c_x a_\beta a_\gamma b_\gamma c_\beta\, (a'b'u) + (a'b'c)\, b_x a_\beta a_\gamma b_\gamma\, \{(cb'u)\, a_\beta' + (a'b'c)\, u_\beta\}$

$\equiv \tfrac{1}{2} a_{\beta'}' c_x a_\beta a_\gamma c_\beta\, (u_{\beta'} a_\gamma' - u_\gamma a_{\beta'}') + (a'b'c)\, (cb'u)\, a_\beta b_\gamma\, (aa' \,.\, \gamma\beta)\, b_x$

$\equiv \tfrac{1}{2} (aa' \,.\, \gamma\beta')\, a_\beta a_\gamma' c_x c_\beta u_{\beta'} + \tfrac{1}{2} (\alpha\gamma\beta)\, (b_\alpha' c_\beta - b_\beta' c_\alpha)\, (cb'u)\, b_\gamma b_x$

$\equiv \tfrac{1}{4} (\alpha\gamma\beta')\, (\alpha\beta\gamma)\, c_x c_\beta u_{\beta'} - \tfrac{1}{2} (\alpha\beta\gamma)\, b_\alpha' c_\beta b_\gamma\, (cb'u)\, b_x$

$\equiv \tfrac{1}{4} (\alpha\gamma\beta)\, (\alpha\beta\gamma) \,.\, c_x c_{\beta'} u_{\beta'} + \overset{\frown}{\beta\beta'} - \tfrac{1}{2} \begin{vmatrix} c_\alpha & c_\beta & c_\gamma \\ b_\alpha' & b_\beta' & b_\gamma' \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} b_\gamma b_\alpha' c_\beta b_x \equiv \tfrac{1}{2} b_\gamma' (c_\alpha u_\beta - c_\beta u_\alpha)\, b_\gamma c_\beta b_\alpha' b_x$

$\equiv \tfrac{1}{2} b_\gamma' b_\alpha' c_\alpha c_\beta u_\beta b_\gamma b_x \equiv \tfrac{1}{2} (bb' \,.\, x\gamma)\, b_\alpha' c_\alpha c_\beta u_\beta b_\gamma \equiv \tfrac{1}{2} (\beta' x\gamma)\, (\beta'\gamma\alpha)\, c_\alpha c_\beta u_\beta$

$\equiv \tfrac{1}{2} (\gamma\alpha x)\, (\beta'\gamma\alpha)\, c_{\beta'} c_\beta u_\beta \equiv \overset{\frown}{\beta\beta'} \equiv 0.$

$20 \equiv (\alpha'\beta\gamma)(\gamma\alpha x)\, u_\beta u_\alpha u_x u_{\alpha'} \equiv u_{\alpha'}^2 . (\alpha\beta\gamma)(\gamma\alpha x)\, \alpha_\beta u_x + \widetilde{\alpha\alpha'}.$

$21 \equiv (\alpha'\beta\gamma)(\gamma\alpha x)\, b_\alpha b_x u_\beta u_{\alpha'} \equiv \frac{1}{2}(\alpha\beta\gamma)(\gamma\alpha x)\, u_\beta . b_{\alpha'} b_x u_{\alpha'} + \frac{1}{2}(\alpha'\beta\alpha)(\gamma\alpha x)\, b_\gamma b_x u_\beta u_{\alpha'}.$

$22 = (\alpha'\beta\gamma)(\gamma\alpha x)(c_\alpha u_\beta - c_\beta u_\alpha)\, c_x u_{\alpha'} \equiv (\alpha\beta\gamma)(\gamma\alpha x)\, u_\beta . c_{\alpha'} c_x u_{\alpha'} + \widetilde{\alpha\alpha'} - (\alpha\beta\gamma)(\gamma\alpha x)\, c_\beta c_x . u_{\alpha'}^2 - \widetilde{\alpha\alpha'}.$

$23 \equiv (\alpha'\beta x)(\gamma\alpha x)(\alpha\beta x)\, b_\alpha b_\gamma \equiv \widetilde{\alpha\alpha'} + \underbrace{(\alpha'\beta x)(\gamma\alpha x)(\alpha'\beta x)}\, b_\alpha b_\gamma.$

$24 \equiv (\alpha'\beta\gamma)(\gamma\alpha x)\, b_\alpha u_\beta b_{\alpha'} \equiv (\alpha\beta\gamma)(\gamma\alpha x)\, u_\beta . b_{\alpha'}^2 + \widetilde{\alpha\alpha'}.$

$25 = (\alpha'\beta\gamma)(\gamma\alpha x)(c_\alpha u_\beta - c_\beta u_\alpha)\, c_{\alpha'} \equiv \widetilde{\alpha\alpha'} + (\alpha\beta\gamma)(\gamma\alpha x)\, u_\beta . c_{\alpha'}^2 - \widetilde{\alpha\alpha'} - (\alpha'\beta\gamma)(\gamma\alpha' x)\, c_\beta u_\alpha c_\alpha$

$\equiv - (\alpha'\beta\gamma)(\gamma\alpha'\beta) . c_x u_\alpha c_\alpha - (\alpha'\underbrace{\beta\gamma)(\gamma\beta x)\, c_{\alpha'} u_\alpha c_\alpha} \equiv \widetilde{\alpha\alpha'} \equiv 0.$

$26 \equiv (\alpha'\beta\gamma)\, a_\gamma \underbrace{u_\alpha a_\beta} u_\alpha u_{\alpha'}.$

$27 \equiv (\alpha'\beta\gamma)(b_\gamma u_\alpha - b_\alpha u_\gamma)\, b_\alpha u_\beta u_{\alpha'} \equiv (\underbrace{\alpha'\beta\gamma)\, b_\gamma b_\alpha} u_\alpha u_\beta u_{\alpha'} \equiv \widetilde{\alpha\alpha'} \equiv 0.$

$28 \equiv a_\gamma b_\gamma b_\alpha c_\alpha c_x a_\beta' (aa' . \beta x) \equiv \frac{1}{2}(\alpha'\beta x)(\alpha'\gamma\beta)\, b_\gamma b_\alpha c_\alpha c_x \equiv \frac{1}{2}(\alpha'\beta\gamma)(\alpha'\gamma\beta) . b_x b_\alpha c_\alpha c_x$

$+ \frac{1}{2}(\gamma\beta x)(\alpha'\gamma\beta)\, b_{\alpha'} b_\alpha c_\alpha c_x \equiv b_{\alpha'}^2 . (\gamma\beta x)(\alpha\gamma\beta)\, c_\alpha c_x + \widetilde{\alpha\alpha'}.$

$29 \equiv a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x' (cc' . x\beta) \equiv \frac{1}{2}(\underbrace{\beta\gamma' x)\, a_\beta a_\gamma b_\gamma} b_\alpha (\gamma'\alpha x) \equiv \frac{1}{2}\widetilde{\gamma\gamma'} + \frac{1}{2}(\beta\gamma x)\, a_\beta a_\gamma . b_{\gamma'} b_\alpha (\gamma'\alpha x).$

$30 \equiv c_\beta a_\gamma b_\gamma b_\alpha c_\alpha (aa'u)\, a_x' u_\beta + a_\gamma b_\gamma b_\alpha c_\alpha (cau) . a_\beta' a_x' u_\beta + a_\gamma b_\gamma b_\alpha c_\alpha (ca'a)\, a_x' . u_\beta^2$

$\equiv \frac{1}{2} u_{\alpha'} (\alpha'\gamma x)\, c_\beta b_\gamma b_\alpha c_\alpha u_\beta \equiv \frac{1}{2}(\alpha\gamma x)\, b_\gamma b_\alpha . c_\beta c_{\alpha'} u_\beta u_{\alpha'} + \widetilde{\alpha\alpha'}.$

$31 = a_\beta a_\gamma b_\gamma b_\alpha c_\alpha (cb'u)\, b_x' u_\beta \equiv a_\beta a_\gamma u_\gamma u_\beta . b_\alpha c_\alpha b_x' (cb'b) + a_\beta a_\gamma b_\gamma' b_\alpha c_\alpha b_x' u_\beta (cbu) \equiv a_\beta a_\gamma b_\gamma' b_x' u_\beta . b_\alpha c_\alpha (cbu).$

$32 = \frac{1}{2} u_{\gamma'} (\gamma' \underbrace{\alpha x)\, a_\beta} a_\gamma b_\gamma b_\alpha u_\beta \equiv \widetilde{\gamma\gamma'} + \frac{1}{2}(\gamma\alpha x)\, b_\gamma b_\alpha . a_\gamma a_\beta u_\beta u_{\gamma'}.$

$33 = a_\beta a_\gamma \underbrace{b_\gamma b_\alpha c_\alpha (ca'u)}\, a_\beta' \equiv a_\beta a_\gamma a_\beta' a_\gamma' . b_\alpha c_\alpha (cbu) + a_\beta \underbrace{a_\gamma u_\gamma b_\alpha c_\alpha (ca'b)\, a_\beta'} \equiv (aa' . \gamma\beta)\, a_\beta u_\gamma b_\alpha c_\alpha (ca'b)$

$\equiv \frac{1}{2}(\alpha'\gamma\beta)\, u_\gamma b_\alpha c_\alpha (b_{\alpha'} c_\beta - b_\beta c_{\alpha'}) \equiv \frac{1}{2}(\alpha'\gamma\beta)\, u_\gamma b_{\alpha'} c_\beta c_\alpha b_\alpha \equiv \frac{1}{2} b_{\alpha'}^2 . (\alpha\gamma\beta)\, c_\alpha c_\beta u_\gamma + \widetilde{\alpha\alpha'}.$

$34 = \frac{1}{2} u_{\gamma'} (\gamma'\alpha\beta)\, a_\gamma b_\gamma a_\beta b_\alpha.$

$35 \equiv (\alpha\beta\gamma)\, b_\gamma c_\beta b_{\alpha'} c_\alpha b_{\alpha'}' b_x' \equiv (\alpha\beta\gamma)\, b_\gamma c_\beta \underbrace{b_{\alpha'}' c_\alpha b_x b_{\alpha'}'} + (bb' . \alpha' x)(\alpha\beta\gamma)\, b_\gamma b_{\alpha'}' c_\beta c_\alpha$

$\equiv \frac{1}{2}(\beta'\alpha' x)(\beta'\gamma\alpha')(\alpha\beta\gamma)\, c_\beta c_\alpha \equiv -\frac{1}{2}(\beta'\alpha' x)\, c_{\beta'} c_{\alpha'} . (\alpha\beta\gamma)^2.$

This completes the system.

§ V. *Forms reducible on multiplication by u_x are*

(303) for $u_x . (abc)\, a_x b_x c_x = (bcu)\, b_x c_x . a_x^2 + (cau)\, c_x a_x . b_x^2 + (abu)\, a_x b_x . c_x^2.$

$(421)_3$ for $u_x . (a'bc)(uca)(uab)\, a_x' = a_x^2 . (bcu)(cau)(abu) + (cau)^2 . (abu)\, a_x b_x$

$+ (abu)^2 . (cau)\, c_x a_x + u_a^2 . (bcu)\, b_x c_x - \frac{1}{2} u_x \{(bcu)\, b_a u_a c_x + (bcu)\, c_a u_a b_x\}.$

$(501)_1$ for $u_x . (abc)\, a_x b_a c_a = (abc)\, a_x b_x u_a c_a + (abc)\, a_x c_x u_a b_a$

$- \{\tfrac{2}{3} a_a^2 . (bcu)\, b_x c_x + b_a^2 . (cau)\, c_x a_x + c_a^2 . (abu)\, a_x b_x - a_x^2 . (bcu)\, b_a c_a\}.$

$(512)_2$ for $u_x . (abc)\, a_\beta u_\beta b_x c_x = a_x a_\beta u_\beta . (bcu)\, b_x c_x + b_x^2 . (cau)\, a_\beta u_\beta c_x + c_x^2 . (abu)\, a_\beta u_\beta b_x.$

$(611)_2$ for $u_x . a_\beta a_\gamma b_\gamma b_x u_\beta \equiv \frac{1}{2}(abu)^2 . (\beta\gamma x)^2$, save for products.

$(630)_1$ for $u_x . (\alpha\beta\gamma)\, u_\alpha u_\beta u_\gamma = u_\alpha^2 . (\beta\gamma x)\, u_\beta u_\gamma + u_\beta^2 . (\gamma\alpha x)\, u_\gamma u_\alpha + u_\alpha^2 . (\alpha\beta x)\, u_\alpha u_\beta.$

$(630)_2$ for $u_x . (abu)\, \alpha_\beta b_\gamma u_\beta u_\gamma = -(abu)^2 . (\beta\gamma x)\, u_\beta u_\gamma$
$+ u_\beta^2 . \{(abu)\, u_\gamma a_\gamma b_x - (abu)\, a_\gamma b_\gamma a_x\} - u_\gamma^2 . (abu)\, a_\beta u_\beta b_x.$

$(630)_3$ for $u_x . (bcu)\, u_\beta u_\gamma b_\gamma c_\beta = -(bcu)^2 . (\beta\gamma x)\, u_\beta u_\gamma + (bcu)\, b_\gamma c_x u_\gamma . u_\beta^2 + (bcu)\, c_\beta b_x u_\beta . u_\gamma^2.$

$(603)_2$ for $u_x . (\beta\gamma x)\, a_x b_x a_\beta b_\gamma = -(\beta\gamma x)^2 . (abu)\, a_x b_x - a_x^2 . (\beta\gamma x)\, b_x b_\gamma u_\beta$
$+ b_x^2 \{(\beta\gamma x)\, a_x a_\gamma u_\beta - (\beta\gamma x)\, a_x a_\beta u_\gamma\}.$

$(603)_3$ for $u_x . (\beta\gamma x)\, b_x c_x b_\gamma c_\beta = -(\beta\gamma x)^2 . (bcu)\, b_x c_x + (\beta\gamma x)\, c_\beta c_x u_\gamma . b_x^2 + (\beta\gamma x)\, b_\gamma b_x u_\beta . c_x^2.$

$(710)_1$ for $u_x . (\alpha\beta\gamma)\, a_\beta a_\gamma u_\alpha = (\alpha\beta\gamma)\, u_\alpha u_\beta a_x a_\gamma + (\alpha\beta\gamma)\, u_\gamma u_\alpha a_x a_\beta - (\alpha\beta x)\, u_\alpha u_\beta . a_\gamma^2$
$- (\gamma\alpha x)\, u_\gamma u_\alpha . a_\beta^2 - \frac{2}{3} a_\alpha^2 . (\beta\gamma x)\, u_\beta u_\gamma.$

$(710)_2$ for $u_x . (bcu)\, a_\beta a_\gamma b_\gamma c_\beta = -(bcu)^2 . (\beta\gamma x)\, a_\beta a_\gamma + u_\beta^2 . (abc)\, a_\gamma b_\gamma c_x$
$+ u_\gamma^2 . (abc)\, c_\beta a_\beta b_x - (cau)\, c_\beta a_\beta . b_\gamma u_\gamma b_x - (abu)\, a_\gamma b_\gamma . c_\beta u_\beta c_x.$

(721) for $u_x . (\alpha\beta\gamma)\, b_\alpha b_x u_\beta u_\gamma = (\beta\gamma x)\, u_\beta u_\gamma . u_\alpha b_\alpha b_x + u_\beta^2 . (\gamma\alpha x)\, b_\alpha b_x u_\gamma + u_\gamma^2 . (\alpha\beta x)\, b_\alpha b_x u_\beta.$

$(801)_1$ for $u_x . (\beta\gamma x)\, b_\gamma c_\beta b_\alpha c_\alpha = -(\beta\gamma x)^2 . (bcu)\, b_\alpha c_\alpha + b_x^2 . (\alpha\beta\gamma)\, c_\alpha c_\beta u_\gamma$
$+ c_x^2 . (\alpha\beta\gamma)\, b_\alpha b_\gamma u_\beta - (\gamma\alpha x)\, b_\gamma b_\alpha . c_\beta c_x u_\beta - (\alpha\beta x)\, c_\alpha c_\beta . b_\gamma b_x u_\gamma.$

$(801)_2$ for $u_x . (a'bc)\, a_\beta a_\gamma b_\gamma c_\beta a_x' = a_x^2 . (bcu)\, a_\beta a_\gamma b_\gamma c_\beta + (abu)\, a_\gamma b_\gamma . c_x a_x c_\beta a_\beta + (cau)\, c_\beta a_\beta . a_x b_x a_\gamma b_\gamma$
$- \frac{1}{2} \{(\gamma\alpha x)\, b_\gamma b_\alpha . u_\beta c_\beta c_x + (\alpha\beta x)\, c_\alpha c_\beta . u_\gamma b_\gamma b_x\}.$

(812) for $u_x . (\alpha'\beta\gamma)\, (\gamma\alpha x)\, (\alpha\beta x)\, u_{\alpha'} = u_\alpha^2 . (\beta\gamma x)\, (\gamma\alpha x)\, (\alpha\beta x) + \frac{4}{3} a_\alpha^2 . a_x^2 . (\beta\gamma x)\, u_\beta u_\gamma$
$+ (\alpha\beta x)^2 . (\gamma\alpha x)\, u_\gamma u_\alpha + (\gamma\alpha x)^2 . (\alpha\beta x)\, u_\alpha u_\beta - \frac{2}{3} a_\alpha^2 . u_x \{(\beta\gamma x)\, a_\beta a_x u_\gamma + (\beta\gamma x)\, a_\gamma a_x u_\beta\}.$

(911) for $u_x . a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x u_\beta = -(cau)\, a_\beta c_x u_\beta . (\gamma\alpha x)\, b_\gamma b_\alpha + \frac{1}{27} u_x^2 . a_\alpha^2 . b_\beta^2 . c_\gamma^2 - \frac{1}{3} c_\gamma^2 . a_x a_\beta u_\beta . u_\alpha b_\alpha b_x$
$+ c_x^2 . u_\alpha u_\beta b_\gamma b_\alpha a_\beta a_\gamma - a_x a_\beta u_\beta . b_\gamma b_\alpha u_\gamma c_\alpha c_x - \frac{1}{9} a_\alpha^2 . b_\beta^2 . u_\gamma^2 . c_x^2.$

$(10 . 1 . 0)$ for $u_x . (\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} = u_\alpha^2 . (\beta\gamma x)\, b_\gamma c_\beta b_\alpha c_\alpha + (\gamma\alpha x)\, b_\gamma b_\alpha . c_\alpha c_\beta u_\alpha u_\beta$
$+ (\alpha\beta x)\, c_\alpha c_\beta . b_\gamma b_\alpha u_\gamma u_\alpha - \frac{2}{3} a_\alpha^2 \{(abu)\, a_\gamma b_\gamma . c_\beta u_\beta c_x + (cau)\, c_\beta a_\beta . b_\gamma u_\gamma b_x\}.$

Thus all but $(421)_3$, $(501)_1$ and $(710)_1$ are expressible by products of terms of lower degree, and these are expressible by forms otherwise occurring in the list of forms.

In regard to the previous table we may remark that, multiplying still further by u_x, we have

$$\left.\begin{array}{l} u_x^2 . (501)_1 \\ u_x^2 . (710)_1 \\ u_x^2 . (801)_2 \\ u_x^2 . (10 . 1 . 0) \\ u_x^2 . (911) \end{array}\right\} \text{ still further reducible, say are "doubly-quasi-reducible,"}$$

and there are, of the 18 forms just given, 13 which are only "singly-quasi-reducible," the reduced forms being expressible by the following 13 "whole" types of forms

$(abc)^2$, b_α^2, a_α^2, a_x^2, $(bcu)\, b_x c_x$, $(bcu)^2$, $u_\alpha b_\alpha b_x$, $(abc)\,(bcu)\, a_x$, $(bcu)\,(cau)\,(abu)$,
$(\alpha\beta\gamma)^2$, u_α^2, $(\beta\gamma x)\, u_\beta u_\gamma$, $(\beta\gamma x)^2$, $(\alpha\beta\gamma)\,(\beta\gamma x)\, u_\alpha$, $(\beta\gamma x)\,(\gamma\alpha x)\,(\alpha\beta x)$,
$(bcu)\, b_\alpha c_\alpha$, $b_\alpha c_\alpha b_x c_x$, $(bcu)\, b_\alpha c_x u_\alpha$, $(bcu)\, b_\gamma c_x u_\gamma$,
$(\beta\gamma x)\, a_\beta a_\gamma$, $u_\beta u_\gamma a_\beta a_\gamma$, $(\beta\gamma x)\, a_\beta a_x u_\gamma$, $(\beta\gamma x)\, c_x c_\beta u_\beta$.

Further, of concomitants of two conics, there is one which is reducible multiplied by u_x, namely $(630)_3 = (bcu)\, u_\beta u_\gamma b_\gamma c_\beta$, and its reciprocal $(603)_3 = (\beta\gamma x)\, b_x c_x b_\gamma c_\beta$.

(Geometrically these represent angular points and sides of self-polar triangle of the two conics.)

Proof of the reductions by multiplication by u_x.

(303) is obvious.

$(421)_3$ $\underbrace{(a'bc)(uca)(uab)\, a'_x u_x} = a_x'^2 \,.\, (bcu)(cau)(abu) + (ca'u)(cau)(abu)\, b_x a_x' + (a'bu)(abu)(cau)\, c_x a_x'$

where $(ca'u)(cau)(abu)\, b_x a_x' = (ca'u)^2 \,.\, (abu)\, a_x b_x + (ca'u)(abu)\, b_x \{(caa')\, u_x - (uaa')\, cx\}$

$= (ca'u)^2 \,.\, (abu)\, a_x b_x + \tfrac{1}{2} (buc)\, u_a b_x \{c_a u_x - c_x u_a\}$

and $(a'bu)(abu)(cau)\, c_x a_x' = (a'bu)^2 \,.\, (cau)\, c_x a_x + (a'bu)(cau)\, c_x \{(uaa')\, b_x - (baa')\, u_x\}$

$= (a'bu)^2 \,.\, (cau)\, c_x a_x + \tfrac{1}{2} (bcu)\, u_a c_x \{u_a b_x - u_x b_a\}$.

$(501)_1$

$(abc)\, a_x b_x u_a c_a = a_a (ubc)\, a_x b_x c_a + (abu)\, a_x b_x \,.\, c_a^2 + (auc)\, a_x b_x b_a c_a$

$= \tfrac{1}{3} a_a^2 \,.\, (ubc)\, b_x c_x + (abu)\, a_x b_x \,.\, c_a^2 + u_x \,.\, (abc)\, a_x b_a c_a + (aub)\, a_x c_x b_a c_a + (buc)\, b_a c_a \,.\, a_x^2$

$= \tfrac{1}{3} a_a^2 \,.\, (bcu)\, b_x c_x + c_a^2 \,.\, (abu)\, a_x b_x + u_x (abc)\, a_x b_a c_a - (abc)\, a_x c_x b_a u_a + (cau)\, c_x a_x \,.\, b_a^2$
$+ (bcu)\, a_x c_x b_a a_a - (bcu)\, b_a c_a \,.\, a_x^2$

or $u_x \,.\, (abc) a_x b_a c_a = (abc)\, a_x b_x u_a c_a + (abc) a_x c_x b_a u_a + a_x^2 \,.\, (bcu) b_a c_a - \tfrac{2}{3} a_a^2 \,.\, (bcu)\, b_x c_x - (cau) c_x a_x \,.\, b_a^2$
$- (abu)\, a_x b_x \,.\, c_a^2$.

$(512)_2$ is obvious.

$(611)_2$ $(abu)^2 (\beta\gamma x)^2 = \{a_\beta b_\gamma u_x - a_\beta b_x u_\gamma + a_\gamma b_x u_\beta - a_\gamma b_\beta u_x + a_x b_\beta u_\gamma - a_x b_\gamma u_\beta\}^2$

$\equiv \{a_\beta b_\gamma u_x - a_\beta b_x u_\gamma + a_\gamma b_x u_\beta - a_x b_\gamma u_\beta\}^2$

$\equiv 2u_x \,.\, a_\beta a_\gamma b_\gamma b_x u_\beta + 2 a_\beta a_x u_\beta \,.\, b_x b_\gamma u_\gamma \equiv 2u_x \,.\, a_\beta a_\gamma b_\gamma b_x u_\beta$.

$(630)_1$ is obvious.

$(630)_2$. Consider $(abu)^2 \,.\, (x\beta\gamma)\, u_\beta u_\gamma$

$$= (abu)\, u_\beta u_\gamma \begin{vmatrix} a_x & b_x & u_x \\ a_\beta & b_\beta & u_\beta \\ a_\gamma & b_\gamma & u_\gamma \end{vmatrix} = u_\beta^2 \,.\, (abu)\, u_\gamma (b_x a_\gamma - b_\gamma a_x) + u_\gamma^2 \,.\, (abu)\, u_\beta (a_x b_\beta - b_x a_\beta) + u_x (abu)\, u_\beta u_\gamma (a_\beta b_\gamma - a_\gamma b_\beta)$$

$= u_x \,.\, (abu)\, a_\beta b_\gamma u_\beta u_\gamma + u_\beta^2 \,.\, \{(abu)\, u_\gamma a_\gamma b_x - (abu)\, u_\gamma b_\gamma a_x\} - u_\gamma^2 \,.\, (abu)\, a_\beta u_\beta b_x$.

$(630)_3$

$$(bcu)^2 \,.\, (\beta\gamma x)\, u_\beta u_\gamma = (bcu)\, u_\beta u_\gamma \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} = b_\beta (bcu)\, u_\beta u_\gamma (c_\gamma u_x - c_x u_\gamma) + c_\beta (b_x u_\gamma - b_\gamma u_x)(bcu)\, u_\beta u_\gamma + u_\beta^2 \,.\, (b_\gamma c_x - b_x c_\gamma)(bcu)\, u_\gamma$$

$= u_\beta^2 \,.\, (bcu)\, b_\gamma c_x u_\gamma + u_\gamma^2 \,.\, (bcu)\, c_\beta b_x u_\beta - u_x \,.\, (bcu)\, u_\beta u_\gamma b_\gamma c_\beta$.

$(603)_2$

$$(\beta\gamma x)^2 \,.\, (abu)\, a_x b_x = (\beta\gamma x)\, a_x b_x \begin{vmatrix} a_\beta & a_\gamma & a_x \\ b_\beta & b_\gamma & b_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} = u_x\, (\beta\gamma x)\, a_\beta b_\gamma a_x b_x - u_x\, (\beta\gamma x)\, a_x b_x a_\gamma b_\beta + a_x^2 \,.\, (\beta\gamma x) b_x (b_\beta u_\gamma - b_\gamma u_\beta) + b_x^2 \,.\, (\beta\gamma x) a_x (a_\gamma u_\beta - a_\beta u_\gamma)$$

$$= u_x \,.\, (\beta\gamma x)\, a_\beta b_\gamma a_x b_x - a_x^2 \,.\, (\beta\gamma x)\, b_x b_\gamma u_\beta + b_x^2 \,.\, \{(\beta\gamma x)\, a_x a_\gamma u_\beta - (\beta\gamma x)\, a_x a_\beta u_\gamma\}.$$

$(603)_3$

$$(\beta\gamma x)^2 \,.\, (bcu)\, b_x c_x = \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} (\beta\gamma x)\, b_x c_x = b_\beta\, (\beta\gamma x)\, b_x c_x\, (c_\gamma u_x - c_x u_\gamma) + (b_\gamma u_\beta c_x - b_\gamma c_\beta u_x)\, (\beta\gamma x)\, b_x c_x + b_x^2 \,.\, (\beta\gamma x)\, c_x\, (c_\beta u_\gamma - c_\gamma u_\beta)$$

$$= (\beta\gamma x)\, b_\gamma b_x u_\beta \,.\, c_x^2 + (\beta\gamma x)\, c_\beta c_x u_\gamma \,.\, b_x^2 - (\beta\gamma x)\, b_x c_x b_\gamma c_\beta \,.\, u_x.$$

$(710)_1$ $\quad u_x \,.\, (\alpha\beta\gamma)\, a_\beta a_\gamma u_\alpha = (\beta\gamma x)\, a_\beta a_\gamma \,.\, u_\alpha^2 + (\gamma\alpha x)\, u_\beta u_\alpha a_\beta a_\gamma + (\alpha\beta x)\, u_\alpha u_\gamma a_\beta a_\gamma$

and $\quad (\gamma\alpha x)\, u_\beta u_\alpha a_\beta a_\gamma = (\alpha\beta\gamma)\, a_\gamma a_x u_\beta u_\alpha + (\beta\alpha x)\, u_\beta u_\alpha \,.\, a_\gamma^2 + (\gamma\beta x)\, u_\beta u_\alpha a_\alpha a_\gamma$

$$= (\alpha\beta\gamma)\, u_\alpha u_\beta a_x a_\gamma - (\alpha\beta x)\, u_\alpha u_\beta \,.\, a_\gamma^2 - \tfrac{1}{3} a_\alpha^2 \,.\, (\beta\gamma x)\, u_\beta u_\gamma$$

$$(\alpha\beta x)\, u_\alpha u_\gamma a_\beta a_\gamma = (\alpha\beta\gamma)\, u_\gamma u_\alpha a_x a_\beta - (\gamma\alpha x)\, u_\gamma u_\alpha \,.\, a_\beta^2 - \tfrac{1}{3} a_\alpha^2 \,.\, (\beta\gamma x)\, u_\beta u_\gamma.$$

$(710)_2$

$$(\beta\gamma x)\, a_\beta a_\gamma \,.\, (bcu)^2 = (bcu)\, a_\beta a_\gamma \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} = \tfrac{1}{3} b_\beta^2 \,.\, (c_\gamma u_x - c_x u_\gamma)\, (acu)\, a_\gamma + c_\beta\, (b_x u_\gamma - b_\gamma u_x)\, (bcu)\, a_\beta a_\gamma + u_\beta\, (b_\gamma c_x - b_x c_\gamma)\, (bcu)\, a_\beta a_\gamma$$

$$= \tfrac{1}{3} b_\beta^2 \,.\, (cau)\, c_x a_\gamma u_\gamma + \tfrac{1}{3} c_\gamma^2 \,.\, (abu)\, b_x a_\beta u_\beta - u_x \,.\, (bcu)\, a_\beta a_\gamma b_\gamma c_\beta + (bcu)\, a_\beta a_\gamma b_x u_\gamma c_\beta + (bcu)\, a_\beta a_\gamma c_x b_\gamma u_\beta,$$

while $\quad (bcu)\, a_\beta u_\gamma a_\beta b_x c_\beta = u_\gamma^2 \,.\, (bca)\, c_\beta a_\beta b_x + (bau)\, c_\gamma u_\gamma a_\beta b_x c_\beta + (acu)\, b_\gamma u_\gamma a_\beta b_x c_\beta$

$$= u_\gamma^2 \,.\, (abc)\, c_\beta a_\beta b_x - \tfrac{1}{3}(abu)\, a_\beta u_\beta b_x \,.\, c_\gamma^2 - (cau)\, c_\beta a_\beta \,.\, b_\gamma u_\gamma b_x$$

and $\quad (bcu)\, a_\beta u_\beta a_\gamma c_x b_\gamma = u_\beta^2 \,.\, (abc)\, b_\gamma a_\gamma c_x - \tfrac{1}{3}(cau)\, a_\gamma u_\gamma c_x \,.\, b_\beta^2 - (abu)\, b_\gamma a_\gamma \,.\, c_\beta u_\beta c_x.$

(721) is obvious.

$(801)_1$

$$(bcu)\, b_\alpha c_\alpha \,.\, (\beta\gamma x)^2 = (\beta\gamma x)\, b_\alpha c_\alpha \begin{vmatrix} b_\beta & b_\gamma & b_x \\ c_\beta & c_\gamma & c_x \\ u_\beta & u_\gamma & u_x \end{vmatrix} = b_\beta\, (c_\gamma u_x - c_x u_\gamma)\, (\beta\gamma x)\, b_\alpha c_\alpha + c_\beta\, (b_x u_\gamma - b_\gamma u_x)\, (\beta\gamma x)\, b_\alpha c_\alpha + u_\beta\, (b_\gamma c_x - b_x c_\gamma)\, (\beta\gamma x)\, b_\alpha c_\alpha$$

$$= \tfrac{1}{3} b_\beta^2 \,.\, (c_\gamma u_x - c_x u_\gamma)(\alpha\gamma x) c_\alpha - (\beta\gamma x) b_\alpha c_\alpha b_\gamma c_\beta \,.\, u_x - \tfrac{1}{3} c_\gamma^2 \,.\, u_\beta b_x (\beta\alpha x) b_\alpha + (\beta\gamma x) b_x b_\alpha c_\alpha c_\beta u_\gamma + (\beta\gamma x) b_\alpha c_\alpha c_x b_\gamma u_\beta,$$

of which $\tfrac{1}{3} b_\beta^2 \,.\, c_\gamma u_x\, (\alpha\gamma x)\, c_\alpha = \tfrac{1}{3} b_\beta^2 \,.\, c_\gamma^2 \,.\, u_x\, (\alpha\alpha x) = 0$;

and $\quad (\beta\gamma x)\, b_x b_\alpha c_\alpha c_\beta u_\gamma = b_x^2 \,.\, (\beta\gamma\alpha)\, c_\alpha c_\beta u_\gamma + (\alpha\gamma x)\, b_x b_\beta c_\alpha c_\beta u_\gamma + (\beta\alpha x)\, b_x b_\gamma c_\alpha c_\beta u_\gamma$

$$= b_x^2 \,.\, (\alpha\beta\gamma)\, c_\alpha c_\beta u_\gamma - \tfrac{1}{3} b_\beta^2 \,.\, (\gamma\alpha x)\, c_x c_\alpha u_\gamma - (\alpha\beta x)\, c_\alpha c_\beta \,.\, b_\gamma b_x u_\gamma$$

$$(\beta\gamma x)\, c_x c_\alpha b_\alpha b_\gamma u_\beta = c_x^2 \,.\, (\beta\gamma\alpha)\, b_\alpha b_\gamma u_\beta + (\beta\alpha x)\, c_x c_\gamma b_\alpha b_\gamma u_\beta + (\alpha\gamma x)\, c_x c_\beta b_\alpha b_\gamma u_\beta$$

$$= c_x^2 \,.\, (\alpha\beta\gamma)\, b_\alpha b_\gamma u_\beta - \tfrac{1}{3} c_\gamma^2 \,.\, (\alpha\beta x)\, b_\alpha b_x u_\beta - (\gamma\alpha x)\, b_\gamma b_\alpha \,.\, c_\beta c_x u_\beta.$$

$(801)_2$ $u_x \,.\, (a'bc)\, a_x' a_\beta a_\gamma b_\gamma c_\beta = (ubc)\, a_\beta a_\gamma b_\gamma c_\beta \,.\, a_x'^2 + (a'bu)\, c_x a_x' a_\gamma a_\beta b_\gamma c_\beta + (a'uc)\, b_x a_x' a_\beta c_\beta a_\gamma b_\gamma$

and $\quad (a'bu)\, c_x a_x' a_\gamma a_\beta b_\gamma c_\beta = (a'bu)\, a_\gamma' b_\gamma \,.\, c_x a_x c_\beta a_\beta + (aa' \,.\, \gamma x)\, (a'bu)\, c_x a_\beta b_\gamma c_\beta$

$$= (a'bu)\, a_\gamma' b_\gamma \,.\, a_x c_x a_\beta c_\beta - \tfrac{1}{2}(\gamma\alpha x)\,(b_\alpha u_\beta - b_\beta u_\alpha)\, c_x b_\gamma c_\beta$$

$$= (a'bu)\, a_\gamma' b_\gamma \,.\, a_x c_x u_\beta c_\beta - \tfrac{1}{2}(\gamma\alpha x)\, b_\gamma b_\alpha \,.\, u_\beta c_\beta c_x + \tfrac{1}{6} b_\beta^2 \,.\, (\gamma\alpha x)\, u_\alpha c_x c_\gamma$$

$$= (a'bu)\, a_\gamma' b_\gamma \,.\, c_x a_x c_\beta a_\beta - \tfrac{1}{2}(\gamma\alpha x)\, b_\gamma b_\alpha \,.\, u_\beta c_\beta c_x\,;$$

also $\quad (a'uc)\, b_x a_x' a_\beta c_\beta a_\gamma b_\gamma = (a'uc)\, a_\beta' c_\beta \,.\, a_x b_x a_\gamma b_\gamma + (aa' \,.\, \beta x)\,(a'uc)\, a_\gamma b_x c_\beta b_\gamma$

$$= (a'uc)\, a_\beta' c_\beta \,.\, a_x b_x a_\gamma b_\gamma + \tfrac{1}{2}(\alpha\beta x)\,(u_\alpha c_\gamma - u_\gamma c_\alpha)\, b_x c_\beta b_\gamma$$

$$= (a'uc)\, a_\beta' c_\beta \,.\, a_x b_x a_\gamma b_\gamma - \tfrac{1}{2}(\alpha\beta x)\, c_\alpha c_\beta \,.\, u_\gamma b_x b_\gamma + \tfrac{1}{6} c_\gamma^2 \,.\, (\alpha\beta x)\, u_\alpha b_x b_\beta$$

$$= (a'uc)\, a_\beta' c_\beta \,.\, a_x b_x a_\gamma b_\gamma - \tfrac{1}{2}(\alpha\beta x)\, c_\alpha c_\beta \,.\, u_\gamma b_x b_\gamma.$$

(812)

$(\alpha'\beta\gamma)(\gamma\alpha x)(\alpha\beta x)\, u_{\alpha'} \,.\, u_x = (\beta\gamma x)(\gamma\alpha x)(\alpha\beta x) \,.\, u_{\alpha'}^2 + (\alpha'\beta x)(\gamma\alpha x)(\alpha\beta x)\, u_{\alpha'} u_\gamma + (\alpha' x\gamma)(\gamma\alpha x)(\alpha\beta x)\, u_{\alpha'} u_\beta$

and $(\alpha'\beta\gamma)(\gamma\alpha x)(\alpha\beta x)\, u_{\alpha'} u_\gamma = (\alpha'\beta x)^2 \,.\, (\gamma\alpha x)\, u_\gamma u_\alpha + (\alpha'\beta x)(\gamma\alpha x)(\alpha\alpha' x)\, u_\beta u_\gamma + (\gamma\alpha x)(\alpha'\beta x)(\alpha\beta\alpha')\, u_x u_\gamma$

$$= (\alpha'\beta x)^2 \,.\, (\gamma\alpha x) u_\gamma u_\alpha + \tfrac{1}{2}(\alpha\alpha' x) u_\beta u_\gamma (\beta\gamma x)(x\alpha\alpha') - \tfrac{1}{2}(\beta\alpha\alpha')(\beta\gamma x)(x\alpha\alpha') u_x u_\gamma$$

$$= (\alpha'\beta\gamma)^2 \,.\, (\gamma\alpha x)\, u_\gamma u_\alpha + \tfrac{2}{3} a_\alpha^2 \,.\, a_x^2 \,.\, (\beta\gamma x)\, u_\beta u_\gamma - \tfrac{2}{3} a_\alpha^2 \,.\, (\beta\gamma x)\, a_\beta u_x u_\gamma \,.\, u_x,$$

while $(\alpha' x\gamma)(\gamma\alpha x)(\alpha\beta x)\, u_{\alpha'} u_\beta = (\gamma\alpha' x)^2 \,.\, (\alpha\beta x)\, u_\alpha u_\beta + (\gamma\alpha' x)(\gamma\alpha\alpha')(\alpha\beta x)\, u_x u_\beta + (\gamma\alpha' x)(\alpha'\alpha x)(\alpha\beta x)\, u_\beta u_\gamma$

$$= (\gamma\alpha' x)^2 \,.\, (\alpha\beta x) u_\alpha u_\beta + \tfrac{1}{2}(\gamma\alpha\alpha')(\beta x\gamma)(x\alpha\alpha') u_x u_\beta + \tfrac{1}{2}(\alpha'\alpha x)(\beta x\gamma)(x\alpha\alpha') u_\beta u_\gamma$$

$$= (\gamma\alpha' x)^2 \,.\, (\alpha\beta x)\, u_\alpha u_\beta + \tfrac{2}{3} a_\alpha^2 \,.\, a_x^2 \,.\, (\beta\gamma x)\, u_\beta u_\gamma - \tfrac{2}{3} a_\alpha^2 \,.\, (\beta\gamma x)\, a_x a_\gamma u_\beta \,.\, u_x.$$

(911)

$$(cau)\, a_\beta c_x u_\beta \,.\, (\gamma\alpha x)\, b_\gamma b_\alpha = b_\gamma b_\alpha a_\beta c_x u_\beta \begin{vmatrix} c_\gamma & c_\alpha & c_x \\ a_\gamma & a_\alpha & a_x \\ u_\gamma & u_\alpha & u_x \end{vmatrix} = \tfrac{1}{3} c_\gamma^2 \,.\, b_\alpha b_x a_\beta u_\beta (a_\alpha u_x - a_x u_\alpha) - a_\gamma (c_\alpha u_x - c_x u_\alpha) b_\gamma b_\alpha a_\beta c_x u_\beta + u_\gamma (c_\alpha a_x - c_x a_\alpha)\, b_\gamma b_\alpha a_\beta c_x u_\beta$$

$$= \tfrac{1}{27} u_x^2 \,.\, a_\alpha^2 \,.\, b_\beta^2 \,.\, c_\gamma^2 - \tfrac{1}{3} c_\gamma^2 \,.\, a_x a_\beta u_\beta \,.\, u_\alpha b_\alpha b_x + c_x^2 \,.\, u_\alpha u_\beta a_\beta a_\gamma b_\gamma b_\alpha - a_x a_\beta u_\beta \,.\, b_\gamma b_\alpha c_\alpha c_x u_\gamma - \tfrac{1}{9} a_\alpha^2 \,.\, b_\beta^2 \,.\, u_\gamma^2 \,.\, c_x^2 - a_\beta a_\gamma b_\gamma b_\alpha c_\alpha c_x u_\beta \,.\, u_x$$

proving the theorem.

While further for $\qquad u_\alpha u_\beta a_\beta a_\gamma b_\gamma b_\alpha$ square $(uab)(\alpha\beta\gamma)$.

$(10\,.\,1\,.\,0)$ $(\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} \,.\, u_x = (x\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha \,.\, u_{\alpha'}^2 + (\alpha' x\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} u_\beta + (\alpha'\beta x)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} u_\gamma,$

and $\quad (\alpha' x\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} u_\beta = (\alpha x\gamma)\, b_\gamma b_\alpha \,.\, c_\beta u_\beta u_{\alpha'} c_{\alpha'} + (\alpha'\alpha\gamma)\, b_\gamma c_\beta b_\alpha c_x u_{\alpha'} u_\beta + (\alpha' x\alpha)\, b_\gamma c_\beta b_\alpha c_\gamma u_{\alpha'} u_\beta$

$$= (\gamma\alpha x) b_\gamma b_\alpha \,.\, c_\beta c_{\alpha'} u_\beta u_{\alpha'} - \tfrac{1}{2}(\alpha\alpha'\gamma)(bu \,.\, \alpha\alpha') b_\gamma \,.\, c_\beta u_\beta c_x + \tfrac{1}{3}(\alpha\alpha' x) b_\beta b_\alpha u_{\alpha'} u_\beta \,.\, c_\gamma^2$$

$$= (\gamma\alpha x)\, b_\gamma b_\alpha \,.\, c_\beta c_{\alpha'} u_\beta u_{\alpha'} - \tfrac{2}{3} a_\alpha^2 \,.\, (abu)\, a_\gamma b_\gamma \,.\, c_\beta u_\beta c_x,$$

and $\quad (\alpha'\beta x)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} u_\gamma = (\alpha'\beta x)\, b_\gamma c_\beta b_\alpha u_\gamma \{c_{\alpha'} u_\alpha + (cu \,.\, \alpha\alpha')\}$

$$= (\alpha'\beta x)\, c_{\alpha'} c_\beta \,.\, b_\gamma b_\alpha u_\gamma u_\alpha + \tfrac{1}{2}(cu \,.\, \alpha\alpha')\,\{(\beta\alpha\alpha')\, b_x - (x\alpha\alpha')\, b_\beta\}\, b_\gamma c_\beta u_\gamma$$

$$= (\alpha'\beta x)\, c_{\alpha'} c_\beta \,.\, b_\gamma b_\alpha u_\gamma u_\alpha + \tfrac{2}{3} a_\alpha^2 \,.\, (cua)\, a_\beta c_\beta \,.\, b_x b_\gamma u_\gamma,$$

omitting $-\tfrac{1}{6}(cu \,.\, \alpha\alpha')(x\alpha\alpha')\, c_\gamma u_\gamma \,.\, b_\beta^2 = 0,$

completing the reduction of the 18 forms on page 84.

§ VI. *Identities and examples.*

The following are given, some because used, others because noteworthy.

1. *The invariant* $t = \eta_\sigma^3$ *of Gundelfinger.*

To establish the identities $-\frac{1}{6}\eta_\sigma^3 = [(abc)^2]^2 + \frac{3}{2}(\alpha\beta\gamma)^2 - (a_\beta^2 \,.\, a_\gamma^2 + b_\gamma^2 \,.\, b_\alpha^2 + c_\alpha^2 \,.\, c_\beta^2)$

$$(a'b'c')(a'bc)(b'ca)(c'ab) = [(abc)^2]^2 + \tfrac{1}{2}(\alpha\beta\gamma)^2 - \tfrac{1}{2}(a_\beta^2 \,.\, a_\gamma^2 + b_\gamma^2 \,.\, b_\alpha^2 + c_\alpha^2 \,.\, c_\beta^2)$$

where we put
$$\eta_x^3 = 6(abc)\,a_x b_x c_x,$$
$$u_\sigma^3 = -6(bcu)(cau)(abu),$$

and these are the definitions of the symbols η_x^3 *and* u_σ^3. These give

$$-u_\sigma v_\sigma w_\sigma = (bcu)(cav)(abw) + (bcu)(caw)(abv) + (cau)(abv)(bcw) + (cau)(abw)(bcv)$$
$$+ (abu)(bcv)(caw) + (abu)(bcw)(cav),$$

$$\therefore\ -\tfrac{1}{6}\eta_\sigma^3 = -(abc)\,a_\sigma b_\sigma c_\sigma = (a'b'c')(a'bc)(b'ca)(c'ab) + (a'b'c')(a'bc)(c'ca)(b'ab)$$
$$+ (caa')(abb')(bcc')(a'b'c') + (caa')(abc')(bcb')(a'b'c') + (a'b'c')(aba')(bcb')(cac')$$
$$+ (a'b'c')(aba')(bcc')(cab'),$$

and

$$(a'b'c')(bca')(cac')(abb') = \tfrac{1}{2}a_\beta(acc')\{(ca'b)(a'c'b') - (ca'b')(a'c'b)\} = -\tfrac{1}{2}a_\beta a_\beta' a_\gamma a_\gamma' = \tfrac{1}{4}(\alpha\beta\gamma)^2 - \tfrac{1}{2}a_\beta^2 \,.\, a_\gamma^2$$

$$(a'b'c')(caa')(bcc')(abb') = \tfrac{1}{2}a_\beta(caa')\{(cc'b)(c'a'b') - (cc'b')(c'a'b)\} = \tfrac{1}{2}a_\beta c_\beta'(caa')(cc'a')$$
$$= \tfrac{1}{4}c_\beta' c_\alpha(c_\alpha c_\beta' - c_\beta c_\alpha') = \tfrac{1}{4}c_\beta^2 c_\alpha^2 + \tfrac{1}{8}(\alpha\beta\gamma)^2 - \tfrac{1}{4}c_\alpha^2 c_\beta^2 = \tfrac{1}{8}(\alpha\beta\gamma)^2$$

$$(a'b'c')(caa')(abc')(bcb') = \tfrac{1}{2}c_\alpha(bcb')\{(bc'a)(b'c'a') - (bc'a')(b'c'a)\} = \tfrac{1}{2}c_\alpha c_\alpha'(bcb')(bb'c') = \tfrac{1}{4}(\alpha\beta\gamma)^2 - \tfrac{1}{2}c_\alpha^2 \,.\, c_\beta^2$$

$$(a'b'c')(aba')(bcb')(cac') = \tfrac{1}{2}b_\alpha(cbb')\{(b'c'a')(c'ca) - (b'c'a)(c'ca')\} = \tfrac{1}{2}b_\alpha c_\alpha'(cbb')(b'cc')$$
$$= \tfrac{1}{4}b_\alpha b_\gamma'(b_\gamma' b_\alpha - b_\alpha' b_\gamma) = \tfrac{1}{4}b_\alpha^2 b_\gamma^2 + \tfrac{1}{8}(\alpha\beta\gamma)^2 - \tfrac{1}{4}b_\gamma^2 b_\alpha^2 = \tfrac{1}{8}(\alpha\beta\gamma)^2$$

$$(a'b'c')(aba')(bcc')(cab') = \tfrac{1}{2}b_\gamma(aba')\{(ab'c)(a'b'c') - (ab'c')(a'b'c)\} = -\tfrac{1}{2}b_\gamma b_\gamma' b_\alpha b_\alpha' = \tfrac{1}{4}(\alpha\beta\gamma)^2 - \tfrac{1}{2}b_\gamma^2 \,.\, b_\alpha^2,$$

from which the result above given immediately follows.

Further

$$(a'b'c')\,a_x' b_y' c_z' - (a'b'c')\,a_x' b_z' c_y' + (a'b'c')\,b_x' c_y' a_z' - (a'b'c')\,b_x' c_z' a_y' + (a'b'c')\,c_x' a_y' b_z' - (a'b'c')\,c_x' a_z' b_y'$$
$$= (a'b'c')^2 \,.\, (xyz).$$

Put herein $x_i,\ y_i,\ z_i = (bc)_i,\ (ca)_i,\ (ab)_i$.

Then $(a'b'c')^2 \,.\, (abc)^2 = (a'b'c')(a'bc)(b'ca)(c'ab) - (a'b'c')(a'bc)(b'ab)(c'ca)$

$$+(a'b'c')(b'bc)(c'ca)(a'ab) - (a'b'c')(b'bc)(c'ab)(a'ca) + (a'b'c')(c'bc)(a'ca)(b'ab) - (a'b'c')(c'bc)(a'ab)(b'ca)$$

from which, by the results given, the above formula follows.

2. *To find the value of* $u_\sigma a_\sigma^2$ *where, as in* 1, $u_\sigma^3 = -6(bcu)(cau)(abu)$.

We have $-\frac{1}{2}u_\sigma v_\sigma^2 = (bcu)(cav)(abv) + (cau)(abv)(bcv) + (abu)(bcv)(cav)$.

whence
$$-\tfrac{1}{2}u_\sigma a_\sigma^2 = (bcu)(caa')(aba') + (cau)(aba')(bca') + (abu)(bca')(caa')$$
$$= -(bcu)\,b_\alpha c_\alpha - \tfrac{1}{2}b_\alpha\{(uca)(bca') - (uca')(bca)\} + \tfrac{1}{2}c_\alpha\{(bua)(bca') - (bua')(bca)\}$$
$$= -(bcu)\,b_\alpha c_\alpha - \tfrac{1}{2}b_\alpha c_\alpha(ubc) + \tfrac{1}{2}c_\alpha b_\alpha(buc)$$

or
$$u_\sigma a_\sigma^2 = 4(bcu)\,b_\alpha c_\alpha,$$

namely, with Gundelfinger, $u_p = 3u_\sigma a_\sigma^2 = 12\,(bcu)\,b_\alpha c_\alpha$
(where Gundelfinger uses u_a for u_p).

So $$u_q = 3u_\sigma b_\sigma^2 = 12\,(cau)\,c_\beta a_\beta,$$
$$u_r = 3u_\sigma c_\sigma^2 = 12\,(abu)\,a_\gamma b_\gamma,$$

and these are the definitions of the points p, q, r.

3. *To find the value of (qrx) in terms of our concomitants.*

$$\tfrac{1}{144}(qrx) = \begin{vmatrix} (ca')_1 & (ca')_2 & (ca')_3 \\ (ab')_1 & (ab')_2 & (ab')_3 \\ x_1 & x_2 & x_3 \end{vmatrix} c_\beta a_\beta' a_\gamma b_\gamma' = \begin{vmatrix} (cab') & (a'ab') \\ c_x & a_x' \end{vmatrix} c_\beta a_\beta' a_\gamma b_\gamma' = (ab'c)\,a_x' c_\beta a_\beta' a_\gamma b_\gamma' + (aa'b')\,c_x c_\beta a_\beta' a_\gamma b_\gamma'$$

$$= (a'bc)\,a_x c_\beta a_\beta a_\gamma' b_\gamma + \tfrac{1}{2} b_\alpha' (\alpha\gamma\beta)\,c_x c_\beta b_\gamma' = (a'bc)\,a_x' a_\beta a_\gamma b_\gamma c_\beta + (aa'\,.\,x\gamma)\,(a'bc)\,c_\beta a_\beta b_\gamma - \tfrac{1}{2}(\alpha\beta\gamma)\,b_\gamma b_\alpha c_x c_\beta$$
$$= (a'bc)\,a_x' a_\beta a_\gamma b_\gamma c_\beta + \tfrac{1}{2}(\gamma\alpha x)\,(b_\alpha c_\beta - b_\beta c_\alpha)\,b_\gamma c_\beta - \tfrac{1}{2}\{(x\beta\gamma)\,b_\alpha c_\alpha b_\gamma c_\beta + (\gamma\alpha x)\,b_\gamma b_\alpha\,.\,c_\beta^2 + (\alpha\beta x)\,b_\gamma b_\alpha c_\gamma c_\beta\}$$
$$= (a'bc)\,a_x' a_\beta a_\gamma b_\gamma c_\beta - \tfrac{1}{6}(\gamma\alpha x)\,c_\alpha c_\gamma\,.\,b_\beta^2 - \tfrac{1}{6}(\alpha\beta x)\,b_\alpha b_\beta\,.\,c_\gamma^2 - \tfrac{1}{2}(\beta\gamma x)\,b_\alpha c_\alpha b_\gamma c_\beta$$
$$= (a'bc)\,a_x' a_\beta a_\gamma b_\gamma c_\beta - \tfrac{1}{2}(\beta\gamma x)\,b_\alpha c_\alpha b_\gamma c_\beta.$$

It is then expressed by the two straight lines $(801)_1$ and $(801)_2$.

4. *To shew that the invariant $(pqr) = s$ is expressible by our concomitants.*

It is afterwards shewn otherwise, after Gundelfinger, that it is $= 8t^2 - 12\,S(\eta_x^3)$, $S(\eta_x^3)$ meaning the quarticinvariant of the ternary cubic η_x^3. But by definition

$$\frac{1}{12}\,.\,s = \frac{1}{1728}(pqr)$$
$$= \begin{vmatrix} (bc')_1 & (bc')_2 & (bc')_3 \\ (ca')_1 & (ca')_2 & (ca')_3 \\ (ab')_1 & (ab')_2 & (ab')_3 \end{vmatrix} b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma b_\gamma' = \begin{vmatrix} (bca'), & (bab') \\ (c'ca'), & (c'ab') \end{vmatrix} b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma b_\gamma' = (a'bc)\,(ab'c')\,b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma b_\gamma' - (abb')\,(a'cc')\,b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma b_\gamma'$$

and $(abb')\,(acc')\,b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma b_\gamma' = \tfrac{1}{4} a_{\beta'} (\beta'\alpha\gamma)\,a_{\gamma'} (\gamma'\beta\alpha)\,a_\gamma a_\beta' \equiv \tfrac{1}{4}(\alpha\beta'\gamma)\,(\alpha\beta\gamma')\,a_{\beta'} a_{\gamma'} a_\gamma a_\beta'$

$$\equiv \tfrac{1}{4}\{(\alpha\beta'\gamma')\,a_\gamma + (\alpha\gamma'\gamma)\,a_{\beta'} + (\gamma'\beta'\gamma)\,a_\alpha\}\{(\alpha\beta'\gamma')\,a_\beta + (\alpha\beta\beta')\,a_{\gamma'} + (\beta'\beta\gamma')\,a_\alpha\}\,a_\gamma a_\beta'$$

which is reducible;

also $(a'bc)\,(ab'c')\,b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma b_\gamma' \equiv \{(a'b'c)\,b_\gamma + (b'bc)\,a_\gamma'\}\,(ab'c')\,b_\alpha c_\alpha' c_\beta a_\beta' a_\gamma$

$$\equiv \{(a'b'c')\,c_\alpha + (a'c'c)\,b_\alpha'\}\,(ab'c')\,b_\gamma b_\alpha c_\beta a_\beta' a_\gamma - \tfrac{1}{2} c_{\beta'} a_\gamma' c_\alpha' c_\beta a_\beta' a_\gamma (c'_{\beta'} a_\alpha - c_\alpha' a_{\beta'})$$
$$(a'b'c')\,\{(a'b'c')\,a_\beta + (ab'a')\,c_\beta'\}\,c_\alpha b_\gamma b_\alpha c_\beta a_\gamma - \tfrac{1}{2} a'_{\gamma'} b_\alpha' b_\gamma b_\alpha a_\beta' a_\gamma (a_{\gamma'} b_\beta' - a_\beta b'_{\gamma'}) - \tfrac{1}{2} c_{\beta'} a_\gamma' c_\alpha' c_\beta a_\beta' a_\gamma (c'_{\beta'} a_\alpha - c_\alpha' a_{\beta'})$$
$$\equiv -\tfrac{1}{2} b'_{\alpha'} (b'_\alpha c_\gamma' - b_\gamma' c'_{\alpha'})\,c_\beta' c_\alpha b_\gamma b_\alpha c_\beta - \tfrac{1}{2} a'_{\gamma'} b_\alpha' b_\gamma b_\alpha a_\beta' a_\gamma (a_\gamma b_\beta' - a_\beta b'_{\gamma'}) - \tfrac{1}{2} c_{\beta'} a_\gamma' c_\alpha' c_\beta a_\beta' a_\gamma (c'_{\beta'} a_\alpha - c_\alpha' a_{\beta'})$$
$$\equiv \tfrac{1}{2} b'_{\alpha'} b_\gamma' c'_{\alpha'} c_\beta' c_\alpha b_\gamma b_\alpha c_\beta + \tfrac{1}{2} a'_{\gamma} b_\alpha' b_\gamma b_\alpha a_\beta' a_\gamma a_\beta b'_{\gamma'}$$
$$= \tfrac{1}{2}(bb'\,.\,\gamma\alpha')\,b_\alpha b_\gamma' c_\alpha c_\beta c'_{\alpha'} c_\beta' + \tfrac{1}{2}(bb'\,.\,\alpha\gamma')\,b_\alpha' b_\gamma a_\beta a_\gamma a_\beta' a'_{\gamma'}$$
$$= \tfrac{1}{4}(\beta'\gamma\alpha')\,(\beta'\alpha\gamma)\,c_\alpha c_\beta c'_{\alpha'} c_\beta' + \tfrac{1}{4}(\beta'\alpha\gamma')\,(\beta'\gamma\alpha)\,a_\beta a_\gamma a_\beta' a'_{\gamma'}$$
$$\equiv \tfrac{1}{4}(\alpha\gamma\alpha')\,(\beta'\alpha\gamma)\,c_{\beta'} c_\beta c'_{\alpha'} c_\beta' + \tfrac{1}{4}(\gamma\alpha\gamma')\,(\beta'\gamma\alpha)\,a_\beta a_{\beta'} a'_\beta a'_{\gamma'}$$
$$\equiv 0.$$

This indicates how its value and thence that of $S(\eta_x^3)$ can be actually found in terms of the 11 fundamental invariants.

5. Putting $$(pqr) f_x^3 = (qrx)\, a_x^2 + (rpx)\, b_x^2 + (pqx)\, c_x^2,$$
(this is the cubic of which the conics are first polars, as will be proved)
and
$$u_p = 12\,(bcu)\, b_\alpha c_\alpha \text{ etc.}$$
$$(\beta\gamma x)^2 = \lambda_x^2 \text{ etc.}$$
$$b_x c_x b_\alpha c_\alpha = r_x^2 \text{ etc.}$$
$$(bcu)\, a_\beta a_\gamma b_\gamma c_\beta = u_f \text{ etc.}$$
$$(\alpha\beta\gamma)\, u_\alpha a_\beta a_\gamma = u_l \text{ etc.}$$

Then it may be shewn that
$$\tfrac{1}{144}(pqr) f_x^3 = \tfrac{1}{12} a_x^2 (\tfrac{1}{2} u_p \,.\, \lambda_x^2 + u_q \,.\, t_x^2 + u_r \,.\, s_x^2) + \text{two similar terms}$$
$$+ (a_x^2)^2 \,.\, u_f + (b_x^2)^2 \,.\, u_g + (c_x^2)^2\, u_h - b_x^2 c_x^2 \,.\, u_l - c_x^2 \,.\, a_x^2 u_m - a_x^2 \,.\, b_x^2 \,.\, u_n,$$
which expresses the cubic in our forms.

6. *To find* $u_\sigma^2 a_\sigma a_x$
$$-\tfrac{1}{2} u_\sigma^2 v_\sigma = (bcv)(cau)(abu) + (cav)(abu)(bcu) + (abv)(bcu)(cau),$$
$$\therefore\ -\tfrac{1}{2} u_\sigma^2 a_\sigma a_x = (a'bc)(cau)(abu)\, a_x' + (caa')(abu)(bcu)\, a_x' + (aba')(bcu)(cau)\, a_x'$$
$$= (a'bc)(cau)(abu)\, a_x' + \tfrac{1}{2}(bcu)\, b_\alpha c_x u_\alpha + \tfrac{1}{2} c_\alpha b_x u_\alpha (bcu) - u_x \,.\, (bcu)\, b_\alpha c_\alpha.$$

7. $$(a'bc)(b'ca)(c'ab)\, a_x' b_y' c_z' = (abc)^2 \,.\, (abc)\, a_x b_y c_z - \tfrac{1}{4}\alpha_\beta^2 \,.\, (\gamma yz)\, a_x a_\gamma + \tfrac{1}{4} b_\alpha^2 \,.\, (\gamma xz)\, b_\gamma b_y$$
$$- \tfrac{1}{4} c_\alpha^2 \,.\, (\beta xy)\, c_\beta c_z + \tfrac{1}{4}(\beta\gamma x)\, a_\beta a_\gamma \,.\, a_x a_y - \tfrac{1}{4}(\gamma\alpha z)\, b_\gamma b_\alpha \,.\, b_x b_y + \tfrac{1}{4}(\beta\alpha y)\, c_\beta c_\alpha \,.\, c_x c_z$$
$$+ \tfrac{1}{8}(\alpha\beta\gamma)(\beta xy)(\gamma\alpha z) - \tfrac{1}{8}(\alpha\beta x)(\beta\gamma y)(\gamma\alpha z)$$
$$+ \tfrac{1}{8}(\alpha\beta\gamma)(\gamma zx)(\alpha\beta y) - \tfrac{1}{8}(\gamma\alpha x)(\alpha\beta y)(\beta\gamma z)$$
$$+ \tfrac{1}{8}(\alpha\beta\gamma)(\gamma yz)(\alpha\beta x).$$

8. Thus $$(a'bc)(b'ca)(c'ab)\, a_x' b_x' c_x' = -\tfrac{1}{4}(\beta\gamma x)(\gamma\alpha x)(\alpha\beta x) + (abc)^2 \,.\, (abc)\, a_x b_x c_x$$
$$+ \tfrac{1}{4}(\beta\gamma x)\, a_\beta a_\gamma \,.\, a_x^2 - \tfrac{1}{4}(\gamma\alpha x)\, b_\gamma b_\alpha \,.\, b_x^2 - \tfrac{1}{4}(\alpha\beta x)\, c_\alpha c_\beta c_x^2,$$
or say $$(\beta\gamma x)(\gamma\alpha x)(\alpha\beta x) \equiv -4\,(a'bc)(b'ca)(c'ab)\, a_x' b_x' c_x' \equiv a_\sigma b_\sigma c_\sigma a_x b_x c_x.$$

9. *Miscellaneous.*

$$(bcu)(cau)\, b_\alpha u_\alpha a_x = -(cau)^2 \,.\, b_\alpha b_x u_\alpha + (abu)^2 \,.\, c_\alpha c_x u_\alpha - u_\alpha^2 \,.\, (abc)(abu)\, c_x$$
$$- \tfrac{1}{2} u_x \{\tfrac{1}{3} a_\alpha^2 \,.\, (bcu)^2 + (abu)^2 \,.\, c_\alpha^2 - (cau)^2 \,.\, b_\alpha^2 - (bcu)^2 \,.\, u_\alpha^2\},$$
$$(bcu)(bca)(b'c'a)\, b_x' c_x' = (bca)^2 \,.\, (bcu)\, b_x c_x + \tfrac{1}{2}(uab)\, a_x b_x \,.\, b_\gamma^2 - \tfrac{1}{2}(uab)\, a_\gamma b_\gamma \,.\, b_x^2$$
$$+ \tfrac{1}{2}(uca)\, c_x a_x \,.\, c_\beta^2 - \tfrac{1}{2}(uca)\, c_\beta a_\beta \,.\, c_x^2 + \tfrac{1}{2} u_x \,.\, (\beta\gamma x)\, a_\beta a_\gamma - \tfrac{1}{4}(\beta\gamma x)\, a_x a_\beta u_\gamma - \tfrac{1}{4}(\beta\gamma x)\, a_x a_\gamma u_\beta,$$
$(abc)\, a_\beta b_\gamma c_x u_\beta u_\gamma \equiv (ubc)\, a_\beta b_\gamma u_\beta a_\gamma c_x \equiv (uba)\, c_\beta b_\gamma u_\beta a_\gamma c_x$ or $(uba)\, a_\gamma b_\gamma \,.\, c_\beta c_x u_\beta \equiv 0,$

$(abc)\, u_\beta u_\gamma a_x b_\gamma c_\beta \equiv 0,$

$(abc')(ab'c)(ubc)(a'b'u)\, a_x' c_x'$ reduces to the forms $(\alpha\beta\gamma)(\beta\gamma x)\, u_\alpha$, save as to products of forms,

$u_\sigma^2 a_\sigma b_x c_x (abc) \equiv (cau)(abu)(a'bc)(a'b'c')\, b_x' c_x' \equiv 0,$

$(a\sigma x)\, u_\sigma b_\sigma b_x \equiv u_\alpha c_\alpha c_\beta a_\beta a_x,$

$(abc')(ab'c)(ubc)(b'a'u)(c'a'u) \equiv (abc)(bcu)\, a_\sigma u_\sigma^2.$

§ VII. *An account of the theory of three conics as given by Gundelfinger, Rosanes, and in Clebsch's lectures.*

§ 1. *Establishment of the cubic of which the conics are first polars.*

For a ternary cubic $f = f_x^3 = g_x^3 = h_x^3 = \ldots$ I write the Hessian, after Clebsch, $H = (fgh)^2 f_x g_x h_x = H_x^3 =$ etc., the quarticinvariant $S = -(fgh)(ghi)(hfi)(fgi)$, the sexticinvariant $T = -(f'g'h')^2 (fgh)(f'gh)(g'hf)(h'fg)$ and the Cayleyan

$$u_s^2 = -(fgh)(ghu)(hfu)(fgu),$$

then we have the known equations

$$f_s^2 f_x u_s = \tfrac{1}{3} S u_x, \quad f_s^3 = S, \quad H_s^3 = T.$$

And since a system of three conics is determined by $3 \cdot 5 =$ a fifteenfold arbitrariness, while a system consisting of a ternary cubic and three points is given by $9 + 3 \cdot 2 =$ also a fifteenfold arbitrariness, it is to be expected that from a system of one kind we can uniquely determine a system of the other: in particular, in order that three conics $a_x^2 = a_x'^2 = \ldots$, b_x^2, c_x^2 should be the polar conics of a ternary cubic f_x^3 in regard to three points p, q, r, it is sufficient that

$$a_x^2 = f_x^2 f_p, \quad b_x^2 = f_x^2 f_q, \quad c_x^2 = f_x^2 f_r,$$

leading to $\qquad (pqr) f_x^2 f_\xi = (qr\xi)\, ax^2 + (pq\xi)\, b_x^2 + (pq\xi)\, c_x^2,$

which gives $u_\sigma^3 = -6\,(bcu)(cau)(abu) = -6\,(ghu)(hfu)(fgu) f_p g_q h_r = 6\,(ghu)(hfu)(fgu) f_p g_r h_q$

$$= -3\,(ghu)(hfu)(fgu) f_p (g_q h_r - g_r h_q)$$
$$= -3\,(hfu)(fgu)(ghu)\, g_p (h_q f_r - h_r f_q)$$
$$= -3\,(fgu)(ghu)(hfu)\, h_p (f_q g_r - f_r g_q)$$
$$=, \text{ by addition, } (pqr)\, u_s^3,$$

or, $\qquad u_\sigma^3 = (pqr)\, u_s^3,$

and therefore $\qquad v_\sigma^2 u_\sigma = (pqr)\, v_s^2 u_s,$

or in particular $\qquad a_\sigma^2 u_\sigma = (pqr)\, a_s^2 u_s = (pqr) f_s^2 f_p u_s = \tfrac{1}{3} (pqr)\, S \,.\, u_p,$

and similarly $\qquad b_\sigma^2 u_\sigma = \tfrac{1}{3} (pqr)\, S \,.\, u_q, \quad c_\sigma^2 u_\sigma = \tfrac{1}{3} (pqr)\, S u_r,$

so that the points p, q, r must in fact be the points $a_\sigma^2 u_\sigma = 0$, $b_\sigma^2 u_\sigma = 0$, $c_\sigma^2 u_\sigma = 0$, and we may take the arbitraries so that $u_p = 3a_\sigma^2 u_\sigma$, $u_q = 3b_\sigma^2 u_\sigma$, $u_r = 3c_\sigma^2 u_\sigma$, $(pqr)\, S = 1$;

while conversely if $\qquad (pqr) f_x^3 = (qrx)\, a_x^2 + (rpx)\, b_x^2 + (pqx)\, c_x^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(i)},$

then $\qquad 3\,(pqr) f_x^2 f_p = (qrp)\, a_x^2 + 2\,(rpx)\, b_x b_p + 2\,(pqx)\, c_x c_p + 2\,(qrx)\, a_x a_p,$

and (as already shewn) $u_p = 12\,(bcu)\, b_\alpha c_\alpha$ so that $b_x b_p = 12\,(bcb')\, b_x' b_\alpha c_\alpha = 6\,(\alpha\beta x)\, c_\alpha c_p = a_x a_q,$

and $\qquad c_\alpha c_p = 6\,(\gamma\alpha x)\, b_\gamma b_\alpha = a_x a_r,$

therefore $\quad 3\,(pqr) f_x^2 f_p = (pqr)\, a_x^2 + 2a_x \{(qrx)\, a_p + (rpx)\, a_q + (pqx)\, a_r\} = 3\,(pqr)\,.\, a_x^2,$

or $\qquad f_x^2 f_p = a_x^2.$

Whence $\qquad f_x^2 f_q = b_x^2, \quad f_x^2 f_r = c_x^2, \quad (pqr) f_x^2 f_\xi = (qr\xi)\, a_x^2 + (rp\xi)\, b_x^2 + (pq\xi)\, c_x^2.$

So that equation (i) properly determines the cubic in question.

§ 2. *Expression of the cubic.*

The cubic (i) may be expressed by our concomitants, for we have shewn

$$\tfrac{1}{144}(qrx) = (a'bc)\, a_x' a_\beta a_\gamma b_\gamma c_\beta - \tfrac{1}{2}(\beta\gamma x)\, b_\alpha c_\alpha b_\gamma c_\beta\,;$$

$$\therefore\ \tfrac{1}{144}(pqr) f_x^3 = a_x^2\,.\,\{(a'bc)\, a_x' a_\beta a_\gamma b_\gamma c_\beta - \tfrac{1}{2}(\beta\gamma x)\, b_\alpha c_\alpha b_\gamma c_\beta\} + + \ldots\ldots\ldots\ldots \text{(ii)}.$$

Or again it may be expressed, after Gundelfinger, in terms of the discriminant, in regard to x, of $\mu_1 a_x^2 + \mu_2 b_x^2 + \mu_3 c_x^2$, $= d_x^2$ say, the discriminant being defined as

$$(dd'd'')^2 = 6 \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}.$$

For putting $\xi_i = p_i\mu_1 + q_i\mu_2 + r_i\mu_3$, $(i = 1, 2, 3)$, so that in fact μ_1, μ_2, μ_3 are the co-ordinates of the point ξ in regard to the triangle p, q, r, we have, solving for μ_1, μ_2, μ_3 in terms of ξ and substituting in the definition equation of d_x^2,

$$d_x^2 = \frac{1}{(pqr)}\{(qr\xi)\, a_x^2 + (rp\xi)\, b_x^2 + (pq\xi)\, ca^2\} = f_x^2 f_\xi,$$

and thence the discriminant $(dd'd'')^2$ is equal to H_ξ^3, namely to the Hessian of f_ξ^3, while $(dd'd'')^2$ is in fact the cubic

$$d_\mu^3 = a_\alpha^2\,.\,\mu_1^3 + b_\beta^2\,.\,\mu_2^3 + c_\gamma^2\,.\,\mu_3^3 + 3b_\alpha^2\,.\,\mu_1^2\mu_2 + 3c_\alpha^2\mu_1^2\mu_3 + 3c_\beta^2\mu_2^2\mu_3 + 3a_\beta^2\mu_2^2\mu_1 + 3a_\gamma^2\mu_3^2\mu_1 + 3b_\gamma^2\mu_3^2\mu_2 + 6\,(abc)^2\,\mu_1\mu_2\mu_3,$$

and therefore, remembering that the Hessian of the Hessian of a ternary cubic is

$$\tfrac{1}{12} S^2 f - \tfrac{1}{3} TH,$$

we see that the Hessian of $(dd'd'')^2$ in regard to μ, namely $H(d_\mu^3)$ is equal to

$$(pqr)^2\,.\,\{\tfrac{1}{12} S^2 f_\xi^3 - \tfrac{1}{3} TH_\xi^3\}:$$

namely [as $(pqr)\,S = 1$

and $\eta_x^3 = 6\,(abc)\, a_x b_x c_x = 6\,(fgh) f_x g_x h_x f_p g_q h_r = (pqr)\, H_x^3$,

whence $t = \eta_\sigma^3 = (pqr)\,\eta_s^3 = (pqr)^2\,.\,H_s^3 = (pqr)^2\, T \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta)$,]

we have $H(d_\mu^3) = \tfrac{1}{12} f_\xi^3 - \tfrac{1}{3} t d_\mu^3$,

that is $f_\xi^2 = 12\, H(d_\mu^3) + 4t\,.\,d_\mu^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (iii),

which gives the value of f_ξ^3 (referred to p, q, r as triangle of co-ordinates and) expressed in terms of the discriminant in regard to x of $\mu_1 a_x^2 + \mu_2 b_x^2 + \mu_3 c_x^2$.

And the 10 invariants $a_\alpha^2, \ldots b_\alpha^2, \ldots, (abc)^2$ are expressible by the cubic,

for $d_\mu^3 = H_\xi^3 = (H_p\,.\,\mu_1 + H_q\,.\,\mu_2 + H_r\,.\,\mu_3)^3$,

so that $a_\alpha^2 = H_p^3 \ldots b_\alpha^2 = H_p^2 H_q \ldots (abc)^2 = H_p H_q H_r$,

with which compare $a_p^2 = f_p^3 \ldots a_q^2 = f_p f_q^2 \ldots a_q a_r = b_r b_p = c_p c_q = f_p f_q f_r$.

Further the conic a_x^2 being in fact $(a_p\lambda_1 + a_q\lambda_2 + a_r\lambda_3)^2$ (where $\lambda_1\lambda_2\lambda_3$ are the current co-ordinates) when referred to the p, q, r triangle, it is seen that the 18 coefficients of the three conics are in fact only **10**, corresponding to some extent to the simplification when two conics are referred to their common self-polar triangle.

While also, remembering that the quartic and sextic invariants of the Hessian of a ternary cubic are in fact $\frac{2}{3}T^2 - \frac{1}{12}S^3, \quad \frac{1}{24}S^3T - \frac{2}{9}T^3,$

it follows from
$$d_\mu^3 = H_\xi^3$$
that
$$S(d_\mu^3) = (pqr)^4\{\tfrac{2}{3}T^2 - \tfrac{1}{12}S^3\} = \tfrac{2}{3}t^2 - \tfrac{1}{12}(pqr),$$
where S, T are, as previously, invariants of f_x^3,

and
$$T(d_\mu^3) = \ldots\ldots \qquad = \tfrac{1}{12}t\,(pqr) - \tfrac{2}{9}t^3;$$
and therefore
$$\left.\begin{aligned}(pqr) &= 8t^2 - 12\,S(d_\mu^3)\\ 2t^3 - 9tS(d_\mu^3) - 18T(d_\mu^3) &= 0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$
and

So that any invariant is a rational function of the ten $a_a^2, \ldots b_a^2, \ldots (abc)^2$ and of t.

The previous mode of expression is Gundelfinger's. Otherwise we may say
$$\eta_\xi^3 = (pqr)\,d_\mu^3 = (pqr)\,\{a_a^2 \,.\, \mu_1^3 + \ldots + 3b_a^2\mu_1^2\mu_2 + \ldots + 6\,(abc)^2\mu_1\mu_2\mu_3\},$$
giving the equation of η_ξ^3 referred to Gundelfinger's triangle,

and
$$H(\eta_\xi^3) = (pqr)\,H(d_\mu^3), \quad S(\eta_\xi^3) = S(d_\mu^3), \quad T(\eta_\xi^3) = T(d_\mu^3),$$
$$(pqr) = 8t^2 - 12S(\eta_\xi^3), \quad 2t^3 - 9tS(\eta_\xi^3) - 18T(\eta_\xi^3) = 0,$$
$$(pqr)\,f_\xi^3 = (pqr)\,\{\mu_1\bar{a}_\mu^2 + \mu_2 b_\mu^2 + \mu_3\bar{c}_\mu^2\} = 12H(\eta_\xi^3) + 4t\eta_\xi^3 \ldots\ldots\ldots\ldots\ldots\text{(v)},$$
giving the expression of f_ξ^3 in terms of η_ξ^3.

And we may see the exact significance of the cubic satisfied by t, by putting
$$S(\eta_\xi^3) = \tfrac{1}{2}g_2,$$
$$T(\eta_\xi^3) = -\tfrac{3}{4}g_3,$$
and
$$4u^3 - g_2u - g_3 = 4\,(u - e_1)(u - e_2)(u - e_3).$$

Then the cubic solves and we obtain $t = -3e_i$ and therefore from (v)
$$(pqr)\,f_\xi^3 = 12\,\{H(\eta_\xi^3) - e_i \,.\, \eta_\xi^3\},$$
namely by a known theory f_ξ^3 is one of the three cubics of which η_ξ^3 is the Hessian, which is right; or, say, f_ξ^3 is a sub-Hessian of η_ξ^3.

And
$$(pqr) = 6\,\{12e_i^2 - g_2\} = 24\,(e_i - e_j)(e_i - e_k) = 12p''\omega_i,$$
where pu is Weierstrass' elliptic function, with g_2, g_3 as invariants, and ω_i a semi-period: and the interpretation of $(pqr) = 0$, $t = 0$ can be deduced.

Note too, the resultant of the three conics, vanishing with the discriminant of the cubic, or $S^3 - 6T^2$, vanishes with $\dfrac{1}{(pqr)^3} - 6\left[\dfrac{t}{(pqr)^2}\right]^2$,

namely with
$$(pqr) - 6t^2,$$
which is therefore the resultant of the three conics.

There is a third way in which we may express the equation of f_x^3.

For $$(pqr) f_x^2 f_\xi = (qr\xi)\, a_x^2 + (rp\xi)\, b_x^2 + (pq\xi)\, c_x^2, \text{ where } u_p = 3a_\sigma^2 u_\sigma$$

$$= \begin{vmatrix} a_x^2 & b_x^2 & c_x^2 & \\ p_1 & q_1 & r_1 & \xi_1 \\ p_2 & q_2 & r_2 & \xi_2 \\ p_3 & q_3 & r_3 & \xi_3 \end{vmatrix} = 27 \begin{vmatrix} a_x^2 & b_x^2 & c_x^2 & \\ a_\sigma^2\sigma_1 & b_\sigma^2\sigma_1 & c_\sigma^2\sigma_1 & \tfrac{1}{3}(uv)_1 \\ a_\sigma^2\sigma_2 & b_\sigma^2\sigma_2 & c_\sigma^2\sigma_2 & \tfrac{1}{3}(uv)_2 \\ a_\sigma^2\sigma_3 & b_\sigma^2\sigma_3 & c_\sigma^2\sigma_3 & \tfrac{1}{3}(uv)_3 \end{vmatrix} \text{ where } (uv)_i = \xi_i$$

$$= 9 \begin{vmatrix} a_x^2 & a_\sigma^2\sigma_1 & a_\sigma^2\sigma_2 & a_\sigma^2\sigma_3 \\ b_x^2 & b_\sigma^2\sigma_1 & b_\sigma^2\sigma_2 & b_\sigma^2\sigma_3 \\ c_x^2 & c_\sigma^2\sigma_1 & c_\sigma^2\sigma_2 & c_\sigma^2\sigma_3 \end{vmatrix} \begin{vmatrix} 0 & u_1 & u_2 & u_3 \\ 0 & v_1 & v_2 & v_3 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 9 \begin{vmatrix} a_\sigma^2 u_\sigma & b_\sigma^2 u_\sigma & c_\sigma^2 u_\sigma \\ a_\sigma^2 v_\sigma & b_\sigma^2 v_\sigma & c_\sigma^2 v_\sigma \\ a_x^2 & b_x^2 & c_x^2 \end{vmatrix} \quad \text{.........(vi)},$$

a form which will be afterwards obtained geometrically. But now using the equation, of which the proof may be momentarily deferred:—

$$\begin{vmatrix} a_x^2 & b_x^2 & c_x^2 \\ a_y^2 & b_y^2 & c_y^2 \\ a_z^2 & b_z^2 & c_z^2 \end{vmatrix} = \tfrac{2}{3}(xyz)\, \eta_x\eta_y\eta_z - \tfrac{1}{3}(yz\sigma)(zx\sigma)(xy\sigma),$$

where $$\eta_x^3 = 6\,(abc)\, a_x b_x c_x,$$
$$u_\sigma^3 = -\,6\,(bcu)\,(cau)\,(abu),$$

we have $$\tfrac{1}{9}(pqr) f_x^2 f_\xi = \tfrac{2}{3}(x\sigma\sigma')\, \eta_x u_\sigma v_{\sigma'} \eta_\sigma \eta_{\sigma'} + \tfrac{1}{3}(\sigma\sigma'\sigma'')\,(\sigma\sigma''x)\,(\sigma\sigma'x)\, u_{\sigma'} v_{\sigma''}$$
$$= \tfrac{1}{3}(x\sigma\sigma')\,(\xi\sigma\sigma')\, \eta_x\eta_\sigma\eta_{\sigma'} + \tfrac{1}{6}(\sigma\sigma'\sigma'')\,(\sigma'\sigma''\xi)\,(\sigma\sigma'x)\,(\sigma\sigma''x)$$

for $$\xi_i = (uv)_i,$$

and thence $$\tfrac{1}{9}(pqr) f_x^3 = \tfrac{1}{3}(x\sigma\sigma')^2\, \eta_x\eta_\sigma\eta_{\sigma'} - \tfrac{1}{6}(\sigma\sigma'\sigma'')\,(\sigma'\sigma''x)\,(\sigma''\sigma x)\,(\sigma\sigma'x) \quad \text{.........(vii)}.*$$

[We may prove the value of $\begin{vmatrix} a_x^2 & b_x^2 & c_x^2 \\ a_y^2 & b_y^2 & c_y^2 \\ a_z^2 & b_z^2 & c_z^2 \end{vmatrix}$ quoted, as follows (after Rosanes, *Math. Ann.* vi. 279)

from $$\left.\begin{aligned} \eta_x^3 &= 6\,(abc)\, a_x b_x c_x \\ u_\sigma^3 &= -\,6\,(bcu)\,(cau)\,(abu) \end{aligned}\right\} \quad \begin{aligned} \tfrac{1}{2}\eta_x^2\eta_y &= (abc)\, \{a_y b_x c_x + b_y c_x a_x + c_y a_x b_x\} \\ -\tfrac{1}{2} v_\sigma^2 u_\sigma &= (bcu)\,(cav)\,(abv) + (cau)\,(abv)\,(bcv) + (abu)\,(bcv)\,(cav). \end{aligned}$$

Therefore $\tfrac{1}{2}\{u_y \,.\, \eta_x^2\eta_y + (xy\sigma)^2 u_\sigma\}$ consists of terms like

$$(bcu)\,\{a_y^2 b_x c_x + a_y b_y a_x c_x + a_y c_y a_x b_x + (ab \,.\, xy)\,(ac \,.\, xy)\},$$

or $$(bcu)\,\{a_y^2 b_x c_x + a_y b_y a_x c_x + a_y c_y a_x b_x + a_x^2 \,.\, b_y c_y - a_y c_y \,.\, a_x b_x - a_y b_y \,.\, a_x c_x + a_y^2 \,.\, b_x c_x\},$$

or $$(bcu)\,\{2a_y^2 b_x c_x + a_x^2 b_y c_y\}.$$

That is

$$\tfrac{1}{2}\{u_y\eta_x^2\eta_y + (xy\sigma)^2 u_\sigma\} = (bcu)(2a_y^2 b_x c_x + a_x^2 b_y c_y) + (cau)(2b_y^2 c_x a_x + b_x^2 c_y a_y) + (abu)(2c_y^2 a_x b_x + c_x^2 a_y b_y).$$

* [From which it follows [since $(pqr)\, S = 1$, $u_\sigma^3 = (pqr)\, u_s^3$, $\eta_x^3 = (pqr)\, H_x^3$], that it must be possible to express a ternary cubic in terms of its Hessian, Cayleyan and quartic-invariant, in the form

$$\tfrac{1}{6} S^2 f_x^3 = \tfrac{1}{3}(xss')^2\, H_x H_s H_{s'} - \tfrac{1}{6}(ss's'')\,(s's''x)\,(s''sx)\,(ss'x),$$

and indeed $(xss')^2 u_s u_{s'} = (HH'u)^2 H_x H_x' + \tfrac{1}{2} S \,.\, (fgu)^2 f_x g_x$, and thence

$$(ss'x)^2 H_x H_s H_{s'} = \tfrac{1}{6} S^2 f - \tfrac{1}{3} TH,$$

and $$(ss's'')\,(s's''x)\,(s''sx)\,(ss'x) = -\tfrac{1}{3} S^2 f - \tfrac{2}{3} TH,$$

(*Math. Annal.* vi. 492. § 15

so that $$\tfrac{2}{3} S^2 f_x^3 = \tfrac{1}{3} S^2 f - \tfrac{2}{3} TH - (-\tfrac{1}{3} S^2 f - \tfrac{2}{3} TH),$$

which is right.]

So

$\frac{1}{2}\{u_x\eta_y^2\eta_x+(xy\sigma)^2u_\sigma\}=(bcu)(a_y^2b_xc_x+2a_x^2b_yc_y)+(cau)(b_y^2c_xa_x+2b_x^2c_ya_y)+(abu)(c_y^2a_xb_x+2c_x^2a_yb_y)$,

whence $\frac{1}{3}\{u_x\,.\,\eta_y^2\eta_x-\frac{1}{2}u_y\,.\,\eta_x^2\eta_y+\frac{1}{2}(xy\sigma)^2u_\sigma\}=a_x^2\,.\,(bcu)\,b_yc_y+b_x^2\,.\,(cau)\,c_ya_y+c_x^2\,.\,(abu)\,a_yb_y$,

whence $\frac{1}{3}\{2u_x\eta_y\eta_z\eta_x-\frac{1}{2}u_y\,.\,\eta_x^2\eta_z-\frac{1}{2}u_z\,.\,\eta_x^2\eta_y+(xy\sigma)(xz\sigma)u_\sigma\}$

$$=a_x^2\,.\,(bcu)(b_yc_z+b_zc_y)+b_x^2\,.\,(cau)(c_ya_z+c_za_y)+c_x^2\,.\,(abu)(a_yb_z+a_zb_y),$$

or putting $u_i=(yz)_i$,

$$\tfrac{1}{3}\{2(xyz)\eta_x\eta_y\eta_z-(yz\sigma)(zx\sigma)(xy\sigma)\}=\begin{vmatrix} a_x^2 & b_x^2 & c_x^2 \\ a_y^2 & b_y^2 & c_y^2 \\ a_z^2 & b_z^2 & c_z^2 \end{vmatrix}.]$$

Theory of conjugate systems.

There is also a theory founded on a relation of a locus of points of the second order (say, shortly, a conic) to a cluster of rays of the second class (say, here, a cluster) [which is an extension of the relation of a conic to two points conjugate thereto or of two lines to a cluster in regard to which they are conjugate], under which relation [either curve may be said to be conjugate to the other or better] the locus may be said to be circumscribed to the cluster and the latter inscribed to the former. It is that poristic relation under which a single infinity of sets of three of the rays of the cluster form a trilateral self-polar in regard to the locus (so that the cluster-conic is in fact inscribed, viz. in a trilateral), and a single infinity of sets of three of the points of the locus form a triangle self-polar in regard to the cluster (so that the locus is in fact circumscribed, viz. to a triangle).

If a_x^2, b_x^2 be two conics, the cluster of tangents of the latter being $u_\lambda^2=u_{\lambda'}^2=\ldots=0$, to the former $u_\alpha^2=u_{\alpha'}^2=\ldots=0$, then the tangents to b_x^2 from the point $v_\alpha u_\alpha=0$, which is the pole of the line v in regard to a_x^2, are $(\lambda x\alpha)(\lambda x\alpha')\,v_\alpha v_{\alpha'}=0$, which are conjugate in regard to $a_x^2=0$ if $(\lambda\alpha\alpha'')(\lambda\alpha'\alpha'')\,v_\alpha v_{\alpha'}=0$.

But $0=(\lambda\alpha\alpha'')(\lambda\alpha'\alpha'')\,v_\alpha v_{\alpha'}=\frac{1}{2}v_{\alpha'}(\lambda\alpha\alpha'')\{(\lambda\alpha\alpha'')\,v_{\alpha'}-(\alpha'\alpha\alpha'')\,v_\lambda\}$

$$=\tfrac{2}{3}a_\alpha^2\{v_\alpha^2a_\lambda^2-a_\alpha v_\lambda a_\lambda v_\alpha\}=\tfrac{2}{3}a_\alpha^2\{v_\alpha^2a_\lambda^2-\tfrac{1}{3}a_\alpha^2v_\lambda^2\}$$

gives in general the cluster $v_\alpha^2a_\lambda^2-\frac{1}{3}a_\alpha^2v_\lambda^2=0$, of which the common tangents of a_x^2 and b_x^2 form part, which cluster coincides with that of the tangents of $b_x^2=0$ provided $a_\lambda^2=0$, and then we have b_x^2 inscribed in a single infinity of self-polar triangles of a_x^2, and also, as may be similarly shewn, a_x^2 circumscribed to a single infinity of self-polar triangles of b_x^2.

Or $a_\lambda^2=0$ is the condition that a_x^2 be circumscribed to u_λ^2.

And it is useful to bear in mind that

1. A conic is circumscribed to a two point cluster provided the points be conjugate in regard thereto—a_x^2 is circumscribed to $u_xu_y=0$ provided $a_xa_y=0$, which is the condition for conjugate points.

2. In particular a conic is circumscribed to a point cluster repeated, when the point is on the conic.

3. A cluster is inscribed to a two-line locus provided the lines be conjugate in regard to the cluster—$p_x q_x$ is circumscribed to u_α^2 provided $p_\alpha q_\alpha = 0$.

4. In particular is inscribed to a line-locus repeated, provided the line be a ray of the cluster.

And, as an example, the equation $f_s^2 f_x u_s = \frac{1}{3} S u_x$, quoted (p. 92), shews that the polar conic of any point in regard to a ternary cubic is circumscribed to the polar cluster in regard to the Cayleyan of any line through the point.

From which we derive the interpretation of equation (vi) of page 95—for if

$$\lambda_1 a_x^2 + \lambda_2 b_x^2 + \lambda_3 c_x^2$$

be the polar conic of a point ξ, where two lines $u_x = 0$, $v_x = 0$ intersect, in regard to a cubic, it must be circumscribed to the polar clusters of $u_x = 0$, $v_x = 0$, in regard to the Cayleyan $u_\sigma^3 = 0$.

Therefore $\lambda_1 a_\sigma^2 u_\sigma + \lambda_2 b_\sigma^2 u_\sigma + \lambda_3 c_\sigma^2 u_\sigma = 0,\ \lambda_1 a_\sigma^2 v_\sigma + \lambda_2 b_\sigma^2 v_\sigma + \lambda_3 c_\sigma^2 v_\sigma = 0,$

from which the equation follows.

Now to be given that a cluster is inscribed to a conic is equivalent to a single linear relation among the six coefficients in its equation, so that a cluster is determined by five circumscribing conics (in particular by five tangents). A 'swarm' (schaar) of clusters (the single infinity $f u_\lambda^2 + g u_\mu^2$), similarly, by four circumscribing conics, and finally a 'web' of clusters (the double infinity $g_1 u_\lambda^2 + g_2 u_\mu^2 + g_3 u_\nu^2$) by three circumscribing conics, or, say, by a circumscribing 'net' of conics $f_1 a_x^2 + f_2 b_x^2 + f_3 c_x^2$ (since $a_\lambda^2 = 0$, $b_\lambda^2 = 0$, $c_\lambda^2 = 0$, require also $g_1 a_\lambda^2 + g_2 b_\lambda^2 + g_3 c_\lambda^2 = 0$), and every cluster of the web is circumscribed to every conic of the net.

The equation of the cluster of this web which is also inscribed in the two arbitrary conics v_x^2, w_x^2 (which we may take to be repeated elements of the cluster, viz. v_x is a straight line as also w_x), is got from

$$a_\lambda^2 = 0,\ b_\lambda^2 = 0,\ c_\lambda^2 = 0,\ u_\lambda^2 = 0,\ v_\lambda^2 = 0,\ w_\lambda^2 = 0,$$

and is therefore

$$0 = \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & a_2a_3 & a_3a_1 & a_1a_2 \\ b_1^2 & b_2^2 & b_3^2 & b_2b_3 & b_3b_1 & b_1b_2 \\ c_1^2 & c_2^2 & c_3^2 & c_2c_3 & c_3c_1 & c_1c_2 \\ v_1^2 & v_2^2 & v_3^2 & v_2v_3 & v_3v_1 & v_1v_2 \\ w_1^2 & w_2^2 & w_3^2 & w_2w_3 & w_3w_1 & w_1w_2 \\ u_1^2 & u_2^2 & u_3^2 & u_2u_3 & u_3u_1 & u_1u_2 \end{vmatrix} \text{ or say, } (abcvwu) = 0,$$

where $a_1^2 \ldots$ are the coefficients of the first conic.

But then from $a_\lambda^2 = 0$, $b_\lambda^2 = 0$, $c_\lambda^2 = 0$, $u_\lambda^2 = 0$ alone, we see that we must have

$$g_1 u_\lambda^2 + g_2 u_\mu^2 + g_3 u_\nu^2 = (abcvwu),$$

where u_λ^2, u_μ^2, u_ν^2 are determinate, and g_1, g_2, g_3 unknown, with also

$$g_1 v_\lambda^2 + g_2 v_\mu^2 + g_3 v_\nu^2 = 0,$$

$$g_1 w_\lambda^2 + g_2 w_\mu^2 + g_3 w_\nu^2 = 0,$$

and therefore after determining a numerical factor

$$(abcvwu) = \begin{vmatrix} u_\lambda^2 & u_\mu^2 & u_\nu^2 \\ v_\lambda^2 & v_\mu^2 & v_\nu^2 \\ w_\lambda^2 & w_\mu^2 & w_\nu^2 \end{vmatrix}.$$

Just so we shall find

$$-8(\lambda\mu\nu xyz) = \begin{vmatrix} a_x^2 & b_x^2 & c_x^2 \\ a_y^2 & b_y^2 & c_y^2 \\ a_z^2 & b_z^2 & c_z^2 \end{vmatrix},$$

and in general the relations between the net and web are mutual.

And we notice another method of writing the equation of the web. The polars of x, $a_x a_y$, $b_x b_y$, $c_x c_y$ are concurrent if $(abc)\,a_x b_x c_x = 0$ and then in $(bc)_i b_x c_x$. This point is then conjugate to x in regard to all the conics of the net—namely, one of the inscribed web is the two-point cluster $u_x\,(bcu)\,b_x c_x$.

So we may therefore write the inscribed web, y, z, t being three arbitrary points on

$$\eta_x^3 = 6\,(abc)\,a_x b_x c_x = 0,$$

$$g_1 \,.\, u_y\,(bcu)\,b_y c_y + g_2 \,.\, u_z\,(bcu)\,b_z c_z + g_3 \,.\, u_t\,(bcu)\,b_t c_t = 0.$$

The Jacobian and Cayleyan of three conics.

We proceed to consider some relations between two derived curves of the net and those of the web.

Defining the Jacobian of the net as the locus of the point x whose polars in regard to three and therefore all the conics are concurrent, we obtain as its equation $\eta_x^3 = 6\,(abc)\,a_x b_x c_x = 0$, the polars of x meeting in $(bcu)\,b_x c_x = 0$ or $(cau)\,c_x a_x = 0$ or $(abu)\,a_x b_x = 0$.

But also there is a single definite conic of the net which consists of two straight lines meeting in x. For $f_1 a_x^2 + f_2 b_x^2 + f_3 c_x^2 = 0$ satisfies the condition, provided simultaneously $f_1 a_x a_i + f_2 b_x b_i + f_3 c_x c_i = 0$—giving the same locus for x—while also

$$f_1 : f_2 : f_3 = (bc)_i b_x c_x : (ca)_i c_x a_x : (ab)_i a_x b_x,$$

and the line pair intersecting in x is

$$(bc)_i b_x c_x \,.\, a_t^2 + (ca)_i c_x a_x \,.\, b_t^2 + (ab)_i a_x b_x \,.\, c_t^2 = 0,$$

(t being the variable)

and therefore making $i = 1, 2, 3$ this line pair is equally

$$(bc \,.\, qr)\,b_x c_x \,.\, a_t^2 + (ca \,.\, qr)\,c_x a_x \,.\, b_t^2 + (ab \,.\, qr)\,a_x b_x \,.\, c_t^2 = 0,$$

where $u_p = 3u_\sigma a_\sigma^2 = 12\,(bcu)\,b_a c_a$ as formerly, and $b_x b_r = c_x c_q = 6\,(\beta\gamma x)\,a_\beta a_\gamma$ and therefore $(ca\,.\,qr)\,c_x a_x = (bc\,.\,rp)\,b_x c_x$, $(ab\,.\,qr)\,a_x b_x = (bc\,.\,pq)\,b_x c_x$, so that the line pair is

$$(qry)\,a_t^2 + (rpy)\,.\,b_t^2 + (pqy)\,c_t^2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(\alpha),$$

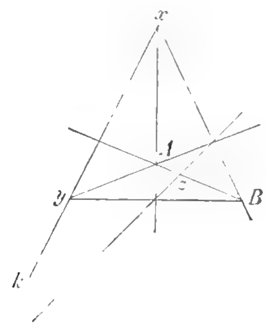
where y is the conjugate point of x, namely $(bcu)\,b_x c_x = 0$, as of course is obvious from our previous determination of the cubic of which the conics are first polars. But the theory of the Jacobian should be independent of the theory of this cubic.

So the line pair intersecting in y is

$$(qrx)\,.\,a_t^2 + (rpx)\,.\,b_t^2 + (pqx)\,.\,c_t^2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(\beta).$$

And if k be any point on the join of x, y the conic

$$(qrk)\,a_t^2 + (rpk)\,b_t^2 + (pqk)\,c_t^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\gamma),$$



—since k is a linear function of x and y—is a linear function of the other two, (α) and (β), and therefore passes through the intersections of these. In particular when k is on the Jacobian (namely is third point of intersection of xy therewith), this conic becomes the line pair through its conjugate point and can therefore only be the diagonals of the quadrilateral formed by other two line pairs, and z their point of intersection must be the conjugate of this third or 'complementary' point of xy on the Jacobian.

Also in general the polar line of ξ in regard to $(qr\xi)\,a_t^2 + (rp\xi)\,b_t^2 + (pq\xi)\,c_t^2 = 0$ is

$$(qr\xi)\,a_\xi a_t + (rp\xi)\,b_\xi b_t + (pq\xi)\,\xi c_t = 0,$$

namely as $\quad (qr\xi)\,a_\xi a_t = (qrt)\,a_\xi^2 + (r\xi t)\,a_\xi a_q + (\xi qt)\,a_\xi a_r \Big\}$ etc.
and $\quad (r\xi t)\,a_\xi a_q = (r\xi t)\,b_\xi b_p = -\,(\xi rt)\,b_\xi b_p$

This polar is simply

$$(qrt)\,a_\xi^2 + (rpt)\,b_\xi^2 + (pqt)\,c_\xi^2 = 0.$$

Thus from harmonic properties of the quadrilateral, the equations of the lines yz, xz are

$$\left.\begin{aligned}(qrt)\,a_x^2 + (rpt)\,b_x^2 + (pqt)\,c_x^2 = 0\\ (qrt)\,a_y^2 + (rpt)\,b_y^2 + (pqt)\,c_y^2 = 0\end{aligned}\right\}\text{ which intersect in } z.$$

Therefore the line pair through the 'complementary' point k, the conjugate of z, which is

$$(qrz)\,a_t^2 + (rpz)\,b_t^2 + (pqz)\,c_t^2 = 0,$$

must pass through x and y and *thus contains xy as one part.*

Thus, purely from the theory of conics, we arrive at the third property of the Jacobian, that the join of every pair of conjugate points thereon is itself part of one of the line pairs contained in the net. And through every point on the Jacobian there pass two line pairs of the system, one having its central point there—but in general through any point of the plane there pass three line pairs, as may be easily seen.

Consider now the Cayleyan—it is the envelope of the joins of conjugate points on the Jacobian, say the envelope of a line cutting the conics in involution, and therefore, from the theory of binary quadratics, its equation is

$$(bcu)(cau)(abu) = 0.$$

But it is, by the theory just given, also the envelope of the lines, or say better, the cluster of lines, into which the polar conics of the system break up. As such however its most natural form of equation is given by a determinant of six rows and columns. Namely we eliminate from equations of the form

$$f_1 a_{ij} + f_2 b_{ij} + f_3 c_{ij} = u_i v_j + u_j v_i,$$

the quantities $f_1, f_2, f_3, v_1, v_2, v_3,$

which determinant is however given from the previous definition by noticing that the conjugate points, considered as a two-point cluster, are inscribed in the conics

$$a_x^2,\ b_x^2,\ c_x^2,\ u_x x_1,\ u_x x_2,\ u_x x_3.$$

We have in fact the following noteworthy identity, after determining a numerical factor:—

$$u_\sigma^3 = -6\,(bcu)(cau)(abu) =$$

$$3\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & 2a_2a_3 & 2a_3a_1 & 2a_1a_2 \\ b_1^2 & b_2^2 & b_3^2 & 2b_2b_3 & 2b_3b_1 & 2b_1b_2 \\ c_1^2 & c_2^2 & c_3^2 & 2c_2c_3 & 2c_3c_1 & 2c_1c_2 \end{vmatrix}\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 & x_2x_3 & x_3x_1 & x_1x_2 \\ 2x_1y_1 & 2x_2y_2 & 2x_3y_3 & x_2y_3+x_3y_2 & x_3y_1+x_1y_3 & x_1y_2+x_2y_1 \\ y_1^2 & y_2^2 & y_3^2 & y_2y_3 & y_3y_1 & y_1y_2 \end{vmatrix}$$

$$= -3\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & 2a_2a_3 & 2a_3a_1 & 2a_1a_2 \\ b_1^2 & b_2^2 & b_3^2 & 2b_2b_3 & 2b_3b_1 & 2b_1b_2 \\ c_1^2 & c_2^2 & c_3^2 & 2c_2c_3 & 2c_3c_1 & 2c_1c_2 \\ u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \end{vmatrix} \quad (\text{where } (xy)_i = u_i).$$

So for the Jacobian, if $x_i = (uv)_i$, x and its conjugate are not only a two-point cluster described in a_x^2, b_x^2, c_x^2, but also in u_x^2, u_xv_x, v_x^2,

and therefore

$$\eta_x^3 = 6\,(abc)\,a_xb_xc_x = 6\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & a_2a_3 & a_3a_1 & a_1a_2 \\ b_1^2 & b_2^2 & b_3^2 & b_2b_3 & b_3b_1 & b_1b_2 \\ c_1^2 & c_2^2 & c_3^2 & c_2c_3 & c_3c_1 & c_1c_2 \end{vmatrix}\begin{vmatrix} x_1 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 & x_1 & 0 \end{vmatrix}$$

$$= 6 \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & a_2a_3 & a_3a_1 & a_1a_2 \\ b_1^2 & b_2^2 & b_3^2 & b_2b_3 & b_3b_1 & b_1b_2 \\ c_1^2 & c_2^2 & c_3^2 & c_2c_3 & c_3c_1 & c_1c_2 \\ u_1^2 & u_2^2 & u_3^2 & u_2u_3 & u_3u_1 & u_1u_2 \\ 2u_1v_1 & 2u_2v_2 & 2u_3v_3 & u_2v_3+u_3v_2 & u_3v_1+u_1v_3 & u_1v_2+u_2v_1 \\ v_1^2 & v_2^2 & v_3^2 & v_2v_3 & v_3v_1 & v_1v_2 \end{vmatrix}.$$

Let now u_H^3, Σ_x^3 represent the Jacobian and Cayleyan of the web. The former will be the envelope of the joins of two-point clusters of the web:—two conjugate points on η_x^3 are however such. Thus u_H^3 will be the same cluster as u_σ^3, and similarly Σ_x^3 as η_x^3. And, in fact, if in

$$8\,(abcuvw) = 8 \begin{vmatrix} u_\lambda^2 & u_\mu^2 & u_\nu^2 \\ v_\lambda^2 & v_\mu^2 & v_\nu^2 \\ w_\lambda^2 & w_\mu^2 & w_\nu^2 \end{vmatrix} \text{ (page 98)},$$

we put

$$u_1^2 = u_1,\quad u_2^2 = 0,\quad u_3^2 = 0,\quad 2u_2u_3 = 0,\quad 2u_3u_1 = 0,\quad 2u_1u_2 = 0,$$

$$v_1^2 = 0,\quad v_2^2 = u_2,\quad v_3^2 = 0,\quad 2v_2v_3 = u_3,\quad 2v_3v_1 = 0,\quad 2v_1v_2 = u_1,$$

$$w_1^2 = 0,\quad w_2^2 = 0,\quad w_3^2 = u_3,\quad 2w_2w_3 = u_2,\quad 2w_3w_1 = u_1,\quad 2w_1w_2 = 0,$$

and use the identity of the previous page, we obtain

$$-\tfrac{1}{3}u_\sigma^3 = |\, u_1\lambda_1^2 + u_2\lambda_1\lambda_2 + \ldots\ldots \,| = 8u_\lambda u_\mu u_\nu \,|\, \lambda_1\lambda_2\lambda_3 \,| = 8u_\lambda u_\mu u_\nu\,(\lambda\mu\nu),$$

namely

$$u_\sigma^3 = -4u_H^3.$$

So

$$\eta_x^3 = 2\Sigma_x^3.$$

Resultant of three conics (see also page 94).

If the three conics a_x^2, b_x^2, c_x^2 meet in y, the point cluster repeated $u_y^2 = 0$, is inscribed in all the conics of the net, namely is one of the point pairs occurring in the inscribed web, with however the speciality that the points coincide. Thus the Jacobian (of the net) must have a double point at y, and therefore all its first polars will pass through this point or be circumscribed to it. Namely each of the conics $\eta_x^2\eta_1$, $\eta_x^2\eta_2$, $\eta_x^2\eta_3$ will be circumscribed to $(u_y^2 = \ \ )\ g_1u_\lambda^2 + g_2u_\mu^2 + g_3u_\nu^2 = 0$,

so that

$$g_1\eta_\lambda^2\eta_1 + g_2\eta_\mu^2\eta_1 + g_3\eta_\nu^2\eta_1 = 0,$$

$$g_1\eta_\lambda^2\eta_2 + g_2\eta_\mu^2\eta_2 + g_3\eta_\nu^2\eta_2 = 0,$$

$$g_1\eta_\lambda^2\eta_3 + g_2\eta_\mu^2\eta_3 + g_3\eta_\nu^2\eta_3 = 0,$$

are consistent. These give

$$\eta_1\eta_2'\eta_3'' \begin{vmatrix} \eta_\lambda^2 & \eta_\mu^2 & \eta_\nu^2 \\ \eta'_\lambda{}^2 & \eta'_\mu{}^2 & \eta'_\nu{}^2 \\ \eta''_\lambda{}^2 & \eta''_\mu{}^2 & \eta''_\nu{}^2 \end{vmatrix} \text{ or say } \tfrac{1}{6}(\eta\eta'\eta'') \begin{vmatrix} \eta_\lambda^2 & \eta_\mu^2 & \eta_\nu^2 \\ \eta'_\lambda{}^2 & \eta'_\mu{}^2 & \eta'_\nu{}^2 \\ \eta''_\lambda{}^2 & \eta''_\mu{}^2 & \eta''_\nu{}^2 \end{vmatrix} = 0.$$

But from the identity proved at the bottom of page 95, this is

$$\tfrac{2}{3}(uvw)\,u_H v_H w_H - \tfrac{1}{3}(vw\Sigma)(wu\Sigma)(uv\Sigma) = 0,$$

where

$$u,\ v,\ w = \eta,\ \eta',\ \eta'',$$

and $u_H{}^3 = -\frac{1}{4}u_\sigma{}^3$ is the Jacobian of the web and $\Sigma_x{}^3 = \frac{1}{2}\eta_x{}^3$ is the Cayleyan of the web, namely, is

$$(uvw)\,u_\sigma v_\sigma w_\sigma + (vw\eta)(wu\eta)(uv\eta) = 0.$$

So that the resultant of the three conics may be written

$$(\eta\eta'\eta'')^2\,\eta_\sigma\eta'_\sigma\eta''_\sigma + (\eta\eta'\eta'')(\eta'\eta''\eta''')(\eta''\eta\eta''')(\eta\eta'\eta''') = 0.$$

And as verification, since $u_\sigma{}^3 = (pqr)\,u_s{}^3$, $\eta_x{}^3 = (pqr)\,H_x{}^3$ (as proved), it should be possible to write the equation and discriminant of a ternary cubic

$$H(H)_{x=s} = S(H),$$

while as

$$H(H) = \tfrac{1}{12}S^2 f - \tfrac{1}{3}TH,\ \ S(H) = \tfrac{2}{3}T^2 - \tfrac{1}{12}S^3,\ f_{x=s} = S,\ \ H_{x=s} = T,$$

this becomes $\frac{1}{6}S^3 - T^2$, *which is right.*

And, as for the net, so for the inscribed web, we can write down a class cubic whereof the first polars coincide with the web.

§ VIII. *Notes on some of the concomitants.*

1. We can find a class cubic of which the clusters $u_\alpha{}^2$, $u_\beta{}^2$, $u_\gamma{}^2$ are the first polars in regard to three straight lines. For the polars of q and r in regard to f_3 and f_2 respectively are the same straight line, namely,

$$b_x b_r = c_x c_q = 6\,(\beta\gamma x)\,a_\beta a_\gamma.$$

Put then

$$b_x b_r = c_x c_q = 6\,(\beta\gamma x)\,a_\beta a_\gamma = \frac{1}{3}\frac{l_x}{a_\alpha{}^2},$$

$$c_x c_p = a_x a_r = 6\,(\gamma\alpha x)\,b_\gamma b_\alpha = \frac{1}{3}\frac{m_x}{b_\beta{}^2},$$

$$a_x a_q = b_x b_p = 6\,(\alpha\beta x)\,c_\alpha c_\beta = \frac{1}{3}\frac{n_x}{c_\gamma{}^2},$$

so that

$$m_\gamma u_\gamma = 3b_\beta{}^2\,.\,u_\gamma c_\gamma c_p = b_\beta{}^2\,.\,c_\gamma{}^2\,.\,u_p = n_\beta u_\beta,$$

and take $(lmn)\,u_t{}^3 = (mnu)\,u_\alpha{}^2 + (nlu)\,u_\beta{}^2 + (lmu)\,u_\gamma{}^2$. Then as previously $u_t{}^2 l_t = u_\alpha{}^2$, etc., and really $(mnu) = 9b_\beta{}^2c_\gamma{}^2\,(aa'u)\,a_q a'_r = \frac{1}{2}\,9b_\beta{}^2c_\gamma{}^2 u_\alpha\,(qr\alpha) = 9b_\beta{}^2c_\gamma{}^2\,(bcu)\,b_p c_p$

$$= -\tfrac{1}{4}\,9b_\beta{}^2c_\gamma{}^2\,\{(\alpha'\beta\gamma)\,u_{\alpha'} b_\alpha c_\alpha b_\gamma c_\beta - \tfrac{2}{3}\,a_\alpha{}^2\,.\,(ubc)\,b_\alpha c_\alpha b_\gamma c_\beta\}.$$

Thus the cubic can be expressed by our concomitants, or in terms of $(\beta\gamma x)(\gamma\alpha x)(\alpha\beta x)$ and $(\alpha\beta\gamma)\,u_\alpha u_\beta u_\gamma$ as before.

One form of its equation is

$$\frac{u_\alpha{}^2}{a_\alpha{}^2}\,.\,(qr\alpha)\,u_\alpha + \frac{u_\beta{}^2}{b_\beta{}^2}\,.\,(rp\beta)\,u_\beta + \frac{u_\gamma{}^2}{c_\gamma{}^2}\,.\,(pq\gamma)\,u_\gamma = 0.$$

2. The conic $b_x c_x b_\alpha c_\alpha = 0$ {or $(402)_1 = 0$} is the locus of a point whose polars in regard to b_x^2 and c_x^2 are conjugate in regard to a_x^2.

These polars meet in the point $(bcu) b_x c_x = 0$, or say $u_\xi = 0$, and we have through this point three lines conjugate in pairs in regard to the three conics. Namely, y being the variable, the pairs are

$(x\xi y) = 0$ and $b_x b_y = 0$ conjugate in regard to b_x^2, or harmonic in regard to tangents from ξ to b_x^2,

$(x\xi y) = 0$ and $c_x c_y = 0$,, ,, ,, c_x^2 ,, ,, ,, ,, ,, ,, ,, c_x^2,

$b_x b_y = 0$ and $c_x c_y = 0$,, ,, ,, a_x^2 ,, ,, ,, ,, ,, ,, ,, a_x^2.

3. In general the condition that the conjugates through ξ, $(\beta\xi y) u_\beta = 0$, $(\gamma\xi y) u_\gamma = 0$, of a line u, in regard to b_x^2 and c_x^2 should be conjugate in regard to a_x^2 is

$$(\alpha\beta\xi)(\alpha\gamma\xi) u_\beta u_\gamma = 0,$$

namely, u touches a conic and there are two such lines u through ξ.

Putting herein, to connect with (2), ξ the conjugate of x or $v_\xi = (bcv) b_x c_x$, we obtain

$$(\alpha\beta \,.\, bc)(\alpha\gamma \,.\, b'c') b_x c_x b_x' c_x' u_\beta u_\gamma = - b_\alpha c_\alpha b_x c_x \{b_\gamma b_x u_\gamma \,.\, c_\beta c_x u_\beta + \tfrac{1}{9} u_x^2 \,.\, b_\beta^2 \,.\, c_\gamma^2\}$$
$$+ \tfrac{1}{3} u_x [c_\gamma^2 \{b_\alpha^2 \,.\, b_x^2 - \tfrac{1}{2}(\alpha\beta x)^2\} \,.\, c_\beta c_x u_\beta + b_\beta^2 \,.\, \{c_\alpha^2 c_x^2 - \tfrac{1}{2}(\gamma\alpha x)^2\} \,.\, b_\gamma b_x u_\gamma],$$

and if x be on $b_\alpha c_\alpha b_x c_x = 0$ the cluster is two pointed, one point being, as predicted, x and the other on the join of the points $b_\gamma b_x u_\gamma = 0$, $c_x c_\beta u_\beta = 0$ (whereof the former is the pole in regard to c_x^2 of the polar of x in regard to b_x^2).

And as x moves on $b_\alpha c_\alpha b_x c_x = 0$ its conjugate $(bcu) b_x c_x = 0$ moves on

$$0 = (\alpha\beta x)(\alpha\gamma x) b_\gamma c_\beta b_x c_x = -(\beta\gamma x)^2 \,.\, b_\alpha c_\alpha b_x c_x + \tfrac{1}{6}\{b_x^2 \,.\, c_\gamma^2 (\alpha\beta x)^2 + c_x^2 \,.\, b_\beta^2 (\gamma\alpha x)^2\}$$
$$+ \tfrac{1}{2}(\beta\gamma x)^2 \,.\, (b_x^2 \,.\, c_\alpha^2 + c_x^2 \,.\, b_\alpha^2) - \tfrac{1}{2} b_x^2 \,.\, c_\beta^2 \,.\, (\gamma\alpha x)^2 - \tfrac{1}{2} c_x^2 \,.\, b_\gamma^2 \,.\, (\alpha\beta x)^2.$$

4. Further in regard to the cluster $(\alpha\beta\xi)(\alpha\gamma\xi) u_\beta u_\gamma = 0$ [which reduces to the concomitant $(611)_1$], the polars of ξ in regard to b_x^2 and c_x^2 are among its rays and for the conjugate through ξ (in regard to b_x^2) of $b_y b_\xi = 0$ we must take the join of ξ to the point $(\alpha\gamma\xi) b_\gamma b_\xi u_\alpha = 0$—which point is the pole of the join of ξ to $b_\gamma b_\xi u_\gamma = 0$ in regard to a_x^2—(it is the concomitant $(512)_1$).

For consider the locus of the poles in regard to b_x^2 of the rays of the cluster $(\alpha\beta\xi)(\alpha\gamma\xi) u_\beta u_\gamma$. Its equation in y is $0 = (\alpha\beta\xi)(\alpha\gamma\xi) b_\beta' b_y' b_\gamma b_y$, or say $(\alpha y\xi)(\alpha\gamma\xi) b_\gamma b_y = 0$, which certainly passes through ξ, and putting $\rho y_i = \xi_i + \kappa z_i$ and then $(z\xi)_i = u_i$, we obtain

$$(\alpha\gamma\xi) u_\alpha b_\gamma b_\xi = 0 \text{ (for } \kappa = 0).$$

5. Consider further the conjugates through ξ in regard to b_x^2 of the rays of the cluster $(\alpha\beta\xi)(\alpha\gamma\xi) u_\beta u_\gamma$ through ξ. They are the joins of ξ to the two points given by

$$\left.\begin{aligned} b_\xi b_y &= 0 \\ (\alpha y\xi)(\alpha\gamma\xi) b_\gamma b_y &= 0 \end{aligned}\right\}.$$

Putting $\rho y_i = \xi_i + \kappa z_i$ in both, we obtain

$$\left.\begin{array}{r} b_\xi^2 + \kappa b_\xi b_z = 0 \\ (\alpha z \xi)(\alpha\gamma\xi)\, b_\gamma b_\xi + \kappa\, (\alpha z \xi)(\alpha\gamma\xi)\, b_\gamma b_z = 0 \end{array}\right\},$$

and

wherefrom

$$(\alpha z \xi)(\alpha\gamma\xi)\, b_\gamma \underline{b_\xi \,.\, b'_\xi b'_z} - (\alpha z \xi)(\alpha\gamma\xi)\, b_\gamma b_z \,.\, b_\xi'^2 = 0,$$

and therefore

$$(\alpha z \xi)(\alpha\gamma\xi)(bb' \,.\, \xi z)\, b_\gamma b'_\xi = 0,$$

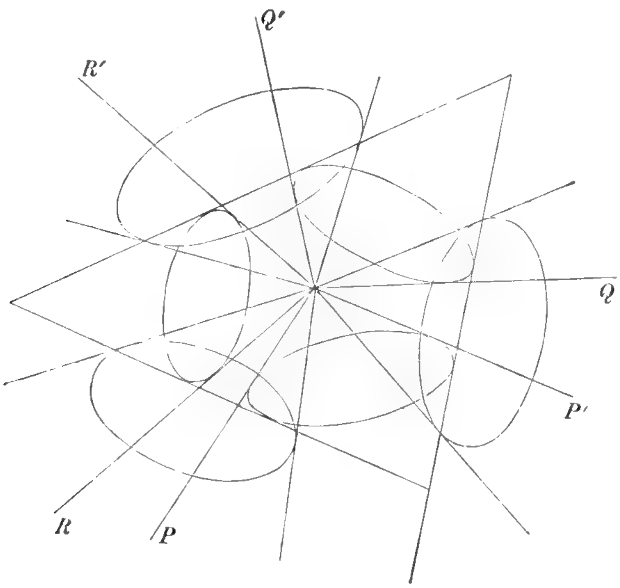
or finally

$$(\alpha z \xi)(\beta z \xi)(\alpha\gamma\xi)(\beta\gamma\xi) = 0,$$

so that the conjugates sought are rays of the cluster $(\alpha\gamma\xi)(\beta\gamma\xi)\, u_\alpha u_\beta = 0$, which is of the same form in regard to c_x^2 as the original in regard to a_x^2.

We have then through ξ six lines OP, OP', OQ, OQ', OR, OR',

such that OQ, OR are conjugate in regard to a_x^2, as are OQ', OR',
OR, OP ,, ,, ,, ,, b_x^2, ,, OR', OP',
OP, OQ ,, ,, ,, ,, c_x^2, ,, OP', OQ'.



6. $c_\beta c_x u_\beta = 0$ is the pole in regard to b_x^2 of the polar of x in regard to c_x^2,
$b_\gamma b_x u_\gamma = 0$ is the pole in regard to c_x^2 of the polar of x in regard to b_x^2.

The join of these points is $(\beta\gamma y)\, b_\gamma c_\beta b_x c_x = 0$.

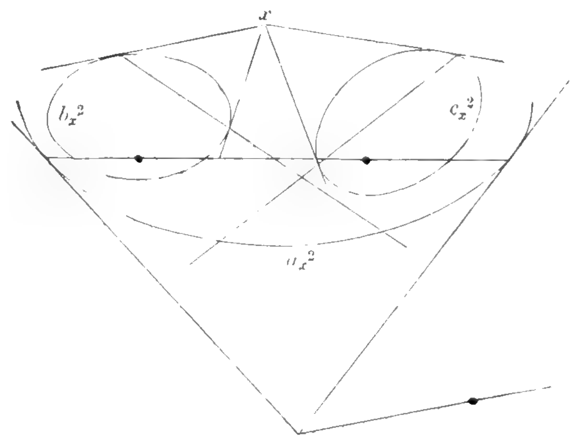
Conversely if this join passes through a fixed point y the point x lies on a conic; which conic is harmonically circumscribed to a_x^2 (or $u_\alpha^2 = 0$) provided y lie on the line

$$(\beta\gamma y)\, b_\gamma c_\beta b_\alpha c_\alpha = 0,$$ [This is the concomitant $(801)_1$],

and then the points x form a single infinity of sets of three, each forming a triangle self-conjugate in regard to a_x^2.

From the equation of this join we derive the equation of the self-polar trilateral of the two conics: namely x must lie on this join, giving the equation

$$0 = (\beta\gamma x)\, b_\gamma c_\beta b_x c_x \qquad [\text{or } (603)_3 = 0].$$



Further, the pole of this join in regard to a_x^2 is the point

$$(\alpha\beta\gamma)\, u_\alpha b_\gamma c_\beta b_x c_x = 0,$$

and if this pole lie on a fixed straight line, x describes a conic, which is harmonically circumscribed to a_x^2 (or u_α^2) provided this line pass through the point

$$(\alpha'\beta\gamma)\, b_\gamma c_\beta b_\alpha c_\alpha u_{\alpha'} = 0 \qquad [\text{which is } (10\,.\,1\,.\,0) = 0].$$

7. The point $(bcu)\, b_x c_x = 0$ is conjugate to x in regard to both b_x^2 and c_x^2, say is the 'conjugate' of x. Conversely the locus of the conjugates of collinear points is a conic, the conversion being allowable because the conjugate of the conjugate of a point x is

$$(bcu)(bb'c')(cb''c'')\, b'_x c'_x b''_x c''_x = -\tfrac{1}{2} u_x \,.\, (\beta\gamma x)\, b_\gamma c_\beta b_x c_x,$$

namely is x itself—the factor $(\beta\gamma x)\, b_\gamma c_\beta b_x c_x$ representing the common self-polar trilateral of the two conics.

The locus of a point x whose joins to its conjugate always pass through a fixed point y is a cubic curve

$$(bc\,.\,xy)\, b_x c_x = b_x^2 \,.\, c_x c_y - c_x^2 \,.\, b_x b_y = 0,$$

which passes through the intersection of the conics, through y itself, through the conjugate of y (this being in fact the "tangential" of y on the cubic), through the points of contact of the tangents from y, and in general may be generated as the locus of the points of contact of tangents from y to the bundle $b_x^2 + \lambda c_x^2 = 0$. And thus, in fact, from a known property, any cubic curve can be thus generated; as also follows from the fact that three collinear points x, y, z on a cubic f_x^3, whereof H_x^3 is the Hessian, satisfy

$$f_x f_y f_z = 0, \quad H_x H_y H_z = 0$$

(as follows from Salmon's identity $(fHu)^3 = 0$).

8. The conic $(bcu)\,b_xc_x = 0$ is also the locus of the poles of the line u in regard to the conics of the bundle $b_x^2 + \lambda c_x^2 = 0$: and, in fact, the pole of $(qrx) = 0$ in regard to the general conic $(qr\xi)\,a_x^2 + (rp\xi)\,b_x^2 + (pq\xi)\,c_x^2 = 0$ is the point $(bcu)\,b_\xi c_\xi = 0$, the point conjugate to ξ. Thus if ξ move on a line v, its three conjugates $(bcu)\,b_\xi c_\xi = 0$, $(cau)\,c_\xi a_\xi = 0$, $(abu)\,a_\xi b_\xi = 0$ move on three conics $(bcv)\,b_xc_x = 0, \ldots$ and these three conics correspond also to $(qrx) = 0$, $(rpx) = 0$, $(pqx) = 0$ respectively, in regard to the general conic $(qr\xi)\,a_x^2 + (rp\xi)\,b_x^2 + (pq\xi)\,c_x^2 = 0$, which is now described about a fixed quadrilateral.

9. Lastly the conic $(bcu)\,b_xc_x = 0$ for the line $(qrx) = 0$ is

$$(bc\,.\,qr)\,b_xc_x = b_qb_x\,.\,c_rc_x - (c_qc_x)^2,$$

namely touches b_qb_x, c_rc_x the polars of q, r in regard to b_x^2 and c_x^2 respectively, on the line $c_qc_x = 0$.

10. The conic $\quad a_y^2\,.\,(bcu)\,b_xc_x + b_y^2\,.\,(cau)\,c_xa_x + c_y^2\,.\,(abu)\,a_xb_x = 0,$

y being the variable, is the conic of the net for which x is the pole of u.

If the line u be $(qrx) = 0$, then since

$$(ca\,.\,qr)\,c_xa_x = (bc\,.\,rp)\,b_xc_x,\quad (ab\,.\,qr)\,a_xb_x = (bc\,.\,pq)\,b_xc_x,$$

the conic is

$$a_y^2\,(qr\xi) + b_y^2\,(rp\xi) + c_y^2\,(pq\xi) = 0,$$

where ξ is the conjugate of x in regard to b_x^2 and c_x^2.

In general the conic passes through x provided

$$u_x\,.\,(abc)\,a_xb_xc_x = 0.$$

Take $u_x = 0$.

Then the conic touches the line u at the point x. It is a line-pair provided u is tangent to a class cubic (for the discriminant of a cubic is of the third degree in its coefficients). Thus through any point x there pass three line-pairs of the net, which, touching the tangents to a certain class cubic at this point, must either have their double points at x (which is excluded) or have the three tangents to the cubic as part of themselves. *Namely the class cubic is the Cayleyan.*

§ IX. The following list of memoirs may be added:

1. Gundelfinger. *Crelle,* LXXX. 1875, 73.
2. Sylvester. *Camb. and Dub. Math. Journ.* t. VIII. p. 256. (1853.)
3. Cayley. *Crelle,* LVII. 139. (1860.)
4. Hermite. *Crelle,* LVII. 371. (1853.)
5. Darboux. *Bulletin des Sciences Math.* t. I. p. 348.
6. Rosanes. *Math. Annal.* VI. S. 264.
7. Schröter. *Math. Annal.* V. S. 50.
8. Smith. *Proceedings Lond. Math. Society,* II. (1868.)
9. *Leçons sur la Géométrie.* Alfred Clebsch. Lindemann. Traduites p. Adolphe Benoist, Paris, 1880. Vol. II. 248.

January, 1889.

IV. *On Sir William Thomson's estimate of the Rigidity of the Earth.* By A. E. H. Love, M.A., St John's College.

[Read April 28, 1890.]

The question really propounded in the articles of Thomson and Tait's *Natural Philosophy* devoted to the discussion of the Earth's rigidity is this:—*Supposing that for purposes of discussion the Earth is replaced by a homogeneous elastic solid sphere of the same mass and diameter, what degree of rigidity must be attributed to such a solid in order that ocean-tides on the sphere may be of the same height as the actual ocean-tides on the Earth?* This rigidity is called the "tidal effective rigidity." As is well known the tides to be considered are the fortnightly tides, as being of sufficiently long period to be capable of adequate discussion on the "equilibrium theory," and at the same time free from certain difficulties which beset the observation and discussion of annual and semi-annual tides. The actual amount of the fortnightly tide on the Earth appears to be still to some extent matter of dispute. For the purpose in hand the estimate of it employed is one made by Professor G. H. Darwin founded on a series of observations chiefly made in the Indian Ocean. According to this estimate, the amount of the fortnightly tide is little less than $\frac{2}{3}$ and certainly much greater than $\frac{2}{5}$ of the true equilibrium height. Now, in the articles of the *Natural Philosophy* referred to, it was shown that if the Earth were replaced by a homogeneous *incompressible* elastic solid sphere of the same mass and diameter, and of rigidity equal to that of steel, the height of the ocean-tide would be reduced by the elastic yielding to about $\frac{2}{3}$ of the equilibrium height, while the reduction would be to about $\frac{2}{5}$ of that height if the rigidity were equal to that of glass. It was concluded that the tidal effective rigidity of the Earth is nearly that of steel, and the conclusion was held to disprove the Geological hypothesis of internal fluidity.

The present paper is not occupied with any attempt to review the evidence used by Professor Darwin as to the amount of the observable fortnightly tide, or to criticise the conclusion of Sir William Thomson from the great tidal effective rigidity of the Earth to the improbability of the hypothesis of internal fluidity*. Its purpose is merely to discover what difference would be made in the tidal effective rigidity if the elastic

* [Note added Sept. 1890. It is proper to mention that Professor G. H. Darwin has in a recent paper, *Proc. Roy. Soc.* Lond. Nov. 1886, expressed an opinion that it is probably impossible to obtain a correct estimate of the Earth's tidal effective rigidity. In all previous calculations it had been supposed that the fortnightly tide obeys with sufficient accuracy the equilibrium law, but it is there pointed out that oceanic tidal friction is probably too great to allow of the application of the equilibrium theory to the fortnightly tide. Sir W. Thomson's estimate of the Earth's tidal effective rigidity is based on such an application.]

solid replacing the Earth were not assumed to be incompressible, but to have its modulus of compression and its rigidity in the same ratio as most hard solids have. It may be premised at once that the difference is very slight. We find ourselves confronted with a particular case of the following problem—A gravitating solid elastic sphere of any finite rigidity and compressibility is subject to the action of bodily forces derivable from a potential expressible in spherical harmonic series, it is required to determine the resulting displacements. Certain problems of the same kind, but less general than this, are solved by Thomson and Tait. These authors consider the case where the elastic solid has any finite compressibility and rigidity but is free from its own gravitation, and the case where the solid is incompressible and gravitating and of any finite rigidity. The solution of the general problem is here obtained, and it is noteworthy that it cannot be derived from these solutions by any method of linear synthesis.

Let W, the disturbing potential, be expanded in a series of spherical solid harmonics in the form $W = \Sigma W_{i+1}$, where i is an integer, and suppose the equation of the deformed free surface expressed in the form $r = a + \Sigma\epsilon_i Q_{i+1}$, where ϵ_i is a small quantity and Q_{i+1} is a spherical solid harmonic of degree $(i+1)$, then among the bodily forces acting at any point are included the attractions of the inequalities. These are derivable from a potential of the form ΣV_{i+1}, where V_{i+1} is in like manner a spherical solid harmonic. The other forces to be taken account of are the attraction of the nucleus and the forces whose potential is W. It is easy to obtain, by using Thomson and Tait's solutions, a general solution of the equations of equilibrium under these sets of forces in a form adapted to satisfy boundary conditions at the deformed surface. The conditions to be fulfilled are those which express that this surface is free from stress. Such solutions contain complementary functions, and particular integrals depending on the bodily forces, and, inasmuch as the harmonic inequalities contain terms depending on the complementary functions, the bodily forces, some of which arise from the attractions of these inequalities, contain similar terms, and thus the particular integrals contain unknown harmonics which occur in the complementary functions. This is one important difference between the present problem and those considered by Thomson and Tait. A second consists in the fact that, the attraction of the nucleus being very great compared with the other forces concerned, it is not sufficient to estimate the surface-tractions to which it gives rise at the surface of the mean sphere, but they must be estimated at the surface of the harmonic inequality. This is done by a method I have employed in a previous paper (*Proc. Lond. Math. Soc.* XIX.). When the complete expressions for the surface-tractions at the deformed surface arising from the complementary functions and particular integrals have been obtained, it is easy by equating them to zero to deduce the expression of all the unknown functions that occur, and thus to express the displacements at any point in terms of the disturbing potential. One result is that the harmonic inequality arising from any spherical harmonic term in the disturbing potential is proportional to that term and contains no other harmonic.

The application to the tidal problem is made by supposing the disturbing potential to consist of a single term which is a spherical solid harmonic of the second order, say W_2, and thus by taking $i = 1$. We have also to take ρ the density of the solid equal to the Earth's mean density. The elasticity of the material composing the sphere will be defined by two constants m and n such that $m-\frac{1}{3}n$ is the resistance to compression, and n the resistance to distortion. By supposing m to become infinite, and n to remain finite and comparable with $g\rho a$, where a is the radius of the sphere (taken equal to the Earth's mean radius), and g is the value of gravity at its surface, we fall again on the case of incompressible material treated by Thomson and Tait, and obtain the same results. This serves as a partial verification of the analysis. If however we suppose m and n both finite and comparable with $g\rho a$, and connected by the relation $m=2n$ which holds nearly enough for most hard solids that have been submitted to experiment, we get a different case. Now it is shown in this paper that in both cases the harmonic inequality is expressible in the form $\epsilon W_2/g$ where ϵ is a number, and that ϵ is a rational function of a second number $\vartheta=\frac{1}{3}g\rho a/n$. This number ϑ is such that $(3\vartheta)^{-\frac{1}{2}}$ is the ratio of the velocity of waves of distortion in the material to that due to falling through half the radius of the sphere under gravity kept constant and equal to that at its surface.

When $n/m=0$, as in the first case, the numerator and denominator are linear in ϑ. When $n/m=\frac{1}{2}$, as in the second case, the numerator and denominator are cubics, neither of which has a positive root. It appears on calculating the values of the two functions for positive values of ϑ that the values of ϵ in the two cases are always very nearly equal for the same value of ϑ. When the rigidity is not less than that of glass ϑ is $\ngtr 5$ and it appears that for all such values of ϑ the value of ϵ given by the second supposition is slightly greater than that given by the first, for some value of ϑ greater than 5 they become equal, and subsequently the value of ϵ given by the first is slightly greater than that given by the second. The differences are always very minute. Thus for the purpose of estimating the tidal effective rigidity of the Earth, Sir William Thomson's method is sufficiently exact. For this purpose we must consider a third case of the problem, viz. we must find the tidal distortion in a sphere of homogeneous liquid of the same mass and diameter as the Earth. This is also expressible in the form $\epsilon W_2/g$ and ϵ is the fraction $\frac{5}{2}$. If then the values of ϵ found by either of the previous calculations be multiplied by $\frac{2}{5}$ we shall have the ratio of the elastic solid yielding to the fluid yielding. The fraction obtained by subtracting this ratio from unity is the ratio of the height of the ocean-tides on the yielding nucleus to the true equilibrium height. As mentioned before, this fraction is about $\frac{2}{3}$ for a tidal effective rigidity equal to the rigidity of steel, and about $\frac{2}{5}$ for a tidal effective rigidity equal to that of glass.

1. Let W be the potential of the external disturbing bodies, and suppose that for space within the sphere W is expanded in a convergent series of spherical solid harmonics in the form

$$W=\Sigma_{i=0}^{i=\infty} W_{i+1} \quad \text{.......................................} (1).$$

Suppose that by the action of the external forces the sphere originally of radius a' is strained so that the equation to its surface becomes

$$r=a+\Sigma_0^\infty \epsilon_i Q_{i+1} \quad \text{.......................................} (2),$$

where ϵ_i is a small quantity and Q_{i+1} a spherical solid harmonic of degree $i+1$. Then the harmonic inequalities $\epsilon_i Q_{i+1}$ will exert an attraction on the mass whose potential we may denote by V, and this potential will, like W, be capable of expansion in a convergent series of spherical solid harmonics in the form

$$V = \Sigma_{i=0}^{i=\infty} V_{i+1} \ldots\ldots\ldots\ldots (3).$$

If ρ be the density of the solid and γ the constant of gravitation the bodily forces will be derivable from a potential

$$-\tfrac{2}{3}\pi\gamma\rho r^2 + V + W \ldots\ldots\ldots\ldots (4),$$

which we shall denote by Y, and the general equations of equilibrium will be three of the form

$$m\frac{\partial\delta}{\partial x} + n\nabla^2\alpha + \rho\frac{\partial Y}{\partial x} = 0 \ldots\ldots\ldots\ldots (5),$$

where α, β, γ are the displacements in the direction of the axes of x, y, z, δ is the cubical dilatation $\partial\alpha/\partial x + \partial\beta/\partial y + \partial\gamma/\partial z$, and m and n are two elastic constants.

2. The solution of the system of equations (5) consists of particular integrals and of complementary functions which satisfy a system identical with (5) when Y is left out. The latter are given in Thomson and Tait, Art. 736 (*e*), in a form adapted to satisfy conditions at the surface of a sphere $r = a$ and this form is equivalent to

$$\alpha = \Sigma_{i=0}^{i=\infty}\left[A_i\frac{r^i}{a^i} + a^2 M_{i+2}\frac{\partial\psi_{i+1}}{\partial x} - r^2 M_i\frac{\partial\psi_{i-1}}{\partial x}\right] \ldots\ldots\ldots\ldots (6),$$

where we have picked out the terms of order i in x, y, z. β and γ are to be derived by cyclical interchanges of the letters (A, B, C), (x, y, z), A_i, B_i, C_i are spherical surface harmonics, and at the surface

$$\alpha = \Sigma A_i,\ \beta = \Sigma B_i,\ \gamma = \Sigma C_i \ldots\ldots\ldots\ldots (7),$$

M_i is the constant

$$\tfrac{1}{2}\frac{m}{m(i-1) + n(2i-1)} \ldots\ldots\ldots\ldots (8),$$

and

$$\psi_{i-1} = \frac{\partial}{\partial x}\left(A_i\frac{r^i}{a^i}\right) + \frac{\partial}{\partial y}\left(B_i\frac{r^i}{a^i}\right) + \frac{\partial}{\partial z}\left(C_i\frac{r^i}{a^i}\right) \ldots\ldots\ldots\ldots (9),$$

which is a spherical solid harmonic of degree $i-1$.

3. For the expression of the surface-tractions at the surface of the mean sphere $r = a$ we have to introduce a new function ϕ_{-i-2} defined by the equation

$$\phi_{-i-2} = \frac{\partial}{\partial x}\left(A_i\frac{a^{i+1}}{r^{i+1}}\right) + \frac{\partial}{\partial y}\left(B_i\frac{a^{i+1}}{r^{i+1}}\right) + \frac{\partial}{\partial z}\left(C_i\frac{a^{i+1}}{r^{i+1}}\right) \ldots\ldots\ldots\ldots (10),$$

then ϕ_{-i-2} is a spherical solid harmonic of degree $-i-2$ and differs from Thomson and Tait's Φ_{i+1} only in being divided through by r^{2i+3}/a^{i+1}. The surface tractions parallel to x, y, z at any point of the mean sphere are calculated in Thomson and Tait, Art. 737, and are equivalent to F, G, H, where

$$F \,.\, r = n\Sigma\left[(i-1)A_i\frac{r^i}{a^i} - \frac{1}{2i+1}\frac{\partial}{\partial x}\left(\frac{r^{2i+3}}{a^{i+1}}\phi_{-i-2}\right) - E_{i+2}\frac{r^{2i+5}}{a^{2i+3}}\frac{\partial}{\partial x}\left(\psi_{i+1}\frac{a^{2i+3}}{r^{2i+3}}\right)\right] \ldots\ldots (11),$$

and we have picked out the terms containing surface harmonics of order i. G and H are to be derived by cyclical interchanges of the letters (A, B, C), (x, y, z), and E_{i+2} is the constant

$$\frac{1}{2i+5}\frac{m(i+4)-n(2i+3)}{m(i+1)+n(2i+3)} \quad\text{.............................(12).}$$

4. We have now to consider the particular integrals of (5). We shall treat first the term of order zero $-\frac{2}{3}\pi\gamma\rho r^2$. The purely radial force $-\frac{4}{3}\pi\gamma\rho r$ hence arising produces a purely radial displacement U whose amount can easily be shown to be

$$U = Ar + Hr^3 \quad\text{..(13),}$$

where A is an arbitrary constant and

$$H = \frac{1}{10}\frac{\rho}{m+n}\frac{4}{3}\pi\gamma\rho \quad\text{......................................(14).}$$

The six strains e, f, g, a, b, c referred to the axes of x, y, z depending on (13) are given by such formulæ as

$$e = H(r^2 + 2x^2) + A, \ldots, \ldots, a = 4Hyz, \ldots \quad\text{.............................(15),}$$

as shown in my previous paper (*Proc. Lond. Math. Soc.* XIX. p. 185), and the surface tractions at the surface $r = a + \Sigma\epsilon_i Q_{i+1}$ are of the form $\lambda P + \mu U + \nu T, \ldots, \ldots$, where ($\lambda$, μ, ν) are the direction cosines of the outward-drawn normal to the surface and

$$P = (m-n)\,\delta + 2ne, \ldots\ S = na, \ldots$$

are the six stresses as calculated from the formulæ (15).

Now neglecting ϵ_i^2, λ is given by the formula

$$\lambda = \frac{x}{r} + \Sigma\epsilon_i\left\{\frac{(i+1)\,x}{r}Q_{i+1} - \frac{\partial Q_{i+1}}{\partial x}\right\} \quad\text{.............................(16),}$$

and for μ and ν we have similar expressions, and we find without difficulty for the part contributed to $F.r$, neglecting ϵ_i^2, the form

$$x\left[Ha^2(5m+n) + A\,(3m-n)\right]\left(1 + \Sigma\frac{\epsilon_i Q_{i+1}}{r}\right) + 2Ha\Sigma\epsilon_i Q_{i+1}(5m+n)\,x$$

$$-a\Sigma\epsilon_i\left[Ha^2(5m-3n)\frac{\partial Q_{i+1}}{\partial x} + 4nH(i+1)\,xQ_{i+1} + (3m-n)\,A\frac{\partial Q_{i+1}}{\partial x}\right] \quad\text{.........(17).}$$

We shall shew hereafter that the term $x\left[Ha^2(5m+n) + A\,(3m-n)\right]$ is the only one not containing a spherical solid harmonic with a small multiplier, like $\epsilon_i Q_{i+1}$, and thus this term will have to vanish, and we find

$$A = -\frac{5m+n}{3m-n}Ha^2 \quad\text{.......................................(18).}$$

This with (13) and (14) gives the mean radial displacement, a matter which need not detain us here.

Using now (18) to simplify (17) we obtain for the typical term contributed to $F.r$

$$2\{5m-(2i+1)\,n\}\,Ha\epsilon_i xQ_{i+1} + 4nHa^3\epsilon_i\frac{\partial Q_{i+1}}{\partial x} \quad\text{.......................(19).}$$

or at the surface of the mean sphere $r=a$, we find by using the identity

$$xQ_{i+1}=\frac{1}{2i+3}\left[r^2\frac{\partial Q_{i+1}}{\partial x}-r^{2i+5}\frac{\partial}{\partial x}\left(\frac{Q_{i+1}}{r^{2i+3}}\right)\right] \quad\text{.........................(20)},$$

that the typical term contributed to $F.r$ may be written

$$2Ha^3\epsilon_i\frac{5m+(2i+5)n}{2i+3}\frac{\partial Q_{i+1}}{\partial x}-2Ha\epsilon_i\frac{5m-(2i+1)n}{2i+3}r^{2i+5}\frac{\partial}{\partial x}\left(\frac{Q_{i+1}}{r^{2i+3}}\right) \quad\text{.......... (21)},$$

as in my previous paper, p. 187, equation (44), with a like verification to that on p. 188 of the same paper.

5. Take next the term of order $i+1$ in (4) and write

$$Y_{i+1}=V_{i+1}+W_{i+1} \quad\text{.......................................(22)}.$$

The particular integral will be found as in Thomson and Tait, Art. 834, by taking

$$\alpha=\frac{\partial\phi}{\partial x},\quad \beta=\frac{\partial\phi}{\partial y},\quad \gamma=\frac{\partial\phi}{\partial z},\quad \delta=\nabla^2\phi \quad\text{.............................(23)}.$$

This reduces equations (5) to the form

$$(m+n)\nabla^2\phi+\Sigma\rho Y_{i+1}=0 \quad\text{.....................................(24)},$$

and a solution is

$$\phi=-\frac{\rho}{m+n}\Sigma\frac{r^2}{2(2i+5)}Y_{i+1} \quad\text{..................................(25)},$$

since Y_{i+1} is a solid harmonic of order $i+1$.

Hence the particular integral for α is of the form

$$\alpha=-\frac{\rho}{m+n}\Sigma\frac{1}{2(2i+5)}\frac{\partial}{\partial x}(r^2Y_{i+1}) \quad\text{............................(26)},$$

or by using the identity (20) with Y in place of Q we find for the typical term of the particular integral for α

$$\frac{\rho}{m+n}\left[\frac{r^{2i+5}}{(2i+3)(2i+5)}\frac{\partial}{\partial x}\left(\frac{Y_{i+1}}{r^{2i+3}}\right)-\frac{1}{2}\frac{r^2}{2i+3}\frac{\partial Y_{i+1}}{\partial x}\right] \quad\text{.....................(27)},$$

and those for β and γ are to be found by cyclical interchanges of the letters (x, y, z), and the complete value of α is to be found by adding the expressions in (27) and (6). This practically agrees with Thomson and Tait's Art. 834, equation (1). The surface-tractions that are contributed by the solutions such as (27) are calculated also in Thomson and Tait's article and the typical term contributed to $F.r$ can be written in the form

$$-\rho\frac{m+n(i+1)}{(m+n)(2i+3)}r^2\frac{\partial Y_{i+1}}{\partial x}+\rho\frac{(2i+5)m-n}{(m+n)(2i+3)(2i+5)}r^{2i+5}\frac{\partial}{\partial x}\left(\frac{Y_{i+1}}{r^{2i+3}}\right) \quad\text{....... (28)}.$$

6. We have now to find V. This is the potential within a sphere of radius a of a distribution of density on its surface equal to the product of the volume-density ρ and the radial displacement $(\alpha x+\beta y+\gamma z)/r$ calculated for the surface $r=a$. The part contributed to the surface-value of $\alpha x+\beta y+\gamma z$ by the complementary functions (6) contains

a typical term which is seen to be

$$a^2\left[\frac{1}{2i+5}\psi_{i+1}-\frac{1}{2i+1}\phi_{-i-2}\right] \quad\text{.............................(29),}$$

where we have picked out the terms containing surface harmonics of order $(i+1)$.

This is obtained by using (7) and observing that in virtue of an identity similar to (20)

$$\frac{r^i}{a^i}(A_ix+B_iy+C_iz)=\frac{r^2}{2i+1}\left[\psi_{i-1}-\frac{r^{2i+1}}{a^{2i+1}}\phi_{-i-2}\right] \quad\text{......................(30).}$$

The part contributed to $\alpha x+\beta y+\gamma z$ by the particular integrals (27) has a typical term whose surface-value is

$$-\frac{\rho a^2}{m+n}\frac{i+2}{2(2i+5)}Y_{i+1} \quad\text{.....................................(31).}$$

Hence the surface-density of which V_{i+1} is the internal potential is

$$\rho a\left(\frac{1}{2i+5}\psi_{i+1}-\frac{1}{2i+1}\phi_{-i-2}-\frac{\rho}{m+n}\frac{i+2}{2(2i+5)}Y_{i+1}\right) \quad\text{...................(32).}$$

We may easily deduce an equation for V_{i+1} in the form

$$V_{i+1}=\frac{4\pi\gamma\rho a^2}{2i+3}\left[\frac{1}{2i+5}\psi_{i+1}-\frac{1}{2i+1}\frac{r^{2i+3}}{a^{2i+3}}\phi_{-i-2}-\frac{\rho}{m+n}\frac{i+2}{2(2i+5)}(V_{i+1}+W_{i+1})\right] \quad\text{... (33).}$$

Hence

$$V_{i+1}=\frac{\dfrac{4\pi\gamma\rho a^2}{2i+3}}{1+\dfrac{2\pi\gamma\rho^2a^2}{m+n}\dfrac{i+2}{(2i+3)(2i+5)}}\left[\frac{1}{2i+5}\psi_{i+1}-\frac{1}{2i+1}\frac{r^{2i+3}}{a^{2i+3}}\phi_{-i-2}-\frac{\rho}{m+n}\frac{i+2}{2(2i+5)}W_{i+1}\right] \text{.(34),}$$

an equation which may be written

$$V_{i+1}=a_iW_{i+1}+b_i\psi_{i+1}+c_ir^{2i+3}\phi_{-i-2} \quad\text{................................(35),}$$

and then

$$Y_{i+1}=(1+a_i)W_{i+1}+b_i\psi_{i+1}+c_ir^{2i+3}\phi_{-i-2} \quad\text{.........................(36).}$$

Thus the potential of the bodily forces contains terms depending on the complementary solutions of the equations (5).

7. The unknown harmonics A_i, ψ_{i+1}, ϕ_{-i-2} are to be determined by adding together the terms contributed to the surface tractions and expressed in (11), (17) and (28) and equating the result to zero. Observing that in (11) and (28) all the terms contain surface harmonics multiplied by small quantities of the order of the amplitude of the harmonic inequality, we see that (18) holds and (17) may be replaced by (21). Also by (29) and (31) we have

$$\epsilon_iQ_{i+1}=(\alpha x+\beta y+\gamma z)/r$$

$$=a\left[\frac{1}{2i+5}\psi_{i+1}-\frac{1}{2i+1}\phi_{-i-2}-\frac{\rho}{m+n}\frac{i+2}{2(2i+5)}Y_{i+1}\right],$$

or

$$\epsilon_iQ_{i+1}=a\left[\frac{1}{2i+5}\psi_{i+1}\left(1-\frac{\rho}{m+n}\frac{i+2}{2}b_i\right)-r^{2i+3}\phi_{-i-2}\left\{\frac{1}{(2i+1)a^{2i+3}}+\frac{\rho}{m+n}\frac{i+2}{2(2i+5)}c_i\right\}\right.$$
$$\left.-\frac{\rho}{m+n}\frac{i+2}{2(2i+5)}(1+a_i)W_{i+1}\right] \quad\text{........(37).}$$

We substitute this in (21), add together the terms of (11), (21) thus modified, and (28) modified by using (36), and equate the result to zero, and find a surface-condition which may be written

$$\Sigma\left[n(i-1)A_i\frac{r^i}{a^i}+P_i\frac{\partial W_{i+1}}{\partial x}+r^{2i+5}Q_i\frac{\partial}{\partial x}\left(\frac{W_{i+1}}{r^{2i+3}}\right)+P_i'\frac{\partial \psi_{i+1}}{\partial x}+r^{2i+5}Q_i'\frac{\partial}{\partial x}\left(\frac{\psi_{i+1}}{r^{2i+3}}\right)\right.$$
$$\left.+P_i''\frac{\partial}{\partial x}(r^{2i+3}\phi_{-i-2})+Q_i''r^{2i+5}\frac{\partial\phi_{-i-2}}{\partial x}\right]=0\text{.....................(38)},$$

when $r=a$. The values of the coefficients P_i, Q_i, P_i', Q_i', P_i'', Q_i'' are given by the equations

$$\left.\begin{aligned}
P_i&=-\frac{a^2\rho}{m+n}(1+a_i)\left[\frac{m+n(i+1)}{2i+3}+Ha^2\frac{5m+(2i+5)n}{(2i+3)(2i+5)}(i+2)\right]\\
Q_i&=\frac{\rho}{m+n}(1+a_i)\left[\frac{(2i+5)m-n}{(2i+3)(2i+5)}+Ha^2\frac{5m-(2i+1)n}{(2i+3)(2i+5)}(i+2)\right]\\
P_i'&=-\frac{a^2\rho}{m+n}b_i\left[\frac{m+n(i+1)}{2i+3}+Ha^2\frac{5m+(2i+5)n}{(2i+3)(2i+5)}(i+2)\right]+2Ha^4\frac{5m+(2i+5)n}{(2i+3)(2i+5)}\\
Q_i'&=\frac{\rho}{m+n}b_i\left[\frac{(2i+5)m-n}{(2i+3)(2i+5)}+Ha^2\frac{5m-(2i+1)n}{(2i+3)(2i+5)}(i+2)\right]\\
&\qquad-2Ha^2\frac{5m-(2i+1)n}{(2i+3)(2i+5)}-nE_{i+2}\\
P_i''&=-\frac{a^2\rho}{m+n}c_i\left[\frac{m+n(i+1)}{2i+3}+Ha^2\frac{5m+(2i+5)n}{(2i+3)(2i+5)}(i+2)\right]\\
&\qquad-2Ha^4\frac{5m+(2i+5)n}{(2i+1)(2i+3)}\frac{1}{a^{2i+3}}-\frac{n}{(2i+1)a^{2i+1}}\\
Q_i''&=\frac{\rho}{m+n}c_i\left[\frac{(2i+5)m-n}{(2i+3)(2i+5)}+Ha^2\frac{5m-(2i+1)n}{(2i+3)(2i+5)}(i+2)\right]\\
&\qquad+2Ha^2\frac{5m-(2i+1)n}{(2i+1)(2i+3)}\frac{1}{a^{2i+3}}
\end{aligned}\right\}\text{...(39)},$$

where E_{i+2} is given by (12), and H by (14).

The other surface-conditions are to be obtained from (38) by cyclical interchanges of the letters (A, B, C) and (x, y, z).

From these equations we are to find $A_i,\ldots,\ \psi_{i+1}\ldots,\ \phi_{-i-2}\ldots$ in terms of W_{i+1} and the other harmonics occurring in the disturbing potential.

8. We may find the solution for each term of the disturbing potential by supposing all the other terms to vanish. We shall therefore suppose that W_{i+1} is the expression of the disturbing potential and proceed to determine the unknowns so far as they depend on it.

Now in (38) the function on the left is finite continuous and one-valued within the region containing the origin, satisfies Laplace's equation, and vanishes at the surface $r=a$. It is therefore identically zero. Take then the equations such as (38) and differentiate them with respect to x, y, z respectively and add, we thus obtain the equation

$$-(2i+5)(i+2)\left[Q_iW_{i+1}+Q_i'\psi_{i+1}+Q_i''r^{2i+3}\phi_{-i-2}\right]+n(i+1)\psi_{i+1}=0\text{ ....(40)},$$

where we have picked out the terms containing surface harmonics of order $(i+1)$.

Again multiply equation (38) and the like equations by x, y, z add and use (30) and we get

$$\{P_i(i+1)-Q_i(i+2)r^2\}W_{i+1}+\{P_i'(i+1)-Q_i'(i+2)r^2\}\psi_{i+1}+\{P_i''(i+1)-Q_i''(i+2)r^2\}r^{2i+3}\phi_{-i-2}$$
$$+\frac{n(i+1)r^2}{2i+5}\psi_{i+1}-n(i-1)\frac{r^{2i+3}}{a^{2i+1}(2i+1)}\phi_{-i-2}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (41),$$

where as before we have picked out the terms containing surface harmonics of order $(i+1)$. Using (40) to simplify (41) we have

$$(i+1)\{P_iW_{i+1}+P_i'\psi_{i+1}+P_i''r^{2i+3}\phi_{-i-2}\}=n(i-1)\frac{r^{2i+3}}{a^{2i+1}(2i+1)}\phi_{-i-2}\quad\ldots\ldots\ldots(42).$$

Equations (40) and (42) determine ψ_{i+1} and ϕ_{-i-2} in terms of W_{i+1} and they shew that each of these functions is simply proportional to W_{i+1}.

To find the A, B, C observe that all the terms of (38) except the $\Sigma(i-1)A_i\frac{r^i}{a}$ contain spherical surface harmonics of order i or else of order $i+2$ so that the only A, B, C that can occur are A_i, A_{i+2} and the like B and C. Thus picking out the terms containing surface harmonics of orders i and $i+2$ separately we have the equations

$$\left.\begin{aligned}-n(i-1)A_i\frac{r^i}{a^i}&=P_i\frac{\partial W_{i+1}}{\partial x}+P_i'\frac{\partial\psi_{i+1}}{\partial x}+P_i''\frac{\partial}{\partial x}(r^{2i+3}\phi_{-i-2})\\ -n(i+1)A_{i+2}\frac{r^{i+2}}{a^{i+2}}&=r^{2i+5}\left[Q_i\frac{\partial}{\partial x}\left(\frac{W_{i+1}}{r^{2i+3}}\right)+Q_i'\frac{\partial}{\partial x}\left(\frac{\psi_{i+1}}{r^{2i+3}}\right)+Q_i''\frac{\partial\phi_{-i-2}}{\partial x}\right]\end{aligned}\right\}\ \ldots\ldots (43).$$

And the displacement α is given by the equations

$$\alpha=A_i\frac{r^i}{a^i}+A_{i+2}\frac{r^{i+2}}{a^{i+2}}+(a^2-r^2)M_{i+2}\frac{\partial\psi_{i+1}}{\partial x}$$
$$-\frac{\rho}{m+n}\frac{1}{2(2i+5)}\frac{\partial}{\partial x}\{r^2(1+a_i)W_{i+1}+r^2b_i\psi_{i+1}+c_ir^{2i+5}\phi_{-i-2}\}\ldots (44),$$

and in like manner the other displacements can be written down.

The amount of the harmonic inequality ϵ_iQ_{i+1} is given by the equation (37), in which as we now see ψ_{i+1} and ϕ_{-i-2} are proportional to W_{i+1} so that to each term in the disturbing potential there corresponds one term in the equation of the surface

$$r=a+\Sigma\epsilon_iQ_{i+1}$$

and these terms contain the same surface harmonic.

9. We proceed to reduce the question to one of arithmetical calculation in two special cases. These will agree in that we shall take W to consist of a single term W_2 which is a spherical solid harmonic of order 2, i.e. we shall take $i=1$. They will also agree in that we shall assume $\rho=5{\cdot}6$ or that the density is about the same as the Earth's mean density. They will differ in that in the first we shall suppose the solid incompressible, i.e. we shall take m great compared with n and great compared with $\pi\gamma\rho^2a^2$ which will be taken of the same order as n, while in the second we shall suppose m and n connected by the relation $m=2n$ which is nearly verified for most solids that have been tested by experiment.

Let us write θ for the number $\frac{4}{3}\pi\gamma\rho^2a^2/(m+n)$ and g for the value of gravity at the surface, i.e. for $\frac{4}{3}\pi\gamma\rho a$. Then

$$\theta = g\rho a/(m+n) \qquad (45).$$

In the first case $\theta = 0$ but θm is finite and $= g\rho a$.

In the second case $\theta = \frac{1}{3}g\rho a/n$ so that θ has in this case the meaning given to the symbol ϑ in the introduction. The two symbols are distinct in the first case.

We shall have for both cases

$$\left.\begin{aligned} \left(Q_1' - \frac{2n}{21}\right)\psi_2 + Q_1'' r^5 \phi_{-3} &= -Q_1 W_2 \\ P_1'\psi_2 + P_1'' r^5 \phi_{-3} &= -P_1 W_2 \end{aligned}\right\} \qquad (46)$$

by (40) and (42); and the equation giving the amount of the harmonic inequality is the surface-value of

$$\epsilon_1 Q_2 = a\frac{70}{70+9\theta}\left(\frac{\psi_2}{7} - \frac{\phi_{-3}}{3}\right) - \frac{W_2}{g}\frac{15\theta}{70+9\theta} \qquad (47).$$

Also the values of the P's and Q's are

$$\left.\begin{aligned}
P_1 &= -\frac{a\theta}{g}\frac{70}{70+9\theta}\left(\frac{m+2n}{5} + \frac{3\theta}{10}\frac{5m+7n}{35}\right) \\
Q_1 &= \frac{\theta}{ag}\frac{70}{70+9\theta}\left(\frac{7m-n}{35} + \frac{3\theta}{10}\frac{5m-3n}{35}\right) \\
P_1' &= -\frac{a\theta}{g}\frac{6ag}{70+9\theta}\left(\frac{m+2n}{5} + \frac{3\theta}{10}\frac{5m+7n}{35}\right) + \frac{\theta a^2}{5}\frac{5m+7n}{35} \\
Q_1' &= \frac{\vartheta}{ag}\frac{6ag}{70+9\theta}\left(\frac{7m-n}{35} + \frac{3\theta}{10}\frac{5m-3n}{35}\right) - \frac{\theta}{5}\frac{5m-3n}{35} - \frac{n}{7}\frac{5m-5n}{2m+5n} \\
P_1'' &= \frac{1}{a^5}\left[\frac{a\theta}{g}\frac{14ag}{70+9\theta}\left(\frac{m+2n}{5} + \frac{3\theta}{10}\frac{5m+7n}{35}\right) - \frac{\theta a^2}{5}\frac{5m+7n}{15} - \frac{na^2}{3}\right] \\
Q_1'' &= \frac{1}{a^5}\left[-\frac{\theta}{ag}\frac{14ag}{70+9\theta}\left(\frac{7m-n}{35} + \frac{3\theta}{10}\frac{5m-3n}{35}\right) + \frac{\theta}{5}\frac{5m-3n}{15}\right]
\end{aligned}\right\} \qquad (48).$$

10. Taking up now the first case putting $\theta = 0$ but $\theta m = g\rho a$ and substituting in (46) we have as is easily verified

$$\left.\begin{aligned} \left(\frac{19n}{42} + \frac{2g\rho a}{175}\right)\psi_2 - \frac{2g\rho a}{75}\frac{r^5}{a^5}\phi_{-3} &= \tfrac{1}{5}\rho W_2 \\ \frac{2g\rho a}{175}\psi_2 - \left(\frac{n}{3} + \frac{2g\rho a}{75}\right)\frac{r^5}{a^5}\phi_{-3} &= \tfrac{1}{5}\rho W_2 \end{aligned}\right\} \qquad (49),$$

from which by solving and substituting in (47) where θ is put $= 0$ we find

$$\epsilon_1 Q_2 = \frac{a\rho W_2}{\dfrac{2g\rho a}{5} + \dfrac{19}{5}n} \qquad (50)$$

and this may be written

$$\epsilon_1 Q_2 = \frac{W_2}{g}\frac{15\vartheta}{6\vartheta + 19} \qquad (51),$$

where $\vartheta = \frac{1}{3}g\rho a/n$.

Hence if Q_2 be taken to be W_2/g and if we write ϵ for ϵ_1, ϵ will give a measure of the amount of the inequality and we have

$$\epsilon = \frac{15\vartheta}{6\vartheta + 19} \quad \text{.......................................(52).}$$

11. Again taking up the second case, putting everywhere $m = 2n$, $\theta = \frac{1}{3}g\rho a/n$, we find

$$P_1 = -\frac{a^2\rho}{15}\frac{280 + 51\theta}{70 + 9\theta},$$

$$Q_1 = \frac{\rho}{15}\frac{130 + 21\theta}{70 + 9\theta},$$

$$P_1' = \frac{2}{3}\frac{g\rho a^3}{70 + 9\theta},$$

$$Q_1' - \frac{2n}{21} = -\frac{g\rho a}{189}\frac{770 + 135\theta}{(70 + 9\theta)\,\theta},$$

$$P_1'' = -\frac{g\rho}{9a^2}\frac{70 + 23\theta}{\theta\,(70 + 9\theta)},$$

$$Q_1'' = \frac{4}{9}\frac{g\rho}{a^4}\frac{1}{70 + 9\theta}.$$

Substituting in (46) we have for the surface-values at $r = a$,

$$\left.\begin{aligned} \frac{770 + 135\theta}{9}\frac{\psi_2}{7} - 4\theta\frac{r^5}{a^5}\frac{\phi_{-3}}{3} &= (130 + 21\theta)\frac{\theta W_2}{5ag} \\ 2\theta\psi_2 - (70 + 23\theta)\frac{r^5}{a^5}\frac{\phi_{-3}}{3} &= (280 + 51\theta)\frac{\theta W_2}{5ag} \end{aligned}\right\} \quad \text{.....................(53),}$$

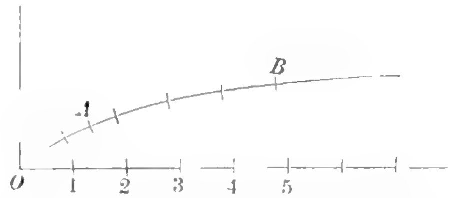
and thence in (47) taking as before $Q_2 = \dfrac{W_2}{g}$ we shall find ϵ_1 or ϵ given by

$$\epsilon = \frac{\vartheta}{70 + 9\vartheta}\,\frac{3356500 + 863100\vartheta + 55485\vartheta^2}{53900 + 27160\vartheta + 2601\vartheta^2} \quad \text{........................(54),}$$

where ϑ is written for θ the two numbers being in this case identical.

12. Now taking the data furnished in Thomson and Tait, Arts. 837, 838 as to the rigidity of steel and glass we shall find that $\vartheta = \frac{3}{2}$ nearly for the rigidity of steel, and $\vartheta = 5$ nearly for that of glass, the density ρ being taken equal to the Earth's mean density 5·6. To see therefore how the inequality ϵ depends on ϑ or on the rigidity it is only necessary to trace the curves (52) and (54) with ϵ for ordinate and ϑ for abscissa. The curve (52) is a rectangular hyperbola passing through the origin and the part ϑ positive of the branch through the origin is the part to be considered. It can be easily seen by calculation that the corresponding part of the curve (54) lies always very near to (52). The tangent lines at the origin to (52) and (54) start out at inclinations of $\tan^{-1}\frac{4}{5}$ and $\tan^{-1}\frac{9}{10}$ nearly so that the points of (54) begin by being slightly above those of (52) which have the same abscissæ. This state of things goes on until $\vartheta > 5$ but the difference is diminishing all the way from $\vartheta = \frac{3}{2}$ to $\vartheta = 5$. When ϑ is infinite the hyperbola touches the asymptote $\epsilon = \frac{5}{2}$ and the curve (54) touches the asymptote $\epsilon = 55485 \div 23409$ which is slightly less than $\frac{5}{2}$. It is difficult without taking a

very large number of points to draw both curves. I have therefore contented myself with a drawing of the hyperbola (52). On the scale to which the figure is drawn it would not be easy to distinguish the two curves.



Of the points A and B, A corresponds to the rigidity of steel and B to that of glass, i.e. A to $\vartheta = \frac{3}{2}$ and B to $\vartheta = 5$. The ordinate of A is about ·803 or nearly $\frac{4}{5}$, that of B is about 1·53 or slightly greater than $\frac{3}{2}$.

To determine the "tidal effective rigidity" we may with sufficient exactness compare the value of ϵ as given by (52) with that which would obtain in a homogeneous liquid sphere of the same mass and diameter. The latter will be found from (50) by making $n = 0$, i.e. it gives $\epsilon = \frac{5}{2}$.

We have seen that for rigidity equal to that of steel ϵ is nearly $\frac{4}{5}$ it follows that the ratio of the elastic solid yielding in this case to the fluid yielding is nearly $\frac{8}{25}$ or about $\frac{1}{3}$. Consequently the height of the ocean-tide will be reduced to about $\frac{2}{3}$ of the true equilibrium amount by the elastic yielding of the nucleus when the "tidal effective rigidity" is that of steel. In like manner it will be reduced to about $\frac{2}{5}$ of the true equilibrium amount when the "tidal effective rigidity" is that of glass.

V. *On Solution and Crystallization. No. III.* By G. D. LIVEING, M.A., Professor of Chemistry in the University of Cambridge.

[Read May 26, 1890.]

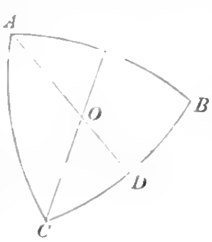
IN my last communication on this subject I made the supposition that all the molecules of the same substance have, on the average, under similar conditions of temperature, pressure, and other external circumstances affecting their mechanical state, similar motions; and that the excursions of the parts of any molecule from the centre of mass of the molecule are, under given conditions, comprised with a certain ellipsoid. This ellipsoid I called for convenience the molecular volume, and assumed it to be of the same average dimensions for all molecules of the same substance under the same circumstances. In passing from the fluid to the crystalline state the molecules will pack themselves as closely in the solid state as is consistent with their molecular volumes, and then, as I shewed, each ellipsoid will be touched by twelve others, and the orientation of the axes will be the same for all of them. It is on this arrangement that I conceive the ordinary properties of crystals to depend.

If the ellipsoids have all their axes equal, that is be spheres, the crystal will belong to the cubic system with the principal cleavage octahedral: if the ellipsoids be oblate spheroids with longest and shortest diameters in the ratio $\sqrt{2} : 1$ and the axes of revolution perpendicular to one of the planes in which the points of contact of each spheroid with its neighbours are four in number (Part II. fig. 2), the crystal will belong to the cubic system but the principal cleavage will be dodecahedral: if the ratio of the greatest and least diameters of the spheroids be $2 : 1$ and the axes of revolution perpendicular to the plane of fig. 1, the crystal will still belong to the cubic system, but the principal cleavage will be cubic. Now if we conceive the spheres and spheroids to be material, instead of being merely the geometric boundaries of the excursions of the parts of the molecules, and to be subject to a uniform stress perpendicular to one plane of the fundamental cube, those originally spheres will be strained to spheroids, and those originally spheroids with axes of revolution perpendicular to the plane of four contacts (fig. 2) will have the ratio of their greatest and least diameters altered, and those with their axes perpendicular to the plane of fig. 1 will become ellipsoids. By any of such changes the arrangement of molecules will lose symmetry in consequence of the strain and the crystal will become pyramidal instead of cubic.

If the stress be in the direction of one diagonal of the cube, the effect will be to convert the crystal from cubic to rhombohedral. In the arrangement indicated in fig. 1, one diagonal of the cube is perpendicular to the plane of the figure and if the stress be in that direction the original spheres will be strained into spheroids with axes of revolution perpendicular to the plane of the figure, and in the case of the spheroids with greatest and least diameters in the ratio 2 : 1, this ratio will be altered; and in both these cases the arrangement of the molecules will be the same as if we supposed space divided into equal and similar rhombohedrons and a molecule placed with its centre in each angular point of the rhombohedrons. In the unstrained system of spheres the arrangement is that which would ensue if space were divided into equal cubes, and spheres were placed so that there should be the centre of one in each corner of the cubes, and also the centre of one in the centre of each face of each cube. The strain which converts the cube into a rhombohedron will leave the spheroids similarly arranged one at each corner of the rhombohedron and one at the centre of each face; but this arrangement can be represented more simply since the planes which pass through one extremity of the axis of a rhombohedron and through the centres of two adjacent faces, will cut up space into rhombohedrons all similar and equal to one another, which will have one spheroid at each angular point and none in any other position. The new rhombohedrons will be more acute than the old. In fact if the unbroken lines in fig. 9 represent the original rhombohedron viewed in the direction of its axis of symmetry, the dotted lines will represent the new rhombohedron, which will have the same axis as the original one and will be placed transversely. There will be four times as many of the new rhombohedrons in a given space as there were rhombohedrons of the original form.

In the remaining case, in which the original cube had the centre of a spheroid in each corner and one in its centre, the spheroids will become strained into ellipsoids, the cube will become a rhombohedron with the centre of an ellipsoid in each angular point and one in its centre. Figure 10 will represent the ellipsoids of one rhombohedron projected on the plane of four contacts (fig. 2, Part II.), the ellipses with unbroken outline representing the ellipsoids with centres *a*, *b*, *c*, *d* in that plane, the ellipse with dotted outline representing the ellipsoid, with centre *e*, lying next above them, and those with broken outline representing the ellipsoids with centres *A*, *B*, *C*, *D* lying above that with dotted outline. Figure 11 represents a section through *ab*, perpendicular to the plane of fig. 10. The ellipsoids with centres in the plane of fig. 10, or in planes parallel to it, will touch each other at the extremities of the equal conjugate diameters, and the diameters through *aA*, *bB* and so on will be conjugate to the plane of fig. 10. These data will suffice to determine the ratio of the axes of the ellipsoids and their orientation when the angle between the axis of the rhombohedron and the normal to one of the faces is given*.

* For if *A*, *B*, *C* be the points where three adjacent edges of the rhombohedron (100), passing through the axis and through the centre of the sphere of projection, meet the surface of that sphere, *O* be the pole of (111), and the angular element, that is, the angle between the normals to 100 and 111, be *D*, *OD* will be $90° - D$. If further, each of the



angles *AC*, *CB*, *BA* be α, and *AD* (which is the angle *Aab* in fig. 11) be ϕ, we have

in triangle *OCD*, the angle at $O = 60°$,

and $$\cos 60° = \cot D \cot(\phi + D - 90°),$$

or $$\tan(\phi + D - 90°) = 2 \cot D \quad \text{.................. (1)},$$

also $$\cos D = \cot 60° \tan \frac{\alpha}{2},$$

or, $$\tan \frac{\alpha}{2} = \sqrt{3} \cos D \quad \text{........................ (2)}.$$

It does not appear that there is any other form and arrangement of the ellipsoids, when packed as closely as possible, which will give rise to the structure of a crystal of the rhombohedral system. At first sight it seems as if these would not suffice to explain the occurrence of what are called hexagonal crystals; but this difficulty vanishes when the following considerations are taken into account. Let us confine our attention for the present to crystals built up of spheroids having their axes of revolution perpendicular to the plane of fig. 1, and let the circles with unbroken outline in that figure, centres *a, b, c, d, e, f, g*, represent spheroids with axes of revolution perpendicular to the plane of the figure and centres in one plane, then the next layer of spheroids may either take the positions indicated by the circles with dotted outline, centres *h, k, l*, or those indicated by the circles with broken outline, centres *m, n, o*. Either of these arrangements equally well fulfils the condition of maximum concentration of the spheroids, and so far either is equally probable. Now in the first case the three planes of the fundamental rhombohedron (100) will be parallel to *hcl, lck* and *kch*, and in the other case they will be parallel to *mco, ocn* and *ncm*. The second rhombohedron will be transverse to the first; or will be in the position of a twin to the first, the twin axis being the axis of the rhombohedron. The crystal may therefore, so far as concentration of molecules is concerned, be built up of alternating layers, of indefinite thicknesses, of such twin crystals. Now what are called hexagonal forms, that is the forms for which the poles lie in great circles bisecting the angles between the three planes which pass each through the axis of symmetry and through one of the three poles of the fundamental rhombohedron, are not in any way affected by this sort of twinning. In fact the forms *hkl*, when $h+k+l=0$, and when $h-2k+l=0$, are identical with the twin forms when the twin axis is the axis of symmetry of the crystal. None of these forms therefore will be at all affected by the alternations of twin layers referred to. It will be otherwise with rhombohedral forms. Any face of such a form which grows when the deposition consists of alternating layers of twins, must either be formed of alternating layers of transverse rhombohedrons, or the face will be ridged and irregular. In the former case the average condensation

Since the plane of fig. 11 is parallel to the stress, it will be a plane of principal section of the ellipsoids and contain two of the axes of the ellipsoids, which will be the axes of the ellipses in that figure and may be called $2x$ and $2z$. The third axis, $2y$, will be perpendicular to that plane and will be the axis parallel to *cd* of the ellipses in fig. 10.

In fig. 11, *ab'* is conjugate to the plane of fig. 10, and *ab', aa'* are conjugate semidiameters of the ellipse with centre *a* in fig. 11. Let *ab'* be z', *aa'* be x' and *ao* which is half of *Aa* be r.

Then $$\left(\frac{AN}{z'}\right)^2+\left(\frac{NP}{x'}\right)^2=1,$$

and since the inscribed parallelogram is half the circumscribed parallelogram, $2AN \cdot NP=z'x'$, and therefore

$$\left(\frac{AN}{z'}\right)^2+\frac{1}{4}\left(\frac{z'}{AN}\right)^2=1;$$

whence $$2\left(\frac{AN}{z'}\right)^2=1,$$

or, $$\frac{r}{z'}=\sqrt{2} \qquad (3).$$

In like manner, $$\frac{aq}{x'}=\sqrt{2}.$$

But by fig. 10,

$$aq=ac\cos\frac{\alpha}{2}=2r\cos\frac{\alpha}{2};$$

so that, $$\sqrt{2}r\cos\frac{\alpha}{2}=x' \qquad (4).$$

Also $$\frac{cq}{y}=\sqrt{2},$$

and $$\sqrt{2}r\sin\frac{\alpha}{2}=y \qquad (5).$$

We have also $$x^2+z^2=x'^2+z'^2 \qquad (6),$$

and $$xz=x'z'\sin\phi \qquad (7).$$

From these equations the ratios $x:y:z$ may be found when D is known, and vice versa.

of the molecules will be the mean of that in the two rhombohedrons transverse to one another: but this will not be a true measure of the surface tension which, for these rhombohedral faces, will change with each alternation of growth. If the alternations took place with perfect regularity, so as to produce alternate layers of each rhombohedron of uniform very small thickness, the effect might be the same as that of a form having the mean condensation. But in fact the alternations will not in general be regular, but determined by causes which depend on the mechanical conditions of the fluid at the points where crystallization occurs; causes which, so far as the forms developed are concerned, may be called accidental. The growth of such faces will therefore be impeded in comparison with the growth of hexagonal forms.

It is obvious that in those cubic crystals in which the molecular volumes are spherical, there will also be the same tendency to grow in alternate layers of twin crystals with the twin axis perpendicular to the octahedral faces. And such alternations have not infrequently been observed. But in the cubic crystal the twinning may take place equally well about any one of the four axes perpendicular to the faces of the octahedron, and in general the only indication of such twinning would be a roughness of the faces. Neither in the hexagonal nor the cubic crystals would the optical and other physical characters be affected, unless the crystal were grown under some stress which gave a peculiar character to those properties.

It is also plain that if the system of spheroids arranged with their axes perpendicular to the plane of fig. 1 be strained in a direction lying in that plane, the spheroids will become ellipsoids and that plane will be a plane of principal section. In this case also alternations of twins will be probable as before.

Similar alternations of growth may also occur when the plane of fig. 1 is not a plane of principal section, because the ellipsoids which represent the molecular volumes may assume in an irregular manner sometimes the positions indicated by the dotted lines and sometimes those of the broken outlines in fig. 1. In these cases the crystals will belong to the less symmetrical systems, and the alternations, though definitely related to one another, will not have the relation of ordinary twins.

Returning to hexagonal forms, if a face has been developed parallel to the plane of fig. 1, that is, a face of the form 111, and the other faces developed be also hexagonal, there will be no cause to interfere with the alternation of twin layers as the crystal grows. But if besides 111 a rhombohedral form, as for example 100, has been developed and the crystal grows by an addition to the face 111, the twinning will cause a discontinuity of the surfaces of 100 at the edges where the forms 111 and 100 intersect. If the transverse form 122 be developed as well as 100, there will be no discontinuity of surface at these edges but some discontinuity of surface tension, which is not the same in the faces of the two forms. This will be a force tending to prevent the twinning or else to prevent the growth of the rhombohedral forms. In most cases the rule that the crystal will grow in such a way that the surface-tension shall, on the whole, be a minimum will, unless the condensation in the rhombohedral form is much greater than in any hexagonal form, ensure the preponderance of the hexagonal forms. These hexagonal forms likewise lend themselves more readily to the formation of nearly globular crystals, that is to crystals with a minimum of total surface.

The cleavages of the hexagonal forms will not be at all affected by the alternations of twins, but cleavages in rhombohedral forms will be rendered difficult and, if they occur at all, will be interrupted. In general the average condensation in a di-rhombohedral pair of forms will be the mean of what it would be in those two forms if there were no twinning. With this consideration we may calculate the relative condensation in the faces of different forms. For this purpose, if p be the perpendicular distance between successive sets of molecules parallel to a face of the form hkl, P the point where the normal to that face meets the sphere of projection, O the corresponding point for the face of the form 111, and X, Y, Z, the traces on that sphere of the crystallographic axes we have, as shewn in Part I.,

$$p = \frac{\cos PX}{h},$$

and

$$\cos PX = \cos PO \cos OX + \sin PO \sin OX \cos POX.$$

Also if D be the angle between the normals to the faces 111, 100

$$\tan POX = \sqrt{3}\frac{k-l}{2h-k-l},$$

$$\tan PO = \frac{\sqrt{\tfrac{1}{2}\{(k-l)^2+(l-h)^2+(h-k)^2\}}}{h+k+l}\tan D,$$

$$\tan OX = 2\cot D,$$

and similar equations with reference to the axes Y and Z.

The hexagonal forms are those for which either POX or PO is 90°, and for these the condensation in the faces is p.

For the other forms it will be $\frac{1}{2}(p+p')$ where p' is the value of p for a face of the transverse form.

For shortness we may designate the form $01\bar{1}$ as a, the form 100 as r, and so on, and the corresponding values of p as p_a, p_r, and so on.

Then taking first hexagonal forms, we have for a or 011, $PO = 90°$, $POY = 30°$,

$$p_a = \sin OY \cos 30° = \frac{\sqrt{3}}{\sqrt{(\tan D)^2+4}},$$

which increases as D diminishes, or as the fundamental rhombohedron is flatter, that is more obtuse.

If $b = 2\bar{1}\bar{1}$, $PO = 90°$, $POX = 0°$, $\quad p_b = \dfrac{\sin OX}{2} = \dfrac{1}{\sqrt{3}}p_a.$

For $x = 210$, $POY = 90°$, $POX = 30°$, $\tan PO = \dfrac{1}{\sqrt{3}}\tan D$,

$$\cos PX = \frac{\sqrt{3}\tan D + \sqrt{3}\tan D}{\sqrt{\{3+(\tan D)^2\}\{(\tan D)^2+4\}}},$$

and

$$p_x = \frac{\cos PX}{2} = \frac{\sqrt{3}\tan D}{\sqrt{\{(\tan D)^2+3\}\{(\tan D)^2+4\}}}.$$

For $z = 311$, $POX = 30°$, $\tan PO = \frac{2}{\sqrt{3}} \tan D$,

$$p_z = \frac{\sqrt{3} \tan D}{\sqrt{\{4 (\tan D)^2 + 3\} \{(\tan D)^2 + 4\}}}.$$

$i = 231$, $POX = 90°$, $\tan PO = \frac{\tan D}{2\sqrt{3}}$,

$$p_i = \frac{\sqrt{3} \tan D}{\sqrt{\{(\tan D)^2 + 12\} \{(\tan D)^2 + 4\}}}.$$

$h = 3\bar{2}\bar{1}$, $PO = 90°$, $\tan POX = \frac{1}{3\sqrt{3}}$,

$$p_h = \frac{\sqrt{3}}{\sqrt{7 \{(\tan D)^2 + 4\}}}.$$

$o = 111$, $PO = 0°$, $p_o = \cos OX = \frac{\tan D}{\sqrt{(\tan D)^2 + 4}}.$

Next for rhombohedral forms.

For $r = 100$, $PO = D$, $POX = 0°$,

$$p_r = \frac{3 \sin D}{\sqrt{(\tan D)^2 + 4}},$$

which increases as D increases up to 45°, and diminishes as D increases from 45° to 90°.

For $r_1 = 122$, the rhombohedron transverse to r, $POX = 180°$,

$$p_{r_1} = \frac{\sin D}{\sqrt{(\tan D)^2 + 4}};$$

whence if we put $p_r' = \frac{1}{2}(p_r + p_{r_1})$,

$$p_r' = \frac{2 \sin D}{\sqrt{(\tan D)^2 + 4}}.$$

For $e = 011$, $\tan PO = \frac{1}{2} \tan D = \cot OX$, $POX = 180°$, $POY = 60°$,

$$p_e = \frac{3 \tan D}{\{(\tan D)^2 + 4\}},$$

which increases as D diminishes.

For $e_1 = 411$, $\tan PO = \frac{1}{2} \tan D = \cot OX$, $POX = 0°$,

$$p_{e_1} = \frac{\tan D}{\{(\tan D)^2 + 4\}} \quad \text{and} \quad p_e' = \frac{2 \tan D}{(\tan D)^2 + 4},$$

for $s = 11\bar{1}$, $\tan PO = 2 \tan D$, $POX = 60°$,

$$p_s = \frac{3 \tan D}{\sqrt{\{4 (\tan D)^2 + 1\} \{(\tan D)^2 + 4\}}}.$$

For $s_1 = 5\overline{1}\overline{1}$, $POX = 0°$, $\tan PO = 2 \tan D$,

$$p_{s_1} = \frac{\tan D}{\sqrt{\{4 (\tan D)^2 + 1\} \{(\tan D)^2 + 4\}}},$$

whence

$$p_s' = \frac{2 \tan D}{\sqrt{\{4 (\tan D)^2 + 1\} \{(\tan D)^2 + 4\}}}.$$

For $n = 211$, $\tan POX = 0°$, $\tan PO = \frac{\tan D}{4}$,

$$p_n = \frac{3 \tan D}{\sqrt{\{(\tan D)^2 + 4\} \{(\tan D)^2 + 16\}}},$$

and for $n_1 = 255$,

$$p_{n_1} = \frac{\tan D}{\sqrt{\{(\tan D)^2 + 4\} \{(\tan D)^2 + 16\}}}.$$

These formulae will help us to compare the relative probability of the occurrence of the several hexagonal forms. For the reasons given above they are not applicable for the comparison of rhombohedral forms with hexagonal; for we cannot say that p_r', which is the average condensation in a plane parallel to a twin face of the form r and of the transverse form r_1, is a measure of the smallness of the surface-tension on such a face, though it indicates a minimum below which that tension will not on the average fall.

From these formulae we get

$p_a : p_o = \sqrt{3} \cot D$, which is greater than unity if D be less than 60°;

$p_a : p_x = \sqrt{1 + 3 (\cot D)^2}$, always greater than unity;

$p_b : p_x = \sqrt{1 + 3 (\cot D)^2} : \sqrt{3}$, which is greater than unity if $\cot D$ be greater than $\sqrt{\frac{2}{3}}$ or D less than 39° 13′;

$p_x : p_z = \sqrt{4 (\tan D)^2 + 3} : \sqrt{(\tan D)^2 + 3}$, which is always greater than unity;

$p_z : p_i = \sqrt{(\tan D)^2 + 12} : \sqrt{4 (\tan D)^2 + 3}$, which is greater than unity if D be less than 60°;

$p_a : p_r' : p_o = \sqrt{3} : 2 \sin D : \tan D$, and p_r' is always intermediate between p_a and p_o.

In crystals having for their molecular volumes spheroids arranged with their axes perpendicular to the plane of fig. 1, we should therefore expect the faces a and o to predominate, and faces to occur in the same zones with the faces of those forms, but the rhombohedral forms to occur rarely. And in fact we find that the distinct cleavages of hexagonal crystals are parallel to either o or a.

If we examine particular cases we find in Apatite, $D = 55° 40'$, and if A be the radius of the principal section of the molecular volume, B the semi-axis,

$$B : A = \frac{\tan D}{2\sqrt{2}} = \cdot 51764.$$

And for hexagonal faces the values of p, which are proportional to the condensation, are for

a,	$01\bar{1}$,	·69877,
o,	111,	·59070,
x,	210,	·45106,
b,	$2\bar{1}\bar{1}$,	·40344,
z,	311,	·30073,
i,	321,	·27206,

and the mean values of p for pairs of transverse rhombohedra are for

rr_1,	100,	$\bar{1}22$,	·49974,
ss_1,	$\bar{1}11$,	$5\bar{1}\bar{1}$,	·47724,
ee_1,	011,	411,	·35746.

The cleavages are parallel to a and o, the former being the more easily obtained.

In the (nearly) isomorphous crystals of Mimetite and Pyromorphite, the most frequent forms are a, o and x; and they have an imperfect cleavage parallel to x. In Vanadinite a and o occur, and Des Cloizeaux gives a figure of a crystal which is exactly like a crystal of Apatite.

In Greenockite, $D = 58° 47'$, the condensations in a and o differ but little, the faces most frequent are all hexagonal, a, o, x, z, i, and the cleavages parallel to a and o.

In Molybdenite the faces occurring are a, o, x and there is a very perfect cleavage parallel to o.

In Polybasite, $D = 71° 31'$, the condensation in o is therefore greater than in a, the cleavage is parallel to o, and the forms which occur are o, a, x.

In Covelline, forms o, a occur and the cleavage is very perfect parallel to faces of o.

In Pyrrhotine, $D = 60° 7'$ so that the condensation in o is slightly greater than in a, and we find that it has a perfect cleavage parallel to o, a less distinct one parallel to b; and the forms which occur most frequently are o, a, b, x, z and the pair r, r_1.

In Graphite the forms developed are hexagonal, the usual forms o, a, and the cleavage parallel to o, but the striation seems to indicate an unsuccessful struggle for the development of rhombohedral forms.

In Ice the usual forms are o and a, and the cleavage parallel to o.

In Brucite forms o and a occur, and the cleavage is very perfect parallel to o, traces parallel to a.

In Hydrargillite, o, a, b occur and there is perfect cleavage parallel to o.

In Emerald, Miller gives $D = 44° 56'$. The most common forms are a and o, then b, x and the pair r, r_1, cleavages o and a, the latter interrupted. With $D = 44° 56'$ we find $p_a : p_o = 1{\cdot}7360$, and we should therefore expect that the cleavage parallel to a would be more perfect than that parallel to o. If however we take the form which Miller assumes to be 100, to be 011, as we are perfectly at liberty to do, we shall get a different value for D, namely 63° 15′, and $p_a : p_o = {\cdot}87302$, and the facts then correspond closely with theory.

In Nepheline $D = 59°\ 10'$, the most frequent forms are o, a, x, z and the cleavages o and a. As D is nearly 60° p_a and p_o are nearly equal.

In Pyrosmalite, $D = 46°\ 42'$, the forms o, a, x, z occur, and the cleavages are o perfect, a less perfect.

In Davyne, $D = 59°\ 15'$ according to Miller, who assumes the most common six-sided pyramid to be the form 231. It seems more reasonable to assume this form to be 120, the other six-sided pyramid which occurs will then be $31\bar{1}$, and $D = 40°\ 2'$. The forms occurring will then be o, a, b, x, z, and the cleavage is perfect parallel to a.

The varieties of Chlorite known as Pennine and Ripidolite appear to me to be hexagonal, or rather to have their molecular volumes spheroids with their axes perpendicular to the plane of fig. 1. Des Cloizeaux taking the acute rhombohedron, which is developed in crystals found on the Rimpfischwänge near Zermatt, as the form 100 finds $D = 76°\ 15'$. Miller makes the corresponding angle 79° 55′. The former angle gives

$$p_a : p_r : p_o = 1{\cdot}732 : 2{\cdot}914 : 4{\cdot}087,$$

the latter gives 1·732 : 2·954 : 5·623. o is the plane of perfect cleavage, a is rarely developed but there are traces of cleavage parallel to it. The rhombohedral faces are usually striated and ridged or undulated parallel to their intersection with o. In large crystals the face o is so dominant that the crystals become six-sided tables. These characters correspond well with theory. The condensation in planes parallel to o is much greater than in any other plane, and it is so large in r that there must be a strong tendency to the development of that form. At the same time the unevenness of the faces r betrays the peculiar growth of hexagonal crystals. Specimens from localities other than Zermatt are much more hexagonal in their appearance, the form 311 and its transverse form occurring frequently, and striated parallel to their intersections with o. The molecular volume will be a prolate spheroid with greatest and least semi-diameters in the ratio 1·444 if we take Des Cloizeaux's measure, or 1·988 if we take Miller's measure, of the angular element. As an illustration of the application of the theory to the facts it does not matter which we take.

Tamarite may very likely have a similar molecular grouping. $D = 71°\ 16'$, and it has a very perfect cleavage parallel to the faces of o, with traces parallel to the faces of r, and the crystals are very thin in a direction perpendicular to o.

In Coquimbite $D = 43°\ 50'$, the forms developed are a, o, x; and it has imperfect cleavages parallel to a and x.

In Parisite the forms which occur are o and z, $D = 81°\ 20'$, and it has a very perfect cleavage parallel to o, and a very imperfect cleavage parallel to r. With so large a value for D the concentration in planes parallel to r is much greater than in planes parallel to a.

Although the twinning which produces hexagonal forms is very likely to occur, yet its occurrence is mainly determined by the more or less accidental circumstances under which the growth of the crystal takes place. The chief obstructive cause to such twinning will be, as stated above, the variations of surface-tension which will occur at the junction of the twin layers where adjacent faces do not belong to faces in the zone oa or the zone ab. In cases in which the condensation in planes parallel to r is much greater than in planes parallel to a, the obstruction to the twinning may suffice to prevent its occurrence. This will be the case when the value of D is large, as in the case of Pennine. And it is probable that those crystals which have a very perfect cleavage parallel to o, but are usually classed as rhombohedral, really have their molecular volumes spheroids and arranged with their axes perpendicular to the plane of fig. 1.

In Bismuth if we take the rhombohedron which in natural crystals is most common, namely that to which Miller assigns the symbol $\bar{1}11$, to be the form 100, we get for D 71° 37′, which differs very little from a cubic form. The forms occurring in natural crystals will then be 111, 100 and 211. There is a very perfect cleavage parallel to 111 or o, less perfect parallel to the faces of the other two forms. The form developed in crystallizing bismuth from fusion will be 011, but there is no cleavage parallel to its faces. The anomalous expansion of bismuth in solidifying indicates a change in the dimensions of the molecular volumes at that temperature, and this circumstance may affect the form assumed by the metal in crystallizing at that temperature.

Antimony is very nearly isomorphous with bismuth, and if we take the form to which Miller assigns the symbol $\bar{1}11$ to be 100, D becomes 71° 40′, and the forms observed are 111, 332 and 011. The cleavages are o very perfect, n distinct, r less distinct, a traces.

Arsenic also is nearly isomorphous with bismuth. Making a similar assumption as to the symbol of the most common rhombohedron namely that it is 011, we find $D = 72° 33′$, the cleavages are parallel to the faces of o, perfect, and parallel to the faces of 211 imperfect; while the faces observed are 111, 011, and $\bar{2}77$. The crystals are of course laboratory preparations.

Spartalite is most probably hexagonal. It has distinct cleavages parallel to o and a, and if we take the form to which Miller assigns the symbol $51\bar{3}$ to be 210 we find for D 71° 57′. If however we take that form to be $31\bar{1}$ we get for D 56° 56′. The latter is perhaps more probable, as it makes the condensation in planes parallel to a and o more nearly equal. We get in that case, $p_a : p_o = 1{\cdot}023$, which agrees well with observation. The natural mineral gives only cleavage faces, as far as I am aware.

Of the isomorphous minerals Haematite, Ilmenite, and Corundum, the last shews a decided tendency to hexagonal forms. The cleavages are parallel to the faces of o and and r, $D = 57° 34′$ and we find $p_a : p_r : p_o = 1 : 1{\cdot}462 : {\cdot}908$. There is a great difference between these values, and they seem inconsistent with the cleavages. But the cleavages are very variable in these minerals, in some specimens seemingly perfect, in others indistinct; the apparently perfect cleavages are sometimes only faces of union of aggregated crystals, so that after all the inconsistency may be more apparent than real.

In specimens of Willemite from Vieille-Montagne near Moresnet there is an easy cleavage parallel to the faces of *o*, a difficult one parallel to the faces of *a*, while in specimens from Franklin in New Jersey, the cleavage is easy parallel to the faces of *a*, according to Des Cloizeaux; and $D = 37° 43'$. Miller gives a different value for *D*, but Dana agrees with Des Cloizeaux. Dana says the rhombohedral faces are seldom smooth, while the prismatic are smooth. It seems therefore probable that in this case also the molecular volumes are spheroids with their axes perpendicular to the plane of fig. 1.

Susannite has an easy cleavage parallel to the faces of *o*, and $D = 68° 38'$.

In Tellurium if we take the form which Miller puts as *b* to be *a*, and those which he puts as rr_1 to be *z*, we find $D = 53° 46'$, and the faces which occur are *o*, *a*, *z*, with a very distinct cleavage parallel to the faces of *a*, and an imperfect one parallel to the faces of *o*.

In Osmiridium, Miller gives the faces which occur as *o*, *a*, *z*, and $D = 58° 27'$. There is a tolerably perfect cleavage parallel to the faces of *o*. If we take the form to which Miller assigns the symbol $31\bar{1}$ to be 210 we shall have $D = 72° 56'$, the forms occurring will be *o*, *a*, *x*, and the condensation greatest in the planes of cleavage.

Breithauptite exhibits forms *o*, *a*, *i*, and $25\bar{1}$, and Kupfernickel the forms *o*, *x*.

Amongst laboratory crystals of hexagonal development we find

Lithium sulphate, with forms *a*, *x*, *o*, with cleavage parallel to *o*, and angular element 73° 26′.

Barium perchlorate, with forms *a*, *x* if crystallized from alcohol and *a*, *z* if crystallized from water, and angular element 52° 57′.

Ethyl-ammonium chloroplatinate, with forms *r*, *o*, *b* hemihedral, with perfect cleavage parallel to *o* and angular element 54° 6′. More probably the forms are *x*, *a*, *o* and angular element 67° 19′, *x* and *a* being hemihedral.

Iodoform, with forms *x* and *o* and angular element 53° 32′.

Ceroso-ceric sulphate, with forms rr_1, *b*, *x*, *o* and angular element 69° 45′; or if we assume the hexagonal prism to be *a*, and the di-rhombohedron rr_1 to be *x*, the forms will be *a*, *x*, *o*, 144, 522, and angular element 77° 58′.

Basic ferric-potassium sulphate, with forms *a*, *o*.

All these agree well with theory if we assume (as I have done) that the six-sided prism is the form 011 and the six-sided pyramid 012.

There are yet two natural crystals which are commonly classed as rhombohedral but to me appear rather to be hexagonal. These are quartz and cinnabar. Both are remarkable for exhibiting asymmetric hemihedry (trapezoidal tetartohedry of some crystallographers) and for their rotation of the plane of polarization of plane polarized light.

To begin with quartz. The most common, I believe the invariable, form is a six-sided prism terminated by a six-sided pyramid with or without other forms. This generally hexagonal appearance is modified frequently by unequal development, and unequal smoothness, of the alternate faces of the terminal pyramids, which is thought to mark them as di-rhombohedral combinations. The cleavages are so difficult to obtain and so interrupted that they hardly help us, but as far as they go they confirm the hexagonal character of the crystal. They are given by Miller, and by Des Cloizeaux, as perpendicular to the axis of the six-sided prism, and parallel to the faces of *both* rhombohedrons of the di-rhombohedral combination, and there is no indication that the cleavage parallel to the faces of one rhombohedron differs in character or facility from that parallel to the faces of the transverse rhombohedron. I know no other case of equal cleavages parallel to the faces of a di-rhombohedral combination, and it appears to me essentially an hexagonal character. Twins are common, almost universal, with the twin axis the axis of the prism. This is very frequent amongst hexagonal crystals, but is not confined to them. If we regard the crystal as hexagonal the difference in size and roughness of the alternate faces of the terminal pyramids will be indications of hemihedral development, or growth under stress, as is the case in many hemihedral crystals when the hemihedry does not extend to the complete suppression of half the faces. The asymmetric hemihedry of quartz is an indication of the formation of the crystal under stress, and there is no reason why both kinds of hemihedry should not coexist. If the crystal be taken as hexagonal the prism will be the form (a) or $01\bar{1}$ and the terminal pyramids the form (x) or 012. We shall then have for the angular element 65° 33′·2, and if $u'v'w'$ be the symbol of a face referred to the new axes and uvw the symbol of the same face referred to the axes assumed by Miller,

$$u' = w + 2u, \quad v' = u + 2v, \quad w' = v + 2w.$$

The abundance of quartz in nature, and the great variety of circumstances in which it has crystallized, have caused a great many combinations of forms to be recorded. The symbols of some of the most frequent forms as referred to the old and new axes are given in the following table:

Miller's Symbol	Hexagonal Symbol	Miller's Symbol	Hexagonal Symbol
$2\bar{1}1$	$10\bar{1}$	$10\bar{1}$	$11\bar{2}$
100 and $\bar{1}22$	210	$7\bar{2}\bar{2}$	$41\bar{2}$
$14\bar{2}$	010	221 and $8\bar{1}\bar{4}$	$\bar{3}24$
011	123	$4\bar{1}\bar{2}$	$62\bar{5}$
$51\bar{1}$ and $\bar{1}11$	$\bar{1}13$	$\overline{14}$, 22, 7	$\bar{7}$, 10, 12
13, 8, 8	618	$\bar{4}52$	$\bar{2}23$

The symbol of the form 111 remains unchanged and though it never occurs except as a cleavage face it is the regular twin-face. This form and the first three forms in the left-hand column have the greatest condensation in their faces, and the supposition that quartz is hexagonal agrees sufficiently well with my molecular theory.

Cinnabar has quite a rhombohedral appearance so far as external form goes, but it has a perfect cleavage parallel to the faces of a hexagonal prism. There is no truly rhombohedral crystal which has such a cleavage, and I infer that the apparently rhombohedral development is due to hemihedry. This inference is confirmed by the fact that cinnabar sometimes shews in its external form an asymmetric hemihedry, and shews by its powerful twisting of the plane of polarization of light that it has this asymmetry in its internal structure. In this respect it presents a striking analogy to the hyposulphates of lead, strontium and calcium, described further on. These three substances are isomorphous, and the strontium hyposulphate has decided hexagonal symmetry, while the crystals of lead hyposulphate resemble those of cinnabar. If we take cinnabar to be hexagonal we must take the cleavage prism to be the form (a) $01\bar{1}$. The most common forms besides the hexagonal prism, are those to which Miller assigns the symbols 111, 100, 522. If we take the last of the three to be the hemihedral development of 012, we get for the form 100 the new symbol $41\bar{2}$, the form 111 retains its symbol, and the less frequent forms become $\bar{1}25$, 741, and 13, 5, 1. The angular element becomes $56^\circ 47'$. The asymmetric hemihedral forms observed by Des Cloizeaux seem to be the alternate faces of $2\bar{1}\bar{1}$ and of a scalenohedron. They are however rare.

We might assume the form 100 of Miller to be 012. We should then get for 522 the new symbol 432, and for the less frequent forms the symbols 123, 543, 753. The numerical values of the indices become a trifle more simple on this assumption, but the angular element, $70^\circ 43'$, would give a smaller value for the condensation in planes parallel to the faces of the hexagonal prism than in planes at right angles to them, and the facility of cleavage in the former planes seems to negative this. Again it might be assumed that the form given as 011 by Miller should be 012. This would give still more simple indices for the forms observed but would still give a greater condensation in planes parallel to 111 than in planes parallel to the faces of the hexagonal prism. On the whole the first supposition corresponds very well with the facts and entirely with my theory. In twin crystals of cinnabar the twin face is 111, as in most hexagonal crystals.

In lead hyposulphate, mentioned above, the forms observed, if we take the crystals as rhombohedral, are r, e, o, a, b, s, and $\bar{1}55$, the first three being most common, and the angular element 60°. If we change the axes and take the form r to be 012 (x), we get the hexagonal forms x, i, o, b, a, z and $\bar{1}37$, and the angular element $71^\circ 34'$. There is no cleavage, and the facts agree well with theory.

Calcium hyposulphate and strontium hyposulphate are isomorphous with the lead salt, but the forms of the strontium hyposulphate are o and x, o being largely developed, and x holohedral but with uneven faces. There is also an imperfect cleavage parallel to o, as we should expect because the maximum concentration (on the hypothesis that the angular element is $71^\circ 34'$) is greatest in the planes parallel to o.

Crystals of sodium periodate with three molecules of water have a very unusual appearance from unequal development of the faces. The forms commonly developed, considered as rhombohedral, are *r*, *e*, *s*, *b*, *o*, *o* being hemimorphic and *b* sometimes hemihedral, and the angular element 51° 38′. They rotate the plane of polarization of light, and besides the hemihedral character of *b*, sometimes shew the alternate edges formed by the intersection of *r* and *e* truncated by a hemihedral scalenohedron. If we assume the crystal to be hexagonal and hemihedral and make the forms *r*, *b*, to be 012, $10\bar{1}$, respectively, we get for *e*, *s*, the symbols 123, $\bar{1}13$, respectively, and for the angular element 65° 26′, which makes the facts and theory agree. The corresponding silver salt appears to be isomorphous with it, or very nearly so, and it exhibits quite as irregular an appearance. It is very likely endowed with the power of rotating the plane of polarization of light, but I am not aware that any one has actually observed this fact. In a few other crystals similar characters have been observed, but they hardly call for a detailed discussion.

Next referring to fig. 2 of Part II., let us consider that the circles with dotted outline *e e e* represent spheres with their centres in the plane of the paper, while those with unbroken outline *bcd*, &c. represent the projections on that plane of the outlines of a set of spheres which touch the former set and have their centres in a plane below the plane of the figure. We may suppose that there is another set of spheres also touching the first set, but lying above them. The projections of their outlines on the plane of the paper will correspond with the circles of unbroken outline, and to distinguish the set lying above the first set we may designate their centres as *B*, *C*, *D* &c., *b* and *B*, *c* and *C*, *d* and *D*, &c. having the same projections, respectively. Then the points *c*, *C*, *c′*, *C′*, *d′*, *D′*, *d*, *D*, lie in the corners of a rectangular parallelopiped with the centre of a sphere *e* in its centre, and the whole space may be cut up into similar and equal parallelopipeds, each having the centre of a sphere at each corner and one in its centre. If the spheres become oblate spheroids with axes perpendicular to the plane of the figure, these parallelopipeds will be cubes if the ratio of the greatest to the least diameter be $\sqrt{2}$. If further we suppose the spheroids to be all strained in the direction of one of the diagonals of the cube the spheroids will become ellipsoids and the cubes will become rhombohedrons. The axes of these rhombohedrons will not be perpendicular to the plane of fig. 1. In fact if the circles with unbroken outline are supposed to have their centres in the plane of the paper, those with dotted outline below, and those with broken outline above, that plane, and *c* be the central sphere, the eight centres which form the corners of the parallelopiped may be *abmnlkfe*, and two of the diagonals *ae*, *bf* lie in the plane of the paper, the others *mk*, *ln* lie in an inclined position. If the parallelopiped become a cube by changing the spheres into spheroids their axes of revolution will be perpendicular to the plane *amnb*. If further the system be subject to a uniform stress in the direction of one of the diagonals of the cube, the spheroids will become strained into ellipsoids and the cube into a rhombohedron with its axis in the direction of the strain. The arrangement of the ellipsoids will be the same as if space were divided into equal rhombohedrons with the centre of an ellipsoid in each angular point and one in the centre of each rhombohedron. This is the same as if two sets of rhombohedrons were superposed, all being equal, similar, and similarly situated, and each having the centre of an ellipsoid at each corner but none in its centre, but one set having its angular points at the centres of the other set. The planes of a set of parallel planes which pass through the corners of one set of rhombohedrons will not in general pass through the corners of the other set, so that, if the arrangement represent the structure of a crystal, the relative condensation of molecules in the direction of the sets of planes will in general be the same as if there were but one set of rhombohedrons with molecules at their corners only. But there are certain cases in which the same plane will pass through the corners of both sets of rhombohedrons, and in such a plane the condensation will be double of what it would otherwise be.

To see what planes have this property, let figure 12 represent the traces on three planes of reference of the planes forming one set of rhombohedrons. Then a plane which passes through z_1 and y_1 and is parallel to the axis OX, will pass through the centres of the rhombohedrons as well as through their corners. This will be the face 011. Also any plane parallel to OX, which passes through $z_n y_m$, where m and n are odd, will also pass through the centres of some of the rhombohedrons. The symbol of the face in this case will be $0hk$ where h and k are both odd numbers. Next if the plane pass through x_l, where l is odd, and also through the intersection of the lines in the plane ZOY drawn parallel to OY and OZ through $z_n y_m$, where m and n are odd numbers, it will pass through the centres of some of the rhombohedrons. That is for such a plane the reciprocals of the indices (reduced to whole numbers) must be one of them an odd number, and the others equimultiples by a power of 2 of some odd numbers; or the indices, without regard to sign, must be of the form

$$2^k(2m+1)(2n+1), \quad (2m+1)(2r+1), \quad (2n+1)(2r+1),$$

where k is an integer, and m, n, r are integers or zero.

Such will be 211, 433, 631, &c.

How to find the relation between the axes of the ellipsoids, and their orientation, when the angular element of the crystal is known, has been already explained. Taking the same notation as before we get in the faces of certain forms double the concentrations which were obtained when there was no molecule in the centre of the rhombohedron.

For a, $01\bar{1}$, $p_a = \dfrac{2\sqrt{3}}{\sqrt{(\tan D)^2 + 4}}$,

b, $\bar{2}\bar{1}\bar{1}$, $p_b = \dfrac{2}{\sqrt{(\tan D)^2 + 4}}$,

e, 011, $p_e = \dfrac{6 \tan D}{(\tan D)^2 + 4}$,

e_1, 411, $p_{e_1} = \dfrac{2 \tan D}{(\tan D)^2 + 4}$,

n, 211, $p_n = \dfrac{6 \tan D}{\sqrt{\{(\tan D)^2 + 4\}\{(\tan D)^2 + 16\}}}$,

$$n_1,\quad 25\bar{5},\quad p_{n_1}=\frac{2\tan D}{\sqrt{\{(\tan D)^2+4\}\{(\tan D)^2+16\}}},$$

$$i,\quad 321\quad p_i=\frac{2\sqrt{3}\tan D}{\sqrt{\{(\tan D)^2+4\}\{(\tan D)^2+12\}}},$$

$$h,\quad 32\bar{1},\quad p_h=\frac{2\sqrt{3}}{\sqrt{7\{(\tan D)^2+4\}}},$$

and so on; while those forms of which the indices do not satisfy one of the conditions above enunciated, will have the same concentration as if there were no molecule at the centre of the rhombohedron.

Comparing the concentration in some of the forms we find

$$\frac{p_a}{p_e}=\frac{\sqrt{(\tan D)^2+4}}{\sqrt{3}\tan D},$$

which is greater than unity if $\tan D$ is less than $\sqrt{2}$ or D less than 54° 45′.

Also $\frac{p_e}{p_r}=\frac{2}{\cos D\sqrt{(\tan D)^2+4}}$, which is always greater than unity; and hence, with this arrangement of molecules, the rhombohedron with the easiest cleavage will be 011 and not 100.

$$\frac{p_e}{p_s}=2\sqrt{\frac{4(\tan D)^2+1}{(\tan D)^2+4}},$$

which is always greater than unity.

Again $\frac{p_a}{p_o}=\frac{2\sqrt{3}}{\tan D}$, and p_a is greater than p_o if $\tan D$ is less than $2\sqrt{3}$, or D less than 73° 54′; and $\frac{p_e}{p_o}=\frac{6}{\sqrt{(\tan D)^2+4}}$, and p_e is greater than p_o if $\tan D$ is less than $4\sqrt{2}$, or D less than 79° 59′.

Now if Calcite have the molecular arrangement now under consideration, the cleavage form must be 011, not 100, and we must change the axes. If we make a change of axes so that form 100 becomes 011, we shall have the new axes parallel to the intersections of every two of the faces of the form $\bar{1}11$, and for a face uvw referred to the original axes we shall have the symbol $u'v'w'$ referred to the new axes, where $u'=v+w$, $v'=u+w$ and $w'=u+v$.

In the case of any face for which $u+v+w=0$ the symbol will remain unchanged. Also for any face for which $2u-v-w=0$ we shall have $2u'-v'-w'=0$.

Form	100 (r)	becomes	011 (e'),	Form	$2\bar{1}0$	becomes	$21\bar{1}$,
,,	011 (e)	,,	211 (n'),	,,	$31\bar{1}$ (z)	,,	210 (x'),
,,	211 (n)	,,	233	,,	321 (i)	,,	345,
,,	$\bar{1}11$ (s)	,,	100 (r'),	,,	$5\bar{1}\bar{1}$ (s_1)	,,	$\bar{1}22$ (r_1'),
,,	$\bar{1}22$ (r_1)	,,	411 (e_1'),	,,	411 (e_1)	,,	255 (n_1'),
,,	210 (x)	,,	123 (i'),	,,	$3\bar{1}\bar{1}$	,,	$\bar{1}11$ (s').

Forms 111, $01\bar{1}$, $2\bar{1}\bar{1}$, $32\bar{1}$, retain their symbols.

Also for the new angular element of the crystal, we have $\tan D' = 2 \tan D$.

Hence for Calcite $D' = 63° 7'·3$, and the relative condensation in the planes of faces of the most common forms are given in the following table:

Symbol referred to old axes	Symbol referred to new axes	Condensation.
100	011	1·00000
$10\bar{1}$	$10\bar{1}$	·82211
$11\bar{1}$	100	·63505
011	211	·62991
$2\bar{1}\bar{1}$	$2\bar{1}\bar{1}$	·47465
111	111	·46823
210	321	·40687
$3\bar{1}\bar{1}$	$11\bar{1}$	·34508
122	411	·33333
511	$\bar{1}22$	·21168

Calculating the ratios of the axes of the ellipsoids representing the molecular volumes we find them as 1 : ·76159 : ·57216.

A similar change of axes will be needed in the case of other crystals which have a perfect rhombohedral cleavage. Most of these are isomorphous, or nearly so, with calcite, and it may be assumed that the anhydrous carbonates of rhombohedral form are all similarly constituted. Nitratine follows them. Pyrargyrite and Proustite both have rhombohedral cleavage, and if we assume the symbol of the cleavage face to be 011 we find the angular element for the former 61° 12′·6 and for the latter 61° 40′·5. The characters of Chabasie, which has a tolerably perfect rhombohedral cleavage, are satisfied by a similar supposition.

Phenakite has a not very distinct rhombohedral cleavage, and also a similar cleavage parallel to the six-sided prism $10\bar{1}$. If we assume the symbol of the cleavage rhombohedron to be 011, as before, the angular element will be 56° 44′, and the condensations in planes parallel to the two faces named will have the ratio ·9525, or nearly one of equality, which agrees with the facts of the case.

Dioptase has a perfect rhombohedral cleavage parallel to the face 011, but as the angular element is 50° 39′ the ratio of the condensations in planes parallel to the faces of $10\bar{1}$ and 011 respectively is 1·109, and we should have expected a cleavage parallel to the faces of $10\bar{1}$ as well as of 011. No such cleavage has been observed, though the form $10\bar{1}$ is almost always developed. The faces of that form are however striated in such a way as to lead to the supposition of some sort of alternations having occurred in the growth of the crystals, which may possibly interfere with the cleavages parallel to those faces.

Millerite has perfect rhombohedral cleavages parallel to the faces 011 and 100. The angular element is however only 20° 51′, which should give the condensation in planes parallel to the faces of the form 011 much greater than in either of the cleavage forms. This form is that which is chiefly developed and the crystals are usually capillary so that it would be hardly possible to observe whether they had a cleavage parallel to the faces of $01\bar{1}$.

The cleavages in tourmaline are imperfect parallel to faces of the forms 100 and 111. If we change the axes as before the symbols for these faces become 011 and 100, and the angular element 45° 57′, which makes the condensation greatest in planes parallel to the form $01\bar{1}$. If however we take the form to which Miller assigns the symbol $\bar{1}11$ to be 011, the form 100 becomes 211 and the angular element 76° 24′. The concentration in the faces of the most common forms then become

	Symbol referred to old axes	Symbol referred to new axes	Concentration
(*s*)	$\bar{1}11$	011	1·00000
(*r*)	100	211	·79839
(*o*)	111	111	·76553
(*a*)	$10\bar{1}$	$10\bar{1}$	·64134
(*y*)	$3\bar{1}\bar{1}$	100	·53984
(*b*)	$2\bar{1}\bar{1}$	$2\bar{1}\bar{1}$	·37028

These figures agree sufficiently with the observed facts. The tendency to the development of the form (*b*) $2\bar{1}\bar{1}$, for which the concentration is much less than for some other forms, seems to be connected with the stress producing hemihedrism (as explained in Part I.), since the form (*b*) $2\bar{1}\bar{1}$ is almost always hemihedral.

Of laboratory crystals not many of rhombohedral character require special mention.

In magnesium sulphite the forms observed are *r*, *e*, *a*, *o*, and the angular element is 50° 29′.

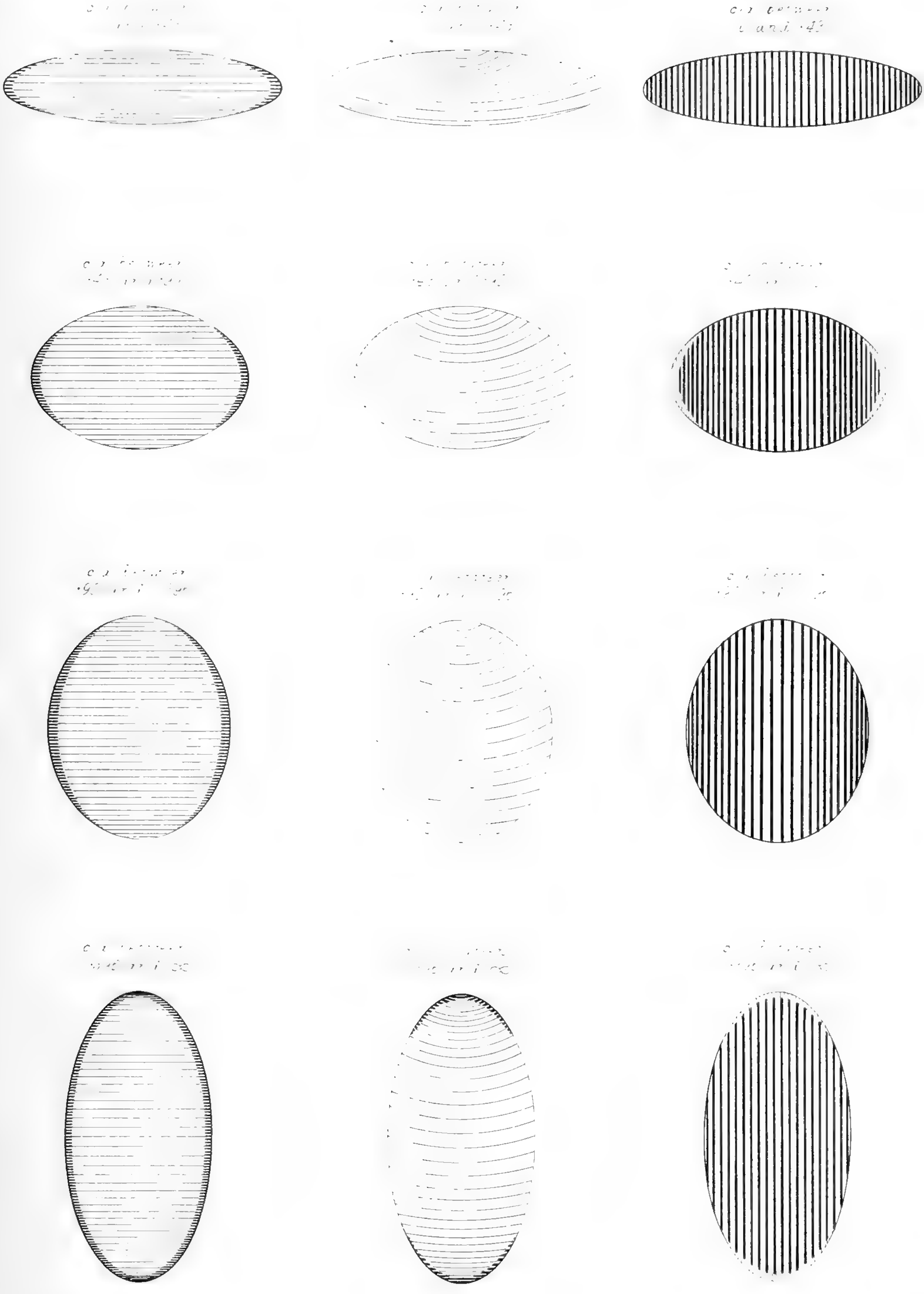
The double ferro-cyanide of barium and potassium has forms *r*, *o* and angular element 61° 7′, and cleavage parallel to the faces of *r*. If we take the cleavage form to be *e* or 011, the angular element becomes 74° 35′, and theory will agree with the facts.

Aldehyd-ammonia has *r*, *e*, *a*, *o*, with cleavage *r*, and angular element 58° 10′. If we take the cleavage form to be *e* or 011, *r* becomes *n* or 211, and the angular element 72° 45′, which agrees well with theory, since with that angular element the condensation is greatest in the faces of *e*, next in *n*, *a*, *o*, in order.

In crystals of sodium chloride with grape sugar and two molecules of water, the faces of *a*, rr_1, *e* and more rarely *b*, *o* have been observed, and the angular element is 63° 15′. This agrees with theory, but the forms rr_1 might be taken as ee_1, when the other forms observed would be *a*, *n*, *b*, *o* and the angular element 75° 51′.

Some may think, in the light of Reusch's experiment in producing the rotation of the plane of polarized light by a pile of plates of mica successively twisted through 60°, to which the twinning of quartz and other hexagonal crystals bears a close resemblance, that such twinning would account for the effect of quartz on plane polarized light. This cause is however, as it seems to me, inadequate. The rotation can hardly be accounted for by any static arrangement of molecules. It is a phenomenon more nearly related to the rotation of the apsides of a planetary orbit, and seems to imply a stress. This view is borne out by the fact that it is produced by some liquids, and that these liquids appear, so far as it is possible to judge of such a fact in a biaxal crystal, to lose their rotatory power when crystallized in asymmetric hemihedral forms; while the asymmetric crystals which have the power of rotation lose that power when liquified. The stress reacts, as it should do, on the external form, because the tendency must always be for the molecules, so far as they are free, to arrange themselves in such a way as to counteract the stress.

On the whole the molecular arrangement for which the principles of mechanics give adequate reason accounts remarkably well for the main features of hexagonal and rhombohedral crystallization. I say the main features, because surface-tension, though the primary and principal cause of crystalline form, is not the only cause which affects the growth of crystals. The other causes mentioned in Part I. have a secondary influence, and produce in some cases disturbing effects, but they are only disturbing not overpowering.



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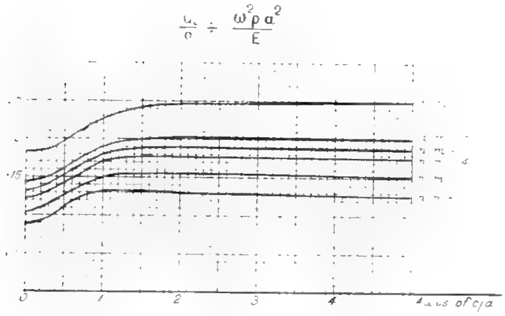

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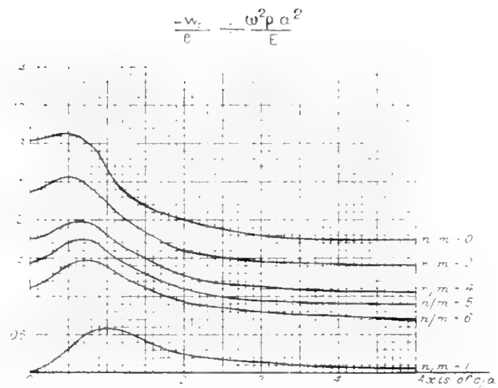

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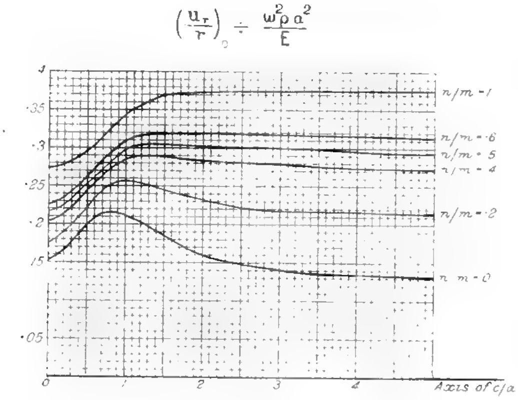

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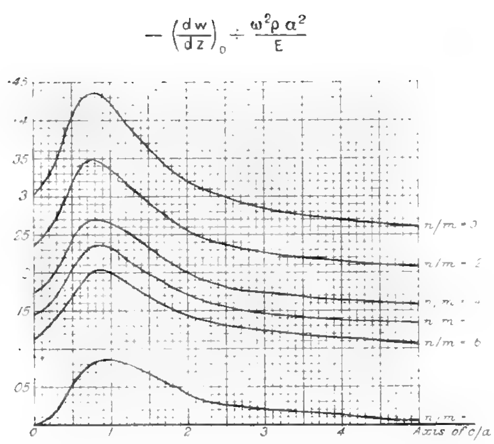

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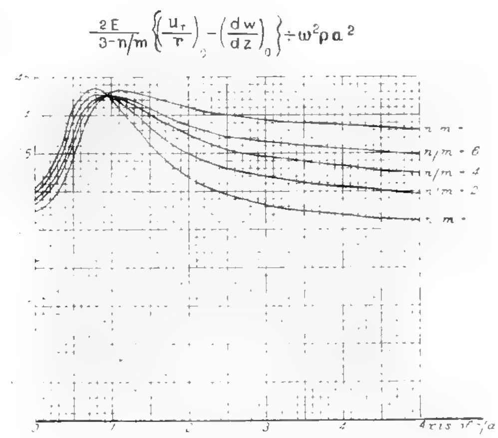

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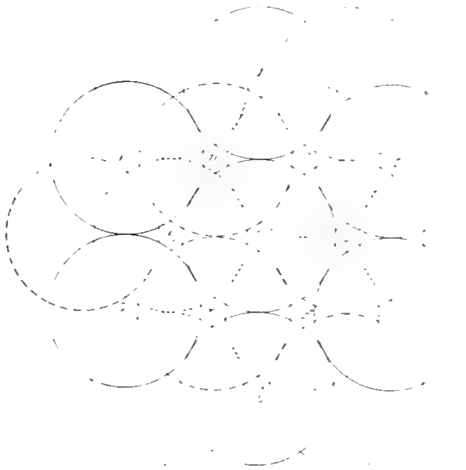

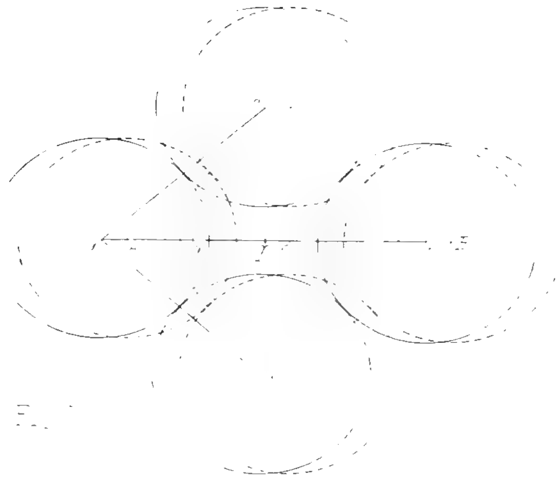

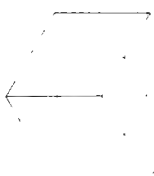

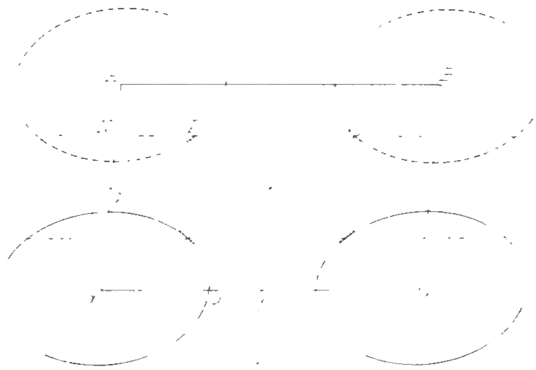

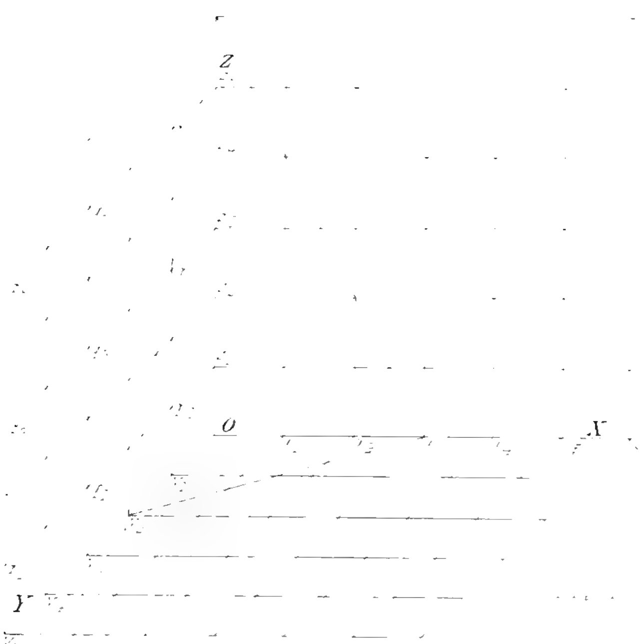

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VI. *On some Compound Vibrating Systems.* By C. Chree, M.A., Fellow of King's College.

By C. Chree, M.A., Fellow of King's College.

INDEX.

SECTION I.

GENERAL PRINCIPLES.

§ 1. IN the most general type of vibrations of an isotropic elastic solid there have to be considered three displacement and three stress components at every point of a surface along three mutually orthogonal directions. In the general case at a common surface of two media there are six necessary conditions, arising from the equality of the displacements and stresses at adjacent points on opposite sides of the surface. In the types of vibration discussed in the present memoir the surfaces limiting the several media are either concentric spheres or coaxial right cylinders, and the displacements are either entirely radial or entirely transversal. In all the cases considered the number of independent conditions to be satisfied at the common surface of two media reduces to two, one arising from the equality of the stresses, the other from the equality of the displacements at adjacent points on opposite sides of the surface. If a surface where no stress exists be termed *free*, and one where the displacement is zero be termed *fixed*, then in the types of vibration treated here, there is at a free surface a single condition expressing the vanishing of the stress, and at a fixed surface a single condition expressing the vanishing of the displacement. The centre of a solid sphere and the axis of a solid cylinder may be regarded as fixed surfaces.

§ 2. For the sake of brevity we shall frequently have occasion to apply the term *layer* to a portion of homogeneous isotropic material limited by two concentric spherical or two coaxial cylindrical surfaces. When one such layer exists alone it will be termed a *simple shell*, while a series of layers one above another will be termed a *compound shell*, provided there be no material at the centre of the sphere or at the axis of the cylinder. When the material extends to the centre of the sphere or the axis of the cylinder, the system will be termed *compound* when more than one medium exists. The inmost material, whose outer surface is of course spherical or cylindrical, will be spoken of as the *core*.

The principal object of this memoir is to determine how the pitch of the several notes of a simple shell or core would be altered by the existence in it of a thin layer differing from the rest of the material. Now the elasticity of a layer can doubtless be altered without altering its volume, but of course the density cannot. For the sake of brevity, however, the term *altered layer* will be applied here whatever be the difference between the structure of the layer and that of the rest of the material. The term merely indicates the existence of a certain definite want of homogeneity, and does not imply that the vibrating system ever was homogeneous. By *the change of pitch due to an altered layer* is meant the difference between the pitches of corresponding notes in two vibrating systems, the only difference between which is the existence in one of them of a layer differing in an assigned way from the rest of the material.

§ 3. A vibrating system is in general capable of producing a large—theoretically an infinite—number of different notes, answering to each of which there appears a separate term in the expression for the displacement. The expression for the *representative* displacement at any point in a layer may be regarded as a product of two factors. One of these is $\cos kt$, where $k/2\pi$ is the frequency of the representative note and t the time. This factor is the same for every point in all the media of a compound system. The other factor is the sum of two functions each multiplied by an arbitrary constant. These functions have for their variable the radial or axial distance r, and contain, in addition to k, the density and one or both of the elastic constants of the medium; they thus vary from layer to layer. In a core one of the above two functions of r must be omitted, as it would become infinite when r vanishes.

In the case of the transverse vibrations of a sphere there exists in general a third factor in the representative displacement. It is, however, a function solely of the angular coordinates. It does not in fact enter into the surface conditions and may for our present purpose be left out of account.

The following remarks apply equally to the radial and to the purely transversal vibrations of spherical and cylindrical systems.

If a compound shell consist of n layers the expressions for the representative displacement contain $2n$ arbitrary constants. At each of the $n-1$ surfaces separating the layers there are two surface conditions, and at each of the bounding surfaces of the shell—whether fixed or free—there is 1 surface condition. There are thus $2n$ equations, of which $2n-1$ suffice to determine the ratios of the $2n$ arbitrary constants. Thus we are left with a single equation from which all the arbitrary constants have been eliminated, and this supplies the frequencies of the vibrations of the given type which can occur in the compound system.

If there be a core and $n-1$ layers there are $2n-1$ arbitrary constants and $2n-1$ equations connecting them, so that the result is exactly the same. In general it will be unnecessary to consider separately the case when a core exists.

§ 4. At the common surface, $r=a_s$, of two media the two surface conditions may be put in the form

$$A_{s-1}F(a_s.\gamma_{s-1})+B_{s-1}F_1(a_s.\gamma_{s-1})=A_sF(a_s.\gamma_s)+B_sF_1(a_s.\gamma_s) \quad \text{..............(1)},$$

$$A_{s-1}G(a_s.\gamma_{s-1})+B_{s-1}G_1(a_s.\gamma_{s-1})=A_sG(a_s.\gamma_s)+B_sG_1(a_s.\gamma_s) \quad \text{..............(2)},$$

the first representing the equality of stress, the second of displacement on the two sides of the surface. Here the A's and B's are arbitrary constants whose absolute magnitudes depend on the amplitude of the vibration. The F's and G's represent certain functions of a_s, of the density and of the elastic properties of the media. For brevity the letter γ is employed to represent all the material properties of the medium, i.e. its density and elastic constants m and n combined. $F(a_s.\gamma_{s-1})$ is of course the same function of ρ_{s-1}, m_{s-1} and n_{s-1} that $F(a_s.\gamma_s)$ is of ρ_s, m_s and n_s.

The right-hand side of (1) is *proportional* to the stress and the right-hand side of (2) to the displacement at the surface $r=a_s$ in the medium γ_s. It must, however, be clearly understood that the expressions in (1) and (2), multiplied by $\cos kt$, need not be the *exact* stresses and displacements themselves.

If $r=a_s$ were the outer bounding surface of a compound shell then the surface condition would be got by equating to zero the left-hand side of (1) or the left-hand side of (2), according as the surface was free or fixed. Similarly, if $r=a_s$ were the inner bounding surface, we should equate to zero the right-hand side of (1) or the right-hand side of (2) according to circumstances.

In a shell, whether simple or compound, there are four fundamental types of vibration,

the *free-free*, the *fixed-free*, the *free-fixed*, the *fixed-fixed*,

where the first term applies to the inner surface.

In what follows it is necessary to adopt some one notation free from ambiguity. Thus a compound shell of, say, three layers, the inmost of material (ρ_1, m_1, n_1)—represented by γ_1—bounded by the surfaces $r=e$ and $r=c$, the middle of material (ρ_2, m_2, n_2), and the outmost of material (ρ_3, m_3, n_3) bounded by $r=b$ and $r=a$, will be spoken of as the shell $(e.\gamma_1.c.\gamma_2.b.\gamma_3.a)$.

The letter f will be invariably employed for the function which equated to zero gives the frequency equation, and inside the accompanying bracket will be given the letters necessary to define the system. If a bounding surface be fixed, then the radius of that surface will appear in the bracket with a horizontal line over it. Thus, for instance,

$$f(e\,.\,\gamma_1\,.\,c\,.\,\gamma_2\,.\,b\,.\,\gamma_3\,.\,\bar{a}) = 0$$

represents the frequency equation of the three-layer compound shell specified above, the inner bounding surface, $r = e$, being free, the outer, $r = a$, being fixed.

From the remarks made on the forms assumed by (1) and (2) at a bounding surface, we find at once for the frequency equations of the four fundamental types in the simple shell $(b\,.\,\gamma\,.\,a)$ the following—

$$f(b\,.\,\gamma\,.\,a) = F(a\,.\,\gamma)\,F_1(b\,.\,\gamma) - F_1(a\,.\,\gamma)\,F(b\,.\,\gamma) = 0 \quad\text{......(3)},$$

$$f(\bar{b}\,.\,\gamma\,.\,a) = F(a\,.\,\gamma)\,G_1(b\,.\,\gamma) - F_1(a\,.\,\gamma)\,G(b\,.\,\gamma) = 0 \quad\text{......(4)},$$

$$f(b\,.\,\gamma\,.\,\bar{a}) = G(a\,.\,\gamma)\,F_1(b\,.\,\gamma) - G_1(a\,.\,\gamma)\,F(b\,.\,\gamma) = 0 \quad\text{......(5)},$$

$$f(\bar{b}\,.\,\gamma\,.\,\bar{a}) = G(a\,.\,\gamma)\,G_1(b\,.\,\gamma) - G_1(a\,.\,\gamma)\,G(b\,.\,\gamma) = 0 \quad\text{......(6)}.$$

The terms in these functions will always be supposed to present themselves in the same order as above.

§ 5. Suppose now we proceed to find the frequency equations for the two-layer shell $(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3)$. For the free-free vibrations we have to eliminate the arbitrary constants from

$$A_1F(a_1\,.\,\gamma_1) + B_1F_1(a_1\,.\,\gamma_1) = 0 \quad\text{......(7)},$$

$$A_1F(a_2\,.\,\gamma_1) + B_1F_1(a_2\,.\,\gamma_1) = A_2F(a_2\,.\,\gamma_2) + B_2F_1(a_2\,.\,\gamma_2) \quad\text{......(8)},$$

$$A_1G(a_2\,.\,\gamma_1) + B_1G_1(a_2\,.\,\gamma_1) = A_2G(a_2\,.\,\gamma_2) + B_2G_1(a_2\,.\,\gamma_2) \quad\text{......(9)},$$

$$0 = A_2F(a_3\,.\,\gamma_2) + B_2F_1(a_3\,.\,\gamma_2) \quad\text{......(10)}.$$

The result of elimination is easily found to be

$$f(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3) = \{F(a_2\,.\,\gamma_1)F_1(a_1\,.\,\gamma_1) - F_1(a_2\,.\,\gamma_1)F(a_1\,.\,\gamma_1)\}\{F(a_3\,.\,\gamma_2)G_1(a_2\,.\,\gamma_2) - F_1(a_3\,.\,\gamma_2)G(a_2\,.\,\gamma_2)\}$$
$$- \{G(a_2\,.\,\gamma_1)F_1(a_1\,.\,\gamma_1) - G_1(a_2\,.\,\gamma_1)F(a_1\,.\,\gamma_1)\}\{F(a_3\,.\,\gamma_2)F_1(a_2\,.\,\gamma_2) - F_1(a_3\,.\,\gamma_2)F(a_2\,.\,\gamma_2)\} = 0.$$

Comparing this with equations (3)—(5) we obviously have

$$f(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3) = f(a_1\,.\,\gamma_1\,.\,a_2)\,f(\bar{a}_2\,.\,\gamma_2\,.\,a_3) - f(a_1\,.\,\gamma_1\,.\,\bar{a}_2)\,f(a_2\,.\,\gamma_2\,.\,a_3) \quad\text{......(11)};$$

similarly we may easily prove

$$f(\bar{a}_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3) = f(a_1\,.\,\gamma_1\,.\,a_2)\,f(\bar{a}_2\,.\,\gamma_2\,.\,a_3) - f(\bar{a}_1\,.\,\gamma_1\,.\,\bar{a}_2)\,f(a_2\,.\,\gamma_2\,.\,a_3) \quad\text{......(12)},$$

$$f(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,\bar{a}_3) = f(a_1\,.\,\gamma_1\,.\,a_2)\,f(\bar{a}_2\,.\,\gamma_2\,.\,\bar{a}_3) - f(a_1\,.\,\gamma_1\,.\,\bar{a}_2)\,f(a_2\,.\,\gamma_2\,.\,a_3) \quad\text{......(13)},$$

$$f(\bar{a}_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,\bar{a}_3) = f(\bar{a}_1\,.\,\gamma_1\,.\,a_2)\,f(\bar{a}_2\,.\,\gamma_2\,.\,\bar{a}_3) - f(\bar{a}_1\,.\,\gamma_1\,.\,\bar{a}_2)\,f(a_2\,.\,\gamma_2\,.\,\bar{a}_3) \quad\text{......(14)}.$$

In each of these identities there is a very obvious physical meaning. For instance, we see from (11) that $f(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3) = 0$ will be satisfied by any value of k which satisfies simultaneously either

$$f(a_1\,.\,\gamma_1\,.\,a_2) = 0, \text{ and } f(a_2\,.\,\gamma_2\,.\,a_3) = 0,$$

or

$$f(a_1\,.\,\gamma_1\,.\,\bar{a}_2) = 0, \text{ and } f(\bar{a}_2\,.\,\gamma_2\,.\,a_3) = 0.$$

This merely signifies that if there be a common frequency of vibration for the two layers existing separately with their common surface either a free or a fixed surface, then this too is the frequency of a vibration which the compound shell can execute.

At first sight it might appear that in (11) we had also the two alternatives

$$f(a_1 . \gamma_1 . a_2) = 0 = f(a_1 . \gamma_1 . a_2),$$
$$f(a_2 . \gamma_2 . a_3) = 0 = f(a_2 . \gamma_2 . a_3).$$

Neither of these alternatives is, however, possible in any case, as might easily be foreseen from the physical meaning of the functions.

§ 6. The relations (11)—(14) are particular cases of a general law which will now be proved.

It will be sufficient to limit our proof to the cases when both surfaces of the compound shell are free or when the outer only is fixed. The method of proof in any other case is practically identical.

Let us assume that for a compound shell $(a_1 . \gamma_1 . a_2 \ldots a_n . \gamma_n . a_{n+1})$ of n layers the frequency equations take the forms

$$f(a_1 . \gamma_1 . a_2 \ldots a_n . \gamma_n . a_{n+1}) = f(a_1 \ldots a_n) f(\bar{a}_n . \gamma_n . a_{n+1}) - f(a_1 \ldots \bar{a}_n) f(a_n . \gamma_n . a_{n+1}) = 0 \ldots (15),$$
$$f(a_1 . \gamma_1 . a_2 \ldots a_n . \gamma_n . \bar{a}_{n+1}) = f(a_1 \ldots a_n) f(\bar{a}_n . \gamma_n . a_{n+1}) - f(a_1 \ldots \bar{a}_n) f(a_n . \gamma_n . \bar{a}_{n+1}) = 0 \ldots (16),$$

where $f(a_1 \ldots a_n) = 0$, and $f(a_1 \ldots \bar{a}_n) = 0$ are the frequency equations in the compound shell $(a_1 \ldots a_n)$ of $n-1$ layers.

Now the difference between the frequency equations

$$f(a_1 \ldots a_{n+1}) = 0, \text{ and } f(a_1 \ldots a_{n+1} . \gamma_{n+1} . a_{n+2}) = 0,$$

is that whereas two arbitrary constants A_n, B_n have in the first case their ratio determined by the single equation

$$A_n F(a_{n+1} . \gamma_n) + B_n F_1(a_{n+1} . \gamma_n) = 0,$$

this ratio is in the second case determined by means of the three equations

$$A_n F(a_{n+1} . \gamma_n) + B_n F_1(a_{n+1} . \gamma_n) = A_{n+1} F(a_{n+1} . \gamma_{n+1}) + B_{n+1} F_1(a_{n+1} . \gamma_{n+1}),$$
$$A_n G(a_{n+1} . \gamma_n) + B_n G_1(a_{n+1} . \gamma_n) = A_{n+1} G(a_{n+1} . \gamma_{n+1}) + B_{n+1} G_1(a_{n+1} . \gamma_{n+1}),$$
$$0 = A_{n+1} F(a_{n+2} . \gamma_{n+1}) + B_{n+1} F_1(a_{n+2} . \gamma_{n+1}).$$

Eliminating A_{n+1} and B_{n+1} from these three equations we find

$$\frac{A_n}{-B_n} = \frac{-f(a_{n+1} . \gamma_{n+1} . a_{n+2}) G_1(a_{n+1} . \gamma_n) + f(\bar{a}_{n+1} . \gamma_{n+1} . a_{n+2}) F_1(a_{n+1} . \gamma_n)}{-f(a_{n+1} . \gamma_{n+1} . a_{n+2}) G(a_{n+1} . \gamma_n) + f(\bar{a}_{n+1} . \gamma_{n+1} . a_{n+2}) F(a_{n+1} . \gamma_n)} \ldots\ldots (17).$$

Thus we obtain $f(a_1 \ldots a_{n+2})$ by replacing in (15) the ratio $F_1(a_{n+1} . \gamma_n) : F(a_{n+1} . \gamma_n)$ by the ratio given by (17) for $A_n : -B_n$.

The only factors in (15) in which $F_1(a_{n+1}.\gamma_n)$ and $F(a_{n+1}.\gamma_n)$ occur are

$$f(\bar{a}_n.\gamma_n.a_{n+1}) \equiv F(a_{n+1}.\gamma_n)\,G_1(a_n.\gamma_n) - F_1(a_{n+1}.\gamma_n)\,G(a_n.\gamma_n),$$
$$f(a_n.\gamma_n.a_{n+1}) \equiv F(a_{n+1}.\gamma_n)\,F_1(a_n.\gamma_n) - F_1(a_{n+1}.\gamma_n)\,F(a_n.\gamma_n).$$

These factors are thus to be replaced, the first by

$$-f(a_{n+1}.\gamma_{n+1}.a_{n+2})\{G(a_{n+1}.\gamma_n)\,G_1(a_n.\gamma_n) - G_1(a_{n+1}.\gamma_n)\,G(a_n.\gamma_n)\}$$
$$+f(\bar{a}_{n+1}.\gamma_{n+1}.a_{n+2})\{F(a_{n+1}.\gamma_n)\,G_1(a_n.\gamma_n) - F_1(a_{n+1}.\gamma_n)\,G(a_n.\gamma_n)\},$$

the second by

$$-f(a_{n+1}.\gamma_{n+1}.a_{n+2})\{G(a_{n+1}.\gamma_n)\,F_1(a_n.\gamma_n) - G_1(a_{n+1}.\gamma_n)\,F(a_n.\gamma_n)\}$$
$$+f(\bar{a}_{n+1}.\gamma_{n+1}.a_{n+2})\{F(a_{n+1}.\gamma_n)\,F_1(a_n.\gamma_n) - F_1(a_{n+1}.\gamma_n)\,F(a_n.\gamma_n)\}.$$

In other words, we obtain $f(a_1...a_{n+2})$ from (15) by substituting

$$-f(a_{n+1}.\gamma_{n+1}.a_{n+2})f(\bar{a}_n.\gamma_n.\bar{a}_{n+1}) + f(\bar{a}_{n+1}.\gamma_{n+1}.a_{n+2})f(\bar{a}_n.\gamma_n.a_{n+1})$$

for $f(\bar{a}_n.\gamma_n.a_{n+1})$, and

$$-f(a_{n+1}.\gamma_{n+1}.a_{n+2})f(a_n.\gamma_n.\bar{a}_{n+1}) + f(\bar{a}_{n+1}.\gamma_{n+1}.a_{n+2})f(a_n.\gamma_n.a_{n+1})$$

for $f(a_n.\gamma_n.a_{n+1})$.

Thus we find

$$f(a_1...a_{n+2}) =$$
$$-\{f(a_1...a_n)f(\bar{a}_n.\gamma_n.\bar{a}_{n+1}) - f(a_1...\bar{a}_n)f(a_n.\gamma_n.\bar{a}_{n+1})\}f(a_{n+1}.\gamma_{n+1}.a_{n+2})$$
$$+\{f(a_1...a_n)f(\bar{a}_n.\gamma_n.a_{n+1}) - f(a_1...\bar{a}_n)f(a_n.\gamma_n.a_{n+1})\}f(\bar{a}_{n+1}.\gamma_{n+1}.a_{n+2}) = 0 \quad ...(18).$$

Hence we find from the assumptions (15) and (16)

$$f(a_1...a_{n+1}.\gamma_{n+1}.a_{n+2}) = f(a_1...a_{n+1})f(\bar{a}_{n+1}.\gamma_{n+1}.a_{n+2}) - f(a_1...\bar{a}_{n+1})f(a_{n+1}.\gamma_{n+1}.a_{n+2}) = 0 ...(19).$$

Similarly we may prove that if (15) and (16) be the proper forms for the frequency equations of an n-layer shell, then

$$f(a_1...a_{n+1}.\gamma_{n+1}.\bar{a}_{n+2}) = f(a_1...a_{n+1})f(\bar{a}_{n+1}.\gamma_{n+1}.\bar{a}_{n+2}) - f(a_1...\bar{a}_{n+1})f(a_{n+1}.\gamma_{n+1}.\bar{a}_{n+2}) = 0 ...(20).$$

Thus if (15) and (16) be correct types of the frequency equations for the free-free and free-fixed vibrations of a compound shell of n layers they are likewise correct types for a compound shell of $n+1$ layers. But they agree with the forms (11) and (13) which we obtained for a shell of two layers, and so their universal truth is established.

We can easily establish in like manner the formulae

$$f(\bar{a}_1.\gamma_1.a_2...a_n.\gamma_n.a_{n+1}) = f(\bar{a}_1...a_n)\,f(\bar{a}_n.\gamma_n.a_{n+1}) - f(\bar{a}_1...\bar{a}_n)f(a_n.\gamma_n.a_{n+1}) = 0...(21),$$
$$f(\bar{a}_1.\gamma_1.a_2...a_n.\gamma_n.\bar{a}_{n+1}) = f(\bar{a}_1...a_n)f(\bar{a}_n.\gamma_n.\bar{a}_{n+1}) - f(\bar{a}_1...\bar{a}_n)f(a_n.\gamma_n.\bar{a}_{n+1}) = 0...(22).$$

§ 7. We can obviously by means of these results obtain very simply the frequency equations of any compound shell in terms of the functions which when equated to zero are the frequency equations of the individual layers. Thus in the case of (15) our next step would be to express $f(a_1\ldots a_n)$ and $f(a_1\ldots\bar{a}_n)$ in terms of $f(a_1\ldots a_{n-1})$, $f(a_1\ldots\bar{a}_{n-1})$, $f(a_{n-1}\,.\,\gamma_{n-1}\,.\,a_n)$, $f(\bar{a}_{n-1}\,.\,\gamma_{n-1}\,.\,a_n)$, $f(a_{n-1}\,.\,\gamma_{n-1}\,.\,a_n)$ and $f(a_{n-1}\,.\,\gamma_{n-1}\,.\,\bar{a}_n)$, and so on.

The final form so obtained for the function which when equated to zero constitutes the frequency equation of a compound shell of n layers is a series of terms each composed of n factors. Each of these factors when equated to zero constitutes a frequency equation of one of the four fundamental types for one of the layers of which the shell is composed, and each layer contributes one factor to each term.

For instance, the frequency equation for the free-free vibrations of the three-layer shell $(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3\,.\,\gamma_3\,.\,a_4)$ is

$$\begin{aligned} f(a_1\,.\,\gamma_1\,.\,a_2\,.\,\gamma_2\,.\,a_3\,.\,\gamma_3\,.\,a_4) &= f(a_1\,.\,\gamma_1\,.\,a_2)f(\bar{a}_2\,.\,\gamma_2\,.\,a_3)f(\bar{a}_3\,.\,\gamma_3\,.\,a_4) \\ &- f(a_1\,.\,\gamma_1\,.\,\bar{a}_2)f(a_2\,.\,\gamma_2\,.\,a_3)f(\bar{a}_3\,.\,\gamma_3\,.\,a_4) + f(a_1\,.\,\gamma_1\,.\,\bar{a}_2)f(a_2\,.\,\gamma_2\,.\,\bar{a}_3)f(a_3\,.\,\gamma_3\,.\,a_4) \\ &- f(a_1\,.\,\gamma_1\,.\,a_2)f(\bar{a}_2\,.\,\gamma_2\,.\,\bar{a}_3)f(a_3\,.\,\gamma_3\,.\,a_4) = 0 \qquad (23). \end{aligned}$$

§ 8. There is a considerable resemblance between the functions we are here dealing with and the sines and cosines of multiple angles. An illustration of this, which is also of importance in itself, is the following:

Instead of converting (18) into (19) we can write it as

$$\begin{aligned} f(a_1\ldots a_{n+2}) &= f(a_1\ldots a_n)\{f(\bar{a}_n\,.\,\gamma_n\,.\,a_{n+1})f(\bar{a}_{n+1}\,.\,\gamma_{n+1}\,.\,a_{n+2}) - f(\bar{a}_n\,.\,\gamma_n\,.\,\bar{a}_{n+1})f(a_{n+1}\,.\,\gamma_{n+1}\,.\,a_{n+2})\} \\ &- f(a_1\ldots\bar{a}_n)\{f(a_n\,.\,\gamma_n\,.\,a_{n+1})f(\bar{a}_{n+1}\,.\,\gamma_{n+1}\,.\,a_{n+2}) - f(a_n\,.\,\gamma_n\,.\,\bar{a}_{n+1})f(a_{n+1}\,.\,\gamma_{n+1}\,.\,a_{n+2})\} = 0, \end{aligned}$$

or

$$f(a_1\ldots a_{n+2}) = f(a_1\ldots a_n)f(\bar{a}_n\,.\,\gamma_n\,.\,a_{n+1}\,.\,\gamma_{n+1}\,.\,a_{n+2}) - f(a_1\ldots\bar{a}_n)f(a_n\,.\,\gamma_n\,.\,a_{n+1}\,.\,\gamma_{n+1}\,.\,a_{n+2}) = 0 \ldots (24),$$

by (11) and (12).

This can easily be extended so as to lead to the result

$$f(a_1\ldots a_n) = f(a_1\ldots a_s)f(\bar{a}_s\ldots a_n) - f(a_1\ldots\bar{a}_s)f(a_s\ldots a_n) = 0 \qquad (25),$$

where a_s is the boundary surface separating any two of the n layers.

The corresponding results for the other three types of vibration are

$$f(a_1\ldots\bar{a}_n) = f(a_1\ldots a_s)f(\bar{a}_s\ldots\bar{a}_n) - f(a_1\ldots\bar{a}_s)f(a_s\ldots\bar{a}_n) = 0 \qquad (26),$$

$$f(\bar{a}_1\ldots a_n) = f(\bar{a}_1\ldots a_s)f(\bar{a}_s\ldots a_n) - f(\bar{a}_1\ldots\bar{a}_s)f(a_s\ldots a_n) = 0 \qquad (27),$$

$$f(\bar{a}_1\ldots\bar{a}_n) = f(\bar{a}_1\ldots a_s)f(\bar{a}_s\ldots\bar{a}_n) - f(\bar{a}_1\ldots\bar{a}_s)f(a_s\ldots\bar{a}_n) = 0 \qquad (28).$$

§ 9. As the results we have obtained for the frequency equations arise from the elimination of arbitrary constants, different methods of elimination may lead to results which can be reduced to our standard forms only through multiplication by some factor, which ought of course to be incapable of vanishing. The existence of factors which can not vanish, and therefore supply no additional roots to the frequency equation, is obviously of no importance.

As this point is a little obscure without an example, let us consider the following case. Let us suppose c to be any length intermediate between a and b. We can regard the shell $(b.\gamma.a)$ as composed of two layers of the same material whose common surface is of radius c. Thus

$$f(b.\gamma.c.\gamma.a) = f(b.\gamma.c)f(\bar{c}.\gamma.a) - f(b.\gamma.\bar{c})f(c.\gamma.a) = 0$$

ought to supply all the roots of $f(b.\gamma.a) = 0$ and no additional roots, but the two functions $f(b.\gamma.c.\gamma.a)$ and $f(b.\gamma.a)$ are not identical.

It is in fact easily proved that

$$f(b.\gamma.c.\gamma.a) = \{F(c.\gamma)G_1(c.\gamma) - F_1(c.\gamma)G(c.\gamma)\}f(b.\gamma.a) \quad \text{.............(29).}$$

Now referring to (4) we see that

$$F(c.\gamma)G_1(c.\gamma) - F_1(c.\gamma)G(c.\gamma) = 0$$

would be the frequency equation for the vibrations of an infinitely thin simple shell of radius c, one of whose surfaces is fixed. But it is subsequently proved in the case of all the forms of vibration treated here that the free-free is the only possible form of vibration in a very thin shell. Thus $f(b.\gamma.c.\gamma.a)$ is the product of $f(b.\gamma.a)$ and a factor which cannot vanish.

The result (29) can easily be extended so as to lead to

$$f(a_1.\gamma.a_2.\gamma...\gamma.a_s.\gamma.a_{s+1}...a_n) = \{F(a_2.\gamma)G_1(a_2.\gamma) - F_1(a_2.\gamma)G(a_2.\gamma)\} \times ...$$
$$\times \{F(a_s.\gamma)G_1(a_s.\gamma) - F_1(a_s.\gamma)G(a_s.\gamma)\} \times ... \times f(a_1.\gamma.a_n) \quad \text{......(30),}$$

where the number of factors such as $F(a_s.\gamma)G_1(a_s.\gamma) - F_1(a_s.\gamma)G(a_s.\gamma)$ is equal to the number of intermediate surfaces whose radii are $a_2 ... a_s ...$. These same factors will also present themselves though one or both of the bounding surfaces $r = a_1$, and $r = a_n$ be fixed.

§ 10. There is another class of general results which regarded as independent facts seem very curious. They present themselves repeatedly, so their explanation at an early stage is advisable.

Suppose we have a simple shell $(b.\gamma.a + \partial a)$, where ∂a is so small that $(\partial a/a)^2$ is negligible. We may write the frequency equation for the free-free vibrations of this shell in the form

$$f(b.\gamma.a.\gamma.a+\partial a) = f(b.\gamma.a)f(\bar{a}.\gamma.a+\partial a) - f(b.\gamma.\bar{a})f(a.\gamma.a+\partial a) = 0,$$

or, since $f(\bar{a}.\gamma.a+\partial a)$ cannot vanish,

$$f(b.\gamma.a) - \frac{f(b.\gamma.\bar{a})}{f(\bar{a}.\gamma.a+\partial a)}f(a.\gamma.a+\partial a) = 0 \quad \text{.....................(31).}$$

This must be equivalent to $f(b.\gamma.a+\partial a) = 0$ and so, as $(\partial a/a)^2$ is negligible, to

$$f(b.\gamma.a) + \partial a\frac{d}{da}f(b.\gamma.a) = 0 \quad \text{.............................(32).}$$

Since the equations (31) and (32) are equivalent we must have

$$\frac{d}{da}f(b.\gamma.a) = -\left\{\frac{f(b.\gamma.\bar{a})}{\partial a.f(\bar{a}.\gamma.a+\partial a)}\right\}f(a.\gamma.a+\partial a) \quad \text{.................(33).}$$

But $f(b\,.\,\gamma\,.\,a) = 0$ is the frequency equation for the free-fixed vibrations of the simple shell $(b\,.\,\gamma\,.\,a)$, and $f(a\,.\,\gamma\,.\,a + \partial a) = 0$ is the frequency equation for the free-free vibrations of a very thin shell of radius a. Thus if we take the function $f(b\,.\,\gamma\,.\,a)$ which when equated to zero gives the frequency of free-free vibrations in a simple shell $(b\,.\,\gamma\,.\,a)$, and differentiate it with respect to the radius a of the outer surface, this differential coefficient equated to zero must supply us with the frequency of the free-fixed vibrations of the shell $(b\,.\,\gamma\,.\,a)$ and with the frequency of the free-free vibrations of an infinitely thin shell of radius a, when we modify it in a suitable way by introducing the facts that—as follows from (32)—$f(b\,.\,\gamma\,.\,a)$ differs from zero only by a term of the order $\partial a/a$ and that $(\partial a/a)^2$ is negligible.

Examples of this result will be found in § 14, Sect. II., § 50, Sect. III., § 64, Sect. IV., § 92, Sect. VI. etc.

A similar treatment of $\frac{d}{db} f(b\,.\,\gamma\,.\,a)$, when the result is equated to zero, leads to the equation

$$f(b\,.\,\gamma\,.\,a)\, f(b - \partial b\,.\,\gamma\,.\,b) = 0.$$

Such a result as this last, in which it is tacitly assumed that b does not vanish, cannot of course be applied to any case in which a core exists, but all the results such as (21) or (22) where no such assumption is latent apply immediately in the case of a core. The result (33) also applies to a core when b is replaced by 0.

§ 11. In so far as the results of the present section are mathematical they may doubtless be deduced from the properties of the determinant which would result from the elimination of the arbitrary constants in the surface conditions treated as simultaneous equations.

The methods of this section are probably the simplest for obtaining the change of pitch due to a thin altered layer in an otherwise homogeneous system, and their application to this object will be found in Sections VI. to IX. which deal with spherical and cylindrical shells. In Sections II. to V., however, a different procedure is adopted in dealing with solid spheres and cylinders in order to determine how the type of vibration changes.

SECTION II.

RADIAL VIBRATIONS IN SOLID SPHERE.

§ 12. In a simple spherical shell of material (ρ, m, n) vibrating radially the representative displacement may be taken as

$$u = \cos kt \left\{ \frac{A}{r} \left(\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r \right) + \frac{B}{r} \left(\frac{\cos k\alpha r}{k\alpha r} + \sin k\alpha r \right) \right\} \quad \ldots\ldots\ldots\ldots\ldots (1)^*,$$

where

$$\alpha = \sqrt{\rho/(m+n)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

* *Cambridge Philosophical Transactions*, Vol. XIV., equation (60), p. 320.

The corresponding radial stress is

$$U = \frac{1}{r^2}\cos kt\left[A\left\{(m+n)\,k\alpha r\sin k\alpha r - 4n\left(\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r\right)\right\}\right.$$

$$\left. + B\left\{(m+n)\,k\alpha r\cos k\alpha r - 4n\left(\frac{\cos k\alpha r}{k\alpha r} + \sin k\alpha r\right)\right\}\right] \quad\ldots\ldots\ldots\ldots(3).$$

Suppose now we have the compound solid sphere $(0\,.\,\alpha\,.\,c\,.\,\alpha_1\,.\,b\,.\,\alpha\,.\,a)$, where $b-c$ is so small its square is negligible. Here we denote $\sqrt{\rho_1/(m_1+n_1)}$ by α_1, supposing ρ_1, m_1, n_1 to be respectively the density and the elastic constants of the thin layer.

The presence of the thin layer will produce only a corresponding small change in the type of vibration throughout the rest of the sphere. We may thus assume for the type of vibration answering to a note of frequency $k/2\pi$,

in the core $$u/\cos kt = \frac{A}{r}\left(\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r\right) \quad\ldots\ldots\ldots\ldots(4);$$

in the layer $$u/\cos kt = \frac{A_1}{r}\left(\frac{\sin k\alpha_1 r}{k\alpha_1 r} - \cos k\alpha_1 r\right) + \frac{B_1}{r}\left(\frac{\cos k\alpha_1 r}{k\alpha_1 r} + \sin k\alpha_1 r\right) \quad\ldots\ldots\ldots\ldots(5);$$

in the material outside the layer

$$u/\cos kt = \frac{A+\partial A}{r}\left(\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r\right) + \frac{\partial B}{r}\left(\frac{\cos k\alpha r}{k\alpha r} + \sin k\alpha r\right) \quad\ldots\ldots\ldots\ldots(6).$$

The several quantities A, A_1, etc. are constants to be connected presently through the surface conditions.

If the layer did not exist the expression (4) would apply to the whole sphere. Thus $\partial A/A$ and $\partial B/A$ must be of the order $b-c$ of small quantities at least.

§ 13. We shall confine our attention entirely to the case when the surface of the sphere is free. The relations connecting the constants of the solution may then be written in the form

$$A\left(\frac{\sin k\alpha c}{k\alpha c} - \cos k\alpha c\right) = A_1\left(\frac{\sin k\alpha_1 c}{k\alpha_1 c} - \cos k\alpha_1 c\right) + B_1\left(\frac{\cos k\alpha_1 c}{k\alpha_1 c} + \sin k\alpha_1 c\right) \quad\ldots\ldots\ldots(7).$$

$$A\left\{(m+n)\,k\alpha c\sin k\alpha c - 4n\left(\frac{\sin k\alpha c}{k\alpha c} - \cos k\alpha c\right)\right\}$$

$$= A_1\left\{(m_1+n_1)\,k\alpha_1 c\sin k\alpha_1 c - 4n_1\left(\frac{\sin k\alpha_1 c}{k\alpha_1 c} - \cos k\alpha_1 c\right)\right\}$$

$$+ B_1\left\{(m_1+n_1)\,k\alpha_1 c\cos k\alpha_1 c - 4n_1\left(\frac{\cos k\alpha_1 c}{k\alpha_1 c} + \sin k\alpha_1 c\right)\right\} \quad\ldots\ldots\ldots\ldots(8).$$

$$(A+\partial A)\left(\frac{\sin k\alpha b}{k\alpha b} - \cos k\alpha b\right) + \partial B\left(\frac{\cos k\alpha b}{k\alpha b} + \sin k\alpha b\right)$$

$$= A_1\left(\frac{\sin k\alpha_1 b}{k\alpha_1 b} - \cos k\alpha_1 b\right) + B_1\left(\frac{\cos k\alpha_1 b}{k\alpha_1 b} + \sin k\alpha_1 b\right) \quad\ldots\ldots\ldots\ldots(9),$$

$$(A+\partial A)\left\{(m+n)\,kab\sin kab-4n\left(\frac{\sin kab}{kab}-\cos kab\right)\right\}$$
$$+\,\partial B\left\{(m+n)\,kab\cos kab-4n\left(\frac{\cos kab}{kab}+\sin kab\right)\right\}$$
$$=A_1\left\{(m_1+n_1)\,k\alpha_1 b\sin k\alpha_1 b-4n_1\left(\frac{\sin k\alpha_1 b}{k\alpha_1 b}-\cos k\alpha_1 b\right)\right\}$$
$$+\,B_1\left\{(m_1+n_1)\,k\alpha_1 b\cos k\alpha_1 b-4n_1\left(\frac{\cos k\alpha_1 b}{k\alpha_1 b}+\sin k\alpha_1 b\right)\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

$$\left(1+\frac{\partial A}{A}\right)\left\{(m+n)\,kaa\sin kaa-4n\left(\frac{\sin kaa}{kaa}-\cos kaa\right)\right\}$$
$$+\frac{\partial B}{A}\left\{(m+n)\,kaa\cos kaa-4n\left(\frac{\cos kaa}{kaa}+\sin kaa\right)\right\}=0\ldots\ldots(11).$$

In equation (7) put $c=b-(b-c)$ and neglect terms in $(b-c)^2$; then subtract the equation from (9) and we find

$$\partial A\left(\frac{\sin kab}{kab}-\cos kab\right)+\partial B\left(\frac{\cos kab}{kab}+\sin kab\right)$$
$$=-A\,\frac{(b-c)}{b}\left\{kab\sin kab-\left(\frac{\sin kab}{kab}-\cos kab\right)\right\}$$
$$+\,A_1\,\frac{b-c}{b}\left\{k\alpha_1 b\sin k\alpha_1 b-\left(\frac{\sin k\alpha_1 b}{k\alpha_1 b}-\cos k\alpha_1 b\right)\right\}$$
$$+\,B_1\,\frac{b-c}{b}\left\{k\alpha_1 b\cos k\alpha_1 b-\left(\frac{\cos k\alpha_1 b}{k\alpha_1 b}+\sin k\alpha_1 b\right)\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (12).$$

Treating (8) and (10) similarly, we deduce

$$\partial A\left\{(m+n)\,kab\sin kab-4n\left(\frac{\sin kab}{kab}-\cos kab\right)\right\}+\partial B\left\{(m+n)\,kab\cos kab-4n\left(\frac{\cos kab}{kab}+\sin kab\right)\right\}$$
$$=-A\,\frac{b-c}{b}\left\{(m+n)\,kab\,(\sin kab+kab\cos kab)-4n\left(kab\sin kab-\frac{\sin kab}{kab}+\cos kab\right)\right\}$$
$$+\,A_1\,\frac{b-c}{b}\left\{(m_1+n_1)\,k\alpha_1 b\,(\sin k\alpha_1 b+k\alpha_1 b\cos k\alpha_1 b)-4n_1\left(k\alpha_1 b\sin k\alpha_1 b-\frac{\sin k\alpha_1 b}{k\alpha_1 b}+\cos k\alpha_1 b\right)\right\}$$
$$+\,B_1\,\frac{b-c}{b}\left\{(m_1+n_1)\,k\alpha_1 b\,(\cos k\alpha_1 b-k\alpha_1 b\sin k\alpha_1 b)-4n_1\left(k\alpha_1 b\cos k\alpha_1 b-\frac{\cos k\alpha_1 b}{k\alpha_1 b}-\sin k\alpha_1 b\right)\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Now as terms in $(b-c)^2$ are negligible we are to determine ∂A and ∂B from (12) and (13) by substituting in these equations the approximate values for A_1/A and B_1/A deduced from (7) and (8) by putting $c=b$, or from (9) and (10) by neglecting ∂A and ∂B. These approximate values are

$$\frac{A_1}{A}(m_1+n_1)\,k\alpha_1 b=\left(\frac{\cos k\alpha_1 b}{k\alpha_1 b}+\sin k\alpha_1 b\right)\left\{(m+n)\,kab\sin kab-4n\left(\frac{\sin kab}{kab}-\cos kab\right)\right\}$$
$$-\left(\frac{\sin kab}{kab}-\cos kab\right)\left\{(m_1+n_1)\,k\alpha_1 b\cos k\alpha_1 b-4n_1\left(\frac{\cos k\alpha_1 b}{k\alpha_1 b}+\sin k\alpha_1 b\right)\right\}\ \ldots\ldots(14),$$

$$\frac{B_1}{A}(m_1+n_1)\,k\alpha_1 b = -\left(\frac{\sin k\alpha_1 b}{k\alpha_1 b} - \cos k\alpha_1 b\right)\left\{(m+n)\,kab\sin kab - 4n\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}$$
$$+\left(\frac{\sin kab}{kab} - \cos kab\right)\left\{(m_1+n_1)\,k\alpha_1 b\sin k\alpha_1 b - 4n_1\left(\frac{\sin k\alpha_1 b}{k\alpha_1 b} - \cos k\alpha_1 b\right)\right\} \quad\ldots\ldots(15).$$

Substituting these values of A_1/A and B_1/A in (12), reducing and arranging the terms, we get

$$\frac{\partial A}{A}\left(\frac{\sin kab}{kab} - \cos kab\right) + \frac{\partial B}{A}\left(\frac{\cos kab}{kab} + \sin kab\right)$$
$$= \frac{b-c}{b}\left\{\frac{4(n_1-n)}{m_1+n_1}\left(\frac{\sin kab}{kab} - \cos kab\right) - \left(1 - \frac{m+n}{m_1+n_1}\right)kab\sin kab\right\} \quad\ldots\ldots(16).$$

The same substitutions enable us in like manner to reduce (13) to

$$\frac{\partial A}{A}\left\{(m+n)\,kab\sin kab - 4n\left(\frac{\sin kab}{kab} - \cos kab\right)\right\} + \frac{\partial B}{A}\left\{(m+n)\,kab\cos kab - 4n\left(\frac{\cos kab}{kab} + \sin kab\right)\right\}$$
$$= \frac{b-c}{b}\left[\left\{(m+n)\,k^2a^2b^2 - (m_1+n_1)\,k^2\alpha_1^2b^2 + \frac{4(n_1-n)(3m_1-n_1)}{m_1+n_1}\right\}\left(\frac{\sin kab}{kab} - \cos kab\right)\right.$$
$$\left. + 4(m+n)\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)kab\sin kab\right] \quad\ldots\ldots(17).$$

Solving (16) and (17) we obtain

$$\frac{\partial A}{A}(m+n)\,kab \div \frac{b-c}{b}$$
$$= \left\{(m+n)\,k^2a^2b^2 - (m_1+n_1)\,k^2\alpha_1^2b^2 - \frac{4n(3m-n)}{m+n} + \frac{4n_1(3m_1-n_1)}{m_1+n_1}\right\}$$
$$\times\left(\frac{\sin kab}{kab} - \cos kab\right)\left(\frac{\cos kab}{kab} + \sin kab\right)$$
$$+\left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)\left\{(m+n)\,kab\sin kab - 4n\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}$$
$$\times\left\{(m+n)\,kab\cos kab - 4n\left(\frac{\cos kab}{kab} + \sin kab\right)\right\}$$
$$+4\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)\left[\left(\frac{\sin kab}{kab} - \cos kab\right)\left\{(m+n)\,kab\cos kab - 4n\left(\frac{\cos kab}{kab} + \sin kab\right)\right\}\right.$$
$$\left.+\left(\frac{\cos kab}{kab} + \sin kab\right)\left\{(m+n)\,kab\sin kab - 4n\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}\right] \quad\ldots\ldots\ldots\ldots(18),$$

$$\frac{\partial B}{A}(m+n)\,kab \div \frac{b-c}{b}$$
$$= -\left\{(m+n)\,k^2a^2b^2 - (m_1+n_1)\,k^2\alpha_1^2b^2 - \frac{4n(3m-n)}{m+n} + \frac{4n_1(3m_1-n_1)}{m_1+n_1}\right\}\left(\frac{\sin kab}{kab} - \cos kab\right)^2$$
$$-\left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)\left\{(m+n)\,kab\sin kab - 4n\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}^2$$
$$-8\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)\left(\frac{\sin kab}{kab} - \cos kab\right)\left\{(m+n)\,kab\sin kab - 4n\left(\frac{\sin kab}{kab} - \cos kab\right)\right\} \quad\ldots(19).$$

§ 14. If the thin layer did not exist the frequency equation would be got by putting $\partial A = 0 = \partial B$ in (11), which would give

$$f(0\,.\,\alpha\,.\,a) = (m+n)\,k\alpha a \sin k\alpha a - 4n\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right) = 0 \quad\text{..............(20)*.}$$

In consequence of the existence of the thin layer, $f(0\,.\,\alpha\,.\,a)$ is no longer zero but is of the order $b-c$. We may thus neglect $\partial A/A$ in (11). Further as $\partial B/A$ is by (19) of order $b-c$, we may introduce into its coefficient in (11) any modification consistent with the supposition that (20) is exactly true. We thus reduce (11) to

$$(m+n)\,k\alpha a \sin k\alpha a - 4n\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right) - \frac{(m+n)\,k\alpha a}{\dfrac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a}\,\frac{\partial B}{A} = 0 \quad\text{.......(21).}$$

Now in this equation $k/2\pi$ is the frequency of the vibration of the compound system. Thus if the presence of the layer has raised the frequency by $\partial k/2\pi$, then $(k-\partial k)/2\pi$ was the frequency of the corresponding note of the simple sphere, and so $k-\partial k$ must be a root of (20).

As ∂k is of order $b-c$ we are thus to substitute $k-\partial k$ in (20) and neglect terms in $(\partial k)^2$. We thus find

$$f(0\,.\,\alpha\,.\,a) - \frac{\partial k}{k}\,k\frac{d}{dk}f(0\,.\,\alpha\,.\,a) = 0 \quad\text{.................................(22).}$$

Now $k\dfrac{d}{dk}f(0\,.\,\alpha\,.\,a) = k\alpha a\,\dfrac{d}{d\,.\,k\alpha a}f(0\,.\,\alpha\,.\,a)$

$$= k\alpha a\left\{(m+n)(\sin k\alpha a + k\alpha a\cos k\alpha a) - 4n\left(\frac{\cos k\alpha a}{k\alpha a} + \sin k\alpha a - \frac{\sin k\alpha a}{(k\alpha a)^2}\right)\right\}.$$

As this occurs in (22) in the coefficient of ∂k we may modify it by any transformation consistent with the hypothesis that (20) is exactly true. We thus easily transform it into

$$k\frac{d}{dk}f(0\,.\,\alpha\,.\,a) = -\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right)\left\{(m+n)\,k^2\alpha^2a^2 - \frac{4n\,(3m-n)}{m+n}\right\} \quad\text{...........(23).}$$

We may thus replace (22) by

$$(m+n)\,k\alpha a\sin k\alpha a - 4n\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right) + \frac{\partial k}{k}\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right)\left\{(m+n)\,k^2\alpha^2a^2 - \frac{4n\,(3m-n)}{m+n}\right\} = 0.$$

This equation being necessarily identical with (21), we obtain

$$\frac{\partial k}{k} = \frac{-(m+n)\,k\alpha a\,\partial B/A}{\left(\dfrac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right)^2\left\{(m+n)\,k^2\alpha^2a^2 - 4n\dfrac{(3m-n)}{m+n}\right\}} \quad\text{..................(24).}$$

As $\partial B/A$ and so $\partial k/k$ is of order $b-c$, we may in this equation regard $k/2\pi$ as the frequency in the simple sphere $(0\,.\,\alpha\,.\,a)$. Thus the ratio of the small change in the frequency of a typical note to the value it possesses in the simple sphere is found by substituting in (24) the value obtained for $\delta B/A$ in (19).

* Cf. *Transactions*, Vol. xiv., equation (55), p. 318.

§ 15. Some preliminary considerations will enable us to give for $\partial k/k$ a comparatively short symbolical expression.

Let $\frac{1}{2\pi}K_{(\alpha \cdot a)}$ denote the frequency of the free-free radial vibrations in an infinitely thin spherical shell of material α and radius a. Then it is known that

$$K^2_{(\alpha \cdot a)}\, a^2 = \frac{4n\,(3m-n)}{\alpha^2\,(m+n)^2} = \frac{4n\,(3m-n)}{\rho\,(m+n)} \quad\text{.........................(25)*.}$$

This result may also be obtained by equating our expression (23) to zero in accordance with the general result established in Sect. I.†

Also let
$$u_r = \frac{1}{r}\left(\frac{\sin kar}{kar} - \cos kar\right) \quad\text{..(26),}$$

$$U_r = \frac{1}{r^2}\left\{(m+n)\,kar\sin kar - 4n\left(\frac{\sin kar}{kar} - \cos kar\right)\right\} \quad\text{..................(27).}$$

These represent respectively the amplitude of a displacement and the corresponding greatest radial stress at a distance r from the centre of a simple sphere of material (ρ, m, n). Whatever be the magnitude of the displacement or the instant considered, the simultaneous displacements at radial distances r and r' are in the ratio $u_r : u_{r'}$, and the ratio of the radial stress at r' to the simultaneous displacement at r is always $U_{r'} : u_r$.

Employing these several abbreviations in (19), and then substituting for $\partial B/A$ in (24), we finally obtain

$$\frac{\partial k}{k} = \frac{b-c}{a}\left\{\frac{\rho\,(k^2 - K^2_{(\alpha \cdot b)}) - \rho_1\,(k^2 - K^2_{(\alpha_1 \cdot b)})}{\rho\,(k^2 - K^2_{(\alpha \cdot a)})}\left(\frac{b}{a}\right)^2\left(\frac{u_b}{u_a}\right)^2\right.$$
$$\left. + \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)\frac{b^2}{a^2\rho(k^2 - K^2_{(\alpha \cdot a)})}\left(\frac{U_b}{u_a}\right)^2 + 8\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)\frac{b}{a^2\rho\,(k^2 - K^2_{(\alpha \cdot a)})}\left(\frac{u_b U_b}{u_a^2}\right)\right\} \quad\text{...(28).}$$

§ 16. In establishing (28) certain assumptions have been made which limit its applicability.

The primary assumption is made in § 12 where $\partial A/A$ and $\partial B/A$ are supposed to be *small* quantities of the order $b-c$. In the proof this is interpreted as meaning that $(b-c)/b$ is small. The form of the expressions (18) and (19) constitute a complete justification of the primary assumption and of the mathematical treatment provided kab be not very small.

If however we were in (18) and (19) to suppose $k\alpha b$ very small, we should find that while $\partial B/A$ varies as $(b-c)\,b^2$, $\partial A/A$ varies as $(b-c)/b$. Now in (11) we are

* *Transactions*, Vol. XIV., equation (67), p. 321.

† See § 10, noticing that $k\frac{d}{dk}f(0\,.\,\alpha\,.\,a) = a\frac{d}{da}f(0\,.\,\alpha\,.\,a)$, and that $\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a = 0$ is by (4) identical with $f(0\,.\,\alpha\,.\,\bar{a}) = 0$.

justified in neglecting $\partial A/A$ only if it be of the same order of small quantities as $\partial B/A$. Thus our method and assumptions are legitimate only when $(b-c)\,b^2/b^3$ as well as $(b-c)\,b^2/a^3$ is a quantity whose square is negligible. In other words the volume of the layer must be small compared to the volume of the mass inside it.

It would thus be unjustifiable to apply (28) to the case when the material (ρ_1, m_1, n_1) forms a core, but by supposing $(b-c)$ sufficiently small it may be applied to any true layer however small its radius may be. When the layer is of infinitely small radius, its thickness being supposed of course of a still higher order of small quantities, it will be designated the *central layer*.

The results obtained for the central layer are practically useful, because as will presently appear, the effect of a given alteration of material is for the central layer either zero or else a numerical maximum. Thus the values obtained for the central layer are asymptotic limits, and they supply very close approximations for practical cases in which the layer has a finite though small radius.

Further discussion of the central layer and core is reserved for § 22.

§ 17. We notice in (28) the separation of the expression for the change of pitch into three distinct terms, the first depending on the square of the displacement at the altered layer, the second on the square of the radial stress, and the third on the product of the displacement and radial stress.

If the layer differ from the remainder of the sphere only in density then the first term alone exists. This is also the case when the position of the layer coincides with the surface of the sphere, or more generally with any *no-stress* surface—i.e. a surface over which the radial stress U vanishes.

If on the other hand the layer occur at a *node* surface—or surface where the displacement u is always zero—then the second term alone exists.

If the material of the layer remain the same, then however its distance from the centre may vary the *signs* of these two terms remain unchanged.

The third term vanishes when the layer coincides either with a node or with a no-stress surface. It differs from the other terms in the important respect that its sign varies with the position of the layer. Another important feature of this term is that it vanishes if $m_1/n_1 = m/n$, a relation which on the uniconstant theory of isotropy is necessarily true.

§ 18. Before entering on a discussion of (28) it will be convenient to consider shortly the type of vibration throughout the sphere. In the core there is no pronounced change of type because (4), with of course a different value for k, would apply equally to a simple sphere. The only consequence of the existence of the layer is that every node, no-stress and *loop* surface—or surface where the displacement is a maximum—alters its radius r according to the law

$$-\partial r/r = \partial k/k \qquad \text{.....................(29).}$$

Substituting in (6) the values of $\partial A/A$ and $\partial B/A$ from (18) and (19) and reducing, we find outside the layer

$$u/A\cos kt = \frac{1}{r}\left(\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r\right)$$

$$+ \frac{b-c}{(m+n)k\alpha r}\left[\{\rho(k^2 - K^2_{(\alpha.b)}) - \rho_1(k^2 - K^2_{(\alpha_1.b)})\} b u_b f(b.\alpha.r) + \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right) U_b f(b.\alpha.\bar{r})\right.$$

$$\left.+ 4\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)\{b^{-1}u_b f(b.\alpha.\bar{r}) + U_b f(b.\alpha.\bar{r})\}\right] \quad\ldots\ldots(30);$$

where

$$\left.\begin{aligned} f(\bar{b}.\alpha.\bar{r}) &= \left(1 + \frac{1}{k^2\alpha^2 rb}\right)\sin k\alpha(r-b) - \frac{1}{k\alpha}\left(\frac{1}{b} - \frac{1}{r}\right)\cos k\alpha(r-b), \\ f(b.\alpha.r) &= (m+n)\left\{\frac{b}{r}\sin k\alpha(r-b) - k\alpha b\cos k\alpha(r-b)\right\} - 4nf(\bar{b}.\alpha.r) \end{aligned}\right\} \quad\ldots\ldots(31).$$

The functions f have the same significations in reality as in Sect. I.

This is easily proved if we notice that

$$\left.\begin{aligned} F\ (b.\alpha) &= (m+n)k\alpha b\sin k\alpha b - 4n\left(\frac{\sin k\alpha b}{k\alpha b} - \cos k\alpha b\right), \\ F_1(b.\alpha) &= (m+n)k\alpha b\cos k\alpha b - 4n\left(\frac{\cos k\alpha b}{k\alpha b} + \sin k\alpha b\right), \\ G\ (b.\alpha) &= \frac{\sin k\alpha b}{k\alpha b} - \cos k\alpha b, \\ G_1(b.\alpha) &= \frac{\cos k\alpha b}{k\alpha b} + \sin k\alpha b \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(32).$$

It will be noticed that $f(b.\alpha.\bar{r})$ vanishes and changes sign as r passes through any value answering to a node surface of a simple shell of material (ρ, m, n) performing radial vibrations of frequency $k/2\pi$, whose inner surface is of radius b and is fixed. Similarly $f(b.\alpha.\bar{r})$ vanishes and changes sign as r passes through any value answering to a node surface of a simple shell of material (ρ, m, n) whose inner surface is of radius b and is free, the frequency of vibration being also $k/2\pi$.

The formula (30) differs from that for the displacement in the core by the addition of the long expression which has $b-c$ for its factor. This expression we shall here call the *change of type*. It consists of three terms corresponding to the three terms in (28).

If the difference between the material of the layer and that of the remainder be such that one or more terms in the expression for the change of frequency vanish, then the corresponding term or terms in the expression for the change of type also vanish. Again if the position of the layer is such that either of the first two terms in the expression for the change of frequency vanishes, then too the corresponding term in the change of type vanishes.

While, however, the third term in the expression for the change of frequency vanishes when the layer occurs either at a node or at a no-stress surface, the third term in the change of type cannot vanish except for a chance value of r, for u_b and U_b cannot be simultaneously zero. It thus appears that, except on the uniconstant theory of isotropy, an alteration of elasticity occurring throughout a thin layer coincident either with a node or with a no-stress surface may produce a change of type to which there is no corresponding change of frequency.

It is also worth noticing that while the first two terms in the expression for $\partial k/k$ depend respectively on the squares of u_b and U_b, the first two terms in the change of type depend for their sign on the position of the layer.

A special interest attaches to the displacement just outside the layer. As $f(\bar{b}.\alpha.\bar{b})$ vanishes and $f(b.\alpha.b) = -(m+n)\,kab$ by (31), the displacement in question is

$$u = A\cos kt\left[u_b - (b-c)\left\{\left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)U_b + 4\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)b^{-1}u_b\right\}\right] \quad\ldots\ldots(33).$$

Now if in crossing the layer the type of vibration existing in the core were maintained, the displacement just outside would be simply

$$u = A\cos kt\,.\,u_b.$$

Thus the coefficient of $b-c$ in (33) is the measure of the change of type met with in crossing the layer.

The displacement in the layer itself may be got very simply from the consideration that it must have the value (33) when $r = b$, and the value

$$A\cos kt\,\frac{1}{c}\left(\frac{\sin kac}{kac} - \cos kac\right)$$

when $r = c$, terms in $(b-c)^2$ being neglected. It is thus given by

$$u/A\cos kt = \frac{1}{b}\left(\frac{\sin kab}{kab} - \cos kab\right) - \frac{b-r}{b}\,\frac{1}{b}\left\{kab\sin kab - 2\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}$$
$$-(r-c)\left\{\left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)U_b + 4\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)b^{-1}u_b\right\} \quad\ldots..(34).$$

The term in $r-c$ in (34) represents the progressive change of type, due to alteration of material alone, met with as we cross the layer from within outwards, and it reaches the value represented by the term in $b-c$ in (33) when the layer is completely crossed.

If the layer differ from the remainder only in density no change of type is met with in crossing it. In other words the layer vibrates as if it formed a portion of the included core.

Any alteration of elasticity will in general produce a progressive change of type in the layer, but this will not be the case when the layer coincides with a no-stress surface if the uniconstant theory be true, or if both constants in the biconstant theory be altered in the same proportion.

§ 19. As continual references to the properties of a simple vibrating sphere are essential for a discussion of (28), and as a good many of these properties have not, so far as I know, been fully discussed elsewhere I shall briefly notice them.

The frequency equation for the simple sphere $(0.\alpha.a)$ is (20).

The roots of this equation answering to the six notes of lowest pitch have been calculated by Professor Lamb* for the values 0, ·25, ·3 and ·$\dot{3}$ of Poisson's ratio $\sigma \equiv (m-n)/2m$.

* *Proceedings of the London Mathematical Society*, Vol. XIII. p. 202.

Answering to $\sigma = 1/2$ the frequency equation is

$$\sin k\alpha a = 0 \quad\quad (35);$$

whence $k\alpha a = i\pi$, where i is any positive integer.

The following table incorporates some of Professor Lamb's results.

TABLE I.

Values of $k\alpha a/\pi$.

Number of note.	$\sigma =$ 0	·25	$\dot{\cdot 3}$	·5
(1)	·6626	·8160	·8733	1
(2)	1·8909	1·9285	1·9470	2
(3)	2·9303	2·9539	2·9656	3
(4)	3·9485	3·9658	3·9744	4
(5)	4·9590	4·9728	4·9796	5
(6)	5·9660	5·9774	5·9830	6

It will be noticed that except in the lowest note or two the frequencies are nearly independent of the value of σ, and that the case $\sigma = \cdot 5$ supplies asymptotic values to which the results in the other cases tend.

As (4) is the type of vibration in the simple sphere the *node* surfaces are the concentric spheres whose radii are given by

$$r = x/k\alpha, \text{ where } \tan x = x \quad\quad (36).$$

The following are the first six roots, taken from p. 266 of Verdet's *Leçons d'Optique Physique*, Tome I.,

$$\frac{x}{\pi} = 0, \quad 1{\cdot}4303, \quad 2{\cdot}4590, \quad 3{\cdot}4709, \quad 4{\cdot}4774, \quad 5{\cdot}4818.$$

The higher roots are approximately odd multiples of $\pi/2$.

The *no-stress* surfaces are likewise concentric spheres, and their radii are supplied by (20) for the note of frequency $k/2\pi$ when the a in that equation is replaced by r. Thus for a given note and a given value of σ, the ratios of the radii of the no-stress surfaces to the radius of the sphere are obtained by dividing the values of $k\alpha a/\pi$ in Table I. for all the notes of less frequency, and for the note itself by the value of $k\alpha a/\pi$ for the note in question, all being taken for the assigned value of σ.

This method of determining the positions of these surfaces is given by Professor Lamb in his p. 197. The surfaces so determined he, however, speaks of as *loop* surfaces. I have here ventured to employ the term in a different sense, defining a *loop* surface as one over which the displacement is a maximum.

I employ the term *no-stress* surface only in default of a better. It must be borne in mind that over a surface so named it is only the *radial* stress that vanishes.

As defined above *loop* surfaces are the loci where $\frac{1}{r}\left(\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r\right)$ numerically considered is a maximum. They are thus concentric spheres whose radii are given by

$$r = x/k\alpha, \text{ where } \sin x - \frac{2}{x}\left(\frac{\sin x}{x} - \cos x\right) = 0 \quad\text{.......................(37).}$$

Now if we write x for $k\alpha a$, and 1 for m/n in (20) we transform it into (37). Thus the radii of the loop surfaces are found by equating $k\alpha r$ to the values ascribed to $k\alpha a$ in Table I. for the value 0 of σ. The loop surfaces accordingly coincide with the no-stress surfaces only when Poisson's ratio is zero. For all other values of Poisson's ratio each loop surface lies inside the corresponding no-stress surface.

The following table gives the positions of the node, loop and no-stress surfaces for the first six notes for the limiting values 0 and ·5 of σ, and the value ·25 of the uni-constant theory.

TABLE II.

Values of r/a over node, loop, and no-stress surfaces.

	$\sigma = 0$		$\sigma = \cdot 25$			$\sigma = \cdot 5$		
Number of note	Node surfaces	Loop and no-stress surfaces	Node surfaces	Loop surfaces	No-stress surfaces	Node surfaces	Loop surfaces	No-stress surfaces
(1)	0	1·0	0	·8120	1·0	0	·6626	1·0
(2)	0	·3504	0	·3436	·4231	0	·3313	·5
	·7564	1·0	·7417	·9805	1·0	·7151	·9454	1·0
(3)	0	·2261	0	·2243	·2762	0	·2209	$\cdot\dot{3}$
	·4881	·6453	·4842	·6401	·6529	·4768	·6303	$\cdot\dot{6}$
	·8392	1·0	·8325	·9920	1·0	·8197	·9768	1·0
(4)	0	·1678	0	·1671	·2058	0	·1656	·25
	·3622	·4789	·3607	·4768	·4863	·3576	·4727	·5
	·6228	·7421	·6201	·7389	·7448	·6147	·7326	·75
	·8790	1·0	·8752	·9956	1·0	·8677	·9871	1·0
(5)	0	·1336	0	·1332	·1641	0	·1325	·2
	·2884	·3813	·2876	·3802	·3878	·2861	·3782	·4
	·4959	·5909	·4945	·5893	·5940	·4918	·5861	·6
	·6999	·7962	·6980	·7940	·7975	·6942	·7897	·8
	·9029	1·0	·9004	·9972	1·0	·8955	·9918	1·0
(6)	0	·1111	0	·1109	·1365	0	·1104	$\cdot 1\dot{6}$
	·2397	·3169	·2393	·3163	·3226	·2384	·3151	$\cdot\dot{3}$
	·4122	·4912	·4114	·4902	·4942	·4098	·4884	·5
	·5818	·6618	·5807	·6606	·6635	·5785	·6581	$\cdot\dot{6}$
	·7505	·8312	·7491	·8296	·8319	·7462	·8265	$\cdot 8\dot{3}$
	·9188	1·0	·9171	·9981	1·0	·9136	·9943	1·0

§ 20. Counting the centre as a node surface and the outer surface as a no-stress surface, the number of the node, loop, or no-stress surfaces is always equal to the number of the note. We shall refer to any such surface by its number, supposing the surface of the same kind of least radius to be number (1).

For the node surfaces $k\alpha r$ is equated to certain numerical quantities independent of σ, viz. the roots of (36). Thus the ratio of the radii of the node surfaces of numbers (i) and (i') in a given sphere, when i and i' are given integers, is the same whatever be the value of σ for the material of the sphere or the number of the note. In like manner for the loop surfaces $k\alpha r$ is equated to certain numerical quantities. Thus the ratio of the radii of the loop surfaces of numbers (i) and (i') in a given sphere is independent of the value of σ or of the number of the note.

For the no-stress surfaces, however, $k\alpha r$ is equated to the values obtained for $k\alpha a$ from the frequency equation, and these vary with the value of σ. It thus appears that while in a sphere of given material the ratio of the radii of the no-stress surfaces of numbers (i) and (i') is the same for all the notes, this ratio is different for materials which differ in the value of Poisson's ratio.

It will be seen from the table that unless σ be small there is a marked difference in the positions of the corresponding loop and no-stress surfaces of least number. Between the loop and no-stress surfaces of high number the difference is obviously very small. Their radii, as well as those of the node surfaces of large number, are but little dependent on σ. As the number of the node surface increases it tends continually to become equidistant from two successive loop or no-stress surfaces.

§ 21. In all the expressions we are about to deal with for the change of frequency there occurs one or other of two factors. The first is

$$\left.\begin{aligned} Q &\equiv \left(\frac{k\alpha}{u_a}\right)^2 \div \{k^2\alpha^2a^2 - 4n\,(3m-n)\,(m+n)^{-2}\}, \\ \text{the second}\qquad \tfrac{1}{3}Q' &\equiv \tfrac{1}{3}k^2\alpha^2a^2 Q \end{aligned}\right\} \quad\text{..........................(38).}$$

As the expressions (38) occur in the coefficient of $b-c$ in the expressions for $\partial k/k$ we may, to the present degree of approximation, simplify them by any transformation which regards $k\alpha a$ as a root of (20) or the quantity tabulated in Table I.

Thus we may take

$$\begin{aligned} (k\alpha)^{-1}u_a &\equiv \frac{1}{k\alpha a}\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right) = \frac{m+n}{4n}\sin k\alpha a, \\ &= \left\{k^2\alpha^2a^2 - \frac{8n\,(m-n)}{(m+n)^2} + \left(\frac{4n}{m+n}\right)^2\frac{1}{k^2\alpha^2a^2}\right\}^{-\frac{1}{2}}; \end{aligned}$$

whence we get the following alternative formulae

$$Q = \{4n\,(m+n)^{-1}\operatorname{cosec} k\alpha a\}^2 \div \{k^2\alpha^2a^2 - 4n\,(3m-n)\,(m+n)^{-2}\} \quad\text{........................(39),}$$

$$= \left\{k^2\alpha^2a^2 - \frac{8n\,(m-n)}{(m+n)^2} + \left(\frac{4n}{m+n}\right)^2(k\alpha a)^{-2}\right\} \div \{k^2\alpha^2a^2 - 4n\,(3m-n)\,(m+n)^{-2}\} \quad\text{......(40).}$$

In the higher notes (40) is much the safer formula to use, because with it any small error in the value attributed to $\kappa a a$ in Table I. has a wholly insignificant effect.

The method by which (40) was deduced requires modification when $\sigma = \cdot 5$. It is easy however independently to prove for this case

$$Q = 1,$$

a result consistent with (40); whence we also get

$$Q' = i^2 \pi^2,$$

where i is an integer equal to the number of the note.

Employing the results in Table I., I find the following values for Q and Q'—

TABLE III.

Number of note	Value of Q			Value of Q'		
	$\sigma = 0$	$\sigma = \cdot 25$	$\sigma = \cdot 5$	$\sigma = 0$	$\sigma = \cdot 25$	$\sigma = \cdot 5$
(1)	2·253	1·369	1	9·762	8·995	9·8696
(2)	1·0635	1·0401	1	37·53	38·18	39·48
(3)	1·0247	1·0161	1	86·84	87·51	88·83
(4)	1·0133	1·0088	1	155·93	156·59	157·91
(5)	1·0084	1·0055	1	244·74	245·415	246·74
(6)	1·0058	1·0038	1	353·31	353·98	355·31

We may regard the case $\sigma = \cdot 5$ as supplying an inferior asymptotic value, viz. 1, for Q, and a superior asymptotic value, viz. $i^2\pi^2$ where i denotes the number of the note, for Q'. Except in the case of note (1) we may in rough calculations treat Q as unity, and regard Q' as varying as the square of the number of the note whatever be the value of σ.

§ 22. We shall first discuss some special cases of (28).

By supposing b/a very small we pass to the case of the *central layer* mentioned in § 16. Supposing V the volume of the whole sphere, ∂V that of the layer, we have

$$\partial V / V = 3(b - c)\, b^2/a^3.$$

Retaining in (28) only the lowest powers of b, and treating the function of $\kappa a a$ in the manner just discussed, we easily find for this case

$$\frac{\partial k_l}{k} = \frac{\partial V}{V} Q' \frac{3m_1 - n_1 - (3m - n)}{9(m_1 + n_1)} \left\{1 + \frac{4}{3}\frac{n_1 - n}{m + n}\right\} \quad \ldots\ldots\ldots\ldots (41_l),$$

where the suffix l signifies that the material $(\rho_1,\ m_1,\ n_1)$ forms a true layer.

As already explained, the case when the material $(\rho_1,\ m_1,\ n_1)$ forms a core cannot be derived from (28). I have, therefore, worked out this case by a rigid method independently. Supposing b the radius of the core and ∂V its volume, so that

$$\partial V / V = b^3/a^3,$$

and retaining only the lowest power of b/a, so that the result assumes the core of very small volume compared to the sphere, I find

$$\frac{\partial k_c}{k} = \frac{\partial V}{V} Q' \frac{3m_1 - n_1 - (3m - n)}{3(3m_1 - n_1 + 4n)} \quad \text{..........} (41_c).$$

The suffix c signifies that the material (ρ_1, m_1, n_1) actually forms a core.

The physical conditions under which (41_l) and (41_c) apply are totally different, so there is no reason to expect an identity between the two results. It will be noticed, however, that when the difference between the material of the layer or core and that of the rest of the sphere is small (41_l) and (41_c) lead to the same result, viz.

$$\frac{\partial k}{k} = \frac{\partial V}{V} Q' \frac{3m_1 - n_1 - (3m - n)}{9(m + n)} \quad \text{..........} (42).$$

Since $\rho_1 - \rho$ appears neither in (41_l) nor (41_c) we see that an alteration of density alone throughout either a central layer or a small core has to the present degree of approximation no effect on the pitch of any note.

In investigating the effects of alteration of elasticity we shall mainly consider the three following special cases:—

*1° when the elastic constant m alone is altered,

2° when the rigidity n alone is altered,

†3° when both elastic constants are altered in the same proportion so that

$$m_1/m = n_1/n = 1 + p \quad \text{..........} (43),$$

where p must of course be algebraically greater than -1.

The relation (43) is on the uniconstant hypothesis necessarily true, but on the bi-constant hypothesis of isotropy there is no *à priori* reason to expect it to hold.

Employing the suffixes l and c as in (41_l) and (41_c), we find for the changes of pitch in the above three cases:—

1° when m alone is altered

$$\frac{\partial k_l}{k} = \frac{\partial k_c}{k} = \frac{\partial V}{V} Q' \frac{m_1 - m}{3(m_1 + n)} \quad \text{..........} (44');$$

2° when n alone is altered

$$\left.\begin{aligned} \frac{\partial k_l}{k} &= -\frac{\partial V}{V} Q' \frac{n_1 - n}{9(m + n_1)} \left\{1 + \frac{4}{3} \frac{n_1 - n}{m + n}\right\}, \\ \frac{\partial k_c}{k} &= -\frac{\partial V}{V} Q' \frac{n_1 - n}{3(3m - n_1 + 4n)} \end{aligned}\right\} \quad \text{.......} (44'');$$

3° when the relation (43) holds

$$\left.\begin{aligned} \frac{\partial k_l}{k} &= \frac{p}{1 + p} \frac{\partial V}{V} Q' \frac{3m - n}{9(m + n)} \left\{1 + \frac{4}{3} p \frac{n}{m + n}\right\}, \\ \frac{\partial k_c}{k} &= p \frac{\partial V}{V} Q' \frac{3m - n}{9(m + n)} \left\{1 + p \frac{3m - n}{3(m + n)}\right\}^{-1} \end{aligned}\right\} \quad \text{....} (44''')$$

(44'), (44''), (44''') together: .... (44).

* This gives the most general alteration of the compressibility, or of Young's modulus, which is accompanied by no change in rigidity.

† This is the most general alteration consistent with the constancy of Poisson's ratio.

We see that an increase in m alone throughout a small volume at or close to the centre raises the pitch and a diminution of m lowers it; also for a given numerical alteration of m the fall of pitch when m is diminished is greater than the rise of pitch when m is increased.

Since $3m - n$ is essentially positive we see that in both forms of (44″) the sign of ∂k is opposite to that of $n_1 - n$. Thus when the rigidity at or close to the centre is altered the pitch is raised or lowered according as the rigidity is diminished or increased. The fall of pitch due to a small increase of rigidity at or close to the centre is greater than the rise of pitch due to an equal small diminution of rigidity.

In the case of the core this is obviously the case whatever be the magnitude of the alteration of rigidity. In the case of the central layer we may regard ∂k_l as composed of two terms, the first varying as $n_1 - n$ and indicating a change of pitch opposite in sign to the alteration of rigidity, the second varying as $(n_1 - n)^2$ and always indicating a fall of pitch.

If the alteration of elasticity satisfy (43), then the pitch is raised or lowered according as the elastic constants are increased or diminished. In the case of the core the rise of pitch due to a given numerical increase in the elastic constants is obviously always less than the fall of pitch due to an equal diminution in the constants. The same is easily proved true for the case of the central layer when the alteration in elasticity is small.

For any alteration of elasticity other than those above considered occurring at or close to the centre, we obtain from inspection of (41_l) and (41_c) the general law that the pitch of all the notes is raised or lowered according as the elastic quantity $m - n/3$—i.e. the *bulk-modulus*—is increased or diminished.

§ 23. When, as necessarily happens on the uniconstant theory of isotropy, only one elastic quantity is involved, the meaning to be attached to the terms *stiffness* and *elasticity* is in general free from ambiguity, and the statement that a local increase in stiffness raises the pitch may be in all cases sufficiently definite to admit of its truth being tested. As applied to the case (43) it is strictly true, and so when proceeding from supporters of uniconstant isotropy is in accordance with the facts here arrived at.

When, however, the statement is made by supporters of the biconstant theory it fails in the present case to have any exact meaning. This is obvious if we consider that the terms stiffness and elasticity might be interpreted to mean the rigidity, the bulk-modulus, Young's modulus, or any other modulus.

Now an increase in the rigidity produces an increase in Young's modulus and a fall in the bulk-modulus, while an increase in m increases both Young's modulus and the bulk-modulus. Thus a given increase in Young's modulus may be accompanied by a rise or by a fall in the bulk-modulus.

Our recent investigation shows that if the term *stiffness* is limited to mean the bulk-modulus the general statement is here in accordance with the facts; whereas if it be supposed equivalent to Young's modulus it may be true or false according to circumstances.

§ 24. As concerns the numerical magnitude of the change of pitch we may regard in the case of the central layer

$$\frac{\partial V}{V}\frac{3m_1-n_1-(3m-n)}{3(m_1+n_1)}\left\{1+\frac{4}{3}\frac{n_1-n}{m+n}\right\}=\partial E_l,$$

and in the case of the core

$$\frac{\partial V}{V}\frac{3m_1-n_1-(3m-n)}{3m_1-n_1+4n}=\partial E_c,$$

as measuring the magnitude of the alteration of elasticity.

The expressions (41_l) and (41_c) may then be written

$$\frac{1}{k}\partial k_l \div \partial E_l=\frac{1}{k}\partial k_c \div \partial E_c=\tfrac{1}{3}Q'.$$

Thus if in Table III. we divide the values given for Q' by 3, and alter the heading from Q' to $\frac{1}{k}\partial k_l \div \partial E_l=\frac{1}{k}\partial k_c \div \partial E_c$, we obtain at once a numerical measure of the changes in the pitch of all the notes considered in that table. The forms taken by ∂E_l and ∂E_c in the special cases when m alone is altered, or n alone is altered, or (43) holds are obvious from equations (44).

The forms given above are convenient when we examine the effect on the pitch due to a given alteration of material occurring throughout a given volume.

We shall also have occasion to deal with layers of *given thickness*, for which $b-c$ is constant. The square of the thickness is supposed in every case negligible, thus the effect on the pitch of any note due to any alteration of material throughout a central layer of *given thickness* or throughout a core of equal small radius, being at least of order $(k\alpha b)^3$, must be held to be zero.

§ 25. A second special case arises when the alteration of material occurs at the surface.

As the proof on which (28) rests assumes that the material (ρ_1, m_1, n_1) has material (ρ, m, n) outside it, its application without further proof to the case when (ρ_1, m_1, n_1) forms a surface layer might be objected to. I have thus worked out independently the case of the two-material compound sphere ($0 . \alpha . b . \alpha_1 . a$), and proceeding to the limit when $\{(a-b)/a\}^2$ is negligible I obtained a result identical with that derived from (28) by supposing $b=a$.

Denoting the thickness of the layer by t, and remembering that in virtue of the surface condition in a simple sphere U_a is zero, we easily obtain from (28)

$$\frac{\partial k}{k} \div \frac{t}{a} = -\frac{k^2 \alpha^2 a^2 \frac{\rho_1 - \rho}{\rho} - \frac{4n(3m-n)}{(m+n)^2}\left\{\frac{n_1(3m_1-n_1)(m_1+n_1)^{-1}}{n(3m-n)(m+n)^{-1}} - 1\right\}}{k^2\alpha^2 a^2 - \frac{4n(3m-n)}{(m+n)^2}} \quad \text{...... .....(45).}$$

The value of $\partial k/k$ when the density at the surface alone is altered is shown in the following table for the first six notes answering to the values 0, ·25 and ·5 of σ.

TABLE IV.

Value of $-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1 - \rho}{\rho}\right)$ *for a surface layer.*

	Number of note	(1)	(2)	(3)	(4)	(5)	(6)
$\sigma =$	0	1·857	1·060	1·024	1·013	1·008	1·006
	·25	1·511	1·064	1·0265	1·0145	1·009	1·006
	·5	1	1	1	1	1	1

Noticing that if we suppose in (45)

$$\frac{n_1(3m_1-n_1)(m_1+n_1)^{-1}}{n(3m-n)(m+n)^{-1}} - 1 = \frac{\rho_1}{\rho} - 1 = q \quad \text{............. ....(46),}$$

it reduces to the wonderfully simple form

$$\frac{\partial k}{k} = -\frac{t}{a}q \quad \text{... (47),}$$

we deduce at once from Table IV. the following results for the change of pitch due to a surface alteration of elasticity alone—

TABLE V.

Value of $\frac{\partial k}{k} \div \left[\frac{t}{a}\left\{\frac{n_1(3m_1-n_1)(m_1+n_1)^{-1}}{n(3m-n)(m+n)^{-1}} - 1\right\}\right]$ *for a surface layer.*

	Number of note	(1)	(2)	(3)	(4)	(5)	(6)
$\sigma =$	0	·857	·060	·024	·013	·008	·006
	·25	·511	·064	·0265	·0145	·009	·006
	·5	0	0	0	0	0	0

From Table IV. we see that in every case of a surface alteration of density the pitch is raised or lowered according as the density is diminished or increased.

The effect of a surface alteration of elasticity whatever be the value of σ is very small in the case of the higher notes, and continually diminishes, as measured by the *percentage* change of pitch, as the number of the note increases. For the limiting value ·5 of σ the effect of a surface alteration of elasticity alone is always zero.

From Tables IV. and V. we see that if a thin surface layer of an isotropic sphere be altered in any manner consistent with its remaining isotropic, the ratios of the frequencies of all the higher notes can only be very slightly affected; but, unless the value of σ for the unaltered material approach the limiting value ·5, or else both density and elasticity be altered in such a way as approximately to satisfy (46), the ratio of the frequency of the fundamental note to that of any of the higher notes may be sensibly disturbed.

If we suppose the relation (43) to hold, then (46) takes the form

$$p = q,$$

or the percentage alterations in the density and in the elastic constants are to be numerically equal and of the same sign.

§ 26. An exhaustive analysis of (28) being out of the question, I propose limiting the investigation to the following cases:

1°. Suppose the layer to differ from the remainder only in density, then remembering (38) and (26) we have

$$-\frac{\partial k}{k} = \frac{t}{a}\frac{\rho_1 - \rho}{\rho} Q\left(\frac{\sin kab}{kab} - \cos kab\right)^2 = \frac{\partial M}{M}\frac{Q'}{3}\left\{\frac{1}{kab}\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}^2 \quad \text{.......(48)},$$

where $t = b - c, \quad M = 4\pi a^3\rho/3, \quad \partial M = 4\pi b^2 (b - c)(\rho_1 - \rho),$

and $\partial M/M$ is supposed small.

The form of (48) to be used is the first or second according as the layer is of given thickness or given volume.

2°. Suppose m alone altered, or the layer to differ from the remainder in all its elastic properties except the rigidity. For this case there are the two alternative formulae

$$\frac{\partial k}{k} = \frac{t}{a}\frac{m_1 - m}{m_1 + n} Q \sin^2 kab = \frac{\partial V}{V}\frac{m_1 - m}{m_1 + n}\frac{Q'}{3}\left(\frac{\sin kab}{kab}\right)^2 \quad \text{.........................(49)},$$

where $V = 4\pi a^3/3, \quad \partial V = 4\pi b^2 (b - c),$

and $\partial V/V$ is supposed small.

*3°. Suppose m constant and n alone altered. The following seems the most convenient way of representing the expression for the change of pitch—

$$\frac{\partial k}{k} = \frac{t}{a}\frac{n_1 - n}{m + n_1} Q\left[\left\{\sin kab - \frac{2}{kab}\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}\left\{\sin kab - \frac{6}{kab}\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}\right.$$
$$\left. - \frac{4(n_1 - n)}{m + n}\left\{\frac{1}{kab}\left(\frac{\sin kab}{kab} - \cos kab\right)\right\}^2\right] \quad \text{...........(50)}.$$

We can obtain an alternative form in ∂V by putting

$$\frac{t}{a} Q = \frac{\partial V}{V}\frac{Q'}{3}\left(\frac{1}{kab}\right)^2 \quad \text{....................................(51)},$$

employing ∂V under the same restriction as in (49).

* For the case where the compressibility is constant while the rigidity is altered, see the note at the end of this Section.

4. Suppose the relation (43) to hold. The formula for the change of pitch is

$$\frac{\partial k}{k} = \frac{t}{a} pQ \left[\frac{4n(3m-n)}{(m+n)^2} \left\{ \frac{1}{k\alpha b} \left(\frac{\sin k\alpha b}{k\alpha b} - \cos k\alpha b \right) \right\}^2 + (1+p)^{-1} \left\{ \sin k\alpha b - \frac{4n(m+n)^{-1}}{k\alpha b} \left(\frac{\sin k\alpha b}{k\alpha b} - \cos k\alpha b \right) \right\}^2 \right] \text{......(52)}.$$

The substitution (51) gives the equivalent form in ∂V, applicable under the usual restriction.

§ 27. Comparing the several expressions (48), (49), (50) and (52) for the change of pitch we see that each is a product of three factors.

The first factor is such as

$$\frac{t}{a} \frac{\rho_1 - \rho}{\rho} \text{ or } \frac{\partial V}{V} \frac{m_1 - m}{m_1 + n},$$

and may be regarded as measuring the magnitude of the alteration in the material. For a given alteration of material the first factor is the same for all notes, and for all positions of the layer. The second factor is either Q or $Q'/3$. These quantities vary with the number of the note and the value of σ, as shown by Table III., but are independent of b. The third factors are such as $\sin^2 k\alpha b$. They determine how the effect on the pitch of a given note of a given alteration of material varies with the position of the altered layer.

In the case of (48) and (49) these third factors do not contain m or n explicitly, and depend on σ only in so far as $k\alpha$ does. They may thus be regarded as functions solely of the variable $k\alpha b$. We thence arrive at a comparatively simple way of treating the subject.

§ 28. We shall first examine the case of (48) and (49).

As an example let us take the first form of (48) and draw a curve B, fig. 1, viz.

$$y = \left(\frac{\sin x}{x} - \cos x \right)^2 \text{..(53)},$$

whose abscissae are the values of $x \equiv k\alpha b$. Then the ordinates of this curve indicate the variation in the magnitude of

$$-\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho} \right)$$

with the radius of the layer of altered density, supposed of given thickness, whatever be the number of the note or the value of σ. The only effect of a variation in the number of the note or in the value of σ is to vary the value of the factor, viz. $(k\alpha a)^{-1}$, by which the abscissae must be multiplied to get the corresponding values of b/a, and the factor, viz. Q, by which the ordinates must be multiplied so as to give the numerical values of

$$-\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho} \right).$$

In the fundamental note for instance the position of the layer to which the abscissa x refers answers to $b/a = x/(\cdot 6626\pi)$ when $\sigma = 0$, and to $b/a = x/(\cdot 8160\pi)$ when $\sigma = \cdot 25$. In the first case the portion of the curve which applies is limited by the abscissae 0 and $\cdot 6626\pi$, whereas in the second case the limiting abscissae are 0 and $\cdot 8160\pi$. In the first case to find the numerical value of

$$-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1 - \rho}{\rho}\right),$$

we must multiply the ordinates by 2·253, whereas in the second case the factor of multiplication is 1·369.

Suppose again we consider one of the higher notes, for instance note (4) when $\sigma = \cdot 25$. Here the position of the layer to which the abscissa x refers answers to

$$b/a = x/(3\cdot 9658\pi),$$

and the whole of the curve between the origin and the point whose abscissa is $3\cdot 9658\pi$ applies. To get the numerical value of

$$-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1 - \rho}{\rho}\right),$$

we must in this case multiply the ordinates by 1·0088.

Still employing the same curve we shall illustrate its application to the determination of relations between the successive positions of the layer when the change of pitch vanishes or is a maximum. Since ∂k vanishes when the ordinate of (53) vanishes, the several positions of the layer when its existence has no effect on the pitch are found by equating kab to the successive roots of equation (36), which are absolute constants independent of k or σ.

In like manner the several positions of the layer when its effect on the pitch is a maximum are found by equating kab to those abscissae which supply the maxima ordinates of (53), i.e. to the successive roots greater than zero of the equation

$$\sin x - \frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right) = 0 \quad\text{.................................(54).}$$

The roots of this equation are likewise numerical quantities. We thus conclude that as ka is constant for a given sphere performing a vibration of given frequency, the radii of the several positions of the layer where its existence has no effect or a maximum effect on the frequency of a given note are to one another in certain constant ratios wholly independent of the number of the note, of the value of σ, or of the magnitude of the sphere.

If we denote the i^{th} positive root in ascending order of (54) by x_i, and the radius of the corresponding position of the layer for the note of frequency $k/2\pi$ by b_i, then

$$b_i/a = x_i/kaa \quad\text{...(55).}$$

Thus the ratio to the radius of the sphere of the radius of the layer when in the position answering to the maximum change of frequency of given number (i), in the note of frequency $k/2\pi$,—the position nearest the centre being held number (1)—varies inversely as the value of $k\alpha a$ for the note and material considered. The same is obviously true of the radii of those positions of the layer where its effect on the pitch vanishes.

Again since the numerical value of $\partial k/k$ for a given note in a given sphere is obtained by multiplying the ordinate of (53) by a constant factor, we find between the maxima changes of pitch of numbers (i) and (j) in a note of frequency $k/2\pi$ and the maxima ordinates of numbers (i) and (j) in the curve (53) the simple relation

$$\partial k_i : \partial k_j :: y_i : y_j \quad\text{...(56).}$$

Now y_i and y_j are certain numerical quantities, thus, whatever be the number of the note or the value of σ, the ratio of the maxima changes of frequency of numbers (i) and (j) is the same. Thus if we desire to compare the relative magnitudes of the successive maxima changes of frequency in the pitch of a note of given number in a given sphere, due to an assigned alteration of density throughout a layer of given small thickness, all we have to do is to compare the lengths of the successive maxima ordinates of the curve B, fig. (1).

Conclusions of the same general character obviously apply to the three following curves—

$$A\text{, fig. 1, viz. } y = \left\{\frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right)\right\}^2 \quad\text{...........................(57),}$$

$$B\text{, fig. 2, ,, } y = \sin^2 x \quad\text{..(58),}$$

$$A\text{, fig. 2, ,, } y = (x^{-1}\sin x)^2 \quad\text{.......................................(59),}$$

which represent the variation of $\partial k/k$ with the value of $k\alpha b$ in the second form of (48), and in the first and second forms of (49) respectively. In the case of (57) and (59), where the layer is supposed of given volume, the restriction of the formula in the case when the radius of the layer becomes very small must be remembered. The ordinates however at the origin give correctly the change of pitch due to a *central layer*.

§ 29. There are various other general conclusions which are easily derived from (48) and (49), in the elucidation of which the curves (53), (57), (58) and (59) are useful.

If we suppose the curves drawn on the same scale, then the value of b/a which answers to a given value of x is, for a given note in a given material, the same in all the curves.

Again if we are considering the effect of an altered layer of given thickness, the second factor, which determines the variation of $\partial k/k$ with the value of σ or with the number of the note, is Q in the first forms of both (48) and (49).

We thus conclude that if the same scale be adopted in the curves, then the quantities $-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1-\rho}{\rho}\right)$ and $\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{m_1-m}{m_1+n}\right)$ for any given note and material are simply in the ratio of those ordinates of the curves B, fig. 1, and B, fig. 2, whose abscissae are found by multiplying the values of b/a for the assigned positions of the layer by that value of $k\alpha a$ which applies to the note and material under consideration.

If we suppose the thicknesses of the layer of altered density and the layer whose elastic constant m is altered the same, and further suppose

$$\frac{\rho_1-\rho}{\rho}=\frac{m_1-m}{m_1+n} \qquad\qquad (60),$$

then the numerical magnitudes of the changes of pitch in the two cases in a given note and material are simply as the lengths of the ordinates of the curves.

Similarly if we are considering the effect of altered layers of given volume, we see from the second forms of (48) and (49) that the second factors are the same, viz. $Q'/3$, whether the alteration be in the density alone, or in the elastic constant m alone. We thus conclude that the magnitudes of the quantities $-\frac{\partial k}{k} \div \left(\frac{\partial M}{M}\right)$ and $\frac{\partial k}{k} \div \left(\frac{\partial V}{V}\frac{m_1-m}{m_1+n}\right)$ for any given note and material are simply in the ratio of the ordinates of the curves A, fig. 1, and A, fig. 2, supplied by the abscissae which correspond to the assigned positions of the layer.

§ 30. The expressions (50) and (52) do not admit of so simple a treatment.

We may, however, regard (50) as composed of two terms, each of which may have its dependence on b represented by a curve whose ordinate is a function solely of $x \equiv k\alpha b$.

When the layer is of given thickness, these curves are

$$y=\psi_1(x)\,\psi_2(x) \qquad\qquad (61),$$

where

$$\left.\begin{aligned}\psi_1(x)&=\sin x-2x^{-1}(x^{-1}\sin x-\cos x)\\ \psi_2(x)&=\sin x-6x^{-1}(x^{-1}\sin x-\cos x)\end{aligned}\right\} \qquad\qquad (62),$$

and (57).

We may then suppose a *compound* curve drawn whose ordinate is the ordinate of (61) diminished by the product of the ordinate of (57) into the quantity

$$4(n_1-n)/(m+n).$$

Since this quantity depends on the value of σ and on the magnitude of $(n_1-n)/n$ the compound curve varies with the value of σ in the material and with the magnitude of the alteration of rigidity.

When the layer is of given volume the two curves are

$$y = x^{-2}\psi_1(x)\,\psi_2(x)\ldots\ldots\ldots\ldots(63),$$

and

$$y = \{x^{-2}(x^{-1}\sin x - \cos x)\}^2\ \ldots\ldots\ldots\ldots(64).$$

A compound curve may be derived from (63) and (64) precisely as one was derived from (61) and (57).

The expression (52) may likewise be regarded as composed of two terms. The first of these may have its dependence on b shown by a curve whose form is independent of σ. This curve is (57) or (64) according as the layer is of given thickness or of given volume. The second term has its dependence on b shown, according as the thickness or volume of the layer is given, by the curves

$$y = \{\sin x - 4n(m+n)^{-1}x^{-1}(x^{-1}\sin x - \cos x)\}^2\ \ldots\ldots\ldots\ldots(65),$$

$$y = x^{-2}\{\sin x - 4n(m+n)^{-1}x^{-1}(x^{-1}\sin x - \cos x)\}^2\ldots\ldots\ldots\ldots(66)$$

respectively.

Compound curves may as before be constructed showing the variation with b of the complete expression (52). These compound curves vary with the value of σ and with the magnitude of the alteration of elasticity.

If we suppose a compound curve drawn in the case either of (50) or (52) answering to a given alteration of elasticity and a given value of σ, then it applies to all possible notes. There are thus *for a given alteration of elasticity and a given value of* σ the same species of relations between the relative positions of the layer when its effect on the pitch is a maximum, and between the magnitudes of the several maxima of $\partial k/k$, as there were in the case of (48) and (49).

§ 31. When the alteration of elasticity and the value of σ remain unchanged then in (50) and (52), precisely as in (48) and (49), the variation of the several maxima of $\partial k/k$ with the number of the note depends only on the factor Q when the layer is of given thickness, and on the factor $Q'/3$ when the layer is of given volume.

Now as appears from Table III., Q differs but little from unity except for note (1); whereas in the higher notes Q' increases at least very approximately as the square of the number of the note. Thus for any one of the four types of alteration of material treated here, the maxima percentage changes of any given number in the frequencies of the several notes above the first are all nearly equal when the layer is of given thickness, but vary approximately as the squares of the numbers of the notes when the layer is of given volume.

§ 32. The evaluation of some of the functions of x represented by the curves being a very laborious process, I have carried none of the calculations beyond the value 3π of x. The results are given in Table VII. This supplies most necessary data for the first three notes in any material, but in the case of the higher notes its scope is limited to positions of the layer which, roughly speaking, lie inside the third loop surface.

The unit abscissa adopted in the table is $\pi/18$. For shortness the functions are represented by $f_1(x) \ldots f_{13}(x)$. Full information as to the first eleven of these headings is supplied in the following table. The entry "p" in the column headed "Property of material altered" means that both elastic constants are supposed altered in the same proportion, as in (43):

TABLE VI.

	Property of material altered	Layer of given	Figure where curve drawn	Letter attached to curve	Values of σ curve applies to
$f_1(x) = \sin^2 x$	m	thickness	2	B	all
	p	„	„	„	·5
$f_2(x) = x^{-2} f_1(x)$	m	volume	2	A	all
	p	„	„	„	·5
$f_3(x) = (x^{-1}\sin x - \cos x)^2$	ρ	thickness	1	B	all
$f_4(x) = x^{-2} f_3(x)$	ρ	volume	1	A	all
	n	thickness	„	„	„
	p	„	„	„	„
$f_5(x) = x^{-4} f_3(x)$	n	volume	1	C	all
	p	„	„	„	„
$f_6(x) = \{\sin x - 2x^{-1}(x^{-1}\sin x - \cos x)\}^2$	p	thickness	4	B_0	0
$f_7(x) = x^{-2} f_6(x)$	p	volume	4	A_0	0
$f_8(x) = \{\sin x - \frac{4}{3} x^{-1}(x^{-1}\sin x - \cos x)\}^2$	p	thickness	4	$B_{\cdot 25}$	·25
$f_9(x) = x^{-2} f_8(x)$	p	volume	4	$A_{\cdot 25}$	·25
$f_{10}(x) = \left\{\sin x - \frac{2}{x}\left(\frac{\sin x}{x} - \cos x\right)\right\}\left\{\sin x - \frac{6}{x}\left(\frac{\sin x}{x} - \cos x\right)\right\}$	n	thickness	3	B	all
$f_{11}(x) = x^{-2} f_{10}(x)$	n	volume	3	A	all

As the first nine functions cannot be negative no signs are attached to their values. In the case of $f_{10}(x)$ and $f_{11}(x)$ signs are attached to those entries which occur next the zero value. Any number without a sign attached has the sign last entered in the column.

The functions $f_{12}(x)$ and $f_{13}(x)$ which appear in the table apply both to the radial and the transverse vibrations of a sphere. Their use in radial vibrations is stated at the end of this section; their form is more fully discussed in Sect. III.

TABLE VII.

Value of x/π	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$	$f_{10}(x)$	$f_{11}(x)$	$f_{12}(x)$	$f_{13}(x)$
0	0	1	0	0	$\cdot\dot{1}$	0	$\cdot\dot{1}$	0	·3086	0	$-\cdot\dot{3}$	0	0
1/18	·03015	·9899	·0001	·0034	·1104	·0033	·1092	·0093	·3047	− ·0101	·3300	·00000	·00000
2/18	·11698	·9600	·0016	·0132	·1085	·0126	·1032	·0356	·2924	·0390	·3199	·00001	·00006
3/18	·25	·9119	·0079	·0288	·1052	·0257	·0939	·0749	·2731	·0832	·3036	·00009	·0003
4/18	·41318	·8477	·0239	·0491	·1007	·0398	·0818	·1206	·2475	·1371	·2813	·00048	·0010
5/18	·58682	·7706	·0552	·0725	·0952	·0517	·0679	·1656	·2175	·1933	·2538	·0017	·0023
6/18	·75	·6839	·1069	·0975	·0889	·0583	·0532	·2022	·1844	·2433	·2219	·0050	·0046
7/18	·88302	·5916	·1824	·1222	·0819	·0578	·0388	·2242	·1502	·2784	·1865	·0119	·0080
8/18	·96985	·4975	·2827	·1450	·0744	·0498	·0256	·2276	·1168	·2902	·1489	·0248	·0127
9/18	1	·4053	·4053	·1643	·0666	·0359	·0146	·2113	·0856	·2712	·1099	·0466	·0189
10/18	·96985	·3184	·5445	·1787	·0587	·0194	·0064	·1773	·0582	·2160	·0709	·0804	·0264
11/18	·88302	·2396	·6914	·1876	·0509	·0054	·0015	·1312	·0356	− ·1219	−·0331	·1293	·0351
12/18	·75	·1710	·8345	·1902	·0434	·0000	·0000	·0809	·0184	+ ·0110	+·0025	·1958	·0446
13/18	·58682	·1140	·9612	·1867	·0363	·0096	·0019	·0361	·0070	·1793	·0348	·2811	·0546
14/18	·41318	·0692	1·0591	·1774	·0297	·0398	·0067	·0066	·0011	·3760	·0630	·3853	·0645
15/18	·25	·0365	1·1173	·1630	·0238	·0946	·0138	·0015	·0002	·5912	·0863	·5059	·0738
16/18	·11698	·0150	1·1282	·1447	·0186	·1753	·0225	·0273	·0035	·8124	·1042	·6386	·0819
17/18	·03015	·0034	1·0886	·1237	·0140	·2805	·0319	·0871	·0099	1·0255	·1165	·7766	·0882
1	0	0	1	·1013	·0103	·4053	·0411	·1801	·0182	1·2158	·1232	·9119	·0924
19/18	·03015	·0027	·8694	·0791		·5417	·0493	·3009	·0274	1·3695	·1245	1·0347	·0941
20/18	·11698	·0096	·7085	·0581		·6794	·0558	·4403	·0361	1·4745	·1210	1·1351	·0932
21/18	·25	·0186	·5323	·0396		·8066	·0600	·5859	·0436	1·5218	·1133	1·2038	·0896
22/18	·41318	·0280	·3584	·0243		·9113	·0618	·7236	·0491	1·5066	·1022	1·2332	·0836
23/18	·58682	·0364	·2043	·0127		·9825	·0610	·8393	·0521	1·4289	·0887	1·2184	·0756

TABLE VII. *continued.*

Value of x/π	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$	$f_{10}(x)$	$f_{11}(x)$	$f_{12}(x)$	$f_{13}(x)$
24/18	·75	·0427	·0860	·0049		1·0121	·0577	·9204	·0525	1·2938	·0737	1·1579	·0660
25/18	·88302	·0464	·0160	·00084		·9955	·0523	·9573	·0503	1·1113	·0584	1·0543	·0554
26/18	·96985	·0471	·0019	·00009		·9326	·0453	·9449	·0459	·8956	·0435	·9142	·0444
27/18	1	·0450	·0450	·0020		·8280	·0373	·8835	·0398	·6641	·0299	·7481	·0337
28/18	·96985	·0406	·1408	·0059		·6910	·0289	·7787	·0326	·4357	·0182	·5693	·0238
29/18	·88302	·0345	·2784	·0109		·5346	·0209	·6411	·0250	·2297	·0090	·3930	·0153
30/18	·75	·0274	·4428	·0161		·3744	·0137	·4852	·0177	+ ·0633	+ ·0023	·2350	·0086
31/18	·58682	·0200	·6152	·0210		·2267	·0077	·3280	·0112	− ·0494	− ·0017	·1096	·0037
32/18	·41318	·0132	·7764	·0249		·1071	·0034	·1870	·0060	·0994	·0032	·0287	·0009
33/18	·25	·0075	·9079	·0274		·0286	·00086	·0781	·0024	·0833	·0025	·0000	·0000
34/18	·11698	·0033	·9947	·0282		·00004	·000001	·0139	·0004	− ·0039	− ·00011		
35/18	·03015	·00081	1·0266	·0275		·0250	·00067	·0023	·00006	+ ·1299	+ ·0035		
2	0	0	1	·0253		·1013	·00257	·0450	·00114	·3040	·0077		
38/18	·11698	·00266	·7888	·01793		·3719	·00845	·2710	·00616	·6986	·0159		
40/18	·41318	·00848	·4542	·00932		·6987	·01433	·5952	·01221	1·0214	·0210		
42/18	·75	·01396	·1458	·00271		·9413	·01752	·8751	·01629	1·1435	·0213		
44/18	·96985	·01645	·0021	·00003		·9933	·01684	·9854	·01671	1·0169	·0172		
46/18	·96985	·01505	·0878	·00136		·8299	·01287	·8753	·01358	·6954	·0108		
48/18	·75	·01069	·3641	·00519		·5213	·00743	·5929	·00845	·3133	·0045		
50/18	·41318	·00543	·7051	·00926		·2028	·00266	·2647	·00348	+ ·0295	+ ·00039		
51/18	·25	·00316	·8504	·01073		·0857	·00108	·1309	·00165	− ·0356	− ·00045		
52/18	·11698	·00142	·9553	·01160		·0160	·00019	·0394	·00048	− ·0385	− ·00047		
53/18	·03015	·00035	1·0072	·01177		·0019	·00002	·0008	·00001	+ ·0207	+ ·00024		
3	0	0	1	·01126		·0450	·00051	·0200	·00022	·1351	·00152		

§ 33. We shall now discuss in some detail the effects of the several alterations of material.

When the layer differs from the remainder only in density the change of pitch is given by (48). The positions of the layer when the pitch of a given note is unaffected coincide with the node surfaces for that note.

When the layer is in any other position the pitch is raised or lowered according as the density is diminished or increased.

When the layer of altered density is of given volume $\partial k/k$ varies simply as u_b^2, as may be seen by comparing (26) with the function of x occurring in curve (57), i.e. A, fig. 1.

The points of this curve whose ordinates vanish answer of course to the node surfaces including the centre. The successive maxima ordinates answer to positions of the layer coincident with the successive loop surfaces.

The number of maxima is always equal to the number of the note. When $\sigma = 0$ the surface of the sphere is always a position supplying a maximum.

We see at once from the curve that the first maximum is far the most important. Thus the effect on the pitch of any note of an alteration of density throughout a layer of given small volume whose radius exceeds that of the first, or at all events the second, node surface is comparatively insignificant. The calculation of the lengths of the maxima ordinates may be simplified by the consideration that since the corresponding abscissae are the roots of (37) we may put

$$\left.\begin{aligned}\left\{\frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right)\right\}^2 = \left(\frac{1}{2}\sin x\right)^2 = \frac{1}{x^2}\left(1 + \frac{4}{x^4}\right)^{-1} \\ = \frac{1}{x^2} - \frac{4}{x^6} + \dots\dots\end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(67),$$

where x/π has the values ascribed in Table I. to the case $\sigma = 0$. For the ratios of the first to the successive maxima ordinates, and so of the first to the successive maxima of $-\frac{\partial k}{k} \div \frac{\partial M}{M}$, I find

$$1 : \cdot 1485 : \cdot 0620 : \cdot 0342 : \cdot 0217 : \cdot 0150 \dots .$$

As already explained the absolute magnitudes of the maxima vary as Q' and so depend on the value of σ and on the number of the note. The following table gives the first and so the largest maximum for the first six notes.

TABLE VIII.

First maximum of $-\frac{\partial k}{k} \div \frac{\partial M}{M}$.

Value of σ	Number of note	(1)	(2)	(3)	(4)	(5)	(6)
0		·619	2·38	5·51	9·89	15·52	22·41
·25		·570	2·42	5·55	9·93	15·56	22·45
·5		·626	2·50	5·63	10·01	15·65	22·53

When $\sigma = \cdot 5$ the first maximum for note (i) is given by

$$-\frac{\partial k}{k} \div \frac{\partial M}{M} = i^2 \times \cdot 6259 \quad\text{.............(68):}$$

and for all values of i above 6, this equation will give a close approximation to the first maximum whatever be the value of σ.

§ 34. When the layer of altered density is of given thickness t, the mode of variation of $\partial k/k$ with $k\alpha b$ is given by curve (53), i.e. B, fig. 1. The successive maxima ordinates diminish slightly as the values of x to which they correspond increase.

The exact values of the abscissae supplying the maxima ordinates are the positive roots of

$$\sin x - \frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right) = 0 \quad\text{.............(69),}$$

excluding zero. It will be noticed that (20) may be made identical with (69) by writing x for $k\alpha a$ and taking $m = 3n$. Thus the roots of (69) are the values assigned to $k\alpha a$ in Table I. in the column for $\sigma = \cdot\dot{3}$. The corresponding positions of the layer thus coincide with the no-stress surfaces when $\sigma = \cdot\dot{3}$, and lie outside or inside these surfaces according as σ is less or greater than this value. It follows that provided σ be not less than $\cdot\dot{3}$ the number of *true* maxima of $\partial k/k$ is equal to the number of the note. If, however, σ be less than $\cdot\dot{3}$ the number of true maxima is less by unity than the number of the note.

This point requires special attention in note (1), as there is here no true maximum if σ be less than $\cdot\dot{3}$. This simply means that when σ is less than $\cdot\dot{3}$ the portion of curve B, fig. 1, which applies to this note does not extend as far as the first maximum ordinate. The value of $\partial k/k$ in such a case increases continually as the layer moves out from the centre. The value arising when the layer is at the surface may be called a maximum, but it must be carefully distinguished from the *true* maxima which answer to the maxima ordinates of curve B.

All the data necessary for calculating the positions of the layer answering to the true maxima in the case of those notes and materials considered here exist in Table I. I have, however, thought it worth while to record the results in the following table. The blanks indicate the absence of *true* maxima.

TABLE IX.

Values of b/a when $-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1-\rho}{\rho}\right)$ is a maximum.

Number of note	Value of σ	Number of maximum (1)	(2)	(3)	(4)	(5)	(6)
(1)	0	——					
	·25	——					
	·5	·8733					
(2)	0	·4618	——				
	·25	·4528	——				
	·5	·4366	·9735				
(3)	0	·2980	·6644	——			
	·25	·2956	·6591	——			
	·5	·2911	·6490	·9885			
(4)	0	·2212	·4931	·7511	——		
	·25	·2202	·4909	·7478	——		
	·5	·2183	·4867	·7414	·9936		
(5)	0	·1761	·3926	·5980	·8015	——	
	·25	·1756	·3915	·5964	·7992	——	
	·5	·1747	·3894	·5931	·7949	·9959	
(6)	0	·1464	·3263	·4971	·6662	·8347	——
	·25	·1461	·3257	·4961	·6649	·8331	——
	·5	·1455	·3245	·4943	·6624	·8299	·9972

In calculating the lengths of the successive maxima ordinates of curve B, fig. 1, we may, since x is a root of (69), replace

$$\left(\frac{\sin x}{x} - \cos x\right)^2 \text{ by } (1 - x^{-2} + x^{-4})^{-1}.$$

For the ratios of the first to the successive maxima ordinates I find

1 : ·908 : ·895 : ·890 : ·888 : ·887....

These are thus the ratios of the first to the successive maxima of $-\partial k/k$ due to a given alteration of density occurring throughout a layer of given thickness in a given sphere, whatever be the density or elasticity of the sphere or the number of the note considered.

The *absolute* values of the maxima vary as Q. In the following table are given the absolute values of the first and so largest maxima for the cases considered here.

TABLE X.

First maximum of $-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1-\rho}{\rho}\right)$.

Value of σ	Number of note	(1)	(2)	(3)	(4)	(5)	(6)
0		1·857*	1·202	1·158	1·145	1·140	1·137
·25		1·511*	1·175	1·148	1·140	1·136	1·135
·5		1·130	1·130	1·130	1·130	1·130	1·130

In the case of note (1) there are no *true* maxima for the values 0 and ·25 of σ. I have, however, given the greatest values which the quantity tabulated can have in these two cases. They answer to positions of the layer coincident with the surface, and are distinguished by asterisks.

As in the case of all quantities varying as Q, it is only in the first few notes that the percentage change of pitch depends to any marked extent on σ. For any note above the sixth in any isotropic material the formula for the limiting case $\sigma = {\cdot}5$, viz.

$$-\frac{\partial k}{k} \div \left(\frac{t}{a}\,\frac{\rho_1 - \rho}{\rho}\right) = 1{\cdot}130 \qquad (70),$$

supplies a very fair approximation to the first maximum.

§ 35. In the second case we are to consider m alone is altered. Mathematically considered this change is very important, as the expressions which occur in the formula for $\partial k/k$ are of extraordinary simplicity.

The change of pitch in this case is given by (49). The positions of the layer when the change of pitch in a note of frequency $k/2\pi$ vanishes are given by the equation

$$x = k\alpha b = i\pi \qquad (71),$$

where i is any positive integer. For all other positions the pitch is raised or lowered according as m is increased or diminished throughout the layer.

Employing (71) we can easily calculate from Table I. the values of b/a for those positions in which the layer does not affect the pitch of the several notes. When $\sigma = {\cdot}5$ these positions coincide with the no-stress surfaces. For other values of σ it seems unnecessary to tabulate them, because they lie exactly midway between the successive positions given in Table XI. where the layer when of given thickness has most effect on the pitch.

Supposing first that the layer is of given volume, we have the variation of $\partial k/k$ with the value of $k\alpha b$ given by the curve (59), i.e. A, fig. 2.

Of the maxima ordinates that at the origin is very much the largest. Thus the maximum change of pitch which arises when the altered material forms a central layer is extremely large compared to the other maxima.

In the present case to obtain the change of pitch due to a central layer, we have only to divide by 3 the values given for Q' in Table III., and to alter the heading from Q' to $\dfrac{\partial k}{k} \div \left(\dfrac{\partial V}{V}\,\dfrac{m_1 - m}{m_1 + n}\right)$.

From (59) we see that the several maxima ordinates have for their abscissae the roots of $\tan x = x$. The corresponding positions of the layer are thus coincident with the node surfaces.

In comparing the lengths of the maxima ordinates it is convenient to notice that since $\tan x = x$,

$$(x^{-1} \sin x)^2 = (1 + x^2)^{-1}.$$

Employing this relation, I find for the ratios of the first to the subsequent maxima ordinates, and so for the ratios of the first maximum change of pitch—answering to a change of m throughout a central layer—to the subsequent maxima

1 : ·04719 : ·01648 : ·00834 : ·00503 : ·00336....

For notes above the sixth a close approximation to the first maximum in any material is supplied by the equation

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{m_1 - m}{m_1 + n}\right) = \frac{i^2 \pi^2}{3} \quad \text{.................................(72)},$$

where i is the number of the note. This is the exact equation for the value ·5 of σ.

§ 36. Suppose next that the layer whose m differs from that of the remainder is of given thickness. The corresponding curve is (58), i.e. B, fig. 2, which is merely a special form of the curve of sines.

The zero ordinates coincide of course with those of curve A, fig. 2. The abscissae supplying the maxima ordinates are found by ascribing positive integral values to i in the equation

$$x = (2i + 1)\, \pi/2.$$

The corresponding values of b/a for the notes and materials treated here are given in the following table:

TABLE XI.

Values of b/a when $\dfrac{\partial k}{k} \div \left\{\dfrac{t}{a} \dfrac{m_1 - m}{m_1 + n}\right\}$ is a maximum.

Number of note	Value of σ	Number of Maximum (1)	(2)	(3)	(4)	(5)	(6)
(1)	0	·7546					
	·25	·6127					
	·5	·5					
(2)	0	·2644	·7933				
	·25	·2593	·7778				
	·5	·25	·75				
(3)	0	·1706	·5119	·8532			
	·25	·1693	·5078	·8463			
	·5	$·1\dot{6}$	·5	$·8\dot{3}$			
(4)	0	·1266	·3799	·6332	·8864		
	·25	·1261	·3782	·6304	·8826		
	·5	·125	·375	·625	·875		
(5)	0	·1008	·3025	·5041	·7058	·9074	
	·25	·1005	·3016	·5027	·7038	·9049	
	·5	·1	·3	·5	·7	·9	
(6)	0	·0838	·2514	·4190	·5867	·7543	·9219
	·25	·0836	·2509	·4182	·5855	·7528	·9201
	·5	$·08\dot{3}$	·25	$·41\dot{6}$	$·58\dot{3}$	·75	$·91\dot{6}$

Comparing the preceding table with Table II. it will be seen that the positions of the layer of given thickness when an alteration in m has most effect on the pitch are, with the exception of the first, only a very small distance outside of the corresponding node surfaces. The distances separating the two sets of surfaces become less and less the higher the note.

The maxima ordinates are all exactly equal. The exact expression for the maxima changes of pitch is

$$\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{m_1 - m}{m_1 + n}\right) = Q \quad \text{.....................................(73)}.$$

Their numerical values are thus given explicitly in Table III. by altering the heading in that table from Q to $\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{m_1 - m}{m_1 + n}\right)$.

§ 37. In the next case we are to consider when the layer differs from the remainder only in rigidity the change of frequency is given by (50).

This may be regarded as composed of two separate terms, one varying as the first, the other as the second power of $n_1 - n$. When the difference between the rigidities of the layer and the remainder is small the second term may be neglected, except for such values of b as make the first term nearly vanish. By supposing the difference of the rigidities sufficiently small we can indefinitely reduce the limits wherein the second term is comparable with the first. We shall thus for the sake of simplicity commence by supposing that $n_1 - n$ is very small and that the term in $(n_1 - n)^2$ is negligible.

The law of variation of $\partial k/k$ with the value of $k\alpha b$ is in this case given by (61) or (63) according as the layer is of given thickness or of given volume. The sign of $\partial k/(n_1 - n)$ is thus the same as that of the product of the functions $\psi_1(x)$ and $\psi_2(x)$ defined in (62).

The ordinate of curve A, fig. 3, is the quantity $x^{-2}\psi_1(x)\psi_2(x)$, or $f_{11}(x)$ of Table VII.; while the ordinate of curve B, fig. 3, is the quantity $\psi_1(x)\psi_2(x)$, or $f_{10}(x)$. Thus the ordinates of these curves are proportional to the changes of pitch when a small alteration in rigidity occurs throughout (1) a given volume, (2) a given thickness.

The sign of ∂k is the same as that of $n_1 - n$ or the opposite according as the ordinates of the curves are positive or negative. The zero ordinates have for their abscissae the roots of the two equations

$$\psi_1(x) = 0 \quad \text{...(74)},$$

$$\psi_2(x) = 0 \quad \text{...(75)}.$$

As x increases through a root of (74) the curves cross from the negative to the positive side of the axis of x, while as x increases through a root of (75) they cross from the positive to the negative side.

Comparing (62) with (20) we see that the roots of (74) are the values ascribed to $k\alpha a$ in Table I. for $\sigma = 0$, the corresponding positions of the layer being coincident with the loop surfaces. For the first two roots of (75), excluding zero, I find approximately $1{\cdot}694\pi$ and $2{\cdot}797\pi$.

If we denote by ${}_1x_i$ and ${}_2x_i$ the i^{th} roots excluding zero of (74) and (75) respectively, then it is easily proved that as i increases the roots ${}_1x_i$ and ${}_2x_{i-1}$ both continually approach $i\pi$. Also ${}_1x_i - {}_2x_{i-1}$ remains positive but continually diminishes as i increases. Thus the breadth of the segments which lie on the negative side of the axis becomes less and less the further they are from the origin, while the breadth of the positive segments approaches π.

For further information as to details the reader may consult the following table, remembering that the term in $(n_1 - n)^2$ is neglected in its conclusions.

TABLE XII.

Sign of $\partial k/(n_1 - n)$, and values of b/a for which its sign changes.

Number of note	$\partial k/(n_1-n)$	$= 0$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0
(1)	$\sigma = 0$	$b/a = 0$		1								
	$\sigma = {\cdot}25$	$b/a = 0$		·812		1						
	$\sigma = {\cdot}5$	$b/a = 0$		·663		1						
(2)	$\sigma = 0$	$b/a = 0$		·350		·896		1				
	$\sigma = {\cdot}25$	$b/a = 0$		·344		·878		·980		1		
	$\sigma = {\cdot}5$	$b/a = 0$		·331		·847		·945		1		
(3)	$\sigma = 0$	$b/a = 0$		·226		·578		·645		·955		1
	$\sigma = {\cdot}25$	$b/a = 0$		·224		·574		·640		·947		·992
	$\sigma = {\cdot}5$	$b/a = 0$		·221		·565		·630		·932		·977
(4)	$\sigma = 0$	$b/a = 0$		·168		·429		·479		·708		·742
	$\sigma = {\cdot}25$	$b/a = 0$		·167		·427		·477		·705		·739
	$\sigma = {\cdot}5$	$b/a = 0$		·166		·424		·473		·699		·733
(5)	$\sigma = 0$	$b/a = 0$		·134		·342		·381		·564		·591
	$\sigma = {\cdot}25$	$b/a = 0$		·133		·341		·380		·562		·589
	$\sigma = {\cdot}5$	$b/a = 0$		·132		·339		·378		·559		·586
(6)	$\sigma = 0$	$b/a = 0$		·111		·284		·317		·469		·491
	$\sigma = {\cdot}25$	$b/a = 0$		·111		·283		·316		·468		·490
	$\sigma = {\cdot}5$	$b/a = 0$		·110		·282		·315		·466		·488

For the fourth and higher notes the table is complete only for positions of the layer inside the third loop surface. The other positions of the layer in which $\partial k/(n_1 - n)$ vanishes in changing from negative to positive, being the same as the loop surfaces above the third, are given for notes (4)—(6) in Table II.

§ 38. For the numerical magnitudes of the changes of pitch we must separately consider the cases when the layer is of given volume and of given thickness. In the former case the curve A of fig. 3 applies. This curve has its largest maximum ordinate at the origin. The numerical magnitude of the first maximum change of pitch may be obtained from §§ 22 and 24. As explained there its values for the several notes and materials treated here may be found by dividing by 9 the values assigned to Q' in Table III. and equating the results to $-\frac{\partial k}{k} \div \left(\frac{n_1 - n}{m+n}\frac{\partial V}{V}\right)$.

Thus the change in pitch due to a given small alteration in n throughout a central layer is numerically equal to one-third the change in pitch due to an equal alteration in m throughout the same central layer. The fact that ∂k is opposite in sign to $n_1 - n$ is thus important practically as well as theoretically.

The abscissae answering to the subsequent maxima ordinates are the roots of a complicated equation. The approximate values of the first few roots can be seen from the figure or from Table VII. As regards the higher roots it is comparatively easy to prove that they split up into two sets, one set approaching the values $(2i+1)\pi/2$, the other set approaching $i\pi$, where i is an integer. Answering to the first set are those maxima for which $\partial k/(n_1 - n)$ is positive, to the second those maxima for which $\partial k/(n_1 - n)$ is negative. The number of negative maxima, including that for the central layer, is equal to the number of the note and exceeds by 1 the number of positive maxima.

It is not difficult to prove that the successive positive maxima ordinates vary approximately as the inverse squares of the corresponding abscissae, while the negative maxima ordinates after the first vary approximately as the inverse fourth powers of the abscissae. No great interest thus attaches to the numerical magnitudes of any but the first positive and negative maxima ordinates which can be approximately derived from the figure or from Table VII.

§ 39. When the layer whose rigidity suffers a given small alteration is of given thickness the variation of $\partial k/k$ with the value of $k\alpha b$ is shown by curve B of fig. 3. The equation determining the abscissae corresponding to the maxima ordinates is very complicated. It is, however, easily proved that there are two sets of roots, the higher roots of the first set being approximately odd multiples, and the higher roots of the second set approximately even multiples of $\pi/2$.

The first set supply the positive, the second the negative maxima ordinates. It is easily proved that the positive maxima changes of pitch which answer to those of the maxima ordinates which are most remote from the origin in the case of the higher notes are all approximately given by

$$\frac{\partial k}{k} \div \left(\frac{b-c}{a}\frac{n_1 - n}{m+n}\right) = Q \quad \text{...(76).}$$

They thus approach to equality amongst themselves and likewise to equality with the maxima of $\frac{\partial k}{k} \div \left(\frac{b-c}{a}\frac{m_1 - m}{m_1 + n}\right)$ in the same notes.

The positions of the layer answering to the $(i-1)^{\text{th}}$ positive maximum in the case of n altered, and to the i^{th} maximum in the case of m altered are also when i is large nearly identical.

The abscissa supplying the first and largest positive maximum ordinate is greater than π; thus the corresponding maximum change of pitch cannot apply to note (1). This ordinate is greater than the maxima ordinates of curve B, fig. 2, by fully 50 per cent. Thus the greatest possible change in the pitch of any note, except the first, due to a given small alteration of n throughout a layer of given thickness is fully 50 per cent. greater than the maximum change of pitch in the same note due to an equal alteration of m throughout a layer of equal thickness.

The abscissa answering to the first and largest maximum negative ordinate is approximately $\cdot 44\pi$, and the corresponding value of

$$-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{n_1 - n}{m+n}\right)$$

slightly exceeds $\cdot 29Q$.

This is a far from insignificant change of pitch, and it applies to all the notes in every material. In the case of note (1) it is the only *true* maximum there is, and when σ is small it is the numerically largest change of pitch which the given alteration of rigidity can produce. If, however, σ approach $\cdot 5$ an equal alteration of rigidity throughout a layer at or near the surface of the sphere is more effective in altering the pitch, and in this position the sign of ∂k is the same as that of $n_1 - n$.

The subsequent maxima negative ordinates rapidly diminish as the corresponding abscissae increase.

§ 40. We must next take into consideration the term in $(n_1 - n)^2$ in (50). Its contribution to the change of pitch is given, writing x for kab, by

$$\left.\begin{aligned} \frac{\partial k}{k} &= -\frac{t}{a}\frac{4(n_1-n)^2}{(m+n_1)(m+n)} Qf_4(x) \\ \text{or}\quad \frac{\partial k}{k} &= -\frac{\partial V}{V}\frac{4(n_1-n)^2}{(m+n_1)(m+n)}\frac{Q'}{3} f_5(x) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(77),$$

according as the layer is of given thickness or of given volume. The term in $(n_1 - n)^2$ indicates a fall in pitch whether the rigidity of the layer be increased or diminished. The curves

$$y = f_4(x), \text{ and } y = f_5(x)$$

are A and C of fig. 1 respectively, the former of which was discussed in § 33. The zero ordinates of both curves answer to positions of the layer coincident with the node surfaces. Of the maxima ordinates of curve A the first is much the largest. The corresponding contribution to the change of pitch in the present case may easily be calculated approximately from the curve and Table III. It is far from being insignificant compared to the contribution of the term in $n_1 - n$ when the alteration in rigidity is large. As the subsequent maxima ordinates of curve A, fig. 1, rapidly diminish as their abscissae increase, while the several maxima ordinates of curve B, fig. 3, remain large, it follows that for an alteration of rigidity throughout a layer of given thickness the relative importance of the term in $(n_1 - n)^2$ rapidly diminishes as the layer moves outwards from the first loop surface.

Exactly similar conclusions for the case when the layer is of given volume follow from a comparison of curve C, fig. 1, and curve A, fig. 3. Of the maxima ordinates of curve C, fig. 1, that at the origin is much the largest. In fact the second maximum is so small that I have not attempted to draw the curve further than the first zero ordinate.

§ 41. Our investigations show that for positions of the layer inside the first loop surface the term in $(n_1 - n)^2$ is in general far from negligible unless the alteration in rigidity be small; but that in the case of the higher notes for positions of the layer outside the first loop surface this term is in general comparatively insignificant even when the alteration in rigidity is large.

It must, however, be remembered that the term in $n_1 - n$ vanishes when the layer coincides with a loop surface, whereas the term in $(n_1 - n)^2$ has its maxima when the layer is at or very close to the loop surfaces. Thus, however small the alteration in rigidity may be, when it occurs in a layer immediately adjacent to a loop surface the term in $(n_1 - n)^2$ is the larger of the two.

We thus arrive at the following conclusions.

There are certain volumes within a sphere performing any given note where *any* alteration in rigidity[1] throughout a thin layer lowers the pitch. As the term in $(n_1 - n)^2$ varies as $(m + n_1)^{-1}$ the corresponding fall of pitch is greater when the rigidity is diminished than when it is increased.

The principal volumes of this kind are in the immediate neighbourhood of the loop surfaces L_1, L_2 .... There are, however, similar volumes in the neighbourhood of the surfaces S_1, S_2, etc. which answer to the roots of (75). The volumes surrounding two adjacent surfaces S_{i-1} and L_i may possibly in some cases when $n_1 - n$ is large become coterminous, but when $n_1 - n$ is small they are certainly separate. An alteration of rigidity throughout a layer within one of these volumes acts to some extent as what is frequently termed a *constraint*.

In general terms it may be said that the existence of the term in $(n_1 - n)^2$ extends the regions wherein an increase of rigidity lowers the pitch, and increases numerically this lowering of pitch. On the other hand it restricts the limits of the regions wherein a diminution of rigidity raises the pitch and reduces numerically this rise of pitch.

§ 42. In our last special case the change of pitch is given by (52). For the limiting value ·5 of σ this assumes the simple form

$$\frac{\partial k}{k} = \frac{t}{a} \frac{p}{1 + p} Q \sin^2 k\alpha b \qquad (78).$$

Now the coefficient of $p/(1 + p)$ in (78) is the same as that of $(m_1 - m)/(m_1 + n)$ in (49). Thus the curves of fig. 2 and the conclusions already come to in the case when m alone varies apply at once with merely a change in phraseology.

[1] i.e. any alteration of elasticity which leaves m unaltered.

Except in this extreme case the coefficient of p on the right-hand side of (52) is the *sum* of two squares.

Further as the equations $\sin x = 0$ and $\tan x = 0$ have no common root other than zero, the two squares cannot simultaneously vanish unless $b = 0$. Thus an alteration of both elastic constants in the same proportion necessarily affects the pitch unless it occur at the centre, and the pitch is raised or lowered according as the constants of the layer are increased or diminished.

It will also be seen from § 22 that when such an alteration of elasticity occurs throughout a core of given volume there is a change of pitch whose sign agrees with that of p. Thus the statement that the change of pitch is of the same sign as the alteration of elasticity is on the uniconstant theory *universally* correct as well as unambiguous.

§ 43. It will be convenient to suppose

$$\partial k = \partial k_1 + \partial k_2,$$

where

$$\frac{\partial k_1}{k} = \frac{t}{a} pQ \frac{4n(3m-n)}{(m+n)^2} \left\{ \frac{1}{kab} \left(\frac{\sin kab}{kab} - \cos kab \right) \right\}^2 \quad \ldots\ldots (79),$$

$$\frac{\partial k_2}{k} = \frac{t}{a} \frac{p}{1+p} Q \left(\frac{1}{kab} \right)^2 \left\{ kab \sin kab - \frac{4n}{m+n} \left(\frac{\sin kab}{kab} - \cos kab \right) \right\}^2 \quad \ldots\ldots (80).$$

The numerical magnitude of ∂k_1 is independent of the sign of p, whereas ∂k_2 is numerically greater for a given negative value of p than for an equal positive value.

Again ∂k_1 depends on the square of the displacement. It thus vanishes when the altered layer is at a node surface, and when the layer is of given thickness it has its maxima when the layer coincides with the loop surfaces. On the other hand ∂k_2 depends on the square of the radial stress. It thus vanishes when the altered layer is at a no-stress surface, and when the layer is of given volume it has its maxima when the layer coincides with those surfaces over which the radial stress is a maximum.

Again the law of variation of $\partial k_1/k$ with kab is wholly independent of the value of σ, but the absolute values of $\partial k_1/k$ diminish rapidly and become inconsiderable as σ approaches near the limiting value ·5. On the other hand the law of variation of $\partial k_2/k$ with kab varies with the value of σ, and this is very conspicuous in the case of the fundamental note, or so long as b/a is small in the case of the higher notes.

Perhaps the most important difference of all is that in the case of the higher notes when the layer, supposed of given thickness, travels outwards from the third node surface $\partial k_1/k$ becomes rapidly insignificant, whereas $\partial k_2/k$ has a succession of important maxima of nearly uniform magnitude and nearly independent of σ. By supposing the layer of given volume we should come to precisely the same conclusion as to the relative preponderance of ∂k_2 when the layer is outside the third node surface. An exception must of course be made of positions of the layer immediately adjacent to the no-stress surfaces where ∂k_2 vanishes.

§ 44. To obtain some idea of the numerical magnitude of the change of pitch we must consider separately the cases when the layer is of given volume and when it is of given thickness.

In the former case, with the usual limitation as to the centre, writing x for $k\alpha b$,

$$\frac{\partial k_1}{k} = p\,\frac{\partial V}{V}\frac{Q'}{3}\,\frac{4n\,(3m-n)}{(m+n)^2}\left\{\frac{1}{x^2}\left(\frac{\sin x}{x} - \cos x\right)\right\}^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(81).$$

The variation of $\partial k_1/k$ with $k\alpha b$ is thus shown by $f_5(x)$ as tabulated in Table VII., and by curve C, fig. 1, for values of x less than $3\pi/2$. This curve has by far its largest maximum ordinate at the origin. This ordinate is by no means insignificant. It has also in the present case to be multiplied by $4n\,(3m-n)\,(m+n)^{-2}$, a quantity which varies between 2 and 9/4 for values of σ less than $\dot{3}$. Thus the corresponding change of pitch is of considerable importance in ordinary isotropic materials. So long in fact as x is less than π the ordinates of curve C, fig. 1, are fairly comparable with the ordinates of the other curves which apply when the layer is of constant volume.

For positions of the layer, however, answering to points beyond the first zero ordinate of curve C, fig. 1, ∂k_1 is always extremely small. It is in fact easily proved that the second maximum ordinate is less than 1/134 of that at the origin.

Still supposing the layer of given volume, we have with the usual limitation, writing x for $k\alpha b$,

$$\frac{\partial k_2}{k} = \frac{p}{1+p}\,\frac{\partial V}{V}\,\frac{Q'}{3}\left[\frac{1}{x^2}\left\{\sin x - \frac{4n}{m+n}\,\frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right)\right\}^2\right] \quad\ldots\ldots\ldots\ldots\ldots(82).$$

The function of x inside the square bracket reduces when $\sigma = \cdot 5$ to $x^{-2}\sin^2 x$, the quantity appearing as $f_2(x)$ in Table VII., and represented by curve A, fig. 2. This curve has been already exhaustively considered. The function is also tabulated for the values 0 and ·25 of σ in Table VII. under the headings $f_7(x)$ and $f_9(x)$ respectively. The corresponding curves are A_0 and $A_{\cdot 25}$ of fig. 4.

The differences between the three curves last mentioned are very conspicuous near the origin.

For small values of x the ordinates of curve C, fig. 1, are comparable with the ordinates of the curves mentioned above. Thus in comparing the changes of pitch due to a given percentage alteration of elasticity for different values of σ we must, at least when the altered layer is inside the second node surface, construct compound curves of the kind mentioned in § 30.

The compound curves showing the variation with $k\alpha b$ of

$$\frac{\partial k}{k} \div \left(\frac{p}{1+p}\,\frac{\partial V}{V}\,\frac{Q'}{3}\right)$$

are found as follows:—

when $\sigma = 0$, multiply the ordinate of curve C, fig. 1, by $2(1+p)$, and add it to the ordinate of curve A_0, fig. 4.

when $\sigma = \cdot 25$, multiply the ordinate of curve C, fig. 1, by $20(1+p)/9$, and add it to the ordinate of curve $A_{\cdot 25}$, fig. 4.

when $\sigma = \cdot 5$ there is the simple curve A, fig. 2.

When the alteration in elasticity is small we may neglect p in forming the compound curves, i.e. replace $1+p$ by 1 simply.

In deducing the numerical value of $\partial k/k$ for a given value of p the ordinate of the corresponding compound curve must be multiplied by that value of $Q'/3$ which applies to the note and material under investigation.

Since the largest maximum ordinate in all the compound curves occurs at the origin, it will be found simplest when the greatest possible change of pitch alone is wanted to apply at once the result obtained in § 24, replacing ∂E_l by

$$\frac{\partial V}{V} \frac{p}{1+p} \frac{3m-n}{3(m+n)} \left\{1 + \frac{4}{3} p \frac{n}{m+n}\right\}.$$

§ 45. The three curves A, fig. 2, A_0 and $A_{\cdot 25}$, fig. 4, become extremely similar when x is large.

The equation for the abscissae supplying the maxima ordinates in these curves is

$$\frac{\sin x}{x} - \cos x + \frac{2(1-2\sigma)}{1-\sigma} \frac{1}{x^2} \left\{x \sin x - 3\left(\frac{\sin x}{x} - \cos x\right)\right\} = 0 \ldots\ldots\ldots\ldots\ldots(83).$$

For $\sigma = \cdot 5$ the roots of (83) are identical with those of $\tan x = x$, and for all other values of σ the higher roots of (83) though less than the roots of $\tan x = x$ are very nearly equal to them.

Thus the more remote positions of the layer answering to the maxima values of $\partial k_2/k$ in the case of the higher notes lie close inside the successive node surfaces, except for the limiting value $\cdot 5$ of σ when they exactly coincide with the node surfaces.

The first root of (83) other than zero varies from $1\cdot 232\pi$ when $\sigma = 0$ to $1\cdot 430\pi$ when $\sigma = \cdot 5$. Thus the position of that maximum ordinate which lies between the first and second zero ordinates varies to an appreciable extent with the value of σ.

There is also an appreciable difference in the lengths of this ordinate in the three curves, these lengths unlike those of the ordinates at the origin increasing as σ diminishes. Beyond the second zero ordinates the curves would lie very close together, so in fig. 4, curve $A_{\cdot 25}$ stops at this point.

For values of x exceeding π, $\partial k_1/\partial k_2$ is very small except for such positions of the layer as make ∂k_2 insignificant. Thus for practical purposes the dependence of $\partial k/k$ on the position of the layer, when close to or outside of the second node surface, is approximately given for the values 0, $\cdot 25$ and $\cdot 5$ of σ by the curves A_0, $A_{\cdot 25}$ of fig. 4 or A, fig. 2, alone.

Except in the case of the first one or two maxima no serious error will be introduced by supposing the positions of the layer which supply the maxima changes of pitch to coincide exactly with the node surfaces.

These maxima are also approximately given by the formula which in strictness applies only when $\sigma = \cdot 5$, viz.

$$\frac{\partial k}{k} = \frac{p}{1+p} \frac{\partial V}{V} \frac{i^2\pi^2}{3(1+x^2)} \quad \text{.................................}(84).$$

Here i is the number of the note and x is that root of (36) answering to the particular node surface, at or close to which the layer is found.

§ 46. We shall next suppose that the layer is of given thickness. We may regard ∂k as consisting of two terms given by (79) and (80). Of these the variation of ∂k_1 with κab is shown by curve A, fig. 1, while the variation of ∂k_2 is shown for the values 0, $\cdot 25$ and $\cdot 5$ of σ by B_0, $B_{\cdot 25}$, fig. 4, and B, fig. 2.

It is obvious from these curves that for values of x exceeding π, ∂k_1 is small compared to ∂k_2, except very near the vanishing positions of the latter quantity, and the value of ∂k_2 depends but little on the value of σ.

The exact positions of the layer supplying the maxima changes of pitch in the limiting case represented by curve B, fig. 2, are the positions given in Table XI. for $\sigma = \cdot 5$. In this case all the maxima for any given note are equal, and their numerical values are obtained at once from the formula

$$\frac{\partial k}{k} = \frac{t}{a} \frac{p}{1+p}.$$

In the third segments there is a difference only of something like 1 per cent. between the lengths of the maxima ordinates of the curves B_0, $B_{\cdot 25}$, fig. 4, and B, fig. 2. Also these maxima are near the zero ordinates of curve A, fig. 1, representing the variation in ∂k_1. Thus by altering the heading of Table III. from Q to $\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{p}{1+p}\right)$ and rejecting the numbers given for notes (1) and (2), we obtain what are extremely good approximations for the third and subsequent maxima, for the values 0 and $\cdot 25$ of σ as well as for $\sigma = \cdot 5$. Even in the case of the second maxima the magnitudes derived from this use of Table III. would not be seriously in error.

When κab is small the dependence of the law of variation of $\partial k_2/k$ on the value of σ is so conspicuous in the figures that further comment is hardly necessary.

§ 47. For even a rough approximation to the change of pitch when the layer is inside or but little outside of the first loop surface we must construct compound curves for the values 0 and $\cdot 25$ of σ. These are formed by combining curve A, fig. 1, with the curves B_0 and $B_{\cdot 25}$ of fig. 4, in precisely the same way as the compound curves in the case of a layer of constant volume were formed by combining curve C, fig. 1, with the curves A_0 and $A_{\cdot 25}$ of fig. 4.

If we suppose p very small the greatest ordinate that either of the compound curves supplies for values of x less than π is very considerably less than 1, which is the approximate value of the subsequent maxima ordinates. Thus for a small alteration of elasticity there is in the case of the higher notes no position of the layer inside of or close to the first loop surface which can produce as great a change of pitch as the positions near the second and subsequent node surfaces. For note (1) however none of the maxima answering to positions near the node surfaces apply.

For $\sigma = 0$, x/π must be less than ·6626 to apply to note (1). Now it is easily found that when p is neglected in the equation, the compound curve for $\sigma = 0$ runs very nearly parallel to the axis of x between the values $·6\pi$ and $·6626\pi$ of x. The corresponding ordinate is approximately ·381, and is greater than any ordinate answering to a smaller value of x.

Also for $\sigma = 0$ the value of Q in note (1) is 2·253. Thus the maximum change of pitch due to a very small alteration of elasticity, in a layer of given thickness, in the case of note (1) for $\sigma = 0$ is approximately given by

$$\frac{\partial k}{k} \div \frac{t}{a} p = ·86.$$

The corresponding position of the layer is at or close to the surface of the sphere. This result is in accordance with Table V.

For $\sigma = ·25$ the compound curve when p is neglected in its equation has a *true* maximum ordinate for a value of x answering to a position of the layer at some distance inside the first loop surface. The length of the ordinate is ·58 roughly. Thus as Q when $\sigma = ·25$ has the value 1·369 for note (1), it follows that the maximum change of pitch in this case for a very small alteration in elasticity throughout a thin layer is approximately given by

$$\frac{\partial k}{k} \div \frac{t}{a} p = ·79.$$

The greatest possible percentage change of pitch in note (1) for given values of p and t is thus less when σ equals ·25 than when it equals 0 or ·5.

When p is large the form of the compound curve near the origin will vary widely from the form it takes when p is small. When p is positive the compound curve is the more influenced by the form of curve A, fig. 1, the larger p is, whereas when p is negative the influence of this curve continually diminishes as p increases numerically.

§ 48. In the case of the higher notes a pretty close approximation to the change of pitch due to any alteration solely in elasticity, occurring in a layer outside the third or fourth node surface and not in the immediate neighbourhood of a no-stress surface, is easily obtained by the following considerations.

Comparing (26) and (27) we see that when kab is large u_b and U_b except when negligible may be replaced respectively by

$$u_b = -b^{-1} \cos kab, \quad U_b = b^{-2}(m+n)\, kab \sin kab.$$

Thus, noticing (25), we see that when the elasticity alone is altered the terms in $(u_b)^2$ and u_bU_b in (28) may in general be neglected when kab is large, and that an approximate expression for the change of pitch is then

$$\frac{\partial k}{k} = \frac{b-c}{a}\left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)(m+n)\,Q\sin^2 kab \text{.....................(85).}$$

Near the no-stress surfaces the terms in $(u_b)^2$ and u_bU_b cease to be small compared to the term in $(U_b)^2$, but their greatest values being small compared to those of the latter term, this limitation to the applicability of (85) is not of much practical importance.

We thus see that in the case of the higher notes when the alteration of elasticity occurs outside of the third or fourth node surface the change of pitch, when of practical importance, may be regarded as depending mainly on the alteration of only one elastic quantity, viz. $m+n$.

It will be remembered that when a small alteration of elasticity occurs near the centre the change of pitch may be regarded as arising from the alteration in the single elastic quantity $m-n/3$; and in the case of note (1), for a surface alteration of material, there is for ordinary values of σ a not inconsiderable change of pitch depending on the alteration of the single elastic quantity $n(3m-n)/(m+n)$.

It thus appears that in any purely verbal explanation of the phenomena such terms as *stiffness* or *elasticity* would require to be used in a very elastic sense.

Note. August 7, 1891.

[When the rigidity is altered while the bulk modulus $m-n/3$, and so the compressibility, is unaltered, the change of pitch is given, writing x for kab, by

$$\partial k/k = \frac{t}{a}\,\frac{n_1-n}{m_1+n_1}\,Q\tfrac{4}{3}f_{12}(x) = \frac{\partial V}{V}\,\frac{n_1-n}{m_1+n_1}\,\frac{Q'}{3}\,\tfrac{4}{3}f_{13}(x)\,;$$

where, as in Table VII., $f_{12}(x) = x^2 f_{13}(x) = \{\sin x - 3x^{-1}(x^{-1}\sin x - \cos x)\}^2$.

So in this case the change of pitch is always of the same sign as the alteration of rigidity.

The variation of $\partial k/k$ with the position of the altered layer is shown by A or by B, fig. 5, according as the layer is of given volume or given thickness. For comparison with the effects of other alterations of material the ordinates of these curves should be increased in the ratio 4 : 3. When so increased the first maximum ordinate of B is the largest ordinate in any of the curves. It answers to an abscissa of $1{\cdot}24\pi$ approximately, and so never applies to note (1). The extremely flat character of these curves near the origin calls for special notice.]

SECTION III.

TRANSVERSE VIBRATIONS IN SOLID SPHERE.

§ 49. By transverse vibrations are here meant vibrations in which there is no radial displacement.

Let ρ be the density, n the rigidity, of an isotropic material, and

$$\beta^2 = \rho/n \qquad (1).$$

Also let $J_{i+\frac{1}{2}}(x)$, $J_{-(i+\frac{1}{2})}(x)$ represent the two solutions of the Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y\left\{1 - x^{-2}(i+\tfrac{1}{2})^2\right\} = 0 \qquad (2),$$

where i is a positive integer.

Then the types of the displacements v and w, respectively in and perpendicular to the meridian plane—or plane containing the line $\theta = 0$—in a transverse vibration of frequency $k/2\pi$ in a simple shell are*

$$v = \cos kt\, r^{-\frac{1}{2}} (\sin\theta)^{-1} \{X_i J_{i+\frac{1}{2}}(k\beta r) + X'_i J_{-(i+\frac{1}{2})}(k\beta r)\} \qquad (3),$$

$$w = \cos kt\, r^{-\frac{1}{2}} \{w_i J_{i+\frac{1}{2}}(k\beta r) + w'_i J_{-(i+\frac{1}{2})}(k\beta r)\} \qquad (4).$$

Here X_i, X'_i are surface spherical harmonics of degree i, while w_i, w'_i are quantities connected with them by the relations

$$\frac{dw_i}{d\phi} = -\frac{dX_i}{d\theta}, \qquad \frac{dw'_i}{d\phi} = -\frac{dX'_i}{d\theta} \qquad (5).$$

The spherical harmonics X_i, X'_i must be of such a type that v is nowhere infinite, and so—at least for a complete shell—must contain $\sin\theta$ raised to some positive power.

Under (3) and (4) we may suppose included the type of vibration

$$\left.\begin{array}{l} v = 0, \\ w = \cos kt\, r^{-\frac{1}{2}} \{w_0 \sin\theta J_{\frac{3}{2}}(k\beta r) + w'_0 \sin\theta J_{-\frac{3}{2}}(k\beta r)\} \end{array}\right\} \qquad (6);$$

in which w_0 and w'_0 are constants, and so $w_0 \sin\theta$ and $w'_0 \sin\theta$ may be regarded as equivalent to the quantities w_1 and w'_1 satisfying (5). This special form of vibration will here be spoken of as the *rotatory*, this term being applied to it by Professor Lamb†.

At the spherical surface separating two isotropic media there are in this case nominally four surface conditions, viz. the equality in the two media of the two displacement components v and w, and likewise of the two stress components $n\left(\frac{dv}{dr} - \frac{v}{r}\right)$ and $n\left(\frac{dw}{dr} - \frac{w}{r}\right)$.

In consequence however of the relation (5) these constitute in reality only two independent equations.

* See *Camb. Phil. Transactions*, Vol. XIV. p. 319, equations (34′) and (35′).

† *Proceedings of the London Mathematical Society*, Vol. XIII. p. 196.

A moment's consideration will also show that the X_i and the X'_i in the v displacement in any layer of a compound solid sphere must be the same function of θ and ϕ, and that this function must be the same for all the other layers and for the core. We may thus represent the w displacements in the typical vibration of frequency $k/2\pi$ in the compound sphere $(0 . \beta . c . \beta_1 . b . \beta . a)$ as follows:

In the core $\quad wr^{\frac{1}{2}}/w_i \cos kt = A_i J_{i+\frac{1}{2}}(k\beta r)$ ....(7).

In the layer $\quad wr^{\frac{1}{2}}/w_i \cos kt = {}_1A_i J_{i+\frac{1}{2}}(k\beta_1 r) + {}_1B_i J_{-(i+\frac{1}{2})}(k\beta_1 r)$ ....(8).

Outside the layer

$$wr^{\frac{1}{2}}/w_i \cos kt = (A_i + \partial A_i) J_{i+\frac{1}{2}}(k\beta r) + \partial B_i J_{-(i+\frac{1}{2})}(k\beta r) \quad\text{..................(9).}$$

Here A_i, ${}_1A_i$, etc. are constants whose relationships are determined by the surface conditions, and w_i is a certain function of θ and ϕ. If we suppose $b-c$ small then $\partial A_i/A_i$ and $\partial B_i/A_i$ are of the order $b-c$ of small quantities, and their squares are negligible when that of $b-c$ is neglected.

It is unnecessary to write down the expressions for the v components in the several media as they lead to precisely the same conditions at the surfaces as the w components.

§ 50. Let us for shortness put

$$J'_{i+\frac{1}{2}}(k\beta r) = \frac{1}{k\beta}\frac{d}{dr} J_{i+\frac{1}{2}}(k\beta r), \quad J'_{-(i+\frac{1}{2})}(k\beta r) = \frac{1}{k\beta}\frac{d}{dr} J_{-(i+\frac{1}{2})}(k\beta r),$$

$$\left.\begin{aligned} F(r . \beta) &= n\{k\beta r J'_{i+\frac{1}{2}}(k\beta r) - \tfrac{3}{2}J_{i+\frac{1}{2}}(k\beta r)\}, \\ F_1(r . \beta) &= n\{k\beta r J'_{-(i+\frac{1}{2})}(k\beta r) - \tfrac{3}{2}J_{-(i+\frac{1}{2})}(k\beta r)\} \end{aligned}\right\} \quad\text{....................(10).}$$

Then we find from the surface conditions

$$\left.\begin{aligned} A_i J_{i+\frac{1}{2}}(k\beta c) &= {}_1A_i J_{i+\frac{1}{2}}(k\beta_1 c) + {}_1B_i J_{-(i+\frac{1}{2})}(k\beta_1 c), \\ A_i F(c . \beta) &= {}_1A_i F(c . \beta_1) + {}_1B_i F_1(c . \beta_1), \\ (A_i + \partial A_i) J_{i+\frac{1}{2}}(k\beta b) + \partial B_i J_{-(i+\frac{1}{2})}(k\beta b) &= {}_1A_i J_{i+\frac{1}{2}}(k\beta_1 b) + {}_1B_i J_{-(i+\frac{1}{2})}(k\beta_1 b), \\ (A_i + \partial A_i) F(b . \beta) + \partial B_i F_1(b . \beta) &= {}_1A_i F(b . \beta_1) + {}_1B_i F_1(b . \beta_1), \\ (A_i + \partial A_i) F(a . \beta) + \partial B_i F_1(a . \beta) &= 0 \end{aligned}\right\} \quad\text{....(11).}$$

Treating the first four of these equations in the usual manner, and putting

$$\Delta(b . \beta . b') = J_{i+\frac{1}{2}}(k\beta b) J'_{-(i+\frac{1}{2})}(k\beta b) - J'_{i+\frac{1}{2}}(k\beta b) J_{-(i+\frac{1}{2})}(k\beta b) \quad\text{............(12),}$$

we find

$$\frac{\partial A_i}{A_i} nk\beta b \Delta(b . \beta . b') \div \frac{b-c}{b}$$

$$= -\{nk^2\beta^2b^2 - n_1k^2\beta_1^2b^2 + (n_1 - n)(i-1)(i+2)\} J_{i+\frac{1}{2}}(k\beta b) J_{-(i+\frac{1}{2})}(k\beta b)$$

$$-\left(\frac{1}{n} - \frac{1}{n_1}\right) n\{k\beta b J'_{i+\frac{1}{2}}(k\beta b) - \tfrac{3}{2}J_{i+\frac{1}{2}}(k\beta b)\}\, n\{k\beta b J'_{-(i+\frac{1}{2})}(k\beta b) - \tfrac{3}{2}J_{-(i+\frac{1}{2})}(k\beta b)\} \quad\text{...(13),}$$

$$\frac{\partial B_i}{A_i} nk\beta b \Delta(b . \beta . b') \div \frac{b-c}{b} = \{nk^2\beta^2b^2 - n_1k^2\beta_1^2b^2 + (n_1 - n)(i-1)(i+2)\}\{J_{i+\frac{1}{2}}(k\beta b)\}^2$$

$$+\left(\frac{1}{n} - \frac{1}{n_1}\right)[n\{k\beta b J'_{i+\frac{1}{2}}(k\beta b) - \tfrac{3}{2}J_{i+\frac{1}{2}}(k\beta b)\}]^2 \quad\text{........................(14).}$$

For the frequency equation of a simple sphere we find from the last of equations (11), putting $\partial B_i = 0$,

$$f(0 \,.\, \beta \,.\, a) \equiv k\beta a J'_{i+\frac{1}{2}}(k\beta a) - \tfrac{3}{2} J_{i+\frac{1}{2}}(k\beta a) = 0 \quad \text{.....................(15)*.}$$

From the properties of the Bessel's function

$$-k\beta a \frac{d}{k\beta da} f(0 \,.\, \beta \,.\, a) = \{k^2\beta^2 a^2 - (i + \tfrac{1}{2})^2\} J_{i+\frac{1}{2}}(k\beta a) + \tfrac{3}{2} k\beta a J'_{i+\frac{1}{2}}(k\beta a) \quad \text{........(16).}$$

Supposing (15) to hold we may reduce (16) to

$$-k\beta a \frac{d}{k\beta da} f(0 \,.\, \beta \,.\, a) = \{k^2\beta^2 a^2 - (i-1)(i+2)\} J_{i+\frac{1}{2}}(k\beta a) \quad \text{............(17)†.}$$

Supposing (15) to hold we also obtain

$$F_1(a \,.\, \beta) = nk\beta a \Delta (a \,.\, \beta \,.\, a') \div J_{i+\frac{1}{2}}(k\beta a).$$

Thus, following the same train of reasoning as in Sect. II., we conclude that if ∂k be the increase in k due to the existence of the layer, the two following equations must be identical—

$$f(0 \,.\, \beta \,.\, a) + \frac{\partial k}{k} \{k^2\beta^2 a^2 - (i-1)(i+2)\} J_{i+\frac{1}{2}}(k\beta a) = 0,$$

$$f(0 \,.\, \beta \,.\, a) + \frac{\partial B_i}{A_i} k\beta a \Delta (a \,.\, \beta \,.\, a') \div J_{i+\frac{1}{2}}(k\beta a) = 0.$$

Thence we find for the change of frequency

$$\frac{\partial k}{k} = \frac{\partial B_i}{A_i} \frac{k\beta a \Delta (a \,.\, \beta \,.\, a') \{J_{i+\frac{1}{2}}(k\beta a)\}^{-2}}{k^2\beta^2 a^2 - (i-1)(i+2)} \quad \text{..........................(18).}$$

Let $\frac{1}{2\pi} K_{(\beta \,.\, a)}$ denote the frequency of the free transverse vibration of the type (3) and (4) in an infinitely thin spherical shell of material $(\rho \,.\, n)$ and radius a; then

$$K^2_{(\beta \,.\, a)} a^2 = (i-1)(i+2)\beta^{-2} = (i-1)(i+2) n/\rho \quad \text{.....................(19)§.}$$

Also let

$$\left. \begin{aligned} w_r &= r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r), \\ W_r &= nr^{-\frac{3}{2}} \{k\beta r J'_{i+\frac{1}{2}}(k\beta r) - \tfrac{3}{2} J_{i+\frac{1}{2}}(k\beta r)\} \end{aligned} \right\} \quad \text{.........................(20);}$$

so that $w_r w_i \cos kt$ represents a w displacement in a simple sphere performing a transverse vibration of frequency $k/2\pi$, and $W_r w_i \cos kt$ the corresponding stress, both quantities referring to points at a distance r from the centre, and w_i being a function derivable from a surface spherical harmonic of degree i through the relation (5).

Employing this notation, introducing in (18) the value of $\partial B_i/A_i$ from (14), and noticing that

$$k\beta b \Delta (b \,.\, \beta \,.\, b') = k\beta a \Delta (a \,.\, \beta \,.\, a') = -C \quad \text{.........................(21),}$$

* Cf. *Transactions*, Vol. XIV. p. 316, equation (47 a). † See Sect. I. § 10.

§ Cf. *Transactions*, Vol. XIV. p. 320, equation (59), and (17) above.

where C is an absolute constant, we finally obtain

$$\frac{\partial k}{k} = \frac{(b-c)}{a}\frac{b^2}{a^2}\frac{1}{\rho(k^2-K^2_{(\beta.a)})}\left[\{\rho(k^2-K^2_{(\beta.b)})-\rho_1(k^2-K^2_{(\beta_1.b)})\}\left(\frac{w_b}{w_a}\right)^2+\left(\frac{1}{n}-\frac{1}{n_1}\right)\left(\frac{W_b}{w_a}\right)^2\right] \quad \text{.....................(22).}$$

This may be applied with the same limitation as in Sect. II. to the case of a *central layer.*

§ 51. Inside the layer there is no change of type other than a shifting of all the node, loop and no-stress surfaces according to the law

$$-\partial r/r = \partial k/k \quad \text{...(23).}$$

Outside the layer we find on substituting in (9) the values of $\partial A_i/A_i$ and $\partial B_i/A_i$ from (13) and (14) and reducing,

$$w/A_iw_i\cos kt = r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)$$
$$+\frac{b-c}{nC}\left(\frac{b}{r}\right)^{\frac{1}{2}}\left[\{\rho(k^2-K^2_{(\beta.b)})-\rho_1(k^2-K^2_{(\beta_1.b)})\}\,bw_bf(b.\beta.\bar{r})\right.$$
$$\left.+\left(\frac{1}{n}-\frac{1}{n_1}\right)W_bf(b.\beta.\bar{r})\right] \quad \text{....................(24);}$$

where

$$\left.\begin{aligned} f(\bar{b}.\beta.r) &= J_{i+\frac{1}{2}}(k\beta r)J_{-(i+\frac{1}{2})}(k\beta b)-J_{-(i+\frac{1}{2})}(k\beta r)J_{i+\frac{1}{2}}(k\beta b),\\ f(\bar{b}.\beta.\bar{r}) &= J_{i+\frac{1}{2}}(k\beta r)F_1(b.\beta)-J_{-(i+\frac{1}{2})}(k\beta r)F(b.\beta)\end{aligned}\right\} \quad \text{..................(25).}$$

The functions f have their usual meaning.

In the layer itself the displacement is given by

$$w/A_iw_i\cos kt = b^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta b)-k\beta(b-r)\{b^{-\frac{1}{2}}J'_{i+\frac{1}{2}}(k\beta b)-(2k\beta b^{\frac{3}{2}})^{-1}J_{i+\frac{1}{2}}(k\beta b)\}$$
$$-(r-c)\left(\frac{1}{n}-\frac{1}{n_1}\right)W_b \quad \text{................(26).}$$

The *change of type* outside the layer, i.e. the coefficient of $b-c$ in (24), consists like the expression (22) for the change of pitch of two terms only. The first terms in each alone exist when the layer differs from the remainder only in density, and they vanish when the layer coincides with a node surface. The second terms vanish when the layer coincides with a no-stress surface. In the special case of the *rotatory* vibrations the second terms alone exist when the layer differs from the remainder only in rigidity.

In the layer itself the change of type is given by the last term of (26). Thus if there be an alteration only in density, or an alteration of rigidity occurring at a no-stress surface, then no progressive change of type appears as we cross the layer; in other words the layer vibrates as if it were of the same structure as the core.

§ 52. Before discussing (22) it is desirable to trace the characteristic features of the transverse vibrations of a simple sphere. The type of such vibrations is given by (3) and (4) with $X'_i = 0 = w'_i$, and the corresponding frequency equation by (15).

If i be a large integer X_i may be any one of a large number of spherical harmonics, but (15) depends solely on i, on the radius of the sphere, and on the material. There may thus be a large number of different forms of vibration which have all the same frequency equation.

The displacements vary, unless $i=1$, with θ and ϕ as well as with r. Thus there is a conical surface, or a series of surfaces, given by

$$X_i = 0 \qquad (27),$$

over which the component of the displacement in the meridian plane vanishes. Similarly there is a conical surface, or series of surfaces, given by

$$w_i = 0 \qquad (28),$$

over which the component at right angles to the meridian plane vanishes. A line of intersection of (27) and (28) is a locus where the resultant displacement is always zero.

While the title node surface might legitimately be applied to the lines or conical surfaces which are the intersection of (27) and (28), it will here be understood to apply solely to the spherical surfaces over which the displacement vanishes. Such surfaces we see from (3) and (4), putting $X'_i = 0$, are obtained by equating $k\beta r$ to the successive roots of

$$J_{i+\frac{1}{2}}(x) = 0 \qquad (29).$$

Thus for a given sphere the positions of these surfaces depend solely on the number i of the spherical harmonic X_i, and in no respect on its form.

In like manner there are spherical loop surfaces, obtained by equating $k\beta r$ to the successive roots of

$$J'_{i+\frac{1}{2}}(x) - \frac{1}{2x} J_{i+\frac{1}{2}}(x) = 0 \qquad (30),$$

where the displacement regarded as a function *solely* of r is numerically a maximum. There are also spherical no-stress surfaces, obtained by equating $k\beta r$ to the successive roots of

$$J'_{i+\frac{1}{2}}(x) - \frac{3}{2x} J_{i+\frac{1}{2}}(x) = 0 \qquad (31),$$

at every point of which the transverse stress is zero.

In a given sphere the radii of the several loop and no-stress surfaces depend, like those of the node surfaces, entirely on the number i, and in no respect on the form of the spherical harmonic X_i.

The above equation (31) is of course identical with (15), but for certain purposes its present form is more useful.

§ 53. Since the equations (29), (30) and (31) do not contain ρ or n it follows that the nature of the material, supposed of course isotropic, has no effect on the ratios of the frequencies of the several notes answering to a given value of i, or on the mutual ratios of the radii of the node, loop, or no-stress surfaces of given number, or on the ratios of these radii to the radius of the sphere.

As regards the form of the Bessel's function $J_{i+\frac{1}{2}}(x)$ we know that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \text{.......................................(32)*},$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right\} \text{ ....(33)*};$$

and between any three consecutive functions there subsists the well-known relation

$$(2i+1)J_{i+\frac{1}{2}}(x) = x\{J_{i-\frac{1}{2}}(x) + J_{i+\frac{3}{2}}(x)\} \text{..............................(34)}.$$

If the value of x be large a close approximation to the value of these functions is supplied by

$$-J_{i+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin\left(\frac{i\pi}{2} - x\right) \text{.................................(35)†}.$$

From (35) we see that the higher roots of (29) are given approximately, j denoting a positive integer, by

$$x = (2j+1)\pi/2 \text{..(36)},$$

or

$$x = j\pi \text{...(37)},$$

according as i is odd or even.

Again, the roots of both (30) and (31) obviously approach more and more nearly the higher they are to the corresponding roots of

$$J'_{i+\frac{1}{2}}(x) = 0 \text{...(38)},$$

and from (35) it is easily seen that the higher roots of (38) lie approximately midway between consecutive roots of (29). Thus the higher roots of both (30) and (31) are more nearly given the higher they are by

$$x = j\pi,$$

or

$$x = (2j+1)\pi/2,$$

according as i is odd or even.

Again, from (35) it follows that those maxima values of $x\{J_{i+\frac{1}{2}}(x)\}^2$ which answer to large values of x are all approximately equal $2/\pi$, and that the corresponding values of x are approximately given by (37) or by (36) according as i is odd or even. In like manner we conclude that the maxima values of $\{J_{i+\frac{1}{2}}(x)\}^2$ which answer to large values of x vary approximately inversely as the corresponding values of x, and these values of x are likewise given approximately by (37) or by (36) according as i is odd or even.

* Lommel, *Studien über die Bessel'schen Functionen*, p. 118.

† See Todhunter's *Functions of Laplace, Lamé, and Bessel*, Arts. 406—7, especially equation (9), p. 313.

§ 54. From the data obtained for the approximate positions of the roots of the equations (29), (30) and (31) we may draw the following conclusions:—

The pitch of the higher notes in a given sphere answering to any given value of i increases approximately in an arithmetical progression with the number of the note. In any one of these higher notes the corresponding no-stress and loop surfaces of higher number lie very close to one another, and are very nearly midway between successive node surfaces. The radii of successive higher surfaces of the same kind, whether node, loop or no-stress surfaces, increase very approximately in arithmetical progression.

§ 55. Before discussing the general application of (22) it will be convenient to consider the special cases when the change of material occurs at or close to the centre, and when it occurs at the surface.

Supposing first the change of material to take place throughout a *central layer*, we require to find the dimensions of the lowest powers of b occurring in (22).

Employing the ordinary formula for the Bessel in ascending powers of the variable, we see that when b is very small the most important terms in the coefficients of $\rho_1 - \rho$ and $n_1 - n$ respectively in (22) are of orders $(b-c)\,b^{2i+2}a^{-(2i+3)}$ and $(b-c)\,b^{2i}a^{-(2i+1)}$. Also $(i-1)$ occurs as a factor of $n_1 - n$. Thus even when $i = 1$, $(\partial k/k) \div (\partial V/V)$ is of the order $(b/a)^2$ of small quantities. Thus to the present degree of approximation no alteration of material whatever, occurring throughout a central layer whether of given thickness or given volume, has any effect on the pitch of any note of any transverse type.

Working out independently the case when the material (ρ_1, n_1) forms a true core, I come to exactly the same conclusion.

Next, making $b = a$ in (22) we obtain the change of pitch due to an alteration of material throughout a surface layer. Putting $b - c = t$, and remembering that $W_a = 0$ for a simple sphere, we find

$$\frac{\partial k}{k} = -\frac{t}{a}\,\frac{k^2\beta^2 a^2 \dfrac{\rho_1 - \rho}{\rho} - (i-1)(i+2)\dfrac{n_1 - n}{n}}{k^2\beta^2 a^2 - (i-1)(i+2)} \qquad \text{.........................(39).}$$

When $i = 1$ the change in frequency depends solely on the alteration of density. For other values of i it may be regarded as composed of two terms, the first giving the effect of a surface alteration of density, the second of a surface alteration of rigidity. The denominator in (39) is essentially positive; thus the pitch is lowered when the density at the surface is increased, and raised, except in the rotatory vibrations, when the rigidity is increased.

Since the values of $k\beta a$ supplied by the frequency equation are the same for all isotropic materials, it follows that the percentage change of pitch due to a given surface alteration of density is quite independent of the rigidity; and similarly the percentage change of pitch due to a given surface alteration of rigidity is independent of the density.

Putting

$$\rho_1/\rho - 1 = q \quad\text{...(40)},$$

$$n_1/n - 1 = p \quad\text{...(41)},$$

we find from (39) when $q = p$

$$\frac{\partial k}{k} = -\frac{t}{a} p \quad\text{...(42)}.$$

In the fundamental note answering to any given value of i greater than 1, the effect on the pitch of equal percentage alterations in the density and in the rigidity are fairly comparable. The higher however the number of the note the smaller is the relative importance of the alteration of rigidity, and the more nearly is the change of pitch given by

$$\frac{\partial k}{k} = -\frac{t}{a}\frac{\rho_1 - \rho}{\rho} \quad\text{...(43)}.$$

In the case $i = 1$ this result is exact for all the notes.

§ 56. We shall next suppose the position of the layer to be any whatever, but the alterations in density and rigidity to occur separately. As in either case the change of pitch vanishes for an altered core, we may without restriction replace

$$(b - c)\, b^2/a^3 \text{ by } \tfrac{1}{3}\partial V/V$$

and

$$\frac{\rho_1 - \rho}{\rho}\frac{(b - c)\, b^2}{a^3} \text{ by } \tfrac{1}{3}\partial M/M.$$

When the density alone is altered in the layer we have, according as the volume or the thickness of the layer is given,—

$$-\frac{\partial k}{k} = \frac{\partial M}{M}\frac{Q'}{3}\left\{(k\beta b)^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta b)\right\}^2 \quad\text{.........................(44 a)},$$

$$-\frac{\partial k}{k} = \frac{t}{a}\frac{\rho_1 - \rho}{\rho} Q\left\{(k\beta b)^{\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta b)\right\}^2 \quad\text{.........................(44 b)};$$

where

$$Q' = \frac{k^2\beta^2 a^2}{k^2\beta^2 a^2 - (i-1)(i+2)}\left\{(k\beta a)^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta a)\right\}^{-2} \quad\text{............(45)},$$

$$Q = Q'/(k\beta a)^2 \quad\text{...(46)}.$$

When the layer differs from the remainder only in rigidity we have, according as it is of given volume or given thickness,

$$\frac{\partial k}{k} = \frac{\partial V}{V}\frac{n_1 - n}{n}\frac{Q'}{3}\Bigg[(i-1)(i+2)\left\{(k\beta b)^{-\frac{3}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta b)\right\}^2$$
$$+ \frac{n}{n_1}\left\{(k\beta b)^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J'_{i+\frac{1}{2}}(k\beta b) - \tfrac{3}{2}(k\beta b)^{-\frac{3}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta b)\right\}^2\Bigg] \quad\text{......(47 a)},$$

$$\frac{\partial k}{k} = \frac{t}{a}\frac{n_1 - n}{n} Q\Bigg[(i-1)(i+2)\left\{(k\beta b)^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta b)\right\}^2$$
$$+ \frac{n}{n_1}\left\{(k\beta b)^{\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J'_{i+\frac{1}{2}}(k\beta b) - \tfrac{3}{2}(k\beta b)^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta b)\right\}^2\Bigg] \quad\text{.......(47 b)}.$$

It will be noticed that the several expressions depend on i but are wholly independent of the *form* of X_i. Thus the change of pitch accompanying any such alteration of material as is here considered is the same for all possible forms of vibration which have the same frequency.

In (44*a*), (44*b*), (47*a*) and (47*b*) the expression for the change of pitch consists, like the expressions in the case of the radial vibrations, of three factors. The first measures the magnitude of the alteration of material, the second is Q or $Q'/3$ according as the layer is of given thickness or given volume, and the third gives the law of variation of the change of pitch with the position of the layer.

The variation of the third factors with $x, \equiv k\beta b$, may be shown by curves which apply to all the notes answering to a given value of i. These curves are as follows:

For a layer of altered density of given volume

$$y = \{x^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(x)\}^2 = f_1(x) \quad \text{.........................(48)}.$$

For a layer of altered density of given thickness

$$y = x^2 f_1(x) = f_2(x) \quad \text{...(49)}.$$

For a layer of altered rigidity of given volume

$$y = (i-1)(i+2)\, x^{-2} f_1(x) + \frac{n}{n_1}\{x^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J'_{i+\frac{1}{2}}(x) - \tfrac{3}{2}x^{-\frac{3}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(x)\}^2 = f_3(x) \quad \text{...........(50)}.$$

For a layer of altered rigidity of given thickness

$$y = x^2 f_3(x) = f_4(x) \quad \text{...(51)}.$$

When the value of i is given, and in the case of (50) and (51) the magnitude of the alteration of material, the lengths of the maxima ordinates of these four curves are numerical quantities which are independent of the number of the note. Thus the maxima percentage changes of pitch of any given number—i.e. the changes answering to a certain definite maximum ordinate—in the different notes which answer to a given value of i, vary as Q' or as Q according as the volume or thickness of the layer is given.

Now the values of $k\beta a$ for the notes of higher number are very near the roots of (38), and so are very close to those values of x which make $\{J_{i+\frac{1}{2}}(x)\}^2$ a maximum. Thus by the same reasoning as in § 53 we conclude that in the notes of higher number $J_{i+\frac{1}{2}}(k\beta a)$ varies more and more nearly as $(k\beta a)^{-\frac{1}{2}}$ the higher the number of the note. For the definition of a Bessel to which (32) and the approximate form (35) relate we get for the higher notes $J_{i+\frac{1}{2}}(k\beta a) = \sqrt{2/\pi k\beta a}$ approximately.

Again the factor $k^2\beta^2a^2 \div \{k^2\beta^2a^2 - (i-1)(i+2)\}$ approaches more and more nearly to 1, the larger $k\beta a$, i.e. the higher the number of the note.

We thus conclude that in the higher notes answering to a given value of i, Q' varies more and more nearly as $(k\beta a)^2$ the higher the number of the note, whereas Q continually approaches a finite constant value. With our definition of a Bessel we have for these approximate values $Q' = k^2\beta^2a^2$, $Q = 1$.

We have also seen that according as i is odd or even the higher values of $k\beta a$ approach to $j\pi$ or to $(2j+1)\pi/2$, where j is a positive integer.

Thus for a given alteration of material throughout a layer of given volume the maxima percentage changes of pitch of any given number in the case of the higher notes answering to a given value of i, vary approximately as $j^2\pi^2$ or $(2j+1)^2\pi^2/4$ according as i is odd or even. In other words the maxima percentage changes of pitch of any given number in the case of the higher notes are such that their square roots increase approximately in an arithmetical progression with the number of the note.

On the other hand for a given alteration of material throughout a layer of given thickness the maxima percentage changes of pitch of any given number in the case of the higher notes answering to a given value of i are all nearly equal.

§ 57. When the layer differs from the remainder only in density we see from (44 a) or (44 b) that the law of variation of the change of pitch with the position of the layer is always independent of the magnitude of the alteration of material.

The change of pitch vanishes when the layer coincides with the node surfaces, and for all other positions the pitch is raised or lowered according as the density is diminished or increased.

When the layer of altered density is of given volume the curve showing the variation of $\partial k/k$ with kab is (48). The abscissae supplying the maxima ordinates are easily seen to be the roots of (30). Thus the positions of the layer supplying the maxima changes of pitch coincide with the loop surfaces.

Since the larger values of x answering to the maxima ordinates approach more and more nearly the larger they are to the roots of (38), our previous reasoning shows that the lengths of the successive maxima ordinates of higher number vary more and more approximately the higher the number as the inverse squares of the corresponding abscissae. Thus the maxima changes of pitch of higher number in any given note diminish very rapidly as the radius of the corresponding position of the layer increases.

From a consideration of (44 b) and (49) we similarly conclude that when the layer of altered density is of given thickness the positions in which it is most effective lie outside of but close to the successive higher loop surfaces. Also the successive maxima changes of pitch of higher number in the case of any given note are all approximately equal.

From the preceding results we may take as approximations to the maxima of higher number in the higher notes answering to any value of i—

for a layer of given volume $\frac{-\partial k}{k} = \frac{1}{3}\frac{\partial M}{M}\left(\frac{a}{b}\right)^2$, where b is the radius of the corresponding position of the layer,

for a layer of given thickness $-\partial k/k = \frac{t}{a}\frac{\rho_1 - \rho}{\rho}$.

§ 58. When the layer differs from the remainder only in elasticity the change of pitch depends solely on the alteration of rigidity.

In this case we see from (47 *a*) or (47 *b*) that, unless $i=1$, the expression for the change of pitch is the *sum* of two squares which cannot simultaneously vanish except when $x=0$. Thus unless in the rotatory vibrations an alteration of rigidity occurring anywhere but at the centre necessarily affects the pitch, and the pitch is always raised or lowered according as the rigidity is increased or diminished.

When the layer of altered rigidity is of given thickness the curve giving the variation of $\partial k/k$ with $k\alpha b$ is (51). The form of the curve, unless $i=1$, is dependent on the nature of the material and varies with the magnitude of the alteration of rigidity. Thus in an exhaustive investigation it would be advisable to construct two simple curves answering to the two terms in (47 *b*). The first curve would be the same as (48), the second would be

$$y = \{x^{\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J'_{i+\frac{1}{2}}(x) - \tfrac{3}{2}x^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J_{i+\frac{1}{2}}(x)\}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (52).$$

Adding the ordinate of (48) multiplied by $(i-1)(i+2)$ to the ordinate of (52) multiplied by n/n_1 we should get a compound curve as on previous occasions.

For small values of x, and so for all positions of the layer in note (1), or for positions near the centre in the case of the higher notes answering to a given value of i, the contributions of (48) and (52) to the compound curve will be of like order of magnitude.

Outside however of the third or fourth node surface in the case of the higher notes answering to a given value of i, the contribution of (48) to the compound curve is always small.

On the other hand when x is large (52) becomes almost identical with the curve

$$y = \{x^{\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} J'_{i+\frac{1}{2}}(x)\}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

and the successive maxima ordinates of higher number of this curve continually approach a finite constant value, viz. 1. The corresponding values of x are close to the higher roots of (29), which answer to the node surfaces. For the maxima changes of pitch of higher number in the higher notes we may practically leave (48) out of account and take as an approximate formula, for all values of i, $\dfrac{\partial k}{k} = \dfrac{t}{a}\,\dfrac{n_1 - n}{n_1}$.

When the layer of altered rigidity is of given volume we come to precisely the same conclusion as to the relative importance of the first and second terms of (47 *a*); and it is easily seen that when the layer is outside of the third or fourth node surface in one of the higher notes answering to a given value of i, there are a series of maxima changes of pitch answering to positions of the layer near the higher node surfaces which depend almost entirely on the second term.

These maxima are however usually insignificant compared to the maxima which depend essentially on the first as well as on the second term of (47 *a*). Thus in the case of a layer of given volume the comparative insignificance of the first term for large values of x is not of much practical importance.

Unless the altered layer coincide with a no-stress surface—when the second terms in (47 a) and (47 b) vanish—a given increase of rigidity has less effect on the pitch than an equal diminution, and this difference becomes more and more important in all but the rotatory vibrations as the radius of the layer increases.

§ 59. For the *rotatory* vibrations we get from (20) and (32)

$$\sqrt{\frac{\pi k\beta}{2}}\, w_r = \frac{1}{r}\left(\frac{\sin k\beta r}{k\beta r} - \cos k\beta r\right),$$

$$\sqrt{\frac{\pi k\beta}{2}}\, W_r = \frac{n}{r^2}\left\{k\beta r \sin k\beta r - 3\left(\frac{\sin k\beta r}{k\beta r} - \cos k\beta r\right)\right\}.$$

Also the frequency equation, obtained by equating W_a to zero, is

$$k\beta a \sin k\beta a - 3\left(\frac{\sin k\beta a}{k\beta a} - \cos k\beta a\right) = 0 \quad\text{.........................(54).}$$

It will be seen that but for the multiplier $\sqrt{\pi k\beta/2}$, w_r and W_r are exactly the same functions of $k\beta r$ and n as u_r and $\frac{3}{4}U_r$ of Sect. II. for the radial vibrations are of $k\alpha r$ and n, if we put $m = n/3$. Also (54) when α is written for β is identical with the frequency equation for the radial vibrations when m is put $= n/3$.

Since the condition for the node surfaces is that w_b vanishes, and the condition for the loop surfaces that w_b^2 is a maximum, it follows that the corresponding values of $k\beta b$ are identical with the values of $k\alpha b$ answering to the node and loop surfaces respectively in the case of the radial vibrations.

The relation $n/m = 3$ is however physically impossible, so that the values of $k\beta a$ for the several rotatory notes cannot be identical with the values of $k\alpha a$ for the radial notes in any isotropic material, and the values of $k\beta b$ for the several no-stress surfaces in the rotatory vibrations are also different from the values of $k\alpha b$ for the no-stress surfaces in the radial vibrations.

It follows that the positions of the several node, loop and no-stress surfaces in the case of a rotatory note in a given sphere cannot be *identical* with the positions of these surfaces in the case of any radial note.

The first four roots of (54) according to Professor Lamb* are given by

$$k\beta a/\pi = 1{\cdot}8346, \quad 2{\cdot}8950, \quad 3{\cdot}9225, \quad 4{\cdot}9385.$$

Comparing these with the results of Table I. Sect. II. it will be seen that the value of $k\beta a$ for the rotatory note of number $(i-1)$ is very near the value of $k\alpha a$ for the radial note of number (i), though always slightly less than the least value of $k\alpha a$, which answers to $\sigma = 0$. Thus in any isotropic sphere, when i is large, the frequencies of the i^{th} radial

* *Proceedings of the London Mathematical Society*, Vol. XIII. p. 197.

note and of the $(i-1)^{\text{th}}$ rotatory note are very approximately in the ratio $\sqrt{m+n} : \sqrt{n}$. In reality in the case of the rotatory vibrations there is a sort of suppressed note of zero frequency as the following investigation shows.

The frequency equations for the radial vibrations, for all values of σ, and for the rotatory vibrations may be included under

$$f(x) \equiv x^{-1} \sin x - q^2 x^{-2} (x^{-1} \sin x - \cos x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(55);$$

where $q^2 = 4n/(m+n)$ for the radial, and $=3$ for the rotatory vibrations. So long as q^2 is less than 3, (55) has a root between 0 and π. This root however diminishes rapidly as q^2 approaches 3 and for this critical value becomes absolutely zero.

In what follows I shall speak of the note answering to $k\beta a/\pi = 1{\cdot}8346$ as note (1).

The positions of all the node, loop and no-stress surfaces for the first four notes are given in the following table. They are calculated from the values given above for $k\beta a$ and from the data already employed in Sect. II.

TABLE I.

Values of r/a over node, loop and no-stress surfaces.

Note (1)			Note (2)			Note (3)			Note (4)		
Node surfaces	No-stress surfaces	Loop surface	Node surfaces	No-stress surfaces	Loop surfaces	Node surfaces	No-stress surfaces	Loop surfaces	Node surfaces	No-stress surfaces	Loop surfaces
0	0	·3612	0	0	·2289	0	0	·1689	0	0	·1342
·7796	1·0		·4941	·6337	·6532	·3646	·4677	·4821	·2896	·3715	·3829
			·8494	1·0		·6269	·7380	·7470	·4979	·5862	·5934
						·8849	1·0		·7028	·7943	·7995
									·9066	1·0	

The centre is at once a node and a no-stress surface, and the number whether of node or of no-stress surfaces is one greater than the number of loop surfaces, which equals the number of the note. The loop surfaces lie outside of the corresponding no-stress surfaces, and not inside them as in the case of the radial vibrations.

A comparison of the above table with Table II. Sect. II. leads to many interesting results as to the relative positions of the node, loop and no-stress surfaces in the radial and rotatory vibrations.

§ 60. We have already seen that an alteration of material at the centre has no effect on the pitch of a rotatory vibration, and that when a surface layer is altered the change of pitch depends only on the alteration of density and is given by (43).

Supposing the layer to differ from the remainder only in density, the general formula for the change of pitch is identical with (48), Sect. II., writing β for α, viz.

$$\frac{\partial k}{k} = -\frac{t}{a}\,\frac{\rho_1-\rho}{\rho}\,Q\left(\frac{\sin k\beta b}{k\beta b} - \cos k\beta b\right)^2 = -\frac{\partial M}{M}\,\frac{Q'}{3}\left\{\frac{1}{k\beta b}\left(\frac{\sin k\beta b}{k\beta b} - \cos k\beta b\right)\right\}^2 \ldots(56).$$

When the layer differs from the remainder only in rigidity we have

$$\frac{\partial k}{k} = \frac{t}{a}\frac{n_1 - n}{n_1} Q \left\{\sin k\beta b - \frac{3}{k\beta b}\left(\frac{\sin k\beta b}{k\beta b} - \cos k\beta b\right)\right\}^2 \text{..........................(57 } a\text{)},$$

$$= \frac{\partial V}{V}\frac{n_1 - n}{n_1}\frac{Q'}{3}\left\{\frac{\sin k\beta b}{k\beta b} - \frac{3}{k^2\beta^2 b^2}\left(\frac{\sin k\beta b}{k\beta b} - \cos k\beta b\right)\right\}^2 \text{....................(57 } b\text{)}.$$

In these formulae t, M, V, etc. have the same significations as previously. The formulae may be applied without any restriction since ∂k vanishes when the alteration of material occurs at the centre.

Convenient expressions for Q and Q' may be obtained from (38) and (40), Sect. II., by writing β for α and supposing $m = n/3$.

This substitution gives

$$Q = 1 + 3(k\beta a)^{-2} + 9(k\beta a)^{-4} \text{................................(58)},$$

$$Q' = (k\beta a)^2 + 3 + 9(k\beta a)^{-2} \text{....................................(59)}.$$

From these formulae and the values given above for $k\beta a$ the values of Q and Q' for the first four notes may be easily calculated. The results are given in the following table:—

TABLE II.

Values of Q and Q'.

	Note (1)	Note (2)	Note (3)	Note (4)
$Q =$	1·098	1·038	1·020	1·013
$Q' =$	36·49	85·83	154·91	243·74

A comparison of this table with Table III. Sect. II. will be found instructive.

§ 61. When the layer differs from the remainder only in density the curves showing the variation of $\partial k/k$ with $k\beta b$ are exactly the same as those which under corresponding conditions show the variation of $\partial k/k$ with $k\alpha b$ in the case of the radial vibrations. They are thus curve A or curve B of fig. 1 according as the layer is of given volume or given thickness.

When the layer is of given volume the positions in which it has most effect on the pitch of a given note coincide with the loop surfaces. The ratios of the first to the subsequent maxima changes of pitch in the case of a given note are the same as in the case of the radial vibrations, viz.

1 : ·1485 : ·0620 : ·0342......

The values of the first maxima are given for the first four notes in the following table:—

TABLE III.

First maximum of $\frac{-\partial k}{k} \div \frac{\partial M}{M}$.

Note (1)	Note (2)	Note (3)	Note (4)
2·314	5·443	9·824	15·457

The number of maxima is equal to the number of the note, and so all the maxima in the first four notes may be calculated from the ratios given above.

For notes above the fourth we obtain a close approximation to the first maximum by means of the following formula, in which i is the number of the note,

$$\frac{-\partial k}{k} \div \frac{\partial M}{M} = (i+1)^2 \times \cdot 6259 \quad \ldots\ldots\ldots\ldots (60).$$

This formula is adapted from (68), Sect. II.

When the layer of altered density is of given thickness the positions in which it has most effect on the pitch of the note of frequency $k/2\pi$ are obtained by equating $k\beta b$ to the values supplied for $k\alpha a$ for the value ·3 of σ in Table I. Sect. II.

These positions are given for the first four notes in the following table:—

TABLE IV.

Values of b/a when $\frac{-\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right)$ is a maximum.

Note (1)	Note (2)	Note (3)	Note (4)
·4760	·3017	·2226	·1768
	·6725	·4964	·3942
		·7560	·6005
			·8048

The ratios of the first to the subsequent maxima changes of pitch are the same as in the corresponding case in the radial vibrations, viz.

1 : ·908 : ·895 : ·890.......

The first maxima for the first four notes are as follows:—

TABLE V.

First maximum of $\frac{-\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right)$.

Note (1)	Note (2)	Note (3)	Note (4)
1·242	1·173	1·153	1·144

From these results and the ratios already given all the maxima may be found for these notes.

As the number of the note increases the formula

$$\frac{-\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right) = 1\cdot130 \text{ ....(61)},$$

applies with continually increasing exactness to the first maximum.

For any maximum of high number in the case of one of the higher notes a close approximation is supplied by

$$\frac{-\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right) = 1\cdot00 \text{ ....(62)}.$$

§ 62. When the layer differs from the remainder only in elasticity, the change of pitch depends only on the alteration of rigidity. In this case we see from (57 a) or (57 b) that the change of pitch of a note vanishes when the layer coincides with a no-stress surface, and that for all other positions of the layer the pitch is raised or lowered according as the rigidity is increased or diminished. For a given numerical alteration of rigidity the effect on the pitch is greater when the rigidity is diminished than when it is increased.

When the layer of altered rigidity is of given volume the curve showing the variation of $\partial k/k$ with $k\beta b$, $\equiv x$, is

$$y = \{x^{-1} \sin x - 3x^{-2} (x^{-1} \sin x - \cos x)\}^2 = f_{13}(x) \text{ ....(63)}.$$

The first segment of this curve appears as curve A in fig. 5, and the corresponding function of x is tabulated in Table VII. Sect. II.

The second and subsequent segments of this curve would lie extremely close to the third and subsequent segments of the curve A of fig. 2. The first segment answers apparently to the first two segments of the curves of fig. 2.

The abscissae supplying the maxima ordinates of curve A, fig. 5, are the roots of the equation

$$1 - 9x^{-2} - x^{-1} (4 - 9x^{-2}) \tan x = 0 \text{ ....(64)},$$

and the lengths of the maxima ordinates are found by substituting the roots of this equation for x in the expression

$$y = x^{-2} \{1 - 2x^{-2} + 9x^{-4} + 81x^{-6}\}^{-1} \text{ ....(65)}.$$

For the first root and the corresponding maximum ordinate I find approximately

$$x = 1\cdot0638\pi, \quad y = \cdot09412.$$

From these results with the assistance of Table II. and the values of $k\beta a$ I have calculated the corresponding positions of the layer and the values of the corresponding maximum change of pitch in the first four notes. They are as follows:—

TABLE VI.

First maximum of $\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{n_1 - n}{n_1}\right)$ and corresponding position of layer.

	Note (1)	Note (2)	Note (3)	Note (4)
$\frac{\partial k}{k} \div \frac{\partial V}{V} \left(\frac{n_1 - n}{n_1}\right) =$	1·145	2·693	4·860	7·647
for $b/a =$	·5799	·3675	·2712	·2154

In passing it may be noticed that the positions of the layer in this table coincide with the first *maximum-stress surface*, i.e. the surface of least radius where the transverse stress W_r is a maximum.

From the consideration that when i is greater than 3 or 4 the value of Q' for note (i) is approximately $(i+1)^2\pi^2$, we obtain as a pretty close approximation to the first maximum in the case of one of the higher notes of number (i)

$$\frac{\partial k}{k} \div \frac{\partial V}{V} \frac{n_1 - n}{n_1} = (i+1)^2 \times \cdot 310 \qquad (66).$$

The first maxima given in the table are considerably the largest for the respective notes.

§ 63. When the layer of altered rigidity is of given thickness the equation to the curve showing the variation of $\partial k/k$ with $k\beta b$, $\equiv x$, is

$$y = x^2 f_{13}(x) = f_{12}(x) \qquad (67).$$

The first segment of this curve appears as curve B in fig. 5 and the corresponding function of x is tabulated in Table VII. Sect. II.

The second and subsequent segments would lie very close to the third and subsequent segments of curves B in fig. 4, and like them continually approach, as x increases, to coincidence with curve B, fig. 2.

The abscissae supplying the maxima ordinates of curve B, fig. 5, are the roots of the equation

$$1 - 6x^{-2} - 3x^{-1}(1 - 2x^{-2}) \tan x = 0 \qquad (68),$$

and the lengths of the maxima ordinates are found by substituting the roots of this equation for x in the expression

$$y = (1 - 3x^{-2} + 36x^{-6})^{-1} \qquad (69).$$

For the first root and the corresponding maximum ordinate I find approximately

$$x = 1\cdot 2319\pi, \quad y = 1\cdot 2339.$$

From these results with the assistance of Table II. and the values of $k\beta a$ I have calculated the corresponding positions of the layer and the values of the corresponding maximum change of pitch in the first four notes, and give them in the following table:—

TABLE VII.

First maximum of $\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{n_1 - n}{n_1}\right)$ and corresponding position of layer.

		Note (1)	Note (2)	Note (3)	Note (4)
	$\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{n_1 - n}{n_1}\right) =$	1·355	1·280	1·259	1·249
for	$b/a =$	·6715	·4255	·3141	·2494

As the value of Q continually approaches unity as the number of the note increases, the first maximum in one of the higher notes is given more and more correctly the higher the number of the note by

$$\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{n_1 - n}{n_1}\right) = 1·234 \quad\text{.................................(70).}$$

It is obvious from (69) that the first maximum ordinate is decidedly the largest, the length of the others approaching more and more nearly to 1 the larger the corresponding value of x. In the case of the higher notes all but the first two or three maxima changes of pitch are given very approximately by

$$\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{n_1 - n}{n_1}\right) = 1·00 \quad\text{.................................(71),}$$

and the corresponding positions of the layer are in the immediate neighbourhood of the node surfaces.

SECTION IV.

RADIAL VIBRATIONS IN SOLID CYLINDER.

§ 64. If $J_1(kx)$, $Y_1(kx)$ represent the two solutions of the Bessel's equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + u\left(k^2 - \frac{1}{x^2}\right) = 0 \quad\text{.................................(1),}$$

then the type of vibration in a cylindrical shell of material (ρ, m, n) performing radial vibrations of frequency $k/2\pi$ is

$$u = \cos kt\,\{AJ_1(k\alpha r) + BY_1(k\alpha r)\} \quad\text{.................................(2)*.}$$

Here, as usual, A and B represent arbitrary constants, and

$$\alpha^2 = \rho/(m+n) \quad\text{.................................(3).}$$

The displacements in a compound solid cylinder $(0.\alpha.c.\alpha_1.b.\alpha.a)$, where $b - c$ is small, are as follows:

* *Transactions*, Vol. XIV. p. 356.

In the core $u/\cos kt = AJ_1(k\alpha r)$ ....(4).

In the thin layer $u/\cos kt = A_1 J_1(k\alpha_1 r) + B_1 Y_1(k\alpha_1 r)$ ....(5).

Outside the layer $u/\cos kt = (A + \partial A) J_1(k\alpha r) + \partial B Y_1(k\alpha r)$ ....(6).

We shall suppose terms in $(b-c)^2$, and so in $(\partial A/A)^2$ and $(\partial B/A)^2$, negligible.

Let us for shortness put

$$\left.\begin{aligned} J_1'(k\alpha r) &= \frac{1}{k\alpha}\frac{d}{dr} J_1(k\alpha r), \\ Y_1'(k\alpha r) &= \frac{1}{k\alpha}\frac{d}{dr} Y_1(k\alpha r) \end{aligned}\right\} \text{..........(7)},$$

$$\left.\begin{aligned} F(r\,.\,\alpha) &= (m+n)\,k\alpha r J_1'(k\alpha r) + (m-n) J_1(k\alpha r), \\ F_1(r\,.\,\alpha) &= (m+n)\,k\alpha r Y_1'(k\alpha r) + (m-n) Y_1(k\alpha r) \end{aligned}\right\} \text{..........(8).}$$

We then find for the relations connecting the arbitrary constants and supplying the frequency equation:

$$AJ_1(k\alpha c) = A_1 J_1(k\alpha_1 c) + B_1 Y_1(k\alpha_1 c) \text{..........(9)},$$

$$AF(c\,.\,\alpha) = A_1 F(c\,.\,\alpha_1) + B_1 F_1(c\,.\,\alpha_1) \text{..........(10)},$$

$$(A+\partial A) J_1(k\alpha b) + \partial B Y_1(k\alpha b) = A_1 J_1(k\alpha_1 b) + B_1 Y_1(k\alpha_1 b) \text{..........(11)},$$

$$(A+\partial A) F(b\,.\,\alpha) + \partial B F_1(b\,.\,\alpha) = A_1 F(b\,.\,\alpha_1) + B_1 F_1(b\,.\,\alpha_1) \text{..........(12)},$$

$$(A+\partial A) F(a\,.\,\alpha) + \partial B F_1(a\,.\,\alpha) = 0 \text{..........(13).}$$

The process of obtaining the frequency equation having been already illustrated in the case of the sphere, no difficulty should be encountered in carrying it out when an eye is kept on the expression

$$\Delta(b\,.\,\alpha_1\,.\,b') \equiv J_1(k\alpha_1 b)\, Y_1'(k\alpha_1 b) - Y_1(k\alpha_1 b)\, J_1'(k\alpha_1 b) \text{..........(14)},$$

which cuts out in the final equations determining $\partial A/A$ and $\partial B/A$. The results I find are as follows:

$$-\frac{\partial A}{A}(m+n)\,k\alpha b\,\Delta(b\,.\,\alpha\,.\,b') \div \frac{b-c}{b}$$

$$= \left\{(m+n)k^2\alpha^2 b^2 - (m_1+n_1)k^2\alpha_1^2 b^2 - \frac{4mn}{m+n} + \frac{4m_1 n_1}{m_1+n_1}\right\} J_1(k\alpha b)\, Y_1(k\alpha b)$$

$$+ \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right) F(b.\alpha) F_1(b.\alpha) + 2\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)\{J_1(k\alpha b) F_1(b.\alpha) + Y_1(k\alpha b) F(b.\alpha)\} \text{...(15)},$$

$$\frac{\partial B}{A}(m+n)\,k\alpha b\,\Delta(b\,.\,\alpha\,.\,b') \div \frac{b-c}{b}$$

$$= \left\{(m+n)k^2\alpha^2 b^2 - (m_1+n_1)k^2\alpha_1^2 b^2 - \frac{4mn}{m+n} + \frac{4m_1 n_1}{m_1+n_1}\right\} \{J_1(k\alpha b)\}^2$$

$$+ \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right) \{F(b\,.\,\alpha)\}^2 + 4\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right) J_1(k\alpha b)\, F(b\,.\,\alpha) \text{..........(16).}$$

It is important to notice that

$$k\alpha b\Delta(b\,.\,\alpha\,.\,b') = -C \quad \text{.........(17)},$$

where C is a constant independent of k, α or b, determined entirely by the definition given of the Bessel's function.

If the layer did not exist the frequency equation would be obtained by putting 0 for $\partial A/A$ and $\partial B/A$ in (13), whence

$$f(0\,.\,\alpha\,.\,a) \equiv F(a\,.\,\alpha) = 0 \quad \text{.........(18)}.$$

In consequence of the existence of the thin layer, $f(0\,.\,\alpha\,.\,a)$ is no longer zero but is of order $b-c$. Thus neglecting ∂A in (13), we find for the frequency equation in the compound cylinder

$$F(a\,.\,\alpha) + \frac{\partial B}{A} F_1(a\,.\,\alpha) = 0 \quad \text{.........(19)}.$$

As terms in $(b-c)^2$ are negligible, we may transform the coefficient of $\partial B/A$ in (19) by any substitution which supposes (18) exactly true. We thus are enabled to replace (19) by

$$F(a\,.\,\alpha) + \frac{\partial B}{A}\,\frac{(m+n)\,k\alpha a\Delta(a\,.\,\alpha\,.\,a')}{J_1(k\alpha a)} = 0 \quad \text{.........(20)}.$$

If the presence of the thin layer has raised the frequency by $\partial k/2\pi$ then $k-\partial k$ must satisfy (18), whence, neglecting terms in ∂k^2, we find

$$F(a\,.\,\alpha) - \partial k \frac{d}{dk} F(a\,.\,\alpha) = 0 \quad \text{.........(21)}.$$

Now $\quad k\alpha a \dfrac{d}{k\alpha da} F(a\,.\,\alpha) = -(m+n)(k^2\alpha^2a^2 - 1)\,J_1(k\alpha a) + (m-n)\,k\alpha a J_1'(k\alpha a).$

As this occurs in the coefficient of ∂k we may substitute for $J_1'(k\alpha a)$ as if (18) were exactly true. Doing so, we get

$$k\alpha a \frac{d}{k\alpha da} F(a\,.\,\alpha) = -(m+n)\left\{k^2\alpha^2a^2 - \frac{4mn}{(m+n)^2}\right\} J_1(k\alpha a) \quad \text{.........(22)}.$$

Substituting this in (21), and then noticing that (19) and (21) must be identical, we find

$$\frac{\partial k}{k} = \frac{\partial B}{A}\,\frac{k\alpha a\Delta(a\,.\,\alpha\,.\,a')}{\{k^2\alpha^2a^2 - 4mn(m+n)^{-2}\}\{J_1(k\alpha a)\}^2} \quad \text{.........(23)}.$$

Let $\dfrac{1}{2\pi} K_{(a.\,a)}$ denote the frequency of free radial vibrations in an infinitely thin shell of material $(\rho,\, m,\, n)$ and radius a, then

$$K^2_{(a.\,a)}\, a^2 = \frac{4mn}{(m+n)^2\alpha^2} = \frac{4mn}{(m+n)\rho} \quad \text{.........(24)*}.$$

* *Transactions, l. c.* p. 356, equations (43) and (43 *a*). Cf. also (22) above.

Also let

$$u_r = J_1 (k\alpha r) \quad\ldots\ldots\ldots(25),$$

$$U_r = \frac{1}{r} F (r \,.\, \alpha) = \frac{1}{r} \{(m+n) k\alpha r\, J_1' (k\alpha r) + (m-n) J_1 (k\alpha r)\} \quad\ldots\ldots\ldots(26),$$

so that $u_r \cos kt$ represents a displacement in a simple cylinder performing radial vibrations of frequency $k/2\pi$ and $U_r \cos kt$ the corresponding radial stress, both quantities referring to points at distance r from the axis.

Employing these substitutions in the value of $\partial B/A$ given by (16), and then substituting in (23) and employing (17), we find

$$\frac{\partial k}{k} = \frac{b-c}{b} \left[\frac{\rho (k^2 - K^2_{(a.b)}) - \rho_1 (k^2 - K^2_{(a_1.b)})}{\rho (k^2 - K^2_{(a.a)})} \left(\frac{b}{a}\right)^2 \left(\frac{u_b}{u_a}\right)^2 \right.$$
$$\left. + \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right) \frac{b^2}{a^2 \rho (k^2 - K^2_{(a.a)})} \left(\frac{U_b}{u_a}\right)^2 + 4\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right) \frac{b}{a^2 \rho (k^2 - K^2_{(a.a)})} \frac{u_b U_b}{u_a^2} \right] \ldots(27).$$

In (27), as in (28), Sect. II., we notice the existence of three distinct terms, the first depending on the square of the displacement of the altered layer, the second on the square of the radial stress, and the third on the product of the displacement and radial stress. The first term alone exists if the layer differ from the remainder of the cylinder only in density, or if it coincide with any no-stress surface. If the layer occur at a node surface then the second term alone exists. The signs of these two terms are independent of the radius of the layer.

The third term vanishes if $m_1/n_1 = m/n$; otherwise its sign as well as its magnitude varies with the position of the layer.

§ 65. In the core there is no change of type due to the existence of the layer other than a displacement of any node, loop, or no-stress surface originally of radius r according to the law

$$-\partial r/r = \partial k/k \quad\ldots\ldots\ldots(28).$$

Outside the layer we find by substituting in (6) the values of $\partial A/A$ and $\partial B/A$ from (15) and (16), and reducing

$$u/A \cos kt = J_1 (k\alpha r)$$
$$+ \frac{b-c}{(m+n)C} \left[\{\rho (k^2 - K^2_{(a.b)}) - \rho_1 (k^2 - K^2_{(a_1.b)})\}\, b u_b f(\bar{b} \,.\, \alpha \,.\, r) + \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right) U_b f(b \,.\, \alpha \,.\, \bar{r}) \right.$$
$$\left. + 2\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right) \{b^{-1} u_b f(b \,.\, \alpha \,.\, \bar{r}) + U_b f(b \,.\, \alpha \,.\, \bar{r})\} \right] \quad\ldots\ldots\ldots(29);$$

where, with our usual notation,

$$\left.\begin{aligned} f(b \,.\, \alpha \,.\, r) &= J_1 (k\alpha r)\, Y_1 (k\alpha b) - Y_1 (k\alpha r)\, J_1 (k\alpha b), \\ f(b \,.\, \alpha \,.\, \bar{r}) &= J_1 (k\alpha r)\, F_1 (b \,.\, \alpha) - Y_1 (k\alpha r)\, F (b \,.\, \alpha) \end{aligned}\right\} \quad\ldots\ldots\ldots(30).$$

The loci where $f(b \,.\, \alpha \,.\, r)$ vanishes and changes sign are what would be the node surfaces of a simple shell of material (ρ, m, n) whose inner surface $r = b$ is fixed and whose frequency of vibration is $k/2\pi$. Similarly the loci where $f(b \,.\, \alpha \,.\, \bar{r})$ vanishes answer to the node surfaces in the vibration of frequency $k/2\pi$ in a simple shell of material (ρ, m, n) whose inner surface $r = b$ is free.

We notice the existence of three terms in the coefficient of $b - c$ in (29) answering to the three terms in (27). The first two terms in (27) and (29) vanish together. The third term however in (27) vanishes when the layer coincides either with a node or a no-stress surface, whereas unless $m_1/n_1 = m/n$ the third term in (29) can vanish only for special values of r wherever the layer may be situated.

Noticing that $f(\bar{b} . \alpha . \bar{b}) = 0$, and $f(b . \alpha . \bar{b}) = -(m+n) C$ ....(31),

we find from (29) for the displacement just outside the layer

$$u = A \cos kt \left[u_b - (b - c) \left\{ \left(\frac{1}{m+n} - \frac{1}{m_1 + n_1} \right) U_b + 2 \left(\frac{n}{m+n} - \frac{n_1}{m_1 + n_1} \right) b^{-1} u_b \right\} \right] \quad \text{......(32).}$$

From (32) we may deduce the following expression for the displacement throughout the layer itself:

$$u/A \cos kt = J_1(kab) - k\alpha(b - r) J_1'(kab)$$

$$- (r - c) \left\{ \left(\frac{1}{m+n} - \frac{1}{m_1 + n_1} \right) U_b + 2 \left(\frac{n}{m+n} - \frac{n_1}{m_1 + n_1} \right) b^{-1} u_b \right\} \quad \text{......(33).}$$

Thus, precisely as in the radial vibrations of a sphere, no change of type manifests itself as we cross the layer if it differ from the remainder only in density, or if while differing in elasticity it coincide with a no-stress surface and the relation $n_1/m_1 = n/m$ hold.

§ 66. For a discussion of (27) we require to know the characteristics of radial vibrations in a simple cylinder.

The type of the displacement is shown in (25). Thus there are a series of node surfaces whose radii, r, for the note of frequency $k/2\pi$ are found by equating kar to the successive roots of

$$J_1(x) = 0 \quad \text{..(34),}$$

viz. 0, 3·832, 7·016, 10·173, 13·323......,

the higher roots being of course only approximate.

The radii of the loop surfaces, where the displacement is a maximum, are found by equating kar to the roots of

$$J_1'(x) = 0 \quad \text{..(35),}$$

whose approximate values are 1·841, 5·331, 8·536, 11·706.......

The radii of the no-stress surfaces are obtained by equating kar to the roots of

$$(m+n) x J_1'(x) + (m-n) J_1(x) = 0 \quad \text{..........................(36);}$$

while by equating kaa to these roots we obtain the frequencies of the several notes the cylinder can produce.

The form of (36) depends on σ. Thus when $\sigma = 0$ it is identical with (35). When $\sigma = \cdot 25$ it becomes

$$3xJ_1'(x) + J_1(x) = 0 \quad\text{..........(37)};$$

whose roots, excluding zero, are approximately 2·069, 5·396, 8·576, 11·735.......

Finally when $\sigma = \cdot 5$ it becomes

$$J_0(x) = 0 \quad\text{..........(38)};$$

whose roots are approximately 2·404, 5·520, 8·654, 11·792.......

For the roots of (34) and (38) I am indebted to Lord Rayleigh's *Theory of Sound*, Vol. I. Table B, p. 274. The roots of (35) and (37) I have calculated from the tables in Lommel's *Studien über die Bessel'schen Functionen*.

Since the roots of (34) and (35) are independent of σ the ratio of the radii of any two node or loop surfaces of given numbers in a given cylinder performing a given note is the same whatever be the number of the note or the value of σ.

The values of $k\alpha a$, however, being the roots of (36), vary with the value of σ; thus the ratios of the radii of the node or loop surfaces to the radius of the cylinder vary with the material. Still in the case of the second and higher notes the value of σ has only a small effect on the absolute positions of the several node and loop surfaces in a cylinder of given radius.

The roots of (36) exceed the corresponding roots of (35) for all values of σ greater than 0. Thus the loop surfaces, while coinciding with the no-stress surfaces when $\sigma = 0$, lie inside them for all other kinds of isotropic material.

In the case of all three equations (34), (35) and (36) the successive higher roots come to differ almost exactly by π, and the corresponding higher roots of (35) and (36) are for all values of σ nearly equal and are approximately half-way between successive roots of (34).

Thus between successive higher notes there is a nearly constant difference of pitch, and between consecutive surfaces of higher number of the same kind—whether node, loop or no-stress surfaces—a nearly constant difference of radius. Also the node surfaces of higher number lie nearly half-way between consecutive loop surfaces.

The positions of the node, loop and no-stress surfaces for the values 0, ·25 and ·5 of σ in the four lowest notes are given in the following table to three places of decimals:—

TABLE I.

Values of r/a over node, loop, and no-stress surfaces.

Number of note	$\sigma = 0$ Node surfaces	$\sigma = 0$ Loop and no-stress surfaces	$\sigma = \cdot 25$ Node surfaces	$\sigma = \cdot 25$ Loop surfaces	$\sigma = \cdot 25$ No-stress surfaces	$\sigma = \cdot 5$ Node surfaces	$\sigma = \cdot 5$ Loop surfaces	$\sigma = \cdot 5$ No-stress surfaces
(1)	0	1·0	0	·890	1·0	0	·766	1·0
(2)	0	·345	0	·341	·384	0	·334	·435
	·719	1·0	·710	·988	1·0	·694	·966	1·0
(3)	0	·216	0	·215	·241	0	·213	·278
	·449	·625	·447	·622	·629	·443	·616	·638
	·822	1·0	·818	·995	1·0	·811	·986	1·0
(4)	0	·157	0	·157	·176	0	·156	·204
	·327	·455	·327	·454	·460	·325	·452	·468
	·599	·729	·598	·727	·731	·595	·724	·734
	·869	1·0	·867	·998	1·0	·863	·993	1·0

A comparison should be made of the above results with those of Table II. Sect. II.

In the table the axis is counted as a node and the surface of the cylinder as a no-stress surface, and under all circumstances the number of node, loop, or no-stress surfaces is equal to the number of the note.

I shall refer to any such surface by its number, regarding the surface of the same kind of least radius as number (1).

§ 67. In all the expressions for the change of pitch there occurs one or other of the two following quantities:

$$\left.\begin{aligned} Q &\equiv \frac{k\alpha a\,\{J_1(k\alpha a)\}^{-2}}{k^2\alpha^2a^2 - 4mn\,(m+n)^{-2}}, \\ \tfrac{1}{2}Q' &\equiv \tfrac{1}{2}k\alpha a Q \end{aligned}\right\} \quad \text{.......................(39).}$$

Employing the results already recorded for the roots of the frequency equation, I have calculated from Lommel's tables the following approximate values for Q and Q':—

TABLE II.

Values of Q and Q'.

	Q (1)	Q (2)	Q (3)	Q (4)	Q' (1)	Q' (2)	Q' (3)	Q' (4)
$\sigma = 0$	2·275	1·623	1·590	1·581	4·189	8·652	13·574	18·507
$\sigma = \cdot 25$	1·868	1·602	1·583	1·577	3·867	8·644	13·573	18·507
$\sigma = \cdot 5$	1·542	1·565	1·568	1·569	3·708	8·637	13·572	18·507

In the higher notes the influence of σ on the value of Q is small and continually diminishes as the number of the note increases. In notes (3) and (4) the variation in the value of Q' with the value of σ is practically insensible. The numbers entered in the table in the two last columns are scarcely to be relied on in the last decimal place. The third decimal place is retained in these columns mainly with the view of showing how remarkably small the influence of the value of σ is.

The following considerations enable pretty close approximations to be found for the values of Q and Q' in the higher notes.

From the general formula for the approximate values of Bessel's functions for large values of the argument, we may when x is large put

$$J_1(x) = \sqrt{\frac{2}{\pi x}} \cos\left(\frac{3\pi}{4} - x\right)$$

approximately, employing the usual definition of the Bessel.

From the above expression we conclude that for large values of x the maxima values of $x\{J_1(x)\}^2$ are all nearly equal, while the maxima of $\{J_1(x)\}^2$ vary approximately as the reciprocals of the corresponding values of x. Also the larger values of x supplying the maxima whether of $x\{J_1(x)\}^2$ or $\{J_1(x)\}^2$ increase very approximately in an arithmetical progression with a common difference π.

If now we write the frequency equation (36) in the form

$$J_1'(x) + \frac{m-n}{m+n} x^{-1} J_1(x) = 0,$$

we see that its higher roots, whatever be the value of σ, must be nearly identical with the higher roots of $J_1'(x) = 0$, i.e. of (35). This is in fact the exact form of the frequency equation when $\sigma = 0$, and the difference between the second root even of (35) and those of (37) and (38)—the frequency equations for the values 0 and ·5 of σ—is, it will be noticed, far from conspicuous.

Thus whatever be the value of σ the values of $k\alpha a$ for the higher notes are nearly identical with those values of x which make $\{J_1(x)\}^2$ a maximum.

Now for notes above the fourth the value of $k\alpha a$ is not less than 14·8, and so $4mn(m+n)^{-2}$ is very small compared to $k^2\alpha^2a^2$.

Thus we see from (39) that for notes above the fourth a close approximation to the value of Q, whatever be the value of σ, is obtained by equating Q to $1 \div \{x^{\frac{1}{2}}J_1(x)\}^2$, where x is one of the higher numbers which make $\{J_1(x)\}^2$ a maximum. It immediately follows from our recent investigation that for notes above the fourth the value of Q is approximately constant and independent of σ. No serious error will arise by ascribing to it the value $\pi/2$.

In the same way we find as an approximation for notes above the fourth

$$Q' = 1/\{J_1(x)\}^2,$$

where x is one of the higher numbers which make $\{J_1(x)\}^2$ a maximum. Consequently Q' varies approximately as these values of x. But we saw that these values of x increase approximately in an arithmetical progression with common difference π, and so the successive values of Q' increase approximately in arithmetical progression with a common difference $\frac{\pi^2}{2}$.

This conclusion is strongly supported by the numbers given in Table II. We are thus entitled to assume that the value of Q' for any note of number (i) greater than 4 is very approximately given for all values of σ by

$$Q' = 18{\cdot}51 + (i - 4) \times (4{\cdot}935) \quad\dots\dots(40).$$

§ 68. As in previous sections I shall, before discussing the general application of the frequency equation, consider briefly two special cases.

In the first of these the material (ρ_1, m_1, n_1) occurs at or close to the axis. By supposing b/a very small, but $(b-c)/b$ still smaller, we pass to the case of a very thin layer close to the axis of the cylinder. This we shall call the *axial layer*.

Writing $$(b-c)\, b/a^2 = \tfrac{1}{2}\partial V/V,$$

we obtain the value of $\partial k/k$ in this case by retaining only the lowest powers of b/a occurring in (27). We easily find, distinguishing this case by the suffix l,

$$\frac{1}{k}\partial k_l = \frac{\partial V}{V}\frac{Q'}{2}\frac{(m_1 - m)(m + n_1)}{(m+n)(m_1+n_1)} \quad\dots\dots(41_l).$$

If the material (ρ_1, m_1, n_1) form a thin core we must proceed by considering the form taken by the frequency equation $f(0\,.\,\alpha_1\,.\,b\,.\,\alpha\,.\,a) = 0$ when b/a is very small.

The application of the method of Sect. I. to this case presents no difficulty when the following data are kept in view.

From the usual formula for the Bessel's functions we obtain at once when x is very small the approximate values

$$J_1(x) = x/2, \qquad J_1'(x) = 1/2.$$

Now for the other solution of the Bessel's equation we have

$$Y_1(x) = -x^{-1}J_0(x) + \log(x)\,J_1(x) - J_1(x) + \text{powers of } x \text{ above the first}^*.$$

But when x is very small approximate values are

$$J_0(x) = 1, \qquad \log(x)\,J_1(x) = 0,$$

and we have as first approximations

$$Y_1(x) = -x^{-1}, \qquad Y_1'(x) = x^{-2}.$$

The numerical value of the constant C of (17) is also required in this case. We may determine it very simply by noticing that when x is very small

$$-C \equiv x\,\{J_1(x)\,Y_1'(x) - J_1'(x)\,Y_1(x)\} = x\left\{\frac{x}{2}\,.\,x^{-2} - \tfrac{1}{2}(-x^{-1})\right\} = 1.$$

* See Neumann's *Theorie der Bessel'schen Functionen*, p. 52, equations (13), (14), and (15).

Supposing the core of radius b and volume ∂V per unit of length, so that

$$b^2/a^2 = \partial V/V,$$

I find, distinguishing this case by the suffix c,

$$\frac{1}{k}\partial k_c = \frac{\partial V}{V}\frac{Q'}{2}\frac{m_1 - m}{m_1 + n} \qquad \text{(41}_c\text{)}.$$

The formulae (41_l) and (41_c) are not in general identical. When however the alteration in elasticity is small they both reduce to

$$\frac{\partial k}{k} = \frac{\partial V}{V}\frac{Q'}{2}\frac{m_1 - m}{m + n} \qquad \text{(41')}.$$

From (41_l) and (41_c) it follows that to the present degree of approximation an alteration only in density does not affect the pitch of any radial note when it occurs at or close to the axis.

In the case of the core the change of pitch depends entirely on the alteration of the elastic constant m, and in the case of the axial layer the sign of the change of pitch depends entirely on the sign of $m_1 - m$ and its magnitude for any ordinary alteration of material would not be greatly modified by the alteration in n.

If the elastic constant m alone is altered, then the formula (41_l) for the axial layer becomes identical with the general formula (41_c) for the core.

If both elastic constants are altered in the same proportion according to the law

$$m_1/m = n_1/n = 1 + p \qquad \text{(42)},$$

the changes of pitch are given by

$$\frac{1}{k}\partial k_l = \frac{p}{1+p}\frac{\partial V}{V}\frac{Q'}{2}\frac{m}{(m+n)}\left\{1 + p\frac{n}{m+n}\right\} \qquad \text{(43}_l\text{)},$$

$$\frac{1}{k}\partial k_c = p\frac{\partial V}{V}\frac{Q'}{2}\frac{m}{(m+n)}\left\{1 + p\frac{m}{m+n}\right\}^{-1} \qquad \text{(43}_c\text{)}.$$

For any alteration whatsoever of elasticity at or close to the axis the pitch is raised or lowered according as the elastic constant m is increased or diminished. Thus m takes the place that the bulk modulus occupies in the corresponding case in the sphere.

§ 69. Next suppose the alteration of material to take place throughout a surface layer of thickness t. Then, remembering that U_a is zero, we easily obtain from (27)

$$\frac{\partial k}{k} \div \frac{t}{a} = -\frac{k^2\alpha^2a^2\dfrac{\rho_1 - \rho}{\rho} - \dfrac{4mn}{(m+n)^2}\left(\dfrac{m^{-1} + n^{-1}}{m_1^{-1} + n_1^{-1}} - 1\right)}{k^2\alpha^2a^2 - \dfrac{4mn}{(m+n)^2}} \qquad \text{(44)}.$$

The values of $\partial k/k$, when the density at the surface alone is altered, are shown in the following table for the first four notes answering to the values 0, ·25 and ·5 of σ:—

TABLE III.

Value of $\frac{-\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1 - \rho}{\rho}\right)$ for a surface layer.

Number of note		(1)	(2)	(3)	(4)
$\sigma =$	0	1·418	1·036	1·014	1·007
	·25	1·262	1·031	1·012	1·006
	·5	1·0	1·0	1·0	1·0

If in (44) we suppose

$$\frac{m^{-1} + n^{-1}}{m_1^{-1} + n_1^{-1}} - 1 = \frac{\rho_1}{\rho} - 1 = q \qquad (45),$$

then it at once reduces to

$$\frac{\partial k}{k} = -\frac{t}{a} q \qquad (46).$$

Thus we derive at once from Table III. the following results for the change of pitch due to a surface alteration of elasticity alone:—

TABLE IV.

Value of $\frac{\partial k}{k} \div \left\{\frac{t}{a}\left(\frac{m^{-1} + n^{-1}}{m_1^{-1} + n_1^{-1}} - 1\right)\right\}$ for a surface layer.

Number of note		(1)	(2)	(3)	(4)
$\sigma =$	0	·418	·036	·014	·007
	·25	·262	·031	·012	·006
	·5	0	0	0	0

A comparison of Tables III. and IV. leads to many interesting results as to the relative importance of surface alterations of density and elasticity in changing the pitch of the fundamental and higher notes.

The most important of these results is that if a thin surface layer of an isotropic cylinder be altered in any way consistent with its remaining isotropic, then the ratios of the frequencies of all the higher notes can only be slightly affected; but, unless the value of σ for the unaltered material be near the limiting value ·5, or else both density and elasticity be altered in such a way as approximately to satisfy (45), the ratio of the frequency of the fundamental note to that of any of the higher notes may be sensibly disturbed.

§ 70. It will be necessary to restrict our discussion of (27) to some special forms of alteration of material. We may in every case modify the function of $\kappa\alpha a$ that appears in the expression for ∂k by any substitution that supposes (18) to be exactly true.

(1) Suppose the layer to differ from the remainder only in density. We have already seen that the change of pitch is then always zero when the layer is axial. We may thus employ without restriction a formula in which the alteration of mass per unit length of cylinder is represented by

$$\partial M = 2\pi b (b - c)(\rho_1 - \rho).$$

Denoting by t the thickness $b-c$ of the layer, and by M the original mass $\pi a^2\rho$ of the cylinder per unit length, we find from (27)

$$\frac{\partial k}{k} = -\frac{t}{a}\frac{\rho_1-\rho}{\rho} Qkab\,\{J_1(kab)\}^2 = -\frac{\partial M}{M}\frac{Q'}{2}\{J_1(kab)\}^2 \quad\text{.....................(47).}$$

(2) Suppose the layer to differ from the remainder only in the value of m. Employing the well-known relations between successive Bessel's functions, we obtain from (27)

$$\frac{\partial k}{k} = \frac{t}{a}\frac{m_1-m}{m_1+n} Qkab\,\{J_0(kab)\}^2 = \frac{\partial V}{V}\frac{m_1-m}{m_1+n}\frac{Q'}{2}\{J_0(kab)\}^2 \quad\text{................(48);}$$

where

$$V=\pi a^2, \qquad \partial V = 2\pi(b-c)\,b.$$

This formula it will be remembered happens to apply for an axial core as well as an axial layer.

(3) Suppose the layer to differ only in the value of n. We find

$$\frac{\partial k}{k} = \frac{t}{a}\frac{n_1-n}{m+n_1} Qkab\,\{J_2(kab)\}^2 = \frac{\partial V}{V}\frac{n_1-n}{m+n_1}\frac{Q'}{2}\{J_2(kab)\}^2 \quad\text{..................(49).}$$

This vanishes for an axial layer.

(4) Suppose both elastic constants to be altered in the same proportion according to (42), then by (27) for any true layer

$$\frac{\partial k}{k} = \frac{t}{a}\frac{p}{1+p} Qkab\left[\left\{J_1'(kab)+\frac{m-n}{m+n}\frac{J_1(kab)}{kab}\right\}^2 + \frac{4mn}{(m+n)^2}(1+p)\left\{\frac{J_1(kab)}{kab}\right\}^2\right] \quad\text{.....(50).}$$

An alternative formula applicable under the usual restriction may be obtained by the substitution

$$\frac{t}{a}\,Q\,kab = \frac{\partial V}{V}\frac{Q'}{2}.$$

§ 71. Comparing the expressions (47), (48), (49) and (50), we notice that each is a product of three factors of the usual kind.

Except in the case of (50), where the third factor is a function of σ and of the magnitude of the alteration of material, we may very easily construct curves *, whose abscissae are the values of x, $\equiv kab$, to represent the variation in the magnitude of $\partial k/k$ with the position of the layer.

The equations to these simple curves are

$$y = x\,\{J_1(x)\}^2 = f_4(x) \quad\text{.......................................(51),}$$

$$y = \{J_1(x)\}^2 = f_3(x) \quad\text{.......................................(52),}$$

$$y = x\,\{J_0(x)\}^2 = f_2(x) \quad\text{.......................................(53),}$$

$$y = \{J_0(x)\}^2 = f_1(x) \quad\text{.......................................(54),}$$

$$y = x\,\{J_2(x)\}^2 = f_8(x) \quad\text{.......................................(55),}$$

$$y = \{J_2(x)\}^2 = f_7(x) \quad\text{.......................................(56).}$$

These curves apply whatever be the value of σ in the material. Full information as to their use is recorded in the following table:—

* On account of the difference in the values of Q for the sphere and cylinder, the ordinates of the curves of Plate V. should be increased in the ratio $\pi : 2$ for comparison with Plate IV.

TABLE V.

Function of x.	Property of material altered	Layer of given	Figure where curve drawn	Letter attached to curve
$f_3(x)$	ρ	volume	6	A
$f_4(x)$	ρ	thickness	6	B
$f_1(x)$	m	volume	7	A
$f_2(x)$	m	thickness	7	B
$f_7(x)$	n	volume	8	A
$f_8(x)$	n	thickness	8	B

After the long discussion of the corresponding curves in the case of a sphere, it is hardly necessary to say more than that the use of the present curves is exactly the same as that of the previous. Each of the curves of Table V. applies to all materials and notes. The ratios of its successive maxima ordinates are the ratios of the several maxima changes of pitch due to the given assigned alteration of material.

Since the factor by which the ordinates of all the curves B are to be multiplied to get the numerical magnitude of the change of pitch is Q, the curves supply us immediately, supposing them drawn on the same scale, with a comparison of the changes of pitch, of any given note in any given cylinder, accompanying independent alterations of material throughout a layer of given thickness such that

$$(\rho_1 - \rho)/\rho = (m_1 - m)/(m_1 + n) = (n_1 - n)/(m + n_1) \qquad \text{.........................(57).}$$

Again for the higher notes the values of Q are nearly constant and independent of σ; thus in any one of the three cases when ρ alone is altered, when m alone is altered, or when n alone is altered throughout a layer of given thickness, the maxima percentage changes of frequency of any given number are approximately the same for all the higher notes and for all isotropic materials.

In the case of all the A curves the factor is $Q'/2$, thus the curves, if drawn on the same scale, supply at once a comparison of the changes of pitch of any given note in any given cylinder accompanying independent alterations of material, satisfying (57), throughout a layer of given volume.

Also since the higher values of Q' increase approximately in arithmetical progression and are practically independent of σ, it follows that when ρ alone is altered, when m alone is altered, or when n alone is altered throughout a layer of given volume, the maximum percentage change of pitch of any number (j) in a note of number (i), which is greater than 2, exceeds the maximum percentage change of pitch of number (j) in the note of number ($i-1$) in the same cylinder by a quantity which is practically independent of i or of σ and may be regarded as depending only on j.

The factors, viz. the reciprocals of $k\alpha a$, by which an abscissa x must be multiplied to supply the corresponding value of b/a are given in the following table:—

TABLE VI.

Values of $1/k\alpha a$.

Number of note	(1)	(2)	(3)	(4)
$\sigma =$ 0	·5431	·1876	·1171	·0854
$\sigma =$ ·25	·4832	·1853	·1166	·0852
$\sigma =$ ·5	·4160	·1812	·1156	·0848

Approximate values of these multipliers in any of the higher notes may be easily derived from the consideration that their reciprocals $k\alpha a$ are nearly independent of σ and increase approximately in an arithmetical progression with a common difference π.

§ 72. The functions of Table V., and several others whose occurrence will subsequently be explained, are tabulated in Table VII. For the data necessary in making the calculations I am indebted to the tables of $J_0(x)$ and $J_1(x)$ in Lommel's work. I have in no case gone beyond the value 15 of x. The necessity of carrying the calculations further may in general be avoided, as the following considerations show.

We have already seen in § 67 that the maxima of $\{J_1(x)\}^2$ when x becomes large vary approximately as the reciprocals of the corresponding values of x, and so tend to become small; while the maxima of $x\{J_1(x)\}^2$ tend to approach a finite constant value. Now the same results may be proved in a similar way for any Bessel's function $J_i(x)$.

Thus a glance at equations (51)—(56) suffices to show that the successive maxima ordinates of any one of the curves A of Table V. diminish rapidly as the radii of the corresponding positions of the layer increase, while the successive maxima of any one of the curves B continually approach to equality. Consequently unless very great accuracy is required it is unnecessary to draw either set of curves for large values of x.

The other functions occurring in Table VII. present themselves in the treatment of (50). The form of $f_5(x)$ is given by (77), of $f_6(x)$ by (78), of $f_9(x)$ by (79) with $\sigma=0$, of $f_{10}(x)$ by (80) with $\sigma=0$, of $f_{11}(x)$ by (81), and of $f_{12}(x)$ by (82).

This last group of functions are also represented by curves, but these must be combined in pairs so as to form *compound* curves, or else apply only for special values of σ. The ordinates of these curves have to be multiplied by Q or $Q'/2$, and their abscissae by the factors given in Table VI. according to circumstances.

TABLE VII.

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$	$f_{10}(x)$	$f_{11}(x)$	$f_{12}(x)$
0	1	0	0	0	·25	0	0	0	·25	0	$\cdot\dot{6}$	0
·1	·9950	·0995	·0025	·0002							·6633	·0663
·2	·9801	·1960	·0099	·0020	·2475	·0495	·00002	·000005	·2426	·0485	·6534	·1307
·4	·9224	·3689	·0384	·0154			·00039	·00016	·2212	·0885	·6150	·2460
·6			·0822	·0493	·2283	·1370	·0019	·0011				
·8	·7162	·5730					·0057	·0046	·1484	·1187	·4794	·3835
·9	·6521	·5869									·4377	·3939
1·0	·5855	·5855	·1936	·1936	·1936	·1936	·0132	·0132	·1057	·1057	·3948	·3948
·2	·4504	·5405					·0254	·0305	·0655	·0786	·3087	·3705
·4							·0430	·0602				
·6	·2074	·3318	·3248	·5196	·1269	·2030	·0660	·1056	·0098	·0158	·1603	·2564
·8			·3382	·6087	·1044	·1879	·0937	·1687				
2·0	·0501	·1002	·3326	·6652	·0832	·1663	·1245	·2490	·0042	·0083	·0749	·1498
·2	·0122	·0268	·3091	·6800	·0639	·1405					·0601	·1323
·4	·00001	·00002	·2706	·6494	·0470	·1127	·1857	·4458	·0459	·1102	·0619	·1486
·6	·0094	·0244									·0765	·1988
·8	·0342	·0959							·1098	·3074	·0989	·2769
3·0			·1150	·3449	·0128	·0383	·2363	·7088				
·2	·1025	·3281					·2338	·7482	·1615	·5168	·1463	·4681
·4			·0321	·1092	·0028	·0094	·2206	·7502	·1739	·5912		
·6	·1535	·5525	·0091	·0328	·0007	·0025	·1979	·7123	·1750	·6299	·1683	·6058
·8	·1621	·6158							·1648	·6262	·1639	·6227
·9	·1615	·6297									·1578	·6154
4·0	·1577	·6309	·0044	·0174	·00027	·00109	·1326	·5304	·1449	·5795	·1493	·5974
·4	·1171	·5154	·0411	·1809		·00934	·0625	·2752	·0877	·3860	·0989	·4353
·8	·0578	·2775	·0891	·4277		·01856	·0135	·0646	·0318	·1525	·0430	·2065

TABLE VII. *continued.*

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$	$f_{10}(x)$	$f_{11}(x)$	$f_{12}(x)$
5·0			·1073	·5365		·02146	·0022	·0108				
·2	·0122	·0632	·1178	·6126		·02266	·0005	·0024	·0020	·0102	·0083	·0430
·4			·1193	·6440		·02209					·0036	·0196
·6	·0007	·0041	·1118	·6260		·01996	·0214	·1200	·0075	·0421	·0076	·0427
·8							·0396	·2296				
6·0	·0227	·1362	·0766	·4593		·01276	·0590	·3539	·0387	·2323		
·4	·0592	·3789	·0330	·2112		·00515	·0900	·5763	·0738	·4724		
·6							·0973	·6421	·0858	·5665		
·7							·0983	·6586				
·8	·0859	·5841	·0043	·0289		·00063	·0975	·6631	·0916	·6230		
7·0	·0900	·6303					·0909	·6360	·0905	·6332		
·2	·0871	·6269	·0030	·0213					·0827	·5953		
·6	·0633	·4811	·0253	·1927			·0440	·3342	·0532	·4043		
8·0	·0295	·2357	·0551	·4404			·0128	·1021	·0203	·1620		
·2							·0035	·0288	·0082	·0675		
·4	·0048	·0402	·0733	·6159			·00002	·00018				
·5			·0746	·6340								
·6	·0002	·0018	·0744	·6398			·0024	·0205				
·8							·0098	·0867				
9·0	·0082	·0734	·0602	·5416			·0210	·1888				
·4	·0312	·2928					·0464	·4362				
·6			·0195	·1869								
·8	·0540	·5287	·0086	·0845			·0631	·6185				
10·0	·0605	·6048	·0019	·0189			·0648	·6484				
·2	·0623	·6355					·0617	·6290				

TABLE VII. *continued.*

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$	$f_{10}(x)$	$f_{11}(x)$	$f_{12}(x)$
10·3	·0614	·6320										
·4	·0592	·6160	·0031	·0320			·0542	·5632				
·6												
·8	·0413	·4459	·0202	·2183			·0313	·3379				
11·0												
·2	·0177	·1981	·0416	·4654			·0093	·1045				
·4							·0026	·0299				
·6	·0020	·0231	·0538	·6243			·00002	·00025				
·7			·0544	·6368								
·8			·0540	·6367								
12·0	·0023	·0273	·0499	·5991								
·2												
·4	·0168	·2081	·0327	·4049								
·6												
·8	·0356	·4558	·0124	·1589								
13·0			·0049	·0643								
·2	·0470	·6198	·0007	·0097								
·3	·0477	·6338										
·6												
·8	·0378	·5212										
14·0												
·2	·0200	·2838										
·4												
·6	·0046	·0672										
·8	·0007	·0109										

§ 73. We may now examine the four special cases in detail.

When the layer differs from the remainder only in density the change of pitch is given by (47). The law of variation of $\partial k/k$ with the position of the layer is thus independent of the magnitude of the alteration of density.

The positions of the layer when the pitch of a given note is unaffected coincide with the node surfaces for that note. When the layer is in any other position the pitch is raised or lowered according as the density is diminished or increased.

When the layer of altered density is of given volume the curve showing the dependence of the change of pitch on the value of kab is A fig. 6, whose equation is (52). The maxima ordinates answer to positions of the layer coincident with the loop surfaces.

The first maximum ordinate is much the largest. For the ratios it bears to the succeeding maxima ordinates, and so for the ratios of the first to the succeeding maxima changes of pitch I find

$$1 : \cdot 3539 : \cdot 2206 : \cdot 1608 \ldots\ldots$$

Employing these ratios, all the maxima in the case of the first four notes can be calculated from the numerical magnitudes of the first maxima which are given in the following table:—

TABLE VIII.

First maximum of $-\dfrac{\partial k}{k} \div \dfrac{\partial M}{M}$.

Value of σ	Number of note	(1)	(2)	(3)	(4)
0		·709	1·465	2·298	3·133
·25		·655	1·463	2·298	3·133
·5		·628	1·462	2·297	3·133

For any of the higher notes approximations to the numerical magnitude of the first maximum change of pitch can easily be obtained by the consideration that these numbers increase approximately in an arithmetical progression with the number of the note. Thus for any note of number (i), greater than 4, a close approximation to the first maximum is given for any value of σ by

$$-\frac{\partial k}{k} \div \frac{\partial M}{M} = 3\cdot 133 + (i-4) \times \cdot 835 \ldots\ldots\ldots\ldots (58).$$

In these higher notes the next three maxima changes of pitch can be obtained from the ratios already given in this paragraph. The maxima of higher number can be obtained to a less close degree of approximation from the consideration that the reciprocals of the successive maxima changes of pitch in a given note are approximately in arithmetical progression. Thus from the values for the ratios of successive maxima already given in this paragraph we find as a fairly close approximation to the maximum change of pitch of number (j) in the note of number (i), supposing i and j both greater than 4,

$$-\frac{\partial k}{k} \div \frac{\partial M}{M} = \frac{3\cdot 133 + (i-4) \times \cdot 835}{6\cdot 22 + (j-4) \times 1\cdot 67} \ldots\ldots\ldots\ldots (59).$$

§ 74. When the layer of altered density is of given thickness the mode of variation of $\partial k/k$ with kab is shown by curve B fig. 6, whose equation is (51).

The abscissae supplying the maxima ordinates are the roots greater than zero of

$$2xJ_1'(x) + J_1(x) = 0 \quad \text{...(60)}.$$

Their approximate values are 2·166, 5·427, 8·595, 11·749...

When $\sigma = \dot{3}$ the equations (60) and (36) are identical, and so the positions of the layer supplying the maxima changes of pitch are coincident with the no-stress surfaces. For other materials these positions lie outside or inside the no-stress surfaces according as σ is less or greater than $\dot{3}$. For all values of σ they lie outside the loop surfaces.

When $\sigma = \dot{3}$ one of the positions supplying a maximum of $\partial k/k$ coincides with the cylindrical surface, and for this and all larger values of σ the number of maxima is equal to the number of the note. For values of σ less than $\dot{3}$ the number of maxima is less by 1 than the number of the note. Thus in note (1) there is no *true* maximum, the value of $\partial k/k$ increasing continually as the layer moves out from the axis to the surface.

The following table gives the positions of the layer corresponding to all the maxima in the case of the first four notes for the values 0, ·25 and ·5 of σ:—

TABLE IX.

Values of b/a supplying maxima of $-\dfrac{\partial k}{k} \div \left(\dfrac{t}{a}\dfrac{\rho_1 - \rho}{\rho}\right)$.

	Note (1)			Note (2)			Note (3)			Note (4)		
$\sigma =$	0	·25	·5	0	·25	·5	0	·25	·5	0	·25	·5
	—	—	·901	·406	·401	·392	·254	·253	·250	·185	·185	·184
				—	—	·983	·636	·633	·627	·464	·462	·460
							—	—	·993	·734	·732	·729
										—	—	·996

The blanks are intended to draw attention to the absence of true maxima. A comparison with Table I. will be found instructive.

For the ratios of the first to the successive maxima ordinates of curve B, and so of the first to the subsequent maxima changes of pitch, I find

1 : ·947 : ·940 : ·938...

The absolute values of the first and largest maxima are given in the following table for the first four notes:—

TABLE X.

First maximum of $-\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right)$.

	Note (1)			Note (2)			Note (3)			Note (4)		
$\sigma =$	0	·25	·5	0	·25	·5	0	·25	·5	0	·25	·5
	1·418*	1·262*	1·050	1·104	1·090	1·065	1·082	1·077	1·067	1·076	1·073	1·068

Asterisks are attached to the entries for the values 0 and ·25 of σ under note (1) to show that they are not *true* maxima. They do not answer to the first maximum ordinate of curve B fig. 6, but to positions of the layer at the surface of the cylinder.

From the results already obtained as to the values of Q in the higher notes and as to the maxima of $x\{J_1(x)\}^2$ answering to large values of x, we are enabled to conclude that, for any note whose number exceeds 4 and for any value of σ, a close approximation to the first maximum of $-\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right)$ is 1·07, and to any maximum whose number exceeds 3 a pretty close approximation is 1·00.

§ 75. In the case when the layer differs from the remainder only in the value of m the change of pitch is given by (48). From this it appears that the law of variation of $\partial k/k$ with the position of the layer is independent of the magnitude of the alteration of elasticity.

The positions of the layer when the change of pitch vanishes are found by equating κab to the roots of (38). They thus coincide with the no-stress surfaces when $\sigma = ·5$, and for all other values of σ they lie outside of the no-stress surfaces though very close to all except the first.

When the layer is of given volume the curve showing the variation of the change of pitch with κab is A fig. 7, the equation to which is (54).

The ordinate at the origin is much the largest in the curve. Thus the change of pitch which arises when the altered material forms an axial layer is far the largest maximum.

The magnitude of the change of pitch due to any assigned alteration of elasticity throughout an axial layer has been already determined in § 68, the necessary formula in the present case coinciding with (41_c). The numerical magnitude is obtained at once by dividing by 2 the values supplied for Q' in Table II. and altering the heading from Q' to $\frac{1}{k}\partial k \div \left(\frac{\partial V}{V} \frac{m_1 - m}{m_1 + n}\right)$. For a note of number (i) above the fourth we obtain from (40) as an approximate formula

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{m_1 - m}{m_1 + n}\right) = \tfrac{1}{2}\{18·51 + (i - 4) \times 4·935\} \qquad (61).$$

The abscissae supplying the subsequent maxima ordinates are the roots of (34). Thus the corresponding positions of the layer coincide with the node surfaces. For the ratios of the first to the subsequent maxima ordinates, and so for the ratios of the first to the subsequent maxima changes of pitch, I find

$$1 : \cdot 162 : \cdot 090 : \cdot 062 \ldots$$

From considerations as to the values of those maxima of $\{J_0(x)\}^2$ which answer to large values of x, of an exactly analogous nature to those discussed in § 67, it may be proved that a fairly close approximation to the maximum change of pitch of number (j) in the note of number (i), i and j being both greater than 4, is supplied by

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V}\frac{m_1 - m}{m_1 + n}\right) = \frac{18\cdot 51 + (i-4) \times 4\cdot 935}{32\cdot 08 + (j-4) \times 9\cdot 87} \qquad (62).$$

In this formula j may equal but cannot exceed i, as the number of maxima, being equal to the number of node surfaces, including the axis, is equal to the number of the note.

§ 76. When the layer whose m differs from that of the remainder is of given thickness the curve showing the variation of the change of pitch with κab is B fig. 7, the equation to which is (53).

The abscissae supplying the maxima ordinates are the roots greater than zero of

$$(2x^2 - 1) J_1(x) - xJ_1'(x) = 0 \qquad (63).$$

For the first two roots I find approximately ·9408 and 3·9594.

It is easily proved that the positions of the layer answering to the maxima changes of pitch whose numbers exceed 2 lie outside of but very close to the corresponding node surfaces. The positions of the layer answering to the first two maxima are given in the following table for the first four notes and the usual values of σ:—

TABLE XI.

Values of b/a where $\dfrac{\partial k}{k} \div \left(\dfrac{t}{a}\dfrac{m_1 - m}{m_1 + n}\right)$ is a maximum.

	Note (1)			Note (2)			Note (3)			Note (4)		
$\sigma =$	0	·25	·5	0	·25	·5	0	·25	·5	0	·25	·5
	·511	·455	·391	·176	·174	·170	·110	·110	·109	·0804	·0802	·0798
				·743	·734	·717	·464	·462	·458	·338	·337	·336

As the second maximum ordinate is very nearly equal to all the subsequent maxima, and is decidedly greater than the first, I have included in the following table the first two maxima changes of pitch. For note (1) of course there is only one maximum.

TABLE XII.

Maxima values of $\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{m_1 - m}{m_1 + n}\right)$.

	Note (1)			Note (2)			Note (3)			Note (4)		
$\sigma =$	0	·25	·5	0	·25	·5	0	·25	·5	0	·25	·5
	1·338	1·099	·907	·954	·942	·920	·935	·931	·922	·930	·928	·923
				1·026	1·012	·989	1·005	1·000	·991	·999	·997	·992

The number of maxima is always equal to the number of the note so that the table gives all the maxima only for the first two notes.

In the higher notes for all values of σ pretty close approximations are

$$\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{m_1 - m}{m_1 + n}\right) = \cdot 924 \quad \text{.......................................(64)}$$

for the first maximum change of pitch, and

$$\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{m_1 - m}{m_1 + n}\right) = 1\cdot 00 \quad \text{.......................................(65)}$$

for the second and all subsequent maxima.

§ 77. In the third special case, when the layer differs from the remainder only in rigidity, the change of pitch is given by (49). This shows that the law of variation of $\partial k/k$ with the position of the layer is independent of the magnitude of the alteration of rigidity.

The positions of the layer when the change of pitch vanishes are found by equating kab to the roots of

$$J_2(x) = 0 \quad \text{...(66),}$$

viz.* 0, 5·135, 8·417, 11·620... .

The higher roots are of course only approximate. The root $x = 0$ applies whether the layer be of given volume or of given thickness, so that the axis is always one of the positions where an alteration in rigidity does not affect the pitch.

Whatever be the value of σ, the second and higher roots of the frequency equation (36) are slightly larger than the second and higher roots of (66). Thus counting the axis, the number of positions of the layer for which ∂k vanishes is always equal to the number of the note. Also these positions commencing with the second are close to but inside of the successive no-stress surfaces commencing with the second. It seems unnecessary to determine these positions more precisely. All the data necessary in the case of the first four notes and the usual values of σ are given above.

* See Lord Rayleigh's *Theory of Sound*, Vol. I. p. 274.

When the layer of altered rigidity is of given volume the curve showing the variation of the change of pitch with the position of the layer is A fig. 8, the equation to which is (56).

There is, it will be noticed, a very close resemblance both in magnitude and position between the segments of this curve which are most remote from the origin and the segments of curve A fig. 7. The first segment however of the present curve would seem to answer to the whole of curve A fig. 7 between the origin and the second zero ordinate.

The abscissae supplying the maxima ordinates of the present curve are the roots greater than zero of

$$(x^2-2)\,J_1(x)+2xJ_1'(x)=0 \qquad \text{.................................(67).}$$

For their approximate values I find 3·054, 6·706, 9·9695, 13·170....

When i is greater than 2 the $(i-1)^{\text{th}}$ root of (67), omitting zero, is near but always less than the i^{th} root of (34), the equation which determines the position of the node surfaces. The first root of (67) is however noticeably less than the second root of (34). The number of true maxima being one less than the number of node surfaces is one less than the number of the note. In particular there is no true maximum for note (1). The following table gives the positions of the layer supplying the true maxima in the first four notes for the values 0 and ·25 of σ:—

TABLE XIII.

Values of b/a where $\dfrac{\partial k}{k} \div \left(\dfrac{\partial V}{V}\,\dfrac{n_1-n}{m+n_1}\right)$ is a maximum.

	Note (2)	Note (3)		Note (4)		
$\sigma = 0$	·573	·358	·786	·261	·573	·852
$\sigma = ·25$	·566	·356	·782	·260	·572	·850

For the ratios of the first to the subsequent maxima ordinates, and so for the ratios of the first to the subsequent true maxima changes of pitch, I find approximately

1 : ·415 : ·274 : ·206......

The numerical values of the first maxima in notes (2), (3) and (4), and of the greatest possible change of pitch in the case of note (1) are given by the following table for the values 0 and ·25 of σ:—

TABLE XIV.

First maximum of $\frac{\partial k}{k} \div \left(\frac{\partial V}{V}\frac{n_1 - n}{m + n_1}\right)$.

	Note (1)		Note (2)		Note (3)		Note (4)	
$\sigma =$	0	·25	0	·25	0	·25	0	·25
	·209*	·262*	1·024	1·023	1·606	1·606	2·190	2·190

The asterisks are intended to draw attention to the fact that the entries under note (1) are not *true maxima*. The influence of σ in the case of the higher notes is practically nil.

As fairly approximate values for the first and for the j^{th} maximum respectively in note (i), supposing i and j both greater than 4, we may take

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V}\frac{n_1 - n}{m + n_1}\right) = 2{\cdot}190 + (i - 4) \times {\cdot}584 \quad\text{.........................(68)},$$

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V}\frac{n_1 - n}{m + n_1}\right) = \frac{2{\cdot}190 + (i - 4) \times {\cdot}584}{4{\cdot}85 + (j - 4) \times 1{\cdot}20} \quad\text{.........................(69)}.$$

These equations hold for all values of σ. For values of j less than 4 the ratios given above should be used.

§ 78. When the layer of altered rigidity is of given thickness the curve showing the variation of $\partial k/k$ with the value of κab is B fig. 8, the equation to which is (55).

In general we see that when i is greater than 2 the $(i - 1)^{\text{th}}$ segment of curve B fig. 8 corresponds pretty closely in position and magnitude of ordinates to the i^{th} segment of curve B fig. 7.

The abscissae supplying the maxima ordinates of curve B fig. 8 are the roots greater than zero of

$$(2x^2 - 3)\, J_1(x) + 3x J_1'(x) = 0 \quad\text{...............................(70)}.$$

For their approximate values I find 3·311, 6·787, 10·0215, 13·209.... These roots are intermediate between those of (34) and (67).

For note (1) there is no true maximum, as the number of maxima is one less than the number of the note. The positions of the layer supplying all the maxima in notes (2), (3) and (4) for the values 0 and ·25 of σ are shown in the following table:—

TABLE XV.

Values of b/a where $\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{n_1 - n}{m + n_1}\right)$ is a maximum.

	Note (2)	Note (3)		Note (4)		
$\sigma = 0$	·621	·388	·795	·283	·580	·856
$\sigma = {\cdot}25$	·614	·386	·791	·282	·578	·854

For the ratios of the first to the two next maxima ordinates, and so of the first to the two next maxima changes of pitch, I find

$$1 : \cdot 880 : \cdot 860.$$

The fourth and subsequent maxima are only very slightly less than the third.

In the following table are given the numerical magnitudes of the first maxima for notes (2), (3) and (4), and of the greatest possible change of pitch in the case of note (1).

TABLE XVI.

First maximum of $\dfrac{\partial k}{k} \div \left(\dfrac{t}{a} \dfrac{n_1 - n}{m + n_1}\right)$.

	Note (1)		Note (2)		Note (3)		Note (4)	
$\sigma =$	0	·25	0	·25	0	·25	0	·25
	·418*	·524*	1·224	1·208	1·199	1·194	1·192	1·189

The asterisks under note (1) indicate as usual that the entries are not *true* maxima.

From the table, with the assistance of the ratios given above, all the maxima in the notes (2), (3) and (4) may be calculated.

In notes above the fourth a pretty close approximation to the first maximum will be given for all values of σ by

$$\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{n_1 - n}{m + n_1}\right) = 1 \cdot 185 \qquad (71).$$

From this and the ratios given above, the values of the two next maxima may be found. For maxima of number greater than (3) in these higher notes we may take approximately

$$\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{n_1 - n}{m + n_1}\right) = 1 \cdot 00 \qquad (72).$$

§ 79. In the fourth special case the change of pitch is given by (50). For the limiting value ·5 of σ this assumes the simple form

$$\frac{\partial k}{k} = \frac{t}{a} \frac{p}{1+p} Q x \{J_0(x)\}^2 = \frac{\partial V}{V} \frac{p}{1+p} \frac{Q'}{2} \{J_0(x)\}^2 \qquad (73),$$

writing x for kab.

This becomes identical with (48) when the factor $(m_1 - m)/(m_1 + n)$ of that equation is replaced by $p/(1+p)$. Thus the conclusions already come to in the case when m alone is altered apply also to the present case for $\sigma = \cdot 5$ with merely a change in phraseology.

Except in this extreme case the expression (50) for the change of pitch is the *sum* of *two* squares, which cannot simultaneously vanish unless $b/a = 0$. Further we see from § 68 that when an alteration of material of this kind takes place throughout an axial layer of given volume the signs of ∂k and p are the same. Thus an alteration of both elastic constants in the same proportion throughout a layer of given volume necessarily affects the pitch wherever it occurs, and the pitch is raised or lowered according as the elasticity of the layer is increased or diminished.

In considering (50) it will be convenient to consider separately the two squares by writing

$$\partial k = \partial k_1 + \partial k_2 \quad\text{..........(74)};$$

where, σ denoting as usual Poisson's ratio,

$$\frac{\partial k_1}{k} = \frac{t}{a} pQ \frac{1-2\sigma}{(1-\sigma)^2} \frac{1}{\kappa ab} \{J_1(\kappa ab)\}^2 \quad\text{..........(75)},$$

$$\frac{\partial k_2}{k} = \frac{t}{a} \frac{p}{1+p} Q\kappa ab \left\{J_1'(\kappa ab) + \frac{\sigma}{1-\sigma} \frac{J_1(\kappa ab)}{\kappa ab}\right\}^2 \quad\text{..........(76)}.$$

The numerical magnitude of ∂k_1 is independent of the sign of p, whereas ∂k_2 is numerically greater for a given negative value of p than for an equal positive value.

Again ∂k_1 depends on the square of the displacement and so vanishes when the altered layer is at a node surface. The more remote positions of the layer supplying the maxima of ∂k_1 in the case of the higher notes are inside of but close to the loop surfaces of higher number whether the layer be of given volume or of given thickness. On the other hand ∂k_2 depends on the square of the radial stress. It thus vanishes when the altered layer is at a no-stress surface, and when the layer is of given volume it has its maxima when the layer coincides with those surfaces over which the radial stress is a maximum.

Further the law of variation of ∂k_1 with κab is independent of the value of σ, but the maxima of ∂k_1 diminish rapidly and become insignificant as σ approaches near to its limiting value ·5. On the other hand so long as κab is small the law of variation of ∂k_2 with κab depends largely on the value of σ.

In the case of notes (1) and (2), or for positions of the layer inside the third node surface in the case of the higher notes, the contribution of ∂k_1 to the change of pitch cannot in general be neglected. For more remote positions of the layer, however, in the case of the higher notes $\partial k_1/\partial k_2$ is always insignificant, except in the immediate neighbourhood of the no-stress surfaces where ∂k_2 vanishes. Thus so far as the maxima changes of pitch are concerned the error introduced by neglecting ∂k_1 is very trifling when the layer lies outside of the third node surface in the case of the higher notes.

It may also be proved from (76) that the value of σ has very little influence on the maxima of ∂k_2 of number higher than 2.

We thus conclude that for practical purposes the change of pitch due to the alteration of elasticity of the kind under discussion is given to a very fair degree of approximation by (73) for all values of σ, provided the layer lie outside of the third node surface of the note considered.

§ 80. When the change of pitch is wanted for positions of the layer answering to small values of kab, it will in general be best to construct separately curves showing the variation of $\partial k_1/k$ and $\partial k_2/k$, and then derive from them compound curves.

For the variation of $\partial k_1/k$ we have the curves

$$y = x^{-2}\{J_1(x)\}^2 = f_5(x) \qquad\qquad (77),$$

or

$$y = x^{-1}\{J_1(x)\}^2 = f_6(x) \qquad\qquad (78),$$

according as the layer is of given volume or given thickness. These curves are those styled C and D respectively in fig. 6.

Between the origin and the next zero ordinate of curve D,—which answer to positions of the layer at the first and second node surfaces respectively—the ordinates of both curves are far from insignificant compared to the ordinates of the other curves.

Beyond the third zero ordinate—which answers to a position of the layer at the third node surface—I have not drawn the curve D. Its successive segments become rapidly flatter, as may be seen at once from the consideration that in fig. 6 the ordinate of curve A is the geometric mean of the ordinates of curves B and D.

The curve C is drawn only as far as its first zero ordinate, answering to the second node surface. An idea of the extreme flatness of the other segments is easily derived from the consideration that the ordinate of curve D is the geometric mean of the ordinates of curves A and C.

For the variation of $\partial k_2/k$ we have the curves

$$y = \left\{J_1'(x) + \frac{\sigma}{1-\sigma}\frac{J_1(x)}{x}\right\}^2 = f_9(x) \qquad\qquad (79),$$

$$y = xf_9(x) = f_{10}(x) \qquad\qquad (80),$$

according as the layer is of given volume or given thickness.

These curves are drawn for the special value 0 of σ in fig. 9 and are styled respectively A and B. Both have zero ordinates answering to positions of the layer at all the loop surfaces. At the origin the ordinate of curve A is precisely equal to that of curve C, fig. 6, and for all other values of x less than 2 the ordinates of the latter curve are the larger. In fact the ordinates of curve A do not markedly predominate over those of curve C, fig. 6, until the layer has passed well outside of the first loop surface.

Curve B fig. 9 has a zero ordinate at the origin, and the first segment lies completely inside the first segment of curve D fig. 6. The great predominance, however, of the second and subsequent maxima ordinates of curve B over the second and subsequent maxima ordinates of curve D fig. 6 is a complete justification of what has been said of the general insignificance of $\partial k_1/\partial k_2$ for positions of the layer outside the third or even the second node surface.

In the case just considered when $\sigma = 0$, the compound curve is constructed, according as the layer is of given volume or of given thickness, by adding the ordinate of curve C fig. 6 multiplied by $1+p$ to the ordinate of curve A fig. 9, or by adding the ordinate of curve D fig. 6 multiplied by $1+p$ to the ordinate of curve B fig. 9. The quantities represented by these compound curves are respectively

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{p}{1+p} \frac{Q'}{2}\right) \text{ and } \frac{\partial k}{k} \div \left(\frac{t}{a} \frac{p}{1+p} Q\right).$$

§ 81. As a complete graphical representation of the law of variation of $\partial k/k$ with small values of kab for some one case when the elastic constants are altered in the same proportion seems desirable, I have considered the most important special case, viz. when p is so small that p^2 is negligible and σ has the value ·25.

In this case for layers of constant volume and of constant thickness respectively, the curves are

$$y = \{J_0(x)\}^2 - \tfrac{4}{3} x^{-1} J_1(x) J_1'(x) = f_{11}(x) \quad \text{.......................(81)},$$

$$y = x f_{11}(x) = f_{12}(x) \quad \text{.......................(82)}.$$

These are styled A and B respectively in fig. 10, and the quantities they represent are

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V} p \frac{Q'}{2}\right) \text{ and } \frac{\partial k}{k} \div \left(\frac{t}{a} pQ\right).$$

The marked differences between the earlier portions of these curves and the corresponding portions of the curves A and B of fig. 9 are well worthy of notice.

§ 82. There is still one point worthy of explicit reference. As we have already pointed out, $x^{-1} J_1(x)$ when x is large is in general negligible compared to $J_1'(x)$. Now if we neglect $x^{-1}J_1(x)$ compared to $J_1'(x)$ and suppose the layer to differ from the remainder only in elasticity, we may throw (27) into the simple form

$$\frac{\partial k}{k} = \frac{t}{a}\left(1 - \frac{m+n}{m_1+n_1}\right) Q\, kab\, \{J_1'(kab)\}^2 \quad \text{.......................(83)},$$

a formula which is exact for positions of the layer coincident with any node surface.

Thus when the layer is outside the third or even the second node surface in the case of one of the higher notes, the change of pitch due to an alteration in elasticity alone may be regarded, when of practical importance, as due very approximately to the alteration in a single elastic quantity, viz. $m+n$. This result should be compared with that found for the radial vibrations of a sphere in § 48 Sect. II.

Note to Section IV.

The ultimate practical coincidence of the corresponding curves of figs. 7 and 8, and the fact that their maxima and zero ordinates ultimately almost coincide in position with the zero and maxima ordinates respectively of the curves of fig. 6 are of course entirely due to the relations between

$$J_0(x), \quad J_1(x) \text{ and } J_2(x).$$

We have already pointed out that the successive values of x, when large, which make any given Bessel zero increase very approximately by π, and each is very nearly equidistant from two consecutive values of x which make the square of the Bessel in question a maximum.

Now from the relations between consecutive Bessel's we have

$$-J_0'(x) = J_1(x) = \frac{x}{2}\{J_0(x) + J_2(x)\}, \quad 2J_1'(x) = J_0(x) - J_2(x).$$

Thus when $J_1(x)$ vanishes $\{J_0(x)\}^2$ is a maximum, and when $\{J_1(x)\}^2$ has either its maxima or its zero values we have $\{J_0(x)\}^2 = \{J_2(x)\}^2$.

Thus the higher values of x which make $\{J_0(x)\}^2$ and $\{J_2(x)\}^2$ maxima, and the higher values which make them zero, respectively coincide with or are very close to those higher values of x which make $\{J_1(x)\}^2$ vanish, and those which make it a maximum. Also corresponding maxima of $\{J_0(x)\}^2$ and $\{J_2(x)\}^2$, except the first one or two, are nearly equal.

[November 14, 1891. If while n is altered the bulk modulus $m - n/3$ remains unaltered, the change of pitch is given, writing x for kab, by

$$\frac{\partial k}{k} = \frac{b-c}{a}\frac{n_1 - n}{m_1 + n_1} Q \frac{4}{3} x \left[\{J_1'(x) - \tfrac{1}{2}x^{-1}J_1(x)\}^2 + \frac{3}{4}\frac{m_1 + n_1}{m + n}\{x^{-1}J_1(x)\}^2\right].$$

It has obviously always the same sign as $n_1 - n$.]

SECTION V.

TRANSVERSE VIBRATIONS IN SOLID CYLINDER.

§ 83. In this form of vibration the displacement is at any point at right angles to the plane which contains the point and the axis of the cylinder. Employing $J_1(x)$ and $Y_1(x)$ for the two solutions of

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y\left(1 - \frac{1}{x^2}\right) = 0,$$

we obtain for the displacement in the typical vibration in a shell

$$v/\cos kt = AJ_1(k\beta r) + BY_1(k\beta r) \quad\ldots\ldots(1)^*:$$

where

$$\beta^2 = \rho/n \quad\ldots\ldots(2),$$

and A and B are constants.

In a compound solid cylinder $(0\,.\,\beta\,.\,c\,.\,\beta_1\,.\,b\,.\,\beta\,.\,a)$ where $b - c$ is small, the typical displacements are as follows:—

In the core $\quad v/\cos kt = AJ_1(k\beta r) \quad\ldots\ldots(3).$

In the layer $\quad v/\cos kt = A_1J_1(k\beta_1 r) + B_1Y_1(k\beta_1 r) \quad\ldots\ldots(4).$

Outside the layer $\quad v/\cos kt = (A + \partial A)J_1(k\beta r) + \partial BY_1(k\beta r) \quad\ldots\ldots(5).$

Terms in $(b-c)^2$, and so those of order $(\partial A/A)^2$ or $(\partial B/A)^2$, are as usual neglected.

* *Transactions*, Vol. XIV. Equation (44), p. 356.

If for shortness we put

$$\left.\begin{aligned} F(r\,.\,\beta) &= n\,\{k\beta r J_1'(k\beta r) - J_1(k\beta r)\},\\ F_1(r\,.\,\beta) &= n\,\{k\beta r Y_1'(k\beta r) - Y_1(k\beta r)\}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(6),$$

then the relations connecting the arbitrary constants and leading to the frequency equation are—

$$\left.\begin{aligned} AJ_1(k\beta c) &= A_1J_1(k\beta_1 c) + B_1Y_1(k\beta_1 c),\\ AF(c\,.\,\beta) &= A_1F(c\,.\,\beta_1) + B_1F_1(c\,.\,\beta_1),\\ (A+\partial A)\,J_1(k\beta b) + \partial B\,Y_1(k\beta b) &= A_1J_1(k\beta_1 b) + B_1Y_1(k\beta_1 b),\\ (A+\partial A)\,F(b\,.\,\beta) + \partial B F_1(b\,.\,\beta) &= A_1F(b\,.\,\beta_1) + B_1F_1(b\,.\,\beta_1),\\ (A+\partial A)\,F(a\,.\,\beta) + \partial B F_1(a\,.\,\beta) &= 0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(7).$$

Referring now to the radial vibrations of a solid cylinder in Sect. IV., we see that the transverse type of displacement differs from the radial only in being a function of $k\beta r$ instead of $k\alpha r$. Also all the surface conditions in the transverse vibrations can be deduced from those holding for the radial vibrations by simply writing β for α and supposing m to vanish. We may thus at once deduce all the results we require for the transverse vibrations by making m zero and writing β for α in the results already obtained for the radial vibrations.

The frequency of transverse vibrations in an infinitely thin shell vanishes, and thus (27) Sect. IV. transforms into

$$\frac{\partial k}{k} = \frac{b-c}{b}\left[-\frac{\rho_1-\rho}{\rho}\left(\frac{b}{a}\right)^2\left(\frac{v_b}{v_a}\right)^2 + \left(\frac{1}{n}-\frac{1}{n_1}\right)\frac{b^2}{k^2\rho a^2}\left(\frac{V_b}{v_a}\right)^2\right]\quad\ldots\ldots\ldots\ldots(8),$$

where

$$\left.\begin{aligned} v_b &= J_1(k\beta b),\\ V_b &= nb^{-1}\{k\beta b J_1'(k\beta b) - J_1(k\beta b)\} = -\,nk\beta J_2(k\beta b)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(9).$$

Obviously $v_b \cos kt$ represents a displacement during a vibration of frequency $k/2\pi$ in a solid cylinder and $V_b \cos kt$ the corresponding transverse stress.

§ 84. In the core the only change in the type due to the existence of the layer consists as usual of a displacement of all the node, loop and no-stress surfaces according to the law

$$-\,\partial r/r = \partial k/k\quad\ldots\ldots\ldots\ldots(10).$$

From (29) Sect. IV. we find for the displacement outside the layer

$$v/A\cos kt = J_1(k\beta r) + \frac{b-c}{nC}\left\{-(\rho_1-\rho)\,k^2bv_b f(b\,.\,\beta\,.\,\bar{r}) + \left(\frac{1}{n}-\frac{1}{n_1}\right)V_b f(b\,.\,\beta\,.\,\bar{r})\right\}\ldots\ldots(11);$$

where C has the same meaning as in (17) Sect. IV., and with our usual notation

$$\left.\begin{aligned} f(b\,.\,\beta\,.\,\bar{r}) &= J_1(k\beta r)\,Y_1(k\beta b) - Y_1(k\beta r)\,J_1(k\beta b),\\ f(b\,.\,\beta\,.\,\bar{r}) &= J_1(k\beta r)\,F_1(b\,.\,\beta) - Y_1(k\beta r)\,F(b\,.\,\beta)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(12).$$

In the layer itself the displacement is given by

$$v/A\cos kt = J_1(k\beta b) - k\beta\,(b-r)\,J_1'(k\beta b) - (r-c)\left(\frac{1}{n}-\frac{1}{n_1}\right)V_b\quad\ldots\ldots\ldots\ldots(13).$$

The *change of type* outside the layer, i.e. the coefficient of $b-c$ in (11), consists like the expression (8) for the change of pitch of two terms only. There is an exact correspondence between the terms in the two equations. The first terms in each depend only on the alteration of density, and simultaneously vanish when the layer is at a node surface. The second terms depend only on the alteration of rigidity, and simultaneously vanish when the layer is at a no-stress surface.

The change of type in the layer itself is the last term of (13). Thus if there be an alteration only in density, or an alteration in rigidity occurring at a no-stress surface, then no progressive change of type manifests itself as we cross the layer, i.e. the layer vibrates as if it were of the same structure as the core.

§ 85. For a discussion of (8) we require to know the characteristics of the transverse vibrations in a simple cylinder.

Taking (3) as the type of vibration, we see that the node surfaces are obtained by equating $k\beta b$ to the roots of

$$J_1(x) = 0 \qquad (14).$$

This is the same as (34) Sect. IV., and its roots are thus already recorded.

The radii of the loop surfaces are found by equating $k\beta b$ to the roots of

$$J_1'(x) = 0 \qquad (15).$$

This is the same as (35) Sect. IV., whose roots have been already given.

The radii of the no-stress surfaces are found by equating $k\beta b$ to the roots of

$$x^{-1} J_1(x) - J_1'(x) \equiv J_2(x) = 0 \qquad (16).$$

This is the same as (66) Sect. IV., whose roots have been already given. Writing $k\beta a$ for x in (16) we get the frequency equation.

Since the equations (14), (15) and (16) do not contain σ explicitly, it follows that, for any note of given number, the ratios borne by the radii of the several node, loop and no-stress surfaces to the radius of the cylinder are the same for all isotropic materials. Also the ratio of the radii of any two surfaces of given numbers, whether node, loop or no-stress surfaces, in a given cylinder performing a given note is the same whatever be the value of σ or the number of the note.

Since (14) and (15) are the same as (34) and (35) Sect. IV., it follows that the ratios subsisting between the radii of the several node and loop surfaces in a cylinder performing one of its transverse vibrations are precisely the same as those subsisting between the radii of the several node and loop surfaces in a cylinder performing one of its radial vibrations.

Since, however, the frequency equation (16) would agree with the frequency equation (36) Sect. IV. only when the physically impossible relation $m/n = 0$ was supposed to exist, it follows that the ratios borne by the radii of the node and loop surfaces to the radius of the cylinder cannot in any isotropic material be the same for a radial and for a transverse vibration. The ratios also between the frequencies of the several notes which are produced by a cylinder vibrating radially cannot possibly be *identical* with the ratios subsisting between the frequencies of the several notes produced by a cylinder vibrating transversely. These latter ratios, it will be observed, are independent of the value of σ, and so the same for all isotropic materials.

Comparing (16) with (36) Sect. IV. we see that when x is large they both approach the form

$$J_1'(x) = 0.$$

Thus the higher roots of the frequency equations, both transversal and radial, approach more and more nearly the larger they are to the roots of (15). Thus the higher notes of the two modes of vibration in a given cylinder correspond to one another in pairs, such that the two sets of node and loop surfaces become *nearly* coincident, and the frequency of the transverse vibration is to that of the radial *approximately* in the constant ratio

$$\sqrt{n} : \sqrt{m+n}.$$

A similar result, it will be remembered, was found in the case of the sphere.

The positions of the several node, loop and no-stress surfaces for the first four notes are given in the following table. It applies to all values of σ.

TABLE I.

Values of r/a over node, loop and no-stress surfaces.

Note (1)			Note (2)			Note (3)			Note (4)		
Node surfaces	No-stress surfaces	Loop surfaces	Node surfaces	No-stress surfaces	Loop surfaces	Node surfaces	No-stress surfaces	Loop surfaces	Node surfaces	No-stress surfaces	Loop surfaces
0	0	·359	0	0	·219	0	0	·158	0	0	·124
·746	1·0		·455	·610	·633	·330	·442	·459	·259	·347	·360
			·834	1·0		·604	·724	·735	·474	·569	·577
						·875	1·0		·688	·785	·791
									·900	1·0	

It will be observed that the number of loop surfaces always equals the number of the note, and is one less than the number of node or of no-stress surfaces. Also the loop surfaces, precisely as in the rotatory vibrations of a sphere, lie outside of the corresponding no-stress surfaces, and not inside them as in the case of radial vibrations both in spheres and cylinders.

The axis has the curious property of being at once a node and a no-stress surface.

In comparing the transverse and radial vibrations it will be found that note $(i-1)$ of the former class corresponds to note (i) of the latter.

§ 86. I shall consider first two special positions of the layer.

Supposing in (8) b/a very small, while $(b-c)/b$ is also very small, we obtain the change of frequency due to the presence of a thin axial layer differing from the rest of the material. It will be found that ∂k vanishes under all conditions. The same result may independently be proved for a core of small radius. Thus, to the present degree of approximation, no change in pitch follows any alteration of material throughout a thin axial layer or core.

§ 87. Putting $b=a$ and $V_b=0$ in (8) we pass to the case of an alteration of material throughout a surface layer of small thickness $t \equiv b-c$. For the change in frequency we get the simple result

$$-\frac{\partial k}{k}=\frac{t}{a}\frac{\rho_1-\rho}{\rho} \quad \ldots\ldots(17).$$

A surface alteration in elasticity has thus no effect on the pitch of any note, and a surface alteration in density alters the pitch of all the notes in the proportion of their original frequencies, and so leaves their ratios unaffected.

§ 88. Let us now consider the general case when the density alone is altered. As the change of pitch vanishes for an altered core we may without restriction put

$$b(b-c)/a^2=\tfrac{1}{2}\partial V/V,$$

$$b(b-c)(\rho_1-\rho)/a^2\rho=\tfrac{1}{2}\partial M/M.$$

From (8) we find for the change of pitch

$$-\frac{\partial k}{k}=\frac{t}{a}\frac{\rho_1-\rho}{\rho}\frac{k\beta b}{k\beta a}\left\{\frac{J_1(k\beta b)}{J_1(k\beta a)}\right\}^2=\frac{\partial M}{M}\frac{1}{2}\left\{\frac{J_1(k\beta b)}{J_1(k\beta a)}\right\}^2 \quad \ldots\ldots(18).$$

The change of pitch vanishes when the layer of altered density coincides with a node surface.

When the layer is of given volume, the curve showing the law of variation of dk/k with $k\beta b$ is

$$y=\{J_1(x)\}^2 \quad \ldots\ldots(19).$$

This is the same curve that applies in the corresponding case of the radial vibrations. It appears as curve A in fig. 6. The function of x appears as $f_3(x)$ in Table VII., Sect. IV.

This curve has been already discussed in § 73 and the ratios of its successive maxima ordinates recorded.

The positions of the layer supplying the maxima, are coincident with the loop surfaces. The first and largest maxima, answering to positions of the layer at the first loop surfaces, are given in the following table for the first four notes:

TABLE II.

First maximum of $-\frac{\partial k}{k} \div \frac{\partial M}{M}$.

Note (1)	Note (2)	Note (3)	Note (4)
1·468	2·299	3·133	3·968

The number of maxima is equal to the number of the note.

The first maximum for the $(i-1)^{\text{th}}$ transverse note is practically identical with that for the i^{th} radial note. Also the ratios of the first to the subsequent maxima are the same in the two cases. Thus from (58) and (59) Sect. IV. we find as pretty close approximations to the first maximum in note (i) and to the j^{th} maximum in the same note respectively, i and j being both greater than 4,

$$-\frac{\partial k}{k} \div \frac{\partial M}{M} = 3{\cdot}968 + (i-4) \times {\cdot}835 \quad \text{(20)},$$

$$-\frac{\partial k}{k} \div \frac{\partial M}{M} = \frac{3{\cdot}968 + (i-4) \times {\cdot}835}{6{\cdot}22 + (j-4) \times 1{\cdot}67} \quad \text{(21)}.$$

Maxima of number less than (5) can be obtained by means of the ratios given in § 73 for any note in which the first maximum is known.

§ 89. When the layer of altered density is of given thickness the curve showing the law of variation of $\partial k/k$ with κab is

$$y = x\,\{J_1(x)\}^2 \quad \text{(22)}.$$

This is the same curve that applies in the corresponding case in the radial vibrations. It appears as curve B in fig. 6, and the corresponding function of x appears as $f_4(x)$ in Table VII., Sect. IV.

This curve has been already discussed in § 74, Sect. IV.

The number of maxima is always equal to the number of the note, and the positions corresponding to the maxima in the first four notes are all shown in the following table:

TABLE III.

Values of b/a supplying maxima of $-\frac{\partial k}{k} \div \left(\frac{t}{a}\frac{\rho_1 - \rho}{\rho}\right)$.

Note (1)	Note (2)	Note (3)	Note (4)
·422	·257	·186	·146
	·645	·467	·367
		·740	·581
			·794

The magnitudes of the first and largest maxima, answering to the positions nearest the axis in the above table, are as follows:

TABLE IV.

First maximum of $-\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right)$.

Note (1)	Note (2)	Note (3)	Note (4)
1·149	1·098	1·084	1·078

As in the case of the radial vibrations we find that in the higher notes a close approximation to the first maximum of $-\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{\rho_1 - \rho}{\rho}\right)$ is 1·07, and to any maximum whose number exceeds (4) a close approximation is 1·00. Maxima of number less than (5) can be obtained by means of the ratios given in § 74 for any note in which the first maximum is known.

§ 90. When the elasticity alone is altered, we find from (8) for the change of pitch

$$\frac{\partial k}{k} = \frac{t}{a} \frac{n_1 - n}{n_1} \frac{k\beta b}{k\beta a} \left\{\frac{J_2(k\beta b)}{J_1(k\beta a)}\right\}^2 = \frac{\partial V}{V} \frac{n_1 - n}{n_1} \frac{1}{2} \left\{\frac{J_2(k\beta b)}{J_1(k\beta a)}\right\}^2 \quad \text{.....................(23).}$$

The change of pitch thus depends solely on the alteration of rigidity. It vanishes when the layer is at any no-stress surface, and has for all other positions of the layer the same sign as $n_1 - n$. Its law of variation with the position of the layer is independent of the magnitude of the alteration in rigidity.

When the layer is of given volume the curve showing the law of variation of $\partial k/k$ with $k\alpha b$ is

$$y = \{J_2(x)\}^2 \quad \text{...(24).}$$

This is the same curve that applies in the case of the radial vibrations when an alteration in rigidity alone takes place throughout a layer of given volume. It appears as curve A in fig. 8, and the corresponding function of x appears as $f_7(x)$ in Table VII., Sect. IV.

This curve has been already discussed in § 77, Sect. IV.

All the positions of the layer supplying maxima in the first four notes are given by the following table. They coincide with those surfaces over which the transverse stress is a maximum.

TABLE V.

Values of b/a where $\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{n_1 - n}{n_1}\right)$ is a maximum.

Note (1)	Note (2)	Note (3)	Note (4)
·595	·363	·263	·206
	·797	·577	·453
		·858	·674
			·890

The first and largest maxima in the case of these notes are as follows:

TABLE VI.

First maximum of $\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{n_1 - n}{n_1}\right)$.

Note (1)	Note (2)	Note (3)	Note (4)
1·026	1·607	2·190	2·774

The first maximum of $\partial k/k$ in the $(i-1)^{\text{th}}$ transverse note in the present case is practically identical with the first maximum of $\partial k/k$ in the i^{th} radial note in the case when the rigidity alone is altered throughout a given volume, and the ratios of the first to the subsequent maxima are the same in the two cases. We thus find, as fairly close approximations for the first and j^{th} maxima respectively in note (i), supposing i and j both greater than 4,

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{n_1 - n}{n_1}\right) = 2{\cdot}774 + (i-4) \times {\cdot}584 \quad\text{.........................(25)},$$

$$\frac{\partial k}{k} \div \left(\frac{\partial V}{V} \frac{n_1 - n}{n_1}\right) = \frac{2{\cdot}774 + (i-4) \times {\cdot}584}{4{\cdot}85 + (j-4) \times 1{\cdot}20} \quad\text{.........................(26)}.$$

Maxima of number less than (5) can be obtained by means of the ratios given in § 77 for any note in which the first maximum is known.

§ 91. When the layer of altered rigidity is of given thickness the curve showing the law of variation of $\partial k/k$ with kab is

$$y = x\,\{J_2(x)\}^2 \quad\text{.....................................(27)}.$$

This is the same curve that applies in the case of the radial vibrations when an alteration in rigidity alone takes place throughout a layer of given thickness. It appears as curve B in fig. 8, and the corresponding function of x appears as $f_8(x)$ in Table VII., Sect. IV.

This curve has been already discussed in § 78, Sect. IV.

All the positions of the layer supplying maxima in the first four notes are recorded in the following table:

TABLE VII.

Values of b/a where $\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{n_1 - n}{n_1}\right)$ is a maximum.

Note (1)	Note (2)	Note (3)	Note (4)
·645	·393	·285	·224
	·806	·584	·459
		·862	·677
			·893

The first and largest maxima in the case of these notes are as follows:

TABLE VIII.

First maximum of $\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{n_1 - n}{n_1}\right)$.

Note (1)	Note (2)	Note (3)	Note (4)
1·273	1·217	1·201	1·195

For all notes of higher number a fairly close approximation to the first maximum change of pitch is given by

$$\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{n_1 - n}{n_1}\right) = 1{\cdot}19 \qquad (28).$$

For all maxima of number greater than (4) we may take as a close approximation

$$\frac{\partial k}{k} \div \left(\frac{t}{a} \frac{n_1 - n}{n_1}\right) = 1{\cdot}00 \qquad (29).$$

Maxima of number less than (4) can be obtained by means of the ratios given in § 78 for any note in which the first maximum is known.

SECTION VI.

RADIAL VIBRATIONS IN SPHERICAL SHELL.

§ 92. I now proceed to apply the method of Sect. I. to determine the frequency of vibration in compound shells.

I shall first consider the radial vibrations of spherical shells.

The type of vibration and of the radial stress in a simple shell are shown in (1) and (3) of Sect. II. From these expressions we may select the following as the values to be assigned to the F, F_1, G, G_1 of (1) and (2), Sect. I.:

$$F(a\,.\,\alpha) = (m+n)\,k\alpha a \sin k\alpha a - 4n\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right) \qquad (1),$$

$$F_1(a\,.\,\alpha) = (m+n)\,k\alpha a \cos k\alpha a - 4n\left(\frac{\cos k\alpha a}{k\alpha a} + \sin k\alpha a\right) \qquad (2),$$

$$G(a\,.\,\alpha) = \frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a \qquad (3),$$

$$G_1(a\,.\,\alpha) = \frac{\cos k\alpha a}{k\alpha a} + \sin k\alpha a \qquad (4).$$

The form of the frequency equations in a simple shell $(b.\alpha.a)$ of the types free-free, fixed-free, free-fixed and fixed-fixed are given in equations (3), (4), (5) and (6), Sect. I. For the present case these lead to:

$$f(b.\alpha.a) \equiv \sin k\alpha(a-b)\left[(m+n)^2k^2\alpha^2ab - 4n(m+n)\frac{a^2+b^2}{ab} + 16n^2\{1+(k^2\alpha^2ab)^{-1}\}\right]$$
$$- k\alpha(a-b)\cos k\alpha(a-b).4n\{m+n+4n(k^2\alpha^2ab)^{-1}\} = 0 \quad\ldots\ldots\ldots(5),$$

$$f(\bar{b}.\alpha.a) \equiv (m+n)\{k\alpha a\cos k\alpha(a-b) + ab^{-1}\sin k\alpha(a-b)\}$$
$$- 4n(k^2\alpha^2ab)^{-1}\{(1+k^2\alpha^2ab)\sin k\alpha(a-b) - k\alpha(a-b)\cos k\alpha(a-b)\} = 0 \quad\ldots\ldots\ldots(6),$$

$$f(b.\alpha.\bar{a}) \equiv (m+n)\{ba^{-1}\sin k\alpha(a-b) - k\alpha b\cos k\alpha(a-b)\}$$
$$- 4n(k^2\alpha^2ab)^{-1}\{(1+k^2\alpha^2ab)\sin k\alpha(a-b) - k\alpha(a-b)\cos k\alpha(a-b)\} = 0 \quad\ldots\ldots\ldots(7),$$

$$f(\bar{b}.\alpha.\bar{a}) \equiv \sin k\alpha(a-b)\{1+(k^2\alpha^2ab)^{-1}\} - k\alpha(a-b)(k^2\alpha^2ab)^{-1}\cos k\alpha(a-b) = 0 \quad\ldots\ldots\ldots(8).$$

The above expressions are the exact forms of $f(b.\alpha.a)$ etc. and are not reduced by division or multiplication by any factor.

If the shell be so thin that terms in $(a-b)^2$ may be neglected the expressions become:

$$f(b.\alpha.a) = k\alpha(a-b)(m+n)\{(m+n)k^2\alpha^2a^2 - 4n(3m-n)(m+n)^{-1}\} \quad\ldots\ldots\ldots(9_a),$$

$$= k\alpha(a-b)(m+n)\rho a^2(k^2 - K^2_{(\alpha.a)}) \quad\ldots\ldots\ldots(9_b),$$

$$f(\bar{b}.\alpha.a) = (m+n)k\alpha a + k\alpha(a-b)(m-3n) \quad\ldots\ldots\ldots(10),$$

$$f(b.\alpha.\bar{a}) = -(m+n)k\alpha a + 2k\alpha(a-b)(m-n) \quad\ldots\ldots\ldots(11),$$

$$f(\bar{b}.\alpha.\bar{a}) = k\alpha(a-b) \quad\ldots\ldots\ldots(12).$$

The meaning of $K_{(\alpha.a)}$, etc. is the same as in Sect. II. In the coefficient of $a-b$ we may of course replace a by b.

Equating the coefficient of $a-b$ in (9_a) to zero we get the frequency equation for a free-free vibration. None of the three other types has in a thin shell a vibration of finite period.

By supposing in (6) and (7) b absolutely equal to a, we find

$$f(\bar{a}.\alpha.a) \equiv F(a.\alpha)G_1(a.\alpha) - F_1(a.\alpha)G(a.\alpha) = (m+n)k\alpha a \quad\ldots\ldots\ldots(13),$$

$$f(a.\alpha.\bar{a}) \equiv G(a.\alpha)F_1(a.\alpha) - G_1(a.\alpha)F(a.\alpha) = -(m+n)k\alpha a \quad\ldots\ldots\ldots(14).$$

These quantities cannot vanish unless k, α, or a vanishes, and thus the occurrence of either as a factor in a frequency equation does not supply a note of possible frequency. This proves for the present case the truth of a statement made in § 9, Sect. I.

Employing the relation (13) in equation (30) Sect. I., we find

$$f(e.\alpha.c.\alpha.b.\alpha.a) = (m+n)^2k^2\alpha^2bcf(e.\alpha.a) \quad\ldots\ldots\ldots(15).$$

We also require the value of $k\frac{d}{dk}f(e.\alpha.a)$ under the condition that $k/2\pi$ is the frequency of a free-free vibration in a simple shell $(e.\alpha.a)$.

Looking on $k\alpha e$ and $k\alpha a$ as independent variables, we may put

$$k\frac{d}{dk}f(e.\alpha.a)=\left[k\alpha e\frac{d}{d.k\alpha e}+k\alpha a\frac{d}{d.k\alpha a}\right]f(e.\alpha.a)\quad\text{.....(16).}$$

A form of $f(e.\alpha.a)$ may be got by writing e for b in (5). It is simpler however in obtaining the above differentials to deal with the unreduced form obtained by the immediate substitution in (3) Sect. I. of the expressions (1) and (2) for F and F_1.

It will suffice to give the work in one case. Thus

$$f(e.\alpha.a)=F(a.\alpha)\left\{(m+n)\,k\alpha e\cos k\alpha e-4n\left(\frac{\cos k\alpha e}{k\alpha e}+\sin k\alpha e\right)\right\}$$
$$-F_1(a.\alpha)\left\{(m+n)\,k\alpha e\sin k\alpha e-4n\left(\frac{\sin k\alpha e}{k\alpha e}-\cos k\alpha e\right)\right\};$$

$$\therefore\ k\alpha e\frac{d}{d.k\alpha e}f(e.\alpha.a)$$
$$=F(a.\alpha)\left[\left(\frac{\cos k\alpha e}{k\alpha e}+\sin k\alpha e\right)\{-(m+n)\,k^2\alpha^2e^2+4n\}+2(m-n)\,k\alpha e\cos k\alpha e\right]$$
$$-F_1(a.\alpha)\left[\left(\frac{\sin k\alpha e}{k\alpha e}-\cos k\alpha e\right)\{-(m+n)\,k^2\alpha^2e^2+4n\}+2(m-n)\,k\alpha e\sin k\alpha e\right],$$
$$=\{-(m+n)\,k^2\alpha^2e^2+4n\}\left\{F(a.\alpha)\left(\frac{\cos k\alpha e}{k\alpha e}+\sin k\alpha e\right)-F_1(a.\alpha)\left(\frac{\sin k\alpha e}{k\alpha e}-\cos k\alpha e\right)\right\}$$
$$+2(m-n)\,k\alpha e\{F(a.\alpha)\cos k\alpha e-F_1(a.\alpha)\sin k\alpha e\}\quad\text{.....(17).}$$

Remembering that $f(e.\alpha.a)$ is supposed equal zero, and employing the expressions supplied by (1) and (2) for $F(e.\alpha)$ and $F_1(e.\alpha)$, we find

$$F(a.\alpha)\cos k\alpha e-F_1(a.\alpha)\sin k\alpha e$$
$$=\frac{4n}{m+n}\frac{1}{k\alpha e}\left\{F(a.\alpha)\left(\frac{\cos k\alpha e}{k\alpha e}+\sin k\alpha e\right)-F_1(a.\alpha)\left(\frac{\sin k\alpha e}{k\alpha e}-\cos k\alpha e\right)\right\}.$$

Substituting thence in the coefficient of $m-n$ in (17) and putting the terms together, we find

$$k\alpha e\frac{d}{d.k\alpha e}f(e.\alpha.a)=-(m+n)\left\{k^2\alpha^2e^2-\frac{4n(3m-n)}{(m+n)^2}\right\}$$
$$\times\left\{F(a.\alpha)\left(\frac{\cos k\alpha e}{k\alpha e}+\sin k\alpha e\right)-F_1(a.\alpha)\left(\frac{\sin k\alpha e}{k\alpha e}-\cos k\alpha e\right)\right\}\quad\text{.....(18).}$$

Finally noticing the forms of G and G_1 in (3) and (4), the expression for $f(\bar{e}.\alpha.a)$ supplied by (4) Sect. I., and the expression (25) Sect. II. for the frequency of radial vibrations in an infinitely thin shell, we obtain

$$k\alpha e\frac{d}{d.k\alpha e}f(e.\alpha.a)=-\rho e^2\left(k^2-K^2_{(a.e)}\right)f(\bar{e}.\alpha.a)\quad\text{.....(19).}$$

In an exactly similar manner it may be proved that

$$k\alpha a \frac{d}{d\,.\,k\alpha a} f(e\,.\,\alpha\,.\,a) = -\rho a^2 (k^2 - K^2_{(\alpha\,.\,a)}) f(e\,.\,\alpha\,.\,\bar{a}) \text{.................(20).}$$

Thus

$$k\frac{d}{dk} f(e\,.\,\alpha\,.\,a) = -\rho \{e^2 (k^2 - K^2_{(\alpha\,.\,e)}) f(\bar{e}\,.\,\alpha\,.\,a) + a^2 (k^2 - K^2_{(\alpha\,.\,a)}) f(e\,.\,\alpha\,.\,\bar{a})\} \text{......(21),}$$

where after differentiation k is treated as a root of the frequency equation $f(e\,.\,\alpha\,.\,a) = 0$.

The results (19) and (20) are particular cases of the general theorem treated in § 10, Sect. I.

§ 93. We now possess all the data necessary for determining the change of pitch in the radial vibrations of a spherical shell due to the existence of a thin layer differing from the rest of the material. Supposing the shell to be $(e\,.\,\alpha\,.\,c\,.\,\alpha_1\,.\,b\,.\,\alpha\,.\,a)$, we have from the general result (23) in Sect. I.

$$f(e\,.\,\alpha\,.\,c\,.\,\alpha_1\,.\,b\,.\,\alpha\,.\,a) = f(\bar{b}\,.\,\alpha\,.\,a)\{f(e\,.\,\alpha\,.\,c) f(\bar{c}\,.\,\alpha_1\,.\,b) - f(e\,.\,\alpha\,.\,\bar{c}) f(c\,.\,\alpha_1\,.\,b)\}$$
$$- f(b\,.\,\alpha\,.\,a)\{f(e\,.\,\alpha\,.\,c) f(\bar{c}\,.\,\alpha_1\,.\,\bar{b}) - f(e\,.\,\alpha\,.\,\bar{c}) f(c\,.\,\alpha_1\,.\,\bar{b})\} \text{ ....(22).}$$

Now supposing the layer $(c\,.\,\alpha_1\,.\,b)$ so thin that terms in $(b-c)^2$ are negligible, let us employ the relations (9_a)—(12) for a thin shell. Then, replacing c by b in the coefficient of $b-c$, we find for the frequency equation

$$\frac{f(e\,.\,\alpha\,.\,c\,.\,\alpha_1\,.\,b\,.\,\alpha\,.\,a)}{(m_1+n_1)\,k\alpha_1 b} = f(e\,.\,\alpha\,.\,c) f(\bar{b}\,.\,\alpha\,.\,a) - f(e\,.\,\alpha\,.\,\bar{c}) f(b\,.\,\alpha\,.\,a)$$
$$+ \frac{b-c}{b}\left(\frac{1}{m_1+n_1}\right)\{(m_1 - 3n_1) f(e\,.\,\alpha\,.\,b) f(\bar{b}\,.\,\alpha\,.\,a) + 2(m_1 - n_1) f(e\,.\,\alpha\,.\,\bar{b}) f(b\,.\,\alpha\,.\,a)\}$$
$$- \frac{b-c}{b}\rho_1 b^2 (k^2 - K^2_{(\alpha_1\,.\,b)}) f(e\,.\,\alpha\,.\,\bar{b}) f(\bar{b}\,.\,\alpha\,.\,a) - \frac{b-c}{b}\frac{1}{m_1+n_1} f(e\,.\,\alpha\,.\,b) f(b\,.\,\alpha\,.\,a) = 0 \text{....(23).}$$

Writing α, m, n for α_1, m_1, n_1 respectively in (23), we get a similar expression for

$$f(e\,.\,\alpha\,.\,c\,.\,\alpha\,.\,b\,.\,\alpha\,.\,a) \div \{(m+n)\,k\alpha b\}.$$

Employing this last expression in (23), we easily find for the frequency equation

$$0 = \frac{f(e\,.\,\alpha\,.\,c\,.\,\alpha_1\,.\,b\,.\,\alpha\,.\,a)}{(m_1+n_1)\,k\alpha_1 b} = \frac{f(e\,.\,\alpha\,.\,c\,.\,\alpha\,.\,b\,.\,\alpha\,.\,a)}{(m+n)\,k\alpha b}$$
$$+ \Big[b^2\{\rho(k^2 - K^2_{(\alpha\,.\,b)}) - \rho_1(k^2 - K^2_{(\alpha_1\,.\,b)})\} f(e\,.\,\alpha\,.\,\bar{b}) f(\bar{b}\,.\,\alpha\,.\,a) + \left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right) f(e\,.\,\alpha\,.\,b) f(b\,.\,\alpha\,.\,a)$$
$$+ \left(\frac{m_1 - 3n_1}{m_1+n_1} - \frac{m-3n}{m+n}\right) f(e\,.\,\alpha\,.\,b) f(\bar{b}\,.\,\alpha\,.\,a) + 2\left(\frac{m_1-n_1}{m_1+n_1} - \frac{m-n}{m+n}\right) f(e\,.\,\alpha\,.\,\bar{b}) f(b\,.\,\alpha\,.\,a)\Big]\frac{b-c}{b} \text{...(24).}$$

Remembering (15) we may in (24) put

$$\frac{f(e\,.\,\alpha\,.\,c\,.\,\alpha\,.\,b\,.\,\alpha\,.\,a)}{(m+n)\,k\alpha b} = (m+n)\,k\alpha c f(e\,.\,\alpha\,.\,a) \text{...........(25).}$$

Suppose now that $\partial k/2\pi$ is the increase in the frequency of a note due to the presence of the layer. Then k being supposed a root of (24), $k-\partial k$ must be a root of $f(e\,.\,\alpha\,.\,a)=0$. Thus assuming ∂k of the order $b-c$, the above equation (24) must be identical with

$$f(e\,.\,\alpha\,.\,a)-\frac{\partial k}{k}k\frac{d}{dk}f(e\,.\,\alpha\,.\,a)=0,$$

i.e. with

$$f(e\,.\,\alpha\,.\,a)+\frac{\partial k}{k}\rho\,\{e^2(k^2-K^2{}_{(\alpha\,.\,e)})f(\bar{e}\,.\,\alpha\,.\,a)+a^2(k^2-K^2{}_{(\alpha\,.\,a)})f(e\,.\,\alpha\,.\,\bar{a})\}=0\ldots\ldots(26).$$

Making the substitution (25) in (24)—replacing c by b since $f(e\,.\,\alpha\,.\,a)$ is of order $b-c$—and then comparing the identical equations (24) and (26), we find

$$\frac{\partial k}{k}(m+n)\,k\alpha b\rho\,\{e^2(k^2-K^2{}_{(\alpha\,.\,e)})f(\bar{e}\,.\,\alpha\,.\,a)+a^2(k^2-K^2{}_{(\alpha\,.\,a)})f(e\,.\,\alpha\,.\,\bar{a})\}\div\frac{b-c}{b}$$

$$=b^2\,\{\rho\,(k^2-K^2{}_{(\alpha\,.\,b)})-\rho_1(k^2-K^2{}_{(\alpha_1\,.\,b)})\}\,f(e\,.\,\alpha\,.\,b)\,f(b\,.\,\alpha\,.\,a)$$

$$+\left(\frac{1}{m+n}-\frac{1}{m_1+n_1}\right)f(e\,.\,\alpha\,.\,b)\,f(b\,.\,\alpha\,.\,a)+\left(\frac{m_1-3n_1}{m_1+n_1}-\frac{m-3n}{m+n}\right)f(e\,.\,\alpha\,.\,b)\,f(\dot{b}\,.\,\alpha\,.\,a)$$

$$+2\left(\frac{m_1-n_1}{m_1+n_1}-\frac{m-n}{m+n}\right)f(e\,.\,\alpha\,.\,\bar{b})\,f(b\,.\,\alpha\,.\,a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

§ 94. Now, as explained previously, the expression for $\partial k/k$ as containing $b-c$ may be modified by any substitution consistent with $f(e\,.\,\alpha\,.\,a)=0$ being exactly true. This enables us to put (27) into a form which brings out more clearly its physical significance.

From (1) to (4) combined with (1) and (3) of Sect. II., we may suppose the displacement u and radial stress U at a distance r from the centre of a simple shell $(e\,.\,\alpha\,.\,a)$, performing a free-free vibration of frequency $k/2\pi$, to be given by

$$ur/\cos kt=ru_r\;=AG(r\,.\,\alpha)+BG_1(r\,.\,\alpha)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

$$Ur^2/\cos kt=r^2U_r=AF(r\,.\,\alpha)+BF_1(r\,.\,\alpha)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29),$$

where A and B are constants independent of r or t.

In virtue of the surface conditions we have

$$AF(e\,.\,\alpha)+BF_1(e\,.\,\alpha)=0=AF(a\,.\,\alpha)+BF_1(a\,.\,\alpha)\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

Thence we get

$$A:B::F_1(e\,.\,\alpha):-F(e\,.\,\alpha)::-F_1(a\,.\,\alpha):F(a\,.\,\alpha)\ldots\ldots\ldots\ldots\ldots(31).$$

Employing these ratios in (28) and (29), it is easy to prove

$$\left.\begin{aligned}f(e\,.\,\alpha\,.\,b)\div f(e\,.\,\alpha\,.\,\bar{a})&=bu_b\;\div au_a,\\ f(\bar{b}\,.\,\alpha\,.\,a)\div f(\bar{e}\,.\,\alpha\,.\,a)&=bu_b\;\div eu_e,\\ f(e\,.\,\alpha\,.\,b)\div f(e\,.\,\alpha\,.\,\bar{a})&=b^2U_b\div au_a,\\ f(b\,.\,\alpha\,.\,a)\div f(\bar{e}\,.\,\alpha\,.\,a)&=b^2U_b\div eu_e\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

We also easily prove

$$f(\bar{e}\,.\,\alpha\,.\,a) = \{F(e\,.\,\alpha)\,G_1(e\,.\,\alpha) - F_1(e\,.\,\alpha)\,G(e\,.\,\alpha)\}\left[\frac{\{F(a\,.\,\alpha)\}^2 + \{F_1(a\,.\,\alpha)\}^2}{\{F(e\,.\,\alpha)\}^2 + \{F_1(e\,.\,\alpha)\}^2}\right]^{\frac{1}{2}}$$

$$= (m+n)\,k\alpha e\left[\frac{\{(m+n)\,k\alpha a - 4n\,(k\alpha a)^{-1}\}^2 + 16n^2}{\{(m+n)\,k\alpha e - 4n\,(k\alpha e)^{-1}\}^2 + 16n^2}\right]^{\frac{1}{2}} \quad\text{..............(33):}$$

and similarly

$$f(e\,.\,\alpha\,.\,\bar{a}) = -(m+n)\,k\alpha a\left[\frac{\{(m+n)\,k\alpha e - 4n\,(k\alpha e)^{-1}\}^2 + 16n^2}{\{(m+n)\,k\alpha a - 4n\,(k\alpha a)^{-1}\}^2 + 16n^2}\right]^{\frac{1}{2}} \quad\text{..............(34).}$$

Thus
$$f(\bar{e}\,.\,\alpha\,.\,a)\times f(e\,.\,\alpha\,.\,\bar{a}) = -(m+n)^2k^2\alpha^2 ea \quad\text{..............................(35).}$$

Employing the results (32) and (35) in (27), we easily deduce

$$\frac{\delta k}{k} \div \frac{b-c}{a} = \frac{-(m+n)\,k\alpha a}{\rho u_e u_a\left[e^2\,(k^2 - K^2_{(\alpha\,.\,e)})\,f(\bar{e}\,.\,\alpha\,.\,a) + a^2\,(k^2 - K^2_{(\alpha\,.\,a)})\,f(e\,.\,\alpha\,.\,\bar{a})\right]}$$

$$\times\left[b^2u_b^2\,\{\rho\,(k^2 - K^2_{(\alpha\,.\,b)}) - \rho_1\,(k^2 - K^2_{(\alpha_1\,.\,b)})\} + b^2U_b^2\left(\frac{1}{m+n} - \frac{1}{m_1+n_1}\right)\right.$$

$$\left. + 8bu_bU_b\left(\frac{n}{m+n} - \frac{n_1}{m_1+n_1}\right)\right] \quad\text{...(36)*.}$$

§ 95. The deduction from (36) of the formula for the special case of a solid sphere requires careful treatment. Thus the term in the denominator containing

$$e^2u_e\,(k^2 - K^2_{(\alpha\,.\,e)})\,f(\bar{e}\,.\,\alpha\,.\,a)$$

is easily seen to vanish with e, but $u_e f(e\,.\,\alpha\,.\,\bar{a})$ assumes the form $0\times\infty$.

To avoid this difficulty we may by means of (35) replace the second of equations (32) by

$$u_e f(e\,.\,\alpha\,.\,\bar{a}) = -(m+n)^2k^2\alpha^2 abu_b \div f(b\,.\,\alpha\,.\,a) \quad\text{........................(37).}$$

Thence proceeding to the limit when e vanishes we easily find

$$u_e f(e\,.\,\alpha\,.\,\bar{a}) = -(m+n)\,k\alpha a\,u_a \quad\text{..............................(38).}$$

This leads to the same result as was obtained in Sect. II.

§ 96. The right-hand side of (36) is the product of two factors of which the second alone is a function of b. It contains u_b and U_b in the same way as does the right-hand side of (28) Sect. II., and the physical significations of u_b and U_b are precisely the same as in the case of the solid sphere. The mathematical expressions for u_b and U_b are however, it must be remembered, different in the two cases, those for the shell being much the more complicated.

As the first factor on the right-hand side of (36) does not contain b, it is for a given note the same in sign and in magnitude wherever the layer may be, or whatever be the nature of its difference from the rest of the material. The law of variation of $\delta k/k$ with the position of the layer in no way depends on it, but only the absolute magnitude and the sign of the change of pitch.

* See the note on p. 266.

For a solid sphere we found the first factor essentially positive. A purely mathematical demonstration that it is always positive in the case of a shell presents considerable difficulties, but is I believe rendered unnecessary by the following physical consideration.

Suppose the layer to differ from the remainder only in density, then we have

$$\frac{\partial k}{k} \div \frac{b-c}{a} = [\text{first factor}] \times b^2 (u_b)^2 (\rho - \rho_1) k^2.$$

Thus, unless an *increase* of density occurring *anywhere* except at the nodes is to *raise* the pitch, the first factor must be positive. This consideration affords I think convincing proof that the first factor is essentially positive, and that such is the case will now be taken for granted.

§ 97. As (36) is in form so exactly analogous to (28) Sect. II. for the solid sphere, a brief discussion will suffice.

When an alteration of density occurs at a node surface of a particular note it does not affect its pitch, but in any other position it lowers the pitch when an increase and raises it when a decrease.

The percentage lowering of frequency due to a given increase of density throughout a given layer is always equal to the percentage rise of frequency due to an equal diminution of density throughout the same layer. The law of variation of the change of pitch, due to a given alteration of density, with the position of the layer is independent of the magnitude of the alteration of density. When the layer of altered density is of given volume the positions in which it has most effect on the pitch of a given note coincide with the loop surfaces for that particular note; when the layer is of given thickness its most effective positions lie slightly outside the loop surfaces.

If the layer differ from the remainder only in elasticity the change of pitch consists of three terms. Of these the first has the same sign as, and is proportional in magnitude to

$$n_1 (3m_1 - n_1)(m_1 + n_1)^{-1} - n (3m - n)(m + n)^{-1}.$$

It vanishes when the layer coincides with a node surface of the note in question.

The second term has the same sign as, and is proportional in magnitude to

$$(m + n)^{-1} - (m_1 + n_1)^{-1}.$$

It vanishes when the layer coincides with a no-stress surface.

The third term varies as

$$n (m + n)^{-1} - n_1 (m_1 + n_1)^{-1},$$

but its sign depends also on the value of b. It vanishes when the layer coincides either with a node or a no-stress surface. It likewise vanishes for all positions of the layer provided

$$m_1/m = n_1/n = 1 + p \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39).$$

Thus if the uniconstant theory be true, or more generally if the relation (39) subsist, the sign of the change of pitch accompanying a given alteration in elasticity is independent of the position of the altered layer, and is the same as that of p. If however the relation (39) do not hold, the sign of the change of pitch may for certain alterations of elasticity vary with the position of the layer.

The positions of the layer whether of given thickness or given volume, when a given alteration of elasticity has most effect on the pitch of a given note would require to be separately determined for each possible alteration of elasticity. The first term—that depending on the alteration of $n(3m-n)(m+n)^{-1}$—is largest when the layer, supposed of given thickness, coincides with a loop surface. The second term—that depending on the alteration of $(m+n)^{-1}$—is largest when the layer, supposed of given volume, coincides with a surface where the radial stress is a maximum. As a function of b the first term varies as $(u_b)^2$, the second as $(bU_b)^2$ and the third as $u_b . bU_b$ when the layer is of given thickness. Now from equations (1)—(4) we see that when $k\alpha b$ is large $F(b.\alpha)$ and $F_1(b.\alpha)$ are of the orders $k\alpha b \sin k\alpha b$ and $k\alpha b \cos k\alpha b$, while $G(b.\alpha)$ and $G_1(b.\alpha)$ are only of the orders $\cos k\alpha b$ and $\sin k\alpha b$. Thus it follows from (28) and (29) that when $k\alpha b$ is large $u_b/(bU_b)$ is of the order $1/k\alpha b$ of small quantities and so is small. Consequently when $k\alpha b$ is large the second term—that depending on the alteration of $(m+n)^{-1}$—is much the most important, and the third term is next in importance.

Thus when the effect on the pitch of one of the higher notes due to an alteration of elasticity is being considered, we obtain in general—unless the alteration occur close to the inner surface and the radius of this surface be small—a close approximation to the value of ∂k by neglecting altogether the first and third terms; and when the change of pitch of one of these higher notes is of practical importance it may be regarded as due approximately to the alteration of the single elastic quantity $(m+n)^{-1}$. The change of pitch is in such a case greatest when the alteration of elasticity occurs at or in the immediate neighbourhood of the surfaces of greatest radial stress.

In the case of the two or three lowest notes serious error might however arise from neglecting the first and third terms, especially when the alteration of elasticity occurs near a no-stress surface, more particularly the inner surface of the shell.

§ 98. I do not purpose an exhaustive investigation of (36), but one or two of the more interesting special cases may be considered without much analysis.

Thus let us suppose the layer to be at the outer surface, so that $b=a$. Then by (10)

$$f(\bar{b}.\alpha.a)=(m+n)\,k\alpha a,$$

and so the second of equations (32) becomes

$$f(\bar{e}.\alpha.a)=(m+n)\,k\alpha a e u_e/(a u_a) \quad \text{.................................(40).}$$

Hence by (35)

$$f(e.\alpha.\bar{a})=-(m+n)\,k\alpha e a u_a/(e u_e) \quad \text{...............................(41).}$$

Again, owing to the surface conditions, $U_b \equiv U_a = 0$. Thus from (36), if the thickness of the layer be t_1 and the change in pitch ∂k_1,

$$\frac{\partial k_1}{k} = \frac{t_1}{a} a^3 (u_a)^2 \{\rho (k^2 - K^2_{(\alpha, a)}) - \rho_1 (k^2 - K^2_{(\alpha_1, a)})\} \div \rho D \quad \text{.................(42)},$$

where $$D = a^3 u_a^2 (k^2 - K^2_{(\alpha, a)}) - e^3 u_e^2 (k^2 - K^2_{(\alpha, e)}) \quad \text{.................(43)}.$$

Similarly if the layer, supposed of thickness t_2 and material (ρ_2, α_2), occur at the inner surface of the shell the change in pitch, ∂k_2, is given by

$$\frac{\partial k_2}{k} = \frac{t_2}{e} e^3 (u_e)^2 \{\rho (k^2 - K^2_{(\alpha, e)}) - \rho_2 (k^2 - K^2_{(\alpha_2, e)})\} \div \rho D \quad \text{.................(44)}.$$

If the layer differ from the remainder only in density, and the mass of the shell be increased by ∂M_1 when the layer is at the outer surface, and by ∂M_2 when the layer is at the inner surface, then putting

$$M_1 = 4\pi a^3 \rho/3, \qquad \partial M_1 = 4\pi a^2 t_1 (\rho_1 - \rho),$$
$$M_2 = 4\pi e^3 \rho/3, \qquad \partial M_2 = 4\pi e^2 t_2 (\rho_2 - \rho),$$

we get $$\left.\begin{aligned} \frac{\partial k_1}{k} &= -\partial M_1 (u_a)^2 k^2 \div D', \\ \frac{\partial k_2}{k} &= -\partial M_2 (u_e)^2 k^2 \div D' \end{aligned}\right\} \quad \text{.................(45)},$$

where $$D' = 3 \{M_1 (u_a)^2 (k^2 - K^2_{(\alpha, a)}) - M_2 (u_e)^2 (k^2 - K^2_{(\alpha, e)})\} \quad \text{.................(46)}.$$

The mass of the shell when of uniform density ρ is of course $M_1 - M_2$. From (45) we have the elegant relation

$$\partial k_1 : \partial k_2 :: \partial M_1 (u_a)^2 : \partial M_2 (u_e)^2 \quad \text{.................(47)}.$$

Thus the changes in the pitch of a given note in a given shell when alterations of density occur at its surfaces are in the ratio of the consequent alterations of the mean values of the kinetic energies resident in the corresponding layers.

Supposing the altered surface layers to differ from the remainder only in elasticity, we find

$$\left.\begin{aligned} \frac{\partial k_1}{k} &= t_1 (u_a)^2 \{4n_1 (3m_1 - n_1)(m_1 + n_1)^{-1} - 4n (3m - n)(m + n)^{-1}\} \div \rho D, \\ \frac{\partial k_2}{k} &= t_2 (u_e)^2 \{4n_2 (3m_2 - n_2)(m_2 + n_2)^{-1} - 4n (3m - n)(m + n)^{-1}\} \div \rho D \end{aligned}\right\} \quad \text{............(48)},$$

where D is given by (43).

Thus the change in pitch is proportional to the alteration in the elastic quantity $n(3m - n)/(m + n)$. We also notice that for equal alterations in the material at the two surfaces

$$\partial k_1 : \partial k_2 :: t_1 (u_a)^2 : t_2 (u_e)^2 \quad \text{.................(49)}.$$

Comparing (47) and (49) we see that the effect on the pitch of a given alteration in elasticity relative to that of a given alteration of density is always more important when the alterations occur at the inner surface of a shell than when they occur at the outer.

§ 99. Supposing the squares of $\partial k_1/k$, $\partial k_2/k$ and $(\partial k_1+\partial k_2)/k$ all negligible, we may take $(\partial k_1+\partial k_2)/k$ for the change in pitch due to alterations in the material existing simultaneously at both surfaces of the shell.

We can also obtain the effect on the pitch of a note of completely removing thin layers of the material from either or both of its surfaces by simply substituting 0 for ρ_1 and ρ_2 in (42) and (44) respectively. When layers of thicknesses t_1 and t_2 are simultaneously removed we have

$$\frac{\partial k}{k}=\left\{\frac{t_1}{a}\,.\,a^3(u_a)^2(k^2-K^2_{(\alpha\,.\,a)})+\frac{t_2}{e}\,.\,e^3(u_e)^2(k^2-K^2_{(\alpha\,.\,e)})\right\}\div D \;\ldots\ldots\ldots\ldots(50).$$

By supposing t_1 or t_2 negative we can obtain the change of pitch due to *adding* an additional layer of thickness t_1 or t_2 to the outer or inner surface respectively. This may be regarded as obvious, supposing it be admitted that the effects of adding and removing equal very thin layers at a surface must be equal and opposite.

As the immediately preceding deductions travel somewhat outside of strict elastic solid principles, the following substantiating evidence may give increased confidence in their validity.

In (50) let us suppose

$$t_2\,e=-t_1\,a\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51),$$

and we get

$$\partial k/k=\quad t_1/a\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52).$$

Thus our latest conclusions tell us that the effect of paring off a thickness t_1 at the outer surface and adding a thickness t_1e/a at the inner surface raises the pitch in the ratio $t_1 : a$; whereas an addition of thickness t_1 at the outer surface and a paring off of thickness t_1e/a at the inner surface lowers the pitch in the same ratio. Now this is obviously a correct conclusion, because in the frequency equation of the simple shell $(e.\alpha.a)$, k presents itself solely in the combinations $k\alpha a$ and $k\alpha e$. Thus the frequency equation remains unchanged if

$$\partial(k\alpha a)=0=\partial(k\alpha e);$$

or, α being constant, if

$$\partial k/k=-\partial a/a=-\partial e/e\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53).$$

Now a negative value of ∂a means a paring off of material at the outer surface, while a negative value of ∂e means an addition of material at the inner surface. Thus equations (52) and (53) are identical.

§ 100. The case when the compound shell itself is very thin may be most easily treated independently. For instance let us consider the compound shell

$$(a_1\,.\,\alpha_1\,.\,a_2\,.\,\alpha_2\,.\,a_3\,.\,\alpha_3\,.\,a_4),$$

where a_4-a_1 is so small that its square is negligible.

By (23) of Sect. I. the frequency equation is

$$f(a_1 . \alpha_1 . a_2) f(\bar{a}_2 . \alpha_2 . a_3) f(\bar{a}_3 . \alpha_3 . a_4) + f(a_1 . \alpha_1 . \bar{a}_2) f(a_2 . \alpha_2 . \bar{a}_3) f(a_3 . \alpha_3 . a_4)$$
$$- f(a_1 . \alpha_1 . \bar{a}_2) f(a_2 . \alpha_2 . a_3) f(\bar{a}_3 . \alpha_3 . a_4) - f(a_1 . \alpha_1 . a_2) f(\bar{a}_2 . \alpha_2 . \bar{a}_3) f(a_3 . \alpha_3 . a_4) = 0 \quad \text{......(54)}.$$

As $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$ are all small, we may apply results answering to equations (9)—(12) for all these functions. Thus neglecting products such as $(a_2 - a_1)(a_3 - a_2)$, we get

$$k\alpha_1 (a_2 - a_1)(m_1 + n_1) \{(m_1 + n_1) k^2 \alpha_1^2 a^2 - 4n_1 (3m_1 - n_1)(m_1 + n_1)^{-1}\} \times (m_2 + n_2) k\alpha_2 a \times (m_3 + n_3) k\alpha_3 a$$
$$+ \text{three other terms} = 0 \quad \text{..........................(55)}.$$

Here a may be regarded as the mean radius of the shell. The last term in (55), viz. that answering to the term in (54) which contains $f(\bar{a}_2 . \alpha_2 . \bar{a}_3)$, is of order

$$(a_2 - a_1)(a_3 - a_2)(a_4 - a_3),$$

and so completely negligible. The remaining terms are of the same type as the first, which alone is shown in (55).

Thus dividing out by the essentially positive quantity

$$(m_1 + n_1)(m_2 + n_2)(m_3 + n_3) k^3 \alpha_1 \alpha_2 \alpha_3 a^4,$$

we obtain from (55) for the frequency equation

$$(a_2 - a_1) \rho_1 (k^2 - K^2_{(a_1 . a)}) + (a_3 - a_2) \rho_2 (k^2 - K^2_{(a_2 . a)}) + (a_4 - a_3) \rho_3 (k^2 - K^2_{(a_3 . a)}) = 0 \quad \text{........}(56_a),$$

where $K_{(a . a)}/2\pi$ represents as usual the frequency of the radial vibrations in a thin shell of radius a and material (ρ, m, n).

Supposing the layers of thicknesses t_1, t_2, t_3 and of masses M_1, M_2, M_3 respectively, we may write (56_a) in either of the alternative forms

$$k^2 = \{t_1 \rho_1 K^2_{(a_1 . a)} + t_2 \rho_2 K^2_{(a_2 . a)} + t_3 \rho_3 K^2_{(a_3 . a)}\} \div (t_1 \rho_1 + t_2 \rho_2 + t_3 \rho_3) \quad \text{..............}(56_b),$$
$$k^2 = \{M_1 K^2_{(a_1 . a)} + M_2 K^2_{(a_2 . a)} + M_3 K^2_{(a_3 . a)}\} \div (M_1 + M_2 + M_3) \quad \text{..............}(56_c).$$

This result may be extended to a thin compound shell of any number of very thin layers, and thus in the limit to a thin shell whose material varies continuously or discontinuously with the distance from the centre. If M denote the entire mass of the shell, a_1 and a_2 the radii of its bounding surfaces, terms of order $(1 - a_2/a_1)^2$ being supposed negligible, and the elastic constants m, n be known functions of the distance r from the centre, we have for the frequency equation

$$k^2 = \left\{\int_{a_1}^{a_2} 4\pi . 4n (3m - n)(m + n)^{-1} dr\right\} \div M \quad \text{........................(57)}.$$

This result for a thin compound shell could doubtless be easily—and probably in the opinion of most authorities satisfactorily—obtained without reference to the surface conditions by applying dynamical principles to some assumed type of vibration. Whether this has been already done or not I do not know.

SECTION VII.

TRANSVERSE VIBRATIONS IN SPHERICAL SHELL.

§ 101. I pass next to a consideration of the transverse vibrations in a spherical shell.

Employing the notation of Sect. I. and the forms given in Sect. III. for the types of displacement and stress in this case, we have

$$\left.\begin{aligned} F(r.\beta) &= n\{k\beta r J'_{i+\frac{1}{2}}(k\beta r) - \tfrac{3}{2}J_{i+\frac{1}{2}}(k\beta r)\},\\ F_1(r.\beta) &= n\{k\beta r J'_{-(i+\frac{1}{2})}(k\beta r) - \tfrac{3}{2}J_{-(i+\frac{1}{2})}(k\beta r)\},\\ G(r.\beta) &= J_{i+\frac{1}{2}}(k\beta r),\\ G_1(r.\beta) &= J_{-(i+\frac{1}{2})}(k\beta r)\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(1).$$

Putting for shortness

$$\left.\begin{aligned} \Delta(a.\beta.b) &= J_{i+\frac{1}{2}}(k\beta a)J_{-(i+\frac{1}{2})}(k\beta b) - J_{-(i+\frac{1}{2})}(k\beta a)J_{i+\frac{1}{2}}(k\beta b),\\ \Delta(a'.\beta.b) &= J'_{i+\frac{1}{2}}(k\beta a)J_{-(i+\frac{1}{2})}(k\beta b) - J'_{-(i+\frac{1}{2})}(k\beta a)J_{i+\frac{1}{2}}(k\beta b),\\ \Delta(a.\beta.b') &= J_{i+\frac{1}{2}}(k\beta a)J'_{-(i+\frac{1}{2})}(k\beta b) - J_{-(i+\frac{1}{2})}(k\beta a)J'_{i+\frac{1}{2}}(k\beta b),\\ \Delta(a'.\beta.b') &= J'_{i+\frac{1}{2}}(k\beta a)J'_{-(i+\frac{1}{2})}(k\beta b) - J'_{-(i+\frac{1}{2})}(k\beta a)J'_{i+\frac{1}{2}}(k\beta b)\end{aligned}\right\}\ \ldots\ldots(2),$$

we find for the frequency equations of the four fundamental types in the simple shell $(b.\beta.a)$:

$$f(b.\beta.a) = n^2\{k^2\beta^2ab\Delta(a'.\beta.b') + \tfrac{9}{4}\Delta(a.\beta.b) - \tfrac{3}{2}k\beta a\Delta(a'.\beta.b) - \tfrac{3}{2}k\beta b\Delta(a.\beta.b')\} = 0 \ldots(3),$$

$$f(\bar{b}.\beta.a) = n\{k\beta a\Delta(a'.\beta.b) - \tfrac{3}{2}\Delta(a.\beta.b)\} = 0 \ldots\ldots(4),$$

$$f(b.\beta.\bar{a}) = n\{k\beta b\Delta(a.\beta.b') - \tfrac{3}{2}\Delta(a.\beta.b)\} = 0 \ldots\ldots(5),$$

$$f(\bar{b}.\beta.\bar{a}) = \Delta(a.\beta.b) = 0 \ldots\ldots(6).$$

These forms of the frequency equations are easily obtained from the general formulae in Sect. I.

For a shell in which $\{(a-b)/a\}^2$ is negligible the functions reduce to the following forms:—

$$f(b.\beta.a) = -\frac{a-b}{a}n^2k\beta a\Delta(a.\beta.a')\{k^2\beta^2a^2 - (i-1)(i+2)\} \ldots\ldots(7),$$

$$f(b.\beta.a) = -nk\beta a\Delta(a.\beta.a')\left(1 - \frac{3}{2}\frac{a-b}{a}\right) \ldots\ldots(8),$$

$$f(b.\beta.\bar{a}) = nk\beta a\Delta(a.\beta.a')\left(1 + \frac{3}{2}\frac{a-b}{a}\right) \ldots\ldots(9),$$

$$f(\bar{b}.\beta.\bar{a}) = -\frac{a-b}{a}k\beta a\Delta(a.\beta.a') \ldots\ldots(10).$$

It has been already pointed out that

$$k\beta a\Delta(a.\beta.a') = -C \ldots\ldots(11),$$

where C is a constant quantity independent of k, β or a.

Equating the several functions to zero we get the frequency equations for the four fundamental types in a thin shell. The free-free vibration is, it will be observed, the only case in which the frequency equation has a finite root.

Supposing b absolutely equal to a we get

$$\left.\begin{aligned} f(\bar{a}\,.\,\beta\,.\,a) &= -f(a\,.\,\beta\,.\,\bar{a}) = nC, \\ f(a\,.\,\beta\,.\,a) &= f(\bar{a}\,.\,\beta\,.\,\bar{a}) \quad = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12).$$

Thus $f(\bar{a}\,.\,\beta\,.\,a)$ and $f(a\,.\,\beta\,.\,\bar{a})$ are quantities which cannot vanish, each being the product of n into an absolute constant.

Employing the result (12) in the general equation (30) of Sect. I., we find

$$f(e\,.\,\beta\,.\,c\,.\,\beta\,.\,b\,.\,\beta\,.\,a) = n^2C^2 f(e\,.\,\beta\,.\,a) \quad \ldots\ldots\ldots\ldots(13).$$

Another result we require is the value when $f(e\,.\,\beta\,.\,a)=0$ of

$$k\frac{d}{dk}f(e\,.\,\beta\,.\,a) \equiv \left[k\beta a\frac{d}{d\,.\,k\beta a} + k\beta e\frac{d}{d\,.\,k\beta e}\right] f(e\,.\,\beta\,.\,a),$$

where $k\beta a$ and $k\beta e$ are to be regarded as independent variables. By work exactly similar in its general outlines to that already indicated in the case of the radial vibrations it is not very difficult to prove

$$\left.\begin{aligned} kbe\frac{d}{d\,.\,k\beta e}f(e\,.\,\beta\,.\,a) &= -\rho e^2\{k^2 - K^2_{(\beta\,.\,e)}\} f(\bar{e}\,.\,\beta\,.\,a), \\ k\beta a\frac{d}{d\,.\,k\beta a}f(e\,.\,\beta\,.\,a) &= -\rho a^2\{k^2 - K^2_{(\beta\,.\,a)}\} f(e\,.\,\beta\,.\,\bar{a}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(14),$$

where $\frac{1}{2\pi}K_{(\beta\,.\,r)}$ is the frequency of free-free transverse vibrations in an infinitely thin shell of radius r and material β.

Thus $$k\frac{d}{dk}f(e\,.\,\beta\,.\,a) = -\rho\,\{e^2(k^2 - K^2_{(\beta\,.\,e)}) f(\bar{e}\,.\,\beta\,.\,a) + a^2(k^2 - K^2_{(\beta\,.\,a)}) f(e\,.\,\beta\,.\,\bar{a})\} \ldots(15).$$

§ 102. We have now all the necessary data for determining the frequency equation for the compound shell $(e\,.\,\beta\,.\,c\,.\,\beta_1\,.\,b\,.\,\beta\,.\,a)$, in which $b-c$ is small.

From the general equation (23) in Sect. I. we have

$$\begin{aligned} f(e\,.\,\beta\,.\,c\,.\,\beta_1\,.\,b\,.\,\beta\,.\,a) &= f(\bar{b}\,.\,\beta\,.\,a)\{f(e\,.\,\beta\,.\,c)f(\bar{c}\,.\,\beta_1\,.\,b) - f(e\,.\,\beta\,.\,\bar{c})f(c\,.\,\beta_1\,.\,b)\} \\ &\quad - f(b\,.\,\beta\,.\,a)\{f(e\,.\,\beta\,.\,c)f(\bar{c}\,.\,\beta_1\,.\,\bar{b}) - f(e\,.\,\beta\,.\,\bar{c})f(c\,.\,\beta_1\,.\,\bar{b})\} = 0 \quad \ldots\ldots\ldots(16). \end{aligned}$$

Now supposing terms in $\{(b-c)/b\}^2$ negligible and employing the results corresponding to (7)—(10), we easily put (16) into the form

$$\begin{aligned} \frac{f(e\,.\,\beta\,.\,c\,.\,\beta_1\,.\,b\,.\,\beta\,.\,a)}{n_1 C} &= f(e\,.\,\beta\,.\,c)f(b\,.\,\beta\,.\,a) - f(e\,.\,\beta\,.\,c)f(b\,.\,\beta\,.\,a) \\ &\quad - \frac{b-c}{b}\rho_1 b^2(k^2 - K^2_{(\beta\,.\,b)}) f(e\,.\,\beta\,.\,b)f(\bar{b}\,.\,\beta\,.\,a) \end{aligned}$$

$$-\frac{b-c}{b}\left[\frac{1}{n_1}f(e\,.\,\beta\,.\,b)f(b\,.\,\beta\,.\,a) + \tfrac{3}{2}f(e\,.\,\beta\,.\,b)f(\bar{b}\,.\,\beta\,.\,a) + \tfrac{3}{2}f(e\,.\,\beta\,.\,\bar{b})f(b\,.\,\beta\,.\,a)\right] = 0 \ldots(17).$$

In the coefficient of $b-c$ in accordance with the hypothesis that $(b-c)^2$ is negligible, c has always been replaced by b.

Writing β, n for β_1, n_1 respectively in (17) we obtain an expression for

$$f(e.\beta.c.\beta.b.\beta.a),$$

employing which we find for the frequency equation

$$\frac{f(e.\beta.c.\beta_1.b.\beta.a)}{n_1 C}$$

$$=\frac{f(e.\beta.c.\beta.b.\beta.a)}{nC}+\frac{b-c}{b}.b^2\{\rho(k^2-K^2_{(\beta.b)})-\rho_1(k^2-K^2_{(\beta_1.b)})\}f(e.\beta.\bar{b})f(b.\beta.a)$$

$$+\frac{b-c}{b}\left(\frac{1}{n}-\frac{1}{n_1}\right)f(e.\beta.b)f(b.\beta.a)=0 \quad \text{..........................(18)}.$$

But if ∂k be the increase in k due to the existence of the layer, this must be identical with

$$f(e.\beta.a)-\frac{\partial k}{k}.k\frac{d}{dk}f(e.\beta.a)=0 \quad \text{..............................(19)}.$$

Thus remembering (13) and (15), we find on comparing (18) and (19),

$$\frac{\partial k}{k}.nC\rho\{e^2(k^2-K^2_{(\beta.e)})f(\bar{e}.\beta.a)+a^2(k^2-K^2_{(\beta.a)})f(e.\beta.\bar{a})\}\div\frac{b-c}{b}$$

$$=b^2\{\rho(k^2-K^2_{(\beta.b)})-\rho_1(k^2-K^2_{(\beta_1.b)})\}f(e.\beta.\bar{b})f(\bar{b}.\beta.a)$$

$$+\left(\frac{1}{n}-\frac{1}{n_1}\right)f(e.\beta.b)f(b.\beta.a) \quad \text{.............(20)}.$$

This formula can be transformed into another of greater physical significance. By methods precisely similar to those employed in the case of the radial vibrations I find when $f(e.\beta.a)=0$:

$$\left.\begin{aligned} f(e.\beta.\bar{b})\div f(e.\beta.\bar{a}) &= (b/a)^{\frac{1}{2}}\times(w_b/w_a),\\ f(\bar{b}.\beta.a)\div f(\bar{e}.\beta.a) &= (b/e)^{\frac{1}{2}}\times(w_b/w_e),\\ f(e.\beta.b)\div f(e.\beta.\bar{a}) &= (b^3/a)^{\frac{1}{2}}\times(W_b/w_a),\\ f(b.\beta.a)\div f(\bar{e}.\beta.a) &= (b^3/e)^{\frac{1}{2}}\times(W_b/w_e)\end{aligned}\right\} \quad \text{.......................(21)},$$

$$f(\bar{e}.\beta.a)\times f(e.\beta.\bar{a})=-n^2C^2 \quad \text{.......................................(22)},$$

where C is the quantity defined in (11), and w and W are the displacement and stress in a simple shell. The form of $b^{\frac{1}{2}}w_b$ may be got by writing b for r and β for β_1 on the right-hand side of (8), Sect. III., and W_b is the corresponding stress.

Employing these relations we transform (20) into

$$\frac{\partial k}{k}\div\frac{b-c}{\sqrt{ae}}=\frac{-nC}{\rho w_a w_e\{e^2(k^2-K^2_{(\beta.e)})f(\bar{e}.\beta.a)+a^2(k^2-K^2_{(\beta.a)})f(e.\beta.\bar{a})\}}$$

$$\times\left[b^2(w_b)^2\{\rho(k^2-K^2_{(\beta.b)})-\rho_1(k^2-K^2_{(\beta_1.b)})\}+b^2(W_b)^2\left(\frac{1}{n}-\frac{1}{n_1}\right)\right] \quad \text{...(23)*}.$$

§ 103. Passing to the limit when e vanishes it may be shown without much difficulty that

$$\left.\begin{aligned} e^{\frac{1}{2}}w_e f(e.\beta.\bar{a}) &= -nCa^{\frac{1}{2}}w_a,\\ e^{\frac{5}{2}}w_e f(\bar{e}.\beta.a) &= 0\end{aligned}\right\} \quad \text{.................................(24)}.$$

* See the note on p. 266.

When these values are substituted (23) becomes identical with the result obtained for the solid sphere, viz. (22) of Sect. III.

§ 104. From the same consideration as was employed in the case of the radial vibrations we conclude that the first factor on the right-hand side of (23) is essentially a positive quantity.

The second factor on the right of (23), which alone varies with b, is identical in *form* with the corresponding factor in (22) Sect. III., giving the change of frequency in a solid sphere, so a brief discussion of its general features will suffice.

When an alteration of density occurs at a node surface of a particular note it does not affect its pitch, but when it occurs elsewhere the pitch is invariably raised or lowered according as the density is diminished or increased. The numerical magnitude of the percentage change of pitch depends solely on the magnitude of the alteration of density and not at all on its sign.

The law of variation with the position of the layer of the change of pitch due to a given alteration of density is independent of the magnitude of the alteration of density. When the layer of altered density is of given volume the positions in which it has most effect on the pitch of a given note coincide with the loop surfaces for that note; when the layer is of given thickness its most effective positions lie slightly outside the loop surfaces.

When the layer differs from the remainder only in elasticity the second factor on the right of (23) reduces to

$$\left[w_b^2(n_1-n)(i-1)(i+2)+b^2W_b^2\left(\frac{1}{n}-\frac{1}{n_1}\right)\right].$$

The change of pitch thus depends solely on the alteration of rigidity. Unless in the case of the rotatory vibrations, for which $i=1$, the above factor is the sum of two squares which cannot simultaneously vanish except for $b=0$. Thus excluding the case of a solid sphere, an alteration of rigidity throughout a thin layer situated anywhere necessarily affects the pitch of any transverse vibration other than one of the rotatory type, and the pitch is raised or lowered according as the rigidity is increased or diminished. In the case of a rotatory vibration the change of pitch when existent has always the same sign as the alteration of rigidity, but it vanishes when the altered layer coincides with a no-stress surface.

In the case of a rotatory vibration the positions in which the layer, when of given volume, has most effect on the pitch coincide with those surfaces over which the transverse stress is a maximum, but this is not exactly true of any other vibration of the transverse type.

§ 105. Some of the more interesting special cases call for a more detailed examination.

Thus suppose the altered layer to be found at the outer surface so that

$$b = a, \quad W_b = W_a = 0.$$

Remembering (12) we find from the second of equations (21)

$$f(\bar{e} \,.\, \beta \,.\, a) = nC\,(e/a)^{\frac{1}{2}}\,(w_e/w_a),$$

whence by (22)

$$f(e \,.\, \beta \,.\, \bar{a}) = -\,nC\,(a/e)^{\frac{1}{2}}\,(w_a/w_e).$$

Thus from (23) if the thickness of the layer be t_1 and the change of pitch ∂k_1,

$$\frac{\partial k_1}{k} = \frac{t_1}{a} \,.\, a^3\,(w_a)^2\,\{\rho\,(k^2 - K^2_{(\beta\,.\,a)}) - \rho_1\,(k^2 - K^2_{(\beta_1\,.\,a)})\} \div \rho D \quad\text{..................(25)},$$

where $$D = a^3\,(w_a)^2\,(k^2 - K^2_{(\beta\,.\,a)}) - e^3\,(w_e)^2\,(k^2 - K^2_{(\beta\,.\,e)}) \quad\text{........................(26)}.$$

Similarly if ∂k_2 be the change of pitch due to the existence of a layer of thickness t_2 and material (ρ_2, n_2) at the inner surface of the shell, we find

$$\frac{\partial k_2}{k} = \frac{t_2}{e} \,.\, e^3\,(w_e)^2\,\{\rho\,(k^2 - K^2_{(\beta\,.\,e)}) - \rho_2\,(k^2 - K^2_{(\beta_2\,.\,e)})\} \div \rho D \quad\text{..................(27)}.$$

If the layer differ from the remainder only in density, and the mass of the shell be increased by ∂M_1 when the layer is at the inner surface and by ∂M_2 when it is at the outer, then putting

$$M_1 = 4\pi a^3\rho/3, \quad \partial M_1 = 4\pi a^2 t_1\,(\rho_1 - \rho),$$
$$M_2 = 4\pi e^3\rho/3, \quad \partial M_2 = 4\pi e^2 t_2\,(\rho_2 - \rho),$$

we find

$$\left.\begin{aligned} \frac{\partial k_1}{k} &= -\,\partial M_1\,(w_a)^2\,k^2 \div D', \\ \frac{\partial k_2}{k} &= -\,\partial M_2\,(w_e)^2\,k^2 \div D' \end{aligned}\right\} \quad\text{................................(28)},$$

where $$D' = 3\,\{M_1\,(w_a)^2\,(k^2 - K^2_{(\beta\,.\,a)}) - M_2\,(w_e)^2\,(k^2 - K^2_{(\beta\,.\,e)})\} \quad\text{.................(29)}.$$

From (28) we get

$$\partial k_1 \,:\, \partial k_2 \,::\, \partial M_1\,(w_a)^2 \,:\, \partial M_2\,(w_e)^2 \quad\text{.............................(30)}.$$

If on the other hand the surface layers differ from the remainder only in elasticity, we find for the corresponding changes of pitch

$$\left.\begin{aligned} \frac{\partial k_1}{k} &= (n_1 - n)\,(i-1)\,(i+2)\,t_1\,(w_a)^2 \div \rho D, \\ \frac{\partial k_2}{k} &= (n_2 - n)\,(i-1)\,(i+2)\,t_2\,(w_e)^2 \div \rho D \end{aligned}\right\} \quad\text{........................(31)},$$

where D is given by (26).

Thus for equal alterations of rigidity at the two surfaces

$$\partial k_1 \,:\, \partial k_2 \,::\, t_1\,(w_a)^2 \,:\, t_2\,(w_e)^2 \quad\text{.................................(32)}.$$

The results (30) and (32) are identical in import with the corresponding results for the radial vibrations, viz. (47) and (49) Sect. VI., and similar conclusions may be drawn.

An exception must however be made of the rotatory vibrations as their pitch is unaffected by an alteration of rigidity occurring at either surface.

On account of this peculiarity in the rotatory vibrations it seems worth while recording the special forms taken in their case by the expressions for the changes of pitch due to surface alterations of material, viz.

$$\left.\begin{aligned}\frac{\partial k_1}{k} &= -t_1(\rho_1-\rho)\,a^2(w_a)^2 \div \{\rho a^3(w_a)^2 - \rho e^3(w_e)^2\},\\ \frac{\partial k_2}{k} &= -t_2(\rho_2-\rho)\,e^2(w_e)^2 \div \{\rho a^3(w_a)^2 - \rho e^3(w_e)^2\}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

§ 106. In the general case we may, provided $(\partial k_1 \pm \partial k_2)/k$ be small, suppose the alterations in the material at the surfaces to exist simultaneously. Also by supposing ρ_1 and ρ_2 to vanish we can obtain the effect on the pitch of removing thin layers from the surfaces. Thus when layers of thicknesses t_1 and t_2 are simultaneously removed the change of pitch is given by

$$\frac{\partial k}{k} = \left\{\frac{t_1}{a}\,.\,a^3(w_a)^2(k^2 - K^2_{(\beta.a)}) + \frac{t_2}{e}\,.\,e^3(w_e)^2(k^2 - K^2_{(\beta.e)})\right\} \div D \ldots\ldots\ldots\ldots(34),$$

where D is given by (26).

Further by writing $-t_1$ for t_1 and $-t_2$ for t_2 we find the effect of adding layers of thicknesses t_1 and t_2 and of the same material as the remainder to the outer and inner surfaces. A verification of these conclusions is supplied by putting in (34)

$$t_2/e = -t_1/a,$$

when it reduces to $\qquad \partial k/k = t_1/a.$

§ 107. For a compound shell of three thin layers we have a frequency equation deducible from (54) Sect. VI. by writing β for α. This leads to a result deducible from (56_b) or (56_c) of that section by writing β for α. It may also be put in the specially neat form

$$k^2 = (i-1)(i+2)(n_1t_1 + n_2t_2 + n_3t_3) \div \{a^2(\rho_1t_1 + \rho_2t_2 + \rho_3t_3)\}\ldots\ldots\ldots\ldots\ldots(35).$$

Here t_1 etc. denote the thicknesses of the thin layers, (ρ_1, n_1) etc. their materials, and a the mean radius of the shell.

We may extend (35) to a thin compound shell of any number of layers, or to one in which the density and rigidity vary in any manner with the distance from the centre. The general formula applicable to all such cases is

$$k^2 = (i-1)(i+2)\int_{a_1}^{a_2} 4\pi n dr \div M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

Here M is the mass of the shell, a_1, a_2 the radii of its bounding surfaces, $(a_2-a_1)/a_1$ being so small its square is negligible, and n is supposed a known function of r, continuous or discontinuous.

SECTION VIII.

RADIAL VIBRATIONS IN CYLINDRICAL SHELL.

§ 108. Employing the notation of Sects. I. and IV. we may take in the case of the radial vibrations of a cylindrical shell:

$$\left.\begin{aligned} F(r.\alpha) &= (m+n)\,k\alpha r J_1'(k\alpha r) + (m-n)\,J_1(k\alpha r),\\ F_1(r.\alpha) &= (m+n)\,k\alpha r Y_1'(k\alpha r) + (m-n)\,Y_1(k\alpha r),\\ G(r.\alpha) &= J_1(k\alpha r),\\ G_1(r.\alpha) &= Y_1(k\alpha r) \end{aligned}\right\} \quad \text{.....................(1).}$$

Putting for shortness

$$\left.\begin{aligned} \Delta\,(a.\alpha.b) &= J_1\,(k\alpha a)\,Y_1\,(k\alpha b) - Y_1\,(k\alpha a)\,J_1\,(k\alpha b),\\ \Delta\,(a'.\alpha.b) &= J_1'(k\alpha a)\,Y_1\,(k\alpha b) - Y_1'(k\alpha a)\,J_1\,(k\alpha b),\\ \Delta\,(a.\alpha.b') &= J_1\,(k\alpha a)\,Y_1'(k\alpha b) - Y_1\,(k\alpha a)\,J_1'(k\alpha b),\\ \Delta\,(a'.\alpha.b') &= J_1'(k\alpha a)\,Y_1'(k\alpha b) - Y_1'(k\alpha a)\,J_1'(k\alpha b) \end{aligned}\right\} \quad \text{.....................(2),}$$

we find for the frequency equations of the four fundamental types in the simple shell $(b.\alpha.a)$:

$$f(b.\alpha.a) = (m+n)^2 k^2\alpha^2 ab\Delta\,(a'.\alpha.b') + (m-n)^2\,\Delta\,(a.\alpha.b)$$
$$+ (m^2-n^2)\,\{k\alpha a\Delta\,(a'.\alpha.b) + k\alpha b\Delta\,(a.\alpha.b')\} = 0 \quad \text{......(3),}$$

$$f(\bar{b}.\alpha.a) = (m+n)\,k\alpha a\Delta\,(a'.\alpha.b) + (m-n)\,\Delta\,(a.\alpha.b) = 0 \quad \text{.................(4),}$$

$$f(b.\alpha.\bar{a}) = (m+n)\,k\alpha b\Delta\,(a.\alpha.b') + (m-n)\,\Delta\,(a.\alpha.b) = 0 \text{.....................(5),}$$

$$f(\bar{b}.\alpha.\bar{a}) = \Delta\,(a.\alpha.b) = 0 \quad \text{...(6).}$$

For a thin shell in which $\{(a-b)/a\}^2$ is negligible the above functions assume the forms:

$$f(b.\alpha.a) = \frac{a-b}{a}\,C\,\{k^2\alpha^2a^2\,(m+n)^2 - 4mn\} \text{.........................(7),}$$

$$f(\bar{b}.\alpha.a) = C\left\{m+n+\frac{a-b}{a}\,(m-n)\right\} \quad \text{...........................(8),}$$

$$f(b.\alpha.\bar{a}) = -\,C\left\{m+n-\frac{a-b}{a}\,(m-n)\right\} \quad \text{...........................(9),}$$

$$f(\bar{b}.\alpha.\bar{a}) = \frac{a-b}{a}\,C \quad \text{...(10).}$$

where

$$C \equiv -\,k\alpha a\,\Delta(a.\alpha.a') \text{.......................................(11)}$$

is an absolute constant, depending only on the definition of the Bessel.

The result

$$F(a.\alpha)\,G_1\,(a.\alpha) - F_1\,(a.\alpha)\,G\,(a.\alpha) = (m+n)\,C \text{.....................(12),}$$

will be found useful in verifying the conclusions arrived at.

The method of obtaining the change of pitch due to the existence of the thin layer $(c.\alpha_1.b)$ in the shell $(e.\alpha.a)$ is precisely the same as that already illustrated in the case of the sphere. The relation

$$k\frac{d}{dk}f(e.\alpha.a) = -\rho\{e^2(k^2-K^2_{(\alpha,e)})f(\bar{e}.\alpha.a) + a^2(k^2-K^2_{(\alpha,a)})f(e.\alpha.\bar{a})\} \quad\text{......(13)}$$

also applies as in the case of the sphere, though of course the actual forms of the functions are different, and the values of $K_{(\alpha,a)}$ and $K_{(\alpha,e)}$ are to be derived from (24) Sect. IV.

Thus it will suffice to record the result of the operations indicated, viz.,

$$\frac{\partial k}{k}(m+n)C\rho\{e^2(k^2-K^2_{(\alpha,e)})f(\bar{e}.\alpha.a) + a^2(k^2-K^2_{(\alpha,a)})f(e.\alpha.\bar{a})\} \div \frac{b-c}{b}$$

$$= b^2\{\rho(k^2-K^2_{(\alpha,b)}) - \rho_1(k^2-K^2_{(\alpha_1,b)})\}f(e.\alpha.\bar{b})f(\bar{b}.\alpha.a) + \left(\frac{1}{m+n}-\frac{1}{m_1+n_1}\right)f(e.\alpha.b)f(b.\alpha.a)$$

$$+2\left(\frac{n}{m+n}-\frac{n_1}{m_1+n_1}\right)\{f(e.\alpha.b)f(\bar{b}.\alpha.a)+f(e.\alpha.\bar{b})f(b.\alpha.a)\} \quad\text{..........(14).}$$

Denoting by $u_r\cos kt$ the displacement, and by $U_r\cos kt$ the corresponding radial stress at an axial distance r, the following relations may be established in precisely the same way as the results (32) and (35) of Sect. VI., the relation $f(e.\alpha.a)=0$ being supposed to hold,

$$\left.\begin{aligned} f(e.\alpha.\bar{b})\div f(e.\alpha.\bar{a}) &= u_b/u_a,\\ f(\bar{b}.\alpha.a)\div f(\bar{e}.\alpha.a) &= u_b/u_e,\\ f(e.\alpha.b)\div f(e.\alpha.\bar{a}) &= bU_b/u_a,\\ f(b.\alpha.a)\div f(\bar{e}.\alpha.a) &= bU_b/u_e \end{aligned}\right\} \quad\text{..............................(15).}$$

$$f(\bar{e}.\alpha.a)\times f(e.\alpha.\bar{a}) = -(m+n)^2C^2 \quad\text{..........................(16).}$$

Employing these results, remembering that in the coefficient of $b-c$ we may suppose $f(e.\alpha.a)$ to vanish, we transform (14) into

$$\frac{\partial k}{k}\div\frac{b-c}{a} = \frac{-(m+n)aC}{\rho u_e u_a\{e^2(k^2-K^2_{(\alpha,e)})f(\bar{e}.\alpha.a)+a^2(k^2-K^2_{(\alpha,a)})f(e.\alpha.\bar{a})\}}$$

$$\times\left[b(u_b)^2\{\rho(k^2-K^2_{(\alpha,b)})-\rho_1(k^2-K^2_{(\alpha_1,b)})\}+b(U_b)^2\left(\frac{1}{m+n}-\frac{1}{m_1+n_1}\right)\right.$$

$$\left.+4u_bU_b\left(\frac{n}{m+n}-\frac{n_1}{m_1+n_1}\right)\right] \quad\text{...........(17)*.}$$

§ 109. In the limiting case when e vanishes it may be shown that

$$\left.\begin{aligned} u_e f(e.\alpha.a) &= -(m+n)Cu_a,\\ e^2u_e(k^2-K^2_{(\alpha,e)})f(\bar{e}.\alpha.a) &= 0 \end{aligned}\right\} \quad\text{.............................(18).}$$

and we thence obtain for the value of $\partial k/k$ in a solid cylinder a result identical with (27) of Sect. IV.

* See the note on p. 266.

§ 110. From the same consideration as before we conclude that the first factor on the right-hand side of (17), which is independent of b, is essentially a positive quantity.

The form of the second factor on the right of (17) leads to the following general conclusions:—

When an alteration of density alone occurs at a node surface of a particular note it does not affect the pitch of that note, but when it occurs elsewhere the pitch is raised or lowered according as the density is diminished or increased. The numerical magnitude of the percentage change of pitch is independent of the sign of a given numerical alteration in density. The law of variation with the position of the layer of the change of pitch due to a given alteration of density is independent of the magnitude of the alteration. When the layer of altered density is of given volume, i.e. when $(b-c)b$ is constant, the positions in which it has most effect on the pitch of a given note coincide with the loop surfaces; when the layer is of given thickness the most effective positions lie slightly outside the loop surfaces.

When the layer differs from the remainder only in elasticity the expression for the change of pitch consists of three terms. Of these the first has the same sign as, and is proportional in magnitude to $m_1n_1(m_1+n_1)^{-1} - mn(m+n)^{-1}$. It vanishes when the layer coincides with a node surface of the note in question.

The second term has the same sign as, and is proportional in magnitude to $(m+n)^{-1} - (m_1+n_1)^{-1}$. It vanishes when the layer coincides with a no-stress surface.

The third term varies as $n(m+n)^{-1} - n_1(m_1+n_1)^{-1}$, but its sign depends also on the value of b. It vanishes when the layer coincides either with a node or a no-stress surface. It vanishes for all positions of the layer provided

$$m_1/m = n_1/n = 1 + p \quad \text{.................................(19).}$$

Thus on the uniconstant theory, or more generally when (19) is true, the sign of the change of pitch following a given alteration of elasticity is the same as that of p and does not vary with the position of the layer. If however (19) do not hold, the sign of the change of pitch may vary for certain alterations of elasticity with the position of the layer.

From the form of the expressions for u_b and U_b it is easily proved that when kab is large the second term in the expression for the change of pitch due to an alteration in elasticity alone is much the most important, and that the third term is more important than the first. Thus in the case of the higher notes the effect of an alteration of elasticity, when of importance, especially when the alteration occurs near the maximum-stress surfaces of greatest radius, depends almost entirely on the term containing U_b^2; and the consequent change of pitch is a maximum when the alteration of elasticity occurs very close to the maximum-stress surfaces.

§ 111. Confining our further remarks to special cases, let us suppose the layer to be at one or other of the bounding surfaces. Remembering that U vanishes at a free surface, we easily find for the two positions of the layer with our usual notation

$$\left.\begin{aligned}\frac{\partial k_1}{k} &= \frac{t_1}{a} \cdot a^2 (u_a)^2 \{\rho (k^2 - K^2_{(a,a)}) - \rho_1 (k^2 - K^2_{(a_1,a)})\} \div \rho D, \\ \frac{\partial k_2}{k} &= \frac{t_2}{e} \cdot e^2 (u_e)^2 \{\rho (k^2 - K^2_{(a,e)} - \rho_2 (k^2 - K^2_{(a_2,e)})\} \div \rho D\end{aligned}\right\} \quad \text{.............(20);}$$

where
$$D = a^2 (u_a)^2 (k^2 - K^2_{(a,a)}) - e^2 (u_e)^2 (k^2 - K^2_{(a,e)}) \quad \text{..........................(21).}$$

When the layer differs from the remainder only in density, let us denote the masses per unit length of cylinders of radii a and e and of density ρ by M_1 and M_2 respectively, and let ∂M_1 and ∂M_2 denote the increases in the mass of the shell per unit length due to the existence of altered layers at its surfaces, so that

$$M_1 = \pi a^2 \rho, \qquad \partial M_1 = 2\pi a t_1 (\rho_1 - \rho),$$
$$M_2 = \pi e^2 \rho, \qquad \partial M_2 = 2\pi e t_2 (\rho_2 - \rho).$$

In this case (20) reduces to

$$\left.\begin{aligned}\frac{\partial k_1}{k} &= - \partial M_1 (u_a)^2 k^2 \div D', \\ \frac{\partial k_2}{k} &= - \partial M_2 (u_e)^2 k^2 \div D'\end{aligned}\right\} \quad \text{..................................(22),}$$

where
$$D' = 2 \{M_1 (u_a)^2 (k^2 - K^2_{(a,a)}) - M_2 (u_e)^2 (k^2 - K^2_{(a,e)})\} \quad \text{....................(23).}$$

From (22) we get

$$\partial k_1 : \partial k_2 :: \partial M_1 (u_a)^2 : \partial M_2 (u_e)^2 \quad \text{....................................(24).}$$

If on the other hand the surface layers differ from the remainder only in elasticity we find

$$\left.\begin{aligned}\frac{\partial k_1}{k} &= \frac{t_1}{a} (u_a)^2 \left\{\frac{4m_1 n_1}{m_1 + n_1} - \frac{4mn}{m + n}\right\} \div \rho D, \\ \frac{\partial k_2}{k} &= \frac{t_2}{e} (u_e)^2 \left\{\frac{4m_2 n_2}{m_2 + n_2} - \frac{4mn}{m + n}\right\} \div \rho D\end{aligned}\right\} \quad \text{..........................(25),}$$

where D is given by (21).

Thus for equal alterations in elasticity at the two surfaces we have

$$\partial k_1 : \partial k_2 :: a^{-1} t_1 (u_a)^2 : e^{-1} t_2 (u_e)^2 \quad \text{.............................(26).}$$

Comparing (24) and (26) we find

$$(\partial k_1/\partial k_2),\ \rho \text{ altered, } : (\partial k_1/\partial k_2), \text{ elasticity altered, } :: a^2 : e^2 \quad \text{...............(27),}$$

supposing the alterations in density and in elasticity to be the same at the two surfaces and to occur there throughout given layers. Thus relatively considered, an alteration of elasticity at the inner surface is more important than a like alteration at the outer surface.

§ 112. Supposing $(\partial k_1 \pm \partial k_2)/k$ small we may suppose the alterations at the surfaces to occur simultaneously. Also by supposing ρ_1 and ρ_2 to vanish we may find the effect of removing thin layers from the surfaces. Thus when layers of thicknesses t_1 and t_2 are simultaneously removed the change of pitch is given by

$$\frac{\partial k}{k} = \left\{\frac{t_1}{a} a^2 (u_a)^2 (k^2 - K^2_{(a,a)}) + \frac{t_2}{e} e^2 (u_e)^2 (k^2 - K^2_{(a,e)})\right\} \div D \quad\text{.................(28)},$$

where D is given by (21).

By changing the signs of t_1 and t_2 in (28) we get the effect of adding layers of thicknesses t_1 and t_2 to the bounding surfaces, the added layers being of the same material as the rest of the shell. As usual a verification is supplied by putting in (28)

$$t_2/e = -t_1/a,$$

when it reduces to

$$\partial k/k = t_1/a.$$

§ 113. For a compound shell of three thin layers the equation (54) Sect. VI. applies without any change in form. From it we easily obtain results identical in form with (56_b) and (56_c) of that section. We may also write the expression for the frequency in the form

$$k^2 = \left(t_1 \frac{4m_1n_1}{m_1+n_1} + t_2 \frac{4m_2n_2}{m_2+n_2} + t_3 \frac{4m_3n_3}{m_3+n_3}\right) \div a^2 (\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3) \quad\text{...........(29)}.$$

This result may be extended to a thin compound shell of any number of layers, or to one in which the density and elasticity vary in any manner with the distance from the axis. The general formula applicable to all such cases is

$$k^2 = \frac{8\pi}{aM}\int_{a_1}^{a_2} \frac{mn}{m+n}\, dr \quad\text{....................................(30)}.$$

Here M is the mass of the shell per unit length, a_1, a_2 the radii of its bounding surfaces, $\{(a_2 - a_1)/a_1\}^2$ being negligible, a the mean radius of the shell, and m, n are supposed known functions of the axial distance r.

SECTION IX.

TRANSVERSE VIBRATIONS IN CYLINDRICAL SHELL.

§ 114. Employing the notation of Sections I. and V., we may take in the case of the transverse vibrations of a cylindrical shell:

$$\left.\begin{aligned} F(r.\beta) &= n\{k\beta r J_1'(k\beta r) - J_1(k\beta r)\}, \\ F_1(r.\beta) &= n\{k\beta r Y_1'(k\beta r) - Y_1(k\beta r)\}, \\ G(r.\beta) &= J_1(k\beta r), \\ G_1(r.\beta) &= Y_1(k\beta r) \end{aligned}\right\} \quad\text{.........................(1)}.$$

Now these expressions and likewise the expressions for the displacements and stresses can be at once derived from the corresponding expressions in the case of the radial vibrations by simply supposing m to vanish and writing β for α. Thus it is unnecessary to go through the mathematical work by which the expression for $\partial k/k$ is arrived at, because with 0 substituted for m and β for α each step of the analysis in the case of the radial vibrations applies to the present case.

The very same constant quantity C that occurred in the case of the radial vibrations occurs here also, though it presents itself under the form

$$C = -k\beta a\{J_1(k\beta a)\,Y_1'(k\beta a) - J_1'(k\beta a)\,Y_1(k\beta a)\} \quad \text{.....................(2).}$$

In transforming the expression (17) Sect. VIII. for the change of pitch it must be remembered that, as shown in Sect. V., $K_{(\beta.a)}$ is zero.

We thus find for the change of pitch in the transverse note of frequency $k/2\pi$ in the shell $(e.\beta.a)$ due to the presence of the thin altered layer $(c.\beta_1.b)$ the equation—

$$\frac{\partial k}{k} \div \frac{b-c}{a} = \frac{-naC}{\rho v_e v_a k^2\{e^2 f(\bar{e}.\beta.\alpha) + a^2 f(e.\beta.\bar{a})\}} \times \left\{-b\,(v_b)^2 k^2(\rho_1 - \rho) + b\,(V_b)^2\left(\frac{1}{n} - \frac{1}{n_1}\right)\right\} \quad \text{...........(3)*.}$$

The forms of v and V are given by

$$v_r = AG(r.\beta) + BG_1(r.\beta),$$
$$rV_r = AF(r.\beta) + BF_1(r.\beta),$$

the value of B/A being determined by one of the surface conditions.

§ 115. For the limiting case when e vanishes we have

$$\left.\begin{aligned} v_e f(e.\beta.\bar{a}) &= -nCv_a, \\ e^2 v_e f(\bar{e}.\beta.a) &= 0 \end{aligned}\right\} \quad \text{....................................(4),}$$

and we thence obtain for $\partial k/k$ a result identical with (8) of Sect. V.

§ 116. The first factor on the right-hand side of (3) is independent of b and may by the same consideration as in the previous types of vibration be seen to be essentially positive. The second factor, which shows the variation of the change of pitch with the position of the layer, consists of only two terms, of which the first depends only on the alteration of density, the second only on the alteration of rigidity.

When an alteration of density alone occurs, the pitch of a given note is unaffected when the layer coincides with one of its node surfaces, but for all other positions of the layer the pitch is raised or lowered according as the density is diminished or increased. The numerical magnitude of the percentage change of pitch is independent of the sign of the alteration of density, and the law of variation with the position of the layer of the change of pitch due to a given alteration of density is independent of the magnitude of the alteration. When the layer of altered density is of given volume per unit length of cylinder, the positions in which it has most effect on the pitch of a given note coincide with its loop surfaces.

When an alteration of elasticity alone occurs, the change of pitch depends solely on the alteration of rigidity. The pitch of a given note is unaffected when the layer coincides with one of its no-stress surfaces, but for all other positions of the layer it is raised or lowered according as the rigidity is increased or diminished. The law of variation with the position of the layer of the change of pitch due to a given alteration of rigidity

* See the note on p. 266.

is independent of the magnitude of the alteration; but a diminution of rigidity is more effective in lowering the pitch than an equal increase is in raising it. For a given alteration of rigidity throughout a given volume the change of pitch has its maxima when the layer is at the maximum-stress surfaces.

§ 117. For the cases when the layer coincides with the surfaces of the shell we have with the usual notation

$$\left.\begin{aligned} \frac{\partial k_1}{k} &= -\frac{t_1}{a}\frac{\rho_1-\rho}{\rho}\frac{a^2(v_a)^2}{D}, \\ \frac{\partial k_2}{k} &= -\frac{t_2}{e}\frac{\rho_2-\rho}{\rho}\frac{e^2(v_e)^2}{D} \end{aligned}\right\} \qquad \text{.......(5)},$$

where

$$D = a^2(v_a)^2 - e^2(v_e)^2 \qquad \text{.......(6)}.$$

A surface alteration of elasticity has thus no effect on the pitch, and if ∂M_1 and ∂M_2 be the alterations in the mass of the shell per unit length due to alterations in the density at the outer and inner surfaces respectively, the corresponding changes of pitch have their ratio given by

$$\partial k_1 : \partial k_2 :: \partial M_1 (v_a)^2 : \partial M_2 (v_e)^2 \qquad \text{.......(7)}.$$

When alterations exist simultaneously at both surfaces we have with the usual limitation

$$\partial k = \partial k_1 + \partial k_2.$$

When layers of thicknesses t_1 and t_2 are simultaneously removed the change of pitch is given by

$$\frac{\partial k}{k} = \left\{\frac{t_1}{a}\,.\,a^2(v_a)^2 + \frac{t_2}{e}\,.\,e^2(v_e)^2\right\} \div D \qquad \text{.......(8)},$$

where D is given by (6).

By changing the signs of t_1 and t_2 we get the effect of adding surface layers of thicknesses t_1 and t_2 of the same material as the remainder.

The frequency of the transverse vibrations of a composite shell when very thin is always zero. In other words no such vibration has a physical existence.

[December 1, 1891. The factors independent of b in the general expressions for $\partial k/k$ in shells can be put into simpler forms. Replace (36) p. 248 by $\partial k/k = (b-c)\,\rho^{-1}D^{-1} \times$ [last factor] ...(a), (23) p. 256 by $\partial k/k = (b-c)\,\rho^{-1}D^{-1} \times$ [last factor]...(b), (17) p. 261 by $\partial k/k = (b-c)\,\rho^{-1}D^{-1} \times$ [last factor]...(c), (3) p. 265 by $\partial k/k = (b-c)\,k^{-2}\rho^{-1}D^{-1} \times$ [last factor]...(d), where D is given: in (a) by (43) p. 251, in (b) by (26) p. 258, in (c) by (21) p. 263, in (d) by (6) p. 266.

The modes of reduction are all similar to the following for case (a). Using the notation of pp. 247—8, we have

$$\frac{f(\bar{e}\,.\,a\,.\,a)}{(m+n)\,kaa} = \frac{F(a\,.\,a)\,G_1(e\,.\,a) - F_1(a\,.\,a)\,G(e\,.\,a)}{F(a\,.\,a)\,G_1(a\,.\,a) - F_1(a\,.\,a)\,G(a\,.\,a)} = \frac{BG_1(e\,.\,a) + AG(e\,.\,a)}{BG_1(a\,.\,a) + AG(a\,.\,a)} = \frac{eu_e}{au_a},$$

and therefore by (35) p. 248, $f(e\,.\,a\,.\,\bar{a}) \div (m+n)\,kae = -\,au_a/eu_e$.

In case (b) use $nC = F(a\,.\,\beta)\,G_1(a\,.\,\beta) - F_1(a\,.\,\beta)\,G(a\,.\,\beta)$, and similarly for ($c$) and ($d$).]

VII. *On Pascal's Hexagram.* By H. W. Richmond, M.A., Fellow of King's College.

In the volume of the *Atti della Reale Accademia dei Lincei*, published in 1877, there are two important memoirs on the subject of the Pascal Hexagram: the first, by Professor Veronese, contains geometrical proofs of all previously known properties of the figure together with a large number of new properties discovered by him. The second memoir, by Cremona, obtains proofs of many of the theorems given by Veronese from a new standpoint, viz. by deriving the hexagram from the projection of the lines which lie on a cubic surface with a nodal point, the nodal point being the origin of projection.

It is my purpose in these pages to attack the subject by the methods of Analysis, adopting Cremona's point of view. I have recently been led to notice a new form of the equation of a nodal cubic surface which has the advantage of giving the equations of the lines on the surface in perfectly symmetrical forms,—that is to say in forms where each line is represented by exactly similar equations: using this form of equation to the surface, I propose to develop briefly a few properties of these lines, and others connected with them, and then by projecting these lines upon an arbitrary plane to obtain analytical proofs of theorems relating to the Pascal Hexagram.

There are three other references which I wish to make to papers on this subject. The second volume of the *American Journal of Mathematics* contains an interesting paper by Miss Christine Ladd, in which the chief properties of Veronese are explained in a concise form and his notation improved and simplified; some new results are given connecting the Pascal Hexagram formed by six points on a conic with the Brianchon Hexagram formed by drawing tangents at those points: in the second place, Professor Cayley has published two papers in the *Quarterly Journal of Mathematics*, Vol. IX., pp. 268 and 348, of which the latter contains some results whose form is strikingly suggestive of the forms obtained here, though the connexion is not apparent: lastly, in the volume of the same periodical for 1888 will be found a short paper written before I had obtained the simpler form to which the equation to the cubic surface can be reduced, which forms the foundation of the present discussion.

The nodal cubic surface.

Let the nodal or conical point O be taken as one vertex of the tetrahedron of reference for a system of four plane coordinates, so that the equation to the surface is of the form

$$(*\between x, y, z)^3 + w(*\between x, y, z)^2 = 0.$$

It is clear that there are six straight lines on the surface which pass through O the nodal point and that these lie on a quadric cone; they are in fact the lines of intersection of the two cones

$$(*\between x, y, z)^2 = 0,$$

and

$$(*\between x, y, z)^3 = 0.$$

Denote these lines by A, B, C, D, E, F; then any plane which contains two of them, as for example C and E, must cut the surface also in a third line which does not pass through the nodal point; this line may be called CE.

We have thus found on the surface six lines which pass through O the nodal point, and fifteen other lines which do not pass through O, and these form the complete system of lines on the surface. For the plane through any line on the surface and the nodal point O must cut the surface also in a curve of the second order having a double point at O, i.e. in two straight lines which pass through O: hence, since only six lines on the surface pass through O, there can only be fifteen other lines on the surface. Two lines such as CD and CE cannot intersect since they both meet the line C; but it may be shewn that any two of the fifteen lines which are not met by the same line through O must intersect. For if we take a series of planes through one of the lines, AB, these cut the surface also in conics which are found to break up into two straight lines for three planes of the system besides the plane OAB; further it is seen that the pairs of points of intersection of these conics with AB are in involution. It is therefore necessary that these three planes which pass through the line AB should contain respectively the pairs of lines CD, EF; CE, DF; CF, DE.

There are therefore fifteen planes, known as tritangent (or triple tangent) planes, which cut the surface in three straight lines and which do not pass through O; three such planes pass through each of the fifteen lines, and moreover the eight points on any line AB where it is met by the lines CD, EF; CE, DF; CF, DE; and by the lines A and B are in involution.

Equation to the surface.

Taking nine lines such as AB, AC, AF, DB, DC, DF, EB, EC, EF, we see that they lie by threes in six tritangent planes;

AF, BD, CE, lie in a tritangent plane $x = 0$,
AC, BE, DF, ... $y = 0$,
AB, CD, EF, ... $z = 0$,
AB, CE, DF, ... $u = 0$,
AC, BD, EF, ... $v = 0$,
AF, BE, CD, ... $w = 0$.

Hence the equation to the surface must be

$$xyz = k \, . \, uvw.$$

Further, since none of these six planes pass through O the nodal point, we are at liberty to assume that at O

$$x = y = z = u = v = w.$$

Therefore $k = 1$ and the equation to the surface is

$$xyz = uvw.$$

The equation to the tangent plane at $(x'y'z'u'v'w')$ is

$$\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = \frac{u}{u'} + \frac{v}{v'} + \frac{w}{w'};$$

if now $(x'y'z'u'v'w')$ be the coordinates of O, this will give an identical relation in $xyzuvw$, viz.

$$x + y + z \equiv u + v + w.$$

But a second identical linear relation must connect these quantities, such as

$$l_1 x + m_1 y + n_1 z + p_1 u + q_1 v + r_1 w \equiv 0,$$

where

$$l_1 + m_1 + n_1 + p_1 + q_1 + r_1 = 0,$$

since at O

$$x = y = z = u = v = w.$$

Hence $\quad (l_1 + \lambda) x + (m_1 + \lambda) y + (n_1 + \lambda) z + (p_1 - \lambda) u + (q_1 - \lambda) v + (r_1 - \lambda) w \equiv 0$ for all values of λ.

We can now find one finite value of λ such that

$$(l_1 + \lambda)(m_1 + \lambda)(n_1 + \lambda) + (p_1 - \lambda)(q_1 - \lambda)(r_1 - \lambda) = 0.$$

Give λ this value and replace

$$l_1 + \lambda, \ m_1 + \lambda, \ n_1 + \lambda, \ p_1 - \lambda, \ q_1 - \lambda, \ r_1 - \lambda,$$

by

$$l, \ m, \ n, \ p, \ q, \ r,$$

and the second linear relation takes the form

$$lx + my + nz + pu + qv + rw \equiv 0,$$

where

$$l + m + n + p + q + r = 0,$$

and

$$lmn + pqr = 0.$$

Equations of the fifteen lines.

It has now been shewn that the equation to the surface can be brought to the form

$$\left.\begin{aligned} xyz &= uvw \ \ldots\ldots\ldots\ldots\ldots\ldots(1), \\ \text{where} \quad x + y + z &\equiv u + v + w \ \ldots\ldots\ldots\ldots(2), \\ lx + my + nz + pu + qv + rw &\equiv 0 \ldots\ldots\ldots(3), \\ l + m + n + p + q + r &= 0 \ \ldots\ldots\ldots\ldots(4), \\ lmn + pqr &= 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(5) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{A}:$$

and at the nodal point O,

$$x = y = z = u = v = w.$$

It is now possible to obtain the equations of all the fifteen lines AB, AC, etc. Nine of them have already been found, viz.

AB, $z=0$, $u=0$;	DB, $x=0$, $v=0$;	EB, $y=0$, $w=0$;
AC, $y=0$, $v=0$;	DC, $z=0$, $w=0$;	EC, $x=0$, $u=0$;
AF, $x=0$, $w=0$;	DF, $y=0$, $u=0$;	EF, $z=0$, $v=0$.

The equations of the remaining six lines are derived from (3): the three planes

$$lx + pu = 0, \qquad my + qv = 0, \qquad nz + rw = 0,$$

intersect in a straight line which lies on the surface, and which meets the lines

$$x=0,\ u=0, \text{ or } EC; \quad y=0,\ v=0, \text{ or } AC; \quad z=0,\ w=0, \text{ or } DC.$$

Hence it is the line BF, and the remaining six lines are identified as follows:—

BF,	$lx + pu = 0$,	$my + qv = 0$,	$nz + rw = 0$;
FC,	$lx + qv = 0$,	$my + rw = 0$,	$nz + pu = 0$;
CB,	$lx + rw = 0$,	$my + pu = 0$,	$nz + qv = 0$;
AD,	$lx + pu = 0$,	$my + rw = 0$,	$nz + qv = 0$;
DE,	$lx + rw = 0$,	$my + qv = 0$,	$nz + pu = 0$;
EA,	$lx + qv = 0$,	$my + pu = 0$,	$nz + rw = 0$.

Also the fifteen tritangent planes are made up of:—

Six such as $x = 0$,

nine such as $lx + pu = 0$.

These equations are obvious modifications of Schläfli's equations for the lines on an ordinary non-singular cubic surface; by means however of a simple transformation it is possible to bring the equations to all the fifteen lines and all the fifteen tritangent planes to absolutely symmetrical forms.

First let

$$2l = b + c, \qquad 2m = c + a, \qquad 2n = a + b,$$

$$2p = e + f, \qquad 2q = f + d, \qquad 2r = d + e,$$

Then

$$a + b + c + d + e + f = 0 \quad \text{.........(i)},$$

and

$$(a + b)(b + c)(c + a) + (d + e)(e + f)(f + d) = 0 \quad \text{.........(ii)}.$$

But by (1)

$$(a + b + c)^3 + (d + e + f)^3 = 0;$$

that is

$$a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a) + d^3 + e^3 + f^3 + 3(d + e)(e + f)(f + d) = 0.$$

Therefore

$$a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0 \quad \text{.........(iii)}.$$

Again let $2lx = \beta + \gamma, \quad 2my = \gamma + \alpha, \quad 2nz = \alpha + \beta.$

$2pu = \epsilon + \zeta, \quad 2qv = \zeta + \delta, \quad 2rw = \delta + \epsilon.$

That is
$$x = \frac{\beta + \gamma}{b + c}, \quad y = \frac{\gamma + \alpha}{c + a}, \text{ etc.}$$

Thus
$$\alpha + \beta + \gamma + \delta + \epsilon + \zeta \equiv 0 \qquad \text{(iv)},$$
and, as in (ii), the equation to the surface
$$(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta) + (\delta + \epsilon)(\epsilon + \zeta)(\zeta + \delta) = 0$$
is equivalent to
$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \zeta^3 = 0 \qquad \text{(v)}.$$

Also
$$\frac{\beta + \gamma}{b + c} + \frac{\gamma + \alpha}{c + a} + \frac{\alpha + \beta}{a + b} = \frac{\delta + \epsilon}{d + e} + \frac{\epsilon + \zeta}{e + f} + \frac{\zeta + \delta}{f + d};$$
$$\therefore \quad \frac{(\alpha + \beta + \gamma)(a + b + c)^2 - a^2\alpha - b^2\beta - c^2\gamma}{(b + c)(c + a)(a + b)} = \frac{(\delta + \epsilon + \zeta)(d + e + f)^2 - d^2\delta - e^2\epsilon - f^2\zeta}{(d + e)(e + f)(f + d)}.$$

The two denominators are equal and opposite, and
$$(a + b + c)^2 = (d + e + f)^2;$$
hence by (iv) this is equivalent to
$$a^2\alpha + b^2\beta + c^2\gamma + d^2\delta + e^2\epsilon + f^2\zeta = 0.$$

Lastly at O the nodal point,
$$x = y = z = u = v = w;$$
that is,
$$\frac{\beta + \gamma}{b + c} = \frac{\gamma + \alpha}{c + a} = \frac{\alpha + \beta}{a + b} = \frac{\epsilon + \zeta}{e + f} = \frac{\zeta + \delta}{f + d} = \frac{\delta + \epsilon}{d + e},$$
or
$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{\delta}{d} = \frac{\epsilon}{e} = \frac{\zeta}{f}.$$

The six planes $\alpha = 0$, $\beta = 0$, etc. appear to have hitherto escaped notice: I shall speak of them as coordinate planes or fundamental planes.

The complete system of equations is now as follows:—

Equation to the surface
$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \zeta^3 = 0 \qquad (1),$$
where
$$\left.\begin{aligned} &\alpha + \beta + \gamma + \delta + \epsilon + \zeta \equiv 0 \qquad (2),\\ &a^2\alpha + b^2\beta + c^2\gamma + d^2\delta + e^2\epsilon + f^2\zeta \equiv 0 \qquad (3),\\ &a + b + c + d + e + f = 0 \qquad (4),\\ &a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0 \qquad (5)\end{aligned}\right\} \qquad \text{B},$$
and at the nodal point
$$\alpha : \beta : \gamma : \delta : \epsilon : \zeta :: a : b : c : d : e : f.$$

Each of the fifteen tritangent planes is now represented by an equation of the form $\alpha + \beta = 0$, $\alpha + \delta = 0$, and each line of the surface by three equations such as
$$\alpha + \beta = \gamma + \delta = \epsilon + \zeta = 0.$$

The equations of the fifteen lines and fifteen tritangent planes are given below:—

AB,	$\alpha+\beta=\gamma+\delta=\epsilon+\zeta=0$,	$\alpha+\beta=0$ contains	AB, CD, EF;
AC,	$\alpha+\gamma=\beta+\epsilon=\delta+\zeta=0$,	$\alpha+\gamma=0$	AC, BE, DF;
AD,	$\alpha+\delta=\beta+\zeta=\gamma+\epsilon=0$,	$\alpha+\delta=0$	AD, BF, CE;
AE,	$\alpha+\epsilon=\beta+\delta=\gamma+\zeta=0$,	$\alpha+\epsilon=0$	AE, BD, CF;
AF,	$\alpha+\zeta=\beta+\gamma=\delta+\epsilon=0$,	$\alpha+\zeta=0$	AF, BC, DE;
BC,	$\alpha+\zeta=\beta+\delta=\gamma+\epsilon=0$,	$\beta+\gamma=0$	AF, BD, CE;
BD,	$\alpha+\epsilon=\beta+\gamma=\delta+\zeta=0$,	$\beta+\delta=0$	AE, BC, DF;
BE,	$\alpha+\gamma=\beta+\zeta=\delta+\epsilon=0$,	$\beta+\epsilon=0$	AC, BF, DE;
BF,	$\alpha+\delta=\beta+\epsilon=\gamma+\zeta=0$,	$\beta+\zeta=0$	AD, BE, CF;
CD,	$\alpha+\beta=\gamma+\zeta=\delta+\epsilon=0$,	$\gamma+\delta=0$	AB, CF, DE;
CE,	$\alpha+\delta=\beta+\gamma=\epsilon+\zeta=0$,	$\gamma+\epsilon=0$	AD, BC, EF;
CF,	$\alpha+\epsilon=\beta+\zeta=\gamma+\delta=0$,	$\gamma+\zeta=0$	AE, BF, CD;
DE,	$\alpha+\zeta=\beta+\epsilon=\gamma+\delta=0$,	$\delta+\epsilon=0$	AF, BE, CD;
DF,	$\alpha+\gamma=\beta+\delta=\epsilon+\zeta=0$,	$\delta+\zeta=0$	AC, BD, EF;
EF,	$\alpha+\beta=\gamma+\epsilon=\delta+\zeta=0$.	$\epsilon+\zeta=0$	AB, CE, DF.

These equations have been arranged in such a way as to shew a certain correspondence between the English and Greek letters; but this correspondence is soon lost sight of in the subsequent work.

This system of equations having been obtained, the properties of the fifteen lines and fifteen planes may be discussed. It should be explained that the names of the various points and lines which present themselves will be borrowed from the projections of those points and lines in the Pascal hexagram.

(1) In each tritangent plane, as $\alpha+\beta=0$, lie three lines AB, CD, EF, which form a triangle denoted by $\Delta_{\alpha\beta}$, or sometimes merely by Δ; the vertices of this triangle are called P points; thus CD, EF intersect in the P point

$$\alpha+\beta=\gamma+\epsilon=\delta+\zeta=\gamma+\zeta=\delta+\epsilon=0,$$

or

$$\alpha+\beta=0, \qquad \gamma=\delta=-\epsilon=-\zeta.$$

There are forty-five of these P points, each lying in five tritangent planes, and on each line lie six of these points, which were seen to fall into three pairs of points in involution.

The fifteen tritangent planes pass by threes through the fifteen lines of the surface, and any plane is met by six others in lines which lie on the surface.

(2) Although the six fundamental planes $\alpha=0$, $\beta=0$, etc. appear to have hitherto escaped notice, yet the fifteen planes given by equations such as $\alpha=\beta$, were known to Plücker, and are usually spoken of as Plücker planes; two Plücker planes pass through

each P point; for example through the P point $\alpha+\beta=0$, $\gamma=\delta=-\epsilon=-\zeta$ pass the two Plücker planes $\gamma=\delta$, and $\epsilon=\zeta$.

Each of the fifteen Plücker planes corresponds to one of the fifteen tritangent planes, thus the Plücker plane $\beta=\gamma$ corresponds to the tritangent plane $\beta+\gamma=0$; two such planes pass through the line of intersection of two of the fundamental planes $\beta=0$, $\gamma=0$, and are harmonically conjugate with respect to those planes.

(3) Two triple tangent planes $\alpha+\beta=0$, $\alpha+\gamma=0$, which do not pass through a common line on the surface, intersect in a line $-\alpha=\beta=\gamma$, which must meet the surface in three points. But the complete intersection of $\alpha+\beta=0$ with the surface is the three lines AB, CD, EF, and the complete intersection of $\alpha+\gamma=0$ with the surface is the three lines AC, BE, DF; hence this line $-\alpha=\beta=\gamma$ must meet AB, CD, EF, the sides of $\Delta_{\alpha\beta}$, in the same three points it meets AC, BE, DF, the sides of $\Delta_{\alpha\gamma}$: hence the line $-\alpha=\beta=\gamma$ must pass through the three P points which are the intersections of AB and DF, CD and BE, EF and AC.

Such a line is called a *Pascal* line or an h line and there are sixty such lines in all, each given by an equation similar to $-\alpha=\beta=\gamma$, and each the common line of intersection of two tritangent planes and one Plücker plane. Eight h lines lie in each tritangent plane, and four in each Plücker plane.

It has been seen that each h or *Pascal* line passes through three P points; thus the h line $-\delta=\epsilon=\zeta$ passes through the three P points

$$-\delta=\epsilon=\zeta=-\alpha, \quad \beta+\gamma=0, \text{ i.e. } AF,\ BD,$$
$$-\delta=\epsilon=\zeta=-\beta, \quad \gamma+\alpha=0, \text{ i.e. } BE,\ AC,$$
$$-\delta=\epsilon=\zeta=-\gamma, \quad \alpha+\beta=0, \text{ i.e. } CD,\ EF.$$

Conversely, through each P point pass four h lines; thus through the intersection of AB, CD, i.e. the P point $\alpha+\beta=0$, $\gamma=-\delta=\epsilon=-\zeta$ pass the four h lines

$$-\gamma=\delta=\zeta;\ -\delta=\gamma=\epsilon;\ -\epsilon=\delta=\zeta;\ -\zeta=\gamma=\epsilon.$$

(4) It is clear that besides intersecting by fours in the P points, the h lines intersect by threes in various other points: thus the three $-\alpha=\beta=\gamma$; $-\alpha=\gamma=\delta$; $-\alpha=\beta=\delta$ are seen to meet in the point

$$-\alpha=\beta=\gamma=\delta.$$

Such points are known as *Kirkman* or H points, and are sixty in number: each lies on three tritangent and three Plücker planes, and through each H point pass three h lines and on each h line lie three H points.

The notation employed being absolutely symmetrical shews that a correspondence exists between the h line $-\alpha=\beta=\gamma$ and the H point $-\alpha=\delta=\epsilon=\zeta$; it is easily verified that if three h lines meet in an H point, the corresponding H points lie on the corresponding h line; but a more convenient method of defining the correspondence is the following:—

The five tritangent planes

$$\alpha+\beta=0,\ \alpha+\gamma=0,\ \alpha+\delta=0,\ \alpha+\epsilon=0,\ \alpha+\zeta=0,$$

contain all fifteen lines of the surface and form a pentahedron which may be called the 'α' pentahedron: there are then six such pentahedra the faces of each being tritangent planes, and any two pentahedra have one face common: any two faces of a pentahedron intersect in an h line, and the three remaining faces are found to intersect in the corresponding H point; thus each of the six pentahedra has ten edges which are h lines, and ten vertices which are the corresponding H points; in other words the sixty h lines and sixty H points may be subdivided into six groups of ten points and ten lines, the lines and points of each group being the edges and vertices of a pentahedron.

(5) There are twenty other points in which three h lines intersect, which complete the system of the intersections of the tritangent planes, viz. points such as

$$\alpha=\beta=\gamma=0.$$

These are known as *Steiner* or G points, and are twenty in number; two such as $\alpha=\beta=\gamma=0$, and $\delta=\epsilon=\zeta=0$ are said to be conjugate to each other, so that the twenty G points fall into ten pairs of conjugate points. The G points are therefore the twenty vertices of the hexahedron formed by the fundamental or coordinate planes $\alpha=0$, $\beta=0$, etc. and must therefore lie by tens in these planes, and must also lie by fours in the edges of the hexahedron.

The *Steiner* or G points therefore lie by fours in fifteen lines such as $\alpha=\beta=0$, called *Steiner-Plücker* lines or i lines, each i line being the intersection of a tritangent plane with the corresponding Plücker plane.

If six lines such as AB, BC, CA, DE, EF, FD, be omitted from the fifteen, the remaining nine lines may be grouped into three plane triangles Δ in two distinct ways: for if the lines be arranged in a square thus,

	$\beta+\zeta$	$\zeta+\gamma$	$\beta+\gamma$
$\alpha+\delta$	AD	BF	CE
$\alpha+\epsilon$	CF	AE	BD
$\delta+\epsilon$	BE	CD	AF

they may be grouped into triangles either by the rows or columns of the square, and the plane of each triangle is shewn at the end of the row or column. The three planes of either group of three triangles intersect in a G point, and those of the other group intersect in the conjugate G point.

(6) It was noticed in (4) that if three h lines meet in an H point, the three corresponding H points lie in an h line; it is also true that if three h lines meet in a G point, the corresponding H points lie in a line. For if we take the G point $\alpha=\beta=\gamma=0$, the three H points are

$$-\alpha=\delta=\epsilon=\zeta;\ -\beta=\delta=\epsilon=\zeta;\ -\gamma=\delta=\epsilon=\zeta,$$

and clearly lie on the line $\delta=\epsilon=\zeta$.

There are twenty of these *Cayley-Salmon* or g lines, each corresponding to one G point; thus the line $\delta=\epsilon=\zeta$ corresponds to the point $\alpha=\beta=\gamma=0$, and moreover the g line which corresponds to a G point passes through the conjugate G point.

When four G points lie in an i line, the corresponding g lines are found to meet in a point: thus corresponding to the four G points which lie on $\alpha=\beta=0$, are the four g lines $\delta=\epsilon=\zeta$; $\gamma=\epsilon=\zeta$; $\gamma=\delta=\zeta$; $\gamma=\delta=\epsilon$; which meet in the *Salmon* point or I point

$$\gamma=\delta=\epsilon=\zeta.$$

There are then fifteen of these I points, through each of them pass six Plücker planes.

The rest of the lines and points of intersection of these systems of planes do not appear to be of sufficient interest to be worthy of separate mention here: their projections are of interest in the theory of the Pascal hexagram, and will be treated of in fuller detail in connexion with that theory; moreover, since it will be found that the development of the theory of the Pascal hexagram is so closely related to that of the lines on a nodal cubic surface, that from each proposition relating to the former theory an analogous proposition relating to the latter is at once deduced, it seems better to obtain the properties of the Pascal hexagram first, and to state where necessary the corresponding properties of the cubic surface as corollaries.

Before passing to the projections of these lines, I wish to mention certain quadrics which pass through sets of six of these lines of the surface.

(7) Any set of six lines such as AD, DE, EA, BC, CF, FB, must be generators of a quadric surface, since each of the first three intersects each of the last three; and the nine planes in which pairs of intersecting lines lie may be concisely shewn by means of the table

	BF,	FC,	CB,
AD	$\alpha+\delta$,	$\beta+\zeta$,	$\gamma+\epsilon$,
AE	$\gamma+\zeta$,	$\alpha+\epsilon$,	$\beta+\delta$,
DE	$\beta+\epsilon$,	$\gamma+\delta$,	$\alpha+\zeta$.

The equation to the quadric is found by equating to zero any minor of the determinant

$$\begin{vmatrix} \alpha+\delta, & \beta+\zeta, & \gamma+\epsilon \\ \gamma+\zeta, & \alpha+\epsilon, & \beta+\delta \\ \beta+\epsilon, & \gamma+\delta, & \alpha+\zeta \end{vmatrix}.$$

Another more symmetrical form of the equation may be deduced; for if

$$(\alpha+\delta)(\alpha+\epsilon)=(\beta+\zeta)(\gamma+\zeta),$$

that is
$$\alpha^2+\alpha\delta+\alpha\epsilon+\delta\epsilon=\zeta^2+\zeta\beta+\zeta\gamma+\beta\gamma,$$

then
$$(\alpha+\delta+\epsilon)^2+\alpha^2-\delta^2-\epsilon^2=(\zeta+\beta+\gamma)^2+\zeta^2-\beta^2-\gamma^2.$$

But $$(\alpha+\delta+\epsilon)^2 \equiv (\zeta+\beta+\gamma)^2.$$

Hence the equation to the quadric may be written

$$\alpha^2+\beta^2+\gamma^2=\delta^2+\epsilon^2+\zeta^2.$$

There are ten quadrics such as this, whose complete intersection with the cubic surface consists of six of the fifteen lines on the surface; any two such quadrics have two common generators, thus the quadric

$$\delta^2+\beta^2+\gamma^2=\alpha^2+\epsilon^2+\zeta^2,$$

which passes through the six lines *AC, CD, DA, BE, EF, FB*, has the two generators *AD, BF* in common with the former quadric. The complete intersection of the two quadrics is contained in the two planes $\alpha \pm \delta = 0$, of which the former contains the two common generators *AD, BF*; hence the remainder of the curve of intersection of the two quadrics consists of the plane conic

$$\alpha=\delta, \quad \beta^2+\gamma^2=\epsilon^2+\zeta^2.$$

THE PASCAL HEXAGRAM.

As has been stated above, Cremona has shewn that by projecting the lines and points derived from the consideration of the lines on a nodal cubic surface, we obtain the figure of the Pascal Hexagram.

Adaptation of equations. The equations we have made use of in discussing the cubic surface are readily transformed into others which are applicable to the plane figure; for since at O, the nodal point

$$\alpha : \beta : \gamma : \delta : \epsilon : \zeta :: a : b : c : d : e : f,$$

we can always find the equation to the plane which passes through O and any line whose equations are known, or to the line that joins O to any point that has been determined.

It is now only necessary to imagine that this system of lines and planes, all of which pass through O, is cut by an arbitrary plane Ψ, and the projection of the three-dimensional figure upon this plane Ψ will have been obtained. It is not desirable that any particular plane should be selected as the plane of projection, but, for the sake of the nomenclature, I shall consider that the section by a plane Ψ has always been made: thus, although $\frac{\alpha}{a}=\frac{\beta+\gamma}{b+c}$ really represents a *plane* which passes through O, the conical point, I shall be justified in speaking of the *line* $\frac{\alpha}{a}=\frac{\beta+\gamma}{b+c}$, if it is always understood that the system of lines and planes is cut by the plane Ψ in a system of points and lines. In the same way, when I speak of a conic, the equation used will really represent a quadric cone whose vertex is at O, the conical point.

The six lines A, B, C, D, E, F, which pass through the conical point O, were found to be the lines of intersection of a cubic cone and a quadric cone: projected from the conical point upon a plane Ψ, they appear as six points A, B, C, D, E, F, which lie on a conic.

The fifteen lines $AB, AC, \ldots$ each of which meets two of the six lines, are projected into the lines which join by pairs the six points A, B, C, D, E, F, and thus furnish the foundation of the figure of the Hexagram.

Equations of the fifteen lines $AB, AC\ldots$

The equation to the plane which passes through O the nodal point and the line AB is

$$\frac{\alpha+\beta}{a+b}=\frac{\gamma+\delta}{c+d}=\frac{\epsilon+\zeta}{e+f},$$

hence this is also the equation of the line AB in the projected figure. Expressions such as $\frac{\alpha+\beta}{a+b}$ and $\frac{\alpha-\beta}{a-b}$ will occur so frequently in subsequent work that it is convenient at once to replace them by simpler symbols.

Let $\frac{\alpha+\beta}{a+b}$ be represented by the symbol $(\alpha\beta)$,

and $\frac{\alpha-\beta}{a-b}$ be represented by the symbol $\chi(\alpha\beta)$.

Thus in three dimensions, each tritangent plane is given by an equation such as $(\alpha\beta)=0$, and each Plücker plane by an equation such as $\chi(\alpha\beta)=0$, and at O the nodal point

$$\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}\ldots=(\alpha\beta)=(\alpha\gamma)\ldots=\chi(\alpha\beta)=\chi(\alpha\gamma)\ldots.$$

The equations of the fifteen lines AB, AC, can be at once derived from those on p. 272; they are

$$\begin{array}{ll}
AB & (\alpha\beta)=(\gamma\delta)=(\epsilon\zeta);\\
AC & (\alpha\gamma)=(\beta\epsilon)=(\delta\zeta);\\
AD & (\alpha\delta)=(\beta\zeta)=(\gamma\epsilon);\\
AE & (\alpha\epsilon)=(\beta\delta)=(\gamma\zeta);\\
AF & (\alpha\zeta)=(\beta\gamma)=(\delta\epsilon);\\
BC & (\alpha\zeta)=(\beta\delta)=(\gamma\epsilon);\\
BD & (\alpha\epsilon)=(\beta\gamma)=(\delta\zeta);\\
BE & (\alpha\gamma)=(\beta\zeta)=(\delta\epsilon);\\
BF & (\alpha\delta)=(\beta\epsilon)=(\gamma\zeta);\\
CD & (\alpha\beta)=(\gamma\zeta)=(\delta\epsilon);\\
CE & (\alpha\delta)=(\beta\gamma)=(\epsilon\zeta);\\
CF & (\alpha\epsilon)=(\beta\zeta)=(\gamma\delta);\\
DE & (\alpha\zeta)=(\beta\epsilon)=(\gamma\delta);\\
DF & (\alpha\gamma)=(\beta\delta)=(\epsilon\zeta);\\
EF & (\alpha\beta)=(\gamma\epsilon)=(\delta\zeta).
\end{array}$$

The equation to the conic on which A, B, C, D, E, F lie is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\epsilon^2 + f\zeta^2 = 0 \quad\text{..............................(1)},$$

and the complete system of equations is

$$\alpha + \beta + \gamma + \delta + \epsilon + \zeta \equiv 0 \quad\text{..............................(2)},$$

$$a^2\alpha + b^2\beta + c^2\gamma + d^2\delta + e^2\epsilon + f^2\zeta \equiv 0 \quad\text{..............................(3)},$$

$$a + b + c + d + e + f = 0 \quad\text{..............................(4)},$$

$$a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0 \quad\text{..............................(5)}.$$

Further, $(\alpha\beta)$ is defined as $\dfrac{\alpha+\beta}{a+b}$ ....(6),

$\chi(\alpha\beta)$ .... $\dfrac{\alpha-\beta}{a-b}$ ....(7).

It follows that

if $\dfrac{\alpha}{a} = \dfrac{\beta}{b}$ each is necessarily also $= (\alpha\beta) = \chi(\alpha\beta)$,

if $(\alpha\beta) = (\gamma\delta)$ .... $= (\epsilon\zeta)$,

if $(\alpha\beta) = (\alpha\gamma)$ .... $= \chi(\beta\gamma)$,

if $\chi(\alpha\beta) = \chi(\alpha\gamma)$.............................. $= \chi(\beta\gamma)$,

and at the nodal point

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} \ldots = (\alpha\beta) = (\alpha\gamma) \ldots = \ldots \chi(\alpha\beta) = \chi(\alpha\gamma) \ldots.$$

Before I pass to the Pascal hexagram, it is convenient to discuss in two lemmas some properties of the figures formed by projecting on any plane the lines of intersection first of five planes and secondly of six planes in three-dimensional space.

I. Take five planes in three-dimensional space,

$$u = 0, \quad v = 0, \quad w = 0, \quad x = 0, \quad y = 0,$$

forming a pentahedron, with ten edges and ten angles; take also a point O not situated on any of these planes as origin of projection.

We may introduce factors into the functions u, v, w, x, y, so that at O,

$$u = v = w = x = y.$$

Further, the five quantities u, v, w, x, y, must be connected by an identical linear relation

$$pu + qv + rw + sx + ty \equiv 0 \quad\text{..............................(1)},$$

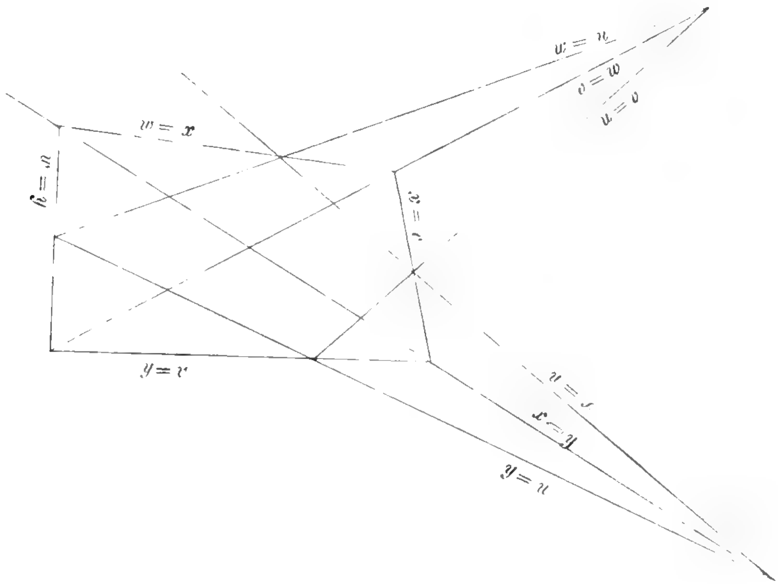
where

$$p + q + r + s + t = 0.$$

Then $u = v$ represents a plane passing through O and the line of intersection of

$$u = 0, \quad v = 0 \text{ etc.}$$

If now we consider the sections of all the planes which pass through O and through one of the ten edges of the pentahedron, by an arbitrary plane Ψ, we obtain the required projection. The figure is shewn below.



It consists of ten lines which meet by threes in ten points, and three of these points lie on each of the ten lines. Selecting any point $u = v = w$, the three lines $u = v$, $v = w$, $w = u$ pass through it; six of the remaining lines form two perspective triangles, viz. $u = x$, $v = x$, $w = x$; and $u = y$, $v = y$, $w = y$, and the tenth line $x = y$ is the line of perspective on which corresponding sides intersect.

There is a certain conic such that each of the ten points is the pole of the corresponding line, viz.

$$pu^2 + qv^2 + rw^2 + sx^2 + ty^2 = 0 \qquad (2).$$

For the polar of the point $(u_0 v_0 w_0 x_0 y_0)$ is

$$puu_0 + qvv_0 + rww_0 + sxx_0 + tyy_0 = 0.$$

If now $u_0 = v_0 = w_0$, the polar is

$$u_0 (pu + qv + rw) + sxx_0 + tyy_0 = 0,$$

or by equation (1)

$$u_0 (-sx - ty) + sxx_0 + tyy_0 = 0,$$

or

$$sx (x_0 - u_0) + ty (y_0 - u_0) = 0.$$

But

$$pu_0 + qv_0 + rw_0 + sx_0 + ty_0 = 0,$$

$$\therefore \quad (p + q + r)(u_0) + sx_0 + ty_0 = 0,$$

or

$$(-s - t) u_0 + sx_0 + ty_0 = 0;$$

$$\therefore \quad s (x_0 - u_0) + t (y_0 - u_0) = 0.$$

Hence the polar of the point $u=v=w$ is the line $x=y$.

The figure may be called a Projected Pentahedron.

II. Taking next six planes

$$u=0,\quad v=0,\quad w=0,\quad x=0,\quad y=0,\quad z=0,$$

we project their intersections from the point O at which

$$u=v=w=x=y=z.$$

The six quantities u, v, w, x, y, z, are connected by two linear relations

$$pu+qv+rw+sx+ty+kz\equiv 0,$$
$$p'u+q'v+r'w+s'x+t'y+k'z=0,$$

where

$$p+q+r+s+t+k=0,$$
$$p'+q'+r'+s'+t'+k'=0.$$

The projection consists of fifteen lines $x=y$, ... which meet by threes in twenty points $x=y=z$, and four of these points lie on each line.

The figure, which may be called the figure of a projected Hexahedron, is shewn below.

If we select any point, e.g. $x=y=z$, through which pass the three lines $x=y$, $y=z$, $z=x$, nine of the other lines group themselves into three perspective triangles, viz. $x=u$, $y=u$, $z=u$; $x=v$, $y=v$, $z=v$; $x=w$, $y=w$, $z=w$; and the three lines of perspective in which corresponding sides of any two triangles intersect are the remaining three lines $u=v$, $v=w$, $w=u$, which meet in the point $u=v=w$.

If we start with the point $u=v=w$, the nine sides of the three perspective triangles are the same nine lines as before, but differently grouped.

(α) Two points such as $x=y=z$, $u=v=w$ are conjugate with respect to any of the conics

$$(p+\lambda p')\,u^2+(q+\lambda q')\,v^2+(r+\lambda r')\,w^2+(s+\lambda s')\,x^2+(t+\lambda t')\,y^2+(k+\lambda k')\,z^2=0.$$

Denote the coefficients by P, Q, R, S, T, K, then two points $(u_0, v_0, w_0, x_0, y_0, z_0)$ and $(u_1, v_1, w_1, x_1, y_1, z_1)$ are conjugate if

$$Pu_0u_1+Qv_0v_1+Rw_0w_1+Sx_0x_1+Ty_0y_1+Kz_0z_1=0.$$

If now $u_0=v_0=w_0$ and $x_1=y_1=z_1$ the condition of conjugacy is

$$u_0\,(Pu_1+Qv_1+Rw_1)+x_1\,(Sx_0+Ty_0+Kz_0)=0.$$

But we know that at any point

$$Pu+Qv+Rw+Sx+Ty+Kz=0,$$

$$\therefore\ (P+Q+R)\,u_0+(Sx_0+Ty_0+Kz_0)=0,$$

and

$$(Pu_1+Qv_1+Rw_1)+(S+T+K)\,x_1=0.$$

Also

$$(P+Q+R)+(S+T+K)=0.$$

Hence the condition is satisfied and the points are conjugate with respect to any conic of the system.

(β) The system of conics above consists of all conics which pass through four fixed points which for the moment may be called P, Q, R, S. If the diagonals of the quadrangle $PQRS$ meet in L, M, N, it follows that the lines from any one of these points such as L to any two conjugate points, as $x=y=z$ and $u=v=w$, form an involution, the double rays being the lines which pass through the four points P, Q, R, S.

If the conic

$$Pu^2+Qv^2+Rw^2+Sx^2+Ty^2+Kz^2=0$$

break up into two straight lines, which intersect in the point $(u_0v_0w_0x_0y_0z_0)$,

then

$$Puu_0+Qvv_0+Rww_0+Sxx_0+Tyy_0+Kzz_0\equiv 0;$$

$$\therefore\ Pu_0=\alpha p+\beta P,\qquad Sx_0=\alpha s+\beta S,$$

$$Qv_0=\alpha q+\beta Q,\qquad Ty_0=\alpha t+\beta T,$$

$$Rw_0=\alpha r+\beta R,\qquad Kz_0=\alpha k+\beta K;$$

hence substituting in

$$p'u_0+q'v_0+\ldots\ldots=0,$$

we have

$$\frac{pp'}{P}+\frac{qq'}{Q}+\frac{rr'}{R}+\frac{ss'}{S}+\frac{tt'}{T}+\frac{kk'}{K}=0,$$

that is $$\frac{pp'}{p+\lambda p'}+\frac{qq'}{q+\lambda q'}+\frac{rr'}{r+\lambda r'}+\frac{ss'}{s+\lambda s'}+\frac{tt'}{t+\lambda t'}+\frac{kk'}{k+\lambda k'}=0,$$

an equation which gives three finite values of λ.

Giving λ these three values in succession, we may find the coordinates of the three points L, M, N.

The six points where any line of the figure $u=v$ is met by the six lines $w=x$, $y=z$; $w=y$, $x=z$; $w=z$, $x=y$; are conjugate in pairs with respect to one conic of the system, viz. that for which

$$P+Q \text{ or } p+\lambda p'+q+\lambda q'=0.$$

For the condition of conjugacy being as before

$$Pu_0u_1+Qv_0v_1+Rw_0w_1+Sx_0x_1+Ty_0y_1+Kz_0z_1=0,$$

if we have

$$u_0=v_0,\ u_1=v_1,\ w_0=x_0,\ y_1=z_1,$$

the condition becomes

$$(P+Q)\,u_0u_1+w_0\,(Rw_1+Sx_1)+y_1\,(Ty_0+Kz_0)=0.$$

Also $$(P+Q)\,u_0+(R+S)\,w_0+(Ty_0+Kz_0)=0,$$
$$(P+Q)\,u_1+(Rw_1+Sx_1)+(T+K)\,y_1=0.$$

If then $P+Q=0$, the condition is satisfied, since

$$(R+S)+(T+K)=0.$$

I now proceed to deduce from the properties proved for the cubic surface the analogous properties of the plane figure.

III. The fifteen lines AB, AC, ... which join by twos the six points A, B, C, D, E, F, group themselves into fifteen triangles Δ, on whose sides lie all the six points A, B, C, D, E, F: such a triangle is AB, CD, EF, to which as in section (1) I give the name $\Delta_{\alpha\beta}$: any line AB belongs to the three triangles AB, CD, EF; AB, CE, DF: AB, CF, DE; and further since the other sides join the four points C, D, E, F, it follows that the six vertices of triangles Δ which lie on AB are in involution.

The vertices of these triangles are called P points and are 45 in number.

IV. From (3) we infer that

AB meets DF, CD meets BE, EF meets AC } in three points which lie on the h or *Pascal* line $(\alpha\beta)=(\alpha\gamma)=\chi(\beta\gamma)$.

And sixty such lines exist.

Consider now the six lines just mentioned: if we arrange them in the order AB, BE, EF, FD, DC, CA, it is clear that they are sides of a hexagon $ABEFDC$ inscribed in the conic, and we have shewn, 'The opposite sides of any hexagon inscribed in a conic intersect in three collinear points.'

There are sixty different hexagons which we can form by joining the six points *A, B, C, D, E, F,* in different ways, and from each hexagon is derived one of the sixty *h* lines.

On each *h* line lie three *P* points, and through each *P* point pass four *h* lines; thus through the intersection of *AB, CD* pass the four *h* lines derived from the hexagons *ABECDF, ABFCDE, ABEDCF, ABFDCE.*

V. The sixty *h* lines intersect by threes in sixty *H* or Kirkman points

$$(\alpha\beta) = (\alpha\gamma) = (\alpha\delta) = \chi(\beta\gamma) = \chi(\gamma\delta) = \chi(\beta\delta);$$

and on each *h* line lie three *H* points.

The three concurrent *h* lines are derived from the hexagons *ABEFDC, ACEBFD, ADCEFB,* respectively: it was pointed out that to each *h* line corresponds one *H* point: now the sides of these three hexagons are composed of nine only of the fifteen lines *AB, AC...*; and the six lines omitted are the sides of the hexagon *AEDBCF* from which is derived the corresponding *h* line $(\alpha\epsilon) = (\alpha\zeta)$.

VI. The edges and angles of each pentahedron are projected into ten *h* lines and ten *H* points, forming a figure of a projected pentahedron discussed in I.: it follows that a conic exists such that each of the *h* lines which form the figure is the polar of the corresponding *H* point.

The sixty *h* lines and sixty *H* points fall into six groups of ten lines and ten points; and with each group is associated a conic such that each *h* line of the group is the polar of the corresponding *H* point (which always belongs to the same group) with respect to it.

There is no difficulty in finding the equation of this conic,

$$(\alpha+\beta)+(\alpha+\gamma)+(\alpha+\delta)+(\alpha+\epsilon)+(\alpha+\zeta) \equiv 4\alpha$$

and $$b^2(\alpha+\beta)+c^2(\alpha+\gamma)+\ldots \equiv (b^2+c^2+d^2+e^2+f^2-a^2)\,\alpha,$$

$$\therefore\ (b^2+c^2+d^2+e^2+f^2-a^2)\{(\alpha+\beta)+(\alpha+\gamma)+(\alpha+\delta)+(\alpha+\epsilon)+(\alpha+\zeta)\}$$

$$\equiv 4[b^2(\alpha+\beta)+c^2(\alpha+\gamma)+d^2(\alpha+\delta)\ldots\ldots].$$

That is $$(a^2+3b^2-c^2-d^2-e^2-f^2)(a+b)(\alpha\beta)+\ldots\ldots=0.$$

Hence the equation to the conic is

$$(a^2+3b^2-c^2-d^2-e^2-f^2)(a+b)(\alpha\beta)^2+\ldots\ldots=0.$$

VII. The sixty *h* lines also intersect by threes in twenty Steiner or *G* points,

$$\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}=(\alpha\beta)=(\alpha\gamma)=(\beta\gamma)=\chi(\alpha\beta)=\chi(\alpha\gamma)=\chi(\beta\gamma).$$

The three concurrent h lines are derived from the following hexagons:

$$(\alpha\beta) = (\alpha\gamma) = \chi(\beta\gamma) \text{ from } ABEFDC,$$
$$(\alpha\beta) = (\beta\gamma) = \chi(\alpha\gamma) \text{ from } AFECDB,$$
$$(\alpha\gamma) = (\beta\gamma) = \chi(\alpha\beta) \text{ from } ACEBDF,$$

in which the first, third and fifth letters are the same, and the second, fourth and sixth are cyclically interchanged.

The twenty G points fall into ten pairs: with the point above is associated the point

$$\frac{\delta}{d} = \frac{\epsilon}{e} = \frac{\zeta}{f} = (\delta\epsilon) = (\epsilon\zeta) = (\delta\zeta) = \chi(\delta\epsilon) = \chi(\epsilon\zeta) = \chi(\delta\zeta),$$

in which intersect the three h lines derived from the hexagons $ABECDF$, $ACEFDB$, $AFEBDC$, where the first, third, and fifth letters are again the same as before, while the second, fourth, and sixth are derived from those of the former hexagons by non-cyclical interchanges.

VIII. We may apply the results of II. to the figure formed by the projection of the intersections of the six tritangent planes

$$(\alpha\beta) = 0,\ (\beta\gamma) = 0,\ (\gamma\alpha) = 0,\ (\delta\epsilon) = 0,\ (\epsilon\zeta) = 0,\ (\zeta\delta) = 0.$$

The figure is simpler than that in II. inasmuch as one of the linear relations connecting

$$(\alpha\beta),\ (\beta\gamma),\ (\gamma\alpha),\ (\delta\epsilon),\ (\epsilon\zeta),\ (\zeta\delta)$$

is
$$(\alpha\beta) + (\beta\gamma) + (\gamma\alpha) \equiv (\delta\epsilon) + (\epsilon\zeta) + (\zeta\delta), \text{ see page (271)},$$

so that the three lines such as

$$(\alpha\beta) = (\delta\epsilon);\ (\beta\gamma) = (\epsilon\zeta);\ (\gamma\alpha) = (\zeta\delta)$$

are concurrent.

The second linear relation is

$$(b + c)(\beta\gamma) + (c + a)(\gamma\alpha) + (a + b)(\alpha\beta) + (d + e)(\delta\epsilon) + (e + f)(\epsilon\zeta) + (f + d)(\zeta\delta) \equiv 0.$$

The system of conics in II. comprises all conics which pass through the four points common to

$$(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (\delta\epsilon)^2 + (\epsilon\zeta)^2 + (\zeta\delta)^2$$

and $(a + b)(\alpha\beta)^2 + (b + c)(\beta\gamma)^2 + (c + a)(\alpha\gamma)^2 + (d + e)(\delta\epsilon)^2 + (e + f)(\epsilon\zeta)^2 + (f + d)(\zeta\delta)^2 = 0.$

The former of these two is the fundamental conic on which the six points A, B, C, D, E, F, lie, and can therefore be reduced to

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\epsilon^2 + f\zeta^2 = 0.$$

Two G points such as

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c}, \text{ and } \frac{\delta}{d} = \frac{\epsilon}{e} = \frac{\zeta}{f},$$

which have been called conjugate G points, are therefore conjugate with respect to the fundamental conic.

By simplification of the second equation, it may be shewn that these two G points are conjugate with respect to all conics which pass through the four points common to

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\epsilon^2 + f\zeta^2 = 0$$

and $$(bc + ca + ab)(\alpha + \beta + \gamma)^2 - bc\alpha^2 - ca\beta^2 - ab\gamma^2 = (de + ef + fd)(\delta + \epsilon + \zeta)^2 - ef\delta^2 - fd\epsilon^2 - de\zeta^2;$$

and, further, nine pairs of P points such as

$$(\alpha\beta) = (\alpha\gamma) = (\epsilon\zeta); \quad (\delta\epsilon) = (\delta\zeta) = (\alpha\beta);$$

the intersections of AB, DF, and of AF, DB are conjugate with respect to all conics of the system.

Again by II. (β), we see that any side such as $(\alpha\beta) = (\epsilon\zeta)$ or AB, is met by the h lines $(\beta\gamma) = (\gamma\alpha)$ and $(\delta\epsilon) = (\delta\zeta)$, which are derived from the hexagons $ACEBDF$, $ACDBEF$, in two points which are conjugate with respect to the fundamental conic, and therefore form with A and B a harmonic range.

On the side AB there must lie six such pairs of conjugate points, each pair forming a harmonic range with the points A and B.

IX. The fifteen lines in the figure of this projected hexahedron are composed of six h lines and nine sides of the triangles Δ, which join two of the six points A, B, C, D, E, F; consider the grouping of the eighteen points where the nine sides of the triangles are met by the h lines.

On each h line, as $(\alpha\beta) = (\alpha\gamma)$, lie three of the points, viz. the points where this line is met by AF, CE, BD, the sides of $\Delta_{\beta\gamma}$. The points fall into two groups of nine, according as the h line they lie on passes through one or other of the G points. Arrange the points thus:

$$\left.\begin{array}{l|l|l} (\alpha\beta) = (\alpha\gamma),\ (\beta\gamma) = (\epsilon\zeta) & (\beta\gamma) = (\beta\alpha),\ (\gamma\alpha) = (\epsilon\zeta) & (\gamma\alpha) = (\gamma\beta),\ (\alpha\beta) = (\epsilon\zeta) \\ (\alpha\beta) = (\alpha\gamma),\ (\beta\gamma) = (\zeta\delta) & (\beta\gamma) = (\beta\alpha),\ (\gamma\alpha) = (\zeta\delta) & (\gamma\alpha) = (\gamma\beta),\ (\alpha\beta) = (\zeta\delta) \\ (\alpha\beta) = (\alpha\gamma),\ (\beta\gamma) = (\delta\epsilon) & (\beta\gamma) = (\beta\alpha),\ (\gamma\alpha) = (\delta\epsilon) & (\gamma\alpha) = (\gamma\beta),\ (\alpha\beta) = (\delta\epsilon) \end{array}\right| \ldots\ldots (A);$$

$$\left.\begin{array}{l|l|l} (\delta\epsilon) = (\delta\zeta),\ (\epsilon\zeta) = (\beta\gamma) & (\epsilon\zeta) = (\epsilon\delta),\ (\zeta\delta) = (\beta\gamma) & (\zeta\delta) = (\zeta\epsilon),\ (\delta\epsilon) = (\beta\gamma) \\ (\delta\epsilon) = (\delta\zeta),\ (\epsilon\zeta) = (\gamma\alpha) & (\epsilon\zeta) = (\epsilon\delta),\ (\zeta\delta) = (\gamma\alpha) & (\zeta\delta) = (\zeta\epsilon),\ (\delta\epsilon) = (\gamma\alpha) \\ (\delta\epsilon) = (\delta\zeta),\ (\epsilon\zeta) = (\alpha\beta) & (\epsilon\zeta) = (\epsilon\delta),\ (\zeta\delta) = (\alpha\beta) & (\zeta\delta) = (\zeta\epsilon),\ (\delta\epsilon) = (\alpha\beta) \end{array}\right| \ldots\ldots (B).$$

Taking either group, the nine points form three triangles, if we take them in rows, and lie by threes on the h lines, if we take them in columns. The conjugates to three points of either group which form a triangle are three points of the other group which lie on an h line.

The sides of the triangles of the first group are

$$(\alpha\beta) + (\alpha\gamma) = (\epsilon\zeta) + (\beta\gamma); \quad (\beta\alpha) + (\beta\gamma) = (\epsilon\zeta) + (\alpha\gamma); \quad (\gamma\alpha) + (\gamma\beta) = (\epsilon\zeta) + (\alpha\beta);$$
$$(\alpha\beta) + (\alpha\gamma) = (\zeta\delta) + (\beta\gamma); \quad (\beta\alpha) + (\beta\gamma) = (\zeta\delta) + (\alpha\gamma); \quad (\gamma\alpha) + (\gamma\beta) = (\zeta\delta) + (\alpha\beta);$$
$$(\alpha\beta) + (\alpha\gamma) = (\delta\epsilon) + (\beta\gamma); \quad (\beta\alpha) + (\beta\gamma) = (\delta\epsilon) + (\alpha\gamma); \quad (\gamma\alpha) + (\gamma\beta) = (\delta\epsilon) + (\alpha\beta).$$

Thus the corresponding sides of any two triangles intersect on an h line which passes through the second G point.

Again, since

$$(\alpha\beta)+(\beta\gamma)+(\gamma\alpha) \equiv (\delta\epsilon)+(\epsilon\zeta)+(\zeta\delta),$$

the equation to each of these lines may be written in a new form: for example

$$(\alpha\beta)+(\alpha\gamma) = (\epsilon\zeta)+(\beta\gamma)$$

is equivalent to

$$2(\beta\gamma) = (\delta\epsilon)+(\delta\zeta).$$

Hence this line passes through the P point

$$(\beta\gamma) = (\delta\epsilon) = (\delta\zeta) = (\alpha\zeta) = (\alpha\epsilon), \text{ i.e. the intersection of } AF, BD,$$

and further it forms with the h line $(\delta\epsilon) = (\delta\zeta)$ and the two sides AF, BD a harmonic pencil.

X. Corresponding to the three h lines which meet in the G point

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c},$$

are the three H points

$$(\alpha\delta) = (\alpha\epsilon) = (\alpha\zeta) = \chi(\delta\epsilon) = \chi(\epsilon\zeta) = \chi(\zeta\delta);$$

$$(\beta\delta) = (\beta\epsilon) = (\beta\zeta) = \chi(\delta\epsilon) = \chi(\epsilon\zeta) = \chi(\zeta\delta);$$

$$(\gamma\delta) = (\gamma\epsilon) = (\gamma\zeta) = \chi(\delta\epsilon) = \chi(\epsilon\zeta) = \chi(\zeta\delta);$$

which are seen to lie on the Cayley-Salmon or g line

$$\chi(\delta\epsilon) = \chi(\epsilon\zeta) = \chi(\zeta\delta).$$

This g line corresponds to the G point above, and passes through the conjugate G point. There are twenty such lines in the hexagram, on each of which lie three H points and one G point.

XI. Four G points such as

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = (\alpha\beta) = (\alpha\gamma) = (\beta\gamma) = \chi(\alpha\beta) = \chi(\alpha\gamma) = \chi(\beta\gamma);$$

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\delta}{d} = (\alpha\beta) = (\alpha\delta) = (\beta\delta) = \chi(\alpha\beta) = \chi(\alpha\delta) = \chi(\beta\delta);$$

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\epsilon}{e} = (\alpha\beta) = (\alpha\epsilon) = (\beta\epsilon) = \chi(\alpha\beta) = \chi(\alpha\epsilon) = \chi(\beta\epsilon);$$

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\zeta}{f} = (\alpha\beta) = (\alpha\zeta) = (\beta\zeta) = \chi(\alpha\beta) = \chi(\alpha\zeta) = \chi(\beta\zeta);$$

lie in one of fifteen Steiner-Plücker or i lines such as

$$\frac{\alpha}{a} = \frac{\beta}{b} = (\alpha\beta) = \chi(\alpha\beta),$$

which pass by threes through the twenty G points.

The twenty G points and fifteen i lines form the figure of a projected hexahedron, discussed in II., viz. the projection of the hexahedron formed by the six fundamental planes

$$\alpha = 0 \ldots \text{etc.}$$

Any two conjugate G points are therefore conjugate with respect to all conics which pass through the four points common to the fundamental conic

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\epsilon^2 + f\zeta^2 = 0$$

and

$$\frac{\alpha^2}{a} + \frac{\beta^2}{b} + \frac{\gamma^2}{c} + \frac{\delta^2}{d} + \frac{\epsilon^2}{e} + \frac{\zeta^2}{f} = 0.$$

The conic

$$\left(a + \frac{\lambda}{a}\right)\alpha^2 + \left(b + \frac{\lambda}{b}\right)\beta^2 + \left(c + \frac{\lambda}{c}\right)\gamma^2 + \left(d + \frac{\lambda}{d}\right)\delta^2 + \left(e + \frac{\lambda}{e}\right)\epsilon^2 + \left(f + \frac{\lambda}{f}\right)\zeta^2 = 0$$

will break into two straight lines if (as may be deduced from II.)

$$\frac{1}{a + \frac{\lambda}{a}} + \frac{1}{b + \frac{\lambda}{b}} + \frac{1}{c + \frac{\lambda}{c}} + \frac{1}{d + \frac{\lambda}{d}} + \frac{1}{e + \frac{\lambda}{e}} + \frac{1}{f + \frac{\lambda}{f}} = 0,$$

or

$$\frac{a}{a^2 + \lambda} + \frac{b}{b^2 + \lambda} + \frac{c}{c^2 + \lambda} + \frac{d}{d^2 + \lambda} + \frac{e}{e^2 + \lambda} + \frac{f}{f^2 + \lambda} = 0;$$

whence

$$5\lambda^3 - 3s_2\lambda^2 + s_4\lambda + s_6 = 0,$$

where

$$s_2 = ab + ac + ad + \ldots;$$

$$s_4 = abcd + abce + \ldots,$$

$$s_6 = abcdef.$$

But if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = 0$, any value of λ satisfies the equation. In this case however, the constants a, b, c, d, e, f are equal and opposite in pairs, and the fundamental conic degenerates into two straight lines.

XII. Corresponding to four G points which lie on the i line

$$\frac{\alpha}{a} = \frac{\beta}{b} = (\alpha\beta) = \chi(\alpha\beta)$$

are four g lines which meet in one of fifteen Salmon or I points

$$\chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\gamma\zeta) = \chi(\delta\epsilon) = \chi(\delta\zeta) = \chi(\epsilon\zeta),$$

and this I point corresponds to the i line above.

XIII. The projection of the figure formed by the five planes

$$\chi(\alpha\beta) = 0,\ \chi(\alpha\gamma) = 0,\ \chi(\alpha\delta) = 0,\ \chi(\alpha\epsilon) = 0,\ \chi(\alpha\zeta) = 0$$

gives the figure of a projected pentahedron discussed in I.

The ten lines are here g lines and the ten points are I points: but each g line is common to three of the six figures and each I point is common to four figures, and in different figures different g lines correspond to the same I points, and different I points to the same g line.

With each figure of ten g lines and ten I points is associated a conic such that each g line is the polar of the I point which corresponds to it in that figure; the equation to the conic is found to be

$$\Sigma (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 6b^2)(a - b)\overline{\chi(\alpha\beta)}|^2 = 0.$$

XIV. In the three-dimensional figure consider the h lines which pass through the vertices of the triangle $\Delta_{\alpha\beta}$ formed by AB, CD, EF.

Through the intersection of CD, EF pass the four h lines

$$-\gamma = \epsilon = \zeta; \quad -\delta = \epsilon = \zeta; \quad -\epsilon = \gamma = \delta; \quad -\zeta = \gamma = \delta.$$

Through the intersection of EF, AB, pass

$$-\gamma = \delta = \epsilon; \quad -\delta = \gamma = \zeta; \quad -\epsilon = \gamma = \zeta; \quad -\zeta = \delta = \epsilon.$$

Through the intersection of AB, CD, pass

$$-\gamma = \delta = \zeta; \quad -\delta = \gamma = \epsilon; \quad -\epsilon = \delta = \zeta; \quad -\zeta = \gamma = \epsilon.$$

These twelve h lines intersect by threes in four H points

$$-\gamma = \delta = \epsilon = \zeta; \quad -\delta = \epsilon = \zeta = \gamma; \quad -\epsilon = \zeta = \gamma = \delta; \quad -\zeta = \gamma = \delta = \epsilon;$$

and in four G points

$$\delta = \epsilon = \zeta = 0; \quad \epsilon = \zeta = \gamma = 0; \quad \zeta = \gamma = \delta = 0; \quad \gamma = \delta = \epsilon = 0,$$

such that the four conjugate G points are collinear.

The H point $-\gamma = \delta = \epsilon = \zeta$ is joined to the G point $\delta = \epsilon = \zeta = 0$ by the g line $\delta = \epsilon = \zeta$, and is joined to each of the other three G points by an h line which passes through a vertex of $\Delta_{\alpha\beta}$: also the four g lines intersect in the I point $\gamma = \delta = \epsilon = \zeta$.

Hence the tetrahedron formed by the H points and that formed by the G points are perspective with respect to four distinct centres, viz. the vertices of $\Delta_{\alpha\beta}$ and the I point $\gamma = \delta = \epsilon = \zeta$.

The corresponding property of the hexagram is, The quadrangles formed by the H points

$$(\gamma\delta) = (\gamma\epsilon) = (\gamma\zeta); \quad (\delta\gamma) = (\delta\epsilon) = (\delta\zeta); \quad (\epsilon\gamma) = (\epsilon\delta) = (\epsilon\zeta); \quad (\zeta\gamma) = (\zeta\delta) = (\zeta\epsilon);$$

and the G points

$$(\delta\epsilon) = (\epsilon\zeta) = (\zeta\delta); \quad (\gamma\epsilon) = (\epsilon\zeta) = (\gamma\zeta); \quad (\gamma\delta) = (\delta\zeta) = (\gamma\zeta); \quad (\gamma\delta) = (\gamma\epsilon) = (\delta\epsilon),$$

respectively, are perspective with regard to four distinct centres of perspective, viz. the vertices of $\Delta_{\alpha\beta}$ and the I point

$$\chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\gamma\zeta) = \chi(\delta\epsilon) = \chi(\delta\zeta) = \chi(\epsilon\zeta).$$

I proceed to consider the complete figure formed by the projection of the lines and points of intersection of the tritangent planes, the Plücker planes, and the six coordinate planes: from this figure are deduced nearly all the properties of the hexagram given by Veronese, and one or two new properties. It will be convenient to treat first of the intersections of the tritangent and Plücker planes, and to introduce the six new coordinate planes later.

XV. Consider the projections of the eight h lines which lie in a tritangent plane $\alpha+\beta=0$; they form two quadrilaterals

$$(\alpha\beta)=(\alpha\gamma);\quad (\alpha\beta)=(\alpha\delta);\quad (\alpha\beta)=(\alpha\epsilon);\quad (\alpha\beta)=(\alpha\zeta),$$
$$(\beta\alpha)=(\beta\gamma);\quad (\beta\alpha)=(\beta\delta);\quad (\beta\alpha)=(\beta\epsilon);\quad (\beta\alpha)=(\beta\zeta).$$

The six vertices of each quadrilateral are H points, and corresponding sides intersect in G points which lie on the i line $\frac{\alpha}{a}=\frac{\beta}{b}=(\alpha\beta)=\chi(\alpha\beta)$; while sides which do not correspond meet in the twelve P points which lie on the sides of $\Delta_{\alpha\beta}$ but are not vertices of that triangle.

The lines which join corresponding vertices of the two quadrilaterals are called v lines; for example the two vertices

$$(\alpha\beta)=(\alpha\gamma)=(\alpha\delta)=\chi(\beta\gamma)=\chi(\beta\delta)=\chi(\gamma\delta),$$
$$(\beta\alpha)=(\beta\gamma)=(\beta\delta)=\chi(\alpha\gamma)=\chi(\alpha\delta)=\chi(\gamma\delta),$$

are joined by the v line

$$(\alpha\beta)=\chi(\gamma\delta).$$

The hexagram contains ninety of these v lines, each the projection of the intersection of a tritangent plane $\alpha+\beta=0$ with a Plücker plane $\gamma-\delta=0$: on each v line lie two H points; through each P point pass two v lines, and through each H point pass three such lines.

The six v lines derived from the two quadrilaterals given above pass by twos through the vertices of the triangle $\Delta_{\alpha\beta}$; their equations are

$$(\alpha\beta)=\chi(\gamma\delta);\quad (\alpha\beta)=\chi(\gamma\epsilon);\quad (\alpha\beta)=\chi(\gamma\zeta);$$
$$(\alpha\beta)=\chi(\epsilon\zeta);\quad (\alpha\beta)=\chi(\zeta\delta);\quad (\alpha\beta)=\chi(\delta\epsilon);$$

and therefore they intersect by threes in four points which for the present I call H_2 points, such as

$$(\alpha\beta)=\chi(\gamma\delta)=\chi(\gamma\epsilon)=\chi(\delta\epsilon),$$

each of which lies on one of the g lines which pass through the I point corresponding to $\Delta_{\alpha\beta}$. It follows that the diagonals of the quadrangle of H_2 points are the sides of $\Delta_{\alpha\beta}$, and hence

The two v lines which pass through any P point form a harmonic pencil with the sides of the triangle Δ which intersect in that P point.

There are sixty H_2 points in the hexagram, lying by threes on the twenty g lines: there is clearly a correspondence between the sixty H_2 points and the sixty h lines and the sixty H points; thus the H_2 point

$$(\alpha\beta) = \chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\delta\epsilon)$$

corresponds to the H point

$$(\zeta\gamma) = (\zeta\delta) = (\zeta\epsilon) = \chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\delta\epsilon),$$

and to the h line

$$(\zeta\alpha) = (\zeta\beta) = \chi(\alpha\beta).$$

Each H_2 point is joined to the corresponding H point by the g line which passes through it.

XVI. It will be seen that the h lines which correspond to two H points of a v line meet in a P point, and are the projections of two h lines which lie in a Plücker plane. The four H points therefore which correspond to four h lines through a P point such as $(\alpha\beta) = (\gamma\epsilon) = (\gamma\zeta) = (\delta\epsilon) = (\delta\zeta)$ lie on two v lines, $(\gamma\delta) = \chi(\alpha\beta)$, $(\epsilon\zeta) = \chi(\alpha\beta)$; and these intersect in a Y point

$$\frac{\alpha}{a} = \frac{\beta}{b} = (\alpha\beta) = (\gamma\delta) = (\epsilon\zeta) = \chi(\alpha\beta),$$

the intersection of the i line which corresponds to the triangle $\Delta_{\alpha\beta}$ with the side of the triangle opposite to the P point.

The Y points number forty-five and lie by threes on each side of a triangle Δ and on each i line.

The six v lines which pass through the intersections of the diagonals of the quadrangle C, D, E, F, are

$$(\alpha\beta) = \chi(\gamma\delta);\quad (\gamma\delta) = \chi(\alpha\beta);\quad (\epsilon\zeta) = \chi(\alpha\beta);$$
$$(\alpha\beta) = \chi(\epsilon\zeta);\quad (\gamma\delta) = \chi(\epsilon\zeta);\quad (\epsilon\zeta) = \chi(\gamma\delta),$$

and intersect by twos in the three Y points of the line AB. Since the v lines through the intersection of CD, EF form a harmonic pencil with CD, EF, it follows that the six P points of any side AB form harmonic ranges with two of the three Y points of that side.

XVII. To the forty-five Y points, where a side of a triangle $\Delta_{\alpha\beta}$ is met by the corresponding i line, correspond forty-five y lines which join the opposite vertex of the triangle to the corresponding I point.

The y lines are seen to be given by equations such as

$$\chi(\gamma\delta) = \chi(\epsilon\zeta),$$

this being the line which corresponds to

$$(\alpha\beta) = (\gamma\delta) = (\epsilon\zeta) = \chi(\alpha\beta).$$

To three Y points which lie in an i line correspond three y lines which meet in an I point; and to three Y points which lie on the side of a triangle Δ, as

$$(\alpha\beta) = (\gamma\delta) = (\epsilon\zeta)$$

correspond three y lines which meet in one of fifteen R points

$$\chi(\alpha\beta) = \chi(\gamma\delta) = \chi(\epsilon\zeta),$$

to which I shall have occasion to return later. The three y lines which meet in the R point which corresponds to the side AB, pass through the intersections of the diagonals of the quadrangle $CDEF$.

Each y line is the projection of the line of intersection of two Plücker planes: through each P point pass two Plücker planes, which intersect in the y line, and each of which contains two h lines, and one v line passing through the P point: the four lines in each plane form a harmonic pencil.

For through the P point $\alpha+\beta=0$, $\gamma=\delta=-\epsilon=-\zeta$ passes the Plücker plane $\epsilon=\zeta$, and this meets the four planes

$$\gamma+\epsilon=0, \quad \delta+\zeta=0, \quad (\gamma+\epsilon)\pm(\delta+\zeta)=0,$$

in four lines which form a harmonic pencil, whose rays are the four lines spoken of.

The projections of these lines also form a harmonic pencil.

XVIII. It was shewn in XV. that the four points H_2

$$\begin{aligned}(\alpha\beta) &= \chi(\delta\epsilon) = \chi(\epsilon\zeta) = \chi(\delta\zeta),\\ (\alpha\beta) &= \chi(\gamma\epsilon) = \chi(\epsilon\zeta) = \chi(\gamma\zeta),\\ (\alpha\beta) &= \chi(\gamma\delta) = \chi(\delta\zeta) = \chi(\gamma\zeta),\\ (\alpha\beta) &= \chi(\gamma\delta) = \chi(\delta\epsilon) = \chi(\gamma\epsilon),\end{aligned}$$

form a quadrangle whose diagonals intersect in the vertices of the triangle $\Delta_{\alpha\beta}$: hence the lines joining these four points to any other point and any two of the lines which join the point to the vertices of $\Delta_{\alpha\beta}$, form three pairs of lines in involution.

In particular, if the point chosen be the I point, which corresponds to $\Delta_{\alpha\beta}$, we have the property that

The four g lines through an I point and any two of the three y lines through the point, form three pairs of lines in involution.

XIX. The y lines intersect by threes in sixty Σ points, such as

$$\chi(\alpha\beta) = \chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\delta\epsilon)$$

which lie by threes on the g lines, and correspond to the sixty h lines and H points.

Consider the quadrangle formed by this Σ point and the three I points

$$\begin{aligned}\chi(\alpha\beta) &= \chi(\alpha\delta) = \chi(\alpha\epsilon) = \chi(\beta\delta) = \chi(\beta\epsilon) = \chi(\delta\epsilon),\\ \chi(\alpha\beta) &= \chi(\alpha\epsilon) = \chi(\alpha\gamma) = \chi(\beta\epsilon) = \chi(\beta\gamma) = \chi(\epsilon\gamma),\\ \chi(\alpha\beta) &= \chi(\alpha\gamma) = \chi(\alpha\delta) = \chi(\beta\gamma) = \chi(\beta\delta) = \chi(\gamma\delta).\end{aligned}$$

The I points are joined by the g lines

$$\chi(\alpha\beta) = \chi(\alpha\gamma) = \chi(\beta\gamma),$$
$$\chi(\alpha\beta) = \chi(\alpha\delta) = \chi(\beta\delta),$$
$$\chi(\alpha\beta) = \chi(\alpha\epsilon) = \chi(\beta\epsilon),$$

and the Σ point is joined to the three I points by the y lines

$$\chi(\alpha\beta)=\chi(\delta\epsilon); \quad \chi(\alpha\beta)=\chi(\epsilon\gamma); \quad \chi(\alpha\beta)=\chi(\gamma\delta).$$

These lines must cut any transversal in involution; take as the transversal the g line

$$\chi(\alpha\beta)=\chi(\alpha\zeta)=\chi(\beta\zeta).$$

Hence the three I points and the three Σ points of any g lines are in involution.

XX. The ninety v lines also intersect by pairs on the h lines in 180 E points

$$(\alpha\beta)=(\alpha\gamma)=\chi(\beta\gamma)=\chi(\delta\epsilon);$$

each E point is the intersection of two v lines, one h line, and one y line; and three E points lie on each h line and four on each y line.

If we take as the transversal which meets the sides of the quadrangle in XIX. the h line $(\alpha\zeta)=(\beta\zeta)=\chi(\alpha\beta)$ we see that the three E points of any h line and the three H points of the line are in involution.

XXI. The only other points furnished by the intersections of the tritangent and Plücker planes are ninety N points, given in the hexagram by equations such as

$$\frac{\alpha}{a}=\frac{\beta}{b}=(\alpha\beta)=\chi(\alpha\beta)=\chi(\gamma\delta),$$

each the intersection of a v line, a y line, and an i line.

XXII. To complete the figure formed by the tritangent, Plücker and coordinate planes, it is necessary to consider only the intersection of one coordinate plane with the planes and lines discussed above; for the line of intersection of two coordinate planes is an i line and has already received notice.

Each coordinate plane, as $\alpha=0$, is met by ten of the tritangent planes $\beta+\gamma=0$ in a line called a σ line $\alpha=\beta+\gamma=0$. The projections of these sixty σ lines

$$\frac{\alpha}{a}=(\beta\gamma),$$

are noticed by Veronese who shews that if three y lines meet in a Σ point, the corresponding Y points lie in a σ line which passes through a G point.

The line in which a coordinate plane $\alpha=0$ is met by a Plücker plane $\beta=\gamma$ may be called a μ line: there are sixty μ lines, each containing three N points and one G point: the projections of these μ lines

$$\frac{\alpha}{a}=\chi(\beta\gamma),$$

have the same property.

XXIII. There are further in the three-dimensional figure

60 F points, $\alpha=0;\quad \beta=\gamma=\delta;$

180 J points, $\alpha=0;\quad \beta=\gamma=-\delta;$

90 K points, $\alpha=0;\quad \beta=\gamma;\quad \delta=\epsilon;$

180 L points, $\alpha=0;\quad \beta+\gamma=0;\quad \delta=\epsilon;$

whose projections give in the hexagram

60 F points, $\frac{\alpha}{a}=\chi(\beta\gamma)=\chi(\gamma\delta)=\chi(\delta\beta),$

180 J points, $\frac{\alpha}{a}=(\beta\delta)=(\gamma\delta)=\chi(\beta\gamma),$

90 K points, $\frac{\alpha}{a}=\chi(\beta\gamma)=\chi(\delta\epsilon),$

180 L points, $\frac{\alpha}{a}=(\beta\gamma)=\chi(\delta\epsilon).$

It will be as well to pause here and enumerate the various lines and points which compose the hexagram as far as we have at present discovered them. There are

15 sides of hexagons ....$(\alpha\beta)=(\gamma\delta)=(\epsilon\zeta)$;

60 h (Pascal) lines..................$(\alpha\beta)=(\alpha\gamma)=\chi(\beta\gamma)$;

20 g (Cayley-Salmon) lines ......$\chi(\alpha\beta)=\chi(\alpha\gamma)=\chi(\beta\gamma)$;

15 i (Steiner-Plücker) lines ......$(\alpha\beta)=\chi(\alpha\beta)=\frac{\alpha}{a}=\frac{\beta}{b}$;

90 v lines..............................$(\alpha\beta)=\chi(\gamma\delta)$;

45 y lines ....$\chi(\alpha\beta)=\chi(\gamma\delta)$;

60 σ lines..............................$\frac{\alpha}{a}=(\beta\gamma)$;

60 μ lines..............................$\frac{\alpha}{a}=\chi(\beta\gamma)$.

6 fundamental points......A, B, C, D, E, F;

45 P points ....$(\alpha\beta)=(\gamma\epsilon)=(\gamma\zeta)=(\delta\epsilon)=(\delta\zeta)=\chi(\gamma\delta)=\chi(\epsilon\zeta)$;

60 H (Kirkman) points....$(\alpha\beta)=(\alpha\gamma)=(\alpha\delta)=\chi(\beta\gamma)=\chi(\beta\delta)=\chi(\gamma\delta)$;

20 G (Steiner) points ......$(\alpha\beta)=(\alpha\gamma)=(\beta\gamma)=\chi(\alpha\beta)=\chi(\alpha\gamma)=\chi(\beta\gamma)=\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}$;

15 I (Salmon) points ......$\chi(\alpha\beta)=\chi(\alpha\gamma)=\chi(\alpha\delta)=\chi(\beta\gamma)=\chi(\beta\delta)=\chi(\gamma\delta)$;

60 H_2 points..................$(\alpha\beta) = \chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\delta\epsilon)$;

45 Y points ....$(\alpha\beta) = (\gamma\delta) = (\epsilon\zeta) = \chi(\alpha\beta) = \frac{\alpha}{a} = \frac{\beta}{b}$;

15 R points ....$\chi(\alpha\beta) = \chi(\gamma\delta) = \chi(\epsilon\zeta)$;

60 Σ points ....$\chi(\alpha\beta) = \chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\delta\epsilon)$;

180 E points ....$(\alpha\beta) = (\alpha\gamma) = \chi(\beta\gamma) = \chi(\delta\epsilon)$;

90 N points ....$(\alpha\beta) = \chi(\alpha\beta) = \chi(\gamma\delta) = \frac{\alpha}{a} = \frac{\beta}{b}$;

60 F points ....$\chi(\beta\gamma) = \chi(\beta\delta) = \chi(\gamma\delta) = \frac{\alpha}{a}$;

180 J points..................$(\beta\delta) = (\gamma\delta) = \chi(\beta\gamma) = \frac{\alpha}{a}$;

90 K points ....$\chi(\beta\gamma) = \chi(\delta\epsilon) = \frac{\alpha}{a}$;

180 L points ....$(\beta\gamma) = \chi(\delta\epsilon) = \frac{\alpha}{a}$.

Of the lines and points derived from the figure formed by the tritangent and Plücker planes, all receive notice in Veronese's Memoir except the N points and R points; but of the intersections of these planes with the coordinate planes only the σ lines are mentioned. In the case of two kinds of points I have altered Veronese's notation, the H_2 points and Σ points being called by him Z_2 points and ζ points respectively; and for the sake of brevity I have spoken of H points, G points and I points, h lines, g lines and i lines where Veronese uses the names of the mathematicians by whom they were discovered.

In the three-dimensional figure, the lines and points of which the projections have been given are as follows:

15 sides of hexagons...$\alpha + \beta = \gamma + \delta = \epsilon + \zeta = 0$;

60 h lines ....$-\alpha = \beta = \gamma$;

20 g lines ....$\alpha = \beta = \gamma$;

15 i lines ....$\alpha = \beta = 0$;

90 v lines ....$\alpha + \beta = 0$; $\gamma = \delta$;

45 y lines ....$\alpha = \beta$; $\gamma = \delta$;

60 σ lines ....$\alpha = 0$; $\beta + \gamma = 0$;

60 μ lines ....$\alpha = 0$; $\beta = \gamma$;

45 P points ....$\alpha+\beta=0$; $\gamma=\delta=-\epsilon=-\zeta$;

60 H points ....$-\alpha=\beta=\gamma=\delta$;

20 G points ....$\alpha=\beta=\gamma=0$;

15 I points..............$\alpha=\beta=\gamma=\delta$;

60 H_2 points ....$\alpha+\beta=0$; $\gamma=\delta=\epsilon$;

45 Y points.............$\alpha=\beta=\gamma+\delta=\epsilon+\zeta=0$;

15 R points ....$\alpha=\beta$; $\gamma=\delta$; $\epsilon=\zeta$;

60 Σ points ....$\alpha=\beta$; $\gamma=\delta=\epsilon$;

180 E points ....$-\alpha=\beta=\gamma$; $\delta=\epsilon$;

90 N points ....$\alpha=\beta=0$; $\gamma=\delta$;

60 F points ....$\alpha=0$; $\beta=\gamma=\delta$;

180 J points ....$\alpha=0$; $\beta=\gamma=-\delta$;

90 K points ....$\alpha=0$; $\beta=\gamma$; $\delta=\epsilon$;

180 L points ....$\alpha=0$; $\beta=\gamma$; $\delta+\epsilon=0$.

There are two sets of lines and points which Veronese has noticed, viz. m lines which are the projection of lines such as

$$\alpha+\beta=0,\quad 2\alpha+\gamma+\delta=0,$$

and T points which are the projections of points such as

$$\alpha+\beta=0,\quad \delta=\epsilon,\quad 2\alpha+\gamma+\delta=0;$$

but these do not appear worthy of further mention.

It was pointed out in (7) that the six lines AD, DE, EA, BC, CF, FB are generators of a quadric surface, viz.

$$\alpha^2+\beta^2+\gamma^2=\delta^2+\epsilon^2+\zeta^2 \quad\text{.................................(1).}$$

It follows that the planes which pass through O the conical point and these six lines touch the enveloping cone from O to this quadric

$$(\alpha^2+\beta^2+\gamma^2-\delta^2-\epsilon^2-\zeta^2)(a^2+b^2+c^2-d^2-e^2-f^2)=(a\alpha+b\beta+c\gamma-d\delta-e\epsilon-f\zeta)^2,$$

or

$$\Sigma(a\delta-d\alpha)^2=\Sigma(a\beta-b\alpha)^2 \quad\text{.................................(2),}$$

where on the left hand side are the nine squares in which one of the three letters α, β, γ, is associated with one of the three δ, ϵ, ζ; and on the right hand side are the remaining six squares.

Hence in the hexagram, the six sides of the triangles ADE, BCF touch the conic (2).

Again, the projection of any plane section of (1) is a conic which has double contact with (2). Hence it is inferred from (7) that the projection of the conic

$$\alpha=\delta,\quad \beta^2+\gamma^2=\epsilon^2+\zeta^2$$

has double contact with the two conics which touch the sides of the triangles *ADE*, *BCF*, and of *ACD*, *BEF*.

This conic clearly passes through the eight points

$$\alpha = \delta, \quad \beta = \pm \epsilon, \quad \gamma = \pm \zeta,$$

and

$$\alpha = \delta, \quad \beta = \pm \zeta, \quad \gamma = \pm \epsilon.$$

Hence it is inferred

The four *P* points $\alpha = \delta = -\beta = -\epsilon; \quad \gamma + \zeta = 0;$ or *CD*, *AE*,

$$\alpha = \delta = -\beta = -\zeta; \quad \gamma + \epsilon = 0; \quad \text{or } BC,\ EF,$$

$$\alpha = \delta = -\gamma = -\epsilon; \quad \beta + \zeta = 0; \quad \text{or } BE,\ CF,$$

$$\alpha = \delta = -\gamma = -\zeta; \quad \beta + \epsilon = 0; \quad \text{or } AC,\ DE,$$

the two *V* points

$$\alpha = \delta = 0; \quad \beta + \epsilon = 0; \quad \gamma + \zeta = 0;$$

$$\alpha = \delta = 0; \quad \beta + \zeta = 0; \quad \gamma + \epsilon = 0;$$

and the two *R* points

$$\alpha = \delta; \quad \beta = \epsilon; \quad \gamma = \zeta;$$

$$\alpha = \delta; \quad \beta = \zeta; \quad \gamma = \epsilon;$$

lie on a conic which has double contact with the conic which touches the sides of the triangles *ADE*, *BCF* and with that which touches the sides of *ACD*, *BEF*.

The remainder of Veronese's memoir, of which I wish now to give the analytical equivalent, treats of certain systems of lines and points (called by him $z_2 z_3 \ldots$ lines and $Z_2 Z_3 \ldots$ points) which correspond in many ways to the *h* lines and *H* points and may be grouped into six sets of ten lines and points in a similar manner: as stated above the Z_2 points of Veronese have been spoken of as H_2 points.

XXIV. It was shewn in XV. that the six *v* lines which pass through the three vertices of a triangle Δ intersect by threes in four points H_2 such as

$$(\alpha\beta) = \chi(\gamma\delta) = \chi(\gamma\epsilon) = \chi(\delta\epsilon),$$

that there are sixty such points in the hexagram which lie by threes on the *g* lines, and that further the point above corresponds to the *H* point

$$(\zeta\gamma) = (\zeta\delta) = (\zeta\epsilon) = \chi(\gamma\delta) = \chi(\delta\epsilon) = \chi(\gamma\epsilon),$$

and to the *h* line

$$(\zeta\alpha) = (\zeta\beta) = \chi(\alpha\beta).$$

In the three-dimensional figure, the six *v* lines which pass through the vertices of a triangle Δ and lie in its plane, intersect by threes in four H_2 points, such as

$$\alpha + \beta = 0, \quad \gamma = \delta = \epsilon,$$

which corresponds to the *H* point $-\zeta = \gamma = \delta = \epsilon$, and to the *h* line $-\zeta = \alpha = \beta$.

'If three *H* points lie in an *h* line, the corresponding H_2 points lie in a line called an h_2 line.'

Taking the three H points which lie on $-\zeta=\alpha=\beta$, the corresponding H_2 points are

$$\delta+\epsilon=0;\quad \alpha=\beta=\gamma;$$

$$\epsilon+\gamma=0;\quad \alpha=\beta=\delta;$$

$$\gamma+\delta=0:\quad \alpha=\beta=\epsilon;$$

and these are seen to lie on the h_2 line

$$\alpha=\beta=\gamma+\delta+\epsilon,$$

which is equivalent to

$$\zeta+3\alpha=\zeta+3\beta=0.$$

Thus corresponding to the h line $\alpha+\beta=\alpha+\gamma=0$, and to the H point

$$\alpha+\delta=\alpha+\epsilon=\alpha+\zeta=0,$$

are the h_2 line

$$\alpha+3\beta=\alpha+3\gamma=0,$$

and the H_2 point

$$\alpha+3\delta=\alpha+3\epsilon=\alpha+3\zeta=0.$$

XXV. Thus, both in the three-dimensional figure and in the Hexagram, the sixty H_2 points and h_2 lines correspond to the H points and h lines; when three h lines meet in an H point the corresponding h_2 lines meet in an H_2 point, and the corresponding H points and H_2 points lie on an h line or h_2 line respectively; while if three h lines meet in a G point, the corresponding h_2 lines meet in the same G point, and the corresponding H points and H_2 points lie on the corresponding g line. Hence the G points and g lines and therefore also the I points and i lines are common to the two systems (1) of h lines and H points, (2) of h_2 lines and H_2 points.

Thus from the figure formed by the five planes

$$\alpha+3\beta=0;\quad \alpha+3\gamma=0;\quad \alpha+3\delta=0;\quad \alpha+3\epsilon=0;\quad \alpha+3\zeta=0;$$

it is clear that in the hexagram the ten h_2 lines and H_2 points which correspond to the ten h lines and H points of a projected pentahedron (as in VI.) themselves form another such figure which has associated with it a conic such that each H_2 point of the ten is the pole of the corresponding h_2 line; and the h_2 lines and H_2 points may be grouped into six such figures.

But the relations between two or more figures of h_2 lines and H_2 points are not identical with those existing between the corresponding figures of h lines and H points; for the latter are derived from the projections of the intersections of fifteen planes, while the h_2 lines and H_2 points cannot be derived from fewer than thirty planes; thus, in the three-dimensional figure each pentahedron of h lines and H points is contained by six out of fifteen planes, and each plane occurs in two pentahedra; but in the case of h_2 lines and H_2 points, each pentahedron is contained by five out of thirty planes, and no plane occurs more than once.

XXVI. The intersections of these thirty planes however furnish a second similar system of lines and points, which for the present may be distinguished by the suffix 3.

Corresponding to two H points which lie on a v line $\alpha+\beta=0$; $\gamma=\delta$; viz. the H points

$$\alpha+\beta=\alpha+\gamma=\alpha+\delta=0; \quad \alpha+\beta=\beta+\gamma=\beta+\delta=0;$$

we have the two h_2 lines

$$\alpha+3\epsilon=\alpha+3\zeta=0; \quad \beta+3\epsilon=\beta+3\zeta=0;$$

which meet in a point called by Veronese a V point (or later a V_{23} point)

$$\alpha : \beta : \epsilon : \zeta :: 3 : 3 : -1 : -1$$

lying on the y line $\alpha=\beta$, $\epsilon=\zeta$.

From these V points may be derived the second system of h_3 lines and H_3 points.

For through each V point pass two h_3 lines of the second system, viz.

$$3\epsilon+\alpha=3\epsilon+\beta=0,$$

$$3\zeta+\alpha=3\zeta+\beta=0.$$

Thus in the projected figure, each line of either system contains three V points; thus the first system determines the V points, and these determine geometrically a second similar system of lines and points which has all the properties of the first system.

XXVII. These results may at once be generalised. Consider the system of thirty planes such as $\lambda\alpha+\mu\beta=0$, where λ and μ are definite constants.

Two of these planes pass through each of the fifteen i lines, and are harmonically conjugate with respect to the tritangent plane and the Plücker plane which intersect in that line.

Let the line in which two of these planes such as

$$\lambda\alpha+\mu\beta=0; \quad \lambda\alpha+\mu\gamma=0$$

intersect, be defined as an $h_{\lambda,\mu}$ line, and let a point such as

$$\lambda\alpha+\mu\delta=\lambda\alpha+\mu\epsilon=\lambda\alpha+\mu\zeta=0$$

be defined as an $H_{\lambda,\mu}$ point; and let their projections in the hexagram bear the same names.

Thus the h_2 lines and H_2 points are equivalent to $h_{1,3}$ lines and $H_{1,3}$ points, and the h_3 lines and H_3 points to $h_{3,1}$ lines and $H_{3,1}$ points.

Then it is clear that we have a system of sixty $h_{\lambda,\mu}$ lines and $H_{\lambda,\mu}$ points which has all the properties mentioned in XXV. as possessed by the h_2 lines and H_2 points and further that a second similar system of $h_{\mu,\lambda}$ lines and $H_{\mu,\lambda}$ points may be deduced in the same way that the h_3 lines and H_3 points were deduced from the h_2 lines and H_2 points.

The $h_{\lambda,\mu}$ lines and $H_{\lambda,\mu}$ points with the $h_{\mu,\lambda}$ lines and $H_{\mu,\lambda}$ points together form a system which may be called the $(\lambda\mu)$ system: each system $(\lambda\mu)$ contains ninety $V_{\lambda,\mu}$ points, such as

$$\lambda\alpha+\mu\epsilon=\lambda\alpha+\mu\zeta=\lambda\beta+\mu\epsilon=\lambda\beta+\mu\zeta=0$$

two of which lie on each y line and are harmonically conjugate with respect to the P point and the I point of that line.

In the hexagram, the projections of these lines and points have analogous properties, and the $V_{\lambda,\mu}$ points serve to connect the $h_{\lambda,\mu}$ lines with the $h_{\mu,\lambda}$ lines.

XXVIII. Veronese connects the systems for different values of $\lambda:\mu$ by a method which leads to a curious analytical equivalent.

The $V_{\lambda,\mu}$ points were obtained as the intersection of two $h_{\lambda,\mu}$ lines which correspond to two H points of a v line. If instead, the corresponding $H_{\lambda,\mu}$ points are taken, the line which joins them may be called a $v_{\lambda,\mu}$ line.

Let the v line be $\alpha+\beta=0$; $\gamma=\delta$; then the two $H_{\lambda,\mu}$ points are

$$\lambda\alpha+\mu\beta=\lambda\alpha+\mu\gamma=\lambda\alpha+\mu\delta=0,$$
$$\lambda\beta+\mu\alpha=\lambda\beta+\mu\gamma=\lambda\beta+\mu\delta=0,$$

and the $v_{\lambda,\mu}$ line which joins them is

$$\frac{\alpha+\beta}{\lambda-\mu}=\frac{\gamma}{\lambda}=\frac{\delta}{\lambda}.$$

Thus there are ninety $v_{\lambda,\mu}$ lines, which intersect by pairs in the forty-five V points, and form harmonic pencils with the v lines and I lines.

But since $\alpha+\beta+\gamma+\delta+\epsilon+\zeta\equiv 0$, the line may be written

$$\frac{\alpha+\beta}{\lambda-\mu}=\frac{\gamma}{\lambda}=\frac{\delta}{\lambda}=\frac{\epsilon+\zeta}{\mu-3\lambda},$$

and hence belongs equally to a system (λ', μ') for which

$$\frac{\mu'-3\lambda'}{\lambda'}=\frac{\lambda-\mu}{\lambda},$$

that is

$$\frac{\mu'}{\lambda'}+\frac{\mu}{\lambda}=4.$$

Thus from the system of $h_{\lambda,\mu}$ lines and $H_{\lambda,\mu}$ points ninety $v_{\lambda,\mu}$ lines are determined, from which in turn a second system of $H_{\lambda',\mu'}$ points is determined, viz. as points of concurrence of three $v_{\lambda,\mu}$ lines, which belong equally to the new system.

Thus in the hexagram, from the system of points and lines distinguished by suffix (λ_1, μ_1) may be deduced by means of V points a second system given by the suffix (μ_1, λ_1).

Hence by means of v lines a new system (λ_2, μ_2) is derived, where

$$\frac{\mu_2}{\lambda_2} + \frac{\lambda_1}{\mu_1} = 4,$$

and hence again a system (μ_2, λ_2).

From this is obtained a system (λ_3, μ_3) where

$$\frac{\mu_3}{\lambda_3} + \frac{\lambda_2}{\mu_2} = 4,$$

and so on ad infinitum.

Again it is possible to reverse the process, hence as a rule from any system a series of other systems extending to an infinite number may be deduced in two ways, the whole forming one complete series of systems of sixty points and sixty lines.

The solution of the equation

$$\frac{\mu_{n+1}}{\lambda_{n+1}} + \frac{\lambda_n}{\mu_n} = 4$$

is

$$\frac{\mu_n}{\lambda_n} = \frac{1}{2}\,\frac{A(\sqrt{3}+1)^{2n+1} + B(\sqrt{3}-1)^{2n+1}}{A(\sqrt{3}+1)^{2n-1} + B(\sqrt{3}-1)^{2n-1}}.$$

Hence whatever system be chosen to start from, the limiting value of $\frac{\mu}{\lambda}$ is always either $2+\sqrt{3}$ or $2-\sqrt{3}$.

From the system of h lines and H points for which $\mu_0 = \lambda_0$, Veronese deduces a series of systems, given by values of the above fraction when $A = B$.

XXIX. There is one special system of the $h_{\lambda,\mu}$ lines and $H_{\lambda,\mu}$ points which has not been noticed, and appears to deserve attention. Corresponding to four h lines which pass through a P point, as for example

$$\gamma+\epsilon = \gamma+\zeta = 0; \quad \epsilon+\gamma = \epsilon+\delta = 0;$$
$$\delta+\epsilon = \delta+\zeta = 0; \quad \zeta+\gamma = \zeta+\delta = 0;$$

are the four $H_{\lambda,\mu}$ points

$$\lambda\gamma + \mu\alpha = \lambda\gamma + \mu\beta = \lambda\gamma + \mu\delta = 0;$$
$$\lambda\delta + \mu\alpha = \lambda\delta + \mu\beta = \lambda\delta + \mu\gamma = 0;$$
$$\lambda\epsilon + \mu\alpha = \lambda\epsilon + \mu\beta = \lambda\epsilon + \mu\zeta = 0;$$
$$\lambda\zeta + \mu\alpha = \lambda\zeta + \mu\beta = \lambda\zeta + \mu\epsilon = 0.$$

The first two are joined by the line

$$\frac{\alpha}{\lambda} = \frac{\beta}{\lambda} = \frac{\gamma+\delta}{\lambda-\mu} = \frac{\epsilon+\zeta}{\mu-3\lambda},$$

and the last two are joined by

$$\frac{\alpha}{\lambda} = \frac{\beta}{\lambda} = \frac{\epsilon+\zeta}{\lambda-\mu} = \frac{\gamma+\delta}{\mu-3\lambda}.$$

If now $\lambda-\mu = \mu-3\lambda$, that is $2\lambda = \mu$, these two lines are identical, and therefore, corresponding to four h lines which meet in a P point, there are four $H_{1,2}$ points which lie on a line conveniently called a p line. Thus to the P point

$$\alpha+\beta = 0, \quad \gamma = \delta = -\epsilon = -\zeta,$$

corresponds the p line

$$\alpha = \beta = -(\gamma+\delta) = -(\epsilon+\zeta).$$

To three P points which are vertices of a triangle $\Delta_{\alpha\beta}$ correspond three p lines which lie in the plane $\alpha = \beta$, and form a triangle whose vertices are R points, while to any P point on one of the fifteen lines AB,

$$\alpha + \beta = \gamma + \delta = \epsilon + \zeta = 0$$

there corresponds a p line which passes through the R point

$$\alpha = \beta, \quad \gamma = \delta, \quad \epsilon = \zeta.$$

The p lines may also be obtained as the lines of intersection of the ten planes such as

$$\alpha + \beta + \gamma = \delta + \epsilon + \zeta = 0,$$

which may be called Φ planes. Each p line, the intersection of two Φ planes, lies also in a Plücker plane; also each h_{12} line

$$\alpha + 2\beta = \alpha + 2\gamma = 0$$

is the line of intersection of one of the ten Φ planes with a Plücker plane

$$\alpha + \beta + \gamma = \delta + \epsilon + \zeta = 0; \quad \beta = \gamma.$$

The H_{12} points are points of intersection of three of the Φ planes; for if

$$\alpha + 2\delta = \alpha + 2\epsilon = \alpha + 2\zeta = 0,$$

then

$$\beta + \gamma + \delta = \beta + \gamma + \epsilon = \beta + \gamma + \zeta = 0.$$

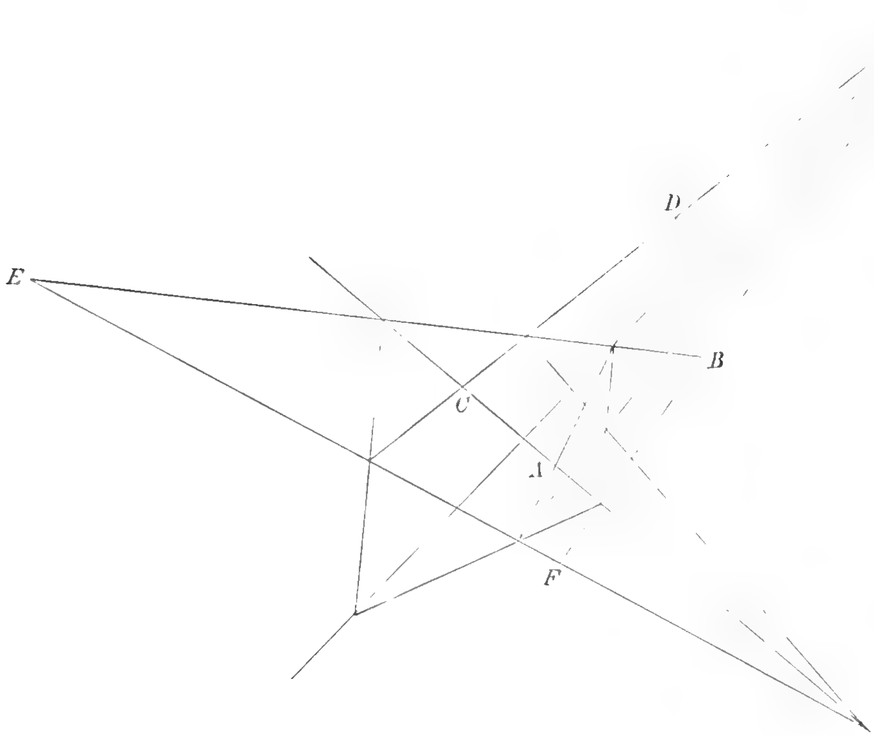
The Φ planes intersect by twos in the forty-five p lines, by threes in the H_{12} points, and by fours in the fifteen R points: each plane contains two conjugate G points and is met by the six coordinate planes in six σ lines.

These Φ planes would furnish by their intersections with one another and with the tritangent, Plücker, and coordinate planes many new lines and points of interest in the theory of the Hexagram; I have however no wish to increase further the already unwieldy number of lines and points of the plane figure. In three dimensions, it has been seen a comparatively small number of planes is sufficient to determine the complete figure, and the confused intricacy of the plane Hexagram is avoided. With this brief mention then of the Φ planes, which appear to stand next in importance to the tritangent, Plücker, and coordinate planes, I shall leave the subject.

It is clear that in the figure of the Hexagram, the lines and points obtained may be grouped into figures of projected pentahedra and hexahedra in a very large number of ways; for if from the planes in the three-dimensional figure any five are selected of which no three intersect in a common line and no four pass through a common point, their intersections will give a figure of a projected pentahedron, and any six planes selected under the same conditions will give a projected hexahedron; should the conditions not be satisfied, the figure of the projection will be modified. It may be worth while to examine one or two of these figures.

(a) Taking the five planes $(\alpha\beta) = 0$, $(\alpha\gamma) = 0$, $(\alpha\delta) = 0$, $(\delta\epsilon) = 0$, $(\delta\zeta) = 0$ the projections of the edges of the pentahedron will be found to consist of six h lines, and the four lines AC, CD, BE, EF; and the ten vertices are the two H points $(\alpha\beta) = (\alpha\gamma) = (\alpha\delta)$; $(\alpha\delta) = (\delta\epsilon) = (\delta\zeta)$ and eight of the nine P points in which the sides of the triangle ACD meet those of BEF, the point of intersection of AD and BF being omitted.

The figure is given below.



Since $$(\alpha+\delta)\equiv(\alpha+\beta)+(\alpha+\gamma)+(\delta+\epsilon)+(\delta+\zeta)$$

or $$(a+d)(\alpha\delta)\equiv(a+b)(\alpha\beta)+(a+c)(\alpha\gamma)+(d+e)(\delta\epsilon)+(d+f)(\delta\zeta)$$

it follows that each of the ten points of the figure is the pole of the opposite line with respect to the conic

$$(a+d)(\alpha\delta)^2=(a+b)(\alpha\beta)^2+(a+c)(\alpha\gamma)^2+(d+e)(\delta\epsilon)^2+(d+f)(\delta\zeta)^2.$$

(*b*) Taking the six planes

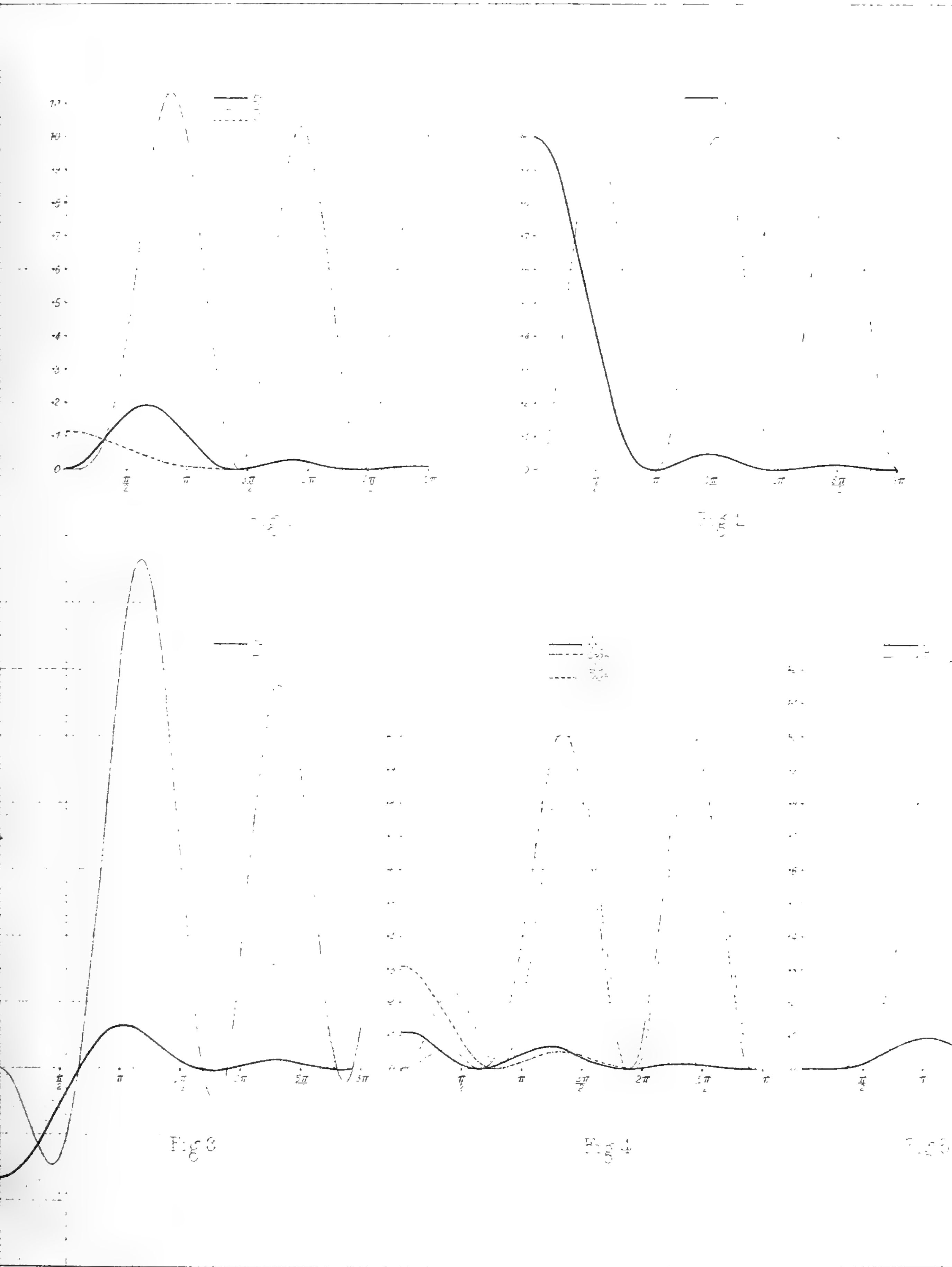
$$(\alpha\beta)=0,\quad(\gamma\delta)=0,\quad(\alpha\gamma)=0,\quad(\beta\delta)=0,\quad(\alpha\delta)=0,\quad(\beta\gamma)=0,$$

we have a figure of a projected hexahedron, whose fifteen sides are made up of the three sides of the triangle $\Delta_{\epsilon\zeta}$, and twelve h lines, and whose twenty vertices are made up of four H points, four G points, and twelve P points which lie on the sides of the triangle $\Delta_{\epsilon\zeta}$ but are not vertices of that triangle. The four H points are conjugate to the four G points, and the P points are conjugate in pairs with respect to any conic which passes through the four points given by

$$(a+b)(\alpha\beta)^2+(c+d)(\gamma\delta)^2=(a+c)(\alpha\gamma)^2+(b+d)(\beta\delta)^2=(a+d)(\alpha\delta)^2+(b+c)(\beta\gamma)^2$$

or $$(a+b)(c+d)[(\alpha\beta)-(\gamma\delta)]^2=(a+c)(b+d)[(\alpha\gamma)-(\beta\delta)]^2=(a+d)(b+c)[(\alpha\delta)-(\beta\gamma)]^2.$$

Veronese also obtains many properties of harmonicism and involution which I pass over, as in no case does the proof present any difficulty.



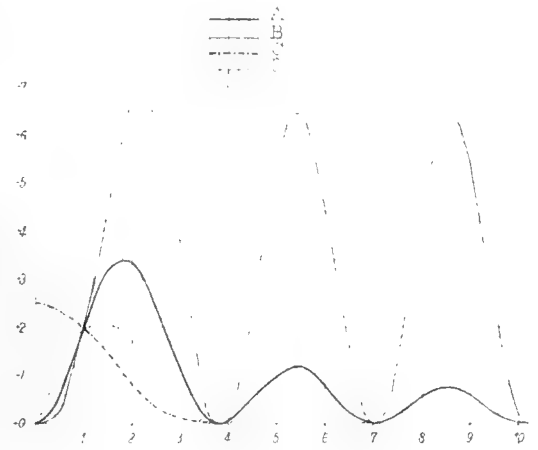


Fig. 6

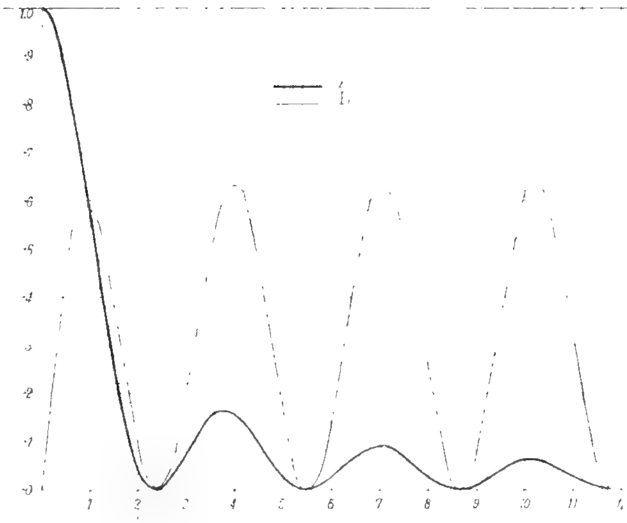


Fig. 7

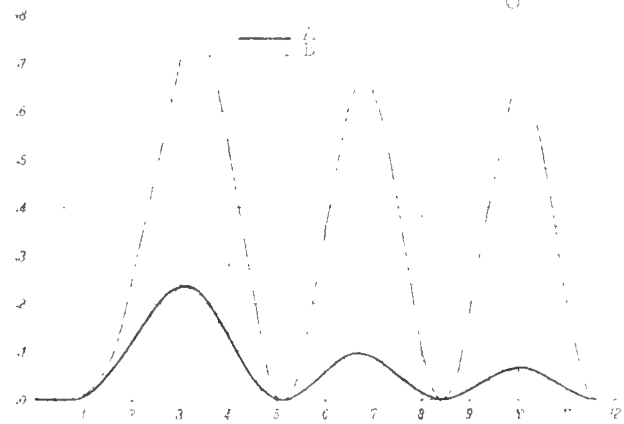


Fig. 8

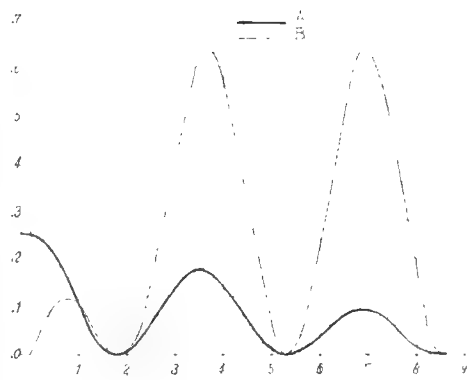


Fig. 9

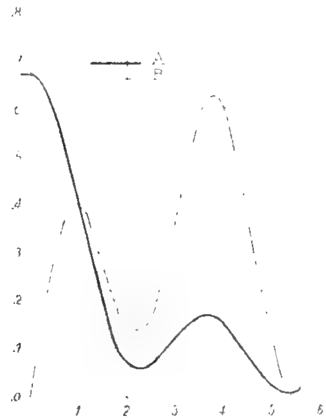


Fig. 10

VIII. *The Self-Induction of Two Parallel Conductors.* By H. M. MACDONALD, Clare College.

IN § 685 of his *Electricity and Magnetism*, Vol. II., Maxwell gives the relation

$$\frac{L}{l} = \tfrac{1}{2}(\mu + \mu') + 2\mu_0 \log \frac{b^2}{aa'},$$

as that existing between the self-induction L of two parallel infinite cylindrical conductors, radii a and a', the distance between their axes being b, μ, μ' their magnetic permeabilities and μ_0 the magnetic permeability of the surrounding medium. It was remarked by Lord Rayleigh in the *Phil. Mag.*, May, 1886, that this expression is only true when $\mu = \mu' = \mu_0$. The following is a solution of the cases when the μ's are not all equal.

1. F, G, H the components of the vector potential at any point x, y, z satisfy the equations

$$\left.\begin{aligned} \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + 4\pi\mu u = 0, \\ \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} + 4\pi\mu v = 0, \\ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} + 4\pi\mu w = 0 \end{aligned}\right\} \quad \text{.................................(1)}$$

throughout space, u, v, w being the components of the total current at the points x, y, z and μ the magnetic permeability of the medium at that point. At the bounding surface of two media for which μ is the same, F, G, H satisfy the equations

$$\left.\begin{aligned} \frac{\partial F}{\partial \nu} + \frac{\partial F'}{\partial \nu'} = 0, \\ \frac{\partial G}{\partial \nu} + \frac{\partial G'}{\partial \nu'} = 0, \\ \frac{\partial H}{\partial \nu} + \frac{\partial H'}{\partial \nu'} = 0 \end{aligned}\right\} \quad \text{...(2),}$$

ν, ν' being the directions of the normals drawn from the bounding surface into the two media.

Equations (1) and (2) were given by Maxwell, *Phil. Trans.*, 1865. The equations which hold at the bounding surface of two media, magnetic permeabilities μ, μ', may be shown to be

$$\left.\begin{aligned}\frac{1}{\mu}\frac{\partial F}{\partial \nu}+\frac{1}{\mu'}\frac{\partial F'}{\partial \nu'}&=0,\\ \frac{1}{\mu}\frac{\partial G}{\partial \nu}+\frac{1}{\mu'}\frac{\partial G'}{\partial \nu'}&=0,\\ \frac{1}{\mu}\frac{\partial H}{\partial \nu}+\frac{1}{\mu'}\frac{\partial H'}{\partial \nu'}&=0\end{aligned}\right\}\quad\text{..........(3)},$$

exactly as the analogous equations at the bounding surface of two media, for which the specific inductive capacities are K and K', are proved in electrostatics, by taking $K\mu=1$, $K'\mu'=1$ and remembering that

$$F=\iiint\frac{\mu\mu'}{r}\,dx'\,dy'\,dz',\ \text{etc.}$$

2. Applying these to the case of two infinite parallel conductors with circular sections, taking as plane of xy a section perpendicular to their lengths, as axis of y the straight line joining the two limiting points of the circles in which they cut the plane, and as axis of x the straight line bisecting this at right angles, we find the equations

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+4\pi\mu w=0 \quad\text{.......... (1)},$$

$$\frac{1}{\mu}\frac{\partial H}{\partial \nu}+\frac{1}{\mu'}\frac{\partial H'}{\partial \nu'}=0 \quad\text{..........(2)}$$

to determine H, while F and G are constant or zero. Transform these equations by the relation

$$x+\iota y=c\tan\tfrac{1}{2}(\xi+\iota\eta),$$

$2c$ being the distance between the limiting points of the circles. Let $\eta=\alpha$ be the bounding surface of one conductor, μ its magnetic permeability, $\eta=-\beta$ the bounding surface of the other, μ' its magnetic permeability, and μ_0 that of the surrounding medium. Equations (1) and (2) become

$$\frac{\partial^2 H}{\partial \xi^2}+\frac{\partial^2 H}{\partial \eta^2}+\frac{4\pi\mu w c^2}{(\cosh\eta+\cos\xi)^2}=0\ldots(3)\ \text{from}\ \eta=\infty\ \text{to}\ \eta=\alpha,$$

$$\frac{\partial^2 H_0}{\partial \xi^2}+\frac{\partial^2 H_0}{\partial \eta^2}=0\ldots(4)\ \text{from}\ \eta=\alpha\ \text{to}\ \eta=-\beta,$$

$$\frac{\partial^2 H'}{\partial \xi^2}+\frac{\partial^2 H'}{\partial \eta^2}+\frac{4\pi\mu' w' c^2}{(\cosh\eta+\cos\xi)^2}=0\ldots(5)\ \text{from}\ \eta=-\beta\ \text{to}\ \eta=-\infty,$$

$$H=H_0\ \text{and}\ \frac{1}{\mu}\frac{\partial H}{\partial \eta}=\frac{1}{\mu_0}\frac{\partial H_0}{\partial \eta}\quad\ldots(6)\ \text{when}\ \eta=\alpha,$$

$$H_0=H'\ \text{and}\ \frac{1}{\mu_0}\frac{\partial H_0}{\partial \eta}=\frac{1}{\mu'}\frac{\partial H'}{\partial \eta}\quad\ldots(7)\ \text{when}\ \eta=-\beta,$$

where H is the vector-potential inside the first conductor, H_0 in the surrounding medium, and H' in the second conductor, w and w' the densities of the currents in the two conductors.

3. To solve these equations assume

$$H = A_0 + B\frac{\sinh(\alpha - \eta)}{\cosh\eta + \cos\xi} + \sum_1^\infty e^{-n\eta}(A_n \cos n\xi + B_n \sin n\xi) \quad\ldots\ldots\ldots(8),$$

$$H_0 = A_0' + B_0'\eta + \sum_1^\infty (A_n' \cosh n\eta \cos n\xi + B_n' \sinh n\eta \sin n\xi + C_n' \cosh n\eta \sin n\xi + D_n' \sinh n\eta \cos n\xi) \ldots(9),$$

$$H' = A_0'' + B''\frac{\sinh(\beta + \eta)}{\cosh\eta + \cos\xi} + \sum_1^\infty e^{n\eta}(A_n'' \cos n\xi + B_n'' \sin n\xi)\ldots\ldots\ldots(10).$$

Equation (8) satisfies (3) and is finite when $\eta = \infty$, (9) satisfies (4), and (10) satisfies (5), and is finite when $\eta = -\infty$. Further by differentiating (8) and (10) and substituting in (3) and (5), we obtain

$$\left.\begin{aligned} B &= -\frac{2\pi\mu w c^2}{\sinh\alpha} \\ B'' &= -\frac{2\pi\mu' w' c^2}{\sinh\beta} \end{aligned}\right\} \ldots\ldots\ldots(11).$$

To determine the remaining constants (6) and (7) give

$$A_0 + \sum_1^\infty e^{-n\alpha}(A_n \cos n\xi + B_n \sin n\xi)$$
$$= A_0' + B_0'\alpha + \sum_1^\infty (A_n' \cosh n\alpha \cos n\xi + B_n' \sinh n\alpha \sin n\xi + C_n' \cosh n\alpha \sin n\xi + D_n' \sinh n\alpha \cos n\xi) \ldots\ldots\ldots(12),$$

$$-\frac{B}{\cosh\alpha + \cos\xi} - \sum_1^\infty n e^{-n\alpha}(A_n \cos n\xi + B_n \sin n\xi)$$
$$= \frac{\mu}{\mu_0}\left\{B_0' + \sum_1^\infty n(A_n' \sinh n\alpha \cos n\xi + B_n' \cosh n\alpha \sin n\xi + C_n' \sinh n\alpha \sin n\xi + D_n' \cosh n\alpha \cos n\xi)\right\} \ldots(13),$$

$$A_0'' + \sum_1^\infty e^{-n\beta}(A_n'' \cos n\xi + B_n'' \sin n\xi)$$
$$= A_0' - B_0'\beta + \sum_1^\infty (A_n' \cosh n\beta \cos n\xi - B_n' \sinh n\beta \sin n\xi + C_n' \cosh n\beta \sin n\xi - D_n' \sinh n\beta \cos n\xi) \ldots\ldots\ldots(14),$$

$$\frac{B''}{\cosh\beta + \cos\xi} + \sum_1^\infty n e^{-n\beta}(A_n'' \cos n\xi + \beta_n'' \sin n\xi)$$
$$= \frac{\mu'}{\mu_0}\left\{B_0' + \sum_1^\infty n(-A_n' \sinh n\beta \cos n\xi + B_n' \cosh n\beta \sin n\xi - C_n' \sinh n\beta \sin n\xi + D_n' \cosh n\beta \cos n\xi)\right\} \ldots(15),$$

whence

$$A_0 = A_0' + B_0'\alpha \qquad (16),$$

$$B_0' = -\frac{\mu_0}{\mu}\frac{B}{\sinh\alpha} \qquad (17),$$

$$A_0'' = A_0' - B_0'\beta \qquad (18),$$

$$B_0' = \frac{\mu_0}{\mu'}\frac{B''}{\sinh\beta} \qquad (19).$$

Now

$$\frac{\sinh\alpha}{\cosh\alpha+\cos\xi} = 1 - 2e^{-\alpha}\cos\xi + 2e^{-2\alpha}\cos 2\xi - \text{etc.},$$

therefore from (12) and (13) we obtain

$$\left.\begin{aligned}
&A_ne^{-n\alpha} = A_n'\cosh n\alpha + B_n'\sinh n\alpha,\\
&B_ne^{-n\alpha} = B_n'\sinh n\alpha + C_n'\cosh n\alpha,\\
&\frac{\mu_0}{\mu}\left(-nA_ne^{-n\alpha} - (-)^ne^{-n\alpha}\cdot\frac{2B}{\sinh\alpha}\right) = n\,(A_n'\sinh n\alpha + D_n'\cosh n\alpha),\\
&\frac{\mu_0}{\mu}\,(-ne^{-n\alpha}B_n) \qquad\qquad\qquad = n\,(B_n'\cosh n\alpha + C_n'\sinh n\alpha)
\end{aligned}\right\},$$

hence

$$\left.\begin{aligned}
&A_n' = A_ne^{-n\alpha}\left(\cosh n\alpha + \frac{\mu_0}{\mu}\sinh n\alpha\right) + (-)^n\frac{2Be^{-n\alpha}}{n\sinh\alpha}\cdot\frac{\mu_0}{\mu}\sinh n\alpha,\\
&B_n' = B_ne^{-n\alpha}\left(-\frac{\mu_0}{\mu}\cosh n\alpha - \sinh n\alpha\right),\\
&C_n' = B_ne^{-n\alpha}\left(\frac{\mu_0}{\mu}\sinh n\alpha + \cosh n\alpha\right),\\
&D_n' = A_ne^{-n\alpha}\left(-\sinh n\alpha - \frac{\mu_0}{\mu}\cosh n\alpha\right) - (-)^n\frac{2Be^{-n\alpha}}{n\sinh\alpha}\cdot\frac{\mu_0}{\mu}\cosh n\alpha
\end{aligned}\right\} \qquad (20),$$

also from (14) and (15)

$$\left.\begin{aligned}
&A_n''e^{-n\beta} = A_n'\cosh n\beta - D_n'\sinh n\beta,\\
&B_n''e^{-n\beta} = -B_n'\sinh n\beta + C_n'\cosh n\beta,\\
&\frac{\mu_0}{\mu'}\left(nA_n''e^{-n\beta} + (-)^n\frac{2B''e^{-n\beta}}{\sinh\beta}\right) = n\,(-A_n'\sinh n\beta + D_n'\cosh n\beta),\\
&\frac{\mu_0}{\mu'}\,nB_n''e^{-n\beta} = n\,(B_n'\cosh n\beta - C_n'\sinh n\beta)
\end{aligned}\right\},$$

whence by (20)

$$\left.\begin{aligned}
&B_n = B_n'' = B_n' = C_n' = 0,\\
&A_n' = A_n''e^{-n\beta}\left(\cosh n\beta + \frac{\mu_0}{\mu}\sinh n\beta\right) + (-)^n\frac{2B''e^{-n\beta}}{n\sinh\beta}\cdot\frac{\mu_0}{\mu'}\sinh n\beta,\\
&D_n' = A_n''e^{-n\beta}\left(\frac{\mu_0}{\mu'}\cosh n\beta + \sinh n\beta\right) + (-)^n\frac{2B''e^{-n\beta}}{n\sinh\beta}\cdot\frac{\mu_0}{\mu'}\cosh n\beta
\end{aligned}\right\} \qquad (21).$$

Solving (20) and (21) for A_n'' and A_n, we find

$$\left.\begin{aligned} A_n e^{-n\alpha} &= \frac{-(-)^n \dfrac{2Be^{-n\alpha}}{n\sinh\alpha}\cdot\dfrac{\mu_0}{\mu}\left\{\dfrac{\mu_0}{\mu'}\sinh n(\alpha+\beta)+\cosh n(\alpha+\beta)\right\}-(-)^n\dfrac{2B''e^{-n\beta}}{n\sinh\beta}\cdot\dfrac{\mu_0}{\mu'}}{\left(1+\dfrac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\dfrac{\mu_0}{\mu}+\dfrac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)}, \\ A_n'' e^{-n\alpha} &= \frac{-(-)^n \dfrac{2Be^{-n\alpha}}{n\sinh\alpha}\cdot\dfrac{\mu_0}{\mu}-(-)^n\dfrac{2B''e^{-n\beta}}{n\sinh\beta}\cdot\dfrac{\mu_0}{\mu'}\left\{\dfrac{\mu_0}{\mu}\sinh n(\alpha+\beta)+\cosh n(\alpha+\beta)\right\}}{\left(1+\dfrac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\dfrac{\mu_0}{\mu}+\dfrac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)} \end{aligned}\right\} \quad\ldots\ldots(22).$$

The current in the $\eta=-\beta$ conductor being the return current to that in the $\eta=\alpha$ conductor, we have

$$\iint w\,dx\,dy+\iint w'\,dx'\,dy'=0,$$

that is
$$w\int_{\infty}^{\alpha}\int_0^{\pi}\frac{d\xi\,d\eta}{(\cosh\eta+\cos\xi)^2}+w'\int_{-\infty}^{-\beta}\int_0^{\pi}\frac{d\xi'\,d\eta'}{(\cosh\eta'+\cos\xi')^2}=0\,;$$

now
$$\int_0^{\pi}\frac{d\xi}{(\cosh\eta+\cos\xi)^2}=-\frac{1}{\sinh\eta}\,\frac{d}{d\eta}\int_0^{\pi}\frac{d\xi}{\cosh\eta+\cos\xi},$$

therefore
$$\left.\begin{aligned}\frac{w}{\sinh^2\alpha}+\frac{w'}{\sinh^2\beta}&=0,\\ \frac{B}{\mu\sinh\alpha}+\frac{B''}{\mu'\sinh\beta}&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23),$$

the latter of the two equations being obtained from (11).

Hence from (20), (21), (22), (23), we have

$$\left.\begin{aligned} A_n &= (-)^n\frac{2B\mu_0}{n\mu\sinh\alpha}\cdot\frac{e^{n(\alpha-\beta)}-\dfrac{\mu_0}{\mu'}\sinh n(\alpha+\beta)-\cosh n(\alpha+\beta)}{\left(1+\dfrac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\dfrac{\mu_0}{\mu}+\dfrac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)},\\ A_n'' &= (-)^n\frac{2B\mu_0}{n\mu\sinh\alpha}\cdot\frac{\dfrac{\mu_0}{\mu}\sinh n(\alpha+\beta)+\cosh n(\alpha+\beta)-e^{n(\beta-\alpha)}}{\left(1+\dfrac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\dfrac{\mu_0}{\mu}+\dfrac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)},\\ A_n' &= (-)^n\frac{2B\mu_0}{n\mu\sinh\alpha}\cdot\frac{e^{-n\beta}\left(\cosh n\alpha+\dfrac{\mu_0}{\mu}\sinh n\alpha\right)-e^{-n\alpha}\left(\cosh n\beta+\dfrac{\mu_0}{\mu'}\sinh n\beta\right)}{\left(1+\dfrac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\dfrac{\mu_0}{\mu}+\dfrac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)}\\ D_n' &= (-)^n\frac{2B\mu_0}{n\mu\sinh\alpha}\cdot\frac{-e^{-n\beta}\left(\sinh n\alpha+\dfrac{\mu_0}{\mu}\cosh n\alpha\right)-e^{-n\alpha}\left(\sinh n\beta+\dfrac{\mu_0}{\mu'}\cosh n\beta\right)}{\left(1+\dfrac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\dfrac{\mu_0}{\mu}+\dfrac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)} \end{aligned}\right\}\quad\ldots\ldots\ldots(24).$$

Therefore from (11), (16), (17), (18), (19), (21) and (24), we find the expressions for H, H_0 and H' to be

$$H = A_0' + \frac{2\pi\mu wc^2}{\sinh^2\alpha} \cdot \alpha - \frac{2\pi\mu wc^2}{\sinh\alpha} \cdot \frac{\sinh(\alpha-\eta)}{\cosh\eta+\cos\xi}$$

$$+ \frac{4\pi\mu_0 wc^2}{\sinh^2\alpha} \sum_1^\infty \frac{(-)^{n-1} e^{-n\eta}}{n} \cdot \frac{e^{n(\alpha-\beta)} - \frac{\mu_0}{\mu'} \sinh n(\alpha+\beta) - \cosh n(\alpha+\beta)}{\left(1+\frac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta) + \left(\frac{\mu_0}{\mu}+\frac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)} \cos n\xi,$$

$$H_0 = A_0' + \frac{2\pi\mu_0 wc^2}{\sinh^2\alpha}\eta + \frac{4\pi\mu_0 wc^2}{\sinh^2\alpha} \sum_1^\infty \frac{(-)^{n-1}}{n} \cdot \frac{e^{-n\beta}\cos n\xi \left\{\cosh n(\alpha-\eta) + \frac{\mu_0}{\mu}\sinh n(\alpha-\eta)\right\}}{\left(1+\frac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta) + \left(\frac{\mu_0}{\mu}+\frac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)}$$

$$- \frac{4\pi\mu_0 wc^2}{\sinh^2\alpha} \sum_1^\infty \frac{(-)^{n-1} e^{-n\alpha}}{n} \cdot \frac{\cosh n(\beta+\eta) + \frac{\mu_0}{\mu'}\sinh n(\beta+\eta)}{\left(1+\frac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta) + \left(\frac{\mu_0}{\mu}+\frac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)} \cos n\xi,$$

$$H' = A_0' - \frac{2\pi\mu_0 wc^2}{\sinh^2\alpha}\beta + \frac{2\pi\mu' wc^2}{\sinh^2\alpha} \frac{\sinh\beta\sinh(\beta+\eta)}{\cosh\eta+\cos\xi}$$

$$+ \frac{4\pi\mu_0 wc^2}{\sinh^2\alpha} \sum_1^\infty \frac{(-)^{n-1} e^{n\eta}}{n} \cdot \frac{\frac{\mu_0}{\mu}\sinh n(\alpha+\beta) + \cosh n(\alpha+\beta) - e^{n(\beta-\alpha)}}{\left(1+\frac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta) + \left(\frac{\mu_0}{\mu}+\frac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)} \cos n\xi.$$

A_0' can be determined from the condition that H_0 vanishes at an infinite distance from the conductors.

4. Let I be the whole current in either conductor, then $I = \pi wc^2/\sinh^2\alpha$, and further let L be the coefficient of self-induction of the current, then

$$LI^2 = \iint Hw\,dx\,dy$$

$$= wc^2 \iint \frac{H d\xi d\eta}{(\cosh\eta+\cos\xi)^2} + w'c^2 \iint \frac{H' d\xi d\eta}{(\cosh\eta+\cos\xi)^2}$$

$$= wc^2 \int_\alpha^\infty \int_0^\pi \frac{d\xi d\eta}{(\cosh\eta+\cos\xi)^2} \left\{A_0' + \frac{2\pi\mu_0 wc^2}{\sinh^2\alpha}\alpha - \frac{2\pi\mu wc^2}{\sinh\alpha} \frac{\sinh(\alpha-\eta)}{\cosh\eta+\cos\xi} + \text{etc.}\right\}$$

$$+ w'c^2 \int_{-\beta}^{-\infty} \int_0^\pi \frac{d\xi d\eta}{(\cosh\eta+\cos\xi)^2} \{A_0' - \text{etc.}\}.$$

Now by (23)

$$wc^2 \iint \frac{d\xi d\eta}{(\cosh\eta+\cos\xi)^2} + w'c^2 \iint \frac{d\xi d\eta}{(\cosh\eta+\cos\xi)^2} = 0.$$

Again

$$wc^2 \int_\alpha^\infty \int_0^\pi \frac{2\pi\mu_0 wc^2\alpha}{\sinh^2\alpha} \frac{d\xi d\eta}{(\cosh\eta+\cos\xi)^2} - w'c^2 \int_{-\beta}^{-\infty} \int_0^\pi \frac{2\pi\mu_0 wc^2\beta}{\sinh^2\alpha} \frac{d\xi d\eta}{(\cosh\eta+\cos\xi)^2}$$
$$= 2\mu_0 I^2(\alpha+\beta),$$

also

$$\int_\alpha^\infty \int_0^\pi \frac{\sinh(\alpha-\eta)}{(\cosh\eta+\cos\xi)^3}d\xi d\eta = \int_\alpha^\infty \frac{\sinh(\alpha-\eta)}{\sinh\eta}\cdot\frac{d}{d\eta}\left\{\frac{1}{\sinh\eta}\frac{d}{d\eta}\left(\frac{\pi}{\sinh\eta}\right)\right\}d\eta = \frac{\pi}{4\sinh^3\alpha},$$

and
$$\int_\alpha^\infty \int_0^\pi \frac{e^{-n\eta}\cos n\xi d\xi d\eta}{(\cosh\eta+\cos\xi)^2} = -\int_\alpha^\infty \frac{e^{-n\eta}}{\sinh\eta}\frac{d}{d\eta}\left(\frac{(-)^n e^{-n\eta}}{\sinh\eta}\right)d\eta = \frac{(-)^n e^{-2n\alpha}}{2\sinh^2\alpha}.$$

Therefore

$$LI^2 = I^2\{2\mu_0(\alpha+\beta)+\tfrac{1}{2}(\mu+\mu')\}$$
$$+ 4\mu_0 I^2 \sum_1^\infty \frac{1}{n}\cdot\frac{(e^{-2n\alpha}+e^{-2n\beta})\cosh n(\alpha+\beta)+\left(\frac{\mu_0}{\mu}e^{-2n\beta}+\frac{\mu_0}{\mu'}e^{-2n\alpha}\right)\sinh n(\alpha+\beta)}{\left(1+\frac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\frac{\mu_0}{\mu}+\frac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)}$$
$$- 8\mu_0 I^2 \sum_1^\infty \frac{1}{n}\cdot\frac{e^{-n(\alpha+\beta)}}{\left(1+\frac{\mu_0^2}{\mu\mu'}\right)\sinh n(\alpha+\beta)+\left(\frac{\mu_0}{\mu}+\frac{\mu_0}{\mu'}\right)\cosh n(\alpha+\beta)},$$

that is

$$L = \tfrac{1}{2}(\mu+\mu')+2\mu_0(\alpha+\beta)$$
$$+ 4\mu_0\sum_1^\infty \frac{e^{n(\alpha-\beta)}\left(1+\frac{\mu_0}{\mu}\right)+e^{n(\beta-\alpha)}\left(1+\frac{\mu_0}{\mu'}\right)+e^{-n(\alpha+3\beta)}\left(1-\frac{\mu_0}{\mu}\right)+e^{-n(3\alpha+\beta)}\left(1-\frac{\mu_0}{\mu'}\right)-e^{-n(\alpha+\beta)}}{n\left\{\left(1+\frac{\mu_0}{\mu}\right)\left(1+\frac{\mu_0}{\mu'}\right)e^{n(\alpha+\beta)}-\left(1-\frac{\mu_0}{\mu}\right)\left(1-\frac{\mu_0}{\mu'}\right)e^{-n(\alpha+\beta)}\right\}}.$$

When $\mu=\mu'=\mu_0$, we have

$$L = \mu_0+2\mu_0(\alpha+\beta)+4\mu_0\sum_1^\infty\frac{e^{n(\alpha-\beta)}+e^{n(\beta-\alpha)}-2e^{-n(\alpha+\beta)}}{2ne^{n(\alpha+\beta)}}$$
$$= \mu_0+2\mu_0(\alpha+\beta)+4\mu_0\sum_1^\infty\frac{e^{-2n\beta}+e^{-2n\alpha}-2e^{-2n(\alpha+\beta)}}{2n}$$
$$= \mu_0+2\mu_0\log\frac{\sinh^2(\alpha+\beta)}{\sinh\alpha\sinh\beta};$$

now if b is the distance between the axes of the conductors, a and a' their radii, then

$$a = c\,\text{cosech}\,\alpha,\quad a'=c\,\text{cosech}\,\beta,\quad b=c\sinh(\alpha+\beta)/\sinh\alpha\sinh\beta;$$

therefore
$$L=\mu_0\left\{1+2\log\frac{b^2}{aa'}\right\};$$

and the force between the conductors tending to increase their distance apart is $2\mu_0 I^2/b$ per unit length. These results agree with those given in § 685 of Maxwell's *Electricity and Magnetism*.

5. When $\mu'=\mu_0$

$$L=\tfrac{1}{2}(\mu+\mu_0)+2\mu_0(\alpha+\beta)+4\mu_0\sum_1^\infty\frac{e^{n(\alpha-\beta)}\left(1+\frac{\mu_0}{\mu}\right)+e^{-n(\alpha+3\beta)}\left(1-\frac{\mu_0}{\mu}\right)+2e^{n(\beta-\alpha)}-4e^{-n(\alpha+\beta)}}{2n\left(1+\frac{\mu_0}{\mu}\right)e^{n(\alpha+\beta)}}$$

$$= \tfrac{1}{2}(\mu + \mu_0) + 2\mu_0(\alpha + \beta) + 4\mu_0 \sum_1^\infty \frac{e^{-2n\beta} + e^{-2n\alpha} - 2e^{-2n(\alpha+\beta)}}{2n}$$

$$+ 4\mu_0 \frac{\mu - \mu_0}{\mu + \mu_0} \cdot \sum_1^\infty \frac{e^{-2n\alpha} + e^{-n(2\alpha+4\beta)} - 2e^{-2n(\alpha+\beta)}}{2n}$$

$$= \tfrac{1}{2}(\mu + \mu_0) + 2\mu_0 \log \frac{\sinh^2(\alpha+\beta)}{\sinh\alpha \sinh\beta} + 2\mu_0 \frac{\mu - \mu_0}{\mu + \mu_0} \log \frac{\sinh^2(\alpha+\beta)}{\sinh\alpha \sinh(\alpha + 2\beta)}$$

$$= \tfrac{1}{2}(\mu + \mu_0) + 2\mu_0 \log \frac{b^2}{aa'} + 2\mu_0 \frac{\mu - \mu_0}{\mu + \mu_0} \log \frac{b^2}{b^2 - a^2}.$$

This gives L when one conductor is iron, the other being any substance whose magnetic permeability is the same as that of the surrounding medium.

The repulsive force between the conductors is

$$2\mu_0 \left(\frac{1}{b} - \frac{\mu - \mu_0}{\mu + \mu_0} \frac{a^2}{b(b^2 - a^2)}\right) I^2 \text{ per unit length.}$$

These results shew that Maxwell's formula makes L too small in this case, the error being of amount

$$2\mu_0 \frac{\mu - \mu_0}{\mu + \mu_0} \log \frac{b^2}{b^2 - a^2},$$

and makes the force between the conductors too large by an amount

$$2\mu_0 \frac{\mu - \mu_0}{\mu + \mu_0} \frac{a^2}{b(b^2 - a^2)} I^2.$$

Taking the case of conductors of equal section, the following table shews how the variable part of the coefficient of induction varies with their distance apart.

b.	$\log \frac{b^2}{aa'}$.	$\frac{\mu - \mu_0}{\mu + \mu_0} \log \frac{b^2}{b^2 - a^2}$.	Increase per cent.	$L - 50{\cdot}5$ Maxwell.	$L - 50{\cdot}5$ from above formula.
$2a$	1·38629	·282007	20·3	2·77258	3·33659
$3a$	2·19722	·117760	5·3	4·39444	4·62996
$4a$	2·77258	·063260	2·2	5·54516	5·67168
$5a$	3·21887	·039829	1·2	6·43774	6·51739
$6a$	3·58351	·027583	·7	7·16702	7·22218
$7a$	3·89164	·020211	·5	7·78328	7·82370
$8a$	4·15888	·015936	·3	8·31776	8·34963
$9a$	4·39425	·012131	·2	8·78850	8·81276
$10a$	4·60517	·009851	·2	9·21034	9·23005

The first column gives the distances between the axes of the conductors, the second the values of half the variable term in Maxwell's formula, the third half the term which has to be added to it, the fourth the increase per cent. of the variable part due to the term neglected by Maxwell, the fifth and sixth the values of the variable part of the induction in both cases; μ_0 being taken to be unity and $\mu = 100$. The table shews that the term neglected is considerable when the conductors are near one another, and decreases rapidly as they move apart at first and afterwards more slowly.

Again taking the conductors touching one another, the following table gives the maximum values of the correction as the radius of the iron conductor increases.

a.	b.	$\log \frac{b^2}{aa'}$.	$\frac{\mu-\mu_0}{\mu+\mu_0}\log\frac{b^2}{b^2-a^2}$.	Increase per cent.	$L-50{\cdot}5$ Maxwell.	$L-50{\cdot}5$ from above formula.
a'	$2a'$	1·38629	·282007	20·3	2·77258	3·33659
$2a'$	$3a'$	1·50407	·576147	38·3	3·00814	4·16043
$3a'$	$4a'$	1·67397	·810307	48·0	3·34794	4·96855
$4a'$	$5a'$	1·83257	1·001419	54·6	3·66514	5·66797
$5a'$	$6a'$	1·97407	1·162144	58·8	3·94814	6·27242
$6a'$	$7a'$	2·10005	1·300593	61·9	4·20010	6·80128
$7a'$	$8a'$	2·21297	1·422097	64·2	4·42594	7·27013
$8a'$	$9a'$	2·31447	1·530317	66·1	4·62894	7·68957
$9a'$	$10a'$	2·40794	1·627843	67·6	4·81598	8·07166
$10a'$	$11a'$	2·49320	1·716587	68·8	4·98640	8·41951

The first column expresses the radius of the iron conductor in terms of that of the other conductor; the remaining columns are as in the preceding table.

The expression for the force between the conductors

$$\frac{2\mu_0}{b}\left(1-\frac{\mu-\mu_0}{\mu+\mu_0}\,\frac{a^2}{b^2-a^2}\right)$$

can be made to change sign by choosing the radii of the conductors so that b^2 is somewhat less than $2a^2$, thus making the force attractive instead of repulsive.

It may be noticed that the part of the above formulas depending on the size of the conductors and their distance apart is but slightly altered whether we suppose μ to be 100 or 1000.

6. When $\mu=\mu'$.

$$L=\mu+2\mu_0(\alpha+\beta)$$

$$+\,4\mu_0\sum_1^\infty\frac{\left(1+\frac{\mu_0}{\mu}\right)\left(e^{n(\alpha-\beta)}+e^{n(\beta-\alpha)}\right)+\left(1-\frac{\mu_0}{\mu}\right)\left(e^{-n(\alpha+3\beta)}+e^{-n(3\alpha+\beta)}\right)-4e^{-n(\alpha+\beta)}}{n\left\{\left(1+\frac{\mu_0}{\mu}\right)^2e^{n(\alpha+\beta)}-\left(1-\frac{\mu_0}{\mu}\right)^2e^{-n(\alpha+\beta)}\right\}}:$$

putting

$$\frac{\mu-\mu_0}{\mu+\mu_0}=\lambda,$$

$$L=\mu+2\mu_0(\alpha+\beta)$$

$$+\,2\mu_0(\lambda+1)\sum^\infty\frac{e^{n(\alpha-\beta)}+e^{n(\beta-\alpha)}+\lambda\left(e^{-n(\alpha+3\beta)}+e^{-n(3\alpha+\beta)}\right)-2(\lambda+1)\,e^{-n(\alpha+\beta)}}{n\left(e^{n(\alpha+\beta)}\right)-\lambda^2e^{-n(\alpha+\beta)}}$$

$$= \mu + 2\mu_0 \log \frac{\sinh^2(\alpha+\beta)}{\sinh\alpha \sinh\beta} + 2\mu_0\lambda \log \frac{\sinh^4(\alpha+\beta)}{\sinh\alpha \sinh\beta \sinh(\alpha+2\beta)\sinh(2\alpha+\beta)}$$

$$+ 4\mu_0\lambda^2 \log \frac{\sinh(\alpha+\beta)\sinh 2(\alpha+\beta)}{\sinh(\alpha+2\beta)\sinh(2\alpha+\beta)}$$

$$+ 2\mu_0\lambda^3 \log \frac{\sinh^4 2(\alpha+\beta)}{\sinh(\alpha+2\beta)\sinh(2\alpha+\beta)\sinh(2\alpha+3\beta)\sinh(3\alpha+2\beta)}$$

$$+ 4\mu_0\lambda^4 \log \frac{\sinh 2(\alpha+\beta)\sinh 3(\alpha+\beta)}{\sinh(2\alpha+3\beta)\sinh(3\alpha+2\beta)} + \text{etc.}$$

From the relations

$$b = c\frac{\sinh(\alpha+\beta)}{\sinh\alpha\sinh\beta},$$

$$a\sinh\alpha = a'\sinh\beta = c,$$

we have

$$L = \mu + 2\mu_0 \log \frac{b^2}{aa'} + 2\mu_0\lambda \log \frac{b^4}{(b^2-a^2)(b^2-a'^2)}$$

$$+ 4\mu_0\lambda^2 \log \frac{b^2(b^2-a^2-a'^2)}{(b^2-a^2)(b^2-a'^2)}$$

$$+ 2\mu_0\lambda^3 \log \frac{b^4(b^2-a^2-a'^2)^4}{(b^2-a^2)(b^2-a'^2)\{(b^2-a^2)^2-a'^2b^2\}\{(b^2-a'^2)^2-a^2b^2\}} + \text{etc.}$$

If we take p and q, so that

$$p + \frac{1}{p} = 2\cosh(\alpha-\beta),$$

$$q + \frac{1}{q} = 2\cosh(\alpha+\beta),$$

then

$$L = \mu - 2\mu_0 \log q + 2\mu_0(\lambda+1)\sum_1^\infty \frac{\left(p^n + \frac{1}{p^n}\right)(1+\lambda q^{2n}) - 2(\lambda+1)q^n}{n\left(\frac{1}{q^n} - \lambda^2 q^n\right)},$$

where

$$p = \frac{a^2+a'^2}{2aa'} - \frac{(a^2-a'^2)}{2b^2aa'} - \frac{(a^2-a'^2)c}{baa'},$$

$$q = \frac{b^2}{2aa'} - \frac{a^2+a'^2}{2aa'} - \frac{bc}{aa'},$$

$$c = \frac{\sqrt{(b^2-a^2-a'^2)^2 - 4a^2a'^2}}{2b}.$$

The repulsive force between the conductors is

$$I^2\left\{\frac{4\mu_0 bq}{(1-q^2)aa'} + \frac{4\mu_0(\lambda+1)(a^2-a'^2)^2p^2}{b^3aa'(1-p^2)}\sum_1^\infty \frac{(1-p^{2n})(1+\lambda q^{2n})q^n}{(1-\lambda^2q^{2n})p^{n+1}}\right.$$

$$\left.\frac{4\mu_0(\lambda+1)bq^2}{aa'(1-q^2)}\sum_1^\infty \frac{4(\lambda+1)q^{2n-1}p^n - (1+p^{2n})q^{n-1}(1+3\lambda q^{2n}+\lambda^2q^{2n}-\lambda^3q^{4n})}{p^n(1-\lambda^2q^{2n})^2}\right\}.$$

IX. *Changes in the dimensions of Elastic Solids due to given systems of forces.* By C. Chree, M.A., Fellow of King's College.

[Read March 7, 1892.]

§ 1. Let e, f, g, a, b, c denote the strains, and $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$ the corresponding stresses in an elastic solid referred to a system of orthogonal Cartesian co-ordinates. Then the most general form of the stress-strain relations is:

$$\left.\begin{aligned}
\widehat{xx} &= c_{11}e + c_{12}f + c_{13}g + c_{14}a + c_{15}b + c_{16}c,\\
\widehat{yy} &= c_{21}e + c_{22}f + c_{23}g + c_{24}a + c_{25}b + c_{26}c,\\
\widehat{zz} &= c_{31}e + c_{32}f + c_{33}g + c_{34}a + c_{35}b + c_{36}c,\\
\widehat{yz} &= c_{41}e + c_{42}f + c_{43}g + c_{44}a + c_{45}b + c_{46}c,\\
\widehat{zx} &= c_{51}e + c_{52}f + c_{53}g + c_{54}a + c_{55}b + c_{56}c,\\
\widehat{xy} &= c_{61}e + c_{62}f + c_{63}g + c_{64}a + c_{65}b + c_{66}c
\end{aligned}\right\} \quad \text{..........................(1)}.$$

where the coefficients c_{rs} and c_{sr} are equal. The notation is that employed by Professor Voigt*. If the solid be homogeneous, in the sense that at every point it has the same properties along directions fixed in space, then the 21 independent coefficients appearing in (1) have everywhere constant values.

Let Π denote the determinant of 6 rows and columns formed by the 21 coefficients, and in it let C_{rs} be the minor answering to c_{rs}, the order of the suffixes being immaterial. Let strains with suffix 1, e.g. e_1, answer to $\widehat{xx} = 1$ with all the other stresses zero, strains with suffix 2 to $\widehat{yy} = 1$ with all the other stresses zero, and so on for each of the other six stresses in order.

Thus for instance answering to $\widehat{zz} = 1$, with all the other stresses zero, we have

$$\left.\begin{aligned}
e_3 = C_{13}/\Pi,\quad f_3 = C_{23}/\Pi,\quad g_3 = C_{33}/\Pi,\\
a_3 = C_{34}/\Pi,\quad b_3 = C_{35}/\Pi,\quad c_3 = C_{36}/\Pi
\end{aligned}\right\} \quad \text{..............................(2)};$$

while answering to $\widehat{yz} = 1$, with all the other stresses zero, we have

$$\left.\begin{aligned}
e_4 = C_{14}/\Pi,\quad f_4 = C_{24}/\Pi,\quad g_4 = C_{34}/\Pi,\\
a_4 = C_{44}/\Pi,\quad b_4 = C_{45}/\Pi,\quad c_4 = C_{46}/\Pi
\end{aligned}\right\} \quad \text{..............................(3)}.$$

* Cf. Wiedemann's *Annalen*, Bd. 34, p. 981, 1888.

We shall also employ the following notation:

$$\left.\begin{array}{llll} e_1 = 1/E_1, & f_2 = 1/E_2, & g_3 = 1/E_3, & \\ a_4 = 1/n_1, & b_5 = 1/n_2, & c_6 = 1/n_3, & \\ e_3/g_3 = -\eta_{31}, & f_3/g_3 = -\eta_{32}, & g_1/e_1 = -\eta_{13}, & g_2/f_2 = -\eta_{23}, \\ a_3/g_3 = -\eta_{34}, & b_3/g_3 = -\eta_{35}, & g_4/a_4 = -\eta_{43}, & g_5/b_5 = -\eta_{53} \\ \ldots\ldots & & & \end{array}\right\} \ldots\ldots\ldots(4).$$

The quantities E_1, E_2, E_3 are Young's moduli for longitudinal traction in directions parallel to the axes of x, y and z respectively; while n_1, n_2, n_3 are moduli of rigidity.

The quantities η, when the suffix does not contain 4, 5 or 6, are values of Poisson's ratio. For instance, η_{31} is the ratio of lateral contraction parallel to x to longitudinal expansion parallel to z for longitudinal traction parallel to z. The order of the suffixes is not in general immaterial in η.

§ 2. Let
$$\chi = \tfrac{1}{2}(ex^2 + fy^2 + gz^2 + ayz + bzx + cxy) \ldots\ldots\ldots(5),$$

and let the suffixes 1, ... 6 attached to the coefficients have the same significations as above. Thus for instance the coefficients in

$$2\chi_3 = e_3x^2 + f_3y^2 + g_3z^2 + a_3yz + b_3zx + c_3xy$$

are the strains answering to $\widehat{zz} = 1$, with all the other stresses zero. The quadric surface

$$\chi = \text{constant} \ldots\ldots\ldots(6)$$

is what is termed the *elongation quadric*. In general the elongation quadric varies in form from point to point of the solid, but when the strains have everywhere constant values a single form of elongation quadric shows the strain at every point. This is the case in the present applications, and we shall suppose the quadric to have its centre at the origin of co-ordinates and may regard its dimensions to alter so as to enable any point we choose to lie on its surface. When the strain is pure, as in the present applications, and is also small, as is required for a legitimate application of the elastic solid equations, the displacements α, β, γ at any point may be derived as follows. Take the elongation quadric (6), where χ has the form (5), supposing its centre at the origin of co-ordinates and its magnitude such that it passes through the point in question, then

$$\alpha = \frac{d\chi}{dx}, \quad \beta = \frac{d\chi}{dy}, \quad \gamma = \frac{d\chi}{dz}.$$

This may be at once verified, as it obviously gives

$$\frac{d\alpha}{dx} = e, \ \ldots \ \frac{d\alpha}{dy} + \frac{d\beta}{dx} = c, \quad \frac{d\alpha}{dy} - \frac{d\beta}{dx} = 0, \text{ etc.}$$

The physical meaning is that the direction of the resultant displacement is along the normal to the elongation quadric.

Answering to $\widehat{zz}=1$, and all the other stresses zero, we have

$$\left.\begin{aligned}\alpha_3 &= \frac{d\chi_3}{dx} = -\frac{1}{E_3}(\eta_{31}x + \tfrac{1}{2}\eta_{36}y + \tfrac{1}{2}\eta_{35}z),\\ \beta_3 &= \frac{d\chi_3}{dy} = -\frac{1}{E_3}(\tfrac{1}{2}\eta_{36}x + \eta_{32}y + \tfrac{1}{2}\eta_{34}z),\\ \gamma_3 &= \frac{d\chi_3}{dz} = \frac{1}{E_3}(-\tfrac{1}{2}\eta_{35}x - \tfrac{1}{2}\eta_{34}y + z)\end{aligned}\right\}\quad\ldots\ldots(7).$$

Similarly when $\widehat{yz}=1$ and all the other stresses are zero,

$$\left.\begin{aligned}\alpha_4 &= \frac{d\chi_4}{dx} = -\frac{1}{n_1}(\eta_{41}x + \tfrac{1}{2}\eta_{46}y + \tfrac{1}{2}\eta_{45}z),\\ \beta_4 &= \frac{d\chi_4}{dy} = \frac{1}{n_1}(-\tfrac{1}{2}\eta_{46}x - \eta_{42}y + \tfrac{1}{2}z),\\ \gamma_4 &= \frac{d\chi_4}{dz} = \frac{1}{n_1}(-\tfrac{1}{2}\eta_{45}x + \tfrac{1}{2}y - \eta_{43}z)\end{aligned}\right\}\quad\ldots\ldots(8).$$

The values of E's, n's and η's may all be expressed as above in terms of the 21 elastic constants occurring in (1).

§ 3. There is another case we require to consider, viz. when there is everywhere a uniform normal tension equal to 1. In this case

$$\widehat{xx} = \widehat{yy} = \widehat{zz} = 1, \quad \widehat{yz} = \widehat{zx} = \widehat{xy} = 0.$$

Let the suffix 0 distinguish the corresponding strains and the corresponding form of χ. Then by (1)

$$\left.\begin{aligned}e_0 &= (C_{11}+C_{12}+C_{13})/\Pi, & f_0 &= (C_{12}+C_{22}+C_{23})/\Pi, & g_0 &= (C_{13}+C_{23}+C_{33})/\Pi,\\ a_0 &= (C_{14}+C_{24}+C_{34})/\Pi, & b_0 &= (C_{15}+C_{25}+C_{35})/\Pi, & c_0 &= (C_{16}+C_{26}+C_{36})/\Pi\end{aligned}\right\}\quad\ldots\ldots(9).$$

The corresponding uniform dilatation Δ_0 is given by

$$\Delta_0 = e_0 + f_0 + g_0 = 1/k \quad\ldots\ldots(10),$$

where

$$k = \Pi/(C_{11}+C_{22}+C_{33}+2C_{12}+2C_{13}+2C_{23}) \quad\ldots\ldots(11).$$

From its physical meaning k, the bulk modulus, is necessarily an invariant whatever be the directions of the co-ordinate axes.

§ 4. Let X, Y, Z denote the component bodily forces at any point *per unit of volume* (including the reversed effective forces

$$-\rho\frac{d^2\alpha}{dt^2},\ -\rho\frac{d^2\beta}{dt^2},\ -\rho\frac{d^2\gamma}{dt^2}$$

where there is vibratory motion), and let F, G, H be the component surface forces per unit of surface. Then the bodily and surface equations in the elastic solid are each 3 in number, of the respective types:

$$\left.\begin{aligned}X &= -\left(\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz}\right),\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}\right\}\quad\ldots\ldots(12),$$

$$\left.\begin{aligned}F &= \lambda\widehat{xx} + \mu\widehat{xy} + \nu\widehat{xz},\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}\right\}\quad\ldots\ldots(13),$$

where λ, μ, ν are the direction cosines of the outwardly-directed normal at points on the surface, or surfaces if there be more than one. The strain energy W per unit volume at any point of the solid is a quadratic function of the 6 strains, and is obtained in terms of the strains by substituting for the stresses from (1) in

$$W = \tfrac{1}{2}\,(e\widehat{xx} + f\widehat{yy} + g\widehat{zz} + a\widehat{yz} + b\widehat{zx} + c\widehat{xy}).$$

Suppose that a second system of bodily and surface forces acting on the same solid, with the accompanying displacements, strains, stresses and energy, are given by dashed letters, $X'\ldots$, $F'\ldots$, $\alpha'\ldots$, $e'\ldots$, $\widehat{xx}'\ldots$, W'. Then Professor Betti* has established the equality of the following four expressions for any two systems of force:

$$\iiint (X\alpha' + Y\beta' + Z\gamma')\,dxdydz + \iint (F\alpha' + G\beta' + H\gamma')\,dS \ldots\ldots\ldots\ldots\ldots\text{(I)},$$

$$\iiint (\widehat{xx}e' + \widehat{yy}f' + \widehat{zz}g' + \widehat{yz}a' + \widehat{zx}b' + \widehat{xy}c')\,dxdydz \ldots\ldots\ldots\ldots\ldots\text{(II)},$$

$$\iiint (\widehat{xx}'e + \widehat{yy}'f + \widehat{zz}'g + \widehat{yz}'a + \widehat{zx}'b + \widehat{xy}'c)\,dxdydz \ldots\ldots\ldots\ldots\ldots\text{(III)},$$

$$\iiint (X'\alpha + Y'\beta + Z'\gamma)\,dxdydz + \iint (F'\alpha + G'\beta + H'\gamma)\,dS \ldots\ldots\ldots\ldots\text{(IV)}.$$

The volume integrals are taken throughout the entire volume occupied by the solid, and the surface integrals over its entire surface, or surfaces if there be more than one. Professor Betti's mode of proof is very simple. Multiply the equations (12) by α', β', γ' respectively. Then integrating the right-hand sides by parts, using (13) and adding, we at once establish the identity of (I) and (II). Then remembering that

$$\widehat{xx} = \frac{dW}{de}\ldots,\quad \widehat{yz} = \frac{dW}{da}\ldots,\quad \widehat{xx}' = \frac{dW'}{de'}\ldots,\quad \widehat{yz}' = \frac{dW'}{da'}\ldots,$$

and that W and W' are quadratic functions of the strains possessed of the same coefficients, we deduce the equality of (II) and (III). Then (III) bears to (IV) the same relation that (II) bears to (I). The equality of (I) and (III), with the reversed effective forces supposed zero, is the relation that is made use of here.

§ 5. In passing, attention may be called to the relation that exists when we suppose the two systems of applied forces the same, so that the dashed and undashed letters are equal. Then (I) gives the work done by the applied bodily and surface forces acting through the displacements answering to the position of statical equilibrium, while (II) represents double the work done by the elastic stresses as the strains increase from zero to their equilibrium values. If then the applied forces suddenly commence to act, the work they have done up to the instant when the body passes through that position of strain which answers to final equilibrium—assuming all elements to reach this position simultaneously—is double the work done by the stresses. Thus the energy communicated to the solid is at this instant half potential energy of strain and half kinetic energy of motion.

* *Annali di Matematica Pura ed Applicata*, Ser. II. Tomo VI. pp. 102–3.

§ 6. The use that is to be made here of the equality of (I) and (III) is in determining the mean values, throughout the volume of an elastic solid, of the equilibrium strains and dilatation answering to any assigned system of bodily and surface forces. Suppose, for instance, we wish to find the mean value of the strain g when the forces X, Y, Z, F, G, H are given, then we have only to put $\widehat{zz}'$ in (III) equal to 1, with all the other stresses zero, and to substitute in (I) the corresponding displacements from (7). Thus representing this mean value by $\bar{g}$, and denoting by v the volume of the solid, we have

$$v\bar{g} \equiv \iiint g\,dx\,dy\,dz = \iiint\left(X\frac{d\chi_3}{dx} + Y\frac{d\chi_3}{dy} + Z\frac{d\chi_3}{dz}\right)dx\,dy\,dz + \iint\left(F\frac{d\chi_3}{dx} + G\frac{d\chi_3}{dy} + H\frac{d\chi_3}{dz}\right)dS \quad \text{..............(14)},$$

where the volume integral is taken throughout the whole space occupied by the material, and the surface integral over its entire surface or surfaces. Sometimes it is convenient to retain the χ, but in other cases it is better to insert at once the expressions for the displacements. Thus we have

$$E_3 v\bar{g} = \iiint[X(-\eta_{31}x - \tfrac{1}{2}\eta_{36}y - \tfrac{1}{2}\eta_{35}z) + Y(-\tfrac{1}{2}\eta_{36}x - \eta_{32}y - \tfrac{1}{2}\eta_{34}z) + Z(-\tfrac{1}{2}\eta_{35}x - \tfrac{1}{2}\eta_{34}y + z)]\,dx\,dy\,dz$$
$$+ \iint [F(\qquad\qquad) + G(\qquad\qquad) + H(\qquad\qquad)]\,dS \quad \text{..................(15)},$$

where the coefficients of F, G and H are respectively the same as those of X, Y and Z. Similarly for the mean value $\bar{a}$ of the shearing strain a, putting $\widehat{yz}' = 1$ and all the other stresses zero in (III), and substituting the corresponding displacements from (8) in (I), we find

$$v\bar{a} = \iiint\left(X\frac{d\chi_4}{dx} + Y\frac{d\chi_4}{dy} + Z\frac{d\chi_4}{dz}\right)dx\,dy\,dz + \iint\left(F\frac{d\chi_4}{dx} + G\frac{d\chi_4}{dy} + H\frac{d\chi_4}{dz}\right)dS \quad \text{.........(16)},$$

or

$$n_1 v\bar{a} = \iiint[X(-\eta_{41}x - \tfrac{1}{2}\eta_{46}y - \tfrac{1}{2}\eta_{45}z) + Y(-\tfrac{1}{2}\eta_{46}x - \eta_{42}y + \tfrac{1}{2}z) + Z(-\tfrac{1}{2}\eta_{45}x + \tfrac{1}{2}y - \eta_{43}z)]\,dx\,dy\,dz$$
$$+ \iint [F(\qquad\qquad) + G(\qquad\qquad) + H(\qquad\qquad)]\,dS \quad \text{..................(17)}.$$

For the mean value $\bar{\Delta}$ of the dilatation

$$\Delta \equiv e + f + g$$

we put $$\widehat{xx}' = \widehat{yy}' = \widehat{zz}' = 1,$$

and $$\widehat{yz}' = \widehat{zx}' = \widehat{xy}' = 0$$

in (III), and substitute in (I) the corresponding displacements from (9). Also we notice

$$v\bar{\Delta} \equiv \iiint \Delta\,dx\,dy\,dz = \delta v \quad \text{......................................(18)},$$

where δv is the increase in the whole volume occupied by the solid. Thus we find

$$v\Delta = \delta v = \iiint\left(X\frac{d\chi_0}{dx} + Y\frac{d\chi_0}{dy} + Z\frac{d\chi_0}{dz}\right)dxdydz$$
$$+\iint\left(F\frac{d\chi_0}{dx} + G\frac{d\chi_0}{dy} + H\frac{d\chi_0}{dz}\right)dS \quad\text{.........................(19)},$$

or $$\delta v = \iiint[X(e_0x + \tfrac{1}{2}c_0y + \tfrac{1}{2}b_0z) + Y(\tfrac{1}{2}c_0x + f_0y + \tfrac{1}{2}a_0z) + Z(\tfrac{1}{2}b_0x + \tfrac{1}{2}a_0y + g_0z)]\,dxdydz$$
$$+\iint[F(\qquad) + G(\qquad) + H(\qquad)]\,dS \quad\text{...(20)},$$

where $e_0 \dots c_0$ are given by (9).

For the case of isotropy the expressions for the mean strains are of course much simpler. Thus

$$E v\bar{g} = \iiint\{Zz - \eta(Xx + Yy)\}\,dxdydz + \iint\{Hz - \eta(Fx + Gy)\}\,dS \quad\text{............(21)},$$

$$nv\bar{a} = \tfrac{1}{2}\iiint(Yz + Zy)\,dxdydz + \tfrac{1}{2}\iint(Gz + Hy)\,dS \quad\text{..............................(22)},$$

$$3k\delta v = \iiint(Xx + Yy + Zz)\,dxdydz + \iint(Fx + Gy + Hz)\,dS \quad\text{..................(23)}.$$

The mean values of the strains in the case of isotropy for given surface forces—i.e. results such as (21) and (22) with $X = Y = Z = 0$—were given I believe by Professor Betti* in his original paper. But this I have unfortunately been unable to consult. I may add that I arrived quite independently at (23) and (20) when unacquainted with Professor Betti's results, having been led to their discovery by what seemed a curious coincidence in the expressions for the changes of volume produced by rotation in certain solids (see (32) below).

§ 7. One very general result as regards the *mean strains*—as we may call $\bar{e}, \dots \bar{a}, \dots \bar{\Delta}$—is obvious from the formulae containing the functions χ. Taking, for instance, the strain g, we see from (14) that $\bar{g}$ vanishes if

$$\left.\begin{aligned} X\frac{d\chi_3}{dx} + Y\frac{d\chi_3}{dy} + Z\frac{d\chi_3}{dz} = 0,\\ F\frac{d\chi_3}{dx} + G\frac{d\chi_3}{dy} + H\frac{d\chi_3}{dz} = 0 \end{aligned}\right\} \quad\text{..............................(24)}.$$

This signifies that if the resultant of the applied forces at every point, both in the interior and at the surface, lies in the tangent plane at the point to the elongation quadric, for the stress $\widehat{zz} = 1$ with all the other stresses zero, which passes through the point and has its centre at the origin, then the mean strain $\bar{g}$ vanishes. A similar result applies for each of the other mean strains. These results obviously follow from the property of the elongation quadric mentioned above in § 2. Attention may specially be called to the fact that (23) implies that the change of volume in an isotropic solid vanishes when the bodily and surface forces have their resultant at every point perpendicular to the radius from the origin.

* *Nuovo Cimento*, 1872.

§ 8. In some cases the formulae for the mean strains can be put into neater forms. For instance, if the applied surface forces be everywhere normal to the surface of the solid, then denoting the normal force by N and its direction by $\mathfrak{n}$, we have

$$\iint\left(F\frac{d\chi_i}{dx}+G\frac{d\chi_i}{dy}+H\frac{d\chi_i}{dz}\right)dS=\iint N\frac{d\chi_i}{d\mathfrak{n}}\,dS \qquad (25).$$

Again, if the bodily forces be derived from a potential V, we obtain, noticing that

$$\frac{d^2\chi_i}{dx^2}=e_i, \text{ etc.},$$

$$\iiint\left(X\frac{d\chi_i}{dx}+Y\frac{d\chi_i}{dy}+Z\frac{d\chi_i}{dz}\right)dxdydz$$

$$=\iint V\frac{d\chi_i}{d\mathfrak{n}}\,dS-\iiint V(e_i+f_i+g_i)\,dxdydz \qquad (26),$$

$$=\iint \chi_i\frac{dV}{d\mathfrak{n}}\,dS-\iiint \chi_i\nabla^2 V dxdydz \qquad (27).$$

The form (26) might prove convenient when the surface of the solid is an equipotential surface for the bodily forces. In applying it to determine the change of volume the relation (10) should be noticed. The form (27) seems likely to prove convenient when V is the potential arising from gravitational forces whose origin lies outside S, for the volume integral would then vanish since $\nabla^2 V=0$.

In the case of the change of volume in isotropy we may replace the volume integral in (23) when a potential V exists by

$$\iint pVdS-3\iiint Vdxdydz,$$

where p is the perpendicular from the origin on the tangent plane to the surface of the solid.

§ 9. Owing to their physical meaning the expressions (I)—(IV) must remain equal however their forms may be altered by changes in the system of coordinates. We may for instance suppose the forces, displacements, strains and stresses occurring therein to refer to any set of orthogonal coordinates,—such for instance as r, θ, ϕ in polars—and may thus, at least in some cases of isotropy, determine the mean values of the corresponding strains throughout the solid. In an aeolotropic material, such as (1) refers to, the constants in the stress-strain relations in coordinates other than Cartesians would vary from point to point, owing to the variation of the directions of the coordinate axes. There may however be some solids in which the values of the elastic constants are the same at different points not for parallel systems of axes as in (1), but for some other orthogonal system. And it is conceivable that in some such cases the mean values of strains referred to this orthogonal system may be obtained by means of the equality of (I) and (III).

Determination of the compressibility.

§ 10. In an isotropic solid we may by means of (23) determine the bulk-modulus, and so the compressibility, by measuring the change of volume produced by any known system of forces in a body of any shape. Suppose, for instance, a block of the material to rest on a perfectly smooth plane and to be subjected to vertical pressure over its upper surface, supposed horizontal. Taking the plane xy through the base of the block, with the origin at any convenient point, and supposing the upper surface at a height h above this, we find from (23), denoting the total pressure by P,

$$-3k\delta v = Ph \qquad (28).$$

If the block have a uniform horizontal section, and p be the mean pressure per unit of area of the upper surface, this becomes

$$-\delta v/v = p/3k \qquad (29).$$

Thus for a given total pressure, δv increases with h, but for a given pressure per unit of surface $\delta v/v$ is independent of h.

§ 11. Since no plane is absolutely smooth it would appear desirable in practice to have the base of the block as small as is consistent with the stress-strain relations remaining everywhere linear, so as to make the value of

$$\iint (Fx + Gy)\, dS$$

taken over the base as small as possible. The general effect of these frictional forces is easily traced, at least in a block of regular shape. Under vertical pressure the solid tends to expand horizontally, and this the frictional forces on the base must oppose. Thus supposing the origin at the C. G. of the base, the frictional forces are on the whole directed towards the origin, or $Fx + Gy$ is negative. Thus the surface integral would add numerically to the right-hand side of (28), and so its omission makes the calculated value of $(-3k\delta v)$ too small. The value of k deduced from (28) and the observed value of $(-\delta v)$ would consequently be too small also. Another source of error would be the want of absolute rigidity in the supporting plane, in consequence of which the points of application of the large surface forces H on the base would not all lie in the plane $z = 0$. This error would be minimised by taking the height of the block great.

§ 12. In any aeolotropic solid the bulk-modulus may be determined as follows. Cut a rectangular block out of the material with its edges l_1, l_2, l_3 in any orthogonal directions. Place it on a smooth unyielding plane with an edge, say l_3, vertical and apply symmetrically a total pressure P_3 over the upper face, measuring the corresponding reduction $(-\delta v_3)$ in volume. Repeat the experiment with the edges l_2 and l_1 successively vertical, applying total pressures P_2 and P_1, and determine the corresponding reductions in volume $(-\delta v_2)$ and $(-\delta v_1)$. Now the origin being at the C. G. of the base, the axis of z vertically upwards, and the pressure being symmetrically applied, it is clear that

$$\iint \tfrac{1}{2} H (xb_0 + ya_0)\, dS$$

vanishes over both faces. Thus we easily deduce

$$-\delta v_3 = l_3 P_3 g_0, \quad -\delta v_2 = l_2 P_2 f_0, \quad -\delta v_1 = l_1 P_1 e_0,$$

where e_0, f_0, g_0 are given by (9). Whence by means of (10) we obtain

$$-\{\delta v_1/l_1P_1 + \delta v_2/l_2P_2 + \delta v_3/l_3P_3\} = e_0 + f_0 + g_0 = 1/k \quad \text{.....................(30).}$$

If in each case we have the same mean pressure p per unit area of face, this becomes

$$-(\delta v_1 + \delta v_2 + \delta v_3)/v = p/k \quad \text{..... ....(31).}$$

Rotating Bodies.

§ 13. Suppose a homogeneous elastic solid to rotate with uniform angular velocity ω about a principal axis of inertia through its C. G., and to be exposed to no forces other than the "centrifugal forces". This motion is dynamically possible, i.e. no constraint is required to preserve the direction of the axis of rotation or to prevent the body travelling off into space. Taking the axis of rotation for axis of x and denoting the density as previously by ρ, we have

$$Y/y = Z/z = \omega^2\rho, \quad X = 0, \quad \text{and} \quad F = G = H = 0.$$

Substituting in (23), we find for any isotropic body

$$3k\delta v = \iiint \omega^2 \rho \,(y^2 + z^2)\, dx\, dy\, dz,$$

or

$$\delta v = \omega^2 I/3k \quad \text{...(32),}$$

where I is the moment of inertia about the axis of rotation. The value of k might of course be deduced by means of this formula, supposing it possible to measure δv.

In the case of an aeolotropic solid, free from surface forces and rotating about a principal axis through the C. G., let us take this axis for that of x, and let the axes of y and z be the two other principal axes at the C. G. Then denoting the angular velocity by ω_1 and the increase in volume by δv_1, we find from (20)

$$\delta v_1 = \iiint \omega_1^2 \rho\,(f_0 y^2 + g_0 z^2)\, dx\, dy\, dz$$

$$= \omega_1^2\,(B' f_0 + C' g_0) \quad \text{...(33),}$$

where A', B' and C' are the moments of inertia with respect to the planes yz, zx and xy. Similarly let δv_2 and δv_3 be the increases in volume when the body rotates with angular velocities ω_2 and ω_3 about the axes of y and z respectively, then

$$\delta v_2 = \omega_2^2\,(A' e_0 + C' g_0), \quad \delta v_3 = \omega_3^2\,(A' e_0 + B' f_0) \quad \text{..........................(34).}$$

Thus we obtain

$$1/k = e_0 + f_0 + g_0 = \tfrac{1}{2}\left\{\left(\frac{1}{C'} + \frac{1}{B'} - \frac{1}{A'}\right)\frac{\delta v_1}{\omega_1^2}\right.$$

$$\left. + \left(\frac{1}{A'} + \frac{1}{C'} - \frac{1}{B'}\right)\frac{\delta v_2}{\omega_2^2} + \left(\frac{1}{B'} + \frac{1}{A'} - \frac{1}{C'}\right)\frac{\delta v_3}{\omega_3^2}\right\} \quad \text{..................(35).}$$

If the body be a sphere of radius R and the three angular velocities be equal, this simplifies to

$$(\delta v_1 + \delta v_2 + \delta v_3)/v = 2\omega^2 \rho R^2/5k \quad \text{.................................(36).}$$

§ 14. The form of rotating body for which the present method supplies most information appears to be a right cylinder, including the right prism. Let the axis of the cylinder be axis of z, the origin being at the middle point, and let the axes of x and y be the two principal axes of the cross section. Denote the area of the cross section by σ and its principal radii of gyration by κ_1 and κ_2, so that

$$\sigma\kappa_1^2 = \iint y^2 dx dy, \quad \sigma\kappa_2^2 = \iint x^2 dx dy \quad \text{.................................(37),}$$

where the integrals are taken over the cross section.

The increments δv_1, δv_2 and δv_3 in the volume $v, \equiv 2l\sigma$, where $2l$ is the length, when the cylinder rotates with angular velocities ω_1, ω_2 and ω_3 about the axes of x, y and z respectively are, by (20) and (37),

$$\delta v_1/\omega_1^2 \rho v = f_0\kappa_1^2 + g_0 l^2/3, \quad \delta v_2/\omega_2^2 \rho v = e_0\kappa_2^2 + g_0 l^2/3, \quad \delta v_3/\omega_3^2 \rho v = e_0\kappa_2^2 + f_0\kappa_1^2 \quad \text{...........(38),}$$

from which k can be found as in (35). For the case of isotropy

$$\delta v_1 = \frac{\omega_1^2 \rho v}{3k}\left(\kappa_1^2 + \frac{l^2}{3}\right), \quad \delta v_2 = \frac{\omega_2^2 \rho v}{3k}\left(\kappa_2^2 + \frac{l^2}{3}\right), \quad \delta v_3 = \frac{\omega_3^2 \rho v}{3k}(\kappa_1^2 + \kappa_2^2) \quad \text{...........(39).}$$

Thus in isotropy, when $\omega_1 = \omega_2 = \omega_3 = \omega$, we have

$$\delta v_1 + \delta v_2 - \delta v_3 = 2(\omega l)^2 \rho v/9k \quad \text{....................................(40),}$$

a relation wholly independent of the shape of the cross section, and which in the case of a very thin disk approximates to the form

$$\delta v_1 + \delta v_2 = \delta v_3 \quad \text{...(41).}$$

§ 15. In the case of any right cylinder we may find the mean change in the length, or what in a thin disk is called the thickness.

For (15) gives the value of

$$v\bar{g} = \iiint g\, dx dy dz \equiv \iiint \frac{d\gamma}{dz}\, dx dy dz \quad \text{................................(42),}$$

taken throughout the volume. But the axis of z being along the axis of the cylinder, this is simply $2\sigma\overline{\delta l}$, where

$$\overline{\delta l} = \sigma^{-1} \iint \gamma_l\, dx dy \quad \text{...(43)}$$

is the mean, taken over the cross section, of the increments in the half length l. Let now the cylinder rotate with angular velocity ω_1 about the axis of x, taken as before along a principal axis of the cross section, then substituting in (15)

$$X = 0 \quad Y/y = Z/z = \omega_1^2 \rho,$$

we find, calling the mean increment in the length of the half axis $\overline{\delta l}_1$,

$$2\sigma\overline{\delta l}_1 = (\omega_1^2\rho/E_3)\iiint(-\eta_{32}y^2 + z^2)\,dx\,dy\,dz,$$

or

$$\overline{\delta l}_1/l = \omega_1^2\rho\,(\tfrac{1}{3}l^2 - \eta_{32}\kappa_1^2)/E_3 \quad\ldots\ldots\ldots\ldots(44).$$

Similarly if $\overline{\delta l}_2$ and $\overline{\delta l}_3$ be the mean increments in the half length of the cylinder for angular velocities ω_2 and ω_3 about the second principal axis of the cross section and the axis of the cylinder respectively, we find

$$\left.\begin{aligned}\overline{\delta l}_2/l &= \omega_2^2\rho\,(\tfrac{1}{3}l^2 - \eta_{31}\kappa_2^2)/E_3,\\ \overline{\delta l}_3/l &= -\,\omega_3^2\rho\,(\eta_{31}\kappa_2^2 + \eta_{32}\kappa_1^2)/E_3\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(45).$$

In any case supposing $\omega_1 = \omega_2 = \omega_3 = \omega$, we find

$$(\overline{\delta l}_1 + \overline{\delta l}_2 - \overline{\delta l}_3)/l = 2\,(\omega l)^2\rho/3E_3 \quad\ldots\ldots\ldots\ldots(46),$$

a very simple relation which for a very thin disk approximates to the form

$$\overline{\delta l}_1 + \overline{\delta l}_2 = \overline{\delta l}_3 \quad\ldots\ldots\ldots\ldots(47).$$

§ 16. When the cylinder rotates about its axis of figure its mean length is certainly reduced when η_{31} and η_{32} are both positive. There is however no reason why one at least of these constants should not be negative in some forms of aeolotropy, for at least some combinations of orthogonal directions. If η_{32} be negative rotation about the axis of x always increases the mean length, and if η_{31} be negative rotation about the axis of y always increases the mean length. But when these quantities are positive the mean length is diminished by rotation about the axis of x when

$$l < \kappa_1\sqrt{3\eta_{32}} \quad\ldots\ldots\ldots\ldots(48),$$

and by rotation about the axis of y when

$$l < \kappa_2\sqrt{3\eta_{31}} \quad\ldots\ldots\ldots\ldots(49).$$

In the case of isotropy $\eta_{31} = \eta_{32} = \eta$, and η would appear to be essentially positive. In a circular isotropic cylinder of radius R, assuming uniconstant isotropy, i.e. $\eta = 1/4$, we find the mean length increased or diminished by rotation about a diameter of the central normal section according as

$$l/R > \text{ or } > 3/\sqrt{48}, \text{ i.e. } 3{\cdot}7 \text{ approximately.}$$

When an isotropic cylinder rotates round its axis, the changes in the volume and in the mean length are connected by a very simple relation, the same for all forms of cross section, viz.

$$(1 - 2\eta)(\overline{\delta l}_3/l) + \eta\,(\delta v_3/v) = 0 \quad\ldots\ldots\ldots\ldots(50).$$

It is also worthy of notice that ultimately in a very thin circular isotropic disk the reduction in the mean thickness is twice as great when it rotates round its axis as when it rotates round a diameter, the angular velocity being the same in the two cases.

§ 17. In the case of any rotating right cylinder we may find the mean change $\bar{\delta\sigma}$ in the area σ of the cross sections by combining the previous data. For $v = 2\sigma l$, so that

$$\bar{\delta\sigma}/\sigma = \bar{\delta v}/v - \bar{\delta l}/l \qquad (51),$$

where the mean values refer to any one case of rotation.

For instance, when an isotropic cylinder rotates first about a principal diameter of the central section, and then about its axis of figure we obtain

$$\bar{\delta\sigma_1}/\sigma = \omega_1^2\rho \left\{\tfrac{1}{3}\left(\frac{1}{3k} - \frac{1}{E}\right) l^2 + \left(\frac{1}{3k} + \frac{\eta}{E}\right)\kappa_1^2\right\}$$

$$= \omega_1^2\rho\,\{-2\eta l^2/3 + (1-\eta)\,\kappa_1^2\}/E \qquad (52),$$

$$\bar{\delta\sigma_3}/\sigma = \omega_3^2\rho \left(\frac{1}{3k} + \frac{\eta}{E}\right)(\kappa_1^2 + \kappa_2^2)$$

$$= \omega_3^2\rho\,(1-\eta)(\kappa_1^2 + \kappa_2^2)/E \qquad (53).$$

The last result it will be noticed is independent of the length of the cylinder. Since every cross section of an isotropic circular cylinder rotating round its axis must remain circular, we may deduce the mean change in the radii of the cross sections from the equation

$$\bar{\delta R}/R = \tfrac{1}{2}\bar{\delta\sigma}/\sigma \qquad (54).$$

When a cylinder rotates about a diameter of the central section the alteration of a radius in any given cross section depends on its inclination to the axis of rotation.

§ 18. In the case of rotating rectangular parallelepipeds certain additional results of interest are easily obtained.

We shall confine our attention to isotropic materials.

Thus suppose the rectangular parallelepiped $2a \times 2b \times 2c$ to rotate about the axis $2c$, taken as axis of z. Then we find the mean change $2\bar{\delta a}$ in the dimension $2a$, supposed parallel to x, from the formula

$$E\iiint \frac{d\alpha}{dx}\,dx\,dy\,dz = \iiint \omega^2\rho\,(x^2 - \eta y^2)\,dx\,dy\,dz,$$

whence

$$\bar{\delta a}/a = \omega^2\rho\,(a^2 - \eta b^2)/3E \qquad (55).$$

Thus this dimension has its mean value increased or diminished according as

$$a/b > \text{ or } < \sqrt{\eta} \qquad (56).$$

The tendency to increase in length in a material line perpendicular to the axis of rotation will thus become reversed when the dimension which is at right angles both to it and to the axis of rotation is sufficiently increased.

Consider next the rectangular parallelepiped $2a \times 2a \times 2c$, one cross section of which, supposed parallel to xy, is a square. Any diameter in the central section xy is a principal axis of inertia, and so may serve for an axis of rotation without the existence of constraints. Take then for axis of rotation a diameter inclined at an angle θ_1 to the axis of x, supposed parallel to an edge 2a, the axis of z being as stated above parallel to 2c. We then have

$$X/\sin\theta_1 = -Y/\cos\theta_1 = \omega^2\rho\,(x\sin\theta_1 - y\cos\theta_1),\quad Z = \omega^2\rho z.$$

Thence we easily deduce for the mean change in the dimension 2a parallel to x—i.e. inclined at an angle θ_1 to the axis of rotation—

$$\overline{\delta a}/a = \omega^2\rho\,\{a^2\sin^2\theta_1 - \eta\,(c^2 + a^2\cos^2\theta_1)\}/3E \quad\ldots\ldots\ldots\ldots\ldots\ldots(57).$$

Thus $\overline{\delta a}$ increases algebraically as θ_1 increases from 0 to $\pi/2$. The mean alteration in the dimension 2c perpendicular to the axis of rotation is easily shown to be independent of θ_1.

Finally consider the cube 2a × 2a × 2a. Here any line through the centre is a principal axis and may serve as an axis of rotation without the application of constraints. Take for coordinate axes the three perpendiculars from the centre O on the faces, and for axis of rotation a line whose direction cosines relative to Ox, Oy, Oz are respectively $\cos\theta_1$, $\cos\theta_2$ and $\cos\theta_3$. Then

$$X = \omega^2\rho\,(x\sin^2\theta_1 - y\cos\theta_1\cos\theta_2 - z\cos\theta_1\cos\theta_3),$$

and the other components of the bodily forces may be written down from symmetry. Employing these values for the component forces, it is easy to find the expression for the mean change in the dimension 2a parallel to Ox, and it may be reduced to the simple form

$$\overline{\delta a}/a = \omega^2\rho a^2\,\{1 - \eta - (1+\eta)\cos^2\theta_1\}/3E \quad\ldots\ldots\ldots\ldots\ldots\ldots(58).$$

The mean change in a dimension parallel to an edge thus depends solely on the angular velocity and on the inclination of the edge to the axis of rotation. Attention may be specially called to the cone of semi-vertical angle

$$\theta_1 = \cos^{-1}\{\sqrt{(1-\eta)/(1+\eta)}\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(59),$$

whose axis is the perpendicular from the centre on two opposite faces. Its generators have the property that when they act as axes of rotation the mean dimension parallel to the axis of the cone is unaltered.

§ 19. To enable a solid to continue rotating about any axis other than a principal axis through its C.G. some constraint must exist. When the axis of rotation is excentric—i.e. does not pass through the C.G.—there must be pressures between the axle and its supports balancing the "centrifugal force" of the mass supposed collected at the C.G. This implies the existence of terms in the surface integrals in (20) and (23). If everything be symmetrical about a plane through the C.G. perpendicular to the axis of rotation, it is obvious from symmetry that if we take this axis for that of z, and neglect friction parallel to z on the axle, the surface force H at the bearings will vanish. If further the diameter of the axle be small compared to diameters of the body perpendicular to the axis of rotation, the coordinates x and y in the surface integrals may be treated as small quantities, and for a first approximation the surface integrals may be neglected. In such a case formula (32) gives as before the change of volume in an isotropic body, but the moment of inertia round the axis of rotation is of course greater than about a parallel axis through the C.G. Thus if κ be the radius of gyration about a parallel to the axis of rotation through the C.G., and $\bar{y}$ be the perpendicular from the C.G. on this axis, we have

$$\delta v = \omega^2 \rho v (\kappa^2 + \bar{y}^2)/3k = \delta v_0 \{1 + (\bar{y}/\kappa)^2\} \quad \text{.........................(60)},$$

where δv_0 is the change of volume for rotation with the same angular velocity about a parallel axis through the C.G. Thus while a displacement of the C.G. from the axis of rotation has but little effect so long as it is small compared to κ, it is most important when comparable with κ.

§ 20. In an aeolotropic solid of form symmetrical with respect to the plane through the C.G. perpendicular to the axis of rotation we in like manner obtain a formula of the general form (33) provided we take for our coordinate planes the principal planes of inertia at the point where the plane of symmetry cuts the axis of rotation. When the principal planes containing the axis of rotation are parallel to principal planes through the C.G. the effect of a displacement of the C.G. from the axis of rotation is as easily traced as in isotropy, but otherwise it must be remembered that the values of the elastic constants vary with the directions of the axes. It might thus in some cases be most convenient to take the two coordinate axes, which are perpendicular to the axis of rotation, parallel to principal axes at the C.G., though this introduce a product of inertia into the formula deduced from (20).

§ 21. When the radius r_1 of the axle, assumed circular, is small compared to the distance of the C.G. from the axis of rotation we can easily find a fairly accurate measure of the correction to the value of δv required on account of the hitherto neglected surface integral. Thus for isotropy, let the axis of rotation be axis of z, and let the C.G. lie on the axis of y at a distance $\bar{y}$ from the origin. Also let θ denote the angle which a radius of the axle makes with the plane yz. We shall suppose the body symmetrical about the plane xy, and neglect friction on the axle parallel to its length, so that there is no component parallel to z in the surface forces. The forces exerted at any point of the axle by a bearing may then be resolved into N along r_1 and T perpendicular to it. Thus supposing there to be two bearings, and assuming N and T the same numerically at $-\theta$ as at $+\theta$, we must have

$$4\int_0^{\frac{\pi}{2}} N \cos\theta r_1 d\theta = \omega^2 \rho v \bar{y} \quad \text{.....................................(61)}.$$

Also since
$$Fx + Gy = -Nr_1,$$
the surface integral in (23) becomes

$$-4\int_0^{\frac{\pi}{2}} N r_1^2 d\theta \quad \text{...(62)}.$$

To evaluate this integral exactly we require the law of distribution of N over the surface of the axle between $\theta = \pm\pi/2$. As this is unknown, I have calculated the correction to δv on three hypotheses. The work is easy so it will suffice to state the hypotheses and quote the results. These are as follows:

Hypothesis	Correction to δv
1° N uniform, i.e. independent of θ,	$-(\omega^2\rho v\bar{y}r_1/3k)\times\pi/2$,
2° $N\propto\cos\theta$,	$-(\omega^2\rho v\bar{y}r_1/3k)\times 4/\pi$,
3° N concentrated at end of diameter $\theta=0$,	$-(\omega^2\rho v\bar{y}r_1/3k)\times 1$.

The true formula of correction will probably vary from one shape of body to another, but the result must lie between those of 1° and 3°, and most likely will in general be not far from the result of 2°. Taking this as the most likely value we have in place of (60)

$$\delta v=\omega^2\rho v\left(\kappa^2+\bar{y}^2-\frac{4}{\pi}\bar{y}r_1\right)\Big/3k \qquad (63).$$

§ 22. The effect on the length of a right cylinder of an excentric position of the axis of rotation is also easily studied provided it be parallel to the axis of the figure, or else be in the central cross section and be perpendicular to an axis of symmetry of that section. It will suffice to give the results for an isotropic material in these two cases, neglecting the correction arising from the surface integral. This correction may however easily be approximated to, just as in the case of the change of volume.

Let the C.G. of the cross section be at a distance $\bar{y}$ from the axis of rotation. Then for the increment δl in the mean half length we find from (21):

1° when the axis of rotation is parallel to the axis of figure

$$\delta l_3/l=-\eta\omega_3^2\rho\,(\kappa_1^2+\kappa_2^2+\bar{y}^2)/E \qquad (64),$$

2° when the axis of rotation lies in the central cross section and is perpendicular to a plane of symmetry

$$\delta l_1/l=\omega_1^2\rho\,\{\tfrac{1}{3}l^2-\eta\,(\kappa_1^2+\bar{y}^2)\}/E \qquad (65).$$

The notation will easily be understood from the previous examples.

The effect of the excentric position is in either case to promote shortening of the mean length.

Gravity at the Earth's Surface.

§ 23. Let a homogeneous elastic solid of any shape be suspended from a point on its surface. The centre of gravity must lie on the vertical through this point, say at a depth h below it. Taking the point of suspension for origin, and the axis of z vertically downwards, and denoting gravity by g so that $Z=g\rho$, we find from (20), for an aeolotropic solid

$$\delta v/v=g\rho h g_0 \qquad (66);$$

whence, or from (23), for an isotropic solid

$$\delta v/v=g\rho h/3k \qquad (67).$$

If on the other hand the solid rest on a smooth horizontal plane—or be supported at one or more points in a horizontal plane—let us take this for xy and let the axis of z be drawn vertically upwards through the C.G. Then putting $Z = -g\rho$ in (20), and noticing that

$$\iint HxdS \text{ and } \iint HydS$$

must vanish owing to the conditions of statical equilibrium, we find for the change $\delta v'$ in the volume of an aeolotropic solid

$$\delta v'/v = -g\rho h' g_0 \quad \text{......(68)}.$$

For an isotropic solid

$$\delta v'/v = -g\rho h'/3k \quad \text{......(69)}.$$

In these two formulae h' is the height of the C.G. above the horizontal plane of support. There may be a number of isolated areas of support, as in a girder bridge, provided all are in one horizontal plane; and in any such case in an isotropic material the volume is diminished or increased according as the C.G. in the position of equilibrium is above or below the level of the supports.

If the same material line be the axis of z in the two cases answering to (66) and (68), and the length of this diameter be d, we find

in the aeolotropic solid $(\delta v - \delta v')/v = g\rho d g_0$ ......(70),

" " isotropic " $(\delta v - \delta v')/v = g\rho d/3k$ ......(71).

The quantity k is essentially positive, and thus in isotropic solids the volume is greater when the body is suspended and less when it is supported on a smooth plane than it would be if the body were free from the earth's attraction. The quantity g_0 is positive as a rule in aeolotropic solids, but there is no obvious reason why in some solids it may not be negative for certain directions of the corresponding axis.

§ 24. To get some idea of the magnitude of this effect in isotropic solids we shall consider some special cases of bodies which may reasonably be regarded as fairly isotropic. In steel* we may regard a *length modulus* of 25×10^7 centimetres as a fair average for E, and may put $\eta = 1/4$. Taking these values, and denoting the densities of steel when suspended and when supported by ρ, ρ' respectively, we find for its density $\bar{\rho}$ if unacted on by the earth's gravitation

$$\bar{\rho} = \rho(1 + 2h/10^9), \quad \bar{\rho} = \rho'(1 - 2h'/10^9),$$

where h and h' are the lengths occurring in (67) and (69) measured in centimetres. If the body were a right cylinder its height would equal $2h$ or $2h'$. Thus the cylinder would require to be 5 metres high before its specific gravities when suspended and when supported differed from one another by one part in a million. Steel, or iron, is however the metal in which the effect is least. In such a metal as lead it is very much greater. Thus if we assign to E in cast lead* a length modulus of 16×10^6 cm. and suppose $\eta = 1/4$, the difference between the specific gravities when suspended and supported would amount to one part in a million in a cylinder about a third of a metre in height, i.e. little over a foot.

* See the table of moduli in Sir W. Thomson's article on *Elasticity* in the *Encyclopaedia Britannica*.

In sheet lead, according to Sir W. Thomson's table, this difference of the specific gravities would arise in a cylinder about 4 inches high. Of course these numerical results are intended merely to give an idea of the magnitude of the effect, and it must not be supposed that the elastic data they are based on—more especially the hypothesis of uni-constant isotropy in sheet lead—possess any great accuracy.

§ 25. In the case of a right cylinder we can also find the alteration in the mean length due to the action of the earth's gravitation. Thus supposing the cylinder first suspended, and then supported on a smooth plane, with its axis of figure, taken as axis of z, vertical we find from (15) for the mean increments $\overline{\delta l_3}$ and $\overline{\delta l_3'}$ in the length l for any elastic material

$$\overline{\delta l_3}/l = -\overline{\delta l_3'}/l = \tfrac{1}{2} g\rho l/E_3 \quad \text{.......(72)}.$$

Here E_3 is Young's modulus for the direction parallel to the axis, and so presumably is essentially a positive quantity.

If again the cylinder be suspended with its axis horizontal in such a way as to prevent flexure—for instance, by a large number of strings attached to points along a generator—and the vertical plane xz contain the C.G., the axis of the cylinder being axis of z, we find from (15) for the increment $\overline{\delta l_1}$ of the mean length

$$\overline{\delta l_1}/l = -\eta_{31} g\rho h_1/E_3 \quad \text{.......(73)},$$

where h_1 is the distance of the C.G. below the horizontal plane through the points of suspension. While if the cylinder rest on a smooth horizontal plane in this position, the increment $\overline{\delta l_1'}$ in the mean length is given by

$$\overline{\delta l_1'}/l = \eta_{31} g\rho h_1'/E_3 \quad \text{.......(74)},$$

where h_1' is the height of the C.G. above the supporting plane.

For an isotropic material we have only to replace E_3 by E and η_{31} by η in the last three formulae.

The general conclusion we are led to is that under the action of gravity any elastic right cylinder lengthens when suspended with its axis vertical and shortens when suspended with its axis horizontal, unless in the latter case η_{31} be negative; but when supported on a smooth horizontal plane it shortens when its axis is vertical and lengthens, unless η_{31} be negative, when its axis is horizontal.

If we suppose the same diameter d vertical in the two cases (73) and (74) we get

$$(\overline{\delta l_1'} - \overline{\delta l_1})/l = \eta_{31} g\rho d/E_3 \quad \text{.......(75)}.$$

Comparing this with (72) written as

$$(\overline{\delta l_3} - \overline{\delta l_3'})/l = g\rho l/E_3 \quad \text{.......(76)},$$

we see how much more effective gravity is in altering the length of a long bar, of small diameter, when its axis is vertical than when it is horizontal. But if the diameter of a long horizontal cylinder be considerable, the effect of gravity on its length is deserving of attention, especially in materials such as lead or gold, and to a smaller extent in silver and platinum.

In any right cylinder the mean change in the cross section in the several cases just treated may be found by combining the results for δv and δl by means of the formula

$$\delta\sigma/\sigma = \delta v/v - \delta l/l \qquad \text{(77)}.$$

§ 26. As the plane supporting a solid is never quite smooth, it is desirable to see what effect the roughness of this plane would have on the previous results. Confining our attention to isotropy, we require to add to the value of δv for a cylinder supported with its axis vertical on the plane $z = 0$, the value of the surface integral

$$\frac{1}{3k}\iint (Fx + Gy)\, dxdy$$

taken over the supported base, where F and G are the components parallel to x and y of the frictional forces.

Let N and T be the components of the frictional force at any point along and perpendicular to the radius vector r from the origin. Then the above integral becomes

$$\frac{1}{3k}\iint Nr dxdy.$$

Now the tendency of the supported solid—whose C.G. is assumed above the supporting plane—shortening under gravity is clearly to expand horizontally, and thus the frictional force is towards the origin, or N is negative. The surface integral is thus negative and from the corrected formula (69), viz.

$$\delta v' = - g\rho h' v/3k + \iint (Nr/3k)\, dxdy \qquad \text{(78)},$$

we see that this correction tends to increase numerically the reduction in volume due to the action of gravity.

The corrected formula (72) under the same conditions is

$$\overline{\delta l_3}' = - g\rho l^2/2E - \iint (\eta Nr/E\sigma)\, dxdy \qquad \text{(79)},$$

where the surface integral is taken over the supported base.

The frictional forces thus tend to reduce numerically the shortening in the cylinder's length due to gravity. The corrections in these two cases are less, ceteris paribus, the smaller the base of the body.

Bodies under the mutual gravitation of their parts.

§ 27. In a gravitating sphere of radius R, volume v and uniform density ρ we have

$$X/x = Y/y = Z/z = - g\rho/R,$$

where g is the acceleration of "gravity" at the surface. Substituting these values in (20) and remembering (10), we deduce for the change of volume in any elastic sphere

$$-\delta v = g\rho R v / 5k \quad \text{.......................................(80).}$$

Knowing the change of volume we can at once deduce the change of radius. If we were to apply this result to a sphere of the earth's size and mass, we should find that unless we assigned to k a much greater value than in any known material, under normal conditions, our formula would imply strains much in excess of those to which the mathematical theory of elasticity is legitimately applicable.

§ 28. To determine the effect of a small deviation from the spherical form, let us consider a homogeneous solid whose surface is given by

$$r = R + \Sigma\,(R_i \sigma_i) \quad \text{.......................................(81),}$$

where $R_i\sigma_i$ represents a term, or a series of terms, involving surface spherical harmonics of degree i, and the ratio of each term to R, or the ratio of the sum of all the terms of all degrees to R, is supposed so small its square is negligible. For such a body the gravitational potential is given by

$$V = -\tfrac{1}{2} g r^2 / R^2 + \Sigma\,\{3 g R_i \sigma_i\,(r/R)^i \div (2i+1)\} \quad \text{.........................(82)*,}$$

where g represents the mean value of "gravity" at the surface.

Supposing the material elastically homogeneous but of the most general aeolotropic character given by (1), we find the change of volume from (20) by substituting

$$X = \rho \frac{dV}{dx}, \quad Y = \rho \frac{dV}{dy}, \quad Z = \rho \frac{dV}{dz}, \quad F = G = H = 0.$$

The sum of the terms independent of σ_i inside the integral is simply $-2g\rho R^{-1}\chi_0$. Thus integrating the terms involving σ_i by parts we find, using (10) and representing the element of normal to the surface by $d\mathfrak{n}$,

$$\begin{aligned} -\delta v / g\rho = &\iiint 2R^{-1}\chi_0\, dx\,dy\,dz \\ &- \Sigma\left[3R^{-i}(2i+1)^{-1} \iint R_i\sigma_i r^i \frac{d\chi_0}{d\mathfrak{n}}\, dS\right] \\ &+ \Sigma\left[3R^{-i}(2i+1)^{-1} \iiint R_i\sigma_i r^i k^{-1}\, dx\,dy\,dz\right] \quad \text{.....................(83),} \end{aligned}$$

where the volume integrals are taken throughout the entire volume, and the surface integral over the whole surface (81).

Now $$\iiint 2R^{-1}\chi_0\, dx\,dy\,dz = 2\iiint R^{-1}(r^{-2}\chi_0)\, r^4 \sin\theta\, dr\, d\theta\, d\phi,$$

and as $r^{-2}\chi_0$ is independent of r this becomes, neglecting terms of order $(R_i\sigma_i/R)^2$,

$$\iiint 2R^{-1}\chi_0\, dx\,dy\,dz = \tfrac{2}{5} R^4 \iint \{1 + 5\Sigma\,(R_i\sigma_i/R)\}\,(r^{-2}\chi_0) \sin\theta\, d\theta\, d\phi.$$

* Cf. Professor Darwin, *Phil. Trans.* 1882, p. 200.

But $$r^{-2}\chi_0 = \tfrac{1}{2}r^{-2}(e_0x^2 + f_0y^2 + g_0z^2 + a_0yz + b_0zx + c_0xy)$$
$$= \tfrac{1}{6}(e_0 + f_0 + g_0) + \text{sum of surface harmonics of 2nd degree} \ldots\ldots\ldots\ldots (84).$$

Thus using (10) and remembering that the integral of a surface harmonic over the surface of the sphere vanishes, we obtain

$$\iiint 2R^{-1}\chi_0 dxdydz = \tfrac{2}{5}R^4\left(\frac{2}{3}\frac{\pi}{k}\right) + 2R^4\iint \Sigma\,(R_i\sigma_i/R)\,(r^{-2}\chi_0)\sin\theta d\theta d\phi.$$

Again in the surface integral in (83) we may replace $r\dfrac{d\chi_0}{dn}$ by $r\dfrac{d\chi_0}{dr}$, or $2\chi_0$, and may then put $r = R$. Also transforming the last volume integral in (83) into polar coordinates, and neglecting terms of order $(R_i\sigma_i/R)^2$, we see that the integral vanishes by the ordinary property of surface harmonics. Thus, combining the several simplifications, we replace (83) by

$$-\delta v/g\rho = Rv/5k + 2R^4\iint \Sigma\left\{\left(1 - \frac{3}{2i+1}\right)\frac{R_i\sigma_i}{R}\right\}(r^{-2}\chi_0)\sin\theta d\theta d\phi \ldots\ldots\ldots\ldots (85).$$

Referring to (84) we see at once from the ordinary properties of surface harmonics that the only terms in $\Sigma(R_i\sigma_i)$ which can contribute anything to δv are those of the second degree. Again, the most general possible form of $R_2\sigma_2$ is given by

$$R_2\sigma_2/R = \tfrac{1}{2}r^{-2}(A_2x^2 + B_2y^2 + C_2z^2 + 2D_2yz + 2E_2zx + 2F_2xy) \ldots\ldots\ldots\ldots (86),$$

where the constants are subject to the one condition

$$A_2 + B_2 + C_2 = 0 \ldots\ldots\ldots\ldots (87).$$

Thus we may replace (85) by

$$-\delta v/g\rho = Rv/5k + \tfrac{1}{3}R^4\iint (A_2, B_2, C_2, D_2, E_2, F_2 \between x, y, z)^2 \times (e_0, f_0, g_0, \tfrac{1}{2}a_0, \tfrac{1}{2}b_0, \tfrac{1}{2}c_0 \between x, y, z)^2\, d\varpi$$
$$\ldots\ldots\ldots\ldots (88),$$

where $d\varpi$ is the element of surface of a sphere of unit radius.

Now it is easy to prove

$$\iint x^4 d\varpi = \ldots = 3\iint y^2z^2 d\varpi = \ldots = 4\pi/5,$$

while the integrals of all terms involving an odd power of x, y or z vanish.

Thus using (87) we obtain from (88)

$$-\delta v = \frac{g\rho Rv}{5k}\left\{1 + \frac{2k}{5}(A_2e_0 + B_2f_0 + C_2g_0 + D_2a_0 + E_2b_0 + F_2c_0)\right\} \ldots\ldots\ldots\ldots (89),$$

where A_2, B_2, C_2 are subject to (87).

This form of the result may be the most convenient under certain conditions, since the stress-strain relations in most kinds of aeolotropy are simplified by taking the axes of coordinates in certain fixed directions, but the physical meaning may be rendered clearer by a change of axes.

By the properties of quadric surfaces we may change the directions of the axes, keeping them orthogonal, so as to transform (86) into

$$R_2\sigma_2/R = \tfrac{1}{2}r'^{-2}(A_2'x'^2 + B_2'y'^2 + C_2'z'^2) \quad \text{.............................(90)},$$

where

$$A_2' + B_2' + C_2' = A_2 + B_2 + C_2 = 0 \quad \text{.............................(91)}.$$

Thus putting

$$C_2' = 2A_2'', \quad B_2' = -(A_2'' + B_2''), \quad A_2' = B_2'' - A_2'' \quad \text{....................(92)},$$

we have

$$R_2\sigma_2/R = \tfrac{1}{2}r'^{-2}\{A_2''(2z'^2 - x'^2 - y'^2) + B_2''(x'^2 - y'^2)\} \quad \text{....................(93)}.$$

Let now e_0', f_0', g_0' be the extensions, for uniform normal unit tension, in the directions of the new axes, then we transform (89) into

$$-\delta v = \frac{g\rho Rv}{5k}\left[1 + \tfrac{2}{5}k\{A_2''(2g_0' - e_0' - f_0') + B_2''(e_0' - f_0')\}\right] \quad \text{..............}(94_a),$$

or

$$-\delta v = \frac{g\rho Rv}{5k}\left[1 + \tfrac{2}{5}\{A_2''(3kg_0' - 1) + kB_2''(e_0' - f_0')\}\right] \quad \text{..................}(94_b).$$

Now a positive value of A_2'' means an increase of that diameter in whose direction g_0' is measured and a diminution of all perpendicular diameters, while a positive value of B_2'' means an increase of that diameter in whose direction e_0' is measured, a diminution of that diameter in whose direction f_0' is measured, and an unchanged length in that diameter in whose direction g_0' is measured. Thus the general result implied in (89) or (94) is that the diminution in volume in the mass due to its own gravitation is greater or less than in a sphere of equal volume according as the longest diameters in the nearly spherical body

$$r = R + R_2\sigma_2$$

are directions in the material along which the reduction of length accompanying uniform normal pressure is above or below the average.

For any isotropic material the reduction in volume has the same value as in a sphere of equal volume. Thus the reduction in volume of a given isotropic mass due to its mutual gravitation is in general either a maximum or a minimum when the bounding surface is spherical.

To determine whether in this case the reduction is a maximum or a minimum we would require to go at least as far as terms of order $(R_i\sigma_i/R)^2$, and it would be necessary to employ a more exact formula for the potential than (82). Such formulae are unknown to me save for ellipsoids, in which case we can go to any required degree of accuracy. As regards harmonic terms of degrees above the second, it seems most likely that for a given maximum value of $R_i\sigma_i$ the effect on the change of volume will in general be less the greater i is. Thus the second harmonic term, unless relatively inconsiderable, may be anticipated to have usually a predominating influence. When the elastic properties of the medium, while showing aeolotropy, vary but little in different directions, the terms in A_2'' and B_2'' in (94) may conceivably be of no greater importance than those depending on the squares of the harmonic terms. It has thus appeared desirable not to assume isotropy in the following treatment of the ellipsoid even when nearly spherical.

Gravitating Ellipsoid.

§ 29. Let a, b, c be the semi-axes of an ellipsoid of uniform density ρ and volume v, of a homogeneous aeolotropic elastic material given by (1), and let

$$\psi = \int_{\infty}^{0} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad\text{..................(95).}$$

Then denoting by μ the gravitational force between two unit masses at unit distance, we find for the bodily forces*

$$X = -\rho A x, \quad Y = -\rho B y, \quad Z = -\rho C z,$$

where $$A = 3\mu\rho v \frac{d\psi}{da^2}, \quad B = 3\mu\rho v \frac{d\psi}{db^2}, \quad C = 3\mu\rho v \frac{d\psi}{dc^2} \quad\text{..................(96).}$$

The surface forces everywhere vanish. Thus from (20) we find for the change in volume of the ellipsoid due to its mutual gravitation

$$-\delta v/\rho = \iiint (x^2 A e_0 + y^2 B f_0 + z^2 C g_0)\, dx\, dy\, dz$$

$$= \tfrac{1}{5} v \{a^2 A e_0 + b^2 B f_0 + c^2 C g_0\} \quad\text{..................(97).}$$

If g_1, g_2, g_3 be the values of "gravity" at the ends of the three principal axes of figure

$$aA = g_1, \quad bB = g_2, \quad cC = g_3;$$

thus $$-\delta v/v = \tfrac{1}{5}\rho\, (a g_1 e_0 + b g_2 f_0 + c g_3 g_0) \quad\text{..................(98),}$$

or for isotropy $$-\delta v/v = \frac{1}{15}\frac{\rho}{k}(a g_1 + b g_2 + c g_3) \quad\text{..................(99).}$$

The quantities A, B, C, or g_1, g_2, g_3 may be expressed as elliptic integrals.

When the ellipsoid is nearly spherical, let

$$b^2/a^2 = 1 - \epsilon_1^2, \quad c^2/a^2 = 1 - \epsilon_2^2 \quad\text{..................(100).}$$

Then expanding $\frac{d\psi}{da^2}$ etc. in powers of ϵ_1 and ϵ_2, and neglecting powers above the fourth, we easily find

$$\left.\begin{aligned} A &= \frac{3\mu\rho v}{a^3}\left(\frac{1}{3} + \frac{\epsilon_1^2 + \epsilon_2^2}{10} + \frac{3\epsilon_1^4 + 2\epsilon_1^2\epsilon_2^2 + 3\epsilon_2^4}{56}\right), \\ B &= \frac{3\mu\rho v}{a^3}\left(\frac{1}{3} + \frac{3\epsilon_1^2 + \epsilon_2^2}{10} + \frac{15\epsilon_1^4 + 6\epsilon_1^2\epsilon_2^2 + 3\epsilon_2^4}{56}\right), \\ C &= \frac{3\mu\rho v}{a^3}\left(\frac{1}{3} + \frac{\epsilon_1^2 + 3\epsilon_2^2}{10} + \frac{3\epsilon_1^4 + 6\epsilon_1^2\epsilon_2^2 + 15\epsilon_2^4}{56}\right) \end{aligned}\right\} \quad\text{..................(101).}$$

Now let R be the radius of a sphere equal in volume and mass to the ellipsoid, and let g be the value of "gravity" at its surface; then

$$\left.\begin{aligned} R^3 &= abc = a^3(1-\epsilon_1^2)^{\frac{1}{2}}(1-\epsilon_2^2)^{\frac{1}{2}}, \\ \mu\rho v/R^2 &= g \end{aligned}\right\} \quad\text{..................(102).}$$

* See Thomson and Tait's *Natural Philosophy*, Vol. I., Part II., p. 47.

Substituting the values of A, B, C from (101) in (97), eliminating a, b, c by means of (100) and (102) and arranging the terms, we find

$$-\delta v = \tfrac{1}{5}g\rho Rv\left[e_0 + f_0 + g_0 + \tfrac{2}{15}\{\epsilon_1^2(e_0 + g_0 - 2f_0) + \epsilon_2^2(e_0 + f_0 - 2g_0)\}\right.$$
$$\left.- \tfrac{1}{315}\{\epsilon_1^4(47f_0 - 13e_0 - 13g_0) - \epsilon_1^2\epsilon_2^2(11e_0 + 5f_0 + 5g_0) + \epsilon_2^4(47g_0 - 13e_0 - 13f_0)\}\right] \text{...(103)}.$$

Employing (10), we may write this in the more convenient form

$$-\delta v = \frac{g\rho Rv}{5k}\left[1 - \frac{\epsilon_1^4 - \epsilon_1^2\epsilon_2^2 + \epsilon_2^4}{45} + \tfrac{2}{15}\{\epsilon_1^2(1 - 3kf_0) + \epsilon_2^2(1 - 3kg_0)\}\right.$$
$$\left.+ \frac{2}{315}\{10\epsilon_1^4(1 - 3kf_0) + \epsilon_1^2\epsilon_2^2(3ke_0 - 1) + 10\epsilon_2^4(1 - 3kg_0)\}\right] \text{.........(104)}.$$

It is easy to show that the terms in ϵ_1^2 and ϵ_2^2 agree with those already obtained in (94).

For any isotropic material we have the simple result

$$-\delta v = \frac{g\rho Rv}{5k}\left(1 - \frac{\epsilon_1^4 - \epsilon_1^2\epsilon_2^2 + \epsilon_2^4}{45}\right) \text{..............................(105)}.$$

Thus in an isotropic nearly spherical ellipsoid the reduction in volume is always less than in a sphere of equal volume, or *the sphere is that form of ellipsoid in which the reduction of volume due to the mutual gravitation of the parts is a maximum.* The smallness however of the terms in (105) depending on the eccentricity seems rather remarkable.

In an aeolotropic material the terms in ϵ_1^4, $\epsilon_1^2\epsilon_2^2$ and ϵ_2^4 which depend on differences of elastic quality in different directions have obviously the same physical import as the terms in ϵ_1^2 and ϵ_2^2; i.e. they signify an increased or diminished reduction of volume relative to that in the sphere according as the longest diameters are directions in which the contraction under uniform normal pressure is above or below the average.

For a prolate spheroid about the axis 2a, putting $\epsilon_2^2 = \epsilon_1^2 = \epsilon^2$ in (104), and using (10), we get

$$-\delta v = \frac{g\rho Rv}{5k}\left[1 - \frac{\epsilon^4}{45} + \frac{2}{15}\epsilon^2\left(1 + \frac{11}{21}\epsilon^2\right)(3ke_0 - 1)\right] \text{.....................(106)}.$$

For an oblate spheroid about the axis 2c, putting $\epsilon_1 = 0$, and $\epsilon_2 = \epsilon'$ in (104), we find

$$-\delta v' = \frac{g\rho Rv}{5k}\left[1 - \frac{\epsilon'^4}{45} + \frac{2}{15}\epsilon'^2\left(1 + \frac{10}{21}\epsilon'^2\right)(1 - 3kg_0)\right] \text{.....................(107)}.$$

As in (104), R denotes the radius of the sphere of equal volume and g gravity at its surface.

We notice that $\delta v' = \delta v$ when $\epsilon' = \epsilon$ in all isotropic materials. In an aeolotropic material when the spheroids have their axes of figure in the same direction in the material, e_0 in (106) and g_0 in (107) are identical. Thus when $\epsilon' = \epsilon$ the effects of aeolotropy in the two spheroids are very nearly equal numerically, though of opposite sign.

§ 30. In this paper our attention has hitherto been confined to the mean values of the strains, but we may obviously from the equality of (I.) and (II.) arrive even more easily at the mean values of the stresses. For instance, to find the mean value of $\widehat{xx}$ answering to a given system of applied forces, viz. X, Y, Z per unit of *volume*, and F, G, H per unit of surface, put $f' = g' = a' = b' = c' = 0$ in (II.), and regard e' as constant. Then for the corresponding displacements we have

$$\alpha' = e'x, \quad \beta' = \gamma' = 0 \qquad \text{(108)},$$

and so from (I.) and (II.), dividing out by e', we find

$$\iiint \widehat{xx}\, dx\,dy\,dz = \iiint Xx\,dx\,dy\,dz + \iint Fx\,dS \qquad \text{(109)},$$

where the volume integrals are taken throughout the whole volume and the surface integral over the entire surface, or surfaces, of the solid.

Again, regarding a' as constant, putting

$$e' = f' = g' = b' = c' = 0 \text{ in (II.)},$$

and substituting in (I.) the corresponding displacements, viz.

$$\alpha' = 0, \quad \beta'/z = \gamma'/y = a'/2 \qquad \text{(110)},$$

we find

$$\iiint \widehat{yz}\, dx\,dy\,dz = \tfrac{1}{2}\iiint (Yz + Zy)\, dx\,dy\,dz + \tfrac{1}{2}\iint (Gz + Hy)\, dS \qquad \text{(111)}.$$

The formulae for the other mean stresses may be written down from symmetry.

The results for the mean stresses are wholly independent of the aeolotropic or isotropic nature of the medium. They may be verified in the simplest manner by direct reference to (12) and (13).

The information derivable from the values of the mean strains and stresses is necessarily in general of an imperfect character, as the law of variation of the strains and stresses throughout the solid is essential to a complete study of an elastic problem. Still the mean strains and stresses may indirectly prove of considerable service in verifying the accuracy of mathematical work, and perhaps occasionally in affording a test of the sufficiency of theories which supply for a definite physical problem a mathematical substitute as to whose approximate equivalence doubts may be entertained.

[April 22, 1892. By ordinary Statics the bodily and surface forces must satisfy three equations such as

$$\iiint X\,dx\,dy\,dz + \iint F\,dS = 0,$$

and three such as

$$\iiint (Zy - Yz)\, dx\,dy\,dz + \iint (Hy - Gz)\, dS = 0.$$

Employing these we can write some of the general formulae in the paper in a variety of equivalent forms. For instance, we may transform (15) into

$$E_3 vg = \iiint [X(-\eta_{31}x - \eta_{36}y - \eta_{35}z) + Y(-\eta_{32}y - \eta_{34}z) + Zz]\, dxdydz$$

$$+ \iint [F(\qquad\qquad\qquad) + G(\qquad\qquad) + Hz]\, dS,$$

and may combine (22) and (111) in the form

$$nv\bar{a} = \iiint \widehat{yz}\, dxdydz = \iiint \{(1-p)Yz + pZy\}\, dxdydz + \iint \{(1-p)Gz + pHy\}\, dS,$$

where p is any constant, including 0.]

X. *The Isotropic Elastic Sphere and Spherical Shell.* By C. Chree, M.A., Fellow of King's College.

[Read February 13, 1893.]

CONTENTS.

Part I.

Equilibrium under given bodily and surface forces.

§ 1. The determination of the displacements, strains and stresses in an isotropic elastic spherical shell is of great interest as one of the few elastic problems of which a mathematically exact solution has been obtained. The problem has been solved in several different ways, but with results rather of mathematical than physical interest. The aim of the present solution may best be indicated by a brief reference to previous solutions.

The first treatment of the problem is due to Lamé*, who considered the case when the surfaces of the shell are acted on by any given forces, but took into account only one or two simple systems of bodily forces. His solution is in polar coordinates, and is an elegant if somewhat lengthy piece of analysis. It obtains expressions for the displacements involving arbitrary constants, and the *method* of determining these from the

* *Liouville's Journal*, Tome 19, pp. 51—87, 1854.

surface conditions is clearly shown. A physicist, however, desirous of applying the solution in practice would probably find the labour of determining these constants sufficiently arduous to deter him from his purpose.

A solution better known in this country is that of Lord Kelvin*. It is in some important respects more complete than Lamé's, as the method of treating bodily forces derivable from a potential is included, and the case of given surface displacements is also considered. In the opinion of Thomson and Tait† the use of Cartesian coordinates in this solution in place of the polar coordinates of Lamé is a great simplification.

This is not an opinion which the author of the present solution can endorse, and it seems to him that for practical purposes Lord Kelvin's solution stands very much in the same position as Lamé's.

Recently the cases of given surface displacements and given surface forces have been solved in a way quite unlike either of the preceding by Cerruti‡. His results in the case of surface displacements are intelligible only to one familiar with what may be called the "potential methods" of solution originated by Betti and Lord Kelvin, and whose best known applications are due to Boussinesq. Judging by the abstract in the '*Beiblatter*' to *Wiedemann's Annalen* § the solution for given surface forces—the original of which the author has not seen—is of the same character. The mathematical difficulties in this form of solution are very great, and the results do not seem of such a character as to lend themselves readily to practical applications.

In 1887 a paper‖ was contributed by the author to the Society, containing *inter alia* a solution in polar coordinates which led by a more direct route than Lamé's to equivalent results.

This paper determined explicitly the arbitrary constants for the case of a solid sphere under given normal surface forces, or with given normal surface displacements, but for other cases the results laboured under similar disadvantages to Lamé's, as the labour of determining the arbitrary constants was left for the reader. This defect it is the primary object of the present paper to remove. It assumes the mathematical work of the previous paper, reproducing only so much as is required to render the results clearly intelligible; it then determines the arbitrary constants for all cases and furnishes an explicit solution applicable without serious trouble to any special problem. The opportunity is also taken of considering in some detail the conclusions to which the solution leads when the shell is very thin.

The results obtained in this case, being independent of any assumptions as to the relative magnitudes of the several stresses, seem not unlikely to be of service in testing the results arrived at by the ordinary treatment of thin shells.

It must of course be borne in mind that there may exist in some other forms of thin shells phenomena widely different from those shown by a complete spherical shell.

* Royal Society's *Transactions* for 1863, p. 583; or Thomson and Tait's *Natural Philosophy*, Part II., pp. 735 *et seq.*

† *Natural Philosophy*, vol. I. Part II., Art. 735.

‡ *Rend. R. Acc. dei Lincei* 5, 2 sem. pp. 189—201, 1889; also *Mem. R. Acc. dei Lincei*, pp. 25—44, 1890.

§ Bd. XV. pp. 630—1.

‖ *Camb. Phil. Trans.* vol. XIV. pp. 250—369.

For example, the strains and stresses produced by the flexure of thin plates with straight or curved edges, especially in the case of narrow strips, or the strains and stresses produced by surface forces at points on a thin shell where the curvature is unusually great, for instance near the ends of the axis of a very prolate spheroid, may follow laws which bear but a slight resemblance to those arrived at here.

§ 2. Employing the ordinary polar coordinates r, θ, ϕ as in my previous paper, and denoting the displacements by u, v, w, we have for the components of strain

$$\left.\begin{aligned}&\frac{du}{dr},\\ &\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta},\\ &\frac{u}{r}+\frac{v}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{dw}{d\phi},\\ &\frac{1}{r\sin\theta}\frac{dv}{d\phi}+\frac{1}{r}\frac{dw}{d\theta}-\frac{w}{r}\cot\theta,\\ &\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi},\\ &\frac{1}{r}\frac{du}{d\theta}+\frac{dv}{dr}-\frac{v}{r}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(1).$$

Of these the first three are in the terminology of Todhunter and Pearson's "*History*", *stretches*, the last three *slides*, i.e. *shearing strains.*

The dilatation δ is given by

$$\delta=\frac{du}{dr}+\frac{2u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{v}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{dw}{d\phi}\quad\ldots\ldots\ldots\ldots(2).$$

The stresses, employing Professor Pearson's notation*, are

$$\left.\begin{aligned}\widehat{rr}&=(m-n)\,\delta+2n\frac{du}{dr},\\ \widehat{\theta\theta}&=(m-n)\,\delta+2n\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right),\\ \widehat{\phi\phi}&=(m-n)\,\delta+2n\left(\frac{u}{r}+\frac{v}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{dw}{d\phi}\right),\\ \widehat{r\theta}&=n\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\right),\\ \widehat{r\phi}&=n\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right),\\ \widehat{\theta\phi}&=n\frac{1}{r}\left(\frac{dw}{d\theta}-w\cot\theta+\frac{1}{\sin\theta}\frac{dv}{d\phi}\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(3),$$

where m and n are Thomson and Tait's elastic constants.

Of the stresses the last three in (3) are the shearing stresses.

* Todhunter and Pearson's *History*, vol. I. pp. 882—3.

For shortness let

$$\left.\begin{aligned}\mathfrak{A} &= \frac{1}{r^2 \sin\theta}\left\{\frac{d}{d\theta}(wr\sin\theta) - \frac{d}{d\phi}(vr)\right\},\\ \mathfrak{B} &= \frac{1}{\sin\theta}\left\{\frac{du}{d\phi} - \frac{d}{dr}(wr\sin\theta)\right\},\\ \mathfrak{C} &= \sin\theta\left\{\frac{d}{dr}(vr) - \frac{du}{d\theta}\right\}\end{aligned}\right\} \quad \text{.........................(4).}$$

Then for an isotropic solid of uniform density ρ, acted on by bodily forces derived from a potential V, the internal equations of equilibrium are

$$\left.\begin{aligned}(m+n)\,r^2\sin\theta\frac{d\delta}{dr} - n\frac{d\mathfrak{C}}{d\theta} + n\frac{d\mathfrak{B}}{d\phi} + \rho r^2\sin\theta\frac{dV}{dr} &= 0,\\ (m+n)\sin\theta\frac{d\delta}{d\theta} - n\frac{d\mathfrak{A}}{d\phi} + n\frac{d\mathfrak{C}}{dr} + \rho\sin\theta\frac{dV}{d\theta} &= 0,\\ (m+n)\operatorname{cosec}\theta\frac{d\delta}{d\phi} - n\frac{d\mathfrak{B}}{dr} + n\frac{d\mathfrak{A}}{d\theta} + \rho\operatorname{cosec}\theta\frac{dV}{d\phi} &= 0\end{aligned}\right\} \quad \text{..................(5).}$$

§ 3. We shall consider first the case of given surface forces.

If over a bounding spherical surface the components of the applied forces along r, θ, ϕ be respectively F, G, H, then the surface conditions are

$$\left.\begin{aligned}\widehat{rr} &= \pm F,\\ \widehat{r\theta} &= \pm G,\\ \widehat{r\phi} &= \pm H\end{aligned}\right\} \quad \text{...(6),}$$

where the + sign is taken at the outer, the − sign at the inner boundary.

The displacements constituting the solution of (5) and (6) for a spherical shell may most conveniently be subdivided into the following three classes:

(i) *Pure radial displacements*, in which there is no displacement perpendicular to the radius;

(ii) *Pure transverse displacements*, in which there is no displacement along the radius;

(iii) *Mixed radial and transverse displacements*.

CLASS I. *Pure radial displacements.*

§ 4. These displacements in practical cases answer to bodily forces derived from a potential

$$Vr^2 + V'r^{-1},$$

where V and V' are constants, and to uniform normal surface forces, say

$$\left.\begin{aligned}\widehat{rr} &= R \text{ over } r = a,\\ \widehat{rr} &= R' \text{ over } r = b\end{aligned}\right\} \quad \text{....................................(7).}$$

Supposing $a > b$, and V, V', R, R' to be positive quantities, the applied forces have the directions and magnitudes shewn in fig. 1, where O is the centre of the sphere

$$OB = b, \quad OA = a.$$

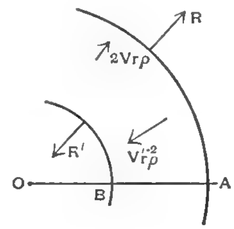


Fig 1

The potential Vr^2 is such as would arise from mutual gravitation in the shell, or from a term in the centrifugal force independent of surface harmonics, if the shell were rotating uniformly about a diameter. The only displacement is along the radius and is of the form

$$u = \frac{1}{3} r Y_0 + r^{-2} Z_{-1} - \frac{1}{5} r^3 \frac{\rho V}{m+n} - \frac{1}{2} \frac{\rho V'}{m+n} \quad \text{.........................(8)},$$

where Y_0 and Z_{-1} are arbitrary constants to be determined by the surface conditions (7). Employing the value of $\widehat{rr}$ given in (3) and noticing that δ reduces to $\frac{du}{dr} + \frac{2u}{r}$, we obtain two simple equations for the determination of Y_0 and Z_{-1}. It is hardly necessary to record the values of these constants. When substituted in (8) they give

$$u = -\frac{1}{5} r^3 \frac{\rho V}{m+n} + \frac{r}{(3m-n)(a^3-b^3)} \left\{ a^3 R - b^3 R' + \frac{5m+n}{5(m+n)} (a^5 - b^5) \rho V + \frac{m-n}{m+n} (a^2 - b^2) \rho V' \right\}$$
$$- \frac{1}{2} \frac{\rho V'}{m+n} + \frac{1}{4n} \frac{r^{-2} a^3 b^3}{a^3 - b^3} \left\{ R - R' + \frac{5m+n}{5(m+n)} (a^2 - b^2) \rho V - \frac{m-n}{m+n} \frac{a-b}{ab} \rho V' \right\} \quad \text{....(9)}.$$

The value of the dilatation is

$$\delta = -r^2 \frac{\rho V}{m+n} + \frac{3}{(3m-n)(a^3-b^3)} \left\{ a^3 R - b^3 R' + \frac{5m+n}{5(m+n)} (a^5 - b^5) \rho V + \frac{m-n}{m+n} (a^2 - b^2) \rho V' \right\}$$
$$- \frac{\rho V'}{m+n} \frac{1}{r} \quad \text{....(10)}.$$

The principal strains are $\frac{du}{dr}$ along r, and two equal strains u/r along any two directions orthogonal to one another and to r. We may suppose θ and ϕ these two directions, and may regard the corresponding principal stresses as $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$. They are given by

$$\widehat{\theta\theta} = \widehat{\phi\phi} = (m-n)\,\delta + 2n \frac{u}{r},$$

and may be found at once from (9) and (10). The other principal stress $\widehat{rr}$ is of more importance for the theory of thin shells, so it is desirable to express it in a form suitable for applications of this kind. This object is secured by the formula

$$\widehat{rr} = \frac{a^3}{r^3} \frac{r^3 - b^3}{a^3 - b^3} R + \frac{b^3}{r^3} \frac{a^3 - r^3}{a^3 - b^3} R' + \frac{5m+n}{5(m+n)} \frac{\{a^3 (a^2 - r^2)(r^3 - b^3) - b^3 (r^2 - b^2)(a^3 - r^3)\}}{r^3 (a^3 - b^3)} \rho V$$
$$- \frac{m-n}{m+n} \frac{a^2 (a-r)(r^2 - b^2) - b^2 (r-b)(a^2 - r^2)}{r^3 (a^3 - b^3)} \rho V' \quad \text{...(11)}.$$

The algebraically greatest strain at any point may be either u/r or $\dfrac{du}{dr}$ according to the nature of the applied forces. The stress-difference is the positive value of

$$S = \widehat{rr} - \widehat{\theta\theta} = \pm 2nr \frac{d}{dr}(u/r) \quad \text{..............................(12).}$$

§ 5. When the shell is very thin we may conveniently put

$$a - b = h, \quad a - r = \xi,$$

so that h denotes the thickness of the shell, and ξ the distance of any point from the outer surface. Retaining the lowest and next lowest powers of h/a and ξ/a in the coefficients of the several terms, we easily deduce from the previous formulae the approximate results:

$$u = \frac{m+n}{4n(3m-n)}\left[\left(1 - 2\frac{m-n}{m+n}\frac{h-\xi}{a}\right)\frac{Ra^2}{h} - \left(1 - 2\frac{h}{a} + 2\frac{m-n}{m+n}\frac{\xi}{a}\right)\frac{R'a^2}{h}\right.$$
$$\left. + 2\left(1 - \frac{1}{2}\frac{5m+n}{m+n}\frac{h}{a} + 2\frac{m-n}{m+n}\frac{\xi}{a}\right)a^3\rho V - \left(1 - \frac{m-n}{m+n}\frac{h-2\xi}{a}\right)\rho V'\right] \text{...(13),}$$

$$\delta = \frac{1}{3m-n}\left[\frac{a}{h}\left(1 + \frac{h}{a}\right)R - \frac{a}{h}\left(1 - 2\frac{h}{a}\right)R' + 2\left(1 - \frac{1}{2}\frac{5m+n}{m+n}\frac{h}{a} + \frac{3m-n}{m+n}\frac{\xi}{a}\right)a^2\rho V\right.$$
$$\left. - \left(1 - \frac{m-n}{m+n}\frac{h}{a} + \frac{3m-n}{m+n}\frac{\xi}{a}\right)a^{-1}\rho V'\right] \text{...(14),}$$

$$\widehat{rr} = \frac{h-\xi}{h}\left(1 + 2\frac{\xi}{a}\right)R + \frac{\xi}{h}\left(1 - 2\frac{h-\xi}{a}\right)R' + \frac{5m+n}{m+n}\frac{\xi(h-\xi)}{a^2}\left(1 - \frac{2}{3}\frac{h}{a} + \frac{4}{3}\frac{\xi}{a}\right)a^2\rho V$$
$$- \frac{m-n}{m+n}\frac{\xi(h-\xi)}{a^2}\left(1 + \frac{1}{3}\frac{h}{a} + \frac{7}{3}\frac{\xi}{a}\right)a^{-1}\rho V' \text{...(15),}$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \frac{1}{2}\frac{a}{h}\left(1 + \frac{\xi}{a}\right)R - \frac{1}{2}\frac{a}{h}\left(1 - \frac{2h-\xi}{a}\right)R' + \left(1 - \frac{1}{2}\frac{5m+n}{m+n}\frac{h}{a} + \frac{3m-n}{m+n}\frac{\xi}{a}\right)a^2\rho V$$
$$- \frac{1}{2}\left(1 - \frac{m-n}{m+n}\frac{h}{a} + \frac{3m-n}{m+n}\frac{\xi}{a}\right)a^{-1}\rho V' \text{...(16).}$$

§ 6. If we denote *Young's modulus* by E, the *bulk modulus* by k and *Poisson's ratio* by η, then

$$E = n(3m-n)/m, \quad k = m - n/3, \quad \eta = (m-n)/2m.$$

Using these, and retaining only lowest powers, we easily find from the results (13)—(16)

$$u/r = \frac{1-\eta}{2E}\frac{a}{h}F \quad \text{..(17),}$$

$$\frac{du}{dr} = -\frac{du}{d\xi} = -\frac{\eta}{E}\frac{a}{h}F \quad \text{..(18),}$$

$$\delta = \frac{1}{3k}\frac{a}{h}F \quad \text{..(19),}$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \frac{1}{2}\frac{a}{h}F \quad \text{.......................................(20)},$$

$$S = \pm\frac{1}{2}\frac{a}{h}F \quad \text{.......................................(21)},$$

where
$$F = R - R' + 2ah\rho V - a^{-2}h\rho V' \text{..............................(22)}.$$

These values of the strains and stresses may, under certain restrictions explained below, be called the "first approximations".

The quantity F is obviously, to the present degree of approximation, the resultant *per unit area of surface* of the entire radial force exerted by combined surface and bodily forces on the shell.

The necessary restrictions to the use of the results as first approximations will be easily grasped by considering the case when there are no bodily forces. In this case we must clearly have $\frac{h}{a}\frac{R}{R-R'}$ a small quantity in order that (17) may be a legitimate first approximation from (13); in other words if R and R' be of the same sign—i.e. both tensions or both pressures,—they must not be so nearly equal that their difference bears to their sum a ratio of the order borne by the thickness of the shell to its radius. The general conclusion is that the results (17)—(21) are not to be employed as first approximations when F is so small compared to the individual bodily and surface forces of which it is composed as to bear to them a ratio of the order h/a.

§ 7. We shall first consider the case when F is of the same order as its greatest components, and consequently (17)—(21) are satisfactory first approximations. The strains are then all approximately constant at every point of the thickness, and the same is true of the principal stresses $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$, whose directions are parallel to the surface. Also the radial stress, while rapidly varying along the thickness is, to a first approximation negligible compared to the other stresses. The important strains and stresses are in fact due to the stretching or shortening of the "fibres" parallel to the surface, which accompanies the increase or diminution of radius produced by the application of F. What the exact mode of application of F may be, whether it consist solely of bodily or solely of surface forces, or partly of both, and whether, if composed of surface forces, it be applied over the outer or the inner surface, is to a first approximation of no consequence. As regards the absolute magnitudes of the strains and stresses in this case, we see from (17)—(21), that they bear to the strains and stresses which a longitudinal traction of intensity F would produce in a long bar of the material ratios of the order $a : h$. This is a very important consideration, as it leads at once to a restriction in the value permissible to F: viz. that the ratio of F to the greatest traction permissible in a long bar of the material must be at most of the order h/a of small quantities. This is obvious at once on the stress-difference theory of rupture from the form of (21). It also follows at once from (17) and (18) from the mathematical condition that the strains must be small.

It also may in general be deduced on the greatest strain theory of rupture from (17) and (18), since either u/r or $\frac{du}{dr}$ must be positive. An exception to the latter proof would however arise if η were very small and F directed inwards.

§ 8. We have next the case when F is so small compared to its components that (17)—(21) cease to be satisfactory approximations, and we must fall back on the more general results (13)—(16). If we suppose that the bodily forces per unit of surface are small compared to the surface forces, or more generally that the resultants of the bodily and surface forces are separately very small, then, with the exception of $\widehat{rr}$ in so far as it depends on the bodily forces, all the strains and stresses are to a first approximation constant throughout the thickness. The fact that $\widehat{rr}$ is now of the same order as the other stresses is also important.

The limits allowable in the strains or stresses depend on the material, or on mathematical restrictions independent of the nature of the applied forces, and so these quantities may be as large in the present case as in the previous. The conclusion to be derived from a consideration of these limits in the present case is that the separate forces R, R' etc. may now be comparable in magnitude with the greatest traction permissible in a long bar of the material. In the present case the alteration of the radius is small and the consequent stretching but trifling, but the direct action of the applied load on its immediate neighbourhood is important.

§ 9. One general conclusion of considerable physical interest is obvious on inspection of (13) and (16). The terms in ξ/a inside all the brackets are positive, and thus the values of u—and so obviously of u/r—and of $\widehat{\theta\theta}$ or $\widehat{\phi\phi}$ are invariably numerically greatest over the inner surface of the shell.

§ 10. The variation in the value of the stresses $\widehat{\theta\theta}$, $\widehat{\phi\phi}$ with the distance from the surfaces is seldom of much consequence, but the variation of $\widehat{rr}$ is interesting in itself and important in the theory of thin shells. We shall consider it in the several cases when there are only surface forces over one of the two surfaces, and when there are only bodily forces.

In this and subsequent occasions certain curves called here "stress-gradient curves" will be found useful. In these the abscissa measures the distance from a surface of the shell, and the ordinate the corresponding value of the stress under consideration. In none of the cases occurring here is there any change in the sign of the stress as the distance from the surface alters, so for convenience the curves are all drawn on the positive side of the axis of abscissae. The same curve thus applies whether a surface force be a tension or a pressure. The rate at which the stress alters with the distance from a surface is measured by the tangent of the inclination to the axis of abscissae of the tangent to the stress-gradient curve. The numerical value of this tangent is here termed the "stress-gradient".

Take for instance the case of a force R' over the inner surface. Then by (15) the first approximation, viz.

$$\widehat{rr} = (\xi/h)\, R',$$

gives for the "stress-gradient curve" a straight line passing through the origin when the abscissa measures the distance from the outer or unstressed surface. The "stress-gradient" is thus to a first approximation uniform, precisely like the temperature gradient in the steady state of heat conduction through an infinite plate. To this degree of approximation each thin layer of the shell bears, as it were, its fair share of the applied surface force. Similar results clearly hold in the case of a normal force R over the outer surface, because $h - \xi$ is now the distance from the inner or unstressed surface.

Taking into account the second approximations we see that in the case of both R and R' the stress-gradient is steepest at the inner surface of the shell, and that the gradient continually diminishes as we approach the outer surface. In the accompanying figures 2 and 3 the thick lines BKE, DHA are the gradient curves in these two cases according to the second approximations, while the dotted straight lines answer to the first approximations.

Surface forces.

Force R.

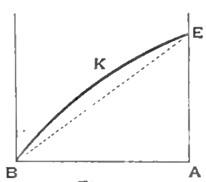


Fig 2

Force R'.

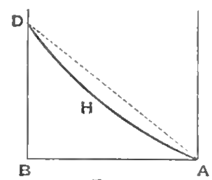


Fig 3

In both figures B represents the inner, A the outer surface, and BA the thickness.

In fig. 2 the force R—represented in magnitude by AE—acts on the outer surface; in fig. 3 the force R'—represented in magnitude by BD—acts on the inner surface. In both cases the dotted lines are parallel to the tangents to the second approximation curves at the point where $\xi = h/2$—or what we may call the "mid-thickness".

The cases when bodily forces act may also be represented by stress-gradient curves. Thus fig. 4 applies to the case of bodily forces derived from a potential Vr^2, and fig. 5 to bodily forces derived from $V'r^{-1}$; in both figures B represents the inner, A the outer surface. The dotted curves in both figures refer to the first approximations. They are parabolas whose vertices answer to the mid-thickness, and whose axes are perpendicular to the axis of abscissae.

The thick line curves BDA answer to the second approximations. In fig. 4 the points where the dotted and thick line curves intersect answers to the mid-thickness.

Bodily forces.

Force from Vr^2

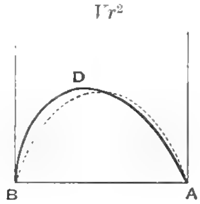


Fig 4

Force from $V'r^{-1}$

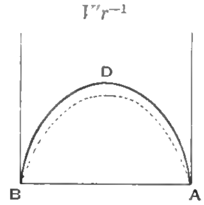


Fig 5

In both cases the gradients are steepest at the two surfaces, where the ordinates are zero, and the gradient at the inner surface B is according to the second approximations slightly greater than that at the outer A. When the shell is very thin the difference between the ordinates of the dotted and thick line curves is much exaggerated in the figures.

§ 11. Before quitting the subject of uniform radial forces a few remarks on the relative magnitudes of the effects of bodily and surface forces may be of service. Let us confine our attention to the terms in V and R, because the same conclusions hold in the case of V' and R'.

The bodily force is to a first approximation, *i.e.* treating r as constant, $2\rho Va$ per unit of volume, or $2\rho Vah$ per unit of surface of the shell. Now from (17)—(22) we see that according to the first approximation all the strains, and likewise the stresses $\widehat{\theta\theta}$, $\widehat{\phi\phi}$, arising from the bodily force bear to those arising from the surface force precisely the ratio $2\rho Vah : R$ that the bodily force measured per unit of surface bears to the surface force. As appears, however, from (15) the radial stress arising from the bodily force bears to that arising from the surface force a ratio of the order $(2\rho Vah)(h/a) : R$.

If then a radial force act over one only of the two surfaces of a thin shell, the strains it produces, and the stresses whose directions are perpendicular to the radius, are precisely of the same order of magnitude as those produced by a bodily force the same in direction at every point of the thickness, whose total amount per unit of surface is the same; the radial stress however due to the surface force is, except in the immediate neighbourhood of the unstressed surface, very much larger than that due to the bodily force.

§ 12. Before considering the two other classes of displacements it is necessary to explain the form under which the surface forces are given. In a complicated problem like the present, in order to avoid cumbrous mathematical analysis, care must be taken to let the solution follow its natural channel. The following method of treatment is very forcibly suggested by the form of the general solution.

Let T_i, $\mathbf{T}_i$ represent surface spherical harmonics of degree i, including constant coefficients. The case when i is fractional does not seem excluded from our general solution, but when, as in the present instance, the spherical surfaces are complete i will be a positive integer. Then if Θ and Φ be the tangential components of the forces applied at one of the surfaces, say $r=a$, in the directions θ, ϕ at the point considered, we are to present Θ and Φ in the respective forms

$$\Theta = \Sigma\left[\frac{dT_i}{d\theta} + \frac{1}{\sin\theta}\frac{d\mathbf{T}_i}{d\phi}\right] \quad\quad (23),$$

$$\Phi = \Sigma\left[\frac{1}{\sin\theta}\frac{dT_i}{d\phi} - \frac{d\mathbf{T}_i}{d\theta}\right] \quad\quad (24).$$

The summation is with respect to i. The surface forces are practically split into two sets, one derivable from a "potential function" ΣT_i, the other from a "stream function" $\Sigma\mathbf{T}_i$.

It will, I believe, be found that in most practical cases the tangential surface forces fall naturally into this shape, but if any difficulty should be experienced in giving them this form recourse may be had to the following results. Multiply (23) by $\sin\theta$ and differentiate with respect to θ, then add to (24) differentiated with respect to ϕ. This eliminates the $\mathbf{T}$ harmonics. Then employing the equation

$$i(i+1)Y_i + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY_i}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Y_i}{d\phi^2} = 0 \quad \text{.....................(25)},$$

satisfied by a surface harmonic Y_i of the ith degree, we find

$$\Sigma[i(i+1)T_i] = -\frac{1}{\sin\theta}\left\{\frac{d}{d\theta}(\Theta\sin\theta) + \frac{d\Phi}{d\phi}\right\} \quad \text{.......................(26).}$$

Next multiply (24) by $\sin\theta$, then differentiate with respect to θ and subtract from (23) differentiated with respect to ϕ. This eliminates the T functions and leads with the help of (25) to

$$\Sigma[i(i+1)\mathbf{T}_i] = \frac{1}{\sin\theta}\left[-\frac{d\Theta}{d\phi} + \frac{d}{d\theta}(\Phi\sin\theta)\right] \quad \text{.......................(27).}$$

Expanding the right-hand sides of (26) and (27) in the ordinary way, and equating harmonics of the same degrees on the two sides of the equations, we have at once the values of all the T and $\mathbf{T}$ functions.

The radial surface forces are supposed presented in the form of surface harmonics and are denoted by ΣR_i.

§ 13. For some purposes it might have been more advantageous to group the radial surface forces along with that part of the tangential surface forces expressed by the T functions, deducing both from a potential

$$\Sigma[(r^i/a^{i-1})Q_i + r^{-i-1}a^{i+2}Q_{-i-1}],$$

where Q_i, Q_{-i-1} are surface harmonics of the ith degree, and r is put equal a after differentiation. The relations between Q_i, Q_{-i-1} and R_i, T_i are simply

$$\left.\begin{aligned} R_i &= iQ_i - (i+1)Q_{-i-1}, \\ T_i &= Q_i + Q_{-i-1} \end{aligned}\right\} \quad \text{.....................................(28).}$$

§ 14. In dealing with the solution of the equations (5) in terms of surface harmonics it will suffice to take as a type the terms which contain harmonics of a single degree. Thus suppose the bodily forces derivable from the potential

$$r^iV_i + r^{-i-1}V_{-i-1},$$

where V_i, V_{-i-1} are surface harmonics of the ith degree.

The typical terms in the displacements are then those given in p. 268 of my previous paper. Slightly altering the notation, we may write

$$\delta = r^i\left(-\frac{\rho}{m+n}V_i + Y_i\right) + r^{-i-1}\left(-\frac{\rho}{m+n}V_{-i-1} + Y_{-i-1}\right) \quad \text{..............(29),}$$

$$u = -r^{i+1}\left\{\frac{i+2}{2(2i+3)}\frac{\rho V_i}{m+n} + \frac{im-2n}{2(2i+3)n}Y_i\right\} + r^{i-1}Z_i$$

$$-r^{-i}\left\{\frac{i-1}{2(2i-1)}\frac{\rho V_{-i-1}}{m+n} + \frac{(i+1)m+2n}{2(2i-1)n}Y_{-i-1}\right\} + r^{-i-2}Z_{-i-1} \text{......(30)},$$

$$v = \frac{d}{d\theta}\left[-\frac{r^{i+1}}{2(2i+3)}\left\{\frac{\rho V_i}{m+n} + \frac{(i+3)m+2n}{(i+1)n}Y_i\right\} + \frac{1}{i}r^{i-1}Z_i + \frac{r^{-i}}{2(2i-1)}\left\{\frac{\rho V_{-i-1}}{m+n} + \frac{(i-2)m-2n}{in}Y_{-i-1}\right\}\right.$$

$$\left.-\frac{1}{i+1}r^{-i-2}Z_{-i-1}\right] + \frac{1}{\sin\theta}\frac{d}{d\phi}\left[r^iX_i + r^{-i-1}X_{-i-1}\right] \text{...(31)},$$

$$w = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{r^{i+1}}{2(2i+3)}\left\{\frac{\rho V_i}{m+n} + \frac{(i+3)m+2n}{(i+1)n}Y_i\right\} + \frac{1}{i}r^{i-1}Z_i\right.$$

$$\left.+\frac{r^{-i}}{2(2i-1)}\left\{\frac{\rho V_{-i-1}}{m+n} + \frac{(i-2)m-2n}{in}Y_{-i-1}\right\} - \frac{1}{i+1}r^{-i-2}Z_{-i-1}\right] - \frac{d}{d\theta}\left[r^iX_i + r^{-i-1}X_{-i-1}\right] \text{...(32)}.$$

Here Y_i, Y_{-i-1}, Z_i, Z_{-i-1}, X_i, X_{-i-1} are surface harmonics of degree i whose form depends on the surface forces, and in the case of the first four harmonics also on V_i and V_{-i-1}. The letters may be regarded as including arbitrary constants to be determined by the surface conditions. In my previous paper dashed letters Y_i' etc. stood in place of Y_{-i-1} etc.; also $X_i \sin\theta$ stood in place of $\frac{1}{\sin\theta}\frac{dX_i}{d\phi}$ and w_i stood for $-\frac{dX_i}{d\theta}$. Thus the present X_i has not precisely the same meaning as that letter bore previously. The present notation has the advantage of replacing two letters—connected through an equation—by a single letter; but it in no respect adds to or takes from the solution as first enunciated.

§ 15. In order to apply the surface conditions (6) we require the typical terms in the expressions for $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{r\phi}$. Referring to (3) we easily deduce from (29)—(32) the following values:

$$\widehat{rr} = -\frac{r^i}{2i+3}\left[\{(2i+3)m + (i^2+i-1)n\}\frac{\rho V_i}{m+n} + \{(i^2-i-3)m+n\}Y_i\right] + 2(i-1)nr^{i-2}Z_i$$

$$+\frac{r^{-i-1}}{2i-1}\left[-\{(2i-1)m - (i^2+i-1)n\}\frac{\rho V_{-i-1}}{m+n} + \{(i^2+3i-1)m+n\}Y_{-i-1}\right] - 2(i+2)nr^{-i-3}Z_{-i-1}$$

$$\text{......(33)},$$

$$\widehat{r\theta} = \frac{d}{d\theta}\left[-\frac{i+1}{2i+3}nr^i\frac{\rho V_i}{m+n} - \frac{i(i+2)m-n}{(i+1)(2i+3)}r^iY_i + \frac{2(i-1)n}{i}r^{i-2}Z_i\right.$$

$$\left.-\frac{i}{2i-1}nr^{-i-1}\frac{\rho V_{-i-1}}{m+n} - \frac{(i^2-1)m-n}{i(2i-1)}r^{-i-1}Y_{-i-1} + \frac{2(i+2)}{i+1}nr^{-i-3}Z_{-i-1}\right]$$

$$+n\frac{1}{\sin\theta}\frac{d}{d\phi}\left[(i-1)r^{i-1}X_i - (i+2)r^{-i-2}X_{-i-1}\right] \text{...(34)},$$

$$\widehat{r\phi} = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{i+1}{2i+3}nr^i\frac{\rho V_i}{m+n} - \frac{i(i+2)m-n}{(i+1)(2i+3)}r^iY_i + \frac{2(i-1)n}{i}r^{i-2}Z_i\right.$$
$$\left. - \frac{i}{2i-1}nr^{-i-1}\frac{\rho V_{-i-1}}{m+n} - \frac{(i^2-1)m-n}{i(2i-1)}r^{-i-1}Y_{-i-1} + \frac{2(i+2)}{i+1}nr^{-i-3}Z_{-i-1}\right]$$
$$- n\frac{d}{d\theta}\left[(i-1)r^{i-1}X_i - (i+2)r^{-i-2}X_{-i-1}\right] \quad \ldots(35).$$

It should be noticed that in the typical terms in the displacements and the stresses, the terms in V_{-i-1}, Y_{-i-1} etc. may be deduced from those in V_i, Y_i etc. by simply writing $(-i-1)$ for $(+i)$ throughout all coefficients and indices; the converse mode of deduction is of course equally correct. This fact is an important aid to simplifying the algebraical work of evaluating the arbitrary constants.

§ 16. The surface values of the stresses (33), (34) and (35) are to be equated to the given surface forces. Thus over $r=a$ we must have

$$\widehat{rr} = R_i;\ \widehat{r\theta} = \frac{dT_i}{d\theta} + \frac{1}{\sin\theta}\frac{d\mathbf{T}_i}{d\phi};\ \widehat{r\phi} = \frac{1}{\sin\theta}\frac{dT_i}{d\phi} - \frac{d\mathbf{T}_i}{d\theta};$$

and over $r=b$

$$\widehat{rr} = R_i';\ \widehat{r\theta} = \frac{dT_i'}{d\theta} + \frac{1}{\sin\theta}\frac{d\mathbf{T}_i'}{d\phi};\ \widehat{r\phi} = \frac{1}{\sin\theta}\frac{dT_i'}{d\phi} - \frac{d\mathbf{T}_i'}{d\theta};$$

where R_i, R_i' etc. are surface harmonics as explained above.

These six equations obviously lead to the following six:—

$$(i-1)na^{i-1}X_i - (i+2)na^{-i-2}X_{-i-1} = \mathbf{T}_i \quad \ldots\ldots(36),$$

$$(i-1)nb^{i-1}X_i - (i+2)nb^{-i-2}X_{-i-1} = \mathbf{T}_i' \quad \ldots\ldots(37),$$

$$-\frac{(i^2-i-3)m+n}{2i+3}a^iY_i + 2(i-1)na^{i-2}Z_i + \frac{(i^2+3i-1)m+n}{2i-1}a^{-i-1}Y_{-i-1} - 2(i+2)na^{-i-3}Z_{-i-1}$$
$$= R_i + \frac{(2i+3)m+(i^2+i-1)n}{2i+3}a^i\frac{\rho V_i}{m+n} + \frac{(2i-1)m-(i^2+i-1)n}{2i-1}a^{-i-1}\frac{\rho V_{-i-1}}{m+n} \quad \ldots(38),$$

$$-\frac{i(i+2)m-n}{(i+1)(2i+3)}a^iY_i + \frac{2(i-1)}{i}na^{i-2}Z_i - \frac{(i^2-1)m-n}{i(2i-1)}a^{-i-1}Y_{-i-1} + \frac{2(i+2)}{i+1}na^{-i-3}Z_{-i-1}$$
$$= T_i + \frac{i+1}{2i+3}na^i\frac{\rho V_i}{m+n} + \frac{i}{2i-1}na^{-i-1}\frac{\rho V_{-i-1}}{m+n} \quad \ldots\ldots(39),$$

$$-\frac{(i^2-i-3)m+n}{2i+3}b^iY_i + 2(i-1)nb^{i-2}Z_i + \frac{(i^2+3i-1)m+n}{2i-1}b^{-i-1}Y_{-i-1} - 2(i+2)nb^{-i-3}Z_{-i-1}$$
$$= R_i' + \frac{(2i+3)m+(i^2+i-1)n}{2i+3}b^i\frac{\rho V_i}{m+n} + \frac{(2i-1)m-(i^2+i-1)n}{2i-1}b^{-i-1}\frac{\rho V_{-i-1}}{m+n} \quad \ldots(40),$$

$$-\frac{i(i+2)m-n}{(i+1)(2i+3)}b^iY_i + \frac{2(i-1)}{i}nb^{i-2}Z_i - \frac{(i^2-1)m-n}{i(2i-1)}b^{-i-1}Y_{-i-1} + \frac{2(i+2)}{i+1}nb^{-i-3}Z_{-i-1}$$
$$= T_i' + \frac{i+1}{2i+3}nb^i\frac{\rho V_i}{m+n} + \frac{i}{2i-1}nb^{-i-1}\frac{\rho V_{-i-1}}{m+n} \quad \ldots\ldots(41).$$

These six equations clearly constitute two independent sets; the first set, comprising (36) and (37), determines the two unknowns X_i and X_{-i-1}; the second set, comprising (38), (39), (40) and (41), determines the four unknowns Y_i, Z_i, Y_{-i-1}, Z_{-i-1}. The equations are to be regarded as simple equations, in which the right-hand sides are known quantities.

There are no terms in V_i or V_{-i-1} on the right of (36) and (37), and so the values of X_i and X_{-i-1} are independent alike of the bodily forces and of the surface forces derivable from a potential T_i.

Again there are no terms in X_i or X_{-i-1} in the expressions (29) and (30) for δ and u; thus the displacements depending on X_i and X_{-i-1} do not contribute to the dilatation and have no radial component. They constitute what were termed above "pure transverse displacements". Owing to their great simplicity it is convenient to regard them as next in order to the pure radial displacements.

CLASS II. *Pure transverse displacements.*

§ 17. From (36) and (37)

$$\left.\begin{aligned} X_i &= (a^{i+2}\mathbf{T}_i - b^{i+2}\mathbf{T}_i') \div \{(i-1)\,n\,(a^{2i+1} - b^{2i+1})\}, \\ X_{-i-1} &= (ab)^{i+2}\,(b^{i-1}\mathbf{T}_i - a^{i-1}\mathbf{T}_i') \div \{(i+2)\,n\,(a^{2i+1} - b^{2i+1})\} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(42).$$

The corresponding displacements are by (31) and (32)

$$v = \frac{1}{n\,(a^{2i+1} - b^{2i+1})}\,\frac{1}{\sin\theta}\,\frac{d}{d\phi}\left[\frac{r^i\,(a^{i+2}\mathbf{T}_i - b^{i+2}\mathbf{T}_i')}{i-1} + \frac{r^{-i-1}(ab)^{i+2}(b^{i-1}\mathbf{T}_i - a^{i-1}\mathbf{T}_i')}{i+2}\right] \ldots\ldots\ldots\ldots(43),$$

$$w = -\frac{1}{n\,(a^{2i+1} - b^{2i+1})}\,\frac{d}{d\theta}\,[\text{same expression as in square brackets in value of } v]\ldots\ldots\ldots\ldots(44).$$

For such displacements δ, as already stated, is zero and the only stresses existent are $\widehat{r\theta}$, $\widehat{r\phi}$ and $\widehat{\theta\phi}$. The two former are given by

$$\widehat{r\theta} = \frac{1}{a^{2i+1} - b^{2i+1}}\,\frac{1}{\sin\theta}\,\frac{d}{d\phi}\,[r^{i-1}\,(a^{i+2}\mathbf{T}_i - b^{i+2}\mathbf{T}_i') - r^{-i-2}\,(ab)^{i+2}\,(b^{i-1}\mathbf{T}_i - a^{i-1}\mathbf{T}_i')]\ldots\ldots(45),$$

$$\widehat{r\phi} = -\frac{1}{a^{2i+1} - b^{2i+1}}\,\frac{d}{d\theta}\,[\text{same expression as in square brackets in value of } \widehat{r\theta}\,]\ldots\ldots(46).$$

Having regard to (3) and (25) we may throw the value of $\widehat{\theta\phi}$ into the form

$$\widehat{\theta\phi} = -\frac{r^{-1}}{a^{2i+1} - b^{2i+1}}\left[i\,(i+1) + 2\,\frac{d^2}{d\theta^2}\right][\text{same expression as in square brackets in (43)}]\ldots(47).$$

The case when $\mathbf{T}_i$, $\mathbf{T}_i'$ are zonal harmonics merits special attention on account of its great simplicity; for it v and $\widehat{r\theta}$ are everywhere zero.

In the case of a thin shell we find from (43) and (44) as approximate values with our previous notation

$$v = \frac{a^2}{(i-1)(i+2)nh} \frac{1}{\sin\theta} \frac{d}{d\phi} \left[\left(1 + \frac{h-\xi}{a}\right) \mathbf{T}_i - \left(1 - \frac{2h+\xi}{a}\right) \mathbf{T}_i' \right] \quad \text{.............(48)},$$

$$w = -\frac{a^2}{(i-1)(i+2)nh} \frac{d}{d\theta} \text{[same expression as in square brackets in (48)]} \quad \text{...(49)},$$

$$\widehat{r\theta} = \frac{1}{\sin\theta} \frac{d}{d\phi} \left[\frac{h-\xi}{h} \left(1 + \frac{2\xi}{a}\right) \mathbf{T}_i + \frac{\xi}{h} \left(1 - 2\frac{h-\xi}{a}\right) \mathbf{T}_i' \right] \quad \text{..........................(50)},$$

$$\widehat{r\phi} = -\frac{d}{d\theta} \text{[same expression as in square brackets in (50)]} \quad \text{... ....(51)},$$

$$\widehat{\theta\phi} = -\frac{a}{(i-1)(i+2)h} \left[i(i+1) + 2\frac{d^2}{d\theta^2} \right] \left[\left(1 + \frac{h}{a}\right) \mathbf{T}_i - \left(1 - \frac{2h}{a}\right) \mathbf{T}_i' \right\} \quad \text{..........(52)}.$$

Owing to the similarity in form we need consider only one of the two displacements and one of the two stresses $\widehat{r\theta}$ and $\widehat{r\phi}$. We may most conveniently select w and $\widehat{r\phi}$, because in the case when $\mathbf{T}_i$ and $\mathbf{T}_i'$ are zonal harmonics v and $\widehat{r\theta}$ vanish.

Attention must be paid to the directions in which the surface forces are measured.

At the outer surface the positive direction along ϕ is that in which ϕ increases, but at the inner surface the positive direction is that in which ϕ diminishes. Thus the applied forces at the two surfaces are in the same or in opposite directions at corresponding points,—*i.e.* points on the same radius vector,—according as

$$\frac{d\mathbf{T}_i}{d\theta} \text{ and } \frac{d\mathbf{T}_i'}{d\theta}$$

are of opposite signs or of the same sign.

§ 18. There are two principal cases, of a character precisely analogous to the two that presented themselves in the case of pure radial displacements. In the first case

$$\frac{d\mathbf{T}_i}{d\theta} - \frac{d\mathbf{T}_i'}{d\theta}$$

is of the same order as the greater of the two $\frac{d\mathbf{T}_i}{d\theta}$ and $\frac{d\mathbf{T}_i'}{d\theta}$; in the second case the former quantity is small compared to the latter. In the first case the statical resultant of the forces applied at corresponding points on the two surfaces is of the same order of magnitude as the greater of the forces applied at these points. In the second case the forces at the two surfaces are approximately equal and opposite. In the first case we get as satisfactory first approximations

$$w = -\frac{a^2}{(i-1)(i+2)nh} \frac{d}{d\theta} (\mathbf{T}_i - \mathbf{T}_i') \quad \text{...(53)},$$

$$\widehat{\theta\phi}/n = -\frac{a}{(i-1)(i+2)nh} \left[i(i+1) + 2\frac{d^2}{d\theta^2} \right] [\mathbf{T}_i - \mathbf{T}_i'] \quad \text{............... ....(54)}.$$

Thus the displacements, and the shearing strain and stress whose axes θ, ϕ are parallel to the surfaces, have nearly constant values throughout the thickness; also this strain and stress bear to the other strains and stresses $\widehat{r\phi}/n$, $\widehat{r\phi}$ etc. ratios of the order $a : h$ and so are relatively very large. In the case of these and similar statements it must be remembered that the magnitude of surface harmonics varies over the surface, so that terms which at most places are far the most important are zero, and may be vanishingly small compared to the other terms, at certain points or along certain curves. In order to avoid the prolixity that the continual reference to such special loci would entail, it will be assumed in what follows that the reader keeps the necessity of such limitations continually in view. He should notice that if either $\mathbf{T}_i - \mathbf{T}_i'$ or its differentials with respect to a variable it contains be everywhere very small, while $\mathbf{T}_i$ and $\mathbf{T}_i'$ themselves have their maxima values considerable, the harmonics must be of the same form and not merely of the same degree. Also near loci where the principal terms in a displacement vanish, the other terms may largely predominate, but the displacement all the same will be but small compared to the values it possesses where the principal terms are largest.

To return to our consideration of the case when $\frac{d}{d\theta}(\mathbf{T}_i - \mathbf{T}_i')$ is not small, we see that the conclusion it leads to is that when in the neighbourhood of a point on the surface there is everywhere a considerable resultant tangential force,—the forces tending to pull round the surface in the same direction,—there is a large displacement in this direction, and the strains and stresses whose directions are parallel to the surface tend to become large. The magnitude of these strains and stresses imposes an obvious limit to the magnitude of the resultant of the applied forces. Noticing that a shearing strain σ is equivalent to an extension $\sigma/2$ and a compression $-\sigma/2$ along the directions bisecting its axes, we should deduce from (54), by means either of the greatest strain theory or of the mathematical condition that the strains must be small, the conclusion that the ratio of the resultant of the tangential forces at corresponding points on the two surfaces to the greatest traction permissible in a long bar of the material may be at most of the order h/a of small quantities.

§ 19. We now pass to the case when $\frac{d}{d\theta}(\mathbf{T}_i - \mathbf{T}_i')$ bears to $\frac{d\mathbf{T}_i}{d\theta}$ a ratio of the order $h : a$ for all values of θ and ϕ, *i.e.* when the tangential forces over the two surfaces are derived from the same harmonics and are at corresponding points nearly equal and opposite. It is easily seen from (48)—(52) that *all* the strains and stresses are now to a first approximation constant along the thickness. The stresses are now also all of the same order of magnitude, and the same is true of course of the strains. The order of magnitude is the same as for the stresses and strains in a long bar of the material subjected to a longitudinal traction of similar magnitude to the force on one of the surfaces of the shell; and thus this force may now be of the same order as the greatest traction permissible in a long bar.

§ 20. As yet nothing has been said as to the influence of the degree of the harmonic from which the surface forces are derived: but this is of considerable interest and claims attention. From (48) and (49) we see that for given maxima values of the surface forces—*i.e.* of $\frac{d\mathbf{T}_i}{d\theta}$ etc.—the displacements vary approximately as i^{-2} when i is large, and so fall off very rapidly as the degree of the harmonic increases. The formulae (50) and (51) do not contain i explicitly; thus $\widehat{r\theta}$, $\widehat{r\phi}$ and the corresponding strains depend to the present degree of approximation only on the *magnitude* of the applied forces. A general law applicable to $\widehat{\theta\phi}$ is not so easily laid down.

In a general way, when i is large we may regard the ratio of the maxima values of $\frac{d\mathbf{T}_i}{d\theta}$ to those of $\mathbf{T}_i$ as being of the order $i:1$. We thus conclude that when i is large the values of $\widehat{\theta\phi}$ and the corresponding strain vary for a given magnitude in the surface forces inversely as i. A large value in i implies a rapid fluctuation in the magnitude and sign—*i.e.* in the direction relative to θ and ϕ—of the resultant of the forces applied over a surface, the area throughout which this resultant retains one sign becoming more and more restricted in the direction parallel to θ as i increases. This consideration explains the rapid diminution in the displacements as i increases. Take for simplicity the case when $\mathbf{T}_i$ is a zonal harmonic, when the surface force is everywhere perpendicular to the axis of the harmonic and has a constant value round the perimeter of any small circle whose plane is perpendicular to this axis. When $i=2$ the surface forces vanish only at what we may call the "poles" and the "equator". The forces over one of the two hemispheres tend to twist the sphere round the axis of the harmonic in one direction, and the forces on the opposite hemisphere have an equal tendency in the opposite direction. It is thus obvious that as we leave the equator, where the displacement will be nil, and travel towards one of the poles along a meridian, the action of the forces over the successive zones into which we may suppose the surface divided by "parallels of latitude" will all conspire, so that each zone will be turned through a small angle relative to the preceding zone in the direction of the forces. To find where the displacement is a maximum we notice that

$$w \propto -\frac{dP_2}{d\theta} \propto 3\sin\theta\cos\theta,$$

so that w is a maximum in latitude 45°. The angular displacement $w/a\sin\theta$ increases, as we have said, right up to the poles, but after latitude 45° the linear displacement falls off owing to the diminution in the radii of the parallels of latitude.

Now if we take for comparison $i=4$, we get

$$\widehat{r\phi} \propto w \propto \sin\theta\cos\theta\,(7\cos^2\theta - 3),$$

so that the direction of the surface forces and of the displacement changes sign not only at the equator but also in the latitudes $\sin^{-1}\sqrt{3/7}$, or a little under 41°. As we travel from the equator to a pole the rotations of the successive elementary zones are in the same direction only till we reach the latitude $\sin^{-1}\sqrt{1/7}$, or about $22\frac{1}{5}$°, where $w/\sin\theta$ is a maximum, and the latitude where the displacement w is a maximum is only about 21°.

There is thus much less room when $i=4$ than when $i=2$ for the cumulative effect of the rotations of the elementary zones to produce a large displacement; and obviously as i increases this is more and more the case, because the parallels of latitude where the surface forces and the displacement vanish and change sign become increasingly numerous.

A general idea of the reason why the stress $\widehat{\theta\phi}$ and the corresponding strain diminish as i increases seems also easily attainable. The strain consists in a shearing of the parallels of latitude on the same surface of the shell relatively to one another. Now suppose a long flat bar of uniform breadth and thickness held at both ends to be acted on in its plane by a series of forces of intensity $+P$ on one half of its length and $-P$ on the other all perpendicular to the length. Then it is easily proved that the maximum shearing force over a cross section diminishes rapidly as l diminishes though P remain the same. This is of course intended only for a very rough illustration of what happens, as the conditions it supposes differ widely from those of the actual case.

As regards $\widehat{r\theta}$, $\widehat{r\phi}$, since at the surfaces they must equal the applied forces, it is obvious *a priori* that the magnitude of their principal terms can not depend on the degree of the harmonic.

§ 21. The stress $\widehat{\theta\phi}$, as we have seen, has under ordinary conditions a nearly constant value throughout the thickness, but the variations of the other stresses along the thickness are always rapid unless the forces at corresponding points on the two surfaces are nearly equal and opposite. To consider the law of these variations, let Θ, Φ denote the *total* components parallel to θ and ϕ of the forces over the outer surface,—these forces being assumed of course to come from one or a series of the $\mathbf{T}$ functions—and let Θ' and Φ' be the corresponding quantities for the inner surface. Then from (50) and (51) we find as our second approximations

$$\widehat{r\theta} = \frac{h-\xi}{h}\left(1+\frac{2\xi}{a}\right)\Theta + \frac{\xi}{h}\left(1-2\frac{h-\xi}{a}\right)\Theta' \quad\text{.........................(55)},$$

$$\widehat{r\phi} = \frac{h-\xi}{h}\left(1+\frac{2\xi}{a}\right)\Phi + \frac{\xi}{h}\left(1-2\frac{h-\xi}{a}\right)\Phi' \quad\text{.........................(56)}.$$

It is certainly noteworthy that the law of variation of these stresses along the thickness is, to so close an approximation, the same for all forces applied over one only of the surfaces, whatever be the degree or degrees of the harmonic term or terms from which they are derived. A similar conclusion as to the variation of the displacements along the thickness follows from (48) and (49), but the *amplitude* of the displacements depends on the degrees of the harmonics as well as on the absolute magnitudes of the surface forces.

Comparing (55) and (56) with (15), we see that the law of variation of $\widehat{r\theta}$ or $\widehat{r\phi}$ along the thickness of a thin shell for a tangential force over either surface is precisely the same as the law of variation of $\widehat{rr}$ in the case of a uniform normal force over the same surface. Thus the stress gradient curve 2, § 10, will apply to the case of tangential forces derived from stream functions over the outer surface, and the curve 3 to the case of tangential forces over the inner surface.

§ 22. Before quitting the subject of pure tangential displacements it is necessary to point out that, in general, surface forces derived from a harmonic of degree 1 must be excluded from our solution. The reason will appear from a consideration of the simplest case, that of the zonal harmonic P_1.

Thus put $\mathbf{T}_1 = \Phi_1 P_1$, $\mathbf{T}_1' = \Phi_1' P_1$, where Φ_1, Φ_1' are constants. Thence, since

$$-\frac{dP_1}{d\theta} = \sin\theta,$$

we have

$$\left.\begin{array}{l} \widehat{r\phi} = \Phi_1 \sin\theta \text{ over } r = a, \\ \widehat{r\phi} = \Phi_1' \sin\theta \quad ,, \quad r = b \end{array}\right\} \qquad (57).$$

The forces over either one of the surfaces clearly all tend to turn the shell in the same direction round $\theta = 0$, the numerical magnitudes of the resultant couples being $\frac{8}{3}\pi a^3 \Phi_1$ for the outer, and $\frac{8}{3}\pi b^3 \Phi_1'$ for the inner surface. Unless these couples be equal and opposite there will not be equilibrium. We shall first show that when there is equilibrium our solution applies.

For equilibrium we must have

$$\mathbf{T}_1'/\mathbf{T}_1 = \Phi_1'/\Phi_1 = (a/b)^3.$$

Substituting in (44), we see that the coefficient of r takes the form $\frac{0}{0}$ and so appears indeterminate. The corresponding terms however in (45), (46) and (47) contribute nothing to the stresses and consequently nothing to the strains, and so this term has nothing whatever to do with the elastic problem. A displacement $w \propto r \sin\theta$ is in fact a rigid body rotation round $\theta = 0$, and the magnitude of such a displacement is fixed by other than elastic conditions.

We need thus consider only the second term in (44), or may take

$$w = -\frac{\Phi}{3n} a^3 r^{-2} \sin\theta \qquad (58).$$

This is the complete answer to the elastic solid problem in the present case.

We have clearly, however, not obtained a complete explanation of the elastic solid aspects of the case $i = 1$.

It is obvious that the resultant couples over the two surfaces need not in a shell always balance one another, while, if the sphere is solid, equilibrium under forces of this kind over the one surface is impossible. When the applied forces are not in equilibrium motion will ensue, but elastic strains and stresses will exist during the motion. Their investigation requires account to be taken of the "reversed effective forces". When this is done it will, I believe, be found that when the initial circumstances are completely given the displacements, strains and stresses at any subsequent time, supposing the limits of perfect elasticity not to be exceeded, are as determinate as in any case of equilibrium.

The problem is an interesting one, but its present consideration would lead us too far afield.

CLASS III. *Mixed radial and transverse displacements.*

§ 23. The displacements are those represented in the formulae (30), (31), (32) by the terms in V_i, V_{-i-1} and the four harmonics Y_i, Z_i, Y_{-i-1}, Z_{-i-1} whose values are determined by the equations (38)—(41). Thus the displacements of this class depend on the bodily forces, the normal surface forces and that part of the tangential surface forces which is derived from the T, or potential, functions.

Let us consider the determinant whose terms are the coefficients of the four unknowns in equations (38), (39), (40), (41), each divided by n. Calling this Π we have

$$\Pi = \begin{vmatrix} -\frac{(i^2-i-3)m+n}{(2i+3)n}a^i, & 2(i-1)a^{i-2}, & \frac{(i^2+3i-1)m+n}{(2i-1)n}a^{-i-1}, & -2(i+2)a^{-i-3} \\ -\frac{i(i+2)m-n}{(i+1)(2i+3)n}a^i, & \frac{2(i-1)}{i}a^{i-2}, & -\frac{(i^2-1)m-n}{i(2i-1)n}a^{-i-1}, & \frac{2(i+2)}{i+1}a^{-i-3} \\ -\frac{(i^2-i-3)m+n}{(2i+3)n}b^i, & 2(i-1)b^{i-2}, & \frac{(i^2+3i-1)m+n}{(2i-1)n}b^{-i-1}, & -2(i+2)b^{-i-3} \\ -\frac{i(i+2)m-n}{(i+1)(2i+3)n}b^i, & \frac{2(i-1)}{i}b^{i-2}, & -\frac{(i^2-1)m-n}{i(2i-1)n}b^{-i-1}, & \frac{2(i+2)}{i+1}b^{-i-3} \end{vmatrix} \quad \text{......(59).}$$

Denote the coefficients of the members of the first row in the expanded determinant by the letters Π_{11}, Π_{12} etc., the coefficients of the members of the second row by Π_{21}, Π_{22} etc. and so on. Also for shortness let

$$\frac{\rho(m+n)^{-1}}{2i+3}\left[\left\{(2i+3)\frac{m}{n}+i^2+i-1\right\}(a^i\Pi_{11}+b^i\Pi_{31})+(i+1)(a^i\Pi_{21}+b^i\Pi_{41})\right]=\varpi_1 \quad \text{......(60),}$$

$$\frac{\rho(m+n)^{-1}}{2i+3}\left[\left\{(2i+3)\frac{m}{n}+i^2+i-1\right\}(a^i\Pi_{12}+b^i\Pi_{32})+(i+1)(a^i\Pi_{22}+b^i\Pi_{42})\right]=\varpi_2 \quad \text{......(61),}$$

$$\frac{\rho(m+n)^{-1}}{2i+3}\left[\left\{(2i+3)\frac{m}{n}+i^2+i-1\right\}(a^i\Pi_{13}+b^i\Pi_{33})+(i+1)(a^i\Pi_{23}+b^i\Pi_{43})\right]=\varpi_3 \quad \text{......(62),}$$

$$\frac{\rho(m+n)^{-1}}{2i+3}\left[\left\{(2i+3)\frac{m}{n}+i^2+i-1\right\}(a^i\Pi_{14}+b^i\Pi_{34})+(i+1)(a^i\Pi_{24}+b^i\Pi_{44})\right]=\varpi_4 \quad \text{......(63),}$$

$$\frac{\rho(m+n)^{-1}}{2i-1}\left[\left\{(2i-1)\frac{m}{n}-(i^2+i-1)\right\}(a^{-i-1}\Pi_{11}+b^{-i-1}\Pi_{31})+i(a^{-i-1}\Pi_{21}+b^{-i-1}\Pi_{41})\right]=\varpi_1' \text{...(64),}$$

$$\frac{\rho(m+n)^{-1}}{2i-1}\left[\left\{(2i-1)\frac{m}{n}-(i^2+i-1)\right\}(a^{-i-1}\Pi_{12}+b^{-i-1}\Pi_{32})+i(a^{-i-1}\Pi_{22}+b^{-i-1}\Pi_{42})\right]=\varpi_2' \text{...(65),}$$

$$\frac{\rho(m+n)^{-1}}{2i-1}\left[\left\{(2i-1)\frac{m}{n}-(i^2+i-1)\right\}(a^{-i-1}\Pi_{13}+b^{-i-1}\Pi_{33})+i(a^{-i-1}\Pi_{23}+b^{-i-1}\Pi_{43})\right]=\varpi_3' \text{...(66),}$$

$$\frac{\rho(m+n)^{-1}}{2i-1}\left[\left\{(2i-1)\frac{m}{n}-(i^2+i-1)\right\}(a^{-i-1}\Pi_{14}+b^{-i-1}\Pi_{34})+i(a^{-i-1}\Pi_{24}+b^{-i-1}\Pi_{44})\right]=\varpi_4' \text{...(67).}$$

Then

$$n\Pi Y_i \quad = R_i\Pi_{11} + T_i\Pi_{21} + R_i'\Pi_{31} + T_i'\Pi_{41} + V_i\varpi_1 + V_{-i-1}\varpi_1' \text{ ....(68)},$$

$$n\Pi Z_i \quad = R_i\Pi_{12} + T_i\Pi_{22} + R_i'\Pi_{32} + T_i'\Pi_{42} + V_i\varpi_2 + V_{-i-1}\varpi_2' \text{ ....(69)},$$

$$n\Pi Y_{-i-1} = R_i\Pi_{13} + T_i\Pi_{23} + R_i'\Pi_{33} + T_i'\Pi_{43} + V_i\varpi_3 + V_{-i-1}\varpi_3' \text{ ....(70)},$$

$$n\Pi Z_{-i-1} = R_i\Pi_{14} + T_i\Pi_{24} + R_i'\Pi_{34} + T_i'\Pi_{44} + V_i\varpi_4 + V_{-i-1}\varpi_4' \text{ ....(71)}.$$

§ 24. This constitutes from a purely mathematical standpoint a complete solution of the problem, but to render it of practical value we must evaluate the determinants. We find

$$\Pi = 4(i-1)(i+2)(ab)^{-2}\,\overline{\Pi} \div \{i^2(i+1)^2(2i-1)(2i+3)\} \text{ ....(72)},$$

where

$$\overline{\Pi} = (ab)^{-2i-2}\left[\left\{(2i^2+4i+3)\frac{m}{n} - (2i+1)\right\}\left\{(2i^2+1)\frac{m}{n} + 2i + 1\right\}(a^{2i-1} - b^{2i-1})(a^{2i+3} - b^{2i+3}) \right.$$
$$\left. - (i-1)\,i\,(i+1)(i+2)(2i-1)(2i+3)(m/n)^2(ab)^{2i-1}(a^2-b^2)^2\right] \text{ ...(73)};$$

also

$$\Pi_{11} = \frac{4(i-1)(i+2)\,a^{-i-3}b^{-2i-4}}{i^2(i+1)(2i-1)}\left[\left\{(2i^2+1)\frac{m}{n} + 2i + 1\right\}a^2(a^{2i-1} - b^{2i-1}) \right.$$
$$\left. + i(i+2)(2i-1)(m/n)\,b^{2i-1}(a^2-b^2)\right] \text{............(74)},$$

$$\Pi_{12} = \frac{2(i+2)\,a^{-i-3}\,b^{-2i-4}}{i(i+1)^2(2i-1)(2i+3)}\left[\left\{(2i^2+1)\frac{m}{n} + 2i + 1\right\}\left\{i(i+2)\frac{m}{n} - 1\right\}(a^{2i+3} - b^{2i+3}) \right.$$
$$\left. + (i-1)(i+1)(2i+3)\frac{m}{n}\left\{(i^2-1)\frac{m}{n} - 1\right\}b^{2i+1}(a^2-b^2)\right] \text{............(75)},$$

$$\Pi_{13} = \frac{-4(i-1)(i+2)\,a^{-i-3}\,b^{-5}}{i(i+1)^2(2i+3)}\left[\left\{(2i^2+4i+3)\frac{m}{n} - (2i+1)\right\}(a^{2i+3} - b^{2i+3}) \right.$$
$$\left. + (i-1)(i+1)(2i+3)(m/n)\,a^{2i+1}(a^2-b^2)\right] \text{............(76)},$$

$$\Pi_{14} = \frac{-2(i-1)\,a^{-i-1}\,b^{-3}}{i^2(i+1)(2i-1)(2i+3)}\left[\left\{(2i^2+4i+3)\frac{m}{n} - (2i+1)\right\}\left\{(i^2-1)\frac{m}{n} - 1\right\}b^2(a^{2i-1} - b^{2i-1}) \right.$$
$$\left. + i(i+2)(2i-1)\frac{m}{n}\left\{i(i+2)\frac{m}{n} - 1\right\}a^{2i-1}(a^2-b^2)\right] \text{............(77)},$$

$$\Pi_{21} = \frac{-4(i-1)(i+2)\,a^{-i-3}\,b^{-2i-4}}{i(i+1)(2i-1)}\left[\left\{(2i^2+1)\frac{m}{n} + 2i + 1\right\}a^2(a^{2i-1} - b^{2i-1}) \right.$$
$$\left. - (i+1)(i+2)(2i-1)\frac{m}{n}\,b^{2i-1}(a^2-b^2)\right] \text{...............(78)},$$

$$\Pi_{22} = \frac{-2(i+2)\,a^{-i-3}\,b^{-2i-4}}{i(i+1)(2i-1)(2i+3)}\left[\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}\left\{(i^2-i-3)\frac{m}{n}+1\right\}(a^{2i+3}-b^{2i+3})\right.$$
$$\left. -(i-1)\,i\,(2i+3)\frac{m}{n}\left\{(i^2+3i-1)\frac{m}{n}+1\right\}b^{2i+1}(a^2-b^2)\right] \quad\text{(79)},$$

$$\Pi_{23} = \frac{-4(i-1)(i+2)\,a^{-i-3}\,b^{-5}}{i(i+1)(2i+3)}\left[\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}(a^{2i+3}-b^{2i+3})\right.$$
$$\left. -(i-1)\,i\,(2i+3)\frac{m}{n}\,a^{2i+1}(a^2-b^2)\right] \quad\text{(80)},$$

$$\Pi_{24} = \frac{-2(i-1)\,a^{-i-1}\,b^{-3}}{i(i+1)(2i-1)(2i+3)}\left[\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}\times\right.$$
$$\left\{(i^2+3i-1)\frac{m}{n}+1\right\}b^2(a^{2i-1}-b^{2i-1})$$
$$\left. -(i+1)(i+2)(2i-1)\frac{m}{n}\left\{(i^2-i-3)\frac{m}{n}+1\right\}a^{2i-1}(a^2-b^2)\right] \quad\text{(81)},$$

$$\left.\begin{array}{l} \Pi_{31} = \Pi_{11} \text{ with } a \text{ and } b \text{ interchanged} \\ \Pi_{32} = \Pi_{12} \quad " \quad " \quad " \\ \Pi_{33} = \Pi_{13} \quad " \quad " \quad " \\ \Pi_{34} = \Pi_{14} \quad " \quad " \quad " \\ \Pi_{41} = \Pi_{21} \quad " \quad " \quad " \\ \Pi_{42} = \Pi_{22} \quad " \quad " \quad " \\ \Pi_{43} = \Pi_{23} \quad " \quad " \quad " \\ \Pi_{44} = \Pi_{24} \quad " \quad " \quad " \end{array}\right\} \quad\text{(82)}.$$

The last 8 relations are obvious, since the third and fourth rows of the complete determinant Π may be deduced from the first and second respectively by writing b for a. An inspection of the determinant also shows that we may deduce Π_{13} from Π_{11}, Π_{14} from Π_{12}, Π_{23} from Π_{21} and Π_{24} from Π_{22} by substituting $(-i-1)$ for $(+i)$. It is thus in reality necessary to calculate only 4 of the 16 minors.

We also find

$$\varpi_1 = \frac{4(i-1)(i+2)(ab)^{-2i-4}}{i^2(i+1)(2i-1)(2i+3)}\frac{\rho n}{m+n}\left[\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}\times\right.$$
$$\left\{(2i+3)\frac{m}{n}-1\right\}(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})$$
$$\left. +i(i+2)(2i-1)(2i+3)\frac{m}{n}\left(\frac{m}{n}+i\right)(ab)^{2i-1}(a^2-b^2)^2\right] \quad\text{(83)},$$

$$\varpi_2 = \frac{2(i+2)\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}}{(i+1)^2(2i-1)(2i+3)}\times$$
$$\left\{(i+2)\frac{m}{n}-1\right\}\rho\,(ab)^{-2i-4}(a^{2i+1}-b^{2i+1})(a^{2i+3}-b^{2i+3}) \quad\text{(84)},$$

$$\varpi_3 = \frac{-4(i-1)(i+2)(2i+1)}{(i+1)^2(2i+3)} \left\{(i+2)\frac{m}{n} - 1\right\} \rho (ab)^{-5} (a^2-b^2)(a^{2i+3} - b^{2i+3}) \text{ ....(85)},$$

$$\varpi_4 = \frac{-2(i-1)(i+2)}{(i+1)(2i+3)} \frac{m}{n} \left\{(i+2)\frac{m}{n} - 1\right\} \rho (ab)^{-3} (a^2-b^2)(a^{2i+1} - b^{2i+1}) \text{ ....(86)},$$

$$\varpi_1' = \frac{-4(i-1)(i+2)(2i+1)}{i^2(2i-1)} \left\{(i-1)\frac{m}{n} + 1\right\} \rho (ab)^{-2i-4} (a^2-b^2)(a^{2i-1} - b^{2i-1}) \text{ ....(87)},$$

$$\varpi_2' = \frac{-2(i-1)(i+2)}{i(2i-1)} \frac{m}{n} \left\{(i-1)\frac{m}{n} + 1\right\} \rho (ab)^{-2i-4} (a^2-b^2)(a^{2i+1} - b^{2i+1}) \text{ ....(88)},$$

$$\varpi_3' = \frac{4(i-1)(i+2)(ab)^{-2i-4}}{i(i+1)^2(2i-1)(2i+3)} \frac{\rho n}{m+n} \left[\left\{(2i^2+4i+3)\frac{m}{n} - (2i+1)\right\} \times \right.$$

$$\left\{(2i-1)\frac{m}{n} + 1\right\} (a^{2i-1} - b^{2i-1})(a^{2i+3} - b^{2i+3})$$

$$\left. - (i-1)(i+1)(2i-1)(2i+3)\frac{m}{n}\left(\frac{m}{n} - i - 1\right)(ab)^{2i-1}(a^2-b^2)^2\right] \text{ ....(89)},$$

$$\varpi_4' = \frac{2(i-1)}{i^2(2i-1)(2i+3)} \left\{(2i^2+4i+3)\frac{m}{n} - (2i+1)\right\} \times$$

$$\left\{(i-1)\frac{m}{n} + 1\right\} \rho (ab)^{-2i-2} (a^{2i-1} - b^{2i-1})(a^{2i+1} - b^{2i+1}) \text{ ....(90)}.$$

§ 25. Substituting the values just found for $\Pi_{11} \ldots \varpi_1 \ldots$ etc. in (68), (69), (70) and (71), we obtain the values of Y_i, Z_i, Y_{-i-1}, Z_{-i-1}; and inserting these in equations (29), (30), (31) and (32) we have the typical terms in the values of the dilatation and displacements explicitly determined. The solution so obtained, it must be remembered, includes only what we have denoted, § 23, "mixed radial and transverse displacements". It answers both to bodily and surface forces; the types of the former are derived from the potential (see § 14)

$$r^i V_i + r^{-i-1} V_{-i-1};$$

the latter have for their types (see § 16):

over $r = a$,

$$\widehat{rr} = R_i, \quad \widehat{r\theta} = \frac{dT_i}{d\theta}, \quad \widehat{r\phi} = \frac{1}{\sin\theta}\frac{dT_i}{d\phi},$$

over $r = b$,

$$\widehat{rr} = R_i', \quad \widehat{r\theta} = \frac{dT_i'}{d\theta}, \quad \widehat{r\phi} = \frac{1}{\sin\theta}\frac{dT_i'}{d\phi}.$$

The dilatation and displacements are as follows, $\overline{\Pi}$ being given by (73):

$$n\overline{\Pi}\delta = R_i\left[\frac{(i+1)(2i+3)r^i}{a^{i+1}b^{2i+2}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)a^2(a^{2i-1}-b^{2i-1})\right.\right.$$

$$\left.+\,i(i+2)(2i-1)(m/n)\,b^{2i-1}(a^2-b^2)\right\}$$

$$-\frac{i(2i-1)r^{-i-1}}{a^{i+1}b^{i}}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.\left.+\,(i-1)(i+1)(2i+3)(m/n)\,a^{2i+1}(a^2-b^2)\right\}\right]$$

$+\ R_i'$ [coefficient obtained from that of R_i by interchanging a and b]

$$+\,T_i\left[-\frac{i(i+1)(2i+3)}{a^{i+1}b^{2i+2}}r^i\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)a^2(a^{2i-1}-b^{2i-1})\right.\right.$$

$$\left.-\,(i+1)(i+2)(2i-1)(m/n)\,b^{2i-1}(a^2-b^2)\right\}$$

$$-\frac{i(i+1)(2i-1)}{a^{i+1}b^{i}}r^{-i-1}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.\left.-\,(i-1)\,i\,(2i+3)(m/n)\,a^{2i+1}(a^2-b^2)\right\}\right]$$

$+\ T_i'$ [coefficient obtained from that of T_i by interchanging a and b]

$$+\,\rho V_i\left[\frac{ir^i}{(ab)^{2i+2}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})\right.\right.$$

$$\left.+\,i(i+1)(i+2)(2i-1)(2i+3)(m/n)(ab)^{2i-1}(a^2-b^2)^2\right\}$$

$$\left.-\,i^2(2i-1)(2i+1)\left\{(i+2)\frac{m}{n}-1\right\}r^{-i-1}(ab)^{-3}(a^2-b^2)(a^{2i+3}-b^{2i+3})\right]$$

$$+\,\rho V_{-i-1}\left[-(i+1)^2(2i+1)(2i+3)\left\{(i-1)\frac{m}{n}+1\right\}r^i(ab)^{-2i-2}(a^2-b^2)(a^{2i-1}-b^{2i-1})\right.$$

$$-\frac{(i+1)r^{-i-1}}{(ab)^{2i+2}}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})\right.$$

$$\left.\left.-\,(i-1)\,i\,(i+1)(2i-1)(2i+3)(m/n)(ab)^{2i-1}(a^2-b^2)^2\right\}\right]\quad\ldots(91),$$

$2n\overline{\Pi}u$

$$=R_i\left[-\frac{(i+1)r^{i+1}}{(ab^2)^{i+1}}\left(i\frac{m}{n}-2\right)\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)a^2(a^{2i-1}-b^{2i-1})+i(i+2)(2i-1)\frac{m}{n}b^{2i-1}(a^2-b^2)\right\}\right.$$

$$+\frac{i}{i-1}\frac{r^{i-1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)\left(i(i+2)\frac{m}{n}-1\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.+\,(i-1)(i+1)(2i+3)\frac{m}{n}\left((i^2-1)\frac{m}{n}-1\right)b^{2i+1}(a^2-b^2)\right\}$$

$$+ i\left((i+1)\frac{m}{n}+2\right)\frac{r^{-i}}{a^{i+1}b^3}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.+(i-1)(i+1)(2i+3)\frac{m}{n}a^{2i+1}(a^2-b^2)\right\}$$

$$-\frac{i+1}{i+2}\frac{r^{-i-2}}{a^{i-1}b}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left((i^2-1)\frac{m}{n}-1\right)b^2(a^{2i-1}-b^{2i-1})\right.$$

$$\left.\left.+i(i+2)(2i-1)\frac{m}{n}\left(i(i+2)\frac{m}{n}-1\right)a^{2i-1}(a^2-b^2)\right\}\right]$$

$+R_i'$ [coefficient obtained from that of R_i by interchanging a and b]

$$+T_i\left[i(i+1)\left(\frac{m}{n}i-2\right)\frac{r^{i+1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)a^2(a^{2i-1}-b^{2i-1})\right.\right.$$

$$\left.-(i+1)(i+2)(2i-1)\frac{m}{n}b^{2i-1}(a^2-b^2)\right\}$$

$$-\frac{i(i+1)}{i-1}\frac{r^{i-1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)\left((i^2-i-3)\frac{m}{n}+1\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.-(i-1)i(2i+3)\frac{m}{n}\left((i^2+3i-1)\frac{m}{n}+1\right)b^{2i+1}(a^2-b^2)\right\}$$

$$+i(i+1)\left(\frac{m}{n}(i+1)+2\right)\frac{r^{-i}}{a^{i+1}b^3}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.-(i-1)i(2i+3)\frac{m}{n}a^{2i+1}(a^2-b^2)\right\}$$

$$-\frac{i(i+1)}{i+2}\frac{r^{-i-2}}{a^{i-1}b}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left((i^2+3i-1)\frac{m}{n}+1\right)b^2(a^{2i-1}-b^{2i-1})\right.$$

$$\left.\left.-(i+1)(i+2)(2i-1)\frac{m}{n}\left((i^2-i-3)\frac{m}{n}+1\right)a^{2i-1}(a^2-b^2)\right\}\right]$$

$+T_i'$ [coefficient obtained from that of T_i by interchanging a and b]

$$+i\rho V_i\left[-\frac{r^{i+1}}{(ab)^{2i+2}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)\left((i+1)\frac{m}{n}-1\right)(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})\right.\right.$$

$$\left.+i(i+1)(i+2)(2i-1)\frac{m}{n}\left(\frac{m}{n}-2\right)(ab)^{2i-1}(a^2-b^2)^2\right\}$$

$$+\frac{i}{i-1}\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{i-1}}{(ab)^{2i+2}}(a^{2i+1}-b^{2i+1})(a^{2i+3}-b^{2i+3})$$

$$+i(2i+1)\left\{(i+1)\frac{m}{n}+2\right\}\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{-i}}{(ab)^3}(a^2-b^2)(a^{2i+3}-b^{2i+3})$$

$$\left.-i(i+1)(2i-1)\frac{m}{n}\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{-i-2}}{ab}(a^2-b^2)(a^{2i+1}-b^{2i+1})\right]$$

$$-(i+1)\rho V_{-i-1}\Bigg[-(i+1)(2i+1)\left(i\frac{m}{n}-2\right)\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{i+1}}{(ab)^{2i+2}}(a^2-b^2)(a^{2i-1}-b^{2i-1})$$

$$+i(i+1)(2i+3)\frac{m}{n}\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{i-1}}{(ab)^{2i+2}}(a^2-b^2)(a^{2i+1}-b^{2i+1})$$

$$+\frac{r^{-i}}{(ab)^{2i+2}}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left(i\frac{m}{n}+1\right)(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})\right.$$

$$\left.-(i-1)i(i+1)(2i+3)\frac{m}{n}\left(\frac{m}{n}-2\right)(ab)^{2i-1}(a^2-b^2)^2\right\}$$

$$-\frac{i+1}{i+2}\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{-i-2}}{(ab)^{2i}}(a^{2i-1}-b^{2i-1})(a^{2i+1}-b^{2i+1})\Bigg]\quad\ldots(92),$$

$$2n\Pi v=\frac{dR_i}{d\theta}\Bigg[-\left\{(i+3)\frac{m}{n}+2\right\}\frac{r^{i+1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)a^2(a^{2i-1}-b^{2i-1})\right.$$

$$\left.+i(i+2)(2i-1)\frac{m}{n}b^{2i-1}(a^2-b^2)\right\}+\frac{1}{i-1}\frac{r^{i-1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)\left(i(i+2)\frac{m}{n}-1\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.+(i-1)(i+1)(2i+3)\frac{m}{n}\left((i^2-1)\frac{m}{n}-1\right)b^{2i+1}(a^2-b^2)\right\}$$

$$-\left\{(i-2)\frac{m}{n}-2\right\}\frac{r^{-i}}{a^{i+1}b^3}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.+(i-1)(i+1)(2i+3)\frac{m}{n}a^{2i+1}(a^2-b^2)\right\}$$

$$+\frac{1}{i+2}\frac{r^{-i-2}}{a^{i-1}b}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left((i^2-1)\frac{m}{n}-1\right)b^2(a^{2i-1}-b^{2i-1})\right.$$

$$\left.+i(i+2)(2i-1)\frac{m}{n}\left(i(i+2)\frac{m}{n}-1\right)a^{2i-1}(a^2-b^2)\right\}\Bigg]$$

$$+\frac{dR_i'}{d\theta}\left[\text{coefficient obtained from that of }\frac{dR_i}{d\theta}\text{ by interchanging }a\text{ and }b\right]$$

$$+\frac{dT_i}{d\theta}\Bigg[i\left\{(i+3)\frac{m}{n}+2\right\}\frac{r^{i+1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)a^2(a^{2i-1}-b^{2i-1})\right.$$

$$\left.-(i+1)(i+2)(2i-1)\frac{m}{n}b^{2i-1}(a^2-b^2)\right\}-\frac{i+1}{i-1}\frac{r^{i-1}}{(ab^2)^{i+1}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)\times\right.$$

$$\left.\left((i^2-i-3)\frac{m}{n}+1\right)(a^{2i+3}-b^{2i+3})-(i-1)i(2i+3)\frac{m}{n}\left((i^2+3i-1)\frac{m}{n}+1\right)b^{2i+1}(a^2-b^2)\right\}$$

$$-(i+1)\left\{(i-2)\frac{m}{n}-2\right\}\frac{r^{-i}}{a^{i+1}b^3}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)(a^{2i+3}-b^{2i+3})\right.$$

$$\left.-(i-1)i(2i+3)\frac{m}{n}a^{2i+1}(a^2-b^2)\right\}$$

$$+\frac{i}{i+2}\frac{r^{-i-2}}{a^{i-1}b}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left((i^2+3i-1)\frac{m}{n}+1\right)b^2(a^{2i-1}-b^{2i-1})\right.$$

$$\left.-(i+1)(i+2)(2i-1)\frac{m}{n}\left((i^2-i-3)\frac{m}{n}+1\right)a^{2i-1}(a^2-b^2)\right\}\Bigg]$$

$$+\frac{dT_i'}{d\theta}\left[\text{coefficient obtained from that of } \frac{dT_i}{d\theta} \text{ by interchanging } a \text{ and } b\right]$$

$$+\rho\frac{dV_i}{d\theta}\left[-\frac{r^{i+1}}{(ab)^{2i+2}}\left\{\left((2i^2+1)\frac{m}{n}+2i+1\right)\left((i+3)\frac{m}{n}-1\right)(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})\right.\right.$$

$$\left.+i(i+2)(2i-1)\frac{m}{n}\left((i+3)\frac{m}{n}+2i\right)(ab)^{2i-1}(a^2-b^2)^2\right\}$$

$$+\frac{i}{i-1}\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{i-1}}{(ab)^{2i+2}}(a^{2i+1}-b^{2i+1})(a^{2i+3}-b^{2i+3})$$

$$-i(2i+1)\left\{(i-2)\frac{m}{n}-2\right\}\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{-i}}{(ab)^3}(a^2-b^2)(a^{2i+3}-b^{2i+3})$$

$$\left.+i^2(2i-1)\frac{m}{n}\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{-i-2}}{ab}(a^2-b^2)(a^{2i+1}-b^{2i+1})\right]$$

$$+\rho\frac{dV_{-i-1}}{d\theta}\left[(i+1)(2i+1)\left\{(i+3)\frac{m}{n}+2\right\}\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{i+1}}{(ab)^{2i+2}}(a^2-b^2)(a^{2i-1}-b^{2i-1})\right.$$

$$-(i+1)^2(2i+3)\frac{m}{n}\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{i-1}}{(ab)^{2i+2}}(a^2-b^2)(a^{2i+1}-b^{2i+1})$$

$$+\frac{r^{-i}}{(ab)^{2i+2}}\left\{\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left((i-2)\frac{m}{n}+1\right)(a^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})\right.$$

$$\left.-(i-1)(i+1)(2i+3)\frac{m}{n}\left((i-2)\frac{m}{n}+2i+2\right)(ab)^{2i-1}(a^2-b^2)^2\right\}$$

$$\left.-\frac{i+1}{i+2}\left((2i^2+4i+3)\frac{m}{n}-(2i+1)\right)\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{-i-2}}{(ab)^{2i}}(a^{2i-1}-b^{2i-1})(a^{2i+1}-b^{2i+1})\right]\ldots\ldots(93).$$

The value of w is obtained from that of v by replacing $\frac{d}{d\theta}$ by $\frac{1}{\sin\theta}\frac{d}{d\phi}$. In any one of the quantities δ, u, v, w, so far as they depend on the surface forces, terms in r^{-i} may be obtained from those in r^{i+1}, and terms in r^{-i-2} from those in r^{i-1}, by simply writing $(-i-1)$ for $(+i)$ in all indices and coefficients. The same substitution deduces the coefficient of V_{-i-1} in each case from that of V_i. The quantity $\overline{\Pi}$ on the left of the equations will be found to transform into itself, *i.e.* to remain unchanged, when $(-i-1)$ is written for i.

The solution just written down may at first sight seem rather cumbrous. It must be remembered however that it contains the answer to innumerable special problems, and that in very few practical applications will there be found anything like so general a system of applied forces as that treated here. Having regard to the actual facts, the comparative brevity of the solution is in reality somewhat remarkable.

§ 26. From these typical terms in the displacements the typical terms in the stresses may be found by means of the general formulae (3). Three only of the stresses, viz. $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{r\phi}$, are given explicitly below. They possess greater inherent interest than the other three, more especially in the case of thin shells. The method by which they were actually calculated was by substituting in (33), (34) and (35) the values deduced from (68), (69),

(70) and (71) for V_i, etc. The expressions are as follows, $\overline{\Pi}$ being as before given by (73):—

$$\overline{\Pi}\widehat{rr} = \frac{1}{2i+1} R_i \left[\left\{ (2i^2+4i+3)\frac{m}{n} - (2i+1) \right\} \left\{ (2i^2+1)\frac{m}{n} + 2i + 1 \right\} \times \right.$$

$$\frac{r^{-i-3}}{(ab^2)^{i+1}} \{(i+1)\, a^2 (a^{2i-1} - b^{2i-1})(r^{2i+3} - b^{2i+3}) + ir^2 (r^{2i-1} - b^{2i-1})(a^{2i+3} - b^{2i+3})\}$$

$$- (i-1)\, i\, (i+1)(i+2)(2i-1)(2i+3)(m/n)^2 \frac{r^{-i-3}}{a^{i+1} b^3} (a^2 - b^2)(r^2 - b^2) \{(i+1)\, a^{2i+1} + ir^{2i+1}\}$$

$$+ i(i+1)(i+2)(2i-1)\frac{m}{n} \left\{ (2i^2+4i+3)\frac{m}{n} - (2i+1) \right\} \frac{r^{-i-3}}{a^{i+1} b^3} \times$$

$$\{b^2 (a^2 - r^2)(r^{2i+1} - b^{2i+1}) - a^2 (r^2 - b^2)(a^{2i+1} - r^{2i+1})\}$$

$$+ (i-1)\, i\, (i+1)(2i+3)\frac{m}{n} \left\{ (2i^2+1)\frac{m}{n} + 2i + 1 \right\} \frac{r^{-i-3}}{a^{i-1} b^{2i+2}} \times$$

$$\left. \{a^{2i-1} (a^2 - r^2)(r^{2i+1} - b^{2i+1}) - b^{2i-1} (r^2 - b^2)(a^{2i+1} - r^{2i+1})\} \right]$$

$+ \dfrac{1}{2i+1} R_i'$ [coefficient obtained from that of R_i inside square bracket by interchanging a and b]

$$+ i(i+1)\, T_i \frac{r^{-i-3}}{(ab^2)^{i+1}} \left[- \left\{ (2i^2+4i+3)\frac{m}{n} - (2i+1) \right\} \times \right.$$

$$\left\{ (i^2+3i-1)\frac{m}{n} + 1 \right\} (a^{2i+1} - b^{2i+1})(a^2 - r^2)(r^{2i+1} - b^{2i+1})$$

$$+ (2i+1) \left\{ (i^2+3i-1)\frac{m}{n} + 1 \right\} \left\{ (i^2-i-3)\frac{m}{n} + 1 \right\} b^{2i-1} (a^2 - b^2)(r^2 - b^2)(a^{2i+1} - r^{2i+1})$$

$$\left. + 2(2i+1)\left(3\frac{m}{n} - 1\right) \left\{ i(i+1)\frac{m}{n} + 1 \right\} \{a^{2i+1} (a^2 - r^2)(r^{2i+1} - b^{2i+1}) - b^{2i+1} (r^2 - b^2)(a^{2i+1} - r^{2i+1})\} \right]$$

$+ i(i+1)\, T_i' \dfrac{r^{-i-3}}{(a^2 b)^{i+1}}$ [coefficient obtained from that of T_i inside square bracket by interchanging a and b]

$$+ \rho V_i i^2 \left\{ (i+2)\frac{m}{n} - 1 \right\} \frac{r^{-i-3}}{(ab)^{2i+2}} \left[\left\{ (2i^2+1)\frac{m}{n} + 2i + 1 \right\} (a^{2i+3} - b^{2i+3}) \right.$$

$$\times \{a^{2i-1} (a^2 - r^2)(r^{2i+1} - b^{2i+1}) - b^{2i-1} (r^2 - b^2)(a^{2i+1} - r^{2i+1})\}$$

$$\left. + (i+1)(i+2)(2i-1)\frac{m}{n} (ab)^{2i-1} (a^2 - b^2) \{b^2 (a^2 - r^2)(r^{2i+1} - b^{2i+1}) - a^2 (r^2 - b^2)(a^{2i+1} - r^{2i+1})\} \right]$$

$$+ \rho V_{-i-1} (i+1)^2 \left\{ (i-1)\frac{m}{n} + 1 \right\} \frac{r^{-i-3}}{(ab)^{2i+2}} \left[\left\{ (2i^2+4i+3)\frac{m}{n} - (2i+1) \right\} (a^{2i-1} - b^{2i-1}) \right.$$

$$\times \{a^2 (r^2 - b^2)(a^{2i+1} - r^{2i+1}) - b^2 (a^2 - r^2)(r^{2i+1} - b^{2i+1})\}$$

$$\left. + (i-1)\, i\, (2i+3)\frac{m}{n} (a^2 - b^2) \{b^{2i-1} (r^2 - b^2)(a^{2i+1} - r^{2i+1}) - a^{2i-1} (a^2 - r^2)(r^{2i+1} - b^{2i+1})\} \right] \text{......(94)},$$

$$\Pi\widehat{r\theta} = \frac{dR_i}{d\theta}\left[\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}\left\{i(i+2)\frac{m}{n}-1\right\}\times\right.$$

$$\frac{r^{-i-3}}{(ab^2)^{i+1}}\{a^{2i+1}(a^2-r^2)(r^{2i+1}-b^{2i+1})-b^{2i+1}(r^2-b^2)(a^{2i+1}-r^{2i+1})\}$$

$$+\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}\left\{(i^2-1)\frac{m}{n}-1\right\}\times$$

$$\frac{r^{-i-3}}{a^{i+1}b}\{(r^2-b^2)(a^{2i+1}-r^{2i+1})-(a^2-r^2)(r^{2i+1}-b^{2i+1})\}$$

$$\left.+(2i+1)\left\{i(i+2)\frac{m}{n}-1\right\}\left\{(i^2-1)\frac{m}{n}-1\right\}\frac{r^{-i-3}}{a^{i+1}b^3}(a^2-b^2)(r^2-b^2)(a^{2i+1}-r^{2i+1})\right]$$

$$+\frac{dR_i'}{d\theta}\left[\text{coefficient obtained from that of } \frac{dR_i}{d\theta} \text{ by interchanging } a \text{ and } b\right]$$

$$+\frac{dT_i}{d\theta}\,\frac{1}{2i+1}\,\frac{r^{-i-3}}{(ab^2)^{i+1}}\left[\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}\times\right.$$

$$\{(i+1)r^2(r^{2i-1}-b^{2i-1})(a^{2i+3}-b^{2i+3})+ia^2(a^{2i-1}-b^{2i-1})(r^{2i+3}-b^{2i+3})\}$$

$$-(i-1)i(i+1)(i+2)(2i-1)(2i+3)(m/n)^2 b^{2i-1}(a^2-b^2)(r^2-b^2)\{(i+1)r^{2i+1}+ia^{2i+1}\}$$

$$+i(i+1)(i+2)(2i-1)\frac{m}{n}\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}b^{2i-1}\times$$

$$\{a^2(r^2-b^2)(a^{2i+1}-r^{2i+1})-b^2(a^2-r^2)(r^{2i+1}-b^{2i+1})\}$$

$$+(i-1)i(i+1)(2i+3)\frac{m}{n}\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}a^2\times$$

$$\left.\{b^{2i-1}(r^2-b^2)(a^{2i+1}-r^{2i+1})-a^{2i-1}(a^2-r^2)(r^{2i+1}-b^{2i+1})\}\right]$$

$$+\frac{dT_i'}{d\theta}\,\frac{1}{2i+1}\,\frac{r^{-i-3}}{(a^2b)^{i+1}}\left[\text{coefficient obtained from that of } \frac{dT_i}{d\theta} \text{ inside square bracket by interchanging } a \text{ and } b\right]$$

$$+\rho\frac{dV_i}{d\theta}\,i\left\{(i+2)\frac{m}{n}-1\right\}\frac{r^{-i-3}}{(ab)^{2i+2}}\left[\left\{(2i^2+1)\frac{m}{n}+2i+1\right\}(a^{2i+3}-b^{2i+3})\times\right.$$

$$\{a^{2i-1}(a^2-r^2)(r^{2i+1}-b^{2i+1})-b^{2i-1}(r^2-b^2)(a^{2i+1}-r^{2i+1})\}$$

$$\left.+i(i+2)(2i-1)\frac{m}{n}(ab)^{2i-1}(a^2-b^2)\{a^2(r^2-b^2)(a^{2i+1}-r^{2i+1})-b^2(a^2-r^2)(r^{2i+1}-b^{2i+1})\}\right]$$

$$+\rho\frac{dV_{-i-1}}{d\theta}(i+1)\left\{(i-1)\frac{m}{n}+1\right\}\frac{r^{-i-3}}{(ab)^{2i+2}}\left[\left\{(2i^2+4i+3)\frac{m}{n}-(2i+1)\right\}(a^{2i-1}-b^{2i-1})\times\right.$$

$$\{b^2(a^2-r^2)(r^{2i+1}-b^{2i+1})-a^2(r^2-b^2)(a^{2i+1}-r^{2i+1})\}$$

$$+(i-1)(i+1)(2i+3)\frac{m}{n}(a^2-b^2)\{b^{2i-1}(r^2-b^2)(a^{2i+1}-r^{2i+1})$$

$$\left.-a^{2i-1}(a^2-r^2)(r^{2i+1}-b^{2i+1})\}\right]\ldots\ldots\ldots\ldots\ldots\ldots(95).$$

The expression for $\widehat{r\phi}$ is obtained from that for $\widehat{r\theta}$ by replacing $\frac{d}{d\theta}$ by $\frac{1}{\sin\theta}\frac{d}{d\phi}$. As in the case of the displacements, the substitution of $(-i-1)$ for $(+i)$ deduces terms in r^{-i} and r^{-i-2} depending on the surface forces from those in r^{i+1} and r^{i-1} respectively, and also the terms containing V_{-i-1} from those containing V_i.

§ 27. The complete expressions for the displacements u, v of the third class are found by summation with respect to i of the typical terms given by (92) and (93), and a similar summation is of course required of the typical terms given in (94) and (95) for $\widehat{rr}$ and $\widehat{r\theta}$. The complete expressions for w and $\widehat{r\phi}$ are derived from the complete expressions for v and $\widehat{r\theta}$ by the substitution of $\frac{1}{\sin\theta}\frac{d}{d\phi}$ for $\frac{d}{d\theta}$. The limits of the summation had better be regarded as $i=2$ and $i=\infty$. The case $i=0$ would answer to forces, such as uniform normal tractions, whose values are independent of the angular coordinates; and the correct solution is in reality derivable from (92). We already, however, have considered it, treating the displacements so produced as of a separate class, and have given the solution in (9). It is in fact easily verified that if in (92) we put $i=0$, and replace R_i, R_i', V_{-i-1} by R, R', V' respectively, we obtain the corresponding terms in (9). The terms in V in (9) are not represented in (92). The potential from which the bodily forces answering to the solution (92) are derived satisfies Laplace's equation $\nabla^2=0$, or answers to forces other than the mutual gravitation of the shell. But the potential Vr^2 answering to (9) includes mutual gravitation and "centrifugal force", neither of which satisfies Laplace's equation.

The case $i=1$ must in general be excluded from the solution for the reasons stated in § 22 in the analogous case in pure transverse displacements. In any particular case where forces involving harmonics of the first degree are distributed over the two surfaces of a shell in such a way that the entire system of forces is in statical equilibrium the solution (91), (92), etc. will give correctly the *elastic* displacements.

§ 28. The forms under which the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$ are presented may seem at first sight rather peculiar. They have been adopted with a view principally to two ends, viz. to afford a ready means of testing the accuracy by reference to the surface conditions, and to facilitate application to the case of thin shells. The coefficients are all constructed on a uniform and very simple plan. Take for instance the values of $\widehat{rr}$ depending on R_i and T_i. In the case of R_i the expression inside the square bracket must by the surface conditions vanish when $r=b$, and a glance shows the occurrence of $r-b$ as a factor in every term. The terms in the last 4 lines contain in addition the factor $a-r$ and so vanish likewise over $r=a$. The first 3 lines inside the square bracket on the other hand when a is substituted for r fall at once into $(2i+1)\,\overline{\Pi}$, and so the surface condition $\widehat{rr}=R_i$ over $r=a$ is seen to be satisfied. The first terms are those which are of most importance near the surface where the corresponding stress is applied, and in the case of a thin shell these terms are of a higher order of magnitude than the subsequent terms which vanish over both surfaces. The expression for $\widehat{rr}$ in terms of T_i has to vanish over both surfaces, and so is arranged to show the factors $a-r$ and $r-b$ in each term.

The terms which vanish at both surfaces can be thrown into a variety of equivalent forms, some more convenient for one purpose, some for another. Use may be made for instance of the identities:

$$\begin{aligned}&a^{2i-1}(a^2-r^2)(r^{2i+1}-b^{2i+1})-b^{2i-1}(r^2-b^2)(a^{2i+1}-r^{2i+1})\\&=r^2\{a^{2i-1}(a^2-r^2)(r^{2i-1}-b^{2i-1})-b^{2i-1}(r^2-b^2)(a^{2i-1}-r^{2i-1})\}\\&=r^2\{(r^{2i-1}-b^{2i-1})(a^{2i+1}-r^{2i+1})-(a^{2i-1}-r^{2i-1})(r^{2i+1}-b^{2i+1})\}.\end{aligned}$$

In what precedes we have always described b as the radius of the inner surface and we shall continue to do so. But from the form of (92), (93), (94), (95) we may clearly in these equations regard a as the radius of the inner surface if we take the undashed letters R_i, T_i to denote the forces applied over that surface. When the outer surface is free of force the reader may find it a saving of time to take this view.

§ 29. We pass now to the consideration of the form taken by the displacements and stresses in a thin shell. The expressions given below for $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{r\phi}$ were calculated directly from (94) and (95); and the values of u, v, w might similarly be derived from (92) and (93). As a matter of fact, however, the displacements were found by inserting in equations (68)—(71) the approximate values found for Π, Π_1, etc. by expanding the expressions (72)—(90) in powers of h/a, where h is the thickness of the shell. To save space these approximate values of Π etc. are not recorded here.

Denoting Young's modulus by E and Poisson's ratio by η, and putting as before $a-b=h$, $a-r=\xi$, we find, retaining the lowest and next lowest powers of h/a and ξ/a, the following results*:

$$\begin{aligned}u=R_i\frac{a^2}{nh}&\left[\frac{(2i^2+2i-1)m-n}{2(i-1)(i+2)(3m-n)}-\eta\frac{n}{E}\frac{h-\xi}{a}\right]\\&-R_i'\frac{a^2}{nh}\left[\frac{(2i^2+2i-1)m-n}{2(i-1)(i+2)(3m-n)}(1-2h/a)+\eta\frac{n}{E}\frac{\xi}{a}\right]\\&+T_i i(i+1)\frac{a^2}{2nh}\left[\frac{1}{(i-1)(i+2)}-\frac{n}{E}\frac{h}{a}\right]-T_i' i(i+1)\frac{a^2}{2nh}\left[\frac{1-2h/a}{(i-1)(i+2)}+\frac{n}{E}\frac{h}{a}\right]\\&+a^{i+1}\frac{\rho V_i}{n}\left[\frac{i\{(2i+1)m-n\}}{2(i-1)(3m-n)}+i\eta\frac{n}{E}\frac{\xi}{a}-\frac{i^2\{(i+2)m-n\}}{2(i-1)(3m-n)}\frac{h}{a}\right]\\&-a^{-i}\frac{\rho V_{-i-1}}{n}\left[\frac{(i+1)\{(2i+1)m+n\}}{2(i+2)(3m-n)}+(i+1)\eta\frac{n}{E}\frac{\xi}{a}+\frac{(i+1)^2\{(i-1)m+n\}}{2(i+2)(3m-n)}\frac{h}{a}\right]\text{............(96)},\end{aligned}$$

$$\begin{aligned}v=\frac{dR_i}{d\theta}\frac{a^2}{2nh}&\left[\frac{1}{(i-1)(i+2)}+\frac{n}{E}\frac{2\xi-h}{a}\right]-\frac{dR_i'}{d\theta}\frac{a^2}{2nh}\left[\frac{1-2h/a}{(i-1)(i+2)}+\frac{n}{E}\frac{2\xi-h}{a}\right]\\&+\frac{dT_i}{d\theta}\frac{a^2}{nh}\left[\frac{1}{(i-1)(i+2)}-\frac{1}{2}\frac{h-\xi}{a}\right]-\frac{dT_i'}{d\theta}\frac{a^2}{nh}\left[\frac{1-2h/a}{(i-1)(i+2)}+\frac{1}{2}\frac{\xi}{a}\right]\\&+a^{i+1}\frac{\rho}{2n}\frac{dV_i}{d\theta}\left[\frac{1}{i-1}+\frac{(2i+3)m-n}{3m-n}\frac{\xi}{a}-\frac{i}{i-1}\frac{(i+2)m-n}{3m-n}\frac{h}{a}\right]\\&-a^{-i}\frac{\rho}{2n}\frac{dV_{-i-1}}{d\theta}\left[\frac{1}{i+2}+\frac{(2i-1)m+n}{3m-n}\frac{\xi}{a}-\frac{i+1}{i+2}\frac{(i-1)m+n}{3m-n}\frac{h}{a}\right]\text{..................(97)},\end{aligned}$$

* The reader must bear in mind that these results answer only to the displacements of our Class (iii), i.e. to the system of applied forces given near the beginning of § 25 above.

$$\widehat{rr} = R_i \frac{h-\xi}{h}\left[1 - \frac{(i-1)(i+2)m-2n}{m+n}\frac{\xi}{a}\right] + R_i' \frac{\xi}{h}\left[1 + \frac{(i-1)(i+2)m-2n}{m+n}\frac{h-\xi}{a}\right]$$

$$- T_i 2i(i+1)\frac{m}{m+n}\frac{\xi(h-\xi)}{ah}\left[1 - \frac{i(i+1)m+n}{3m}\frac{h}{a} + \frac{(i^2+i+15)m+n}{6m}\frac{\xi}{a}\right]$$

$$+ T_i' 2i(i+1)\frac{m}{m+n}\frac{\xi(h-\xi)}{ah}\left[1 + \frac{(i^2+i-9)m+n}{6m}\frac{h}{a} + \frac{(i^2+i+15)m+n}{6m}\frac{\xi}{a}\right]$$

$$- a^i\rho V_i i^2 \frac{(i+2)m-n}{m+n}\frac{\xi(h-\xi)}{a^2}\left[1 - \frac{(i^2+6i+2)m-in}{3(3m-n)}\frac{h-2\xi}{a} - (i-2)\frac{\xi}{a}\right]$$

$$+ a^{-i-1}\rho V_{-i-1}(i+1)^2 \frac{(i-1)m+n}{m+n}\frac{\xi(h-\xi)}{a^2} \times$$

$$\left[1 - \frac{(i^2-4i-3)m+(i+1)n}{3(3m-n)}\frac{h-2\xi}{a} + (i+3)\frac{\xi}{a}\right] \quad \text{.................(98)},$$

$$\widehat{r\theta} = \frac{dR_i}{d\theta}\frac{m\{(2i^2+2i-1)m-n\}}{(m+n)(3m-n)}\frac{\xi(h-\xi)}{ah}\left[1 + \frac{3m^2+(2i^2+2i+1)mn-2n^2}{3m\{(2i^2+2i-1)m-n\}}\frac{h}{a}\right.$$

$$\left. + \frac{3(5i^2+5i-3)m^2-(i^2+i+8)mn+n^2}{3m\{(2i^2+2i-1)m-n\}}\frac{\xi}{a}\right]$$

$$- \frac{dR_i'}{d\theta}\frac{m\{(2i^2+2i-1)m-n\}}{(m+n)(3m-n)}\frac{\xi(h-\xi)}{ah}\left[1 - \frac{3(3i^2+3i-1)m^2+(i^2+i-4)mn-n^2}{3m\{(2i^2+2i-1)m-n\}}\frac{h}{a}\right.$$

$$\left. + \frac{3(5i^2+5i-3)m^2-(i^2+i+8)mn+n^2}{3m\{(2i^2+2i-1)m-n\}}\frac{\xi}{a}\right]$$

$$+ \frac{dT_i}{d\theta}\frac{h-\xi}{h}\left[1 + \frac{(i^2+i+2)m+2n}{m+n}\frac{\xi}{a}\right] + \frac{dT_i'}{d\theta}\frac{\xi}{h}\left[1 - \frac{(i^2+i+2)m+2n}{m+n}\frac{h-\xi}{a}\right]$$

$$+ a^i\rho\frac{dV_i}{d\theta} i \frac{\{(i+2)m-n\}\{(2i+1)m+n\}}{(m+n)(3m-n)}\frac{\xi(h-\xi)}{a^2} \times$$

$$\left[1 - \frac{i\{3(i+1)m+n\}}{3\{(2i+1)m+n\}}\frac{h-2\xi}{a} - (i-2)\frac{\xi}{a}\right]$$

$$- a^{-i-1}\rho\frac{dV_{-i-1}}{d\theta}(i+1)\frac{\{(i-1)m+n\}\{(2i+1)m-n\}}{(m+n)(3m-n)}\frac{\xi(h-\xi)}{a^2} \times$$

$$\left[1 + \frac{(i+1)(3im-n)}{3\{(2i+1)m-n\}}\frac{h-2\xi}{a} + (i+3)\frac{\xi}{a}\right] \quad \text{......(99)}.$$

The value of w may be got from that of v, and the value of $\widehat{r\phi}$ from that of $\widehat{r\theta}$, by substituting $\frac{1}{\sin\theta}\frac{d}{d\phi}$ for $\frac{d}{d\theta}$.

§ 30. Noticing that $\frac{d}{dr} = -\frac{d}{d\xi}$ we find, retaining only the algebraically lowest powers of h/a,

$$\frac{du}{dr} = -\frac{\eta}{E}\frac{a}{h}F_i \quad \text{...(100)},$$

$$\delta = \frac{1}{3k}\frac{a}{h}F_i \quad \text{...(101)},$$

where $$F_i = R_i - R_i' + h\{ia^{i-1}\rho V_i - (i+1)a^{-i-2}\rho V_{-i-1}\} \quad \text{......................(102)},$$

and k as before is the bulk modulus.

Obviously F_i is the total radial force per unit of surface, at the element considered, arising from all the bodily and surface forces which contain harmonics of degree i.

With the exception, as explained below, of cases in which i is very large, (100) and (101) will be satisfactory first approximations unless F_i be small compared to the individual forces R_i, R_i', etc., of which it is composed. These results are the exact equivalents of the results (18) and (19) for uniform normal forces.

§ 31. Before examining more minutely these and similar results, it is convenient to form some idea of the magnitude of the strains and stresses. The actual determination of the greatest strain and the stress-difference is complicated by the fact that the directions of the principal strains and stresses at a point will not in general coincide with the fundamental directions r, θ, ϕ, and also by the fact that the magnitudes of all the terms involved fluctuate over the surface. Exact determinations are apparently possible only for particular cases treated individually. Without actually calculating the greatest strain it is, however, fairly obvious that it will in general be a quantity of the same order of magnitude as the greater of the two expressions u/r and $\frac{1}{r}\frac{dv}{d\theta}$ whose sum constitutes the stretch along θ.

This consideration enables us to reach some important conclusions for the cases when all the forces act on the surfaces. Let

$$\left.\begin{aligned} R_i - R_i' &= F_i, \\ \frac{d}{d\theta}(T_i - T_i') &= \Theta_i, \\ \frac{1}{\sin\theta}\frac{d}{d\phi}[T_i - T_i'] &= \Phi_i \end{aligned}\right\} \quad \text{..................................(103)},$$

so that F_i, Θ_i, Φ_i are the components along r, θ, ϕ of the resultant of the forces on both surfaces derived from harmonics of degree i. Then, retaining only the algebraically lowest power of h/a, we find

$$\left.\begin{aligned} u/r &= F_i\frac{a}{nh}\frac{(2i^2+2i-1)m-n}{2(i-1)(i+2)(3m-n)} + (T_i - T_i')\frac{a}{2nh}\frac{i(i+1)}{(i-1)(i+2)}, \\ \frac{1}{r}\frac{dv}{d\theta} &= \frac{d^2F_i}{d\theta^2}\frac{a}{2nh}\frac{1}{(i-1)(i+2)} + \frac{d\Theta_i}{d\theta}\frac{a}{nh}\frac{1}{(i-1)(i+2)}, \\ \frac{1}{r\sin\theta}\frac{dw}{d\phi} &= \frac{1}{\sin^2\theta}\frac{d^2F_i}{d\phi^2}\frac{a}{2nh}\frac{1}{(i-1)(i+2)} + \frac{1}{\sin\theta}\frac{d\Phi_i}{d\phi}\frac{a}{nh}\frac{1}{(i-1)(i+2)} \end{aligned}\right\} \quad \text{...........(104)}.$$

These quantities must in general not exceed the order of magnitude permissible to strains in the material, and this condition clearly cannot be satisfied all over the surface unless F_i, Θ_i, Φ_i and their resultant be kept so small that their ratios to the greatest longitudinal traction permissible in a long bar of the material be, at most, small quantities of the order h/a. This condition will of course be satisfied for the components along r, θ, ϕ if it is satisfied for their resultant.

The condition that the resultant must be small must clearly also hold though bodily forces act in addition; and, as the resultant of bodily forces per unit of surface will usually be very small in the case of a really thin shell, even when their direction is the same all along the thickness, this condition will in general be sufficient. The condition will, however, cease to be sufficient if the bodily forces are so intense that their resultant per unit of surface bears a ratio of the order h/a to the greatest traction permissible in a long bar of the material. This follows from the fact that the principal terms in the displacements and strains depending on V_i and V_{-i-1} do not cut out when

$$ia^{i-1}\rho V_i - (i+1)\, a^{-i-2}\rho V_{-i-1} = 0.$$

Unless the bodily forces be of unusual intensity we may for a first approximation neglect the terms containing h and ξ in the coefficients of V_i and V_{-i-1} in (96) and (97); but if the resultant of all the applied forces along the thickness be small compared to the resultant for one only of the surfaces, we must retain all the terms in these expressions depending on surface forces. In such a case the individual forces R_i etc. over either of the two surfaces may be of the same order of magnitude as the greatest traction permissible in a long bar of the material.

§ 32. One of the most striking features of (96) and (97) is brought out by a comparison of the terms in R_i and T_i, regarding these as quantities of the same order of magnitude. According to the first approximation the term in u depending on R_i is of the same order of magnitude as that depending on T_i, and the terms in v depending on R_i and T_i are likewise of the same order of magnitude. These latter terms are in fact precisely equal if $R_i = 2T_i$. Similar results follow a comparison of the principal terms in R_i' and T_i'.

From these considerations we see that the magnitude of the maxima values of a *displacement* whether radial or tangential depends rather on the magnitude than the direction of those of the applied forces which vary harmonically. It should, however, be noticed that, since for instance $\frac{dR_i}{d\theta}$ and $\frac{dR_i}{d\phi}$ vanish when R_i is a maximum, the tangential displacements due to the normal surface forces derived from a particular harmonic vanish where the radial displacements are a maximum. Also the radial displacements due to the tangential surface forces derived from a particular harmonic will have their maxima values at points where these forces themselves and the tangential displacements vanish.

§ 33. We have next to consider the nature of the terms in h/a and ξ/a inside the square brackets in the expressions (96) and (97) for the displacements. Supposing that the resultant per unit of surface of the applied forces is a quantity of the same order as the resultant of the forces applied over one of the surfaces, these terms—at least when i is not very large—are to be regarded as of secondary importance. Being linear in ξ, these terms have necessarily their mean values at the mid surface. Again the coefficient of ξ is in every case positive. Thus to a second approximation the displacements numerically considered, when they vary with ξ, have their maxima values at the inner surface, their mean values at the mid surface.

The fact that the radial displacements arising from tangential surface forces are, even to a second approximation, the same at all points along the thickness is worthy of notice. It shows that while, as we have seen, the radial *displacement* arising from tangential surface forces is similar in order of magnitude to that arising from equal radial forces, the radial *strain* in the former case is small compared to that in the latter.

It will be noticed that when surface forces alone act, even if the total components F_i, Θ_i, Φ_i for the two surfaces absolutely vanish, the values of u, v and w—and consequently of all the strains whose directions are parallel to the surface—are approximately constant all along the thickness. The values of these strains are in general of a higher order of magnitude than those of the three strains $\frac{du}{dr}$, $\frac{\widehat{r\theta}}{n}$ and $\frac{\widehat{r\phi}}{n}$, but this ceases to be the case when the forces at corresponding points on the two surfaces are nearly equal and opposite.

§ 34. We have next to consider the influence of the degree of the harmonic on the values of the displacements. When i is large we shall regard $\frac{dR_i}{d\theta}$ as of the order iR_i etc.; and we shall regard R_i, R_i', $\frac{dT_i}{d\theta}$ and $\frac{dT_i'}{d\theta}$ as of given magnitude.

From (96) we see that the radial displacements arising from radial surface forces have neither their "principal" nor their "secondary" terms much affected by the value of i: but when i is large the radial displacements depending on tangential surface forces have their "principal" terms varying inversely and their "secondary" terms directly as i. This latter law applies also to the tangential displacements arising from radial surface forces. The influence of the degree of the harmonic on the tangential displacements arising from tangential surface forces is even more important, for when i is large the magnitude of the "principal" terms varies inversely as i^2. We notice that in the case of surface forces the "secondary" terms in the tangential displacements when i is large bear to the "principal" terms ratios of the order i^2h/a, and that the same law applies to the radial displacements derived from tangential forces. Thus, except for the radial displacements derived from radial forces, the importance of the "secondary" terms relative to the "principal" increases very rapidly with the degree of the harmonic from which the surface forces are derived. In fact when i is very large i^2h/a ceases to be small and the "secondary" terms may be of as great or even greater importance than the "principal". In such a case we ought not to rely on (96) and (97), but must have recourse to (92) and (93) to ensure that we do not neglect terms of the same order as we have retained.

In the case of bodily forces when i is large, if we treat iV_i, $\frac{dV_i}{d\theta}$, iV_{-i-1} and $\frac{dV_{-i-1}}{d\theta}$ as of given magnitude, we see that the "principal" terms in u are nearly independent of i, while the "principal" terms in v vary inversely as i. The "secondary" terms in both u and v increase rapidly in importance relatively to the "principal" terms as i increases.

§ 35. We have next to consider the stresses. Of these the three $\widehat{\theta\theta}$, $\widehat{\phi\phi}$, $\widehat{\theta\phi}$ have "principal" terms independent of ξ. Thus unless the resultant force over the thickness of the shell be small compared to the resultants for the two surfaces separately, or else i be so large that terms in ξ/a become important, these stresses have nearly constant values throughout the thickness. The "principal" terms in these stresses may easily be derived from the displacements, the relation (101) being employed in the formulae for $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$ unless F_i be small. These stresses unless F_i, Θ_i, Φ_i be small are of a higher order of magnitude than $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{r\phi}$; but they are of less interest in the theory of thin shells, and further, owing to the variety of the differential coefficients they contain, they can hardly be considered satisfactorily except by treating each individual case by itself. It is thus sufficient to point out that the conclusions to be derived from them, through the maximum stress-difference they supply, as to the magnitudes permissible in the applied forces, are of the same character as we arrived at by considering the strains.

We now pass to the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$, and since the two latter are exactly similar in form we need not consider $\widehat{r\phi}$ separately. We shall as before speak of the terms containing the algebraically least powers of h/a as the "first approximation", but in almost every case it must be borne in mind that when i is so large that i^2h/a ceases to be small the "secondary" terms may be of as great or even greater importance.

In the special case when there are no bodily forces and when the surface forces at corresponding points on the two surfaces are exactly equal and opposite, the "principal" terms in $\widehat{rr}$ depending on the radial forces, and the "principal" terms in $\widehat{r\theta}$ and $\widehat{r\phi}$ depending on the tangential forces are constant throughout the thickness. In the same case the principal terms in $\widehat{rr}$ depending on the tangential forces, and the principal terms in $\widehat{r\theta}$ and $\widehat{r\phi}$ depending on the radial forces vanish. Thus all three stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$ show a remarkable approach to constancy along the thickness.

In general, however, when the forces at corresponding points on the two surfaces give a moderate resultant, the rate of variation of $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{r\phi}$ along the thickness is very rapid. The law of variation when forces of one *type* only—i.e. either radial forces alone, or tangential forces alone—act over one only of the two surfaces, is conveniently shown as in previous cases by stress-gradient curves. The only novelty is that two curves are now required for each type of forces, one, the "radial" curve, representing the variation of $\widehat{rr}$ with ξ, the other, the "tangential" curve, the variation of $\widehat{r\theta}$ and $\widehat{r\phi}$.

As regards both types of surface forces, we see that to a first approximation the stress of the same type as the applied force—$\widehat{rr}$ being a radial, $\widehat{r\theta}$ and $\widehat{r\phi}$ tangential stresses relative to the surface—has for its gradient curve a straight line whose zero ordinate answers to the unstressed surface. Also the gradient, to this degree of approximation, depends only on the local magnitude of the force and not on the degree of the harmonic it comes from. The stress-gradient curves of the opposite type to the applied surface forces are to a first approximation parabolas, the maximum ordinates answering to the mid-thickness, the zero ordinates to the two surfaces.

In the case of the bodily forces arising either from V_i or V_{-i-1} the radial and tangential stress-gradient curves are to a first approximation parabolas symmetrical about the maxima ordinates, which answer to the mid-thickness, and with zero ordinates answering to the two surfaces of the shell.

§ 36. When we take into account the "secondary" terms, and notice that $m-n$ is positive in all known materials and i is not less than 2, we find that in the case of radial surface forces the radial stress-gradient curve lies below or above the straight line given by the first approximation according as the forces act over the outer or the inner surface. These curves are shown in figs. 6 and 7, the dotted line referring to the first, the thick line to the second approximation.

Radial Forces.
Radial curves.

R_i

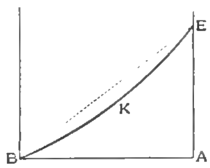


Fig 6

R'

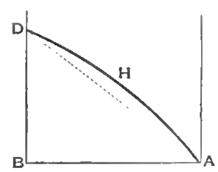


Fig 7

As in previous curves B refers to the inner, A to the outer surface. In both the thick line curves the gradient is steepest at the outer surface. This it will be remembered is the opposite of what happens when the radial forces are of constant magnitude over the surface (see § 10).

When the radial forces act over the outer surface the tangential stress gradient curve given by the second approximation lies, as shown by fig. 8, above the parabola given by the first approximation; but when the forces act over the inner surface the second approximation curve, as shown by fig. 9, lies above the parabola given by the first approximation only near the inner surface.

Radial Forces.
Tangential curves.

R_i

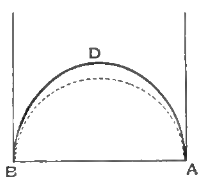


Fig 8

R'_i

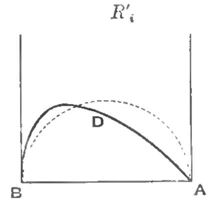


Fig 9

The mode of distinguishing the first and second approximation curves is the same as before.

The radial * and tangential gradient curves answering to the tangential surface forces

Tangential Forces.
Radial curves.

T_i

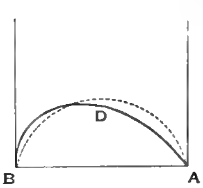


Fig 10

T'_i

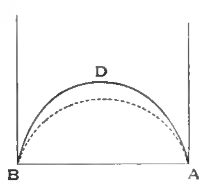


Fig 11

Tangential Forces.
Tangential curves.

T_i

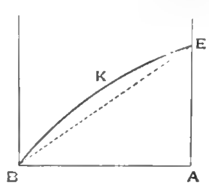


Fig 12

T'_i

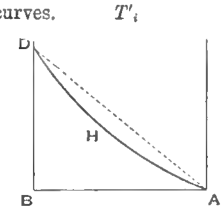


Fig 13

* In Fig. 10 the thick line curve will lie completely below the dotted curve if $i \nless 4$.

are shown in figs. 10—13. The notation and mode of representation is the same as in the other curves. The tangential gradient curves, as in the case of Class (ii) displacements, are of the same general form as the gradient curves 2 and 3 for uniform radial forces.

The radial and tangential* gradient curves for the bodily forces are similarly represented in figs. 14—17.

In both the radial curves the stress gradient according to the second approximation curves is steeper at the inner surface and less steep at the outer surface than according to the first approximation, or dotted line, curves.

BODILY FORCES.

V_i Radial Curves. V_{-i-1} V_i Tangential Curves. V_{-i-1}

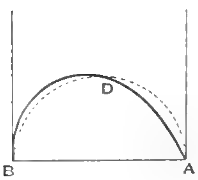


Fig 14

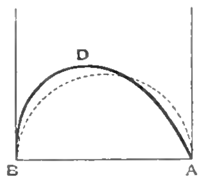


Fig 15

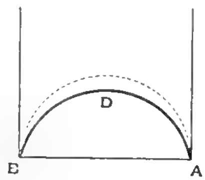


Fig 16

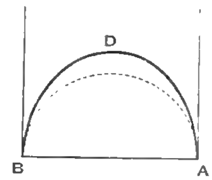


Fig 17

In the case of each curve it is to be kept in view that what is shown is the relative magnitude of a single stress at different distances from the surface along a single radius vector. The law of variation as ξ varies in the value say of $\widehat{rr}$ in terms of R_i is the same for all radii vectores, but the absolute value and the sign of $\widehat{rr}$ vary with the values of θ and ϕ.

Again the maxima values of the radial and tangential stresses arising from one and the same type of surface forces are of different orders of magnitude in h/a. Thus the "principal" term in the approximation to the stress opposite in type to the applied surface force is only of the same order of magnitude as the "secondary" terms in the approximation to the stress of the same type as the applied force. In other words the stress opposite in type to the applied surface force is to a first approximation negligible compared to the stress of the same type. It should also be noticed that the "principal" terms in the stresses arising from the bodily forces will be of the same order of magnitude as the "secondary" terms in a stress arising from a surface force of its own type only when the bodily forces per unit of surface are of the same order of magnitude as the surface forces.

In the preceding remarks on the gradient curves we have assumed "secondary" terms small compared to those containing algebraically lower powers of h/a. As i increases, however, the "secondary" terms in those stresses that are of the same type as the applied surface forces rapidly increase in relative importance, and they cease to be small compared to the "principal" terms when i^2h/a ceases to be small. Moreover when i becomes very big the stress opposite in type to the applied surface force ceases to be

* In Fig. 16 the thick line curve will lie above the dotted curve close to B if $i<5$.

small relative to the stress that is of the same type. Thus for a complete investigation of what happens in any instance when i^2h/a is not small recourse should be had to the general formulae (94) and (95).

An approximation to what happens when i is very large in the case of both displacements and stresses may be found by retaining only the highest powers of i in (92), (93), (94) and (95). Thus, for instance, on the left of these equations we may take $\overline{\Pi}$ as given by the following simplified form of (73):

$$\Pi = 4i^4 (m/n)^2 (ab)^{-2i-2} \{(a^{2i-1} - b^{2i-1})(a^{2i+3} - b^{2i+3}) - i^2 (ab)^{2i-1} (a^2 - b^2)^2\}.$$

The course then to be adopted depends on how big i and h/a actually are. Until this is known we are rather in the dark as to the relative importance of the two terms in the above expression for Π, or of the several terms in the coefficients of R_i etc. on the right of equations (92)—(95).

§ 37. Before quitting the subject of thin shells it may be well to give a brief summary of the results we have established for all forms of applied forces, whether the displacements they lead to be of the first, second or third class. As previously a denotes the radius, h the thickness of the shell, and h/a is very small. Our conclusions are as follows:

(1°) The resultant per unit of surface of all the forces applied along a radius—whether these be bodily or surface forces, or both combined—must be small compared to the greatest longitudinal traction* permissible in a long bar of the material. The ratio borne by the former quantity to the latter may be at most of the order h/a of small quantities.

If, however, the surface forces at corresponding points on the two surfaces be nearly equal and opposite, the resultant of *either* set may be of the same order of magnitude as the limiting longitudinal traction in the bar.

(2°) If the resultant of the forces applied along a radius do not vary very rapidly in magnitude or direction relative to r, θ, ϕ—i.e. if there be no surface harmonics of high degrees with large numerical coefficients—and if this resultant be not small compared to the resultant of the forces applied over one only of the surfaces, then approximate values to the radial strain and dilatation at all points in the shell are

$$\frac{du}{dr} = -\frac{\eta}{E}\frac{a}{h}F,$$

$$\delta = \frac{1}{3k}\frac{a}{h}\overline{F},$$

where F is the radial component per unit of surface of all the applied forces acting along the radius through the point considered, while η is Poisson's ratio, E Young's modulus and k the bulk modulus.

(3°) Under the same conditions as in 2°, the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$, usually assumed negligible in theories of thin shells, are in reality small compared to the other stresses,

* Measured of course per unit of cross section.

to which they bear ratios of the order h/a of small quantities. In this case the stretching of the shell is the important factor in the values of the principal strains and stresses.

(4°) If there be no very intense bodily forces, and if the surface forces at corresponding points all over the two surfaces be nearly equal and opposite, the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$ lose their inferiority relative to the other stresses. This also happens in any case when the magnitude, or direction relative to r, θ, ϕ, of the applied forces varies rapidly from point to point of the surface.

(5°) If a force of given type—radial or tangential—whose rate of variation with the angular coordinates is not very excessive, be applied over one only of the surfaces, the stress of the corresponding type has to a first approximation a straight line for its gradient curve, and the stress of opposite type—tangential or radial—unless it absolutely vanishes has for its gradient curve according to the first approximation a parabola whose vertex and maximum ordinate answer to the mid-thickness.

(6°) The displacements, strains and stresses arising from a bodily force are in general* of the same order of magnitude as those arising from a surface force when the two forces measured per unit of *surface* are of equal magnitudes. In practice this means that in a very thin shell the effects of bodily forces must be very small unless these forces be of extremely great intensity.

Solid Sphere.

§ 38. The *displacements* in the solid sphere may be derived from the corresponding results for the shell by omitting all terms containing b raised to a positive power. We shall represent all three classes of displacements simultaneously. With our previous notation answering to

$$\left.\begin{array}{l}\text{bodily forces from the potential } r^2V+\Sigma r^iV_i,\\ \text{surface forces}\left\{\begin{aligned}\widehat{rr}&=R+\Sigma R_i,\\ \widehat{r\theta}&=\Sigma\left[\frac{dT_i}{d\theta}+\frac{1}{\sin\theta}\frac{d\mathbf{T}_i}{d\phi}\right],\\ \widehat{r\phi}&=\Sigma\left[\frac{1}{\sin\theta}\frac{dT_i}{d\phi}-\frac{d\mathbf{T}_i}{d\theta}\right]\end{aligned}\right.\end{array}\right\}\qquad\text{(105)},$$

we get

$$\begin{aligned}u=\frac{rR}{3m-n}&+\frac{1}{5}\frac{\rho V}{m+n}\left\{\frac{5m+n}{3m-n}a^2r-r^3\right\}\\ &+\Sigma\frac{1}{2n\{(2i^2+4i+3)m-(2i+1)n\}}\left[i\rho V_i\left\{\frac{i}{i-1}\{(i+2)m-n\}a^2r^{i-1}-\{(i+1)m-n\}r^{i+1}\right\}\right.\\ &\qquad+R_i\left\{\frac{i}{i-1}\{i(i+2)m-n\}\frac{r^{i-1}}{a^{i-2}}-(i+1)(mi-2n)\frac{r^{i+1}}{a^i}\right\}\\ &\qquad\left.+i(i+1)T_i\left\{(mi-2n)\frac{r^{i+1}}{a^i}-\frac{1}{i-1}\{(i^2-i-3)m+n\}\frac{r^{i-1}}{a^{i-2}}\right\}\right]\qquad\text{(106)},\end{aligned}$$

* There are exceptions amongst the strains and stresses; compare for instance terms in T_i and V_i in (99).

$$v = \frac{1}{n \sin\theta} \sum \left[\frac{1}{i-1} \frac{d\mathbf{T}_i}{d\phi} \frac{r^i}{a^{i-1}} \right]$$

$$+ \sum \frac{1}{2n\{(2i^2+4i+3)m-(2i+1)n\}} \left[\rho \frac{dV_i}{d\theta} \left\{ \frac{i}{i-1} \{(i+2)m-n\} a^2 r^{i-1} - \{(i+3)m-n\} r^{i+1} \right\} \right.$$

$$+ \frac{dR_i}{d\theta} \left\{ \frac{1}{i-1} \{i(i+2)m-n\} \frac{r^{i-1}}{a^{i-2}} - \{(i+3)m+2n\} \frac{r^{i+1}}{a^i} \right\}$$

$$\left. + \frac{dT_i}{d\theta} \left\{ i\{(i+3)m+2n\} \frac{r^{i+1}}{a^i} - \frac{i+1}{i-1} \{(i^2-i-3)m+n\} \frac{r^{i-1}}{a^{i-2}} \right\} \right] \quad \text{.........(107)},$$

$$w = -\frac{1}{n} \sum \left[\frac{1}{i-1} \frac{d\mathbf{T}_i}{d\theta} \frac{r^i}{a^{i-1}} \right]$$

$$+ \sum \frac{1}{2n\{(2i^2+4i+3)m-(2i+1)n\}} \left[\frac{\rho}{\sin\theta} \frac{dV_i}{d\phi} \left\{ \frac{i}{i-1} \{(i+2)m-n\} a^2 r^{i-1} - \{(i+3)m-n\} r^{i+1} \right\} \right.$$

$$+ \frac{1}{\sin\theta} \frac{dR_i}{d\phi} \left\{ \frac{1}{i-1} \{i(i+2)m-n\} \frac{r^{i-1}}{a^{i-2}} - \{(i+3)m+2n\} \frac{r^{i+1}}{a^i} \right\}$$

$$\left. + \frac{1}{\sin\theta} \frac{dT_i}{d\phi} \left\{ i\{(i+3)m+2n\} \frac{r^{i+1}}{a^i} - \frac{i+1}{i-1} \{(i^2-i-3)m+n\} \frac{r^{i-1}}{a^{i-2}} \right\} \right] \text{...(108)}.$$

The summations run from $i=2$ to $i=\infty$. The value $i=1$ is incompatible with the preservation of equilibrium.

§ 39. It must be carefully noticed that though we may thus deduce the displacements for a solid sphere from those for a shell, the strains and stresses due to given forces over the outer surface are not the same in a solid sphere as in a shell whose outer boundary is the same, however small the radius of the inner surface may be. In the solid sphere we omit in the displacements all terms vanishing with b, and deduce the strains and stresses from the terms left; but in a shell a displacement $b^{i+1}r^{-i}$, while itself negligible however small r may be, will supply a strain varying as $(b/r)^{i+1}$. Such a strain will be very small except near the inner surface, but close to that surface it may be very large. Thus the strains and stresses near the centre of the solid sphere and near the inner surface of the nearly solid shell may be, and in fact generally are, widely different*.

§ 40. In the case of purely surface forces derived from a potential $(r^i/a^{i-1})\,Q_i$, as in § 13, the results (106), (107) and (108) take the remarkably simple forms

$$\left.\begin{aligned} u &= \frac{d}{dr}\left\{\frac{r^i/a^{i-2}}{2n(i-1)} Q_i\right\}, \\ v &= \frac{1}{r}\frac{d}{d\theta}\left\{\frac{r^i/a^{i-2}}{2n(i-1)} Q_i\right\}, \\ w &= \frac{1}{r\sin\theta}\frac{d}{d\phi}\left\{\frac{r^i/a^{i-2}}{2n(i-1)} Q_i\right\} \end{aligned}\right\} \quad \text{.................................(109)}.$$

In this case the dilatation δ obviously vanishes, as Q_i is a surface harmonic.

* For an explanation of this seeming discontinuity see the Society's *Proceedings*, Vol. VII. pp. 285–6, 1892.

Nearly solid shell.

§ 41. There is considerable interest attaching to the action of forces applied over the inner surface of a *nearly solid* shell, i.e. a shell for which b/a is very small. The method of treating this case will perhaps be sufficiently illustrated by the deduction of the radial displacement answering to the purely radial force R_i'. To find this we employ (92), retaining in the coefficient of each power of r only the lowest power of b. The result is of course only a first approximation, neglecting higher powers of b/a than those retained. It is

$$2n\{(2i^2+4i+3)m-(2i+1)n\}\{(2i^2+1)m+(2i+1)n\}u/R_i'$$

$$= i(i+1)(i+2)(2i-1)m(im-2n)(br)^{i+1}a^{-2i-1}$$

$$-\frac{i(2i+1)}{i-1}[(i^4+2i^3-i^2-2i+3)m^2+2mn-n^2]r^{i-1}b^{i+1}a^{-2i+1}$$

$$+\{(2i^2+4i+3)m-(2i+1)n\}\left[\frac{i+1}{i+2}\{(i^2-1)m-n\}\frac{b^{i+3}}{r^{i+2}}-i\{(i+1)m+2n\}\frac{b^{i+1}}{r^i}\right] \text{......(110)}.$$

Near the inner surface, i.e. when r is of the order b, we may obviously neglect the terms in r^{i+1} and r^{i-1} compared to those in r^{-i} and r^{-i-2}, and so get the approximation

$$u=\frac{R_i'}{2n\{(2i^2+1)m+(2i+1)n\}}\left[\frac{i+1}{i+2}\{(i^2-1)m-n\}\frac{b^{i+3}}{r^{i+2}}-i\{(i+1)m+2n\}\frac{b^{i+1}}{r^i}\right] \text{.........(111)}.$$

This result, it will be observed, may be derived from the term in R_i in (106) by substituting b for a and writing $(-i-1)$ for $(+i)$ in all indices and coefficients. The same substitution applies in the case of any displacement for any surface force. Thus if we want the displacements, strains or stresses near the inner surface of a nearly solid shell arising from forces applied over that surface, we have only to transform the corresponding results for a solid sphere, acted on over its surface by forces following the same law, by replacing a by b, and i by $-(i+1)$ in all indices and coefficients. When i is large u diminishes with extreme rapidity as r increases so long as (111) remains a satisfactory first approximation. A similar result holds for the other displacements and for the strains and stresses.

The formula (111) applies only when r is of order b. On the other hand when r becomes of the order a the terms retained in (111) are negligible, and the terms in r^{i+1} and r^{i-1} in (110) then constitute the first approximation. In this case it will suffice to point out the physical consequences.

Regarding r in (110) as of order a we obviously have u/r of the order $(b/a)^{i+1}$, and the same result holds for all the strains and stresses due to R_i' or to tangential forces derived from a potential T_i'. In the corresponding case of tangential surface forces derived from a "stream function" $\mathbf{T}_i'$ the rate of diminution in the strains and stresses as i increases when r is of order a is measured by $(b/a)^{i+2}$. Thus in all cases the strains and stresses due to surface forces derived from surface harmonics of high degrees are comparatively insignificant except close to the inner surface. At very moderate distances from this surface the strains and stresses will be almost entirely due to those forces which are constant or which vary but slowly over the surface. Regarding the strains and stresses as propagated outwards from the surface, the effects transmitted from adjacent parts of the surface where the applied forces are oppositely directed tend to neutralise one another, and thus the action of the medium is to obliterate the effects of any want of uniformity in the distribution of the surface forces. This damping out of the effects of the forces derived from the high harmonics relative to the effects of the constant forces does not however, it should be noticed, increase with the distance, after this has reached the limit at which the terms in r^{i+1} and r^{i-1} in (110) constitute a satisfactory approximation.

Part II.

Equilibrium under given surface displacements.

§ 42. The previous solution may also be applied to a shell whose surfaces are subjected to given displacements. These displacements must of course be of such a character as not to strain the shell beyond the limits permissible in the material. All rigid body displacements may be excluded. As the case of given surface displacements seems of much less physical interest than that of given surface forces it calls for less fulness of treatment.

The displacements may most conveniently be considered under the three classes of Part I.

Class (i). *Pure radial displacements.*

The two constants of the solution

$$u = \tfrac{1}{3} r Y_0 + r^{-2} Z_{-1} \quad \text{.......... (1)}$$

are to be determined from the data

$$\left.\begin{aligned} u &= U \quad \text{over } r = a, \\ u &= U' \quad \text{over } r = b \end{aligned}\right\} \quad \text{.......... (2)},$$

where U and U' are constants.

The solution obviously becomes

$$u = \{r(a^2U - b^2U') + a^2b^2r^{-2}(aU' - bU)\} \div (a^3 - b^3) \quad \text{.......... (3)},$$

$$\delta = 3(a^2U - b^2U') \div (a^3 - b^3) \quad \text{.......... (4)},$$

$$\widehat{rr} = \{(3m - n)(a^2U - b^2U') - 4na^2b^2r^{-3}(aU' - bU)\} \div (a^3 - b^3) \quad \text{.......... (5)},$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \{(3m - n)(a^2U - b^2U') + 2na^2b^2r^{-3}(aU' - bU)\} \div (a^3 - b^3) \quad \text{.......... (6)}.$$

For a thin shell, putting $a - b = h$, $a - r = \xi$, we get the approximate values:

$$u = U\frac{h - \xi}{h}\left(1 + \frac{\xi}{a}\right) + U'\frac{\xi}{h}\left(1 - \frac{h - \xi}{a}\right) \quad \text{.......... (7)},$$

$$\widehat{rr} = \frac{U}{h}\left\{(m+n)\left(1+\frac{h}{a}\right) - 4n\frac{h-\xi}{a}\right\} - \frac{U'}{h}\left\{(m+n)\left(1-\frac{h}{a}\right) + 4n\frac{\xi}{a}\right\} \quad\text{.............(8)},$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \frac{U}{h}\left\{(m-n)\left(1+\frac{h}{a}\right) + 2n\frac{h-\xi}{a}\right\} - \frac{U'}{h}\left\{(m-n)\left(1-\frac{h}{a}\right) - 2n\frac{\xi}{a}\right\} \quad\text{.............(9)}.$$

Two important conclusions as to the necessary limits to be assigned to the surface displacements in thin shells are easily deduced. From (7) we have the approximate results

$$u/r = u(1+\xi/a)/a = \frac{U}{a}\frac{h-\xi}{h}\left(1+\frac{2\xi}{a}\right) + \frac{U'}{a}\frac{\xi}{h}\left(1+\frac{2\xi-h}{a}\right) \quad\text{.................(10)},$$

$$\frac{du}{dr} = -\frac{du}{d\xi} = \frac{U-U'}{h}\left(1-\frac{h-2\xi}{a}\right) \quad\text{..............................(11)}.$$

Now u/r and $\frac{du}{dr}$ are strains, and thus U/a, U'/a and $(U-U')/h$ must be small quantities of the order permissible to strains in the material. The last limitation, which is fairly obvious *a priori*, must be kept in view in judging of the accuracy of approximations. It shows that terms in $U-U'$ may be of less importance than terms in Uh/a.

If $U'=U$ the strains and stresses have their values very nearly constant along the thickness, the approximate values of the stresses being

$$\left.\begin{aligned}\widehat{rr} &= 2(m-n)\,U/a,\\ \widehat{\theta\theta} = \widehat{\phi\phi} &= 2mU/a\end{aligned}\right\} \quad\text{..............................(12)}.$$

CLASS (ii). *Pure transverse displacements.*

§ 43. Here we have to determine the X_i, X_{-i-1} of (31) and (32) Part I. from the conditions

$$\left.\begin{aligned} v &= \frac{1}{\sin\theta}\frac{d\mathbf{T}_i}{d\phi}, \quad w = -\frac{d\mathbf{T}_i}{d\theta} \quad \text{over } r=a,\\ v &= \frac{1}{\sin\theta}\frac{d\mathbf{T}_i'}{d\phi}, \quad w = -\frac{d\mathbf{T}_i'}{d\theta} \quad \text{over } r=b \end{aligned}\right\} \quad\text{..............................(13)},$$

where $\mathbf{T}_i$, $\mathbf{T}_i'$ are surface harmonics of degree i.

We easily find

$$v = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left\{r^i\left(a^{i+1}\mathbf{T}_i - b^{i+1}\mathbf{T}_i'\right) + \left(\frac{ab}{r}\right)^{i+1}\left(a^i\mathbf{T}_i' - b^i\mathbf{T}_i\right)\right\} \div \left(a^{2i+1} - b^{2i+1}\right)\right] \quad\text{...........(14)},$$

$$w = -\frac{d}{d\theta}\,[\text{same expression as inside square brackets in (14)}] \quad\text{....................(15)}.$$

In a thin shell approximate values are

$$w = -\frac{d\mathbf{T}_i}{d\theta}\frac{h-\xi}{h}\left(1+\frac{\xi}{a}\right) - \frac{d\mathbf{T}_i'}{d\theta}\frac{\xi}{h}\left(1-\frac{h-\xi}{a}\right) \quad\text{..............................(16)},$$

$$\widehat{r\phi}/n = -\frac{d\mathbf{T}_i}{d\theta}\frac{1}{h}\left(1+\frac{h}{a}-3\frac{h-\xi}{a}\right)+\frac{d\mathbf{T}_i'}{d\theta}\frac{1}{h}\left(1-\frac{h}{a}+\frac{3\xi}{a}\right) \quad \text{.......................................(17)},$$

$$\widehat{\theta\phi}/n = -\left\{i(i+1)\mathbf{T}_i+2\frac{d^2\mathbf{T}_i}{d\theta^2}\right\}\frac{1}{a}\frac{h-\xi}{h}\left(\frac{1+2\xi}{a}\right)$$

$$-\left\{i(i+1)\mathbf{T}_i'+2\frac{d^2\mathbf{T}_i'}{d\theta^2}\right\}\frac{1}{a}\frac{\xi}{h}\left(1-\frac{h-2\xi}{a}\right) \quad \text{.......................(18)}.$$

The value of v may be got from that of w and the value of $\widehat{r\theta}$ from that of $\widehat{r\phi}$ by writing $\frac{1}{\sin\theta}\frac{d}{d\phi}$ for $-\frac{d}{d\theta}$. The reason for writing down the value of w rather than that of v is that w alone exists when $\mathbf{T}_i$ and $\mathbf{T}_i'$ are zonal harmonics.

Since $\widehat{r\phi}/n$ and $\widehat{\theta\phi}/n$ are strains we see that the displacement at either surface divided by the radius, and the difference in the displacements at corresponding points on the two surfaces in the same direction divided by the thickness, must be quantities not exceeding in order of magnitude the limits permissible to strains in the material. When the displacements are equal over the two surfaces, all the strains and stresses have to a first approximation constant values along the thickness.

CLASS (iii). *Mixed radial and transverse displacements.*

§ 44. Here we determine the Y_i, Y_{-i-1}, Z_i, Z_{-i-1} of the formulae (30)—(32) Part I.—in which V_i, V_{-i-1} are now supposed zero—from the conditions

$$\left.\begin{aligned} u=U_i,\ \ v=\frac{dT_i}{d\theta},\ \ w=\frac{1}{\sin\theta}\frac{dT_i}{d\phi}\ \text{ over } r=a,\\ u=U_i',\ \ v=\frac{dT_i'}{d\theta},\ \ w=\frac{1}{\sin\theta}\frac{dT_i'}{d\phi}\ \text{ over } r=b \end{aligned}\right\} \quad \text{.............................(19)},$$

where U_i, U_i', T_i, T_i' are surface harmonics of degree i.

These conditions give

$$-\frac{im-2n}{2(2i+3)n}a^{i+1}Y_i+a^{i-1}Z_i-\frac{(i+1)m+2n}{2(2i-1)n}a^{-i}Y_{-i-1}+a^{-i-2}Z_{-i-1}=U_i \quad \text{...................(20)},$$

$$-\frac{(i+3)m+2n}{2(i+1)(2i+3)n}a^{i+1}Y_i+\frac{1}{i}a^{i-1}Z_i$$

$$+\frac{(i-2)m-2n}{2i(2i-1)n}a^{-i}Y_{-i-1}-\frac{1}{i+1}a^{-i-2}Z_{-i-1}=T_i \quad \text{...................(21)},$$

$$-\frac{im-2n}{2(2i+3)n}b^{i+1}Y_i+b^{i-1}Z_i-\frac{(i+1)m+2n}{2(2i-1)n}b^{-i}Y_{-i-1}+b^{-i-2}Z_{-i-1}=U_i' \quad \text{..................(22)},$$

$$-\frac{(i+3)m+2n}{2(i+1)(2i+3)n}b^{i+1}Y_i+\frac{1}{i}b^{i-1}Z_i$$

$$+\frac{(i-2)m-2n}{2i(2i-1)n}b^{-i}Y_{-i-1}-\frac{1}{i+1}b^{-i-2}Z_{-i-1}=T_i' \quad \text{.................(23)}.$$

The method of treating these equations employed in my original paper* for the case of normal displacements seems the simplest way of solving the above. For shortness let

$$\left.\begin{aligned} U_i - iT_i &= A_i, \quad U_i + (i+1)\,T_i = B_i, \\ U_i' - iT_i' &= A_i', \quad U_i' + (i+1)\,T_i' = B_i' \end{aligned}\right\} \qquad (24).$$

Then putting

$$\Pi = \frac{\{im + (2i+1)\,n\}\,\{(i+1)\,m + (2i+1)\,n\}}{i\,(i+1)(2i-1)(2i+3)\,n^2}\,(ab)^{-2i+1}\,(a^{2i-1} - b^{2i-1})\,(a^{2i+3} - b^{2i+3})$$
$$- \left(\frac{m}{2n}\right)^2 (a^2 - b^2)^2 \qquad (25),$$

we easily find

$$\Pi Y_i = \frac{(i+1)\,m + (2i+1)\,n}{i\,(2i-1)\,n}\,\frac{a^{2i-1} - b^{2i-1}}{(ab)^{2i-1}}\,\{a^{i+2}A_i - b^{i+2}A_i'\}$$
$$+ \frac{m}{2n}\,(a^2 - b^2)\,\{a^{-i+1}B_i - b^{-i+1}B_i'\} \qquad (26),$$

$$\Pi Z_i = \frac{i}{2i+1}\,(ab)^2 \left[\frac{m}{2n}\,\frac{(i+1)\,m + (2i+1)\,n}{i\,(2i-1)\,n}\,\frac{a^{2i+1} - b^{2i+1}}{(ab)^{2i+1}}\,(a^{i+2}A_i - b^{i+2}A_i')\right.$$
$$+ \frac{\{im + (2i+1)\,n\}\,\{(i+1)\,m + (2i+1)\,n\}}{i\,(i+1)\,(2i-1)\,(2i+3)\,n^2}\,\frac{a^{2i+3} - b^{2i+3}}{(ab)^{2i+1}}\,(a^iB_i - b^iB_i')$$
$$\left. + \left(\frac{m}{2n}\right)^2 (a^2 - b^2)\,(a^{-i-1}B_i - b^{-i-1}B_i')\right] \qquad (27),$$

$$\Pi Y_{-i-1} = \frac{m}{2n}\,(a^2 - b^2)\,(a^{i+2}A_i - b^{i+2}A_i')$$
$$+ \frac{im + (2i+1)\,n}{(i+1)\,(2i+3)\,n}\,(a^{2i+3} - b^{2i+3})\,(a^{-i+1}B_i - b^{-i+1}B_i') \qquad (28),$$

$$\Pi Z_{-i-1} = \frac{i+1}{2i+1}\,(ab)^2 \left[\frac{\{im + (2i+1)\,n\}\,\{(i+1)\,m + (2i+1)\,n\}}{i\,(i+1)\,(2i-1)\,(2i+3)\,n^2}\,\frac{a^{2i-1} - b^{2i-1}}{(ab)^{i-1}}\,(a^{i+1}A_i' - b^{i+1}A_i)\right.$$
$$+ \left(\frac{m}{2n}\right)^2 (a^2 - b^2)\,(a^iA_i - b^iA_i')$$
$$\left. + \frac{m}{2n}\,\frac{im + (2i+1)\,n}{(i+1)\,(2i+3)\,n}\,(a^{2i+1} - b^{2i+1})\,(a^{-i+1}B_i - b^{-i+1}B_i')\right] \qquad (29).$$

The substitution of $(-i-1)$ for $(+i)$ in all indices and coefficients transforms Π into itself, and deduces the values of Y_{-i-1} and Z_{-i-1} from those of Y_i and Z_i respectively.

§ 45. Substituting the above values of Y_i, Z_i, Y_{-i-1}, Z_{-i-1} in equations (30) and (31) Part I., and writing $U_i - iT_i$ for A_i etc., we find

$$\Pi u = (U_i - iT_i)\left(\frac{a}{r}\right)^{i+2} \left[\frac{i+1}{2i+1}\left\{\frac{\{im + (2i+1)\,n\}\,\{(i+1)\,m + (2i+1)\,n\}}{i\,(i+1)\,(2i-1)\,(2i+3)\,n^2}\,\frac{(a^{2i-1} - b^{2i-1})\,(r^{2i+3} - b^{2i+3})}{(ab)^{2i-1}}\right.\right.$$
$$\left. - \left(\frac{m}{2n}\right)^2 (a^2 - b^2)\,(r^2 - b^2)\right\}$$

* *Camb. Trans.*, Vol. XIV., pp. 305, 306.

$$+\frac{m}{2n}\frac{(i+1)m+(2i+1)n}{(2i-1)(2i+1)n}\left(b^{-2i+1}(a^2-r^2)(r^{2i+1}-b^{2i+1})-a^{-2i+1}(r^2-b^2)(a^{2i+1}-r^{2i+1})\right)\Bigg]$$

$+(U_i'-iT_i')\left(\frac{b}{r}\right)^{i+2}$ [coefficient obtained from that of U_i-iT_i inside square brackets by interchanging a and b]

$$+\{U_i+(i+1)T_i\}\left(\frac{a}{r}\right)^i\Bigg[\frac{i}{2i+1}\Bigg\{\frac{\{im+(2i+1)n\}\{(i+1)m+(2i+1)n\}}{i(i+1)(2i-1)(2i+3)n^2}\frac{(a^{2i+3}-b^{2i+3})(r^{2i-1}-b^{2i-1})}{(ab)^{2i-1}}$$

$$-\left(\frac{m}{2n}\right)^2\left(\frac{r}{a}\right)^{2i-1}(a^2-b^2)(r^2-b^2)\Bigg\}$$

$$+\frac{m}{2n}\frac{im+(2i+1)n}{(2i+1)(2i+3)n}\frac{1}{a^{2i-1}r^2}\left(b^2(a^2-r^2)(r^{2i+1}-b^{2i+1})-a^2(r^2-b^2)(a^{2i+1}-r^{2i+1})\right)\Bigg]$$

$+\{U_i'+(i+1)T_i'\}\left(\frac{b}{r}\right)^i$ [coefficient obtained from that of $U_i+(i+1)T_i$ inside square brackets by interchanging a and b] ....(30),

$$\Pi v=-\frac{1}{2i+1}\frac{d}{d\theta}(U_i-iT_i)\left(\frac{a}{r}\right)^{i+2}\Bigg[\Bigg\{\frac{\{im+(2i+1)n\}\{(i+1)m+(2i+1)n\}}{i(i+1)(2i-1)(2i+3)n^2}\times$$

$$\frac{(a^{2i-1}-b^{2i-1})(r^{2i+3}-b^{2i+3})}{(ab)^{2i-1}}-\left(\frac{m}{2n}\right)^2(a^2-b^2)(r^2-b^2)\Bigg\}$$

$$-\left(\frac{m}{2n}\right)\frac{(i+1)m+(2i+1)n}{i(2i-1)n}\left(b^{-2i+1}(a^2-r^2)(r^{2i+1}-b^{2i+1})-a^{-2i+1}(r^2-b^2)(a^{2i+1}-r^{2i+1})\right)\Bigg]$$

$-\frac{1}{2i+1}\frac{d}{d\theta}(U_i'-iT_i')\left(\frac{b}{r}\right)^{i+2}$ [coefficient obtained from that of $\frac{d}{d\theta}(U_i-iT_i)$ inside square brackets by interchanging a and b]

$$+\frac{1}{2i+1}\frac{d}{d\theta}\{U_i+(i+1)T_i\}\left(\frac{a}{r}\right)^i\Bigg[\Bigg\{\frac{\{im+(2i+1)n\}\{(i+1)m+(2i+1)n\}}{i(i+1)(2i-1)(2i+3)n^2}\times$$

$$\frac{(a^{2i+3}-b^{2i+3})(r^{2i-1}-b^{2i-1})}{(ab)^{2i-1}}-\left(\frac{m}{2n}\right)^2\left(\frac{r}{a}\right)^{2i-1}(a^2-b^2)(r^2-b^2)\Bigg\}$$

$$+\frac{m}{2n}\frac{im+(2i+1)n}{(i+1)(2i+3)n}\frac{1}{a^{2i-1}r^2}\left(a^2(r^2-b^2)(a^{2i+1}-r^{2i+1})-b^2(a^2-r^2)(r^{2i+1}-b^{2i+1})\right)\Bigg]$$

$+\frac{1}{2i+1}\frac{d}{d\theta}\{U_i'+(i+1)T_i'\}\left(\frac{b}{r}\right)^i$ [coefficient obtained from that of $\frac{d}{d\theta}\{U_i+(i+1)T_i\}$ inside square brackets by interchanging a and b] ....(31).

The value of w may be deduced from that of v by replacing $\frac{d}{d\theta}$ by $\frac{1}{\sin\theta}\frac{d}{d\phi}$. The substitution of $(-i-1)$ for $(+i)$ in all indices and coefficients derives coefficients of $U_i+(i+1)T_i$ and $U_i'+(i+1)T_i'$ from those of U_i-iT_i and $U_i'-iT_i'$ respectively.

§ 46. The form of the results suggests the deduction of the surface displacements from two potential functions after the manner indicated for the surface forces in § 13. Thus if the sum of these functions, for the surface $r=a$, be

$$(r^i/a^{i-1})\,Q_i + (r^{-i-1}a^{i+2})\,Q_{-i-1},$$

where Q_i, Q_{-i-1} are surface harmonics of degree i, we should put

$$\left.\begin{aligned} U_i &= \frac{d}{dr}\left[(r^i/a^{i-1})\,Q_i + (r^{-i-1}a^{i+2})\,Q_{-i-1}\right],\\ \frac{dT_i}{d\theta} &= \frac{1}{r}\frac{d}{d\theta}\,[\text{same expression}],\\ \frac{1}{\sin\theta}\frac{dT_i}{d\phi} &= \frac{1}{r\sin\theta}\frac{d}{d\phi}\,[\text{same expression}] \end{aligned}\right\} \quad\text{..........(32)},$$

where a is substituted for r after differentiation. The relations between U_i, T_i and Q_i, Q_{-i-1} take the simple forms

$$\left.\begin{aligned} \{U_i+(i+1)\,T_i\}/(2i+1) &= Q_i,\\ \{U_i-iT_i\}/(2i+1) &= -Q_{-i-1} \end{aligned}\right\} \quad\text{..........(33)}.$$

The expressions for the displacements are obviously much simplified if either Q_i or Q_{-i-1} is zero, and the form in which (30) and (31) are presented was chosen partly with a view to bring this out. Other reasons for selecting this form were that it affords a ready means of testing the accuracy of the results and that it lends itself readily to applications to thin shells.

§ 47. The arrangement in (30) and (31) is analogous to that adopted in (94) and (95) for the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$. Thus in (30) in the coefficients of both $U_i - iT_i$ and $U_i + (i+1)\,T_i$ the expressions inside the { } brackets obviously vanish over $r=b$ and take the value Π over $r=a$, while the last lines of these coefficients clearly vanish over both surfaces of the shell.

For the thin shell, putting $a-b=h$, $a-r=\xi$, as before, we easily deduce the following approximate results:

$$u = U_i\frac{h-\xi}{h}\left(1+\frac{\xi}{a}\right) + U_i'\frac{\xi}{h}\left(1-\frac{h-\xi}{a}\right) - (T_i - T_i')\,\frac{i(i+1)}{2}\,\frac{m}{m+n}\,\frac{\xi(h-\xi)}{ah} \quad\text{..........(34)},$$

$$v = \frac{d}{d\theta}(U_i - U_i')\,\frac{m}{2n}\,\frac{\xi(h-\xi)}{ah} + \frac{dT_i}{d\theta}\,\frac{h-\xi}{h}\left(1+\frac{\xi}{a}\right) + \frac{dT_i'}{d\theta}\,\frac{\xi}{h}\left(1-\frac{h-\xi}{a}\right) \quad\text{..........(35)},$$

$$\begin{aligned} \widehat{rr} = {} & \frac{U_i}{h}\left\{(m+n)\left(1+\frac{h}{a}\right) - 4n\,\frac{h-\xi}{a}\right\} - \frac{U_i'}{h}\left\{(m+n)\left(1-\frac{h}{a}\right) + 4n\,\frac{\xi}{a}\right\}\\ & - \frac{T_i}{a}\,\frac{i(i+1)}{2}\left(m-2n\,\frac{h-\xi}{h}\right) - \frac{T_i'}{a}\,\frac{i(i+1)}{2}\left(m-2n\,\frac{\xi}{h}\right) \quad\text{..........(36)}, \end{aligned}$$

$$\begin{aligned} \widehat{r\theta} = {} & \frac{1}{a}\,\frac{dU_i}{d\theta}\,\frac{m}{2}\left\{1-2\,\frac{m-n}{m}\,\frac{h-\xi}{h}\right\} + \frac{1}{a}\,\frac{dU_i'}{d\theta}\,\frac{m}{2}\left\{1-2\,\frac{m-n}{m}\,\frac{\xi}{h}\right\}\\ & + \frac{1}{h}\,\frac{dT_i}{d\theta}\,n\left(1+\frac{h}{a}-3\,\frac{h-\xi}{a}\right) - \frac{1}{h}\,\frac{dT_i'}{d\theta}\,n\left(1-\frac{h}{a}+3\,\frac{\xi}{a}\right) \quad\text{..........(37)}, \end{aligned}$$

$$\widehat{\theta\phi} = \frac{2n}{a}\frac{d^2}{d\theta d\phi}\left[\frac{1}{\sin\theta}\left\{(U_i - U_i')\frac{m}{2n}\frac{\xi(h-\xi)}{ah}\right.\right.$$

$$\left.\left. + T_i\frac{h-\xi}{h}\left(1+2\frac{\xi}{a}\right) + T_i'\frac{\xi}{h}\left(1-\frac{h-2\xi}{a}\right)\right\}\right] \quad\text{..................(38).}$$

The values of w and $\widehat{r\phi}$ may be found from those of v and $\widehat{r\theta}$ respectively by substituting $\frac{1}{\sin\theta}\frac{d}{d\phi}$ for $\frac{d}{d\theta}$.

The limitations in the magnitudes permissible to the displacements over either surface, and to the difference between the displacements at corresponding points on the two surfaces, are precisely similar to those established in the two previous classes of displacements.

§ 48. When the surface displacements have no tangential component i does not appear in the coefficients in (34) and (35), and the coefficients of U in (7) and U_i in (34) are identical. Thus to the present degree of approximation if radial displacements $\bar{U}$ and $\bar{U}'$ be applied over the surfaces of a thin shell according to *any* law whatsoever—consistent of course with the limitations as to the magnitudes of the strains—we have

$$u = \bar{U}\frac{h-\xi}{h}\left(1+\frac{\xi}{a}\right) + \bar{U}'\frac{\xi}{h}\left(1-\frac{h-\xi}{a}\right) \quad\text{..............................(39),}$$

$$v = \frac{d}{d\theta}(\bar{U} - \bar{U}')\frac{m}{2n}\frac{\xi(h-\xi)}{ah} \quad\text{..(40),}$$

$$w = \frac{1}{\sin\theta}\frac{d}{d\phi}(\bar{U} - \bar{U}')\frac{m}{2n}\frac{\xi(h-\xi)}{ah} \quad\text{..................................(41).}$$

The coefficients in the expressions for the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{r\phi}$, $\widehat{\theta\phi}$ do not in this case contain i either, and the coefficients of U and U' in (8) are the same as those of U_i and U_i' in (36); thus the expressions for these stresses may be found by putting $T_i = T_i' = 0$ in (36), (37) and (38) and replacing U_i by $\bar{U}$ and U_i' by $\bar{U}'$. It must be remembered however that if $(\bar{U} - \bar{U}')/\bar{U}$ be very small, terms involving higher powers of h/a than those retained in (40) and (41) may be of equal or greater importance. A similar limitation would apply to the expressions deduced for $\widehat{rr}$ and $\widehat{\theta\phi}$.

§ 49. We notice that the coefficients of $\frac{dT_i}{d\theta}$ and $\frac{dT_i'}{d\theta}$ in both (35) and (37) do not contain i, and by referring to (16) and (17) it will be seen that the same factors, e.g.

$$1 + \frac{h}{a} - 3\frac{h-\xi}{a} \text{ and } 1 - \frac{h}{a} + 3\frac{\xi}{a} \text{ in (17) and (37),}$$

occur in the two cases. Now the total components parallel to θ, ϕ of the tangential displacements on the two surfaces are given by

$$\left.\begin{aligned}\overline{V} &= \Sigma\left[\frac{dT_i}{d\theta} + \frac{1}{\sin\theta}\frac{d\mathbf{T}_i}{d\phi}\right], \quad \overline{W} = \Sigma\left[\frac{1}{\sin\theta}\frac{dT_i}{d\phi} - \frac{d\mathbf{T}_i}{d\theta}\right], \\ \overline{V}' &= \Sigma\left[\frac{dT_i'}{d\theta} + \frac{1}{\sin\theta}\frac{d\mathbf{T}_i'}{d\phi}\right], \quad \overline{W}' = \Sigma\left[\frac{1}{\sin\theta}\frac{dT_i'}{d\phi} - \frac{d\mathbf{T}_i'}{d\theta}\right]\end{aligned}\right\} \quad \text{.........(42).}$$

Thus we obviously have, for the most general tangential displacements consistent with the limits permissible in the magnitudes of the strains, the approximate results

$$v = \overline{V}\frac{h-\xi}{h}\left(1+\frac{\xi}{a}\right) + \overline{V}'\frac{\xi}{h}\left(1-\frac{h-\xi}{a}\right) \quad \text{.........(43),}$$

$$w = \overline{W}\frac{h-\xi}{h}\left(1+\frac{\xi}{a}\right) + \overline{W}'\frac{\xi}{h}\left(1-\frac{h-\xi}{a}\right) \quad \text{.........(44),}$$

$$\widehat{r\theta} = \frac{\overline{V}}{h}n\left(1+\frac{h}{a}-3\frac{h-\xi}{a}\right) - \frac{\overline{V}'}{h}n\left(1-\frac{h}{a}+3\frac{\xi}{a}\right) \quad \text{.........(45),}$$

$$\widehat{r\phi} = \frac{\overline{W}}{h}n\left(1+\frac{h}{a}-3\frac{h-\xi}{a}\right) - \frac{\overline{W}'}{h}n\left(1-\frac{h}{a}+3\frac{\xi}{a}\right) \quad \text{.........(46).}$$

As before, it should be noticed that when the difference between the displacements at corresponding points is very small compared to the displacement for one of the surfaces, terms containing higher powers of h/a may have to be retained.

§ 50. Let us suppose that one only of the two surfaces is displaced, say the outer. We then see from (39), (43) and (44), that the way in which $u/\overline{U}$, $v/\overline{V}$, and $w/\overline{W}$ vary with ξ is precisely the same. Thus to the present degree of approximation we see that the same "displacement-gradient curve"—i.e. a curve whose abscissae measure the distance from a surface of the shell and whose ordinates give the corresponding magnitude of a particular displacement—would apply in all cases when there is no radial surface displacement, or when there is no tangential surface displacement, to the displacement which is of the same type as the given surface displacement.

A similar result obviously applies in the case of displacements applied over the inner surface only. The curve is in the case of either surface a straight line according to the first approximation, whose zero ordinate answers to the undisplaced surface. The curves according to the second approximations are of the forms of those in fig. 2 or fig. 3, § 10, according as the outer or inner surface is that displaced. The gradients in both cases are steepest at the inner surface.

From (34) and (35) we see that the gradient curves for the displacements which are opposite in type to the given surface displacement are to a first approximation parabolas symmetrical about their maximum ordinate, which answers to the mid surface, and with zero ordinates answering to the two surfaces of the shell.

There is one important distinction between the displacements which are of the same type as the given surface displacement and those which are of the opposite type. The magnitude of the former depends, to the present degree of approximation, only on the local magnitude of the applied displacement, but the latter increase somewhat rapidly with the degree of the harmonic from which the displacements are derived. This is obvious when i is large, as we are then to regard $\frac{dT_i}{d\theta}$ and $\frac{dU_i}{d\theta}$ as of orders iT_i and iU_i; thus for a given magnitude of $\frac{dT_i}{d\theta}$ the corresponding term in u in (34) varies as i, and for a given magnitude of U_i the corresponding term in v in (35) varies as i. When i is small the displacements opposite in type to the given surface displacement bear to those of the same type a ratio of the order h/a, and so to a first approximation may be neglected; but as i increases their relative importance increases, and they may not be neglected even to a first approximation when ih/a ceases to be small.

If we suppose i so small, or the shell so thin, that ih/a is negligible, we have to a first approximation for simultaneous displacements $\bar{U}$, $\bar{V}$, $\bar{W}$ over the outer surface only

$$u/\bar{U} = v/\bar{V} = w/\bar{W} = (1 - \xi/h) \qquad (47).$$

This signifies that the resultant displacement at any point of the thickness is parallel to the applied surface displacement, and proportional in magnitude to the distance from the inner surface. A corresponding result holds under like conditions for displacements over the inner surface only.

When ih/a ceases to be small it would be wise to employ the exact results (14), (15), (30) and (31) to ensure that terms are not omitted equal in magnitude to those retained in the above approximations. This is especially the case when the difference of the displacements at corresponding points on the two surfaces is small compared to the displacement over either surface.

It must also be borne in mind that taking the displacements over a surface zero is equivalent to supposing that surface held by the surface forces requisite to prevent displacement. Thus the cases treated above where the displacements are given over one surface only, and the other surface is supposed undisplaced, answer to a totally different set of matters from that arising when the one surface is displaced in an assigned arbitrary way and the other is left free of forces. This latter case seems not unlikely to be the more interesting of the two in practice and we shall briefly consider it presently.

§ 51. Before doing so, however, it may be as well to point out that the solution for a solid sphere subjected to given arbitrary surface displacements may be deduced from that for a shell precisely as in the case of given forces. To get the displacements for the solid sphere we have only to put $b = 0$ in (3), (14), (15), (30) and (31), noticing in the two latter equations the occurrence of b^{-2i+1} as a factor in Π.

In the case of a nearly solid shell approximate solutions may be deduced by retaining only the lowest powers of b/a in the coefficients of the several powers of r. This would be very easily done for the first two classes of displacements as given by (3), (14) and (15).

The formulae (30) and (31) for the third class are not so convenient for this purpose, and it might be found simpler to substitute in the formulae (30)—(32) Part I. the values found for Y_i, Z_i, Y_{-i-1}, Z_{-i-1} in (26)—(29) by retaining only lowest powers of b/a. Little interest seems to attach to these results except in so far as they show that when the inner surface of a nearly solid shell is arbitrarily displaced, the outer surface remaining fixed, those displacements, strains, and stresses, which depend on the surface displacements deduced from high harmonics, fall off at first very rapidly in relative importance as the distance from the inner surface increases, so that at a considerable distance from this surface the effects of irregularities in the distribution of the surface displacements have largely disappeared.

One surface arbitrarily displaced, the other free.

§ 52. We need only indicate the method of treating this problem. Take for instance the case when the surface $r = a$ is subjected to displacements of the third class, given say by the first equation of (19), the surface $r = b$ being free of all forces. Then we may treat the problem independently by determining Y_i, Z_i, Y_{-i-1}, Z_{-i-1} from equations (20) and (21) § 44 combined with (40) and (41) of Part I. In the latter two equations we are to suppose the right hand sides to be zero. The solution in this case might also be deduced by taking (30) and (31) as they stand, but regarding U_i', T_i' as unknown quantities to be found by equating to zero the values of $\widehat{rr}$ and $\widehat{r\theta}$, or $\widehat{r\phi}$, supplied by this solution over $r = b$.

Here we shall only determine the solution for a thin shell. Suppose $r = a$ the surface subjected to given displacements, $r = a - h$ the free surface. Then, using the second method indicated above, it is easy to deduce the approximations:

$$u = \bar{U}\left(1 + 2\frac{m-n}{m+n}\frac{\xi}{a}\right) - \frac{m-n}{m+n}\frac{\xi}{a}\sum_{i=2}^{i=\infty}\left[i\,(i+1)\,T_i\right] \quad\text{.........................(48)},$$

$$v = \bar{V}\left(1 - \frac{\xi}{a}\right) + \frac{d\bar{U}}{d\theta}\frac{\xi}{a} \quad\text{.........................(49)},$$

$$w = \bar{W}\left(1 - \frac{\xi}{a}\right) + \frac{1}{\sin\theta}\frac{dU}{d\phi}\frac{\xi}{a} \quad\text{.........................(50)}.$$

Here $\bar{U}$, $\bar{V}$, $\bar{W}$ are the *total* components along r, θ, ϕ of the given arbitrary displacements on the outer surface, and T_i is the term containing surface harmonics of degree i in the potential from which arise the tangential displacements occurring under class (iii). First approximations to the stresses $\widehat{\theta\theta}$, $\widehat{\phi\phi}$, $\widehat{\theta\phi}$ may be derived from these results. The complete difference between these results and those obtained for the case of one surface fixed and the other subjected to given displacements should be noticed.

If the outer were the free surface and the inner that displaced, the only change required in (48), (49), (50) would be the substitution of $(-h + \xi)$ for ξ, taking $\bar{U}$ etc. as now the displacements over the inner surface.

XI. *On the Kinematics of a Plane, and in particular on Three-bar Motion: and on a Curve-tracing Mechanism.* By Professor CAYLEY. (Plates VI., VII.)

THE first part of the present paper, On the Kinematics of a Plane, and on Three-bar Motion, is purely theoretical: the second part contains a brief description of a Curve-tracing Mechanism, which has been at my suggestion constructed by Prof. Ewing for the Engineering Laboratory, Cambridge.

PART I.

1. The theory of the motion of a plane when two given points thereof describe given curves has been considered by Mr S. Roberts in his paper, "On the motion of a plane under given conditions," *Proc. Lond. Math. Soc.* t. III. (1871), pp. 286—318, and he has shown if for the given curves the order, class, number of nodes, and of cusps, are (m, n, δ, κ) and $(m', n', \delta', \kappa')$ respectively $(n = m^2 - m - 2\delta - 3\kappa,\ n' = m'^2 - m' - 2\delta' - 3\kappa')$, then for the curve described by any fixed point of the plane:

$$\begin{aligned}
&\text{order} &&= 2mm',\\
&\text{class} &&= 2(mm' + mn' + nm'),\\
&\text{number of nodes} &&= mm'(2mm' - m - m') + 2(m\delta' + m'\delta),\\
&\text{number of cusps} &&= 2(m\kappa' + m'\kappa),
\end{aligned}$$

but he remarks that these formulæ require modification when the directrices or either of them pass through the circular points at infinity. And he has considered the case where the two directrices become one and the same curve.

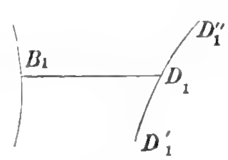
2. It will be convenient to speak of the line joining the two given points as the link; the two given points, say B and D, are then the extremities of the link; and I take the length of the link to be $= c$, and the two directrices to be b and d; we have thus the link $c = BD$ moving in suchwise that its extremity B describes the curve b of the order m, and its extremity D the curve d of the order m': in Mr Roberts' problem the locus is that described by a point P rigidly connected with the link, or say by a point P the vertex of the triangle PBD.

3. The points B, D describe of course the directrices b, d respectively: taking on b a point B_1 at pleasure, then if B be at B_1 the corresponding positions of D are the intersections of d by the circle centre B_1 and radius c, viz. there are thus $2m'$ positions of D: and similarly taking on d a point D_1 at pleasure, then if D be at D_1 the cor-

responding positions of B are the intersections of b by the circle centre D_1 and radius c, viz. there are thus $2m'$ positions of B. The motion thus establishes a $(2m, 2m')$ correspondence between the points of the directrices b and d, viz. to a given point on b there correspond $2m'$ points on d, and to a given point on d there correspond $2m$ points on b. Of course for a given point on either directrix the corresponding points on the other directrix may be any or all of them imaginary; and thus it may very well be that for either directrix not the whole curve but only a part or detached parts thereof will be actually described in the course of the motion. In saying that a part is described, we mean described by a continuous motion; say that the point B (the point D remaining always on a part of d) is capable of describing continuously a part of b; it may very well happen that the point B (the point D remaining always on a different part of d) is capable of describing continuously a different part of b, but that it is not possible for B to pass from the one to the other of these parts of b without removing D from the one part and placing it on the other part of d, and thus that we have on b detached parts each of them continuously described by B; and similarly we may have on d detached parts each of them continuously described by D.

4. But dropping for the moment the question of reality, to a given position of B on b there correspond as was mentioned $2m'$ positions of D on d, or say $2m'$ positions of the link c: in the entire motion of the link it must assume each of these $2m'$ positions, and for each of them the point B comes to assume the position in question on b; the directrix b is thus described $2m'$ times, that is the locus described by B, will be the directrix b repeated $2m'$ times, or say a curve of the order $m \times 2m'$, $= 2mm'$. Similarly the locus described by D will be the directrix d repeated $2m$ times, or say a curve of the order $m' \times 2m$, $= 2mm'$.

5. In general if B_1D_1 be any position of the link and if B moves from B_1 along b in a determinate sense, then D will move from D_1 along d in a determinate sense; and if B moves from B_1 along b in the opposite sense, then also D will move from D_1 along d in the opposite sense. Or what is the same thing we may have B moving in a determinate sense through B_1, and D moving in a determinate sense through D_1, and reversing the sense of B's motion we reverse also the sense of D's motion. But there are certain critical positions of the link, viz. we have a critical position when the link is a normal at B_1 to the directrix b, or a normal at D_1 to the directrix d. Say first the link is a normal at B_1 to the directrix b. The infinitesimal element at B_1 may be regarded as a straight line at right angles to the link; hence if for a moment D_1 is regarded as a fixed point the link may rotate in either direction round D_1, that is B may move from B_1 along b in either of the two opposite senses, say B_1 is a "two-way point." But if on d we take on opposite sides of D_1 the consecutive points D_1' and D_1'', say $D_1'D_1$ cuts D_1B_1 at an acute angle and $D_1''D_1$ cuts it at an obtuse angle, then D_1' will be nearer to b than was D_1, and thus the circle centre D_1' and radius c will cut b in two real points B_1' and B_1'' near to and



on opposite sides of B_1; or as D moves to D_1', B will move from B_1 indifferently to B_1' or B_1''. Contrariwise D_1'' is further from b than was D_1, and thus the circle centre D_1'' and radius c, will not meet b in any real point near to B_1, and hence D is incapable of moving from D_1 in the sense D_1D_1''. Or what is the same thing the described portion of d, which includes a point D_1' will terminate at D_1, or say D_1 is a "summit" on the directrix d. We have thus a summit on d, corresponding to the two-way point on b. And of course in like manner if the link is a normal at D_1 to the directrix d, then D_1 is a two-way point on d, and the corresponding point B_1 is a summit on b.

6. If the link is at the same time a normal at B_1 to b and at D_1 to d, then each of the points B_1, D_1 is a two-way point and also a summit; or more accurately each of them is a two-way point and also a pair of coincident summits.

But the case requires further investigation. Considering the position B_1D_1 as given, we may take the axis of x coincident with this line, and the origin O in suchwise

that OB_1, OD_1 are each positive and $OD_1 > OB_1$; say we have $OD_1 = \delta$, $OB_1 = \beta$, and therefore $\delta - \beta = c$. The equation of the curve b in the neighbourhood of B_1 is $y^2 = 2\rho(x - \beta)$, where ρ is the radius of curvature at B_1, assumed to be positive when the curve is convex to O, or what is the same thing when the centre of curvature R lies to the right of B_1 $(OR - OB_1 = +)$; and similarly the equation of d in the neighbourhood of D_1 is $y^2 = 2\sigma(x - \delta)$ where σ is the radius of curvature at D_1 assumed to be positive when the curve is convex to O or what is the same thing when the centre of curvature S lies to the right of D_1 $(OS - OD_1 = +)$.

Consider now (x_1, y_1) the coordinates of a point on b in the neighbourhood of B_1, $y_1^2 = 2\rho(x_1 - \beta)$, and taking B at this point, let (x_2, y_2) be the coordinates of the corresponding point D on d in the neighbourhood of D_1, $y_2^2 = 2\sigma(x_2 - \delta)$. We have

$$c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

and here
$$x_1 = \beta + \frac{y_1^2}{2\rho}, \quad x_2 = \delta + \frac{y_2^2}{2\sigma},$$

whence
$$x_1^2 = \beta^2 + \frac{\beta y_1^2}{\rho}, \quad x_1x_2 = \beta\delta + \frac{1}{2}\frac{\delta y_1^2}{\rho} + \frac{1}{2}\frac{\beta y_2^2}{\sigma}, \quad x_2^2 = \delta^2 + \frac{\delta y_2^2}{\sigma}.$$

The equation thus becomes

$$(\delta - \beta)^2 + \frac{y_1^2}{\rho}(\beta - \delta) + \frac{y_2^2}{\sigma}(\delta - \beta) + (y_1 - y_2)^2 = c^2,$$

that is
$$y_1^2\left(1 + \frac{\beta - \delta}{\rho}\right) - 2y_1y_2 + y_2^2\left(1 + \frac{\delta - \beta}{\sigma}\right) = 0,$$

a quadric equation between y_1 and y_2. Evidently if we had taken D a point on d, coordinates (x_2, y_2) in the neighbourhood of D_1 and had sought for the coordinates (x_1, y_1) of the corresponding point B on b in the neighbourhood of B_1, we should have found the same equation between y_1 and y_2.

7. The equation will have real roots if

$$1 > \left(1 + \frac{\beta - \delta}{\rho}\right)\left(1 + \frac{\delta - \beta}{\sigma}\right),$$

viz. ρ, σ the same sign, this is $\rho\sigma > (\rho + \beta - \delta)(\sigma + \delta - \beta)$,
but ρ, σ opposite signs, then $\rho\sigma < (\rho + \beta - \delta)(\sigma + \delta - \beta)$.

These conditions may be written

$$(OR - OB_1)(OS - OD_1) - (OS - OB_1)(OR - OD_1) > \text{ or } < O,$$

that is

$$(OS - OR)(OD_1 - OB_1) > \text{ or } < O.$$

But we have $OD_1 - OB_1 = +$, and therefore, ρ, σ the same sign, the condition of reality is $OS > OR$, *i.e.* S to the right of R; but ρ, σ opposite signs, the condition of reality is $OS < OR$, *i.e.* S to the left of R. Observe that S lying to the left of R, we cannot have $\rho = -$, $\sigma = +$, and that the second alternative thus is $\rho = +$, $\sigma = -$, then $OS < OR$, or S lies to the left of R.

The condition was investigated as above in order to exhibit more clearly the geometrical signification, but of course the original form or say the equation

$$1 - \left(1 + \frac{\beta - \delta}{\rho}\right)\left(1 + \frac{\delta - \beta}{\sigma}\right) = +$$

gives at once

$$\frac{\delta - \beta}{\rho\sigma}(\delta + \sigma - \beta - \rho) = +.$$

8. Writing the quadric equation in the form

$$y_1^2\left(1 - \frac{c}{\rho}\right) - 2y_1y_2 + \left(1 + \frac{c}{\sigma}\right)y_2^2 = 0,$$

we have

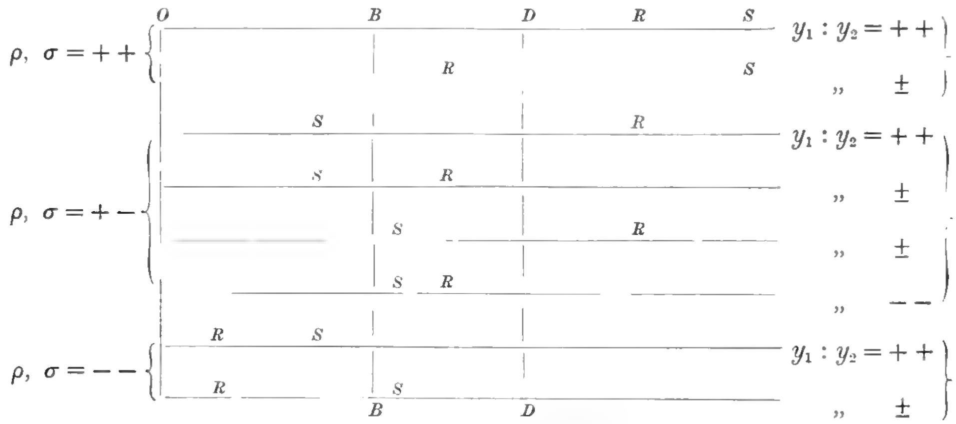
$$\left(1 - \frac{c}{\rho}\right)y_1 = \left\{1 \pm \sqrt{\frac{c}{\rho\sigma}(c + \sigma - \rho)}\right\}y_2;$$

the two values of $y_1 : y_2$ will have the same sign or opposite signs according as $1 - \frac{c}{\rho}$ and $1 + \frac{c}{\sigma}$ have the same sign or opposite signs, and in the case where these have the same sign, then this is also the sign of each of the two values of $y_1 : y_2$. Or what is the same thing if $1 - \frac{c}{\rho}$ and $1 + \frac{c}{\sigma}$ are each of them positive, then the two values of $y_1 : y_2$ are each of them positive; if $1 - \frac{c}{\rho}$ and $1 + \frac{c}{\sigma}$ are each of them negative then the two values of $y_1 : y_2$ are each of them negative; and if $1 - \frac{c}{\rho}$ and $1 + \frac{c}{\sigma}$ have opposite signs then the two values of $y_1 : y_2$ have opposite signs. Considering the different cases ρ, $\sigma = + +$, $+ -$, $- -$, we find

ρ, $\sigma = + +$, then values of $y_1 : y_2$ are $+ +$ or $- -$, according as DR, BS are $+ +$ or $- -$.
ρ, $\sigma = + -$ " " " " " DR, SB " "
ρ, $\sigma = - -$ " " " " " RD, SB " "

and in each case values of $y_1 : y_2$ are $+-$ if the two distances referred to have opposite signs: $DR = +$ means that R is to the right of, or beyond, D, and so in other cases.

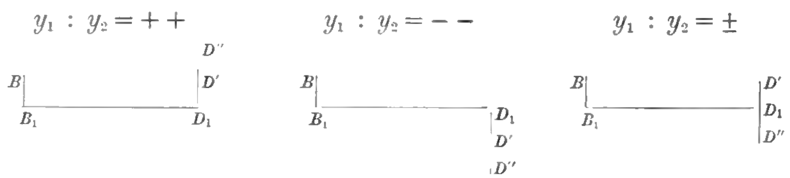
9. The different cases, two real roots as above, are



Obviously the cases $\rho,\ \sigma = - -$, correspond exactly to the cases $\rho,\ \sigma = +, +$; the only difference is that the concavities, instead of the convexities, of the two curves are turned towards the point O.

10. If the two roots of the quadratic equation are imaginary, then B_1D_1 is a conjugate or isolated position of the link, and B_1, D_1 are isolated points on the curves b and d respectively.

11. If the roots are real, then the three cases $y_1 : y_2 = + +$, $- -$ and $+ -$, may be delineated as in the annexed figures, viz. taking in each case y_1 as positive, that is imagining B to move upwards from B_1 through an infinitesimal arc of b, then D moves from D_1 through either of two infinitesimal arcs of d, both upwards, both downwards, or the one upwards and the other downwards, as shown in the figures

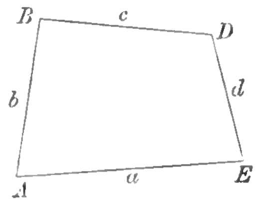


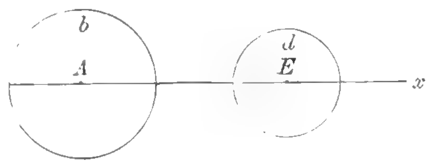
and where it is to be observed that reversing the sense of the motion of B from B_1 we reverse also the senses of the motion of D from D_1: moreover that considering D as moving through an infinitesimal arc of d from D, we have the like relations thereto of the two infinitesimal arcs of b described by B from B_1. Thus the points B_1 and D_1 are singular points of like character.

If $y_1 : y_2 = + +$, we may say that B_1 (or D_1) is a for-forwards point; if $y_1 : y_2 = - -$, then that B_1 (or D_1) is a back-backwards point; and if $y_1 : y_2 = \pm$, then that B_1 (or D_1) is a back-forwards point.

12. The separating case between two imaginary roots and two real roots is that of two equal real roots: the condition for this is $\delta + \sigma = \beta + \rho$, that is $OS = OR$, or the two centres of curvature are coincident; the characters of the points B_1 and D_1 would in this case depend on the aberrancies of curvature of the curves b and d at these points respectively. If each of the curves is a circle, then the curves are concentric circles, and the link BD moves in suchwise that its direction passes always through the common centre of the two circles—or say so that BD is always a radius of the annulus formed by the two circles—and for any position of BD, the two extremities B, D are related to each other in like manner with the points B_1 and D_1. Thus in this case there are no singular points B_1 and D_1 to be considered.

13. In the case where the curves b, d are circles we have three-bar motion: say the figure is as here shown; I take in it b, d for the radii of the two circles respectively and a for the distance of their centres; viz. we have the link $BD = c$, pivoted at its extremities to the arms or radii $AB = b$, and $ED = d$, which rotate about the fixed centres A, E at a distance from each other $= a$. Here a, b, c, d are each of them positive; a, b, d may have any values, but then c is at most $= a + b + d$, and if $a > b + d$ then c is at least $= a - b - d$; but if $a =$ or $< b + d$, then c may be $= 0$, viz. it may have any value from 0 to $a + b + d$. And in either case there will be critical values of c. The cases are very numerous. To make an exhaustive enumeration, we may assume d at most $= b$, and in each of the two cases $d < b$ and $d = b$, considering the centre of the circle d as moving from the right of the centre of the circle b towards this centre, we may in the first instance divide as follows:





$d < b$

$\odot$ d exterior to $\odot$ b,
,, touches it externally,
,, cuts it,
,, touches it internally,
,, lies within it,
,, is concentric with it,

$d = b$

$\odot$ d exterior to $\odot$ b,
,, touches it externally,
,, cuts it,
,, is concentric and thus coincident with it;

and then, in each of these cases, give to the length c of the link its different admissible values.

14. Considering the case $d < b$, then we have (see Plate VI.), exterior series, the figures 1, 1—2, 2, 2—3, 3, 3—4, 4, viz.

fig.	1,	$c = a - b - d$,
	1—2,	„ intermediate,
	2,	$c = a - b + d$,
	2—3,	„ intermediate,
	3,	$c = a + b - d$,
	3—4,	„ intermediate,
	4,	„ $= a + b + d$.

15. In figure 1, the curves described by the extremities B and D respectively are each of them a mere point.

In figure 1—2, we have $a + d > b + c$ and $a + b > d + c$. Hence in the course of the motion the arms b, c come into a right line, giving a position B_1D_1' of the link, where B_1 is a two-way point on b and D_1' a summit on d; or rather there are two such positions symmetrically situate on opposite sides of the axis Ax. And again in the course of the motion the arms d, c come into a right line, giving a position $B_1'D_1$, where D_1 is a two-way point on d and B_1' a summit on b; or rather there are two such positions symmetrically situate on opposite sides of the axis Ax. Only an arc of the circle b is described, viz. the arc adjacent to d included between the two summits B_1' on b; and in like manner only an arc of the circle d is described, viz. the arc adjacent to b included between the two summits D_1' on d. The described portions on b and d respectively are to be regarded each of them as a double line or indefinitely thin bent oval: and it is to be observed that for a given position of B (or D) there are two positions of the link BD, each of these positions being assumed by the link in the course of its motion.

16. In figure 2 the two positions B_1D_1' of the link come to coincide together in a single axial position BD, but we still have the other two positions $B_1'D_1$ of the link, where B_1' is a summit on b, and D_1 a two-way point on d. As regards BD, this is the configuration ρ, $\sigma = --$, R, B, S, D : $y_1 : y_2 = \pm$, and thus each of the axial points B, D is a back-and-forwards point. Thus only the arc $B'B_1'$ of the circle b is described by the point B, but the whole circumference of the circle d is described by the point D. If we further examine the motion it will appear that as B moves from the axial point B say to the upper summit B_1' and returns to B, then D starting from the axial point D may describe (*and that in either sense*, viz. $y_1 = +$, then we have $y_2 = \pm$) the entire circumference of d, returning to the axial point D; and similarly as B moves from the axial point B to the lower summit B_1' and returns to B, then D starting as before from the axial point D may describe (*and that in either sense*, viz. $y_1 = -$, then we have $y_2 = \pm$) the entire circumference of d, returning to the axial point D. It is thus not the entire arc $B_1'B_1'$ but each of the half-arcs BB_1' which corresponds, and that in either of two ways, to the circumference of d.

17. In figure 2—3, there are four critical positions $B_1'D_1$ (forming two pairs, those of the same pair situate symmetrically on opposite sides of the axis Ax), where as before

B_1' is a summit on b, and D_1 a two-way point on d. The described portions of b are the detached arcs $B_1'B_1'$ between the two upper summits, and $B_1'B_1'$ between the two lower summits: the described portion of d is the whole circumference. In fact attending to one of the arcs on b, say the upper arc $B_1'B_1'$, as B moves from one of the summits, say the left-hand summit B_1', and then returns to the left-hand summit B_1', then D, starting from the corresponding two-way point D_1, may describe, *and that in either sense*, the entire circumference of d, returning to the same point D_1; and similarly as B describes the lower arc $B_1'B_1'$, starting from and returning to a summit, then D, starting from the corresponding two-way point D_1, may describe, *and that in either sense*, the entire circumference of d, returning to the same two-way point D_1.

18. In figure 3, two of the positions $B_1'D_1$ have come to coincide together in the axial position BD, but we still have the other two positions $B_1'D_1$, where B_1' is a summit on b, and D_1 a two-way point on d. As regards the axial points B, D, this is the configuration $\rho, \sigma = + +$; B, R, D, S; $y_1 : y_2 = \pm$, viz. each of the points B, D is a back-and-forwards point. The two detached arcs $B_1'B_1'$ of b have united themselves into a single arc $B_1'B_1'$, which is the described portion of b; the described portion of d is as before the entire circumference. It is to be observed (as in fig. 2) that properly it is not the entire arc $B_1'B_1'$ but each of the half-arcs BB_1' which corresponds to the entire circumference of d.

19. The figure 3—4 closely corresponds to fig. 1—2, the only difference being that the arcs $B_1'B_1'$ and $D_1'D_1'$ which are the described portions of b and d respectively (instead of being the nearer portions, or those with their convexities facing each other) are the further portions, or those with their concavities facing each other, of the two circles respectively.

Finally in fig. 4, the described portions of the two circles reduce themselves to the axial points B and D respectively.

20. Still assuming $d < b$, and passing over the case of external contact, we come to that in which the circles intersect each other; but this case has to be subdivided: since the circles intersect we have $b + d > a$, consistently herewith we may have

b, d each $< a$,	A, E each outside the lens common to the two circles,
$b = a$, $d < a$,	A outside, E on boundary of the lens,
$b > a$, $d < a$,	A outside, E inside the lens,
$b > a$, $d = a$,	A on boundary of, E inside the lens,
b, d each $> a$,	A, E, each inside the lens;

and in each case we have to consider the different admissible values of c. I omit the discussion of all these cases.

21. Still assuming $d < b$, and passing over the case of internal contact, we come to that of the circle d included within the circle b: we have here again a subdivision of

cases; viz. we may have $d > A$, that is A inside d, $d = A$, that is A on the circumference of d, or $d < a$, that is A outside d. The critical values of c arranged in order of increasing magnitude in these three cases respectively are

$d > a$	$d = a$	$d < a$
$b - d - a$,	$b - 2d$,	$b - d - a$,
$b - d + a$,	b,	$b + d - a$,
$b + d - a$,	b,	$b - d + a$,
$b + d + a$,	$b + 2d$,	$b + d + a$.

I attend only to the first case; we have here (see Plate VII.), interior series, the figures 1, 1—2, 2, 2—3, 3, 3—4, 4, viz.

fig. 1	$c = b - d - a$,	
1—2	„	intermediate,
2	$c = b - d + a$,	
2—3	„	intermediate,
3	$c = b + d - a$,	
3—4	„	intermediate,
4	$c = b + d + a$.	

22. In figure 1 the curves described by the points B_1D are each of them a mere point. In figure 1—2, we have two critical positions $B_1'D_1$ situate symmetrically on opposite sides of the axis, B_1' being a summit on b, and D_1 a two-way point on d, and moreover two critical positions B_1D_1' situate symmetrically on opposite sides of the axis, B_1 being a two-way point on b, and D_1' a summit on d. The described portion of b is the arc $B_1'B_1'$, and the described portion of d is the arc $D_1'D_1'$, these two arcs being thus the nearer portions of the two circles respectively.

23. In figure 2, the four critical positions coalesce all of them in the axial position BD; the described portions are thus the entire circumferences of the two circles respectively. This is a remarkable case. The configuration is $\rho, \sigma = + +$; B, D, R, S; $y_1 : y_2 = + +$. Imagine D to move from the axial point D in a given sense round the circle d, say with uniform velocity, then B moves from the axial point B in the same sense *but with either of two velocities* round the circle b; one of these velocities is at first small but ultimately increases rapidly, the other is at first large but ultimately decreases rapidly, so that the two revolutions of B from the axial point B round the entire circumference to the axial point B correspond each of them to the revolution of D from the axial point D round the entire circumference to the axial point D. And similarly if we imagine B to move in a given sense from the axial point B round the circle b, say with uniform velocity, then D moves from the axial point D in the same sense but with either of two velocities round the circle d: one of these velocities is at first small but ultimately increases rapidly, the other is at first large but ultimately

decreases rapidly, so that the two revolutions from the axial point D round the entire circumference of d to the axial point D correspond each of them to the revolution from the axial point B round the entire circumference of b to the axial point B.

24. In figure 2—3 there are no critical positions, the described portions of the circles b, d are the entire circumferences of the two circles respectively, these being described in the same sense, by the points B and D respectively. It is to be observed that to a given position of B on b, there correspond two positions of D on d, or say two positions of the link, but the link does not in the course of its motion pass from one of these positions to the other; the motions are separate from each other, and may be regarded as belonging to different configurations of the system. And of course in like manner to a given position of D on d, there correspond two positions of B on b, or say two positions of the link: we have thus the same two separate motions.

25. In figure 3 the critical axial position BD of the link makes its appearance, the described portions are still the entire circumferences of the two circles respectively. As the point D is here to the left of the point B we must take the origin O to the right of B, and reverse the direction of the axis Ox; the configuration is thus $\rho, \sigma = + -$, B, S, R, D; $y_1 : y_2 = - -$. Everything is the same as in fig. 2 except (the signs of $y_1 : y_2$ being, as just mentioned, $- -$) that the motions in the circles b and d instead of being in the same sense are in opposite sense, viz. as D moves from the axial point D in a given sense round the circle d to the axial point D say with uniform velocity, then B moves from the axial point B round the circle b in the opposite sense, *and with either of two velocities;* and similarly as B moves from the axial point B in a given sense round the circle b say with uniform velocity, then D moves from the axial point D round the circle d in the opposite sense, *and with either of two velocities.*

26. In figure 3—4 we have again the two critical positions $B_1'D_1$ symmetrically situate on opposite sides of the axis, B_1' a summit on b, D_1 a two-way point on d: and also the two critical positions B_1D_1' symmetrically situate on opposite sides of the axis, B_1 a two-way point on b, D_1' a summit on d. The described portion of b is the arc $B_1'B_1'$, and the described portion of d the arc $D_1'D_1'$, these arcs being thus the further portions of the two circles respectively.

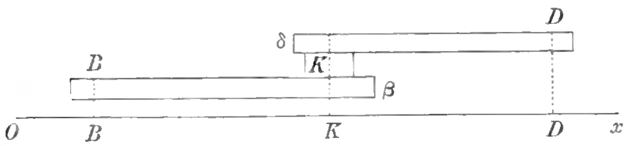
Finally, in figure 4 the described portions reduce themselves to the two points B, D respectively.

27. The several forms for $d = b$ can be at once obtained from those for $d < b$; the only difference is that several intermediate forms disappear, and the entire series of divisions is thus not quite so numerous.

Part II.

1. The curve-tracing mechanism was devised with special reference to the curves of three-bar motion, viz. the object proposed was that of tracing the curve described by a point K of the link BD, the extremities whereof B and D describe given circles respectively, or more generally by a point K, the vertex of a triangle KBD, whereof the other vertices B and D describe given circles respectively, and that in suchwise that the points B and D might be free to describe the two entire circumferences respectively: but the principle applies to other motions, and I explain it in a general way as follows.

2. Imagine the cranked link BD, composed of the bars $B\beta$ and $D\delta$, rigidly attached $B\beta$ to the top and $D\delta$ to the bottom of the cylindrical disk K (this same letter K is used to denote the axis of the disk), and where $B\beta$ and $D\delta$ may be either parallel or inclined to each other at any given angle, so that referring the points B, K, D to a hori-

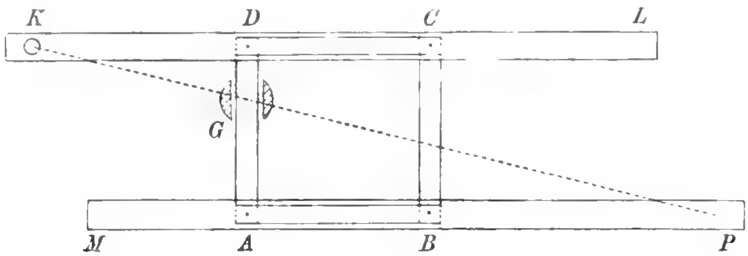


Cranked link with disk: elevation.

Arm of Pentagraph: plan.

zontal plane BKD is either a right line, or else K is the vertex of a triangle the other vertices whereof are B and D. The disk K, with the attached bars $B\beta$ and $D\delta$, moves in a horizontal plane: and if the motion of the point B be regulated in any manner by a mechanism lying wholly below B and supported by the bed of the entire mechanism, and similarly if the motion of the point D be regulated in any manner by a mechanism lying wholly above D and supported by a bridge of sufficient length (resting on the bed of the entire mechanism), then the disk K moves in its own horizontal plane unimpeded by other parts of the mechanism: and if we fit the disk K so as to move smoothly within a circular aperture in the arm of a pentagraph, then the pencil of the pentagraph will trace out on a sheet of paper the curve described by the point K on the axis of the disk, or say by the point K of the beam BKD. Of course for the three-bar motion, all that is required is that the point B shall describe a circle, viz. it must be pivoted on to an arm AB, which is itself pivoted at A to the bed: and that the point D shall describe a circle, viz. it must be pivoted on to an arm DE, which is itself pivoted at E to the bridge. Special arrangements are required to enable the variation of the several lengths AB, BK, KD, DE and ED, and the mechanism thus unavoidably assumes a form which appears complicated for the object intended to be thereby effected.

3. The form of Pentagraph which I use consists of a parallelogram $ABCD$, pivoted together at the points A, B, C, D, the bars AD and BC being above AD and BC. There is a cradle G, rotating about a fixed centre, and which carries between guides the arm AD, which has a sliding motion, so that the lengths GD and GA may be made to have



any given ratio to each other. Above the bar DC and sliding along it we have the arm KL (where K is the circular aperture which fits on to the disk K of the cranked link): and above AB and sliding along it we have the arm MP which carries the pencil P: of course in order that the pentagraph may be in adjustment the points K, G, P must be *in lineâ*.

XII *Examples of the application of Newton's polygon to the theory of singular points of algebraic functions.* By H. F. BAKER, M.A., Fellow of St John's College.

INTRODUCTION.

APART from its interest in the theory of plane curves, the theory of the multiple points is a convenient preliminary to the study of algebraic functions. We may of course suppose every algebraic curve to be beforehand transformed into one possessing only ordinary double points. But this transformation is one which it is not in general possible to carry out practically.

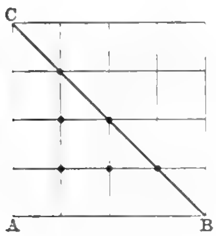
Cayley's rules for any singularity whatever have been amply justified in many subsequent papers. But in all these a good deal of calculation is necessary to obtain the series used and the final result. We naturally seek to find a method for evaluating a multiple point which shall appeal more directly to the explicitly given coefficients of the curve upon which these series depend. The following paper gives some rules which are effective in a very large number of cases—founded upon *a consideration of Newton's parallelogram.* The deficiency of a curve and the equivalent number for any multiple point is determined by counting the number of unit points within a certain polygon which can be immediately constructed from the equation of the curve. I have sought to give typical examples used in other papers as illustrations of other methods and shew the application of the present rules to them.

For convenience the paper is separated into six parts. In the first part it is shewn that Abel's determination of the deficiency of a curve admits of an immediate graphical interpretation. In the second part that this graphical result is in accord with the theory of Abelian integrals—the deficiency being defined by the number of integrals of the first kind that are linearly independent and the explicit form of these integrals determined. Cayley's rules appear thus as following from Riemann's number associated with the connectivity of his surface. The general values of the coefficients of the curve thus far accepted are in Part III subjected to certain restrictions of frequent occurrence and a graphical rule given for the necessary correction. These rules are applied in Part IV to various examples; among them is a consideration of Weierstrass' normal form of curve of which the corresponding Riemann surface has a branch point at infinity in which all the sheets are included. And it is proved that the number of orders of integral algebraic functions that are not integrally expressible is the same as the number of double points of the normal curve.

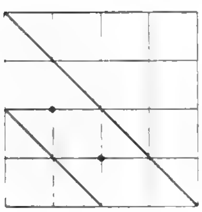
This part may serve as the beginning of a commentary on Kronecker's paper (Crelle, 91).

Part V is devoted to shewing that the quadratic transformation applied by Cramer and Noether is in direct connection with the graphical rules previously given. A particular monomial transformation obtainable by a succession of such quadratic transformations is employed in Part VI, to the example given by Noether in exposition of his own method, and to transform any curve to one whose only singularities are at infinity.

The main result of Part II, found in August 1892, was given, not quite correctly, in the Mathematical Tripos, Part II. of this year. This result enables us in all cases to specify immediately an upper limit to the deficiency of any given curve and a lower limit to the equivalent numbers of any of its multiple points. Cayley's rules of course apply to all possible cases—the rules given here for the exact values of the deficiency etc. may fail for particular values of the coefficients of the curve. In the simple case of a curve wherein all the terms are present, say for instance the quartic curve the deficiency 3 is the same as the number of unit points entirely within the triangle ABC in the diagram which represents all the terms of the curve in Newton's manner.



In case the constant term and the terms x, y, be absent, in which case there is a double point at the origin and the deficiency is 2, we have the second figure, having as before a number of interior points equal to the deficiency. The same is true when the terms in xy, y^2 are absent. This illustrates the general rule obtained here.



It may be remarked that Part I. is added for the sake of completeness: and the results of it not assumed in what follows.

October, 1893.

PART I.

On Abel's expression for the least number of sequent intersections of a curve with a variable curve.

In the *Phil. Trans. of the Royal Society* [1881] Mr Rowe has given an exposition of Abel's great paper (*Collected Works*, 1881, page 145) upon the sums of integrals related to a plane curve. Part of this paper is occupied with the determination of what Prof. Cayley, in an appendix to Mr Rowe's paper, proves to be, in general, the deficiency of the fundamental curve. The subtlety of the method employed by Abel in this part of his paper will justify the following diagrammatic interpretation of the algebra employed. It would not be wonderful indeed if some such method were in the mind of Abel. I have preferred to give by the way enough account of Abel's method to make the advantage of the present representation obvious.

If we have a curve

$$\chi(y) = y^n + p_{n-1} y^{n-1} + p_{n-2} y^{n-2} + \dots + p_0,$$

and any associated curve, this latter can in all cases so far as its intersections with $\chi(y)$ are concerned be taken in the form

$$\theta(y) = q_{n-1} y^{n-1} + \dots + q_0,$$

wherein $q_{n-1}, q_{n-2}, \dots, q_0$ are integral functions of x of at present unassigned order, whose coefficients are to be regarded as variable and independent,

then, denoting by $y_1, y_2, \dots y_n$ the n roots of $\chi(y) = 0$ for any value of x, the expression

$$E = \theta(y_1) \,.\, \theta(y_2) \dots \theta(y_n)$$

gives the abscissae of the (finite) intersections of these curves, and the number of these intersections is equal to the degree of E in x. If then one of the roots of $\chi(y)$, when expanded in descending powers (supposed positive), begin with the term in x^σ, and $\overline{\theta y}$ denote the highest power of x in θy when x^σ is written for y, this degree of E may be denoted by $\Sigma\overline{\theta y}$.

In what follows we desire to determine how many of the intersections of χy and θy are determined by the others. It is clear in fact that as many points of θy, upon χy, can be determined as there are assignable constants in θy, and that the remaining intersections of θy with χy are determined by the values assigned to these coefficients in θy, and these remaining intersections alter in a definite way when the coefficients of θy are altered. Since now there are in θy effectively

$$q_{n-1} + 1 + q_{n-2} + 1 + \dots + q_0 + 1 - 1$$
$$= \Sigma\bar{q} + n - 1 \text{ coefficients,}$$

where q means the degree of q in x, it follows that the intersections of θy with χy which are determined by the others are in number

$$\Sigma\theta y - \Sigma q - n + 1.$$

In what follows we seek by a proper choice of the terms and degrees in θy to make this expression as small as possible.

Suppose that the initial terms in the expansions of $y_1, y_2, \dots y_n$, consist of

$n_1\mu_1$ terms of the forms $A_1x^{\sigma_1}, A_2x^{\sigma_1}, \dots, A_{n_1\mu_1}x^{\sigma_1}$,

$n_2\mu_2$ terms of the forms $y = B_1x^{\sigma_2}, B_2x^{\sigma_2}, \dots, B_{n_2\mu_2}x^{\sigma_2}$

&c.

where $n_1\mu_1 + n_2\mu_2 + \dots = n$,

$\sigma_1 > \sigma_2 > \sigma_3 > \dots$.

Then when we substitute in θy, $y = x^{\sigma_1}$, there will be in general one term wherein the resulting power of x is highest.

Denote this term by $x^{[\rho_1]}\, y^{\rho_1}$ where $[\rho_1]$ is another notation for the highest power of x in q_{ρ_1}. In the same way the term which gives the highest power of x, when in θy x^{σ_2} is written for y, is denoted by $x^{[\rho_2]}\, y^{\rho_2}$, and so on.

Abel proved that it is possible to arrange the degrees and the coefficients in θy so that

ρ_1 is one of the indices $n-1, n-2, \dots, n-n_1\mu_1$,

ρ_2 is one of the indices $n-n_1\mu_1-1, n-n_1\mu_1-2, \dots, n-n_1\mu_1-n_2\mu_2$,

....

and he works out the least value of the number of 'sequent' intersections of χy and θy on this hypothesis. We shall follow him.

Imagine that we have a plane of rectangular coordinate axes, the positive quadrant of which is ruled with lines parallel to the axes at unit distances apart, and let every term of θy be represented on this chart, the term $x^h y^k$ being represented by the point whose abscissa is h and whose ordinate is k. Thus we shall have $\overline{q_{n-1}} + 1$ terms on a line parallel to the axis of x at distance $n-1$ from it, representing the terms in θy which were written $q_{n-1}y^{n-1}$, and so on. Of these points we shall only here be concerned with those, on the various lines parallel to the axis of x, which are furthest from the axis of y. The power of x arising from any term in θy when x^σ is written for y is easily constructed graphically by drawing through the point of the chart that represents that term of θy a line whose positive direction makes with the negative direction of the axis of y the angle $\left(\text{for the present assumed to be between } 0 \text{ and } \frac{\pi}{2}\right) \tan^{-1}\sigma$. The distance of the point in which this line meets the axis of x from the origin is the power of x arising from the term of θy considered: and to say that the term $x^{[\rho_1]}y^{\rho_1}$ gives the highest power of x when y is written x^{σ_1}, is to say that, if through the point of the chart whose coordinates are $[\rho_1]$, ρ_1, there be drawn a line whose positive direction makes with the negative direction of the axis of y the angle $\tan^{-1}\sigma_1$ $\left(\text{between } 0 \text{ and } \frac{\pi}{2}\right)$, and a parallel line be drawn through every other representative point on the chart the first drawn line will meet the axis of x further from the origin than all the points in which the other lines meet the axis of x. Let the point $[\rho_1]$, ρ_1 be called R_1, $[\rho_2]$, ρ_2 be called R_2, and so on, and let the line parallel to the axis of x at distance l be called l_k. Then as we have said Abel shews that the point R_1 may be taken to be on one of the lines $l_{n-1}, l_{n-2}, \ldots l_{n-n_1\mu_1}$. These lines we shall call the first set—and so for each of the following sets. Suppose now that a line σ_1 is drawn through R_1, and a line σ_2 drawn through R_2, and so on. [By a line σ_1 we mean a line making an angle $\tan^{-1}\sigma_1$ with the axis of y, as previously explained.] These lines form with the two axes of coordinates a closed polygon, and it is obvious that the expression of the characteristic property of the points $R_1, R_2, \ldots$ is that all the lines R_1R_2, $R_2R_3, \ldots$ shall lie within this polygon. This is the expression of Abel's conditions

$$\sigma_1 > \tau_1 > \sigma_2 > \tau_2 > \ldots$$

$$[\rho_r] = [\rho_1] + \sum_{\kappa=1}^{\kappa=r-1} \tau_\kappa (\rho_\kappa - \rho_{\kappa-1}),$$

τ_r being the tangent of the angle which R_rR_{r+1} makes with the negative direction of the axis of y, and is obviously sufficient to ensure that the term corresponding to R_1 gives a higher power of x than either of the terms corresponding to $R_2, R_3, \ldots$ for $y = x^{\sigma_1}$, and that the term corresponding to R_2 gives a higher power of x than the terms corresponding to $R_1, R_3, \ldots$ for $y = x^{\sigma_2}$, and so on.

Consider now R_r. We have to ensure that for $y = x^{\sigma_r}$ this shall give not only a higher power of x than the term corresponding to R_s, but shall also give a higher power of x than every other term in the set to which R_s belongs. Abel shews that the analytic condition for this can be reduced to the two following criteria:

(1) that the term R_r for $y = x^{\sigma_r}$ gives a higher power of x than $y = x^{\sigma_r}$ gives in each of the terms of the following set $(r+1)$ only;

(2) that $y = x^{\sigma_r}$ gives for the term R_r a higher power of x than it gives for any of the terms of the previous set $(r-1)$.

And it is easy to see that these conditions are sufficient. For suppose the first satisfied. Imagine lines drawn through all the points of the set $(r+1)$ parallel to σ_r and a line parallel to these drawn through R_r. By hypothesis this last line meets the axis of x at a point further from the origin than any of the other lines do. If now all these lines be turned to a greater inclination with the negative direction of the axis of y, into the direction σ_{r-1}, each about the point through which it was drawn, this statement will remain true—namely, the line drawn through the point R_r parallel to σ_{r-1} is further from the origin than the parallel lines drawn through the points of the set $(r+1)$.

Therefore the line through the point R_{r-1} parallel to σ_{r-1} which by hypothesis is further from the origin than the parallel line through R_r is also further from the origin than the parallel lines through the points of the set $(r+1)$. Continuing thus we can shew that for all values of s less than $r+1$, the line through R_s is further from the origin than the parallel lines through the points of the set $(r+1)$. Supposing next that the second condition is satisfied, namely that the line through R_r parallel to σ_r is further from the origin than the parallel lines through all the points of the set $(r-1)$, and supposing all these lines turned about their respective points to a less inclination with the negative direction of the axis of y, so as to become parallel to σ_{r+1}, the line through R_r will remain the furthest from the origin. But now the line parallel to σ_{r+1} is by hypothesis further from the origin than the line through R_r and is therefore also further from the origin than the parallel lines through all the points of the set $(r-1)$. Continuing thus we can shew that the line through R_s where $s>r-1$ is further from the origin than the parallel lines through the set $(r-1)$.

Thus we have only to consider how to satisfy conditions (1) and (2) for all values of r. The first condition clearly is that all the points in the set $(r+1)$ lie on the same side of the line through R_r parallel to σ^r as does the origin. While the condition that R_{r+1} corresponds to the highest term in x for $y=x^{\sigma_{r+1}}$, of the set $(r+1)$, requires that all the terms of the set $(r+1)$ lie on the same side of the line σ_{r+1} through R_{r+1} as does the origin. We see in fact that the conditions only are that all the points must lie within the polygon, and further that this is perfectly obvious geometrically without the cumbrous interposition of the conditions (1) and (2).

Considering now again our expression

$$E=\theta y_1 \,.\, \theta y_2 \ldots \theta y_n$$

it is clear that the first $n_1\mu_1$ factors give rise to the same power of x as their highest power of x, namely the power $[\rho_1]+\rho_1\sigma_1$. For x^{σ_1} was the highest power of x in each of $y_1, y_2, \ldots y_{n_1\mu_1}$. We shall therefore have in the summation $\Sigma\overline{\theta y}$, $n_1\mu_1$ terms each equal to $[\rho_1]+\rho_1\sigma_1$. But it is convenient to write each of these $n_1\mu_1$ terms in a different way, thus

Let $q_{\alpha_1}.y^{\alpha_1}$ denote the general term corresponding to the first set on our chart, so that α_1 is in turn equal to $n-1, n-2, \ldots n-n_1\mu_1$. The degree of the term $q_{\alpha_1}.y^{\alpha_1}$ for $y=x^{\sigma_1}$ is $[\alpha_1]+\alpha_1\sigma_1$ where $[\alpha_1]$ means the degree of q_{α_1}. We denote the difference

$$[\rho_1]+\rho_1\sigma_1-[\alpha_1]-\alpha_1\sigma_1 \text{ by } D_{\alpha_1},$$

which gives

$$[\rho_1]+\rho_1\sigma_1=[\alpha_1]+\alpha_1\sigma_1+D_{\alpha_1},$$

and this is the substitution which for the $n_1\mu_1$ values of α_1 we make for the $n_1\mu_1$ terms of the form $[\rho_1]+\rho_1\sigma_1$ arising in $\Sigma\overline{\theta y}$. And the part contributed by these terms to the summation $\Sigma\theta y-\Sigma\overline{q}$, since the values of $\overline{q}$ entering here have also been denoted by $[\alpha_1]$, is

$$\sigma_1 \sum_{n-n_1\mu_1}^{n-1} \alpha_1+\Sigma D_{\alpha_1}.$$

The whole expression $\Sigma\overline{\theta y} - \Sigma\overline{q}$ can therefore be written

$$\Sigma_r\sigma_r \,.\, \Sigma_a\alpha_r + \Sigma D_{\alpha_r}.$$

And $\Sigma_r\sigma_r \,.\, \Sigma_a\alpha_r$, writing $m_r = \sigma_r\mu_r$,

$$= n_1\mu_1\left[n - \frac{n_1\mu_1 + 1}{2}\right]$$

$$+ n_2\mu_2\left[n - n_1\mu_1 - \frac{n_2\mu_2 + 1}{2}\right]$$

$$+ \ldots\ldots$$

$$+ n_r m_r\left[n - n_1\mu_1 - \ldots - n_{r-1}\mu_{r-1} - \frac{n_r\mu_r + 1}{2}\right]$$

$$+ \ldots\ldots$$

Consider now ΣD_{α_r}.

We have $$D_{\alpha_r} = [\rho_r] - [\alpha_r] - (\rho_r - \alpha_r)\,\sigma_r$$

$$= [\rho_r] - [\alpha_r] + \text{integral part of } (\rho_r - \alpha_r)\frac{m_r}{\mu_r}$$

$$+ \text{fractional part of } \frac{(\rho_r - \alpha_r)\, m_r n_r}{\mu_r n_r},$$

this fractional part being taken positive.

And D_{α_r} may be constructed graphically by drawing a line through the point $[\alpha_r]$, α_r parallel to σ_r to meet the line $y = \rho_r$, say in A_r. The line $A_r R_r$ is D_{α_r} (and by the definition of R_r is necessarily positive and has a positive integral part). Since now it is our endeavour to make $\Sigma\overline{\theta y} - \Sigma\overline{q}$ as small as possible, and since the other part in the expression for this, namely $\Sigma_r\sigma_r\Sigma_a\alpha_r$ has a definite value prescribed by the curve χ we shall make our summation $\Sigma\overline{\theta y} - \Sigma\overline{q}$ as small as possible if we make the part ΣD_{α_r} as small as possible. We may agree then first of all that the integer part of D_{α_r} shall vanish, and we may notice here that this uniquely prescribes the chart-point (of θy) upon the line $y = \alpha_r$ lying furthest from the axis of y, namely thus,—imagine a line keeping always parallel to σ_r to move from the position in which it passes through R_r towards the origin, then the first unit point it reaches upon the line $y = \alpha_r$ is the point prescribed.

And Σ fractional parts of

$$\frac{(\rho_r - \alpha_r)\, m_r n_r}{\mu_r n_r} = \Sigma n_r\,\frac{\mu_r - 1}{2},$$

and this has a value dependent on the curve χ only.

Thus on the whole

$$\Sigma\overline{\theta y} - \Sigma\overline{q} - n + 1$$

has for its least possible value

$$\sum_{s>r}^{r=1} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n\mu - \tfrac{1}{2}\Sigma n + 1$$

and $$n = \Sigma n\mu,$$

and this is the number of 'sequent' intersections; and we may bear in mind that this number was diminished by taking the curve θy such that the quantities D_{a_r} were all less than 1.

But it should be noticed that this enumeration takes no count of possible infinite intersections. It is in fact to be afterwards shewn that our conditions $D_{a_r} < 1$ are equivalent to prescribing a certain number of points at infinity on θy. So that the curve θy is not only specialised by the supposed prescribed values given to the $\Sigma \bar{q} + n - 1$ coefficients left in it, but also by the prescription of these infinite points.

Returning now to the polygon formed by the lines $\sigma_1, \sigma_2, \ldots$ its construction contains necessarily a very large amount of arbitrariness. Writing for shortness

$$r_1, r_2, \ldots \text{ for } [\rho_1], [\rho_2], \ldots \text{ respectively,}$$

the points (r_1, ρ_1), (r_2, ρ_2), ... are first to be taken arbitrarily, save only that ρ_1 is to be one of the numbers $n-1$, $n-2$, ..., $n - n_1\mu_1$, ρ_2 one of the numbers

$$n - n_1\mu_1 - 1, \ldots, n - n_1\mu_1 - n_2\mu_2, \text{ etc.,}$$

and r_1 is to be sufficiently great for the line σ_1 through R_1 or (r_1, ρ_1) to meet the axis of y beyond the point $(0, n-1)$—though the contrary only means that q_{n-1} is identically zero. Then (r_2, ρ_2) must be taken consistently with a certain condition that may be thus expressed:

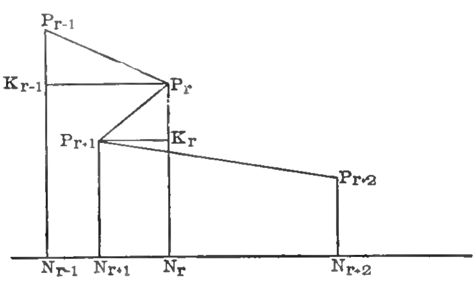
Denote the intersection of the σ_1 line through R_1 and the σ_2 line through R_2 by K_1—and suppose first that the line $y = n - n_1\mu_1 - 1$ meets these σ_1, σ_2 lines at points further from the axis of x than K_1 is, say in A_1, A_2 respectively. Then A_1A_2 must be less than 1. This is required by the condition, which was necessary to make our quantities $D_{a_r} < 1$, that the curve points of θ furthest from the axis of y and belonging to the second set should all be at less than unit distance measured parallel to the axis of x, from the σ_2 line, combined with the condition that these points must be within the polygon. Or supposing next that the line $y = n - n_1\mu_1 + 1$ meets the σ_1, σ_2 lines in points not so distant from the axis of x as the point K_1 is, say B_1, B_2 respectively, then B_1B_2 must be less than 1, for a similar reason. This condition ensures that the points in which $y = n - n_1\mu_1$ meets the σ_1, σ_2 lines shall not be beyond a certain limit of distance from the point K_1. It is of course easy to express this condition analytically—and a similar condition must obviously be satisfied at each angular point of the polygon.

We should next remark that the conditions $D_{a_r} < 1$ together with the other conditions for θy are really equivalent to prescribing that our curve θy shall behave as an 'adjoint' curve at the multiple points of χy that lie at infinity. This is really obvious from Cayley's proof that the number of 'sequent' points given above is the deficiency of the curve χy. For we know that the number of sequent intersections of a non-adjoint curve with a curve χy having δ double points and k cusps is the deficiency $+\delta+k$. And it is assumed in Cayley's proof that the curve χy has no finite singularities. We shall moreover give an independent proof of the fact that the expression found above for the number of sequent points is in general the deficiency of χy (Part II. of present paper).

We proceed now to shew that the above expression for the number of sequent points is capable of a certain graphical interpretation.

Suppose as before a plane of rectangular axes to have its positive quadrant ruled with lines at unit distance apart parallel to the coordinate axes. Let the intersections of these lines be called unit points. Join now the points (o, n), $(n_1m_1, n-n_1\mu_1)$ by a straight line. This will be parallel to the σ_1 line before spoken of and will contain, counting the end points, n_1+1 unit points. We shall denote the coordinates of the extremities of this line by (x_0, y_0) and (x_1, y_1) and call them P_0, P_1. Join P_1 to the point (x_2, y_2) where $x_2 = n_1m_1 + n_2m_2$, $y_2 = n - n_1\mu_1 - n_2\mu_2$. Denote (x_2, y_2) by P_2. P_1P_2 will be parallel to the σ_2 line before mentioned, and contains, counting the end points, n_2+1 unit points. Proceeding thus we shall get a polygon whose sides are the two axes of coordinates, and lines parallel to the σ_1, σ_2, σ_3, ... lines. We may call the number of these latter lines $k+1$, so that the last of them is P_kP_{k+1}, P_{k+1} being (x_{k+1}, y_{k+1}) and y_{k+1} being 0—and $\Sigma nm = x_{k+1}$ $\Sigma n\mu = n$. Then what we proceed to prove is that our number previously found for the number of sequent points is the same as the number of unit points within the polygon.

In proving this we shall not, except at first, need to assume that $\sigma_1 > \sigma_2 > \sigma_3 \ldots$ or that σ_1, σ_2, ... are positive.



Consider one side $P_{r-1}P_r$ of our polygon.

Let $P_{r-1}N_{r-1}$, P_rN_r be the ordinates from its ends to the axis of x, and let P_rK_{r-1} be drawn parallel to the axis of x to meet $P_{r-1}N_{r-1}$ in K_{r-1}. Then the number of unit points actually within the triangle $P_rK_{r-1}P_{r-1}$ together with the number of those (except P_{r-1} and P_r) upon the side $P_{r-1}P_r$ is

$$\frac{1}{2}(n_r^2m_r\mu_r - n_rm_r - n_r\mu_r + n_r),$$

as is easily seen by completing the rectangle of which P_rP_{r-1} is the diagonal and remembering that the number of unit points on P_rP_{r-1} is n_{r-1}. [This gives a graphical proof of the theorem

$$\sum_{x=1}^{x=A-1} \text{integer part} \left(x\frac{B}{A}\right) = \frac{1}{2}(AB - A - B + d)$$

where A, B are positive integers and d is the greatest factor common to both.]

Also the number of unit points within the rectangle $P_rK_{r-1}N_{r-1}N_r$ together with the number of those (other than P_r, K_{r-1}) upon the line P_rK_{r-1} is $y_r(x_r - x_{r-1} - 1)$—and the number of unit points on the line $P_{r-1}N_{r-1}$, other than N_{r-1}, is y_{r-1}. Adding these three numbers, and subtracting the number of unit points upon $P_{r-1}P_r$ other than P_r, namely n_r, and putting $y_{r-1} - y_r = n_r\mu_r$, $x_r - x_{r-1} = n_rm_r$, and $y_r = \sum\limits_{s=r+1} n_s\mu_s$, we obtain as the number of unit points actually within the trapezium $P_rP_{r-1}N_{r-1}N_r$ and upon the side $P_{r-1}N_{r-1}$ (other than P_{r-1}) the result

$$n_r\mu_r + n_rm_r \sum_{s=r+1} n_s\mu_s + \frac{1}{2}[n_r^2m_r\mu_r - n_r\mu_r - n_rm_r + n_r] - n_r,$$

which because $n = \sum\limits_{s=1} n_s\mu_s$ is equal to

$$n_r\mu_r + n_rm_r[n - n_1\mu_1 - n_2\mu_2 \dots - n_r\mu_r] + \frac{1}{2}n_rm_r[n_r\mu_r - 1] - \frac{1}{2}n_r\mu_r - \frac{1}{2}n_r$$

$$= n_rm_r\left[n - n_1\mu_1 \dots - n_{r-1}\mu_{r-1} - \frac{n_r\mu_r + 1}{2}\right] + \frac{1}{2}n_r[\mu_r - 1] \;\dots\dots\dots\text{(i)},$$

and if we assume for the present that all the quantities σ_1, σ_2, ... are positive, it is obvious that the whole number of points within the polygon is merely the arithmetic sum of such expressions as these, except that we must subtract from this sum, in order to exclude the unit points on the axis of y which occur for the trapezium $P_0P_1N_1N_0$, the number $n-1$. If this arithmetic sum be formed it will be found to agree with our number. But without this it is sufficient to notice that the expression (i) found above is identical with the value before found for $\sigma_r\Sigma\alpha_r + \Sigma_a$ fractional parts of D_{a_r}, and to recall that our number was defined as the value of

$$\sum_r [\sigma_r\sum_a \alpha_r + \sum_a \text{fractional part of } D_{a_r}] - n + 1.$$

The geometrical interpretation of the formula is then established in case σ_1, σ_2, ... be all positive.

In case however some of them be negative, e.g. σ_{r+1} in the figure [p. 411], it will be found that the contribution corresponding to the trapezium $P_rP_{r+1}N_{r+1}N_r$ has the same form as a function of the quantities n_1, m_1, μ_1, n_2, m_2, μ_2, ... as if σ_{r+1} were positive. In fact having calculated the number of points as above for the trapezium $P_{r-1}P_rN_rN_{r-1}$, we must subtract the number of points within the trapezium $P_rP_{r+1}N_{r+1}N_r$ and also the number of points upon the sides $N_{r+1}P_{r+1}$, $P_{r+1}P_r$ (other than P_r). If after this σ_{r+2} should also be negative it will be needful to subtract an exactly similar quantity for the trapezium $P_{r+1}P_{r+2}N_{r+2}N_{r+1}$; while if σ_{r+2} be positive we shall have to add an expression for this trapezium which is to be calculated in exactly the same way as was the contribution for the trapezium $P_{r-1}P_rN_rN_{r-1}$. But the subtractive contribution above corresponding to the trapezium $P_rP_{r+1}N_{r+1}N_r$ is

$$-\left[\frac{1}{2}(-n^2_{r+1}m_{r+1}\mu_{r+1}+n_{r+1}m_{r+1}-n_{r+1}\mu_{r+1}-n_r)+(-n_{r+1}m_{r+1}-1)\sum_{s=r+2} n_s\mu_s+\sum_{s=r+2} n_s\mu_s\right],$$

which is exactly equal to

$$n_{r+1}\mu_{r+1}+n_{r+1}m_{r+1}\sum_{s=r+2} n_s\mu_s+\frac{1}{2}[n^2_{r+1}m_{r+1}\mu_{r+1}-n_{r+1}\mu_{r+1}-n_{r+1}m_{r+1}+n_{r+1}]-n_{r+1},$$

and this has exactly the same form as a function of $r+1$ as had the expression found above for the contribution of the trapezium $P_{r-1}P_rN_rN_{r-1}$ as a function of r. Thus our geometrical interpretation is completely justified.

Part II.

A priori proof of the significance of the number of points within Newton's polygon.

Taking once more our positive quadrant of rectangular axes ruled with lines at unit distance apart and any arbitrary curve whatever, $F=0$, mark on the chart, corresponding to the term $A_{r,s}x^ry^s$ of the curve F, the point whose coordinates are $x=r$, $y=s$. This will be called a curve point, the original points being called merely unit points. Then it is possible to form a polygon each of whose sides shall begin and end in a curve point and which shall be everywhere convex and have all the curve points (other than those on its sides) in its interior. And in fact starting from the curve point on the axis of y which is furthest from the origin, say the point P_0 at distance n from the origin, let a line passing through P_0 and coinciding with the positive axis of y turn about P_0 in a clockwise direction until it again contains a curve point. In this position it may contain several curve points. In any case let P_1 denote the curve point on this line which is furthest from P_0. Let n_1m_1 be the abscissa of P_1 and $n-n_1\mu$ its ordinate, m_1 and μ_1 being coprime and μ_1 possibly negative. Put σ_1 for $\frac{m_1}{\mu_1}$ and notice there are n_1+1 unit points upon P_0P_1. In the same way let a line pivot in a clockwise direction about P_1 from coincidence with the continuation of P_0P_1 until it again contain curve points, P_2 being then the curve point furthest from P_1, the coordinates of P_2 being $x_2=n_1m_1+n_2m_2$, $y_2=n-n_1\mu_1-n_2\mu_2$ where m_2, μ_2 are coprime; use $\sigma_2=\frac{m_2}{\mu_2}$. And so on until we ultimately come to a point P_{k+1} on the axis of x, this being the curve point on the axis of x which is furthest from the origin. In a similar way let P'_0 be the curve point on the axis of y which is nearest to the origin, at a distance n' say—and proceed from this to obtain in succession the straight sides

$$P_0'P_1',\ P_1'P_2',\dots P'_{k'}P'_{k'+1},$$

the successive rotations being now all in a counter clockwise direction. It is obvious that all the fractions σ_1', σ_2' ... thus obtained are positive.

Then if in the equation of the curve we make a substitution

$$y = Ax^{\sigma} + \text{infinite descending series of powers of } x$$

the highest power of x arising from any term $A_{rs}x^r y^s$ of our curve is the abscissa of the point in which the axis of x is met by the line drawn from the curve point (r, s) in the direction making with the negative axis of y the angle $\tan^{-1}\sigma$. If then σ_1, σ_2, ... be all positive, the terms in the curve corresponding to the unit points upon the side $P_{i-1}P_i$ become, for the substitution $y = Ax^{\sigma_i} + \ldots$, of the same order in x, this order being higher than that arising for this substitution in any other terms of the equation of the curve. Hence the curve has a series of infinite branches whose equations are of the form

$$y = Ax^{\sigma_i} + \ldots,$$

the values of A being obtained by arranging the terms of the curve corresponding to the curve points upon $P_{i-1}P_i$, in the form

$$Cx^{x_{i-1}}y^{y_i}\,[y^{\mu_i} - k_1 x^{m_i}] \ldots\ldots [y^{\mu_i} - k_{n_i} x^{m_i}]$$

(where x_{i-1}, y_{i-1}, x_i, y_i are the coordinates of P_{i-1} and P_i).

In what follows we assume that each of σ_1, σ_2, ... are positive. The method of proof does not otherwise apply without considerably more detail in explanation. Various examples are however given in which the main result obtained here holds when some of σ_1, σ_2, ... are negative. But the consideration of this case is never necessary in practice, because by the substitutions $x = \xi + c\eta$, $y = \eta + c'\xi$, it is always possible to reduce the equation to one in which the highest powers of ξ and η that enter have, both, constant coefficients—in which case all of σ_1, σ_2, ... are positive.

In the same way as for the infinite branches, the diagram enables us to state the first terms of the expansions

$$y = Ax^{\sigma'} + \text{infinite ascending series of higher powers of } x,$$

of the curve near the origin, here supposed to be a multiple point.

Naturally we confine ourselves in the first instance to the most general curve represented by the diagram—in that case its singularity at the origin and at infinity is competently represented by the diagram. It is afterwards shewn how to represent diagrammatically the corrections needful when the coefficients of the highest or lowest terms in the equation are subject to certain particular relations, which are those of most common occurrence.

Proposition. Consider all the unit points entirely within the polygon and write down a curve with perfectly general coefficients whose curve points are just these unit points. Since no one of these unit points has a zero abscissa, or a zero ordinate, the equation of this curve will be divisible by xy.—Denote the curve then by $xy\phi$. Then I say that ϕ is of order $N-3$, where N is the order of the original curve F, and that it is 'adjoint' to F at the origin and at each of the singularities at infinity. Limiting ourselves to the case when all of $\sigma_1 \dots$ are positive the only exceptional case is when there are only two of these, $\sigma_1 = \infty$, $\sigma_2 = 0$. Then ϕ is of order $N-4$. This is the case in Riemann's canonical form for the equation of his surface. In this case ϕ is to be interpreted as $z\phi$ where $z=1$, and $z=0$ is the equation of the line at infinity: then $z\phi = 0$ is the most general adjoint curve of order $N-3$.

From this proposition it will follow that the number of unit points entirely within the curve polygon is $p+\delta+\kappa$, where p is the deficiency of F and $\delta+\kappa$ the number of simple double points and cusps to which the finite singularities of F other than the origin are equivalent. This follows from the known number of linearly independent adjoint curves of order $N-3$. *And if the curve have no finite singularities other than the origin the number of interior points will be exactly equal to its deficiency.*

To prove that the order of $xy\phi$ is $N-1$ we remark that if $P_r P_s$ with coordinates x_r, y_r and x_s, y_s, be the ends of the side of the polygon which represents the terms of F which are of highest aggregate order, so that either $s=r$ or else $s=r+1$ (in which case $P_r P_{r+1}$ is inclined at 45° to the negative axis of y), and if Q_r be the unit point $(x_r - 1, y_r)$, Q_s be the unit point $(x_s, y_s - 1)$, then the side $Q_r Q_s$ contains the points representing the highest terms of the curve $xy\phi$ and these terms are clearly of order $N-1$. The only exceptional case is the Riemann curve just mentioned in which $Q_r Q_s$ are not points for the $xy\phi$ curve—being on the sides of the F polygon. But the modification and verification of the result stated is obvious.

To prove that ϕ is 'adjoint' at the origin and infinity it is sufficient to prove that the integral

$$\int \phi \frac{x\,dy - y\,dx}{\dfrac{\partial F}{\partial z}} = \int \frac{xy\phi}{\dfrac{\partial F}{\partial z}} \left(\frac{dy}{y} - \frac{dx}{x} \right),$$

where z, $=1$, is introduced into the equation F to make it homogeneous, is finite on all the branches at infinity and at the origin.

Consider the infinite branches and consider first the case where as above there is a side $P_r P_{r+1}$ of the polygon inclined at 45° to the negative axis of y. Then the curve has branches at infinity, $y = Ax +$ lower powers of x, along which (for $x = r\cos\theta$, $y = r\sin\theta$)

$$\frac{dy}{y} - \frac{dx}{x} = \frac{d\theta}{\sin\theta\cos\theta}$$

is zero of the same order as $d\theta$. The terms entering in $\dfrac{\partial F}{\partial z}$ can be represented in our chart and will give rise to exactly the same curve points as F with the exception only of the points on the line $P_r P_{r+1}$. The points $Q_r Q_{r+1}$ mentioned above, namely the points whose coordinates are $(x_r - 1, y_r)$, $(x_{r+1}, y_{r+1} - 1)$, which represent the effectively highest terms of the curve $xy\phi$ for a substitution of the form $y = Ax + \dots$, will be outside points of the polygon representing the terms of $\dfrac{\partial F}{\partial z}$. Hence $\dfrac{xy\phi}{\dfrac{\partial F}{\partial z}}$ is finite on this branch and so therefore the integral. With the exception of points on this line $Q_r Q_{r+1}$, all other points arising from the curve $xy\phi$ lie within the polygon representing the terms of $\frac{\partial F}{\partial z}$. In fact if P_{r-1} be the angular point of the F polygon before P_r, and P_{r+2} the angular point after P_{r+1}, $P_{r-1}Q_rQ_{r+1}P_{r+2}$ are outside points of the $\frac{\partial F}{\partial z}$ polygon. Hence for any substitution $y = Ax^\sigma$ in which $\sigma > 1$, $\frac{xy\phi}{\frac{\partial F}{\partial z}}$ will be zero like some positive power of $\frac{1}{x} = \frac{C}{x^r}$ say, and

$$\int \frac{xy\phi}{\frac{\partial F}{\partial z}} \left(\frac{dy}{y} - \frac{dx}{x}\right) = \int C(\sigma - 1) \frac{dx}{x^{r+1}} + \text{integral of higher powers of } \frac{1}{x},$$

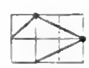
will be finite.

Exactly similar remarks apply to the case when there is no infinite branch for which $\sigma = 1$, and to the case of the singularity at the origin, at which the $\frac{\partial F}{\partial z}$ polygon entirely encloses the $xy\phi$ polygon.

Hence our proposition is completely proven.

We may give the following examples of the case when all the $\sigma_1, \sigma_2, \ldots$ are not positive—in both cases the curve ϕ obtained by the interior points of the polygon is adjoint at infinity and the origin.

(1) $F = y^2x + y(x, 1)_3 + (x, 1)_1 = 0.$



Here the points inside the polygon give

$$xy\phi = xy\ (A + Bx)$$

and in fact, if $\eta = yx + \frac{1}{2}(x, 1)_3$, the equation becomes

$$\eta^2 = (x, 1)_6$$

which is known to be of deficiency 2, the adjoint curve which gives rise to integrals of first kind being $A + Bx$—in fact $\int (A + Bx) \frac{dx}{\eta}$ is always finite, and this is, for our original form

$$\int (A + Bx) \frac{dx}{\frac{1}{2}\frac{\partial F}{\partial y}}.$$

(2) $F = y^3x^3 + y^2(x, 1)_2 + y(x, 1)_1 + (x, 1)_1 = 0.$

The diagram gives

$$xy\phi = xy\ (A + By + Cxy).$$

And in fact, by $x = \frac{1}{\xi}$, the curve becomes

$$\xi^3 F = f = y^3 + y^2\xi(1, \xi)_2 + y\xi^2(1, \xi)_1 + \xi^2(1, \xi)_1 = 0,$$

shewing that, by the demonstration given,

$$\int \frac{y\xi(A\xi + By\xi + Cy)}{\frac{\partial f}{\partial z}}\left(\frac{d\xi}{\xi} - \frac{dy}{y}\right)$$

$$= - \int \frac{A\xi + By\xi + Cy}{\frac{\partial f}{\partial y}} d\xi$$

$$= + \int \frac{(A + By + Cxy)/x}{\frac{1}{x^3}\frac{\partial F}{\partial y}} \frac{dx}{x^2}$$

$$= \int \frac{A + By + Cxy}{\frac{\partial F}{\partial y}} dx,$$

is everywhere finite. So that $A + By + Cxy$ is 'adjoint' as desired.

The proof thus furnished that the curve ϕ is an adjoint curve of order $N - 3$, gives then, in the case in which the origin is not a multiple point, another proof of the theorem proved by Professor Cayley in the addition to Rowe's memoir referred to.

But more; it gives an evaluation of the number, $\delta + \kappa$, of simple double points and cusps to which our complex singularity at the origin is to be reckoned as equivalent. For this equivalence is required only to be such as will give the proper value for the deficiency of the curve: the value of κ itself is independently determined by reference to the number of cycles arising by all the branches at the origin—say by the number of branch points at $x = 0$ on the Riemann surface representing the equation F other than those that arise by tangents of the curve parallel to the axis of y—which number is clearly, in the notation explained, $\Sigma n_r' (\mu_r' - 1)$, provided the expansions are of the form $y =$ (integral series in $x^{\frac{1}{\mu_r'}}$) and none of σ_1', σ_2' ... are < 1; and this is the number given by Cayley (*Quart. Jour.* Vol. vii.). Considering then what are the additional points of our polygon when the origin ceases to be a multiple point we have the

Proposition. The multiple point at the origin furnishes a contribution to the total $\delta + \kappa$ of the curve F which is equal to the number of unit points between the axes and the sides $P_0'P_1'$, ... $P'_{k'+1}$ plus the number of those, other than P_0' and $P'_{k'+1}$, upon these lines.

We proceed to verify that this is the number obtained by applying Cayley's rules (*Quart. Jour.*, vol. vii.) to the expansions of the branches of the curve at the origin.

We have to consider the number of intersections of all the branches corresponding for instance to the side $P'_{r-1}P'_r$ among themselves, and the intersections of all the branches corresponding to $P'_{r-1}P'_r$ with all the branches corresponding to $P'_{s-1}P'_s$ for all values of $s > r$. For brevity we may be allowed for the present to drop the dashes, and assume that each of σ_1', σ_2', ... is > 1.

Then a branch $y = A_1 x^{\sigma_r}$ intersects a branch $y = A_2 x^{\sigma_r}$ in σ_r points, in Cayley's nomenclature. And the number of such pairs corresponding to σ_r is $\frac{1}{2} n_r \mu_r (n_r \mu_r - 1)$. So that on the whole we get $\frac{1}{2} m_r (n_r \mu_r - 1)$ intersections. The number of intersections of $y = Ax^{\sigma_r}$ and $y = Bx^{\sigma_s}$, where $s > r$ and therefore $\sigma_r < \sigma_s$, is σ_r, and the number of such pairs is $n_r \mu_r \,.\, n_s \mu_s$. So that on the whole we obtain $\sum\limits_{s>r} n_r n_s m_r \mu_s$ intersections. Thus Cayley's rules give the formula

$$\delta + \frac{3}{2} \kappa = \sum_{s>r} n_r n_s m_r \mu_s + \frac{1}{2} \Sigma n_r m_r (n_r \mu_r - 1),$$

and hence, by $\kappa = \Sigma n_r (\mu_r - 1)$

$$\delta + \kappa = \sum_{s>r} n_r n_s m_r \mu_s + \frac{1}{2} \Sigma n_r m_r (n_r \mu_r - 1) - \frac{1}{2} \Sigma n_r (\mu_r - 1).$$

Using now the result before obtained for the number of unit points between the axes and the sides $P_0 \ldots P_{k+1}$, and remembering that the number of unit points on these sides is $\Sigma n_r - 1$ (excluding P_0, P_{k+1}), the accuracy of our proposition above is verified.

The proof we have given of the Proposition makes it evident that it is not needful to regard all of σ_1', σ_2', ... as greater than unity. And it is easy to see that this result is equally obtainable by Cayley's rules: we divide, for this purpose, the sides into two sets $\sigma'_1 \ldots \sigma'_{r-1}$ all < 1, and $\sigma'_r = 1$ and $\sigma'_{r+1} \ldots \sigma'_{k+1}$ all > 1. The work is quite similar to that given by Cayley in the addition to Rowe's *Memoir*—but its expression is simplified by the use of the diagram. The κ of the point is in this case

$$= \sum_{t=r}^{t=r-1} n_t' (m_t' - 1) + \sum_{t=r} n_t' (\mu_t' - 1).$$

We may notice that the contribution arising from a single branch $y = Ax^{\sigma_r}$ to $\delta + \kappa$, being $\frac{1}{2} n_r \mu_r (n_r \mu_r - 1) \sigma_r - \frac{1}{2} n_r (\mu_r - 1)$ is capable of geometric representation. In fact if from P_r, $P_r K_r$ be drawn perpendicular to the ordinate of P_{r-1}, the contribution is equal to the number of unit points inside the triangle $P_r K_r P_{r-1}$ plus the number on $P_r P_{r-1}$ other than P_r and P_{r-1}. And the number of the intersections of this branch with all following branches being $n_r m_r \sum\limits_{s>r} n_s \mu_s$, is equal to the whole number of unit points within the rectangle $P_r N_{r-1}$ plus the number on the sides $P_r K_r N_{r-1}$, where $P_{r-1} N_{r-1}$ is the ordinate of P_{r-1}.

PART III.

Extension of foregoing to more particular forms of singular points.

In the previous cases we have assumed that the equation corresponding to any side of the polygon for the origin has all its roots different. In particular we have assumed that the branches which do not touch either the axis of x or the axis of y have separated tangents. This it is by no means necessary to assume. Moreover in counting the number of cusps we have assumed that there are terms in the equation of the curve corresponding to all the unit points within the polygon. This restriction also we proceed to remove.

In fact, considering the branches that correspond to a side σ of our polygon at the origin, if a line coinciding with this σ line move parallel to itself away from the origin until it next contain unit points, and the point in which it intersects the axis of x in this new position be called T_1, while its original position meets the axis of x in a point T, then $TT_1 = \frac{1}{\mu}$. We have practically assumed that the unit points upon this new position of the line are curve points. In what follows we assume that the first position of a line parallel to the σ line which contains curve points meets the axis of x in a point which is at a distance from T equal to $\frac{t}{\mu}$. It will be found that the value of t has an influence upon the number of cusps corresponding to our singularity. (See for instance the examples, pp. 424, 425.)

It is necessary to consider the expansions with some particularity.

Consider the curve in the most general form possible

$$x^h y^k (y^\mu - a_1 x^m)^{N_1} \dots (y^\mu - a_\lambda x^m)^{N_\lambda} + x^{h_1} y^{k_1} (y^\mu,\ x^m)^{r_1} + x^{h_2} y^{k_2} (y^\mu,\ x^m)^{r_2} + \dots\dots$$

where
$$h + \sigma k + mn < h_1 + \sigma k_1 + r_1 m < h_2 + \sigma k_2 + r_2 m < \dots\dots$$
$$n = N_1 + N_2 + \dots + N_\lambda$$

and $(y^\mu,\ x^m)^r$ means an integral polynomial homogeneously of degree r in the quantities $y^\mu,\ x^m$; so that the terms are arranged to correspond to curve points on lines parallel to the σ-side.

Put $\xi = x^{\frac{1}{\mu}}$, a definitely assigned value for each value of x, and $y = v\xi^m = vx^\sigma$.

$$\therefore\ v^k (v^\mu - a_1)^{N_1} \dots\dots (v^\mu - a_\lambda)^{N_\lambda} + v^{k_1} \xi^{t_1} (v^\mu,\ 1)^{r_1} + v^{k_2} \xi^{t_2} (v^\mu,\ 1)^{r_2} + \dots\dots$$

where

$$\frac{t_1}{\mu} = h_1 - h + \sigma(k_1 - k) + m(r_1 - n),$$

$$\frac{t_2}{\mu} = h_2 - h + \sigma(k_2 - k) + m(r_2 - n)$$

....

$$\therefore\ (v^\mu - a_1)^{N_1} = \xi^{t_1} \phi_1(v) + \xi^{t_2} \phi_2(v) + \dots\dots,$$

where

$$\phi_1(v) = \frac{v^{k_1} (1,\ v^\mu)^{r_1}}{v^k (v^\mu - a_2)^{N_2} \dots (v^\mu - a_\lambda)^{N_\lambda}}$$

is a rational function of v which does not become infinite in the neighbourhood of $v = \sqrt[\mu]{a_1}$—and similarly for $\phi_2(v)$, etc. For the present I assume that $\phi_1(v)$ does not become zero in the neighbourhood of $v = \sqrt[\mu]{a_1}$*. Then t_1 is the t spoken of above as determined by T_1.

* Otherwise we proceed quite similarly with the first ϕ which does not vanish, and the corresponding t. See an example in the Corollary to Part VI.

Then we may write

$$(v^\mu - a_1)^{N_1} = \xi^{t_1}\phi_1(v)\left[1 + \xi^{t_2-t_1}\frac{\phi_2(v)}{\phi_1(v)} + \xi^{t_3-t_1}\frac{\phi_3(v)}{\phi_1(v)} + \ldots\ldots\right]$$

$$= \xi^t\phi(v)\left[1 + \xi^{u_1}f_1(v) + \xi^{u_2}f_2(v) + \ldots\ldots\right] \text{ say,}$$

where $f_1(v)$, $f_2(v)$ ... are rational in v and not infinite near $v = \sqrt[\mu]{a_1}$.

Then

$$v^\mu - a_1 = \xi^{\frac{t}{N_1}}\sqrt[N_1]{\phi(v)}\left[1 + \xi^{u_1}f_1(v) + \xi^{u_2}f_2(v) + \ldots\ldots\right]^{\frac{1}{N_1}}$$

of which all the values for which ξ is small are given by

$$v^\mu - a_1 = \omega_{N_1}\xi^{\frac{t}{N_1}}\sqrt[N_1]{\phi(v)}\left[1 + \xi^{u_1}\frac{1}{N_1}f_1(v) + \ldots\ldots\right]$$

where ω_{N_1} is in turn equal to all the N_1th roots of unity,

$$\text{say } v^\mu = a_1\left[1 + \omega_{N_1}x^{\frac{t}{\mu N_1}}P(x^{\frac{1}{\mu}}, v)\right],$$

where $x^{\frac{1}{\mu}}$, $x^{\frac{1}{\mu N_1}}$ have definite meanings and $P(x^{\frac{1}{\mu}}, v)$ is a one-valued power series in $x^{\frac{1}{\mu}}$, whose coefficients are rational functions of v, this power series not vanishing for $x = 0$, and the coefficients not becoming infinite for $v = \sqrt[\mu]{a_1}$.

If now δ be the greatest common divisor of N_1 and t, so that $N_1 = A\delta$, $t = B\delta$, and we put $u = x^{\frac{1}{A\mu}} = x^{\frac{\delta}{N_1\mu}}$, then our equation becomes

$$v^\mu = a_1\left[1 + \omega_{N_1}u^B P(u^A, v)\right].$$

Here A and B have no common factor.

It follows then that v can be expressed as an ascending series of positive *integral* powers of u, and cannot be expressed in integral powers of any root of u. And all the values of v near to $u = 0$ are given by

$$v = \omega_\mu a_1^{\frac{1}{\mu}}\left[1 + \frac{1}{\mu}\omega_{N_1}u^B P(u^A, v) + \frac{1}{2}\frac{1}{\mu}\left(\frac{1}{\mu} - 1\right)u^{2B}\omega^2{}_{N_1}P^2(u^A, v) + \ldots\right]$$

and the continued substitution of this value of v in the right hand leads to the value of v as a power series in u,

$$v = \omega_\mu a_1^{\frac{1}{\mu}} + K\omega_\mu\omega_{N_1}u^B + \ldots + \text{higher ascending powers of } u.$$

To find the value of K we recall that

$$P(u^A, v) = P(x^{\frac{1}{\mu}}, v)$$

is equal to

$$\frac{1}{a_1}\sqrt[N_1]{\phi(v)}\left[1 + \xi^{u_1}\frac{1}{N_1}f_1(v) + \ldots\right],$$

$$\therefore\ K = \frac{a_1^{\frac{1}{\mu}}}{\mu a_1} \sqrt[N_1]{\phi\left(\omega_\mu a_1^{\frac{1}{\mu}}\right)},$$

where
$$\phi(v) = \frac{v^{k_1-k}(1,\ v^\mu)^{r_1}}{(v^\mu - a_2)^{N_2} \dots (v^\mu - a_\lambda)^{N_\lambda}},$$

and
$$\sigma(k_1 - k) = \frac{t}{\mu} - (h_1 - h) - m(r_1 - n) = \frac{t}{\mu} + L, \text{ say};$$

so that
$$\omega_\mu^{k_1-k} = \omega_\mu^{\frac{L\mu}{m} + \frac{t}{m}} = \epsilon, \text{ say},$$

and
$$\sqrt[N_1]{\phi\left(\omega_\mu a_1^{\frac{1}{\mu}}\right)} = \omega_\mu^{\frac{t}{mN_1} + \frac{L\mu}{mN_1}} \cdot a_1^{\frac{L}{mN_1} + \frac{t}{m\mu N_1}} \cdot \frac{(1,\ a_1)^{r_1}}{(a_1 - a_2)^{N_2} \dots (a_1 - a_\lambda)^{N_\lambda}}$$

$$= \mu C \omega_\mu^{\frac{t}{mN_1} + \frac{L\mu}{mN_1}} \cdot a_1^{1 - \frac{1}{\mu}}, \text{ say}$$

where C has a definite value.

So we obtain

$$v = \omega_\mu a_1^{\frac{1}{\mu}} + C\omega_\mu \omega_\mu^{\frac{t}{mN_1} + \frac{L\mu}{mN_1}} \omega_{N_1} u^B + \text{higher ascending powers of } u,$$

$$= \text{power series in } x^{\frac{1}{z}},$$

where $z = A\mu$,

and $u^B = x^{\frac{t}{\mu N_1}}$;

$$\therefore\ y = x^\sigma \omega_\mu a_1^{\frac{1}{\mu}} + x^{\sigma + \frac{t}{\mu N_1}} C\omega_\mu \omega_{N_1} \omega_\mu^{\frac{t}{mN_1} + \frac{L\mu}{mN_1}} + \dots\dots$$

But in this series the coefficients are in general functions of the ω_μ and ω_{N_1} chosen—and certainly not always merely in multiplicative powers—see examples [on pp. 424 and 425]. From this we are to obtain $N_1\mu$ values of y.

These are in fact, arranging them in μ rows each of N_1 values,

$$y_{1,\,1} = x^\sigma \omega_\mu a_1^{\frac{1}{\mu}} + x^{\sigma + \frac{t}{\mu N_1}} C\omega_\mu \omega_\mu^{\frac{t + L\mu}{mN_1}} \omega_{N_1}, \dots\dots\ y_{1,\,i} = x^\sigma \omega_\mu a_1^{\frac{1}{\mu}} + x^{\sigma + \frac{t}{\mu N_1}} C\omega_\mu \omega_\mu^{\frac{t + L\mu}{mN_1}} \omega_{N_1}{}^i; \dots\dots$$

....

$$y_{j,\,1} = x^\sigma \omega_\mu{}^j a_1^{\frac{1}{\mu}} + x^{\sigma + \frac{t}{\mu N_1}} C\omega_\mu{}^{j\left[1 + \frac{t + L\mu}{mN_1}\right]} \omega_{N_1}, \dots\dots\ y_{j,\,i} = x^\sigma \omega_\mu{}^j a_1^{\frac{1}{\mu}} + x^{\sigma + \frac{t}{\mu N_1}} C\omega_\mu{}^{j\left[1 + \frac{t + L\mu}{mN_1}\right]} \omega_{N_1}{}^i, \dots\dots$$

....

(where if we mean $\omega_{N_1}{}^i$ as a i-th power we must assume ω_{N_1} was a primitive N_1th root of unity, etc.).

Suppose that underneath these μ rows we write down the $(\lambda - 1)\mu$ similar rows belonging to the other roots $a_2, \dots a_\lambda$. It is easy to count the intersections of these μn branches among themselves.

The intersections of any row are in number

$$\frac{1}{2}N_1(N_1-1)\left(\sigma+\frac{t}{\mu N_1}\right),$$

giving in all

$$\sum_1^\lambda \mu\frac{1}{2}N_1(N_1-1)\left(\sigma+\frac{t}{\mu N_1}\right),$$

while any one of the branches belonging to the first μ rows intersects each of the $\mu n-N_1$ branches, which are not in the same row with it, in σ points, giving then

$$\frac{1}{2}\sum_1^\lambda \sigma\mu N_1(\mu n-N_1)$$

since each branch is thus counted twice.

Thus on the whole we have

$$\frac{1}{2}n\mu m\Sigma N_1-\frac{1}{2}m\Sigma N_1^2+\frac{1}{2}m(\Sigma N_1^2-\Sigma N_1)+\frac{1}{2}t\Sigma(N_1-1) \text{ intersections};$$

that is

$$\frac{1}{2}n^2\mu m-\frac{1}{2}mn+\frac{1}{2}t(n-\lambda) \text{ intersections.}$$

The first μ rows give either one branch point of order μN_1 or N_1 branch points of order μ, or possibly f_1 branch points of order $\frac{N_1\mu}{f_1}$.

(Thus $f_1=1$ or N_1),

and counting then $f_1\left(\frac{N_1\mu}{f_1}-1\right)$ cusps, so that the first μN_1 branches give $n\mu-\Sigma f_1$ cusps we obtain

$$\delta+\kappa=\frac{1}{2}n^2m\mu-\frac{1}{2}nm+\frac{1}{2}t(n-\lambda)-\frac{1}{2}n\mu t\frac{1}{2}\Sigma f_1$$

and this is greater than the normal value

$$\frac{1}{2}n^2m\mu-\frac{1}{2}nm-\frac{1}{2}n\mu+\frac{1}{2}n$$

by

$$\frac{1}{2}t(n-\lambda)-\frac{1}{2}\left[n-\sum_1^\lambda f_1\right],$$

which, when there is one branch point of order μN, is

$$\frac{1}{2}(t-1)(n-\lambda),$$

and when there are for each N_λ, branch points of order μ, is

$$\frac{1}{2}t(n-\lambda).$$

The quantity f_1 above must in fact be equal to δ. For, if taking one of the $N_1\mu$ series and thinking of the corresponding Riemann's surface, we allow x to describe a closed contour on one of the sheets round $x=0$, the new value of the series must clearly be another of the $N_1\mu$ series. To see this we have only to notice that the original equation remains completely unaltered, and we may imagine the $N_1\mu$ series calculated from it in its new form. One of these newly calculated series will be the changed value of the series first considered.

Thus the μN_1 series consist of one or more cycles.

But in fact, since they are all of them rational in the quantity $x^{\frac{1}{A\mu}}$, revolutions of x round $x=0$ can only change any given one of the series into $\mu A - 1$ other series. There will therefore be $\frac{\mu N_1}{\mu A} = \delta$ cycles.

Substituting then δ for f_1 in the previous formula the excess there found is equal to

$$\frac{1}{2}t(n-\lambda) - \frac{1}{2}\left[n - \sum_{k=1}^{k=\lambda} \delta\left(\overset{t}{N_k}\right)\right].$$

Putting $t = B_k\delta_k$, $N_k = A_k\delta_k$, where A_k, B_k are coprime, this excess is

$$\frac{1}{2}\left\{t\Sigma(N_k - 1) - \Sigma\left[N_k - \delta\left(\overset{t}{N_k}\right)\right]\right\}$$

$$= \frac{1}{2}\Sigma\left[B_k\delta_k(A_k\delta_k - 1) - A_k\delta_k + \delta_k\right]$$

$$= \frac{1}{2}\Sigma\delta_k\left[A_kB_k\delta_k - B_k - A_k + 1\right]$$

$$= \frac{1}{2}\Sigma\left[\delta_k^2A_kB_k - B_k\delta_k - A_k\delta_k + \delta_k\right] = \frac{1}{2}\Sigma\left[N_kt - t - N_k + \delta_k\right].$$

And the quantity within the square bracket here is easily susceptible of a graphical representation—thus, take in a plane, whose positive quadrant is ruled with unit lines as before, a point on the axis of x at distance $= t_k$ from the origin, and a point on the axis of y at distance N_k from the origin, and join these points.

The number within the square brackets is equal to the number of unit points within the right-angled triangle so formed, plus the number on the hypotenuse, less two.

As an example of the previous, consider the curve

$$y(y^2 - ax)^4(y^2 - bx) + yx^2(y^2,\ x)^4 + x^2(y^2,\ x)^5 = 0.$$

It can be shewn that the branches of this corresponding to $(y^2 - ax)^4$ are of the form

$$y = \epsilon\sqrt{a}\,.\,x^{\frac{1}{2}} + \epsilon\omega x^{\frac{3}{4}}\alpha + \epsilon\omega^2 x\beta + \epsilon\omega x^{\frac{5}{4}}\left[\epsilon\gamma + \omega^2\delta\right] + \ldots,$$

where ϵ is a square root of unity, and ω is a fourth root of unity, and where α, β, γ, δ are perfectly definite.

Giving then to ϵ and ω all their possible values we obtain the eight expansions:

$$y = \sqrt{a}\,x^{\frac{1}{2}} + x^{\frac{3}{4}}\alpha + x\beta + x^{\frac{5}{4}}(\gamma + \delta) \qquad \text{(1)},$$

$$y = \sqrt{a}\,x^{\frac{1}{2}} - x^{\frac{3}{4}}\alpha + x\beta - x^{\frac{5}{4}}(\gamma + \delta) \qquad \text{(2)},$$

$$y = \sqrt{a}\, x^{\frac{1}{2}} + i x^{\frac{3}{4}} \alpha - x\beta + i x^{\frac{5}{4}} (\gamma - \delta) \quad \ldots\ldots (3),$$

$$y = \sqrt{a}\, x^{\frac{1}{2}} - i x^{\frac{3}{4}} \alpha - x\beta - i x^{\frac{5}{4}} (\gamma - \delta) \quad \ldots\ldots (4),$$

$$y = -\sqrt{a}\, x^{\frac{1}{2}} - x^{\frac{3}{4}} \alpha - x\beta - x^{\frac{5}{4}} (-\gamma + \delta) \quad \ldots\ldots (5),$$

$$y = -\sqrt{a}\, x^{\frac{1}{2}} + x^{\frac{3}{4}} \alpha - x\beta + x^{\frac{5}{4}} (-\gamma + \delta) \quad \ldots\ldots (6),$$

$$y = -\sqrt{a}\, x^{\frac{1}{2}} - i x^{\frac{3}{4}} \alpha + x\beta + i x^{\frac{5}{4}} (\gamma + \delta) \quad \ldots\ldots (7),$$

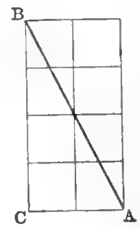
$$y = -\sqrt{a}\, x^{\frac{1}{2}} + i x^{\frac{3}{4}} \alpha + x\beta - i x^{\frac{5}{4}} (\gamma + \delta) \quad \ldots\ldots (8).$$

And if we allow x to make a circuit on the Riemann's surface round $x = 0$, which changes $x^{\frac{1}{4}}$ into $ix^{\frac{1}{4}}$, these series break up into the two cycles

$$(1,\ 7,\ 2,\ 8,\ 1),$$

$$(3,\ 6,\ 4,\ 5,\ 3).$$

In fact here $\mu = 2,\ N_1 = 4,\ t = 2,$



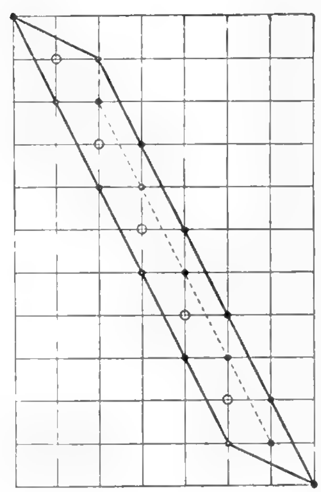
$\therefore\ \delta_1 = 2$, and the excess in the value of $\delta + \kappa$ due to the facts that N_1 is not equal to 1, and t is not equal to 1, is

$$\frac{1}{2} 2 (4 - 1) - \frac{1}{2} (4 - 2)$$

$$= 2,$$

which is the number of unit points within the triangle ABC and upon the hypotenuse other than the points A, B.

The diagram for the curve is as follows:—



Here the circles round the unit points indicate that they are not curve points. In fact $t = 2$. From this diagram, taking count of the correction, we infer that for the origin $\delta + \kappa = 27$: and that the deficiency is 8.

We may remark that if in

$$y (y^2 - ax)^4 (y^2 - bx) + yx^2 (y^2,\ x)^4 + x^2 (y^2,\ x)^5 = 0,$$

we put

$$\xi = \frac{x}{y^2},\quad \eta = \frac{(y^2 - ax)^2}{xy^2},$$

leading to

$$x = \frac{(1 - a\xi)^4}{\eta^2 \xi},\quad y = \frac{(1 - a\xi)^2}{\eta\xi},$$

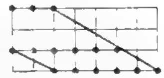
we obtain

$$\eta^3 \xi (1 - b\xi) + \eta\xi (1,\ \xi)^4 + (1 - a\xi)^2 (1,\ \xi)^5 = 0,$$

which, writing y for η and x for $1 - a\xi$ is of the form

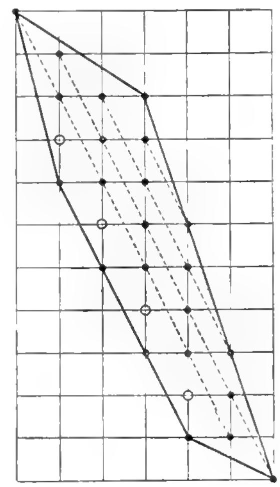
$$y^3 u_1 v_1 + y u_1 u_4 + x^2 u_5 = 0,$$

where u_1, v_1, u_4, u_5 are polynomials in x of the degree indicated by the suffixes.

The figure for this form is  which gives 8 for the deficiency—and indicates that the general finite integral is, in these new coordinates (see Part II.)

$$\int [A + Bx + Cy + Dx^2 + Exy + Fx^2y + Gx^3 + Hx^4] \frac{dx}{u_1(3y^2v_1 + u_4)}.$$

Another example of the theory is the curve represented by the diagram—



The equation of this curve is

$$xy(y^2 - ax)^2 (y^2 - bx) + y(y^2,\ x)^5 + x(y^2,\ x)^5 + x^2y^3(y^2,\ x)^3 + y^6x^3(y^2,\ x) + y^9x^3 = 0.$$

Here
$$\left.\begin{array}{lll} m_1 = 1, & \mu_1 = 4, & n_1 = 1 \\ m_2 = 1, & \mu_2 = 2, & n_2 = 3 \\ m_3 = 2, & \mu_3 = 1, & n_3 = 1 \end{array}\right| \quad t_2 = 2, \quad \delta_2 = 2.$$

The values of y corresponding to the factor $(y^2 - ax)^2$ are given by

$$y = \epsilon\sqrt{a}\,x^{\frac{1}{2}} + \epsilon\zeta x\alpha + x^{\frac{3}{2}}\epsilon\zeta[\epsilon\beta + \zeta\gamma] + \dots,$$

where ϵ, ζ are square roots of unity, and α, β, γ are definite functions of the original coefficients.

Thus the four values of y are

$$y = \sqrt{a}\,x^{\frac{1}{2}} + x\alpha + x^{\frac{3}{2}}[\beta + \gamma] + \dots \quad\dots\dots(1),$$

$$y = \sqrt{a}\,x^{\frac{1}{2}} - x\alpha - x^{\frac{3}{2}}[\beta - \gamma] + \dots \quad\dots\dots(2),$$

$$y = -\sqrt{a}\,x^{\frac{1}{2}} - x\alpha + x^{\frac{3}{2}}[\beta - \gamma] + \dots \quad\dots\dots(3),$$

$$y = -\sqrt{a}\,x^{\frac{1}{2}} + x\alpha - x^{\frac{3}{2}}[\beta + \gamma] + \dots \quad\dots\dots(4).$$

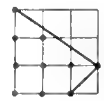
And if we make x describe a contour round $x = 0$, so that $x^{\frac{1}{2}}$ changes into $-x^{\frac{1}{2}}$, then the series (1) changes into the series (4), and the series (4) changes into (1), while also the series (2) changes into the series (3), and the series (3) into the series (2). So that there are two cycles, as there should be according to our theory.

Various other examples of the rules of this Part are given below.

PART IV.

Some examples of the foregoing theory. Consideration of the normal form of any curve given by Weierstrass.

1. In the paper by Rowe referred to in Part I., the deficiency of the curve



$$y^3 + y^2 (x,\ 1)_1 + y (x,\ 1)_3 + (x,\ 1)_2 = 0$$

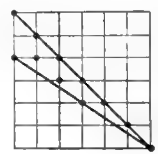
is determined. (=3.)

The result is immediately obvious on inspection of the figure.

2. In the *Math. Annal.* ix. p. 174, Noether gives as example of his method of reduction the curve

$$y^4 + y^2 (x,\ y)_3 + (x,\ y)_6 = 0,$$

and obtains that the multiple point at the origin is equivalent to a quadruple point and two double points, that is in all that $\delta + \kappa = 8$ (beside that $\kappa = 2$).



This result is obvious from the figure.

We shall have further occasion for this Example in Part V.

Our diagram gives moreover the deficiency = 2. Hence the curve can be transformed to $\eta^2 = (1,\ \xi)_6$. Put in fact

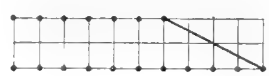
$$\xi = \frac{x}{y},$$

$$\eta = \frac{1}{y}.$$

3. The hyperelliptic curve can always be put in the form

$$y^2 (x,\ 1)_{p+2-r} = (x,\ 1)_{p+r}$$

wherein r is arbitrary.



The number of unit points within its polygon is p.

The figure is drawn for $p = 7$, $r = 3$.

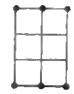
The figure gives, according to the theory here developed, the adjoint curves of order

$$n - 3, \quad \text{viz.} \quad 1,\ x,\ \ldots,\ x^{p-1}.$$

I believe that in all cases in which the deficiency of a hyperelliptic curve is accurately given by the number of unit points within its polygon, these unit points will be collinear, whatever be the form of the curve polygon.

4. An example is quoted by Forsyth (*Theory of Functions*, page 355) from Burnside (London Math. Society, May 14, 1891).

The curve
$$y^3 (x, 1)_2 = [x, 1]_2$$
has deficiency two.



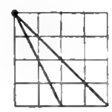
This is obvious from the figure.

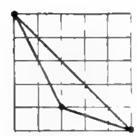
We see further, from previous work, that the finite integrals are
$$\int \frac{dx}{y^2 (x, 1)_2}, \quad \int \frac{dx}{y (x, 1)_2}.$$

5. In a paper (in the *Journal de l'École Polytechnique*) Raffy has given three examples of a method there developed by him for determining the deficiency of a curve.

Two of these are
$$y^4 - x^2 (x^2 + x + 1) = 0, \quad y^5 + x^5 - 5x^2 y = 0,$$
having respectively deficiencies 1 and 2.

These results are obvious from the figures.





The other of these examples is
$$y^5 - 5y^3 (x^2 + x + 1) + 5y (x^2 + x + 1)^2 - 2x (x^2 + x + 1)^2 = 0,$$
for which Raffy obtains $p = 0$. The equation can indeed, by an obvious transformation, be made to take the form of a conic. But the equation is hyperelliptic and this transformation not reversible.

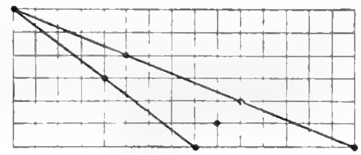
But by putting
$$\xi = \frac{x^2 + x + 1}{y^2}, \quad \eta = -\frac{\omega (x + 2) (x^2 + x + 1)^2}{2y^5},$$
leading to
$$x = \frac{\omega^2 \left[\eta + \frac{1 - \omega^2}{4} (1 - 5\xi + 5\xi^2)\right]^2 - \omega\xi^5}{2\eta^2 + \eta \frac{1 - \omega^2}{2} (1 - 5\xi + 5\xi^2) + \omega\xi^5}, \text{ where } \omega^3 = 1,$$
$$y = -\frac{(\omega - \omega^2) \xi^2 \left[\eta + \frac{1 - \omega^2}{4} (1 - 5\xi + 5\xi^2)\right]}{2\eta^2 + \eta \frac{1 - \omega^2}{2} (1 - 5\xi + 5\xi^2) + \omega\xi^5},$$
we can transform to
$$\eta^2 = \left(\frac{1 - \omega^2}{4}\right)^2 (1 - 5\xi + 5\xi^2)^2 + \omega^2 \xi^5.$$

Thus the curve has $p = 2$.

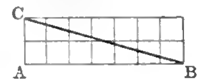
This curve forms a good example of the failure of our rule owing to the very exceptional forms of the coefficients. (It is treated by these rules in Corollary to Part VI.)

6. The following example is given by Cayley (*Quarterly Journal*, Vol. vii., p. 217):

$$(y^3 - x^4)^2 - 3x^5y\,(y^3 + 2x^4) + x^{10}\,(3y^2 - x^5) = 0.$$



The value of the singularity at the origin is obtainable from the figure with the help of the rule developed in Part III.



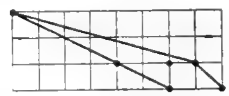
Here $m=4$, $\mu=3$, $t=7$, $N=2$, and in addition to the 20 given by the first diagram there is to be counted 1, given by the second diagram, where

$$AB = t = 7, \quad AC = N = 2,$$

giving on the whole $\delta + \kappa = 21$.

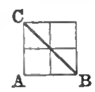
(See the expansions given by Cayley.)

7. The following example is quoted from Miss Scott by Harkness and Morley (*Theory of Functions*, p. 147), and furnishes another example of Part III.



$$(y + 2x^2)\,(y - x^2)^2 - x^6\,(y + 2x^2) + 9x^7y = 0.$$

Here $m=2$, $\mu=1$, $t=2$, $N_1=1$, $N_2=2$, and we have a correction $= 1$, given by the second diagram, where



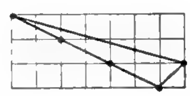
$$AB = t = 2, \quad AC = N = 2;$$

$$\therefore \quad \delta + \kappa = 7.$$

Also the curve has $p=2$ and can be transformed to $\eta^2 = (\xi, 1)_6$.

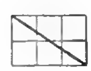
8. In case the curve be

$$(y + 2x^2)\,(y - x^2)^2 + 9x^7y = 0,$$



the figures are slightly modified. But as in (7) there is a correction $=1$. The difference is that in this latter case there is a branch point.

Here $N=2$, $t=3$,



and $\delta + \kappa = 7$, as before. (See the expansions in Harkness and Morley.)

9. *Of Weierstrass' normal curve.*

If g_a be the algebraic function of lowest order which is only infinite at one point A of a plane curve, and g_r be the function of next order prime to a, the equation of the curve can be transformed to

$$F = g_r^{\,a} + g_r^{\,a-1}\,(g_a,\ 1)_{\lambda_1} + \ldots + (g_a,\ 1)_r = 0.$$

Every algebraic function can be rationally expressed by g_a and g_r. Every expression which is integral in g_a and g_r becomes infinite only at A. But conversely there exist in general algebraic functions only becoming infinite at A where g_a and g_r are infinite, which are nevertheless not expressible integrally by g_a and g_r. We can indeed prove the

Proposition. Of algebraic functions which become infinite only at the point A where g_a and g_r become infinite, there exist functions of as many different orders (of infinity at A), which are not integrally expressible by g_a and g_r, as there exist simple double points and cusps of the curve F above; in other words, the part of the $\delta+\kappa$ of the curve F above other than that furnished by the place $g_a=\infty$, $g_r=\infty$, is equal to this number of different orders of existent functions.

In order to prove this we notice that a function of order z cannot be expressed integrally in g_a and g_r unless we can find positive integers x and y such as to make

$$ax+ry=z,$$

and thence put

$$g_z=Cg_a{}^x g_r{}^y+\ldots.$$

And this equation being

$$x=\frac{z-ry}{a},$$

wherein we may suppose $y<a$, requires, for any value of y,

$$z=ry,\quad ry+a,\quad ry+2a,\ \ldots,$$

and therefore cannot be satisfied by those values of $z\equiv ry$ (mod. a) which are $<ry$—that is, cannot be satisfied by

$$z=ry-a,\quad ry-2a,\quad ry-3a,\ \ldots.$$

The number of these values is $E\left(\frac{ry}{a}\right)$, the greatest integer in $\frac{ry}{a}$.

The number of values of z thus excluded is

$$\sum_{y=1}^{y=a-1} E\left(\frac{ry}{a}\right),$$

which is equal to $\frac{1}{2}(r-1)(a-1)$, as we see by noticing that it is equal to the number of unit points inside a right-angled triangle having one side $=r$ and the other equal to a. Any value of z other than these of the form $ry-a$, can be expressed in the form $ax+ry$ —so that for such values of z a function $g_z=Cg_a{}^x g_r{}^y$ certainly exists, and the most general function of this order, infinite only at A, is of the form $Cg_a{}^x g_r{}^y+g_{z'}$, where z' is $<z$ and $g_{z'}$ is, possibly, not expressible integrally by g_a and g_r.

Of the not integrally expressible orders, in number $\frac{1}{2}(r-1)(a-1)$, there are, as we know, (see note at end of this paper), just p which correspond to actually non-existent functions.

Hence there remain just

$$\frac{1}{2}(a-1)(r-1)-p$$

orders, of functions which exist, are infinite only at A, and are not expressible integrally by g_a and g_r.

Consider now the function $\frac{\partial F}{\partial g_r}$. It is a function of degree $a-1$ in g_r and therefore of order $r(a-1)$; and vanishes therefore at $r(a-1)$ points of the original curve. These points consist of (1) those at which dg_a is zero of the second order, namely those which become the branch points of the Riemann surface which represents g_r as a function of g_a and are therefore in number $=2a+2p-2$, of which $a-1$ fall at A where the a values of g_r are all infinite, and (2) of those which become multiple points of the curve F or of the Riemann surface, the number of these for any multiple point other than those already counted among the branch points being $2\delta+2\kappa$ (δ, κ being Cayley's equivalent numbers of double points and cusps for the multiple point).

Hence $\delta+\kappa$ for the whole curve F is

$$\frac{1}{2}\{r(a-1)-[2a+2p-2-\overline{a-1}]\}$$

$$=\frac{1}{2}(r-1)(a-1)-p.$$

The comparison of this number with that previously obtained for the not integrally expressible functions, proves our proposition.

Hence also

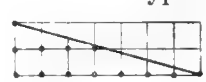
$$p+\delta+\kappa=\frac{1}{2}(r-1)(a-1)$$

= whole number of unit points with the curve polygon of F, this curve polygon being a right-angled triangle of sides r, a, if we do not take count of finite multiple points. This verifies the general proposition of Part II.

Before considering how these exceptional functions are to be expressed we may consider as examples the cases $p=3$, $p=4$.

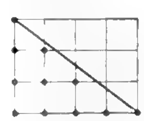
For $p=3$, we may have

(1) $a=2$, $r=7$. The orders of non-existent functions being 1, 3, 5. This is the hyper-elliptic case, the number of moduli being 5: the equation is



$$g_7^2+g_7(g_2,\ 1)_3+(g_2,\ 1)_7=0.$$

(2) $a=3$, $r=4$. The orders of non-existent functions are 1, 2, 5. This is the case of a point of undulation on a plane quartic. The number of moduli is 5. The equation is



$$g_4^3+g_4^2(g_3,\ 1)_1+g_4(g_3,\ 1)_2+(g_3,\ 1)_4=0,$$

reducible to $$h_4^3+h_4(g_3,\ 1)_2+(g_3,\ 1)_4=0,$$

or, say, $$\eta^3\xi+\eta\xi(\eta,\ \xi)_2+(\eta,\ \xi)_4=0,$$

which for $\eta=0$ gives $\xi^4=0$.

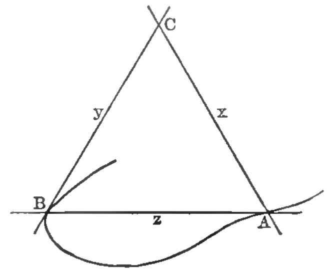
(3) $a=3$, $r=5$. The orders of non-existent functions are 1, 2, 4. There is a function

g_7 not integrally expressible by g_3 and g_5. The (g_3, g_5) curve has therefore a double point. Its equation is (cf. Schottky. Crelle, 83)

$$F = g_5^3 + g_5^2 (g_3, 1)_1 + g_5 g_3 (g_3, 1)_2 + g_3^2 (g_3, 1)_3 = 0$$

and depends on six moduli. The double point is at

$$g_3 = g_5 = 0.$$



In fact by taking for triangle of reference of a plane quartic

$z = 0$ any inflexional tangent,

$y = 0$ the tangent at the remaining point B where the inflexional tangent meets the curve,

$x = 0$ any line through A,

we may put

$$g_3 = \frac{y}{z}, \quad g_5 = \frac{xy}{z^2},$$

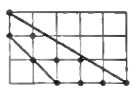
these being infinite at A in the orders indicated, and so reduce the quartic, which takes the form

$$f = x^3 y + x^2 z (y, z)_1 + xz (y, z)_2 + z (y, z)_3 = 0$$

immediately to the form above, with

$$F = \frac{y^2}{z^6} f.$$

The diagram for F is



Notice $7 = (a-1) r - a$.

There is no need to consider cases in which $a > 3$. On every curve for which $p = 3$ there exist points for which g_3 exists.

Considering next $p = 4$, there are five possibilities.

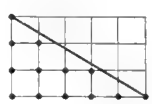
(1) $a = 2$, $r = 9$. The non-existent orders are 1, 3, 5, 7. The equation is

$$g_9^2 + g_9 (g_2, 1)_4 + (g_2, 1)_9 = 0.$$

(2) $a = 3$, $r = 5$. The non-existent orders are 1, 2, 4, 7. Equation is

$$g_5^3 + g_5^2 (g_3, 1)_1 + g_5 (g_3, 1)_3 + (g_3, 1)_5 = 0.$$

Figure is

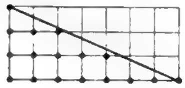


(3) $a = 3$, $r = 7$. Non-existent orders are 1, 2, 4, 5. There exist functions g_8, g_{11}, which are not expressible integrally by g_3 and g_7, so that the (g_3, g_7) curve has two double points.

The equation is

$$g_7^3 + g_7^2\beta_2 + g_7\alpha_2\gamma_2 + \alpha_2^2\alpha_3 = 0,$$

where α_2, β_2, γ_2, α_3 represent integral expressions in g_3 of the order given by their suffixes

For this form the figure is 

and the polygon contains $p + \delta + \kappa = 4 + 2 = 6$ points, as it should.

But by putting $\eta = \dfrac{g_7}{\alpha_2}$ we obtain

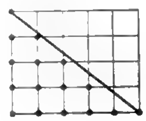
$$\eta^3\alpha_2 + \eta^2\beta_2 + \eta\gamma_2 + \alpha_3 = 0.$$

For this form the polygon contains only $p = 4$ points.

We notice $\qquad 8 = (a-1)r - 2a, \quad 11 = (a-1)r - a.$

(4) $a = 4$, $r = 5$. Non-existent orders are 1, 2, 3, 6. There exist functions g_7, g_{11} which are not integrally expressible by g_4 and g_5, so that the (g_4, g_5) curve has two double points. Its equation is

$$g_5^4 + g_5^3\gamma_1 + g_5^2\beta_2 + g_5\beta_1\gamma_2 + \alpha_1\gamma^2 = 0.$$

For this the figure is 

and polygon has $p + \delta + \kappa = 4 + 2 = 6$ interior points.

But if we put $\gamma_2 = \xi\eta$, $g_5 = 1$, the equation becomes

$$(\xi, \eta)_1 [1 + (\xi, \eta)_1 + (\xi, \eta)_2 + (\xi, \eta)_1 \xi\eta] + (\xi, \eta)_1 \xi^2\eta^2 = 0$$

for which the figure is 

and now the polygon contains only $p = 4$ points.

We notice that $11 = (a-1)r - a$, $7 = (a-1)r - 2a$.

(5) $a = 4$, $r = 7$. Here non-existent orders are 1, 2, 3, 5. There exist

$$g_6, g_9, g_{10}, g_{13}, g_{17}$$

which are not integrally expressible by g_4 and g_7. Thus there are five double points on the (g_4, g_7) curve. We notice that

$$17 = (a-1)r - a, \quad 13 = (a-1)r - 2a, \quad 9 = (a-1)r - 3a,$$

$$10 = (a-2)r - a, \quad 6 = (a-2)r - 2a.$$

Passing from these particular cases to the consideration of the forms of these not integrally expressible functions, we see first that we can always build such a function corresponding to a double point. For if O denote the point of the original curve at which g_a and g_r are infinite, namely the point which becomes the infinite point on the (g_a, g_r) curve and d_1 denote the double point supposed to be reached from one branch of the double point, the other point being denoted by d_2, and P_{od_1} be the integral of the third kind which is once logarithmically infinite at O and at the double point on this first branch, which is therefore finite on the other branch at this double point, then

$$f'(g_r)\frac{\partial P_{od_1}}{\partial g_a}$$

where $f(g_a, g_r)$ is the (g_a, g_r) equation and

$$f'(g_r) = \frac{\partial}{\partial g_r} f(g_a, g_r),$$

is not infinite at d_1, for $\frac{\partial P_{od_1}}{\partial g_a}$ is once algebraically infinite there and $f'(g_r)$ is once zero, and is infinite at O to an order $r(a-1)-(a+1)+1 = r(a-1)-a$.

From this remark, recalling the ordinary method of expressing P_{od_1}, we have a rule for forming this function as a rational expression in g_a and g_r. Viz. it is

$$\frac{\Omega}{L_{od}}$$

where L_{od} represents a linear function in g_a and g_r which vanishes at O and for the values which g_a, g_r have at the places which become the double point, and Ω is for the equation $f(g_a, g_r)$ an adjoint curve which touches the branch d_2 at the double point and passes through the $a-2$ finite points other than O and d, at which L_{od} meets the curve f. We know that such a curve can be expressed as $\Omega_1 + L_{od}\phi$, where Ω_1 is a special curve of the kind and ϕ an integral function in g_a and g_r such that

$$\int \phi \frac{dg_a}{f'(g_r)}$$

is an everywhere finite integral: and one form for Ω_1 is immediately obvious—viz. let t_2 be the tangent to the branch d_2 at the double point of the curve f and ψ be such an integral expression in g_a and g_r that $\int \psi \frac{dg_a}{f'(g_r)}$ is finite at all the double points of f other than the one under consideration, and such that ψ, while not vanishing at this double point, vanishes at the $a-2$ points other than d and O at which L_{od} meets the curve f. The multiplicity of such a curve ψ after passing through all the other double points, is known to be $p+1$, and to prescribe that it passes through $a-2$ points of the line L_{od} leaves it with a multiplicity $p+1-(a-2)$, which is certainly not negative. Hence, noticing that since O is at $\frac{g_r}{g_a} = \infty$, $g_r = \infty$, we may take $L_{od} = g_a - D$, we may write our function

$$G_1 = \frac{t_2\psi}{g_a - D}.$$

In the same way we obtain another such function

$$G_1' = \frac{t_1\psi}{g_a - D},$$

and, attaching proper numerical multipliers to them we may write

$$G_1 - G_1' = \frac{t_2 - t_1}{g_a - D}\psi = \psi = f'(g_r)\frac{\partial P_{d_1d_2}}{\partial g_a}.$$

This representation is in accord with the previous results. If the most general integral expression in g_a and g_r formed by such powers as are represented, in accordance with Part II. of the present paper, by the points within the polygon of the (g_a, g_r) curve, be represented by $g_a g_r \Phi$, we know (see for instance Clebsch and Gordan, *Abelian Functions*, page 16), since Φ is of order $N-3$ (see Part II.), that

$$\int \Phi \frac{dg_a}{f'(g_r)} = C_1 P_{d_1d_2} + \ldots + C_{\delta+\kappa} P_{e_1e_2} + \lambda v_1 + \ldots + \lambda_p v_p + \mu,$$

where e_1, e_2 refer to the $(\delta+\kappa)$th double point, and $v_1 \ldots v_p$ are the everywhere finite integrals, namely

$$\Phi = C_1(G_1 - G_1') + \ldots + C_{\delta+\kappa}(G_{\delta+\kappa} - G'_{\delta+\kappa}) + \phi + \mu,$$

where ϕ is the general adjoint curve of order $N-3$, or

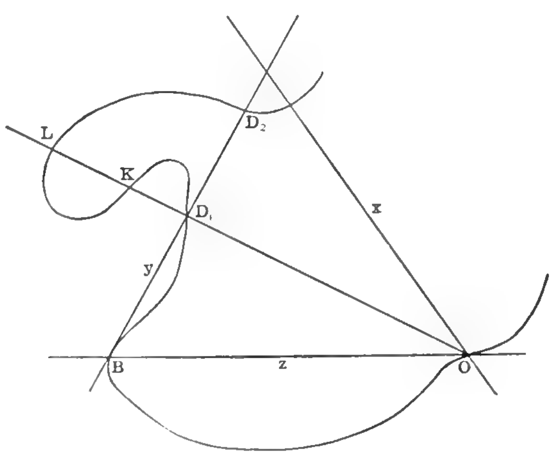
$$\Phi = C_1\psi_1 + \ldots + C_{\delta+\kappa}\psi_{\delta+\kappa} + \phi + \mu.$$

Of course on the other hand, the form of G_1 can be variously altered. For instance, in the example previously considered where $p=3$, $a=3$, $r=5$,

$$g_3 = \frac{y}{z}, \quad g_5 = \frac{xy}{z^2},$$

the double point of the (g_3, g_5) curve arises from the points D_1, D_2, where the quartic is cut by the tangent at B. And we may write

$$g_7 = \frac{yU}{z^2(x-\lambda z)},$$



where $x - \lambda z$ is the line OD_1, and U is an arbitrary conic through K and L: then, easily,

$$g_7 \frac{dg_3}{f'(g_5)} = \frac{U}{x - \lambda z} \frac{zdy - ydz}{\frac{\partial F}{\partial x}},$$

where

$$F = x^3 y + x^2 z (y, z)_1 + xz (y, z)_2 + z (y, z)_3,$$

and this is in agreement with the remark on page 433.

The expression $\frac{t_1 \psi}{g_a - D}$ above can be put into the form

$$\frac{(g_r - E)(g_r - c_1)(g_r - c_2) \dots (g_r - c_{a-2})}{g_a - D} + \text{integral expression in } g_r, g_a,$$

whence as $\frac{t_1 \psi}{g_a - D}$ and the integral expression in g_a, g_r only become infinite when g_a and g_r are infinite, we see that $g_a = D$, $g_r = E$ is the double point and $g_r = c_1, \dots$ are the values of g_r at the points other than the double point in which $g_a - D = 0$ meets the (g_a, g_r) curve. We may thence put

$$g_{(a-1)r-a} = \frac{(g_r - E)(g_r - c_1) \dots (g_r - c_{a-2})}{g_a - D},$$

and this is obviously only infinite when g_a and g_r are infinite.

We might expect to be able to form thence functions of order $(a-1)r - 2a$, etc. for, since $g^2_{(a-1)r-a}$ has an order which is $\equiv 2a \pmod{r}$ we might expect to put

$$g^2_{(a-1)r-a} = \text{integral expression in } g_a \text{ and } g_r + \frac{g_r^{a-1}(g_a, 1)_1 + g_r^{a-2}(g_a, 1)_1 + \dots}{(g_a - D)^2},$$

and thence, putting $(g_a, 1)_1 = \lambda (g_a - D) + \mu$, to obtain

$$g^2_{(a-1)r-a} - \lambda g_{(a-1)r-a} = \text{integral expression in } g_a \text{ and } g_r + \frac{\mu g_r^{a-1} + g_r^{a-2}(g_a, 1)_1 + \dots}{(g_a - D)^2},$$

and thence be able to infer the existence of a function

$$\frac{\mu g_r^{a-1} + g_r^{a-2}(g_a, 1)_1 + \dots}{(g_a - D)^2}$$

only becoming infinite for g_a, g_r infinite, obviously of order $(a-1)r - 2a$, *which is not integrally expressible by g_a and g_r.* But in fact this function will sometimes be integrally expressible by g_a and g_r. For instance, when $p = 3$, $a = 3$, $r = 5$, the curve being

$$g_5^3 + g_5^2 (g_3 - c) + g_5 g_3 (g_3, 1)_2 + g_3^2 (g_3, 1)_3 = 0,$$

though

$$g_7 = \frac{g_5 (g_5 - c)}{g_3}$$

is not integrally expressible, yet we can easily verify that

$$g_7^2 + g_7 (g_3, 1)_2 = g_5^2 + g_5 [(g_3, 1)_2 - (g_3, 1)_3] + (g_3 + c)(g_3, 1)_3,$$

or again, when $p = 4, \quad a = 3, \quad r = 7,$

the curve being $g_7^3 + g_7^2 \beta_2 + g_7 \alpha_2 \gamma_2 + \alpha_2^2 \alpha_3 = 0,$

and $\alpha_2 = c (g_3 - k_1)(g_3 - k_2), \quad \beta_2 = (g_3 - k_1) f_1 + b_1 = c (g_3 - k_2) h_1 + b_2,$

where f_1, h_1 are of the first order in g_3 and c, b_1, b_2 are constants, though

$$g_{11} = \frac{g_7(g_7+b_1)}{g_3-k_1}$$

is not integrally expressible, we can easily verify that

$$g_{11}^2 + g_{11}c\gamma_2(g_3-k_2) = g_7^2 f_1 + g_7[cf_1(g_3-k_2) - c^2\alpha_3(g_3-k_2)^2] + c^2\alpha_3(g_3-k_2)^2(\beta_2-2b_1),$$

and similarly that

$$\frac{g_7^2(g_7+b_1)(g_7+b_2)}{c(g_3-k_1)(g_3-k_2)} = g_7^2 f_1 h_1 - (g_7\gamma_2+\alpha_2\alpha_3)(g_7-\beta_2+b_1-b_2).$$

That such expression as given by these examples should be possible in case of a curve having only *one* double point, is obvious from our proposition that the number of orders of existent not-integrally-expressible "integral" functions is the same as of double points—for we have shewn how to form a function $g_{(a-1)r-a} = \frac{(g_r, 1)_{a-1}}{(g_a, 1)_1}$ corresponding to that double point.

But we can form functions of order $(a-1)r-2a$ etc. in another way.

In the case of a curve having two double points and known to have a not-integrally-expressible function $g_{(a-1)r-2a}$, we may form the difference

$$g_{(a-1)r-a} - g'_{(a-1)r-a}$$

of the two such functions formed as above for the two double points. This will be at most of order $(a-1)r-a-1$ or $r(a-1-A)+a(R-1)$, where A, R are integers less respectively than a and r such that $Ar-Ra=1$. Subtracting from this difference a proper multiple of $g_a^{R-1}g_r^{a-1-A}$ we shall obtain a function of lower order. Proceeding thus we may expect to arrive at an equation

$$g_{(a-1)r-a} - g'_{(a-1)r-a} = \text{integral expression in } g_a,\ g_r + g_{(a-1)r-2a}.$$

For instance, in the example just cited, $p=4$, $a=3$, $r=7$,

$$g_{11} - g'_{11} = \frac{g_7(g_7+b_1)}{g_3-k_1} - \frac{g_7(g_7+b_2)}{g_3-k_2} = (k_1-k_2)\frac{g_7(g_7+\beta_2)}{(g_3-k_1)(g_3-k_2)},$$

so that we may take, unless $k_1=k_2$,

$$g_8 = \frac{g_7(g_7+\beta_2)}{\alpha_2},$$

of which other forms are, in this case,

$$\frac{g_{11}g'_{11}}{b_1cg_7} + \frac{h_1}{b_1}g_{11} + \frac{\alpha_2\alpha_3+\gamma_2 g_7}{b_1},$$

or

$$g_7\frac{f_1h_1}{b_1} + \gamma_2\frac{\beta_2-b_1-b_2}{b_1} + \frac{\alpha_2\alpha_3(\beta_2-b_1-b_2)}{g_7}.$$

In the same way for a curve with any number of double points we can, from any λ of these double points, form a function of order $r(a-1)-\lambda a$, namely

$$\frac{\overset{(1)}{g}_{r(a-1)-a}}{(k_1-k_2)\dots(k_1-k_\lambda)}+\frac{\overset{(2)}{g}_{r(a-1)-a}}{(k_2-k_1)\dots(k_2-k_\lambda)}+\dots+\frac{\overset{(\lambda)}{g}_{r(a-1)-a}}{(k_\lambda-k_1)\dots(k_\lambda-k_{\lambda-1})}$$

$$=\frac{g_r^{a-1}}{(g_a-k_1)\dots(g_a-k^\lambda)}+g_r^{a-2}[\dots]+\dots,$$

where
$$\overset{(1)}{g}_{r(a-1)-a}=\frac{(g_r,\ 1)_{a-1}}{g_a-k_1},\ \text{etc.},$$

as before explained, and the double points are at $g_a=k_1,\ k_2,\ \dots$, these $k_1,\ k_2,\dots$ being supposed different. The function thus obtained is necessarily only infinite when g_a and g_r are so, and it is not expressible integrally, since such integral expression must be of the form $Pg_r^{a-1}+\dots$, where P is integral in g_a.

Thus in the case of a curve with no higher multiple points than double points of which no two have the same value of g_a, we can always express in this way as many not integrally expressible functions of orders of the form $r(a-1)-\lambda a$, as there are double points. Since however every number $r(a-1)-\lambda a$ is prime to a, we see that we must have $r(a-1)-\lambda a>r$, namely $\lambda \not> r-1-E\left(\frac{2r}{a}\right)$. Hence if $(\delta+\kappa)_1$ be the number of the double points

$$(\delta+\kappa)_1 \not> r-1-E\left(\frac{2r}{a}\right),$$

and this is verified in all the examples considered (pages 430 and 432). For instance when

$$p=4,\ a=4,\ r=7,\quad r-1-E\left(\frac{2r}{a}\right)=3,$$

and we found that there were functions $g_9,\ g_{13},\ g_{17}$. The other two $g_6,\ g_{10}$ are of orders

$$(a-2)r-a,\quad (a-2)r-2a.$$

In the case of a curve having double points for which the values of g_a are not all different, we may suppose the previous expression applied only to those double points for which the values of g_a *are* different. We obtain thus as many not integrally expressible functions as the number of these. If then there be a value $g_a=k$, for which there are μ separated double points at $g_r=E_1,\ g_r=E_2,\dots g_r=E_\mu$, there exists a function

$$\frac{(g_r-E_1)\dots(g_r-E_\mu)(g_r,\ 1)_{a-2\mu}}{g_a-k}$$

of order $r(a-\mu)-a$, which is 'integral' and not integrally expressible, the function $(g_r,\ 1)_{a-2\mu}$ being determined to vanish at all the points for which $g_a=k$ other than the double points. The consideration of how we should proceed to obtain functions of other $\mu-1$ orders may be omitted. Especially as the orders of the existent functions do not necessarily determine the nature of the curve. For instance the function $g_{10} = g_{(a-2)\,r-a}$ above might arise as $\frac{(g_r - E_1)(g_r - E_2)}{g_r - k}$ where $g_r = k$ is a double tangent touching the curve at $g_r = E_1$ and $g_r = E_2$. In accordance with Kronecker's theory (Crelle, 91) there is no need in general to consider the normal curve to have higher singularities than double points. The examples here given should be compared with his theory.

PART V.

On the Graphical Meaning of Noether's (Cramer's) Resolution of the Multiple Singularity at the origin, by means of the Quadratic Transformation.

We use the same notation as in Part II. save that for σ_r', m_r', μ_r' we write σ_r, m_r, μ_r; l being the actual degree (= degree in x + degree in y) of the lowest terms in the equation of the curve. So that if the side of the polygon for which $\sigma = 1$ be present, l = distance from the origin of the point in which this side meets the axis of y. And if this side be not present, l = distance of P_0 from the origin. Then according to Noether the singularity is resoluble into a simple multiple point of order l + an additional number of multiple points which happen to be coincident with the multiple point of order l— and these latter in their turn are similarly resoluble. This result is arrived at by a particular case of a reversible quadratic transformation, as follows—

Substitute in the equation of the curve $x = \xi\eta$, $y = \xi\eta_1$, where η, η_1 are connected by a linear relation $p\eta + q\eta_1 = 1$. Then in the transformed curve we may either substitute for η_1 in terms of η and regard ξ, η as the new coordinates, or substitute for η in terms of η_1 and regard ξ, η_1 as the new coordinates. The inverse substitution is

$$\xi = px + qy, \quad \eta = \frac{x}{px + qy}, \quad \eta_1 = \frac{y}{px + qy},$$

so that to a point near the origin and on a branch $y \propto x^\sigma$ corresponds a point near the axis $\xi = 0$ for which

when $\sigma < 1$
$$\eta = \frac{x^{1-\sigma}}{px^{1-\sigma} + q}, \quad \eta_1 = \frac{1}{px^{1-\sigma} + q},$$

when $\sigma > 1$
$$\eta = \frac{1}{p + qx^{\sigma-1}}, \quad \eta_1 = \frac{x^{\sigma-1}}{p + qx^{\sigma-1}}.$$

For $\sigma < 1$ we shall regard ξ, η as the new variables, and for $\sigma > 1$ we shall regard ξ, η_1 as the new variables. Then the part of our singularity for which $\sigma < 1$ becomes a singularity at $\xi = 0$, $\eta = 0$, and the part of our singularity for which $\sigma > 1$ becomes a singularity at $\xi = 0$, $\eta_1 = 0$. The part for which $\sigma = 1$, say $y \propto kx$, becomes a singularity at $\xi = 0$, $\eta_1 = k\eta$. If then there be t branches for which $\sigma = 1$, we obtain $t + 2$ singularities corresponding to our original singularity. And since the transformation is reversible every point on these new singular branches corresponds to a point at the original singularity. Noether uses the substitution in one of the forms in which either p or q is zero, but when this is chosen to be effective for a branch for which $\sigma < 1$, it is ineffective for a branch for which $\sigma > 1$. In the form here no finite point of the original curve (except the points other than the origin upon the line $px + qy = 1$) becomes represented by an infinite point of the new curve. Also there is no multiple point on the new curve arising by transformation from a simple point of the original curve. For if

$$f(x, y) = f(\xi\eta, \xi\eta_1) = \xi^t F(\xi, \eta)$$

the equations

$$0 = \frac{\partial F}{\partial \xi} = \left(\eta \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y}\right) \xi^{-t},$$

$$0 = \frac{\partial F}{\partial \eta} = \left(\frac{\partial f}{\partial x} - \frac{p}{q}\frac{\partial f}{\partial y}\right) \xi^{1-t}$$

give

$$\frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} = 0.$$

We imagine now the polygon constructed for the new curves and each of the $t+2$ new singular points obtained. We proceed first to enquire what the values of the σ's will be at these new points. And, defining provisionally the word 'multiplicity,' applied to our original singularity, as the number of unit points within and upon the origin-polygon, save those upon the axes of coordinates, we shew that this is equal to

$$\frac{1}{2} l(l-1) + \text{the sum of the multiplicities arising from the } t+2 \text{ new points.}$$

The reapplication of this theorem to the new singularities obtained, and so on continually, enables us to give a geometrical meaning to the number which we call the multiplicity.

Consider then the effect of $x = \xi\eta$, $y = \xi\eta_1$, where η_1 is regarded as a linear function of $\eta\,(= a + b\eta)$, upon the branches at the original singularity for which $\sigma < 1$. The lowest terms in the new equation will be of the same dimensions as if we put $x = \xi\eta$, $y = \xi$. From a term $x^f y^g$ there arises a term $\xi^{f+g}\eta^f\eta_1^g$, so that the whole equation divides by ξ^l, and this term becomes effectively $\xi^{f+g-l}\eta^f$. For instance corresponding to the point P_0 in the diagram of the original curve, for which $f = 0$, we obtain in the new curve the term ξ^{g-l}, which gives on the representative chart a point lying on the axis of x. And corresponding to the points (f, g), (f', g') in the old diagram, wherein $f < f'$, and $g > g'$, we obtain in the new diagram the points $(f+g-l, f)$ and $(f'+g'-l, f')$, wherein $f'+g'-l < f+g-l$ and $f' > f$. And the σ' of the corresponding side in the new figure *reckoned away from the axis of* ξ is

$$\sigma' = \frac{f'-f}{(f+g)-(f'+g')} = \frac{m}{\mu - m},$$

where $\dfrac{m}{\mu} = \dfrac{f'-f}{g-g'}$ is the σ of the original figure.

If then in the new diagram all the points are marked corresponding to the points

$P_0, P_1, \dots P_{r-1}$ (where $\sigma_1, \sigma_2, \dots \sigma_{r-1}$ are each < 1, $\sigma_r = 1$, $\sigma_{r+1}, \sigma_{r+2}, \dots$ are all > 1),

the point corresponding to P_{r-1} for which the sum of the coordinates $= l$, that is to say in the notation above $f+g=l$, will be on the axis of η. We shall not mark in this diagram the points corresponding to $P_r, P_{r+1}, \dots$. We desire only to obtain the number of points within and upon the polygon $Q_0 \dots Q_{r-1}$ which corresponds to the part $P_0 \dots P_{r-1}$ of the old. Call this number A and notice that the greatest common measure, say n_t, of the quantities $f'-f$, $g-g'-(f'-f)$, is equal to the G.C.M. of $f-f'$ and $g'-g$. A is formed from the quantities n_t, m_t, $\mu_t - m_t$ in the same way as was our original number from the quantities n_t, m_t, μ_t—and in the new polygon t varies from 1 to $r-1$. Considering next the points of the transformed curve corresponding to the n_r branches for which $\sigma = 1$ on the original curve, the effect of our hypothesis, that in the corresponding n_r expansions of the form $y = Ax + \dots$ all the coefficients $A \dots$ are different, namely that the n_r branches have separated tangents, is that on the transformed curve we have n_r simple points lying on the axis $\xi = 0$, and the multiplicity of these is zero. With reference finally to the branches for which $\sigma > 1$ we imagine η expressed as a linear function of η_1, and regard ξ, η_1 as our new coordinates. So that so far as regards the lowest terms of the new equation, our substitution is equivalent to $x = \xi$, $y = \xi\eta_1$. The effect of this upon a term $x^f y^g$ is to transform it to $\xi^{f+g}\eta_1{}^g$, which after division of the equation by ξ^l becomes $\xi^{f+g-l}\eta_1{}^g$. So that for instance to the term $x^{x_r}y^{y_r}$ where $x_r + y_r = l$ corresponds the term $\xi^0\eta_1{}^{y_r}$. And to the terms $x^f y^g$, $x^{f'}y^{g'}$ correspond in the representative diagram of the new curve, the points $(f+g-l,\ g)$, $(f'+g'-l,\ g')$, giving

$$\sigma' = \frac{f'+g'-(f+g)}{g-g'} = \frac{f'-f-(g-g')}{g-g'} = \frac{m-\mu}{\mu},$$

where $\dfrac{m}{\mu} = \sigma = \dfrac{f'-f}{g-g'}$. We have to determine the multiplicity B given by the new polygon which is formed from the quantities n_t, $m_t - \mu_t$, μ_t, as was our number from the original polygon with the quantities n_r, m_r, μ_r, t having here the values

$$r+1, \dots\dots, k+1.$$

It may be noticed that the total number of sides other than the axes in the two polygons corresponding to the summations A and B is either equal to, or less by one than the number of sides other than the axes in the original polygon. With these explanations, and putting

$$C = \frac{1}{2}\Sigma nm(\Sigma n\mu - 1) + \frac{1}{2}\sum_{s>r} n_r n_s (m_r\mu_s - m_s\mu_r) - \frac{1}{2}\Sigma n(\mu - 1)$$

which, as is easily seen, is another way of writing the number previously obtained of the unit points within our original polygon and upon the sides other than upon the axes, and writing this in the abbreviated form

$$\frac{1}{2}\Sigma a\Sigma b - \frac{1}{2}\Sigma a - \frac{1}{2}\Sigma b + \frac{1}{2}\sum_{s>r}(a_r b_s - a_s b_r) + \frac{1}{2}\Sigma n,$$

where
$$a_r = n_r m_r,$$
$$b_r = n_r \mu_r,$$
the theorem is
$$C = \frac{1}{2} l (l-1) + A + B.$$

In the same way, making the assumption that in the multiple point at $\xi = 0 = \eta$, the branches which do not touch either $\xi = 0$ or $\eta = 0$ have all simple contact with their tangents, we can write
$$A = \frac{1}{2} m (m-1) + A' + B',$$
and similarly at $\xi = 0 = \eta_1$
$$B = \frac{1}{2} m' (m'-1) + A'' + B'',$$
and therefore
$$C = \frac{1}{2} l (l-1) + \frac{1}{2} m (m-1) + \frac{1}{2} m' (m'-1) + A' + B' + A'' + B''$$
and so on continually—and it is perfectly obvious geometrically that the polygons corresponding to $A'B'A''B''$ diminish indefinitely as their number increases, and eventually correspond to only simple points, in which case the corresponding multiplicities are zero. We thus resolve our compound singularity into a coincidence of simple singularities so far as the "multiplicity" is concerned, and are thus able to shew that this multiplicity is really to be interpreted as the contribution to $\delta + \kappa$ which is due to the singularity. It is immediately obvious that the κ of the singularity $= \Sigma_1 n (m-1) + \Sigma_2 n (\mu - 1)$ is the sum of the values of the κ due to the simple singularities into which it is so resolved. Thus we again prove Cayley's rules.

The proof of the equation stated is as follows—the work is quite similar to that of Cayley in the addition to Mr Rowe's memoir. Putting $a_r = n_r m_r$, $b_r = n_r \mu_r$, denoting the number of points on the side for which $\sigma = 1$ by $\nu + 1$, and the corresponding values of a_r, b_r by a_λ, b_λ (each of these being in fact $= \nu$), putting also $\overset{p}{\Sigma}_1$ to denote a summation extending from $r = p$ to $r = \lambda - 1$, and $\overset{p}{\Sigma}_2$ to denote a summation extending from $r = p$ to $r = k + 1$, it being understood that when the p is absent the summation Σ_1 begins with $r = 1$ and the summation Σ_2 begins with $r = \lambda + 1$, we have
$$\Sigma a = \Sigma_1 a + \Sigma_2 a + \nu = \Sigma_1 a + \Sigma_2 (a - b) + \Sigma_2 b + \nu$$
$$\Sigma b = \Sigma_1 b + \Sigma_2 b + \nu = \Sigma_1 (b - a) + \Sigma_2 b + \Sigma_1 a + \nu$$
$$\Sigma n = \Sigma_1 n + \Sigma_2 n + \nu$$
$$2C = \Sigma a \Sigma b - \Sigma a - \Sigma b + \Sigma n + \sum_{s>r} (a_r b_s - a_s b_r)$$
$$2A = \Sigma_1 a \Sigma_1 (b - a) - \Sigma_1 a - \Sigma_1 (b - a) + \Sigma_1 n + \sum_{1\, s>r} (a_r b_s - a_s b_r)$$
$$2B = \Sigma_2 (a - b) \Sigma_2 b - \Sigma_2 (a - b) - \Sigma_2 b + \Sigma_2 n + \sum_{2\, s>r} (a_r b_s - a_s b_r).$$

For the values of σ corresponding in these two cases are

$$\frac{m}{\mu - m}, \quad \frac{m-\mu}{\mu},$$

the former being reckoned in a particular way.

And $$\Sigma a \Sigma b - \Sigma a - \Sigma b + \Sigma n - [\Sigma_1 a \Sigma_1 (b-a) - \Sigma_1 a - \Sigma_1 (b-a) + \Sigma_1 n]$$
$$- [\Sigma_2 (a-b) \Sigma_2 b - \Sigma_2 (a-b) - \Sigma_2 b + \Sigma_2 n]$$

$$= \nu^2 + \nu (\Sigma_1 a + \Sigma_1 b + \Sigma_2 a + \Sigma_2 b) + \Sigma_1 a \Sigma_1 b + \Sigma_2 a \Sigma_2 b + \Sigma_1 a \Sigma_2 b + \Sigma_1 b \Sigma_2 a - \Sigma a - \Sigma b$$
$$+ \nu - \Sigma_1 a \Sigma_1 b + (\Sigma_1 a)^2 + \Sigma_1 b - \Sigma_2 a \Sigma_2 b + (\Sigma_2 b)^2 + \Sigma_2 a$$

(i) $$= \nu^2 + \nu (\Sigma_1 a + \Sigma_1 b + \Sigma_2 a + \Sigma_2 b - 1) + (\Sigma_1 a)^2 + (\Sigma_2 b)^2 + \Sigma_1 a \Sigma_2 b + \Sigma_1 b \Sigma_2 a - \Sigma_1 a - \Sigma_2 b,$$

while

$$\sum_{s>r} a_r b_s = a_1 (\overset{2}{\Sigma_1} b + \nu + \Sigma_2 b) + a_2 (\overset{3}{\Sigma_1} b + \nu + \Sigma_2 b) + \ldots + a_{\lambda-1} (\nu + \Sigma_2 b) + a_\lambda \Sigma_2 b + \sum_{2\, s>r} a_r b_s$$
$$= \sum_{1\, s>r} a_r b_s + \nu (\Sigma_1 a + \Sigma_2 b) + \Sigma_1 a \Sigma_2 b + \sum_{2\, s>r} a_r b_s;$$

(ii) $$\therefore \sum_{s>r} (a_r b_s - b_r a_s) - \sum_{1\, s>r} (a_r b_s - b_r a_s) - \sum_{2\, s>r} (a_r b_s - b_r a_s)$$
$$= \nu (\Sigma_1 a + \Sigma_2 b - \Sigma_1 b - \Sigma_2 a) + \Sigma_1 a \Sigma_2 b - \Sigma_1 b \Sigma_2 a.$$

Adding this to the expression above we obtain

$$\nu^2 + 2\nu (\Sigma_1 a + \Sigma_2 b) - \nu + (\Sigma_1 a)^2 + (\Sigma_2 b)^2 + 2\Sigma_1 a \Sigma_2 b - \Sigma_1 a - \Sigma_2 b$$

and $$l = \Sigma_1 a + \Sigma_2 b + \nu;$$

$\therefore$ this is $$l^2 - l;$$

$$\therefore\ C = \frac{1}{2} l (l-1) + A + B.$$

It would, I imagine, be easy to give a similar interpretation of Noether's work for the case in which the ν roots of the equation corresponding to the line for which $\sigma = 1$ are not all different—for instance, to investigate the branches that correspond to a repeated factor $y - kx$ we must put $y - kx = \xi\eta_2$ and $x = \xi\eta$ where η_2 is a linear function of η.

As an example of this method we proceed to determine the $\delta + \kappa$ of the singularity at the origin for the curve

$$y^{15} + y^{11} (y, x)^5 + y^7 (y, x)^{10} + y^4 (y, x)^{14} + y^2 (y, x)^{17} + y^2 (y, x)^{18} + y (y, x)^{20} + (y, x)^{22} = 0.$$

If the polygon be drawn, the angular points nearest the origin are

(0, 15), (5, 11), (10, 7), (14, 4), (17, 2), (22, 0)

of which the first three are upon one straight line. The number of points between the sides given by these points and the axes, with those upon the sides that are not upon the axes, is 130—so that

$$\delta + \kappa = 130.$$

Also $\sigma_1 = \frac{5}{4},\ n_1 = 2;\ \sigma_2 = \frac{4}{3},\ n_2 = 1;\ \sigma_3 = \frac{3}{2},\ n_3 = 1;\ \sigma_4 = \frac{5}{2},\ n_4 = 1;$

so that $$\kappa = \Sigma n(\mu - 1) = 10.$$

We proceed to prove that this is in accordance with the results given by Noether's method.

I. All the σ's being greater than unity, we put as explained

$$x = x_1$$
$$y = x_1 y_1$$

and obtain after division by x_1^{15}

$$y_1^{15} + x_1 y_1^{11}(y,\ 1)^5 + x_1^2 y_1^7 (y_1,\ 1)^{10} + x_1^3 y_1^4 (y_1,\ 1)^{14}$$
$$+ x_1^4 y_1^2 (y_1,\ 1)^{17} + x_1^5 y_1^2 (y_1,\ 1)^{18} + x_1^6 y_1 (y_1,\ 1)^{20} + x_1^7 (y_1,\ 1)^{22} = 0,$$

wherein $$\sigma_1 = \frac{1}{4},\ \sigma_2 = \frac{1}{3},\ \sigma_3 = \frac{1}{2},\ \sigma_4 = \frac{3}{2}.$$

II. Putting now $x_1 = \xi\eta$, $y_1 = \xi(E + \eta)$, we shall have three branches at $\xi = \eta = 0$, and one branch at $\xi = 0 = E + \eta$. At this latter point σ will be $\frac{1}{2}$, viz. $E + \eta \propto \xi^{\frac{1}{2}}$, that is, we have an ordinary contact with $\xi = 0$, and the "multiplicity" as defined will be 0. Considering then only $\xi = 0 = \eta$ and putting v for $E + \eta$, we obtain, after division by ξ^6,

$$\xi^9 v^{15} + \xi^6 v^{11}\eta(1,\ \xi v)^5 + \xi^3\eta^2 v^7 (1,\ \xi v)^{10} + \xi\eta^3 v^4 (1,\ \xi v)^{14} + \eta^4 v^2 (1,\ \xi v)^{17}$$
$$+ \xi\eta^5 v^2 (1,\ \xi v)^{18} + \xi\eta^6 v (1,\ \xi v)^{20} + \xi\eta^7 (1,\ \xi v)^{22} = 0,$$

and the values of σ are $\sigma_1 = 3$, $\sigma_2 = 2$, $\sigma_3 = 1$ (reckoned from the axis of ξ).

III. Putting now $\xi = \xi_1\eta_1$, $\eta = \xi_1\eta_1'$,

we shall have a simple point corresponding to $\xi = 0 = E + \eta$ and $E + \eta \propto \xi^{\frac{1}{2}}$,

two branches at $\xi_1 = 0 = \eta_1'$

one branch at $\xi_1 = 0 = \eta_1' - k\eta$ corresponding to the terms

$$\eta^3 v^2 [\xi v^2 (1,\ \xi v)^{14} + \eta(1,\ \xi v)^{17}].$$

I assume that this is a simple tangent to $\eta_1' - k\eta$ and put in consequence, simply

$$\xi = \xi_1(E_1 + \eta_1),\quad \eta = \xi_1\eta_1.$$

Then at these two branches at $\xi_1 = 0 = \eta_1$ (reckoning σ from the axis of ξ_1, as in II.)

$$\sigma_1 = 2,\quad \sigma_2 = 1,$$

and putting v_1 for $E_1 + \eta_1$ we obtain, after division by η_1^4,

$$\xi_1^5 v_1^9 v^{15} + \xi_1^3\eta_1 v_1^6 v^{11}(1,\quad)^5 + \xi_1\eta_1^2 v_1^3(\quad)^{10} + \eta_1^3 v_1(\quad)^{14}$$
$$+ \eta_1^4(\quad)^{17} + \xi_1^2\eta_1^5(\quad)^{18} + \xi_1^3\eta_1^6(\quad)^{20} + \xi_1^4\eta_1^7(\quad)^{22} = 0.$$

IV. Putting now

$$\xi_1 = \xi_2 (E_2 + \eta_2), \quad \eta_1 = \xi_2\eta_2$$

we obtain after division by ξ_2^3 a curve having a double point at $\xi_2 = 0 = \eta_2$.

V. And thence putting $\xi_2 = \xi_3 (E_3 + \eta_3)$, $\eta_2 = \xi_3\eta_3$ we obtain after division by ξ_3^2 two simple points on $\xi_3 = 0$.

Reckoning now the $\delta + \kappa$ as indicated in the general theory given, by the indices of the factors that have divided out, we obtain

$$\delta + \kappa = \frac{1}{2} [15 (15 - 1) + 6 . 5 + 4 . 3 + 3 . 2 + 2 . 1]$$
$$= 130, \text{ as before.}$$

The transformations are

$$x = x_1 \;, \quad x_1 = \xi\eta \quad, \quad \xi = \xi_1 (\eta_1 + E_1), \quad \xi_1 = \xi_2 (E_2 + \eta_2), \quad \xi_2 = \xi_3 (E_3 + \eta_3)$$
$$y = x_1 y_1, \quad y_1 = \xi (E + \eta), \quad \eta = \xi_1\eta_1 \quad, \quad \eta_1 = \xi_2\eta_2 \quad, \quad \eta_2 = \xi_3\eta_3.$$

PART VI.

On a particular monomial transformation.

We give now an identity which is useful in a particular kind of transformation—It will be seen that it leads to a resolution of the same kind as Noether's.

Let
$$K_1 + \frac{1}{K_2 +}\frac{1}{K_3 +} \dots \frac{1}{+ K'' +}\frac{1}{K' +}\frac{1}{K +} \dots \frac{1}{+ K_{2m} +}\frac{1}{K_{2m+1}}$$

be any continued fraction, and let the convergents corresponding to the elements K'', K', K be

$$\frac{p''}{q''}, \; \frac{p'}{q'}, \; \frac{p}{q}.$$

Then if A, B be any quantities

$$(qA + pB - 1)(q'A + p'B - 1) - (q'A + p'B - 1)(q''A + p''B - 1) = (q'A + p'B - 1)(Kq'A + Kp'b)$$
$$= K [(q'A + p'B)^2 - (q'A + p'B)]$$

or, if $k = qA + pB$, etc.

(I) $$(k - 1)(k' - 1) - (k' - 1)(k'' - 1) = K (k'^2 - k') = Kk' (k' - 1).$$

Take now A, B, so that $\qquad A = P'\mathrm{a} + P\mathrm{b}, \; B = Q\mathrm{b} - Q'\mathrm{a},$

where $\frac{P'}{Q'}, \frac{P}{Q}$ are the two actual last convergents of our continued fraction, so that

$$\mathrm{a} = QA + PB, \quad \mathrm{b} = QA + P'B,$$

so that a is the last, b is the last but one of the quantities k'', k', k ... and notice that if our fraction begin with

$$K_1 + \frac{1}{K_2 +} \frac{1}{K_3},$$

so that $$k_1 = A + K_1 B, \quad k_2 = K_2 A + (K_1 K_2 + 1) B$$

and we put $$k_0 = B,$$

then $$(k_2 - 1)(k_1 - 1) - (k_1 - 1)(B - 1) = (k_1 - 1) K_2 (A + K_1 B)$$
$$= K_2 k_1 (k_1 - 1)$$

and $$(k_1 - 1)(B - 1) - (B - 1)(A - 1) = (B - 1) K_1 B$$
$$= K_1 B (B - 1)$$
$$= K_1 k_0 (k_0 - 1).$$

Therefore adding all the equations of the form (I.) and using these initial forms of that equation we have

$$(\mathrm{a} - 1)(\mathrm{b} - 1) - (A - 1)(B - 1) = K_1 k_0 (k_0 - 1) + K_2 k_1 (k_1 - 1) + \ldots + K_{2m+1} k_{2m} (k_{2m} - 1),$$

where in fact $$\mathrm{b} = k_{2m}.$$

If now $$\mathrm{a} = \Sigma a_r = \Sigma n_r m_r,$$

$$\mathrm{b} = \Sigma b_r = \Sigma n_r \mu_r$$

and we put $$a_r = Q a_r' + P b_r', \quad b_r = Q' a_r' + P' b_r'$$

leading to $$a_r' = P a_r' - P b_r, \quad b_r' = Q b_r - Q a_r'$$

and $$A = \Sigma a_r', \quad B = \Sigma b_r',$$

and $$\sum_{s>r} (a_r b_s - a_s b_r) = \sum_{s>r} (a_r' b_s' - a_s' b_r'),$$

we obtain the identity in question (wherein $n_r = n_r'$, since clearly any divisor common to a_r, b_r is common to a_r', b_r'; and conversely)

$$\Sigma nm (\Sigma n\mu - 1) + \sum_{s>r} n_r n_s (m_r \mu_s - m_s \mu_r) - \Sigma n (\mu - 1)$$
$$- [\Sigma n'm' (\Sigma n'\mu' - 1) + \sum_{s>r} n_r' n_s' (m_r' \mu_s' - m_s' \mu_r') - \Sigma n (\mu' - 1)]$$
$$= K_1 k_0 (k_0 - 1) + K_2 k_1 (k_1 - 1) + \ldots\ldots + K_{2m+1} k_{2m} (k_{2m} - 1)$$

where, as may be recalled,

$$\frac{P}{Q} = K_1 + \frac{1}{K_2} + \ldots\ldots + \frac{1}{K_{2m+1}}$$

$$\frac{P'}{Q'} = K_1 + \frac{1}{K_2} + \ldots\ldots + \frac{1}{K_{2m}}$$

$$\frac{p_r}{q_r} = K_1 + \frac{1}{K_2} + \ldots\ldots + \frac{1}{K_r}$$

$$k_0 = \Sigma b', \quad k_r = q_r \Sigma a' + p_r \Sigma b' \qquad , \quad k_{2m} = \Sigma b$$
$$= q_r (P' \Sigma a - P \Sigma b) + p_r (-Q' \Sigma a + Q \Sigma b)$$
$$= (q_r P' - p_r Q') \Sigma a - (q_r P - p_r Q) \Sigma b.$$

If now we make the substitution

$$x = \xi^{P'} \eta^{Q'}, \quad y = \xi^P \eta^Q$$

equivalent to

$$\xi = x^{+Q} y^{-Q'}, \quad \eta = x^{-P} y^{+P'},$$

since
$$P'Q - PQ' = 1,$$

this being the result of a combination of such substitutions as

$$x = \xi\eta, \quad y = \eta, \quad x = \xi, \quad y = \eta\xi,$$

the terms $x^f y^g$, $x^{f'} y^{g'}$ of our original equation become

$$\xi^{fP' + gP} \eta^{fQ' + gQ}, \quad \xi^{f'P' + g'P} \eta^{f'Q' + g'Q}$$

and corresponding to

$$m = f' - f, \quad \mu = g - g', \quad \sigma = \frac{m}{\mu}$$

we have
$$m' = f'P' + g'P - (fP' + gP) = mP' - \mu P,$$
$$\mu' = fQ' + gQ - (f'Q' + g'Q) = \mu Q - mQ',$$

and thus
$$m = Qm' + P\mu', \qquad \mu = Q'm' + P'm',$$

which are in accordance with the equations of the previous page,

and
$$\sigma' = \frac{m'}{\mu'} = \frac{\sigma P' - P}{Q - \sigma Q'}, \text{ is positive if } \frac{Q}{Q'} > \sigma > \frac{P}{P'}$$

$$\text{and is negative if } \frac{Q}{Q'} > \sigma > \frac{P}{P'},$$

while
$$\sigma_2' - \sigma_1' = \frac{\sigma_2 - \sigma_1}{(Q - \sigma_1 Q')(Q - \sigma_2 Q')},$$

so that, if $\sigma_1 < \sigma_2$, then $\sigma_1' < \sigma_2'$ if σ_1, σ_2 are (both greater or) both less than $\frac{Q}{Q'}$.

Also
$$\frac{P}{P'} = K_{2m+1} + \frac{1}{K_{2m} +} \cdots\cdots + \frac{1}{K_2 +} \frac{1}{K_1}$$
$$\frac{Q}{Q'} = K_{2m+1} + \frac{1}{K_{2m} +} \cdots\cdots + \frac{1}{K_2}.$$

Noticing now that $\xi = x^Q y^{-Q'}$, $\eta = x^{-P} y^{P'}$ give when $y \propto x^\sigma$, $\xi \propto x^{Q - \sigma Q'}$, $\eta \propto x^{\sigma P' - P}$

we see that the points of our branch $y \propto x^\sigma$ that are near the origin will not be projected to infinity provided

$$\frac{Q}{Q'} \not< \sigma \not< \frac{P}{P'}.$$

If *now for instance there be only one* σ $\left(=\frac{nm}{n\mu}\right)$ *for the singularity at the origin* and we put $\frac{m}{\mu}$ into a continued fraction

$$=K_{2m+1}+\frac{1}{K_{2m}}+\ldots\ldots+\frac{1}{K_1}$$

$$=\frac{P}{P'},$$

then the transformed value m' is equal to 0, and corresponding to the $n\mu$ branches

$$y=A_1x^\sigma,\quad y=A_1\omega x^\sigma,\ \ldots\ldots\ y=A_1\omega^{n-1}x^\sigma$$

$$y=A_2x^\sigma,\quad y=A_2\omega x^\sigma,\ \ldots\ldots\ y=A_2\omega^{n-1}x^\sigma$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y=A_nx^\sigma,\quad y=A_n\omega x^\sigma,\ \ldots\ldots\ y=A_n\omega^{n-1}x^\sigma$$

where

$$\omega=e^{\frac{2\pi i}{\mu}},$$

we have points where

$$\eta=A_1+\ldots,\quad \eta=A_1\omega+\ldots,\quad \ldots\ldots\ \eta=A_1\omega^{n-1}+\ldots$$

$$\eta=A_2+\ldots,\quad \eta=A_2\omega+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

that is, $n\mu$ branches cutting the axis $\xi=0$ at, in general, different points. (When these points are not different the transformation can be reapplied.) And the transformed value of our expression

$$\Sigma nmn\mu+\ldots\ldots$$

is 0, and the original singularity consists of

K_1 k_0-ple points

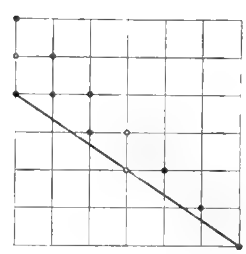
with

K_2 k_1-ple points

etc.

For instance Noether's example (*Math. Annal.* IX. p. 174)

$$y^4+y^2(x,\ y)^3+(x,\ y)^6+\ldots\ldots$$



Here

$$\sigma=\frac{3}{2}=1+\frac{1}{1}\frac{1}{+1}=\frac{P}{P'},$$

$$\frac{Q}{Q'}=1+\frac{1}{1}=\frac{2}{1},$$

$$\frac{P}{Q}=\frac{3}{2}=1+\frac{1}{1}\frac{1}{+1},$$

$$\frac{p_1}{q_1}=\frac{1}{1},\quad \frac{p_2}{q_2}=\frac{2}{1},\quad \frac{p_3}{q_3}=\frac{3}{2}.$$

$$\Sigma a' = P'\Sigma a - P\Sigma b, \quad \Sigma b' = -Q'\Sigma a + Q\Sigma b$$
$$= 6P' - 4P \qquad\qquad = -6Q' + 4Q$$
$$= 0 \qquad\qquad = 8 - 6$$
$$= 2;$$
$$\therefore\ k_0 = 2, \quad k_1 = 2, \quad k_2 = 4,$$

namely, our singularity is resolvable into two double points and one quadruple point—(which gives 8 as the contribution to $\delta + \kappa$; as is obvious from the figure).

Corollary. An Application of the preceding transformation.

If
$$\sigma_1 < \sigma_2 < \ldots\ldots < \sigma_{k+1}$$
be the values of σ for a multiple point at the origin, and we make the transformation
$$x = \xi^{P'}\eta^{Q'}, \quad y = \xi^P\eta^Q$$
taking care only to choose
$$\frac{P}{P'} > \sigma_{k+1},$$
and therefore
$$\frac{Q}{Q'} > \sigma_{k+1}$$
the branch $y \propto x^\sigma$, leading to $\xi \propto x^{Q-\sigma Q'}$, $\eta \propto x^{P'-P}$ becomes always represented by a point at infinity on the axis of η, for all the values $\sigma = \sigma_1, \sigma_2, \ldots \sigma_{k+1}$. Namely on the new curve the singularity corresponding to the singularity at the origin on the old curve is entirely at $\xi = 0$, $\eta = \infty$. If the old curve be
$$f(x, y) = f(\xi^{P'}\eta^{Q'}, \ \xi^P\eta^Q) = \xi^\lambda\eta^\mu F(\xi, \eta) \text{ say,}$$
where $F(\xi, \eta)$ is the new curve, the conditions for a singularity on the new curve, viz.
$$0 = \frac{\partial F}{\partial \xi} = \xi^{-\lambda}\eta^{-\mu}\left[P'\frac{\partial f}{\partial x}\xi^{P'-1}\eta^{Q'} + P\frac{\partial f}{\partial y}\xi^{P-1}\eta^Q\right]$$
$$0 = \frac{\partial F}{\partial \eta} = \xi^{-\lambda}\eta^{-\mu}\left[Q'\frac{\partial f}{\partial x}\xi^{P'}\eta^{Q'-1} + Q\frac{\partial f}{\partial y}\xi^P\eta^{Q-1}\right]$$
give
$$(P'Q - PQ')\frac{\partial f}{\partial x}\xi^{P'}\eta^{Q'}\,.\,\xi^{-\lambda}\eta^{-\mu} = 0$$
$$(P'Q - PQ')\frac{\partial f}{\partial y}\xi^P\eta^Q\,.\,\xi^{-\lambda}\eta^{-\mu} = 0,$$
and can only be satisfied, unless $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, and excluding infinite values of ξ and η for the present, by $\xi = 0$ or $\eta = 0$ or both, namely at points arising from $x = 0 = y$—at which both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are by hypothesis zero. So that the new curve has no finite singularity that does not arise from a singularity on the old curve. The infinite values of ξ and η that are possible, can, since
$$x = \xi^{P'}\eta^{Q'}, \quad y = \xi^P\eta^Q,$$
only have arisen from points $x = \infty$, $y = \infty$. Now suppose that after the transformation

above has been applied to the singularity of the old curve at the origin, we transform the axes of ξ, η by writing $\xi=\xi_1+A$, $\eta=\eta_1+B$, to a point (A, B) which is a singular point of the curve $F(\xi, \eta)$. By a similar transformation to that just applied, viz. writing

$$\xi_1 = X^{M'}Y^{N'} \quad \eta_1 = X^{M}Y^{N}$$
$$X = \xi_1^{N}\eta_1^{-N'} \quad Y = \xi_1^{-M}\eta^{M'}$$

we can transform this singularity to be at $X=0$, $Y=\infty$. The singularity of

$$F(\xi, \eta)=0$$

which is at $$\xi=0, \quad \eta=\infty,$$

that is, also, at $\xi_1=0$, $\eta_1=\infty$, changes to $X=0$, $Y=\infty$—viz. our new curve in X and Y has the singularities corresponding to the two already considered, both at $X=0$, $Y=\infty$. Let this process of changing axes and subsequent transformation be continued.—Hence* *we at length obtain a curve whose only singular points are on the line infinity*—there being a very complex singularity at the infinite end of the axis of zero abscissae and, beside, possible singularities at other points of the line infinity which have persisted throughout. For instance, Raffy's example previously discussed,

$$x^5 - 5x^3(y^2+y+1) + 5x(y^2+y+1)^2 - 2y(y^2+y+1)^2 = 0$$

becomes by $$x=\xi\eta, \quad y=\omega+\xi^3\eta^4$$

$$1 - 5\xi\eta^2(\xi^3\eta^4+c) + 5\xi^2\eta^4(\xi^3\eta^4+c)^2 - 2\xi\eta^3(\omega+\xi^3\eta^4)(c+\xi^3\eta^4)^2 = 0$$

where $$c=\omega-\omega^2.$$

All the singularity of this curve is on the line infinity of the ξ, η plane.

Note. We may put further $\eta=\dfrac{1}{x}$, $\xi=\dfrac{y}{x}$;

and hence obtain

(i) $$x^{25} - 5x^{15}y(y^3+cx^7) + 5x^5y^2(y^3+cx^7)^2 - 2y(\omega x^7+y^3)(cx^7+y^3)^2 = 0$$

which we may treat by the rules of Part III. Putting

$$\xi = x^{\frac{1}{3}}, \quad y = v\xi^7$$

we obtain $$v(c+v^3)^2[2(\omega+v^3) - 5v\xi] + 5v\xi^3(c+v^3) - \xi^5 = 0$$

and here, for the branch $v=\sqrt[3]{-c}+\ldots$, we are to count $t=5$ (see page 419, note) while $N=2$; the correction is therefore 2; the diagram for the curve (i) above gives 102 as the number $\delta+\kappa$ for the singularity at the origin, with 4 interior points. Hence, admitting the correction, we see that, for the origin, $\delta+\kappa=104$ and the deficiency is 2, as previously obtained. The value 104 for $\delta+\kappa$ can be verified by expansions. The curve (i) gives six expansions of the form

$$y = -\vartheta c^{\frac{1}{3}}x^{\frac{7}{3}} \pm \frac{1}{\sqrt{2}}\omega c^{-\frac{1}{6}}x^{\frac{7}{3}+\frac{5}{6}}\ldots\ldots$$

* If, in such a curve, y be an integral function of x, all integral functions are expressible integrally.

where $$\vartheta^3 = 1,$$

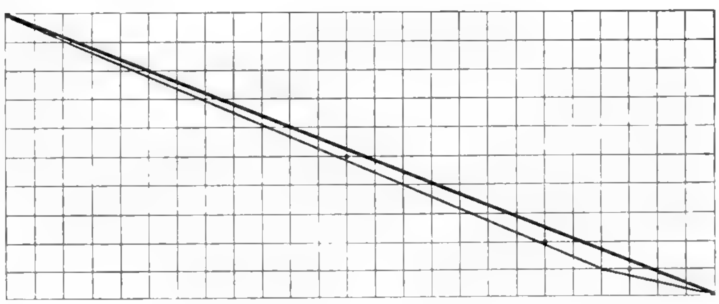
beside three expansions of y in powers of $x^{\frac{1}{3}}$ with different initial coefficients, each series beginning with the term $x^{\frac{7}{3}}$, and one series for y in integral powers of x, beginning with x^4. Hence by Cayley's rules the total number of intersections is

$$\delta + \frac{3}{2}\kappa = 7 + 7 + 3\left(\frac{7}{3} + \frac{5}{6}\right) + 3\left(\frac{7}{3} + \frac{7}{3}\right)$$
$$+ 7$$
$$+ 18\left(\frac{7}{3}\right)$$
$$+ 9\left(\frac{7}{3}\right)$$
$$= 104 + \frac{5}{2} + 1.$$

The first six expansions give $\kappa = 5$, and the second three expansions give $\kappa = 2$.

$$\therefore\ \frac{1}{2}\kappa = \frac{7}{2};$$
$$\therefore\ \delta + \kappa = 104.$$

The figure for the curve (i) is



Notwithstanding the crucial nature of this example and that at the end of Part V. as tests of the method of this paper, the change of the origin of coordinates used in this Corollary may quite well render the coefficients in the resulting equation so mutually dependent that the method of counting the deficiency by the number of interior points of the curve polygon becomes inoperative. For instance the deficiency of

$$(y-a)(y-b) + cxy^2 + dx^2y^3 + fxy + gx^2y^2 + hx^3y^3 + kx^4y^4 = 0$$

is quite properly given by the diagram as 1. But by putting $y - a = \eta$ we obtain a curve having eighteen terms, among the coefficients of which there are nine quadratic relations; and the polygon of this latter contains seven unit points.

Re p. 427. Cf. Noether, Crelle, 97, p. 224. Also a paper by Hensel, Crelle, 109—which I had not seen when this paper was written. His results are not universally true. But they enable us to write down the integral functions when, by some such method as here, we can write down the finite integrals. Or conversely.

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Prof. Cayley on Three-bar Motion.

Fig 1 $c=a-b-d$

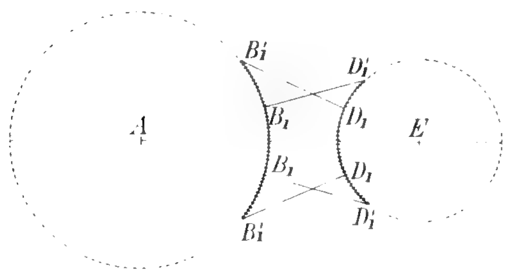

Fig 1 2

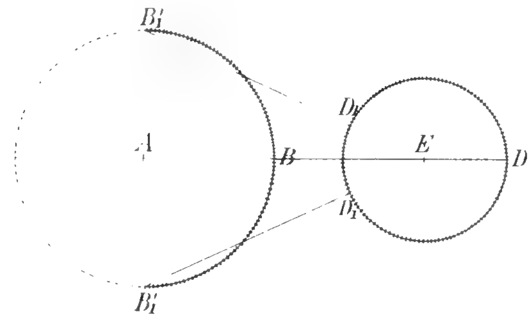

Fig 2 $c=a-b+d$

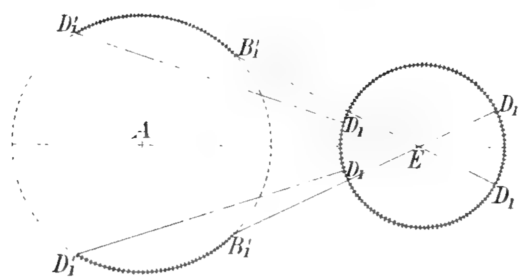

Fig 2-3

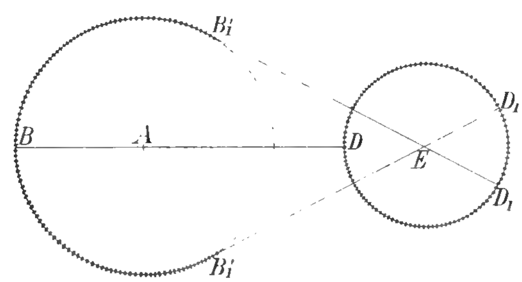

Fig 3 $c=a+b-d$.

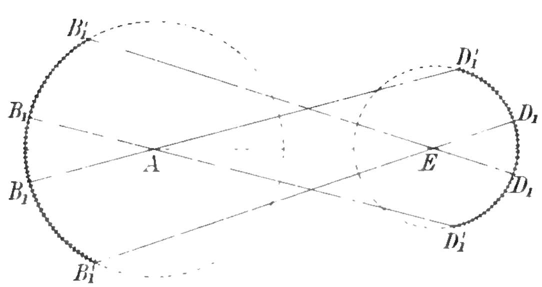

Fig 4. $c=a+b+d$.

Fig 3-4

Exterior Series.

Prof. Cayley on Three-bar Motion

Fig 1

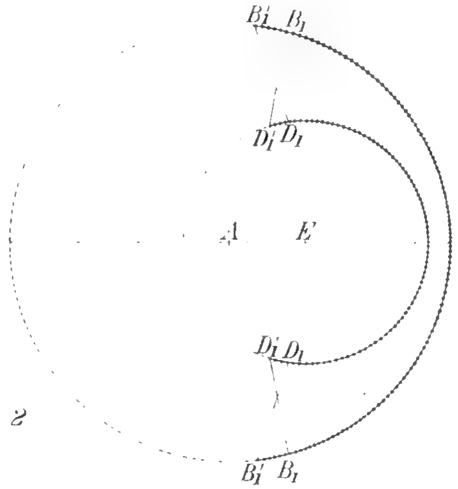

Fig 1 2

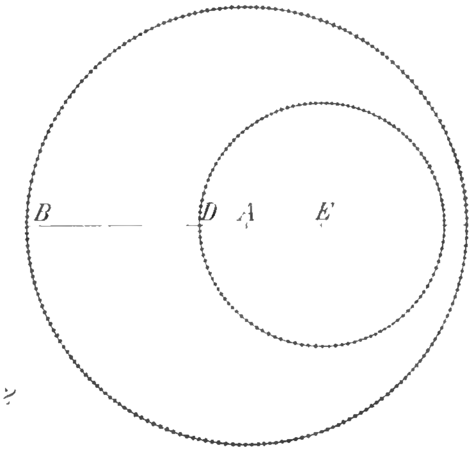

Fig 2

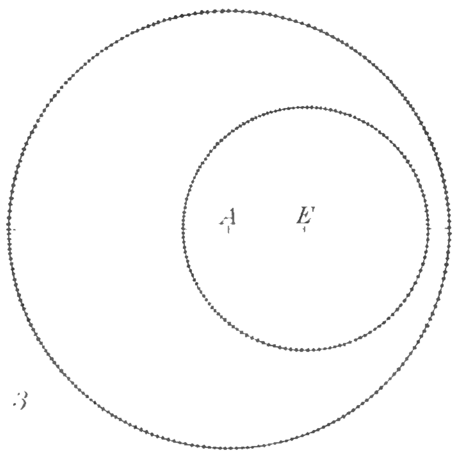

Fig 2 3

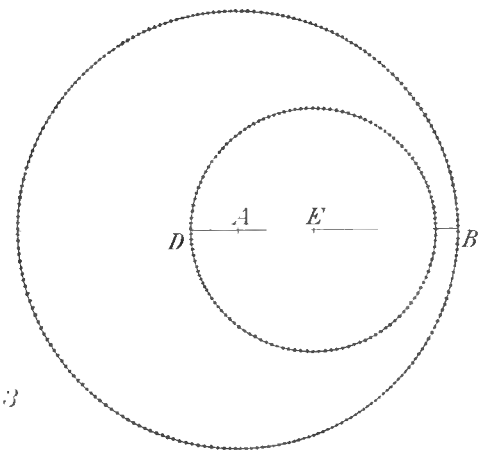

Fig 3

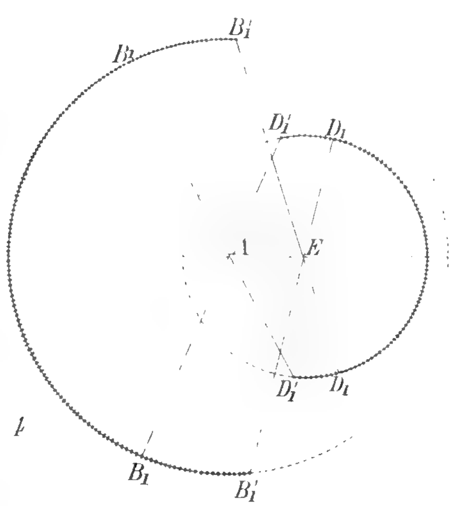

Fig 3 4

Fig 4

Interior Series.