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CONTENTS.

I. *An Algebraically complete system of Quaternariants.* By DAVID B. MAIR, B.A., Fellow of Christ's College.

[Read 26 February, 1894.]

In the *Transactions of the Cambridge Philosophical Society*, Vol. XIV. Part IV. Dr Forsyth discusses the differential equations satisfied by the concomitants of quaternary forms. As point-, plane-, and line-variables are taken

$$x_1,\ x_2,\ x_3,\ x_4,$$
$$u_1,\ u_2,\ u_3,\ u_4,$$
$$p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6;$$

the line-variables being expressible in terms of two sets v, w of plane-variables in the form

$$p_1 = v_1w_4 - v_4w_1 \qquad p_6 = v_2w_3 - v_3w_2$$
$$p_2 = v_2w_4 - v_4w_2 \qquad p_5 = v_3w_1 - v_1w_3$$
$$p_3 = v_3w_4 - v_4w_3 \qquad p_4 = v_1w_2 - v_2w_1.$$

The leading coefficient of a quaternariant, i.e. the coefficient of the term containing only x_1, p_1, and u_1, satisfies four differential equations, which Dr Forsyth writes

$$N_2 = 0, \quad M_3 = 0, \quad N_4 = 0, \quad M_4 = 0.$$

Of these either of the last two may be omitted as it is satisfied in virtue of the remaining three: and any solution of these equations, integral in the coefficients of the quantic, is the leading coefficient of a concomitant, determined except for an additive multiple of

$$p_1p_6 + p_2p_5 + p_3p_4.$$

1. The present object is to derive from the differential equations a complete system of concomitants for any quantic, the number in a complete system being less by 5 than the number of coefficients of the quantic. For this purpose it is convenient to use N_4 rather than M_4, and to introduce a new differential equation $\Phi = 0$ which is satisfied in virtue of the original equations. Solutions ϕ of $N_2 = 0$ are first found, next functions ψ of the quantities ϕ which satisfy $M_3 = 0$, then functions χ of the quantities ψ which satisfy $\Phi = 0$, and finally functions ω of the quantities χ which satisfy $N_4 = 0$ and are therefore leading coefficients. The method is first applied to the cubic.

The cubic being

$$\begin{array}{l|l}
\quad a_0x_1^3 \;+ 3a_1x_1^2x_3 + 3a_2x_1x_3^2 + a_3x_3^3 \;+ 3x_4 & \quad a_0'x_1^2 + 2a_1'x_1x_3 + a_2'x_3^2 \\
+ 3b_0x_1^2x_2 + 6b_1x_1x_3x_2 + 3b_2x_3^2x_2 & + 2b_0'x_1x_2 + 2b_1'x_3x_2 \\
+ 3c_0x_1x_2^2 + 3c_1x_3x_2^2 & + \;\; c_0'x_2^2 \\
+ \;\; d_0x_2^2 &
\end{array}$$

$$+ 3x_4^2 \qquad \begin{array}{l} a_0''x_1 + a_1''x_3 \\ + b_0''x_2 \end{array} \; + a_0'''x_4^3$$

we take as first three solutions of $N_2 = 0$

$$\theta_0 = a_0, \quad \theta_1 = a_0', \quad \theta_2 = a_1.$$

Also $$N_2 b_0 = a_1;$$

this quantity b_0 is taken as variable of reference.

As third solution we take $$\theta_3 = a_2.$$

For the fourth, since $N_2 b_1 = a_2$,

$$b_1 N_2 b_0 - b_0 N_2 b_1 = (b_1,\ a_2 \between a_1,\ -b_0) = \theta_4 \text{ say;}$$

now $N_2\theta_4 = 0$, and we take as fourth solution

$$\theta_4 = (b_1,\ a_2 \between a_1,\ -b_0).$$

Again $N_2 c_0 = 2b_1$, and therefore

$$2c_0 N_2 b_0 - b_0 N_2 c_0 = 2\,(c_0,\ b_1 \between a_1,\ -b_0);$$

calling this expression $2\theta_5'$, we have

$$\theta_5' N_2 b_0 - b_0 N_2 \theta_5' = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2;$$

as this last satisfies $N_2 = 0$, we take as fifth solution

$$\theta_5 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2.$$

As sixth solution we take $$\theta_6 = a_1'.$$

And since $N_2 b_0' = a_1'$, we form the expression

$$b_0' N_2 b_0 - b_0 N_2 b_0' = (b_0',\ a_1' \between a_1,\ -b_0),$$

which gives the seventh solution

$$\theta_7 = (b_0',\ a_1' \between a_1,\ -b_0).$$

In the same way we derive four solutions from a_3, b_2, c_1, d_0, three from a_2', b_1', c_0', two from a_1'', b_0'', and have besides the solutions a_0'' and a_0'''.

The solutions θ are not in a convenient form. If each be multiplied by an appropriate power of θ_2, there results the set of 19 solutions ϕ, given in Table I, such that $M_3\phi_r$ (for all values of r except 2) is a solution of $N_2 = 0$.

2. The solutions of $M_3 = 0$ are found in a similar manner. The effect on the functions ϕ of the operator $a_1^2 M_3$, which will be called Δ, are given in Table I.

For three solutions we take

$$\psi_0 = \phi_0, \quad \psi_1 = \phi_1, \quad \psi_2 = \phi_5.$$

Also $$\Delta\phi_4 = \phi_5;$$
the function ϕ_4 is taken as variable of reference.

We have next, since $\Delta\phi_3 = 2\phi_4$

$$2\phi_3\Delta\phi_4 - \phi_4\Delta\phi_3 = 2(\phi_3, \phi_4 \between \phi_5, -\phi_4);$$

this expression satisfies $\Delta = 0$ and we take as third solution

$$\psi_3 = (\phi_3, \phi_4 \between \phi_5, -\phi_4).$$

As fourth solution we take $$\psi_4 = \phi_7.$$

And from $\Delta\phi_9 = \phi_7$ we have

$$\phi_6\Delta\phi_4 - \phi_4\Delta\phi_6 = (\phi_6, \phi_7 \between \phi_5, -\phi_4),$$

which is a solution so that we put

$$\psi_5 = (\phi_6, \phi_7 \between \phi_5, -\phi_4).$$

In this way are found the 17 solutions ψ which are given in Table II.

3. The operator Φ is, with the present notation,

$$a_1 M_4 - b_0 N_4.$$

Its effect on the functions ϕ is given in the first table, the effect on the functions ψ is thence calculated and given in the second table. It appears that $\Phi\psi_r$ is also a function ψ.

Hence, as before, by the use of ψ_4 as variable of reference, are deduced the 16 solutions χ of $\Phi = 0$, which are given in Table III.

The effect of the operator $\frac{1}{a_1} N_4$ on the quantities ϕ, ψ, χ is calculated in succession, as shewn in the tables, and it appears that $\frac{1}{a_1} N_4 \chi_r$ is also a function χ.

From these, using χ_1 as variable of reference, we find 15 solutions ω of $N_4 = 0$.

4. Since the solutions of each equation are expressed in terms of solutions of the preceding equation, it follows that the quantities ω satisfy the three equations

$$N_2 = 0, \quad M_3 = 0, \quad N_4 = 0.$$

Also, although the ϕ's are not integral in the coefficients, the ω's are integral functions, as is proved by expressing them in the symbolical form. Thirdly, to see that the ω's are independent, consider the system ϕ. The coefficient a_0 is introduced into the system by ϕ_0, a_0' by ϕ_1, a_1 by ϕ_2, and

$$a_2, b_1, c_0, \quad a_1', b_0', a_0'', \quad a_3, b_2, c_1, d_0, \quad a_2', b_1', c_0', \quad a_1'', b_0'', \quad a_0''',$$

by $$\phi_3, \phi_4, \phi_5, \quad \phi_6, \quad \phi_7, \quad \phi_8, \quad \phi_9, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \phi_{14}, \phi_{15}, \phi_{16}, \phi_{17}, \phi_{18},$$

respectively. Since every function added to the system introduces a new coefficient there can be no relation among the functions ϕ. The set

$$\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \psi_{10}, \psi_{11}, \psi_{12}, \psi_{13}, \psi_{14}, \psi_{15}, \psi_{16}$$

introduce $$\phi_0, \phi_1, \phi_5, \phi_3, \phi_7, \phi_6, \phi_8, \phi_{12}, \phi_{11}, \phi_{10}, \phi_9, \quad \phi_{15}, \phi_{14}, \phi_{13}, \quad \phi_{17}, \phi_{16}, \phi_{18},$$

respectively, and are therefore independent. The set

$$\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}$$

introduce $\psi_0, \psi_1, \psi_2, \psi_3, \psi_5, \psi_6, \quad \psi_7, \psi_{11}, \psi_{14}, \psi_{16}, \quad \psi_8, \psi_{12}, \psi_{15}, \quad \psi_9, \psi_{13}, \quad \psi_{10}$,

respectively, and so are independent. The final system

$$\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}, \omega_{14}$$

introduce in succession

$$\chi_0, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \quad \chi_{10}, \chi_7, \quad \chi_{13}, \chi_{11}, \chi_8, \quad \chi_{15}, \chi_{14}, \chi_{12}, \chi_9,$$

and are therefore independent.

Lastly the cubic has 20 coefficients, and we have obtained 15 independent solutions of the equations

$$N_2 = M_3 = N_4 = 0,$$

they are therefore the leading coefficients of a complete set of concomitants.

5. The complete concomitant belonging to each leading coefficient may be found from other differential equations given in Dr Forsyth's paper. A shorter method is to express the leading coefficients in symbolical form. For this purpose the functions ϕ, ψ, χ are first expressed symbolically; they are given in the tables with the use of the contractions

$$(\alpha\beta) = \alpha_3\beta_2 - \alpha_2\beta_3$$

$$\alpha_\xi = \alpha_2 a_1 - \alpha_3 b_0 = -(\alpha\beta)\beta_1^2$$

$$\alpha_\beta = \alpha_4\beta_\xi - \alpha_\xi\beta_4$$

$$(\alpha\beta\gamma) = \begin{vmatrix} \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_2, & \beta_3, & \beta_4 \\ \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix}$$

and the identities

$$\alpha_3\beta_\xi - \alpha_\xi\beta_3 = (\alpha\beta)\, a_1$$

$$\alpha_\beta - (\alpha\beta)\, a_0' = (\alpha\beta\gamma)\,\gamma_1^2.$$

In Table IV. the symbolical form of the leading coefficients is given, the contraction α_ξ being retained for shortness.

To obtain the complete concomitants it is now only necessary to replace

$$\alpha_1, \quad \alpha_\xi, \quad (\alpha\beta), \quad (\alpha\beta\gamma)$$

by

$$\alpha_x, \quad (\alpha\beta p)\,\beta_x^2, \quad -(\alpha\beta p), \quad -(\alpha\beta\gamma u),$$

where

$$\alpha_x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4,$$

$$(\alpha\beta\gamma u) = \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \\ u_1, & u_2, & u_3, & u_4 \end{vmatrix}$$

$$(\alpha\beta p) = (\alpha_2\beta_3 - \alpha_3\beta_2)\, p_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\, p_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\, p_3$$
$$+ (\alpha_1\beta_4 - \alpha_4\beta_1)\, p_6 + (\alpha_2\beta_4 - \alpha_4\beta_2)\, p_5 + (\alpha_3\beta_4 - \alpha_4\beta_3)\, p_4.$$

The degree in x, p, and u may be then seen by inspection; the leading terms of the 15 concomitants U are given in Table IV.

6. It may be mentioned that two of functions ω may be replaced by simpler forms. Since

$$(\alpha\beta)\,\gamma_\xi + (\beta\gamma)\,\alpha_\xi + (\gamma\alpha)\,\beta_\xi = 0,$$

we have

$$(\alpha\beta)(\beta\gamma)\,\alpha_\xi^2\gamma_\xi\beta_1\gamma_1 = \tfrac{1}{2}(\beta\gamma)\,\alpha_\xi^2\beta_1\gamma_1\{(\alpha\beta)\,\gamma_\xi - (\alpha\gamma)\,\beta_\xi\}$$
$$= \tfrac{1}{2}(\beta\gamma)\,\alpha_\xi^2\beta_1\gamma_1\{-(\beta\gamma)\,\alpha_\xi\}$$
$$= -\,\omega_2\omega_5.$$

Hence

$$\omega_8 = (\alpha\beta)\,\beta_1\gamma_1\alpha_\xi\gamma_\xi\{(\beta\gamma)\,\alpha_\xi + (\alpha\beta)\,\gamma_\xi\}$$
$$= -\,\omega_2\,.\,\omega_5 + \omega_1\,.\,(\alpha\beta)^2\,\alpha_\xi\beta_1,$$

so that ω_8 may be replaced by

$$(\alpha\beta)^2\,\alpha_\xi\beta_1.$$

Similarly

$$\omega_{11} = -\,\omega_2\,.\,\omega_8 + \omega_1\,.\,(\alpha\beta)^2(\alpha\gamma)\,\beta_1\gamma_1\gamma_\xi,$$

so that ω_4 may be replaced by

$$(\alpha\beta)^2(\alpha\gamma)\,\beta_1\gamma_1\gamma_\xi.$$

7. In Table V. are given the leading coefficients for the quartic in symbolical form, the contraction α_ξ being used for $(\alpha_3\beta_2 - \alpha_2\beta_3)\,\beta_1^3$.

Table VI. gives the leading coefficients for the simultaneous concomitants of two quadratics. In the literal form small letters denote the coefficients of the first quadratic, capitals those of the second; in the symbolical form undashed letters denote the former, dashed letters the latter. This set is however quite unsymmetrical with respect to the two quadratics, and a more symmetrical set will be found.

A system of two quadratics.

8. The equations $N_2 = 0$, $M_3 = 0$, satisfied by the leading coefficient of the two quaternary quadratics, shew the coefficient to be a concomitant of two binary quadratics with c_0, b_1, a_2, and C_0, B_1, A_2, as coefficients and with four sets of variables $a_1, -b_0$; $a_1', -b_0'$; $A_1, -B_0$; $A_1', -B_0'$. The equations are also satisfied by the six quantities

$$q_1 = a_0, \quad q_2 = A_0,$$
$$q_1' = a_0', \quad q_2' = A_0',$$
$$q_1'' = a_0'', \quad q_2'' = A_0''.$$

Consider then the functions

$$v_1 = (c_0.\ b_1,\ a_0 \between a_1,\ -b_0)^2, \qquad v_2 = (C_0,\ B_1,\ A_2 \between A_1,\ -B_0)^2,$$
$$w_1 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between A_1,\ -B_0), \qquad w_2 = (C_0,\ B_1,\ A_2 \between A_1,\ -B_0 \between a_1,\ -b_0),$$
$$z_1 = (c_0,\ b_1,\ a_2 \between A_1,\ -B_0)^2, \qquad z_2 = (C_0,\ B_1,\ A_2 \between a_1,\ -b_0)^2,$$
$$v_1' = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between a_1',\ -b_0'), \qquad v_2' = (C_0,\ B_1,\ A_2 \between A_1,\ -B_0 \between A_1',\ -B_0'),$$
$$v_1'' = (c_0,\ b_1,\ a_2 \between a_1',\ -b_0')^2, \qquad v_2'' = (C_0,\ B_1,\ A_2 \between A_1',\ -B_0')^2,$$
$$w_1' = (c_0,\ b_1,\ a_2 \between A_1,\ -B_0 \between a_1',\ -b_0'), \qquad w_2' = (C_0,\ B_1,\ A_2 \between a_1,\ -b_0 \between A_1',\ -B_0'),$$
$$h_1 = c_0a_2 - b_1^2, \qquad h_2 = C_0A_2 - B_1^2,$$
$$h_{12} = c_0A_2 + C_0a_2 - 2b_1B_1$$
$$f_{12} = (B_0,\ A_1 \between a_1,\ -b_0).$$

Of these all containing no dashed letters are solutions of $N_4=0$, and for the rest

$$N_4v_1' = a_1h_1, \qquad N_4v_2' = A_1h_2,$$
$$N_4v_1'' = 2a_1'h_1, \qquad N_4v_2'' = 2A_1'h_2.$$
$$N_4w_1' = A_1h_1, \qquad N_4w_2' = a_1h_2.$$

From these we see that

$$r_1 = v_1' - q_1'h_1, \qquad r_2 = v_2' - q_2'h_2,$$
$$s_1 = v_1'' - q_1''h_1, \qquad s_2 = v_2'' - q_2''h_2,$$
$$t_{12} = w_1' - q_2'h_1, \qquad t_{21} = w_2' - q_1'h_2,$$

are solutions of $N_4=0$.

We can now choose a complete set of 15 solutions; we take

$$q_1,\ h_1,\ v_1,\ r_1,\ s_1,\ t_{12};$$
$$q_2,\ h_2,\ v_2,\ r_2,\ s_2,\ t_{21}.$$
$$h_{12},\ f_{12},\ w_1.$$

The five q_1, h_1, v_1, r_1, s_1, belong to the first quadratic and are independent since they introduce a_0, a_2, a_1, a_0', a_0'', in succession. Then A_0 is introduced by q_2; A_1 and B_0 by w_1 and f, which are independent of one another since the elimination of A_1 from

$$f = (B_0,\ A_1 \between a_1,\ -b_0)$$
$$w_1 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between A_1,\ -B_0)$$

does not eliminate B_0; C_0, B_1, A_2 are introduced by v_2, h_{12}, h_2, which are independent for a similar reason; A_0' by t_{12}; B_0', A_1' by r_2, t_{21}; and A_0'' by s_2.

The corresponding concomitants are

$$\begin{aligned}
Q_1 &= \alpha_x^2 &&= q_1x_1^2 + \dots\\
V_1 &= (\alpha\beta p)(\alpha\gamma p)\beta_x\gamma_x &&= v_1x_1^2p_1^2 + \dots\\
H_1 &= \tfrac{1}{2}(\alpha\beta p)^2 &&= h_1p_1^2 + \dots\\
R_1 &= \tfrac{1}{2}(\alpha\beta\gamma u)(\alpha\beta p)\gamma_x &&= r_1x_1p_1u_1 + \dots\\
S_1 &= -\tfrac{1}{6}(\alpha\beta\gamma u)^2 &&= s_1u_1^2 + \dots\\
T_{12} &= \tfrac{1}{2}(\alpha\beta\alpha' u)(\alpha\beta p)\alpha_x' &&= t_{12}x_1p_1u_1 + \dots
\end{aligned}$$

with the symmetrical six

$$Q_2,\ V_2,\ H_2,\ R_2,\ S_2,\ T_{21},$$

and

$$\begin{aligned}
H_{12} &= (\alpha\alpha' p)^2 &&= h_{12}p_1^2 + \dots\\
F_{12} &= -(\alpha\alpha' p)\alpha_x\alpha'_x &&= f_{12}x_1^2p_1 + \dots\\
W_1 &= (\alpha\beta p)(\alpha\alpha' p)\beta_x\alpha_x' &&= w_1x_1^2p_1^2 + \dots.
\end{aligned}$$

There remain three solutions z_1, z_2, w_2 not used as members of the complete set. They are given in terms of the set by

$$w_1^2 - v_1z_1 + h_1f_{12}^2 = 0,$$
$$w_2^2 - v_2z_2 + h_2f_{12}^2 = 0,$$
$$h_{12}f_{12}^2 + 2w_1w_2 - v_1v_2 - z_1z_2 = 0.$$

By reason of these relations w_1 may be replaced by w_2, z_1 or z_2. Or a symmetrical but less simple set is obtained by replacing h_{12} by w_2.

9. The unsymmetrical set of solutions of Table VI. are expressed in terms of the present set as follows, the functions w_2, z_1, z_2 being retained for simplicity:

$$\omega_0 = q_1,$$
$$\omega_1 = v_1,$$
$$\omega_2 = h_1,$$
$$\omega_3 = r_1,$$
$$h_1\omega_4 = r_1^2 - s_1 v_1,$$
$$\omega_5 = q_2,$$
$$\omega_6 = f_{12},$$
$$\omega_7 = w_1,$$
$$h_1\omega_8 = r_1 w_1 - v_1 t_{12},$$
$$\omega_9 = z_2,$$
$$f_{12}\omega_{10} = w_1 z_2 - v_1 w_2,$$
$$\omega_{11} = h_{12}v_1 - h_1 z_2,$$
$$h_1 h_2 f_{12}\omega_{12} = -t_{21}h_1 v_1 w_2 - t_{12}h_2 v_1 z_2 + r_2 h_1 v_1 z_2 + r_1 h_2 w_1 z_2,$$
$$h_1 h_2 f_{12}^2\omega_{13} = (v_1 w_2 - w_1 z_2)\phi + (v_1 v_2 - w_1 w_2)\psi,$$
$$h_1^2 h_2^2 f_{12}^2\omega_{14} = -f_{12}^2 h_1^2 h_2 v_1^2 s_2 + z_2\phi^2 - 2w_2\phi\psi + v_2\psi^2,$$

where

$$\phi = t_{12}v_1 h_2 - r_2 v_1 h_1 - r_1 w_1 h_2,$$
$$\psi = t_{21}v_1 h_1.$$

A system of three quadratics.

10. The concomitants of three quadratics might be found by the general method but not in a symmetrical form.

We take with the two quadratics of § 9 a third with Greek letters as coefficients. We have then as solutions of the equations functions of the types

$$q_1,\ v_1,\ r_1,\ s_1,\ h_1,\ f_{12},\ h_{12},\ t_{12},$$

to which may be added

$$k = \begin{vmatrix} c_0 & b_1 & a_2 \\ C_0 & B_1 & A_2 \\ \gamma_0 & \beta_1 & \alpha_2 \end{vmatrix}, \qquad l = \begin{vmatrix} a_1 & b_0 & a_0' \\ A_1 & B_0 & A_0' \\ \alpha_1 & \beta_0 & \alpha_0' \end{vmatrix}.$$

Also the binary variables in v_1 may be replaced in whole or in part by A_1, $-B_0$, and α_1, $-\beta_0$; this will be denoted by additional suffixes, e.g.

$$v_{1,2} = (c_0, b_1, a_2 \between a_1, -b_0 \between A_1, -B_0),$$
$$v_{1,2,3} = (c_0, b_1, a_2 \between A_1, -B_0 \between \alpha_1, -\beta_0).$$

The 25 leading coefficients of the complete system of concomitants are obtained in the same way as for two quadratics. We take

$$q_1,\ v_1,\ r_1,\ s_1,\ h_1$$

for the first quadratic, and with these

q_2	introducing the coefficient	A_0,
q_3	....	α_0,
f_{12}, v_{12}	....	A_1, B_0.
f_{13}, f_{23}	....	α_1, β_0,
v_2, h_2, h_{12}	....	A_2, B_1, C_0.
v_3, h_3, h_{13}	....	α_2, β_1, γ_0.
t_{12},	....	A_0'.
t_{13}.	....	α_0',
r_2, t_{21},	....	A_1', B_0',
r_3, t_{31},	....	α_1', β_0',
s_2,	....	A_0'',
s_3,	....	α_0''.

11. Now we have the relations

$$v_{122}v_1 = v_{12}^2 + h_1 f_{12}^2$$
$$v_{133}v_1 = v_{13}^2 + h_1 f_{13}^2$$
$$v_{233}v_2 = v_{23}^2 + h_2 f_{23}^2$$
$$v_{322}v_3 = v_{32}^2 + h_3 f_{23}^2$$
$$v_{123}f_{12} = v_{122}f_{13} - v_{12}f_{23}$$
$$v_{13}f_{12} = -v_1 f_{23} + v_{12}f_{13}$$
$$h_{23}f_{23}^2 = v_2v_3 + v_{233}v_{322} - 2v_{23}v_{32}$$
$$h_{12}f_{23}^2 = v_{133}v_2 + v_{122}v_{233} - 2v_{123}v_{23}$$
$$h_{13}f_{23}^2 = v_{122}v_3 + v_{133}v_{322} - 2v_{123}v_{32}.$$

These give an expression for h_{23} in terms of members of the complete system among which v_{12} occurs. The equation may therefore be looked on as expressing v_{12} in terms of the remaining members of the system and h_{23}, so that v_{12} may be replaced by h_{23}.

Again

$$t_{23}f_{12} + t_{21}f_{23} + r_2 f_{31} + h_2 l = 0$$
$$t_{31}f_{23} + t_{32}f_{31} + r_3 f_{12} + h_3 l = 0$$
$$t_{12}f_{31} + t_{13}f_{12} + r_1 f_{23} + h_1 l = 0,$$

so that t_{12}, t_{13}, t_{21}, t_{31} may be replaced by various other sets of four t's or by l and a set of three.

Lastly, k is given in terms of the system by relations of the type

$$kf_{12}^2 = v_3 j_{12}(a^2) + v_{311}j_{12}(A^2) - 2v_{312}j_{12}(aA)$$
$$j_{12}(a^2) = -v_1v_{21} + v_{12}v_{211}$$
$$j_{12}(aA) = -v_1v_2 + v_{122}v_{211},$$

where
$$j_{12}(a^2) = \left| \begin{matrix} c_0 B_1 & c_0 A_2 & b_1 A_2 \\ -C_0 b_1 & -C_0 a_2 & -B_1 a_2 \end{matrix} \right| \between (a_1, -b_0)^2$$

the Jacobian of v_1 and v_{211}, and $j_{12}(aA)$, etc., denote the polars of $j_{12}(a^2)$ in the other sets of binary variables $A_1, -B_0$; $\alpha_1, -\beta_0$. The quantity k might be used as one of the complete set, but the set would not be symmetrical.

The simplest and most symmetrical has then for leading coefficients

$$\begin{matrix} q_1, & q_2, & q_3, \\ v_1, & v_2, & v_3, \\ r_1, & r_2, & r_3, \\ s_1, & s_2, & s_3, \\ h_1, & h_2, & h_3, \\ f_{23}, & f_{31}, & f_{12}, \\ h_{23}, & h_{31}, & h_{12}, \\ t_{12}, & t_{23}, & t_{31}, \\ & l. & \end{matrix}$$

TABLE I. Solutions of $N_2 = 0$.

Solutions of $N_2=0$	Effect of operator $a_1^2 M_3$	Effect of Φ	Effect of $\frac{1}{a_1} N_4$	Solutions in semi-symbolical form
$\phi_0 = a_0$	0	0	0	α_1^3
$\phi_1 = a_0'$	0	0	1	$\alpha_1^2 \alpha_4$
$\phi_2 = a_1$				$\alpha_1^2 \alpha_3$
$\phi_3 = a_1^{-2} a_2$	$2\phi_4$	0	0	$a_1^{-2} \alpha_1 \alpha_3^2$
$\phi_4 = a_1^{-1}(b_1, a_2 \between a_1, -b_0)$	ϕ_5	0	0	$a_1^{-1} \alpha_1 \alpha_3 \alpha_\xi$
$\phi_5 = (c_0, b_1, a_2 \between a_1, -b_0)^2$	0	0	0	$\alpha_1 \alpha_\xi^2$
$\phi_6 = a_1^{-1} a_1'$	ϕ_7	ϕ_4	ϕ_3	$a_1^{-1} \alpha_1 \alpha_4 \alpha_3$
$\phi_7 = (b_0', a_1' \between a_1, -b_0)$	0	ϕ_5	ϕ_4	$\alpha_1 \alpha_4 \alpha_\xi$
$\phi_8 = a_0''$	0	$2\phi_7$	$2\phi_6$	$\alpha_1 \alpha_4^2$
$\phi_9 = a_1^{-3} a_3$	$3\phi_{10}$	0	0	$a_1^{-3} \alpha_3^3$
$\phi_{10} = a_1^{-2}(b_2, a_3 \between a_1, -b_0)$	$2\phi_{11}$	0	0	$a_1^{-2} \alpha_3^2 \alpha_\xi$
$\phi_{11} = a_1^{-1}(c_1, b_2, a_3 \between a_1, -b_0)^2$	ϕ_{12}	0	0	$a_1^{-1} \alpha_3 \alpha_\xi^2$
$\phi_{12} = (d_0, c_1, b_2, a_3 \between a_1 - b_0)^3$	0	0	0	α_ξ^3
$\phi_{13} = a_1^{-2} a_2'$	$2\phi_{14}$	ϕ_{10}	ϕ_9	$a_1^{-2} \alpha_4 \alpha_3^2$
$\phi_{14} = a_1^{-1}(b_1', a_2' \between a_1, -b_0)$	ϕ_{15}	ϕ_{11}	ϕ_{10}	$a_1^{-1} \alpha_4 \alpha_3 \alpha_\xi$
$\phi_{15} = (c_0', b_1', a_2' \between a_1 - b_0)^2$	0	ϕ_{12}	ϕ_{11}	$\alpha_4 \alpha_\xi^2$
$\phi_{16} = a_1^{-1} a_1''$	ϕ_{17}	$2\phi_{14}$	$2\phi_{13}$	$a_1^{-1} \alpha_4^2 \alpha_3$
$\phi_{17} = (b_0'', a_1'' \between a_1, -b_0)$	0	$2\phi_{15}$	$2\phi_{14}$	$\alpha_4^2 \alpha_\xi$
$\phi_{18} = a_0'''$	0	$3\phi_{17}$	$3\phi_{16}$	α_4^3

TABLE II. Solutions of $M_3 = 0$.

Solutions of $M_3=0$	Effect of Φ	Effect of $\frac{1}{a_1}N_4$	Symbolical form
$\psi_0 = \phi_0$	0	0	α_1^3
$\psi_1 = \phi_1$	0	1	$\alpha_1^2\alpha_4$
$\psi_2 = \phi_5$	0	0	$\alpha_1\alpha_\xi^2$
$\psi_3 = (\phi_3, \phi_4 \between \phi_5, -\phi_4)$	0	0	$\frac{1}{2}\alpha_1\beta_1(\alpha\beta)^2$
$\psi_4 = \phi_7$	ψ_2	ϕ_4	$\alpha_1\alpha_4\alpha_\xi$
$\psi_5 = (\phi_6, \phi_7 \between \phi_5, -\phi_4)$	0	ψ_3	$\frac{1}{2}\alpha_1\beta_1(\alpha\beta)\alpha_\beta$
$\psi_6 = \phi_8$	$2\psi_4$	$2\phi_5^{-1}(\psi_5 + \psi_4\phi_4)$	$\alpha_1\alpha_4^2$
$\psi_7 = \phi_{12}$	0	0	α_ξ^3
$\psi_8 = (\phi_{11}, \phi_{12} \between \phi_5, -\phi_4)$	0	0	$\alpha_\xi^2\beta_\xi\beta_1(\alpha\beta)$
$\psi_9 = (\phi_{10}, \phi_{11}, \phi_{12} \between \phi_5, -\phi_4)^2$	0	0	$\alpha_\xi\beta_\xi\gamma_\xi\beta_1\gamma_1(\alpha\beta)(\alpha\gamma)$
$\psi_{10} = (\phi_9, \phi_{10}, \phi_{11}, \phi_{12} \between \phi_5, -\phi_4)^3$	0	0	$\beta_\xi\gamma_\xi\delta_\xi\beta_1\gamma_1\delta_1(\alpha\beta)(\alpha\gamma)(\alpha\delta)$
$\psi_{11} = \phi_{15}$	ψ_7	$\phi_5^{-1}(\psi_8 + \psi_7\phi_4)$	$\alpha_4\alpha_\xi^2$
$\psi_{12} = (\phi_{14}, \phi_{15} \between \phi_5, -\phi_4)$	ψ_8	$\phi_5^{-1}(\psi_9 + \psi_8\phi_4)$	$\alpha_4\alpha_\xi\beta_\xi\beta_1(\alpha\beta)$
$\psi_{13} = (\phi_{13}, \phi_{14}, \phi_{15} \between \phi_5, -\phi_4)^2$	ψ_9	$\phi_5^{-1}(\psi_{10} + \psi_9\phi_4)$	$\alpha_4\beta_\xi\gamma_\xi\beta_1\gamma_1(\alpha\beta)(\alpha\gamma)$
$\psi_{14} = \phi_{17}$	$2\psi_{11}$	$2\phi_5^{-1}(\psi_{12} + \psi_{11}\phi_4)$	$\alpha_4^2\alpha_\xi$
$\psi_{15} = (\phi_{16}, \phi_{17} \between \phi_5, -\phi_4)$	$2\psi_{12}$	$2\phi_5^{-1}(\psi_{13} + \psi_{12}\phi_4)$	$\alpha_4^2\beta_\xi\beta_1(\alpha\beta)$
$\psi_{16} = \phi_{18}$	$3\psi_{14}$	$3\phi_5^{-1}(\psi_{15} + \psi_{14}\phi_4)$	α_4^3

TABLE III. Solutions of $\Phi \equiv a_1 M_4 - b_0 N_4 = 0$.

Solutions of $\Phi=0$	Effect of $\frac{1}{a_1}N_4$	Symbolical form
$\chi_0 = \psi_0$	0	α_1^3
$\chi_1 = \psi_1$	1	$\alpha_1^2\alpha_4$
$\chi_2 = \psi_2$	0	$\alpha_1\alpha_\xi^2$
$\chi_3 = \psi_3$	0	$\frac{1}{2}\alpha_1\beta_1(\alpha\beta)^2$
$\chi_4 = \psi_5$	χ_3	$\frac{1}{2}\alpha_1\beta_1(\alpha\beta)\alpha_\beta$
$\chi_5 = (\psi_6, \psi_4 \between \psi_2, -\psi_4)$	$2\chi_4$	$\frac{1}{2}\alpha_1\beta_1\alpha_\beta^2$
$\chi_6 = \psi_7$	0	α_ξ^3
$\chi_7 = (\psi_{11}, \psi_7 \between \psi_2, -\psi_4)$	χ_{10}	$\alpha_\xi^2\beta_\xi\beta_1\alpha_\beta$
$\chi_8 = (\psi_{14}, \psi_{11}, \psi_7 \between \psi_2, -\psi_4)$	$2\chi_{11}$	$\alpha_\xi\beta_\xi\gamma_\xi\beta_1\gamma_1\alpha_\beta\alpha_\gamma$
$\chi_9 = (\psi_{16}, \psi_{14}, \psi_{11}, \psi_7 \between \psi_2, -\psi_4)^2$	$3\chi_{12}$	$\beta_\xi\gamma_\xi\delta_\xi\beta_1\gamma_1\delta_1\alpha_\beta\alpha_\gamma\alpha_\delta$
$\chi_{10} = \psi_8$	0	$\alpha_\xi^2\beta_\xi\beta_1(\alpha\beta)$
$\chi_{11} = (\psi_{12}, \psi_8 \between \psi_2, -\psi_4)$	χ_{13}	$\alpha_\xi\beta_\xi\gamma_\xi\beta_1\gamma_1(\alpha\beta)\alpha_\gamma$
$\chi_{12} = (\psi_{15}, \psi_{12}, \psi_8 \between \psi_2, -\psi_4)^2$	$2\chi_{14}$	$\beta_\xi\gamma_\xi\delta_\xi\beta_1\gamma_1\delta_1(\alpha\beta)\alpha_\gamma\alpha_\delta$
$\chi_{13} = \psi_9$	0	$\alpha_\xi\beta_\xi\gamma_\xi\beta_1\gamma_1(\alpha\beta)(\alpha\gamma)$
$\chi_{14} = (\psi_{13}, \psi_9 \between \psi_2, -\psi_4)$	χ_{15}	$\beta_\xi\gamma_\xi\delta_\xi\beta_1\gamma_1\delta_1(\alpha\beta)(\alpha\gamma)\alpha_\delta$
$\chi_{15} = \psi_{10}$	0	$\beta_\xi\gamma_\xi\delta_\xi\beta_1\gamma_1\delta_1(\alpha\beta)(\alpha\gamma)(\alpha\delta)$

TABLE IV. Solutions of $N_4 = 0$.

Solutions of $N_4=0$	Symbolical form	Corresponding concomitants
$\omega_0 = \chi_0$	α_1^3	$U_0 = \omega_0 x_1^3 + \ldots$
$\omega_1 = \chi_2$	$\alpha_1 \alpha_\xi^2$	$U_1 = \omega_1 x_1^5 p_1^2 + \ldots$
$\omega_2 = \chi_3$	$\frac{1}{2}\alpha_1 \beta_1 (\alpha\beta)^2$	$U_2 = \omega_2 x_1^2 p_1^2 + \ldots$
$\omega_3 = (\chi_4, \chi_3 \between 1, -\chi_1)$	$\frac{1}{2}\alpha_1 \beta_1 \gamma_1^2 (\alpha\beta)(\alpha\beta\gamma)$	$U_3 = \omega_3 x_1^4 p_1 u_1 + \ldots$
$\omega_4 = (\chi_5, \chi_4, \chi_3 \between 1, -\chi_1)$	$\frac{1}{2}\alpha_1 \beta_1 \gamma_1^2 \delta_1^2 (\alpha\beta\gamma)(\alpha\beta\delta)$	$U_4 = \omega_4 x_1^6 u_1^2 + \ldots$
$\omega_5 = \chi_6$	α_ξ^3	$U_5 = \omega_5 x_1^6 p_1^3 + \ldots$
$\omega_6 = \chi_{10}$	$\alpha_\xi^2 \beta_\xi \beta_1 (\alpha\beta)$	$U_6 = \omega_6 x_1^7 p_1^4 + \ldots$
$\omega_7 = (\chi_7, \chi_{10} \between 1, -\chi_1)$	$\alpha_\xi^2 \beta_\xi \beta_1 (\alpha\beta\gamma)\gamma_1^2$	$U_7 = \omega_7 x_1^9 p_1^3 u_1 + \ldots$
$\omega_8 = \chi_{13}$	$\alpha_\xi \beta_\xi \gamma_\xi \beta_1 \gamma_1 (\alpha\beta)(\alpha\gamma)$	$U_8 = \omega_8 x_1^8 p_1^5 + \ldots$
$\omega_9 = (\chi_{11}, \chi_{13} \between 1, -\chi_1)$	$\alpha_\xi \beta_\xi \gamma_\xi \beta_1 \gamma_1 \delta_1^2 (\alpha\beta)(\alpha\gamma\delta)$	$U_9 = \omega_9 x_1^{10} p_1^4 u_1 + \ldots$
$\omega_{10} = (\chi_8, \chi_{11}, \chi_{13} \between 1, -\chi_1)^2$	$\alpha_\xi \beta_\xi \gamma_\xi \beta_1 \gamma_1 \delta_1^2 \epsilon_1^2 (\alpha\beta\delta)(\alpha\gamma\epsilon)$	$U_{10} = \omega_{10} x_1^{12} p_1^3 u_1^2 + \ldots$
$\omega_{11} = \chi_{15}$	$\beta_\xi \gamma_\xi \delta_\xi \beta_1 \gamma_1 \delta_1 (\alpha\beta)(\alpha\gamma)(\alpha\delta)$	$U_{11} = \omega_{11} x_1^9 p_1^6 + \ldots$
$\omega_{12} = (\chi_{14}, \chi_{15} \between 1, -\chi_1)$	$\beta_\xi \gamma_\xi \delta_\xi \beta_1 \gamma_1 \delta_1 (\alpha\beta)(\alpha\gamma)(\alpha\delta\eta)\eta_1^2$	$U_{12} = \omega_{12} x_1^{11} p_1^5 u_1 + \ldots$
$\omega_{13} = (\chi_{12}, \chi_{14}, \chi_{15} \between 1, -\chi_1)^2$	$\beta_\xi \gamma_\xi \delta_\xi \beta_1 \gamma_1 \delta_1 (\alpha\beta)(\alpha\gamma\xi)(\alpha\delta\eta)\xi_1^2 \eta_1^2$	$U_{13} = \omega_{13} x_1^{13} p_1^4 u_1^2 + \ldots$
$\omega_{14} = (\chi_9, \chi_{12}, \chi_{14}, \chi_{15} \between 1, -\chi_1)^3$	$\beta_\xi \gamma_\xi \delta_\xi \beta_1 \gamma_1 \delta_1 (\alpha\beta\epsilon)(\alpha\gamma\xi)(\alpha\delta\eta)\epsilon_1^2 \xi_1^2 \eta_1^2$	$U_{14} = \omega_{14} x_1^{15} p_1^3 u_1^3 + \ldots$

TABLE V. Leading coefficients of concomitants of quartic in symbolical form.

α_1^4

$\alpha_1^2\alpha_\xi^2$

$\frac{1}{2}\alpha_1^2\beta_1^2(\alpha\beta)^2$

$\frac{1}{2}\alpha_1^2\beta_1^2(\alpha\beta)(\alpha\beta\gamma)\gamma_1^3$

$\frac{1}{2}\alpha_1^2\beta_1^2(\alpha\beta\gamma)(\alpha\beta\delta)\gamma_1^3\delta_1^3$

$\alpha_1\alpha_\xi^3$

$\alpha_1\alpha_\xi^2\beta_1^2\beta_\xi(\alpha\beta)$

$\alpha_1\alpha_\xi^2\beta_1^2\beta_\xi(\alpha\beta\gamma)\gamma_1^3$

$\alpha_1\beta_1^2\gamma_1^2\alpha_\xi\beta_\xi\gamma_\xi(\alpha\beta)(\alpha\gamma)$

$\alpha_1\beta_1^2\gamma_1^2\alpha_\xi\beta_\xi\gamma_\xi(\alpha\beta)(\alpha\gamma\epsilon)\epsilon_1^3$

$\alpha_1\beta_1^2\gamma_1^2\alpha_\xi\beta_\xi\gamma_\xi(\alpha\beta\delta)(\alpha\gamma\epsilon)\delta_1^3\epsilon_1^3$

$\alpha_1\beta_1^2\gamma_1^2\delta_1^2\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta)$

$\alpha_1\beta_1^2\gamma_1^2\delta_1^2\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta\eta)\eta_1^3$

$\alpha_1\beta_1^2\gamma_2^2\delta_1^2\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta)(\alpha\gamma\xi)(\alpha\delta\eta)\xi_1^3\eta_1^3$

$\alpha_1\beta_1^2\gamma_1^2\delta_1^2\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta\epsilon)(\alpha\gamma\xi)(\alpha\delta\eta)\epsilon_1^3\xi_1^3\eta_1^3$

α_ξ^4

$\beta_1^2\alpha_\xi^3\beta_\xi(\alpha\beta)$

$\beta_1^2\alpha_\xi^3\beta_\xi(\alpha\beta\gamma)\gamma_1^3$

$\beta_1^2\gamma_1^2\alpha_\xi^2\beta_\xi\gamma_\xi(\alpha\beta)(\alpha\gamma)$

$\beta_1^2\gamma_1^2\alpha_\xi^2\beta_\xi\gamma_\xi(\alpha\beta)(\alpha\gamma\epsilon)\epsilon_1^3$

$\beta_1^2\gamma_1^2\alpha_\xi^2\beta_\xi\gamma_\xi(\alpha\beta\delta)(\alpha\gamma\epsilon)\delta_1^3\epsilon_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\alpha_\xi\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta)$

$\beta_1^2\gamma_1^2\delta_1^2\alpha_\xi\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta\eta)\eta_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\alpha_\xi\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta)(\alpha\gamma\xi)(\alpha\delta\eta)\xi_1^3\eta_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\alpha_\xi\beta_\xi\gamma_\xi\delta_\xi(\alpha\beta\epsilon)(\alpha\gamma\xi)(\alpha\delta\eta)\epsilon_1^3\xi_1^3\eta_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\epsilon_1^2\beta_\xi\gamma_\xi\delta_\xi\epsilon_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta)(\alpha\epsilon)$

$\beta_1^2\gamma_1^2\delta_1^2\epsilon_1^2\beta_\xi\gamma_\xi\delta_\xi\epsilon_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta)(\alpha\epsilon\kappa)\kappa_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\epsilon_1^2\beta_\xi\gamma_\xi\delta_\xi\epsilon_\xi(\alpha\beta)(\alpha\gamma)(\alpha\delta\theta)(\alpha\epsilon\kappa)\theta_1^3\kappa_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\epsilon_1^2\beta_\xi\gamma_\xi\delta_\xi\epsilon_\xi(\alpha\beta)(\alpha\gamma\eta)(\alpha\delta\theta)(\alpha\epsilon\kappa)\eta_1^3\theta_1^3\kappa_1^3$

$\beta_1^2\gamma_1^2\delta_1^2\epsilon_1^2\beta_\xi\gamma_\xi\delta_\xi\epsilon_\xi(\alpha\beta\xi)(\alpha\gamma\eta)(\alpha\delta\theta)(\alpha\epsilon\kappa)\xi_1^3\eta_1^3\theta_1^3\kappa_1^3$

TABLE VI. Leading coefficients of concomitants of two quadratics.

$$\omega_0 = \alpha_1^3 = a_0$$

$$\omega_1 = \alpha_\xi^2 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2$$

$$\omega_2 = \tfrac{1}{2}(\alpha\beta)^2 = c_0 a_2 - b_1^2$$

$$\omega_3 = \tfrac{1}{2}\gamma_1(\alpha\beta)(\alpha\beta\gamma) = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between a_1',\ -b_0') - a_0'\omega_2$$

$$\omega_4 = \tfrac{1}{2}\gamma_1\delta_1(\alpha\beta\gamma)(\alpha\beta\delta) = a_0''\omega_1 - 2a_0'\omega_3 - a_0'^2\omega_2 - (b_0'a_1 - a_1'b_0)^2$$

$$\omega_5 = \alpha_1'^3 = A_0$$

$$\omega_6 = \alpha_1\alpha_1'(\alpha\alpha') = (B_0,\ A_1 \between a_1,\ -b_0)$$

$$\omega_7 = -\alpha_\xi\alpha_1'(\alpha\alpha') = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between A_1,\ -B_0)$$

$$\omega_8 = \alpha_\xi\alpha_1'\beta_1(\alpha\beta\alpha') = A_0'\omega_1 - a_0'\omega_7 - (b_0'a_1 - a_1'b_0)\,\omega_6$$

$$\omega_9 = \alpha_\xi'^2 = (C_0,\ B_1,\ A_2 \between a_1,\ -b_0)^2$$

$$\omega_{10} = -\alpha_\xi\alpha_\xi'(\alpha\alpha') = \left|\begin{array}{c|c|c} c_0 B_1 & c_0 A_2 & b_1 A_2 \\ -C_0 b_1 & -C_0 a_2 & -B_1 a_2 \end{array}\right| \between a_1,\ -b_0)^2$$

$$\omega_{11} = \alpha_\xi\beta_\xi(\alpha\alpha')(\beta\alpha') = \left|\begin{array}{c|c|c} C_0 b_1^2 & C_0 b_1 a_2 & C_0 a_2^2 \\ -2B_1 c_0 b_1 & -B_1(c_0 a_2 + b_1^2) & -2B_1 b_1 a_2 \\ +A_2 c_0^2 & +A_2 c_0 b_1 & +A_2 b_1^2 \end{array}\right| \between a_1,\ -b_0)^2$$

$$\omega_{12} = \alpha_\xi\alpha_\xi'\beta_1(\alpha\beta\alpha') = \omega_1(B_0'a_1 - A_1'b_0) - \omega_9(b_0'a_1 - a_1'b_0) - \omega_{10}a_0'$$

$$\omega_{13} = -\alpha_\xi\beta_\xi\gamma_1(\alpha\alpha')(\beta\gamma\alpha') = \omega_1(c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between A_1',\ -B_0') - \omega_{10}(b_0'a_1 - a_1'b_0) - \omega_{11}a_0'$$

$$\omega_{14} = \alpha_\xi\beta_\xi\gamma_1\delta_1(\beta\gamma\alpha')(\alpha\delta\alpha') = \omega_1^2 A_0'' - 2\omega_1(B_0'a_1 - A_1'b_0)(b_0'a_1 - a_1'b_0)$$
$$- 2\omega_1 a_0'(c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between A_1',\ -B_0') + 2\omega_{10}a_0'(b_0'a_1 - a_1'b_0) + \omega_{11}a_0'^2.$$

II. *Forced Vibrations in isotropic elastic solid spheres and spherical shells.*

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INDEX.

§ 1. The free vibrations in an isotropic elastic sphere or spherical shell have been treated in some detail by several writers, but comparatively little attention has been given to the motion which accompanies the application of periodic forces. In Vol. XIV. of the Society's 'Transactions'* I wrote down the equations determining the arbitrary constants whose substitution in the general solution gives the amplitude of the vibrations corresponding to given systems of surface forces.

In the 'Proceedings' of the London Mathematical Society, Vol. XIX., Mr Love arrived at equations for determining the forced vibrations of a spherical shell containing a given mass of liquid. In his Treatise on Elasticity† Mr Love has also considered the subject of forced vibrations in a solid sphere due to bodily forces derivable from a potential; illustrating his method by application to the interesting case when the potential involves only a spherical harmonic of the second degree. Mr Love's method is based on Professor Lamb's‡ well-known solution in Cartesian Coordinates. Here, as in my previous treatment of the sphere, I adhere to polar coordinates.

* *l. c.* pp. 315—6. The *method* of treating the spherical shell is described on p. 319.

† Vol. I., pp. 324—8.

‡ *Proceedings London Math. Soc.*, Vols. XIII. and XIV.

The most fundamental division of forced vibrations is into those which have, and those which have not, the same frequency as one of the free vibrations of the same type. In the former case the mathematical theory of elasticity makes the amplitude become infinite. In the latter case in an elastic solid the expressions for the displacements, even in a sphere or spherical shell, are usually too complicated to convey much information except through numerical application in particular cases. There are, however, two classes of cases in which results of a general character are obtainable which are at once elegant and of obvious physical significance.

The first class consists of the vibrations of a solid sphere due to forces whose frequency is small compared to that of the fundamental free vibration of the same type; the second class comprises the forced vibrations of any frequency in a very thin spherical shell. It is to these two classes that attention is almost exclusively devoted in the present paper.

As the whole investigation is based on my general solution* of the elastic solid equations of motion, it is convenient to reproduce these equations and their solution with some slight improvements in the notation.

§ 2. Polar coordinates r, θ, ϕ are used, θ being the 'polar distance' and ϕ the 'azimuth.' The elements dr, $rd\theta$, $r\sin\theta d\phi$ at any point are the *fundamental* directions along which are taken the displacements u, v, w. The dilatation is denoted by Δ, or

$$\Delta = \frac{1}{r^2}\left\{\frac{d}{dr}(ur^2) + \frac{1}{\sin\theta}\frac{d}{d\theta}(vr\sin\theta) + \frac{1}{\sin^2\theta}\frac{d}{d\phi}(wr\sin\theta)\right\} \quad\text{..................(1).}$$

The stresses in the notation of Todhunter and Pearson's 'History' are

$$\left.\begin{aligned}
\widehat{rr} &= (m-n)\Delta + 2n\frac{du}{dr},\\
\widehat{\theta\theta} &= (m-n)\Delta + 2n\left(\frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}\right),\\
\widehat{\phi\phi} &= (m-n)\Delta + 2n\left(\frac{u}{r} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right),\\
\widehat{r\theta} &= n\left(\frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta}\right),\\
\widehat{r\phi} &= n\left(\frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right),\\
\widehat{\theta\phi} &= n\left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)
\end{aligned}\right\} \quad\text{....................(2);}$$

where m, n are the elastic constants in the notation of Thomson and Tait's 'Natural Philosophy.'

Supposing periodic bodily forces to act derivable from a potential V satisfying Laplace's equation, we may present this potential in the form

$$V = \Sigma\,(r^i V_i + r^{-i-1}V_{-i-1})\cos kt \quad\text{..............................(3);}$$

* *Camb. Phil. Soc. Trans.*, Vol. xiv., pp. 308 et seq.

where V_i, V_{-i-1} are surface spherical harmonics of the same degree, i, and t is the time counted from any convenient epoch. Here and in what follows Σ denotes summation with respect to i.

The internal or "body-stress" equations are

$$\left.\begin{aligned}(m+n)r^2\sin\theta\frac{d\Delta}{dr}-n\frac{d\mathfrak{C}}{d\theta}+n\frac{d\mathfrak{B}}{d\phi}+\rho r^2\sin\theta\left(\frac{dV}{dr}-\frac{d^2u}{dt^2}\right)=0,\\ (m+n)\sin\theta\frac{d\Delta}{d\theta}-n\frac{d\mathfrak{A}}{d\phi}+n\frac{d\mathfrak{C}}{dr}+\rho r\sin\theta\left(\frac{1}{r}\frac{dV}{d\theta}-\frac{d^2v}{dt^2}\right)=0,\\ (m+n)\operatorname{cosec}\theta\frac{d\Delta}{d\phi}-n\frac{d\mathfrak{B}}{dr}+n\frac{d\mathfrak{A}}{d\theta}+\rho r\left(\frac{1}{r\sin\theta}\frac{dV}{d\phi}-\frac{d^2w}{dt^2}\right)=0\end{aligned}\right\}\quad\text{...........(4)},$$

where for shortness

$$\left.\begin{aligned}\mathfrak{A}&=\frac{1}{r^2\sin\theta}\left\{\frac{d}{d\theta}(wr\sin\theta)-\frac{d}{d\phi}(vr)\right\},\\ \mathfrak{B}&=\frac{1}{\sin\theta}\left\{\frac{du}{d\phi}-\frac{d}{dr}(wr\sin\theta)\right\},\\ \mathfrak{C}&=\sin\theta\left\{\frac{d}{dr}(vr)-\frac{du}{d\theta}\right\}\end{aligned}\right\}\quad\text{........................(5)}.$$

It is convenient to concentrate attention on the terms actually appearing in (3), taking them as a type. Differentiations with respect to t need not then appear explicitly, since for instance

$$\frac{d^2u}{dt^2}=-k^2u.$$

The representative term in (3) involving r^{-i-1} can occur of course only in a spherical shell, i being regarded here as a positive integer.

The surface forces are conveniently grouped under three classes. Thus in a spherical shell we may regard the forces over the outer surface $r=a$ as consisting of:—

(i) Pure radial forces $\Sigma(R_i\cos kt)$;

(ii) Tangential forces derivable from a *potential*, whose components are

$$\Sigma\left(\frac{dT_i}{d\theta}\cos kt\right)\text{ along }a\,d\theta,\quad \Sigma\left(\frac{1}{\sin\theta}\frac{dT_i}{d\phi}\cos kt\right)\text{ along }a\sin\theta\,d\phi;$$

(iii) Tangential forces derivable from a *stream function*, whose components are

$$\Sigma\left(\frac{1}{\sin\theta}\frac{d\tau_i}{d\phi}\cos kt\right)\text{ along }a\,d\theta,\quad \Sigma\left(-\frac{d\tau_i}{d\theta}\cos kt\right)\text{ along }a\sin\theta\,d\phi.$$

The letters R_i, T_i, τ_i represent spherical surface harmonics of degree i, R_0 being a constant and occurring in the case of uniform normal pressure.

Over the inner surface $r=b$

we may suppose similar surface forces to act, distinguished by the dashed letters R_i', T_i', τ_i'.

The letters R, Θ, Φ without a suffix are employed to denote the total components of the forces on the outer surface along the fundamental directions, the same letters dashed having a like application to the inner surface of a shell. Thus

$$\left.\begin{aligned} R &= \Sigma (R_i \cos kt), \\ \Theta &= \Sigma \left(\frac{dT_i}{d\theta} + \frac{1}{\sin\theta}\frac{d\tau_i}{d\phi}\right)\cos kt, \\ \Phi &= \Sigma \left(\frac{1}{\sin\theta}\frac{dT_i}{d\phi} - \frac{d\tau_i}{d\theta}\right)\cos kt \end{aligned}\right\} \qquad (6).$$

In the general case there may of course be any number of applied forces, whether bodily or surface, with different frequencies and epochs, but as the effects of each are independent of the existence of the others no confusion can arise through $\cos kt$ being made to do duty for the time factor in every case.

The surface conditions which must be satisfied by the solution for the spherical shell are the following six:—

$$\left.\begin{aligned} &\text{over } r = a & &\text{over } r = b \\ &\widehat{rr} = R, & &\widehat{rr} = R', \\ &\widehat{r\theta} = \Theta, & &\widehat{r\theta} = \Theta', \\ &\widehat{r\phi} = \Phi, & &\widehat{r\phi} = \Phi' \end{aligned}\right\} \qquad (7).$$

In the solid sphere there are of course only the first three.

§ 3. In my original treatment of the vibration problem bodily forces were not supposed to act. Thus the complete solution of (4) requires the addition to my previous solution of terms which constitute a particular solution when V exists. This presents no difficulty, for by (1) and (5) we see that Δ, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ all vanish for values of u, v, w of the form

$$u = M\frac{dV}{dr}, \quad v = M\frac{1}{r}\frac{dV}{d\theta}, \quad w = M\frac{1}{r\sin\theta}\frac{dV}{d\phi},$$

where M is any constant, so long as V satisfies Laplace's equation

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2V}{d\phi^2} = 0.$$

Thus a particular solution of (4) is obviously

$$u = -\frac{1}{k^2}\frac{dV}{dr}, \quad v = -\frac{1}{k^2}\frac{1}{r}\frac{dV}{d\theta}, \quad w = -\frac{1}{k^2}\frac{1}{r\sin\theta}\frac{dV}{d\phi} \qquad (8).$$

This is practically the equivalent in polar coordinates of Mr Love's* treatment in Cartesians.

Putting for shortness

$$\frac{\rho}{m+n} = \alpha^2, \quad \frac{\rho}{n} = \beta^2 \qquad (9),$$

* *Treatise on Elasticity*, Vol. I., Arts 139 and 201.

we may represent the complete solution of (4) by the typical terms:—

$$u = \cos kt\left[-\frac{1}{k^2}\frac{d}{dr}(r^iV_i + r^{-i-1}V_{-i-1})\right.$$
$$+\frac{1}{k^2\alpha^2}\left\{\tfrac{1}{2}r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(k\alpha r)\right\}Y_i + r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(k\beta r)\,Z_i$$
$$\left.+\frac{1}{k^2\alpha^2}\left\{\tfrac{1}{2}r^{-\frac{3}{2}}J_{-i-\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}}\frac{d}{dr}J_{-i-\frac{1}{2}}(k\alpha r)\right\}Y_{-i-1} + r^{-\frac{3}{2}}J_{-i-\frac{1}{2}}(k\beta r)Z_{-i-1}\right]\text{.......(10)},$$

$$v = \cos kt\left[\frac{d}{d\theta}\left\{-\frac{1}{k^2r}(r^iV_i + r^{-i-1}V_{-i-1}) - \frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\left(J_{i+\frac{1}{2}}(k\alpha r)\,Y_i + J_{-i-\frac{1}{2}}(k\alpha r)\,Y_{-i-1}\right)\right.\right.$$
$$\left.+\frac{r^{-1}}{i(i+1)}\frac{d}{dr}\cdot r^{\frac{1}{2}}\left(J_{i+\frac{1}{2}}(k\beta r)\,Z_i + J_{-i-\frac{1}{2}}(k\beta r)\,Z_{-i-1}\right)\right\}$$
$$\left.+\frac{1}{\sin\theta}\frac{d}{d\phi}\{r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)\,W_i + r^{-\frac{1}{2}}J_{-i-\frac{1}{2}}(k\beta r)\,W_{-i-1}\}\right]\text{..................(11)},$$

$$w = \cos kt\left[\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{-\frac{1}{k^2r}(r^iV_i + r^{-i-1}V_{-i-1}) - \frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\left(J_{i+\frac{1}{2}}(k\alpha r)\,Y_i + J_{-i-\frac{1}{2}}(k\alpha r)\,Y_{-i-1}\right)\right.\right.$$
$$\left.+\frac{r^{-1}}{i(i+1)}\frac{d}{dr}\cdot r^{\frac{1}{2}}\left(J_{i+\frac{1}{2}}(k\beta r)\,Z_i + J_{-i-\frac{1}{2}}(k\beta r)\,Z_{-i-1}\right)\right\}$$
$$\left.-\frac{d}{d\theta}\{r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)\,W_i + r^{-\frac{1}{2}}J_{-i-\frac{1}{2}}(k\beta r)\,W_{-i-1}\}\right]\text{....................(12)};$$

answering to which

$$\Delta = \cos kt\,\{r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\alpha r)\,Y_i + r^{-\frac{1}{2}}J_{-i-\frac{1}{2}}(k\alpha r)\,Y_{-i-1}\}\text{..................(13)}.$$

In these expressions Y_i, Z_i, Y_{-i-1}, Z_{-i-1}, W_i, W_{-i-1} represent surface harmonics of degree i with constant coefficients.

The *form* of these harmonics depends solely on the harmonics appearing in the bodily and surface forces; their constant coefficients are determined by the surface conditions.

If for instance there be only a bodily force derivable from a potential $r^iV_i\cos kt$, then the surface harmonic appearing in, say, Y_i is the same as that occurring in V_i, and the ratio $Y_i : V_i$ is found from the surface conditions.

As usual $J_{i+\frac{1}{2}}(z)$ and $J_{-i-\frac{1}{2}}(z)$, where $z = k\alpha r$ or $k\beta r$, represent the two solutions of the Bessel's equation

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} + \left\{1 - \frac{(i+\frac{1}{2})^2}{r^2}\right\}z = 0,$$

and their differential coefficients will be denoted by $J'_{i+\frac{1}{2}}(z)$ and $J'_{-i-\frac{1}{2}}(z)$.

For our present work it is convenient to write

$$\left.\begin{aligned} J_{i+\frac{1}{2}}(z) &= L_iz^{i+\frac{1}{2}}\left\{1 - \frac{z^2}{2(2i+3)} + \frac{z^4}{2.4(2i+3)(2i+5)} - \ldots\right\},\\ J_{-i-\frac{1}{2}}(z) &= L_i'z^{-i-\frac{1}{2}}\left\{1 + \frac{z^2}{2(2i-1)} + \frac{z^4}{2.4(2i-1)(2i-3)} + \ldots\right\}\end{aligned}\right\}\text{............(14)},$$

the values of the constants L_i, L_i' being immaterial.

§ 4. In dealing with the stresses it is convenient to use the following abbreviations:—

$$\left.\begin{aligned}
r^{-\frac{1}{2}}\left[\frac{(k\beta r)^2-2(i-1)(i+2)}{(k\alpha r)^2}J_{i+\frac{1}{2}}(k\alpha r)+\frac{2}{k\alpha r}\left\{2J'_{i+\frac{1}{2}}(k\alpha r)-\frac{3}{k\alpha r}J_{i+\frac{1}{2}}(k\alpha r)\right\}\right] &= {}_rA_i,\\
r^{-\frac{3}{2}}k\beta r\left\{2J'_{i+\frac{1}{2}}(k\beta r)-\frac{3}{k\beta r}J_{i+\frac{1}{2}}(k\beta r)\right\} &= {}_rB_i,\\
-r^{-\frac{1}{2}}\frac{1}{k\alpha r}\left\{2J'_{i+\frac{1}{2}}(k\alpha r)-\frac{3}{k\alpha r}J_{i+\frac{1}{2}}(k\alpha r)\right\} &= {}_rC_i,\\
-r^{-\frac{3}{2}}\frac{k\beta r}{i(i+1)}\left[\frac{(k\beta r)^2-2(i-1)(i+2)}{k\beta r}J_{i+\frac{1}{2}}(k\beta r)+2J'_{i+\frac{1}{2}}(k\beta r)-\frac{3}{k\beta r}J_{i+\frac{1}{2}}(k\beta r)\right] &= {}_rD_i,\\
r^{-\frac{1}{2}}\left[\left(\frac{(k\beta r)^2-2}{(k\alpha r)^2}-2\right)J_{i+\frac{1}{2}}(k\alpha r)-\frac{1}{k\alpha r}\left\{2J'_{i+\frac{1}{2}}(k\alpha r)-\frac{3}{k\alpha r}J_{i+\frac{1}{2}}(k\alpha r)\right\}\right] &= {}_rE_i.\\
2r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(k\beta r) &= {}_rF_i,\\
-\frac{2r^{-\frac{1}{2}}}{(k\alpha r)^2}J_{i+\frac{1}{2}}(k\alpha r) &= {}_rG_i,\\
r^{-\frac{3}{2}}\frac{k\beta r}{i(i+1)}\left[\frac{4}{k\beta r}J_{i+\frac{1}{2}}(k\beta r)+2J'_{i+\frac{1}{2}}(k\beta r)-\frac{3}{k\beta r}J_{i+\frac{1}{2}}(k\beta r)\right] &= {}_rH_i
\end{aligned}\right\} \quad \ldots\ldots(15).$$

The expressions obtained by writing $-i-1$ for $+i$ on the left-hand sides of equations (15) will be denoted by ${}_rA_{-i-1}\ldots{}_rH_{-i-1}$ respectively. This substitution, it will be noticed, leaves the values of $i(i+1)$ and $(i-1)(i+2)$ unaltered.

Using these abbreviations we have for the typical terms in the stresses:—

$$\widehat{rr}=n\cos kt\left[-\frac{2}{k^2}\{i(i-1)r^{i-2}V_i+(i+1)(i+2)r^{-i-3}V_{-i-1}\}\right.$$
$$\left.+{}_rA_iY_i+{}_rB_iZ_i+{}_rA_{-i-1}Y_{-i-1}+{}_rB_{-i-1}Z_{-i-1}\right] \quad \ldots\ldots\ldots\ldots(16),$$

$$\widehat{\theta\theta}=n\cos kt\left[-\frac{2}{k^2}\left\{\left(i+\frac{d^2}{d\theta^2}\right)r^{i-2}V_i-\left(i+1-\frac{d^2}{d\theta^2}\right)r^{-i-3}V_{-i-1}\right\}\right.$$
$$+{}_rE_iY_i+{}_rF_iZ_i+{}_rE_{-i-1}Y_{-i-1}+{}_rF_{-i-1}Z_{-i-1}$$
$$+\frac{d^2}{d\theta^2}\left\{{}_rG_iY_i+{}_rH_iZ_i+{}_rG_{-i-1}Y_{-i-1}+{}_rH_{-i-1}Z_{-i-1}\right\}$$
$$\left.+\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{r\left({}_rF_iW_i+{}_rF_{-i-1}W_{-i-1}\right)\right\}\right] \quad \ldots\ldots\ldots\ldots(17),$$

$$\widehat{\phi\phi}=n\cos kt\left[-\frac{2}{k^2}\left\{\left(i+\cot\theta\frac{d}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}\right)V_i-\left(i+1-\cot\theta\frac{d}{d\theta}-\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}\right)V_{-i-1}\right\}\right.$$
$$+{}_rE_iY_i+{}_rF_iZ_i+{}_rE_{-i-1}Y_{-i-1}+{}_rF_{-i-1}Z_{-i-1}$$
$$+\left(\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}+\cot\theta\frac{d}{d\theta}\right)\left\{{}_rG_iY_i+{}_rH_iZ_i+{}_rG_{-i-1}Y_{-i-1}+{}_rH_{-i-1}Z_{-i-1}\right\}$$
$$\left.-\frac{d^2}{d\theta d\phi}\left\{r\operatorname{cosec}\theta\left({}_rF_iW_i+{}_rF_{-i-1}W_{-i-1}\right)\right\}\right] \quad \ldots\ldots\ldots\ldots(18),$$

$$\widehat{r\theta} = n\cos kt\left[-\frac{2}{k^2}\frac{d}{d\theta}\{(i-1)r^{i-2}V_i-(i+2)r^{-i-3}V_{-i-1}\}\right.$$

$$+\frac{d}{d\theta}\{{}_rC_iY_i+{}_rD_iZ_i+{}_rC_{-i-1}Y_{-i-1}+{}_rD_{-i-1}Z_{-i-1}\}$$

$$\left.+\frac{1}{\sin\theta}\frac{d}{d\phi}\{\tfrac{1}{2}r({}_rB_iW_i+{}_rB_{-i-1}W_{-i-1})\}\right] \quad\text{.........................(19)},$$

$$\widehat{r\phi} = n\cos kt\left[-\frac{2}{k^2}\frac{1}{\sin\theta}\frac{d}{d\phi}\{(i-1)r^{i-2}V_i-(i+2)r^{-i-3}V_{-i-1}\}\right.$$

$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\{{}_rC_iY_i+{}_rD_iZ_i+{}_rC_{-i-1}Y_{-i-1}+{}_rD_{-i-1}Z_{-i-1}\}$$

$$\left.-\frac{d}{d\theta}\{\tfrac{1}{2}r({}_rB_iW_i+{}_rB_{-i-1}W_{-i-1})\}\right] \quad\text{.................................(20)},$$

$$\widehat{\theta\phi} = n\cos kt\left[-\frac{2}{k^2}\frac{d^2}{d\theta d\phi}\{\operatorname{cosec}\theta(r^{i-2}V_i+r^{-i-3}V_{-i-1})\}\right.$$

$$+\frac{d^2}{d\theta d\phi}\{\operatorname{cosec}\theta({}_rG_iY_i+{}_rH_iZ_i+{}_rG_{-i-1}Y_{-i-1}+{}_rH_{-i-1}Z_{-i-1})\}$$

$$\left.-\left(\frac{d^2}{d\theta^2}+\frac{i(i+1)}{2}\right)\{r({}_rF_iW_i+{}_rF_{-i-1}W_{-i-1})\}\right] \quad\text{.....................(21)}.$$

§ 5. To get rid of the troublesome prefix r in the surface conditions, we shall write

A_i, B_i etc. for ${}_aA_i$, ${}_aB_i$ etc.,

A'_i, B'_i etc. for ${}_bA_i$, ${}_bB_i$ etc.

Referring to (16), (19) and (20) we see that the six surface conditions (7) lead to the following six equations:—

$$A_iY_i+B_iZ_i+A_{-i-1}Y_{-i-1}+B_{-i-1}Z_{-i-1}$$

$$=\frac{2}{k^2}\{i(i-1)a^{i-2}V_i+(i+1)(i+2)a^{-i-3}V_{-i-1}\}+\frac{1}{n}R_i \quad\text{........................(22)},$$

$$C_iY_i+D_iZ_i+C_{-i-1}Y_{-i-1}+D_{-i-1}Z_{-i-1}$$

$$=\frac{2}{k^2}\{(i-1)a^{i-2}V_i-(i+2)a^{-i-3}V_{-i-1}\}+\frac{1}{n}T_i \quad\text{................................(23)},$$

$$A_i'Y_i+B_i'Z_i+A'_{-i-1}Y_{-i-1}+B'_{-i-1}Z_{-i-1}$$

$$=\frac{2}{k^2}\{i(i-1)b^{i-2}V_i+(i+1)(i+2)b^{-i-3}V_{-i-1}\}+\frac{1}{n}R_i' \quad\text{.......................(24)},$$

$$C_i' Y_i + D_i' Z_i + C'_{-i-1} Y_{-i-1} + D'_{-i-1} Z_{-i-1}$$

$$= \frac{2}{k^2} \{(i-1) b^{i-2} V_i - (i+2) b^{-i-3} V_{-i-1}\} + \frac{1}{n} T_i' \text{ ....(25)},$$

$$aB_i W_i + aB_{-i-1} W_{-i-1} = \frac{2}{n} \tau_i \text{ ....(26)},$$

$$bB_i' W_i + bB'_{-i-1} W_{-i-1} = \frac{2}{n} \tau'_i \text{ ....(27)}.$$

These equations constitute two independent sets. The first set, consisting of the first four equations, determines the four unknowns Y_i, Z_i, Y_{-i-1}, Z_{-i-1}. These have to do either with the bodily forces or the surface forces of the first two classes. The second set, consisting of the remaining two equations, determines W_i and W_{-i-1}. These have to do exclusively with the surface forces of the third class.

The values of Y_i, Z_i, Y_{-i-1}, Z_{-i-1} may of course in any case be easily written down in the shape of determinants, the denominators having the common value Π_i, where

$$\Pi_i = \begin{vmatrix} A_i, & B_i, & A_{-i-1}, & B_{-i-1} \\ C_i, & D_i, & C_{-i-1}, & D_{-i-1} \\ A_i', & B_i', & A'_{-i-1}, & B'_{-i-1} \\ C_i', & D_i', & C'_{-i-1}, & D'_{-i-1} \end{vmatrix} \text{ ....(28)}.$$

The values of the determinants are however somewhat complicated, and the deduction of numerical results answering to given numerical values of k, a, b, m, n would entail a good deal of labour.

The expressions given by (26) and (27) for W_i and W_{-i-1} are comparatively short, and numerical values would not be hard to deduce, supposing tables of the Bessel's functions with arguments $\pm(i+\frac{1}{2})$ existent.

§ 6. Before proceeding further it is convenient to establish one very general relation between the displacements due to bodily forces and those due to surface forces when the elastic material is supposed incompressible. By an *incompressible* material is meant one in which the bulk-modulus $m - \frac{1}{3}n$ is infinite, while the rigidity and Young's modulus are finite; in other words, while n is finite n/m is zero.

Referring to (9) we see that in such a material $k\alpha a$ vanishes compared to $k\beta a$. Here we assume k and a, and so $k\beta a$, finite.

The general relation is as follows:—

The displacements at any point of a sphere or spherical shell of incompressible material, due to bodily forces derivable from a potential V satisfying Laplace's equation, are identical with those due to pure radial surface forces equal to the product of the density ρ into the surface values of V.

For instance, in a spherical shell of radii a and b the displacements due to the bodily forces derivable from the potential $r^i V_i \cos kt$ are identical with those due to the combined action of the radial surface forces

$$R = \rho a^i V_i \cos kt \text{ over } r = a,$$
$$R' = \rho b^i V_i \cos kt \quad ,, \quad r = b.$$

It will be sufficient to prove the relation for the terms depending on surface harmonics of degree i.

Since by hypothesis $k\alpha a$ is vanishingly small, we may in any expression neglect all but the algebraically lowest power.

We thus get from (15),

$$\left.\begin{aligned} A_i &= L_i a^{-\frac{1}{2}} (k\alpha a)^{i-\frac{3}{2}} \{(k\beta a)^2 - 2i(i-1)\}, \\ C_i &= - L_i a^{-\frac{1}{2}} (k\alpha a)^{i-\frac{3}{2}} \times 2(i-1), \\ A_{-i-1} &= L_i' a^{-\frac{1}{2}} (k\alpha a)^{-i-\frac{5}{2}} \{(k\beta a)^2 - 2(i+1)(i+2)\}, \\ C_{-i-1} &= L_i' a^{-\frac{1}{2}} (k\alpha a)^{-i-\frac{5}{2}} \times 2(i+2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

Taking $R_i = 0$, and using the above values of A_i and A_{-i-1}, we see that (22) may be written in the form

$$A_i \left\{ Y_i + \frac{1}{k^2 L_i} (k\alpha)^{-i+\frac{3}{2}} V_i \right\} + B_i Z_i + A_{-i-1} \left\{ Y_{-i-1} + \frac{1}{k^2 L_i'} (k\alpha)^{i+\frac{5}{2}} V_{-i-1} \right\} + B_{-i-1} Z_{-i-1}$$

$$= \frac{2}{k^2} \{ i(i-1) a^{i-2} V_i + (i+1)(i+2) a^{-i-3} V_{-i-1} \}$$

$$+ \frac{1}{k^2 L_i} (k\alpha)^{-i+\frac{3}{2}} V_i L_i a^{-\frac{1}{2}} (k\alpha a)^{i-\frac{3}{2}} \{(k\beta a)^2 - 2i(i-1)\}$$

$$+ \frac{1}{k^2 L_i'} (k\alpha)^{i+\frac{5}{2}} V_{-i-1} L_i' a^{-\frac{1}{2}} (k\alpha a)^{-i-\frac{5}{2}} \{(k\beta a)^2 - 2(i+1)(i+2)\},$$

$$= \frac{1}{n} (V_i \rho a^i + V_{-i-1} \rho a^{-i-1})$$

after reduction, using (9).

Similarly (23), (24) and (25), in the absence of surface forces, may be written

$$C_i \left\{ Y_i + \frac{1}{k^2 L_i} (k\alpha)^{-i+\frac{3}{2}} V_i \right\} + D_i Z_i + C_{-i-1} \left\{ Y_{-i-1} + \frac{1}{k^2 L_i'} (k\alpha)^{i+\frac{5}{2}} V_{-i-1} \right\} + D_{-i-1} Z_{-i-1} = 0,$$

$$A_i' \left\{ Y_i + \frac{1}{k^2 L_i} (k\alpha)^{-i+\frac{3}{2}} V_i \right\} + B_i' Z_i + A'_{-i-1} \left\{ Y_{-i-1} + \frac{1}{k^2 L_i'} (k\alpha)^{i+\frac{5}{2}} V_{-i-1} \right\} + B'_{-i-1} Z_{-i-1}$$

$$= \frac{1}{n} (V_i \rho b^i + V_{-i-1} \rho b^{-i-1}),$$

$$C_i' \left\{ Y_i + \frac{1}{k^2 L_i} (k\alpha)^{-i+\frac{3}{2}} V_i \right\} + D_i' Z_i + C'_{-i-1} \left\{ Y_{-i-1} + \frac{1}{k^2 L_i'} (k\alpha)^{i+\frac{5}{2}} V_{-i-1} \right\} + D'_{-i-1} Z_{-i-1} = 0.$$

Comparing these equations with (22), (23), (24) and (25), we see that the values of $Y_i + \frac{1}{k^2 L_i}(k\alpha)^{-i+\frac{3}{2}} V_i$, Z_i, $Y_{-i-1} + \frac{1}{k^2 L_i'}(k\alpha)^{i+\frac{5}{2}} V_{-i-1}$ and Z_{-i-1} when there act only bodily forces derivable from the potential $(r^i V_i + r^{-i-1} V_{-i-1}) \cos kt$ are identical with the values of Y_i, Z_i, Y_{-i-1}, Z_{-i-1} respectively when there act only the pure radial surface forces

$$\left.\begin{aligned} R &= \rho\,(a^i V_i + a^{-i-1} V_{-i-1}) \cos kt \text{ over } r = a, \\ R' &= \rho\,(b^i V_i + b^{-i-1} V_{-i-1}) \cos kt \quad ,, \quad r = b \end{aligned}\right\} \qquad (30).$$

Again, retaining only the algebraically lowest power of α, we may write (10) in the form

$$u = \cos kt \left[- r^{i-1} i L_i (k\alpha)^{i-\frac{3}{2}} \left\{ Y_i + \frac{1}{k^2 L_i}(k\alpha)^{-i+\frac{3}{2}} V_i \right\} + r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) Z_i \right.$$
$$\left. + r^{-i-2}(i+1) L_i' (k\alpha)^{-i-\frac{5}{2}} \left\{ Y_{-i-1} + \frac{1}{k^2 L_i'}(k\alpha)^{i+\frac{5}{2}} V_{-i-1} \right\} + r^{-\frac{1}{2}} J_{-i-\frac{1}{2}}(k\beta r) Z_{-i-1} \right].$$

This shows that when there act bodily forces derivable from the potential

$$(r^i V_i + r^{-i-1} V_{-i-1}) \cos kt$$

the expression for u is the same as when there act instead the surface forces required to give to Y_i, Z_i, Y_{-i-1} and Z_{-i-1} respectively the values which belong to $Y_i + \frac{1}{k^2 L_i}(k\alpha)^{-i+\frac{3}{2}} V_i$, Z_i, $Y_{i-1} + \frac{1}{k^2 L_i'}(k\alpha)^{i+\frac{5}{2}} V_{-i-1}$ and Z_{-i-1} when the bodily forces act. The requisite system of surface forces as we have just seen is (30).

Our theorem is thus established for the displacement u. Its proof for v and w proceeds on the same lines and is even more easy, it being noticed that W_i and W_{-i-1} in (11) and (12) vanish.

The proof for the solid sphere is really included in the above; it is also easily given independently.

The theorem, it need hardly be said, is not confined to forces varying with the time. If its deduction for the case of equilibrium, by regarding equilibrium as the limiting form of vibration when k vanishes, should seem questionable, it will be found a simple matter to deduce it directly from the equations of equilibrium, or to verify it in the explicit solutions I have given for the general case of equilibrium of the sphere* and spherical shell†.

A particular instance of the theorem was noticed by Professor G. H. Darwin‡ as long ago as 1879, and its truth in the general case of equilibrium of a solid sphere was established by myself in 1887§. In the future it is not unlikely a still more comprehensive result may be established applicable to all shapes of bodies.

* *Camb. Phil. Soc. Trans.*, Vol. XIV., equations (36) to (38), pp. 264—5.

† *Camb. Phil. Soc. Trans.*, Vol. XV., equations (92) and (93), pp. 362—5.

‡ *Phil. Trans.* for 1879, pp. 6 *et seq.*, and *Phil. Trans.* for 1882, p. 200.

§ *Camb. Phil. Soc. Trans.*, Vol. XIV., p. 265.

SOLID SPHERE.

PURE RADIAL VIBRATIONS.

§ 7. The case of pure radial vibrations accompanying the application of the pure radial surface forces

$$R = R_0 \cos kt,$$

where R_0 is a constant, can be deduced from the general case of the radial surface forces

$$R = R_i \cos kt,$$

where R_i is a spherical harmonic of degree i.

It is desirable however to treat the pure radial vibrations independently, both on account of their importance and because they may accompany the action of a type of bodily force not provided for by the general solution. The type in question consists of a radial force

$$2V_0 r \cos kt,$$

where V_0 is a constant.

The corresponding body-stress equations are found by writing $V_0 r^2 \cos kt$ for V in (4). They thus answer to a species of potential, which does not however satisfy Laplace's equation. Forces of this kind would arise in the case of rotation about an axis, if the angular velocity were a periodic function of the time. For supposing this angular velocity to be

$$\omega \cos k't,$$

and to take place about $\theta = 0$, we may regard the "centrifugal" forces as answering to a potential

$$V = \tfrac{1}{2}\omega^2 r^2 \sin^2\theta \cos^2 k't = \tfrac{1}{6}(\omega^2 r^2 - \omega^2 r^2 P_2)(1 + \cos 2k't),$$

where P_2 is the second zonal harmonic.

The potential

$$\tfrac{1}{6}(\omega^2 r^2 - \omega^2 r^2 P_2)$$

is a form considered in my equilibrium solution*; the potential

$$-\tfrac{1}{6}\omega^2 r^2 P_2 \cos 2k't$$

comes under the general case of bodily forces considered presently; and the remaining term in the potential

$$\tfrac{1}{6}\omega^2 r^2 \cos 2k't$$

is a special case of the problem we are just entering on, with

$$V_0 = \tfrac{1}{6}\omega^2, \quad k = 2k'.$$

If from any cause gravity were supposed to contain any periodic terms, the corresponding forces would also be of the type specified.

* *Camb. Phil. Soc. Trans.*, Vol. XIV., pp. 286, et seq.

The problem proposed is to find the forced vibrations in a sphere of radius a due to the simultaneous or independent action of

radial bodily forces $2V_0 r\cos kt$,

„ surface „ $R_0\cos kt$,

where V_0 and R_0 are constants.

Replacing V in (4) by $V_0 r^2\cos kt$, we find that a particular solution is

$$u = -\frac{2}{k^2}V_0 r\cos kt \text{ ....(31)};$$

for this makes Δ constant, while $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ all vanish.

The complete solution is thus

$$u = \cos kt\left[-\frac{2}{k^2}V_0 r + \frac{A}{(k\alpha r)^2}r\left\{\frac{\sin k\alpha r}{k\alpha r} - \cos k\alpha r\right\}\right] \text{ ....(32)},$$

where A is a constant determined by

$$A\left[(m+n)\frac{\sin k\alpha a}{k\alpha a} - \frac{4n}{(k\alpha a)^2}\left(\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right)\right] = R_0 + 2(3m-n)\frac{V_0}{k^2} \text{ ....(33)}.$$

Substituting in (32) the value of A determined by (33) we obtain the solution in its complete form. From a mathematical standpoint this is all that is wanted, but a complicated mathematical expression such as ensues can be made to yield the sort of information a physicist desires only when definite numerical values are ascribed to k, m/n, ρ and a. One can not foresee what individual cases are likely to be of most use, and the construction of elaborate tables for a large variety of values of k, m/n &c. might be a waste of time. Further attention is thus confined to the case when the frequency of the applied forces is small compared to that of the fundamental note of the pure radial type of free vibrations.

The frequency equation of this type is obtained by equating to zero the coefficient of A in (33). Denoting Poisson's ratio $(m-n)/2m$ by η, we know that for the fundamental vibration* $k\alpha a/\pi$ increases from ·6626 when $\eta = 0$, to 1 when $\eta = \cdot 5$. Our hypothesis thus amounts to assuming $k\alpha a$ a small fraction, so that the trigonometrical series for $\sin k\alpha a$ and $\cos k\alpha a$ are rapidly convergent.

This being our first example, the method of treatment will be shown in some detail.

Expanding in powers of $k\alpha a$ we transform (33) into

$$\frac{A}{3}(3m-n)\left\{1 - (k\alpha a)^2\frac{5m+n}{10(3m-n)} + (k\alpha a)^4\frac{7m+3n}{280(3m-n)}\dots\dots\right\} = R_0 + 2(3m-n)\frac{V_0}{k^2}.$$

* See Prof. Lamb in *Proc. Lond. Math. Soc.*, Vol. XIII., p. 202.

Thus we find from (32)

$$\frac{u}{\cos kt}\frac{3m-n}{3}\left\{1-(k\alpha a)^2\frac{5m+n}{10(3m-n)}+(k\alpha a)^4\frac{7m+3n}{280(3m-n)}\dots\dots\right\}$$

$$=-\frac{2}{k^2}\frac{3m-n}{3}V_0 r\left\{1-(k\alpha a)^2\frac{5m+n}{10(3m-n)}+(k\alpha a)^4\frac{7m+3n}{280(3m-n)}\dots\dots\right\}$$

$$+\left\{R_0+2(3m-n)\frac{V_0}{k^2}\right\}\frac{r}{3}\left\{1-\frac{(k\alpha r)^2}{10}+\frac{(k\alpha r)^4}{280}\dots\dots\right\}$$

$$=\tfrac{1}{3}R_0 r\left\{1-\tfrac{1}{10}(k\alpha r)^2\dots\dots\right\}$$

$$-\tfrac{2}{3}\frac{3m-n}{k^2}V_0 r\left[1-1-\tfrac{1}{10}k^2\alpha^2\left(\frac{5m+n}{3m-n}a^2-r^2\right)+\tfrac{1}{280}k^4\alpha^4\left(\frac{7m+3n}{3m-n}a^4-r^4\right)\dots\dots\right].$$

It is important to notice that the principal terms in the coefficient of V_0 cut out. But for this, the approximation need not have been carried so far, as the fourth and higher powers of $k\alpha a$ are neglected in our final result. Taking the coefficient of u to the other side of the equation and reducing, we have to the specified degree of approximation

$$u=\frac{R_0 r\cos kt}{3m-n}\left[1+\tfrac{1}{10}k^2\rho\frac{(5m+n)a^2-(3m-n)r^2}{(m+n)(3m-n)}\right]$$

$$+\frac{V_0\rho r\cos kt}{5(m+n)(3m-n)}\Big[(5m+n)a^2-(3m-n)r^2$$

$$+\frac{k^2\rho}{140(m+n)(3m-n)}\{(245m^2+130mn+29n^2)a^4$$

$$-14(3m-n)(5m+n)a^2r^2+5(3m-n)^2r^4\}\Big]\dots\dots\dots(34).$$

For the value u_a of the displacement at the surface we find

$$u_a=\frac{aR_0\cos kt}{3m-n}\left(1+\tfrac{1}{5}\frac{k^2a^2\rho}{3m-n}\right)+\frac{2a^3V_0\rho\cos kt}{5(3m-n)}\left\{1+\tfrac{2}{35}\frac{k^2a^2\rho(5m+3n)}{(m+n)(3m-n)}\right\}\dots\dots\dots\dots(35).$$

When the terms containing k^2 in (34) are neglected, we obtain for the displacement an expression identical with that supplied by the equilibrium theory. This is I think obvious *a priori*, and merely serves as a confirmation so far of the accuracy of the work.

When we have only surface forces we may, to the present degree of approximation, start at once by neglecting terms in $(k\alpha a)^4$ in the coefficient of A in (33).

When however there are bodily forces it is quite different. The particular solution (31), as containing k^2 in the denominator, becomes infinite when $k=0$, and so the complementary solution is bound to supply a term in k^{-2} to cut it out. Thus if in substituting for A in (32) one went only as far as the $(k\alpha a)^2$ term, one would arrive only at the equilibrium value of u. Terms in k^2 it is true would appear in the denominator, answering to the coefficient of A in (33), but in the absence of the terms of the same degree which should appear in the numerator their presence would be absolutely useless, if not misleading.

Unless the approximation is carried so far as to give correctly the terms of order $(k\alpha a)^2$ in u it is impossible to form a trustworthy estimate of the degree of accuracy of the equilibrium theory. Supposing for instance

$$k\alpha a = 1/10,$$

it is quite true that $(k\alpha a)^2$ itself is small compared to 1, but until one knows the size of the numerical coefficients of the terms of order $(k\alpha a)^2$ it is illegitimate to characterise them as negligible. Strictly speaking, even when the terms of order $(k\alpha a)^2$ are determined, one is hardly justified in drawing physical conclusions without having regard to the possible importance of terms containing higher powers of k. That these terms must in reality be very small may however be readily seen by reference to the rapidity with which $x^{-1}\sin x$ and $x^{-2}(x^{-1}\sin x - \cos x)$ converge when x is small.

Returning to (34), we see that the coefficient of k^2 is positive for all possible values of r in the case both of R_0 and V_0. Thus the displacement is always and everywhere greater than according to the equilibrium theory. An idea of the magnitude of the difference between the dynamical and equilibrium theories is most easily derived from the surface value (35) of the displacement.

In terms of Poisson's ratio we thence deduce for the ratio of the dynamical to the equilibrium value:—

in the case of R_0

$$1 + \frac{k^2\rho a^2}{n}\frac{1-2\eta}{10(1+\eta)} : 1, \text{ or } 1 + (k\alpha a)^2\frac{1-\eta}{5(1+\eta)} : 1;$$

in the case of V_0

$$1 + \frac{k^2\rho a^2}{n}\frac{(1-2\eta)(4-3\eta)}{35(1-\eta^2)} : 1, \text{ or } 1 + (k\alpha a)^2\frac{2(4-3\eta)}{35(1+\eta)} : 1.$$

Taking $k^2\rho a^2/n$ as constant, we see that as η increases from 0 to ·5 the term in k^2, or what may be called the *dynamical correction*, diminishes from $\frac{1}{10}\frac{k^2\rho a^2}{n}$ to 0 in the case of the surface forces, and from $\frac{4}{35}\frac{k^2\rho a^2}{n}$ to 0 in the case of the bodily forces.

In cases where the frequency $k/2\pi$ is compared with the frequency $K/2\pi$ of the fundamental free radial vibration the following table will be found instructive. The quantity tabulated is $\dfrac{\text{dynamical value of } u_a}{\text{equilibrium value of } u_a}$.

TABLE I.

	$\eta =$ 0	·25	·3	·5
Case of Surface forces $R_0 \cos kt$	$1 + 0·867\,\frac{k^2}{K^2}$	$1 + 0·789\,\frac{k^2}{K^2}$	$1 + 0·753\,\frac{k^2}{K^2}$	$1 + 0·658\,\frac{k^2}{K^2}$
Case of Bodily forces $2V_0 r \cos kt$	$1 + 0·990\,\frac{k^2}{K^2}$	$1 + 0·976\,\frac{k^2}{K^2}$	$1 + 0·968\,\frac{k^2}{K^2}$	$1 + 0·940\,\frac{k^2}{K^2}$

The dynamical correction is always more important for the bodily than the surface forces. In the case of the bodily forces the coefficient of k^2/K^2 is wonderfully constant.

SOLID SPHERE.

MIXED RADIAL AND TRANSVERSE VIBRATIONS.

§ 8. The typical vibrations are those answering to the bodily forces derivable from the potential $r^i V_i \cos kt$, and to the surface forces

$$R = R_i \cos kt, \quad \Theta = \frac{dT_i}{d\theta} \cos kt, \quad \Phi = \frac{1}{\sin\theta} \frac{dT_i}{d\phi} \cos kt.$$

To determine the values of Y_i and Z_i in the general solution (10), (11), (12), we have by (22) and (23)

$$\left.\begin{aligned} A_i Y_i + B_i Z_i &= \frac{2}{k^2} i (i-1) a^{i-2} V_i + \frac{1}{n} R_i, \\ C_i Y_i + D_i Z_i &= \frac{2}{k^2} (i-1) a^{i-2} V_i + \frac{1}{n} T_i \end{aligned}\right\} \quad \text{.........................(36).}$$

Thus

$$\left.\begin{aligned} Y_i &= \left\{\frac{2}{k^2} (i-1)(iD_i - B_i) a^{i-2} V_i + \frac{1}{n} (R_i D_i - T_i B_i)\right\} \div \Pi_i, \\ Z_i &= \left\{\frac{2}{k^2} (i-1)(A_i - iC_i) a^{i-2} V_i + \frac{1}{n} (T_i A_i - R_i C_i)\right\} \div \Pi_i \end{aligned}\right\} \quad \text{............(37),}$$

where

$$\Pi_i = A_i D_i - B_i C_i.$$

A_i, B_i, C_i, D_i, being obtained by writing a for r in equations (15), are known quantities; thus the substitution in (10), (11) and (12) of the values of Y_i and Z_i given by equations (37) supplies the complete mathematical solution of the problem proposed.

We shall confine our further attention to the case when the frequency of the forced vibrations is small compared to that of the fundamental free vibration of the type mixed radial and transverse. This is equivalent to assuming $k\alpha a$ and $k\beta a$ small compared to unity. For shortness we shall write x for $k\alpha a$ and y for $k\beta a$.

Taking (14) for the definition of the Bessel, we find

$$A_i = L_i a^{-\frac{1}{2}} x^{i-\frac{3}{2}} \left[-2i(i-1) + y^2 + \frac{i^2 - i - 4}{2i+3} x^2 - \frac{1}{2(2i+3)} y^2 x^2 - \frac{i^2 - i - 8}{4(2i+3)(2i+5)} x^4 \right.$$
$$\left. + \frac{1}{8(2i+3)(2i+5)} y^2 x^4 + \frac{(i-4)(i+3)}{24(2i+3)(2i+5)(2i+7)} x^6 \ldots \right] \text{......(38),}$$

$$B_i = L_i 2a^{-\frac{3}{2}} y^{i+\frac{1}{2}} \left[i - 1 - \frac{i+1}{2(2i+3)} y^2 + \frac{i+3}{8(2i+3)(2i+5)} y^4 \right.$$
$$\left. - \frac{i+5}{48(2i+3)(2i+5)(2i+7)} y^6 \ldots \right] \text{.................................(39).}$$

$$C_i = L_i 2a^{-\frac{1}{2}}x^{i-\frac{3}{2}}\left[-(i-1) + \frac{i+1}{2(2i+3)}x^2 - \frac{i+3}{8(2i+3)(2i+5)}x^4 + \frac{i+5}{48(2i+3)(2i+5)(2i+7)}x^6 \ldots\right] \quad \ldots\ldots(40),$$

$$D_i = L_i \frac{2a^{-\frac{1}{2}}y^{i+\frac{1}{2}}}{i(i+1)}\left[(i-1)(i+1) - \frac{i(i+2)}{2(2i+3)}y^2 + \frac{i^2+4i+5}{8(2i+3)(2i+5)}y^4 - \frac{i^2+6i+14}{48(2i+3)(2i+5)(2i+7)}y^6 \ldots\right] \quad \ldots\ldots(41).$$

whence

$$A_i - iC_i = L_i 2a^{-\frac{1}{2}}x^{i-\frac{3}{2}}\left[\tfrac{1}{2}y^2 - \frac{i+2}{2i+3}x^2 - \frac{1}{4(2i+3)}y^2x^2 + \frac{i+2}{2(2i+3)(2i+5)}x^4 + \frac{(2i+7)y^2x^4 - 2(i+2)x^6}{16(2i+3)(2i+5)(2i+7)}\ldots\right] \quad \ldots\ldots(42),$$

$$iD_i - B_i = L_i a^{-\frac{1}{2}}y^{i+\frac{1}{2}}\frac{y^2}{(i+1)(2i+3)}\left[1 + \frac{1}{2(2i+5)}y^2 - \frac{3}{8(2i+5)(2i+7)}y^4\ldots\right] \quad \ldots\ldots(43),$$

$$\Pi_i = (L_i)^2 \frac{2a^{-3}x^{i-\frac{3}{2}}y^{i+\frac{1}{2}}}{i(i+1)(2i+3)}\left[(i-1)(2i^2+4i+3)y^2 - 2(i-1)(i+1)(i+2)x^2 - \frac{i(2i^2+10i+9)}{2(2i+5)}y^4 + \tfrac{1}{2}(3i+1)y^2x^2 + \frac{(i-1)(i+1)(i+2)}{2i+5}x^4 + \frac{2i^3+18i^2+35i+35}{8(2i+5)(2i+7)}y^6 + \frac{i^3+4i^2-2i-10}{4(2i+3)(2i+5)}y^4x^2 - \frac{2i^3+14i^2+21i+3}{8(2i+3)(2i+5)}y^2x^4 - \frac{(i-1)(i+1)(i+2)}{4(2i+5)(2i+7)}x^6\ldots\right] \quad \ldots\ldots(44).$$

Substituting these values of $A_i, \ldots \Pi_i$ in equations (37), we get the values of Y_i and Z_i to a close degree of approximation. These values of Y_i and Z_i are then to be substituted in the expressions supplied by (10), (11) and (12) for the solid sphere when $k\alpha r$ and $k\beta r$ are treated as small quantities, viz.

$$u = \cos kt\left[-\frac{i}{k^2}r^{i-1}V_i - Y_iL_i\left(\frac{x}{a}\right)^{i-\frac{3}{2}}r^{i-1}\left\{i - \frac{i+2}{2(2i+3)}(k\alpha r)^2 + \frac{i+4}{8(2i+3)(2i+5)}(k\alpha r)^4\ldots\right\} + Z_iL_i\left(\frac{y}{a}\right)^{i+\frac{1}{2}}r^{i-1}\left\{1 - \frac{1}{2(2i+3)}(k\beta r)^2 + \frac{1}{8(2i+3)(2i+5)}(k\beta r)^4\ldots\right\}\right] \quad \ldots\ldots(45),$$

$$v = \cos kt\frac{d}{d\theta}\left[-\frac{1}{k^2}r^{i-1}V_i - Y_iL_i\left(\frac{x}{a}\right)^{i-\frac{3}{2}}r^{i-1}\left\{1 - \frac{1}{2(2i+3)}(k\alpha r)^2 + \frac{1}{8(2i+3)(2i+5)}(k\alpha r)^4\ldots\right\} + Z_iL_i\left(\frac{y}{a}\right)^{i+\frac{1}{2}}\frac{r^{i-1}}{i(i+1)}\left\{i+1 - \frac{i+3}{2(2i+3)}(k\beta r)^2 + \frac{i+5}{8(2i+3)(2i+5)}(k\beta r)^4\ldots\right\}\right]\ldots(46),$$

$$w = \cos kt \frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{1}{k^2} r^{i-1} V_i\right.$$

$$- Y_i L_i \left(\frac{x}{a}\right)^{i-\frac{3}{2}} r^{i-1}\left\{1 - \frac{1}{2(2i+3)}(k\alpha r)^2 + \frac{1}{8(2i+3)(2i+5)}(k\alpha r)^4 \ldots\right\}$$

$$\left. + Z_i L_i \left(\frac{y}{a}\right)^{i+\frac{1}{2}} \frac{r^{i-1}}{i(i+1)}\left\{i+1 - \frac{i+3}{2(2i+3)}(k\beta r)^2 + \frac{i+5}{8(2i+3)(2i+5)}(k\beta r)^4 \ldots\right\}\right] \ldots (47).$$

By (37), (39), (41) and (43), and again by (37), (38), (40) and (42), we see that $(L_i)^2 x^{i-\frac{3}{2}} y^{i+\frac{1}{2}}/\Pi_i$ appears as a factor in both $Y_i L_i x^{i-\frac{3}{2}}$ and $Z_i L_i y^{i+\frac{1}{2}}$, while by (44) we see that Π_i contains the factor $(L_i)^2 x^{i-\frac{3}{2}} y^{i+\frac{1}{2}}$. This factor thus cuts out in the expressions for the displacements. When this factor is removed it is easily seen that the terms containing k^2 in the denominator in (45), (46) and (47) cut out, precisely as in the corresponding case in § 7. It is this that necessitates the carrying the approximation so far as in (44) to get correctly the terms of order $(k\alpha a)^2$ and $(k\beta a)^2$ in the displacements.

Again the terms containing lowest powers of x and y in $A_i D_i - B_i C_i$ cut out, so that for the degree of accuracy attained in (44) we require to find A_i, B_i etc. to the degree of approximation shown by (38), (39) etc.

When the expressions (45), (46), (47) are reduced as far as possible, we still have occurring in the denominator the complicated expression

$$(i-1)\left[(2i^2+4i+3)(m+n) - 2(i+1)(i+2)n\right.$$

$$-\frac{k^2\rho}{2(i-1)(2i+5)n(m+n)}\{i(2i^2+10i+9)(m+n)^2 - (2i+5)(3i+1)n(m+n)$$

$$\left. - 2(i^2-1)(i+2)n^2\}\right].$$

By putting this into what is to the present degree of approximation the equivalent form,

$$(i-1)\{(2i^2+4i+3)m - (2i+1)n\} \times$$

$$\left[1 + k^2\rho \frac{i(2i^2+10i+9)m^2 + (4i^3+14i^2+i-5)mn - (6i+1)n^2}{2\{(2i^2+4i+3)m - (2i+1)n\}(i-1)(2i+5)n(m+n)}\right]^{-1},$$

we get all the terms containing k^2 into the numerator.

It is obvious, however, that the resulting expressions for the displacements must be in general cumbrous, and I have not thought it worth while to work out and record them. In any specified case the values of i and of m/n will be given, and the labour required to obtain the solution in its most convenient form from (45), (46) and (47) by using the particular values of Y_i and Z_i deduced from (37) will not much exceed that required to convert into figures the general formulae resulting from the substitution of the general values of Y_i and Z_i.

§ 9. I have worked out three cases explicitly. The first is that of incompressible material, i.e. material for which n/m, and so x/y, is negligible. In this case we have for all integral values of i not less than 2,

$$u = \frac{(R_i + \rho a^i V_i) \cos kt\, ir^{i-1}a^{-i}}{2n(i-1)(2i^2+4i+3)} \Bigg[i(i+2)a^2 - (i^2-1)r^2$$

$$+ \frac{(k^2\rho/n)}{4(i-1)(2i+5)(2i^2+4i+3)} \{(2i^5 + 18i^4 + 41i^3 + 33i^2 + 17i + 15)a^4$$

$$-2(i-1)i(i+1)(2i^2+10i+9)a^2r^2 + (i-1)^2(i+1)(2i^2+4i+3)r^4\} \Bigg]$$

$$+ \frac{T_i \cos kt\, i(i+1) r^{i-1}a^{-i}}{2n(i-1)(2i^2+4i+3)} \Bigg[-(i^2-i-3)a^2 + i(i-1)r^2$$

$$- \frac{(k^2\rho/n)}{4(i-1)(2i+5)(2i^2+4i+3)} \{i(2i^4 + 8i^3 - 19i^2 - 72i - 45)a^4$$

$$-2(i-1)(2i^4+6i^3-9i^2-26i-15)a^2r^2 + i(i-1)^2(2i^2+4i+3)r^4\} \Bigg] \quad\text{..................(48),}$$

$$v = \frac{\frac{d}{d\theta}(R_i + \rho a^i V_i)\cos kt r^{i-1}a^{-i}}{2n(i-1)(2i^2+4i+3)} \Bigg[i(i+2)a^2 - (i-1)(i+3)r^2$$

$$+ \frac{(k^2\rho/n)}{4(i-1)(2i+5)(2i^2+4i+3)} \{(2i^5 + 18i^4 + 41i^3 + 33i^2 + 17i + 15)a^4$$

$$-2(i-1)i(i+3)(2i^2+10i+9)a^2r^2 + (i-1)^2(i+5)(2i^2+4i+3)r^4\} \Bigg]$$

$$+ \frac{\frac{dT_i}{d\theta}\cos kt r^{i-1}a^{-i}}{2n(i-1)(2i^2+4i+3)} \Bigg[-(i+1)(i^2-i-3)a^2 + (i-1)i(i+3)r^2$$

$$- \frac{(k^2\rho/n)}{4(i-1)(2i+5)(2i^2+4i+3)} \{i(i+1)(2i^4 + 8i^3 - 19i^2 - 72i - 45)a^4$$

$$-2(i-1)(i+3)(2i^4+6i^3-9i^2-26i-15)a^2r^2 + (i-1)^2 i(i+5)(2i^2+4i+3)r^4\} \Bigg] \quad\text{... (49),}$$

$$w = \left[\text{Expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in (49)} \right] \quad\text{..............................(50).}$$

When i is a given integer the somewhat long algebraic functions of i become concise numerical quantities. When i is big the terms containing k^2 bear to the others—i.e. to the equilibrium terms—a ratio of the order $1 : i$. Thus the dynamical correction is relatively of less and less importance as i increases.

The surface displacements, being the only ones admitting of direct observation, claim special attention. Distinguishing them by the suffix $_a$, we have

$$u_a = \frac{(R_i + \rho a^i V_i)\cos kt\, i(2i+1)\,a}{2n(i-1)(2i^2+4i+3)}\left\{1 + \frac{k^2\rho a^2}{n}\,\frac{3(4i^3+8i^2+6i+3)}{2(i-1)(2i+1)(2i+5)(2i^2+4i+3)}\right\}$$

$$+\frac{T_i\cos kt\, 3i(i+1)\,a}{2n(i-1)(2i^2+4i+3)}\left\{1 - \frac{k^2\rho a^2}{n}\,\frac{4i^3-20i^2-32i-15}{6(i-1)(2i+5)(2i^2+4i+3)}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (51),$$

$$v_a = \frac{\dfrac{d}{d\theta}(R_i + \rho a^i V_i)\cos kt\, 3a}{2n(i-1)(2i^2+4i+3)}\left\{1 - \frac{k^2\rho a^2}{n}\,\frac{4i^3-20i^2-32i-15}{6(i-1)(2i+5)(2i^2+4i+3)}\right\}$$

$$+\frac{\dfrac{dT_i}{d\theta}\cos kt\,(2i^2+i+3)\,a}{2n(i-1)(2i^2+4i+3)}\left\{1 + \frac{k^2\rho a^2}{n}\,\frac{4i^4-8i^3+22i^2+63i+45}{2(i-1)(2i+5)(2i^2+i+3)(2i^2+4i+3)}\right\}\ \ldots\ldots (52),$$

$$w_a = \left[\text{Expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v_a\right]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (53).$$

We see that the radial displacement answering either to the bodily forces or the radial surface forces, and the tangential displacements answering to the tangential surface forces, are invariably greater on the dynamical than on the equilibrium theory. Since however $4i^3 - 20i^2 - 32i - 15$ is negative only so long as i is less than 7, the radial displacement due to tangential surface forces and the tangential displacements due to radial surface forces are greater on the dynamical than on the equilibrium theory only so long as i does not exceed 6.

§ 10. For all values of k, ρ, n, a there exists between the radial surface displacement due to tangential surface forces and the tangential surface displacements due to radial surface forces the simple relation

$$\frac{u_a \text{ when tangential forces alone act}}{T_i} = i(i+1)\frac{v_a \text{ when radial forces alone act}}{\dfrac{dR_i}{d\theta}}$$

$$= i(i+1)\frac{w_a \text{ when radial forces alone act}}{\dfrac{1}{\sin\theta}\dfrac{dR_i}{d\phi}}$$

$$= \frac{i(i+1)\sqrt{(v_a)^2+(w_a)^2} \text{ when radial forces alone act}}{\sqrt{\left(\dfrac{dR_i}{d\theta}\right)^2 + \left(\dfrac{1}{\sin\theta}\dfrac{dR_i}{d\phi}\right)^2}}\ \ldots\ldots (54).$$

An interesting interpretation of this result is obtained by the aid of the following lemma:—

if σ_i be any surface harmonic of degree i, in which the azimuth ϕ occurs only with integral coefficients,

$$\int_0^\pi\!\!\int_0^{2\pi}\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\sin\theta\, d\theta\, d\phi = i(i+1)\int_0^\pi\!\!\int_0^{2\pi}(\sigma_i)^2\sin\theta\, d\theta\, d\phi\ \ldots\ldots\ldots (55).$$

To prove this, write $\cos\theta = \mu$ so that σ_i satisfies the equation

$$i(i+1)\sigma_i + \frac{d}{d\mu}(1-\mu^2)\frac{d\sigma_i}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2\sigma_i}{d\phi^2} = 0.$$

Then

$$\int_0^\pi\int_0^{2\pi}\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\sin\theta\, d\theta\, d\phi$$

$$= \int_{-1}^{+1}\int_0^{2\pi}\left\{\frac{d\sigma_i}{d\mu}\cdot(1-\mu^2)\frac{d\sigma_i}{d\mu} + \frac{d\sigma_i}{d\phi}\cdot\frac{1}{1-\mu^2}\frac{d\sigma_i}{d\phi}\right\}d\mu\, d\phi$$

$$= \int_0^{2\pi} d\phi\left[\sigma_i(1-\mu^2)\frac{d\sigma_i}{d\mu}\right]_{\mu=-1}^{\mu=+1} + \int_{-1}^{+1} d\mu\left[\sigma_i\frac{1}{1-\mu^2}\frac{d\sigma_i}{d\phi}\right]_{\phi=0}^{\phi=2\pi}$$

$$-\int_{-1}^{+1}\int_0^{2\pi}\sigma_i\left\{\frac{d}{d\mu}(1-\mu^2)\frac{d\sigma_i}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2\sigma_i}{d\phi^2}\right\}d\mu\, d\phi.$$

The single integrals obviously both vanish under the specified conditions of the problem, and the double integral reduces by means of Laplace's equation to

$$i(i+1)\int_{-1}^{+1}\int_0^{2\pi}(\sigma_i)^2\, d\mu\, d\phi,$$

which proves the lemma.

Now by (54), the value of the fractions being independent of θ and ϕ,

$$\frac{\displaystyle\int_0^\pi\int_0^{2\pi}(u_a)^2\sin\theta\, d\theta\, d\phi \text{ when tangential forces alone act}}{\displaystyle\int_0^\pi\int_0^{2\pi}(T_i)^2\sin\theta\, d\theta\, d\phi}$$

$$= i^2(i+1)^2\frac{\displaystyle\int_0^\pi\int_0^{2\pi}\{(v_a)^2+(w_a)^2\}\sin\theta\, d\theta\, d\phi \text{ when radial forces alone act}}{\displaystyle\int_0^\pi\int_0^{2\pi}\left\{\left(\frac{dR_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dR_i}{d\phi}\right)^2\right\}\sin\theta\, d\theta\, d\phi},$$

and by (55) this is equivalent to

$$\frac{\displaystyle\int_0^\pi\int_0^{2\pi}(u_a)^2\sin\theta\, d\theta\, d\phi \text{ when tangential forces alone act}}{\displaystyle\int_0^\pi\int_0^{2\pi}\{(v_a)^2+(w_a)^2\}\sin\theta\, d\theta\, d\phi \text{ when radial forces alone act}}$$

$$= \frac{\displaystyle\int_0^\pi\int_0^{2\pi}\left\{\left(\frac{dT_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dT_i}{d\phi}\right)^2\right\}\sin\theta\, d\theta\, d\phi}{\displaystyle\int_0^\pi\int_0^{2\pi}(R_i)^2\sin\theta\, d\theta\, d\phi} \quad\text{.........................(56).}$$

Now the resultant tangential displacement and the resultant tangential force at any point of the surface are respectively $\sqrt{(v_a)^2+(w_a)^2}$ and $\sqrt{\left(\frac{dT_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dT_i}{d\phi}\right)^2}\cos kt$.

Thus (56) signifies that the mean square of the surface radial displacements due to tangential surface forces bears to the mean square of the surface tangential displacements due to radial surface forces the same ratio that the mean square of the former set of forces bears to the mean square of the latter. As the result (56) holds equally when $k^2 \sin^2 kt$ is written for $\cos^2 kt$ in both numerator and denominator of the left hand side, we may regard the theorem as holding for the kinetic energies of the radial and tangential surface motions in the two cases instead of for the mean squares of the displacements.

CASE $i = 2$.

§ 11. The second case of mixed radial and transverse forced vibrations I have worked out is that when $i = 2$, there being no restriction on the value of n/m.

The applied forces consist of

bodily forces from the potential $r^2 V_2 \cos kt$,

and the surface forces

$$R = R_2 \cos kt,$$

$$\Theta = \frac{dT_2}{d\theta} \cos kt,$$

$$\Phi = \frac{1}{\sin\theta} \frac{dT_2}{d\phi} \cos kt.$$

Putting $i = 2$ in (38) to (44), and substituting in (37), we find Y_2 and Z_2; then employing these values in (45), (46) and (47) we ultimately obtain for the displacements

$$\begin{aligned} u = \frac{\rho V_2 \cos kt\, r}{n(19m-5n)} \Big[& 2(4m-n)a^2 - (3m-n)r^2 \\ & + \frac{k^2\rho}{84n(m+n)(19m-5n)} \{(2009m^3 + 1047m^2n - 855mn^2 + 123n^3)a^4 \\ & - 4(259m^3 + 113m^2n - 141mn^2 + 21n^3)a^2r^2 + (19m-5n)(7m^2+6mn-3n^2)r^4\} \Big] \\ + \frac{R_2 \cos kt\, ra^{-2}}{n(19m-5n)} \Big[& (8m-n)a^2 - 3(m-n)r^2 \\ & + \frac{k^2\rho}{12n(m+n)(19m-5n)} \{(287m^3 + 217m^2n - 82mn^2 + 12n^3)a^4 \\ & - 4(37m^3 + 34m^2n - 28mn^2 - n^3)a^2r^2 + (19m-5n)(m^2+2mn-2n^2)r^4\} \Big] \\ + \frac{T_2 \cos kt\, ra^{-2}}{n(19m-5n)} \Big[& 3(m-n)(a^2+2r^2) \\ & + \frac{k^2\rho}{6n(m+n)(19m-5n)} \{(169m^3 + 5m^2n - 122mn^2 + 18n^3)a^4 \\ & - (m-n)(23m^2 - 44mn - 19n^2)a^2r^2 - (19m-5n)(m^2+2mn-2n^2)r^4\} \Big] \quad \text{......(57)}, \end{aligned}$$

$$v=\frac{\rho\dfrac{dV_2}{d\theta}\cos kt\, r}{2n(19m-5n)}\Bigg[2(4m-n)a^2-(5m-n)r^2$$

$$+\frac{k^2\rho}{252n(m+n)(19m-5n)}\{3(2009m^3+1047m^2n-855mn^2+123n^3)a^4$$

$$-4(1295m^3+782m^2n-474mn^2+63n^3)a^2r^2+(19m-5n)(49m^2+42mn-9n^2)r^4\}\Bigg]$$

$$+\frac{\dfrac{dR_2}{d\theta}\cos kt\, ra^{-2}}{2n(19m-5n)}\Bigg[(8m-n)a^2-(5m+2n)r^2$$

$$+\frac{k^2\rho}{36n(m+n)(19m-5n)}\{3(287m^3+217m^2n-82mn^2+12n^3)a^4$$

$$-2(370m^3+402m^2n-21mn^2+19n^3)a^2r^2+(19m-5n)(7m^2+14mn+4n^2)r^4\}\Bigg]$$

$$+\frac{\dfrac{dT_2}{d\theta}\cos kt\, ra^{-2}}{2n(19m-5n)}\Bigg[3(m-n)a^2+2(5m+2n)r^2$$

$$+\frac{k^2\rho}{18n(m+n)(19m-5n)}\{3(169m^3+5m^2n-122mn^2+18n^3)a^4$$

$$-(115m^3-345m^2n-309mn^2+7n^3)a^2r^2-(19m-5n)(7m^2+14mn+4n^2)r^4\}\Bigg] \quad\text{.........(58)},$$

$$w=\left[\text{Expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v\right]\quad\text{...............(59)}.$$

For the corresponding value of the dilatation we have

$$\Delta=\frac{r^2\cos kt}{19m-5n}\Bigg[2\rho V_2\left\{1+k^2\rho\frac{7(31m^2+33mn-6n^2)a^2-3n(19m-5n)r^2}{42n(m+n)(19m-5n)}\right\}$$

$$+21R_2a^{-2}\left\{1+k^2\rho\frac{(62m^2+259mn+29n^2)a^2-9n(19m-5n)r^2}{126n(m+n)(19m-5n)}\right\}$$

$$-42T_2a^{-2}\left\{1+k^2\rho\frac{(5m^2+217mn+44n^2)a^2-9n(19m-5n)r^2}{126n(m+n)(19m-5n)}\right\}\Bigg]\quad\text{.............(60)}.$$

The coefficients of $T_2k^2a^2r^2$ in (57), $\dfrac{dT_2}{d\theta}k^2a^2r^2$ in (58) and $\dfrac{1}{\sin\theta}\dfrac{dT_2}{d\phi}k^2a^2r^2$ in (59) are the only ones whose sign alters as n/m varies from 0 to 1.

The surface values of the displacements are rendered more concise by the employment of Poisson's ratio $\eta\equiv(m-n)/2m$. Thus we get

$$u_a=\frac{\rho V_2\cos kt(2+\eta)a^3}{n(7+5\eta)}\left\{1+\frac{k^2\rho a^2}{n}\frac{91+81\eta+27\eta^2}{42(2+\eta)(7+5\eta)}\right\}$$

$$+\frac{R_2\cos kt(7-4\eta)a}{2n(7+5\eta)}\left\{1+\frac{k^2\rho a^2}{n}\frac{35-4\eta+26\eta^2}{6(7-4\eta)(7+5\eta)}\right\}$$

$$+\frac{T_2\cos kt\,9\eta a}{n(7+5\eta)}\left\{1+\frac{k^2\rho a^2}{n}\frac{7+55\eta-11\eta^2}{54\eta(7+5\eta)}\right\}\quad\text{..................................(61)}.$$

$$v_a = \frac{\rho \frac{dV_2}{d\theta} \cos kt\,(1+\eta)\,a^3}{2n\,(7+5\eta)} \left\{1 + \frac{k^2\rho a^2}{n} \frac{91 + 222\eta + 81\eta^2}{126\,(1+\eta)(7+5\eta)}\right\}$$

$$+ \frac{dR_2}{d\theta} \frac{\cos kt\, 3\eta a}{2n\,(7+5\eta)} \left\{1 + \frac{k^2\rho a^2}{n} \frac{7 + 55\eta - 11\eta^2}{54\eta\,(7+5\eta)}\right\}$$

$$+ \frac{\frac{dT_2}{d\theta} \cos kt\,(7-\eta)\,a}{2n\,(7+5\eta)} \left\{1 + \frac{k^2\rho a^2}{n} \frac{49 - 2\eta + 67\eta^2}{18\,(7-\eta)(7+5\eta)}\right\} \quad\ldots\ldots\ldots(62),$$

$$w_a = \left[\text{Expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v\right] \quad\ldots\ldots\ldots(63).$$

The tangent of the angle which the resultant displacement at any point of the surface makes with the normal is given

for the bodily forces by

$$\frac{\sqrt{v_a^2 + w_a^2}}{u_a} = \frac{\rho \sqrt{\left(\frac{dV_2}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dV_2}{d\phi}\right)^2}}{V_2} \frac{1+\eta}{2\,(2+\eta)} \left\{1 - \frac{k^2\rho a^2}{n} \frac{13 - 12\eta}{126\,(1+\eta)(2+\eta)}\right\} \quad\ldots\ldots(64),$$

for the radial surface forces by

$$\frac{\sqrt{v_a^2 + w_a^2}}{u_a} = \frac{\sqrt{\left(\frac{dR_2}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dR_2}{d\phi}\right)^2}}{R_2} \frac{3\eta}{7 - 4\eta} \left\{1 + \frac{k^2\rho a^2}{n} \frac{7 + \eta - 38\eta^2}{54\eta\,(7-4\eta)}\right\} \quad\ldots\ldots(65),$$

for the tangential surface forces by

$$\frac{\sqrt{v_a^2 + w_a^2}}{u_a} = \frac{\sqrt{\left(\frac{dT_2}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dT_2}{d\phi}\right)^2}}{T_2} \frac{7-\eta}{18\eta} \left\{1 - \frac{k^2\rho a^2}{n} \frac{7 + 28\eta - 38\eta^2}{54\eta\,(7-\eta)}\right\} \quad\ldots\ldots(66).$$

The coefficients of $k^2\rho a^2/n$ inside the square brackets in (61), (62) and (63) are obviously positive for all values of η from 0 to ·5, which we shall regard as limiting values. Thus in every case the dynamical correction supplies an increase to the numerical values of the surface displacements.

The coefficient of T_2 in the value of u_a bears to the coefficients of $\frac{dR_2}{d\theta}$ and $\frac{1}{\sin\theta}\frac{dR_2}{d\phi}$ in the values of v_a and w_a the ratio 6 : 1, or 2 × 3 : 1. The results established in § 10 for incompressible material thus hold for all values of η when $i = 2$.

An idea of the size of the dynamical correction to the surface values of the displacements in the several cases will perhaps be most easily derived from the following Table II. The quantity tabulated is

$$\frac{\text{dynamical value of displacement}}{\text{equilibrium value of displacement}}.$$

TABLE II.

Force acting	Displacement	$\eta=$ 0	·25	·3	·5
Bodily	radial	$1+\cdot1548\,\frac{k^2\rho a^2}{n}$	$1+\cdot1449\,\frac{k^2\rho a^2}{n}$	$1+\cdot1425\,\frac{k^2\rho a^2}{n}$	$1+\cdot1386\,\frac{k^2\rho a^2}{n}$
	tangential	$1+\cdot1032\,\frac{k^2\rho a^2}{n}$	$1+\cdot1166\,\frac{k^2\rho a^2}{n}$	$1+\cdot1195\,\frac{k^2\rho a^2}{n}$	$1+\cdot1238\,\frac{k^2\rho a^2}{n}$
Surface radial	radial	$1+\cdot1190\,\frac{k^2\rho a^2}{n}$	$1+\cdot1199\,\frac{k^2\rho a^2}{n}$	$1+\cdot1241\,\frac{k^2\rho a^2}{n}$	$1+\cdot1386\,\frac{k^2\rho a^2}{n}$
	tangential	∞	$1+\cdot1801\,\frac{k^2\rho a^2}{n}$	$1+\cdot1546\,\frac{k^2\rho a^2}{n}$	$1+\cdot1238\,\frac{k^2\rho a^2}{n}$
Surface tangential	radial	∞	$1+\cdot1801\,\frac{k^2\rho a^2}{n}$	$1+\cdot1546\,\frac{k^2\rho a^2}{n}$	$1+\cdot1238\,\frac{k^2\rho a^2}{n}$
	tangential	$1+\cdot0\dot{5}\,\frac{k^2\rho a^2}{n}$	$1+\cdot0526\,\frac{k^2\rho a^2}{n}$	$1+\cdot0536\,\frac{k^2\rho a^2}{n}$	$1+\cdot0583\,\frac{k^2\rho a^2}{n}$

Regarding $\frac{k^2\rho a^2}{n}$ as constant, it will be seen that the influence of the value of η on the size of the dynamical correction is comparatively small, except in the case of the radial displacement arising from tangential surface forces and the tangential displacement arising from radial surface forces. These are cases in which the equilibrium values of the displacements absolutely vanish with η.

In the case of the radial surface forces the dynamical correction to u_a passes through a minimum when η is ·11 approximately, and in the case of the tangential surface forces the corrections to v_a and w_a pass through a minimum when η is ·20 approximately.

From (64), (65) and (66) we see that the dynamical correction makes the direction of the resultant displacement at any point of the surface approach the normal in the case of the bodily forces and the tangential surface forces, and likewise in the case of the radial surface forces when η exceeds ·443.

§ 12. The action of the bodily forces of the present case in a sphere of incompressible material has, as stated in § 1, been already considered in some detail by Mr A. E. H. Love. The result at which Mr Love arrives, l. c. p. 327, is with our present notation

$$u_a = \frac{2aV_2\cos kt}{k^2}\left\{-1+\frac{-\frac{38}{21}+\frac{k^2\rho a^2}{n}\frac{29}{189}}{-\frac{38}{21}+\frac{k^2\rho a^2}{n}\frac{74}{189}}\right\} \quad\text{.......................}\quad (67).$$

From this result Mr Love draws the following conclusion:—

"For a sphere of the mass and diameter of the earth, and of the rigidity of steel or iron, executing vibrations of the species considered with a semi-diurnal period, we have, in C. G. S. units

$$2\pi/k = 12 \times 60 \times 60,\ \rho = 5{\cdot}6,\ n = 800 \times 10^9,\ a = 640 \times 10^6,$$

so that $ka\sqrt{\rho/n} = 1/4$ nearly. It follows from this that the neglect of $(ka\sqrt{\rho/n})^4$ would be fairly justifiable in the case of such a body. We conclude that in the case of an elastic solid earth the bodily tides would follow the equilibrium law." (l. c. p. 328.)

As explained in a parallel case in § 7, the result (67) does not really proceed beyond the equilibrium value; what we have to consider is the magnitude not of $(ka\sqrt{\rho/n})^4$ but of the term in $(ka\sqrt{\rho/n})^2$ which actually occurs in u_a. For this purpose we refer to Table II., and taking Mr Love's hypothetical value $ka\sqrt{\rho/n} = 1/4$ we find that the ratio of the dynamical to the equilibrium value of u_a is

$$1{\cdot}009 : 1 \text{ approximately};$$

so that the dynamical correction is slightly under 1 per cent.

The application to the earth of results obtained by the mathematical theory of homogeneous isotropic elastic solids is of course highly speculative. The best value to assign to the rigidity n in such an application is largely a matter of opinion. The elastic moduli of iron, however, are much higher than those of most known substances. It would thus perhaps be better—especially as the maximum error involved in the equilibrium theory is of more interest than the minimum—to assign to n a considerably lower value than Mr Love does. If there is any reason to suppose that the earth but for its rotation would be a true sphere, then a value such as 32×10^7 grammes weight per square centimetre has something to commend it*, and it is at least a fair average value for known materials. With this alteration in Mr Love's data we find for the ratio of the dynamical to the equilibrium value of u_a the considerably higher value

$$1{\cdot}02 : 1.$$

Even with these figures, however, the approximation supplied by the equilibrium theory is still very close, so that Mr Love's conclusion appears less open to criticism than the reasoning on which he based it.

As the absolute size of the tidal disturbance due to the moon's attraction in a hypothetical earth of this sort may possess some interest, I have evaluated (61) and (62) taking $\rho = 5{\cdot}5$, $n = 32 \times 10^7$ grammes wt. per sq. cm., $2\pi/k = 12 \times 60 \times 60$, $a = 64 \times 10^7$, accepting for V_2 the estimate given in Thomson and Tait's 'Natural Philosophy' Part II. Art. 812. The departure of the earth from a spherical form and its mutual gravitation are left out of account, and the dynamical correction is neglected. Taking the foot as

* See *Phil. Mag.* Sept. 1891, p. 250.

unit of length, and supposing the line joining the centres of the earth and moon taken as axis of the harmonic, I find approximately

$$u_a = \tfrac{10}{3} P_2 \cos kt,$$

$$v_a = -\tfrac{3}{2} \sin 2\theta \cos kt.$$

The amplitude of the displacements due to the sun's action would be about half as big.

Supposing simultaneous astronomical observations to proceed at two distant stations on the earth's surface, there might under favourable conditions, under the joint influence of the sun and moon, be apparent fluctuations in their relative latitudes such as might possibly suggest a displacement of the polar axis. A second of arc on the earth's surface answers to nearly 100 feet, so that judging by the preceding figures any effect of the kind must be extremely small; still those conducting the very delicate observations by which a displacement of the earth's axis is attempted to be measured might do well to arrange their experiments so as to secure the elimination so far as possible of any effect of the kind.

Surface forces derivable from a potential.

§ 13. In the third case of mixed radial and transverse vibrations referred to in § 9 the surface forces, radial as well as tangential, are derivable from a potential of the form

$$r^i a^{-i+1} S_i \cos kt,$$

where S_i is a surface harmonic of degree i. This gives in terms of our previous notation

$$V_i = 0,\ R_i = iS_i,\ T_i = S_i \quad \text{.................................... (68).}$$

The solution is easily obtained without any restriction to the values of i or m/n by means of the following artifice. The equilibrium terms* in the displacements are known to be

$$\left.\begin{aligned} u &= \frac{ir^{i-1} a^{-i+2}}{2n(i-1)} S_i \cos kt, \\ \frac{v}{\frac{dS_i}{d\theta}} &= \frac{w}{\frac{1}{\sin\theta}\frac{dS_i}{d\phi}} = \frac{r^{i-1} a^{-i+2}}{2n(i-1)} \cos kt \end{aligned}\right\} \quad \text{.............................. (69),}$$

and so are derivable by differentiation from the potential

$$\frac{r^i a^{-i+2}}{2n(i-1)} S_i \cos kt.$$

Suppose now for a little we employ fixed cartesian coordinates x, y, z, the displacements relative to these being α, β, γ. Also let

$$\alpha = \alpha_0 + k^2 \alpha_2,$$

$$\text{...............,}$$

* *Camb. Phil. Soc. Trans.*, Vol. xv., Equations (109), p. 379.

where α_0 is the equilibrium value. Then as we have just seen

$$\frac{\alpha_0}{\frac{d}{dx}} = \frac{\beta_0}{\frac{d}{dy}} = \frac{\gamma_0}{\frac{d}{dz}} = \frac{r^i a^{-i+2}}{2n(i-1)} S_i \cos kt \qquad (70).$$

We notice that α_0, β_0, γ_0 themselves are solid spherical harmonics, and that the corresponding dilatation Δ_0 vanishes. There being no bodily forces, the first body stress equation is

$$m\frac{d\Delta}{dx} + n\nabla^2\alpha - \rho\frac{d^2\alpha}{dt^2} = 0.$$

The terms independent of k^2 vanish, and the terms in k^2 give

$$m\frac{d\Delta_2}{dx} + n\nabla^2\alpha_2 + \rho\alpha_0 = 0.$$

Thus substituting the value of α_0 from (70), we have

$$m\frac{d\Delta_2}{dx} + n\nabla^2\alpha_2 + \rho\frac{d}{dx}\left\{\frac{a^{-i+2}}{2n(i-1)} r^i S_i \cos kt\right\} = 0 \qquad (71).$$

This is identical with the ordinary equilibrium equation

$$m\frac{d\Delta}{dx} + n\nabla^2\alpha + \rho\frac{dV}{dx} = 0,$$

for the case of bodily forces derivable from a potential V, provided

$$V = \frac{a^{-i+2}}{2n(i-1)} r^i S_i \cos kt.$$

Again there are no terms in k^2 in the surface forces derived from the potential

$$r^i a^{-i+2} S_i \cos kt;$$

thus the terms containing k^2 in the expressions for the displacements must by themselves satisfy the equations for a free surface.

The terms $k^2\alpha_2$ &c. in the displacements arising from the given system of surface forces thus satisfy the same body-stress equations, and the same surface equations, as the displacements supplied by the equilibrium theory for the case when there act bodily forces derivable from the potential

$$\frac{k^2 a^{-i+2}}{2n(i-1)} r^i S_i \cos kt.$$

Consequently the terms in k^2 in the displacements of the present problem must be identical with the displacements in the specified case of equilibrium. They are thus deducible at once from my general solution for the equilibrium of a solid sphere*.

* *Camb. Phil. Soc. Trans.*, Vol. XIV., Equations (36) to (39), pp. 264—265.

The expressions to which we are thus led are

$$u = \frac{S_i \cos kt\, i r^{i-1} a^{-i+2}}{2n(i-1)} \left[1 + \frac{k^2\rho}{n} \frac{i\{(i+2)m - n\} a^2 - (i-1)\{(i+1)m - n\} r^2}{2(i-1)\{(2i^2+4i+3)m - (2i+1)n\}}\right] \quad \text{...(72)}$$

$$\frac{v}{\frac{dS_i}{d\theta}} = \frac{w}{\frac{1}{\sin\theta}\frac{dS_i}{d\phi}} = \frac{\cos kt\, r^{i-1} a^{-i+2}}{2n(i-1)} \left[1 + \frac{k^2\rho}{n} \frac{i\{(i+2)m - n\} a^2 - (i-1)\{(i+3)m - n\} r^2}{2(i-1)\{(2i^2+4i+3)m - (2i+1)n\}}\right] \quad \text{......(73).}$$

The corresponding dilatation is given by

$$\Delta = \frac{k^2\rho}{n} \frac{i r^i a^{-i+2} S_i \cos kt}{2(i-1)\{(2i^2+4i+3)m - (2i+1)n\}} \quad \text{....................(74).}$$

It is zero on the equilibrium theory.

For the surface values of the displacements we have

$$u_a = \frac{iaS_i \cos kt}{2n(i-1)} \left[1 + \frac{k^2\rho a^2}{n} \frac{(2i+1)m - n}{2(i-1)\{(2i^2+4i+3)m - (2i+1)n\}}\right] \quad \text{...........(75).}$$

$$\frac{v_a}{\frac{dS_i}{d\theta}} = \frac{w_a}{\frac{1}{\sin\theta}\frac{dS_i}{d\phi}} = \frac{a\cos kt}{2n(i-1)} \left[1 + \frac{k^2\rho a^2}{n} \frac{3m - n}{2(i-1)\{(2i^2+4i+3)m - (2i+1)n\}}\right] \quad \text{......(76).}$$

The dynamical correction tends as usual to increase the surface displacements; it is relatively more important for the radial than the tangential displacements. Its importance, for a given frequency of vibration, diminishes rapidly as i increases.

The results (72) to (76) may be verified for the case $n/m = 0$ by putting $R_i/i = T_i = S_i$ in (48) to (53), and for the case $i = 2$ by putting $R_2/2 = T_2 = S_2$ in (57) to (63). It was in fact a study of the solutions found in these two cases that led me to the train of reasoning by which the results (72) to (76) are deduced here.

§ 14. Before quitting the subject of mixed radial and transverse vibrations, it is worth noticing that near the centre of the sphere in all the preceding cases the displacements are deducible, to a close degree of approximation, from a species of potential function Q, such that

$$u = \frac{dQ}{dr}, \quad v = \frac{1}{r}\frac{dQ}{d\theta}, \quad w = \frac{1}{r\sin\theta}\frac{dQ}{d\phi} \quad \text{..............................(77).}$$

This is easily verified in the several formulae, retaining in each only the lowest power of r.

The values of Q in the several cases are as follows:—
for incompressible material, with i any value,

$$Q = \frac{\cos kt\, r^i a^{-i+2}}{2n(i-1)(2i^2+4i+3)} \left[(R_i + \rho a^i V_i)\left\{i(i+2) + k^2\rho a^2 \frac{2i^5 + 18i^4 + 41i^3 + 33i^2 + 17i + 15}{4n(i-1)(2i+5)(2i^2+4i+3)}\right\}\right.$$
$$\left. - T_i(i+1)\left\{i^2 - i - 3 + k^2\rho a^2 \frac{i(2i^4 + 8i^3 - 19i^2 - 72i - 45)}{4n(i-1)(2i+5)(2i^2+4i+3)}\right\}\right] \quad \text{.........(78);}$$

for case $i = 2$, with n/m any value,

$$Q = \frac{\cos kt\, r^2}{2n(19m-5n)}\left[\rho a^2 V_2 \left\{2(4m-n) + k^2\rho a^2 \frac{(2009m^3+1047m^2n-855mn^2+123n^3)}{84n(m+n)(19m-5n)}\right\}\right.$$
$$+ R_2\left\{8m-n+k^2\rho a^2\frac{287m^3+217m^2n-82mn^2+12n^3}{12n(m+n)(19m-5n)}\right\}$$
$$\left.+ T_2\left\{3(m-n)+k^2\rho a^2\frac{169m^3+5m^2n-122mn^2+18n^3}{6n(m+n)(19m-5n)}\right\}\right] \quad \text{.......(79)};$$

for case of surface forces derivable from potential $r^i a^{-i+1} S_i \cos kt$,

$$Q = \frac{S_i \cos kt\, r^i a^{-i+2}}{2n(i-1)}\left[1 + k^2\rho a^2 \frac{i\{(i+2)m-n\}}{2n(i-1)\{(2i^2+4i+3)m-(2i+1)n\}}\right] \quad \text{.......(80)}.$$

SOLID SPHERE.

Pure Transverse Vibrations.

§ 15. The typical surface forces are

$$\Theta = \frac{1}{\sin\theta}\frac{d\tau_i}{d\phi}\cos kt, \quad \Phi = -\frac{d\tau_i}{d\theta}\cos kt,$$

see § 2. For the value of W_i in the general solution (11) and (12) we have by (26)

$$aB_i W_i = \frac{2}{n}\tau_i.$$

When the frequency of the forced vibrations is small compared to that of the fundamental vibration of the pure transverse type depending on a harmonic of degree i, we may employ the approximation (39) for B_i. Doing so we find eventually, retaining only terms in k^2 in addition to the equilibrium values,

$$\frac{v}{\frac{1}{\sin\theta}\frac{d\tau_i}{d\phi}} = \frac{w}{-\frac{d\tau_i}{d\theta}} = \frac{\cos kt\, r^i a^{-i+1}}{n(i-1)}\left[1 + k^2\rho\frac{(i+1)a^2-(i-1)r^2}{2n(i-1)(2i+3)}\right] \quad \text{.......(81)}.$$

The dynamical correction obviously increases the numerical value of the displacements for all values of r; this increase, relatively considered, diminishes however as r increases.

For the surface values of the displacements we have

$$\frac{v_a}{\frac{1}{\sin\theta}\frac{d\tau_i}{d\phi}} = \frac{w_a}{-\frac{d\tau_i}{d\theta}} = \frac{a\cos kt}{n(i-1)}\left\{1 + \frac{k^2\rho a^2}{n}\frac{1}{(i-1)(2i+3)}\right\} \quad \text{.......(82)}.$$

For given values of k, ρ, a and n the relative importance of the dynamical correction falls off rapidly as i increases.

SPECIES $i=1$ OF PURE TRANSVERSE VIBRATIONS.

§ 16. This species, called by Prof. Lamb the *rotatory*, claims special attention.

Taking for simplicity the axis of the harmonic as the line $\theta=0$, we have the applied forces given by

$$\Phi = \tau_1 \sin\theta \cos kt \quad \text{......................................(83)},$$

where τ_1 is a constant. Such a force system is not in statical equilibrium except when $\cos kt=0$, but has a resultant couple

$$\tfrac{8}{3}\pi a^3 \tau_1 \cos kt$$

about the line $\theta=0$.

If the time factor did not exist, the couple would produce a continually accelerated angular velocity about $\theta=0$, and the displacements might be regarded as tending to become infinite. When the time factor exists, however, this ceases to be the case. Treating the sphere as a rigid body, the azimuth ϕ, relative to a plane fixed in space, of any plane fixed in the body and containing $\theta=0$, satisfies, it will be found, the differential equation

$$\frac{d^2\phi}{dt^2} = \frac{5}{\rho a^2}\tau_1 \cos kt.$$

If we suppose $\qquad \phi = 0 \text{ when } t=\pi/2k$

we get $\qquad \phi = -\dfrac{5\tau_1 \cos kt}{\rho a^2 k^2}.$

This answers, so long as k is not zero, to a simple harmonic oscillation about a mean position corresponding to $\phi=0$. The displacement of the point (r, θ) from its mean position, measured along the arc of the small circle on which the point moves, is

$$r\sin\theta\,.\,\phi \quad \text{or} \quad -5r\sin\theta\cos kt\,\tau_1/\rho a^2k^2 \quad \text{..........................(84)}.$$

The formulae (81) and (82), if in them we put $i=1$, lead to the obviously erroneous result that the displacements are infinite. This is due to the mathematical treatment, which assumed the value (39) of B_i to be replaceable by

$$B_i = L_i 2a^{-\frac{1}{2}} y^{i+\frac{1}{2}}(i-1) \div \left\{1 - \frac{i+1}{2(i-1)(2i+3)}y^2\right\}.$$

This is satisfactory unless $i=1$, but in that case we have instead

$$B_1 = -L_1 2a^{-\frac{1}{2}} y^{\frac{3}{2}}\,.\,\tfrac{2}{15}y^2\left(1 - \tfrac{1}{14}y^2 + \tfrac{1}{504}y^4\right) \quad \text{.............................(85)}.$$

Using this, we find in place of (81)

$$w = -\tau_1 \sin\theta\cos kt \frac{5r}{\rho a^2k^2}\left[1 - \frac{k^2\rho}{n}\frac{7r^2-5a^2}{70} + \left(\frac{k^2\rho}{n}\right)^2 \frac{63r^4 - 126r^2a^2 + 55a^4}{17640}\right] \quad \text{........(86)}.$$

and in place of (82)

$$w_a = -\tau_1 \sin\theta\cos kt \frac{5a}{\rho a^2k^2}\left[1 - \tfrac{1}{35}\frac{k^2\rho a^2}{n} - \tfrac{1}{2205}\left(\frac{k^2\rho a^2}{n}\right)^2\right] \quad \text{...............(87)}.$$

The principal term in (86) taken alone would give

$$w = -5\tau_1 \frac{r \sin\theta \cos kt}{\rho a^2 k^2} \quad \text{.......................................(88).}$$

Answering to this, however, we see by reference to (2) that the stresses are all zero. In other words (88) must represent a rigid body displacement, and comparing it with (84) we see it must stand for the displacement supplied by ordinary Rigid Dynamics. Omitting the rigid body displacement, we get for the true elastic displacement

$$w = \tau_1 \sin\theta \cos kt \frac{ra^{-2}}{14n} \left\{ 7r^2 - 5a^2 - \frac{k^2\rho}{n} \frac{63r^4 - 126r^2a^2 + 55a^4}{252} \right\} \quad \text{................(89).}$$

and for its surface value

$$w_a = \tau_1 \sin\theta \cos kt \frac{a}{7n} \left\{ 1 + \tfrac{1}{63} \frac{k^2 \rho a^2}{n} \right\} \quad \text{.................................(90).}$$

If these results hold when k is small,—and the proof seems pretty satisfactory,—it is on physical grounds difficult to see how the results

$$\left. \begin{aligned} w &= \tau_1 \sin\theta \, r(7r^2 - 5a^2)/14na^2, \\ w_a &= \tau_1 \sin\theta \, a/7n \end{aligned} \right\} \quad \text{.................................(91)}$$

can fail to hold for the elastic displacements in the sphere under the action of the surface force

$$\Phi = \tau_1 \sin\theta.$$

We thus appear to have hit on the solution of a problem which seemed insoluble when approached from the ordinary equilibrium equations. Our solution throws light on an aspect of the case left dark by ordinary Rigid Dynamics, viz. the mode in which the influence of the surface forces is transmitted inwards. We now see that the surface material forges ahead, following the lead of the applied forces, while the central material lags behind. The total displacement is in fact by (91) greater or less than the rigid body displacement according as the point considered lies outside or inside of the spherical surface

$$r = a\sqrt{5/7}.$$

The rigid body rotation gives origin to "centrifugal forces" and to elastic strains and stresses depending thereon, but these may be separately treated. They prescribe a limit to the application of the elastic solid theory.

The dynamical correction is seen by (90) to increase as usual the surface value of the elastic displacement.

THIN SPHERICAL SHELL.

§ 17. We now proceed to consider the second class of forced vibrations referred to in § 1, viz. the vibrations of any frequency in a thin shell. By a *thin* shell is meant one whose thickness h bears to a, the radius of the outer surface, a ratio whose lowest power only need be retained in any mathematical expression occurring in the solution. This may imply of course a limitation in some of the results.

Supposing there to be only surface forces, given by (7), acting on the outer surface we find that equations (22) to (27) may be written

$$\left.\begin{aligned}
&A_i Y_i + B_i Z_i + A_{-i-1} Y_{-i-1} + B_{-i-1} Z_{-i-1} = \frac{1}{n} R_i,\\
&C_i Y_i + D_i Z_i + C_{-i-1} Y_{-i-1} + D_{-i-1} Z_{-i-1} = \frac{1}{n} T_i,\\
&\left(A_i - \frac{h}{a} a \frac{dA_i}{da}\right) Y_i + \left(B_i - \frac{h}{a} a \frac{dB_i}{da}\right) Z_i + \left(A_{-i-1} - \frac{h}{a} a \frac{dA_{-i-1}}{da}\right) Y_{-i-1}\\
&\qquad + \left(B_{-i-1} - \frac{h}{a} a \frac{dB_{-i-1}}{da}\right) Z_{-i-1} = 0,\\
&\left(C_i - \frac{h}{a} a \frac{dC_i}{da}\right) Y_i + \left(D_i - \frac{h}{a} a \frac{dD_i}{da}\right) Z_i + \left(C_{-i-1} - \frac{h}{a} a \frac{dC_{-i-1}}{da}\right) Y_{-i-1}\\
&\qquad + \left(D_{-i-1} - \frac{h}{a} a \frac{dD_{-i-1}}{da}\right) Z_{-i-1} = 0
\end{aligned}\right\}\quad\ldots(92)$$

$$\left.\begin{aligned}
&aB_i W_i + aB_{-i-1} W_{-i-1} = \frac{2}{n} \tau_i,\\
&\left\{aB_i - \frac{h}{a} a \frac{d}{da}(aB_i)\right\} W_i + \left\{aB_{-i-1} - \frac{h}{a} a \frac{d}{da}(aB_{-i-1})\right\} = 0
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(93).$$

The first four and the last two of these equations form independent systems. Taking first equations (92), we find for the value of the determinant Π_i formed of the coefficients as in (28),

$$\Pi_i = \left(\frac{h}{a}\right)^2 \begin{vmatrix} A_i, & B_i, & A_{-i-1}, & B_{-i-1}\\ C_i, & D_i, & C_{-i-1}, & D_{-i-1}\\ a\frac{dA_i}{da}, & a\frac{dB_i}{da}, & a\frac{dA_{-i-1}}{da}, & a\frac{dB_{-i-1}}{da}\\ a\frac{dC_i}{da}, & a\frac{dD_i}{da}, & a\frac{dC_{-i-1}}{da}, & a\frac{dD_{-i-1}}{da} \end{vmatrix} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(94).$$

In finding the values of Π_i and of Y_i, Z_i, Y_{-i-1}, Z_{-i-1} use is made of the following results obtainable from the definitions (15):—

$$\left.\begin{aligned}
a\frac{dA_i}{da} &= A_i \frac{y^2 - 4x^2 + 6(i-1)(i+2)}{y^2 - 2(i-1)(i+2)}\\
&\quad - \tfrac{1}{2} C_i \frac{y^4 - 4y^2(i^2+i+3) + 16x^2 + 4(i-1)i(i+1)(i+2)}{y^2 - 2(i-1)(i+2)},\\
a\frac{dB_i}{da} &= \frac{2i(i+1) D_i \{y^2 - (i-1)(i+2)\} - 2B_i\{y^2 - 3(i-1)(i+2)\}}{y^2 - 2(i-1)(i+2)},\\
a\frac{dC_i}{da} &= \frac{2A_i\{x^2 - (i-1)(i+2)\} - 4C_i\{y^2 - x^2 - (i-1)(i+2)\}}{y^2 - 2(i-1)(i+2)},\\
a\frac{dD_i}{da} &= -D_i \frac{y^2 - 4(i-1)(i+2)}{y^2 - 2(i-1)(i+2)}\\
&\quad - B_i \frac{y^4 - 2y^2(2i^2 + 2i - 1) + 4(i-1)i(i+1)(i+2)}{2i(i+1)\{y^2 - 2(i-1)(i+2)\}}
\end{aligned}\right\}\quad\ldots(95).$$

where as before

$$x^2 = k^2\alpha^2a^2, \quad y^2 = k^2\beta^2a^2.$$

Writing $A_{-i-1},\ldots D_{-i-1}$ for $A_i,\ldots D_i$ respectively in the above, we obtain without further change the values of $a\dfrac{dA_{-i-1}}{da},\ldots a\dfrac{dD_{-i-1}}{da}$ in terms of $A_{-i-1},\ldots D_{-i-1}$. This follows from the fact that the substitution of $-i-1$ for $+i$ leaves unaltered $i(i+1)$ and $(i-1)(i+2)$.

Writing for shortness

$$\left.\begin{aligned}\frac{A_iC_{-i-1}-C_iA_{-i-1}}{y^2-2(i-1)(i+2)}&=M_i,\\ \frac{D_iB_{-i-1}-B_iD_{-i-1}}{y^2-2(i-1)(i+2)}&=N_i\end{aligned}\right\} \quad\ldots\ldots(96),$$

we obtain from (95) and the corresponding formulae with the suffix $-i-1$ the following results:—

$$A_{-i-1}a\frac{dA_i}{da}-A_ia\frac{dA_{-i-1}}{da}=\tfrac{1}{2}M_i\{y^4-4y^2(i^2+i+3)+16x^2+4(i-1)(i+2)i(i+1)\}\quad\ldots\ldots(97),$$

$$A_{-i-1}a\frac{dC_i}{da}-A_ia\frac{dC_{-i-1}}{da}=4M_i\{y^2-x^2-(i-1)(i+2)\}\quad\ldots\ldots(98),$$

$$C_{-i-1}a\frac{dA_i}{da}-C_ia\frac{dA_{-i-1}}{da}=M_i\{y^2-4x^2+6(i-1)(i+2)\}\quad\ldots\ldots(99),$$

$$C_{-i-1}a\frac{dC_i}{da}-C_ia\frac{dC_{-i-1}}{da}=2M_i\{x^2-(i-1)(i+2)\}\quad\ldots\ldots(100),$$

$$a\frac{dA_i}{da}a\frac{dC_{-i-1}}{da}-a\frac{dA_{-i-1}}{da}a\frac{dC_i}{da}=M_i\{y^2x^2-y^2(i^2+i+2)-2x^2(i-1)(i+2)$$
$$+2(i-1)(i+2)(i-2)(i+3)\}\quad\ldots\ldots(101),$$

$$B_{-i-1}a\frac{dB_i}{da}-B_ia\frac{dB_{-i-1}}{da}=2i(i+1)N_i\{y^2-(i-1)(i+2)\}\quad\ldots\ldots(102),$$

$$B_{-i-1}a\frac{dD_i}{da}-B_ia\frac{dD_{-i-1}}{da}=-N_i\{y^2-4(i-1)(i+2)\}\quad\ldots\ldots(103),$$

$$D_{-i-1}a\frac{dB_i}{da}-D_ia\frac{dB_{-i-1}}{da}=2N_i\{y^2-3(i-1)(i+2)\}\quad\ldots\ldots(104),$$

$$D_{-i-1}a\frac{dD_i}{da}-D_ia\frac{dD_{-i-1}}{da}=\frac{1}{2i(i+1)}N_i\{y^4-2y^2(2i^2+2i-1)+4(i-1)(i+2)i(i+1)\}\ldots(105),$$

$$a\frac{dD_i}{da}a\frac{dB_{-i-1}}{da}-a\frac{dD_{-i-1}}{da}a\frac{dB_i}{da}=N_i\{y^4-y^2(3i^2+3i-2)+2(i-1)(i+2)(i-2)(i+3)\}\ldots(106).$$

Returning now to (94), we have

$$(a/h)^2\,\Pi_i = \left(A_{-i-1}a\frac{dA_i}{da} - A_i a\frac{dA_{-i-1}}{da}\right)\left(D_{-i-1}a\frac{dD_i}{da} - D_i a\frac{dD_{-i-1}}{da}\right)$$
$$+\left(B_{-i-1}a\frac{dB_i}{da} - B_i a\frac{dB_{-i-1}}{da}\right)\left(C_{-i-1}a\frac{dC_i}{da} - C_i a\frac{dC_{-i-1}}{da}\right)$$
$$-\left(A_{-i-1}a\frac{dC_i}{da} - A_i a\frac{dC_{-i-1}}{da}\right)\left(D_{-i-1}a\frac{dB_i}{da} - D_i a\frac{dB_{-i-1}}{da}\right)$$
$$-\left(C_{-i-1}a\frac{dA_i}{da} - C_i a\frac{dA_{-i-1}}{da}\right)\left(B_{-i-1}a\frac{dD_i}{da} - B_i a\frac{dD_{-i-1}}{da}\right)$$
$$+\left(A_iC_{-i-1} - C_iA_{-i-1}\right)\left(a\frac{dD_i}{da}a\frac{dB_{-i-1}}{da} - a\frac{dB_i}{da}a\frac{dD_{-i-1}}{da}\right)$$
$$+\left(D_iB_{-i-1} - B_iD_{-i-1}\right)\left(a\frac{dA_i}{da}a\frac{dC_{-i-1}}{da} - a\frac{dC_i}{da}a\frac{dA_{-i-1}}{da}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(107).$$

Substituting from equations (97)...(106), we find on reduction

$$\Pi_i = \frac{h^2}{a^2}\frac{M_iN_i}{i(i+1)}y^2\{(i-1)(i+2)(3y^2-4x^2) - \tfrac{1}{2}(2i^2+2i+5)y^4$$
$$+(i^2+i+4)y^2x^2 + \tfrac{1}{4}y^6\}\ldots\ldots\ldots\ldots(108).$$

This expression is convenient for our present purposes. When desirable, however, the values of M_i and N_i are easily substituted. For by the definitions (15), we get

$$M_i = \frac{2}{ax^4}\cdot x\{J'_{i+\frac{1}{2}}(x)J_{-i-\frac{1}{2}}(x) - J_{i+\frac{1}{2}}(x)J'_{-i-\frac{1}{2}}(x)\}.$$

Thus using the definition (14) of the Bessel's, with the corresponding result

$$x\{J'_{i+\frac{1}{2}}(x)J_{-i-\frac{1}{2}}(x) - J_{i+\frac{1}{2}}(x)J'_{-i-\frac{1}{2}}(x)\} = (2i+1)L_iL_i',$$

we have

$$M_i = \frac{2(2i+1)}{ax^4}L_iL_i'.$$

Similarly

$$N_i = \frac{2(2i+1)}{i(i+1)a^5}L_iL_i'.$$

Hence finally

$$\frac{M_iN_i}{i(i+1)} = \left\{\frac{2(2i+1)L_iL_i'}{i(i+1)a^3x^2}\right\}^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(109).$$

The employment of (109) in (108) supplies an elegant value for Π_i, showing exactly how it depends on the definition of the Bessel's functions.

§ 18. As we intend retaining only lowest powers of h/a we may regard the displacements as constant throughout the thickness, and so write a for r in the general formulae (10), (11), (12). We can then express the coefficients of Y_i, Z_i, Y_{-i-1}, Z_{-i-1} in these formulae in terms of A_i, B_i etc.

For instance the coefficient of $Y_i \cos kt$ in (10) is

$$\tfrac{1}{2}a\left\{C_i - \frac{2(A_i+2C_i)}{(k\beta a)^2 - 2(i-1)(i+2)}\right\}.$$

Treating each coefficient in (10) in this way, and writing y^2 for $(k\beta a)^2$, we find after reduction

$$\{a^{-1}u/\cos kt\}\{y^2-2(i-1)(i+2)\} = -(A_iY_i + B_iZ_i + A_{-i-1}Y_{-i-1} + B_{-i-1}Z_{-i-1})$$
$$- i(i+1)(C_iY_i + D_iZ_i + C_{-i-1}Y_{-i-1} + D_{-i-1}Z_{-i-1}) + \tfrac{1}{2}y^2(C_iY_i + C_{-i-1}Y_{-i-1}).$$

Whence by means of the two first surface conditions (92), we get

$$u = a\cos kt\left[-\frac{1}{n}\frac{R_i + i(i+1)T_i}{y^2-2(i-1)(i+2)} + \tfrac{1}{2}y^2\frac{C_iY_i + C_{-i-1}Y_{-i-1}}{y^2-2(i-1)(i+2)}\right] \text{............(110).}$$

Treating (11) and (12) in a similar fashion, we find

$$v = a\cos kt\,\frac{d}{d\theta}\left[-\frac{1}{n}\frac{R_i+2T_i}{y^2-2(i-1)(i+2)} + \frac{y^2}{2i(i+1)}\frac{B_iZ_i + B_{-i-1}Z_{-i-1}}{y^2-2(i-1)(i+2)}\right] \text{......(111),}$$

$$w = \left[\text{expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v\right] \text{.........(112).}$$

Taking now the four equations (92), and combining the determinants arising in the values of Y_i and Y_{-i-1}, we find without serious difficulty

$$\frac{C_iY_i + C_{-i-1}Y_{-i-1}}{y^2-2(i-1)(i+2)} = \frac{h}{na}\frac{M_iN_i}{\Pi_i}\left[-\frac{1}{2i(i+1)}R_i\{y^4 - 2y^2(2i^2+2i-1) + 4i(i+1)x^2\}\right.$$
$$\left. + T_i(3y^2-4x^2)\right] \text{............(113).}$$

Similarly we find

$$\frac{B_iZ_i + B_{-i-1}Z_{-i-1}}{y^2-2(i-1)(i+2)} = \frac{h}{na}\frac{M_iN_i}{\Pi_i}[R_i(3y^2-4x^2) - \tfrac{1}{2}T_i(y^4-12y^2+16x^2)] \text{......(114).}$$

Substituting in (113) and (114) the value of $\dfrac{M_iN_i}{\Pi_i}$ from (108), and then introducing the resulting expressions in (110), (111) and (112), we find

$$u = \frac{a\cos kt}{n}\left[-\frac{R_i + i(i+1)T_i}{y^2-2(i-1)(i+2)}\right.$$
$$\left. + \frac{a}{h}\frac{R_i\{2(2i^2+2i-1)y^2 - 4i(i+1)x^2 - y^4\} + 2i(i+1)T_i(3y^2-4x^2)}{4(i-1)(i+2)(3y^2-4x^2) + 4y^2x^2(i^2+i+4) - 2y^4(2i^2+2i+5) + y^6}\right] \text{...(115),}$$

$$v = \frac{a\cos kt}{n}\frac{d}{d\theta}\left[-\frac{R_i+2T_i}{y^2-2(i-1)(i+2)}\right.$$
$$\left. + \frac{a}{h}\frac{2R_i(3y^2-4x^2) + T_i\{4(3y^2-4x^2) - y^4\}}{4(i-1)(i+2)(3y^2-4x^2) + 4y^2x^2(i^2+i+4) - 2y^4(2i^2+2i+5) + y^6}\right] \text{...(116),}$$

$$w = \left[\text{expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v\right] \text{.....................(117).}$$

In determining the values of $C_iY_i + C_{-i-1}Y_{-i-1}$ and $B_iZ_i + B_{-i-1}Z_{-i-1}$ we neglected all but the algebraically lowest power of h/a, and thus to be consistent we must omit the terms

$$-an^{-1}\cos kt\,\{R_i + i(i+1)T_i\}\{y^2 - 2(i-1)(i+2)\}^{-1} \text{ in the value of } u,$$

$$-an^{-1}\cos kt\,\frac{d}{d\theta}(R_i + 2T_i)\{y^2 - 2(i-1)(i+2)\}^{-1} \quad ,, \quad ,, \quad ,, \quad v,$$

$$-an^{-1}\cos kt\,\frac{1}{\sin\theta}\frac{d}{d\phi}(R_i + 2T_i)\{y^2 - 2(i-1)(i+2)\}^{-1} \quad ,, \quad ,, \quad ,, \quad w.$$

The terms left in the values of the displacements are of the order

$$\frac{a^2}{h} \times \frac{\text{applied force}}{n}.$$

To put the expressions for the displacements into an immediately serviceable form, write in their values for x^2 and y^2, and divide out above and below by k^2. We then find after some simplification

$$u = \frac{a^2}{2nh}\frac{\cos kt}{(i-1)(i+2)}\,\frac{R_i\dfrac{(2i^2+2i-1)m-n}{3m-n}\left(1 - \dfrac{k^2\rho a^2}{2n}\dfrac{m+n}{(2i^2+2i-1)m-n}\right) + i(i+1)T_i}{1 - \dfrac{k^2\rho a^2}{n}\dfrac{(2i^2+2i+5)m-3n}{2(i-1)(i+2)(3m-n)} + \left(\dfrac{k^2\rho a^2}{n}\right)^2\dfrac{m+n}{4(i-1)(i+2)(3m-n)}} \quad \text{...(118)},$$

$$v = \frac{a^2}{2nh}\frac{\cos kt}{(i-1)(i+2)}\,\frac{\dfrac{d}{d\theta}\left\{R_i + 2T_i\left(1 - \dfrac{k^2\rho a^2}{4n}\dfrac{m+n}{3m-n}\right)\right\}}{1 - \dfrac{k^2\rho a^2}{n}\dfrac{(2i^2+2i+5)m-3n}{2(i-1)(i+2)(3m-n)} + \left(\dfrac{k^2\rho a^2}{n}\right)^2\dfrac{m+n}{4(i-1)(i+2)(3m-n)}} \quad \text{...(119)},$$

$$w = \left[\text{expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v\right] \quad \text{.....................(120)}.$$

§ 19. The displacements can be thrown into a form which is shorter and more suggestive physically, by the employment of the roots of the various types of free vibrations in the thin shell.

The denominator in the equations (118), (119), (120) when equated to zero is of course the frequency equation for free vibrations of the type mixed radial and transverse depending on surface harmonics of degree i. This equation may be written

$$f(k^2) \equiv k^4 - 2k^2\frac{n}{\rho a^2}\frac{(2i^2+2i+5)m-3n}{m+n} + 4(i-1)(i+2)\left(\frac{n}{\rho a^2}\right)^2\frac{3m-n}{m+n} = 0 \text{......(121)}.$$

This differs from the equation originally given by Lamb* only in the notation.

Regarding this as a quadratic equation in k^2 we shall denote the roots, in ascending order of magnitude, by K_1^2 and K_2^2. We shall also make use of K_0^2 and K_3^2, where $K_0/2\pi$ is the frequency of free vibrations of the pure radial type, and $K_3/2\pi$ that of

* *Proc. London Math. Soc.*, Vol. XIV.

free vibrations of the pure transverse type answering to displacements which contain surface harmonics of degree i. For these quantities we have the expressions

$$K_0^2 = \frac{4n(3m-n)}{\rho a^2(m+n)} \quad\text{.......................(122)*},$$

$$K_3^2 = (i-1)(i+2)\frac{n}{\rho a^2} \quad\text{.......................(123)†}.$$

As pointed out by Lamb, K_0^2 is the real root supplied by (121) when $i=0$, but its value is got most simply by treating the radial vibrations separately.

Defining $f(k^2)$ as in (121) we easily find

$$\frac{f(K_0^2)}{K_0^2} = \frac{f(K_3^2)}{K_3^2} = -\tfrac{1}{4}i(i+1)K_0^2 \quad\text{.......................(124)},$$

$$K_1^2K_2^2 = K_0^2K_3^2 \quad\text{.......................(125)}.$$

$$K_1^2 + K_2^2 - (K_0^2 + K_3^2) = \tfrac{1}{4}i(i+1)K_0^2 \quad\text{.......................(126)}.$$

From (124) we see that K_1^2 is less and K_2^2 greater than either K_0^2 or K_3^2.

The denominator in (118), (119) and (120) is of course simply

$$\left(1-\frac{k^2}{K_1^2}\right)\left(1-\frac{k^2}{K_2^2}\right):$$

and employing (122), (123), (125) and (126) we easily throw these equations into the forms:—

$$u = \frac{a^2}{2nh}\frac{\cos kt}{(i-1)(i+2)}\,\frac{\dfrac{(2i^2+2i-1)m-n}{3m-n}\left(1-\dfrac{k^2}{K_1^2+K_2^2-K_0^2}\right)R_i + i(i+1)T_i}{\left(1-\dfrac{k^2}{K_1^2}\right)\left(1-\dfrac{k^2}{K_2^2}\right)} \quad\text{...(127)},$$

$$v = \frac{a^2}{2nh}\frac{\cos kt}{(i-1)(i+2)}\,\frac{\dfrac{d}{d\theta}\left\{R_i + 2\left(1-\dfrac{k^2}{K_0^2}\right)T_i\right\}}{\left(1-\dfrac{k^2}{K_1^2}\right)\left(1-\dfrac{k^2}{K_2^2}\right)} \quad\text{.......................(128)},$$

$$w = \left[\text{expression obtained by writing } \frac{1}{\sin\theta}\frac{d}{d\phi} \text{ for } \frac{d}{d\theta} \text{ in value of } v\right] \quad\text{...........(129)}.$$

As no assumption has been made as to the magnitude of k, it may have any value which does not lead to infinite values for the displacements. These results are thus in one respect much more general than those found for the solid sphere.

Putting $k=0$ we obtain results identical with those found by retaining only the algebraically lowest power of h/a in my solution of the equilibrium problem‡. This seems so far a satisfactory test of the accuracy of both dynamical and equilibrium solutions. The reservation made in obtaining the equilibrium solution that i^2h/a was small§ is equally necessary in the present case.

* Lamb, l. c. p. 50. See also *Camb. Phil. Soc. Trans.*, Vol. XIV. p. 321.

† *Ibid.* p. 320, or Lamb l. c.

‡ *Camb. Phil. Soc. Trans.*, Vol. XV. Equations (96) and (97) on p. 369.

§ l. c. p. 373.

From (127), (128), (129) we have at once

$$\frac{u}{i(i+1)T_i}, \text{ when } R_i \text{ vanishes,}$$

$$= \frac{v}{\frac{dR_i}{d\theta}} = \frac{w}{\frac{1}{\sin\theta}\frac{dR_i}{d\phi}} = \sqrt{\frac{v^2+w^2}{\left(\frac{dR_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dR_i}{d\phi}\right)^2}}, \text{ when } T_i \text{ vanishes} \quad\text{.............(130)}$$

This is the identical relation met with in § 10 in the case of the solid sphere.

In discussing the influence of the value of k on the displacements, we shall call a displacement *direct* or *reversed* according as its sign is or is not the same as it would be on the equilibrium theory.

The radial displacement depending on the radial surface force is direct when

$$k^2 < K_1^2,$$

and also when $\quad K_1^2 + K_2^2 - K_0^2 < k^2 < K_2^2;$

it is reversed when $\quad K_1^2 < k^2 < K_1^2 + K_2^2 - K_0^2,$

and also when $\quad k^2 > K_2^2.$

It vanishes when $\quad k^2 = K_1^2 + K_2^2 - K_0^2,$

a value less than K_2^2, but exceeding $K_0^2 + K_3^2$ when i is equal to or greater than 2.

When $k^2 < K_1^2$, the radial displacement, being direct, is always greater than on the equilibrium theory; but though still direct it is less than on the equilibrium theory when k^2 only slightly exceeds $K_1^2 + K_2^2 - K_0^2$.

The radial displacement depending on tangential surface forces, and the tangential displacements depending on radial surface forces, are reversed when k lies between K_1 and K_2; otherwise they are direct.

When direct they are greater or less than on the equilibrium theory according as k^2 is less or greater than $K_1^2 + K_2^2$.

The tangential displacements depending on the tangential surface forces are direct when

$$k < K_1,$$

and also when $\quad K_0 < k < K_2;$

they are reversed when $\quad K_1 < k < K_0,$

and also when $\quad k > K_2.$

They vanish, as is well worth noticing, when

$$k = K_0,$$

i.e. when the frequency equals that of the free radial vibrations.

When direct these tangential displacements are greater than on the equilibrium theory except when k lies between

$$K_0 \text{ and } K_0\{1 + \tfrac{1}{4}i(i+1)\}^{\frac{1}{2}}.$$

When k/K_2 is large all the displacements are numerically very small compared to their values on the equilibrium theory.

PURE RADIAL VIBRATIONS.

§ 20. For the pure radial vibrations, answering to the uniform surface force

$$R = R_0 \cos kt$$

over $r=a$, we find—either directly or by putting $i=0$ in (118)—

$$u = \frac{a^2}{nh} \frac{m+n}{4(3m-n)} \frac{R_0 \cos kt}{1 - \frac{1}{4}\frac{k^2\rho a^2}{n}\frac{m+n}{3m-n}} \quad \text{.........................(131);}$$

or, using (122),

$$u = \frac{a^2}{nh} \frac{m+n}{4(3m-n)} \frac{R_0 \cos kt}{1 - \frac{k^2}{K_0^2}} \quad \text{.........................(132).}$$

The displacement is direct or reversed according as the frequency of the applied forces is less or greater than that of the free radial vibrations.

When direct, the displacement is always greater than on the equilibrium theory; when reversed, it diminishes as k increases from K_0, becoming very small when k is very large.

PURE TRANSVERSE VIBRATIONS.

§ 21. The values of W_i, W_{-i-1} in the formulae (11) and (12) of the general solution are given by (93). Retaining only algebraically lowest powers of h/a we thence obtain

$$\left.\begin{aligned} W_i &= \frac{2}{n}\frac{\tau_i}{\Pi_i'} aB_{-i-1}, \\ W_{-i-1} &= -\frac{2}{n}\frac{\tau_i}{\Pi_i'} aB_i \end{aligned}\right\} \quad \text{.........................(133),}$$

where

$$\Pi_i' = -\frac{h}{a}\begin{vmatrix} aB_i, & aB_{-i-1} \\ a\frac{d}{da}(aB_i), & a\frac{d}{da}(aB_{-i-1}) \end{vmatrix} \quad \text{.........................(134).}$$

Employing (95) we find

$$\Pi_i' = 2i(i+1)ha\frac{D_iB_{-i-1} - B_iD_{-i-1}}{y^2 - 2(i-1)(i+2)}\{y^2-(i-1)(i+2)\} \quad \text{.................(135),}$$

where $y \equiv k\beta a$ as usual.

Writing a for r in (11) we find that the terms in W_i and W_{-i-1} may be written

$$v = -a^2 \frac{\frac{1}{\sin\theta}\frac{d}{d\phi}[\{B_i + i(i+1)D_i\}W_i + \{B_{-i-1} + i(i+1)D_{-i-1}\}W_{-i-1}]}{y^2 - 2(i-1)(i+2)}.$$

Substituting for W_i and W_{-i-1} from (133), and thereafter for

$$\frac{D_iB_{-i-1} - B_iD_{-i-1}}{y^2 - 2(i-1)(i+2)} \frac{i(i+1)}{\Pi_i'}$$

from (135), we find

$$v = \frac{a^2}{nh}\frac{\frac{1}{\sin\theta}\frac{d\tau_i}{d\phi}\cos kt}{(i-1)(i+2) - y^2} \quad \text{.........................(136),}$$

and similarly we have

$$w = -\frac{a^2}{nh}\frac{\dfrac{d\tau_i}{d\theta}\cos kt}{(i-1)(i+2)-y^2} \quad \text{.......................} (137).$$

Equating to zero the denominator in (136) or (137) we obtain of course the frequency equation (123) for the free vibrations of the pure transverse type depending on harmonics of degree i. Employing the value of K_3^2 we may replace (136) and (137) by

$$\frac{v}{\dfrac{1}{\sin\theta}\dfrac{d\tau_i}{d\phi}} = \frac{w}{-\dfrac{d\tau_i}{d\theta}} = \frac{a^2}{nh}\frac{\cos kt}{(i-1)(i+2)}\frac{1}{1-\dfrac{k^2}{K_3^2}} \quad \text{.......................} (138).$$

The displacements are thus direct or reversed according as the frequency of the applied forces is less or greater than the frequency of free transverse vibrations depending on harmonics of the same degree as the applied forces. When direct, the displacements are always greater than on the equilibrium theory; when reversed, they fall off as k increases from K_3.

The exact analogy of the conclusions for pure radial and pure transverse vibrations is worthy of notice.

§ 22. For facility of reference I collect the results obtained for the several species of forces in a thin shell of thickness h. The forces are supposed to act over $r = a$, and to be

radial $\qquad R = R_0 \cos kt + R_i \cos kt,$

tangential, along meridian, $\Theta = \dfrac{dT_i}{d\theta}\cos kt + \dfrac{1}{\sin\theta}\dfrac{d\tau_i}{d\phi}\cos kt,$

" , perp. to " , $\Phi = \dfrac{1}{\sin\theta}\dfrac{dT_i}{d\phi}\cos kt - \dfrac{d\tau_i}{d\theta}\cos kt;$

where R_0 is a constant, R_i, T_i, τ_i surface harmonics of degree i. The frequencies of the free vibrations in the shell are

$K_0/2\pi$ pure radial,

$K_1/2\pi$ and $K_2/2\pi$ mixed radial and transverse, depending on surface harmonics of degree i,

$K_3/2\pi$ pure transverse, depending on surface harmonics of degree i.

The displacements are as follows:—

$$u = \frac{a^2}{nh}\cos kt\left[\frac{m+n}{4(3m-n)}\frac{R_0}{1-k^2/K_0^2} + \frac{\dfrac{(2i^2+2i-1)m-n}{3m-n}\left(1-\dfrac{k^2}{K_1^2+K_2^2-K_0^2}\right)R_i + i(i+1)T_i}{2(i-1)(i+2)(1-k^2/K_1^2)(1-k^2/K_2^2)}\right] \quad \text{......} (139).$$

$$v = \frac{a^2}{nh}\frac{\cos kt}{(i-1)(i+2)}\left[\frac{\dfrac{d}{d\theta}\{\tfrac{1}{2}R_i + (1-k^2/K_0^2)T_i\}}{(1-k^2/K_1^2)(1-k^2/K_2^2)} + \frac{\dfrac{1}{\sin\theta}\dfrac{d\tau_i}{d\phi}}{1-k^2/K_3^2}\right] \quad \text{.......................} (140).$$

$$w = \frac{a^2}{nh} \frac{\cos kt}{(i-1)(i+2)} \left[\frac{\dfrac{1}{\sin\theta} \dfrac{d}{d\phi} \{\tfrac{1}{2} R_i + (1 - k^2/K_0^2)\, T_i\}}{(1 - k^2/K_1^2)(1 - k^2/K_2^2)} - \frac{\dfrac{d\tau_i}{d\theta}}{1 - k^2/K_3^2} \right] \quad \text{.........................(141).}$$

The corresponding value of the dilatation is

$$\Delta = \frac{a}{h} \frac{\cos kt}{3m - n} \left[\frac{R_0}{1 - k^2/K_0^2} + \frac{(1 - k^2/K_3^2) R_i + \frac{1}{2} i (i+1)(k^2/K_3^2) T_i}{(1 - k^2/K_1^2)(1 - k^2/K_2^2)} \right] \quad \text{.........(142).}$$

Only the algebraically lowest power of h/a is retained, and the results are not to be trusted unless $i^2 h/a$ is small.

Strains and stresses whose expressions contain no differential coefficients with respect to r may be deduced at once from the values of the displacements, and like them have, to a first approximation, the same value at all points on the same normal to the shell.

The value of the radial strain over $r = a$ is given correctly to the present degree of approximation by the relation

$$\left(\frac{du}{dr}\right)_a = -\frac{m - n}{2n} \Delta.$$

If forces act over both the outer and inner surfaces, the above formulae will still hold when $R_0 \cos kt$, $R_i \cos kt$, &c. are taken to represent the algebraical *resultants* of the forces applied over corresponding unit elements of the two surfaces, provided these resultants be of the same order of magnitude as the separate forces acting over the two surfaces.

§ 23. Looking at (122) we see that in the case of pure radial surface forces,

$$R = R_0 \cos kt,$$

the expression (132) for the displacement may be written

$$u = \frac{R_0 \cos kt}{\rho h (K_0^2 - k^2)} = \frac{R}{\rho h (K_0^2 - k^2)} \quad \text{..............................(143).}$$

Similarly we see from (123) that in the case of the pure transverse surface forces

$$\Theta = \frac{1}{\sin\theta} \frac{d\tau_i}{d\phi} \cos kt, \quad \Phi = -\frac{d\tau_i}{d\theta} \cos kt,$$

the formula (138) for the displacements may be written

$$\frac{v}{\dfrac{1}{\sin\theta} \dfrac{d\tau_i}{d\phi}} = \frac{w}{-\dfrac{d\tau_i}{d\theta}} = \frac{\cos kt}{\rho h (K_3^2 - k^2)} \quad \text{..............................(144),}$$

or

$$v = \frac{\Theta}{\rho h (K_3^2 - k^2)}, \quad w = \frac{\Phi}{\rho h (K_3^2 - k^2)} \quad \text{.........................(145).}$$

When the surface forces are of the mixed radial and transverse type the expressions (127), (128), (129) for the displacements do not naturally fall into such simple forms. For one thing the denominator contains the two factors $K_1^2 - k^2$ and $K_2^2 - k^2$. The simplicity, however, of the results in the other cases led me to try whether something similar might not be effected for the mixed radial and transverse vibrations by separating the two vibrations whose frequencies are combined in (121).

Eventually the following line of reasoning produced the desired result:—

The one common feature of the pure radial and the pure transverse vibrations is that the direction of the displacement coincides with that of the applied force. Is it possible for this phenomenon to occur with mixed radial and transverse displacements: i.e. can we have

$$\frac{u}{R} = \frac{v}{\Theta} = \frac{w}{\Phi}?$$

Putting $k = 0$ in (118), (119), (120) we see this relation is satisfied in the case of equilibrium if

$$\frac{(2i^2 + 2i - 1)\,m - n}{3m - n} + i\,(i + 1)\,\frac{T_i}{R_i} = \frac{R_i}{T_i} + 2 \quad \text{.......................(146).}$$

Employing (122), (123), (125) and (126), we find we can write (146) in the form

$$\left(\frac{R_i}{T_i}\right)^2 - 2\,\frac{R_i}{T_i}\,\frac{K_1^2 + K_2^2 - 2K_0^2}{K_0^2} + 4\,\frac{(K_1^2 - K_0^2)(K_2^2 - K_0^2)}{K_0^4} = 0,$$

or

$$\left(\frac{R_i}{T_i} - 2\,\frac{K_1^2 - K_0^2}{K_0^2}\right)\left(\frac{R_i}{T_i} - 2\,\frac{K_2^2 - K_0^2}{K_0^2}\right) = 0 \quad \text{....................... (147).}$$

The directions of the resultants of the displacements and the applied forces of the mixed radial and transverse type thus coincide when either

$$R_i/T_i = 2\,(K_2^2 - K_0^2)/K_0^2 \quad \text{..................................(148),}$$

or

$$R_i/T_i = 2\,(K_1^2 - K_0^2)/K_0^2 \quad \text{..................................(149).}$$

In the general case when R_i and T_i are independent we split the forces into two sets by making

$$R_i = R_i' + R_i'', \quad T_i = T_i' + T_i'' \quad \text{.............................. (150);}$$

where

$$\frac{R_i'}{2\,(K_2^2 - K_0^2)} = \frac{T_i'}{K_0^2} = \frac{\tfrac{1}{2}K_0^2\,R_i - (K_1^2 - K_0^2)\,T_i}{K_0^2\,(K_2^2 - K_1^2)} \quad \text{.......................(151),}$$

$$\frac{R_i''}{2\,(K_1^2 - K_0^2)} = \frac{T_i''}{K_0^2} = \frac{-\tfrac{1}{2}K_0^2\,R_i + (K_2^2 - K_0^2)\,T_i}{K_0^2\,(K_2^2 - K_1^2)} \quad \text{..... .... (152).}$$

Substituting for R_i and T_i in terms of R_i', R_i'', T_i' and T_i'', and using (122) &c., we easily replace (127), (128) and (129) by

$$u = \frac{R_i'\cos kt}{\rho h\,(K_1^2 - k^2)} + \frac{R_i''\cos kt}{\rho h\,(K_2^2 - k^2)} \quad \text{.................................... (153),}$$

$$v = \frac{\frac{dT_i'}{d\theta}\cos kt}{\rho h(K_1^2 - k^2)} + \frac{\frac{dT_i''}{d\theta}\cos kt}{\rho h(K_2^2 - k^2)} \quad \text{.........(154)},$$

$$w = \frac{\frac{1}{\sin\theta}\frac{dT_i'}{d\phi}\cos kt}{\rho h(K_1^2 - k^2)} + \frac{\frac{1}{\sin\theta}\frac{dT_i''}{d\phi}\cos kt}{\rho h(K_2^2 - k^2)} \quad \text{.........(155).}$$

The object in view is obviously fully accomplished. We have split the applied forces of the mixed radial and transverse type into two sets. The first has for its components

$$R' = R_i'\cos kt,\ \Theta' = \frac{dT_i'}{d\theta}\cos kt,\ \Phi' = \frac{1}{\sin\theta}\frac{dT_i'}{d\phi}\cos kt,$$

where R_i', T_i' are given by (151); and the corresponding displacements u', v', w' are given by

$$\frac{u'}{R'} = \frac{v'}{\Theta'} = \frac{w'}{\Phi'} = \frac{1}{\rho h(K_1^2 - k^2)} \quad \text{.........(156).}$$

The second set has for its components

$$R'' = R_i''\cos kt,\quad \Theta'' = \frac{dT_i''}{d\theta}\cos kt,\quad \Phi'' = \frac{1}{\sin\theta}\frac{dT_i''}{d\phi}\cos kt,$$

where R_i'', T_i'' are given by (152); and the corresponding displacements u'', v'', w'' are given by

$$\frac{u''}{R''} = \frac{v''}{\Theta''} = \frac{w''}{\Phi''} = \frac{1}{\rho h(K_2^2 - k^2)} \quad \text{.........(157).}$$

Since u', v', w' become infinite when $k = K_1$, while u'', v'', w'' become infinite when $k = K_2$, it might be assumed as practically certain on physical grounds that the directions of the resultant displacements in the two cases coincide with those of the resultant displacements in the free vibrations, of frequencies $K_1/2\pi$ and $K_2/2\pi$ respectively, which depend on surface harmonics of the specified forms.

It is, however, unnecessary to rely on physical grounds alone, because the mathematical proof is easily obtainable. Thus take the equations (92) and put

$$R_i = 0 = T_i.$$

Suppose the vibration frequency to be $k/2\pi$, and the surface harmonic appearing in the displacements to be S_i. Then we find without serious trouble

$$\frac{u}{S_i} = \frac{v}{\frac{1}{2}\frac{K_0^2}{K_1^2 - K_0^2}\frac{k^2 - K_0^2}{K_2^2 - K_0^2}\frac{dS_i}{d\theta}} = \frac{w}{\frac{1}{2}\frac{K_0^2}{K_1^2 - K_0^2}\frac{k^2 - K_0^2}{K_2^2 - K_0^2}\frac{1}{\sin\theta}\frac{dS_i}{d\phi}} \quad \text{.........(158).}$$

Taking $k = K_1$, we get

$$\frac{u}{S_i} = \frac{v}{\frac{K_0^2}{2(K_2^2 - K_0^2)}\frac{dS_i}{d\theta}} = \frac{w}{\frac{K_0^2}{2(K_2^2 - K_0^2)}\frac{1}{\sin\theta}\frac{dS_i}{d\phi}};$$

and, supposing $S_i \propto \frac{1}{2}K_0^2 R_i - (K_1^2 - K_0^2)\,T_i,$

we recognise from (151) the identity in the directions of the resultant displacement in this free vibration and the forced vibration (156).

Taking on the other hand $k = K_2$ in (158) we get a free vibration which, supposing

$$S_i \propto -\tfrac{1}{2} K_0^2 R_i + (K_2^2 - K_0^2)\, T_i,$$

has clearly the same direction for its resultant displacement as the forced vibration (157).

The conclusions we have reached for the thin shell may be presented as follows:—

The applied forces may be split into:

Pure radial forces		one set:
Pure transverse forces	giving for representative harmonic of degree i	one set:
Mixed radial and transverse forces	giving for representative harmonic of degree i	two sets.

In each set we have:

Resultant displacement along the same direction at every point as the resultant force, and

$$\text{displacement} = \frac{\text{force}}{\rho h (K^2 - k^2)} \quad \text{.............................(159)};$$

where

$k/2\pi$ = frequency of applied periodic force,
= 0 for equilibrium;

$K/2\pi$ = frequency of free vibration of corresponding type (whose direction of motion coincides with the line of action of the applied force at every point).

In the case of equilibrium $\rho h K^2$ may be regarded as measuring the elastic resistance to the displacement. It is a quantity varying as the mass of the shell per unit area of surface, and as the square of the frequency in that species of free vibration in which the displacements involve the same surface harmonics and have the same direction for their resultant as the equilibrium displacement in question.

This relationship between the phenomena of equilibrium and motion appears of great physical interest. So far as I know, no case of it has been previously noticed in elastic solids.

III. *Distribution of Solar Radiation on the Surface of the Earth, and its dependence on Astronomical Elements.* By R. HARGREAVES, M.A., formerly Fellow of St John's College.

[Read *Jan.* 27, 1896.]

THE object of the following paper is to express in the form of a harmonic series the amount of heat due to the earth, in any latitude or for a zone of any extent, from solar radiation at any period of the year. In the main part of the paper, the earth's atmosphere is taken to be diathermanous, but afterwards absorption is admitted according to a law of some generality, and the same methods are adapted to this case also. The coefficients are expressed in finite form by means of complete elliptic integrals of the three kinds, and also by series of zonal harmonics, and numerical results are tabulated for every ten degrees of latitude. Special attention is paid to the way in which the various terms are affected by changes in the values of the astronomical elements, obliquity of ecliptic, eccentricity of orbit, and longitude of perihelion. The harmonic form is suitable for application to meteorological questions, or the question of underground temperature near the surface of the earth, or to such secular changes of climate as are discussed in the theory of glacial epochs.

As many are interested in these questions who would be unwilling to follow the manipulation of elliptic integrals, I have given a full outline of argument and conclusions apart from the technical work. In this way and by the numerical results, obtained by somewhat laborious calculations, I hope to have made the material accessible for purposes of application, to those who do not care to face the mathematical work.

It seems proper to mention that I have found in Ferrel's tract on 'Temperature of the Atmosphere and Earth's Surface' a table similar to table (B) below for latitudes up to 60°. He refers to Haughton's Lectures on Physical Geography for the method, which is one of approximation by series of slow convergence. He does not appear to have considered specially the influence of changes in values of the astronomical constants. Also Sir Robert Ball's book on Glacial Epochs contains a result for the hemisphere, which is a particular case of results given here for any latitude or for a zone of any extent. I may add that it was the feeling that results for the average of a hemisphere would

NOTE, *Jan.* 20. I have discovered that a paper by Meech (date 1857) in Vol. IX. of *Smithsonian Contributions to Knowledge* covers a certain section of this paper, my results being in agreement with his.

lead to an understatement of the case, that induced me to attempt the more general problem.

§ 1. *General Outline.* The annual variation in the amount of heat received from solar radiation in any latitude depends on two causes, the ellipticity of the orbit, and its inclination to the equatorial plane. In consequence of the first, the distance of the sun varies, in consequence of the second its declination, on which depend both the duration of daylight and the altitude attained by the sun. The heat-supply thus subject to an annual variation may be expressed in Fourier's manner by a harmonic series, and this will contain a non-periodic term, an annual, a semi-annual term, &c.

Denoting by H/r^2 the amount of heat falling on unit surface exposed perpendicularly to the sun's rays for unit time at distance r, the element of heat-supply is

$$\frac{Hd\theta}{\pi h}(L_0 + L_1 \sin\theta + L_2 \cos 2\theta + L_4 \cos 4\theta + \ldots) \text{ or } \frac{Hdt}{\pi r^2}(L_0 + L_1 \sin\theta + \ldots),$$

t being mean time, and θ the orbital angle of the sun measured from the spring equinox. The formula gives the total variation due to the combined action of the two causes. The coefficient L_1 takes the simple form $\frac{\pi}{2}\sin\lambda\sin\epsilon$, λ being latitude, ϵ obliquity of the ecliptic; and has opposite signs in the two hemispheres. The other coefficients L_0, L_2 ... are also functions of λ and ϵ only, but do not change in passing from northern to southern hemisphere; they require for their expression in finite form, complete elliptic integrals of the three kinds, or they may be expressed in series of zonal harmonics with $\sin\lambda$ as argument, and zonal harmonics with associated functions with $\cos\epsilon$ as argument. The astronomical constant h is introduced through the equation $r^2\frac{d\theta}{dt} = h$, and with a year as unit of time its value is $2\pi ab$, a and b being semi-axes of the earth's orbit. Since h varies as the minor axis it is dependent on the eccentricity, to a very minute extent however, as the square of the eccentricity is involved. Apart from this factor the amount of heat received while the sun travels through a fixed angle in its apparent orbit, is quite clear of the influence of eccentricity. The importance of this last element emerges when the results are transferred to mean time.

If summer and winter denote the times between the equinoxes, summer and winter totals of heat-supply on unit area in latitude λ are

$$\frac{H}{h}(L_0 + \sin\lambda\sin\epsilon) \text{ and } \frac{H}{h}(L_0 - \sin\lambda\sin\epsilon), \text{ and the annual total } \frac{2HL_0}{h}.$$

The numerical values of the coefficients as far as L_4 and their differential coefficients with regard to λ and ϵ are given in table (A). As regards L_0 it is sometimes convenient to have its values expressed in percentage of the mean of the globe; these values are:—

$\lambda=0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
$L_0 = 122·4$,	120·7,	115·7,	107·5,	96·7,	83·7,	69·6,	60·0,	52·5,	50·7.

Also the proportions in which these amounts are divided between summer and winter are as follows:—

	$\lambda=0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
Summer	50,	53·65,	57·51,	62·08,	66·88,	73·24,	81·59,	91·15,	97·65,	100
Winter	50,	46·35,	42·49,	37·92,	33·12,	26·76,	18·41,	8·85,	2·35,	0

Analogous formulæ are proved for zones of any extent. The summer and winter heat-supplies for a polar cap extending to latitude λ take the form

$$\frac{\pi H c^2}{h}(Z_0 \pm \cos^2\lambda \sin\epsilon),$$

where c is radius of the earth, and Z_0 a function of λ and ϵ which increases from zero at the pole to $\frac{\pi}{2}$ at the equator (this last being Sir R. Ball's case).

If we take the three zones into which latitudes 30° and 60° divide a hemisphere, the proportions of summer and winter heat-supplies are 55·4 to 44·6, 69 to 31, and 89·9 to 10·1 respectively; while the total annual supplies for the same are 58·6, 33·4, 8 respectively in percentages of the total for the hemisphere: or 117·3, 91·3, 60·4 per unit area where the mean of the globe is 100.

§ 2. The way in which L_0, the quantity determining the annual total, depends on latitude and obliquity of the ecliptic deserves a special study. If ϵ were zero the value of L_0 would reduce to $\cos\lambda$, varying from unity at the equator to zero at the poles. In

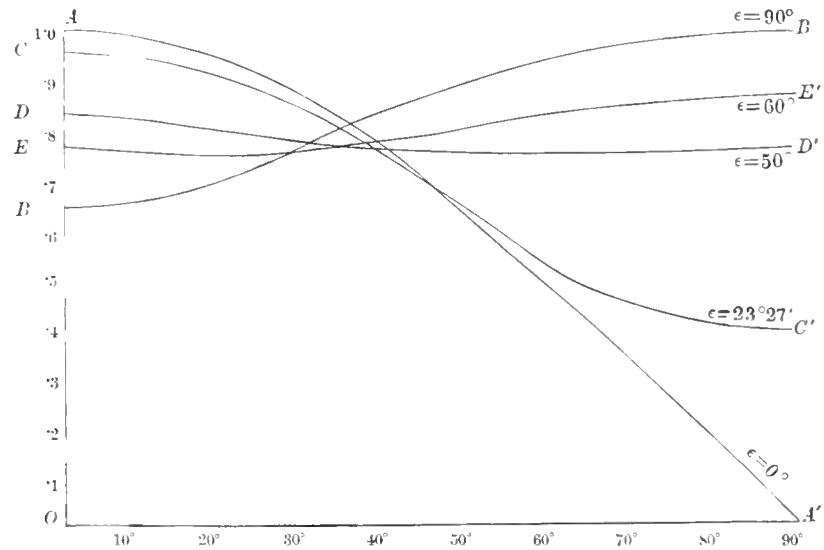


Fig. 1.

Fig. 1, AA' corresponds to this case; the abscissæ represent latitudes and the ordinates corresponding values of L_0. CC' represents the course of L_0 for $\epsilon = 23° 27'$ taken as the present value, and it is clear that for a middle range of latitude, the values of L_0 differ little from those in which $\epsilon = 0$, but near the equator are somewhat less, and near the pole much greater. If ϵ is further increased, the position of the pole becomes constantly more favourable, that of the equator less favourable. In the extreme case when $\epsilon = 90°$, so that the arctic region has grown till it embraces the globe, the equator being the final position of the arctic circle, BB' represents the course of L_0 which increases continuously from equator to pole. $\left(\text{In this case } L_0 = \frac{2E}{\pi}\right.$ where $\cos\lambda$ is the parameter of the elliptic integral, and so L_0 ranges from $\frac{2}{\pi}$ or ·6366 to 1.$\left.\right)$

The various curves all have the tangent for $\lambda = 0$ parallel to OA', and the value of L_0 either a maximum or a minimum; a maximum for values of ϵ less than 65° 20′, beyond that a minimum.

Again, excluding the case $\epsilon = 0$, the tangent for $\lambda = 90°$ is parallel to OA', and the value of L_0 is a minimum up to $\epsilon = 45°$, beyond that a maximum. For values of ϵ less than 45°, L_0 increases continuously as we pass from pole to equator, for values of ϵ greater than 65° 20′ diminishes continuously. But for intermediate values of ϵ both equator and pole have maximum values, and consequently there is an intermediate minimum, which in fact starting when $\epsilon = 45°$ at the polar end, shifts gradually across, till for $\epsilon = 65° 20'$ it reaches the equator. The curves DD', EE' shew two of these cases, one with an arctic, the other a non-arctic intermediate minimum; and the locus of these minima is the curve UZU' of Fig. 2.

§ 3. There exists a curious correlation in the way in which L_0 depends on the two elements ϵ and λ, viz. if each is changed to the complement of the other L_0 is unchanged. For example L_0 is the same for $\epsilon = 20°$, $\lambda = 50°$ as for $\epsilon = 40°$, $\lambda = 70°$; the latitude being arctic in the one case, non-arctic in the other.

Accordingly the statement $L_0 = \cos\lambda$ for $\epsilon = 0°$ has for its correlative that $L_0 = \sin\epsilon$ for the pole $\lambda = 90°$. Thus taking any ordinate in AA' for which $\epsilon = 0$, say for latitude 50°, this is also the proper value for the pole with $\epsilon = 40°$. In exactly the same way the curve BB' gives the values of L_0 at the equator for different values of ϵ. The curves AA', BB' cross in latitude 36° 7′, and the correlative statement is that for $\epsilon = 53° 53'$ the value of L_0 is the same for pole and equator. The correlative of the theorem as to intermediate minima within the range 45° to 65° 20′ for ϵ is, that for values of λ less than 24° 40′, as ϵ increases from 0° to 90°, L_0 diminishes from a maximum value on AA' to a minimum on BB'; for values of λ greater than 45°, exactly the opposite is the case, AA' giving a minimum, BB' a maximum; while for values of λ between 24° 40′ and 45° both curves give maxima values, and there exists for each latitude a minimum value. The curve XY in Fig. 2 represents the locus of these intermediate minima, hence for a latitude between 24° 40′ and 45° we begin for $\epsilon = 0$ with a maximum value Q on AA', drop to a minimum R on XY, and then rise to a final maximum S on BB' where $\epsilon = 90°$.

§ 4. Having dealt with the course of values of L_0 in the general case, it remains to notice the amount of variation that would accompany such secular changes as are thought possible by astronomers.

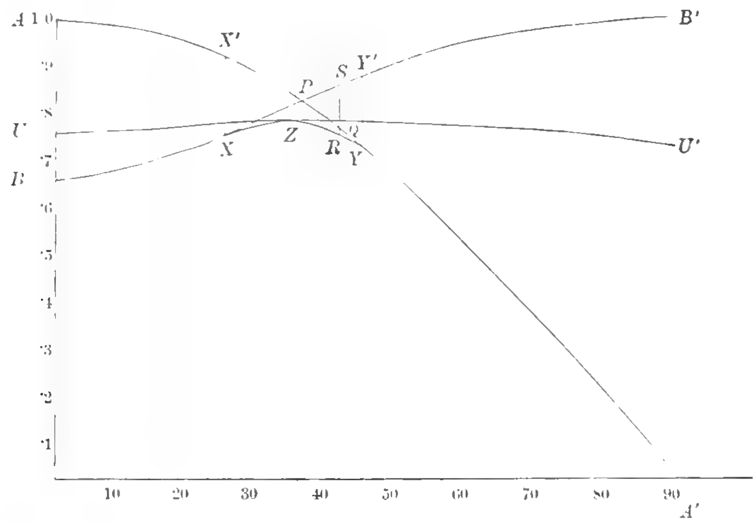


Fig. 2.

The effect of a small departure from the present value of ϵ is shewn by Table A which gives differential coefficients with regard to ϵ.

It appears that for low latitudes L_0 is diminished by an increase in ϵ, and increased by a fall in ϵ, the amount of the change diminishing from equator to latitude 43° 20′; for higher latitudes the effects are reversed. Stockwell's limits for the possible range of ϵ are 21° 58′ 36″ and 24° 36′. With these the total ranges in the value of L_0 expressed in percentages of L_0 are:—

	$\lambda=0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
Range	·93,	·90,	·79,	·59,	·21,	·50,	2·05,	6·51,	9·55,	10·56.

As the present value (taken at 23° 27′ in the calculations) lies between these limits, the range is partly above, and partly below, the present value. The amounts are inconsiderable below latitude 60°, but beyond that seem competent to produce sensible climatic changes.

The mean value of L_0 for a hemisphere or for the globe, as seems obvious *à priori*, is independent of ϵ and $=\frac{\pi}{4}$. In fact the smaller changes over the large area in latitudes below 43° 20′, exactly balance the much greater changes over the smaller area in higher latitudes, when the mean is taken.

The corresponding range in the value of L_1 or $\frac{\pi}{2}\sin\lambda\sin\epsilon$ the annual term is in every latitude 10½ per cent., and increase of ϵ everywhere causes increase in L_1. On the equator L_1 vanishes, the appearance of the sun north of the equator in summer, here giving rise to a semi-annual term which has its maxima at the equinoxes, when the sun is in the zenith at midday. The semi-annual term diminishes in value as we recede from the equator, vanishing about 44°, changing sign and increasing with some rapidity towards the pole. It is generally much smaller than L_0 and L_1, with the exception as to L_1 at the equator just mentioned, and the exclusion of very high latitudes. Its changes with ϵ are on the same scale roughly through most of the range, as those of L_0, and are therefore much greater in proportion.

§ 5. The equation for transferring to mean time is

$$\theta + C = 2\pi t + 2e\sin 2\pi t + \frac{5e^2}{2}\sin 4\pi t \ldots .$$

The constant C depends on the position of perihelion with regard to the first point of Aries, θ has been measured from $E♈$ as initial line, and t will be taken to be zero at perihelion P. The transformation made for the case in which $C = 79°$ gives results which are tabulated in (B) for every ten degrees of latitude north and south. It will be remarked at once that the symmetry between the northern and southern hemispheres has disappeared. So far as secular changes of climate are concerned the cases of most interest are those of Figs. 3 and 4; in the former, summer has its maximum duration, in the latter, winter. As the amounts of heat received in summer and winter have for each latitude values which are independent of their relative duration, it is plain that when summer is longest the division is most equal, and when winter is longest most unequal. In so far as this is a cause for glacial and genial epochs, Fig. 3, in which $C = \frac{\pi}{2}$ corresponds to the genial case, and if squares of e be neglected, the element of heat being

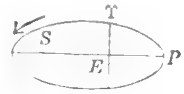


Fig. 3.

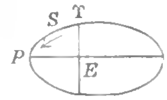


Fig. 4.

$$\frac{2H}{h}Q'dt, \quad Q' = L_0 - (L_1 - 2eL_0 - eL_2)\cos 2\pi t - (L_2 + 2eL_1)\cos 4\pi t \ldots ,$$

while for the glacial case in which $C = -\frac{\pi}{2}$,

$$Q' = L_0 + (L_1 + 2eL_0 + eL_2)\cos 2\pi t - (L_2 - 2eL_1)\cos 4\pi t \ldots$$

both for the northern hemisphere. Each of these formulæ is derivable from the other by changing the sign of L_1, which is precisely the change by which we pass from northern to southern hemisphere. Hence so far as this cause is efficient, the northern hemisphere is in a glacial state when the southern is in a genial state, and *vice-versâ.*

Again as L_1 increases from zero at the equator steadily towards the pole, while L_0 diminishes, the modification produced in the annual term by the eccentricity is greatest absolutely, and all the more relatively, in low latitudes. The difference between the two states is obviously wider, the greater the eccentricity. The maximum limit allowed by astronomers to the eccentricity in the course of secular changes is ·07, and with this extreme value the coefficients of the annual term are:—

	$\lambda=0°$	10°	20°	30°	40°	50°	60°	70°
For extreme glacial epoch	·1372,	·2437,	·3430,	·4322,	·5084,	·5797,	·6143,	·6420.
" " genial "	− ·1372,	− ·0367,	+ ·0946,	·1930,	·2952,	·3681,	·4683,	·5328.

In the lower row the signs are reversed so as to make the midsummer of the hemisphere in question the zero of time in each case. Noticeable is the change of sign which implies that the maxima fall together for low latitudes on opposite sides of the equator, instead of half a year apart as for higher latitudes. The reason for this is that the fact of the sun's being north of the equator in summer and south of it in winter, which generally produces the main part of the annual term, *at* the equator gives rise to a semi-annual term and *near* the equator produces only a small annual term. Hence the secondary influence of the change of distance predominates at and near the equator, and this influence is the same for north as for south. Near the equator, as at 10° say, we have the two influences concurring on one side of it to produce a sensible maximum and minimum, on the other side opposing each other and giving a small resultant term. The differences between these extreme cases seem to me sufficiently remarkable. For example at 70° N.L. in the genial epoch the annual term is about the same as at 43° in the glacial epoch, or in the southern hemisphere at the same time (= that of 62° N.L. at present); so also 50° in the genial corresponds to 22° in the glacial (39° N.L. at present); and 35° in the genial corresponds to 10° in the glacial (26° N.L. at present). For completeness the coefficients of the semi-annual term are added for the same extreme cases

	$\lambda=0°$	10°	20°	30°	40°	50°	60°	70°
Glacial epoch	− ·0294,	− ·0126,	+ ·0076,	+ ·0309,	+ ·0575,	+ ·0888,	+ ·1302,	+ ·2103.
Genial "	− ·0294,	− ·0430,	− ·0522,	− ·0567,	− ·0551,	− ·0452,	− ·0214,	+ ·0359.

When the upper row applies to north latitude, the lower applies to south latitude, and *vice-versâ*.

Croll, in judging of the effects of eccentricity, assumed temperatures proportional to midsummer and midwinter receipts of radiation in any latitude. The inference from heat-supplies to temperatures is a very difficult one owing to the variety of modifying conditions; but even when the problem is stated in its simplest form, the solution of the conduction equation requires the separation of non-periodic and the several periodic terms, these terms are affected with different factors in the integration, and the periodic terms suffer a modification of phase. It seems to me, therefore, that a proper basis for argument on the question of secular climatic changes is afforded by comparing non-periodic terms in the two epochs, annual terms in the two epochs, and superposing the mean temperatures and annual variations separately deduced. For the purpose of such rough comparison as is possible between climates in distant epochs, the semi-annual term may be ignored. But until the comparison between heat-supplies and temperatures is put on a better footing as regards the present state of the earth, a considerable degree of uncertainty must attach to any such comparison.

§ 6. Comparing briefly the influence of the astronomical elements on non-periodic and annual terms:

(1) Eccentricity alters to a minute extent all terms, the minor axis of the orbit occurring as a divisor to the whole formula.

(2) Otherwise the *non-periodic* term is not affected by the eccentricity.

(3) The influence of eccentricity in modifying the *annual* term depends on longitude of perihelion. The positions of greatest influence are when the major axis of the orbit is perpendicular to the line of equinoxes, and when the eccentricity has a value at all approaching its maximum, the changes are quite considerable. In north and south hemispheres the effects are in opposite directions at the same epoch. In higher latitudes where the normal annual term is considerable, these effects are a sensible increase or diminution of the amplitude; in lower latitudes, the normal annual term being much smaller, and the modifying term greater, the difference between the two hemispheres is quite remarkable, the place of zero amplitude being shunted from the equator greatly to the genial side. For example we may have a zero amplitude in 15° S.L., the amplitude increasing as we recede from this in both directions, so that at 15° N.L. it may be of notable dimensions. The character of this influence is obscured by taking a mean for either hemisphere.

(4) All the effects due to obliquity of the ecliptic are in the same direction in the two hemispheres.

(5) The non-periodic term is affected by this cause, and for latitudes higher than 60°, the influence of alterations produced by the usually admitted secular changes in this element, is very sensible.

These effects do not appear in the mean of either hemisphere.

(6) The normal annual term, by which is meant L_1 unmodified by terms depending on eccentricity, is affected similarly in all latitudes by changes in the obliquity of the ecliptic; but the effects on the annual coefficient in mean time are of a more complex character.

In the sketch of absorption, the coefficient of transmission is taken to be of the form $e_0 + e_1 \cos I + e_2 \cos^2 I + \ldots\ldots$, where I is the angle between the sun's rays and the zenith, and it is shewn that the results for each term admit of exact expression in the same forms as before, viz. either by complete elliptic integrals, or by series of zonal harmonics. One of the effects of the absorption is shewn to be a large relative increase of the periodic part in low latitudes, gradually tailing off when the pole is approached.

MATHEMATICAL THEORY.

§ 7. When the absorption of the earth's atmosphere is ignored, the formal supply of heat or light on the surface of the earth depends on the strength of solar radiation, the distance from the sun, and the angle of exposure to the sun's rays. Take for the element per unit area $Hdt \times \cos I/r^2$, dt being time-element, r the distance from the sun and I the angle between the normal to the surface and the direction of the sun's rays. In latitude λ this angle is given by $\cos I = \sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi$ where δ is the sun's declination and ψ the hour-angle changing in the course of the day uniformly from $-\psi_1$ to $+\psi_1$, ψ_1 being the hour-angle at sunset. As the change of ψ is uniform we may put $\frac{dt}{d\psi} = \frac{\Delta t}{2\pi}$ where Δt is a day; then the heat-supply for a day

$$= \frac{H}{r^2} \int \cos I \times dt$$

$$= \frac{H\Delta t}{2\pi r^2} \int_{-\psi_1}^{+\psi_1} (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)\, d\psi$$

$$= \frac{H\Delta t}{\pi r^2} (\psi_1 \sin\lambda \sin\delta + \cos\lambda \cos\delta \sin\psi_1),$$

ψ_1 or $\frac{\pi}{2} + \phi$ say, being determined by $\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi_1 = 0$ or $\sin\phi = \tan\lambda \tan\delta$, ϕ being positive in summer, negative in winter. The integral for the day then assumes the form

$$\frac{H\Delta t}{\pi r^2} \left\{ \left(\frac{\pi}{2} + \phi\right) \sin\lambda \sin\delta + \sqrt{\cos^2\lambda - \sin^2\delta} \right\} \text{.......................I}\,(a).$$

If θ is the orbital angle of the sun measured from the first point of Aries and ϵ the obliquity of the ecliptic $\sin\delta = \sin\epsilon \sin\theta$, and if further we use $h\Delta t = r^2 d\theta$ the well-known astronomical relation, we obtain a second form of the element

$$\frac{Hd\theta}{\pi h} \left\{ \left(\frac{\pi}{2} + \phi\right) \sin\lambda \sin\epsilon \sin\theta + \sqrt{\cos^2\lambda - \sin^2\epsilon \sin^2\theta} \right\} \text{.................I}\,(b).$$

This is taken as element of a continuous heat-supply through the year. We integrate in fact for the time of daylight ignoring changes of declination, and regard the result as a supply distributed uniformly over a complete day, the declination changing continuously in the formula thus obtained. During the period of total day in the polar regions, the integration above is between the limits $-\pi$ and $+\pi$, and the resulting formula $\frac{Hd\theta}{h} \sin\lambda \sin\epsilon \sin\theta$, or in effect the bracket is replaced by $\pi \sin\lambda \sin\epsilon \sin\theta$; while during the period of total night the bracket is null. The comparison of supplies at particular times of the year in the same or different latitudes, is easily made by (I), but to obtain a general view of the annual variation we must express the bracket in (I), call it Q, by a harmonic series. Thus the element of heat-supply being $\frac{HQd\theta}{\pi h}$, it will appear that Q admits of expansion in the form $L_0 + L_1 \sin\theta + L_2 \cos 2\theta + L_4 \cos 4\theta +$ and this again is readily transformed to a series depending on mean time.

§ 8. We begin with the non-arctic case $\lambda < \left(\frac{\pi}{2} - \epsilon\right)$. The following notation is used:—

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta, \quad K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad \Pi = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - \sin^2\epsilon \sin^2\theta)\sqrt{1 - k^2\sin^2\theta}},$$

$$E_2 = \int_0^{\frac{\pi}{2}} \cos 2\theta \sqrt{1 - k^2 \sin^2\theta}\, d\theta, \quad K_2 = \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta\, d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad \Pi_2 = \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta\, d\theta}{(1 - \sin^2\epsilon \sin^2\theta)\sqrt{1 - k^2\sin^2\theta}},$$

involving the relations

$$\sqrt{1 - k^2\sin^2\theta} = \frac{4}{\pi}\left(\frac{E}{2} + E_2 \cos 2\theta + E_4 \cos 4\theta + \ldots\ldots\right)$$

$$1/\sqrt{1 - k^2\sin^2\theta} = \frac{4}{\pi}\left(\frac{K}{2} + K_2 \cos 2\theta + \ldots\ldots\right)$$

$$1/(1 - \sin^2\epsilon\sin^2\theta)\sqrt{1 - k^2\sin^2\theta} = \frac{4}{\pi}\left(\frac{1}{2}\Pi + \Pi_2 \cos 2\theta + \ldots\ldots\right).$$

Thus the last term in Q viz.

$$\sqrt{\cos^2\lambda - \sin^2\epsilon \sin^2\theta} = \frac{4\cos\lambda}{\pi}\left(\frac{1}{2}E + E_2\cos 2\theta + E_4 \cos 4\theta + \ldots\ldots\right),$$

where $k = \sin\epsilon \sec\lambda$. For the expansion of ϕ we have $\sin\phi = \dfrac{\tan\lambda \sin\epsilon \sin\theta}{\sqrt{1 - \sin^2\epsilon\sin^2\theta}}$ and therefore

$$\frac{d\phi}{d\theta} = \frac{\tan\lambda \sin\epsilon\cos\theta}{(1 - \sin^2\epsilon\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} = \frac{4\tan\lambda\sin\epsilon\cos\theta}{\pi}\left(\frac{1}{2}\Pi + \Pi_2\cos 2\theta + \Pi_4 \cos 4\theta + \ldots\ldots\right)$$

$$= \frac{2\tan\lambda\sin\epsilon}{\pi}\{(\Pi + \Pi_2)\cos\theta + (\Pi_2 + \Pi_4)\cos 3\theta + \ldots\ldots\},$$

and so

$$\phi = \frac{2\tan\lambda\sin\epsilon}{\pi}\left\{(\Pi + \Pi_2)\sin\theta + \frac{1}{3}(\Pi_2 + \Pi_4)\sin 3\theta + \frac{1}{5}(\Pi_4 + \Pi_6)\sin 5\theta + \ldots\ldots\right\},$$

no constant being required as ϕ vanishes with $\theta = 0$ or π.

Hence

$$\phi\sin\theta = \frac{\tan\lambda\sin\epsilon}{\pi}\left\{(\Pi + \Pi_2) + \left(\frac{1}{3}\overline{\Pi_2 + \Pi_4} - \overline{\Pi + \Pi_2}\right)\cos 2\theta + \left(\frac{1}{5}\overline{\Pi_4 + \Pi_6} - \frac{1}{3}\overline{\Pi_2 + \Pi_4}\right)\cos 4\theta + \ldots\ldots\right\},$$

and the whole value of Q is

$$\left.\begin{aligned} &\frac{\pi}{2}\sin\lambda\sin\epsilon\sin\theta + \frac{4\cos\lambda}{\pi}\left(\frac{1}{2}E + E_2\cos 2\theta + \ldots\ldots\right) \\ &+ \frac{\sin^2\lambda\sin^2\epsilon}{\pi\cos\lambda}\left\{(\Pi + \Pi_2) + \left(\frac{1}{3}\overline{\Pi_2 + \Pi_4} - \overline{\Pi + \Pi_2}\right)\cos 2\theta \right. \\ &\left. \qquad + \left(\frac{1}{5}\overline{\Pi_4 + \Pi_6} - \frac{1}{3}\overline{\Pi_2 + \Pi_4}\right)\cos 4\theta + \ldots\ldots\right\} \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{II.}$$

Further by differentiation and a little reduction it may be shewn that

$$\left.\begin{aligned}\frac{dQ}{d\epsilon}\tan\epsilon - Q &= \frac{\sin^2\lambda}{\cos\lambda\,(1-\sin^2\epsilon\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} - \frac{1}{\cos\lambda\sqrt{1-k^2\sin^2\theta}}\\ \frac{dQ}{d\lambda}\tan\lambda - Q &= -\sec\lambda\sqrt{1-k^2\sin^2\theta}\end{aligned}\right\}\text{...III }(a),$$

or

$$\left.\begin{aligned}\frac{dQ}{d\epsilon}\tan\epsilon - Q &= \frac{4\sin^2\lambda}{\pi\cos\lambda}\left(\frac{1}{2}\Pi + \Pi_2\cos 2\theta + \Pi_4\cos 4\theta + \ldots\ldots\right)\\ &\quad - \frac{4}{\pi\cos\lambda}\left(\frac{1}{2}K + K_2\cos 2\theta + K_4\cos 4\theta + \ldots\ldots\right)\\ \text{and}\qquad \frac{dQ}{d\lambda}\tan\lambda - Q &= -\frac{4}{\pi\cos\lambda}\left(\frac{1}{2}E + E_2\cos 2\theta + E_4\cos 4\theta + \ldots\ldots\right)\end{aligned}\right\}\text{.........III }(b).$$

We have thus in harmonic form the values of Q, $\frac{dQ}{d\epsilon}$, and $\frac{dQ}{d\lambda}$. The annual term is the only one with simple trigonometrical coefficient, and is also the only one which changes sign with λ, that is in passing from north to south latitude. The total amount of heat received within any range of θ is given by $\int_{\theta'}^{\theta''}\frac{Hd\theta}{\pi h}\left(L_0 + L_1\sin\theta + L_2\cos 2\theta + \ldots\ldots\right)$, and as the various coefficients L depend only on λ and ϵ, the result is independent of the relation between θ and mean time, only depending on the eccentricity through the constant h which varies as the minor axis. Further the difference between north and south hemispheres only appears in the term L_1. If summer and winter be defined by the equinoxes, their total heat-supplies are $\frac{H}{h}(L_0 \pm \sin\lambda\sin\epsilon)$ respectively, and for the southern hemisphere the contemporaneous values have the signs crossed.

§ 9. The calculation of the integrals E_2, K_2...... is effected by means of the sequence equations :—

$$\left.\begin{aligned}(2n+3)E_{2n+2} + (2n-3)E_{2n-2} + 4n\,E_{2n}(2-k^2)/k^2 &= 0\\ (2n+1)K_{2n+2} + (2n-1)K_{2n-2} + 4n\,K_{2n}(2-k^2)/k^2 &= 0\end{aligned}\right\}\text{...............IV }(a),$$

or if both are required, more conveniently from the cross-equations

$$\left.\begin{aligned}8nE_{2n} &= k^2(K_{2n-2} - K_{2n+2})\\ \text{and}\qquad (2n+3)E_{2n+2} - (2n-1)E_{2n} &= K_{2n} + K_{2n+2}\end{aligned}\right\}\text{.......................IV }(b),$$

the last true to $n=0$, the rest to $n=1$. These with $(K-K_2)k^2 = 2(K-E)$ admit of easy proof and together determine the whole series in terms of E and K. The advantage of IV (b) is that for the two functions only one division by k^2 is wanted for each step forward. The quantities K, K_2...... are alternately positive and negative, converge rapidly at first, and ultimately in the ratio $-\tan^2\frac{\phi}{2}$ where $\sin\phi = k$. Of the quantities E, E_2...... a similar statement may be made, but E_4 is the first negative term.

This ratio is -1 when $k=1$, and the K's all become infinite. It will appear presently that this gives rise to no difficulty in the formulæ used. For the Π's the series relation

$$\frac{1}{2}K + K_2 \cos 2\theta + \ldots\ldots = (1 - \sin^2 \epsilon \sin^2 \theta)\left(\frac{1}{2}\Pi + \Pi_2 \cos 2\theta + \ldots\ldots\right),$$

gives by equating coefficients of the various cosines

$$\Pi_2 = \Pi - \frac{\rho}{2}(\Pi - K), \quad \Pi_4 = 2\Pi_2 - \Pi - \rho(\Pi_2 - K_2), \quad \Pi_6 = 2\Pi_4 - \Pi_2 - \rho(\Pi_4 - K_4), \ldots\ldots$$

the form of the relation remaining the same after the first, and ρ standing for $4/\sin^2 \epsilon$.

On reduction $$(\Pi_2 + \Pi)\sin^2 \epsilon = 2(K - \Pi \cos^2 \epsilon)$$

$$(\Pi_4 + \Pi_2)\sin^4 \epsilon = 8E\cos^2\lambda - 8K(\cos^2\lambda - \sin^2\epsilon) - 2(4 - \sin^2\epsilon)(K - \Pi\cos^2\epsilon)$$

$$(\Pi_6 + \Pi_4)\sin^6 \epsilon = 2(16 - 12\sin^2\epsilon + \sin^4\epsilon)(K - \Pi\cos^2\epsilon) + \frac{8K}{3}(12 - 9\sin^2\epsilon + 8\cos^2\lambda)(\cos^2\lambda - \sin^2\epsilon)$$
$$- \frac{8E\cos^2\lambda}{3}(12 - 13\sin^2\epsilon + 8\cos^2\lambda).$$

For the K's the corresponding expressions for the opening terms are:—

$$K_2 \sin^2\epsilon = 2E\cos^2\lambda - K(2\cos^2\lambda - \sin^2\epsilon)$$

$$3K_4\sin^4\epsilon = -8E\cos^2\lambda(2\cos^2\lambda - \sin^2\epsilon) + 16K\cos^2\lambda(\cos^2\lambda - \sin^2\epsilon) + 3K\sin^4\epsilon,$$

and for the E's

$$3E_2\sin^2\epsilon = E(2\cos^2\lambda - \sin^2\epsilon) - 2K(\cos^2\lambda - \sin^2\epsilon)$$

$$15E_4\sin^4\epsilon = -E\sin^4\epsilon - 16E\cos^2\lambda(\cos^2\lambda - \sin^2\epsilon) + 8K(\cos^2\lambda - \sin^2\epsilon)(2\cos^2\lambda - \sin^2\epsilon).$$

With the help of these we obtain for non-arctic regions

$$L_0 = \frac{2}{\pi\cos\lambda}\{E\cos^2\lambda + \sin^2\lambda(K - \Pi\cos^2\epsilon)\}, \qquad L_1 = \frac{\pi}{2}\sin\lambda\sin\epsilon,$$

$$L_2 = \frac{4}{3\pi\sin^2\epsilon\cos\lambda}\{E\cos^2\lambda(2 - \sin^2\epsilon) - 2K(\cos^2\lambda - \sin^2\epsilon)$$
$$- \sin^2\lambda(2 + \sin^2\epsilon)(K - \Pi\cos^2\epsilon)\},$$

$$\frac{dL_0}{d\lambda} = \frac{2\sin\lambda}{\pi}(K - E - \Pi\cos^2\epsilon), \qquad \frac{dL_1}{d\lambda} = \frac{\pi}{2}\cos\lambda\sin\epsilon,$$

$$\frac{dL_2}{d\lambda} = \frac{4\sin\lambda}{3\pi\sin^2\epsilon}\{E\sin^2\epsilon - (2 + \sin^2\epsilon)(K - \Pi\cos^2\epsilon)\}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{V.}$$

$$\frac{dL_0}{d\epsilon} = \frac{2\cos\epsilon}{\pi\sin\epsilon\cos\lambda}\{E\cos^2\lambda - K(\cos^2\lambda - \sin^2\epsilon) - \sin^2\epsilon(K - \Pi\sin^2\lambda)\},$$

$$\frac{dL_1}{d\epsilon} = \frac{\pi}{2}\sin\lambda\cos\epsilon,$$

$$\frac{dL_2}{d\epsilon} = \frac{4\cos\epsilon}{3\pi\sin^3\epsilon\cos\lambda}\{-E\cos^2\lambda(4 + \sin^2\epsilon) + K\sin^2\epsilon(\cos^2\lambda - \sin^2\epsilon)$$
$$+ (K - \Pi\sin^2\lambda)(4 - 2\sin^2\epsilon + \sin^4\epsilon)\}$$

For later terms the direct expression by means of E, K and Π gives too lengthy formulæ. It is better to apply numerical values to the successive sequence equations (IV). In each of these integrals it will be remembered $k = \sin\epsilon\sec\lambda$ and the second parameter of Π is $-\sin^2\epsilon$. On the Arctic circle we have the limiting case $\lambda = \frac{\pi}{2} - \epsilon$, and therefore $k = 1$, $E = 1$, and K and Π both infinite. Now K occurs multiplied by $(\cos^2\lambda - \sin^2\epsilon)$, when the product vanishes, and also in conjunction with Π in the form $K - \Pi\cos^2\epsilon$. But in the limiting case

$$K - \Pi\cos^2\epsilon = \int_0^{\frac{\pi}{2}} \frac{\sin^2\epsilon\cos\theta\,d\theta}{1 - \sin^2\epsilon\sin^2\theta} = \frac{\sin\epsilon}{2}\log_e\frac{1+\sin\epsilon}{1-\sin\epsilon},$$

and with this particular value, all the formulæ in (V) remain valid. As regards (II) and (III) it may also be noted that for the limiting case $\Pi_{2n} + \Pi_{2n+2} = \int_0^{\frac{\pi}{2}} \frac{2\cos(2n+1)\theta d\theta}{1 - \sin^2\epsilon\sin^2\theta}$, and $\Pi_{2n}\sin^2\lambda - K_{2n} = -\int_0^{\frac{\pi}{2}} \frac{\sin^2\epsilon\cos\theta\cos 2n\theta d\theta}{1-\sin^2\epsilon\sin^2\theta}$, both finite. The values for Q, $\frac{dQ}{d\epsilon}$ and $\frac{dQ}{d\lambda}$ are all finite at the limit. When $\lambda = 0$, Π reduces to $\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-\sin^2\epsilon\sin^2\theta)^{\frac{3}{2}}}$ or $E\sec^2\epsilon$. The values of L_0, L_1...... are tabulated with their differential coefficients with regard to ϵ and λ for every ten degrees of latitude. It will be seen that L_4 is very small except in Arctic latitudes, and subsequent terms are smaller still.

§ 10. When we seek a similar expansion for the Arctic regions, the discontinuity in the form of Q needs attention; viz. for periods of partial day, it retains the same form as before, Q_1 say; for the period of total day it is $\pi\sin\lambda\sin\epsilon\sin\theta$, Q_2 say; and for total night it vanishes. Q_1 and Q_2 have the same value for the transition, and also Q_1 merges into zero at the other transition. If the expansion is denoted as before by $L_0 + L_1\sin\theta + L_2\cos 2\theta + ...$ we have

$$2\pi L_0 = \int Q_1 d\theta + \int Q_2 d\theta, \quad \pi L_1 = \int Q_1 d\theta + \int Q_2 d\theta, \;......$$

To find limits for the integrations put $\cos\lambda = \sin\epsilon\sin\tau$, then the periods of partial day are from $\theta = 0$ to τ, from $\theta = \pi - \tau$ to $\pi + \tau$, and from $\theta = 2\pi - \tau$ to 2π. The period of total day is from $\theta = \tau$ to $\pi - \tau$, and that of total night from $\pi + \tau$ to $2\pi - \tau$. Q_1 is integrated through the periods of partial day, Q_2 through the period of total day.

L_1 will be found to retain its original form $\frac{\pi}{2}\sin\lambda\sin\epsilon$.

$$L_0 = \sin\lambda\sin\epsilon\cos\tau + \frac{2}{\pi}\int_0^\tau \sqrt{\cos^2\lambda - \sin^2\epsilon\sin^2\theta}\,d\theta + \frac{2\sin\lambda\sin\epsilon}{\pi}\int_0^\tau \phi\sin\theta d\theta.$$

$$L_2 = -\sin\lambda\sin\epsilon\left(\cos\tau - \frac{1}{3}\cos 3\tau\right) + \frac{4}{\pi}\int_0^\tau \cos 2\theta\sqrt{\cos^2\lambda - \sin^2\epsilon\sin^2\theta}\,d\theta$$
$$+ \frac{4\sin\lambda\sin\epsilon}{\pi}\int_0^\tau \phi\sin\theta\cos 2\theta d\theta$$

To transform these to complete integrals put $\sin\theta = \sin\tau\sin\psi$ and so $\dfrac{d\theta}{d\psi} = \dfrac{\sin\tau\cos\psi}{\Delta}$ where $\Delta = \sqrt{1-\sin^2\tau\sin^2\psi}$. As $\sqrt{\cos^2\lambda - \sin^2\tau\sin^2\theta} = \cos\lambda\cos\psi$, we get

$$L_0 = \sin\lambda\sin\epsilon\cos\tau + \frac{2\cos\lambda\sin\tau}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^2\psi\, d\psi}{\Delta} + \frac{2\sin\lambda\sin\epsilon\sin^2\tau}{\pi}\int_0^{\frac{\pi}{2}}\frac{\phi\sin\psi\cos\psi\, d\psi}{\Delta},$$

$$L_2 = -\sin\lambda\sin\epsilon\left(\cos\tau - \frac{1}{3}\cos 3\tau\right) + \frac{4\cos\lambda\sin\tau}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos 2\theta\cos^2\psi\, d\psi}{\Delta}$$
$$+\frac{4\sin\lambda\sin\epsilon\sin^2\tau}{\pi}\int_0^{\frac{\pi}{2}}\frac{\phi\cos 2\theta\sin\psi\cos\psi\, d\psi}{\Delta}.$$

The first integral in L_0 is

$$\frac{2\sin\epsilon}{\pi}(E - K\cos^2\tau) \quad \text{or} \quad \frac{2}{\pi\sin\epsilon}\{E\sin^2\epsilon - K(\sin^2\epsilon - \cos^2\lambda)\}.$$

The first integral in L_2 is

$$\frac{4\cos\lambda}{\pi\sin\tau}\int_0^{\frac{\pi}{2}}\frac{(\Delta^2 - \cos^2\tau)(2\Delta^2 - 1)}{\Delta}\, d\phi = \frac{4\sin\epsilon}{\pi}\{2D - E(1 + 2\cos^2\tau) + K\cos^2\tau\},$$

where $\quad D = \int_0^{\frac{\pi}{2}}(1 - \sin^2\tau\sin^2\psi)^{\frac{3}{2}}\, d\psi$ and so $3D = 2E(1+\cos^2\tau) - K\cos^2\tau$

and the integral $\quad = \dfrac{4}{3\pi\sin\epsilon}\{E(2\cos^2\lambda - \sin^2\epsilon) + K(\sin^2\epsilon - \cos^2\lambda)\}.$

Obviously any term of this type may be integrated by expanding $\cos 2n\theta$ in powers of $\sin^2\theta$ or $\sin^2\tau\sin^2\psi$ and so of Δ^2, i.e. $(1 - \sin^2\tau\sin^2\psi)$.

For the ϕ terms we have $\sin\phi = \dfrac{\sin\lambda\sin\psi}{\sqrt{1-\cos^2\lambda\sin^2\psi}}$, and therefore $\dfrac{d\phi}{d\psi} = \dfrac{\sin\lambda}{1-\cos^2\lambda\sin^2\psi}$,

giving $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\phi\sin^2\tau\sin\psi\cos\psi\, d\psi}{\Delta} = -\Big[\phi\Delta\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{\Delta\sin\lambda\, d\psi}{1-\cos^2\lambda\sin^2\psi}$

$$= -\frac{\pi}{2}\cot\tau + \frac{\sin\lambda}{\sin^2\epsilon}(K - \Pi\cos^2\epsilon), \text{ where } \Pi = \int_0^{\frac{\pi}{2}}\frac{d\psi}{(1-\cos^2\lambda\sin^2\psi)\,\Delta}.$$

Hence $\quad L_0 = \dfrac{2}{\pi\sin\epsilon}\{E\sin^2\epsilon - K(\sin^2\epsilon - \cos^2\lambda) + \sin^2\lambda(K - \Pi\cos^2\epsilon)\}$

$$= \frac{2}{\pi\sin\epsilon}\{E\sin^2\epsilon + \cos^2\epsilon(K - \Pi\sin^2\lambda)\}.$$

Similarly $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\phi\sin^2\tau\cos 2\theta\sin\psi\cos\psi\, d\psi}{\Delta} = -\frac{\pi}{3}\cos^3\tau + \frac{\pi}{2}\cos\tau + \int_0^{\frac{\pi}{2}}\frac{\sin\lambda\, d\psi}{1-\cos^2\lambda\sin^2\psi}\left(\frac{2}{3}\Delta^3 - \Delta\right),$

in which the integrated section exactly cancels the first expression in L_2. To transform the integral, a factor Δ is introduced in the denominator, and in the numerator $1 - \cos^2\lambda\sin^2\psi = \Delta^2\sin^2\epsilon + \cos^2\epsilon$ is used. The integral then

$$= \frac{4\sin^2\lambda}{3\pi\sin^3\epsilon}\{2E\sin^2\epsilon - (2 + \sin^2\epsilon)(K - \Pi\cos^2\epsilon)\},$$

and the whole value of L_2 is

$$\frac{4}{3\pi \sin^3 \epsilon} \{E \sin^2 \epsilon (2 - \sin^2 \epsilon) - 2K (\sin^2 \epsilon - \cos^2 \lambda) - \cos^2 \epsilon (2 + \sin^2 \epsilon)(K - \Pi \sin^2 \lambda)\}.$$

To find $\frac{dQ}{d\epsilon}$ we may either differentiate or argue as follows: $\frac{dQ}{d\epsilon} \tan \epsilon - Q$ is a function which by III $(a) = \frac{1}{\cos \lambda \sqrt{1 - \sin^2 \epsilon \sec^2 \lambda \sin^2 \theta}} \left(\frac{\sin^2 \lambda}{1 - \sin^2 \epsilon \sin^2 \theta} - 1\right)$ for the periods of partial day, and vanishes for total day or night, Q being $= \pi \sin \lambda \sin \epsilon \sin \theta$ for total day. If this be expanded in the form $m_0 + m_2 \cos 2\theta + \ldots\ldots$,

$$2\pi m_0 = 4 \int_0^{\tau} \frac{d\theta}{\cos \lambda \sqrt{1 - \sin^2 \epsilon \sec^2 \lambda \sin^2 \theta}} \left(\frac{\sin^2 \lambda}{1 - \sin^2 \epsilon \sin^2 \theta} - 1\right),$$

or transforming by $\sin \theta = \sin \tau \sin \psi$

$$m_0 = \frac{2}{\pi \sin \epsilon} \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - \sin^2 \tau \sin^2 \psi}} \left(\frac{\sin^2 \lambda}{1 - \cos^2 \lambda \sin^2 \psi} - 1\right) = \frac{2}{\pi \sin \epsilon} (\Pi \sin^2 \lambda - K).$$

So also $m_2 = \frac{4}{\pi \sin^3 \epsilon} \{-2E \sin^2 \epsilon + 2K (\sin^2 \epsilon - \cos^2 \lambda) + (2 - \sin^2 \epsilon)(K - \Pi \sin^2 \lambda)\}.$

The same method applied to expand $\frac{dQ}{d\lambda} \tan \lambda - Q$ in the form $n_0 + n_2 \cos 2\theta + \ldots\ldots$ gives

$$n_0 = -\frac{2}{\pi \cos^2 \lambda \sin \epsilon} \{E \sin^2 \epsilon - K (\sin^2 \epsilon - \cos^2 \lambda)\}$$

$$n_2 = -\frac{4}{3\pi \cos^2 \lambda \sin \epsilon} \{E \cos^2 \lambda + (K - E)(\sin^2 \epsilon - \cos^2 \lambda)\}.$$

For $\lambda = \frac{\pi}{2}$ the m's all vanish leaving $\frac{dQ}{d\epsilon} \tan \epsilon - Q = 0$, and the n's are all finite leaving $\frac{dQ}{d\lambda} = 0$. Collecting results for Arctic regions we have :—

$$\left.\begin{aligned}
L_0 &= \frac{2}{\pi \sin \epsilon} \{E \sin^2 \epsilon + \cos^2 \epsilon (K - \Pi \sin^2 \lambda)\}, \quad L_1 = \frac{\pi}{2} \sin \lambda \sin \epsilon, \\
L_2 &= \frac{4}{3\pi \sin^3 \epsilon} \{E \sin^2 \epsilon (2 - \sin^2 \epsilon) - 2K (\sin^2 \epsilon - \cos^2 \lambda) \\
&\qquad - \cos^2 \epsilon (2 + \sin^2 \epsilon)(K - \Pi \sin^2 \lambda)\}, \\
\frac{dL_0}{d\epsilon} &= \frac{2 \cos \epsilon}{\pi} (E - K + \Pi \sin^2 \lambda), \quad \frac{dL_1}{d\epsilon} = \frac{\pi}{2} \sin \lambda \cos \epsilon, \\
\frac{dL_2}{d\epsilon} &= \frac{4 \cos \epsilon}{3\pi \sin^4 \epsilon} \{-E \sin^2 \epsilon (4 + \sin^2 \epsilon) + 4K (\sin^2 \epsilon - \cos^2 \lambda) \\
&\qquad + (4 - 2 \sin^2 \epsilon + \sin^4 \epsilon)(K - \Pi \sin^2 \lambda)\}, \\
\frac{dL_0}{d\lambda} &= \frac{2 \sin \lambda}{\pi \sin \epsilon \cos \lambda} \{-E \sin^2 \epsilon + K (\sin^2 \epsilon - \cos^2 \lambda) + \cos^2 \lambda (K - \Pi \cos^2 \epsilon)\}, \quad \frac{dL_1}{d\lambda} = \frac{\pi}{2} \cos \lambda \sin \epsilon, \\
\frac{dL_2}{d\lambda} &= \frac{4 \sin \lambda}{3\pi \sin^3 \epsilon \cos \lambda} \{E \sin^4 \epsilon - K \sin^2 \epsilon (\sin^2 \epsilon - \cos^2 \lambda) - \cos^2 \lambda (2 + \sin^2 \epsilon)(K - \Pi \cos^2 \epsilon)\}
\end{aligned}\right\} \text{VI.}$$

In these formulæ $k = \frac{\cos\lambda}{\sin\epsilon}$, and the second parameter of Π is $-\cos^2\lambda$, whereas for non-arctic regions we had $\frac{\sin\epsilon}{\cos\lambda}$ and $-\sin^2\epsilon$. On the Arctic circle the parameters agree, and the values of L_0, $\frac{dL_0}{d\epsilon}$, $\frac{dL_0}{d\lambda}$, L_1,...... will be found to agree. At the other limit $\lambda = 90°$, $Q = \pi \sin\epsilon \sin\theta$ from $\theta = 0$ to π and $= 0$ from $\theta = \pi$ to 2π. The expansion of such a function by Fourier's theorem is $\sin\epsilon\left(1 + \frac{\pi}{2}\sin\theta - \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta \ldots\ldots\right)$. This limiting form, which will be found to result also from using $E = K = \Pi = \frac{\pi}{2}$ in the above, is also the case in which the convergence of coefficients is slowest.

§ 11. The way in which L_0 depends on λ and ϵ presents some interesting features which we proceed to discuss. On comparing the formulæ for arctic and non-arctic regions, it is clear that $\sin\epsilon$ plays the same part in the one, as $\cos\lambda$ in the other. Hence if other values λ' and ϵ' be taken so that λ' is the complement of ϵ, and ϵ' of λ, the arctic formula of each is transformed to the non-arctic formula of the other. Also if $\lambda + \epsilon < 90°$, so that λ is non-arctic for ϵ, $\lambda' + \epsilon'$ is $> 90°$, and therefore λ' arctic for ϵ', making the correlation complete. The use of this theorem of correlation is both convenient and suggestive. Thus when $\epsilon = 0$ the value of L_0 takes the simple form $\cos\lambda$; therefore when $\lambda = 90°$, $L_0 = \sin\epsilon$, a result already noticed. Again on the equator $L_0 = \frac{2E}{\pi}$ where $\sin\epsilon$ is the parameter; hence for $\epsilon = 90°$, $L_0 = \frac{2E}{\pi}$ where $\cos\lambda$ is the parameter, giving the form for L_0 in the extreme case when the earth's axis is supposed to lie in the plane of the orbit (BB' in Fig. 1). In this last case L_0 increases continuously from equator to pole, the first being a minimum, the second a maximum, whereas for $\epsilon = 0$ the equator has a maximum.

Now for all values of ϵ, $\frac{dL_0}{d\lambda} = 0$ for $\lambda = 0$, and for all except $\epsilon = 0$, the same is true at the pole, and the question is suggested, where does maximum change into minimum at each end? Differentiating III (a), we get

$$\frac{d^2Q}{d\lambda^2} + Q = \frac{1}{\cos\lambda}\left(\frac{1}{\Delta} - \Delta\right),$$

but in arctic regions the right-hand member is replaced by zero for total day or night. Hence

$$\frac{d^2L_0}{d\lambda^2} + L_0 = \frac{2}{\pi\cos\lambda}(K - E) \text{ or } \frac{2\sin\epsilon}{\pi\cos^2\lambda}(K - E),$$

for non-arctic and arctic regions respectively, k being $\frac{\sin \epsilon}{\cos \lambda}$ or $\frac{\cos \lambda}{\sin \epsilon}$. On the equator

$$\frac{d^2 L_0}{d\lambda^2} = \frac{2}{\pi}(K - 2E) \text{ parameter } \sin \epsilon.$$

Up to $\epsilon = 65° \, 20'$, $K < 2E$, after that $> 2E$; hence up to $65° \, 20'$ the value at the equator is a maximum, but beyond that a minimum. At the critical point

$$\frac{d^3 L_0}{d\lambda^3} = 0 \text{ and } \frac{d^4 L_0}{d\lambda^4} = \frac{2E}{\pi} \sec^2 \epsilon,$$

so that the point counts as a minimum. At the pole $L_0 = \sin \epsilon$ and $K - E$ vanishes. Expanding the right-hand member, we get

$$\frac{d^2 L_0}{d\lambda^2} + L_0 = \frac{1}{2 \sin \epsilon}\left(1 + \frac{3 \cos^2 \lambda}{8 \sin^2 \epsilon} + \dots\right).$$

Hence $\frac{d^2 L_0}{d\lambda^2}$ is positive up to $\epsilon = 45°$, and after that negative, so that the pole gives a minimum up to $\epsilon = 45°$, and after that a maximum. For the critical case

$$\frac{d^3 L_0}{d\lambda^3} = 0 \text{ and } \frac{d^4 L_0}{d\lambda^4} = \frac{3\sqrt{2}}{4},$$

and the point is a minimum. Thus from $\epsilon = 0$ to $45°$, L_0 has a maximum at the equator, and a minimum value at the pole, while from $\epsilon = 65° \, 20'$ to $90°$ the conditions are reversed, but in the range $\epsilon = 45°$ to $65° \, 20'$, equator and pole are both maxima and an intermediate minimum is suggested. The correlative statement is that from $\lambda = 0°$ to $24° \, 40'$, $\epsilon = 0$ gives a maximum, and $\epsilon = 90°$ a minimum value; from $\lambda = 45°$ to $90°$ the conditions are reversed, while between $\lambda = 24° \, 40'$ and $45°$ these are both maxima and an intermediate minimum is suggested.

§ 12. Take this statement first, and examine the points for which $\frac{dL_0}{d\epsilon} = 0$.

For *arctic* regions $\frac{dL_0}{d\epsilon} = \frac{2 \cos \epsilon}{\pi}(E - K + \Pi \sin^2 \lambda)$, by (VI) and vanishes for $\epsilon = 90°$, giving the curve BB' of which BX is the minimum, and XB' the maximum section; or for

$$E - K + \Pi \sin^2 \lambda = 0 \quad \dots\dots\dots\dots (a),$$

which reduces L_0 to $\frac{2E}{\pi \sin \epsilon}\left(\text{parameter } \frac{\cos \lambda}{\sin \epsilon}\right)$, this constituting with (a) the equation to the arctic section XZ of the curve of intermediate minima.

For this curve $\frac{dL_0}{d\lambda} = \frac{2 \sin \lambda}{\pi \sin \epsilon \cos \lambda}\{(K - E) \sin^2 \epsilon - \Pi \cos^2 \epsilon \cos^2 \lambda\}$ by (VI), or substituting from (a)

$$= \frac{2(K - E)(\sin^2 \epsilon - \cos^2 \lambda)}{\pi \sin \epsilon \sin \lambda \cos \lambda}.$$

The arctic range must be taken from $\epsilon = 90^\circ$ to the boundary of arctic and non-arctic regions given by $\sin\epsilon = \cos\lambda$, and $\frac{dL_0}{d\lambda}$ is always positive in this range, diminishing from

$$\frac{2(K-E)\tan\lambda}{\pi}, \text{ (parameter } \cos\lambda),$$

when $\epsilon = 90^\circ$, to zero when $\sin\epsilon = \cos\lambda$. When $\epsilon = 90^\circ$, $\Pi\sin^2\lambda = E$, and therefore by (a), $K = 2E$ (parameter $\cos\lambda$), the value of λ is therefore $24^\circ\ 40'$, and the values of L_0 and $\frac{dL_0}{d\lambda}$ are the same as for the curve BB', so that the curve starts at X in the figure and touches BB' at that point. At the arctic boundary $E = 1$ and therefore by (a)

$$1 = K - \Pi\sin^2\lambda = \frac{\sin\epsilon}{2}\log_e\frac{1+\sin\epsilon}{1-\sin\epsilon} = \cos\lambda\log_e\cot\frac{\lambda}{2}.$$

This is satisfied by $\lambda = 33^\circ\ 20\tfrac{1}{2}'$ or $\epsilon = 56^\circ\ 39\tfrac{1}{2}'$, and makes $L_0 = \frac{2}{\pi\sin\epsilon}$ at the point Z where the tangent is parallel to OA'.

For the *non-arctic* section

$$\frac{dL_0}{d\epsilon} = \frac{2\cos\epsilon}{\pi\sin\epsilon\cos^2\lambda}\{\Pi\sin^2\epsilon\sin^2\lambda - (K-E)\cos^2\lambda\},$$

and the range of ϵ is from zero to $\sin\epsilon = \cos\lambda$. $\frac{dL_0}{d\epsilon}$ is zero firstly for $\epsilon = 0$ which makes both terms vanish, giving the curve AA' of which YA' is the minimum, YA the maximum section; and secondly for

$$\Pi\sin^2\epsilon\sin^2\lambda = (K-E)\cos^2\lambda \quad \text{.................................}(b).$$

This condition, with the value of L_0 from (V), gives the equation to the non-arctic section ZY of the curve of intermediate minima. For this curve, quoting (V) again,

$$\frac{dL_0}{d\lambda} = \frac{2\sin\lambda}{\pi}(-E + K - \Pi\cos^2\epsilon) = -\frac{2(K-E)(\cos^2\lambda - \sin^2\epsilon)}{\pi\sin\lambda\sin^2\epsilon}$$

by means of (b); this quantity vanishes at the arctic boundary and after that is negative, attaining its greatest numerical value when $\epsilon = 0$. In the limit when $\epsilon = 0$, $\Pi = \frac{\pi}{2}$, and $(K-E)\cos^2\lambda = \frac{\pi}{4}\sin^2\epsilon$, therefore by ($b$), $\sin^2\lambda = \frac{1}{2}$, and $\lambda = 45^\circ$, and $\frac{dL_0}{d\lambda}$ = limit of

$$-\frac{\sqrt{2}}{\pi}\frac{K-E}{\sin^2\epsilon} = -\frac{1}{\sqrt{2}},$$

so that the non-arctic section ranges from Y, where it touches the curve AA' in latitude 45°, to Z where it has a tangent parallel to OA', and is continuous with XZ.

It appears then that AA' and BB' represent maxima and minima values, AY and $B'X$ being maxima, $A'Y$ and BX minima, while XY is a curve of minima. For any latitude between 24° 40′ and 45°, the value of L_0 is a maximum for $\epsilon = 0$ on curve AA', falls with increasing ϵ to a minimum on curve XY, and with further increase rises to a maximum for $\epsilon = 90°$, that is, on curve BB'. Each curve L_0 touches XY at some point, Y for $\epsilon = 0$, Z for $\epsilon = 56° \, 39\frac{1}{2}'$, and X for $\epsilon = 90°$. P is the point for which the total range is a minimum, viz. latitude 36° 7′, there being a correlative theorem that for $\epsilon = 53° \, 53'$ L_0 is the same for equator as for pole, the total range for this value of ϵ being the least possible.

By starting from $\frac{dL_0}{d\lambda} = 0$, we may obtain the locus UZV (Fig. 2) of the intermediate minima for the bushel of L_0 curves from $\epsilon = 45°$ to 65° 20′.

The ordinates are the same as those of the curve XY, and the latitude to which any ordinate belongs is the complement of the value of ϵ to which the same ordinate refers in XY. For example, U and X have the same ordinate, one referring to $\lambda = 0$, the other to $\epsilon = 90°$; so also Y and V, Z is common to both, and the tangents at U, Z and V are parallel to OA'. If the relation between λ and ϵ were explicit, one curve could readily be deduced from the other. Ordinates at special points are:—B ·6366, C or X ·7389, Z ·7620, P ·8078, Y or U ·7070.

The movements of L_0 with small range on both sides of its present value are readily followed with Table A and the statements in the outline. The percentage values given are

$$\frac{100}{L_0}\frac{dL_0}{d\epsilon}(\Delta'\epsilon - \Delta''\epsilon),$$

where $\epsilon + \Delta\epsilon'$ is greatest value, $\epsilon + \Delta''\epsilon$ least value of ϵ.

§ 13. The element of radiation intercepted by the whole earth is

$$\frac{Hdt}{r^2} \times \pi c^2 \quad \text{or} \quad \frac{Hd\theta}{h} \times \pi c^2, \text{ or per unit area } \frac{Hd\theta}{4h},$$

and therefore the year's total per unit area $= \frac{H\pi}{2h} = \frac{H}{4ab}$, whereas for a particular latitude it is $\frac{2HL_0}{h}$. Hence the mean value of L_0 for sphere or hemisphere is $\frac{\pi}{4}$. (It may be worth remarking that $4\pi H$ represents the total radiation of the sun's surface in a year.)

The mean value of L_0 may also be obtained by direct integration. In the integral

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\epsilon \sec^2\lambda \sin^2\theta} \, d\theta,$$

and other integrals used for the non-arctic case, write $\sin\phi = \sin\epsilon\sin\theta$, then

$$\frac{\pi}{2} L_0 = \int_0^{\epsilon} \frac{(\cos^4\phi - \sin^2\lambda\cos^2\epsilon)\, d\phi}{\cos\phi\sqrt{(\sin^2\epsilon - \sin^2\phi)(\cos^2\lambda - \sin^2\phi)}} .$$

In the arctic regions write $\sin\phi = \cos\lambda\sin\theta$, then

$$\frac{\pi}{2} L_0' = \int_0^{\frac{\pi}{2}-\lambda} \frac{(\cos^4\phi - \sin^2\lambda\cos^2\epsilon)\, d\phi}{\cos\phi\sqrt{(\sin^2\epsilon - \sin^2\phi)(\cos^2\lambda - \sin^2\phi)}} .$$

The mean of L_0 for the hemisphere

$$= \int_0^{\frac{\pi}{2}-\epsilon} L_0 \cos\lambda\, d\lambda + \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} L_0' \cos\lambda\, d\lambda$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}-\epsilon} \cos\lambda\, d\lambda \int_0^{\epsilon} \frac{(\cos^4\phi - \sin^2\lambda\cos^2\epsilon)\, d\phi}{\cos\phi\sqrt{(\sin^2\epsilon - \sin^2\phi)(\cos^2\lambda - \sin^2\phi)}}$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \cos\lambda\, d\lambda \int_0^{\frac{\pi}{2}-\lambda} \frac{(\cos^4\phi - \sin^2\lambda\cos^2\epsilon)\, d\phi}{\cos\phi\sqrt{(\sin^2\epsilon - \sin^2\phi)(\cos^2\lambda - \sin^2\phi)}}$$

$$= \frac{2}{\pi}\int_0^{\epsilon} \frac{d\phi}{\cos\phi}\int_0^{\frac{\pi}{2}-\phi} \frac{\cos\lambda\,(\cos^4\phi - \sin^2\lambda\cos^2\epsilon)\, d\lambda}{\sqrt{(\sin^2\epsilon - \sin^2\phi)(\cos^2\lambda - \sin^2\phi)}}$$

$$= \frac{1}{2}\int_0^{\epsilon} \frac{\cos\phi\, d\phi}{\sqrt{\sin^2\epsilon - \sin^2\phi}} \{\cos^2\epsilon + 2(\sin^2\epsilon - \sin^2\phi)\} = \frac{\pi}{4} .$$

It is clear that the same process of integration is possible where any even power of $\sin\lambda$ occurs multiplying $L_0\cos\lambda$ under the integral sign, and therefore also where $P_{2n}(\sin\lambda)$ occurs multiplying $L_0\cos\lambda$ under the integral sign, P_{2n} being a zonal harmonic.

We might therefore by this method determine the coefficients in the expansion of L_0 in zonal harmonics of even order. This expansion may, however, be obtained in a more general way, giving also L_2 ... in this form, as follows. The value of Q is

$$\left(\frac{\pi}{2} + \phi\right)\sin\lambda\sin\delta + \sqrt{\cos^2\lambda - \sin^2\delta},$$

and by application of III (a) it is easily shewn that

$$\frac{d^2Q}{d\lambda^2} - \tan\lambda\frac{dQ}{d\lambda} = \frac{d^2Q}{d\delta^2} - \tan\delta\frac{dQ}{d\delta} .$$

For the period of total day in the arctic regions $Q = \pi\sin\lambda\sin\delta$, which satisfies the same differential equation. Now if any term in the expansion of Q by zonal harmonics be $M_nP_n(\sin\lambda)$, then for this term

$$\frac{d^2Q}{d\delta^2} - \tan\delta\frac{dQ}{d\delta} = -n(n+1)M_nP_n(\sin\lambda) \quad \text{or} \quad \frac{d^2M_n}{d\delta^2} - \tan\delta\frac{dM_n}{d\delta} = -n(n+1)M_n ,$$

and therefore $M_n \propto P_n(\sin\delta)$, and the expression now stands

$$Q = b_0 + \frac{\pi}{2} P_1(\sin\lambda) P_1(\sin\delta) + b_2 P_2(\sin\lambda) P_2(\sin\delta) + b_4 P_4(\sin\lambda) P_4(\sin\delta) + \ldots\ldots \text{VII};$$

the annual term being the only zonal of odd order. When

$$\epsilon = 0, \quad P_1(\sin\delta) = 0, \quad P_2 = -\frac{1}{2}, \quad P_4 = \frac{3}{8}, \quad \ldots, \quad P_{2n} = (-1)^n \frac{1 \,.\, 3 \ldots \overline{2n-1}}{2 \,.\, 4 \ldots 2n};$$

but Q then reduces to $\cos\lambda$, and this, expanded by even zonals of $\sin\lambda$, is

$$\frac{\pi}{2}\left\{\frac{1}{2} - 5\left(\frac{1}{2}\right)^2 \frac{1}{4} P_2(\sin\lambda) - 9\left(\frac{1}{2\,.\,4}\right)^2 \frac{3}{6} P_4 - 13\left(\frac{1\,.\,3}{2\,.\,4\,.\,6}\right)^2 \frac{5}{8} P_6 \ldots\right\}.$$

Hence

$$b_0 = \frac{\pi}{4}, \quad b_2 = \frac{5\pi}{16}, \quad b_4 = -\frac{3\pi}{32}, \quad \ldots, \quad b_{2n} = (-1)^{n+1}(4n+1)\frac{1\,.\,3 \ldots \overline{2n-3}}{2\,.\,4 \ldots \overline{2n+2}} \times \frac{\pi}{2}.$$

Each term $P_{2n}(\sin\delta)$ may be expanded in cosines of multiples of θ.

For this purpose put

$$\sin\delta = \cos\epsilon\cos\frac{\pi}{2} + \sin\epsilon\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2} - \theta\right),$$

so that θ appears as an azimuthal angle; and apply the general theorem

$$P_i(\cos\gamma) = \sum_{s=0}^{s=i} 2 \times \frac{\lfloor i-s}{\lfloor i+s} \cos s(\phi - \phi') \times \nu^s \nu'^s P_i^s(\mu) P_i^s(\mu'),$$

in which $P_i^s(\mu)$ denotes $\frac{d^s}{d\mu^s} P_i(\mu)$, $\cos\gamma = \mu\mu' + \nu\nu'\cos(\phi - \phi')$, and the factor 2 is omitted for $s = 0$. In the present case $i = 2n$, $\mu' = 0$, $\nu' = 1$, $\mu = \cos\epsilon$, $\nu = \sin\epsilon$; therefore as $P_{2n}^{2s+1}(0) = 0$ and

$$P_{2n}^{2s}(0) = \frac{(-1)^{n-s}}{2^{2n}} \frac{\lfloor 2n+2s}{\lfloor n+s \lfloor n-s} \quad \text{and} \quad \cos 2s\left(\frac{\pi}{2} - \theta\right) = (-1)^s \cos 2s\theta,$$

$$P_{2n}(\sin\delta) = \sum_{s=0}^{s=n} 2 \times (-1)^n \frac{\lfloor 2n-2s}{2^{2n} \lfloor n+s \lfloor n-s} \sin^{2s}\epsilon P_{2n}^{2s}(\cos\epsilon)\cos 2s\theta;$$

the factor 2 omitted for $s = 0$. The substitution of these values in (VII) gives the complete expansion of Q in cosines of multiples of θ, and the coefficients are series of zonal harmonics of $\sin\lambda$, with these and associated functions of $\cos\epsilon$. Thus writing

$$\sqrt{1-\mu^2} = a_0 + a_2 P_2(\mu) + a_4 P_4(\mu) + \ldots,$$

$$L_0 = a_0 + a_2 P_2(\sin\lambda) P_2(\cos\epsilon) + a_4 P_4(\sin\lambda) P_4(\cos\epsilon) + \dots$$

$$\left.\begin{aligned}
L_2 &= \sin^2\epsilon \left\{\frac{1}{2} a_2 P_2(\sin\lambda) P_2^2(\cos\epsilon) + \frac{a_4}{9} P_4(\sin\lambda) P_4^2(\cos\epsilon)\right. \\
&\qquad \left. + \frac{a_6}{20} P_6(\sin\lambda) P_6^2(\cos\epsilon) + \dots \frac{a_{2n} P_{2n}(\sin\lambda) P_{2n}^2(\cos\epsilon)}{(n+1)(2n-1)} + \dots\right\} \\
L_4 &= \sin^4\epsilon \left\{\frac{a_4}{72} P_4(\sin\lambda) P_4^4(\cos\epsilon) + \frac{a_6}{600} P_6(\sin\lambda) P_6^4(\cos\epsilon)\right. \\
&\qquad \left. + \frac{a_8}{2100} P_8 P_8^4 + \dots \frac{a_{2n} P_{2n}(\sin\lambda) P_{2n}^4(\cos\epsilon)}{2(n+1)(n+2)(2n-1)(2n-3)} + \dots\right\} \\
L_6 &= \sin^6\epsilon \left\{\frac{a_6}{7200} P_6(\sin\lambda) P_6^6(\cos\epsilon) + \frac{a_8}{88200} P_8(\sin\lambda) P_8^6(\cos\epsilon)\right. \\
&\qquad \left. + \dots \frac{a_{2n} P_{2n}(\sin\lambda) P_{2n}^6(\cos\epsilon)}{2^2(n+1)(n+2)(n+3)(2n-1)(2n-3)(2n-5)} + \dots\right\}
\end{aligned}\right\} \dots \text{VIII.}$$

As before $L_1 = \frac{\pi}{2}\sin\lambda\sin\epsilon$. The coefficients a_0, a_2, ..., converge rather slowly; with $\epsilon = 23^\circ 27'$ the zonal expansion of L_0 is

$$\cdot7854 - \cdot3743 P_2(\sin\lambda) - \cdot0351 P_4(\sin\lambda) + \cdot0064 P_6(\sin\lambda) + \cdot0109 P_8(\sin\lambda) + \cdot0068 P_{10}(\sin\lambda) + \cdot0018 P_{12}(\sin\lambda) + \dots;$$

the reason of the set-back being that $P_8(\cos\epsilon) = -\cdot3827$, while $P_6(\cos\epsilon) = -\cdot1277$.

The values of $P_{2n}(\cos\epsilon)$ with increasing n become ultimately indefinitely small, though the diminution is not steady from term to term, but accompanied by a rocking to and fro, according as $\cos\epsilon$ lies near roots or maxima or minima of the equation $P_{2n}(\mu) = 0$.

§ 14. For most purposes, I think, the formulæ for a particular latitude give information as useful as those for finite zones, and excepting the case of the hemisphere are simpler in form. I propose therefore to shew how the zone formulæ may be obtained, but with less detail.

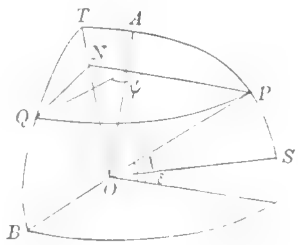


Fig. 5.

The method given is that used by Sir Robert Ball for the hemisphere, viz. we project the area illuminated at any moment by the sun, on a plane perpendicular to the sun's rays, and distribute this evenly in longitude over the whole zone.

Consider the polar cap extending to latitude λ ($< 90^\circ - \epsilon$ unless otherwise mentioned). In Fig. 5 OS is the direction of the sun, PQ is a small circle of latitude λ, ST a quadrant, and PQT represents half the illuminated portion of the cap with the sun at S. We require the projection of the surface $2PTQ$ on the plane BOT at right angles to

OS. In Fig. 6 this projection is represented by $TQMQ'$ for summer, and by $TQM'Q'$ for winter, where $QMQ'M'$ is an ellipse, whose semi-axes are $c\cos\lambda$ and $c\cos\lambda\sin\delta$.

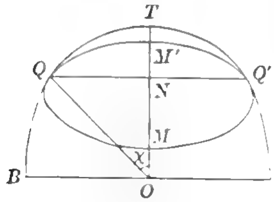


Fig. 6.

The area $2TNQ = c^2(\chi - \sin\chi\cos\chi)$ where $\sin\chi = \cos\lambda\sin\psi$. The area QMQ' is the projection of $2PNQ$ on the plane, and therefore

$$= c^2\cos^2\lambda(\psi - \sin\psi\cos\psi)\sin\delta,$$

giving for the total projection

$$c^2\{\chi - \sin\chi\cos\chi + \sin\delta\cos^2\lambda(\psi - \sin\psi\cos\psi)\};$$

ψ is the hour-angle at sunset determined by $\sin\lambda\sin\delta + \cos\lambda\cos\delta\cos\psi = 0$ for which, as before, we write $\frac{\pi}{2} + \phi$ so that ϕ is positive for summer and negative for winter.

In any non-arctic latitude $\frac{\pi}{2} - \lambda$ is the maximum value of χ, its range is small in low latitudes, greatest on the arctic circle, viz. from 0 to ϵ. Using the relations

$$\sin\delta = \sin\epsilon\sin\theta, \quad \sin\phi = \tan\lambda\tan\delta, \quad \cos\chi = \sin\lambda\sec\delta$$

the above may be written:—

$$c^2\left\{\frac{\pi}{2}\cos^2\lambda\sin\epsilon\sin\theta + \chi + \phi\sin\epsilon\sin\theta\cos^2\lambda - \sin\lambda\cos\lambda\sqrt{1 - \sin^2\epsilon\sec^2\lambda\sin^2\theta}\right\} \text{......IX.}$$

The element of heat-supply for the cap is got by multiplying this projected area by

$$\frac{Hdt}{r^2} \text{ or } \frac{Hd\theta}{h}, \text{ giving } \frac{Hc^2}{h}Zd\theta,$$

where Z is the bracket. When $\lambda > (90^\circ - \epsilon)$, Z takes the value $\pi\cos^2\lambda\sin\epsilon\sin\theta$ for periods of total day, and vanishes for total night, the range of χ being from 0 to $\frac{\pi}{2} - \lambda$ for partial day.

On an indefinitely small zone the amount is $\frac{c^2Hd\theta}{h}\left(-\frac{dZ}{d\lambda}d\lambda\right)$, and this must equal the result previously obtained, namely

$$\frac{HQd\theta}{\pi h}(2\pi c^2\cos\lambda\, d\lambda), \text{ or } \frac{dZ}{d\lambda} = -2Q\cos\lambda,$$

which of course admits of easy verification. We may write

$$Z + Q\sin\lambda = \frac{\pi}{2}\sin\epsilon\sin\theta + \chi + \phi\sin\epsilon\sin\theta.$$

§ 15. The only new element with which we have to deal is χ, which is expanded in a manner analogous to that for ϕ, with this difference, that in integrating the series for $\frac{d\chi}{d\theta}$, it is to be remembered that $\chi = \frac{\pi}{2} - \lambda$ for $\theta = 0$.

Result for non-arctic regions

$$\chi = \frac{\pi}{2} - \lambda + \frac{4 \tan \lambda}{\pi \sin^2 \epsilon} (1 - \cos 2\theta) \{E \cos^2 \lambda - K (\cos^2 \lambda - \sin^2 \epsilon) - (K - \Pi \cos^2 \epsilon)\}$$

$$+ \frac{4 \tan \lambda}{\pi \sin^4 \epsilon} (1 - \cos 4\theta) \left\{(K - \Pi \cos^2 \epsilon)(2 - \sin^2 \epsilon) + \frac{K}{3} (\cos^2 \lambda - \sin^2 \epsilon)(6 - 3 \sin^2 \epsilon + 4 \cos^2 \lambda) - \frac{E \cos^2 \lambda}{3} (6 - 5 \sin^2 \epsilon + 4 \cos^2 \lambda)\right\} + \dots .$$

For arctic regions

$$\chi = \frac{\pi}{2} - \lambda + \frac{4 \sin \lambda}{\pi \sin^3 \epsilon} (1 - \cos 2\theta) \{E \sin^2 \epsilon - (K - \Pi \cos^2 \epsilon)\}$$

$$+ \frac{4 \sin \lambda}{\pi \sin^5 \epsilon} (1 - \cos 4\theta) \left\{(K - \Pi \cos^2 \epsilon)(2 - \sin^2 \epsilon) - \frac{2K}{3} (\sin^2 \epsilon - \cos^2 \lambda) - \frac{E \sin^2 \epsilon}{3} (6 - 5 \sin^2 \epsilon + 4 \cos^2 \lambda)\right\} + \dots .$$

From previous work, for non-arctic regions

$$\phi \sin \epsilon \sin \theta = \frac{2 \tan \lambda}{\pi} (K - \Pi \cos^2 \epsilon) + \frac{4 \tan \lambda}{3\pi \sin^2 \epsilon} \cos 2\theta \{-(K - \Pi \cos^2 \epsilon)(2 + \sin^2 \epsilon) - 2K (\cos^2 \lambda - \sin^2 \epsilon) + 2E \cos^2 \lambda\}$$

$$+ \frac{4 \tan \lambda}{15\pi \sin^4 \epsilon} \cos 4\theta \{(K - \Pi \cos^2 \epsilon)(24 - 8 \sin^2 \epsilon - \sin^4 \epsilon) + 8K (\cos^2 \lambda - \sin^2 \epsilon)(3 - \sin^2 \epsilon + 2 \cos^2 \lambda) - 8E \cos^2 \lambda (3 - 2 \sin^2 \epsilon + 2 \cos^2 \lambda)\} + \dots ,$$

and for arctic regions

$$\phi \sin \epsilon \sin \theta = \frac{2 \sin \lambda}{\pi \sin \epsilon} (K - \Pi \cos^2 \epsilon) + \frac{4 \sin \lambda}{3\pi \sin^3 \epsilon} \cos 2\theta \{-(K - \Pi \cos^2 \epsilon)(2 + \sin^2 \epsilon) + 2E \sin^2 \epsilon\}$$

$$+ \frac{4 \sin \lambda}{15\pi \sin^5 \epsilon} \cos 4\theta \{(K - \Pi \cos^2 \epsilon)(24 - 8 \sin^2 \epsilon - \sin^4 \epsilon) - 8K (\sin^2 \epsilon - \cos^2 \lambda) \sin^2 \epsilon - 8E \sin^2 \epsilon (3 - 2 \sin^2 \epsilon + 2 \cos^2 \lambda)\} + \dots .$$

Summer and winter heat-supplies are, for caps extending at least as far as the polar circle,

$$\frac{2Hc^2}{h} \left(\pm \frac{\pi}{2} \cos^2 \lambda \sin \epsilon - E \sin \lambda \cos \lambda + \sin \lambda \cos \lambda (K - \Pi \cos^2 \epsilon) + \frac{\pi}{2} \chi_0 \right),$$

and for caps reaching not farther than the polar circle

$$\frac{2Hc^2}{h} \left(\pm \frac{\pi}{2} \cos^2 \lambda \sin \epsilon - E \sin \lambda \cos \lambda + \frac{K \sin \lambda}{\sin \epsilon} (\sin^2 \epsilon - \cos^2 \lambda) + \frac{\sin \lambda \cos^2 \lambda}{\sin \epsilon} (K - \Pi \cos^2 \epsilon) + \frac{\pi}{2} \chi_0 \right).$$

where χ_0 is the non-periodic term in the expansion of χ, and in the first formula $k=\dfrac{\sin\epsilon}{\cos\lambda}$, in the second $k=\dfrac{\cos\lambda}{\sin\epsilon}$; in the first formula the second parameter of Π is $-\sin^2\epsilon$, in the second, $-\cos^2\lambda$.

Or with Z expanded in the form

$$Z_0+\frac{\pi}{2}\cos^2\lambda\sin\epsilon\sin\theta+Z_2\cos 2\theta+\ldots,$$

the summer and winter heat-supplies are

$$\frac{\pi Hc^2}{h}(Z_0\pm\cos^2\lambda\sin\epsilon).$$

It may be shewn that for non-arctic regions

$$\frac{dZ_0}{d\epsilon}=-\frac{2E}{\pi}\cot\epsilon\sin\lambda\cos\lambda+\frac{2\sin\lambda\cos\lambda\cot\epsilon}{\pi}(K-\Pi\cos^2\epsilon)+\frac{2\tan\lambda\cot\epsilon}{\pi}(K-\Pi\sin^2\lambda),$$

and for arctic regions

$$\frac{dZ_0}{d\epsilon}=-\frac{2E}{\pi}\sin\lambda\cos\epsilon+\frac{2\cos\epsilon\sin\lambda}{\pi}(K-\Pi\sin^2\lambda)+\frac{2\cos\epsilon\sin\lambda}{\pi\sin^2\epsilon}(K-\Pi\cos^2\epsilon),$$

with the usual parameters for the two cases. These last give the alteration in the non-periodic term, due to change of obliquity of the ecliptic, for any polar cap. When a complete hemisphere is taken $\lambda=0$, and therefore $\phi=0$, $\chi=\dfrac{\pi}{2}$, and the formula for the heat-supply reduces to

$$\frac{\pi Hc^2dt}{2r^2}(1+\sin\epsilon\sin\theta)\quad\text{or}\quad\frac{\pi Hc^2d\theta}{2h}(1+\sin\epsilon\sin\theta).$$

In this case summer and winter supplies are $\dfrac{\pi Hc^2}{h}\left(\dfrac{\pi}{2}\pm\sin\epsilon\right)$ respectively, or per unit area $\dfrac{H}{2h}\left(\dfrac{\pi}{2}\pm\sin\epsilon\right)$, and for the year $\dfrac{\pi H}{2h}$; Sir Robert Ball's results.

The mean supply per unit area in latitude λ was denoted by $\dfrac{2H}{h}L_0$. Hence the mean value of L_0 for the hemisphere is $\dfrac{\pi}{4}$ (belonging to a latitude 36° 35′ approximately), as was proved by direct integration above. Z_0 increases from 0 when $\lambda=90°$ to $\dfrac{\pi}{2}$ when $\lambda=0$; $\dfrac{dZ_0}{d\epsilon}$ is always positive, vanishes for $\lambda=0°$ and 90°, and has a maximum where $\dfrac{dL_0}{d\epsilon}=0$, that is, about latitude 43° 20′. (See Table A.)

§ 16. The values for *zones* being got by subtracting one polar cap from another, it is clear that for zones both of whose margins are below this latitude, $\frac{dZ_0}{d\epsilon}$ is negative, for zones with both margins above this latitude, positive; while zones which embrace this latitude may have $\frac{dZ_0}{d\epsilon}$ positive, negative, or zero according to the extent to which the margins pass to the two sides of latitude 43° 20′. As an illustration, I give figures for the three zones into which the northern hemisphere is divided by the parallels of 30° and 60°.

The polar zone here extends beyond the arctic circle even with ϵ at its maximum value. The heat-supplies for the three zones are given by the following series, multiplied by $\frac{c^2Hd\theta}{h}$, namely for

Tropical zone	$\cdot 9208 + \cdot 1563 \sin\theta + \cdot 0356 \cos 2\theta \ldots$,
Temperate zone	$\cdot 5246 + \cdot 3126 \sin\theta - \cdot 0017 \cos 2\theta \ldots$,
Polar zone	$\cdot 1249 + \cdot 1563 \sin\theta - \cdot 0339 \cos 2\theta \ldots$;

or as the areas of these zones are πc^2, $\pi c^2(\sqrt{3}-1)$, $\pi c^2(2-\sqrt{3})$, the mean supplies per unit area are $\frac{Hd\theta}{\pi h}$ multiplied by series for

Tropical zone	$\cdot 9208 + \cdot 1563 \sin\theta + \cdot 0356 \cos 2\theta \ldots$,
Temperate zone	$\cdot 7168 + \cdot 4270 \sin\theta - \cdot 0023 \cos 2\theta \ldots$,
Polar zone	$\cdot 4740 + \cdot 5833 \sin\theta - \cdot 1265 \cos 2\theta \ldots$.

The values of $\frac{dZ_0}{d\epsilon}$ for $\lambda = 0°$, 30° and 60° are respectively 0, ·1693, and ·1559; hence for the zones the values − ·1693, + ·0134 and ·1559 in the order tropical, temperate, polar. With Stockwell's limits for ϵ the total changes in Z_0 are ·007754, ·000623, and ·00714, or for the non-periodic term of the second series ·0078, ·00085, ·0266; or in percentages ·84 per cent. for the tropical zone, hardly appreciable for the temperate zone, and for the polar zone nearly $5\frac{3}{4}$ per cent. These figures give the whole range up and down from the present value; by an increase in ϵ the tropical zone loses, the temperate very slightly gains, and the polar zone gains considerably. The annual term, exactly like that for a single latitude, has a total range of $10\frac{1}{2}$ per cent.

The application of the zonal harmonic method gives results much simpler in form, but the convergence is a little slow. We have $\frac{dZ}{d\lambda} = -2Q\cos\lambda$, and therefore

$$Z = \int_{\lambda}^{\frac{\pi}{2}} 2Q \cos\lambda \, d\lambda.$$

Integrate the whole zonal expansion of Q given in VII., p. 78; we obtain an expression for Z, viz. $P_1(\sin\lambda)$ is replaced by $\cos^2\lambda$, P_2 by $\sin\lambda\cos^2\lambda$, P_4 by $\frac{\sin\lambda\cos^2\lambda}{4}(7\sin^2\lambda-3)$,

$$P_6 \text{ by } \frac{\sin\lambda\cos^2\lambda}{8}(33\sin^4\lambda-30\sin^2\lambda+5),$$

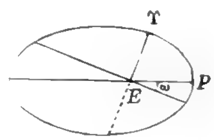
$$P_8 \text{ by } \frac{\sin\lambda\cos^2\lambda}{64}(715\sin^6\lambda-100)(\sin^4\lambda+385\sin^2\lambda-35)\ldots,$$

and generally $\quad P_{2n}$ by $\frac{2}{4n+1}(P_{2n-1}-P_{2n+1})$, *i.e.* $\frac{\cos^2\lambda}{n(2n+1)}P'_{2n}(\sin\lambda)$.

For example, the integration of the non-periodic term in VIII. gives

$$Z_0=\frac{\pi}{2}(1-\sin\lambda)+a_2P_2(\cos\epsilon)\sin\lambda\cos^2\lambda+\frac{a_4}{4}P_4(\cos\epsilon)\sin\lambda\cos^2\lambda\,(7\sin^2\lambda-3)\ldots.$$

§ 17. The transformation of the various results to mean time presents no difficulties. We use the relation



$$\theta+\frac{\pi}{2}-\omega=\psi+2e\sin\psi+\frac{5e^2}{4}\sin 2\psi\ldots,$$

where $\psi=2\pi t$, t measured from perihelion, θ from ϒ the spring equinox. Taking into account e^2L_0, e^2L_1, eL_2 but not e^2L_2, and omitting eL_4, $\frac{Hd\theta}{\pi h}(L_0+L_1\sin\theta\ldots)$ transforms to $\frac{2H}{h}dt$ multiplied by

$$L_0+\cos(\psi+\omega)\left\{-L_1\left(1-\frac{7e^2}{8}\right)+e\cos\omega\,(2L_0+L_2)\right\}+\sin(\psi+\omega)(2L_0-L_2)\,e\sin\omega$$

$$+\cos 2(\psi+\omega)\left\{-L_2+\frac{5e^2}{2}L_0\cos 2\omega-2eL_1\cos\omega\right\}+\sin 2(\psi+\omega)\left(\frac{5e^2}{2}L_0\sin 2\omega-2eL_1\sin\omega\right)$$

$$+\cos(3\psi+2\omega)\left\{-\frac{27e^2}{8}L_1\cos\omega-3eL_2\right\}-\frac{27e^2}{8}L_1\sin\omega\sin(3\psi+2\omega)$$

$$+L_4\cos 4(\psi+\omega)\ldots.$$

For numerical results see Table B.

In this formula L_1 changes sign in passing to the southern hemisphere.

If e is ignored, the formula is

$$L_0-L_1\cos(\psi+\omega)-L_2\cos 2(\psi+\omega)+L_4\cos 4(\psi+\omega)\ldots.$$

Regarding this as a normal form, it is most modified by the ellipticity of the orbit, when, in the course of secular changes, ω has the values 0 and π.

To the same order as above, the heat-supply being $\frac{2HQ'}{h}dt$, we have for $\omega = 0$ perihelion in midwinter

$$Q' = L_0 - \left\{L_1\left(1 - \frac{7e^2}{8}\right) - 2eL_0 - eL_2\right\}\cos 2\pi t - \left\{L_2 - \frac{5e^2}{2}L_0 + 2eL_1\right\}\cos 4\pi t$$
$$- \left(3eL_2 + \frac{27e^2}{8}L_1\right)\cos 6\pi t + L_4\cos 8\pi t \ldots,$$

and for $\omega = \pi$, perihelion in midsummer, the sign of L_1 must be changed.

As in the outline $\omega = 0$ belongs to the genial period, $\omega = \pi$ to the glacial; and a genial period in one hemisphere due to this cause corresponds to a glacial one in the other. The values $\omega = \frac{\pi}{2}, \frac{3\pi}{2}$ make summer and winter of equal length, and give the minimum departure from standard form; for the first, perihelion in conjunction with vernal equinox

$$Q' = L_0 + L_1\left(1 - \frac{7e^2}{8}\right)\sin 2\pi t + e(2L_0 - L_2)\cos 2\pi t + \left(L_2 + \frac{5e^2}{2}L_0\right)\cos 4\pi t + 3eL_2\cos 6\pi t$$
$$+ \frac{27e^2}{8}L_1\sin 6\pi t + L_4\cos 8\pi t \ldots.$$

As an example of zone formulæ transferred to mean time take first the hemisphere, for which the element of supply per unit area is $\frac{Hd\theta}{4h}(1 + \sin\epsilon\sin\theta)$ or $\frac{\pi Hdt}{2h}$ multiplied by:—

for genial epoch $1 - \left\{\sin\epsilon\left(1 - \frac{7e^2}{8}\right) - 2e\right\}\cos 2\pi t - \left(2e\sin\epsilon - \frac{5e^2}{2}\right)\cos 4\pi t$
$$- \frac{27e^2}{8}\sin\epsilon\cos 6\pi t \ldots,$$

" glacial " $1 + \left\{\sin\epsilon\left(1 - \frac{7e^2}{8}\right) + 2e\right\}\cos 2\pi t + \left(2e\sin\epsilon + \frac{5e^2}{2}\right)\cos 4\pi t$
$$+ \frac{27e^2}{8}\sin\epsilon\cos 6\pi t \ldots,$$

" present position $1 - \left\{\sin\epsilon\left(1 - \frac{7e^2}{8}\right) - 2e\cos\omega\right\}\cos(2\pi t + \omega) + 2e\sin\omega\sin(2\pi t + \omega)$
$$- \left(2e\sin\epsilon - \frac{5e^2}{2}\cos\omega\right)\cos(4\pi t + \omega) + \frac{5e^2}{2}\sin\omega\sin(4\pi t + \omega) - \frac{27e^2}{8}\sin\epsilon\cos(6\pi t + \omega)\ldots.$$

In figures these are:—

for genial extreme $1 - {\cdot}2562\cos 2\pi t - {\cdot}0435\cos 4\pi t - {\cdot}0066\cos 6\pi t \ldots,$

" glacial " $1 + {\cdot}5362\cos 2\pi t + {\cdot}0680\cos 4\pi t + {\cdot}0066\cos 6\pi t \ldots,$

" present position $1 - {\cdot}3669\cos(2\pi t + 11^\circ\ 56') - {\cdot}0128\cos(4\pi t + 11^\circ\ 37')$
$$- {\cdot}0004\cos(6\pi t + 11^\circ)\ldots,$$

with $\omega = 11^\circ$, $e = \cdot 0168$ for present position, and with $e = \cdot 07$ for the extremes. In every case time measurements start from perihelion.

As a further example the second type of series on p. 83, referring to the three zones 0° to 30°, 30° to 60°, 60° to 90°, are in mean time for

$$\text{Tropical zone} \quad \cdot 9208 + \begin{matrix} \cdot 2871 \\ -\cdot 0241 \end{matrix}\Bigg\} \cos 2\pi t \begin{matrix} -\cdot 0025 \\ -\cdot 0463 \end{matrix}\Bigg\} \cos 4\pi t \begin{matrix} -\cdot 0049 \\ -\cdot 0101 \end{matrix}\Bigg\} \cos 6\pi t \ldots,$$

$$\text{Temperate zone} \quad \cdot 7168 + \begin{matrix} \cdot 5255 \\ -\cdot 3251 \end{matrix}\Bigg\} \cos 2\pi t \begin{matrix} +\cdot 0708 \\ -\cdot 0488 \end{matrix}\Bigg\} \cos 4\pi t \begin{matrix} +\cdot 0075 \\ -\cdot 0065 \end{matrix}\Bigg\} \cos 6\pi t \ldots,$$

$$\text{Polar zone} \quad \cdot 4740 + \begin{matrix} \cdot 6561 \\ -\cdot 5057 \end{matrix}\Bigg\} \cos 2\pi t \begin{matrix} +\cdot 2140 \\ +\cdot 0506 \end{matrix}\Bigg\} \cos 4\pi t \begin{matrix} +\cdot 0356 \\ +\cdot 0176 \end{matrix}\Bigg\} \cos 6\pi t \ldots;$$

the upper figures belonging to the extreme glacial, the lower to the extreme genial epoch.

By comparison of these results for broad zones with those just given for the hemisphere, it is seen what an unsatisfactory view of the effects of eccentricity is given by taking the average for the hemisphere.

§ 18. *Absorption.* It is usual to allow for absorption of light or heat in passing through an absorbing medium by the use of a formula e^{-mz}, where z is the thickness traversed. For a considerable range from the zenith, when the earth's atmosphere is in question, z is taken proportional to $\sec I$, I being the angle between the zenith and the sun's rays, but near the horizon, the formula is modified and z made to approach a limit depending on the value assumed for the height of the atmosphere. The formula is easy of application when the object is to compare the amounts of absorption at different times of the day, but seems to present considerable difficulties, if we wish to integrate for the annual supply. Moreover it involves the assumption that rays after passing through a mile of the earth's atmosphere, experience the same proportionate absorption in passing through the second mile. I suggest the use of a formula $e_0 + e_1 \cos I + e_2 \cos^2 I + \ldots$ to represent the proportion of heat or light transmitted, and that this formula be compared directly with observation at different zenith distances. This would give the relative values of e_0, $e_1 \ldots$ which is sufficient for all terrestrial problems. For the determination of the solar constant, absolute values are needed, and these involve some such hypothesis as to absorption as is given above. The exact integration of each term of the above can be effected by the methods hitherto used, and some answer can be given to the interesting enquiry, how far the proportions of the coefficients of the non-periodic and various periodic terms are affected in different latitudes.

Thus for example the e_1 term gives for the day's integral

$$\frac{e_1 H \Delta t}{2\pi r^2} \int_{-\psi_1}^{+\psi_1} (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^2 \, d\psi \quad \text{or} \quad \frac{e_1 H Q_1 d\theta}{\pi h},$$

where $$Q_1 = \frac{1}{2}\left(\frac{\pi}{2} + \phi\right)\left\{\cos^2\lambda + \sin^2\epsilon\left(\frac{3}{2}\sin^2\lambda - \frac{1}{2}\right)(1 - \cos 2\theta)\right\}$$
$$+ \frac{3}{2}\sin\lambda \cos\lambda \sin\epsilon \sin\theta \sqrt{1 - \sin^2\epsilon \sec^2\lambda \sin^2\theta}.$$

Here $\cos 2\theta$ is the only even multiple of θ that occurs, this and the non-periodic term being simple trigonometrical functions. The expansion of ϕ has sines of odd multiples of θ, and the last term contributes the same type, both expansions being the same as in previous work. The types alternate, the transcendental terms contributing for Q, $Q_2 \ldots$ to the non-periodic term and even multiples of θ (cosines), for Q_1, $Q_3 \ldots$ to the odd multiples of θ (sines); the other set of terms in each case being purely trigonometrical. Instead of pursuing this method which for succeeding terms becomes heavy, though presenting no special difficulties, I propose to apply the method of expansion by zonal harmonics.

§ 19. Let
$$\chi_p = \int_0^{\psi_1} (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^p \, d\psi,$$

where ψ_1 is the hour-angle at sunset, so that
$$\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi_1 = 0,$$

for non-arctic regions, but for arctic regions ψ_1 is π during the period of total day, and zero for the period of total night, giving at the pole $\chi_p = \pi \sin^p \delta$ for summer when $\sin\delta$ is positive, and zero for the winter when $\sin\delta$ is negative.

Since
$$\frac{d}{d\psi}\{\sin\psi (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^{p-1}\}$$
$$= \cos\psi (\sin\lambda \sin\delta + \ldots)^{p-1} - (p-1)\cos\lambda \cos\delta \sin^2\psi (\sin\lambda \sin\delta + \ldots)^{p-2},$$
$$\therefore \quad (p-1)\cos\lambda \cos\delta \int_0^{\psi_1} \sin^2\psi (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^{p-2} \, d\psi$$
$$= \int_0^{\psi_1} \cos\psi (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^{p-1} \, d\psi,$$

or
$$(p-1)\cos^2\lambda \cos^2\delta \int_0^{\psi_1} \sin^2\psi (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^{p-2} \, d\psi + \chi_{p-1} \sin\lambda \sin\delta = \chi_p.$$

Also
$$\cos^2\lambda \cos^2\delta \int_0^{\psi_1} \cos^2\psi (\sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\psi)^{p-2} \, d\psi$$
$$= \chi_p - 2\chi_{p-1} \sin\lambda \sin\delta + \chi_{p-2} \sin^2\lambda \sin^2\delta.$$

Therefore by addition and reduction
$$(p-1)\chi_{p-2}(\cos^2\lambda - \sin^2\delta) = p\chi_p - (2p-1)\chi_{p-1} \sin\lambda \sin\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

which is the sequence equation. Again, it is easy to shew that
$$\cos\lambda \frac{d\chi_p}{d\lambda} + p\chi_p \sin\lambda = p\chi_{p-1} \sin\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

and by a second differentiation
$$\frac{d^2\chi_p}{d\lambda^2} - \tan\lambda \frac{d\chi_p}{d\lambda} + p(p+1)\chi_p = \frac{p}{\cos^2\lambda}\{p\chi_p - (2p-1)\chi_{p-1} \sin\lambda \sin\delta + (p-1)\chi_{p-2} \sin^2\delta\}$$
$$= p(p-1)\chi_{p-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (c).$$

All these results remain true in arctic regions up to the pole for the values assumed in total day. Each function is thus connected with the alternate one by an equation suited for the comparison of corresponding zonals; and identical results hold for differentiation with regard to δ. Now χ_1 is the same as Q which was expanded in VII., p. 78, and this gives us the starting-point for the odd functions. We seek, therefore, an expansion for χ_0 to furnish the starting-point for the even series. In non-arctic regions

$$\chi_0 = \frac{\pi}{2} + \phi,$$

but in arctic regions during the period of total day $\chi_0 = \pi$ and for total night $= 0$. We prove

$$\frac{d\phi}{d\lambda}\cos\lambda = \frac{\sin\delta}{\sqrt{\cos^2\lambda - \sin^2\delta}},$$

and therefore

$$\frac{d^2\phi}{d\lambda^2} - \tan\lambda\frac{d\phi}{d\lambda} = \frac{\sin\lambda\sin\delta}{(\cos^2\lambda - \sin^2\delta)^{\frac{3}{2}}} = \frac{d^2\phi}{d\delta^2} - \tan\delta\frac{d\phi}{d\delta},$$

which suggests

$$\chi_0 = \frac{\pi}{2} + \phi = \frac{\pi}{2} + c_1 P_1(\sin\lambda)\, P_1(\sin\delta) + c_3 P_3(\sin\lambda)\, P_3(\sin\delta) + \ldots.$$

But when $\lambda = 0$,

$$\frac{d\phi}{d\lambda} = \tan\delta = \frac{\pi}{2}\left\{3\times\frac{1}{2}P_1(\sin\delta) + 7\left(\frac{1}{2}\right)^2\frac{3}{4}P_3(\sin\delta) + 11\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{5}{6}P_5(\sin\delta)\ldots\right\},$$

quoting the expansion of $\frac{\mu}{\sqrt{1-\mu^2}}$ in Todhunter's *Laplace's Functions*, p. 115.

Also $\qquad P_1'(0) = 1,\quad P_3'(0) = -\frac{3}{2},\quad \ldots\quad P'_{2n+1}(0) = (-1)^n\frac{3\,.\,5\ldots\overline{2n+1}}{2\,.\,4\ldots 2n};$

$$\therefore\quad c_1 = \frac{3\pi}{4},\quad c_3 = -\frac{7\pi}{16},\quad c_5 = \frac{11\pi}{32},\quad\ldots\quad c_{2n+1} = \frac{\pi}{2}(4n+3)\frac{1\,.\,3\ldots\overline{2n-1}}{2\,.\,4\ldots\overline{2n+2}}(-1)^n,$$

or this is

$$c_{2n+1} = (4n+3)\,\frac{\pi}{2}\int_0^1 P_{2n+1}(\mu)\,d\mu.$$

But if we expand by odd harmonics $P_{2n+1}(\sin\delta)$ a function which $= \frac{\pi}{2}$ for positive values of $\sin\delta$, and $-\frac{\pi}{2}$ for negative values of $\sin\delta$, this is the coefficient required. Now when $\lambda = \frac{\pi}{2}$, $P_n(\sin\lambda) = 1$, and so the value of χ_0 given by the series reduces at the pole to π for summer and zero for winter, as it should do.

§ 20. The coefficient ${}_nb_p$ of any term $P_n(\sin\lambda)\, P_n(\sin\delta)$ in χ_p is derived from the corresponding one in χ_{p-2} by the factor $\frac{-p(p-1)}{(n-p)(n+p+1)}$ as appears from substituting in equation (c), and a repeated application brings us to the term in χ_0 or χ_1 whose coefficient is known. But this leaves that of P_p in χ_p undetermined in each case, this being the highest of the odd or even set of terms, according as p is odd or even. To determine terms of this type compare the coefficients of $\sin^p\delta$ in the values of χ_p, and in the series, when $\lambda=0$, p being even; and in the value of $\frac{d\chi}{d\lambda}$, p being odd. Thus to take the even case when $\lambda=0$,

$$\chi_{2p}=\int_0^{\frac{\pi}{2}}\cos^{2p}\delta\,.\,\cos^{2p}\psi\,d\psi=\frac{1\,.\,3\,\ldots\,2p-1}{2\,.\,4\,\ldots\,2p}\frac{\pi}{2}\cos^{2p}\delta,$$

and in the series the only term containing $\sin^{2p}\delta$ is ${}_{2p}b_{2p}P_{2p}(0)\,P_{2p}(\sin\delta)$. Now

$$P_{2p}(0)=\frac{1\,.\,3\,\ldots\,\overline{2p-1}}{2\,.\,4\,\ldots\,2p}(-1)^p,$$

and the highest power of $\sin\delta$ in $P_{2p}(\sin\delta)$ is

$$\frac{1\,.\,3\,\ldots\,4p-1}{\lfloor 2p}\sin^{2p}\delta,$$

$$\therefore\quad\frac{\pi}{2}={}_{2p}b_{2p}\frac{1\,.\,3\,\ldots\,4p-1}{\lfloor 2p}.$$

Treat the odd term with the help of

$$\frac{d}{d\lambda}\chi_{2p+1}=(2p+1)\,\chi_{2p}\sin\delta,$$

when $\lambda=0$ derived from (b), and it appears that the formula $\frac{\pi}{2}={}_pb_p\frac{1\,.\,3\,\ldots\,2p-1}{\lfloor p}$ holds whether p is odd or even.

When the difference of p and n is even we have for the coefficient ${}_nb_p$ in χ_p

$${}_nb_p=\frac{\pi}{2}\frac{(2n+1)\,\lfloor p}{2\,.\,4\,\ldots\,\overline{(p-n)}\,.\,1\,.\,3\,\ldots\,\overline{(p+n+1)}}.$$

In this case n does not exceed p, and when $n=p$, $2\,.\,4\,\ldots\,\overline{p-n}$ must be taken $=1$.

When the difference of p and n is odd,

$${}_nb_p=\frac{\pi}{2}\frac{(2n+1)\,\lfloor p\,.\,1\,.\,3\,\ldots\,(n+p)}{2\,.\,4\,\ldots\,(n+p+1)\,.\,(n-p)\,(n-p+2)\,\ldots\,\overline{(n+p)}}\times(-1)^{\frac{p-n-(-1)^n}{2}},$$

and in this case either p or n may be the greater, and the denominator may contain negative terms.

The opening terms are:—

$$\left.\begin{aligned}
\chi_0&=\frac{\pi}{2}+\frac{3\pi}{4}P_1(\sin\lambda)\,P_1(\sin\delta)-\frac{7\pi}{16}P_3(\sin\lambda)\,P_3(\sin\delta)+\frac{11\pi}{32}P_5(\sin\lambda)\,P_5(\sin\delta)\ldots\\
\chi_1&=\frac{\pi}{4}+\frac{\pi}{2}P_1(\sin\lambda)\,P_1(\sin\delta)+\frac{5\pi}{16}P_2(\sin\lambda)\,P_2(\sin\delta)-\frac{3\pi}{32}P_4(\sin\lambda)\,P_4(\sin\delta)\\
&\qquad+\frac{13\pi}{256}P_6(\sin\lambda)\,P_6(\sin\delta)\ldots\\
\chi_2&=\frac{\pi}{6}+\frac{3\pi}{8}P_1(\sin\lambda)\,P_1(\sin\delta)+\frac{\pi}{3}P_2(\sin\lambda)\,P_2(\sin\delta)+\frac{7\pi}{48}P_3P_3-\frac{11\pi}{384}P_5P_5\ldots\\
\chi_3&=\frac{\pi}{8}+\frac{3\pi}{10}P_1P_1+\frac{5\pi}{16}P_2P_2+\frac{\pi}{5}P_3P_3+\frac{9\pi}{128}P_4P_4-\frac{13\pi}{1280}P_6P_6\ldots\text{ (the rest even)}\\
\chi_4&=\frac{\pi}{10}+\frac{\pi}{4}P_1P_1+\frac{2\pi}{7}P_2P_2+\frac{7\pi}{32}P_3P_3+\frac{4\pi}{35}P_4P_4+\frac{11\pi}{320}P_5P_5\ldots\text{ (the rest odd)}
\end{aligned}\right\}\ldots(d);$$

χ_3 for example contains the infinite series of even zonals, and the odd series up to P_3, χ_4 contains the infinite series of odd zonals, and the even series up to P_4.

In χ_3 all the coefficients are derivable from those in χ_1 by the factor

$$-\frac{6}{(n-3)(n+4)}$$

except that of P_3, for which we are thrown back on the special method, but in any case the coefficients are all determined without reference to any previous series by the two formulæ above.

In Arctic regions, at the transitions from total to partial day or night, the series are continuous, the first differential coefficient of χ_0, the second of χ_1, the third of χ_2, ... with regard to either variable being discontinuous.

With these values we have, when $e_0 + e_1 \cos I + e_2 \cos^2 I + \dots$ represents the amount of light or heat transmitted for the inclination I, the element of heat-supply

$$= \frac{Hd\theta}{\pi h}(e_0\chi_1 + e_1\chi_2 + e_2\chi_3 + \dots) \text{ in lieu of } \frac{Hd\theta}{\pi h}\chi_1$$

with no absorption. The first term χ_0, which does not appear here, and was introduced for analytical purposes, is the supply-function as it would be if with the existing duration of daylight, light or heat came with equal strength that of the zenith during the whole day.

It might be applied to the heating of a cloud in mid-air presenting an equal surface to the sun through the course of the day, the meaning being clear when we remember that the use of one factor $\cos I$ was necessitated by the exposure of a *surface* at a varying angle.

To transform $P_n(\sin\delta)$ into a sine or cosine series with regard to θ, for even values of n, the result is given above, p. 78; for odd values it is

$$P_{2n+1}(\sin\delta) = \sum_{s=0}^{s=n} \frac{2(-1)^n \, \underline{|2n-2s}}{2^{2n+1}\underline{|n+s+1}\,\underline{|n-s}} \sin^{2s+1}\epsilon P_{2n+1}^{2s+1}(\cos\epsilon)\sin(2s+1)\theta,$$

and this completes the expression of any member of the group in the standard form

$$L_0 + L_1 \sin\theta + L_2 \cos 2\theta + L_3 \sin 3\theta + L_4 \cos 4\theta + \dots .$$

For χ_{2p} the cosines go as far as $\cos 2p\theta$, and the sines of odd multiples take all values, for χ_{2p+1} the sines go as far as $\sin(2p+1)\theta$, and the cosines of all even multiples occur.

§ 21. At the pole $\chi_p = \pi \sin^p \delta$ or $\pi \sin^p \epsilon \sin^p \theta$ from $\theta = 0$ to π, and vanishes from $\theta = \pi$ to 2π. The expansion of such a function may be effected independently, and the opening terms are:—

$$
\left.\begin{aligned}
\chi_0 &= \frac{\pi}{2} + 2\sin\theta + \frac{2}{3}\sin 3\theta + \frac{2}{5}\sin 5\theta \ldots \\
\chi_1 &= \sin\epsilon\left(1 + \frac{\pi}{2}\sin\theta - \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta \ldots\right) \\
\chi_2 &= \sin^2\epsilon\left(\frac{\pi}{4} + \frac{4}{3}\sin\theta - \frac{\pi}{4}\cos 2\theta - \frac{4}{15}\sin 3\theta - \frac{4}{105}\sin 5\theta \ldots\right) \\
\chi_3 &= \sin^3\epsilon\left(\frac{2}{3} + \frac{3\pi}{8}\sin\theta - \frac{4}{5}\cos 2\theta - \frac{\pi}{8}\sin 3\theta + \frac{4}{35}\cos 4\theta \ldots\right) \\
\chi_4 &= \sin^4\epsilon\left(\frac{3\pi}{16} + \frac{16}{15}\sin\theta - \frac{\pi}{4}\cos 2\theta - \frac{16}{35}\sin 3\theta + \frac{\pi}{16}\cos 4\theta \ldots\right)
\end{aligned}\right\} \ldots\ldots\ldots (e).
$$

If in the general forms (d) λ be put $= \frac{\pi}{2}$, and the comparison of a term of type $\sin(2s+1)\theta$, or $\cos 2s\theta$, be made with the simplified form (e), the result is an expansion of the type $\sin^m\epsilon$ in terms of $P_{2n}^{2s}(\cos\epsilon)$ or $P_{2n+1}^{2s+1}(\cos\epsilon)$ or P_{2n}; s being fixed in the series in question, and n not less than s. The series are finite if m is even, infinite in the other case, and form a generalization of formulæ already quoted in Todhunter's *Laplace's Functions*, pp. 114, 115.

The means of the successive functions $\chi_1, \chi_2 \ldots$ for the globe or for a hemisphere are

$$\frac{\pi}{4},\ \frac{\pi}{6},\ \frac{\pi}{8},\ \frac{\pi}{10} \ldots;$$

for succeeding terms of series (d) the coefficients diminish at a less rate or even increase, and from this we infer that the periodic terms, and the element in non-periodic term dependent on latitude and obliquity of the ecliptic, increase relatively as we pass from χ_1 to the succeeding functions.

It is not easy to judge at sight the effect in any particular latitude, say, on the non-periodic term, because this derives a section from each of the zonals of even degree. But at the pole, as appears from (e), we have

$$\sin\epsilon,\ \frac{\pi}{4}\sin^2\epsilon,\ \frac{2}{3}\sin^3\epsilon,\ \frac{3\pi}{16}\sin^4\epsilon \ldots,$$

where the means for hemisphere are

$$\frac{\pi}{4},\ \frac{\pi}{6},\ \frac{\pi}{8},\ \frac{\pi}{10} \ldots$$

suggesting a rate of decrease growing as we pass from equator to pole. This may be proved generally, but with a view of shewing it more readily, I give numerical values in Table (C) for latitudes 0°, 30°, 45°, 60°, 90°.

As regards the annual term which derives a section from each odd zonal, we have in low latitudes absolute increase, and so a conspicuous relative increase. In latitude 30°, the figures for the successive functions are nearly the same, for higher latitudes the annual term diminishes in absolute value, but still at a less rate than the non-periodic term, and finally at the pole there is little difference in the rates at which these two terms diminish in passing from the normal function to those with absorption.

§ 22. We readily obtain summer or winter totals of radiation received by finding

$$\int_0^\pi P_n(\sin\delta)\,d\theta.$$

The opening terms are

$$\int_0^\pi P_1(\sin\delta)\,d\theta = 2\sin\epsilon,\quad \int_0^\pi P_2 d\theta = \left(\frac{3}{4}\sin^2\epsilon - \frac{1}{2}\right)\pi,\quad \int_0^\pi P_3 d\theta = -3\sin\epsilon + \frac{10}{3}\sin^3\epsilon,$$

$$\int_0^\pi P_4 d\theta = \frac{3\pi}{8}\left(1 - 5\sin^2\epsilon + \frac{35}{8}\sin^4\epsilon\right),\quad \int_0^\pi P_5 d\theta = \frac{3\pi}{8}\left(1 - 5\sin^2\epsilon + \frac{35}{8}\sin^4\epsilon\right)\dots.$$

For winter the terms of even order are the same, those of odd order have the sign reversed.

Also average values of the various terms for the hemisphere may be obtained by integrating

$$\int_0^{\frac{\pi}{2}} P_n(\sin\lambda)\,d\lambda.$$

Even terms except that of zero order vanish,

$$\int_0^{\frac{\pi}{2}} P_1 d\lambda = \frac{1}{2},\quad \int_0^{\frac{\pi}{2}} P_3 d\lambda = -\frac{1}{8},\quad \int_0^{\frac{\pi}{2}} P_5 d\lambda = \frac{1}{16}\dots.$$

Again the series in (d) may be transformed into zonal formulæ (or more properly polar-cap formulæ) by the substitution given at the end of § 16, and noticing that the polar-cap element is $\frac{c^2HZd\theta}{h}$ while the ordinary element in latitude λ which is per unit area is $\frac{Hd\theta}{\pi h}$, it appears that the factor πc^2 is also required. With these changes we have values of polar-cap formulæ down to any latitude for the various absorption functions used, and by subtraction formulæ for a zone of any extent.

The constants e_0, e_1 ... may be different for different wave-lengths. Also if the absorption is different at different seasons of the year for the same angle of incidence, the quantities e_0 ... contain annual or semi-annual terms, and these should be introduced in the expression

$$\frac{Hd\theta}{\pi h}(e_0\chi_1 + e_1\chi_2 + \dots)$$

when transforming to mean time. Or if the absorption is different in different latitudes for the same angle of incidence, then e_0 ... are functions of latitude. The discussion of such points would involve a survey of radiation with a bolometer in different latitudes, and at different seasons of the year, as well as at different times of the day.

(A) Table of coefficients in the expansion of Q in the form

$$L_0 + L_1 \sin\theta + L_2 \cos 2\theta + L_4 \cos 4\theta + \dots,$$

and of their differential coefficients with regard to ϵ and λ. ($\epsilon = 23^\circ\ 27'$.)

λ	L_0	L_1	L_2	L_4	$\frac{dL_0}{d\epsilon}$	$\frac{dL_1}{d\epsilon}$	$\frac{dL_2}{d\epsilon}$	$\frac{dL_4}{d\epsilon}$	$\frac{dL_0}{d\lambda}$	$\frac{dL_1}{d\lambda}$	$\frac{dL_2}{d\lambda}$	$\frac{dL_4}{d\lambda}$
0°	·9591	·0	·0412	−·0004	−·1950	·0	·1987	−·0051	·0	·6251	·0	·0
10°	·9458	·1085	·0394	−·0004	−·1856	·2502	·1897	−·0042	−·1520	·6156	−·0221	·0004
20°	·9065	·2138	·0334	−·0003	−·1570	·4928	·1600	−·0032	−·2973	·5874	−·0456	·0008
30°	·8429	·3126	·0232	−·0001	−·1088	·7205	·1088	−·0007	−·4306	·5413	−·0715	·0014
40°	·7577	·4018	·0081	+·0002	−·0348	·9263	·0330	+·0025	−·5419	·4789	−·1034	·0024
50°	·6557	·4789	−·0138	+·0009	+·0722	1·1039	−·0807	+·0092	−·6193	·4018	−·1509	·0049
60°	·5455	·5413	−·0477	+·0024	+·2441	1·2480	−·2688	+·0274	−·6260	·3126	−·4795	·0289
70°	·4543	·5874	−·1225	+·0116	+·6462	1·3541	−·7357	+·0604	−·3547	·2138	−·6437	·0250
80°	·4112	·6156	−·2255	−·0207	+·8569	1·4192	−·6873	−·1671	−·1546	·1085	−·4499	−·2797
90°	·3979	·6251	−·2653	−·0531	+·9174	1·4411	−·6116	−·1223	·0	·0	·0	·0

The mean value of L_0 for the globe is $\frac{\pi}{4}$.

Unit value of L_0 is what it would be at the equator if ϵ were $= 0$, in which case L_0 would reduce to the form $\cos\lambda$.

L_1 is negative for southern hemisphere, L_0, L_2, ... alike for both hemispheres.

(C) Table of coefficients when the absorption functions are used.

λ	L_0 for				$L_1 (\sin\theta)$ for				$L_2 (\cos 2\theta)$ for			
	χ_1	χ_2	χ_3	χ_4	χ_1	χ_2	χ_3	χ_4	χ_1	χ_2	χ_3	χ_4
0°	·9591,	·7232,	·5893,	·5012	·0	·0	·0	·0	·0412,	·0622,	·0764,	·0859
30°	·8429,	·5735,	·4316,	·3446	·3126,	·3260,	·3191,	·3039	·0232,	·0155,	·0020,	−·0018
45°	·7084,	·4230,	·2868,	·2107	·4420,	·3803,	·3176,	·2640	−·0018,	−·0311,	−·0514,	−·0621
60°	·5455,	·2741,	·1602,	·1013	·5413,	·3460,	·2272,	·1453	−·0477,	−·0777,	−·0761,	−·0645
90°	·3979,	·1244,	·0419,	·0148	·6251,	·2115,	·0841,	·0268	−·2653,	−·1244,	−·0504,	−·0195

λ	$L_3 (\sin 3\theta)$ for				$L_4 (\cos 4\theta)$ for			
	χ_1	χ_2	χ_3	χ_4	χ_1	χ_2	χ_3	χ_4
0°	·0,	·0,	·0,	·0	−·0004,	·0,	·0009,	·0018
30°	·0,	·0063,	·0108,	·0133	−·0001,	·0,	−·0006,	−·0014
45°	·0,	·0055,	·0044,	·0006	+·0005,	·0,	−·0010,	−·0020
60°	·0,	−·0007,	−·0080,	−·0144	+·0024,	·0,	−·0007,	+·0001
90°	·0,	−·0422,	−·0247,	−·0165	−·0531,	·0,	+·0072,	+·0049

In this table χ_1 is the normal case of no absorption, $\chi_1 = Q$.

In χ_2 absorption is proportional to $\cos I$,

χ_3 ....$\cos^2 I$,

χ_4 ....$\cos^3 I$.

(B) Expansion of Q in mean time $\psi = 2\pi t$, t measured from perihelion, unit time 1 year. Eccentricity $e = \cdot 0168$. Longitude of perihelion 79°.

λ		
0°		$\cdot 9591 + \cdot 0328 \cos(\psi + 0^\circ 27') - \cdot 0406 \cos(2\psi + 22^\circ 54') - \cdot 0021 \cos(3\psi + 22^\circ)$,
10°	N.	$\cdot 9458 - \cdot 0769 \cos(\psi + 15^\circ 25') - \cdot 0424 \cos(2\psi + 21^\circ 24') - \cdot 0021 \cos(3\psi + 21^\circ 37')$,
	S.	„ $+ \cdot 1403 \cos(\psi + 8^\circ 35') - \cdot 0352 \cos(2\psi + 23^\circ 34') - \cdot 0019 \cos(3\psi + 22^\circ 36')$,
20°	N.	$\cdot 9065 - \cdot 1834 \cos(\psi + 12^\circ 47') - \cdot 0399 \cos(2\psi + 21^\circ 51') - \cdot 0019 \cos(3\psi + 20^\circ 48')$,
	S.	„ $+ \cdot 2442 \cos(\psi + 9^\circ 40') - \cdot 0257 \cos(2\psi + 22^\circ 24') - \cdot 0015 \cos(3\psi + 23^\circ 33')$,
30°	N.	$\cdot 8429 - \cdot 2845 \cos(\psi + 12^\circ 4') - \cdot 0330 \cos(2\psi + 16^\circ 37') - \cdot 0015 \cos(3\psi + 19^\circ 42')$,
	S.	„ $+ \cdot 3407 \cos(\psi + 10^\circ 6') - \cdot 0128 \cos(2\psi + 37^\circ 55') - \cdot 0009 \cos(3\psi + 25^\circ 49')$,
40°	N.	$\cdot 7577 - \cdot 3766 \cos(\psi + 11^\circ 44') - \cdot 0214 \cos(2\psi + 15^\circ 18') - \cdot 0008 \cos(3\psi + 16^\circ 15')$,
	S.	„ $+ \cdot 4268 \cos(\psi + 10^\circ 21') + \cdot 0068 \cos(2\psi - 3^\circ 26') + \cdot 0001 \sin(3\psi + 22^\circ)$,
50°	N.	$\cdot 6557 - \cdot 4574 \cos(\psi + 11^\circ 32') - \cdot 0079 \cos(2\psi + 0^\circ 42') + \cdot 0016 \cos(3\psi + 64^\circ)$,
	S.	„ $+ \cdot 5001 \cos(\psi + 10^\circ 31') + \cdot 0360 \cos(2\psi + 16^\circ 51') + \cdot 0013 \cos(3\psi + 18^\circ 6')$,
60°	N.	$\cdot 5455 - \cdot 5240 \cos(\psi + 11^\circ 24') + \cdot 0304 \cos(2\psi + 28^\circ 13' \cdot 5) + \cdot 0019 \cos(3\psi + 25^\circ 2')$,
	S.	„ $+ \cdot 5585 \cos(\psi + 10^\circ 37' \cdot 5) + \cdot 0660 \cos(2\psi + 7^\circ 52') + \cdot 0029 \cos(3\psi + 20^\circ 2')$,
70	N.	$\cdot 4543 - \cdot 5726 \cos(\psi + 11^\circ 20') + \cdot 1035 \cos(2\psi + 23^\circ 56') + \cdot 0057 \cos(3\psi + 23^\circ 6')$,
	S.	„ $+ \cdot 6019 \cos(\psi + 10^\circ 41') + \cdot 1422 \cos(2\psi + 20^\circ 31') + \cdot 0067 \cos(3\psi + 21^\circ 4')$,
80°	N.	$\cdot 4112 - \cdot 6056 \cos(\psi + 11^\circ 19') + \cdot 2055 \cos(2\psi + 23^\circ 24') + \cdot 0109 \cos(3\psi + 22^\circ 34')$,
	S.	„ $+ \cdot 6253 \cos(\psi + 10^\circ 41' \cdot 6) + \cdot 2461 \cos(2\psi + 21^\circ 3') + \cdot 0119 \cos(3\psi + 21^\circ 28')$,
90°	N.	$\cdot 3979 - \cdot 6162 \cos(\psi + 11^\circ 19') + \cdot 2450 \cos(2\psi + 22^\circ 55') + \cdot 0128 \cos(3\psi + 22^\circ 32')$,
	S.	„ $+ \cdot 6337 \cos(\psi + 10^\circ 42') + \cdot 2862 \cos(2\psi + 21^\circ 11') + \cdot 0140 \cos(3\psi + 21^\circ 30')$.

IV. *The Contact Relations of certain systems of Circles and Conics.* By Prof. W. McF. Orr, M.A., Fellow of St John's College.

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CHAPTER I.

A Fundamental Theorem.

SECTION I.

Enunciation.

1. I propose to show that if Fig. 1 represents four circles on a sphere intersecting so that either the four points A, B, C, D (and therefore also A', B', C', D') or the four A, B, C', D' (and therefore also A', B', C, D) are concyclic [a condition which may be otherwise expressed by saying that the difference of the arcual angles OAB and OBA equals the difference of the arcual angles OCD and ODC irrespective of sign] then the incircles of each of the following tetrads of arcual triangles are touched by two other circles besides OAC and OBD:—

OAB,	$O'AB$,	OCD,	$O'CD$	(1),
OAB,	$O'AB$,	$OC'D'$,	$O'C'D'$	(2),
$OA'B'$,	$O'A'B'$,	OCD,	$O'CD$	(3),
$OA'B'$,	$O'A'B'$,	$OC'D'$,	$O'C'D'$	(4),
OAB',	$OA'B$,	OCD',	$OC'D$	(5),
OAB',	$OA'B$,	$O'CD'$,*	$O'C'D$	(6),
$O'AB'$,*	$O'A'B$,	OCD',	$OC'D$	(7),
$O'AB'$,*	$O'A'B$,	$O'CD'$,*	$O'C'D$	(8),
OAB,	$O'AB$,	OCD',	$OC'D$	(9),
OAB,	$O'AB$,	$O'CD'$,*	$O'C'D$	(10),
$OA'B'$,	$O'A'B'$,	OCD',	$OC'D$	(11),
$OA'B'$,	$O'A'B'$,	$O'CD'$,*	$O'C'D$	(12),
OAB',	$OA'B$,	OCD,	$O'CD$	(13),
$O'AB'$,*	$O'A'B$,	OCD,	$O'CD$	(14),
OAB',	$OA'B$,	$OC'D'$,	$O'C'D'$	(15),
$O'AB'$,*	$O'A'B$,	$OC'D'$,	$O'C'D'$	(16).

* "The incircle of $O'CD'$" means the circle touching OAC, OBD, $CDC'D'$ and containing within it these three circles. This notation seems natural from analogy with the case in which these three circles are great circles. A similar remark applies to "the incircle of $O'AB'$."

SECTION II.

TWO LEMMAS.

2. I shall use as Lemmas a particular case of a theorem given by Mr Jessop in the *Quarterly Journal of Mathematics*, Vol. XXIII., and its converse. His theorem is enunciated:—"The sum or difference of the angles any two fixed generating circles of the same family of a Bicircular Quartic make with a variable generating circle of any other given family is constant." It might also be enunciated:—"If P, Q, R, S be four generating circles of a Bicircular Quartic, P, Q belonging to one family and R, S to another, then a circle can be drawn through one of each pair of intersections of P with R, P with S, Q with R, and Q with S (and therefore another circle through the remaining four of the same intersections)." The same theorem of course follows, by inversion, or stereographic projection, for Sphero-Quartics also. The particular case referred to is that in which the Quartic consists of two circles on a sphere (or in a plane). As Mr Jessop's proof does not readily enable us to select the concyclic points I give a proof for the particular case depending on Casey's relation among the angles of intersection of four circles that touch a fifth. (It may be worth while pointing out how to write down this relation. Let the points of contact be K, L, M, N as in Fig. 2: write down the analogue of Ptolemy's theorem, viz.:—

$$\sin \tfrac{1}{2} KM \,.\, \sin \tfrac{1}{2} LN = \sin \tfrac{1}{2} KL \,.\, \sin \tfrac{1}{2} MN + \sin \tfrac{1}{2} KN \,.\, \sin \tfrac{1}{2} LM,$$

and substitute for each great arc joining two of the points K, L, M, N the supplement of that angle between the tangent circles at its extremities within which the circle $KLMN$ lies.) Let Fig. 3 represent two circles of each family touching the two circles S and S'. Since the four circles touch S we have

$$\sin \frac{\alpha}{2} \cos \frac{\delta}{2} + \cos \frac{\omega}{2} \cos \frac{\omega'}{2} = \sin \frac{\gamma}{2} \cos \frac{\beta}{2} \quad \text{.........................(1)};$$

and since they touch S' we have

$$\cos \frac{\alpha}{2} \sin \frac{\delta}{2} = \sin \frac{\beta}{2} \cos \frac{\gamma}{2} + \cos \frac{\omega}{2} \cos \frac{\omega'}{2} \quad \text{..........................(2)}.$$

From these relations we obtain the equation

$$\sin \frac{\alpha + \delta}{2} = \sin \frac{\beta + \gamma}{2},$$

therefore either $\alpha + \delta = \beta + \gamma$ or $\alpha + \delta = 2\pi - \beta - \gamma$.

therefore either $\alpha - \beta = \gamma - \delta$ or $\alpha + \beta = 2\pi - \gamma - \delta$.

If the former relation hold A, B, C', D' are concyclic, if the latter, A', B, C', D, Mr Jessop's theorem thus being established on either supposition.

It is however necessary for the present purpose to show that it is the former of these alternatives which is true and not the latter. If a third circle be drawn touching S and S', of the same family as AB and CD and having a position intermediate between them, and if θ, ϕ be the angles that correspond to γ and δ, we have

either $\alpha-\beta=\gamma-\delta$ or $\alpha+\beta+\gamma+\delta=2\pi$;

and at the same time

either $\alpha-\beta=\theta-\phi$ or $\alpha+\beta+\theta+\phi=2\pi$:

and at the same time

either $\gamma-\delta=\theta-\phi$ or $\gamma+\delta+\theta+\phi=2\pi$.

If the former alternative be not taken in the first case it must either be taken in one only of the other cases or not be taken in any case, as if taken in both the other cases from the equations $\alpha-\beta=\theta-\phi$, $\gamma-\delta=\theta-\phi$, we deduce the equation $\alpha-\beta=\gamma-\delta$. Now if we take the former alternative in the second case only, we deduce $\alpha=\theta$, $\beta=\phi$. Thus the equation

$$\sin\frac{\alpha}{2}\cos\frac{\phi}{2}+\cos\frac{\omega}{2}\cos\frac{\Omega'}{2}=\sin\frac{\theta}{2}\cos\frac{\beta}{2},$$

which corresponds to (1) for the circles concerned, reduces to

$$\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\omega}{2}\cos\frac{\Omega'}{2}=\sin\frac{\alpha}{2}\cos\frac{\beta}{2},$$

which cannot be true since neither ω nor Ω' is π. Similarly we cannot take the former alternative in the third case only.

Again, if we take the latter alternative in every case, we deduce

$$\alpha+\beta=\gamma+\delta=\theta+\phi=\pi.$$

Equation (1) then reduces to

$$\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\cos\frac{\omega}{2}\cos\frac{\omega'}{2}=\sin\frac{\gamma}{2}\sin\frac{\alpha}{2},$$

which cannot be true since neither ω nor ω' is π. The former alternative must therefore hold in the first case; (the same argument shows that it must hold in all three cases).

The points $C'D'$ having been proved concyclic with A and B, if the circle $CDC'D'$ be made to vary continuously, touching S and S', it is obvious that in any position the instantaneous positions of C' and D' are always concyclic with A and B, it being noted that C and C' (or D and D') interchange as the varying circle passes through the position in which they coincide. When the concyclic points have been determined for the intersection of two given circles P and Q of one family with any two of the other family this consideration suffices to determine them for the intersection of P and Q with any other two.

3. Conversely if in Fig. 3 COC', DOD' be two circles of one family touching S and S' and $CDC'D'$ be a circle of the other family and A, B be two points one on each of the circles COC', DOD' concyclic with C', D', then through A and B there can be described a circle touching S and S' of the same family as $CDC'D'$. For through A two circles of that family can be described. If a circle of that family be made to vary

continuously from coincidence with $CDC'D'$ first into coincidence with one of these circles and then, restarting from the position $CDC'D'$, into coincidence with the other, in one case C, and in the other C', will move continuously into coincidence with A. The former of the circles will pass through B, because from what has been proved if it do not pass through B but cut DOD' in E and E' the circle $C'D'A$ will pass either through E or E', which is impossible, since by hypothesis the circle $C'D'A$ cuts DOD' in B: therefore the result stated is true.

SECTION III.

PROOF OF THE FUNDAMENTAL THEOREM.

4. In applying the Lemmas to the proof of the theorems stated in Art. 1 the following notation is adopted: the incircle of the triangle OAB is denoted by 4 and the escribed circles of this triangle opposite the angles A, B, O are denoted by 1, 2, 3, and the inverses of 1, 2, 3, 4 with respect to the circle cutting OA, OB, and AB orthogonally are denoted by 1′, 2′, 3′, 4′ respectively: the incircle of the triangle OCD is denoted by IV and the escribed circles of this triangle opposite the angles C, D, O are denoted by I, II, III respectively, while the inverses of I, II, III, IV with respect to the circle cutting OC, OD and CD orthogonally are denoted by I′, II′, III′, IV′ respectively.

I have found it impracticable to get good figures for all the tetrads without varying the sizes and positions of the four circles OA, OB, AB and CD. In all the figures (Figs. 4—12) however O lies within and O' without the circles AB and CD; the points of intersection of these circles lie in the angles AOD', BOC'; and A, B, C', D' lie on a circle; this circle is not drawn. Of the two circles whose existence the theorem asserts, in each case one only is drawn, the dotted circle in the figures.

Fig. 4 shows the circles of tetrad (1) denoted by the numbers 3, 4, III, IV, placed at their centres. Two circles can be drawn having contact of the same kind with the circles 3, IV and of a different kind with 4: let $FEE'F'$ be one of these circles. Then since $ABA'B'$ and $FEE'F'$ both touch 3 and 4, by the first Lemma A, B, E, F are concyclic and

$$\angle OAB - \angle OBA = \angle OFE - \angle OEF$$

(angles between small circles being meant in every case), but

$$\angle OFE - \angle OEF = \angle OE'F' - \angle OF'E',$$

since F, E, F', E' are concyclic, and $\angle OAB - \angle OBA = \angle OCD - \angle ODC$ since A, B, C', D' are concyclic; therefore $\angle OE'F' - \angle OF'E' = \angle OCD - \angle ODC$ and therefore (by the second Lemma) the circle $FEE'F'$ (and not the other circle through E', F' touching IV) touches the circle III. In a similar manner the other circle having contact of one kind with 3, IV, and of a different kind with 4, can be shown to touch III.

The eight circles that can be drawn to touch three given circles consist of four circles and their inverses with respect to the circle cutting the given three orthogonally; there are two species of tetrads of these eight circles which are touched by a fourth circle; a tetrad of one species consists of two circles and their inverses with respect to the orthogonal circle of the given three; a tetrad of the other consists of circles analogous to the inscribed and escribed circles of a plane triangle (the radical centre of the three given circles being supposed to lie within each of them). No distinctive names appear to be in use for the two species; in the present paper the former species will be called inverse tetrads, and the term Hart tetrad restricted to the latter species.

The existence of a second circle touching 3, 4, III and IV, thus indeed follows from that of the circle $FEE'F'$ as the circles 3 4 III IV form an inverse tetrad of the circles touching OAC, OBD and $FEE'F'$. A similar remark applies in the cases of tetrads (2)—(8).

If A, B, C, D be concyclic instead of A, B, C', D' a figure for that case can be obtained by erasing the circle CD and redrawing it, of the same family touching III and IV but so that A, B, C, D are concyclic, C and D denoting points on OA and OB respectively, both to the right of O. In a similar manner, in the cases of the other fifteen tetrads, the case in which A, B, C, D are concyclic may be deduced from that in which A, B, C', D' are concyclic or *vice versa*.

If without altering the figure the letters A and D', B and C', C and B', D and A' be interchanged in the figure and proof, the figure and proof apply to tetrad (4).

Fig. 5 shows the circles of tetrad (2); the proof is precisely similar.

If in proof and figure A and A', B and B', C and C', D and D' are interchanged we establish tetrad (3). This particular interchange of letters might be objected to as a mode of deducing one tetrad from another in other cases on the ground that it changes a figure in which the order of points on AC is O, A, C, O', and on BD, O, B, D, O' into one in which the order on AC is O, C, A, O', and on BD, O, D, B, O'. In this case however if the portions drawn of the circles AB and $C'D'$ did not extend to their point of intersection there would be nothing to show in which order the points actually occur, and therefore a proof valid for one order is valid for the other.

Fig. 6 shows the circles of tetrad (5); the proof is similar except that A, B, E', F' are concyclic instead of A, B, E, F.

Fig. 7 shows the circles of tetrad (6); in this case also A, B, E', F' are concyclic instead of A, B, E, F.

If in figure and proof we interchange A and D', B and C', C and B', D and A' we establish tetrad (7).

Fig. 8 shows the circles of tetrad (8); it has not appeared practicable in this case to draw the circles so that both 1′ and I′ contain within them the three circles they are respectively drawn to touch, as they would do if drawn in Fig. 1. In this figure also A, B, E', F' are concyclic.

Fig. 9 shows the circles of tetrad (9). In this case A, B, E, F are concyclic.

The existence of a second circle touching 1, 2, III, IV follows from that of the circle $EFF'E'$ as the circles 1, 2, III, IV form a Hart tetrad of the circles touching OAC, OBD and $EFF'E'$. A similar remark applies in the cases of tetrads (10)—(16).

If A and D', B and C', A' and D, B' and C be interchanged we establish tetrad (15).

Fig. 10 shows the circles of tetrad (13). In this figure also A, B, E, F are concyclic.

If in figure and proof A and D', B and C', A' and D, B' and C be interchanged we establish tetrad (11).

Fig. 11 shows the circles of tetrad (10). In this figure also A, B, E, F are concyclic.

If in figure and proof A and D', B and C', A' and D, B' and C be interchanged we establish tetrad (16).

Fig. 12 shows the circles of tetrad (14). In this figure also A, B, E, F are concyclic.

If in figure and proof A and D', B and C', A' and D, B' and C be interchanged we establish tetrad (12).

The fundamental theorem stated in Section I. has only been established for one configuration of the four original circles, but by the principle of continuity must be true for all modifications of the figure.

5. Tetrads (1)—(8) are analogous to inverse tetrads and each is in fact such a tetrad of circles touching four others as has been shown in Art. 4. If as a particular case the circle $CDC'D'$ coincide with $ABA'B'$, C coinciding with A and D with B tetrads (1), (4), (5) and (8) merely consist of two circles taken twice, tetrads (2) and (3) become the same inverse tetrad of circles touching OA, OB and AB, and tetrads (6) and (7) become another inverse tetrad of circles touching OA, OB and AB. In the case of each of these last-mentioned four cases one of the two common tangent circles whose existence has been established, becomes the circle AB, and the other becomes the fourth circle that touches an inverse tetrad.

Tetrads (9)—(16) are analogous to Hart tetrads and each is in fact such a tetrad of circles touching four others as has been shown in Art. 4. If as a particular case, the circle $CDC'D'$ coincide with $ABA'B'$, C coinciding with A and D with B, tetrads (9) and (13) become the same Hart tetrad of circles touching OA, OB and AB; so also do (10) and (14), (11) and (15), (12) and (16). In each case, one of the two common tangent circles whose existence has been established, becomes the circle AB, and the other becomes the Hart circle which touches that Hart tetrad.

6. With the notation that has been adopted, the tetrads analogous to inverse tetrads are 1 2 I II, 1 2 I′ II′, 1′ 2′ I II, 3 4 III IV, 3 4 III′ IV′, 3′ 4′ III IV, and 3′ 4′ III′ IV′; and the

tetrads analogous to Hart tetrads are $1\,2\,\mathrm{III}\,\mathrm{IV}$, $1\,2\,\mathrm{III}'\,\mathrm{IV}'$, $1'\,2'\,\mathrm{III}\,\mathrm{IV}$, $1'\,2'\,\mathrm{III}'\,\mathrm{IV}'$, $3\,4\,\mathrm{I}\,\mathrm{II}$, $3\,4\,\mathrm{I}'\,\mathrm{II}'$, $3'\,4'\,\mathrm{I}\,\mathrm{II}$, and $3'\,4'\,\mathrm{I}'\,\mathrm{II}'$. (The order of the tetrads here is not the same as in Section I.) Thus however the circles be situated when we know the Hart tetrads of the circles that touch OA, OB, AB, and the Hart tetrads of those that touch OA, OB, CD, and have identified a single tetrad of either kind of the sixteen above, the others of each kind can be easily identified.

7. There also exist of course, among the circles touching AB, CD, OA, and AB, CD, OB respectively, sixteen tetrads of circles touched by two others besides AB and CD.

CHAPTER II.

CONTACT RELATIONS AMONG THE CIRCLES TOUCHING TRIADS OF THE EIGHT THAT TOUCH THREE GIVEN CIRCLES.

SECTION IV.

THE CIRCLES TOUCHING TRIADS OF A HART GROUP.

8. In Fig. 13 let AB, BC, CA represent three circles intersecting on a sphere or in a plane, let $A'B'C'$ be their other points of intersection which do not appear in the figure, and let 1, 2, 3, 4 be the Hart tetrad of circles touching AB, BC, CA, which are escribed and inscribed to the triangle ABC (4 has not been drawn). By Mr Jessop's theorem or otherwise, two of the points P, P', Q, Q' are concyclic with A and B; which two are they? If the Hart circle change continuously, still touching 1 and 2, it can pass into the position AB without either P coinciding with P' or Q with Q' in any intermediate position, and when it does come into the position AB the point P' coincides with A and Q' with B. Therefore by the concluding paragraph of Art. 2, A, B, P, Q are the concyclic points. In the same way if the Hart circle change continuously, still touching 1 and 3, before it can pass into the position CA, P and P' would coincide and interchange, viz. at the point of contact of BC with 1. Hence P, R, A, C are concyclic. If again the Hart circle change continuously, still touching 2 and 3, before it can come into the position BC both Q and Q', R and R' would interchange, so that B, C, Q, R are concyclic. This is *one* way of getting these results. Hence the four circles BC, CA, AB, PQR satisfy *triply* the condition of Section I.

9. Let us find in which arc of the Hart circle its point of contact with 4 is situated. Let $\overline{1\,2}$ denote the direct and $\overline{1'2}$ or $\overline{1\,2'}$ the transverse common tangent of

1 and 2, with a similar notation in the case of the other circles. By Casey's relation we then have from the circle *BC*,

$$\sin\tfrac{1}{2}(\overline{1\,2'})\sin\tfrac{1}{2}(\overline{3\,4})+\sin\tfrac{1}{2}(\overline{1\,4'})\sin\tfrac{1}{2}(\overline{2\,3})=\sin\tfrac{1}{2}(\overline{1\,3'})\sin\tfrac{1}{2}(\overline{2\,4}),$$

from *CA*,

$$\sin\tfrac{1}{2}(\overline{2\,3'})\sin\tfrac{1}{2}(\overline{1\,4})=\sin\tfrac{1}{2}(\overline{2\,4'})\sin\tfrac{1}{2}(\overline{1\,3})+\sin\tfrac{1}{2}(\overline{1\,2'})\sin\tfrac{1}{2}(\overline{3\,4}),$$

from *AB*,

$$\sin\tfrac{1}{2}(\overline{2\,3'})\sin\tfrac{1}{2}(\overline{1\,4})=\sin\tfrac{1}{2}(\overline{1\,3'})\sin\tfrac{1}{2}(\overline{2\,4})+\sin\tfrac{1}{2}(\overline{3\,4'})\sin\tfrac{1}{2}(\overline{1\,2}).$$

Adding the first and second equations, subtracting the third and omitting terms common to each side, the resulting equation is

$$\sin\tfrac{1}{2}(\overline{1\,4'})\sin\tfrac{1}{2}(\overline{2\,3})=\sin\tfrac{1}{2}(\overline{2\,4'})\sin\tfrac{1}{2}(\overline{1\,3})-\sin\tfrac{1}{2}(\overline{3\,4'})\sin\tfrac{1}{2}(\overline{1\,2}),$$

showing that the points of contact of 1, 2, 3, 4 with the Hart circle form a quadrilateral of which the second and fourth lie on a diagonal; that is to say, the point of contact sought for lies between *P* and *R'*.

10. Let us now apply the theorem of Chap. I. to obtain contact relations among the circles touching *BC*, *CA*, *AB* and the Hart circle of *ABC* (denoted respectively by *a*, *b*, *c* and *d*) in sets of three. Suppose the concavities of *BC*, *CA*, *AB* are towards *A*, *B*, *C* respectively; denote by *abc* the circle which has contact of the same kind with *a*, *b* and *c* and is neither 1, 2, 3 nor 4; denote by *abc'* the circle which has contact of the same kind with *a* and *b* and of the opposite kind with *c'* and is neither 1, 2, 3 nor 4, and adopt a corresponding notation in the case of other circles. Consider circles touching *a*, *b*, *c* and circles touching *a*, *b*, *d*; since *A*, *B*, *P*, *Q* are concyclic, of the *abd* circles *abd* and *abd'* (i.e. the circle escribed to the triangle *CP'Q'* opposite the angle *C* and the circle inscribed in the same triangle) and of the *abc* circles 3, 4 form an inverse tetrad. Hence writing down the *abc* circles as

1, 2, 3, 4,
and *a'bc*, *ab'c*, *abc'*, *abc*,

the first row forming a Hart group and each circle in the first row being the inverse of the one below it with respect to the circle cutting *a*, *b* and *c* orthogonally,

and the *abd* circles as

1, 2, 3, 4,
and *ab'd*, *a'bd*, *abd'*, *abd*,

which we do in accordance with the rule indicated in Art. 6, the first two circles of the second and fourth rows form an inverse tetrad, as also do the last two circles of these rows, and the first two circles in either form with the last two in the other a Hart tetrad. Each of the other twelve tetrads of Chapter I. either consists of two of the circles 1, 2, 3, 4 taken twice or is a tetrad of circles touching three circles, i.e. either *a*, *b*, *c*, or *a*, *b*, *d*. Thus only four of the sixteen tetrads are new. The notation alone is sufficient to enable us to write down the two new tetrads of each kind. For an inverse tetrad consists of four circles of one family touching *a*, *b*, and there are only two such tetrads which are new, and a Hart tetrad consists of two circles touching *a*, *b*, *c* and belonging to the same family of circles touching *a*, *b*, and two circles touching *a*, *b*, *d* and belonging to the other family of circles touching *a*, *b*, and of these also only two are new.

So too considering the *abc* circles and the *acd* circles, writing down the former as

$$\begin{matrix} 1, & 2, & 3, & 4, \\ a'bc, & ab'c, & abc', & abc, \end{matrix}$$

and the latter as

$$\begin{matrix} 1, & 2, & 3, & 4, \\ ac'd, & acd', & a'cd, & acd, \end{matrix}$$

we see that the first and third circles of the second and fourth rows form an inverse tetrad as also do the second and fourth circles of the same rows, while the first and third in either with the second and fourth in the other form a Hart tetrad.

And considering the *abc* circles and the *bcd* circles, writing down the former as

$$\begin{matrix} 1, & 2, & 3, & 4, \\ a'bc, & ab'c, & abc', & abc. \end{matrix}$$

and the latter as

$$\begin{matrix} 1, & 2, & 3, & 4, \\ bcd', & bc'd, & b'cd, & bcd, \end{matrix}$$

we see that of the second and fourth rows the first and fourth circles form an inverse tetrad as also do the second and third, while the second and third of either with the first and fourth of the other form a Hart tetrad.

Again if we consider the *cda* circles and the *cdb* circles, a consideration of the figure shows that if we write the former as

$$\begin{matrix} 1, & 2, & 3, & 4, \\ ac'd, & acd', & a'cd, & acd, \end{matrix}$$

and the latter as

$$\begin{matrix} 1, & 2, & 3, & 4. \\ bcd', & bc'd, & b'cd, & bcd, \end{matrix}$$

of the second and fourth rows the first and second circles form an inverse tetrad, as also do the third and fourth, while the first and second of either with the third and fourth of the other form a Hart tetrad.

Similarly for the *cda* circles and the *abd* circles, and for the *abd* circles and the *bcd* circles.

11. Hence:—If we take any Hart tetrad of circles touching three others and describe circles touching them in threes we get four sets of four circles [exclusive of the original three and of another which with them in every case forms a Hart tetrad of circles touching the Hart group with which we started], each set of course being a Hart tetrad; we can form twenty-four tetrads of circles each consisting of two out of one of the above sets and two out of another such that each tetrad is touched by two circles besides the two which they have been constructed to touch in common; twelve of these tetrads are Hart tetrads and twelve are inverse.

12. All these tetrads can be shown in a table as follows. Write down the numbers 1, 2, 3, 4 and underneath each the letters denoting the conjugate one of the other four touching each of the four triads.

1	2	3	4
bcd'	$bc'd$	$b'cd$	bcd
$ac'd$	acd'	$a'cd$	acd
$ab'd$	$a'bd$	abd'	abd
$a'bc$	$ab'c$	abc'	abc

Consider any two horizontal rows of this table (other than the first); they represent four circles touching in common two of the four circles a, b, c, d; take out of one row two belonging to one family of circles that touch the common circles of a, b, c, d; take out of the other row two belonging to one family (but either) of circles touching the same; these circles form a tetrad touched by two other circles besides two of the four a, b, c, d; if all are of one family it is an inverse tetrad, but if two are of one family and two of another it is a Hart tetrad. Each inverse tetrad consists of two circles and two vertically under them, each Hart tetrad of two circles and two not vertically over or under either of them.

13. As a particular case, if we take the inscribed and escribed circles of a plane or spherical triangle and describe circles touching them in threes we get four sets of four circles [besides the sides of the original triangle and its Hart circle], each set of course being a Hart tetrad; we can form twenty-four tetrads of circles each consisting of two out of one of the above sets and two out of another such that each tetrad is touched by two circles besides the two which they have been constructed to touch in common; twelve of these tetrads are Hart tetrads and twelve are inverse.

These tetrads can be found by the rule given in the preceding section from the annexed table, the notation being the same as in the previous one:

a	b	c	d
1′23	12′3	123′	123
234	2′34	23′4	234′
1′34	134	13′4	134′
1′24	12′4	124	124′

SECTION V.

The Circles touching Triads of a Group consisting of three members of a Hart Group and the Conjugate or Inverse of the Fourth.

14. Suppose as before that a, b, c, d are a Hart tetrad of circles touching 1, 2, 3 (three circles in a plane or on a sphere) and that their inverses with respect to the circle cutting 1, 2, 3 orthogonally are denoted by A, B, C, D respectively; then the circles a, b, c, D like a, b, c, d satisfy triply the condition of Section I. (Fig. 14 represents these circles, D being the circle $XY'ZX'YZ'$.) For B, C, Y, Z'; C, A, Z, X'; A, B, X, Y' are respectively concyclic by Mr Jessop's theorem. Let abc, abc' denote the same circles as before, let abD, abD' denote respectively the circles touching a, b and D with contact of the same kind, and touching a and b with contact of the same kind, but D with contact of the other kind, and which are neither 1, 2, 3 nor the abD circle which forms a Hart group with 1, 2, 3. Let this last circle, the incircle of $C'X'Y$, be denoted by $[abD']$.

Then considering circles touching a, b, c and circles touching a, b, D, since A, B, X, Y' are concyclic 1, 2 of the abc circles and 3, $[abD']$ of the abD circles form a Hart tetrad; and writing down the abc circles as 1, 2, 3, 4,

and $a'bc$, $ab'c$, abc', abc,

and the abD circles as 1, 2, 3, $[abD']$,

and $a'bD$, $ab'D$, abD, abD',

we see that of the sixteen tetrads given by the theorem of Chapter I., and formed by taking the first and second or the third and fourth circles in either the first or second rows with the first and second or third and fourth in either the third or fourth rows, all are new except those in which 1 and 2 occur. Therefore we obtain five new inverse and four new Hart tetrads. Exclusive of those in which 3 occurs there are two new inverse and two new Hart tetrads.

Similarly among circles touching a, b, c and a, c, D respectively we obtain two new inverse and two Hart tetrads exclusive of those in which 2 occurs. We write the abc circles as 1, 2, 3, 4,

and $a'bc$, $ab'c$, abc', abc,

and the acD circles as 1, 2, 3, $[acD']$,

and $a'cD$, acD, $ac'D$, acD'.

Similarly there are two new inverse and two new Hart tetrads (excluding those which contain 1) among circles touching a, b, c and circles touching b, c, D; the same number among circles touching a, b, D and circles touching a, c, D; the same number among circles touching a, b, D and circles touching b, c, D; and the same number among circles touching a, c, D and circles touching b, c, D.

15. These tetrads are shown in the annexed table, very similar to that of Art. 12, and formed as follows:—

1	2	3	
bcD	$b'cD$	$bc'D$	bcD'
$a'cD$	acD	$ac'D$	acD'
$a'bD$	$ab'D$	abD	abD'
$a'bc$	$ab'c$	abc'	abc

Write down in a row the numbers 1, 2, 3, and underneath each the letters denoting the conjugate circles touching the various triads; in the fourth column write down the letters denoting the circle which forms a Hart tetrad with the other three in the same row. The rule for writing down the tetrads is the same as that given in Art. 12. There are twelve of each kind. There are other tetrads not given by this table in each of which however one at least of the circles 1, 2, 3 occurs.

SECTION VI.

THE CIRCLES TOUCHING TRIADS OF A GROUP CONSISTING OF THREE MEMBERS OF A HART GROUP AND THE CONJUGATE OR INVERSE OF ONE OF THEM.

16. Let Fig. 15 represent the circles 1, 2, 3; a, b, c, B; the last four consisting of three members of a Hart tetrad and the inverse of one of them with respect to the circle cutting 1, 2, 3 orthogonally. The circles a, b, c, B satisfy *singly* the condition of Section I., A, C, Z, X' being concyclic. Therefore considering the circles that touch a, b, c, and those that touch a, B, c; 1, 3 of the former and 2, [acB'] of the latter form a Hart tetrad ([acB'] denotes the circle touching a, c, B, and forming a Hart tetrad with 1, 2, 3, i.e. the incircle of $B'X'Z$); therefore writing down the abc circles

	as	1,	2,	3,	4,
	and	$a'bc$,	$ab'c$,	abc',	abc,
and the aBc circles	as	1,	2,	3,	[$aB'c$],
	and	aBc',	aBc,	$a'Bc$,	$aB'c$,

we see that of the sixteen tetrads given by the theorem of Chapter I. and formed by taking the first and third or second and fourth circles in either the first or second rows with the first and third or second and fourth in the third or fourth rows, those in which 1 and 3 do not occur appear to be new. But the circle [$aB'c$] is the same circle as [acD'] in Section V., since a, c, B and D form a Hart tetrad of circles touching 1, 2, 3, and the fourth circle touching them is [$aB'c$] or [acD']. Hence any tetrad in which 2 and [acB'] occur has been already considered. Thus there are only three new tetrads of each kind; or excluding those in which 1, 2, or 3 occurs there are two of each kind.

17. Again, let us consider circles touching b, B, a, and circles touching b, B, c. We shall have in this case to adopt a notation to distinguish any circle touching three from the conjugate one (its inverse with respect to the circle cutting the three orthogonally). So let bBa' for instance now denote the circle touching b, B, a which lies outside b and B but inside a, and similarly in other cases. Since CX' and AZ are drawn across the arcual angle $X'Y'C$, so that C, X', A and Z are concyclic, we see that 2 and abB' (the incircle of $CX'Y'$) as being abB circles and 2 and bcB' (the incircle of AYZ) as being bcB circles form an inverse tetrad. So writing the abB circles

as $\quad abB',\quad 2,\quad ab'B',\quad abB,$

and $\quad 3,\quad 1,\quad a'bB,\quad a'b'B',$

the former row being the inscribed and escribed circles of $CX'Y'$; and the bcB circles

as $\quad bcB',\quad 2,\quad b'c'B',\quad bc'B,$

and $\quad 1,\quad 3,\quad bcB,\quad b'cB',$

the former row being the inscribed and escribed circles of AZY'; we obtain four new Hart tetrads, 2 occurring in all, and five new inverse tetrads in one of which 2 occurs. Omitting those in which 1, 2, or 3 occurs there are four new inverse tetrads but no new Hart.

SECTION VII.

The Circles touching Triads of an Inverse Group.

18. Let Fig. 16 represent the circles 1, 2, 3; a, b, A, B. In this figure the relative positions of the circles 1, 2, 3 have been altered, as otherwise the figure seems somewhat puzzling. The circle CPC' is b, DPD' is B, CQC' is a, and XQX' is A; the former two being supposed concave below. These circles satisfy the condition of Section I. *doubly*, C, D, X, Y and also C, D', X', Y being concyclic.

Consider first circles that touch a, A, b and circles that touch a, A, B. Since across the angle CQX, CX and DY are drawn so that C, D, X, Y are concyclic, 1, 3 of the aAb circles and 1, 3 of the aAB circles form an inverse tetrad. Hence writing the aAb circles with the notation of the previous article

as $\quad aAb'$ or 1, $\quad a'A'b'$ or 3, $\quad aA'b',\quad aA'b,$

and $\quad a'A'b$ or 2, $\quad aAb,\quad a'Ab,\quad a'Ab',$

the former row being the inscribed and escribed circles of the triangle CQX; and the aAB circles

as $\quad aAB$ or 1, $\quad a'A'B$ or 3, $\quad a'AB,\quad a'AB',$

and $\quad a'A'B'$ or 2, $\quad aAB',\quad aA'B',\quad aA'B,$

the former row being the inscribed and escribed circles of DQY; and noting that the second circles in the second and fourth lines are the same, we see that all the Hart tetrads we obtain are merely Hart tetrads of circles that all touch a, A, b or that all touch a, A, B. We obtain four new inverse tetrads by taking the third and fourth circles in either the first or second row with the third and fourth circles in either the third or fourth row.

19. Again, since across the angle CQY, CX' and $D'Y$ are drawn so that C, D', X', Y are concyclic, we see that 2, 3 of the aAb circles and 2, 3 of the aAB circles form an inverse tetrad. Hence writing the aAb circles

as $a'A'b$ or 2, $a'A'b'$ or 3, $aA'b'$, $a'Ab'$,

and aAb' or 1, aAb, $a'Ab$, $aA'b$,

the former row being the inscribed and escribed circles of CQX'; and the aAB circles

as $a'A'B'$ or 2, $a'A'B$ or 3, $aA'B$, $a'AB$,

and aAB or 1, aAB', $a'AB'$, $aA'B'$,

the former row being the inscribed and escribed circles of $D'QY$; we see that we obtain no new Hart tetrads but four new inverse tetrads by taking the third and fourth circles of either the first or second row with the third and fourth circles of the third or fourth row.

20. From the two last articles we see that from the two squares

$$\begin{matrix} aA'b', & aA'b, \\ a'Ab, & a'Ab', \end{matrix} \quad \text{and} \quad \begin{matrix} a'AB, & a'AB', \\ aA'B', & aA'B, \end{matrix}$$

we can obtain eight new inverse tetrads by taking either horizontal row of the first with either horizontal row of the second and either diagonal of the first with either diagonal of the second.

Similarly by considering circles that touch b, B, a and circles that touch b, B, A it can be shown that from the two squares

$$\begin{matrix} b'B'a, & b'B'a', \\ bBa', & bBa, \end{matrix} \quad \text{and} \quad \begin{matrix} bBA', & bBA, \\ b'B'A, & b'B'A', \end{matrix}$$

we can obtain eight new inverse tetrads by taking either horizontal row of the first with either horizontal row of the second and either diagonal of the first with either diagonal of the second.

21. Furthermore the four circles A, B, a, b cut the same circle orthogonally. Hence if any circle touches three of these circles its inverse with respect to the above circle also touches them. But any two circles and their inverses with respect to any circle form an inverse tetrad touched by four other circles. Therefore if we take two inverse circles (with respect to the circle cutting A, B, a, b orthogonally) touching any three of the four circles A, B, a, b and two inverse circles touching any other three of the four, they form an inverse tetrad touched by two other circles besides the two they have been constructed to touch in common. As before tetrads containing 1, 2 or 3 are only the known inverse tetrads of circles touching three given circles. Exclusive

of these we get twenty-four inverse tetrads, being four for each combination in pairs of sets of three of the four circles A, B, a, b. For the sets Aab and ABa the tetrads are obtained by taking either vertical column of the first square in Art. 20 with either vertical column of the second square. And similarly for the sets Bab and ABb.

Thus if we take any inverse tetrad of circles touching three given circles, to touch them in threes there can be drawn four sets of four circles (exclusive of the original three and of one which with them forms an inverse tetrad); by taking two circles out of one set with two out of another we can form forty inverse tetrads, such that each tetrad is touched by two other circles besides the two they have been constructed to touch in common.

SECTION VIII.

General Statement of the Theorems of this Chapter.

22. From the preceding articles we obtain the following result:—Eight circles can be described to touch three given circles; these eight circles form fifty-six triads; to touch any triad we can describe a set of four circles exclusive of the original three and of one which with them forms either a Hart tetrad or an inverse tetrad; each set is known to form a Hart tetrad or an inverse tetrad; by taking two out of one set and two out of another drawn to touch triads which have two members common, we can form in addition two hundred and eighty-eight Hart tetrads and seven hundred and twenty inverse tetrads, each touched by two circles besides the two they have been constructed to touch in common.

23. These tetrads are classified in the following table:

Type of a group of four circles touching three given circles.	Number of tetrads among circles touching different threes of the group.		Number of groups of the stated type.	Total number of new tetrads thus obtained.	
	Hart.	Inverse.		Hart.	Inverse.
Hart group, $abcd$	12	12	8	96	96
Three circles of a Hart group and the inverse of the fourth with respect to the circle cutting 1, 2, 3 orthogonally, $abcD$	12	12	8	96	96
Three circles of a Hart tetrad and the inverse of either of them, $abcB$ ...	2	6	48	96	288
Inverse tetrad, $ABab$ ...	0	40	6	0	240
				288	720

CHAPTER III.

EXTENSION TO CONES AND CONICS.

SECTION IX.

THE FUNDAMENTAL THEOREM.

24. Every two antipodal circles being the intersection of the sphere with a right circular cone, when the four circles OA, OB, AB, CD in Fig. 1 are great circles by combining each of the tetrads in Art. 1 with the tetrad composed of their antipodals we get the theorem:—If four planes P, Q, X and Y passing through a common point are such that a right circular cone can be described through the intersections of P with X, P with Y, Q with X, and Q with Y, then there can be formed eight tetrads of right circular cones each consisting of two touching X, Y, P, and two touching X, Y, Q, such that each tetrad has two common tangent circular cones (besides the planes X, Y); four of these tetrads are analogous to Hart tetrads and four to inverse tetrads. A similar theorem is of course true of cones touching P, Q, X and P, Q, Y.

The enunciation of the reciprocal theorem is obvious.

25. Hence by projection, using the term "U-conic" to denote a conic having double contact with a given one, U:—If four straight lines P, Q, X, Y, are such that through the intersections of P with X, P with Y, Q with X, and Q with Y, there can be described a U-conic, then there can be formed eight tetrads of U-conics each consisting of two touching X, Y, P, and two touching X, Y, Q, such that each tetrad has two common tangent U-conics (besides the lines X, Y); four of these tetrads are analogous to Hart tetrads and four to inverse tetrads. A similar theorem is of course true of conics touching P, Q, and X, and P, Q, and Y.

The enunciation of the reciprocal theorem is obvious.

26. Let us next extend to cones and conics the fundamental theorem when the four circles OA, OB, AB, CD are small circles. Any two covertical right circular cones intersect in four lines two of which lie in each of two planes perpendicular to the plane containing the axes of the cones. Let us restrict to these planes the title "planes of intersection" of the cones. If P, Q, X be any three covertical circular cones there are four sets of planes of intersection of P and Q, Q and X, X and P which pass through a common line. If P, Q, X, Y be any four covertical circular cones, we can take in eight ways planes of intersection of P and X, Q and X, and of P and Y, Q and Y such that through the intersection of the first two and the intersection of the last two can be drawn a plane of intersection of P and Q; and if the cones be those obtained from the circles OA, OB, AB, CD of Fig. 1 and their antipodals, we can choose these planes of

intersection in one way such that in them lie pairs of intersections of P and X, Q and X, P and Y, Q and Y respectively, possessing the property that four of them, being one of each pair, lie on a circular cone (and therefore the other four lie on another circular cone). We obtain the theorem that in such a case out of certain eight of the thirty-two circular cones touching X, Y, P, and certain eight of those touching X, Y, Q, there can be chosen sixteen tetrads each consisting of two touching X, Y, P, and two touching X, Y, Q, such that each tetrad has two common tangent circular cones (besides X, Y); eight of these tetrads are analogous to Hart tetrads and eight to inverse. A similar theorem of course holds for cones touching P, Q, X, and P, Q, Y.

The enunciation of the reciprocal theorem is obvious.

27. By projection we obtain from the four right circular cones of the previous Art. four U-conics possessing a certain property. Any two U-conics meet in four points and two of their common chords pass through the intersection of their chords of contact with U; let us restrict to these two the title "chords of intersection" of the conics. If P, Q, X be any three U-conics there are four points called "radical centres" of the three conics in each of which there meet a chord of intersection of P and Q, a chord of intersection of Q and X, and a chord of intersection of X and P. (The chords of intersection are the six lines joining the radical centres.) If P, Q, X, Y be any four U-conics we can take in eight ways chords of intersection of P and X, Q and X, and of P and Y, Q and Y, such that through the intersection of the first two and the intersection of the last two there passes a chord of intersection of P and Q, and we can in the case in point choose these chords of intersection in one way such that of the four pairs of intersections of P and X, P and Y, Q and X, Q and Y, which lie on them, four points (being one of each pair) lie on a U-conic (and therefore the other four on another U-conic). We obtain the theorem that when this condition holds, out of certain eight of the thirty-two U-conics touching X, Y, P, and certain eight of those touching X, Y, Q, there can be formed sixteen tetrads each consisting of two touching X, Y, P, and two touching X, Y, Q, such that each tetrad has two common tangent U-conics (besides X, Y); eight of these tetrads are analogous to Hart tetrads and eight to inverse. A similar theorem of course holds for U-conics touching P, Q, X, and P, Q, Y.

The enunciation of the reciprocal theorem is obvious.

28. If from any radical centre of three U-conics pairs of tangents be drawn to them, through the six points of contact a U-conic can be drawn. This conic has been called by Casey (among others?) a conic "orthogonal" to the given three. (See "Memoir on Bicircular Quartics," Chap. v., *Transactions, Royal Irish Academy*, Vol. XXIV.)

As there are four radical centres, there are four orthogonal conics. The thirty-two U-conics that touch the three conics consist of sixteen pairs, the members of each pair and some one of the orthogonal conics being the projections of two small circles and a circle with respect to which the one is the inverse of the other. Four pairs are so related to each orthogonal conic. There are thus four sets each consisting of four pairs. The eight conics touching X, Y, P that enter into the tetrads of the theorem stated in the last Art. are the set corresponding to the particular orthogonal conic of X, Y, P derived from the radical centre which is the intersection of the particular chords of intersection which satisfy the condition of the theorem. A similar statement holds for the conics touching X, Y, Q.

A statement of analogous character holds for the tetrads of cones in Art. 26.

SECTION X.

AN EXTENSION OF THE THEOREMS OF SECTIONS IV., V., IN A PARTICULAR CASE.

29. Suppose the three original circles of Chap. II. are great circles; A and a, B and b, C and c, D and d are then antipodals. Let us see how many tetrads of each kind we obtain among two sets of circles, the first touching one of each of the pairs A and a, B and b, C and c, and the second touching one of each of the pairs A and a, B and b, D and d. From each of the following pairs of triads there can be obtained, as in Sections IV., V., two tetrads of each kind, each consisting of two circles touching one triad of the pair and two touching the other triad, viz.:—

ABC and ABD,
ABC and ABd,
ABc and ABD,
ABc and ABd,
AbC and AbD,
AbC and Abd,
Abc and AbD,
Abc and Abd,

and as many more their antipodals by interchanging A and a, B and b, C and c, D and d; there are thus thirty-two Hart tetrads and thirty-two inverse.

30. By joining the circles in the last article to the centre of the sphere by right circular cones we obtain the following result:—If four circular cones be described touching three given planes, to touch any three of these we can describe sixteen other circular cones besides the three original planes and four cones each of which touches *all* the given four; we thus get four sets of sixteen cones; besides the tetrads of cones having a common tangent cone which we can form by taking four cones out of the same set, we can form thirty-two tetrads by taking two cones out of any one set and two out of any other, such that each tetrad is touched by two cones besides the two they have been constructed to touch in common; sixteen are analogous to Hart tetrads and sixteen to inverse tetrads; and as the four sets can be combined in six ways we obtain ninety-six tetrads of each kind.

31. Hence by projection and reciprocation:—If four U-conics be described touching three given lines or passing through three given points, to touch any three of these we can describe sixteen other U-conics (exclusive of the original lines or points and of four

conics each of which touches all the given four); we thus obtain four sets of sixteen U-conics; besides the tetrads of conics touched by another U-conic which we can form by taking four conics out of the same set, we can form thirty-two tetrads by taking two out of any one set with two out of any other, such that each tetrad is touched by two other U-conics besides the two they have been constructed to touch in common; sixteen are analogous to Hart tetrads and sixteen to inverse tetrads: and as the four sets can be combined in six ways we obtain ninety-six tetrads of each kind.

SECTION XI.

An Extension of the General Theorem of Chapter II.

32. Supposing the three original circles of Chap. II. to be small circles the following result may be obtained from the general theorem stated in Art. 22, by combining antipodal circles in pairs and then projecting. Take three U-conics, U_1, U_2, U_3, and consider the eight U-conics touching them which correspond to any definite one of the four orthogonal conics. To touch any three of these there can be described thirty-two U-conics; these consist of four sets of eight (corresponding to the four orthogonal conics of the chosen three); in one set of eight there occur the original three conics and one which with them forms either a Hart or inverse tetrad; consider the remaining four of this set of eight. We have four such conics touching every triad of the eight touching U_1, U_2, U_3. It is already known that every such set of four forms either a Hart or an inverse tetrad; we can however obtain in addition two hundred and eighty-eight Hart tetrads and seven hundred and twenty inverse, each consisting of two conics touching one triad of the eight, and two touching another triad, the two triads having two members common.

SECTION XII.

A Further Extension of the General Theorem of Chapter II.

33. The results of Chap. II. can be further extended to circular cones and to conics having double contact with a given one, but the results as will be seen are too complicated and indefinite to be of much interest.

Any two circles of one family touching two given circles and any two of the other satisfy the condition of Art. 1.

Let us consider the eight circles touching any three circles 1, 2, 3, and the eight touching 1, 2, and any fourth circle 4; let P, Q be any two circles of the former eight and X, Y be any two of the latter eight, such that, of the circles touching 1, 2, P, Q belong to one family and X, Y to the other. Among the circles touching P, Q, X, and the circles touching P, Q, Y, we obtain by Chap. I. sixteen tetrads touched by two circles besides P and Q. But 1, 2, as touching P, Q, X, and 1, 2, as touching P, Q, Y, will be found to constitute one of these tetrads. No tetrad into which 1 and 2 enter is new, being merely a Hart or inverse tetrad touching either P, Q, X, or P, Q, Y. We thus obtain four new Hart tetrads and five new inverse. We obtain as many from the circles touching X, Y, P, and the circles touching X, Y, Q. And we can so choose P, Q, X, Y in seventy-two ways.

Again, if of the circles touching 1, 2, P and X belong to one family and Q and Y to the other, we obtain four new Hart tetrads and five new inverse from circles touching P, X, Q, and circles touching P, X, Y, and as many more from circles touching Q, Y, P and circles touching Q, Y, X. And we can choose P, Q, X, Y in two hundred and fifty-six ways.

Thus we obtain in all $(328 \times 8 =)$ 2624 Hart tetrads and $(328 \times 10 =)$ 3280 inverse by taking all combinations of two circles touching 1, 2, 3 and two touching 1, 2, 4.

34. Again, if P, Q, R be three circles touching 1, 2, 3, and X a circle touching 1, 2, 4, such that, of the circles touching 1, 2, P, Q belong to one family and R, X to the other, we obtain, by Chap. I., sixteen tetrads each consisting of two circles touching P, Q, R, and two touching P, Q, X.

If P, Q, R be three members of a Hart tetrad touching 1, 2, 3, then 1, 2, 3 are members of a Hart tetrad touching P, Q, R, and excluding as before of the sixteen tetrads those in which 1, 2, or 3 occurs, there remain three new Hart tetrads and three new inverse. And there remain four new Hart tetrads and five inverse from among the circles touching P, R, X and the circles touching Q, R, X. P, Q, R, X can be so chosen in one hundred and twenty-eight ways.

If on the other hand P, Q, R be three members of an inverse tetrad of circles touching 1, 2, 3, then 1, 2, 3 are members of an inverse tetrad touching P, Q, R, and excluding those of the sixteen tetrads in which 1, 2, or 3 occurs, we obtain two Hart tetrads and four inverse from the circles touching P, Q, R and the circles touching P, Q, X; and four Hart and five inverse tetrads from the circles touching P, R, X and the circles touching Q, R, X. And we can so choose P, Q, R, X in thirty-two ways.

Thus from all combinations of three circles touching 1, 2, 3 and one touching 1, 2, 4, we obtain $(7 \times 128 + 6 \times 32 =)$ 1088 Hart tetrads and $(8 \times 128 + 9 \times 32 =)$ 1312 inverse.

35. Now let us take on a sphere any three circles 1, 2, 3, and their antipodals which we will call I, II, III. We can describe sixty-four circles antipodal in pairs, touching one of each of the pairs 1 and I, 2 and II, 3 and III. Let us take four of these sixty-four circles such that none is the antipodal of any other and see how many Hart and inverse tetrads we obtain among circles touching two different triads of these four. If the four circles all touch 1, 2, 3 we obtain [Art. 22] 288 Hart tetrads and 720 inverse; as many if all touch 1, II, III, or if all touch I, 2, III, or if all touch I, II, 3. Thus we obtain 1152 Hart tetrads and 2880 inverse and as many more, their antipodals.

By taking different fours, of which three touch 1, 2, 3 and the fourth touches I, 2, 3, we obtain, as shown in Art. 34, 1088 Hart tetrads and 1312 inverse among the circles touching different triads of the four, and by combining the groups 1 2 3, I 2 3, 1 II 3, 1 2 III, 1 II III, I 2 III, I II 3, I II III in pairs which have two circles common, we obtain $(1088 \times 12 =) 13056$ Hart tetrads and $(1312 \times 12 =) 15744$ inverse, with as many more their antipodals.

And by taking different fours, of which two touch 1, 2, 3 and two touch I, 2, 3, we obtain, as shown in Art. 33, 2624 Hart tetrads and 3280 inverse among circles touching different triads of the four, and by combining the different groups in pairs having two members common, we obtain $(2624 \times 6 =) 15744$ Hart tetrads and $(3280 \times 6 =) 19680$ inverse tetrads, and as many more their antipodals.

36. By joining the circles of the last Art. to the centre by right circular cones and projecting we obtain the following results:—

Take three U-conics, U_1, U_2, U_3 and consider the thirty-two U-conics that touch them. By taking various fours of these corresponding to the same indefinite one of the orthogonal conics, we obtain among the U-conics touching one triad of the four and the U-conics touching another triad of the four and belonging to that set of eight which includes the three original conics U_1, U_2, U_3, 1152 tetrads analogous to Hart tetrads and 2880 analogous to inverse. (This is merely the theorem of Section XI. repeated, all the orthogonal conics being now considered instead of some definite one.)

By taking various fours consisting of three that correspond to an indefinite one of the orthogonal conics and one that corresponds to another indefinite one, we obtain among the U-conics touching one triad and the U-conics touching another triad and belonging to that set of eight which includes two or more of the original conics U_1, U_2, U_3, 13056 tetrads analogous to Hart tetrads and 15744 analogous to inverse.

And by taking various fours consisting of two that correspond to an indefinite one of the orthogonal conics and two that correspond to another indefinite one, we obtain among the U-conics touching one triad of the four and the U-conics touching another triad and belonging to that set of eight which includes two or more of the original conics U_1, U_2, U_3, 15744 tetrads analogous to Hart tetrads and 19680 analogous to inverse.

SECTION XIII.

A Method of Further Extension.

37. By polarizing Hart and inverse tetrads of circles on a sphere Mr A. Larmor has obtained new contact relations among systems of circles and has extended the results to cones and conics ("On the contacts of Systems of Circles," *Proc. Lond. Math. Soc.* Vol. XXIII.). The results of Chaps. I. and II. might also be extended in this manner. Apparently however the process would involve a careful examination of the nature of the contact of many of the circles considered, and would not lead to results which could be expressed simply.

V. *Change of the Independent Variable in a Differential Coefficient.*

By E. G. Gallop, M.A.

Let y and u be functions of an independent variable x. The problem to be considered is that of expressing $\frac{d^nu}{dy^n}$ in terms of $\frac{du}{dx}, \frac{d^2u}{dx^2}, \dots$ and $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$

The problem is equivalent to that of the reversion of series or, what is the same thing, the expansion of one function in powers of another. A solution, though in a very undeveloped form, is therefore afforded by Burmann's theorem, as usually given in treatises on the Differential Calculus. If we put $u=f(x)$ and $y=\phi(x)$, the solution may be expressed in the form

$$\frac{d^nu}{dy^n}=\left[\frac{d^{n-1}}{d\xi^{n-1}}\left\{\frac{f'(\xi)(\xi-x)^n}{\{\phi(\xi)-\phi(x)\}^n}\right\}\right]_{\xi=x},$$

where after the differentiations have been effected ξ is put equal to x.

More developed solutions have been given in four different forms by Sylvester*, Schlömilch†, Hess‡ and Leudesdorf§. Sylvester's result is expressed in a fully expanded form with the coefficient of each term evaluated. The proof which he gives is inductive, but the result can be obtained directly from a formula due to Jacobi‖. Herr Schlömilch's result may be regarded as a development of Burmann's form; though not well adapted for the purpose, it can be made to produce Sylvester's expanded formula. (See § 12.) Herr Hess has used the same equations as Schlömilch, and obtained a result in the form of an elegant determinant, the elements of which are calculated by a simple rule. Mr Leudesdorf's form is very important in connexion with reciprocants, being purely symbolical and expressed in terms of an operator which in the particular case when $u=x$ reduces to V, the annihilator of pure reciprocants.

In 1855 Sylvester communicated without proof to the Royal Society (Proceedings) a fully expanded formula for the change of any number of independent variables, and the results were reprinted with corrections, but again without proof, in the *Quarterly Journal*

* *Phil. Mag.*, Vol. VIII. 1854, p. 535.

† *Compendium der höheren Analysis*, Bd. II. pp. 16—20, and *Sitzungsberichte der Königl. sächsischen Gesellschaft der Wissenschaften zu Leipzig*, 1857.

‡ *Zeitschr. f. Mathem. u. Phys.*, Thl. XVII. p. 1.

§ *Proc. Lond. Math. Soc.*, Vols. XVII. p. 197 and p. 329 and XVIII. p. 235.

‖ *Crelle's Journal*, VI. 1830, p. 257. "De resolutione aequationum per series infinitas."

of Mathematics, Vol. I., 1857. Treating the question as the reversion of series Cayley* deduced Sylvester's results from the theorem of Jacobi already mentioned.

This communication is restricted to the case of one independent variable, though it is hoped that it may be possible to extend the results to two or more variables. An expression is obtained for $\frac{d^n u}{dy^n}$, which is closely allied in form to that given by Schlömilch, and may be deduced from it. (See § 12.) At the same time it leads at once to Sylvester's formula, and may indeed be regarded as a concise expression for it; whilst it also leads naturally to Mr Leudesdorf's symbolical result and introduces the operator V in a form convenient for transformation. The formula was originally obtained by induction, and an inductive proof is given in § 5; but a better insight into the nature of the solution is obtained by following Cayley's method, and establishing the formula directly, as in § 6.

The following notation is used throughout the paper. The differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ are denoted by y_1, $y_2 \ldots$, $\frac{du}{dx}, \frac{d^2u}{dx^2}, \ldots$ by u_1, u_2, ..., and D stands for $\frac{d}{dx}$. The result of suppressing all terms of an expression in which y occurs explicitly is indicated by a zero suffix; thus $[D^n y^r]_0$ and $[D^n(uy^r)]_0$ represent the result of expressing $D^n y^r$ and $D^n(uy^r)$ in terms of y, y_1, $y_2 \ldots$, and then suppressing the terms in which y occurs explicitly. Similarly $\Delta^n y^r$ and $\Delta^n(uy^r)$ represent the result of suppressing all terms which contain y and y_1 in $D^n y^r$ and $D^n(uy^r)$, whilst $\Delta_0{}^n(uy^r)$ is the result of suppressing the terms which contain y, y_1, u and u_1 in $D^n(uy^r)$.

The functions thus defined play an important part in the theory of the change of the independent variable, for not only do they enable the results to be expressed in a compact form but they appear as coefficients in all the operators connected with ordinary reciprocants. In fact, the general multilinear operator investigated by Major MacMahon†, which includes as particular cases the operators of the theories of reciprocants and invariants, has for its coefficients numerical multiples of $D^n y^r$ if y, y_1, y_2, ... are replaced by a, b, $2!\,c$, $3!\,d$, ... There is distinct advantage in expressing the coefficients in this way, especially in the more complicated operators, as the transformations of the operators are often considerably simplified thereby. See §§ 9, 10.

§ 1. Consider the function $D^n(u_1 y^r)$, where $u_1 = \frac{du}{dx}$. The coefficients of the terms in Lagrange's theorem on expansions are of this type. The expanded form may be easily obtained by direct differentiation, but it is still simpler to proceed as follows. It is obvious by Taylor's Theorem that, if

$$\left(u_1 + u_2 z + u_3 \frac{z^2}{2!} + \ldots\right)\left(y + y_1 z + y_2 \frac{z^2}{2!} + y_3 \frac{z^3}{3!} + \ldots\right)^r$$

* "Note sur une formule pour la réversion des séries," *Crelle's Journal*, tom. LII. 1856, and "Deuxième note &c.", *Crelle*, tom. LIV. 1857. Sylvester's results are proved in the second note. *Collected Works*, Vol. IV. 229 and 234.

† *Proc. Lond. Math. Soc.* Vol. XVIII.

is expanded in powers of z, the coefficient of z^n will be

$$\frac{D^n(u_1 y^r)}{n!}.$$

Hence by the multinomial theorem

$$\frac{D^n(u_1 y^r)}{n!} = \Sigma \frac{r!}{a!\, b!\, c! \ldots} \frac{u_h}{(h-1)!} y^a \left(\frac{y_1}{1!}\right)^b \left(\frac{y_2}{2!}\right)^c \quad \text{.........................(1)},$$

where summation extends to all terms of degree one in u's, of degree r in y's and of weight $n+1$ in u's and y's together; on the understanding that weight r is assigned to u_r and y_r. Zero values are admissible for a, b, c, ... but not for h. The same relation may be written

$$\frac{D^n(u_1 y^r)}{r!} = \Sigma \frac{n!}{a!\, b!\, c! \ldots} \frac{u_h}{(h-1)!} y^a \left(\frac{y_1}{1!}\right)^b \left(\frac{y_2}{2!}\right)^c \quad \text{..........................(2)}.$$

Putting $u_1 = 1$, we have also

$$\frac{D^n y^r}{r!} = \Sigma \frac{n!}{a!\, b!\, c! \ldots} y^a \left(\frac{y_1}{1!}\right)^b \left(\frac{y_2}{2!}\right)^c \quad \text{...................................(3)},$$

where summation extends to all terms of degree r and weight n.

The expanded forms for $[D^n(u_1 y^r)]_0$, $[D^n y^r]_0$ will be obtained by omitting terms containing y explicitly, whilst to get $\Delta^n(u_1 y^r)$ and $\Delta^n(y^r)$ all terms containing y or y_1 must be suppressed.

To obtain $\Delta_0^n(u_1 y^r)$ we must suppress all terms containing y, y_1 or u_1.

Since $[D^n(u_1 y^r)]_0$ is equal to the coefficient of z^n in

$$(u_1 + u_2 z + \ldots)\left(y_1 z + y_2 \frac{z^2}{2} + \ldots\right)^r,$$

it is obvious that $[D^n(u_1 y^r)]_0$ vanishes when n is less than r. Similarly $\Delta^n(u_1 y^r)$ vanishes when n is less than $2r$.

The general result of this section may be stated

$$\left(u_1 + u_2 z + u_3 \frac{z^2}{2!} + \ldots\right)\left(y + y_1 z + y_2 \frac{z^2}{2!} + \ldots\right)^r = \sum_{s=0}^{s=\infty} \frac{D^s(u_1 y^r)}{s!} z^s \quad \text{.................(4)}.$$

§ 2. To obtain an expression for the differential coefficient of $\Delta^n(u_1 y^r)$ write

$$U = u_1 + u_2 z + u_3 \frac{z^2}{2!} + \ldots,$$

$$Y = y_2 \frac{z^2}{2!} + y_3 \frac{z^3}{3!} + \ldots,$$

and regard z as independent of x. Then by (4)

$$\Sigma \frac{z^n}{n!} \Delta^n(u_1 y^r) = UY^r \quad \text{..(5)}.$$

Differentiate with respect to x; then

$$\Sigma \frac{z^n}{n!}\frac{d}{dx}\Delta^n(u_1y^r) = Y^r\left(u_2 + u_3z + u_4\frac{z^2}{2!} + \ldots\right) + rUY^{r-1}\left(y_3\frac{z^2}{2!} + y_4\frac{z^3}{3!} + \ldots\right).$$

Differentiate (5) with respect to z; then

$$\Sigma \frac{z^{n-1}}{(n-1)!}\Delta^n(u_1y^r) = Y^r\left(u_2 + u_3z + u_4\frac{z^2}{2!} + \ldots\right) + rUY^{r-1}\left(y_2z + y_3\frac{z^2}{2!} + y_4\frac{z^3}{3!} + \ldots\right).$$

Hence by subtraction

$$\Sigma \frac{z^n}{n!}\frac{d}{dx}\Delta^n(u_1y^r) - \Sigma \frac{z^{n-1}}{(n-1)!}\Delta^n(u_1y^r) = -ry_2zUY^{r-1} = -ry_2z\Sigma\frac{z^n}{n!}\Delta^n(u_1y^{r-1}).$$

Whence, comparing coefficients of z^n, we have

$$\frac{d}{dx}\frac{\Delta^n(u_1y^r)}{n!} - \frac{\Delta^{n+1}(u_1y^r)}{n!} = -ry_2\frac{\Delta^{n-1}(u_1y^{r-1})}{(n-1)!},$$

or

$$\frac{\Delta^{n+1}(u_1y^r)}{r!} = \frac{d}{dx}\frac{\Delta^n(u_1y^r)}{r!} + ny_2\frac{\Delta^{n-1}(u_1y^{r-1})}{(r-1)!} \qquad\ldots\ldots(6).$$

Putting $u = x$, we have

$$\frac{\Delta^{n+1}y^r}{r!} = \frac{d}{dx}\frac{\Delta^ny^r}{r!} + ny_2\frac{\Delta^{n-1}y^{r-1}}{(r-1)!} \qquad\ldots\ldots(7).$$

Again, (6) may be written

$$u_1\frac{\Delta^{n+1}y^r}{r!} + \frac{\Delta_0^{n+1}(u_1y^r)}{r!} = \frac{d}{dx}\left[u_1\frac{\Delta^ny^r}{r!} + \frac{\Delta_0^n(u_1y^r)}{r!}\right] + ny_2\left[u_1\frac{\Delta^{n-1}y^{r-1}}{(r-1)!} + \frac{\Delta_0^{n-1}(u_1y^{r-1})}{(r-1)!}\right],$$

and therefore by (7)

$$\frac{\Delta_0^{n+1}(u_1y^r)}{r!} = \frac{d}{dx}\frac{\Delta_0^n(u_1y^r)}{r!} + u_2\frac{\Delta^ny^r}{r!} + ny_2\frac{\Delta_0^{n-1}(u_1y^{r-1})}{(r-1)!} \qquad\ldots\ldots(8).$$

In the same way as (6) was proved it may be shown that

$$\frac{[D^{n+1}(u_1y^r)]_0}{r!} = \frac{d}{dx}\frac{[D^n(u_1y^r)]_0}{r!} + y_2\frac{[D^{n-1}(u_1y^{r-1})]_0}{(r-1)!} \qquad\ldots\ldots(9)$$

and

$$\frac{[D^{n+1}y^r]_0}{r!} = \frac{d}{dx}\frac{[D^ny^r]_0}{r!} + y_2\frac{[D^{n-1}y^{r-1}]_0}{(r-1)!} \qquad\ldots\ldots(10).$$

§ 3. The partial differential coefficients of $D^n(u_1y^r)$ with respect to $y, y_1, y_2, \ldots$ are easily obtained from the equation

$$\Sigma\frac{z^n}{n!}D^n(u_1y^r) = \left(u_1 + u_2z + u_3\frac{z^2}{2!} + \ldots\right)\left(y + y_1z + y_2\frac{z^2}{2!} + \ldots\right)^r.$$

Differentiate with respect to y_s; then

$$\Sigma\frac{z^n}{n!}\frac{\partial}{\partial y_s}D^n(u_1y^r) = \frac{z^s}{s!}\cdot r(y + y_1z + \ldots)^{r-1}(u_1 + u_2z + \ldots)$$

$$= r\frac{z^s}{s!}\Sigma\frac{z^n}{n!}D^n(u_1y^{r-1}).$$

Comparing coefficients of z^n, we have

$$\frac{\partial}{\partial y_s} \frac{D^n (u_1 y^r)}{n!} = r \frac{D^{n-s} (u_1 y^{r-1})}{s! (n-s)!},$$

or

$$\frac{\partial}{\partial y_s} D^n (u_1 y^r) = r (n)_s D^{n-s} (u_1 y^{r-1}) \quad \text{.....(11)},$$

where

$$(n)_s = \frac{n!}{s! (n-s)!}.$$

The result is of course zero if $s > n$.

Similarly

$$\frac{\partial}{\partial u_s} D^n (u_1 y^r) = (n)_{s-1} D^{n-s-1} y^r \quad \text{.....(12)}.$$

Particular cases of these results are obtained by replacing D by Δ or Δ_0, or by putting $u_1 = 1$. Thus

$$\frac{\partial}{\partial y_s} \Delta^n (u_1 y^r) = r (n)_s \Delta^{n-s} (u_1 y^{r-1}) \quad \text{.....(13)},$$

$$\frac{\partial}{\partial u_s} \Delta^n (u_1 y^r) = (n)_{s-1} \Delta^{n-s+1} y^r \quad \text{.....(14)},$$

$$\frac{\partial}{\partial y_s} \Delta_0^n (u_1 y^r) = r (n)_s \Delta_0^{n-s} (u_1 y^{r-1}) \quad \text{.....(15)},$$

$$\frac{\partial}{\partial u_s} \Delta_0^n (u_1 y^r) = (n)_{s-1} \Delta_0^{n-s-1} y^r \quad \text{.....(16)},$$

$$\frac{\partial}{\partial y_s} \Delta^n y^r \quad = r (n)_s \Delta^{n-s} y^{r-1} \quad \text{.....(17)}.$$

In particular

$$\frac{\partial}{\partial y_s} \Delta^n (y y_1) = \frac{\partial}{\partial y_s} \frac{\Delta^{n-1} y^2}{2}$$

$$= (n+1)_s \Delta^{n-s+1} y$$

$$= (n+1)_s y_{n-s+1} \quad \text{.....(18)}.$$

§ 4. Various expansions for the functions may be obtained in powers of y, or of y and y_1, or of y, y_1 and y_2, &c. The method will be sufficiently evident from the following example, which will be required later in § 12. We will prove that

$$\frac{[D^n y^r]_0}{r!} = \frac{\Delta^n y^r}{r!} + n y_1 \frac{\Delta^{n-1} y^{r-1}}{(r-1)!} + (n)_2 y_1^2 \frac{\Delta^{n-2} y^{r-2}}{(r-2)!} + (n)_3 y_1^3 \frac{\Delta^{n-3} y^{r-3}}{(r-3)!} + \dots \quad \text{.....(19)}.$$

It is evident by Taylor's Theorem that the coefficient of y_1^s is equal to the result of suppressing terms containing y_1 in

$$\frac{1}{s!}\left(\frac{\partial}{\partial y_1}\right)^s \frac{[D^n y^r]_0}{r!};$$

that is, in
$$\frac{1}{s!} n(n-1)\dots(n-s+1)\frac{[D^{n-s}y^{r-s}]_0}{(r-s)!},$$

by repeated applications of 11; the coefficient is therefore

$$(n)_s \frac{\Delta^{n-s}y^{r-s}}{(r-s)!}.$$

§ 5. Equations (6) and (7) may be used for verifying the formulæ for the change of the independent variable.

It is easy to verify by direct differentiation for small values of n that

$$\frac{d^n x}{dy^n} = -\frac{\Delta^n y}{y_1^{n+1}} + \frac{1}{y_1^{n+2}}\frac{\Delta^{n+1}y^2}{2!} - \frac{1}{y_1^{n+3}}\frac{\Delta^{n+2}y^3}{3!} + \dots$$
$$+ \frac{(-1)^r}{y_1^{n+r}}\frac{\Delta^{n+r-1}y^r}{r!} + \dots + \frac{(-1)^{n-1}}{y_1^{2n-1}}\frac{\Delta^{2n-2}y^{n-1}}{(n-1)!} \quad\dots\dots(20).$$

The result for general values of n follows at once by induction with the help of (7).

In the same way it may be verified by direct differentiation that for small values of n

$$\frac{d^n u}{dy^n} = \frac{1}{y_1^n}u_n - \frac{1}{y_1^{n+1}}\Delta^n(u_1 y) + \frac{1}{y_1^{n+2}}\frac{\Delta^{n+1}(u_1 y^2)}{2!}$$
$$- \frac{1}{y_1^{n+3}}\frac{\Delta^{n+2}(u_1 y^3)}{3!} + \dots + \frac{(-1)^r}{y_1^{n+r}}\frac{\Delta^{n+r-1}(u_1 y^r)}{r!} + \dots$$
$$+ \frac{(-1)^{n-1}}{y_1^{2n-1}}\frac{\Delta^{2n-2}(u_1 y^{n-1})}{(n-1)!} \quad\dots\dots(21);$$

and as before the result for general values of n is proved by induction with the help of (6).

If now the functions $\Delta^{n+1}y^2$, $\Delta^{n+2}y^3$, ... $\Delta^n(u_1 y)$, $\Delta^{n+1}(u_1 y^2)$, ... are expanded by formulæ (3) and (2), we obtain the fully-developed forms given by Sylvester (*Phil. Mag.* Vol. VIII., 1854).

From (20) we have

$$\frac{d^n x}{dy^n} = \Sigma \frac{(-1)^r}{y_1^{n+r}}\frac{(n+r-1)!}{a!\,b!\,c!\dots}\left(\frac{y_2}{2!}\right)^a\left(\frac{y_3}{3!}\right)^b\left(\frac{y_4}{4!}\right)^c \quad\dots\dots(22),$$

where $r = a+b+c+\dots$, and the summation extends to all sets of $a, b, c, \dots$ which satisfy the equation $a+2b+3c+\dots = n-1$.

From (21)

$$\frac{d^n u}{dy^n} = \Sigma \frac{(-1)^r}{y_1^{n+r}} \frac{(n+r-1)!}{a!\, b!\, c! \ldots} \frac{u_h}{(h-1)!} \left(\frac{y_2}{2!}\right)^a \left(\frac{y_3}{3!}\right)^b \quad \text{.........................(23)},$$

where $r = a + b + c + \ldots$, and summation extends to all sets of h, a, b, ... which satisfy the equation $h + a + 2b + 3c + \ldots = n$, zero values of h being excluded.

Another form may be given to equation (21). Expand $\Delta^{n+1}(u_1 . y)$, $\Delta^{n+1}(u_1 . y^2)$, ... by Leibnitz' theorem for the differentiation of a product. We then find

$$\frac{d^n u}{dy^n} = X_1^n \frac{du}{dx} + X_2^n \frac{d^2 u}{dx^2} + \ldots + X_n^n \frac{d^n u}{dx^n} \quad \text{...............................(24)},$$

where

$$X_r^n = -\frac{1}{y_1^{n+1}} (n)_{r-1} \Delta^{n-r+1} y + \frac{1}{y_1^{n+2}} (n+1)_{r-1} \frac{\Delta^{n-r+2} y^2}{2!}$$

$$- \frac{1}{y_1^{n+3}} (n+2)_{r-1} \frac{\Delta^{n-r+3} y^3}{3!} + \ldots + \frac{(-1)^s}{y_1^{n+s}} (n+s-1)_{r-1} \frac{\Delta^{n-r+s} y^s}{s!} + \ldots$$

$$+ \frac{(-1)^{n-r}}{y_1^{2n-r}} (2n-r-1)_{r-1} \frac{\Delta^{2n-2r} y^{n-r}}{(n-r)!} \quad \text{..........(25)}.$$

Obviously

$$X_1^n = \frac{d^n x}{dy^n}.$$

Symbolically we may write

$$X_r^n = \left(\frac{\partial}{\partial \Delta}\right)^{r-1} X_1^n,$$

where X_1^n is expressed as in (20).

§ 6. A direct proof of the formulæ of the previous section can be obtained from the theorem of Jacobi already mentioned. As the proof of the theorem is very simple in the case of one independent variable, it is reproduced here for the sake of completeness.

Let η be a quantity given in terms of ξ by the equation

$$\eta = a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + \ldots.$$

Denote the right-hand of this equation by X, and put $X = a_1 \xi (1 + P)$.

Let $F(\xi)$ be any function of ξ expansible in powers of ξ, which it is required to expand in powers of η. Suppose the result to be

$$F(\xi) = b_0 + b_1 \eta + b_2 \eta^2 + \ldots.$$

Now, if $f(\xi)$ is any function containing ξ only in the form of powers, positive or negative,

$$\left[\frac{d}{d\xi} f(\xi)\right]_{\xi^{-1}} = 0,$$

where $[\ ]_{\xi^{-1}}$ denotes the coefficient of ξ^{-1} in the development of the function enclosed in brackets.

Hence, if m be any quantity except -1,

$$\left[X^m \frac{dX}{d\xi}\right]_{\xi^{-1}} = \frac{1}{m+1}\left[\frac{d}{d\xi}X^{m+1}\right]_{\xi^{-1}} = 0,$$

where it is understood that X^m is developed in the form

$$X^m = (a_1\xi)^m\left[1 + mP + \frac{m(m-1)}{2!}P^2 + \dots\right],$$

and P, P^2, ... are expanded in powers of ξ.

Again, $$\left[\frac{1}{X}\frac{dX}{d\xi}\right]_{\xi^{-1}} = \left[\frac{d}{d\xi}\log X\right]_{\xi^{-1}}$$

$$= \left[\frac{d}{d\xi}\log\{a_1\xi(1+P)\}\right]_{\xi^{-1}} = \left[\frac{1}{\xi} + \frac{d}{d\xi}\log(1+P)\right]_{\xi^{-1}} = 1,$$

since $\log(1+P)$ contains only powers of ξ.

Now the equation $$F(\xi) = b_0 + b_1X + b_2X^2 + \dots$$

is an identity. Differentiate and then divide by X^n. Therefore

$$\frac{F'(\xi)}{X^n} = \frac{dX}{d\xi}\left[\frac{b_1}{X^n} + \frac{2b_2}{X^{n-1}} + \dots + \frac{nb_n}{X} + (n+1)b_{n-1} + (n+2)b_{n-2}X + \dots\right],$$

and hence, by the preceding work,

$$nb_n = \left[\frac{F'(\xi)}{X^n}\right]_{\xi^{-1}} \quad\text{...(26).}$$

To apply this result, let $y = \phi(x)$, $u = f(x)$.

Let x be increased by ξ, and let the consequent increments in y and u be η and v, where

$$\eta = y_1\xi + y_2\frac{\xi^2}{2!} + y_3\frac{\xi^3}{3!} + \dots = X, \text{ say;}$$

and $$v = u_1\xi + u_2\frac{\xi^2}{2!} + u_3\frac{\xi^3}{3!} + \dots.$$

Then when v is developed in powers of η, the coefficient of η^n is by (26)

$$\left[\frac{1}{X^n}\frac{dv}{d\xi}\right]_{\xi^{-1}}.$$

But the coefficient of η^n is $\frac{1}{n!}\frac{d^nu}{dy^n}$. Therefore

$$\frac{1}{n!}\frac{d^nu}{dy^n} = \frac{1}{n}\left[\frac{dv}{d\xi}X^{-n}\right]_{\xi^{-1}}.$$

Now write

$$U = u_1 + u_2\xi + u_3\frac{\xi^2}{2!} + \dots,$$

$$Y = y_2\frac{\xi^2}{2!} + y_3\frac{\xi^3}{3!} + \dots,$$

so that $$X = y_1\xi + Y.$$

Therefore
$$\frac{d^n u}{dy^n} = (n-1)!\left[U(y_1\xi)^{-n}\left(1+\frac{Y}{y_1\xi}\right)^{-n}\right]_{\xi^{-1}}$$
$$= \frac{(n-1)!}{y_1^n}\left[\xi^{-n}\left(U - n\frac{UY}{y_1\xi} + (n+1)_2\frac{UY^2}{y_1^2\xi^2} - (n+2)_3\frac{UY^3}{y_1^3\xi^3} + \dots\right)\right]_{\xi^{-1}}.$$

Hence, expanding UY, UY^2, UY^3, ... by (5) of § 2, we have $\frac{d^n u}{dy^n}$ = coefficient of ξ^{n-1} in

$$\frac{(n-1)!}{y_1^n}U - \frac{n!}{y_1^{n-1}}\Sigma\frac{\Delta^s(u_1 y)}{s!}\xi^{s-1} + \frac{(n+1)!}{2!\,y_1^{n-2}}\Sigma\frac{\Delta^s(u_1 y^2)}{s!}\xi^{s-2} - \frac{(n+2)!}{3!\,y_1^{n+3}}\Sigma\frac{\Delta^s(u_1 y^3)}{s!}\xi^{s-3} + \dots,$$

and therefore
$$\frac{d^n u}{dy^n} = \frac{u_n}{y_1^n} - \frac{1}{y_1^{n-1}}\frac{\Delta^n(u_1 y)}{1!} + \frac{1}{y_1^{n-2}}\frac{\Delta^{n+1}(u_1 y^2)}{2!} - \frac{1}{y_1^{n+3}}\frac{\Delta^{n+2}(u_1 y^3)}{3!} + \dots.$$

This is the formula (21), and putting $u = x$ we obtain (20).

§ 7. Mr Leudesdorf's symbolical forms for these results may now be easily deduced.

First consider the form (20) for $\frac{d^n x}{dy^n}$. We have

$$\Delta^{m+1}y^{r-1} = (r+1)\,\Delta^m(y^r y_1)$$
$$= (r+1)\,\Delta^m(yy_1\,.\,y^{r-1})$$
$$= (r+1)\left[(m)_3\Delta^3(yy_1)\,.\,\Delta^{m-3}y^{r-1} + (m)_4\Delta^4(yy_1)\,.\,\Delta^{m-4}y^{r-1} + (m)_5\Delta^5(yy_1)\,.\,\Delta^{m-5}y^{r-1} + \dots\right]$$
$$= \frac{(r+1)}{r}\left[\Delta^3(yy_1)\frac{\partial}{\partial y_3} + \Delta^4(yy_1)\frac{\partial}{\partial y_4} + \Delta^5(yy_1)\frac{\partial}{\partial y_5} + \dots\right]\Delta^m y^r$$

by (17) of § 3. Therefore

$$\frac{\Delta^{m-1}y^{r-1}}{(r+1)!} = \frac{1}{r}\left[\Delta^3(yy_1)\frac{\partial}{\partial y_3} + \Delta^4(yy_1)\frac{\partial}{\partial y_4} + \dots\right]\frac{\Delta^m y^r}{r!}$$
$$= \frac{1}{r}V\,.\,\frac{\Delta^m y^r}{r!},$$

where
$$V = \Delta^3(yy_1)\frac{\partial}{\partial y_3} + \Delta^4(yy_1)\frac{\partial}{\partial y_4} + \Delta^5(yy_1)\frac{\partial}{\partial y_5} + \dots \quad\text{..................(27)},$$

and therefore V is the annihilator of pure reciprocants, that is,

$$V = 3y_2^2\frac{\partial}{\partial y_3} + 10y_2 y_3\frac{\partial}{\partial y_4} + \dots.$$

See Mr Leudesdorf's paper, *Proc. Lond. Math. Soc.*, Vol. XVII., p. 199.

The expression for V may also be written

$$V = \tfrac{1}{2}\left[\Delta^4 y^2\frac{\partial}{\partial y_3} + \Delta^5 y^2\frac{\partial}{\partial y_4} + \dots\right] \quad\text{..............................(28)}.$$

Hence also

$$\frac{\Delta^{m+1}y^{r+1}}{(r+1)!} = \frac{1}{r(r-1)}V^2 \cdot \frac{\Delta^{m-1}y^{r-1}}{(r-1)!}$$

$$= \frac{1}{r!}V^r\Delta^{m-r+1}y$$

$$= \frac{1}{r!}V^r \cdot y_{m-r+1} \qquad (29).$$

The formula (20) may now be written

$$\frac{d^n x}{dy^n} = -\frac{1}{y_1^{n+1}}y_n + \frac{1}{y_1^{n+2}}\frac{Vy_n}{1!} - \frac{1}{y_1^{n+3}}\frac{V^2 y_n}{2!} + \frac{1}{y_1^{n+3}}\frac{V^3 y_n}{3!} - \dots$$

$$= -\frac{1}{y_1^{n+1}}e^{-\frac{V}{y_1}}y_n \qquad (30).$$

This is Mr Leudesdorf's result (*Proc. Lond. Math. Soc.*, XVII., p. 208). As he shows in a second paper in the same volume of the *Proceedings*, p. 333, this result is fundamental in the theory of pure reciprocants. For, if $f(y_2, y_3, \dots)$ is any homogeneous isobaric function of degree i and weight w, it is easily deduced that

$$f(x_2, x_3, \dots) = (-1)^i \frac{1}{y_1^{w-i}} e^{-\frac{V}{y_1}} f(y_2, y_3, \dots).$$

§ 8. The formula (21) for $\frac{d^n u}{dy^n}$ may be transformed in a similar manner. We have

$$\frac{d^n u}{dy^n} = \frac{1}{y_1^n}\Delta^{n-1}u_1 - \frac{1}{y_1^{n+1}}\Delta^n(u_1 y) + \frac{1}{y_1^{n+2}}\frac{\Delta^{n+1}(u_1 y^2)}{2!} - \frac{1}{y_1^{n+3}}\frac{\Delta^{n+2}(u_1 y^3)}{3!} + \dots$$

$$= u_1 X_1^n + \frac{1}{y_1^n}\Delta_0^{n-1}u_1 - \frac{1}{y_1^{n+1}}\Delta_0^n(u_1 y) + \frac{1}{y_1^{n+2}}\frac{\Delta_0^{n+1}(u_1 y^2)}{2!} - \frac{1}{y_1^{n+3}}\frac{\Delta_0^{n+2}(u_1 y^3)}{3!} + \dots,$$

where

$$X_1^n = \frac{d^n x}{dy^n} = -\frac{1}{y_1^{n+1}}e^{-\frac{V}{y_1}}y_n,$$

and $\Delta_0^m(u_1 y^r)$ denotes the result of suppressing y, y_1 and u_1 in $D^m(u_1 y^r)$.

Now $\Delta_0^{m+1}(u_1 y^{r+1}) = \Delta_0^{m+1}(u_1 y \cdot y^r)$

$$= (m+1)_3\Delta_0^3(u_1 y) \cdot \Delta^{m-2}y^r + (m+1)_4\Delta_0^4(u_1 y) \cdot \Delta^{m-3}y^r + \dots$$

$$= (m)_2\Delta_0^3(u_1 y) \cdot \Delta^{m-2}y^r + (m)_3\Delta_0^4(u_1 y) \cdot \Delta^{m-3}y^r + \dots$$

$$+ (m)_3\Delta_0^3(u_1 y) \cdot \Delta^{m-2}y^r + (m)_4\Delta_0^4(u_1 y) \cdot \Delta^{m-3}y^r + \dots.$$

since

$$(m)_{r-1} + (m)_r = (m+1)_r.$$

Now by (16) of § 3 the first line of the last expression may be written

$$\left[\Delta_0^3(u_1 y)\frac{\partial}{\partial u_3} + \Delta_0^4(u_1 y)\frac{\partial}{\partial u_4} + \Delta_0^5(u_1 y)\frac{\partial}{\partial u_5} + \dots\right]\Delta_0^m(u_1 y^r) = W_1 \cdot \Delta_0^m(u_1 y^r),$$

where W_1 denotes the operator in square brackets.

The second line is equal to

$$r\left[(m)_3\Delta_0^3(u_1y)\,\Delta^{m-3}(y^{r-1}y_1)+(m)_4\Delta_0^4(u_1y)\,\Delta^{m-4}(y^{r-1}y_1)+\ldots\right]$$

$$=r\Delta_0^m(u_1y\,.\,y^{r-1}y_1)$$

$$=r\Delta_0^m(yy_1\,.\,u_1y^{r-1})$$

$$=r\left[(m)_3\Delta^3(yy_1)\Delta_0^{m-3}(u_1y^{r-1})+(m)_4\Delta^4(yy_1)\Delta_0^{m-4}(u_1y^{r-1})+\ldots\right]$$

$$=\left[\Delta^3(yy_1)\frac{\partial}{\partial y_3}+\Delta^4(yy_1)\frac{\partial}{\partial y_4}+\ldots\right]\Delta_0^m(u_1y^r)$$

$$=V\,.\,\Delta_0^m(u_1y^r).$$

See equation (15) § 3.

Now write $W=W_1+V$, so that W is the operator considered by Mr Leudesdorf (*Proc. Lond. Math. Soc.*, Vol. XVIII., p. 239), allowance being made for difference of notation.

In order to make the notation agree with that used in the paper just quoted we should have to write

$$y_n=\frac{1}{n!}\frac{d^ny}{dx^n},\ \&c.,$$

and substitute y, z, x for u, x, y.

We have therefore

$$\Delta_0^{m-1}(u_1y^{r-1})=W\,.\,\Delta_0^m(u_1y^r)$$

$$=W^2\Delta_0^{m-1}(u_1y^{r-1})$$

$$=W^{r-1}\Delta_0^{m-r}u_1$$

$$=W^{r-1}u_{m-r+1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

The formula for $\dfrac{d^nu}{dy^n}$ therefore becomes

$$\frac{d^nu}{dy^n}=-\frac{1}{y_1^{n-1}}e^{-\frac{V}{y_1}}y_n\,.\,u_1+\frac{1}{y_1^n}\left[1-\frac{W}{y_1}+\frac{1}{2!}\left(\frac{W}{y_1}\right)^2-\frac{1}{3!}\left(\frac{W}{y_1}\right)^3+\ldots\right]u_n$$

$$=-\frac{1}{y_1^{n+1}}e^{-\frac{V}{y_1}}y_n\,.\,u_1+\frac{1}{y_1^n}e^{-\frac{W}{y_1}}u_n$$

$$=\frac{1}{y_1^{n+1}}e^{-\frac{W}{y_1}}(y_1u_n-u_1y_n)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32);$$

since W, when operating on y's only, is equivalent to V.

This is Mr Leudesdorf's result, which, as he shows, may be generalized like the previous one and is fundamental in the theory of certain extensions of the ordinary theory of reciprocants considered by him in the paper referred to.

§ 9. It appears that the usual operators of the theory of ordinary reciprocants can be conveniently expressed in terms of the functions considered in this paper. The formulae necessary for the transformation of the operators when written in this form are given in §§ 2, 3. There seems to be considerable gain in simplicity and directness by the use of this method. As illustrations two important transformations of operators are given in this and the next section. The first will be used to prove a theorem established by Mr Leudesdorf in the paper last referred to.

Defining W as in § 8, the operator W' is defined by the equation

$$W' = \Delta_0^3(uy_1)\frac{\partial}{\partial y_3} + \Delta_0^4(uy_1)\frac{\partial}{\partial y_4} + \Delta_0^5(uy_1)\frac{\partial}{\partial y_5} + \dots$$
$$+ \Delta_0^3(uu_1)\frac{\partial}{\partial u_3} + \Delta_0^4(uu_1)\frac{\partial}{\partial u_4} + \Delta_0^5(uu_1)\frac{\partial}{\partial u_5} + \dots,$$

so that W' is obtained from W by interchanging u and y.

The theorem is that W and W' are commutative.

We proceed to form the product WW'. Write

$$WW' = W \,.\, W' + W * W',$$

where $W \,.\, W'$ denotes the product as formed by ordinary multiplication, and $W * W'$ denotes the result of operating with W on the coefficients of W'. The expression for W is

$$W = \Delta_0^3(u_1y)\frac{\partial}{\partial u_3} + \Delta_0^4(u_1y)\frac{\partial}{\partial u_4} + \Delta_0^5(u_1y)\frac{\partial}{\partial u_5} + \dots$$
$$+ \Delta_0^3(yy_1)\frac{\partial}{\partial y_3} + \Delta_0^4(yy_1)\frac{\partial}{\partial y_4} + \Delta_0^5(yy_1)\frac{\partial}{\partial y_5} + \dots.$$

Hence the coefficient of $\frac{\partial}{\partial u_r}$ in $W * W'$ is

$$\left[\Delta_0^3(u_1y)\frac{\partial}{\partial u_3} + \Delta_0^4(u_1y)\frac{\partial}{\partial u_4} + \Delta_0^5(u_1y)\frac{\partial}{\partial u_5} + \dots\right]\Delta_0^r(uu_1)$$
$$= (r+1)_3\Delta_0^3(u_1y)\,u_{r-2} + (r+1)_4\Delta_0^4(u_1y)\,u_{r-3} + (r+1)_5\Delta_0^5(u_1y)\,u_{r-4} + \dots,$$

by (18) of § 3,

$$= \Delta_0^{r+1}(u_1y\,.\,u) = \Delta_0^{r+1}(yuu_1).$$

Again, the coefficient of $\frac{\partial}{\partial y_r}$ in $W * W'$ is by (15) and (16), after interchange of u and y in these equations,

$$\left[\Delta_0^3(u_1y)\frac{\partial}{\partial u_3} + \dots + \Delta_0^3(yy_1)\frac{\partial}{\partial y_3} + \dots\right]\Delta_0^r(uy_1)$$
$$= \Delta_0^3(u_1y)\,.\,(r)_3\Delta_0^{r-3}y_1 + \Delta_0^4(u_1y)\,.\,(r)_4\Delta_0^{r-4}y_1 + \Delta_0^5(u_1y)\,.\,(r)_5\Delta_0^{r-5}y_1 + \dots$$
$$+ \Delta_0^3(yy_1)\,.\,(r)_2\Delta_0^{r-2}u + \Delta_0^4(yy_1)\,.\,(r)_3\Delta_0^{r-3}u + \Delta_0^5(yy_1)\,.\,(r)_4\Delta_0^{r-4}u + \dots.$$

The first line is equal to $\Delta_0^r(u_1yy_1)$.

In the second line writing $(r)_2 = (r+1)_3 - (r)_3$, &c. we obtain

$$(r+1)_3 \Delta_0^3 (yy_1) \Delta_0^{r-2} u + (r+1)_4 \Delta_0^4 (yy_1) \Delta_0^{r-3} u + \dots$$

$$- [(r)_3 \Delta_0^3 (yy_1) \Delta_0^{r-3} u_1 + (r)_4 \Delta_0^3 (yy_1) \Delta_0^{r-4} u_1 + \dots]$$

$$= \Delta_0^{r+1} (yy_1 . u) - \Delta_0^r (yy_1 . u_1).$$

The coefficient of $\frac{\partial}{\partial y_r}$ is therefore $\Delta_0^{r+1} (uyy_1)$.

Hence
$$WW' = W . W' + \Delta_0^5 (yuu_1) \frac{\partial}{\partial u_4} + \Delta_0^6 (yuu_1) \frac{\partial}{\partial u_5} + \dots$$

$$+ \Delta_0^5 (uyy_1) \frac{\partial}{\partial y_4} + \Delta_0^6 (uyy_1) \frac{\partial}{\partial y_5} + \dots \quad \dots\dots\dots(33).$$

This result is, I believe, new. Being symmetrical with respect to u and y it shows that $WW' = W'W$.

The transformations of

$$V \frac{d}{dx} - \frac{d}{dx} V, \quad W \frac{d}{dx} - \frac{d}{dx} W,$$

and the other operators of Mr Leudesdorf's paper can be effected in the same way.

§ 10. In this section we consider the operator $(\mu, \nu; m, n)$ discussed by Major MacMahon (*Proc. Lond. Math. Soc.*, vol. XVIII., p. 61). By definition

$$(\mu, \nu; m, n) = \Sigma_{s=0}^{s=\infty} (\mu + s\nu) A_{s, m} \frac{\partial}{\partial a_{n+s}},$$

where
$$A_{s, m} = \Sigma \frac{(m-1)!}{\kappa_0! \, \kappa_1! \, \kappa_2! \dots} a_0^{\kappa_0} a_1^{\kappa_1} a_2^{\kappa_2} \dots,$$

and summation extends to all terms of degree m and weight s. If we write

$$a_0 = y, \quad a_1 = y_1/1!, \quad a_2 = y_2/2!, \dots$$

$$A_{s, m} = \frac{1}{m} \frac{D^s y^m}{s!},$$

and
$$(\mu, \nu; m, n) = \frac{1}{m} \Sigma_{s=0}^{s=\infty} (\mu + s\nu)(n+s)! \frac{D^s y^m}{s!} \frac{\partial}{\partial y_{n+s}}.$$

Now the product of two such operators

$$(\mu', \nu'; m', n')(\mu, \nu; m, n)$$

consists of two parts, one formed by ordinary multiplication, the other by operating with $(\mu', \nu'; m', n')$ on the coefficients of $(\mu, \nu; m, n)$. The latter is denoted by

$$(\mu', \nu'; m', n') * (\mu, \nu; m, n).$$

The coefficient of $(\mu + s\nu)\dfrac{\partial}{\partial a_{n+s}}$ in the last expression is

$$\frac{1}{mm'}\left[\Sigma_{s'=0}^{s'=\infty}(\mu' + s'\nu')(n' + s')!\,\frac{D^{s'}y^{m'}}{s'!}\,\frac{\partial}{\partial y_{n'+s'}}\right]\frac{D^{s}y^{m}}{s!}$$

$$= \frac{1}{m'}\,\Sigma_{s'=0}^{s'=s-n'}(\mu' + s'\nu')\,\frac{D^{s'}y^{m'}}{s'!}\,\frac{D^{s-n-s'}y^{m-1}}{(s-n'-s')!}.$$

The coefficient of μ' is, by Leibnitz' Theorem,

$$\frac{1}{m'}\,\frac{D^{s-n'}y^{m+m'-1}}{(s-n')!} = \frac{m+m'-1}{m'}\,A_{s-n',\ m+m'-1}.$$

The coefficient of ν' is

$$\Sigma_{s'=1}^{s'=s-n'}\frac{D^{s'-1}(y^{m'-1}y_1)}{(s'-1)!}\,\frac{D^{s-n'-s'}y^{m-1}}{(s-n'-s')!}$$

$$= \frac{D^{s-n'-1}(y^{m+m'-2}y_1)}{(s-n'-1)!}$$

$$= \frac{1}{m+m'-1}\,\frac{D^{s-n'}y^{m+m'-1}}{(s-n'-1)!}$$

$$= (s-n')\,A_{s-n',\ m+m'-1}.$$

The coefficient sought is therefore

$$\left\{\frac{m+m'-1}{m'}\mu' + (s-n')\nu'\right\}A_{s-n',\ m+m'-1}.$$

Hence $\quad(\mu',\ \nu';\ m',\ n') * (\mu,\ \nu;\ m,\ n)$

$$= \Sigma_{s=n'}^{s=\infty}\left\{\frac{m+m'-1}{m'}\mu' + (s-n')\nu'\right\}(\mu + s\nu)\,A_{s-n',\ m+m'-1}\frac{\partial}{\partial a_{n+s}}.$$

This is the fundamental result of Major MacMahon's paper.

§ 11. The formula (20) can be established more directly from (7) than in § 5 by the following process.

Equation (7) may be written

$$\frac{d}{dx}\Delta^{n}y^{r} = \Delta^{n+1}y^{r} - nry_2\Delta^{n-1}y^{r-1},$$

which may be formally expressed as

$$\left(\Delta - y_2\frac{\partial}{\partial\Delta}\frac{\partial}{\partial y}\right)\Delta^{n}y^{r}.$$

Hence, if f denotes any integral function,

$$\frac{d}{dx}f(\Delta,\ y) = \left(\Delta - y_2\frac{\partial}{\partial\Delta}\frac{\partial}{\partial y}\right)f(\Delta,\ y).$$

Again,
$$\frac{d}{dx}\frac{\Delta^n y^r}{y_1^{n+1}} = \frac{1}{y_1^{n-1}}\left(\Delta - y_2\frac{\partial}{\partial\Delta}\frac{\partial}{\partial y} - \frac{n+1}{y_1}y_2\right)\Delta^n y^r$$
$$= \left[\Delta - y_2\frac{\partial}{\partial\Delta}\left(\frac{\partial}{\partial y} + \frac{\Delta}{y_1}\right)\right]\frac{\Delta^n y^r}{y_1^{n+1}}.$$

Therefore
$$\frac{d}{dy}\frac{\Delta^n y^r}{y_1^{n+1}} = \left[\frac{\Delta}{y_1} - \frac{y_2}{y_1}\frac{\partial}{\partial\Delta}\left(\frac{\partial}{\partial y} + \frac{\Delta}{y_1}\right)\right]\frac{\Delta^n y^r}{y_1^{n+1}};$$

and if m is any positive integer and f an integral function,
$$\frac{d}{dy}\left[\frac{\Delta^m}{y_1^{m+1}}f\left(\frac{\Delta y}{y_1}\right)\right] = \left[\frac{\Delta}{y_1} - \frac{y_2}{y_1}\frac{\partial}{\partial\Delta}\left(\frac{\partial}{\partial y} + \frac{\Delta}{y_1}\right)\right]\frac{\Delta^m}{y_1^{m+1}}f\left(\frac{\Delta y}{y_1}\right).$$

Now choose f so that
$$\left(\frac{\partial}{\partial y} + \frac{\Delta}{y_1}\right)f\left(\frac{\Delta y}{y_1}\right) = 0,$$

and therefore
$$f\left(\frac{\Delta y}{y_1}\right) = e^{-\frac{\Delta y}{y_1}}.$$

The last formula then gives
$$\frac{d}{dy}\left(\frac{\Delta^m}{y_1^{m+1}}e^{-\frac{\Delta y}{y_1}}\right) = \frac{\Delta^{m+1}}{y_1^{m+2}}e^{-\frac{\Delta y}{y_1}}.$$

Now
$$\frac{dx}{dy} = \frac{1}{y_1} = \frac{1}{y_1}e^{-\frac{\Delta y}{y_1}};$$

therefore
$$\frac{d^2x}{dy^2} = \frac{\Delta}{y_1^2}e^{-\frac{\Delta y}{y_1}},$$

and
$$\frac{d^nx}{dy^n} = \frac{\Delta^{n-1}}{y_1^n}e^{-\frac{\Delta y}{y_1}} \quad\text{...(34)}$$
$$= \left(-\frac{\partial}{\partial y}\right)^n \Delta^{-1}e^{-\frac{\Delta y}{y_1}} \quad\text{.................................(35),}$$

which leads to the expanded form
$$\frac{d^nx}{dy^n} = \frac{\Delta^{n-1}}{y_1^n}\left[1 - \frac{\Delta y}{y_1} + \frac{\Delta^2y^2}{2!\,y_1^2} - \frac{\Delta^3y^3}{3!\,y_1^3} + \dots\right]$$
$$= -\frac{1}{y_1^{n+1}}\Delta^n y + \frac{1}{y_1^{n+2}}\frac{\Delta^{n+1}y^2}{2!} - \frac{1}{y_1^{n+3}}\frac{\Delta^{n+2}y^3}{3!} + \dots,$$

which is equation (20). It is to be noted that since $\Delta^n y^r = 0$ when $n < 2r$, the last term of the series will be
$$(-1)^{n-1}\frac{1}{y_1^{2n-1}}\frac{\Delta^{2n-2}y^{n-1}}{(n-1)!}.$$

From (34) it follows that
$$f\left(\frac{d}{dy}\right)x = f\left(\frac{\Delta}{y_1}\right).\,\Delta^{-1}e^{-\frac{\Delta y}{y_1}}$$
$$= f\left(-\frac{\partial}{\partial y}\right).\,\Delta^{-1}e^{-\frac{\Delta y}{y_1}}.$$

The formula (24) for $\frac{d^n u}{dy^n}$ may be obtained in precisely the same way. In symbolical form it will be found that

$$\frac{d^n u}{dy^n} = \frac{\Delta^{n-1}}{y_1^n} e^{-\frac{\Delta y}{y_1}} u_1 = \left(-\frac{\partial}{\partial y}\right)^n . \Delta^{-1} e^{-\frac{\Delta y}{y_1}} u_1,$$

and

$$f\left(\frac{d}{dy}\right) u = f\left(\frac{\Delta}{y_1}\right) . \Delta^{-1} e^{-\frac{\Delta y}{y_1}} u$$

$$= f\left(-\frac{\partial}{\partial y}\right) . \Delta^{-1} e^{-\frac{\Delta y}{y_1}} u_1.$$

§ 12. The connexion of these results with Herr Schlömilch's form will be shown by deducing the formula (21) from his results. In his notation [*Compendium der höheren Analysis*, Bd. II., pp. 19, 20, equations (35) and (36)],

$$X^n_{n-r} = (n-1)_r P_r$$

$$= -\frac{n!}{r!\,(n-r-1)!} \frac{(n+r)_r}{y_1^n}$$

$$\times \left\{ \frac{(r)_1}{n+1} \frac{1}{y_1} \frac{D^{r+1} y}{r+1} - \frac{(r)_2}{n+2} \frac{1}{y_1^2} \frac{[D^{r+2} y^2]_0}{(r+1)(r+2)} \right.$$

$$\left. + \frac{(r)_3}{n+3} \frac{1}{y_1^3} \frac{[D^{r+3} y^3]_0}{(r+1)(r+2)(r+3)} - \dots \text{ to } r \text{ terms} \right\}.$$

Expanding the functions $[D^{r+2} y^2]_0$, &c., by formula (19), we find that the coefficient of $\frac{1}{y_1^p}$ in the bracket is equal to $(-1)^{p-1} \Delta^{r+p} y^p$ multiplied by

$$\frac{(r)_p}{n+p} \frac{1}{(r+1)(r+2)\dots(r+p)} - \frac{(r)_{p+1}}{n+p+1} \frac{(r+p+1)(p+1)}{(r+1)\dots(r+p+1)} \frac{1}{1!}$$

$$+ \frac{(r)_{p+2}}{n+p+2} \frac{(r+p+2)(r+p+1)(p+2)(p+1)}{(r+1)(r+2)\dots(r+p+2)} \frac{1}{2!} - \dots$$

$$= \frac{1}{p!\,(r+1)\dots(r+p)} \left[\frac{(r)_p}{n+p} 1 . 2 \dots p - \frac{(r)_{p+1}}{n+p+1} 2 . 3 \dots p+1 \right.$$

$$\left. + \frac{(r)_{p+2}}{n+p+2} 3 . 4 \dots p+2 - \dots \right].$$

Now $$1 - (r)_1 x + (r)_2 x^2 + \dots = (1-x)^r.$$

Differentiate p times; therefore

$$(-1)^p [(r)_p 1 . 2 . 3 \dots p - (r)_{p+1} 2 . 3 \dots (p+1) . x + (r)_{p+2} 3 . 4 \dots (p+2) x^2 - \dots]$$

$$= (-1)^p r (r-1) \dots (r-p+1) (1-x)^{r-p}.$$

Multiply by x^{n+p-1}, and integrate between limits 0 and 1 for x; therefore

$$\frac{(r)_p}{n+p} 1 . 2 . 3 \ldots p - \frac{(r)_{p+1}}{n+p+1} 2 . 3 \ldots (p+1) + \ldots$$

$$= r(r-1) \ldots (r-p+1) \int_0^1 x^{n+p-1} (1-x)^{r-p} \, dx$$

$$= \frac{r! \, (n+p-1)!}{(n+r)!}.$$

Hence the coefficient of $\frac{1}{{y_1}^{n+p}}$ in X^n_{n-r} is

$$(-1)^p \Delta^{r+p} y^p \frac{n! \, (n+r)!}{r! \, (n-1-r)! \, r! \, n!} \frac{r!}{p! \, (r+p)!} \cdot \frac{r! \, (n+p-1)!}{(n+r)!}$$

$$= (-1)^p (n+p-1)_{n-r-1} \frac{\Delta^{r+p} y^p}{p!},$$

which is the same as the coefficient in X^n_{n-r} as given by (25).

Tides, on the 'equilibrium theory.' By C. Chree, Sc.D.

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SECTION II. CORE AND LAYER OF DIFFERENT HOMOGENEOUS SOLIDS.

SECTION I.

Homogeneous Solid Core and Ocean.

§ 1. The first problem treated here is the influence of disturbing forces from an external source acting on a non-rotating "Earth," which consists of a homogeneous isotropic solid core and a completely enveloping liquid ocean. The forces arise from a potential represented by a single term which involves a surface harmonic of degree i. When $i = 2$ the problem becomes that of the equilibrium theory of the tides. This problem is dealt with in Thomson and Tait's *Natural Philosophy**, but not I think altogether satisfactorily. It is doubtful whether Thomson and Tait absolutely limited their conclusions to the case when the solid is incompressible, but Professor Karl Pearson† in his discussion of Lord Kelvin's researches in Elasticity shows that the elastic solid part of their work is satisfactory only on this limitation. Though a great limitation mathematically, this is seemingly unimportant so far as concerns numerical estimates of tides on the actual earth. Further, the problem, *as presented by Thomson and Tait,* has been solved by Professor Pearson‡ himself without any assumption as to the compressibility of the solid.

* Art. 842.

† Todhunter and Pearson's *History of... Elasticity...*, Vol. II., Part II., Art. 1724.

‡ Todhunter and Pearson's *History of... Elasticity...*, Vol. II., Art. 1723 et seq.

The problem discussed here is more general than that solved by Lord Kelvin or Professor Pearson; but the chief occasion for the present work is that Thomson and Tait's presentation of the tidal problem seems to possess two distinct defects. Of the first the authors were fully conscious, they "neglect the mutual attraction of the waters." In their Art. 815 they had calculated for the case of a *rigid* core the influence of the gravitational action of the ocean itself on the height of the tide, and in Art. 817 they speak of this as a correction of the order of 10 per cent. which may be neglected owing to the numerous uncertainties prevailing in the problem as presented by nature. Presumably in treating the elastic solid "earth" they took the same view of the uncertainties, and did not think it necessary to make the calculations requisite to allow for the liquid's gravitation.

The second defect, though somewhat more important, has I think hitherto escaped detection. It is simply that the tidal ellipticities in the ocean and solid core being different, the liquid pressure on the surface of the core is not uniform and must be taken into account. This conclusion is obvious enough, when pointed out, but I was led to it by no *a priori* considerations, but from having to assure myself that a somewhat conspicuous discrepancy between the result I obtained by a straightforward analytical treatment and the result built up by Thomson and Tait was not due to error on my part.

§ 2. Let $\rho+\rho'$ and ρ be the respective densities of the solid core and ocean, m and n the elastic constants of the core in Thomson and Tait's notation.

It is supposed that in the absence of the disturbing forces the surfaces of the core and ocean would be spherical—though this is merely for brevity—and that the liquid completely covers the solid.

If then the potential of the disturbing forces be represented by

$$r^i V_i' \sigma_i,$$

where σ_i is a surface harmonic of integral degree i, and V_i' a constant, the equations to the equilibrium forms of the common surface of the core and liquid and the outer liquid surface will be respectively

$$r = b + b_i\sigma_i \qquad\qquad (1),$$

$$r = a + a_i\sigma_i \qquad\qquad (2).$$

Here b_i/b and a_i/a are very small, and their squares and product will be neglected.

Under these conditions if V_1 be the potential in the core, V_2 in the liquid, we have

$$V_1 = 2\pi\rho a^2 + 2\pi\rho' b^2 - \tfrac{2}{3}\pi(\rho+\rho')r^2 + r^i V_i'\sigma_i + \frac{4\pi r^i\sigma_i}{2i+1}(\rho a^{-i+1}a_i + \rho' b^{-i+1}b_i) \qquad (3),$$

$$V_2 = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho r^2 + \tfrac{4}{3}\pi\rho'\frac{b^3}{r} + r^i V_i'\sigma_i + \frac{4\pi\sigma_i}{2i+1}(\rho r^i a^{-i+1}a_i + \rho' r^{-i-1}b^{i+2}b_i) \qquad (4).$$

Let u, v, w be the elastic displacements at the point r, θ, ϕ in the core, in the directions of the elements dr, $rd\theta$ and $r\sin\theta d\phi$ respectively; and let $\widehat{rr}$, &c. denote the stresses, in the notation of Todhunter and Pearson's *History of Elasticity*.

Also let

$$\left.\begin{aligned}\Delta &= \frac{du}{dr} + \frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dw}{d\phi},\\ \mathfrak{A} &= \frac{1}{r^2\sin\theta}\left(\frac{d}{d\theta}wr\sin\theta - \frac{d}{d\phi}vr\right),\\ \mathfrak{B} &= \frac{1}{\sin\theta}\left(\frac{du}{d\phi} - \frac{d}{dr}wr\sin\theta\right),\\ \mathfrak{C} &= \sin\theta\left(\frac{dvr}{dr} - \frac{du}{d\theta}\right)\end{aligned}\right\} \qquad (5).$$

Then the body-stress equations in the solid are

$$\left.\begin{aligned}(m+n)r^2\sin\theta\frac{d\Delta}{dr} - n\frac{d\mathfrak{C}}{d\theta} + n\frac{d\mathfrak{B}}{d\phi} &= -(\rho+\rho')r^2\sin\theta\frac{dV_1}{dr},\\ (m+n)\sin\theta\frac{d\Delta}{d\theta} - n\frac{d\mathfrak{A}}{d\phi} + n\frac{d\mathfrak{C}}{dr} &= -(\rho+\rho')\sin\theta\frac{dV_1}{d\theta},\\ (m+n)\operatorname{cosec}\theta\frac{d\Delta}{d\phi} - n\frac{d\mathfrak{B}}{dr} + n\frac{d\mathfrak{A}}{d\theta} &= -(\rho+\rho')\operatorname{cosec}\theta\frac{dV_1}{d\phi}\end{aligned}\right\} \qquad (6).$$

The equations to be satisfied at the surface of the core are

$$\widehat{rr} - (b_i/b)\frac{d\sigma_i}{d\theta}\widehat{r\theta} - (b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{r\phi} = -p \qquad (7),$$

$$\widehat{r\theta} - (b_i/b)\frac{d\sigma_i}{d\theta}\widehat{\theta\theta} - (b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{\theta\phi} = p(b_i/b)\frac{d\sigma_i}{d\theta} \qquad (8),$$

$$\widehat{r\phi} - (b_i/b)\frac{d\sigma_i}{d\theta}\widehat{\theta\phi} - (b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{\phi\phi} = p(b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi} \qquad (9),$$

where p is the pressure at the common surface of the core and ocean.

It has been tacitly assumed that the equilibrium (not the undisturbed) surface is that where the surface equations apply.

The body-stress equations are satisfied by

$$u = \tfrac{1}{3}rY_0 + \tfrac{2}{15}\frac{\pi(\rho+\rho')^2r^3}{m+n} + r^{i-1}Z_i\sigma_i$$

$$-\frac{r^{i+1}\sigma_i}{2(2i+3)}\left\{\frac{(i+2)(\rho+\rho')}{m+n}V_i' + \frac{4\pi(i+2)(\rho+\rho')}{2i+1}\frac{(\rho a^{-i+1}a_i+\rho' b^{-i+1}b_i)}{m+n} + \frac{im-2n}{n}Y_i\right\} \ldots (10),$$

$$v\Big/\frac{d\sigma_i}{d\theta} = w\Big/\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}$$

$$= \frac{1}{i}r^{i-1}Z_i - \frac{r^{i+1}}{2(2i+3)}\left\{\frac{(\rho+\rho')V_i'}{m+n} + \frac{4\pi(\rho+\rho')(\rho a^{-i+1}a_i+\rho' b^{-i+1}b_i)}{(2i+1)(m+n)} + \frac{(i+3)m+2n}{(i+1)n}Y_i\right\} \ldots (11);$$

where Y_0, Y_i and Z_i are arbitrary constants to be determined from the surface conditions. Of these constants two, Y_i and Z_i, are of the order a_i/a or b_i/b.

§ 3. As a preliminary to determining the constants, we require the formulae for the stresses, and the value of p.

The stresses are as follows:

$$\widehat{rr} = \tfrac{1}{3}(3m-n)\,Y_0 + \tfrac{2}{15}\pi(\rho+\rho')^2\frac{5m+n}{m+n}r^2 + 2(i-1)\,nr^{i-2}Z_i\sigma_i$$

$$-\frac{r^i\sigma_i}{2i+3}\left[\frac{(2i+3)\,m+(i^2+i-1)\,n}{m+n}(\rho+\rho')\left\{V_i' + \frac{4\pi(\rho a^{-i+1}a_i+\rho' b^{-i+1}b_i)}{2i+1}\right\}\right.$$

$$\left. + \{(i^2-i-3)\,m+n\}\,Y_i\right] \quad\text{......(12)},$$

$$\widehat{r\theta} \Big/ \frac{d\sigma_i}{d\theta} = \widehat{r\phi} \Big/ \operatorname{cosec}\theta\,\frac{d\sigma_i}{d\phi} = \frac{2(i-1)}{i}\,nr^{i-2}Z_i$$

$$-\frac{r^i}{2i+3}\left[\frac{(i+1)\,n(\rho+\rho')}{m+n}\left\{V_i' + \frac{4\pi}{2i+1}(\rho a^{-i+1}a_i+\rho' b^{-i+1}b_i)\right\} + \frac{i(i+2)\,m-n}{i+1}\,Y_i\right] \text{...(13)},$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \tfrac{1}{3}(3m-n)\,Y_0 + \tfrac{2}{15}\pi(\rho+\rho')^2\frac{5m-3n}{m+n}r^2 + \text{terms of order } a_i/a \text{ ....(14)},$$

$$\widehat{\theta\phi} = \text{terms of order } a_i/a \text{ .... (15)}.$$

In the surface equations $\widehat{\theta\theta}$, $\widehat{\theta\phi}$ and $\widehat{\phi\phi}$ occur multiplied by b_i/b, so that we require only the terms appearing in (14) and (15).

We have next to find the value of p at the surface of the core.

The hydrostatical equations in the liquid require

$$p = \rho V_2 + C \text{..(16)},$$

where C is a constant. At the surface (2) p must vanish exactly, and so the constant terms and the terms containing σ_i must vanish separately. We have thus

$$C = -\tfrac{4}{3}\pi(\rho a^2+\rho' b^3/a) \text{...(17)},$$

$$0 = a^iV_i' - \tfrac{4}{3}\pi a\,a_i(\rho+\rho' b^3a^{-3}) + \frac{4\pi a}{2i+1}\{\rho a_i+\rho'(b/a)^{i+2}b_i\} \text{.................(18)}.$$

Employing in (16) the value of C supplied by (17), and the value of V_2 obtained by writing $b+b_i\sigma_i$ for r in (4), we have the required value of p.

Depth of ocean small compared to "earth's" radius.

§ 4. Thus far no restriction has been put on the depth of the ocean. In the actual earth, however, the depth is very small compared to the radius, and our further attention will be limited to the case when $(a-b)/a$ is very small.

In this case (18) becomes

$$0 = a^iV_i' - \tfrac{4}{3}\pi(\rho+\rho')\,a a_i + \frac{4\pi a}{2i+1}(\rho a_i+\rho' b_i) \text{...............................(19)},$$

and the liquid pressure at the surface of the core is given by

$$p = \tfrac{4}{3}\pi\rho(\rho+\rho')\{(a_i - b_i)\sigma_i + a - b\} \quad \ldots\ldots (20).$$

For our present purpose the part of p independent of σ_i may be omitted. It would merely add to the value presently found for Y_0 a term of the order $(a-b)/a$, which would be negligible in (28), the only equation which depends on the value of Y_0.

Writing $b + b_i\sigma_i$ for r in the expressions for the stresses, substituting in (7), (8) and (9), and employing the value of p given by (20), we obtain three equations determining Y_0, Y_i and Z_i. Of these equations one comes from the constant terms and one from the terms containing σ_i in (7), while the third comes from either (8) or (9). Terms of order $a-b$ being neglected when terms of order a exist, these equations give

$$Y_0 = -\tfrac{2}{5}\pi(\rho+\rho')^2 a^2 (5m+n) \div \{(m+n)(3m-n)\} \quad \ldots\ldots (21),$$

$$2(i-1)na^{i-2}Z_i - \frac{(i^2-i-3)m+n}{2i+3}a^iY_i = P \quad \ldots\ldots (22),$$

$$\frac{2(i-1)}{i}na^{i-2}Z_i - \frac{i(i+2)m-n}{(i+1)(2i+3)}a^iY_i = Q \quad \ldots\ldots (23),$$

where for shortness

$$\begin{aligned} P \equiv\ & \frac{(2i+3)m+(i^2+i-1)n}{(2i+3)(m+n)}(\rho+\rho')a^iV_i' \\ & - \frac{4\pi\rho(\rho+\rho')a(a_i-b_i)}{3(2i+1)(2i+3)(m+n)}\{2(i-1)(2i+3)m+(i+2)(i+3)n\} \\ & - \frac{4\pi(\rho+\rho')^2(i-1)ab_i}{15(2i+1)(2i+3)(m+n)}\{10(2i+3)m-(11i+18)n\} \quad \ldots\ldots (24), \end{aligned}$$

$$\begin{aligned} Q = \ & \frac{i+1}{2i+3}\frac{n}{m+n}(\rho+\rho')\left\{a^iV_i' + \frac{4\pi\rho a(a_i-b_i)}{2i+1}\right\} \\ & - \frac{4\pi(\rho+\rho')^2(i-1)(8i+9)nab_i}{15(2i+1)(2i+3)(m+n)} \quad \ldots\ldots (25) \end{aligned}$$

The solution of (22) and (23) is

$$Y_i = (i+1)(2i+3)a^{-i}(P-iQ) \div \{(2i^2+4i+3)m-(2i+1)n\} \quad \ldots\ldots (26),$$

$$Z_i = \frac{ia^{-i+2}[\{i(i+2)m-n\}P-(i+1)\{(i^2-i-3)m+n\}Q]}{2(i-1)n\{(2i^2+4i+3)m-(2i+1)n\}} \quad \ldots\ldots (27).$$

§ 5. By (21), Y_0 is determined explicitly to the required degree of approximation; but (26) and (27) do not yet give Y_i and Z_i explicitly because the values (24) and (25) of P and Q contain the still unknown quantities $a_i - b_i$ and b_i.

To determine $a_i - b_i$ and b_i we have as yet only the one equation (19). A second is easily got as follows:

By hypothesis the undisturbed surface of the core is

$r = \bar{b}$, where $\bar{b}$ is a constant.

Thus the equation to the equilibrium surface when the disturbing forces act is

$$r = \bar{b} + u \text{ (with } r = b + b_i\sigma_i),$$

but it is also

$$r = b + b_i\sigma_i;$$

consequently b_i = coefficient of σ_i in the value of u when for r we write $b + b_i\sigma_i$.

We thus find

$$b_i = \tfrac{1}{3} b_i Y_0 + \tfrac{2}{15} \frac{\pi(\rho+\rho')^2 3b^2 b_i}{m+n} - \frac{(i+2)(\rho+\rho')b^{i+1}V_i'}{2(2i+3)(m+n)}$$
$$- \frac{2\pi(i+2)(\rho+\rho')b^{i+1}}{(2i+1)(2i+3)} \frac{\rho a^{-i+1}a_i + \rho' b^{-i+1} b_i}{m+n}$$
$$- \frac{im-2n}{2(2i+3)n} \overline{b^{i+1}Y_i + b^{i-1}Z_i} \quad \text{.....................................(28).}$$

Neglecting $(a-b)/a$ as before, substituting for Y_0 from (21), and combining terms, we convert (28) into

$$b_i = -\frac{(i+2)(\rho+\rho')a^{i+1}V_i'}{2(2i+3)(m+n)} - \frac{4\pi\rho(\rho+\rho')(i+2)a^2(a_i-b_i)}{2(2i+1)(2i+3)(m+n)}$$
$$+ 2\pi(\rho+\rho')^2 a^2 b_i \frac{(16i^2-13i-78)m - (16i^2+17i-18)n}{15(2i+1)(2i+3)(m+n)(3m-n)}$$
$$- \frac{im-2n}{2(2i+3)n} \overline{a^{i+1}Y_i + a^{i-1}Z_i} \quad \text{.....................................(29).}$$

Again from (24), (25), (26) and (27) we find

$$- \frac{im-2n}{2(2i+3)n} \overline{a^{i+1}Y_i + a^{i-1}Z_i}$$
$$= \frac{a^{i+1}(\rho+\rho')V_i'\{i(2i+1)(2i+3)m^2 + (2i^4+10i^3+9i^2-5i-6)mn - (2i^3+5i^2-2)n^2\}}{2(i-1)(2i+3)n(m+n)\{(2i^2+4i+3)m-(2i+1)n\}}$$
$$- \frac{2\pi\rho(\rho+\rho')a^2(a_i-b_i)\{2i(2i+1)(2i+3)m^2 + (10i^3+12i^2-11i-12)mn + (8i^3+26i^2+31i+12)n^2\}}{3(2i+1)(2i+3)n(m+n)\{(2i^2+4i+3)m-(2i+1)n\}}$$
$$- \frac{2\pi(\rho+\rho')^2 a^2 b_i \{10i(2i+1)(2i+3)m^2 + (42i^3+44i^2-61i-60)mn - (16i^4+48i^3+26i^2-49i-36)n^2\}}{15(2i+1)(2i+3)n(m+n)\{(2i^2+4i+3)m-(2i+1)n\}}$$
$$\text{............(30).}$$

Substituting in (29) and reducing, we finally obtain

$$b_i = \frac{a^{i+1}(\rho+\rho')iV_i'\{(2i+1)m-n\}}{2(i-1)n\{(2i^2+4i+3)m-(2i+1)n\}}$$
$$- \frac{4\pi\rho(\rho+\rho')a^2(a_i-b_i)\{i(2i+1)m+(2i^2+2i+1)n\}}{3(2i+1)n\{(2i^2+4i+3)m-(2i+1)n\}}$$
$$- \frac{4\pi(\rho+\rho')^2 a^2 b_i\{15i(2i+1)m^2 - (8i^3+6i^2-2i-9)mn + (4i^3-2i^2-3i-3)n^2\}}{15(2i+1)n(3m-n)\{(2i^2+4i+3)m-(2i+1)n\}} \quad \text{......(31).}$$

This is to be taken with (19), thrown most conveniently into the form

$$a^{i+1}V_i' - 4\pi a^2 (a_i - b_i) \left\{\tfrac{1}{3}(\rho+\rho') - \frac{1}{2i+1}\rho\right\} - \tfrac{4}{3}\pi(\rho+\rho')a^2 \frac{2(i-1)}{2i+1} b_i = 0 \quad \ldots\ldots\ldots (32).$$

From (31) and (32) we find

$$(a_i - b_i)\tfrac{4}{3}\pi(\rho+\rho')a^2 \div \left[1 - \tfrac{4}{3}\pi(\rho+\rho')^2 a^2 \frac{(4i^2-4i-9)m - (2i^2-2i-3)n}{5(3m-n)\{(2i^2+4i+3)m-(2i+1)n\}}\right]$$

$$= b_i \div \left[\frac{(\rho+\rho')i\{(2i+1)m-n\} - \rho\{i(2i+1)m+(2i^2-i-2)n\}}{2(i-1)n\{(2i^2+4i+3)m-(2i+1)n\}}\right]$$

$$= a^{i+1}V_i' \div \left[1 - \frac{3}{2i+1}\frac{\rho}{\rho+\rho'} + \left\{\frac{4\pi(\rho+\rho')a^2}{15(2i+1)n(3m-n)\big((2i^2+4i+3)m-(2i+1)n\big)}\right\}\right.$$

$$\times\{(\rho+\rho')\big(15i(2i+1)m^2 - (8i^3+6i^2-2i-9)mn + (4i^3-2i^2-3i-3)n^2\big)$$

$$\left. - \rho\big(15i(2i+1)m^2 + (8i^2-8i-3)mn - (4i^2+i-1)n^2\big)\}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

If the material though of finite rigidity be incompressible, we have n finite but m infinite, and so

$$(a_i - b_i)\tfrac{4}{3}\pi(\rho+\rho')a^2 = b_i \frac{2(i-1)(2i^2+4i+3)n}{i(2i+1)\rho'}$$

$$= \frac{a^{i+1}V_i'}{1 - \dfrac{3}{2i+1}\dfrac{\rho}{\rho+\rho'} + \tfrac{4}{3}\dfrac{\pi(\rho+\rho')\rho' a^2 i}{n(2i^2+4i+3)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

Case $i=2$, luni-solar tides.

§ 6. In the case of most physical interest, when the disturbing forces are due to the action of the moon or sun on the earth, $i=2$. Also if M be the mass, R the distance of the disturbing body, E the earth's mass, and g "gravity" at the earth's surface (neglecting "centrifugal force"),

$$V_2' = g(M/E)(a^2/R^3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

to the present degree of approximation.

Consistently with our previous work, which neglects $(a-b)/a$, we may put

$$\tfrac{4}{3}\pi(\rho+\rho')a = g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36).$$

Thus for the lunar or solar tides we get for the general case of isotropy from (33)

$$(a_2 - b_2) \div \left\{1 + \frac{g(\rho+\rho')a(m+n)}{5(3m-n)(19m-5n)}\right\}$$

$$= b_2 n(19m-5n) \div \left\{g(\rho+\rho')a\left(5m - n - \frac{\rho}{\rho+\rho'}(5m+2n)\right)\right\}$$

$$= \frac{a(M/E)(a/R)^3}{1 - \tfrac{3}{5}\dfrac{\rho}{\rho+\rho'} + \dfrac{3g(\rho+\rho')a\left\{10m^2 - 5mn + n^2 - \tfrac{1}{15}\dfrac{\rho}{\rho+\rho'}(150m^2+13mn-17n^2)\right\}}{5n(3m-n)(19m-5n)}} \quad \ldots\ldots (37).$$

When the core is incompressible, this becomes

$$a_2 - b_2 = \frac{19n}{5g\rho' a} b_2 = \frac{a\,(M/E)\,(a/R)^3}{1 - \frac{3}{5}\frac{\rho}{\rho + \rho'} + \frac{2g\rho' a}{19n}} \quad \text{.........................(38).}$$

The equations to the equilibrium surfaces of the liquid and solid are respectively

$$r = a + a_2 (3\cos^2\theta - 1)/2,$$
$$r = b + b_2 (3\cos^2\theta - 1)/2,$$

the disturbing body being in the direction $\theta = 0$.

Thus the extreme height of the *apparent* ocean tide (high to low water) is

$$3\,(a_2 - b_2)/2,$$

and the extreme height of the true solid tide is

$$3b_2/2.$$

If in (37) or (38) we suppose the solid rigid, i.e. of infinitely large elastic constants, we have

$$b_2 = 0,$$
$$a_2 = a\,(M/E)\,(a/R)^3 \div \left(1 - \tfrac{3}{5}\frac{\rho}{\rho + \rho'}\right) \quad \text{.............................(39),}$$

agreeing with Thomson and Tait and the result (XII.) on p. 367 of Prof. H. Lamb's *Hydrodynamics*.

§ 7. The result found by Thomson and Tait in place of (38) is equivalent to

$$a_2 - b_2 = a\,(M/E)\,(a/R)^3 \div \{1 + 2g\bar{\rho}a/19n\},$$

where $\bar{\rho}$ = the earth's mean density

$= \rho + \rho'$ to the present degree of approximation.

The simplest way of stating the case is that Thomson and Tait's result neglects the density of the ocean relative to the mean density of the earth.

We have approximately

$$\rho'/4{\cdot}5 = \rho = (\rho + \rho')/5{\cdot}5,$$

and it will be found that Thomson and Tait's estimated height of the tide is about 12 °/。 too small when $g\bar{\rho}a/n$ is negligible, and about 22 °/。 too small when $g\bar{\rho}a/n$ is infinite. These are the two extremes for incompressible material.

In the earth we have approximately

$$g\bar{\rho}a = 35 \times 10^3 \text{ grammes wt. per sq. cm.}$$

The value to ascribe to n is largely hypothetical. If the accepted ellipticity of the earth be due to its rotation we have some reason to regard

$$n = 11 \times 10^7 \text{ grammes wt. per sq. cm.}$$

as an inferior limit to the rigidity.

Taking these values we should find Thomson and Tait's estimate nearly 20 °/。 too small.

§ 8. As no material can well be wholly incompressible, considerable interest attaches to the influence of a slight compressibility on the height of the apparent tide. This we find from (37) by retaining terms in n/m while neglecting those in $(n/m)^2$, &c. Thus we get

$$a_2 - b_2 = \frac{a\,(M/E)\,(a/R)^3}{1 - \frac{3}{5}\frac{\rho}{\rho+\rho'} + \frac{2g\rho' a}{19n}\left\{1 - \frac{3}{95}\frac{n}{m}\left(19\frac{\rho}{\rho'} - 2 + \frac{g(\rho+\rho')a}{9n}\right)\right\}} \quad \text{..............(40).}$$

In the true earth

$$19\frac{\rho}{\rho'} - 2 = \tfrac{20}{9} \text{ approx.,}$$

so that the coefficient of n/m in the denominator in (40) is necessarily negative. Thus the rigidity n being supposed constant, the apparent tide is greater for a slightly compressible than for a wholly incompressible earth. The difference is, however, extremely small under any probable contingency.

Thus take the figures suggested by seismological phenomena*

$$n/m = 1/24,$$

$$n = 35 \times 10^7 \text{ grammes wt. per sq. cm.;}$$

with

$$g(\rho+\rho')a = 35 \times 10^8 \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

$$\rho/\rho' = 2/9.$$

These data give

$$\tfrac{3}{95}\frac{n}{m}\left(19\frac{\rho}{\rho'} - 2 + \frac{g(\rho+\rho')a}{9n}\right) = \tfrac{1}{228},$$

and the corresponding increase in $a_2 - b_2$, relative to the value for absolute incompressibility, would be little over 1 part in 500.

Under the same conditions as in (40) we find

$$\frac{b_2}{a_2 - b_2} = \tfrac{5}{19}\frac{g\rho' a}{n}\left\{1 - \tfrac{3}{95}\frac{n}{m}\left(19\frac{\rho}{\rho'} - 2 + \frac{g(\rho+\rho')a}{9n}\right)\right\} \quad \text{.....................(41),}$$

showing that for a given rigidity the tide in the solid decreases slightly relative to the apparent ocean tide as the resistance to compression diminishes from an infinite value.

§ 9. The relative importance of the tide in the solid is, I think, not in general sufficiently appreciated, thus attention may be called to a few numerical results obtained for the case of the incompressible material.

Ascribing to $g\rho' a$ the value 285×10^7 grammes wt. per sq. cm., a close approximation in the case of the true earth, we have the following results:

n (in grammes wt. per sq. cm.) =	80×10^7	35×10^7	11×10^7
$b_2/(a_2 - b_2)$ =	·94	2·1	6·8

* See *Phil. Mag.*, March 1897, p. 200.

According to these figures the true tide in the solid earth may be very considerably larger and is not likely to be much less than the apparent tide in the ocean.

Case when i large.

§ 10. The general equation (33) is complicated unless numerical values be ascribed to the several quantities it contains. When, however, i is very large there is a simple first approximation, viz.

$$a^i V_i'/g = a_i - b_i = \frac{2im\,(b_i/a)}{g\,(\rho' m - \rho n)}\left\{1 - \frac{g\,(\rho + \rho')\,a\,(2m-n)}{5m\,(3m-n)}\right\} \qquad (42).$$

According to this, as i increases $(a_i - b_i)/a^i V_i'$ tends to a constant value, while b_i/a_i tends to vanish. It would thus appear, at least if inertia be neglected, that an external disturbing influence which is either local, or very variable with the angular coordinates, is likely to have a much larger tidal influence on the ocean than on the solid earth.

SECTION II.

CORE AND LAYER OF DIFFERENT HOMOGENEOUS SOLIDS.

§ 11. In treating the solid part of the earth as homogeneous, we make so large a departure from known facts that it seems worth while to try to form some idea of the influence of heterogeneity. The simplest heterogeneous solid consists of a core and enveloping layer, each homogeneous in itself, but differing the one from the other.

The second problem considered here is the influence on such a solid of disturbing forces from the potential

$$r^i V_i' \sigma_i,$$

the notation being as before.

The addition of an enveloping ocean would make the problem resemble more closely that presented by the earth, and so would enhance the physical interest of the results. At the same time the present discussion will be found, I think, to throw considerable light on the actual problem presented by the earth, and to be at the same time quite sufficiently complicated.

In the preliminary work the materials of the core and layer are supposed to be any two different elastic solids, and the surface conditions first presented are perfectly general. These equations are not, however, solved in their most general form. To do so would have entailed very laborious, though not intrinsically difficult, analysis, and in view of certain considerations which will be duly explained, I did not feel disposed myself to devote the necessary time.

§ 12. It will be supposed that in the absence of the disturbing forces both surfaces of the layer are truly spherical, and that under the action of these forces the equations to the outer and inner surfaces become respectively

$$r = a + a_i \sigma_i \qquad (1),$$

$$r = b + b_i \sigma_i \qquad (2).$$

Here a, a_i, b, b_i are constants, while a_i/a and b_i/b are so small their squares and product may be neglected.

The density is ρ in the layer, $\rho+\rho'$ in the core; the elastic constants m, n in the layer, m', n' in the core.

The potentials being V_1 in the core, V_2 in the layer, we have

$$V_1 = 2\pi(\rho a^2+\rho' b^2) - \tfrac{2}{3}\pi(\rho+\rho')r^2 + r^i V_i'\sigma_i + \frac{4\pi r^i\sigma_i}{2i+1}(\rho a^{-i+1}a_i + \rho' b^{-i+1}b_i) \quad\text{...........(3)},$$

$$V_2 = 2\pi\rho(a^2-\tfrac{1}{3}r^2) + \tfrac{4}{3}\pi\rho'\frac{b^3}{r} + r^i V_i'\sigma_i + \frac{4\pi\sigma_i}{2i+1}(\rho r^i a^{-i+1}a_i + \rho' r^{-i-1}b^{i+2}b_i) \quad\text{..............(4)}.$$

The body-stress equations in the core are obtained by writing m', n' for m, n in equations (6) of Sect. I.; while the corresponding equations in the layer require the substitution in these equations of ρ for $\rho+\rho'$ and V_2 for V_1.

The equations to be satisfied at the outer surface (1) are

$$\left.\begin{aligned} &\widehat{rr} - (a_i/a)\frac{d\sigma_i}{d\theta}\widehat{r\theta} - (a_i/a)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{r\phi} = 0,\\ &\widehat{r\theta} - (a_i/a)\frac{d\sigma_i}{d\theta}\widehat{\theta\theta} - (a_i/a)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{\theta\phi} = 0,\\ &\widehat{r\phi} - (a_i/a)\frac{d\sigma_i}{d\theta}\widehat{\theta\phi} - (a_i/a)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{\phi\phi} = 0 \end{aligned}\right\}\quad\text{........................(5)}.$$

At the common surface (2) of the core and layer there must be continuity in the values of the displacements u, v, w, and also of the stress components

$$\widehat{rr} - (b_i/b)\frac{d\sigma_i}{d\theta}\widehat{r\theta} - (b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{r\phi},$$

$$\widehat{r\theta} - (b_i/b)\frac{d\sigma_i}{d\theta}\widehat{\theta\theta} - (b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{\theta\phi}.$$

$$\widehat{r\phi} - (b_i/b)\frac{d\sigma_i}{d\theta}\widehat{\theta\phi} - (b_i/b)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi}\widehat{\phi\phi}.$$

Of the surface equations six occur in pairs, each pair furnishing only one independent equation. We have for instance a pair of equations of the type

$$f(a,\ m,\ n\ldots)\frac{d\sigma_i}{d\theta} = 0 = f(a,\ m,\ n)\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi},$$

holding all over a spherical surface, from which we obtain but the one equation

$$f(a,\ m,\ n\ldots) = 0.$$

§ 13. The body-stress equations in the core are satisfied by

$$u = \tfrac{1}{3}rY_0' + \tfrac{2}{15}\frac{\pi(\rho+\rho')^2 r^3}{m'+n'} + r^{i-1}Z_i'\sigma_i$$

$$-\frac{r^{i+1}\sigma_i}{2(2i+3)}\left\{\frac{(i+2)(\rho+\rho')}{m'+n'}V_i' + \frac{4\pi(i+2)(\rho+\rho')(\rho a^{-i+1}a_i+\rho' b^{-i+1}b_i)}{(2i+1)(m'+n')} + \frac{im'-2n'}{n'}Y_i'\right\}\quad\text{...(6)},$$

$$v \Big/ \frac{d\sigma_i}{d\theta} = w \Big/ \operatorname{cosec}\theta \, \frac{d\sigma_i}{d\phi}$$

$$= \frac{1}{i} r^{i-1} Z_i' - \frac{r^{i+1}}{2(2i+3)} \left\{ \frac{(\rho+\rho')}{m'+n'} V_i' + \frac{4\pi(\rho+\rho')(\rho a^{-i+1} a_i + \rho' b^{-i+1} b_i)}{(2i+1)(m'+n')} + \frac{(i+3)m'+2n'}{(i+1)n'} Y_i' \right\} \ldots\ldots (7),$$

where Y_0', Y_i', Z_i' are constants to be found from the surface conditions.

In like manner the body-stress equations in the layer are satisfied by

$$u = \tfrac{1}{3} r Y_0 + r^{-2} Z_{-1} + \tfrac{2}{15} \frac{\pi \rho^2 r^3}{m+n} - \tfrac{2}{3} \frac{\pi \rho \rho' b^3}{m+n} + r^{i-1} Z_i \sigma_i$$

$$- \frac{r^{i+1}\sigma_i}{2(2i+3)} \left\{ \frac{(i+2)\rho}{m+n} V_i' + \frac{4\pi (i+2) \rho^2 a^{-i+1} a_i}{(2i+1)(m+n)} + \frac{im-2n}{n} Y_i \right\}$$

$$- \frac{r^{-i}\sigma_i}{2(2i-1)} \left\{ \frac{4\pi (i-1) \rho\rho' b^{i+2} b_i}{(2i+1)(m+n)} + \frac{(i+1)m+2n}{n} Y_{-i-1} \right\} + r^{-i-2} Z_{-i-1} \sigma_i \ldots\ldots\ldots\ldots (8),$$

$$v \Big/ \frac{d\sigma_i}{d\theta} = w \Big/ \operatorname{cosec}\theta \, \frac{d\sigma_i}{d\phi}$$

$$= \frac{1}{i} r^{i-1} Z_i - \frac{r^{i+1}}{2(2i+3)} \left\{ \frac{\rho V_i'}{m+n} + \frac{4\pi\rho^2 a^{-i+1} a_i}{(2i+1)(m+n)} + \frac{(i+3)m+2n}{(i+1)n} Y_i \right\}$$

$$+ \frac{r^{-i}}{2(2i-1)} \left\{ \frac{4\pi\rho\rho' b^{i+2} b_i}{(2i+1)(m+n)} + \frac{(i-2)m-2n}{in} Y_{-i-1} \right\} - \frac{1}{i+1} r^{-i-2} Z_{-i-1} \ldots\ldots (9).$$

The terms containing σ_i or its differential coefficients are all of the order a_i/a or b_i/b. The terms independent of σ_i would alone exist if the disturbing forces were absent and the surfaces (1) and (2) truly spherical.

To the present degree of approximation we may neglect subsidiary terms when they have a multiplier a_i/a or b_i/b; hence in dealing with the surface equations we need consider only principal terms in $\widehat{\theta\theta}$, $\widehat{\phi\phi}$ or $\widehat{\theta\phi}$.

Thus the stresses in the core, *so far as required in the surface equations*, are given by

$$\widehat{rr} = \tfrac{1}{3}(3m'-n') Y_0' + \tfrac{2}{15}\pi(\rho+\rho')^2 \frac{5m'+n'}{m'+n'} r^2 + 2(i-1) n' r^{i-2} Z_i' \sigma_i$$

$$- \frac{r^i \sigma_i}{2i+3} \left[\frac{(2i+3)m' + (i^2+i-1)n'}{m'+n'} (\rho+\rho') \left\{ V_i' + 4\pi \frac{(\rho a^{-i+1} a_i + \rho' b^{-i+1} b_i)}{2i+1} \right\} \right.$$

$$\left. + \{(i^2-i-3)m' + n'\} Y_i' \right] \ldots\ldots\ldots\ldots (10),$$

$$\widehat{r\theta} \Big/ \frac{d\sigma_i}{d\theta} = \widehat{r\phi} \Big/ \operatorname{cosec}\theta \, \frac{d\sigma_i}{d\phi} = \frac{2(i-1)}{i} n' r^{i-2} Z_i'$$

$$- \frac{r^i}{2i+3} \left[\frac{(i+1) n' (\rho+\rho')}{m'+n'} \left\{ V_i' + \frac{4\pi (\rho a^{-i+1} a_i + \rho' b^{-i+1} b_i)}{2i+1} \right\} + \frac{i(i+2)m' - n'}{i+1} Y_i' \right] \ldots (11),$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \tfrac{1}{3}(3m'-n') Y_0' + \tfrac{2}{15}\pi(\rho+\rho')^2 \frac{5m'-3n'}{m'+n'} r^2 \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

$$\widehat{\theta\phi} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Similarly the stresses in the layer, *so far as required in the surface conditions*, are

$$\widehat{rr} = \tfrac{1}{3}(3m-n)Y_0 - 4nr^{-3}Z_{-1} + \tfrac{2}{15}\pi\rho^2\frac{(5m+n)}{m+n}r^2 - \tfrac{4}{3}\pi\rho\rho'\frac{m-n}{m+n}\frac{b^3}{r} + 2(i-1)nr^{i-2}Z_i\sigma_i$$

$$-\frac{r^i\sigma_i}{2i+3}\left[\frac{(2i+3)m+(i^2+i-1)n}{m+n}\rho\left(V_i'+\frac{4\pi\rho a^{-i+1}a_i}{2i+1}\right)+\{(i^2-i-3)m+n\}Y_i\right]$$

$$-\frac{r^{-i-1}\sigma_i}{2i-1}\left[\frac{(2i-1)m-(i^2+i-1)n}{m+n}\frac{4\pi\rho\rho' b^{i+2}b_i}{2i+1} - \{(i^2+3i-1)m+n\}Y_{-i-1}\right]$$

$$-2(i+2)nr^{-i-3}Z_{-i-1}\sigma_i \;\ldots\ldots(14),$$

$$\widehat{r\theta}\Big/\frac{d\sigma_i}{d\theta} = \widehat{r\phi}\Big/\operatorname{cosec}\theta\frac{d\sigma_i}{d\phi} = \frac{2(i-1)}{i}nr^{i-2}Z_i$$

$$-\frac{r^i}{2i+3}\left\{\frac{(i+1)n\rho}{m+n}\left(V_i'+\frac{4\pi\rho a^{-i+1}a_i}{2i+1}\right)+\frac{i(i+2)m-n}{i+1}Y_i\right\}$$

$$-\frac{r^{-i-1}}{2i-1}\left\{\frac{4\pi\rho\rho' inb^{i+2}b_i}{2i+1}+\frac{(i^2-1)m-n}{i}Y_{-i-1}\right\}+2\frac{i+2}{i+1}nr^{-i-3}Z_{-i-1}\;\ldots\ldots(15),$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \tfrac{1}{3}(3m-n)Y_0 + 2nr^{-3}Z_{-1} + \tfrac{2}{15}\pi\rho^2\frac{5m-3n}{m+n}r^2 - \tfrac{4}{3}\pi\rho\rho'\frac{m}{m+n}\frac{b^3}{r}\;\ldots\ldots(16),$$

$$\widehat{\theta\phi} = 0 \;\ldots\ldots(17).$$

§ 14. If the disturbing forces were absent and the surfaces (1) and (2) truly spherical, the only arbitrary constants would be Y_0, Z_{-1} and Y_0', and their values—deduced from the vanishing of $\widehat{rr}$ over the outer surface, and the continuity of u and $\widehat{rr}$ over the common surface of the core and layer—would be given by

$$Y_0[(3m-n)(4n+3m'-n')a^3 + 4n\{3m'-n'-(3m-n)\}b^3]$$

$$= -\tfrac{2}{5}\frac{\pi\rho^2}{m+n}\{(5m+n)(4n+3m'-n')a^5 - 4n(5m+n-3m'+n')b^5\}$$

$$+\frac{4\pi\rho\rho' b^3}{m+n}[(m-n)(4n+3m'-n')a^2 + 2n\{3m'-n'-2(m-n)\}b^2] - \tfrac{16}{5}\pi(\rho+\rho')^2nb^5\ldots(18),$$

$$Z_{-1}[(3m-n)(4n+3m'-n')a^3 + 4n\{3m'-n'-(3m-n)\}b^3]$$

$$= \tfrac{2}{15}\frac{\pi\rho^2a^3b^3}{m+n}[(5m+n)\{3m'-n'-(3m-n)\}a^2 + (3m-n)\{5m+n-(3m'-n')\}b^2]$$

$$+\tfrac{2}{3}\frac{\pi\rho\rho' a^2b^5}{m+n}[(3m-n)\{3m'-n'-2(m-n)\}a + 2(m-n)\{3m-n-(3m'-n')\}b]$$

$$-\tfrac{4}{15}\pi(\rho+\rho')^2(3m-n)a^3b^5 \;\ldots\ldots(19),$$

$$Y_0'[(3m-n)(4n+3m'-n')a^3 + 4n\{3m'-n'-(3m-n)\}b^3]$$

$$= -\tfrac{2}{5}\pi\rho^2\{3(5m+n)a^5 - 5(3m-n)a^3b^2 - 8nb^5\}$$

$$-4\pi\rho\rho' b^2\{(3m-n)a^3 - 3(m-n)a^2b - 2nb^3\}$$

$$-\tfrac{2}{5}\frac{\pi(\rho+\rho')^2b^2}{m'+n'}[(3m-n)(4n+5m'+n')a^3 + 4n\{5m'+n'-(3m-n)\}b^3]\;\ldots\ldots(20).$$

Substituting these values in (6) and (8), and neglecting all terms in σ_i, we should get the elastic displacements in a gravitating truly spherical "earth" consisting of a core and outer layer of different materials whose common surface is spherical.

When the disturbing forces act, the values of these 3 constants contain terms of order a_i/a, &c.; but these subsidiary terms would not be required for the determination of a_i and b_i.

The remaining 6 constants V_i, &c., appearing in terms containing σ_i explicitly, may be determined in terms of V_i' by means of the surface equations whatever values we attribute to m, n, m' and n'. Unless, however, we ascribed definite numerical values to the elastic constants, and to b/a, the resulting expressions would be very cumbrous; and if the numerical values of these quantities were known it would probably be simplest to insert them at once in the surface equations.

Materials highly incompressible and of equal rigidities.

§ 15. Partly for this reason, further consideration of the problem is limited to the case when $n'=n$, these quantities being finite, while n/m and n/m' are negligible.

The presumably enormous pressures under which the earth's deep-seated materials exist seem a probable cause of wholly exceptional resistance to change of volume, whether we suppose the material to be wholly elastic or partly "set"; but there is no obvious reason why the resistance to change of shape should be exceptionally large. Thus on physical grounds alone, we should be disposed to suppose n' of moderate size, but n'/m' exceptionally small; and unless the layer were very thin similar reasoning would apply to n and n/m. A perhaps even more important consideration, leading to the same restriction, is that unless much larger values than any hitherto found, even for steel, be ascribed to the constant m'—and to m also, except in a very thin layer—the numerical values deduced for the strains and displacements are too large to be consistent with the fundamental hypothesis of the mathematical theory of linear elasticity.

From the above considerations we should regard an increase in m with the depth as the most plausible hypothesis for an elastic solid earth; but there does not appear the same reason for expecting an increase in n.

The principal reason, however, for supposing $n'=n$ in the rest of this investigation is the great simplification thus introduced in the mathematical work.

§ 16. The physical conditions presupposed in the remainder of the paper are briefly that the core and layer have the same rigidity but different densities; and that the resistances to compression though not equal are both very large.

Putting for brevity

$$\rho a^i V_i' - \frac{8(i-1)\pi\rho^2 a a_i}{3(2i+1)} - \tfrac{4}{3}\pi\rho\rho' b^3 a^{-2} a_i + \frac{4\pi\rho\rho' b^{i+2} a^{-i-1} b_i}{2i+1} = P \quad \text{.................(21)},$$

$$-\rho' b^i V_i' - \frac{4\pi\rho\rho' a (b/a)^i a_i}{2i+1} + \tfrac{4}{3}\pi\rho\rho' b b_i + \frac{8(i-1)\pi\rho'^2 b b_i}{3(2i+1)} = Q \quad \text{.....................(22)};$$

we have for surface conditions

$$2(i-1)a^{i-2}Z_i - \frac{i^2-i-3}{2i+3}\left(\frac{m}{n}\right)a^iY_i + \frac{i^2+3i-1}{2i-1}\left(\frac{m}{n}\right)a^{-i-1}Y_{-i-1} - 2(i+2)a^{-i-3}Z_{-i-1} = P/n \quad\text{.............. (23)},$$

$$\frac{2(i-1)}{i}a^{i-2}Z_i - \frac{i(i+2)}{(i+1)(2i+3)}\left(\frac{m}{n}\right)a^iY_i - \frac{i^2-1}{i(2i-1)}\left(\frac{m}{n}\right)a^{-i-1}Y_{-i-1} + \frac{2(i+2)}{i+1}a^{-i-3}Z_{-i-1} = 0 \quad\text{.................. (24)},$$

$$2(i-1)b^{i-2}Z_i - \frac{i^2-i-3}{2i+3}\left(\frac{m}{n}\right)b^iY_i + \frac{i^2+3i-1}{2i-1}\left(\frac{m}{n}\right)b^{-i-1}Y_{-i-1} - 2(i+2)b^{-i-3}Z_{-i-1} - 2(i-1)b^{i-2}Z_i' + \frac{i^2-i-3}{2i+3}\left(\frac{m'}{n}\right)b^iY_i' = Q/n \quad\text{.......................... (25)},$$

$$\frac{2(i-1)}{i}b^{i-2}Z_i - \frac{i(i+2)}{(i+1)(2i+3)}\left(\frac{m}{n}\right)b^iY_i - \frac{i^2-1}{i(2i-1)}\left(\frac{m}{n}\right)b^{-i-1}Y_{-i-1} + \frac{2(i+2)}{i+1}b^{-i-3}Z_{-i-1} - \frac{2(i-1)}{i}b^{i-2}Z_i' + \frac{i(i+2)}{(i+1)(2i+3)}\left(\frac{m'}{n}\right)b^iY_i' = 0 \quad\text{........................ (26)},$$

$$b^{i-2}Z_i - \frac{i}{2(2i+3)}\left(\frac{m}{n}\right)b^iY_i - \frac{i+1}{2(2i-1)}\left(\frac{m}{n}\right)b^{-i-1}Y_{-i-1} + b^{-i-3}Z_{-i-1} - b^{i-2}Z_i' + \frac{i}{2(2i+3)}\left(\frac{m'}{n}\right)b^iY_i' = 0 \quad\text{.................................... (27)},$$

$$\frac{1}{i}b^{i-2}Z_i - \frac{i+3}{2(i+1)(2i+3)}\left(\frac{m}{n}\right)b^iY_i + \frac{i-2}{2i(2i-1)}\left(\frac{m}{n}\right)b^{-i-1}Y_{-i-1} - \frac{1}{i+1}b^{-i-3}Z_{-i-1} - \frac{1}{i}b^{i-2}Z_i' + \frac{i+3}{2(i+1)(2i+3)}\left(\frac{m'}{n}\right)b^iY_i' = 0 \quad\text{............................ (28)}.$$

It will be noticed that the constants Y_i, Y_{-i-1} and Y_i' in these equations have for multiplier (m/n) or (m'/n), quantities which by the present hypothesis are extremely large.

This implies of course that Y_i, for instance, is very small compared to Z_i, and any term in which Y_i appeared would be negligible compared to one in which Z_i appeared provided the other factors were of like order of magnitude in the two cases. As appears, however, by reference to the formulæ (6) to (9) the coefficients of the Y constants in the expressions for the displacements bear to those of the Z constants ratios of the order $(m/n) : 1$; so that the terms depending on the Y and Z constants are really of like importance.

§ 17. The equations (23) to (28) are satisfied by

$$(m/n)\,a^i Y_i = \frac{(i+1)(2i+3)}{2i^2+4i+3}\frac{P}{n}$$
$$-\frac{i(i+1)(i+2)\{2i+3-(2i+1)(b/a)^2\}(b/a)^{i+1}}{(2i+1)(2i^2+4i+3)}\frac{Q}{n} \quad \ldots\ldots(29),$$

$$a^{i-2}Z_i = \frac{i^2(i+2)}{2(i-1)(2i^2+4i+3)}\frac{P}{n}$$
$$-\frac{i(b/a)^{i+1}}{2(2i+1)(2i^2+4i+3)}\left\{\frac{2i^5+5i^4-5i^2+4i+3}{(i-1)(2i-1)}-i(i+1)(i+2)(b/a)^2\right\}\frac{Q}{n} \quad \ldots\ldots(30).$$

$$(m/n)b^{-i-1}Y_{-i-1} = \frac{i}{2i+1}\frac{Q}{n} \quad \ldots\ldots(31),$$

$$b^{-i-3}Z_{-i-1} = \frac{i(i+1)}{2(2i+1)(2i+3)}\frac{Q}{n} \quad \ldots\ldots(32),$$

$$(m'/n)b^i Y_i' = \frac{(i+1)(2i+3)}{2i^2+4i+3}\left(\frac{b}{a}\right)^i\frac{P}{n}$$
$$-\frac{i+1}{2i+1}\left\{1+\frac{i(i+2)(2i+3-(2i+1)(b/a)^2)(b/a)^{2i+1}}{2i^2+4i+3}\right\}\frac{Q}{n} \quad \ldots\ldots(33),$$

$$b^{i-2}Z_i' = \frac{i^2(i+2)(b/a)^{i-2}}{2(i-1)(2i^2+4i+3)}\frac{P}{n}$$
$$-\frac{i}{2(2i+1)}\left\{\frac{i+1}{2i-1}+\frac{(2i^5+5i^4-5i^2+4i+3)}{(i-1)(2i-1)(2i^2+4i+3)}(b/a)^{2i-1}\right.$$
$$\left.-\frac{i(i+1)(i+2)(b/a)^{2i+1}}{2i^2+4i+3}\right\}\frac{Q}{n} \quad \ldots\ldots(34).$$

The values of P and Q are given explicitly by (21) and (22) when the values of a_i and b_i are known.

To determine a_i and b_i in terms of V_i' we proceed as in Section I. We substitute in (8) $a+a_i\sigma_i$ for r, and note that the coefficient of σ_i in the resulting equation must be a_i; similarly we substitute $b+b_i\sigma_i$ for r, and equate the coefficient of σ_i to b_i.

In this way we find, for n/m and n/m' negligible,

$$a_i = a^{i-1}Z_i - \frac{i}{2(2i+3)}\left(\frac{m}{n}\right)a^{i+1}Y_i - \frac{i+1}{2(2i-1)}\left(\frac{m}{n}\right)a^{-i}Y_{-i-1} + a^{-i-2}Z_{-i-1} \quad \ldots\ldots(35),$$

$$b_i = b^{i-1}Z_i - \frac{i}{2(2i+3)}\left(\frac{m}{n}\right)b^{i+1}Y_i - \frac{i+1}{2(2i-1)}\left(\frac{m}{n}\right)b^{-i}Y_{-i-1} + b^{-i-2}Z_{-i-1} \quad \ldots\ldots(36).$$

Substituting for Y_i, &c. from equations (29) to (32), and inserting their values (21) and (22) for P and Q, we are left finally with two simple equations from which to determine a_i and b_i in terms of V_i'. These two equations are true for all values of b/a, and the explicit determination of a_i in the general case has no difficulty except in the length of the expressions. Through considerations of time I have limited myself to the most interesting case when the thickness $a-b$ of the layer is small compared to a. Before passing to the solution in this case, we may however draw one interesting conclusion in the general case. Neither m nor m' appears in the values (21) and (22) of P and Q, and both are likewise absent from the expressions (29) to (32) for $(m/n)Y_i$, Z_i, $(m/n)Y_{-i-1}$ and Z_{-i-1}. It is thus clear from (35) and (36) that the values of a_i and b_i do not contain m, m' or m/m'. Consequently so long as the layer and core are both highly incompressible, a difference in their resistances to compression has no appreciable influence on the shape of either surface of the layer whatever be the nature of the disturbing forces.

Special case of relatively thin layer.

§ 18. Putting $a-b=t$, and neglecting $(t/a)^2$ we find

$$a_i/a=\frac{i(2i+1)}{2(i-1)(2i^2+4i+3)}\frac{P-Q}{n}+\frac{t}{a}\frac{3i(i+1)}{2(i-1)(2i^2+4i+3)}\frac{Q}{n} \quad\ldots\ldots(37).$$

$$b_i/b=\frac{i(2i+1)}{2(i-1)(2i^2+4i+3)}\frac{P-Q}{n}+\frac{t}{a}\frac{3i}{2(i-1)(2i^2+4i+3)}\frac{iP+Q}{n} \quad\ldots\ldots(38).$$

Now $P-Q$ does not vanish with t, thus to a first approximation

$$b_i/b=a_i/a=\frac{i(2i+1)}{2(i-1)(2i^2+4i+3)}\frac{P-Q}{n} \quad\ldots\ldots(39).$$

Likewise subtracting (37) from (38), we have

$$b_i/b-a_i/a=\frac{(t/a)3i^2}{2(i-1)(2i^2+4i+3)}\frac{P-Q}{n} \quad\ldots\ldots(40).$$

Combining (39) and (40), we get

$$b_i/b=(a_i/a)\left(1+\frac{3i}{2i+1}\frac{t}{a}\right) \quad\ldots\ldots(41),$$

or

$$b_i=a_i\left(1+\frac{i-1}{2i+1}\frac{t}{a}\right) \quad\ldots\ldots(41'),$$

a result independent of the densities of the core and layer.

To obtain the absolute values of a_i and b_i we substitute their values for P and Q in (37) or (38), or preferably in an equation obtained by combining the two. In the latter way I find

$$(\rho a_i+\rho' b_i)\left\{1+\frac{4\pi(\rho+\rho')^2a^2i}{3(2i^2+4i+3)n}\right\}=\frac{i(2i+1)(\rho+\rho')^2a^{i+1}V_i'}{2(i-1)(2i^2+4i+3)n}$$
$$\times\left[1-\frac{1}{2i+1}\frac{(t/a)(\rho'/(\rho+\rho'))}{1+\dfrac{4\pi(\rho+\rho')^2a^2i}{3(2i^2+4i+3)n}}\left\{2i^2+3i+4+\frac{4\pi(\rho+\rho')^2a^2(i-1)i(2i+1)}{3(2i^2+4i+3)n}\right\}\right] \ldots(42).$$

In obtaining (42) use has been made of the fact that in terms multiplied by (t/a) b_i and a_i are interchangeable.

Finally combining (42) and (41′), we find

$$a_i/a = \left\{1 + \frac{4\pi(\rho+\rho')^2 a^2 i}{3(2i^2+4i+3)n}\right\}^{-1} \frac{i(2i+1)(\rho+\rho')a^i V_i'}{2(i-1)(2i^2+4i+3)n}$$

$$\times\left[1 - \frac{1}{2i+1}\,\frac{(t/a)(\rho'/(\rho+\rho'))}{1+\dfrac{4\pi(\rho+\rho')^2a^2i}{3(2i^2+4i+3)n}}\left\{2i^2+4i+3+\frac{8\pi(\rho+\rho')^2a^2(i-1)i(i+1)}{3(2i^2+4i+3)n}\right\}\right] \ldots(43).$$

After determining a_i/a from (43) we obtain b_i/b immediately from (41).

If we neglect t/a altogether we obtain

$$a_i/a = \frac{i(2i+1)(\rho+\rho')a^iV_i'}{2(i-1)(2i^2+4i+3)n} \div \left\{1 + \frac{4\pi(\rho+\rho')^2a^2i}{1+3(2i^2+4i+3)n}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(44).$$

This refers of course to the case of a homogeneous elastic solid "earth" of uniform density $\rho+\rho'$, and agrees with the result I obtained directly in a recent paper* when we allow for the difference in notation and neglect n/m.

A comparison of (43) and (44) shows how much less a_i/a is owing to the lesser density of the layer than if the whole "earth" had possessed the greater density of the core.

§ 19. A more instructive comparison may be made by reference to the case when the homogeneous "earth" has the same mass, and so mean density, as the composite one.

If this mean density be $\bar{\rho}$, then to the present degree of approximation

$$\bar{\rho} = \rho + \rho' - 3\rho' t/a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45);$$

also if g be "gravity" at the surface

$$g = \tfrac{4}{3}\pi\bar{\rho}a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

Of course g is the same as in the hypothetical composite earth. We may now write (43) in the form

$$a_i/a = \left\{1 + \frac{(g\bar{\rho}a/n)i}{2i^2+4i+3}\right\}^{-1} \frac{i(2i+1)\bar{\rho}a^iV_i'}{2(i-1)(2i^2+4i+3)n}$$

$$\times\left[1 - \frac{(t/a)(\rho'/\bar{\rho})\,2i(i-1)}{(2i+1)\left(1+\dfrac{(g\bar{\rho}a/n)i}{2i^2+4i+3}\right)}\left\{1+\frac{(g\bar{\rho}a/n)(2i^2+6i+1)}{2(i-1)(2i^2+4i+3)}\right\}\right] \ldots\ldots\ldots\ldots(47).$$

As the coefficient of t/a in (47) is essentially negative, we see that the lesser density of the layer always makes the disturbing forces less effective in altering the spherical form than if the density were uniform.

* *Phil. Mag.* March, 1897, p. 193, equation (52).

Case $i=2$, luni-solar tides.

§ 20. In the case of most physical interest when the disturbing forces represent the tidal influence of the sun or moon

$$i = 2,$$
$$V_2' = g(M/E)(a^2/R^3),$$

in the notation of equation (35) of Sect. I.

In this case by (47)

$$a_2/a = \frac{\frac{5}{19}(g\bar{\rho}a/n)(M/E)(a/R)^3}{1+\frac{2}{19}(g\bar{\rho}a/n)}\left[1-\frac{\frac{1}{5}(t/a)(\rho'/\bar{\rho})(1+\frac{21}{38}g\bar{\rho}a/n)}{1+\frac{2}{19}g\bar{\rho}a/n}\right] \quad \text{.........} \quad (48)$$

Here the *percentage* reduction in a_2/a due to the lesser density of the layer is

$$80\,(t/a)(\rho'/\bar{\rho})\{1+\tfrac{21}{38}g\bar{\rho}a/n\} \div \{1+\tfrac{2}{19}g\bar{\rho}a/n\} \quad \text{..........................} (49).$$

In the actual earth we have approximately

$$g\bar{\rho}a = 35\times 10^8 \text{ grammes wt. per sq. cm.,}$$

and this we shall employ in the following estimates.

Our work assumes t/a small, so as a convenient example suppose

$$t/a = 1/20.$$

Suppose likewise as an approximation to actual conditions

$$(\rho'/\bar{\rho}) = \tfrac{1}{2}(20/19)^3 = {\cdot}583 \text{ approx.}:$$

and as representative of extreme and mean rigidities take the three cases

(i) $n = 80\times 10^7$ grammes wt. per sq. cm.,

(ii) $ = 35\times 10^7$ " "

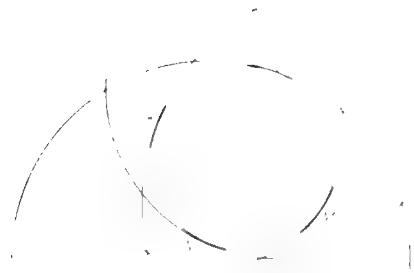
(iii) $ = 11\times 10^7$ " "

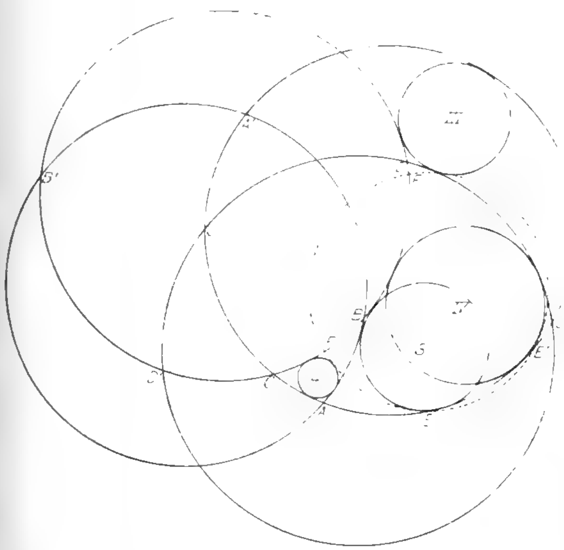
Then by (49) for the percentage reductions in the value of a_2/a due to the lesser density of the layer, we have the following approximate values:

Case	i	ii	iii
Percentage reduction	5	7	10

As the above hypothetical layer would in the case of the true earth be nearly 200 miles thick, and the effect varies directly as the thickness, we may expect variations of density within 30 or 40 miles of the earth's surface to have but little influence on the lunar or solar tides.

In the case of disturbing forces which are local, or vary rapidly with the angular coordinates, variability in the surface strata is probably more important, for the coefficient of t/a inside the bracket in equation (43) tends in general to increase with i.





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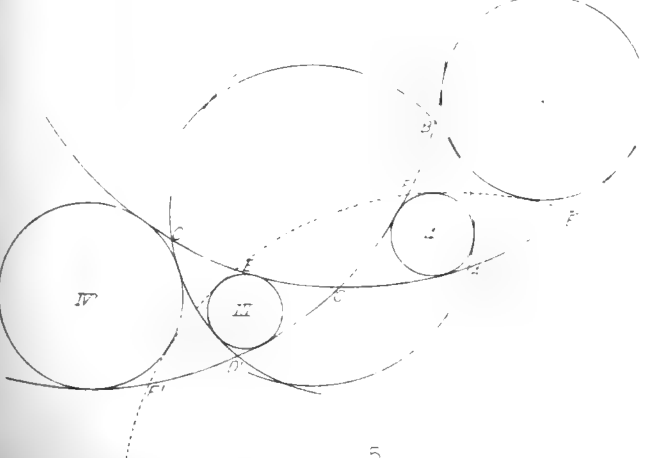

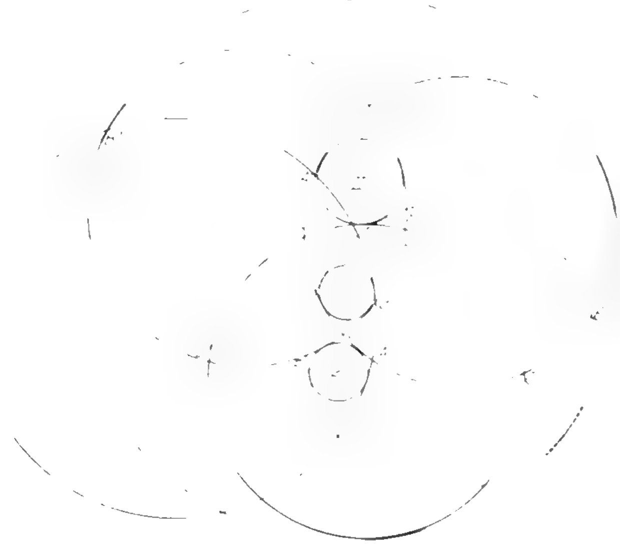

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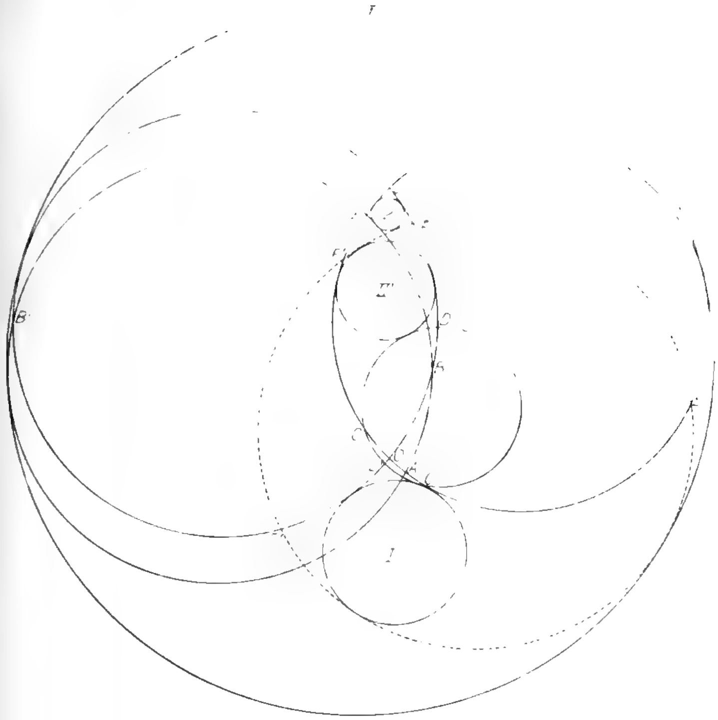

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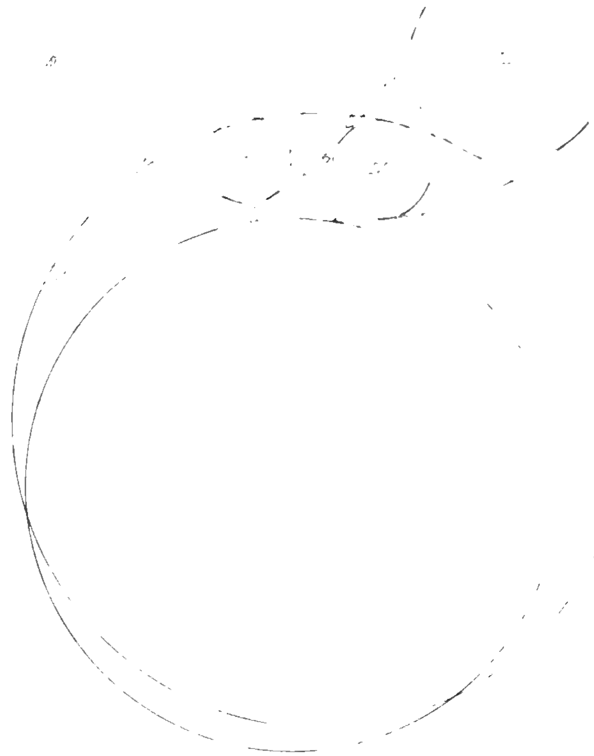

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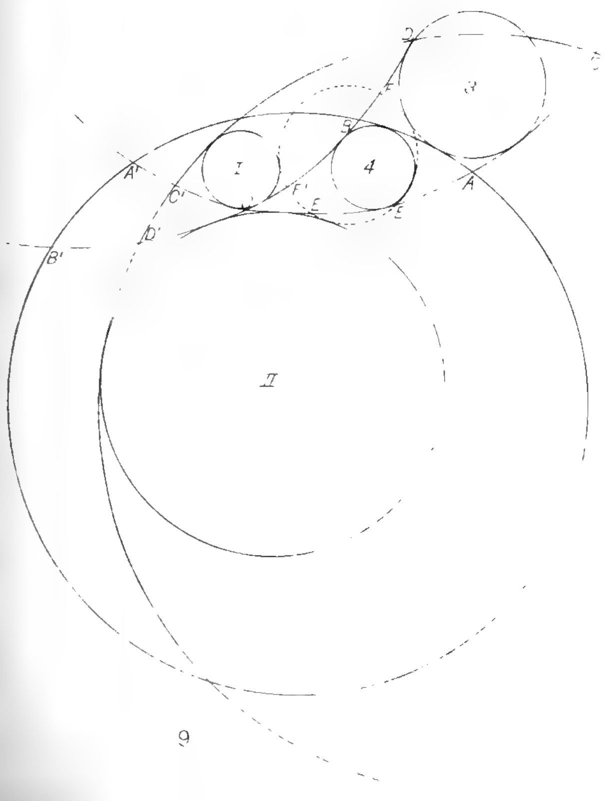

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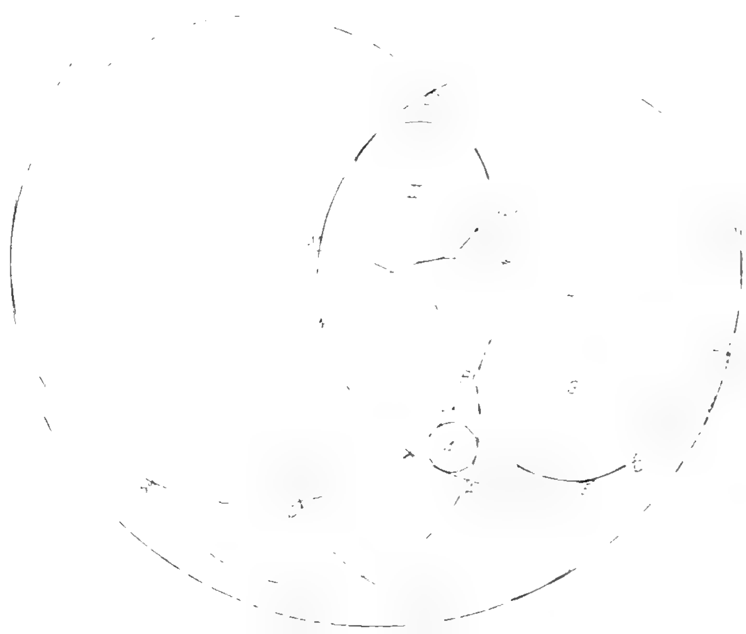
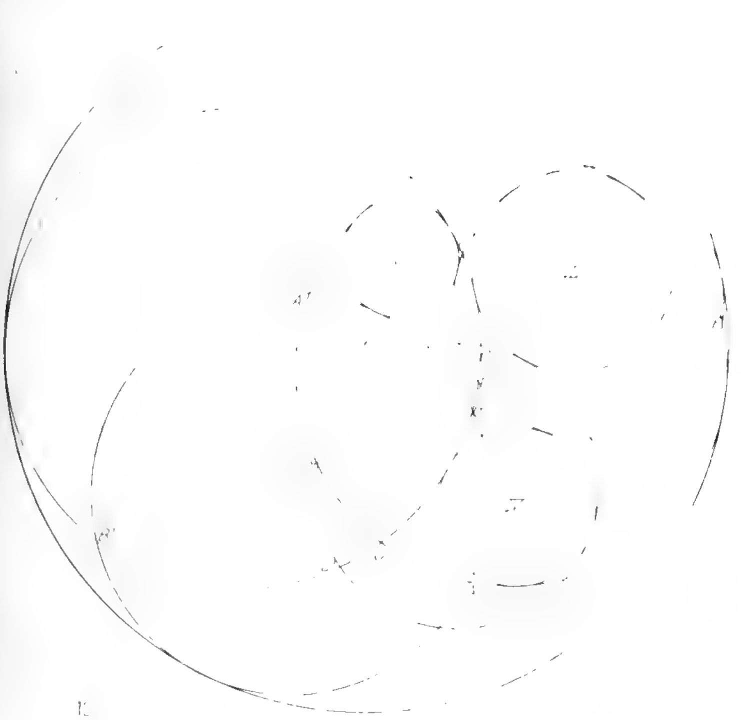




VII. *Circles, Spheres, and Linear Complexes.* By Mr J. H. Grace, B.A., Fellow of St Peter's College, Cambridge.

[*Received*, July, 1897. *Read*, 25 October, 1897.]

In this paper there are discussed certain theorems concerning circles and spheres, and analogous theorems concerning linear complexes.

The whole is divided into seven sections.

In I. we discuss certain relations between systems of linear complexes; these we apply to the exposition of the analogy, due to Klein, between line geometry and sphere geometry in four dimensions; and also to the transformation of Lie in which a straight line corresponds to a sphere and two intersecting straight lines to two spheres which touch.

In II. the theorem, that the circum-circles of the triangles formed by four lines meet in a point, is proved by a method depending on the theory of curves; a similar method is applied to shew that there is no corresponding result in three dimensions and again to prove the analogous theorem in four dimensions.

Then the theorem, that, given three points one on each side of a triangle we get three circles meeting in a point, is extended to three and four dimensions. Finally the results are transformed by both methods explained in I.

In III. we prove the set of theorems given by Clifford ("Synthetic proof of Miquel's theorem"), and also another set of theorems, of which the first are particular cases, viz.:

"Given three coplanar lines and a point on each of them, we obtain three circles meeting in a point."

"Given four lines and four *concyclic* points, one on each of them, we have four sets of three lines and the points derived from these four sets, by the first theorem, lie on a circle."

"Given five lines and five *concyclic* points, we have five sets of four and from each set of four a circle; these five circles meet in a point." And so on *ad inf.*

IV. contains the proof of a corresponding set of theorems in space of three dimensions, as follows:—

"Given three planes and a point on each line of intersection, we have four points in all and they determine a sphere."

"Given four planes and a point on each line of intersection, we have four sets of three planes, and from each set of three a sphere: the four spheres so obtained meet in a point."

"Given five planes and a point on each line of intersection, we have five sets of four, and from each set of four a point; the five points so obtained lie on a sphere." And so on *ad inf.*

In V. the general configuration of points and spheres derived from the theorems in IV. is discussed.

If we take n planes we obtain a system of 2^{n-1} spheres, 2^{n-1} O-points and $\frac{n(n-1)(n-2)}{6} 2^{n-3}$ A-points; each sphere contains n O-points and $\frac{n(n-1)(n-2)}{6}$ A-points; each O-point lies on n spheres and each A-point on 4 spheres.

In VI. certain symmetrical systems of points and spheres are obtained, viz.:

(i) A set of sixteen points lying by eights on ten spheres, there being five spheres through each point.

(ii) A set of seventy-two points lying by sixteens on twenty-seven spheres, there being six spheres through each point.

(iii) A set of 576 points lying by 32's on 126 spheres, there being seven spheres through each point.

Similar sets of points probably exist in which there are eight, nine...spheres through each point, but as the number of points increases the difficulties as to notation become very great.

Finally in VII. some of the results are transformed in accordance with the methods of I.

SECTION I.

Systems of Linear Complexes.

1. In line geometry, as the subject has been treated by Plücker and others, a line is determined by six homogeneous coordinates connected by a quadratic relation: the line in fact depends really on four quantities, but two others are introduced for convenience in analysis.

Supposing that $x_1y_1z_1\omega_1$ and $x_2y_2z_2\omega_2$ are two points on the line; $l_1m_1n_1r_1$, $l_2m_2n_2r_2$ two planes passing through the line; then l, m, n, λ, μ, ν, its six coordinates, are defined by

$$l : m : n : \lambda : \mu : \nu = x_1\omega_2 - x_2\omega_1 : y_1\omega_2 - y_2\omega_1 : z_1\omega_2 - z_2\omega_1 : y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1$$
$$= m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1 : l_1r_2 - l_2r_1 : m_1r_2 - m_2r_1 : n_1r_2 - n_2r_1,$$

the equivalence of the two sets being easily proved.

[It has been considered generally by Pasch, *Crelle* LXXV. p. 108.]

The coordinates are connected by the relation

$$l\lambda + m\mu + n\nu = 0$$

and are independent of the particular pair of points or planes chosen.

Further, two such lines intersect if only

$$l_1\lambda_2 + l_2\lambda_1 + m_1\mu_2 + m_2\mu_1 + n_1\nu_2 + n_2\nu_1 = 0.$$

2. If the coordinates of a line satisfy a linear relation

$$X\lambda + Y\mu + Z\nu + Ll + Mm + Nn = 0,$$

the line belongs to a Linear Complex of which L, M, N, X, X, Z are called the coordinates.

As regards a linear complex $X_1Y_1Z_1L_1M_1N_1$, $L_1X_1 + M_1Y_1 + N_1Z_1$ is called the invariant and we shall denote it by $\frac{1}{2}\varpi_{11}$.

For two linear complexes we have the mutual invariant

$$L_1X_2 + L_2X_1 + M_1Y_2 + M_2Y_1 + N_1Z_2 + N_2Z_1,$$

denoted by ϖ_{12}. If this vanish the two are said to be *in involution*.

Thus if the invariant of a linear complex vanish its coordinates are those of a line and the lines of the complex are the lines meeting this line.

If the mutual invariant of two lines be zero then the lines intersect, and if the mutual invariant of a line and a complex be zero the line belongs to the complex.

3. If we consider l, m, n, λ, μ, ν each replaced by a linear function of six variables x_1, x_2, x_3, x_4, x_5, x_6 we get a new system of coordinates, in which the x's are connected by a quadratic relation

$$\Phi = 0$$

when they are the coordinates of a line.

Further, whereas in the original system the vanishing of a coordinate meant that the line in question met one of the edges of the fundamental tetrahedron, the vanishing of one now indicates that the line belongs to a certain linear complex $x_1 = 0$ for the x's are linear functions of l, m, n, λ, μ, ν. (Klein, *Math. Ann.* II.)

The complexes $x_1 = 0$, $x_2 = 0$, $x_3 = 0 \ldots x_6 = 0$, are called the fundamental complexes, and we shall now obtain the relation $\Phi = 0$ in a form which involves only the mutual invariants of these complexes.

4. For this purpose suppose

$$L_1,\ M_1,\ N_1,\ X_1,\ Y_1,\ Z_1,\ \&c.$$

are the coordinates of fourteen linear complexes

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\ ;\ 1',\ 2',\ 3',\ 4',\ 5',\ 6',\ 7',$$

then we have

$$0 = \begin{vmatrix} L_1 & M_1 & N_1 & X_1 & Y_1 & Z_1 & 0 \\ L_2 & M_2 & N_2 & X_2 & Y_2 & Z_2 & 0 \\ L_3 & \cdots & & & & & \\ L_4 & \cdots & & & & & \\ L_5 & \cdots & & & & & \\ L_6 & \cdots & & & & & \\ L_7 & \cdots & & & & & \end{vmatrix} \times \begin{vmatrix} X_{1'} & Y_{1'} & Z_{1'} & L_{1'} & M_{1'} & N_{1'} & 0 \\ X_{2'} & \cdots & & & & & \\ X_{3'} & \cdots & & & & & \\ X_{4'} & \cdots & & & & & \\ X_{5'} & \cdots & & & & & \\ X_{6'} & \cdots & & & & & \\ X_{7'} & \cdots & & & & & \end{vmatrix}$$

$$= \begin{vmatrix} \varpi_{11'} & \varpi_{12'} & \cdots & \varpi_{17'} \\ \varpi_{21'} & \varpi_{22'} & \cdots & \varpi_{27'} \\ \varpi_{31'} & \varpi_{32'} & \cdots & \varpi_{37'} \\ \varpi_{41'} & \cdots & & \\ \varpi_{51'} & \cdots & & \\ \varpi_{61'} & \cdots & & \\ \varpi_{71'} & \cdots & & \varpi_{77'} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(A),$$

the relation between the mutual invariants of two sets of seven linear complexes.

Suppose now that $1'$, $2'$, $3'$, $4'$, $5'$, $6'$ are the same as 1, 2, 3, 4, 5, 6 respectively, and denote the others by a, b, then the foregoing relation obviously gives ϖ_{ab} in terms of $\varpi_{a1} \ldots$, $\varpi_{b1} \ldots$, i.e. it gives the mutual invariant in terms of the coordinates. By making $\varpi_{ab} = 0$ we get the condition that the complexes a and b should be in involution, and by making b the same as a, and then making $\varpi_{aa} = 0$, we get the relation between the six coordinates of a line, the coefficients being functions of the mutual invariants of the six fundamental complexes.

The relation (A) is precisely similar to that which occurs in the theory of circles and spheres (Lachlan, *Phil. Trans.*, 1886), and is obtained in the same manner.

By supposing certain of the mutual invariants to vanish the relation may be made to take a simpler form; two cases will be considered,

(I) when all invariants ϖ_{mn} in which $m \neq n$ vanish, i.e. when the six fundamental complexes are mutually in involution;

(II) when all invariants ϖ_{mn} except ϖ_{56} vanish where $m \neq n$ and $\varpi_{55}=0$, $\varpi_{66}=0$.

5. In (I) the relation becomes

$$\varpi_{ab} = \frac{\varpi_{a1}\varpi_{b1}}{\varpi_{11}} + \frac{\varpi_{a2}\varpi_{b2}}{\varpi_{22}} \dots \frac{\varpi_{a6}\varpi_{b6}}{\varpi_{66}{}^2},$$

and if we replace ϖ_{a1} by $x_1\sqrt{\varpi_{11}} \dots$,

ϖ_{b1} by $y_1\sqrt{\varpi_{11}} \dots$,

we get $\varpi_{xy} = x_1y_1 + x_2y_2 \dots x_6y_6$,

and consequently with these coordinates the condition for a line is

$$x_1^2 + x_2^2 \dots x_6^2 = 0.$$

Now in the geometry of circles a circle is given by four coordinates x_1, x_2, x_3, x_4, and when the four fundamental circles are mutually orthogonal the condition for a point may be written

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0;$$

also the condition that two circles should cut orthogonally is

$$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0.$$

In the geometry of spheres we have five coordinates x_1, x_2, x_3, x_4, x_5; the condition for a point is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$

and the condition that two spheres should cut orthogonally is

$$x_1y_1 + x_2y_2 + \dots + x_5y_5 = 0.$$

Similarly in space of four dimensions we have a geometry of hyperspheres in which a sphere is given by six coordinates

$$x_1, x_2 \dots x_6,$$

the condition for a point being

$$x_1^2 + x_2^2 \dots x_6^2 = 0,$$

and for orthogonal section

$$x_1y_1 + x_2y_2 \dots x_6y_6 = 0,$$

and a point lies on a sphere if the coordinates of two satisfy the conditions for orthogonal section.

6. Thus comparing these results we see that there is an exact analogy between the geometry of linear complexes, and geometry of spheres in four dimensions, in which

(i) a linear complex corresponds to a hypersphere;

(ii) a straight line corresponds to a point;

(iii) two linear complexes in involution correspond to two spheres cutting at right angles;

(iv) a complex and a straight line belonging to it correspond to a hypersphere and a point on it. (Klein, *Math. Ann.* v.)

Thus if we can prove any theorem involving the stated geometrical relations in the one set, we can immediately infer a corresponding one in the other. For example, to the result that four such hyperspheres have two points in common corresponds the fact that four linear complexes have two lines in common.

To a linear complex and two polar lines correspond a hypersphere and two inverse points, for on the one hand every linear complex containing the given lines is in involution with the given complex, and, on the other, every sphere passing through the two points cuts the given sphere orthogonally.

7. If we suppose the coordinate x_6 to vanish, the condition for a line is

$$x_1^2 + x_2^2 \dots x_5^2 = 0,$$

and the geometry is that of lines in a linear complex and complexes in involution with it.

Thus geometry in a linear complex is equivalent to the geometry of spheres and points in three-dimensional space, the correspondence being exactly the same as in the last article, with the exception that all linear complexes in this system are in involution with the given one.

So, if the coordinates x_5 and x_6 vanish, we find that geometry in a linear congruence is equivalent to the geometry of circles and points in a plane.

It is to be remarked that there is nothing in line geometry corresponding to the element at infinity in sphere geometry, and consequently all propositions involving planes in the sphere geometry must be inverted so that the planes become spheres before they can be transformed. The element at infinity is in fact replaced by a line which has no particular property with reference to other lines.

8. In case (II), Art. 4, by taking suitable multiples of ϖ_{a1}, &c. we can obtain an equation of the form

$$\varpi_{xy} = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - \tfrac{1}{2}(x_5y_6 + x_6y_5) = 0,$$

and the condition for a straight line is

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5x_6 = 0.$$

a form which is most useful for this article.

Now supposing a sphere to be given by its equation in ordinary Cartesian coordinates

$$x^2 + y^2 + z^2 + 2bx + 2cy + 2dz + e = 0,$$

we have
$$r^2 = b^2 + c^2 + d^2 - e,$$

or introducing a quantity a to make the equation homogeneous, we have

$$b^2 + c^2 + d^2 - ae - r^2 = 0,$$

and the sphere is given by the six coordinates a, b, c, d, e, r which are connected by the same relation as above.

Thus we have an analogy between the geometry of spheres and straight lines, but we have still to discover the analogy to a linear complex and the fact of such a complex containing a given line. Also while a line is replaced by a unique sphere, a sphere corresponds to *two* lines because the sphere is unaltered by changing the sign of r, whereas a line is changed by this process into its polar line with respect to the complex corresponding to $r = 0$.

9. To complete the discussion we observe that if two spheres a, b, c, d, e, r; a', b', c', d', e', r' cut each other at an angle α, then

$$2bb' + 2cc' + 2dd' - ae' - a'e - 2rr' \cos\alpha = 0,$$

where α is so taken that

$$\cos\alpha = +1 \text{ for external contact,}$$

and
$$\cos\alpha = -1 \text{ for internal contact.}$$

Thus if $\alpha = 0$, we find

$$2bb' + 2cc' + 2dd' - ae' - a'e - 2rr' = 0$$

as the condition that two spheres should touch externally.

Now the first sphere is replaced by the two lines

$$a, b, c, d, e, +r,$$

$$a, b, c, d, e, -r,$$

and similarly for the second; consequently to *two spheres which touch externally correspond two pairs of lines, such that each of one pair intersects one of the other pair.* As each pair of lines are conjugate with respect to a linear complex one intersection is a necessary consequence of the other.

For internal contact the $+r$ line meets the $-r'$ line, and the $-r$ line meets the $+r'$ line. (Lie, *Math. Ann.* V.)

10. To illustrate this, consider the problem of describing a sphere to touch four given spheres. We have in the transformation to draw a line meeting four given lines. Suppose we have to get a sphere having like contacts with the four given ones, then we take the four lines a, b, c, d corresponding to the $(+r\text{'s})$ and the four a', b', c', d' corresponding to the $(-r\text{'s})$.

There are two lines meeting the former four and two meeting the latter four, viz., the polars of the first two with respect to the fundamental linear complex; therefore there are two spheres having like contacts with the four given ones.

If the sphere sought has to have like contacts with a, b, c, and the opposite kind with d, we take the two quartettes $\begin{matrix} a\,b\,c\,d' \\ a'b'c'd \end{matrix}$ and thus get two more spheres.

For the condition for external contact being

$$2bb' + 2cc' + 2dd' - ae' - a'e + 2rr' = 0$$

simply expresses the fact that the $\pm r$ line meets the $-r'$ line and *vice versâ*.

Hence as we can choose the quartettes

$$\begin{matrix} a\,b\,c\,d', \\ a'b'c'd \end{matrix}$$

in $\frac{2^4}{2} = 8$ ways, we have 8 pairs of spheres touching the four given ones, as ought to be the case.

11. We have so far obtained no meaning for the transformation of a linear complex, but the equation

$$2bb' + 2cc' + 2dd' - ae' - a'e - 2rr' \cos \alpha = 0$$

gives us the interpretation at once, since it shews that if a line belongs to a linear complex the corresponding sphere cuts a given sphere at a given angle; but a system of spheres cutting a fixed sphere at a constant angle and the system which cut it at the supplementary angle correspond to the same system of linear complexes. In fact a sphere is represented by two lines which are polar lines with respect to the linear complex, and therefore taking a sphere and an angle associated with it the spheres which cut the given sphere at the associated angle are represented by the totality of lines belonging to two linear complexes which are inverse (we may say) with respect to the fundamental one, viz., each is the locus of the polar lines of the other.

Changing the sign of $\cos \alpha$ only interchanges these complexes, and so the lines which are the transformations of the two sets of spheres are identical.

Taking b, c, d, a, e, r as the coordinates of the line they satisfy the relation

$$2b'(b) + 2c'(c) + 2d'(d) - e'(a) - a'(e) - r(2r' \cos \alpha) = 0,$$

the line b, c, d, a, e, $-r$ satisfies

$$2b'(b) + 2c'(c) + 2d'(d) - e'(a) - a'(e) - (-r)(-2r' \cos \alpha) = 0,$$

the inverse complex to the foregoing and as stated interchanging the sign of α interchanges these two complexes.

12. If we take five lines and their polar lines, the pair of complexes containing them become the sphere cutting the corresponding spheres at equal angles, where the five lines are $(+r)$ lines and the five polar lines are $(-r)$ lines; if we take a complex containing four of the $(+r)$ lines and one $(-r)$ line and the inverse complex, we get a sphere cutting four of the given spheres at equal angles and the remaining one at the supplementary angle, and so on.

There will thus be in the extended sense of the word 16 spheres cutting five given ones at equal angles, where an angle and its supplement are not taken to differ.

13. To enable us to translate theorems regarding spheres which pass through a fixed point we must remark that if a sphere becomes a point $r=0$ and the pair of lines in this case coincide, and further the united line belongs to a fixed linear complex corresponding to $r=0$, which has been called the fundamental linear complex.

Finally a sphere is replaced by two lines which are conjugate with respect to $r=0$, and it has with the totality of complexes having this pair of polar lines for conjugate lines a relation which may be stated as follows; viz. if we associate our sphere with various angles in the sense already stated, then the singly infinite set of linear complexes correspond to the sphere and the angles thus associated.

14. If the equation of a circle in ordinary Cartesian coordinates be

$$x^2+y^2+2bx+2cy+e=0,$$

then its radius r is given by

$$r^2=b^2+c^2-e,$$

or, introducing a quantity a to make this equation homogeneous, we find that a circle may be considered as having five coordinates a, b, c, e and r connected by the equation

$$b^2+c^2-ae-r^2=0:$$

and if two circles cut each other at an angle α, then we have

$$2bb'+2cc'-ae'-a'e-2rr'\cos\alpha=0,$$

where α is so taken that

$$\alpha=0,\quad \cos\alpha=+1 \text{ for internal contact,}$$

$$\alpha=\pi,\quad \cos\alpha=-1 \text{ for external contact.}$$

Thus, to put the matter briefly, we see that

(1) The geometry of circles is equivalent to that of lines in a linear complex (A).

(2) A given circle is represented by two lines which are conjugate to a complex (B) in involution with the one (A) in which the system lies.

(3) To circles which touch correspond lines which intersect.

(4) Being given four lines and their polars with respect to (B), we have a complex containing the four and in involution with (A), and another containing the

polars and in involution with (A); corresponding to this system in the circle geometry, we have four circles and a circle cutting them at equal angles, exactly analogous to the corresponding proposition in sphere geometry.

Reckoning an angle and its supplement as being not distinct, then there are eight circles cutting four given ones at equal angles; but there is only one which really cuts them at equal angles, the remaining seven cut some of the circles at the one angle and the other circles at the supplementary angle.

Thus, when we have given any theorem regarding lines in a linear complex, and linear complexes in involution with the given one, we can at once derive a proposition concerning circles in one plane, where the complex containing four lines is replaced by the circle cutting four given circles at equal angles.

We have already shewn that there is an exact connection between the geometry of spheres and points in three dimensions and geometry in a linear complex: hence we see that there is a connection between theorems regarding spheres and points in space and circles in a plane.

SECTION II.

15. There is a theorem in plane geometry to the effect that, being given four straight lines, the circumcircles of the triangles formed by omitting each line in turn meet in a point; we are naturally prompted to inquire whether there is a corresponding proposition in three dimensions and also in four, since the latter case leads to theorems regarding lines and linear complexes.

For the purpose of this inquiry it is convenient to regard the proposition in plano as a particular case of the theorem that all cubic curves that pass through eight fixed points pass through a ninth. In fact take as the eight points the six vertices of the quadrangle and the two circular points at infinity, then since each circumcircle and the corresponding omitted line form a cubic through the eight points, the four such cubics have a ninth point in common, consequently the four circumcircles have a point in common.

16. In three dimensions we consider cubic surfaces passing through the circle at infinity (cubic cyclides, in fact) and also through all the 10 vertices formed by five planes, and observe that the circumsphere of the tetrahedron formed by four planes and the remaining plane is such a cubic.

These cubics satisfy $10 + 7 = 17$ linear conditions, for the section at infinity instead of being a general cubic is a known conic and a variable line, and as a cubic can be made to pass through only 19 arbitrary points, all these cubics pass through the points of intersection of any three of them.

To find how many points three such cubics (cubic cyclides) have in common, we consider the case in which each of them is a sphere and a plane and thus see at once that the number is

$$2 + 2 \cdot 3 + 2 \cdot 3 + 1 = 15.$$

(This is in fact Schubert's principle of the fixity of number, ***Abzahlende Geometrie***, passim.)

Hence as our systems have already 10 points in common they have five further points common to all, but by the line theorem just proved the four circumspheres which are got by always including a certain one of the planes meet in a point on that plane and therefore the five further points lie one on each of the planes, and we infer at once that there is no corresponding theorem in three dimensions.

17. This does not preclude the possibility of there being such a theorem in four dimensions, and in fact there is one which we proceed to prove.

We take six hyperplanes in four dimensions; any four of them meet in a point, consequently omitting one of them we get five points through which there is a hypersphere, then the six hyperspheres so obtained by omitting each hyperplane in turn meet in a point.

For this purpose we consider cubics in four dimensions passing through the imaginary sphere at infinity and through the $\frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1} = 15$ vertices of the six "solids" so formed.

The section by the plane (hyperplane) at infinity is a known sphere and a variable plane therefore involving only *three* instead of *nineteen* constants, hence the cubics are subjected to

$$15 + 16 = 31 \text{ linear conditions.}$$

Such a cubic can be made to pass through $\frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} - 1 = 34$ points, and therefore all through 31 fixed points pass through a number of other fixed points.

To find the number of these we have to find the number of the intersections of four such cubics by considering each to be a sphere and a plane, thus the number is

$$2 + 2 \cdot 4 + 2 \cdot \frac{4 \cdot 3}{2 \cdot 1} + 2 \cdot 4 + 1 = 31,$$

and therefore such cubics pass through $31 - 15 = 16$ other fixed points.

Now we get one such point clearly on the plane of intersection of two hyperplanes* (by the line theorem); as there are 15 points of this nature there is one other common point and hence the six hyperspheres meet in a point, which is the theorem we set out to prove.

* Because the section of the figure by the plane of intersection of two such hyperplanes will be four lines and the circles circumscribing the triangles formed by them in threes, hence the application of the line theorem.

18. Translating this proposition into the language of linear complexes and straight lines it manifestly becomes the following: "taking six complexes having a line in common then any four of them have another line in common, and therefore from a set of five of them we get five lines through which one linear complex may be made to pass; then from the six complexes we get six sets of five, and as from each five we get another complex, we thus derive *six* new complexes, then the theorem is that these *six* complexes have one common line."

19. Now in the hyperspace system we had 31 points such that all hypercubic surfaces of the particular kind mentioned which pass through 15 of them pass through the complete system.

Inversion gives us a set of 32 points such that all hyperquartics passing* twice through the imaginary sphere at infinity which pass through 16 of the points pass through the remainder, hence in the line geometry we get a system of 32 lines such that all quadratic complexes through 16 of them pass through the remaining 16.

To prove the theorem without having recourse to the geometry of four dimensions, we would remark that all quadratic complexes through 16 lines pass through 16 other fixed lines (for any four quadratic complexes have 32 lines in common); so we consider the system of quadratic complexes through the first line, and the common lines 15 in number of the six complexes taken four at once; this gives 16 lines common to the complexes, and proceeding in this way we get the theorem, but it seems clearer to first state it for four-dimensional space because then we have the analogues in two and three dimensions to guide us.

In hyperspace, we may mention finally that we have 32 points lying by 16's on 12 spheres, so that there are six spheres through each point, and

In line geometry we have 32 lines lying by 16's on 12 linear complexes, there being six complexes through each line.

20. We remarked that in the hypersphere geometry there was one common point of the cubics on each plane of intersection of two hyperplanes; this corresponds in the line geometry to the following:—

Denote by a, b, c, d, e, f the six original complexes, and by A, B, C, D, E, F the complexes derived by omitting each of the original ones in turn, then such sets as A, B, C, D, e, f have a line in common.

Thus we get 32 sets of six having a line in common, viz.

$a\,b\,c\,d\,e\,f$ ... 1 of this type
$a\,b\,c\,d\,E\,F$ ... 15 „ „
$A\,B\,C\,D\,e\,f$ ... 15 „ „
$A\,B\,C\,D\,E\,F$ ... 1 „ „

shewing the complete symmetry of the system.

* Such a quartic is represented by the general equation of the second degree in hyperspherical coordinates in a manner exactly analogous to the bicircular quartic in circular coordinates, hence it is the analogue of a quadratic complex in line coordinates.

21. A particular case of the foregoing theorem may be here mentioned.

If we have five lines meeting a given line, as in the figure, a, b, c, d, e meet O, then any four of them as a, b, c, d have another line in common: thus we get five such lines and from the theory of the *double sixers* of a cubic surface we know that these five lines are met by another line F.

Now our theorem comes in, viz., if we take six lines a, b, c, d, e, f, then from each set of five we get a line like F, and the property is that these six lines are all met by one and the same straight line.

22. The transformation of these results by means of Lie's method, explained previously, gives rise to several results which, so far as I know, have not been noticed before.

In the first place it is to be remarked that in any such example of the general transformation two systems of lines conjugate with respect to the fundamental complex are really transformed from the line geometry, but as the descriptive properties of two such systems are identical, we need only concern ourselves with one of them.

23. Thus then to five lines meeting a given line we have five spheres touching a given one in prescribed senses, then for any four such spheres there is another tangent sphere associated with the given one. Thus from five spheres touching a given one we derive five others, viz., they are the inverses of the given sphere with respect to the five spheres cutting the tangent spheres at right angles in sets of four, and then the theorem is that these spheres are all touched by one and the same sphere.

Taking six spheres touching a given one we get six sets of five, from each set of five we derive by the preceding a sphere; our main theorem then shews that these six spheres so derived are touched by one and the same sphere.

24. Again suppose we have four spheres A, B, C, D passing through a common point; then since the representative lines for a point coincide and belong to the fundamental complex, the four pairs of lines corresponding to the four spheres meet one and the same straight line.

Let them be a, a', b, b', c, c', d, d' and apply the theorem of Art. 21, to the five lines a, a', b, c, d; then the sphere corresponding to the line $(abcd)$, i.e. the other line meeting a, b, c and d, is a sphere touching our four original spheres (A, B, C, D) in certain prescribed senses, and the sphere corresponding to the line $(a'bcd)$ touches BCD in the same sense as $(abcd)$ does, but touches A in the opposite sense.

The line $aa'bc$ meets also b' and c' since it belongs to the fundamental complex R of which a and a' are conjugate lines, hence the representative sphere is simply the other point of intersection of ABC.

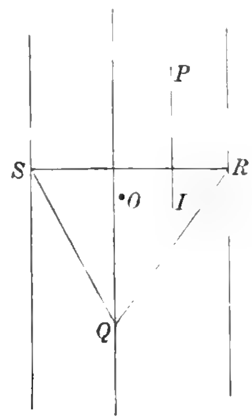
Similarly $aa'bd$ gives the other point of section of ABD,

and $aa'cd$,, ,, ,, ,, ,, ACD.

Hence, taking A, B, C, D to be planes, we infer at once that if $PQRS$ be a tetrahedron the eight spheres touching its faces are divisible into four pairs such that each pair touch the same sphere through the vertices Q, R, S, and similarly for any other three vertices.

25. The manner of dividing the eight touching spheres into the four pairs for the vertices Q, R, S is this (as follows from the above); viz. take any sphere touching the faces in prescribed senses, then the sphere paired with it is the sphere which touches the faces q, r, s on the same side as the original one, and the face p on the opposite side.

Thus in the usual sense of the word the inscribed sphere and the escribed sphere opposite P touch the same sphere through QRS.



The verification in the case where QRS is any triangle and P is at an infinite distance, is immediate; for let O be the circumcentre, I the incentre and $IP=r$, then the theorem is that the sphere, with O as centre and radius OQ, touches the sphere, centre P and radius IP, and this is so for

$$\begin{aligned} OP^2 &= OI^2 + IP^2 \\ &= OQ^2 - 2OQ \,.\, r + r^2 \\ &= (OQ - r)^2, \end{aligned}$$

which gives the verification required.

This is trivial, but it shews how to derive the expression for OI^2 from purely descriptive properties.

Taking five planes, or, what is practically equivalent, five spheres through the same point, we have in the line system five pairs of lines aa', bb', cc', dd', ee', meeting one and the same straight line; we apply our six line theorem to the lines a, a', b, c, d, e, and then translating our result into the language of spheres we get the following theorems.

"Given five planes they form five tetrahedra and taking definite sides of the planes we derive an inscribed sphere from each of these tetrahedra, the five spheres so derived are touched by one and the same sphere."

(This is in fact the translation of the *double sixer* theorem.)

"Taking one of the planes A, and definite sides of the others $BCDE$, then we derive from each set like $ABCD$ two inscribed spheres touching the same sphere through the circle circumscribing the $\triangle$ formed by the lines S, γ, δ in which B, C, D meet A; so we get four spheres, call them S_2, S_3, S_4, S_5; then also by the translation of the *double sixer* theorem, by taking one side of A we get one sphere, and by taking the other side a different sphere; we can assert in virtue of our theorem that these two latter spheres touch one of the spheres tangent to the spheres S derived above."

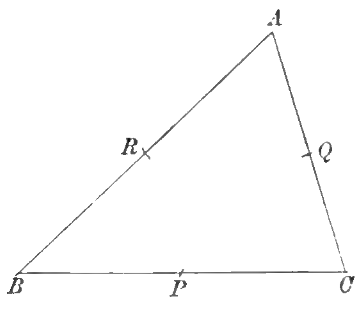
Again, taking the six lines a, a', b, c, d, e, we know from our theorems that the four lines derived from the sets

$$aa'bcd, \quad aa'bde, \quad aa'cde, \quad aa'bce,$$

and the lines a, a' are met by one and the same straight line. Now a line meeting both a and a' belongs to the fundamental complex and hence corresponds to a point, so we infer forthwith that the spheres S_2, S_3, S_4, S_5 above meet in a point on the plane A.

(This is really just the application of the four line theorem to the lines $S, \gamma, \delta, \epsilon$, but the foregoing serves to explain the theory.)

26. Taking the elementary proposition in plane geometry to the effect that if ABC be a triangle and P, Q, R points on the sides taken in order, then the circles AQR, BRP, CPQ meet in a point. I proceed to prove it and to develope similar propositions in three and four dimensions.



For this purpose we remark that all circular cubics through A, B, C, P, Q, R pass through one other fixed point; hence as three such cubics are

(1) the circle AQR, and the line BC,

(2) ,, BRP, ,, ,, CA,

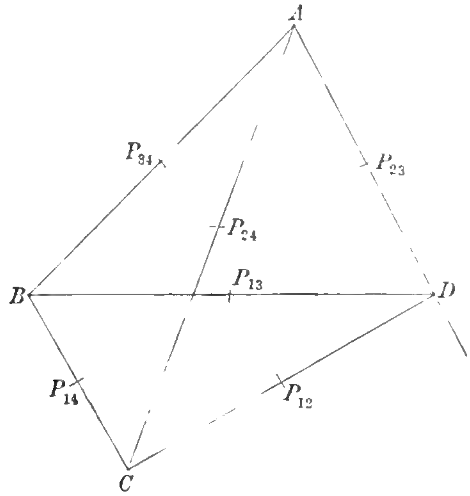
(3) ,, CPQ, ,, ,, AB,

we at once infer that the three circles meet in a point.

27. To prove the corresponding property of a tetrahedron take A, B, C, D for the vertices, a point P_{12} on the edge in which the faces opposite A and B intersect, and so on, then I proceed to shew that the spheres through the four sets of points

$$AP_{23}P_{34}P_{24},\quad BP_{14}P_{13}P_{34},\quad CP_{14}P_{12}P_{24},\quad DP_{12}P_{13}P_{23},$$

meet in a point.



In fact all cubic cyclides through $ABCD$ and the points P (i.e. through 10 points) pass through five other fixed points; but taking the sphere through A and the face opposite A as one such cyclide, and similarly for all four vertices, we find that as the 2nd, 3rd, and 4th spheres meet in the A plane, there is one common point in each face: hence there is one other common point, and it at once follows that the four spheres pass through the same point; a theorem due to Mr S. Roberts (*Proc. L. M. S.* 1890).

28. For the property in four dimensions we take five hyperplanes meeting in sets of four in five points and a point on the line of intersection of each three (10 points in all), then the five hyperspheres, each of which passes through a vertex and the points (four in number) on the edges meeting in that point, have one common point.

To prove this we consider cubics in four dimensions passing through the imaginary sphere at infinity and notice that all through the 15 points mentioned above pass through 16 other fixed points. (Cf. Art. 17.)

We take now the five degenerate cubics and remark that, by the last theorem, they have one common point in each hyperplane (by the 2nd theorem) and one in each plane of intersection of two hyperplanes (by the 1st theorem), 15 common points, and therefore there is one other common point, and it at once follows that the five hyperspheres meet in a point.

29. The theorem concerning hyperspheres and points may be translated into the language of linear complexes and straight lines as follows:—

Take five linear complexes having a line in common, each set of four have another line in common, thus we get five more lines. Then on the ruled surfaces common to each three of the linear complexes take any straight line (10 lines in all).

In each of the first five lines, these meet four complexes, then in each set of three of these, there is a line. Therefore there are four lines of the second set connected with any one of the first set and determining with it a linear complex; the theorem is that the five complexes so obtained have a line in common.

Of course, if the second set of 10 lines all lie on the same complex containing the original line, this reduces to the theorem already proved.

SECTION III.

30. Starting from the proposition regarding the circumcircles of the triangles formed by four lines, Clifford, in his paper on Miquel's Theorem, has shewn that this theorem is really the first of an infinite series of such theorems, viz., taking five lines we get five sets of four, from each set of four a point, therefore we get in all five such points, and these points lie on a circle; then taking six lines we get six circles by taking each set of five lines, these circles meet in a point, and so on *ad inf.*

The theorems are proved by considerations depending on parabolas of class $n+1$ touching the line at infinity n times, but as there seems to be no exact analogy to these parabolas in space of higher dimensions, the proof has to be modified before we can hope to find any corresponding propositions for spheres.

In the first place we remark that inasmuch as there is no corresponding theorem for the five circumspheres of the tetrahedra formed by five planes, there are no exactly analogous theorems for space; but as will appear presently, there is an infinite series of theorems in space to which there are not similar ones in the plane.

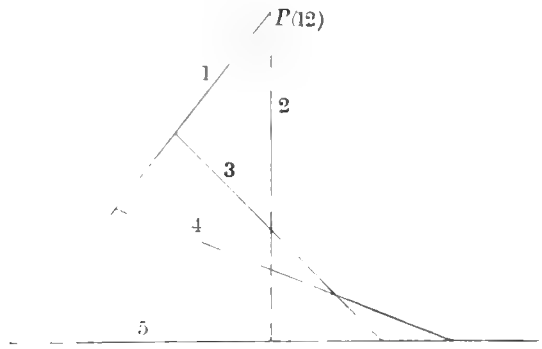
31. To explain the general principle of the proof adopted here, we remark that the theorem concerning the four lines becomes on inversion a theorem relating to four circles A, B, C, D meeting in a point, viz. the circles AB, BC, CA have each pair of them another point of section, and through the three points of section there is a circle, then the theorem derived by inversion is manifestly that the four circles so derived meet in a point.

Thus we have in the figure *eight points* lying by fours on eight circles, there being four circles through each point; in this form there is consequently complete symmetry, as has been remarked by several writers (S. Roberts, Cox, de Longchamps, &c.).

32. In like manner the five-line theorem gives 16 points lying by 5's on 16 circles, and so on.

But to prove the five-line theorem we have only to apply the four-line theorem in the inverted form.

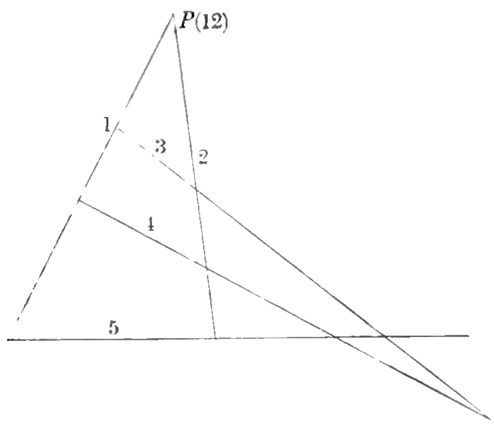
In fact denote the lines by 1, 2, 3, 4, 5, by $C(123)$ the circumcircle of the triangle



formed by 123, by $P(12)$ the intersection of 1 and 2, and by $O(5)$ the point derived from the lines 1, 2, 3, 4 by the line theorem.

I apply the inverted four-line theorem to the

line 1, and the circles $C(123)$, $C(124)$, $C(125)$ meeting in P_{12}.



Taking the trio 1, $C(123)$, $C(124)$ the derived circumcircle passes through $P(13)$, $P(14)$ and $O(5)$, therefore it is $C(134)$.

Similarly from 1, $C(124)$, $C(125)$ we derive $C(145)$;

and 1, $C(125)$, $C(123)$ „ $C(135)$;

while from $C(123)$, $C(124)$, $C(125)$ we derive the circle through $O(3)$, $O(4)$, $O(5)$; but the former three circles meet in $O(2)$, therefore $O(2)$, $O(3)$, $O(4)$, $O(5)$ lie on a circle, and similarly $O(1)$ lies on the same circle, which is the five-line theorem.

For six lines we apply the four-circle theorem to the circles $C(123)$, $C(124)$, $C(125)$, $C(126)$; this shews that four of the five-line circles meet in a point, and hence they all meet in the same point.

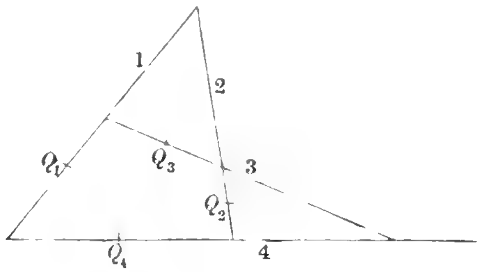
For seven lines we apply the five-circle theorem to the five circles

$$C(123),\quad C(124)\ldots C(127),$$

and so on *ad inf.*

Thus we get Clifford's set by means of the first only.

33. Taking now the more general theorem for the triangle, viz. that if points PQR are taken on the sides, the circles AQR, BRP, CPQ meet in a point, we proceed to inquire whether under any, and if so under what, circumstances the four points so derived from four lines lie on one circle.



Denoting the lines by 1, 2, 3, 4, the points on them by Q_1, Q_2, Q_3, Q_4, their points of section by P_{12}, &c., the point derived from the triangle 123 by $O(4)$, we then see that the following six sets of four points are concyclic, viz.:—

$$Q_1,\ Q_2,\ O(3),\ O(4):\quad Q_1,\ Q_3,\ O(2),\ O(4):\quad Q_2,\ Q_3,\ O_1,\ O_4:\quad Q_3,\ Q_4,\ O(1),\ O(2):$$
$$Q_2,\ Q_4,\ O(1),\ O(3);\quad Q_1,\ Q_4,\ O_2,\ O_3;$$

therefore the eight points

$$Q_1,\ Q_2,\ Q_3,\ Q_4,\ O_1,\ O_2,\ O_3,\ O_4$$

are such that any bicircular quartic through seven of them passes through the eighth, and hence the necessary and sufficient condition that O_1, O_2, O_3, O_4 should be on a circle is that Q_1, Q_2, Q_3, Q_4 should be on one circle.

34. Taking the points Q_1, Q_2, Q_3, Q_4 to be on a circle, I proceed to prove the result just arrived at, by means of the theorem for three lines.

In fact through the point Q_1 we have three circles, viz.

1, through P_{12}, Q_1, Q_2 say a.
2, ... P_{13}, Q_1, Q_3 ... b.
3, ... Q_1, Q_2, Q_3, Q_4 ... c,

a and b meet in O_4, } a point on c is Q_4.
b ...... c .... Q_3, } .... a ... O_3.
c ...... a .... Q_2, } .... b ... O_2.

Hence the three circles through

$$O_2,\ O_3,\ O_4:\quad Q_3,\ Q_4,\ O_2:\quad Q_2,\ Q_4,\ O_3$$

meet in a point.

The latter pair meet again in O_1,

therefore O_1, O_2, O_3 and O_4 are concyclic.

35. Next take five lines such that the five Q points on them are concyclic, and apply the three-circle theorem to the circles $C(12)$, $C(13)$, $C(14)$.

Three points on these circles are

$$O(125),\quad O(135),\quad O(145),$$

where $O(abc)$ is the point derived from the triangle a, b, c; and the circles intersect by pairs in the points

$$O(134),\quad O(124),\quad O(123).$$

Hence the circles through

$$O(134),\ O(135),\ O(145);\quad O(124),\ O(125),\ O(145):\quad O(123),\ O(125),\ O(135)$$

meet in a point.

The first of these circles is the circle derived from the lines 1, 3, 4, 5 by the four-line theorem, say $C(\bar{2})$, therefore $C\bar{2}$, $C\bar{3}$, $C\bar{4}$ meet in a point.

Similarly by considering

$$C(12),\quad C(13),\quad C(15)$$

we find that $C(\bar{2})$, $C(\bar{3})$, $C(\bar{5})$ meet in a point. One point of section of $C\bar{2}$, $C\bar{3}$ is plainly $O(145)$, and $C(\bar{4})$ does not pass through this, so it at once follows that the five circles

$$C(\bar{1}),\quad C(\bar{2}),\quad C(\bar{3}),\quad C(\bar{4}),\quad C(\bar{5})$$

meet in a point.

Thus given five lines and five concyclic points on them we get five sets of four, from each set of four a circle, and the five circles meet in a point.

36. Next take six lines with six concyclic points one on each of them and apply the four-line theorem to the circles $C(12)$, $C(13)$, $C(14)$, $C(15)$ through Q_1, there being on them the four concyclic points

$$O(126),\quad O(136),\quad O(146),\quad O(156),$$

viz., these points lie on the circle $C(16)$; but as this latter circle also passes through Q_1 we practically apply Clifford's *five*-line theorem to

$$C_{12},\quad C_{13},\quad C_{14},\quad C_{15},\quad C_{16}:$$

the point derived from the first four circles is the point derived from the lines (1, 2, 3, 4, 5) by the five-line theorem, hence the application tells us that *given six lines we have six sets of five, from each set of five a point, and the six points so derived lie on a circle.*

Then taking seven lines we apply Clifford's six-line theorem to

$$C(12),\quad C(13), \ldots C(17)$$

and we get the next theorem, viz.

Given seven lines, and seven concyclic points on them, we get seven sets of six, and from each set of six a point by the last theorem, the seven points so derived are concyclic.

Then taking eight lines, and eight concyclic points one on each line, we get eight sets of seven and from each set of seven a circle, these eight circles meet in a point.

And so on *ad inf.* as in Clifford's theorems, viz., we prove the theorem for n lines by applying Clifford's $n-1$ line theorem to the $n-1$ circles,

$$C_{12},\quad C_{13}, \ldots C_{14}.$$

This set of theorems reduces to Clifford's if we allow the circle on which the Q points lie, to degenerate into a straight line.

SECTION IV.

37. Adopting a method similar to that already used for the triangle we proceed to investigate similar extensions of the tetrahedron theorem.

For this purpose consider five planes 1, 2, 3, 4, 5 and a point P_{12} on each line of intersection.

Let the planes 1, 2, 3 meet the line 45 in the points L, M, N and apply the inverted tetrahedron theorem to the four spheres meeting in P_{45}, viz.,

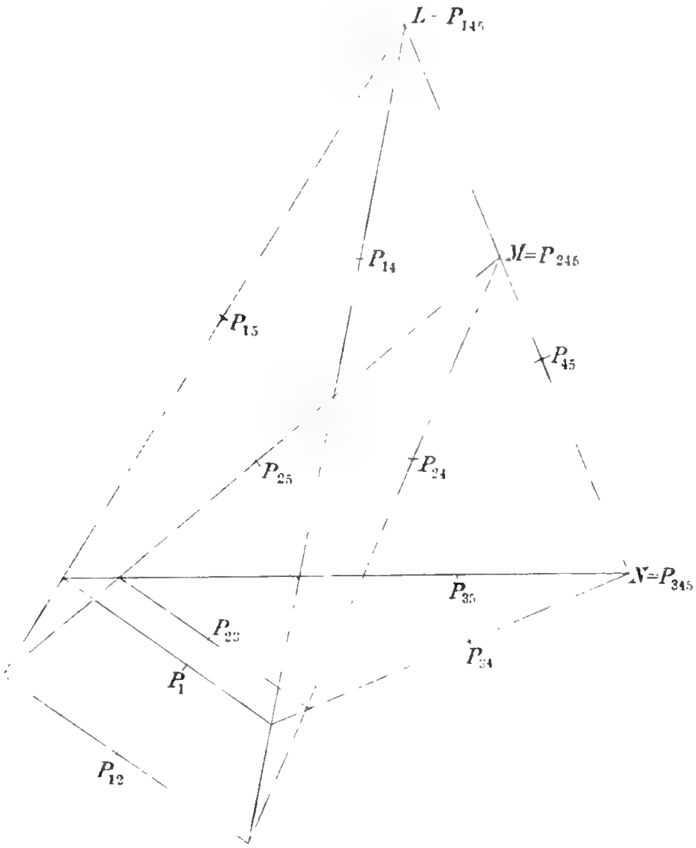
a, the sphere through L, P_{14}, P_{15}, P_{45},
b, ,, ,, ,, M, P_{24}, P_{25}, P_{45},
c, ,, ,, ,, N, P_{34}, P_{35}, P_{45},
d, the plane 4.

Then a, b, c meet in a point Q again,
a, b, d ,, ,, ,, $O(125)4$,
a, c, d ,, ,, ,, $O(135)4$,
b, c, d ,, ,, ,, $O(235)4$.

A point on a and b is $O(1245)$,
,, ,, ,, b, c ,, $O(2345)$,
,, ,, ,, c, a ,, $O(3145)$,
,, ,, ,, a, d ,, P_{14},
,, ,, ,, b, d ,, P_{24},
,, ,, ,, c, d ,, P_{34}.

And the notation shall now be explained.

In fact $O(125)4$ denotes the point derived from the triangle in which the planes 1, 2 and 5 meet the plane 4, by means of the triangle theorem.



Viz., it is the point of intersection of the three circles

$$L,\ P_{14},\ P_{45};\quad M,\ P_{24},\ P_{45};\quad P_{245},\ P_{14},\ P_{24},$$

denoting by P_{abc} the point where the planes a, b, c meet so that

$$L = P_{145},\quad M = P_{245},\quad N = P_{345}.$$

Then $O(1245)$ is the point derived from the tetrahedron formed by the planes 1, 2, 4, 5 by means of the tetrahedron theorem; in fact it is the point where meet the four spheres,

$$P_{145},\ P_{15},\ P_{14},\ P_{45};\quad P_{245},\ P_{25},\ P_{24},\ P_{45};\quad P_{124},\ P_{14},\ P_{24},\ P_{12};\quad P_{125},\ P_{15},\ P_{25},\ P_{12},$$

and for shortness we may denote these spheres by

$$S_{145},\quad S_{245},\quad S_{124},\quad S_{125},$$

respectively.

Thus, having explained the notation, we find that the four spheres

abc,	ab,	bc,	ca,
Q,	$O(1245)$,	$O(2345)$,	$O(3145)$,
abd,	ab,	bd,	da,
$O(125)4$,	$O(1245)$,	P_{24},	P_{14},
bcd,	bc,	cd,	ab,
$O(235)4$,	$O(2345)$,	P_{34},	P_{24},
cad,	ca,	ad,	cd,
$O(315)4$,	$O(3145)$,	P_{14},	P_{34},

meet in a point.

Consider the second of these spheres, its section by the plane 4 is the circle P_{14}, P_{24}, $O(125)4$, which manifestly passes through P_{124}, and since the sphere passes through P_{124}, P_{14}, P_{24}, and $O(1245)$, it is simply the sphere P_{124}, P_{14}, P_{24}, P_{12}, or S_{124}.

Similarly for the third and fourth, and hence we find that the four spheres

$$Q,\ O(1245),\ O(2345),\ O(3145);\quad S_{124},\ S_{234},\ S_{314},\ \text{meet in a point:}$$

and as the last three meet again in $O(1234)$ we infer that

$$O(1234),\quad O(3145),\quad O(2345),\quad O(1245),\ \text{and}\ Q$$

are cospheric.

Now Q is a point which may be denoted by Q_{45}, viz., it is the point of intersection of S_{145}, S_{245}, S_{345}.

Therefore by considering the plane 5 instead of 4 in our application of the tetrahedron theorem, we find that

$$O(1235),\quad O(3145),\quad O(2345),\quad O(1245)\ \text{and}\ Q_{45},$$

are cospheric, hence the five points,

$$O(1234),\quad O(1235),\quad O(1245),\quad O(1345),\quad O(2345),$$

lie on a sphere which passes through Q_{45}, and therefore by symmetry through all such Q points which are ten in number.

Hence we get the first extension, viz., *taking five planes and a point on each line of intersection, we have five sets of four, from each set of four a point is derived and the five points so obtained lie on one sphere.*

38. Consider now 6 planes, let the first four 1, 2, 3 and 4 meet the line 56 in L, M, N, R, and apply the tetrahedron theorem to the four spheres

$$S_{156},\quad S_{256},\quad S_{356},\quad S_{456},$$

meeting in P_{56}, then we denote by $Q(56)4$ the point of intersection of S_{156}, S_{256}, S_{356}, and observe that this point lies on the sphere S_4 obtained from the five planes 1, 2, 3, 5, 6 by the preceding theorem.

Calling the four spheres a, b, c, d for brevity, our theorem for the tetrahedron in its inverted form tells us that if we take a point on each circle of intersection of these four taken in pairs, then drawing a sphere through P_{abc}, P_{ab}, P_{bc}, P_{ca} the four spheres derived in this manner meet in a point.

Now for P_{ab} we may take $O(1256)$,

P_{ca} " " " $O(3456)$, and so on.

Consequently the four following spheres are concurrent,

$$Q(56)\,4,\quad O(1256),\quad O(2356),\quad O(3156),$$
$$Q(56)\,3,\quad O(1256),\quad O(2456),\quad O(4156),$$
$$Q(56)\,2,\quad O(1356),\quad O(3456),\quad O(4156),$$
$$Q(56)\,1,\quad O(2356),\quad O(3456),\quad O(4256),$$

but the first of these spheres is simply S_4 since all its four points lie on S_4 and are not coplanar, similarly for the others, and therefore S_1, S_2, S_3, S_4 meet in a point.

Consequently of the six spheres S_1, S_2, S_3, S_4, S_5, S_6 any four meet in a point, and it at once follows that they all meet in a point.

We may remark that the three spheres S_1, S_2, S_3, meet on S_{456} in virtue of the triangle theorem, so that this accounts for the other intersections of the six spheres taken in threes.

39. Taking now seven planes and a point on each line of intersection we apply the five-plane theorem in its inverted form to the 5 spheres S_{167}, S_{267} ... S_{567}, thus we find immediately that the points derived from the sets of six obtained by omitting 1, 2, 3, 4 and 5 in turn lie on a sphere.

Hence given seven planes we have seven sets of six, from each set of six a point is derived and the seven points so obtained lie on one sphere. &c., &c.

40. Hence we get the following infinite set of theorems analogous to Clifford's for plane geometry but possessing greater generality.

Given four planes, and a point on each line of intersection, we get four spheres meeting in a point.

[Before this we might place the obvious fact that given three planes and a point on each line of intersection we get a sphere through four points.]

Then given five planes and a point on each line of intersection, we have five sets of four each giving rise to a point, the five points are on a sphere.

Given six planes and a point on each line of intersection, we have six sets of five each giving rise to a sphere, the six spheres meet in a point.

Given seven planes, we have seven sets of six, from each of these we get a point, and the seven points so obtained lie on a sphere.

Given eight planes, we have eight sets of seven, the eight spheres so obtained meet in a point.

Given nine planes, we get nine points on a sphere, and so on ad inf., viz., to prove the theorem for n planes we apply the inverted form of that for $n-2$ planes to the $n-2$ spheres $S_{1,\,n-1,\,n}$, $S_{2,\,n-1,\,n} \ldots S_{n-2,\,n-1,\,n}$.

41. As has been already remarked, greater symmetry is given to the results just obtained, by inversion with respect to any point. In fact the planes we start from now become spheres through a given point, and the edges become their respective circles of intersection. Let us briefly consider the system derived from five planes.

Retaining our previous notation we perceive that a plane of the system contains 15 points, viz., the point at infinity, six vertices, four P-points and four points of the type $O(pqr)s$.

Again a sphere of the type S_{abc} contains 15 points, viz., one vertex, three P-points, two points of the type $O(abcd)$, six points of the type $O(abc)d$, and three of the type Q_{ab}.

The final sphere contains

five points of the type $O(abcd)$, and ten of the type Q_{ab}.

Hence we have in all 16 spheres, each containing 15 points.

The total number of points is 56, viz., one at infinity, ten vertices, ten P-points, 20 points of the type $O(abc)d$, ten of the type Q_{ab}, and five of the type $O(abcd)$.

Through the following there pass five spheres:—

The one at infinity, the P-points, and the points $O(abcd)$, and through the rest, viz.,

The vertices, the points $O(abc)d$, and the points Q_{ab}, there pass four.

Thus the whole system consists of 56 points lying by fifteens on sixteen spheres, there being five spheres through sixteen of the points and four through the remaining thirty.

SECTION V.

42. I proceed now to the discussion of the system of points derived in like manner from any number of planes. The processes involved are hardly more than mechanical, when once a comprehensive and luminous notation for the points and spheres of the system has been fixed upon.

Such a notation I shall now endeavour to explain.

43. The planes with which we start are denoted by the letter S with a single suffix, as S_a, &c.

Then, from three planes we derive a sphere which we call S_{abc}.

From five we derive a sphere S_{abcde}, and so on.

Again in the line of intersection of two planes we have a point O_{ab}.

From four planes we derive a point $O\,(abcd)$,

... six ....$O\,(abcdef)$,

and so on from any even number of planes.

Further* S_{abc}, S_{abd}, S_{acd} meet on S_a (line theorem),

S_{abc}, S_{abd}, S_{abe} .... S_{abcde} (five-plane theorem).

Also, in our proof of the five-plane theorem we may write

$$S'_c = S_{abc}, \quad S'_d = S_{abd}, \text{ and so on,}$$

and then inasmuch as S'_{cde}, S'_{cdf}, S'_{def} meet on S'_c,

we find that S_{abcde}, S_{abcdf}, S_{abcef} meet on S_{abc}.

Since also S'_{cde}, S'_{cdf}, S'_{cdg} meet on S_{cdefg},

we see that S_{abcde}, S_{abcdf}, S_{abcdg} meet on $S_{abcdefg}$.

Then applying these two results to the accented system we infer that

$S_{abcdefg}$, $S_{abcdegh}$, $S_{abcdehf}$ meet on S_{abcde},

$S_{abcdefg}$, $S_{abcdefh}$, $S_{abcdefi}$ meet on $S_{abcdefghi}$,

and so on *ad inf.*

44. The points where these sets of four spheres meet, lie on no other spheres of the system we are considering; but as regards an O-point, there are always n spheres through it, as we see as follows:

Through a point O_{ab} we have 2 of the type S_a and $n-2$ of the type S_{abc},

........................O_{abcde} .... 4 ....S_{abc} ... $n-4$ .... S_{abcde},

........................O_{abcdef} .... 6 ....S_{abcde} ... $n-6$ .... $S_{abcdefg}$,

and so on, n being the number of planes with which we start.

On any sphere there lie n O-points.

The sphere S_a contains O_0 (at infinity) and $n-1$ points of the type O_{ab},

.... S_{abc} .... 3 of the type O_{ab} ... $n-3$ .... O_{abcd},

.... S_{abcde} .... 5 ....O_{abcd} ... $n-5$ .... O_{abcdef},

and so on generally.

* $S_a\, S_b\, S_c$ meet on S_{abc} by hypothesis.

The number of O-points is

$$1+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)(n-3)}{4!}+\dots$$
$$=2^{n-1}.$$

The number of spheres is

$$n+\frac{n(n-1)(n-2)}{3!}+\dots$$
$$=2^{n-1}.$$

There are n O-points on each sphere, and n spheres through each such point.

We denote a point where meet S_a, S_b, S_c by A_0 (type),

.... S_{abc}, S_{abd}, S_{abe} ... A_2 ...

.... S_{abcde}, S_{abcdf}, S_{abcdg} ... A_4 ...

and so on. { Thus A_0 lies on one sphere of the type S_{abc},
...... A_2 .... S_{abcde},
and so on.

Also we denote a point where meet S_{abc}, S_{abd}, S_{acd} by A_1 (type),

.... S_{abcde}, S_{abcdf}, S_{abcef} ... A_3 ...

and so on. { A_1 lies on one sphere S_a,
A_3 .... S_{abc}.

Here is to be noticed that each A-point lies on four spheres.

The point of intersection considered being the one through which only four spheres pass.

The number of points A_0 is $\dfrac{n(n-1)(n-2)}{3!}$,

.... A_2 ... $\dfrac{n(n-1)}{2!}\,\dfrac{(n-2)(n-3)(n-4)}{3!}$,

.... A_4 ... $\dfrac{n(n-1)(n-2)(n-3)}{4!}\,\dfrac{(n-4)(n-5)(n-6)}{3!}$,

.... &c. ....

.... A_1 ... $n\,.\,\dfrac{(n-1)(n-2)(n-3)}{3!}$,

.... A_3 ... $\dfrac{n(n-1)(n-2)}{3!}\,\dfrac{(n-3)(n-4)(n-5)}{3!}$,

.... &c. ....

and therefore the total number of A-points is

$$\frac{n(n-1)(n-2)}{3!}\left\{1+n-3+\frac{(n-3)(n-4)}{2!}+\frac{(n-3)(n-4)(n-5)}{3!}\ldots\right\}$$

$$=\frac{n(n-1)(n-2)}{3!}2^{n-3}.$$

45. Finally each sphere contains $\frac{n(n-1)(n-2)}{2!}$ A-points, as we shall now prove.

The sphere S_a contains the following, viz.:

$$\frac{(n-1)(n-2)}{2!} \text{ points } A_0,$$

$$\frac{(n-1)(n-2)(n-3)}{3!} \quad ,, \quad A_1,$$

or

$$\frac{n(n-1)(n-2)}{3!} \quad ,, \quad \text{in all.}$$

The sphere S_{abc} contains

$$1 \text{ point } A_0,$$

$$3(n-3) \quad ,, \quad A_1,$$

$$\frac{3(n-3)(n-4)}{2!} \quad ,, \quad A_2,$$

$$\frac{(n-3)(n-4)(n-5)}{3!} \quad ,, \quad A_3,$$

viz.

$$\frac{n(n-1)(n-2)}{3!} \quad ,, \quad \text{in all.}$$

The sphere S_{abcde} contains

$$\frac{5.4.3}{1.2.3} \text{ points } A_2,$$

$$\frac{5.4}{1.2}(n-5) \quad ,, \quad A_3,$$

$$5.\frac{(n-5)(n-6)}{2!} \quad ,, \quad A_4,$$

$$\frac{(n-5)(n-6)(n-7)}{3!} \quad ,, \quad A_5,$$

$$\text{in all} = \frac{n(n-1)(n-2)}{3!}.$$

A sphere with $(2r+1)$ suffixes contains

$$\frac{(2r+1)(2r)(2r-1)}{3!} \quad \text{points } A_{2r-2},$$

$$\frac{(2r+1)\,2r}{2!}(n-2r-1) \quad ,, \quad A_{2r-1},$$

$$(2r+1)\frac{(n-2r-1)(n-2r-2)}{2!} \quad ,, \quad A_{2r},$$

$$\frac{(n-2r-1)(n-2r-2)(n-2r-3)}{3!} \quad ,, \quad A_{2r+1},$$

or

$$\frac{n(n-1)(n-2)}{3!} \quad ,, \quad \text{in all.}$$

46. Thus, to sum up, we have a system consisting of 2^{n-1} spheres, 2^{n-1} O-points, and $\frac{n(n-1)(n-2)}{3!}2^{n-3}$ A-points; each sphere contains n O-points and $\frac{n(n-1)(n-2)}{3!}$ A-points; each O-point lies on n spheres, and each A-point lies on four spheres.

E.g., $n=5$ gives us 16 O-points and 40 A-points, and there are 15 points on each sphere.

$n=6$ gives us 32 O-points and 160 A-points, and there are 26 points on each sphere.

The O-points form a system analogous to the whole system in Clifford's Theorems, viz., there are 2^{n-1} of them lying by n's on n spheres.

SECTION VI.

47. In the systems of points and spheres already considered, there are, it will be observed, two classes of points; through the smaller class of points in the general case there pass n spheres, while through the other only four spheres pass.

There is not, then, complete symmetry in any system, except that derived from four planes, and here we have five points, through each point pass four spheres, and on each sphere there lie four points.

I proceed now to explain how systems of points may be derived from more than four planes such that through each point pass the same number of spheres, the complete set of points and spheres being analogous to the inverted form of Clifford's Theorems.

48. For this purpose I remark that in the case of n planes, already considered, we took a point on each line of intersection; if these points be taken in one plane we get a system of points derived from $n+1$ planes, then taking each plane in turn as the additional one we have the foundation of the symmetrical system of points.

The general method is to apply the inverted form of theorems obtained previously to the case of the spheres passing through the point where three of the planes meet. The general systems seem rather complicated, though the difficulties are perhaps superficial rather than essential, so in the present pages only the complete systems derived from *five*, *six*, and *seven* planes will be considered.

49. It may be convenient to state the results at once and obtain them afterwards. They are as follows:—

From five planes we get a system of 16 points lying by 8's on 10 spheres, there being five spheres through each point.

From six planes we get a system of 72 points lying by 16's on 27 spheres, there being six spheres through each point.

From seven planes we derive a system of 576 points lying by 32's on 126 spheres, there being seven spheres through each point.

In the case of eight planes the system will consist of a number of points lying by 64's on spheres and there will be eight through each point, but I do not go into this fully at present.

50. I. *Five planes.*

Here all is well known, viz., if the planes be a, b, c, d, e, the circumspheres of the *four* tetrahedra formed by e and the others meet on e, and similarly for a, b, c and d. Thus we have 10 vertices and one point in each plane and then inverting with respect to any point we obtain the complete system as indicated.

51. II. In the case of *six* planes we use the following notation:—

P_{123} for the point of intersection of the planes 1, 2, 3; Q_{123} for the other point of intersection of the three circumspheres through this point; S_6 for the sphere derived from the first five planes by taking the subsidiary points on the 6th, and so on. Also $O(abcd)e$ means the point derived from a, b, c, d when the subsidiary points lie on e.

Thus through the point Q_{123} there pass six spheres, viz., three circumspheres and the three spheres S_1, S_2, S_3. So far then we have 71 points and through each of them pass 6 spheres, viz.

One at infinity through which the six planes pass.

20 vertices through which pass three planes and three circumspheres.

20 Q-points.

30 points, five in each plane, the five in fact derived by taking the other planes in sets of four. Through these pass one plane, one S_a, and four circumspheres.

We shall now shew that the spheres S_1, S_2, S_3, S_4, S_5, S_6 meet in a point.

In fact, apply the inverted form of the tetrahedron theorem to the four spheres S_{1235}, S_{2345}, S_{3415}, S_{4125}, which meet in a point on the plane 5. Calling them a, b, c, d for shortness, we remark that

a, b, c meet in the point $O\,(1235)\,4$,

on a, b	is the	point	Q_{345},
,, b, c	,,	,,	Q_{145},
,, c, a	,,	,,	Q_{245},

hence the sphere derived from a, b, c is simply S_4.

Consequently S_1, S_2, S_3, S_4, meet in a point, and in like manner so do any four of the six, consequently the six meet in one point.

Now we have 72 points, and there are 27 spheres, viz.,

6 planes, 15 circumspheres and S_1, S_2, S_3, S_4, S_5, S_6.

On a plane lie ∞, 10 vertices, and 5 O-points.

On a circumsphere lie 4 vertices, 8 O-points, and 4 Q-points.

On an S sphere lie 5 O-points, 10 Q-points, and the point last obtained.

Thus inverting we have the complete system already indicated, and we may remark that starting from the six spheres meeting in any point we can derive the whole system in exactly the same way as we have derived it from the six planes.

52. III. Taking now the case of seven planes we denote them by 1, 2, 3, 4, 5, 6, 7.

$S\,(1234)$ denotes the circumsphere of the tetrahedron formed by the planes 1, 2, 3, 4.

$S\,(\bar{a}, \underline{b})$ denotes the sphere derived by omitting the plane a and taking the subsidiary points on the plane $\underline{b}$, and in general $\bar{a}$ means that the plane a is omitted; $\underline{b}$ means that the subsidiary points are taken on b.

The three planes 1, 2, 3 meet in P_{123}, and the four circumspheres through this point meet again in sets of three in four new points.

The sphere through these points belongs to our system, and we denote it by $S\,(123)$.

The further notation is explained as it is introduced.

We apply some of our previous results to the system

$$1,\ 2,\ 3,\ S\,(12\overset{\alpha}{3}4),\ \ S\,(12\overset{\beta}{3}5),\ \ S\,(12\overset{\gamma}{3}6),\ \ S\,(12\overset{\delta}{3}7),$$

and denote the four latter, for brevity, by α, β, γ, δ respectively.

There is little difficulty in seeing that

$$S(\alpha\beta\gamma\delta) = S(123),$$
$$S(1\beta\gamma\delta) = S(\bar{4}\ \underline{1}),$$
$$S(2\beta\gamma\delta) = S(\bar{4}\ \underline{2}),$$

and so on.

Hence as $S(1\alpha\beta\delta)$, $S(1\beta\gamma\delta)$, $S(1\gamma\alpha\delta)$, $S(\alpha\beta\gamma\delta)$ meet on δ, we infer that

S_{123} passes through the point on $S(1237)$ in which concur $S(\bar{4}\underline{1})$, $S(\bar{5}\underline{1})$, $S(\bar{6}\underline{1})$.

By symmetry, then, through the same point also pass $S(137)$, $S(127)$, and hence through all such points as this we have seven spheres.

From our six-plane theorems we infer that the spheres

$$S(1\beta\gamma\delta),\quad S(2\beta\gamma\delta),\quad S(\alpha\beta\gamma\delta),\quad S(\bar{3}\underline{\beta}),\quad S(\bar{3}\underline{\gamma}),\quad S(\bar{3}\underline{\delta})$$

meet in a point; we proceed to shew that $S(\bar{3}\underline{\beta})$ is simply the sphere $S(125)$.

One point on $S(\bar{3}\underline{\beta})$ is the point where meet $S(12\beta\alpha)$, $S(12\beta\gamma)$, $S(12\beta\delta)$.

Now
$$S(12\beta\alpha) = S(1245),$$
$$S(12\beta\gamma) = S(1256),$$
$$S(12\beta\delta) = S(1257),$$

these being easy deductions from the theorem regarding the circumcircles of the triangle formed by four lines.

Thus one point on $S(\bar{3}\underline{\beta})$ is the point where meet $S(1254)$, $S(1256)$, $S(1257)$, or as we may call it, $Q(125\bar{3})$.

Another point on $S(\bar{3}\underline{\beta})$ is the point on which meet

$$S(12\alpha\beta),\quad S(12\gamma\beta),\quad S(1\alpha\gamma\beta),\quad S(2\alpha\gamma\beta),\quad \alpha,\quad \beta.$$

Now these are respectively

$$S(1245),\quad S(1265),\quad S(\bar{7}\underline{1}),\quad S(\bar{7}\underline{2})\ \text{and}\ S(1235);$$

therefore on $S(\bar{3}\underline{\beta})$ is the point where meet

$$S(1254),\quad S(1256),\quad S(1253);$$

that is the point
$$Q(125\bar{7}).$$

Similarly the points $Q(125\bar{4})$, $Q(125\bar{6})$ are on the sphere, and therefore it is the sphere $S(125)$.

In like manner
$$S(\bar{3}\underline{\gamma}) = S(126),$$
$$S(\bar{3}\underline{\delta}) = S(127),$$

and hence we infer that

$$S(\bar{4}\underline{1}),\quad S(\bar{4}\underline{2}),\quad S(123),\quad S(125),\quad S(126),\quad S(127)$$

meet in a point.

This point may be called $R(12\bar{4})$ without confusion.

53. Again $S(\beta\gamma\delta)$ passes through the point on $S(\alpha\beta\gamma\delta)$ in which concur

$$S(\bar{1}\underline{\beta}),\quad S(\bar{2}\underline{\beta}),\quad S(\bar{3}\underline{\beta}),$$

i.e. through the point in which meet

$$S(123),\quad S(125),\quad S(315),\quad S(235).$$

Similarly through the point on $S(1\beta\gamma\delta)$ in which meet $S(\bar{2}\beta)$, $S(\bar{3}\underline{\beta})$, $S(\bar{\alpha}\beta)$.

Now
$$S(1\beta\gamma\delta) = S(\bar{4}1),$$
$$S(\bar{2}\beta) \quad = S(135),$$
$$S(\bar{3}\underline{\beta}) \quad = S(125),$$

and I proceed to prove that $S(\bar{\alpha}\underline{\beta}) = S(\bar{4}\underline{5})$ so that in this point there meet

$$S(\bar{4}1),\quad S(\bar{4}\underline{5}),\quad S(135),\quad S(125):$$

therefore through the same* point pass $S(165)$, $S(175)$.

Consequently $S(bcd)$ passes through the following points:

$$R(15\bar{4}),\quad R(16\bar{4}),\quad R(17\bar{4}),$$
$$R(25\bar{4}),\quad R(26\bar{4}),\quad R(27\bar{4}),$$
$$R(35\bar{4}),\quad R(36\bar{4}),\quad R(37\bar{4}).$$

If we interchange 3 and 5 four of these points are unaltered so the sphere is unaltered, and thus if any of the numerals 1, 2, 3, 5, 6, 7 are interchanged the sphere is unaltered; therefore this sphere may be consistently denoted by $S(4)$ for it is symmetrically situated with respect to 1, 2, 3, 5, 6, 7.

54. We have now to identify

$$S(\bar{\alpha}\underline{\beta}) \text{ and } S(\bar{4}\underline{5}).$$

A point on $S(\bar{\alpha}\underline{\beta})$ is where meet

$$S(123\beta),\quad S(12\beta\gamma),\quad S(13\beta\gamma),\quad S(23\beta\gamma),$$

i.e. where meet 5, $S(1256)$, $S(1356)$, $S(2356)$ and $S(1235)$,
we shall call this point $O(1236)\,5$.

Another such point is manifestly where meet

$$S(123\beta),\quad S(12\beta\delta),\quad S(13\beta\delta),\quad S(23\beta\delta) \text{ or } O(1237)\,5.$$

A third point is where meet

$$S(12\gamma\beta),\quad S(12\delta\beta),\quad S(1\gamma\delta\beta),\quad S(2\gamma\delta\beta),$$

or the point where meet

$$S(1256),\quad S(1257),\quad S(\bar{4}1),\quad S(\bar{4}\underline{2}):$$

and through this same point pass also

$$S(1253) \text{ and } S(\bar{4}\underline{5})$$

for it must be the point $Q(125\,\bar{4})$.

* This point is $R(154)$.

A fourth point is where meet

$$S(13\beta\gamma),\quad S(13\delta\beta),\quad S(1\gamma\delta\beta),\quad S(2\gamma\delta\beta),$$

and passing through this point we therefore have

$$S(1356),\quad S(1357),\quad S(\bar{4}\underline{1}),\quad S(\bar{4}\underline{2}),\quad S(1352),\quad S(\bar{4}\underline{5}).$$

Similarly, we obtain a fifth point, and so on, and as all lie on $S(\bar{4}\underline{5})$, we have

$$S(\bar{\alpha}\underline{\beta}) = S(\bar{4}\underline{5}).$$

Now we have proved that

$$S(\beta\gamma\delta) = S(4),$$

so in like manner

$$S(\gamma\delta\alpha) = S(5),$$
$$S(\delta\alpha\beta) = S(6),$$
$$S(\alpha\beta\gamma) = S(7).$$

But we have seen that $S(\beta\gamma\delta)$ passes through the point on $S(\alpha\beta\gamma\delta)$ in which meet

$S(\bar{1}\underline{\beta}),\quad S(\bar{2}\underline{\beta}),\quad S(\bar{3}\underline{\beta})$ (see p. 184, line 7),

i.e. $S(4)$ passes through the point in which meet

$$S(123),\quad S(235),\quad S(315),\quad S(125).$$

Therefore four spheres of the types just written down meet in a point, and this point by symmetry is on S_4, S_6 and S_7.

Hence $$S(\beta\gamma\delta),\quad S(\gamma\delta\alpha),\quad S(\delta\alpha\beta),\quad S(\alpha\beta\gamma)$$

meet in a point and therefore the spheres

$$S(4),\quad S(5),\quad S(6),\quad S(7)$$

meet in a point.

Thus any four of the seven S_1, S_2, S_3, S_4, S_5, S_6, S_7 meet in a point, hence they all meet in a point and this point completes the system.

55. We shall shew the connection of the final points in the six-plane system with our present configuration. Viz., we know that

$$S(1\beta\gamma\delta),\quad S(2\beta\gamma\delta),\quad S(3\beta\gamma\delta),$$
$$S(\dot{\alpha}\underline{\beta}),\quad S(\alpha\gamma),\quad S(\bar{\alpha}\delta),$$

meet in a point which is on $S(\beta\gamma\delta)$.

Consequently

$$S(\bar{4}\underline{1}),\quad S(\bar{4}\underline{2}),\quad S(\bar{4}\underline{3}),\quad S(\bar{4}\underline{5}),\quad S(\bar{4}\underline{6}),\quad S(\bar{4}\underline{7}),$$

meet in a point which is on $S(4)$, and this point is manifestly the point derived from the six planes 1, 2, 3, 5, 6, 7. Calling it N_4, we see that $S(r)$ passes through $N(r)$ for each value of r.

56. The complete system of points obtained then is as follows:—

A. **1 at ∞** through which pass the seven planes.

B. $\mathbf{\frac{7 . 6 . 5}{1 . 2 . 3} = 35}$ **vertices** through which pass three planes and four circumspheres.

C. $\mathbf{7 \times \frac{6 . 5}{1 . 2} = 105}$ **points** of the type $O(1234)5$ through which pass one plane, four circumspheres, and two spheres $S(6\underline{5})$, $S(\bar{7}5)$.

D. $\mathbf{7 \times 1 = 7}$ **points** of the type $O(123456)7$, through which pass one plane and the 6 spheres

$$S(\bar{1}\underline{7}),\quad S(\bar{2}\underline{7})\ldots\ldots S(\bar{6}\underline{7}).$$

E. $\mathbf{7 \times \frac{6 . 5 . 4}{3 . 2 . 1} = 140}$ **points** of the type $Q(123\bar{7})$, through which pass three circumspheres,

$$S(123),\quad S(\bar{7}1),\quad S(72),\quad S(\bar{7}\underline{3}).$$

F. $\mathbf{7 \times \frac{6 . 5 . 4}{1 . 2 . 3} = 140}$ **points** in which meet such sets of spheres as

$$S(123),\quad S(127),\quad S(137),\quad S(1237),\quad S(\bar{4}1),\quad S(\bar{5}\underline{1}),\quad S(\bar{6}\underline{1}).$$

G. $\mathbf{7 \times \frac{6 . 5}{1 . 2} = 105}$ **points** of the type $R(12\bar{4})$, through which pass

$$S(\bar{4}1),\quad S(\bar{4}\underline{2}),\quad S(123),\quad S(125),\quad S(126),\quad S(127).$$

H. $\mathbf{\frac{7 . 6 . 5}{3 . 2 . 1} = 35}$ **points** in which meet such sets of spheres as

$$S(123),\quad S(234),\quad S(341),\quad S(412),\quad S(5),\quad S(6), \text{ and } S(7).$$

I. **7 points N** in which meet such sets as

$$S(\bar{7}\underline{1}),\quad S(7\underline{2}),\quad S(\bar{7}\underline{3}),\quad S(\bar{7}\underline{4}),\quad S(7\underline{5}),\quad S(\bar{7}\underline{6}) \text{ and } S(7).$$

J. **A final point** in which meet

$$S_1,\quad S_2,\quad S_3,\quad S_4,\quad S_5,\quad S_6 \text{ and } S_7.$$

In all we have

$$1 + 35 + 105 + 7 + 140 + 140 + 105 + 35 + 7 + 1 = 576 \text{ points.}$$

Of spheres there are 126 made up as follows: 7 planes, 35 circumspheres, 35 of the type $S(123)$, 42 of the type, $S(\bar{4}\underline{1})$ and 7 of the type $S(r)$.

Also each sphere contains 32 points; we shall enumerate the points for each class of sphere in a table; for this purpose we denote the classes of points by the capital letters opposite them.

	A	B	C	D	E	F	G	H	I	J	Total
Plane contains	1	15	15	1							32
Circumsphere		4	12		12	4					32
$S(123)$					4	12	12	4			32
$S(\bar{4}1)$			5	1	10	10	5		1		32
$S(r)$							15	15	1	1	32
Total number of points of each class	1	35	105	7	140	140	105	35	7	1	576

SECTION VII.

57. In virtue of the general principles explained early in this paper, all our propositions relating to spheres and points may be immediately transformed in two different ways.

I. All points are replaced by lines belonging to a given linear complex, and all spheres by complexes in involution with the given one.

II. Then we may replace all the lines by circles and the linear complexes by circles with associated angles in such a way that two intersecting lines correspond to two circles which touch, and a line belonging to a linear complex becomes a circle cutting a given circle at a given angle.

58. I will not trouble to translate all the propositions obtained above in this way, but give only some general considerations, which will afford an idea as to the nature of the propositions obtained in each of the two classes.

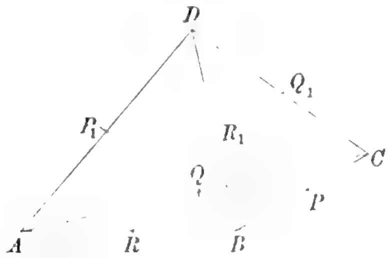
59. Let us take the tetrahedron theorem, viz., points PQR, $P_1Q_1R_1$ are taken on the sides of a tetrahedron as in the figure, then the spheres

$$AQRP_1,\quad BRPQ_1,\quad DP_1Q_1R_1,\quad CPQR_1$$

meet in a point.

Taking a fixed linear complex R, the four planes become complexes in involution with it and having one of their common lines belonging to R. Any three of them and

R have a further line common, the four lines so obtained correspond to A, B, C, D, call them a, b, c, d; also in the ruled surface common to each two and R we take a



line, the six lines so obtained correspond to P, Q, R, P_1, Q_1, R_1 and are denoted by p, q, r, p_1, q_1, r_1; then d, p_1, q_1, r_1 and similar quartettes determine a linear complex in involution with R, and the theorem is that one of the two lines common to the four linear complexes so obtained belongs to R.

Taking five linear complexes, in involution with a given one R and containing a given line of it, and a line in the ruled surface common to each pair and R, we get from each set of four a line belonging to R, these five lines belong to the same complex in involution with R.

Then taking six complexes (still in involution with R) we get six sets of five, from each set of five is derived a linear complex by the last theorem, and the six complexes so obtained are such that they have a common line belonging to R: and so on *ad inf.*

Further, starting from five complexes in involution with R we can build up a set of **16** lines belonging to R lying by 8's in **10** linear complexes in involution with R, there being **5** complexes containing each line.

Then, starting from six complexes, we find a system of **72** lines in a linear complex lying by **16**'s in **27** linear complexes in involution with the given one.

Starting from seven complexes we find a system of **576** lines in a linear complex lying by **32**'s in **126** linear complexes in involution with the given one.

60. In circle geometry we get the following:—

Four circles L, M, N, R cut a given one O at angles λ, μ, ν, ρ respectively, cutting L, M, N at angles λ, μ, ν we have an associated circle (viz. the inverse of O with respect to the orthogonal circle of L, M, N); call this R' and derive $L'M'N'$ in a similar manner.

Then take any circle P_2 cutting M and N at angles μ, ν

....	Q_2	N ...	L	ν, λ
....	R_2	L ...	M	λ, μ
....	P_1	L ...	R	λ, ρ
....	Q_1	M ...	R	μ, ρ
....	R_1	N ...	R	ν, ρ.

We have a circle D cutting $R'P_2Q_2R_2$ at equal angles δ (say)

.... A .... $P'P_2Q_1R_1$ .... α ...

.... B .... $Q'Q_2R_1P_1$ .... β ...

.... C .... $R'R_2P_1Q_1$ .... γ ...

and the theorem is that there exists a circle cutting A, B, C, D at angles α, β, γ, δ respectively.

61. In this connection there arises a difficulty, which it is not very easy to satisfactorily explain, viz., as to which of the eight circles (in the extended sense of the word) cutting $R'PQR$ at equal angles is to be selected.

To settle this point we must remark that circles cutting a given circle at a given angle correspond to the same linear complex as those which cut it at the supplementary angle, but in the one case their radius is taken positively, and in the other negatively.

If now R' cuts L, M, N at the same angles *precisely* as O does, then the radius of R' is to be taken positively, but if it cuts them at the supplementary angles the radius must be taken negatively.

Similar considerations apply to $L'M'N'$ and also to the circles $P_2Q_2R_2$, $P_1Q_1R_1$.

Then the circle cutting $R'P_2Q_2R_2$ at equal angles in our theorem is that one which cuts those of negative radius at the one angle and those of positive radius at the supplementary angle, so that in point of fact there is no ambiguity about the theorem, though it is difficult to state it precisely and concisely.

Like considerations must settle the sign to be given to the sign of the radius of the final circle.

62. Then taking five circles cutting the given one at angles α, β, γ, δ, ϵ, we derive from each set of four a circle by the foregoing, and its radius sign can be determined also, then the five circles so obtained are "cut at equal angles" by one and the same circle, and so on *ad inf.*

63. Starting from five circles we can build up a system of **16** circles, such that they are cut in sets of **8** at equal angles by ten circles.

From six circles we find a system of **72** circles, cut in sets of **16** at equal angles by 27 circles.

From seven circles we build up a system of **576** circles, cut in sets of **32** at equal angles by 126 other circles, each of the **576** being cut by seven of the latter system of circles.

In these enunciations an angle and its supplement are taken to be identical; it would take too long to make these theorems precise from this point of view.

VIII. *Partial Differential Equations of the Second Order, involving three independent variables and possessing an intermediary integral.* By Professor A. R. Forsyth.

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1. The theory of partial differential equations of the first order in one dependent variable and any number of independent variables may be regarded as fairly complete: and, though in a slighter degree, the same may be said of partial differential equations of the second order in one dependent variable and two independent variables. But for equations of order higher than the first involving more than two independent variables, the amount of progress made is slight as compared with what has been secured in the cases already mentioned.*

The present paper deals with those partial differential equations of the second order, involving one dependent variable (say v) and three† independent variables (say x, y, z), and possessing an intermediary integral of the first order. The derivatives of v of the first order with regard to x, y, z are taken to be l, m, n; those of the second order are taken to be a, b, c, f, g, h.

2. The memoir by Vivanti already quoted deals with such equations as in their intermediary integral contain an arbitrary function of two arguments. The general integral of this intermediary equation will introduce another arbitrary function of two arguments; and thus, in the primitive, a couple of arbitrary functions, each of two arguments, will occur—a result which is a particular case of a more general theorem.

* The chief memoirs upon the subject with which I am acquainted are the following:—

Bäcklund, *Math. Ann.*, t. xi. (1877), pp. 199—241.
,, *ib.*, t. xiii. (1878), 68—108.
,, *ib.*, t. xiii. (1878), 411—428.
Beudon, *Comptes Rendus*, t. cxxi. (1895), pp. 808—811.
Hamburger, *Crelle*, t. c. (1887), pp. 390—404.
Sersawy, *Wiener Denkschr.*, t. xlix. (1885), pp. 1—104.
Tanner, *Proc. Lond. Math. Soc.*, t. vii. (1876), pp. 43—60.
,, ,, *ib.*, t. vii. (1876), pp. 75, 90.
,, ,, *ib.*, t. ix. (1878), pp. 41—61.
,, ,, *ib.*, t. ix. (1878), pp. 76—90.
Vivanti, *Math. Ann.*, t. xlviii. (1897), pp. 474—513.
v. Weber, *Math. Ann.*, t. xlvii. (1896), pp. 230—262.

And it should be added that, in the development of the subject, I am indebted to the memoir by Imschenetsky, *Grunert's Archiv*, t. liv. (1872), pp. 209—360, and the memoir by Goursat, *Acta Mathematica*, t. xix. (1895), pp. 285—340: both of which deal with partial differential equations of the second order in two independent variables. The present paper gives the extended form of several of their results.

† Many of the results can immediately be generalised to the case when the number of independent variables is n; it has not seemed necessary to state these explicitly.

Vivanti's investigation is, in fact, the extension to three independent variables of the Monge-Boole problem in two independent variables; as he enters into considerable detail, it will be discussed only briefly here and by a different method.

It need, however, hardly be remarked that the generalisation thus effected is not the only possible source of an equation of the second order having an intermediary integral. For example, if

$$U = 0,$$

be an equation involving v, l, m, n, x, y, z, then a combination of

$$U = 0, \quad \frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0,$$

leads to an equation or equations of the second order having $U = 0$ for an intermediary integral. It is unnecessary to specify the mode of combination; it might arise from the elimination of three arbitrary constants, or from the elimination of one arbitrary functional form and one arbitrary constant, or in other ways. But in view of the developments effected in connection with the Monge-Boole form, it is natural to begin with equations of a corresponding form.

We accordingly, in the first place, assume the existence of an intermediary integral of the form

$$F(\theta, \phi, \psi) = 0,$$

where F is an arbitrary functional form and θ, ϕ, ψ are definite functions of v, l, m, n, x, y, z. Then in order to construct an equation of the second order having $F = 0$ for an intermediary integral, it is sufficient to eliminate $\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \phi}, \frac{\partial F}{\partial \psi}$, between the three equations

$$\left.\begin{aligned} \frac{\partial F}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dx} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dx} &= 0 \\ \frac{\partial F}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dy} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dy} &= 0 \\ \frac{\partial F}{\partial \theta}\frac{d\theta}{dz} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dz} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dz} &= 0 \end{aligned}\right\},$$

where $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ respectively denote

$$\frac{\partial}{\partial x} + l\frac{\partial}{\partial v} + a\frac{\partial}{\partial l} + h\frac{\partial}{\partial m} + g\frac{\partial}{\partial n},$$

$$\frac{\partial}{\partial y} + m\frac{\partial}{\partial v} + h\frac{\partial}{\partial l} + b\frac{\partial}{\partial m} + f\frac{\partial}{\partial n},$$

$$\frac{\partial}{\partial z} + n\frac{\partial}{\partial v} + g\frac{\partial}{\partial l} + f\frac{\partial}{\partial m} + c\frac{\partial}{\partial n}.$$

The result of the elimination is

$$\begin{aligned} &D\Delta \\ &+PA+QB+RC+2SF+2TG+2UH \\ &+Ia+Jb+Kc+2Lf+2Mg+2Nh+W=0, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2.$$

$$A=bc-f^2, \quad B=ca-g^2, \quad C=ab-h^2,$$

$$F=gh-af, \quad G=hf-bg, \quad H=fg-ch.$$

Moreover, the coefficients of the various combinations of a, b, c, f, g, h depend upon θ, ϕ, ψ; in fact, we have

$$\begin{aligned} &D=(lmn), \qquad W=(xyz), \\ &P=(xmn), \quad Q=(lyn), \quad R=(lmz), \\ &I=(lyz), \quad J=(xmz), \quad K=(xyn), \\ &2S=(lzn)+(lmy), \quad 2L=(xnz)+(xym), \\ &2T=(zmn)+(lmx), \quad 2M=(nyz)+(xyl), \\ &2U=(ymn)+(lxn), \quad 2N=(myz)+(xlz), \end{aligned}$$

where $(\alpha\beta\gamma)$ denotes the Jacobian of θ, ϕ, ψ with regard to α, β, γ, for the various combinations: the derivatives with regard to l, m, n being purely partial, and those with regard to x, y, z being $\frac{\partial}{\partial x}+l\frac{\partial}{\partial v}$, $\frac{\partial}{\partial y}+m\frac{\partial}{\partial v}$, $\frac{\partial}{\partial z}+n\frac{\partial}{\partial v}$ respectively.

It is manifest that, as the fourteen coefficients are dependent upon θ, ϕ, ψ, certain relations among them, some algebraical and some functional, must be satisfied.

3. But though the form obtained for the equation of the second order is the only form which can possess an intermediary integral of the general functional type assumed, it does not follow that an equation of that form necessarily possesses such an intermediary integral, or indeed any intermediary integral; as already pointed out, conditions must be satisfied in order to ensure the existence of any intermediary integral. To obtain these conditions, we proceed as follows.

Let an intermediary integral be supposed to exist, say in the form

$$u=u(v,\ x,\ y,\ z,\ l,\ m,\ n)=0.$$

Then writing

$$u_x=\frac{\partial u}{\partial x}+l\frac{\partial u}{\partial v}, \quad u_l=\frac{\partial u}{\partial l},$$

$$u_y=\frac{\partial u}{\partial y}+m\frac{\partial u}{\partial v}, \quad u_m=\frac{\partial u}{\partial m},$$

$$u_z=\frac{\partial u}{\partial z}+n\frac{\partial u}{\partial v}, \quad u_n=\frac{\partial u}{\partial n},$$

we have

$$\left.\begin{aligned} u_x + au_l + hu_m + gu_n &= 0 \\ u_y + hu_l + bu_m + fu_n &= 0 \\ u_z + gu_l + fu_m + cu_n &= 0 \end{aligned}\right\}.$$

The given differential equation must be satisfied in virtue of these equations; in other words, it must become an identity when taken in connection with them.

Hence, solving the three equations for a, b, c respectively and substituting in the equation

$$Ia + Jb + Kc + 2Lf + 2Mg + 2Nh$$
$$+ PA + QB + RC + 2SF + 2TG + 2UH + D\Delta + W = 0,$$

the last equation must become an identity and the coefficients of the various combinations of f, g, h must vanish: *the conditions of evanescence are the partial differential equations of the first order determining* u. We have

$$a = -\frac{u_m u_n}{u_l}\left[\frac{u_x}{u_m u_n} + \frac{g}{u_m} + \frac{h}{u_n}\right];$$

writing

$$\frac{u_x}{u_m u_n} = X, \quad \frac{f}{u_l} = \phi,$$

$$\frac{u_y}{u_n u_l} = Y, \quad \frac{g}{u_m} = \gamma,$$

$$\frac{u_z}{u_l u_m} = Z, \quad \frac{h}{u_n} = \eta,$$

the equation is

$$a = -\frac{u_m u_n}{u_l}(X + \gamma + \eta);$$

and there are two similar equations for b and c. Again,

$$A = bc - f^2$$
$$= u_l^2\{YZ + \phi(Y + Z) + \gamma Y + \eta Z + \phi\gamma + \gamma\eta + \eta\phi\},$$

with two others; and

$$F = gh - af$$
$$= u_m u_n\{\phi X + \phi\gamma + \gamma\eta + \eta\phi\},$$

with two others. Also

$$\Delta = -u_l u_m u_n\{(\phi\gamma + \gamma\eta + \eta\phi)(X + Y + Z) + XYZ$$
$$+ \phi(XY + XZ) + \gamma(YX + YZ) + \eta(ZX + ZY)\}.$$

Substituting and equating to zero the coefficients of the various combinations of ϕ, γ, η, which is the manifest equivalent of equating to zero the coefficients of the various combinations of f, g, h, we have the following equations.

From the coefficient of $\phi\gamma$, or $\gamma\eta$, or $\eta\phi$, there arises the single equation

$$- Du_l u_m u_n(X + Y + Z)$$
$$+ Pu_l^2 + Qu_m^2 + Ru_n^2 + 2Su_m u_n + 2Tu_n u_l + 2Uu_l u_m = 0.$$

From the coefficient of ϕ, there arises the single equation

$$-Du_lu_mu_n(XY+XZ)-J\frac{u_nu_l}{u_m}-K\frac{u_lu_m}{u_n}+2Lu_l$$
$$+Pu_l^2(Y+Z)+Qu_m^2X+Ru_n^2X+2Su_mu_nX=0;$$

from that of γ, the equation

$$-Du_lu_mu_n(YX+YZ)-I\frac{u_mu_n}{u_l}-K\frac{u_lu_m}{u_n}+2Mu_m$$
$$+Pu_l^2Y+Qu_m^2(X+Z)+Ru_n^2Y+2Tu_nu_lY=0;$$

and from that of η, the equation

$$-D\,u_lu_mu_n(ZX+ZY)-I\frac{u_mu_n}{u_l}-J\frac{u_nu_l}{u_m}+2Nu_n$$
$$+Pu_l^2Z+Qu_m^2Z+Ru_n^2(X+Y)+2Uu_lu_mZ=0.$$

Finally, from the term independent of ϕ, γ, η, there arises the equation

$$-Du_lu_mu_nXYZ+Pu_l^2YZ+Qu_m^2ZX+Ru_n^2XY$$
$$-I\frac{u_mu_n}{u_l}X-J\frac{u_nu_l}{u_m}Y-K\frac{u_lu_m}{u_n}Z+W=0.$$

4. These equations must be solved so as to obtain simpler algebraical forms, before proceeding to express the conditions of coexistence and to determine integral equivalents; and a convenient form is that in which u_x, u_y, u_z, are expressed in terms of u_l, u_m, u_n.

The first of the equations being

$$Pu_l^2+Qu_m^2+Ru_n^2+2Su_mu_n+2Tu_nu_l+2Uu_mu_l=D(u_lu_x+u_mu_y+u_nu_z),$$

we introduce six new unknown quantities α, α'; β, β'; γ, γ'; defined by the relations

$$\left.\begin{aligned} Du_x&=Pu_l+\gamma'u_m+\beta u_n\\ Du_y&=\gamma u_l+Qu_m+\alpha'u_n\\ Du_z&=\beta'u_l+\alpha u_m+Ru_n\end{aligned}\right\};$$

there being no initial assumption that the new quantities are independent of u_l, u_m, u_n. When these are substituted, it appears that the above equation is identically satisfied, provided

$$\left.\begin{aligned}\gamma+\gamma'&=2U\\ \beta+\beta'&=2T\\ \alpha+\alpha'&=2S\end{aligned}\right\}.$$

These relations will accordingly be assumed: and there will thus be three unknown quantities left.

Next, substitute the values of u_x, u_y, u_z in the second equation: it becomes, on reduction,

$$(2LD+2SP-\beta\gamma-\beta'\gamma')u_l-(\beta\beta'+DJ-PR)\frac{u_lu_n}{u_m}-(\gamma\gamma'+DK-PQ)\frac{u_lu_m}{u_n}=0.$$

Assume
$$\beta\beta' = PR - DJ,$$
$$\gamma\gamma' = PQ - DK\,;$$
then, as u_l is not zero, we must have
$$\beta\gamma + \beta'\gamma' = 2LD + 2SP.$$

There is still one relation possible in order to the determination of the six quantities. Again, substituting in the third equation, we have
$$(2MD + 2TQ - \alpha\gamma - \alpha'\gamma')\,u_m - (\gamma\gamma' + DK - PQ)\,\frac{u_m u_l}{u_n} - (\alpha\alpha' + DI - QR)\,\frac{u_m u_n}{u_l} = 0.$$
We assume
$$\alpha\alpha' = QR - DI\,;$$
and then, as u_m is not zero, it follows that
$$\alpha\gamma + \alpha'\gamma' = 2MD + 2TQ.$$
The six quantities α, β, γ, α', β', γ' can now be considered known.

Substituting in the fourth equation, and using the condition that u_n is not zero, we find that
$$\alpha\beta + \alpha'\beta' = 2ND + 2UR:$$
and the equation is then identically satisfied.

Lastly, substituting in the fifth equation, and using the preceding relations, we find that it reduces to
$$\alpha'\beta'\gamma' + \alpha\beta\gamma = 2PQR - PID - QJD - RKD + D^2W.$$

It thus appears that the system of five equations can be replaced by a number of sets of three homogeneous linear equations, each set being of the form
$$\left.\begin{aligned} Du_x &= Pu_l + \gamma' u_m + \beta u_n \\ Du_y &= \gamma u_l + Qu_m + \alpha' u_n \\ Du_z &= \beta' u_l + \alpha u_m + Ru_n \end{aligned}\right\}:$$
the coefficients are determined by the equations
$$\left.\begin{aligned} \alpha + \alpha' &= 2S, & \alpha\alpha' &= QR - DI \\ \beta + \beta' &= 2T, & \beta\beta' &= PR - DJ \\ \gamma + \gamma' &= 2U, & \gamma\gamma' &= PQ - DK \end{aligned}\right\};$$
and there must be satisfied four relations obtained by substituting for α, β, γ, α', β', γ' in
$$\begin{aligned} \beta\gamma + \beta'\gamma' &= 2LD + 2SP, \\ \gamma\alpha + \gamma'\alpha' &= 2MD + 2TQ, \\ \alpha\beta + \alpha'\beta' &= 2ND + 2UR, \\ \alpha\beta\gamma + \alpha'\beta'\gamma' &= D^2W - D(PI + QJ + RK) + 2PQR. \end{aligned}$$

These equations are less restricted than Vivanti's (*l.c.*).

5. But conversely, a solution of any one of the systems of equations which are satisfied by u, should lead to the differential equation of the second order. Let such a solution be

$$u = u(v, l, m, n, x, y, z) = \text{constant};$$

then we have

$$u_x + au_l + hu_m + gu_n = 0,$$

that is, substituting for u_x from the partial differential equations satisfied by u, we have

$$(aD + P)u_l + (hD + \gamma')u_m + (gD + \beta)u_n = 0.$$

And similarly for the other two derivatives, with regard to y and z respectively. Eliminating $u_l : u_m : u_n$ between these three equations, we have

$$\begin{vmatrix} aD + P, & hD + \gamma', & gD + \beta \\ hD + \gamma, & bD + Q, & fD + \alpha' \\ gD + \beta', & fD + \alpha, & cD + R \end{vmatrix} = 0.$$

The term independent of the derivatives a, b, c, f, g, h is

$$= PQR - P\alpha\alpha' + \gamma\alpha\beta - R\gamma\gamma' + \beta'\gamma'\alpha' - Q\beta\beta'$$
$$= PDI + \gamma\alpha\beta + \beta'\gamma'\alpha' - 2PQR + QDJ + RDK$$
$$= D^2 W,$$

by means of the equations satisfied by $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$.

Again, the coefficient of a is

$$= D(QR - \alpha\alpha') = D^2 I.$$

The coefficient of h is

$$= D(\alpha\beta - R\gamma + \alpha'\beta' - R\gamma')$$
$$= 2ND^2 + 2URD - 2URD = D^2 . 2N.$$

The coefficient of $ab - h^2$, that is, of C, is

$$= D^2 R;$$

and so on for the others. Substituting and dividing out by D^2, we have the original equation: which accordingly is satisfied by each solution of the subsidiary system of homogeneous equations of the first order determining u.

6. The only relations so far considered are of a purely numerative character: it will be assumed that they are satisfied, as preliminary conditions for the existence of an intermediary integral. It is now necessary to consider other conditions in order that the three partial differential equations for u may have a common solution or common solutions.

Let

$$\left.\begin{aligned} \Delta_1 &= \frac{\partial}{\partial x} + l\frac{\partial}{\partial v} - \frac{1}{D}\left(P\frac{\partial}{\partial l} + \gamma'\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial n}\right) \\ \Delta_2 &= \frac{\partial}{\partial y} + m\frac{\partial}{\partial v} - \frac{1}{D}\left(\gamma\frac{\partial}{\partial l} + Q\frac{\partial}{\partial m} + \alpha'\frac{\partial}{\partial n}\right) \\ \Delta_3 &= \frac{\partial}{\partial z} + n\frac{\partial}{\partial v} - \frac{1}{D}\left(\beta'\frac{\partial}{\partial l} + \alpha\frac{\partial}{\partial m} + R\frac{\partial}{\partial n}\right) \end{aligned}\right\};$$

then the equations satisfied by u are the linear set

$$\Delta_1 u = 0, \quad \Delta_2 u = 0, \quad \Delta_3 u = 0.$$

These must satisfy the Poisson-Jacobi conditions

$$(\Delta_1 \Delta_2) = 0, \quad (\Delta_2 \Delta_3) = 0, \quad (\Delta_3 \Delta_1) = 0,$$

which are

$$0 = \frac{\gamma - \gamma'}{D}\frac{\partial u}{\partial v} + \left(\Delta_2 \frac{P}{D} - \Delta_1 \frac{\gamma}{D}\right)\frac{\partial u}{\partial l} + \left(\Delta_2 \frac{\gamma'}{D} - \Delta_1 \frac{Q}{D}\right)\frac{\partial u}{\partial m} + \left(\Delta_2 \frac{\beta}{D} - \Delta_1 \frac{\alpha'}{D}\right)\frac{\partial u}{\partial n},$$

$$0 = \frac{\alpha - \alpha'}{D}\frac{\partial u}{\partial v} + \left(\Delta_3 \frac{\gamma}{D} - \Delta_2 \frac{\beta'}{D}\right)\frac{\partial u}{\partial l} + \left(\Delta_3 \frac{Q}{D} - \Delta_2 \frac{\alpha}{D}\right)\frac{\partial u}{\partial m} + \left(\Delta_3 \frac{\alpha'}{D} - \Delta_2 \frac{R}{D}\right)\frac{\partial u}{\partial n},$$

$$0 = \frac{\beta - \beta'}{D}\frac{\partial u}{\partial v} + \left(\Delta_1 \frac{\beta'}{D} - \Delta_3 \frac{P}{D}\right)\frac{\partial u}{\partial l} + \left(\Delta_1 \frac{\alpha}{D} - \Delta_3 \frac{\gamma'}{D}\right)\frac{\partial u}{\partial m} + \left(\Delta_1 \frac{R}{D} - \Delta_3 \frac{\beta}{D}\right)\frac{\partial u}{\partial n},$$

respectively.

Manifestly no one of these is satisfied in virtue of any linear combination of $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$; hence each of them is either an identity or it is a new equation.

7. Suppose, first, that each of the equations is an identity. Then each of the coefficients of $\frac{\partial u}{\partial v}, \frac{\partial u}{\partial l}, \frac{\partial u}{\partial m}, \frac{\partial u}{\partial n}$, in each of the equations must vanish. We thus have

$$\gamma = \gamma' = U,$$

$$\beta = \beta' = T,$$

$$\alpha = \alpha' = S,$$

and also

$$\left.\begin{array}{lll} \Delta_2 \frac{P}{D} = \Delta_1 \frac{U}{D}, & \Delta_2 \frac{U}{D} = \Delta_1 \frac{Q}{D}, & \Delta_2 \frac{T}{D} = \Delta_1 \frac{S}{D} \\ \Delta_3 \frac{U}{D} = \Delta_2 \frac{T}{D}, & \Delta_3 \frac{Q}{D} = \Delta_2 \frac{S}{D}, & \Delta_3 \frac{S}{D} = \Delta_2 \frac{R}{D} \\ \Delta_1 \frac{T}{D} = \Delta_3 \frac{P}{D}, & \Delta_1 \frac{S}{D} = \Delta_3 \frac{U}{D}, & \Delta_1 \frac{R}{D} = \Delta_3 \frac{T}{D} \end{array}\right\}.$$

When all these relations are satisfied, the system of equations $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$, is a complete system; as there are seven independent variables for u, it follows that there are four functionally independent integrals, say

$$u_1 = u_1(v, x, y, z, l, m, n),$$

$$u_2 = u_2(v, x, y, z, l, m, n),$$

$$u_3 = u_3(v, x, y, z, l, m, n),$$

$$u_4 = u_4(v, x, y, z, l, m, n).$$

The conditions to be satisfied that this may be the case are, (i) the foregoing set of nine differential relations, (ii) an algebraical set, which can easily be obtained in the form

$$\left.\begin{aligned} S^2 &= QR - DI \\ T^2 &= PR - DJ \\ U^2 &= PQ - DK \end{aligned}\right\}, \qquad \left.\begin{aligned} TU &= LD + SP \\ US &= MD + TQ \\ ST &= ND + UR \end{aligned}\right\},$$

$$D^2 W = \begin{vmatrix} P, & U, & T \\ U, & Q, & S \\ T, & S, & R \end{vmatrix}.$$

The last seven equations may, in fact, be regarded as expressing I, J, K, L, M, N, W in terms of P, Q, R, S, T, U. Using them for this purpose, we can, for the present case, write the differential equation in the form

$$\begin{vmatrix} aD + P, & hD + U, & gD + T \\ hD + U, & bD + Q, & fD + S \\ gD + T, & fD + S, & cD + R \end{vmatrix} = 0.$$

From the form of the equations of which u_1, u_2, u_3, u_4 are functionally independent solutions, it is manifest that any functional combination of them is also a solution, say $\Phi(u_1, u_2, u_3, u_4)$. But it has been seen that any solution is an intermediary integral of the original equation; and so there is an intermediary integral of the form

$$\Phi = 0,$$

where Φ is the most general arbitrary functional form.

8. This is, however, an equation of the first order. In the present case we can, without further integration, actually obtain an integral of the original equation: all that is necessary for the purpose is to eliminate l, m, n between the four equations

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

where a_1, a_2, a_3, a_4 are arbitrary constants. In order to establish this result, we must prove that any two of the integral equations $u = a$ (say they are θ = constant, ϕ = constant) can be taken as coexisting independent equations. Now the condition that this may be the case is

$$\Sigma \begin{pmatrix} \theta, & \phi \\ x, & l \end{pmatrix} + \Sigma l \begin{pmatrix} \theta, & \phi \\ v, & l \end{pmatrix} = 0,$$

which, on substitution for $\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ from the differential equations identically satisfied by them, becomes

$$(\gamma - \gamma')\left(\frac{\partial\theta}{\partial l}\frac{\partial\phi}{\partial m} - \frac{\partial\phi}{\partial l}\frac{\partial\theta}{\partial m}\right) + (\beta - \beta')\left(\frac{\partial\theta}{\partial n}\frac{\partial\phi}{\partial l} - \frac{\partial\phi}{\partial n}\frac{\partial\theta}{\partial l}\right) + (\alpha - \alpha')\left(\frac{\partial\theta}{\partial m}\frac{\partial\phi}{\partial n} - \frac{\partial\phi}{\partial m}\frac{\partial\theta}{\partial n}\right) = 0.$$

But, under the present hypothesis, we have $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$; the condition therefore is satisfied. Accordingly the four equations $u = a$ are four coexisting independent equations.

Eliminating l, m, n between them, we have, in general, a relation involving v, x, y, z and the four arbitrary constants. Suppose it to be of the form

$$v = F(x, y, z, a_1, a_2, a_3, a_4).$$

The fact that the Jacobi-Poisson condition of coexistence for each pair of equations $u = a$ is satisfied, enables us to infer, as in the case of two independent variables, that the values of l, m, n, v deduced algebraically from those equations are such as to give

$$l = \frac{\partial v}{\partial x}, \quad m = \frac{\partial v}{\partial y}, \quad n = \frac{\partial v}{\partial z}.$$

Now the complete system of differential equations

$$D\left(\frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v}\right) - \left(P\frac{\partial u}{\partial l} + U\frac{\partial u}{\partial m} + T\frac{\partial u}{\partial n}\right) = 0,$$

$$D\left(\frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v}\right) - \left(U\frac{\partial u}{\partial l} + Q\frac{\partial u}{\partial m} + S\frac{\partial u}{\partial n}\right) = 0,$$

$$D\left(\frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v}\right) - \left(T\frac{\partial u}{\partial l} + S\frac{\partial u}{\partial m} + R\frac{\partial u}{\partial n}\right) = 0.$$

is associable with the four equations

$$\left.\begin{aligned} dv &= ldx + mdy + ndz \\ -Ddl &= Pdx + Udy + Tdz \\ -Ddm &= Udx + Qdy + Sdz \\ -Ddn &= Tdx + Sdy + Rdz \end{aligned}\right\},$$

as the customary equivalent in differential elements.

Hence it appears that the solution

$$v = F(x, y, z, a_1, a_2, a_3, a_4)$$

of the original equation is such that

$$D\frac{\partial^2 F}{\partial x^2} + P = 0, \quad D\frac{\partial^2 F}{\partial x \partial y} + U = 0, \quad D\frac{\partial^2 F}{\partial x \partial z} + T = 0,$$
$$D\frac{\partial^2 F}{\partial y^2} + Q = 0, \quad D\frac{\partial^2 F}{\partial y \partial z} + S = 0,$$
$$D\frac{\partial^2 F}{\partial z^2} + R = 0.$$

Moreover, the verification that it satisfies the differential equation is immediate by taking this equation in the form obtained in § 7.

9. Now the solution which has been constructed is one that involves four arbitrary constants and so it is not a complete integral. But its importance lies in the fact that it can at once be changed so as to give the most general integral of the equation: a result due to the proposition that *if* $\phi = \phi(\alpha, \beta)$ *and* $\psi = \psi(\alpha, \beta)$ *denote*

two arbitrary functions, then replacing a_1, a_2, a_3, a_4 *by* α, β, ϕ, ψ *respectively, an integral of the equation is given by the elimination of* α *and* β *between*

$$\left.\begin{aligned} v &= F(x, y, z, \alpha, \beta, \phi, \psi) \\ 0 &= \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial \alpha} + \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial \alpha} \\ 0 &= \frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial \beta} + \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial \beta} \end{aligned}\right\};$$

and this integral is the general integral of the equation because it involves two arbitrary functions each of two arguments.

The proposition will be proved by shewing that the postulated integral equations satisfy the differential equation. We take the integral in the form

$$v = F,$$

$$\frac{\partial F}{\partial \alpha} = 0, \quad \frac{\partial F}{\partial \beta} = 0;$$

the two latter implying complete derivatives with regard to α and to β respectively.

From the second and the third equations, we have

$$\left.\begin{aligned} 0 &= \frac{\partial^2 F}{\partial \alpha \partial x} + \frac{\partial^2 F}{\partial \alpha^2}\alpha_x + \frac{\partial^2 F}{\partial \alpha \partial \beta}\beta_x \\ 0 &= \frac{\partial^2 F}{\partial \beta \partial x} + \frac{\partial^2 F}{\partial \alpha \partial \beta}\alpha_x + \frac{\partial^2 F}{\partial \beta^2}\beta_x \end{aligned}\right\},$$

and two similar pairs for derivation with regard to y and to z respectively.

Now from $v = F$, we have

$$l = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \alpha}\alpha_x + \frac{\partial F}{\partial \beta}\beta_x = \frac{\partial F}{\partial x},$$

$$m = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial \alpha}\alpha_y + \frac{\partial F}{\partial \beta}\beta_y = \frac{\partial F}{\partial y},$$

$$n = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \alpha}\alpha_z + \frac{\partial F}{\partial \beta}\beta_z = \frac{\partial F}{\partial z};$$

so that l, m, n have their form the same as when α and β are arbitrary constants. Next,

$$\begin{aligned} a &= \frac{\partial l}{\partial x} \\ &= \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \alpha \partial x}\alpha_x + \frac{\partial^2 F}{\partial \beta \partial x}\beta_x \\ &= \frac{\partial^2 F}{\partial x^2} - \left(\frac{\partial^2 F}{\partial \alpha^2}, \frac{\partial^2 F}{\partial \alpha \partial \beta}, \frac{\partial^2 F}{\partial \beta^2}\right)(\alpha_x, \beta_x)^2. \end{aligned}$$

Writing

$$p,\ q,\ r = \frac{\partial^2 F}{\partial \alpha^2},\ \frac{\partial^2 F}{\partial \alpha \partial \beta},\ \frac{\partial^2 F}{\partial \beta^2},$$

we have

$$a = F_{xx} - (p,\ q,\ r)(\alpha_x,\ \beta_x)^2;$$

and similarly

$$\begin{aligned}
h &= F_{xy} - (p,\ q,\ r)(\alpha_x,\ \beta_x)(\alpha_y,\ \beta_y),\\
g &= F_{xz} - (p,\ q,\ r)(\alpha_x,\ \beta_x)(\alpha_z,\ \beta_z),\\
b &= F_{yy} - (p,\ q,\ r)(\alpha_y,\ \beta_y)^2,\\
f &= F_{yz} - (p,\ q,\ r)(\alpha_y,\ \beta_y)(\alpha_z,\ \beta_z),\\
c &= F_{zz} - (p,\ q,\ r)(\alpha_z,\ \beta_z)^2.
\end{aligned}$$

These values of v: $l,\ m,\ n$; $a,\ b,\ c,\ f,\ g,\ h$; are to be substituted in

$$\Theta = \begin{vmatrix} aD + P, & hD + U, & gD + T \\ hD + U, & bD + Q, & fD + S \\ gD + T, & fD + S, & cD + R \end{vmatrix}.$$

The differences from the case when α and β are arbitrary constants arise solely through the quantities $a,\ b,\ c,\ f,\ g,\ h$ and not through the coefficients of those quantities. Now we have seen that

$$\begin{aligned}
DF_{xx} + P = 0,\quad & DF_{xy} + U = 0,\quad & DF_{xz} + T = 0,\\
& DF_{yy} + Q = 0,\quad & DF_{yz} + S = 0,\\
& & DF_{zz} + R = 0;
\end{aligned}$$

hence, on substitution, we have

$$\Theta = \begin{vmatrix} (p,\ q,\ r)(\alpha_x,\ \beta_x)^2, & (p,\ q,\ r)(\alpha_y,\ \beta_y)(\alpha_x,\ \beta_x), & (p,\ q,\ r)(\alpha_z,\ \beta_z)(\alpha_x,\ \beta_x) \\ (p,\ q,\ r)(\alpha_x,\ \beta_x)(\alpha_y,\ \beta_y), & (p,\ q,\ r)(\alpha_y,\ \beta_y)^2, & (p,\ q,\ r)(\alpha_z,\ \beta_z)(\alpha_y,\ \beta_y) \\ (p,\ q,\ r)(\alpha_x,\ \beta_x)(\alpha_z,\ \beta_z), & (p,\ q,\ r)(\alpha_y,\ \beta_y)(\alpha_z,\ \beta_z), & (p,\ q,\ r)(\alpha_z,\ \beta_z)^2 \end{vmatrix}.$$

Take two quantities λ and μ such that

$$\alpha_x + \lambda\alpha_y + \mu\alpha_z = 0,\quad \beta_x + \lambda\beta_y + \mu\beta_z = 0;$$

and multiplying the second column by λ and the third by μ, add both to the first. Each constituent in that column is zero; and therefore Θ vanishes, or the differential equation is satisfied by the integral equation given.

10. The results can be summarised as follows:—

To solve the differential equation

$$\begin{vmatrix} aD + P, & hD + U, & gD + T \\ hD + U, & bD + Q, & fD + S \\ gD + T, & fD + S, & cD + R \end{vmatrix} = 0,$$

the coefficients satisfying certain conditions, we construct the subsidiary system

$$\left.\begin{aligned}\Delta_1 u &= \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v} - \frac{1}{D}\left(P\frac{\partial u}{\partial l} + U\frac{\partial u}{\partial m} + T\frac{\partial u}{\partial n}\right) = 0\\ \Delta_2 u &= \frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v} - \frac{1}{D}\left(U\frac{\partial u}{\partial l} + Q\frac{\partial u}{\partial m} + S\frac{\partial u}{\partial n}\right) = 0\\ \Delta_3 u &= \frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v} - \frac{1}{D}\left(T\frac{\partial u}{\partial l} + S\frac{\partial u}{\partial m} + R\frac{\partial u}{\partial n}\right) = 0\end{aligned}\right\},$$

which is a complete Jacobian system and therefore possesses four functionally independent integrals. Let these be u_1, u_2, u_3, u_4. Then from

$$u_1 = a_1,\quad u_2 = a_2,\quad u_3 = a_3,\quad u_4 = a_4,$$

we eliminate l, m, n and obtain a relation between v, x, y, z and the four constants, say

$$v = F(x, y, z, a_1, a_2, a_3, a_4).$$

Let $\phi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ be two arbitrary functions, each of two arguments; then eliminating α and β among the three equations

$$v = F\{x, y, z, \alpha, \beta, \phi(\alpha, \beta), \psi(\alpha, \beta)\},$$

$$0 = \frac{\partial F}{\partial \alpha},\quad 0 = \frac{\partial F}{\partial \beta},$$

we obtain the general integral of the differential equation. And the conditions to be satisfied by the coefficients are

$$\left.\begin{aligned}\Delta_2\frac{P}{D} = \Delta_1\frac{U}{D},\quad \Delta_2\frac{U}{D} = \Delta_1\frac{Q}{D},\quad \Delta_2\frac{T}{D} = \Delta_1\frac{S}{D}\\ \Delta_3\frac{U}{D} = \Delta_2\frac{T}{D},\quad \Delta_3\frac{Q}{D} = \Delta_2\frac{S}{D},\quad \Delta_3\frac{S}{D} = \Delta_2\frac{R}{D}\\ \Delta_1\frac{T}{D} = \Delta_3\frac{P}{D},\quad \Delta_1\frac{S}{D} = \Delta_3\frac{U}{D},\quad \Delta_1\frac{R}{D} = \Delta_3\frac{T}{D}\end{aligned}\right\}.$$

It may be added that

$$\begin{vmatrix} a+\lambda, & h+\gamma, & g+\beta \\ h+\gamma, & b+\mu, & f+\alpha \\ g+\beta, & f+\alpha, & c+(\alpha\gamma-\beta\mu)\,l+(\beta\gamma-\alpha\lambda)\,m+(\lambda\mu-\gamma^2)\,n \end{vmatrix} = 0,$$

where α, β, γ, λ, μ are any constants, is a particular example of this case: as is also

$$\begin{vmatrix} 1+l^2+av, & lm+hv, & ln+gv \\ lm+hv, & 1+m^2+bv, & mn+fv \\ ln+gv, & mn+fv, & 1+n^2+cv \end{vmatrix} = 0.$$

And another example is given by Professor Tanner* in the form

$$xyz \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} - lyz \begin{vmatrix} b, & f \\ f, & c \end{vmatrix} - mzx \begin{vmatrix} a, & g \\ g, & c \end{vmatrix} - nxy \begin{vmatrix} a, & h \\ h, & b \end{vmatrix}$$
$$+ xmna + ynlb + zlmc - lmn = 0.$$

11. It remains to consider the various alternatives to the hypothesis that the Jacobi-Poisson conditions are all satisfied identically.

Any such condition, not satisfied identically, is a new equation; and accordingly the various cases for consideration are when these new equations are

i. One new equation,

ii. Two new equations,

iii. Three new equations.

First, when there is one new equation which arises from the conditions of co-existence of $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$. Let it be $\Delta_4 = 0$.

This can occur in various ways. (*a*) Two of the conditions may be satisfied identically, and the third then gives the new equation. (*b*) One of the conditions may be satisfied identically; and the other two give new equations which, in effect, are equivalent to one another. (*c*) No one of the conditions may be satisfied identically; the three are new equations which, in effect, are equivalent to one another.

In general, we have several subsidiary systems: for the equations determining α, α'; β, β'; γ, γ' in general lead to two sets of values for each pair. If, however, a Jacobi-Poisson condition is identically evanescent, it is at once obvious from the form of the condition that the corresponding values are equal; thus if, in § 6, the condition containing the term in $(\alpha - \alpha')\frac{\partial u}{\partial v}$ is evanescent, then $\alpha = \alpha' = S$; and the number of systems of subsidiary equations for u is diminished.

It is simple to take account of the various ways in which we thus far have four equations in the system. Thus for (*a*), we can have two systems; for (*b*), we can have four systems; for (*c*), we can have eight systems. But though this number of systems can arise in the respective cases, it does not always arise of necessity: for the pair of sets of values of say α, α' can be the same without the other conditions being satisfied and so, in (*a*), we might have only a single system.

12. There are now four equations in each system: but additional Jacobi-Poisson conditions must now be satisfied, viz.

$$(\Delta_4\Delta_1) = 0, \quad (\Delta_4\Delta_2) = 0, \quad (\Delta_4\Delta_3) = 0.$$

* *Proc. Lond. Math. Soc.*, t. VII. p. 89.

If these are satisfied, either identically or in virtue of the four equations already established, the system is complete. It then has three functionally independent solutions, say u_1, u_2, u_3; and the most general solution of the system, which is then a general intermediary integral of the original equation, is of the form

$$\Phi(u_1, u_2, u_3) = 0.$$

It may happen that such an intermediary integral can be deduced from another system, given by different sets of values of α, α'; β, β'; γ, γ'; let it be

$$F(U_1, U_2, U_3) = 0,$$

where F is perfectly arbitrary and U_1, U_2, U_3 are the three functionally independent solutions. We proceed to consider under what circumstances (if any) Φ and F can be treated as simultaneous equations.

Since u_1, u_2, u_3 are solutions of a simultaneous system, Φ is also a solution of that system: that is, we have

$$\left.\begin{aligned} D\Phi_x &= P\Phi_l + \gamma'\Phi_m + \beta\Phi_n, \quad \text{where } \Phi_x = \frac{\partial\Phi}{\partial x} + l\frac{\partial\Phi}{\partial v} \\ D\Phi_y &= \gamma\Phi_l + Q\Phi_m + \alpha'\Phi_n, \quad\quad ,, \quad\quad \Phi_y = \frac{\partial\Phi}{\partial y} + m\frac{\partial\Phi}{\partial v} \\ D\Phi_z &= \beta'\Phi_l + \alpha\Phi_m + R\Phi_n, \quad\quad ,, \quad\quad \Phi_z = \frac{\partial\Phi}{\partial z} + n\frac{\partial\Phi}{\partial v} \end{aligned}\right\}.$$

Now for the subsidiary system satisfied by F, let A, A'; B, B'; Γ, Γ' be the corresponding coefficients: so that

A, A' is either α, α' or α', α:
B, B' .... β, β' ... β', β:
Γ, Γ' .... γ, γ' ... γ', γ:

the set of first alternatives in each case giving the system for Φ. Then F satisfies

$$\left.\begin{aligned} DF_x &= PF_l + \Gamma'F_m + BF_n \\ DF_y &= \Gamma F_l + QF_m + A'F_n \\ DF_z &= B'F_l + AF_m + RF_n \end{aligned}\right\}.$$

Now in order that F and Φ may be treated as simultaneous equations, we must have

$$F_x\Phi_l - F_l\Phi_x + F_y\Phi_m - F_m\Phi_y + F_z\Phi_n - F_n\Phi_z = 0.$$

Substituting from the above systems and collecting terms, we find

$$(\Gamma - \gamma')F_l\Phi_m + (\Gamma' - \gamma)F_m\Phi_l + (B - \beta')F_n\Phi_l + (B' - \beta)F_l\Phi_n + (A - \alpha')F_m\Phi_n + (A' - \alpha)F_n\Phi_m = 0,$$

evidently identically satisfied when

$$A, A' = \alpha', \alpha; \quad B, B' = \beta', \beta; \quad \Gamma, \Gamma' = \gamma', \gamma.$$

Hence we have the theorem:—

If all the conditions for the possession of three functionally independent solutions be satisfied for each of the systems

$$\left.\begin{aligned}\Phi_x &= \frac{1}{D}(P\Phi_l + \gamma'\Phi_m + \beta\Phi_n)\\ \Phi_y &= \frac{1}{D}(\gamma\Phi_l + Q\Phi_m + \alpha'\Phi_n)\\ \Phi_z &= \frac{1}{D}(\beta'\Phi_l + \alpha\Phi_m + R\Phi_n)\end{aligned}\right\}, \qquad \left.\begin{aligned}F_x &= \frac{1}{D}(PF_l + \gamma F_m + \beta' F_n)\\ F_y &= \frac{1}{D}(\gamma' F_l + QF_m + \alpha F_n)\\ F_z &= \frac{1}{D}(\beta F_l + \alpha' F_m + RF_n)\end{aligned}\right\},$$

then the general intermediary integrals

$$\Phi = 0, \quad F = 0,$$

deduced from the respective systems, can be treated as simultaneous equations.

Further, it can be established *that the linear equations in differential elements equivalent to $F = 0$ are included in the Charpit-system subsidiary to the integration of $\Phi = 0$ as a partial differential equation of the first order.*

The simplest instance of all in the present case arises when two of the Jacobi-Poisson conditions $(\Delta_1\Delta_2) = 0$, $(\Delta_2\Delta_3) = 0$, $(\Delta_3\Delta_1) = 0$ are satisfied identically, and the third is a new equation containing a term in $\dfrac{\partial u}{\partial v}$: and when, further, the full system is complete. Of the three pairs of quantities α, α'; β, β'; γ, γ'; two contain equal members, and the third contains unequal members. There are then two subsidiary systems; and thus we should have used all the subsidiary systems. I pass over, for the present, the discussion of the relation to one another of integrals derived through subsidiary systems not chosen according to the restriction in the proposition just established.

An instance of this case is furnished by the equation

$$\begin{vmatrix} a+P, & h+U, & g+T \\ h+U, & b+Q, & f+S \\ g+T, & f+S, & c+R \end{vmatrix} + \theta^2(c+R) = 0,$$

where P, Q, R, S, T, U, θ are constants. There are two subsidiary systems; and the intermediary integrals obtained can be treated as simultaneous equations.

13. If the Jacobi-Poisson conditions

$$(\Delta_4\Delta_1) = 0, \quad (\Delta_4\Delta_2) = 0, \quad (\Delta_4\Delta_3) = 0$$

are not satisfied in virtue of $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$, $\Delta_4 = 0$, so that the system of equations is not complete, the new equations that arise through this set of conditions must be associated with the former four. We proceed as before and render the system ultimately a complete Jacobian system; and if in this state, the system contains n equations, there are $7 - n$ functionally independent solutions of the system.

Of the remaining two cases, viz. those in which the conditions for the coexistence of $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$ lead to two new equations and to three new equations respectively, it is unnecessary to say much in general detail. The process is the same as in the last case; the system must be made complete. If it contain more than six equations in this state, there is no common solution and so there is no intermediary integral; but if it contain n equations, n being less than 7, then it possesses $7 - n$ functionally independent solutions and an intermediary integral exists.

14. In the preceding investigation, the most general form of the prescribed type has been taken initially. Instead of making the necessary modifications for simpler forms, it is better to apply the method at once to the simpler forms. For example, in the case of the equation

$$x^2a + 2xyh + y^2b + 2xzg + 2yzf + z^2c = 0,$$

we substitute for a, b, c from

$$\left.\begin{aligned} u_x + au_l + hu_m + gu_n &= 0 \\ u_y + hu_l + bu_m + fu_n &= 0 \\ u_z + gu_l + fu_m + cu_n &= 0 \end{aligned}\right\};$$

then we equate to zero the coefficients of f, g, h and the term independent of these quantities. Solving the resulting equations, we find only a single system of simultaneous equations determining u, viz.

$$\left.\begin{aligned} \frac{u_l}{x} = \frac{u_m}{y} &= \frac{u_n}{z} \\ xu_x + yu_y + zu_z &= 0 \end{aligned}\right\},$$

where u_x denotes $\dfrac{\partial u}{\partial x} + l\dfrac{\partial u}{\partial v}$, and so for u_y and u_z.

It is easy to prove that this system is complete, and that the four functionally independent solutions can be taken in the form

$$\frac{y}{x},\ \frac{z}{x},\ \frac{lx + my + nz}{x},\ v - (lx + my + nz).$$

Hence there are two intermediary integrals of the respective forms

$$\frac{lx + my + nz}{x} = \phi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

$$v - (lx + my + nz) = \psi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

where ϕ and ψ are arbitrary functions. Moreover, by § 12, these can be treated as simultaneous equations, for the initial system is complete as obtained; hence we have

$$v = x\phi\left(\frac{y}{x},\ \frac{z}{x}\right) + \psi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

as a primitive, and manifestly it is the general primitive.

15. Taking now more generally the case in which an equation of the second order possesses an intermediary integral, though not of the functional form previously considered, we have an equation

$$F = 0,$$

satisfied in virtue of derivatives from

$$u(v, x, y, z, l, m, n) = 0,$$

that is, in virtue of

$$lu_v + u_x + au_l + hu_m + gu_n = 0,$$
$$mu_v + u_y + hu_l + bu_m + fu_n = 0,$$
$$nu_v + u_z + gu_l + fu_m + cu_n = 0.$$

Hence when we substitute for a, b, c in $F = 0$, the resulting equation must be evanescent: and therefore the coefficients of all combinations of f, g, h that occur in the modified form must vanish, so that a number of relations will arise. Each such relation is homogeneous in the derivatives $lu_v + u_x$, $mu_v + u_y$, $nu_v + u_z$, u_l, u_m, u_n; hence there cannot be more than five algebraically independent relations. On the other hand, there must, in general, be at least three relations; for if the result of the substitution is to give

$$T + Pf + Qg + Rh + \ldots\ldots = 0,$$

then we must have $T = 0$, $P = 0$, $Q = 0$, ... If these were equivalent to only one relation, this would occur through a common factor that vanishes, say

$$lu_v + u_x = \theta(mu_v + u_y, nu_v + u_z, u_l, u_m, u_n),$$

where θ is homogeneous of the first degree in $mu_v + u_y$, $nu_v + u_z$, u_l, u_m, u_n. We thus have, for the construction of the equation of the second order,

$$\theta + au_l + hu_m + gu_n = 0,$$
$$mu_v + u_y + \qquad\qquad = 0,$$
$$nu_v + u_z + \qquad\qquad = 0,$$

three equations involving four ratios $u_y : u_z : u_l : u_m : u_n$ not homogeneously. The equations are insufficient for this elimination: and therefore, in general, the present case will not arise.

Further if the relations are algebraically equivalent to two only, they may be taken in the form

$$P(lu_v + u_x, mu_v + u_y, nu_v + u_z, u_l, u_m, u_n) = 0,$$
$$Q(lu_v + u_x, mu_v + u_y, nu_v + u_z, u_l, u_m, u_n) = 0,$$
$$lu_v + u_x + au_l + hu_m + gu_n = 0,$$
$$mu_v + u_y + hu_l + bu_m + fu_n = 0,$$
$$nu_v + u_z + gu_l + fu_m + cu_n = 0,$$

five equations involving five ratios $lu_v + u_x : mu_v + u_y : nu_v + u_z : u_l : u_m : u_n$ not homogeneously. The equations are insufficient for the elimination: and therefore, in general, the present case will not arise. Hence there must, in general, be at least three equations, algebraically independent of one another.

We thus have three cases to consider, according as the number of algebraically independent relations is

(i) three in all,

(ii) four in all,

(iii) five in all.

These cases will be taken in turn.

16. *Three algebraically independent equations.* Suppose the equations solved for (say) $lu_v + u_x$, $mu_v + u_y$, $nu_v + u_z$, in terms of u_l, u_m, u_n: if, in a particular case, it proved possible or convenient to solve only for some other combination, a tangential transformation could be effected so as to transfer it to the above form. Let it therefore be

$$\left.\begin{aligned} L &= u_x + lu_v + \lambda(u_l, u_m, u_n) = 0 \\ M &= u_y + mu_v + \mu(u_l, u_m, u_n) = 0 \\ N &= u_z + nu_v + \nu(u_l, u_m, u_n) = 0 \end{aligned}\right\},$$

where λ, μ, ν are homogeneous of the first order in u_l, u_m, u_n and the coefficients in λ, μ, ν are (or may be) functions of v, x, y, z, l, m, n.

But though these are, in the present case, the aggregate of algebraically independent equations thus derivable, they must satisfy the Jacobi-Poisson differential conditions of coexistence. Writing $v, x, y, z, l, m, n = x_1, x_2, x_3, x_4, x_5, x_6, x_7$; and $u_v, \ldots, u_n = p_1, \ldots, p_7$ similarly; we form the combinations

$$(L, M) = \sum_{i=1}^{7} \left(\frac{\partial L}{\partial x_i} \frac{\partial M}{\partial p_i} - \frac{\partial L}{\partial p_i} \frac{\partial M}{\partial x_i} \right),$$

for the three pairs; and these must each vanish. Now we have

$$\begin{aligned} L &= x_5 p_1 + p_2 + \lambda(x_1, \ldots, x_7, p_5, p_6, p_7) = 0, \\ M &= x_6 p_1 + p_3 + \mu(\qquad\qquad) = 0, \\ N &= x_7 p_1 + p_4 + \nu(\qquad\qquad) = 0, \end{aligned}$$

with λ, μ, ν homogeneous of the first degree in p_5, p_6, p_7: and so

$$(L, M) = p_1 \left(\frac{\partial \mu}{\partial p_5} - \frac{\partial \lambda}{\partial p_6} \right) + x_6 \frac{\partial \lambda}{\partial x_1} - x_5 \frac{\partial \mu}{\partial x_1} + \frac{\partial \lambda}{\partial x_3} - \frac{\partial \mu}{\partial x_2} + \begin{pmatrix} \lambda, \mu \\ x_5, p_5 \end{pmatrix} + \begin{pmatrix} \lambda, \mu \\ x_6, p_6 \end{pmatrix} + \begin{pmatrix} \lambda, \mu \\ x_7, p_7 \end{pmatrix} = 0.$$

This manifestly is not satisfied in virtue of $L=0$, $M=0$, $N=0$; and therefore it is either a new equation or an identity.

In order that it may be an identity, the term in p_1 must vanish, for p_1 occurs nowhere else in the equation: hence, as a first condition, we have

$$\frac{\partial \mu}{\partial p_5} = \frac{\partial \lambda}{\partial p_6}.$$

Similarly, if $(M, N) = 0$, $(N, L) = 0$, are identities, we deduce, as first conditions

$$\frac{\partial \nu}{\partial p_6} = \frac{\partial \mu}{\partial p_7},$$

$$\frac{\partial \lambda}{\partial p_7} = \frac{\partial \nu}{\partial p_5},$$

respectively. In order that these may be satisfied, a function Θ of p_5, p_6, p_7 and $x_1, \ldots, x_7$ must exist such that

$$\lambda = \frac{\partial \Theta}{\partial p_5}, \quad \mu = \frac{\partial \Theta}{\partial p_6}, \quad \nu = \frac{\partial \Theta}{\partial p_7}.$$

Evidently Θ must be homogeneous of the second degree in p_5, p_6, p_7. Taking

$$p_5 = \theta p_7, \quad p_6 = \phi p_7,$$

we have

$$\Theta = p_7^2 A(x_1, x_2, x_3, x_4, x_5, x_6, x_7, \theta, \phi) = p_7^2 A,$$

where there now is no restriction upon A, the function of θ and ϕ: and then

$$\lambda = p_7 \frac{\partial A}{\partial \theta}, \quad \mu = p_7 \frac{\partial A}{\partial \phi}, \quad \nu = p_7 \left(2A - \theta \frac{\partial A}{\partial \theta} - \phi \frac{\partial A}{\partial \phi}\right).$$

But these are not the full aggregate of conditions: thus, from $(L, M) = 0$, we also have

$$x_6 \frac{\partial \lambda}{\partial x_1} - x_5 \frac{\partial \mu}{\partial x_1} + \frac{\partial \lambda}{\partial x_3} - \frac{\partial \mu}{\partial x_2} + \left(\frac{\lambda, \mu}{x_5, p_5}\right) + \left(\frac{\lambda, \mu}{x_6, p_6}\right) + \left(\frac{\lambda, \mu}{x_7, p_7}\right) = 0.$$

Substituting the values of λ and μ just obtained, and removing the factor p_7, we have

$$x_6 \frac{\partial^2 A}{\partial x_1 \partial \theta} - x_5 \frac{\partial^2 A}{\partial x_1 \partial \phi} + \frac{\partial^2 A}{\partial x_3 \partial \theta} - \frac{\partial^2 A}{\partial x_2 \partial \phi} + \frac{\partial^2 A}{\partial x_5 \partial \theta}\frac{\partial^2 A}{\partial \theta \partial \phi} - \frac{\partial^2 A}{\partial x_5 \partial \phi}\frac{\partial^2 A}{\partial \theta^2}$$

$$+ \frac{\partial^2 A}{\partial x_6 \partial \theta}\frac{\partial^2 A}{\partial \phi^2} - \frac{\partial^2 A}{\partial x_6 \partial \phi}\frac{\partial^2 A}{\partial \theta \partial \phi} + \frac{\partial^2 A}{\partial x_7 \partial \theta}\left(\frac{\partial A}{\partial \phi} - \theta \frac{\partial^2 A}{\partial \theta \partial \phi} - \phi \frac{\partial^2 A}{\partial \phi^2}\right)$$

$$- \frac{\partial^2 A}{\partial x_7 \partial \phi}\left(\frac{\partial A}{\partial \theta} - \theta \frac{\partial^2 A}{\partial \theta^2} - \phi \frac{\partial^2 A}{\partial \theta \partial \phi}\right) = 0.$$

Similarly, from $(M, N) = 0$, we have the further condition

$$x_7 \frac{\partial \mu}{\partial x_1} - x_6 \frac{\partial \nu}{\partial x_1} + \frac{\partial \mu}{\partial x_4} - \frac{\partial \nu}{\partial x_3} + \left(\frac{\mu, \nu}{x_5, p_5}\right) + \left(\frac{\mu, \nu}{x_6, p_6}\right) + \left(\frac{\mu, \nu}{x_7, p_7}\right) = 0;$$

and from $(N, L)=0$, the further condition

$$x_5\frac{\partial\nu}{\partial x_1}-x_7\frac{\partial\lambda}{\partial x_1}+\frac{\partial\nu}{\partial x_2}-\frac{\partial\lambda}{\partial x_4}+\left(\frac{\nu,\ \lambda}{x_5,\ p_5}\right)+\left(\frac{\nu,\ \lambda}{x_6,\ p_6}\right)+\left(\frac{\nu,\ \lambda}{x_7,\ p_7}\right)=0\,;$$

each of these, when substitution takes place for λ, μ, ν, leading to another equation of the second order satisfied by A. The former equation is, on rejecting a factor p_7,

$$x_7\frac{\partial^2 A}{\partial x_1\partial\phi}-x_6\left(2\frac{\partial A}{\partial x_1}-\theta\frac{\partial^2 A}{\partial x_1\partial\theta}-\phi\frac{\partial^2 A}{\partial x_1\partial\phi}\right)+\frac{\partial^2 A}{\partial x_4\partial\phi}-\left(2\frac{\partial A}{\partial x_3}-\theta\frac{\partial^2 A}{\partial x_3\partial\theta}-\phi\frac{\partial^2 A}{\partial x_3\partial\phi}\right)$$

$$+\frac{\partial^2 A}{\partial x_5\partial\phi}\left(\frac{\partial A}{\partial\theta}-\theta\frac{\partial^2 A}{\partial\theta^2}-\phi\frac{\partial^2 A}{\partial\theta\partial\phi}\right)-\frac{\partial^2 A}{\partial\theta\partial\phi}\left(2\frac{\partial A}{\partial x_5}-\theta\frac{\partial^2 A}{\partial x_5\partial\theta}-\phi\frac{\partial^2 A}{\partial x_5\partial\phi}\right)$$

$$+\frac{\partial^2 A}{\partial x_6\partial\phi}\left(\frac{\partial A}{\partial\phi}-\theta\frac{\partial^2 A}{\partial\theta\partial\phi}-\phi\frac{\partial^2 A}{\partial\phi^2}\right)-\frac{\partial^2 A}{\partial\phi^2}\left(2\frac{\partial A}{\partial x_6}-\theta\frac{\partial^2 A}{\partial x_6\partial\theta}-\phi\frac{\partial^2 A}{\partial x_6\partial\phi}\right)$$

$$+\frac{\partial^2 A}{\partial x_7\partial\phi}\left(2A-2\theta\frac{\partial A}{\partial\theta}-2\phi\frac{\partial A}{\partial\phi}+\theta^2\frac{\partial^2 A}{\partial\theta^2}+2\theta\phi\frac{\partial^2 A}{\partial\theta\partial\phi}+\phi^2\frac{\partial^2 A}{\partial\phi^2}\right)$$

$$-\left(\frac{\partial A}{\partial\phi}-\theta\frac{\partial^2 A}{\partial\theta\partial\phi}-\phi\frac{\partial^2 A}{\partial\phi^2}\right)\left(2\frac{\partial A}{\partial x_7}-\theta\frac{\partial^2 A}{\partial x_7\partial\theta}-\phi\frac{\partial^2 A}{\partial x_7\partial\phi}\right)=0\,;$$

and similarly for the other.

These three equations must be satisfied by A: and when any common solution is obtained, then we can construct the corresponding partial differential equation of the second order which has an intermediary integral. For the equations are

$$-(lu_v+u_x)=p_7\frac{\partial A}{\partial\theta}\,;$$

but

$$lu_v+u_x+au_l+hu_m+gu_n=0,$$

that is,

$$x_5p_1+p_2+ap_5+hp_6+gp_7=0,$$

in the present notation; and so, substituting for u_x, we have

$$-\frac{\partial A}{\partial\theta}+a\theta+h\phi+g=0.$$

Similarly

$$-\frac{\partial A}{\partial\phi}+h\theta+b\phi+f=0\,;$$

and

$$-2A+\theta\frac{\partial A}{\partial\theta}+\phi\frac{\partial A}{\partial\phi}+g\theta+f\phi+c=0.$$

the last of which, in connection with the other two, can be replaced by

$$-2A+a\theta^2+2h\theta\phi+b\phi^2+2g\theta+2f\phi+c=0.$$

Eliminating θ and ϕ between this equation and the other two, or what is the same thing, forming the discriminant of the equation regarded as involving two variables θ and ϕ, we have the equation of the second order which possesses an intermediary integral.

17. It is not difficult to verify that the three equations of the second order determining A all are satisfied when for A we substitute any function that involves θ and ϕ only and not the other variables. In this case, which corresponds to the case treated by Goursat (§ 1, *note*) for the case of two independent variables, the partial differential equation of the second order is obtained by equating to zero the discriminant of

$$-2A(\theta, \phi) + a\theta^2 + 2h\theta\phi + b\phi^2 + 2g\theta + 2f\phi + c = 0.$$

And the differential equations, satisfied by and determining the intermediary integral, are

$$\left.\begin{aligned} F_1 &= 0 = p_2 + x_5 p_1 + p_7 \frac{\partial A}{\partial \theta} \\ F_2 &= 0 = p_3 + x_6 p_1 + p_7 \frac{\partial A}{\partial \phi} \\ F_3 &= 0 = p_4 + x_7 p_1 + p_7 \left(2A - \theta \frac{\partial A}{\partial \theta} - \phi \frac{\partial A}{\partial \phi}\right) \end{aligned}\right\},$$

a system in involution. They must have four functionally independent common solutions: the more simply these are chosen, the more direct will be the construction of the intermediary integral. It is easy to see that

$$(F_1, p_r) = 0, \quad (F_2, p_r) = 0, \quad (F_3, p_r) = 0,$$

for $r = 1, 2, 3, 4$; so that we can take p_1, p_2, p_3, p_4 as the common solutions.

We therefore combine

$$p_1 = a_1, \quad p_2 = a_2, \quad p_3 = a_3, \quad p_4 = a_4,$$

with $F_1 = 0$, $F_2 = 0$, $F_3 = 0$; so that we have

$$p_5 = \theta p_7, \quad p_6 = \phi p_7,$$

where θ, ϕ, and p_7 are determined by

$$\left.\begin{aligned} p_7 \frac{\partial A}{\partial \theta} &= -a_2 - a_1 x_5 \\ p_7 \frac{\partial A}{\partial \phi} &= -a_3 - a_1 x_6 \\ p_7 \left(2A - \theta \frac{\partial A}{\partial \theta} - \phi \frac{\partial A}{\partial \phi}\right) &= -a_4 - a_1 x_7 \end{aligned}\right\}.$$

Now

$$du = p_1 dx_1 + \dots\dots\dots + p_7 dx_7,$$

so that

$$-d(u - a_1 x_1 - a_2 x_2 - a_3 x_3 - a_4 x_4) = -p_7 (dx_7 + \theta dx_5 + \phi dx_6).$$

The right-hand side must be a perfect differential, say $= dU$. In order to evaluate U, we change the variables so that they are θ, ϕ and x_7; writing

$$\frac{\partial A}{\partial \theta} = A_1, \quad \frac{\partial A}{\partial \phi} = A_2, \quad 2A - \theta A_1 - \phi A_2 = \Delta,$$

$$A_1 = \Delta B_1, \quad A_2 = \Delta B_2,$$

we have

$$a_2 + a_1 x_5 = (a_4 + a_1 x_7) B_1,$$

$$a_3 + a_1 x_6 = (a_4 + a_1 x_7) B_2,$$

and therefore

$$dU = \frac{a_4 + a_1 x_7}{\Delta}(dx_7 + \theta dx_5 + \phi dx_6)$$

$$= \frac{A}{\Delta^2} d\frac{(a_4 + a_1 x_7)^2}{a_1} + \frac{(a_4 + a_1 x_7)^2}{a_1}\frac{\theta dB_1 + \phi dB_2}{\Delta}.$$

It is easy to verify that

$$\frac{\theta dB_1 + \phi dB_2}{\Delta} = d\frac{A}{\Delta^2},$$

and so we have

$$dU = d\left[\frac{A}{\Delta^2}\frac{(a_4 + a_1 x_7)^2}{a_1}\right].$$

Consequently $$u = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 - \frac{(a_4 + a_1 x_7)^2}{a_1}\frac{A}{\Delta^2} - c',$$

where c' is an arbitrary constant. Now the intermediary integral is $u = 0$, or $u =$ constant, so that c' may be dropped; that is, dividing by a_1, and writing α, β, γ for a_2/a_1, a_3/a_1, a_4/a_1, we have

$$v + \alpha x + \beta y + \gamma z - (\gamma + n)^2 \frac{A}{\Delta^2} = 0,$$

where $$\Delta = 2A - \theta\frac{\partial A}{\partial\theta} - \phi\frac{\partial A}{\partial\phi},$$

A is any function of θ and ϕ, and the derivatives of v, viz., l, m, n, are given by

$$\frac{\dfrac{\partial A}{\partial\theta}}{\alpha + l} = \frac{\dfrac{\partial A}{\partial\phi}}{\beta + m} = \frac{2A - \theta\dfrac{\partial A}{\partial\theta} - \phi\dfrac{\partial A}{\partial\phi}}{\gamma + n}.$$

These equations determine the intermediary integral of the equation of the second order given by

$$\text{Discrt}_{\theta\phi}(-2A + a\theta^2 + 2h\theta\phi + b\phi^2 + 2g\theta + 2f\phi + c) = 0.$$

18. In order to obtain a primitive of the equation, we note that the variables v, x, y, z occur only in the combination $v + \alpha x + \beta y + \gamma z$, the quantities θ and ϕ implicitly involving l, m, n. Now of the system, subsidiary to the integration of the intermediary integral, two equations are

$$\frac{-dl}{\alpha + l} = \frac{-dm}{\beta + m} = \frac{-dn}{\gamma + n},$$

so that we have

$$\frac{l + \alpha}{n + \gamma} = \text{constant} = \rho,$$

$$\frac{m + \beta}{n + \gamma} = \text{constant} = \sigma,$$

these equations satisfying the proper Jacobian conditions for coexistence with one another and with the intermediary integral: they therefore can be used for substitution in

$$dv = l\,dx + m\,dy + n\,dz.$$

But when these equations are used, θ and ϕ (as also A and Δ) are constants; and so

$$\gamma + n = A^{-\frac{1}{2}}\Delta\,(v + \alpha x + \beta y + \gamma z)^{\frac{1}{2}},$$

$$l + \alpha = \rho\,(\gamma + n),$$

$$m + \beta = \sigma\,(\gamma + n).$$

Substituting and integrating, we have

$$2\,(v + \alpha x + \beta y + \gamma z)^{\frac{1}{2}} = A^{-\frac{1}{2}}\Delta\,(\rho x + \sigma y + z + \tau),$$

where α, β, γ, ρ, σ, τ are arbitrary constants, $\Delta = 2A - \theta\dfrac{\partial A}{\partial\theta} - \phi\dfrac{\partial A}{\partial\phi}$, A is any function of θ and ϕ, and θ and ϕ are determined in terms of ρ and σ by the equations

$$\frac{1}{\rho}\frac{\partial A}{\partial\theta} = \frac{1}{\sigma}\frac{\partial A}{\partial\phi} = 2A - \theta\frac{\partial A}{\partial\theta} - \phi\frac{\partial A}{\partial\phi}.$$

It will be observed that the primitive contains six arbitrary constants, whereas a complete primitive should contain nine.

19. The intermediary integral which was obtained, viz.

$$\left.\begin{aligned} V = v + \alpha x + \beta y + \gamma z - (n+\gamma)^2\,\frac{A}{\left(2A - \theta\dfrac{\partial A}{\partial\theta} - \phi\dfrac{\partial A}{\partial\phi}\right)^2} = 0 \\ \frac{\dfrac{\partial A}{\partial\theta}}{l+\alpha} = \frac{\dfrac{\partial A}{\partial\phi}}{m+\beta} = \frac{2A - \theta\dfrac{\partial A}{\partial\theta} - \phi\dfrac{\partial A}{\partial\phi}}{n+\gamma} \end{aligned}\right\},$$

contains three arbitrary constants α, β, γ. That these three can be eliminated by forming the derivatives, can actually be verified as follows. We have

$$l + \alpha = \xi\frac{\partial A}{\partial\theta},$$

and therefore

$$m + \beta = \xi\frac{\partial A}{\partial\phi},$$

$$n + \gamma = \xi\left(2A - \theta\frac{\partial A}{\partial\theta} - \phi\frac{\partial A}{\partial\phi}\right),$$

$$v + \alpha x + \beta y + \gamma z = \xi^2 A.$$

From the last equation, it follows that

$$l + \alpha = \xi^2\left(\frac{\partial A}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial A}{\partial\phi}\frac{\partial\phi}{\partial x}\right) + 2A\xi\frac{\partial\xi}{\partial x},$$

$$m + \beta = \xi^2\left(\frac{\partial A}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial A}{\partial\phi}\frac{\partial\phi}{\partial y}\right) + 2A\xi\frac{\partial\xi}{\partial y},$$

$$n + \gamma = \xi^2\left(\frac{\partial A}{\partial\theta}\frac{\partial\theta}{\partial z} + \frac{\partial A}{\partial\phi}\frac{\partial\phi}{\partial z}\right) + 2A\xi\frac{\partial\xi}{\partial z}.$$

But from the other equations, we have

$$a = \xi\left(\frac{\partial^2 A}{\partial\theta^2}\frac{\partial\theta}{\partial x} + \frac{\partial^2 A}{\partial\theta\partial\phi}\frac{\partial\phi}{\partial x}\right) + \frac{\partial A}{\partial\theta}\frac{\partial\xi}{\partial x},$$

$$h = \xi\left(\frac{\partial^2 A}{\partial\theta\partial\phi}\frac{\partial\theta}{\partial x} + \frac{\partial^2 A}{\partial\phi^2}\frac{\partial\phi}{\partial x}\right) + \frac{\partial A}{\partial\phi}\frac{\partial\xi}{\partial x},$$

$$g = \xi\left\{\left(\frac{\partial A}{\partial\theta} - \theta\frac{\partial^2 A}{\partial\theta^2} - \phi\frac{\partial^2 A}{\partial\theta\partial\phi}\right)\frac{\partial\theta}{\partial x} + \left(\frac{\partial A}{\partial\phi} - \theta\frac{\partial^2 A}{\partial\theta\partial\phi} - \phi\frac{\partial^2 A}{\partial\phi^2}\right)\frac{\partial\phi}{\partial x}\right\} + \left(2A - \theta\frac{\partial A}{\partial\theta} - \phi\frac{\partial A}{\partial\phi}\right)\frac{\partial\xi}{\partial x}.$$

and therefore
$$a\theta + h\phi + g = \xi\left(\frac{\partial A}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial A}{\partial\phi}\frac{\partial\phi}{\partial x}\right) + 2A\frac{\partial\xi}{\partial x}$$

$$= \frac{l+\alpha}{\xi} = \frac{\partial A}{\partial\theta}.$$

Similarly differentiating the equations with regard to y, and to z, we find

$$h\theta + b\phi + f = \frac{\partial A}{\partial\phi},$$

$$g\theta + f\phi + c = 2A - \theta\frac{\partial A}{\partial\theta} - \phi\frac{\partial A}{\partial\phi},$$

respectively. These, when combined, lead to the required differential equation of the second order.

But the intermediary integral can be generalised. Suppose that α, β, γ are considered functions of x, y, z instead of being constants: they must be subject to the limitation that the final differential equation must be the same in both cases. Now this final differential equation arises from the elimination of θ and ϕ among three equations, one of which is

$$a\theta + h\phi + g = \frac{l+\alpha}{\xi};$$

and therefore these three equations must keep this form under the changed hypothesis. Now the effect of the change is to add, to the left-hand side of the equation quoted, terms

$$\theta\frac{\partial\alpha}{\partial x} + \phi\frac{\partial\beta}{\partial x} + \frac{\partial\gamma}{\partial x},$$

and, to the right-hand side, terms

$$\frac{1}{\xi}\left(x\frac{\partial\alpha}{\partial x} + y\frac{\partial\beta}{\partial x} + z\frac{\partial\gamma}{\partial x}\right).$$

Hence we have
$$\left(\theta - \frac{x}{\xi}\right)\frac{\partial\alpha}{\partial x} + \left(\phi - \frac{y}{\xi}\right)\frac{\partial\beta}{\partial x} + \left(1 - \frac{z}{\xi}\right)\frac{\partial\gamma}{\partial x} = 0.$$

Similarly
$$\left(\theta - \frac{x}{\xi}\right)\frac{\partial\alpha}{\partial y} + \left(\phi - \frac{y}{\xi}\right)\frac{\partial\beta}{\partial y} + \left(1 - \frac{z}{\xi}\right)\frac{\partial\gamma}{\partial y} = 0,$$

$$\left(\theta - \frac{x}{\xi}\right)\frac{\partial\alpha}{\partial z} + \left(\phi - \frac{y}{\xi}\right)\frac{\partial\beta}{\partial z} + \left(1 - \frac{z}{\xi}\right)\frac{\partial\gamma}{\partial z} = 0.$$

These three shew that a functional relation subsists between α, β, γ quà functions of x, y, z; say

$$\gamma = \Gamma(\alpha, \beta).$$

And then we have

$$\left(\theta - \frac{x}{\xi}\right) d\alpha + \left(\phi - \frac{y}{\xi}\right) d\beta + \left(1 - \frac{z}{\xi}\right) d\gamma = 0,$$

that is, as the quantities α and β are independent of one another, we have

$$\theta - \frac{x}{\xi} + \left(1 - \frac{z}{\xi}\right) \frac{\partial \Gamma}{\partial \alpha} = 0,$$

$$\phi - \frac{y}{\xi} + \left(1 - \frac{z}{\xi}\right) \frac{\partial \Gamma}{\partial \beta} = 0.$$

These equations, together with

$$\gamma - \Gamma(\alpha, \beta) = 0,$$

$$v + \alpha x + \beta y + \gamma z - \xi^2 A = 0,$$

$$l + \alpha - \xi \frac{\partial A}{\partial \theta} = 0,$$

$$m + \beta - \xi \frac{\partial A}{\partial \phi} = 0,$$

$$n + \gamma - \xi \left(2A - \theta \frac{\partial A}{\partial \theta} - \phi \frac{\partial A}{\partial \phi}\right) = 0,$$

lead by the elimination of the quantities α, β, γ, θ, ϕ, ξ to the generalised intermediary integral involving one arbitrary function of two arguments.

20. I leave on one side, for the present, the question of generalising the primitive which has already been obtained: as also the wider question of generalising a primitive of any equation of the second order in three independent variables, when the primitive contains more than three arbitrary constants.

Lastly, the preceding investigations are based upon the assumption that the initial system of three algebraical equations is a complete system. In the alternative assumption, the system must be rendered complete by the association of such new equations as arise out of the Jacobi-Poisson conditions: it will then contain more than three equations in each such case, and so effectively is included in the remaining possibilities of § 15, as yet unconsidered.

21. *Four algebraically independent equations.* Suppose the equations solved for say u_x, u_y, u_z, u_l in terms of u_m, u_n. The expressions for each must be homogeneous of the first degree in u_m and u_n: or if we take $u_m = \theta u_n$, so that θ denotes $u_m \div u_n$ for brevity, then we have

$$\begin{aligned}
R &= x_5 p_1 + p_2 + p_7 \rho_1 (x_1, \ldots, x_7, \theta) = 0, \\
S &= x_6 p_1 + p_3 + p_7 \sigma_1 (x_1, \ldots, x_7, \theta) = 0, \\
T &= x_7 p_1 + p_4 + p_7 \tau_1 (x_1, \ldots, x_7, \theta) = 0, \\
P &= p_5 + p_7 \pi_1 (x_1, \ldots, x_7, \theta) = 0, \\
& \quad p_6 - p_7 \theta = 0.
\end{aligned}$$

Forming the function (R, P), we require that it shall vanish: and therefore we have an equation of the form

$$p_1 + \text{terms independent of } p_2,\ p_3,\ p_4 = 0,$$

so that we must have a new equation, say

$$p_1 + p_7\kappa\,(x_1, \ldots, x_7, \theta) = 0.$$

Let the others be transformed by means of this new equation, so that

$$\begin{aligned} p_2 + p_7\rho\,(\qquad\qquad) &= 0,\\ p_3 + p_7\sigma\,(\qquad\qquad) &= 0,\\ p_4 + p_7\tau\,(\qquad\qquad) &= 0,\\ p_5 + p_7\pi\,(\qquad\qquad) &= 0. \end{aligned}$$

Then we must form all the combinations (A, B) in pairs: and they must all vanish either identically, or in virtue of a single equation which determines θ as a function of $x_1, \ldots, x_7$.

In the former case, the system is complete: and so it possesses two integrals functionally independent of one another. We thus can construct an intermediary integral involving two arbitrary constants.

If, in the latter case, the system is complete, there is one solution; and we can deduce an intermediary integral involving one arbitrary constant. If the system is not complete, there is no intermediary integral.

It should however be noted that, though the original four equations are deduced from a given equation of the second order, the latter is not the only equation of the second order satisfied in connection with them. In fact, we have

$$u_x + au_l + hu_m + gu_n = 0,$$

that is, on dividing by $p_7\,(= u_n)$,

$$-\rho_1 - a\pi_1 + h\theta + g = 0;$$

and similarly

$$-\sigma_1 - h\pi_1 + b\theta + f = 0,$$

$$-\tau_1 - g\pi_1 + f\theta + c = 0.$$

When θ is eliminated, two equations of the second order (and not one alone) result: and the supposed given equation is satisfied in virtue of those two.

22. *Five algebraically independent equations.* When there are five equations, they can be solved for (say) u_x, u_y, u_z, u_l, u_m in terms of u_n: and the values will be of the form

$$\left.\begin{aligned} u_x &= \alpha u_n\\ u_y &= \beta u_n\\ u_z &= \gamma u_n\\ u_l &= \delta u_n\\ u_m &= \epsilon u_n \end{aligned}\right\},$$

where α, β, γ, δ, ϵ are functions of v, x, y, z, l, m, n only. But assuming that these equations can coexist as a system of simultaneous partial differential equations, the supposed equation of the second order can be replaced by three equations of the second order, linear in the highest derivatives: these equations being, in fact,

$$\left.\begin{aligned} \alpha + a\delta + h\epsilon + g &= 0 \\ \beta + h\delta + b\epsilon + f &= 0 \\ \gamma + g\delta + f\epsilon + c &= 0 \end{aligned}\right\}.$$

The case is thus of highly restricted generality.

23. Though only particular classes of equations have been considered, the methods indicated enable us to construct equations possessing an intermediary integral and also to obtain the intermediary integral for such equations as possess it. If, however, an equation of one of the proper forms be given but should not satisfy the necessary conditions, or if an equation not of any of the proper forms be given, the general method is inapplicable: the equation does not possess an intermediary integral, and some other process that may lead to a primitive must be adopted. The discussion of this part of the subject is reserved for another paper.

IX. *Reduction of a certain Multiple Integral.* By ARTHUR BLACK.

*Communicated by Professor M. J. M. HILL, M.A., Sc. D., F.R.S.

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1. To evaluate the multiple integral

$$\int_n V(\exp - U)\, dx_1 \ldots dx_n,$$

where U and V are homogeneous quadratic functions of the n variables $x_1 \ldots x_n$ and a constant x_0, and all the integrations are from $-\infty$ to $+\infty$, it being further supposed that U is essentially positive.

If

$$U = \sum_0^n a_{r,r} x_r^2 + 2\sum_0^n a_{r,s} x_r x_s,$$

$$V = \sum_0^n \kappa_{r,r} x_r^2 + 2\sum_0^n \kappa_{r,s} x_r x_s,$$

(where $a_{r,s} = a_{s,r}$ and $\kappa_{r,s} = \kappa_{s,r}$),

if Δ be the discriminant of U, regarded as a quadratic function of $x_0 x_1 \ldots x_n$,

if $A_{r,s}$ be the coefficient of $a_{r,s}$ in Δ, and $B_{r,s}$ the coefficient of $a_{r,s}$ in

$$\begin{vmatrix} a_{11} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n,1} & \ldots\ldots & a_{n,n} \end{vmatrix},$$

then the integral in question is equal to

$$\sqrt{\frac{\pi^n}{A_{00}^5}} \exp\left(-\frac{\Delta x_0^2}{A_{00}}\right)\left[x_0^2\left\{\sum_0^n \kappa_{rr} A_{0r}^2 + 2\sum_0^n \kappa_{r,s} A_{0r} A_{0s}\right\} + \tfrac{1}{2} A_{00}\left\{\sum_1^n \kappa_{rr} B_{rr} + 2\sum_1^n \kappa_{r,s} B_{r,s}\right\}\right].$$

* The Multiple Integral, to which this paper relates, is useful in the applications of the Theory of Probabilities to Statistics. It was evaluated by the late Mr Arthur Black, who died in 1893, in a manuscript work on the application of Mathematics to the Theory of Evolution, but not exactly in the form here presented, the discussion being spread over different portions of the manuscript, and the notation being slightly different to that here adopted. No alteration has been made in any essential point of the work or of the method. The whole of Mr Black's manuscript work is in the hands of his sister, Mrs Constance Garnett, of The Cearne, Kent Hatch, near Edenbridge, Kent, and can be seen by any one interested in the subject.

M. J. M. HILL.

2. The first step is to evaluate an expression of the form

$$I = \int_n dx_1 \ldots dx_n (c_{\nu 0}x_0 + c_{\nu 1}x_1 + \ldots + c_{\nu n}x_n) \exp(-U).$$

$$U = a_{nn}x_n^2 + 2x_n \sum_{r=0}^{r=n-1} a_{nr}x_r + \sum_0^{n-1} a_{rr}x_r^2 + 2\sum_0^{n-1} a_{rs}x_rx_s;$$

$$\therefore\ a_{nn}U = \left(a_{nn}x_n + \sum_0^{n-1} a_{nr}x_r\right)^2 + \sum_0^{n-1} x_r^2 \begin{vmatrix} a_{rr} & a_{rn} \\ a_{nr} & a_{nn} \end{vmatrix} + 2\sum_0^{n-1} \begin{vmatrix} a_{rs} & a_{rn} \\ a_{ns} & a_{nn} \end{vmatrix} x_rx_s,$$

$$I = \int_n dx_1 \ldots dx_n \frac{1}{a_{nn}} \left[c_{\nu n}\sum_0^n (a_{nr}x_r) + \sum_0^{n-1} x_r \begin{vmatrix} c_{\nu r} & c_{\nu n} \\ a_{nr} & a_{nn} \end{vmatrix}\right] \exp(-U).$$

Put

$$a_{nn}x_n + \sum_0^{n-1} a_{nr}x_r = z\sqrt{a_{nn}};^*$$

$$\therefore\ I = \int_n dx_1 \ldots dx_{n-1}dz \frac{1}{\sqrt{a_{nn}^3}} \left[c_{\nu n}z\sqrt{a_{nn}} + \sum_0^{n-1} x_r \begin{vmatrix} c_{\nu r} & c_{\nu n} \\ a_{nr} & a_{nn} \end{vmatrix}\right] \exp(-z^2 - U'),$$

where

$$U' = \frac{1}{a_{nn}}\left[\sum_0^{n-1} x_r^2 \begin{vmatrix} a_{rr} & a_{rn} \\ a_{nr} & a_{nn} \end{vmatrix} + 2\sum_0^{n-1} \begin{vmatrix} a_{rs} & a_{rn} \\ a_{ns} & a_{nn} \end{vmatrix} x_rx_s\right].$$

After integrating with regard to z, the result is

$$I = \int_{n-1} dx_1 \ldots dx_{n-1} \sqrt{\frac{\pi}{a_{nn}^3}} \exp(-U') \,.\, \sum_0^{n-1} x_r \begin{vmatrix} c_{\nu r} & c_{\nu n} \\ a_{nr} & a_{nn} \end{vmatrix}.$$

This is an expression of similar form to the one from which the integration commenced.

Write for brevity

$$\delta_{\nu, r} = \begin{vmatrix} c_{\nu, r} & c_{\nu, n} \\ a_{n, r} & a_{n, n} \end{vmatrix}, \qquad r = 1 \text{ to } n-1,$$

$$b_{r, r} = \frac{1}{a_{n, n}} \begin{vmatrix} a_{rr} & a_{rn} \\ a_{nr} & a_{nn} \end{vmatrix}, \qquad b_{rs} = \frac{1}{a_{nn}} \begin{vmatrix} a_{rs} & a_{rn} \\ a_{ns} & a_{nn} \end{vmatrix}.$$

Then the result of the next integration with regard to x_{n-1} is

$$I = \int_{n-2} dx_1 \ldots dx_{n-2} \sqrt{\frac{\pi}{a_{nn}^3}} \sqrt{\frac{\pi}{b_{n-1, n-1}^3}} \left[\sum_0^{n-2} x_r \begin{vmatrix} \delta_{\nu, r} & \delta_{\nu, n-1} \\ b_{n-1, r} & b_{n-1, n-1} \end{vmatrix}\right] \exp(-U''),$$

where

$$U'' = \frac{1}{b_{n-1, n-1}} \left[\sum_0^{n-2} x_r^2 \begin{vmatrix} b_{rr} & b_{r, n-1} \\ b_{n-1, r} & b_{n-1, n-1} \end{vmatrix} + 2\sum_0^{n-2} x_rx_s \begin{vmatrix} b_{r, s} & b_{r, n-1} \\ b_{n-1, s} & b_{n-1, n-1} \end{vmatrix}\right].$$

Now writing down the determinants

$$\begin{vmatrix} c_{\nu, r} & c_{\nu, n-1} & c_{\nu, n} \\ a_{n-1, r} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, r} & a_{n, n-1} & a_{n, n} \end{vmatrix} \text{ and } \begin{vmatrix} a_{r, s} & a_{r, n-1} & a_{r, n} \\ a_{n-1, s} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, s} & a_{n, n-1} & a_{n, n} \end{vmatrix}$$

* In order that U may be essentially positive, it is necessary that

$$\Delta,\ A_{00},\ B_{11},\ \begin{vmatrix} a_{33} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots a_{nn} \end{vmatrix},\ \begin{vmatrix} a_{44} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots a_{nn} \end{vmatrix},\ \ldots \begin{vmatrix} a_{n-1, n-1} & a_{n-1, n} \\ a_{n, n-1} & a_{nn} \end{vmatrix},\ a_{nn}$$

should all be positive.

it follows from known properties of the adjugates of these that

$$\begin{vmatrix} \delta_{\nu, r} & \delta_{\nu, n-1} \\ b_{n-1, r} & b_{n-1, n-1} \end{vmatrix} = \begin{vmatrix} c_{\nu, r} & c_{\nu, n-1} & c_{\nu, n} \\ a_{n-1, r} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, r} & a_{n, n-1} & a_{n, n} \end{vmatrix},$$

and

$$\begin{vmatrix} b_{r, s} & b_{r, n-1} \\ b_{n-1, s} & b_{n-1, n-1} \end{vmatrix} = \frac{1}{a_{n, n}} \begin{vmatrix} a_{r, s} & a_{r, n-1} & a_{r, n} \\ a_{n-1, s} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, s} & a_{n, n-1} & a_{n, n} \end{vmatrix};$$

$$\therefore\ I = \int_{n-2} dx_1 \ldots dx_{n-2} \frac{\sqrt{\pi^2}}{\left\{\begin{vmatrix} a_{n-1, n-1} & a_{n-1, n} \\ a_{n, n-1} & a_{n, n} \end{vmatrix}\right\}^{\frac{3}{2}}} \sum_0^{n-2} x_r \begin{vmatrix} c_{\nu, r} & c_{\nu, n-1} & c_{\nu, n} \\ a_{n-1, r} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, r} & a_{n, n-1} & a_{n, n} \end{vmatrix} \exp(-U''),$$

where

$$\begin{vmatrix} a_{n-1, n-1} & a_{n-1, n} \\ a_{n, n-1} & a_{n, n} \end{vmatrix} U'' = \sum_0^{n-2} x_r^2 \begin{vmatrix} a_{r, r} & a_{r, n-1} & a_{r, n} \\ a_{n-1, r} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, r} & a_{n, n-1} & a_{n, n} \end{vmatrix} + 2\sum_0^{n-2} x_r x_s \begin{vmatrix} a_{r, s} & a_{r, n-1} & a_{r, n} \\ a_{n-1, s} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, s} & a_{n, n-1} & a_{n, n} \end{vmatrix}.$$

After integrating with regard to x_{n-2},

$$I = \int_{n-3} dx_1 \ldots dx_{n-3} \frac{\sqrt{\pi^3}}{\left\{\begin{vmatrix} a_{n-2, n-2} & a_{n-2, n-1} & a_{n-2, n} \\ a_{n-1, n-2} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, n-2} & a_{n, n-1} & a_{n, n} \end{vmatrix}\right\}^{\frac{3}{2}}} \sum_0^{n-3} x_r \begin{vmatrix} c_{\nu, r} & c_{\nu, n-2} & c_{\nu, n-1} & c_{\nu, n} \\ a_{n-2, r} & a_{n-2, n-2} & a_{n-2, n-1} & a_{n-2, n} \\ a_{n-1, r} & a_{n-1, n-2} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, r} & a_{n, n-2} & a_{n, n-1} & a_{n, n} \end{vmatrix} \exp(-U'''),$$

where

$$\begin{vmatrix} a_{n-2, n-2} & a_{n-2, n-1} & a_{n-2, n} \\ a_{n-1, n-2} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, n-2} & a_{n, n-1} & a_{n, n} \end{vmatrix} U'''$$

$$= \sum_0^{n-3} x_r^2 \begin{vmatrix} a_{r, r} & a_{r, n-2} & a_{r, n-1} & a_{r, n} \\ a_{n-2, r} & a_{n-2, n-2} & a_{n-2, n-1} & a_{n-2, n} \\ a_{n-1, r} & a_{n-1, n-2} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, r} & a_{n, n-2} & a_{n, n-1} & a_{n, n} \end{vmatrix} + 2\sum_0^{n-3} x_r x_s \begin{vmatrix} a_{r, s} & a_{r, n-2} & a_{r, n-1} & a_{r, n} \\ a_{n-2, s} & a_{n-2, n-2} & a_{n-2, n-1} & a_{n-2, n} \\ a_{n-1, s} & a_{n-1, n-2} & a_{n-1, n-1} & a_{n-1, n} \\ a_{n, s} & a_{n, n-2} & a_{n, n-1} & a_{n, n} \end{vmatrix},$$

and so on, until after integrating with regard to x_2,

$$I = \int dx_1 \frac{\sqrt{\pi^{n-1}}}{\left\{\begin{vmatrix} a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n, 2} & \cdots\cdots & a_{n, n} \end{vmatrix}\right\}^{\frac{3}{2}}} \sum_0^{1} x_r \begin{vmatrix} c_{\nu, r} & c_{\nu, 2} \cdots\cdots & c_{\nu, n} \\ a_{2, r} & a_{2, 2} \cdots\cdots & a_{2, n} \\ \cdots & \cdots & \cdots \\ a_{n, r} & a_{n, 2} \cdots\cdots & a_{n, n} \end{vmatrix} \exp(-U^{(n-1)}),$$

where

$$\begin{vmatrix} a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} U^{(n-1)} = \sum_0^1 x_r^2 \begin{vmatrix} a_{r,r} & a_{r,2} & \cdots\cdots & a_{r,n} \\ a_{2,r} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,r} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} + 2\sum_0^1 x_r x_s \begin{vmatrix} a_{r,s} & a_{r,2} & \cdots\cdots & a_{r,n} \\ a_{2,s} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,s} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}$$

$$= x_0^2 \begin{vmatrix} a_{0,0} & a_{0,2} & \cdots\cdots & a_{0,n} \\ a_{2,0} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} + 2x_0x_1 \begin{vmatrix} a_{0,1} & a_{0,2} & \cdots\cdots & a_{0,n} \\ a_{2,1} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} + x_1^2 \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}$$

$$= A_{11}x_0^2 - 2A_{01}x_0x_1 + A_{00}x_1^2$$

$$= A_{00}\left[\left(x_1 - \frac{A_{01}}{A_{00}}x_0\right)^2 + x_0^2\frac{A_{00}A_{11} - A_{01}^2}{A_{00}^2}\right]$$

$$= A_{00}\left(x_1 - \frac{A_{01}}{A_{00}}x_0\right)^2 + x_0^2\frac{\Delta}{A_{00}} \begin{vmatrix} a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix};$$

$$\therefore\ U^{(n-1)} = \frac{A_{00}}{\begin{vmatrix} a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}}\left(x_1 - \frac{A_{01}}{A_{00}}x_0\right)^2 + x_0^2\frac{\Delta}{A_{00}}.$$

To complete the integration put

$$x_1 - \frac{A_{01}}{A_{00}}x_0 = z\sqrt{\frac{1}{A_{00}}\begin{vmatrix} a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}}.$$

Then

$$I = \int_{-\infty}^{\infty} dz \frac{\sqrt{\pi^{n-1}}}{\sqrt{A_{00}}} \frac{1}{\begin{vmatrix} a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}} Q$$

where $$Q = \left\{x_0 \begin{vmatrix} c_{\nu,0} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,0} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} + \frac{A_{01}}{A_{00}}x_0 \begin{vmatrix} c_{\nu,1} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,1} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}\right\} \exp\left[-z^2 - x_0^2\frac{\Delta}{A_{00}}\right],$$

because $$\int_{-\infty}^{\infty} ze^{-z^2}dz = 0.$$

$$\therefore\ I = \sqrt{\frac{\pi^n}{A_{00}}}\exp\left(-\frac{x_0^2\Delta}{A_{00}}\right) \frac{x_0}{A_{00}\begin{vmatrix} a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}} \left[A_{00}\begin{vmatrix} c_{\nu,0} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,0} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} + A_{01}\begin{vmatrix} c_{\nu,1} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,1} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}\right].$$

Now writing down the determinant

$$\begin{vmatrix} c_{\nu 0} & c_{\nu 1} & c_{\nu 2} & \cdots\cdots & c_{\nu n} \\ a_{10} & a_{11} & a_{12} & \cdots\cdots & a_{1,n} \\ a_{2,0} & a_{21} & a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix} = E,$$

it appears that

$$A_{00}\begin{vmatrix} c_{\nu,0} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,0} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,0} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} + A_{01}\begin{vmatrix} c_{\nu,1} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,1} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix} = \begin{Bmatrix} (\text{coefficient of } c_{\nu,0})\,(\text{coefficient of } a_{11}) \\ +\,(\text{coefficient of } c_{\nu,1})\,(-\text{coefficient of } a_{10}) \end{Bmatrix}$$

$$= E\begin{vmatrix} a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}.$$

$$\therefore I = x_0\sqrt{\frac{\pi^n}{A_{00}^3}}\left[\exp\left(-\frac{x_0^2\Delta}{A_{00}}\right)\right]\begin{vmatrix} c_{\nu 0} & c_{\nu 1} & c_{\nu 2} & \cdots\cdots & c_{\nu n} \\ a_{10} & a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}$$

$$= \int_n dx_1 \ldots dx_n\,(c_{\nu 0}x_0 + c_{\nu 1}x_1 + \ldots + c_{\nu n}x_n)\exp(-U).$$

3. The particular case in which

$$c_{\nu,0} = 1,\quad c_{\nu,1} = c_{\nu,2} = \cdots\cdots = c_{\nu,n} = 0,\quad x_0 = 1$$

is very important.

It gives
$$\int_n dx_1 \cdots\cdots dx_n \exp(-U) = \sqrt{\frac{\pi^n}{A_{00}}}\exp\left(-\frac{\Delta}{A_{00}}\right).$$

4. The next step is to evaluate the integral

$$J = \int_n dx_1 \cdots\cdots dx_n x_1 (c_{\nu 0}x_0 + c_{\nu 1}x_1 + \cdots\cdots + c_{\nu n}x_n).$$

The work proceeds as in the evaluation of I up to the end of the integration with regard to x_2.

Hence
$$J = \int x_1 dx_1 \frac{\sqrt{\pi}^{n-1}}{\begin{Bmatrix}\begin{vmatrix} a_{22} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots \\ a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix}\end{Bmatrix}^{\frac{3}{2}}} \sum_0^1 x_r \begin{vmatrix} c_{\nu,r} & c_{\nu,2} & \cdots\cdots & c_{\nu,n} \\ a_{2,r} & a_{2,2} & \cdots\cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,r} & a_{n,2} & \cdots\cdots & a_{n,n} \end{vmatrix} \exp(-U^{(n-1)})$$

$$= \frac{\sqrt{\pi}^{n-1}}{\left\{ \begin{vmatrix} a_{22} \ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots a_{n,n} \end{vmatrix} \right\}^{\frac{1}{2}}} \exp\left(-\frac{\Delta x_0^2}{A_{00}}\right) R$$

where
$$R = \int x_1\, dx_1 \left\{ x_0 \begin{vmatrix} c_{\nu,0} & c_{\nu,2} & \ldots\ldots & c_{\nu,n} \\ a_{2,0} & a_{2,2} & \ldots\ldots & a_{2,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,0} & a_{n,2} & \ldots\ldots & a_{n,n} \end{vmatrix} + x_1 \begin{vmatrix} c_{\nu,1} & c_{\nu,2} & \ldots\ldots & c_{\nu,n} \\ a_{2,1} & a_{2,2} & \ldots\ldots & a_{2,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,1} & a_{n,2} & \ldots\ldots & a_{n,n} \end{vmatrix} \right\} \exp - \frac{A_{00}\left(x_1 - \dfrac{A_{01}}{A_{00}} x_0\right)^2}{\begin{vmatrix} a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}}.$$

Putting
$$x_1 = \frac{A_{01}}{A_{00}} x_0 + z \sqrt{\frac{1}{A_{00}} \begin{vmatrix} a_{22} \ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\; a_{nn} \end{vmatrix}},$$

and neglecting the terms which vanish in virtue of the relation $\int_{-\infty}^{\infty} z e^{-z^2} dz = 0$,

$$J = \frac{\sqrt{\pi}^{n-1}}{\sqrt{A_{00}}} \frac{1}{\begin{vmatrix} a_{22} \ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots a_{nn} \end{vmatrix}} \left(\exp - \frac{\Delta x_0^2}{A_{00}}\right) \int_{-\infty}^{\infty} dz\, e^{-z^2} \left[x_0^2 \frac{A_{01}}{A_{00}} \begin{vmatrix} c_{\nu 0} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{20} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + x_0^2 \frac{A_{01}^2}{A_{00}^2} \begin{vmatrix} c_{\nu 1} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \right.$$

$$\left. + \frac{z^2}{A_{00}} \begin{vmatrix} a_{22} \ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots a_{nn} \end{vmatrix} \begin{vmatrix} c_{\nu 1} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \right].$$

$$\therefore J = \frac{\sqrt{\pi}^{n}}{\sqrt{A_{00}}} \frac{1}{\begin{vmatrix} a_{22} \ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots a_{nn} \end{vmatrix}} \left(\exp - \frac{\Delta x_0^2}{A_{00}}\right) \left[\frac{x_0^2 A_{01}}{A_{00}^2} \left\{ A_{00} \begin{vmatrix} c_{\nu 0} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{20} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + A_{01} \begin{vmatrix} c_{\nu 1} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \right\} \right.$$

$$\left. + \frac{1}{2A_{00}} \begin{vmatrix} a_{22} \ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots a_{nn} \end{vmatrix} \begin{vmatrix} c_{\nu 1} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \right]$$

$$= \frac{\sqrt{\pi}^{n}}{\sqrt{A_{00}}} \left(\exp - \frac{\Delta x_0^2}{A_{00}}\right) \left[\frac{x_0^2 A_{01}}{A_{00}^2} \begin{vmatrix} c_{\nu 0} & c_{\nu 1} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{10} & a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + \frac{1}{2A_{00}} \begin{vmatrix} c_{\nu 1} & c_{\nu 2} & \ldots\ldots & c_{\nu n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \right]$$

$$= \int_n dx_1 \ldots\ldots dx_n\, x_1 (c_{\nu 0} x_0 + c_{\nu 1} x_1 + \ldots\ldots + c_{\nu n} x_n) \exp(-U).$$

5. Now let

$$V = \sum_0^n \kappa_{r,r} x_r^2 + 2\sum_0^n \kappa_{r,s} x_r x_s.$$

Then

$$\begin{aligned} V = \; & x_0 (\kappa_{00} x_0 + \kappa_{01} x_1 + \ldots\ldots + \kappa_{0n} x_n) \\ & + x_1 (\kappa_{10} x_0 + \kappa_{11} x_1 + \ldots\ldots + \kappa_{1n} x_n) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + x_n (\kappa_{n0} x_0 + \kappa_{n1} x_1 + \ldots\ldots + \kappa_{nn} x_n). \end{aligned}$$

$$\therefore \int_n V \exp(-U)\, dx_1 \ldots\ldots dx_n$$

$$= \sqrt{\frac{\pi^n}{A_{00}^5}} \left[\exp\left(-\frac{\Delta x_0^2}{A_{00}}\right)\right] [L x_0^2 + M],$$

where

$$L = A_{00} \begin{vmatrix} \kappa_{00} & \kappa_{01} & \kappa_{02} & \ldots\ldots & \kappa_{0n} \\ a_{10} & a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + A_{01} \begin{vmatrix} \kappa_{10} & \kappa_{11} & \kappa_{12} & \ldots\ldots & \kappa_{1n} \\ a_{10} & a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + \ldots\ldots + A_{0n} \begin{vmatrix} \kappa_{n0} & \kappa_{n1} & \kappa_{n2} & \ldots\ldots & \kappa_{nn} \\ a_{10} & a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}$$

$$= \sum_0^n \kappa_{rr} A_{0r}^2 + 2\sum_0^n \kappa_{r,s} A_{0,r} A_{0,s}$$

and

$$M = \tfrac{1}{2} A_{00} \left[\begin{vmatrix} \kappa_{11} & \kappa_{12} & \ldots\ldots & \kappa_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ \kappa_{21} & \kappa_{22} & \ldots\ldots & \kappa_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} + \ldots\ldots + \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1,1} & a_{n-1,2} & \ldots\ldots & a_{n-1,n} \\ \kappa_{n1} & \kappa_{n2} & \ldots\ldots & \kappa_{nn} \end{vmatrix} \right];$$

or denoting by $B_{r,s}$ the coefficient of a_{rs} in

$$\begin{vmatrix} a_{11} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots\ldots & a_{nn} \end{vmatrix}$$

$$M = \tfrac{1}{2} A_{00} \left[\sum_1^n \kappa_{rr} B_{r,r} + 2\sum_1^n \kappa_{r,s} B_{r,s}\right].$$

X. *On the Fifth Book of Euclid's Elements.* By M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London.

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Abstract.

Art. 1. The objects of this paper are

(I.) To draw attention to the indirect character of the argument in the Fifth Book of Euclid's Elements.

(II.) To reconstruct the argument showing how the indirectness may be removed.

(III.) To develop the theory of ratio from the reconstructed argument.

Art. 2. The indirectness of the argument arises in this way.

Amongst the definitions of the Fifth Book there occurs one (No. 7) which furnishes a test for *unequal* ratios.

This test plays no independent part in Euclid's Elements, being merely used to prove certain properties of *equal* ratios.

Now if the test for *equal* ratios, given in the fifth definition of the Fifth Book, be a sound and complete one, it ought to be possible to deduce all properties of *equal* ratios from it, without employing the test for *unequal* ratios.

This is in fact the case, as is shown in the reconstructed argument, which is given in the second part of this paper.

The developments of the theory of ratio in the third part of the paper are

(1) The proof of the fundamental proposition that two magnitudes of the same kind taken in a definite order determine a real number.

This real number is defined to be the measure of the ratio of the first magnitude to the second.

(It may be noted that this is the first occasion on which the term "ratio" appears in the theory as presented in this paper.)

(2) The proof of the fact that the definitions of the processes of adding and compounding ratios must in every case lead to consistent results; and that the commutative, associative, and distributive laws hold good for these processes.

(3) The proof of the fact that the definition of the measure of a ratio and the definition of the addition of ratios lead to the result that the measure of the ratio, which is the sum of two ratios, is the sum of the measures of these ratios.

From this it follows that the multiplicity of ratios is *measurable*.

(4) One ratio being defined to be greater than another when the measure of the first is greater than that of the second, the conditions which must be satisfied in order that one ratio may be greater than a second are deduced in the form given in the seventh definition of the Fifth Book, so that this definition is treated as a proposition to be proved, and is not laid down as a definition to start with.

When this has been done it becomes possible to *order* the multiplicity of ratios.

I. *The Indirectness of the Argument in the Fifth Book of Euclid's Elements.*

Art. 3. This will be seen from the following account of the contents of the book.

The edition employed by the writer of this paper is the Oxford Edition edited by Gregory and dated 1703.

It is convenient not to follow Euclid's order.

The contents of the book may be grouped as follows:—

(1) There are five Propositions Nos. 1, 2, 3, 5, 6 which relate to magnitudes and their multiples but are not concerned with ratios.

They relate to simple cases of the commutative, associative, and distributive laws.

(2) Of the definitions only three are important. No. 3, which defines ratio, is only sufficient to distinguish ratio from absolute magnitude. No. 5, which furnishes a test for *equal* ratios. No. 7, which furnishes a test for *unequal* ratios.

The 7th definition is only used twice, viz. in the proof of Propositions 8 and 13.

(3) All the remaining propositions Nos. 4 and 7—25 deal with properties of ratios. These may be divided into three groups.

(4) The first group consisting of Propositions 4 (with its very important corollary), 7, 11, 12, 15 and 17, express properties of *Equal* Ratios, and are deduced directly from the Test for *Equal* Ratios.

(5) The second group consists of Propositions 8, 10, and 13, which express properties of *Unequal* Ratios, and depend on the Test for *Unequal* Ratios.

This group of propositions is used in the Fifth Book to prove properties of *Equal* Ratios, but nowhere else in Euclid's Elements.

It follows that the Test for *Unequal* Ratios plays an indirect part only in Euclid's Elements.

(6) The third group consists of Propositions 9, 14, 16 and 18—25. All these deal with properties of *Equal* Ratios, but their proofs depend directly or indirectly on Propositions 8, 10, and 13, and therefore ultimately on the Test for *Unequal* Ratios.

II. *Reconstruction of the Argument.*

Identity.

Art. 4. Two objects are said to be identical when everything that can be said of one can also be said of the other (except that they occupy the same space at the same time).

Two objects are said to be identical in respect of a particular property when everything that can be said concerning the possession of that property by one object can be said concerning its possession by the other.

Number.

Art. 5. When several objects are under consideration, all those which possess a certain property may be distinguished by saying that they constitute together a species, and that this property is characteristic of the species.

One of the objects thus distinguished will be, in regard to this property, a unit of the species.

Recognising the characteristic property in successive units, the simple conception of the whole number is obtained.

Two units of the same species are equal, i.e. equivalent in respect of the specific property.

In this paper, except where otherwise stated, the word "Number" will be used as an abbreviation for "Positive whole number."

Notation for Number.

Art. 6. A Number will always be denoted by a *small* letter.

Assumptions with regard to Magnitude.

Art. 7. (1) If one magnitude is given, it is possible to find any number of others identical with it.

(2) It is possible to unify into a whole any number of identical magnitudes.

The whole is then called a "Multiple" of any one of the identical magnitudes.

Notation for Magnitude.

Art. 8. A magnitude will be denoted throughout this paper by a *capital* letter.

Homogeneous Magnitude.

Art. 9. A homogeneous magnitude is one which can be regarded as consisting of any integral number of identical parts.

It is to be understood that the integral number of parts may be any integral number whatever; that the "identical parts" are objects which are identical in respect of one and the same property, and that the object is identical with its parts when unified into a whole.

Magnitudes of the same kind.

Art. 10. Two homogeneous magnitudes are said to be of the same kind, if they can both be conceived as containing portions which are identical.

Assumptions with regard to magnitudes of the same kind.

Art. 11. (1) If two magnitudes of the same kind are given, it is possible to determine whether one is greater than, equal to, or less than the other.

(2) If two magnitudes of the same kind are given, it is possible to form a multiple of the smaller which is greater than the larger.

Equimultiples.

Art. 12. If the same multiple be taken of each of two magnitudes A and B, these are called equimultiples of A and B.

Scale of Multiples, or Multiple Scale.

Art. 13. There exists a set of magnitudes depending on A, all of which are known when A is known; viz.—

$$A,\ 2A,\ 3A,\ 4A,\ \ldots\ldots,\ rA,\ \ldots\ldots$$

which can be carried on to any extent. These may be distinguished from all other magnitudes by calling them multiples of A (the first being called the first multiple of A for this purpose).

The above set of magnitudes may be called collectively *the scale of the multiples of A*, or more briefly the *Multiple Scale of A*.

Art. 14. If A and B be two magnitudes of the same kind, then however small A may be, or however great B may be, it follows from Art. 11 (2) that the multiples in the scale

$$A,\ 2A,\ 3A,\ 4A,\ \ldots\ldots,\ rA,\ \ldots\ldots$$

will, after a certain multiple, all exceed B.

In like manner, after a certain multiple, they will all exceed $2B$; and so on, multiples can be found, which will exceed $3B$, $4B$, ..., sB, ...

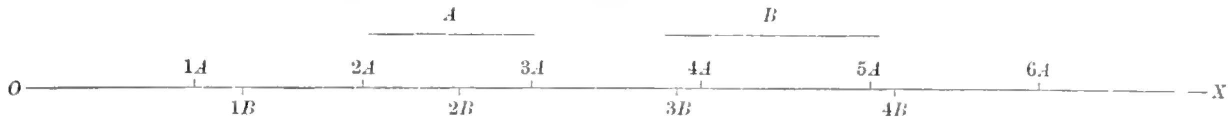
Hence it is possible to determine the positions of the magnitudes

$$B,\ 2B,\ 3B,\ 4B,\ \ldots\ldots,\ sB,\ \ldots\ldots$$

with reference to the multiples in the Multiple Scale of A.

Relative Multiple Scale of Two Magnitudes.

Art. 15. It is possible to arrange in a single series the magnitudes occurring in the multiple scales of two magnitudes A and B of the same kind; e.g. take the two lengths A and B, and an indefinite straight line OX.



Starting from a fixed point O on this line mark off lengths equal to A above it, and lengths equal to B below it.

With the above values of A and B the following magnitudes are in order of magnitude

$$1A,\ 1B,\ 2A,\ 2B,\ 3A,\ 3B,\ 4A,\ 5A,\ 4B,\ 6A,\ \ldots\ldots$$

and this may be continued to any extent.

Now let vertical lines be drawn between consecutive multiples and let the multiples of A be moved upwards on to the line above, there being no horizontal motion. Then let the letters A and B be suppressed, and let A be placed at the commencement of the upper line, B at the commencement of the lower line.

Then there remains the following:—

A	1		2		3		4	5		6
B		1		2		3			4	

and this can be continued to any extent.

It appears therefore that the integers 1, 2, 3, ..., r, ... all appear on each line; and the integer r on the upper line will be on the left of, above, or to the right of the integer s on the lower line, according as rA is less than, equal to, or greater than sB.

The above arrangement of integers is called the *relative multiple scale** *of* A, B; or more briefly, when no confusion is possible, the *scale of* A, B.

* De Morgan remarks in his treatise on the Connexion of Number and Magnitude that the Theory of Relative Multiple Scales must have been known to Euclid. For the form in which they are used in this paper I am indebted to Mr A. E. H. Love.

The abbreviation $[A, B]$ for the *scale of* A, B is very convenient.

It is to be particularly noted that the order of the letters A, B cannot (unless $A = B$) be changed without altering the scale.

Conditions which hold when the scale of A, B is the same as that of C, D.

Art. 16. In order that the scale of A, B may be the same as that of C, D it is necessary and sufficient that for all possible values of the integers r, s the following conditions be satisfied:—

(1) If rA be greater than sB, then must rC be greater than sD.

(2) If rA be equal to sB, then must rC be equal to sD.

(3) If rA be less than sB, then must rC be less than sD.

The fact that the scale of A, B is the same as that of C, D can be conveniently expressed thus:—

$$[A,\ B] \doteq [C,\ D].$$

Art. 17. The proofs of the following propositions, not concerned with ratios, present no difficulty, and will therefore be assumed.

(1) $r(A + B) = rA + rB$. (Euc. v. 1.)

(2) $(r + s)A = rA + sA$. (Euc. v. 2.)

(3) If $A > B$, then $r(A - B) = rA - rB$. (Euc. v. 5.)

(4) If $r > s$, then $(r - s)A = rA - sA$. (Euc. v. 6.)

(5) $r(sA) = rs(A) = sr(A) = s(rA)$.

(6) $rA \gtreqless rB$, according as $A \gtreqless B$, and conversely.

PROPOSITION 1. (Euc. v. 15.)

Art. 18. To prove that

$$[A,\ B] \doteq [nA,\ nB].$$

For

$$sA \gtreqless rB,$$

according as

$$s(nA) \gtreqless r(nB);$$

$$\therefore [A,\ B] \doteq [nA,\ nB].$$

(It may be noted that, since n may be any integer, nA and nB represent an infinite number of pairs of magnitudes having the same scale as A, B.

Hence the scale does not determine the magnitudes corresponding to it, though the magnitudes determine the scale.)

PROPOSITION 2. (Euc. V. 11.)

Art. 19. If $[C, D] \simeq [A, B]$,

and $[E, F] \simeq [A, B]$,

then $[C, D] \simeq [E, F]$.

This is evident.

PROPOSITION 3. (Corollary to Euc. V. 4.)

Art. 20. If $[A, B] \simeq [C, D]$,

to prove that $[B, A] \simeq [D, C]$.

The scale of B, A is obtained from that of A, B by writing the lower line of the scale of A, B above the upper without displacing the figures horizontally.

Now the scale of C, D is the same as that of A, B.

Hence the altered scale will be the scale of D, C as well as that of B, A.

$$\therefore [B, A] \simeq [D, C].$$

PROPOSITION 4 (i). (Euc. V. 7. First Part.)

Art. 21. If $A = B$,

to prove that $[A, C] \simeq [B, C]$.

If $A = B$,

then $rA = rB$,

$$\therefore rA \gtreqless sC, \text{ according as } rB \gtreqless sC.$$

$$\therefore [A, C] \simeq [B, C].$$

PROPOSITION 4 (ii). (Euc. V. 7. Second Part.)

Art. 22. If $A = B$,

to prove that $[C, A] \simeq [C, B]$.

If $A = B$,

then $sA = sB$,

$$\therefore rC \gtreqless sA, \text{ according as } rC \gtreqless sB.$$

$$\therefore [C, A] \simeq [C, B].$$

PROPOSITION 5 (i). (Euc. v. 9. First Part.)

Art. 23. If $[A, C] \doteq [B, C]$,

to prove that $A = B$.

If possible let A be not equal to B. Then one of them is greater than the other. Let A be the greater.

Then $A - B$ is a magnitude of the same kind as C.

Then by Art. 11 (2) it is possible to find an integer n such that

$$n(A - B) > C,$$

$$\therefore \quad nA > nB + C.$$

Hence some multiple of C, say rC, lies between nA and nB.

Let $nA > rC > nB$ ....(I).

But since $[A, C] \doteq [B, C]$,

it follows that if $nA > rC$,

then $nB > rC$ ....(II).

Now (II) and (I) are contradictory.

Hence A and B are not unequal,

$$\therefore \quad A = B.$$

PROPOSITION 5 (ii). (Euc. v. 9. Second Part.)

Art. 24. If $[C, A] \doteq [C, B]$,

to prove that $A = B$.

If $[C, A] \doteq [C, B]$,

then $[A, C] \doteq [B, C]$ (Prop. 3),

$\therefore \quad A = B$ (Prop. 5 (i)).

PROPOSITION 6. (Euc. v. 16.)

Art. 25. If A, B, C, D are four magnitudes of the same kind, and if

$$[A, B] \doteq [C, D],$$

to prove that $[A, C] \doteq [B, D]$.

Take any multiples of A and C, say rA and sC.

Then there are three alternatives, according as

$$rA \gtreqless sC.$$

If $rA < sC$, an integer n exists, such that

$$n(sC - rA) > B,$$

$$\therefore \quad nsC > nrA + B.$$

Hence some multiple of B, say tB, exists such that

$$nsC > tB > nrA.$$

Since $$[A, B] \doteq [C, D],$$

and $$nrA < tB,$$

$$\therefore \quad nrC < tD,$$

$$\therefore \quad s(nrC) < stD,$$

$$\therefore \quad r(nsC) < t(sD).$$

But $$tB < nsC,$$

$$\therefore \quad rtB < r(nsC),$$

$$\therefore \quad rtB < t(sD),$$

$$\therefore \quad rB < sD.$$

Hence if $rA < sC$, then $rB < sD$ ....(I).

In like manner if $rA > sC$, then $rB > sD$ ....(II),

if $rB < sD$, then $rA < sC$ ....(III),

and if $rB > sD$, then $rA > sC$....................................(IV).

From (I), (II), (III), (IV) it will follow that

if $rA = sC$, then $rB = sD$ ....(V),

and if $rB = sD$, then $rA = sC$....................................(VI).

Suppose if possible that when $rA = sC$, rB is not equal to sD, then by (III) and (IV) the fact that rB is not equal to sD involves the conclusion that rA is not equal to sC, which is inconsistent with the hypothesis,

$$\therefore \text{ if } \quad rA = sC, \text{ then } rB = sD.$$

In like manner (VI) follows from (I) and (II).

From (I)—(VI) it follows that

$$[A, C] \doteq [B, D].$$

Note. The latter part of this proposition, viz. that (I)—(IV) involve the conclusion

$$[A, C] \doteq [B, D],$$

is very useful, as it is required in some of the succeeding propositions.

Corollary. Hence, the symbols being the same as in Prop. 6, $A \gtreqless C$ according as $B \lesseqgtr D$. (Euc. v. 14.)

PROPOSITION 7 (i). (Euc. v. 18.)

Art. 26. If $[A, B] \simeq [C, D]$,

to prove that $[A + B, B] \simeq [C + D, D]$.

$$\because [A, B] \simeq [C, D],$$

$$\therefore rA \gtreqless sB \text{ according as } rC \gtreqless sD.$$

$$\therefore r(A + B) \gtreqless (r + s) B \text{ according as } r(C + D) \gtreqless (r + s) D.$$

$$\therefore [A + B, B] \simeq [C + D, D].$$

PROPOSITION 7 (ii). (Euc. v. 17.)

Art. 27. If $[A, B] \simeq [C, D]$,

to prove that $[A \sim B, B] \simeq [C \sim D, D]$.

There are two cases:

(1) If $A > B$, then $C > D$.

$$\because [A, B] \simeq [C, D],$$

$$\therefore rA \gtreqless sB \text{ according as } rC \gtreqless sD,$$

$\therefore$ provided $r < s$, which is all that need be considered,

$$r(A - B) \gtreqless (s - r) B \text{ according as } r(C - D) \gtreqless (s - r) D.$$

$$\therefore [A - B, B] \simeq [C - D, D].$$

(2) If $A < B$, then $C < D$.

$$\because [A, B] \simeq [C, D],$$

$$rA \gtreqless sB \text{ according as } rC \gtreqless sD,$$

$\therefore$ provided $r > s$, which is all that need be considered,

$$r(B - A) \lesseqgtr (r - s) B \text{ according as } r(D - C) \lesseqgtr (r - s) D,$$

$$\therefore [B - A, B] \simeq [D - C, D].$$

PROPOSITION 8. (Euc. v. 4.)

Art. 28. If $[A, B] \simeq [C, D]$,

to prove that $[rA, sB] \simeq [rC, sD]$.

$$\because [A, B] \simeq [C, D],$$

$$\therefore (pr) A \gtreqless (qs) B \text{ according as } (pr) C \gtreqless (qs) D,$$

$$\therefore p(rA) \gtreqless q(sB) \text{ according as } p(rC) \gtreqless q(sD),$$

$$\therefore [rA, sB] \simeq [rC, sD].$$

PROPOSITION 9. (EUC. V. 12.)

Art. 29. If $[A_1, B_1] \simeq [A_2, B_2] \simeq \dots [A_n, B_n]$,
all the magnitudes $A_1, A_2, \dots A_n, B_1, B_2, \dots B_n$

being of the same kind, then it is required to prove that

$$[A_1 + A_2 + \dots + A_n, B_1 + B_2 + \dots + B_n] \simeq [A_1, B_1].$$

$$\because [A_2, B_2] \simeq [A_1, B_1],$$

$$\therefore rA_2 \gtreqless sB_2 \text{ according as } rA_1 \gtreqless sB_1;$$

$$\because [A_3, B_3] \simeq [A_1, B_1],$$

$$\therefore rA_3 \gtreqless sB_3 \text{ according as } rA_1 \gtreqless sB_1,$$

....;

$$\because [A_n, B_n] \simeq [A_1, B_1],$$

$$\therefore rA_n \gtreqless sB_n \text{ according as } rA_1 \gtreqless sB_1.$$

Consequently $$r(A_1 + A_2 + A_3 + \dots + A_n) \gtreqless s(B_1 + B_2 + B_3 + \dots + B_n),$$

according as $$rA_1 \gtreqless sB_1;$$

$$\therefore [A_1 + A_2 + A_3 + \dots + A_n, B_1 + B_2 + B_3 + \dots + B_n] \simeq [A_1, B_1].$$

PROPOSITION 10. (EUC. V. 19.)

Art. 30. If A, B, C, D are magnitudes of the same kind, and if $[A, B] \simeq [C, D]$, to prove that

$$[A \sim C, B \sim D] \simeq [A, B].$$

Of the two magnitudes A, C one is the greater.

Let A be greater than C, then by the Corollary to Prop. 6, B is greater than D.

$$\because [A, B] \simeq [C, D],$$

$$\therefore [A, C] \simeq [B, D] \quad \text{(Prop. 6)},$$

$$\therefore [A - C, C] \simeq [B - D, D] \quad \text{(Prop. 7 (ii))},$$

$$\therefore [A - C, B - D] \simeq [C, D] \quad \text{(Prop. 6)},$$

$$\therefore [A - C, B - D] \simeq [A, B] \quad \text{(Prop. 2)}.$$

The case in which C is greater than A can be dealt with in like manner.

PROPOSITION 11. (Euc. v. 25.)

Art. 31. If A, B, C, D are four magnitudes of the same kind, and if $[A, B] \doteq [C, D]$, then the greatest and least of the four magnitudes are together greater than the sum of the other two.

Suppose A the greatest of the four magnitudes,

$$\because \; A > B,$$

and

$$[A, B] \doteq [C, D],$$

$$\therefore \; C > D;$$

$$\because \; [A, B] \doteq [C, D],$$

and the magnitudes are of the same kind,

$$\therefore \text{ by Prop. 6, } \; [A, C] \doteq [B, D];$$

$$\text{but } \; A > C,$$

$$\therefore \; B > D.$$

Hence D is the least magnitude.

Now, by Prop. 10,

$$[A - C, \; B - D] \doteq [A, B].$$

$$\text{But } \; A > B;$$

$$\therefore \; A - C > B - D,$$

$$\therefore \; A + D > B + C.$$

Art. 32. In the preceding propositions Nos. 1—11, *one* scale only is supposed to be given, and from it, in most cases, a new scale is derived.

The three important propositions which next follow are of a more complicated nature, inasmuch as they show how to derive a definite scale from *two* given scales.

PROPOSITION 12. (Euc. v. 22.)

Art. 33. If A, B, C are magnitudes of the same kind;

if T, U, V are magnitudes of the same kind;

if $\qquad [A, B] \doteq [T, U],$

and if $\qquad [B, C] \doteq [U, V],$

to prove that $\qquad [A, C] \doteq [T, V].$

As in Prop. 6 there are three alternatives, according as

$$rA \gtreqless sC.$$

If $$rA < sC,$$

integers n, t, exist such that

$$nsC > tB > nrA.$$

$$\because \ nrA < tB,$$

and $$[A, B] \simeq [T, U],$$

$$\therefore \ nrT < tU.$$

Since $$tB < nsC,$$

and $$[B, C] \simeq [U, V],$$

$$\therefore \ tU < nsV,$$

$$\therefore \ nrT < nsV,$$

$$\therefore \ rT < sV.$$

Hence if $rA < sC$, then $rT < sV$ ....(I).

In like manner if $rA > sC$, then $rT > sV$ ....(II),

if $rT < sV$, then $rA < sC$ ....(III),

and if $rT > sV$, then $rA > sC$ ....(IV).

And now, as in the latter part of Prop. 6, it follows from (I)—(IV) that

$$[A, C] \simeq [T, V].$$

COROLLARY. (Euc. V. 20.)

To show that, with the notation of Proposition 12, $A \gtreqless C$ according as $T \gtreqless V$.

This follows immediately from

$$[A, C] \simeq [T, V].$$

PROPOSITION 13. (Euc. V. 23.)

Art. 34. If A, B, C be three magnitudes of the same kind: if T, U, V be three magnitudes of the same kind,

if $$[A, B] \simeq [U, V],$$

and if $$[B, C] \simeq [T, U],$$

to prove that $$[A, C] \simeq [T, V].$$

As in Prop. 6 there are three alternatives, according as

$$rA \gtreqless sC.$$

If $$rA < sC,$$

then integers n, t exist such that

$$nsC > tB > nrA;$$

$$\because \quad nrA < tB,$$

and $$[A, B] \simeq [U, V],$$

$$\therefore \quad nrU < tV.$$

$$\because \quad tB < nsC,$$

and $$[B, C] \simeq [T, U],$$

$$\therefore \quad tT < nsU,$$

$$\therefore \quad rtT < rnsU,$$

$$\therefore \quad rtT < s(nrU) < stV,$$

$$\therefore \quad rT < sV.$$

Hence if $rA < sC$, then $rT < sV$ ....(I).

In like manner if $rA > sC$, then $rT > sV$......................................(II),

if $rT < sV$, then $rA < sC$ ....(III),

and if $rT > sV$, then $rA > sC$......................................(IV).

And now, as in the latter part of Prop. 6, it follows from (I)—(IV) that

$$[A, C] \simeq [T, V].$$

COROLLARY. (Euc. V. 21.)

To prove that, with the notation of Prop. 13, $A \gtreqless C$ according as $T \gtreqless V$. This follows immediately from

$$[A, C] \simeq [T, V].$$

PROPOSITION 14. (Euc. V. 24.)

Art. 35. If $$[A, C] \simeq [X, Z],$$

and if $$[B, C] \simeq [Y, Z],$$

to prove that $$[A + B, C] \simeq [X + Y, Z].$$

$$\because [B, C] \simeq [Y, Z]$$

$$\therefore [C, B] \simeq [Z, Y] \quad \text{(Prop. 3).}$$

And $$[A, C] \simeq [X, Z]$$

$$\therefore [A, B] \simeq [X, Y] \quad \text{(Prop. 12).}$$

$$\therefore [A + B, B] \simeq [X + Y, Y] \quad \text{(Prop. 7 (i)).}$$

But $$[B, C] \simeq [Y, Z]$$

$$\therefore [A + B, C] \simeq [X + Y, Z] \quad \text{(Prop. 12).}$$

III. *Development of the Theory of Ratio.*

Art. 36. It will have been noticed that throughout the preceding section on the reconstruction of the argument of Euclid's Fifth Book the word "Ratio" has not been used. Nor has the idea of Ratio been employed. Demonstrations of all those propositions of the Fifth Book which express properties of equal ratios have now been given in a form expressing the sameness of two relative multiple scales.

It is now necessary to show how the idea of ratio is introduced.

Preliminary Discussion of Differing Relative Multiple Scales.

Art. 37. Explanation of Terminology.

If X, Y, A be three magnitudes of the same kind, and X greater than Y, then X is said to occupy a more advanced position amongst the multiples of A than Y does; and Y is said to occupy a less advanced position amongst the multiples of A than X does.

PROPOSITION 15.

Art. 38. To determine the ways in which differing relative multiple scales can differ.

Let the scale of A, B differ from that of C, D. Take any multiple of A, say rA; and any multiple of B, say sB.

Then there are three alternatives

(1) $rA > sB$,

or (2) $rA = sB$,

or (3) $rA < sB$.

Each of these alternatives is inconsistent with the other two.

In like manner in the scale of C, D there are three alternatives

(4) $rC > sD$,

or (5) $rC = sD$,

or (6) $rC < sD$.

On comparing the scales, no difference is shown between them if (1) and (4), or if (2) and (5), or if (3) and (6) coexist.

On the other hand, any other combination of one of the alternatives (1), (2), (3) with one of the alternatives (4), (5), (6) shows a difference in the scales.

Hence the cases to be considered are the combinations of

(1) and (5) giving $rA > sB,\quad rC = sD$ .... (7),

(1) and (6) giving $rA > sB,\quad rC < sD$ .... (8),

(2) and (6) giving $rA = sB,\quad rC < sD$ .... (9),

(2) and (4) giving $rA = sB,\quad rC > sD$ ....(10),

(3) and (4) giving $rA < sB,\quad rC > sD$ ....(11),

(3) and (5) giving $rA < sB,\quad rC = sD$ ....(12).

Art. 39. It will first be shown that if (7) or (9) exist, the existence of a relation of the form (8) with different values of the integers r, s is necessarily implied.

Take (7) in which

$$rA > sB,\quad rC = sD.$$

It is always possible to find an integer n such that

$$n(rA - sB) > B,$$

$$\therefore\ nrA > nsB + B.$$

Hence at least one multiple of B falls between nrA and nsB.

Let tB be such a multiple.

$$\therefore\ nrA > tB > nsB.$$

$$\because\ tB > nsB,$$

$$\therefore\ t > ns,$$

$$\therefore\ tD > nsD,$$

$$\therefore\ tD > nrC.$$

Hence $$(nr)A > tB,\quad (nr)C < tD,$$

which is of the form (8) with r changed into (nr) and s changed into t.

Taking next (9) in which

$$rA = sB,\quad rC < sD,$$

let n be so large that at least one multiple of D, say tD, lies between nrC and nsD.

$$\therefore\ nrC < tD < nsD.$$

$$\because\ tD < nsD,$$

$$t < ns,$$

$$tB < nsB,$$

$$\therefore\ tB < nrA.$$

Hence $$nrA > tB,\quad nrC < tD,$$

which is of the form (8).

Hence the cases (7), (8), (9) are represented by the single form (8).

Observing next that (10), (11), (12) may be obtained from (7), (8), (9) respectively by interchanging A and C, B and D, it follows that cases (10), (11), (12) are represented by the single form (11).

Art. 40. The next point is to determine whether the scale of A, B can differ from that of C, D in one part in the manner indicated by (8), and in another part in the manner indicated by (11).

*This will be shown to be impossible.

The proposition to be proved is this:—

If $$rA > sB, \quad rC < sD \text{ ....(8)},$$

then no integers r', s' can exist such that

$$r'A < s'B, \quad r'C > s'D \text{ ....(13)},$$

which is a relation of the form (11).

If possible let (8) and (13) coexist.

From (8) $$rs'A > ss'B \text{ ....(14)},$$
$$rs'C < ss'D \text{ ....(15)}.$$

From (13) $$r'sA < ss'B \text{ ....(16)},$$
$$r'sC > ss'D \text{ ....(17)}.$$

From (14) and (16) $$rs'A > r'sA$$
$$\therefore\ rs' > r's \text{ ....(18)}.$$

From (15) and (17) $$rs'C < r'sC$$
$$\therefore\ rs' < r's \text{ ....(19)}.$$

But (18) and (19) are contradictory.

Hence (8) and (13) cannot coexist.

Now (13) is of the same form as (11).

Hence the two ways in which two scales can differ indicated by (8) and (11) are exclusive of one another.

But (8) represents (7), (8) and (9); whilst (11) represents (10), (11) and (12).

Hence if two scales differ in any part in one of the ways represented by (7), (8), or (9); then they cannot differ in any other part in one of the ways represented by (10), (11) or (12).

Art. 41. Now in (7), (8) or (9), rA occupies a *more* advanced position amongst the multiples of B than rC does amongst the multiples of D; which may also be expressed thus:—rC occupies a *less* advanced position amongst the multiples of D than rA does amongst the multiples of B.

On the other hand in (10), (11) or (12), rA occupies a *less* advanced position amongst the multiples of B than rC does amongst the multiples of D; which may also be expressed thus:—rC occupies a *more* advanced position amongst the multiples of D than rA does amongst the multiples of B.

* Compare the proof in De Morgan's unpublished Tracts, of which there is a manuscript copy in the Library of University College, London.

The above distinctions may be conveniently expressed thus:—

In (7), (8) or (9) the scale of A, B is above that of C, D: which may also be stated thus:—the scale of C, D is below that of A, B.

In (10), (11) or (12) the scale of A, B is below that of C, D; which may also be stated thus:—the scale of C, D is above that of A, B.

PROPOSITION 16.

(Fundamental Proposition in the Theory of Ratio.)

Art. 42. To prove that if there be three magnitudes, of which the first and second are of the same kind, then there exists one and only one fourth magnitude of the same kind as the third magnitude, such that the relative multiple scale of the first and second magnitudes is the same as that of the third and fourth magnitudes.

Let the given magnitudes be A, B and C, of which A and B are of the same kind.

It is required to prove the existence of a fourth magnitude D, such that the scale of A, B is the same as that of C, D.

(1) By Prop. 3 it is sufficient to find D, such that the scale of D, C is the same as that of B, A.

(2) To show how, if D exist, it is possible to determine two magnitudes between which D must lie.

Take any multiple of A, say rA.

Then find an integer s such that

$$sB > rA \qquad \text{(1)}.$$

Then it is always possible to find a magnitude E such that

$$sE < rC \qquad \text{(2)}.$$

Hence E is any magnitude such that the scale of E, C is below that of B, A.

Next find two positive integers t, u such that

$$uB < tA \qquad \text{(3)}.$$

Then it is always possible to find a magnitude F such that

$$uF > tC \qquad \text{(4)}.$$

Hence F is any magnitude such that the scale of F, C is above that of B, A.

The magnitudes E and F will be shown to possess the required property.

It is necessary to prove first that E is less than F.

From (1) and (3)

$$stB > rtA > ruB,$$

$$\therefore\ st > ru,$$

$$\therefore\ stF > ruF,$$

but $$ruF > rtC \text{ by (4)}:$$

$$\therefore\ stF > rtC,$$

but $$rtC > stE \text{ by (2)}:$$

$$\therefore\ stF > stE,$$

$$\therefore\ F > E.$$

It will now follow that D, if it exist, must lie between E and F.

If possible let $$D < E;$$

$$\therefore\ sD < sE,$$

but $$sE < rC \text{ by (2)};$$

$$\therefore\ sD < rC,$$

whilst $$sB > rA \text{ by (1)}.$$

Hence the scale of D, C differs from that of B, A.

Next, if possible, let $D > F$.

$$\therefore\ uD > uF,$$

but $$uF > tC \text{ by (4)}:$$

$$\therefore\ uD > tC,$$

whilst $$uB < tA.$$

Hence the scale of D, C differs from that of B, A.

Further D is not equal to E or F, because in neither case would the scale of D, C be the same as that of B, A.

Hence D, if it exist, must lie between E and F.

(3) It will next be shown that there cannot be two different values of D.

If possible let G and H be two different values of D both satisfying the required condition;

$$\therefore\ [G, C] \simeq [B, A],$$

and $$[H, C] \simeq [B, A].$$

Hence $$[G, C] \simeq [H, C] \qquad \text{(Prop. 2)}.$$

Hence $$G = H \qquad \text{(Prop. 5 (i))}.$$

Hence G and H cannot be different.

Hence, if D exist, it can have only one value.

(4) To show how to obtain closer limits for D.

This is done by showing that if E be a magnitude such that the scale of E, C is below that of B, A; then a magnitude greater than E, say E', exists such that the scale of E', C is below that of B, A; and by further showing that if F be a magnitude such that the scale of F, C is above that of B, A; then a magnitude less than F, say F', exists, such that the scale of F', C is above that of B, A.

Suitable values of E' and F' are given by

$$sE' = rC,$$

$$uF' = tC.$$

For $$sE' = rC,$$

$$rC > sE \text{ by (2)};$$

$$\therefore\ sE' > sE,$$

$$\therefore\ E' > E.$$

Also $$sE' = rC,\ sB > rA,$$

so that the scale of E', C is below that of B, A.

Further $$uF' = tC < uF \text{ by (4)},$$

$$\therefore\ F' < F.$$

Also $$uF' = tC,\ uB < tA,$$

$\therefore$ the scale of F', C is above that of B, A.

This is a process which can be continued for ever, for if $sE' = rC$, $sB > rA$: then other integers p, q are known by Art. 39 to exist such that

$$pE' < qC,\ pB > qA.$$

Then by taking E'' so that $$pE'' = qC,$$

it follows that $$E'' > E',$$

and since $$pE'' = qC,\ pB > qA,$$

the scale of E'', C is below that of B, A.

In this way the magnitudes between which D, if it exist, is shown to lie, continually approach one another.

This result may be stated thus:—

There is no greatest magnitude E such that the scale of E, C is below that of B, A.

There is no smallest magnitude F such that the scale of F, C is above that of B, A.

(5) Suppose that it is found by carrying on the process above described that, if D exist, then

$$X < D < Y.$$

Then every magnitude Z between X and Y must be such that one of the following alternatives hold :—

(i) The scale of Z, C is below that of B, A.

In this case let Z be called a magnitude of the lower class.

(ii) The scale of Z, C is above that of B, A.

In this case let Z be called a magnitude of the upper class.

(iii) The scale of Z, C is the same as that of B, A.

In this case there is only one possible value of Z, if one exist at all. If such a value exist let Z be said to belong to neither class.

It has been proved that each magnitude of the lower class is less than each magnitude of the upper class, and less than D, if it exist.

It has also been proved that each magnitude of the upper class is greater than each magnitude of the lower class, and greater than D, if it exist.

It has also been proved that there can be *at most* but one magnitude D, such that the scale of D, C is the same as that of B, A; but it has not yet been proved that there is any such magnitude.

(6) It remains to prove that there is such a magnitude.

Suppose if possible no such magnitude exists.

Then every magnitude between X and Y must belong to the lower or upper class.

But if *all* the magnitudes between X and Y be divided into two classes, such that every magnitude of one class is less than every magnitude of the other class, the following are the only two possible alternatives*.

(i) There is a magnitude R such that the magnitudes of the lower class are not greater than R, whilst the magnitudes of the upper class are greater than R.

In this case the lower class has a greatest magnitude, viz.—R.

(ii) There is a magnitude S, such that the magnitudes of the lower class are less than S, whilst the magnitudes of the upper class are not less than S.

In this case the upper class has a least magnitude S.

Hence if the magnitude D such that the scale of D, C is the same as that of B, A do not exist, then all the magnitudes between X and Y fall either into the lower or into the upper class: and either the lower class has a greatest magnitude or the upper class has a least magnitude, both of which alternatives have been shown to be impossible.

Hence one and only one magnitude D exists such that the scale of D, C is the same as that of B, A; and therefore such that the scale of A, B is the same as that of C, D.

* See the note at the end of the paper.

Note on Proposition 16.

Art. 43. There is a certain resemblance between the line of argument adopted in the proof of Prop. 16 and that employed by Dedekind in his definition of a real number in his tract entitled "Stetigkeit und irrationale Zahlen."

The following proposition may be compared with the basis of the argument in the second article of the First Chapter of Jordan's *Cours d'Analyse.*

If B and A are two magnitudes of the same kind, and C any other magnitude, then there exist two magnitudes E and F of the same kind as C, possessing the following properties:

(i) The scale of E, C is below that of B, A.

(ii) The scale of F, C is above that of B, A.

(iii) The magnitude $F - E$ is less than any given magnitude K (however small) of the same kind as C.

There are two separate cases to consider.

Case (I). If B and A are commensurable,

let $$B = rG,$$
$$A = sG,$$
$$\therefore\ rA = sB.$$

Take H so that $$sH = rC.$$

Then take E and F so that
$$H - \tfrac{1}{2}K < E < H;$$
and $$H < F < H + \tfrac{1}{2}K.$$
Then $$F - E < K, \text{ which is (iii).}$$
Also $$sE < sH,$$
$$\therefore\ sE < rC, \quad sB = rA.$$
$$\therefore \text{ the scale of } E, C \text{ is below that of } B, A, \text{ which is (i).}$$

Further, $$sF > sH,$$
$$\therefore\ sF > rC, \quad sB = rA.$$
$$\therefore \text{ the scale of } F, C \text{ is above that of } B, A, \text{ which is (ii).}$$

Hence quantities E, F satisfying all the required conditions have been found.

Case (II). Let B and A be incommensurable. Then as in Art. 42 (2) take any integers r, s, t, u and any quantities E, F such that

$$sB > rA \quad \text{.........(1),}$$
$$sE < rC \quad \text{.........(2),}$$
$$uB < tA \quad \text{.........(3),}$$
$$uF > tC \quad \text{.........(4),}$$

then E satisfies the 1st condition, and F the 2nd condition, but the 3rd condition will not be satisfied unless $F - E < K$.

If the third condition is not satisfied, it is always possible to find an integer n such that
$$C < nK.$$

Now take magnitudes L, M, such that
$$C = nL,$$
$$A = nM.$$

Then from (1), $$suB > runM,$$
and from (3), $$suB < stuM.$$

Now suppose that* $$pM < suB < (p+1)M.$$

Then since $$runM < suB < stuM,$$
it follows that $$runM \leqq pM < (p+1)M \geqq stuM.$$
$$\therefore\ run \leqq p < p+1 \geqq stu.$$

Now from (2), $$suE < runL,$$
$$\therefore\ suE < pL,$$
whilst $$suB > pM,$$
so that the scale of E, L is below that of B, M.

Also from (4), $$suF > stuL,$$
$$\therefore\ suF > (p+1)L,$$
whilst $$suB < (p+1)M;$$
$$\therefore \text{ the scale of } F,\ L \text{ is above that of } B,\ M.$$

Now take E', F' such that $$suE' = pL,$$
$$suF' = (p+1)L;$$

$$\therefore\ suE' = pL, \qquad suB > pM,$$
$$suF' = (p+1)L, \qquad suB < (p+1)M.$$
$$\therefore\ (sun)E' = p(nL), \qquad (sun)B > p(nM),$$
$$(sun)F' = (p+1)(nL), \qquad (sun)B < (p+1)(nM);$$
$$\therefore\ (sun)E' = pC, \qquad (sun)B > pA,$$
$$(sun)F' = (p+1)C, \qquad (sun)B < (p+1)A;$$

∴ the scale of E', C is below that of B, A, which is (i);

and the scale of F', C is above that of B, A, which is (ii).

Further, $$su(F' - E') = L,$$
and $$nL = C < nK.$$

* suB cannot be equal to a multiple of M, for then $sunB$ would be a multiple of A, and then B and A would be commensurable.

$$\therefore\ L < K,$$
$$\therefore\ su(F' - E') < K,$$
$$\therefore\ F' - E' < K.$$

which is (iii).

Hence the existence of the two magnitudes B and A of the same kind renders it possible to separate all magnitudes of the same kind as C into two classes.

The first or lower class contains all magnitudes such as E, having the property that the scale of E, C is below that of B, A.

The second or upper class contains all magnitudes such as F, having the property that the scale of F, C is above that of B, A.

Every magnitude E of the lower class is less than every magnitude F of the upper class.

Further it is possible to find a magnitude F of the upper class, and a magnitude E of the lower class, such that $F-E$ is less than any magnitude K (however small) of the same kind as C.

Under such circumstances, the statement that the two classes define a certain magnitude would correspond exactly with Jordan's definition of a real number. That the magnitude so defined is the magnitude D such that the scale of D, C is the same as that of B, A has been proved in Proposition 16.

Definition of Ratio.

Art. 44. It has been shown that a magnitude D exists such that the scale of D, C is the same as that of B, A: where A and B are any two magnitudes of the same kind, and C is any magnitude (Art. 42).

If C be taken to be the unit of number, then D is a magnitude of the same kind as the unit of number, and may therefore be called a real number.

Hence corresponding to the magnitudes B, A of the same kind there exists a single real number ρ, such that the scale of B, A is the same as that of ρ, 1.

This real number ρ is taken to be the measure of the relative magnitude or ratio of B to A.

The ratio of B to A is written shortly $B : A$.

Consequently $B : A$ is the same as $\rho : 1$, which may be written $B : A = \rho : 1$.

When this relation holds, ρ is the measure of the ratio $B : A$.

The measure of the ratio then is a real number, and may be distinguished from the ratio itself.

Since this real number ρ is entirely determined by the scale of B, A it follows that any two other magnitudes having the same scale as B, A will determine the same number.

Two ratios are considered to be equal when their measures are equal.

If therefore the scale of A, B be the same as the scale of C, D; then the measure of the ratio of A to B is the same as the measure of the ratio of C to D, and therefore the ratio of A to B is equal to the ratio of C to D.

Hence any of the preceding propositions in which it has been proved that the scale of A, B is the same as the scale of C, D, may be referred to as expressing the fact that the ratio of A to B is the same as that of C to D.

Unequal Ratios.

PROPOSITION 17.

Art. 45. To obtain the conditions which must be satisfied in order that one ratio may be greater than another.

Let ρ be the real number corresponding to the ratio $A : B$.

In like manner let σ correspond to the ratio $C : D$.

Then $A : B$ is said to be greater than, equal to, or less than $C : D$ according as ρ is greater than, equal to, or less than σ.

In order to make practical use of this condition, its form must be altered so as to depend, not on the measures, but on the terms, of the ratios.

If $$\rho > \sigma,$$
then $\rho - \sigma$ is a magnitude of the same kind as the unit of number.

Hence an integer r exists such that
$$r(\rho - \sigma) > 1,$$
and $$\therefore\ r\rho > r\sigma + 1.$$
Hence some integer s lies between $r\rho$ and $r\sigma$.
$$\therefore\ r\rho > s(1) > r\sigma.$$

Now $$A : B = \rho : 1,$$
but $$r\rho > s(1),$$
$$\therefore\ rA > sB.$$

Again, $$C : D = \sigma : 1,$$
and $$r\sigma < s(1),$$
$$\therefore\ rC < sD.$$

Consequently $$A : B > C : D,$$
if integers r, s exist such that $rA > sB$, $rC < sD$.

It is necessary to see if the converse proposition is true, before the test can be used; i.e. it must be proved that if

$$rA > sB, \quad rC < sD:$$

then
$$A : B > C : D.$$

Let ρ, σ be the real numbers corresponding to $A : B$ and $C : D$ respectively.

$$\therefore\ A : B = \rho : 1,$$

and
$$rA > sB;$$

$$\therefore\ r\rho > s(1).$$

Also
$$C : D = \sigma : 1,$$

and
$$rC < sD;$$

$$\therefore\ r\sigma < s(1).$$

$$\therefore\ r\rho > s > r\sigma.$$

$$\therefore\ \rho > \sigma.$$

$$\therefore\ A : B > C : D.$$

Now the condition found which must hold in order that $A : B$ may be greater than $C : D$, viz. that integers r, s, exist such that $rA > sB$, $rC < sD$, is precisely (8) of Art. 38.

Hence, by Art. 39, the condition will hold whenever integers r, s exist such that any one of the conditions (7), (8), (9) of Art. 38 hold.

Hence by Art. 41 it follows that

$$A : B > C : D$$

whenever the scale of A, B is above that of C, D.

In like manner
$$A : B < C : D$$

whenever the scale of A, B is below that of C, D.

PROPOSITION 18.

Art. 46. It follows from the Test for Unequal Ratios exactly as in Propositions 8 and 10 of Euclid's Fifth Book that

$$\text{if } A \gtrless B, \text{ then } A : C \gtrless B : C, \text{ and conversely;}$$

and
$$\text{if } A \gtrless B, \text{ then } C : A \lessgtr C : B, \text{ and conversely.}$$

PROPOSITION 19.

Art. 47. To deduce from the test for unequal ratios that if $A : B > C : D$, and $C : D > E : F$, then
$$A : B > E : F.$$

Since
$$A : B > C : D,$$

integers r, s exist such that
$$rA > sB, \quad rC < sD.$$

Since $C : D > E : F$,

integers t, u exist such that $tC > uD, \quad tE < uF$.

$\because rC < sD$.

$\therefore ruC < suD$ ....(1)

$\because uD < tC$.

$\therefore suD < stC$ ....(2).

From (1) and (2) $ruC < stC$.

$\therefore ru < st$ ....(3).

$\because rA > sB$.

$\therefore ruA > suB$ ....(4).

From (3) $stA > ruA$.

$\therefore stA > suB$.

$\therefore tA > uB$,

but $tE < uF$.

$\therefore A : B > E : F$.

Note. The result of this proposition renders it possible to *order* the multiplicity of ratios.

Addition of Ratios.

Art. 48. 1st stage. The general idea at the root of the process of adding ratios is this:—

When it is desired to find the ratio of one magnitude to a second it is permissible to break up the first magnitude into parts, then to find the ratio of each part to the second magnitude, and then to add up the results.

(It should be carefully noted that it is the first magnitude, not the second, which may be broken up.)

2nd stage. To make the idea quite precise, the following definition is necessary.

Let the sum of the ratios $X : Z$ and $Y : Z$ be defined to be $X + Y : Z$.

(This is the same fact as that expressed in Euclid's 22nd Datum.)

3rd stage. To apply this definition to the addition of any two ratios $A : B$ and $C : D$, the following process is to be followed.

Take any arbitrary magnitude Z, and then find two others X and Y (Prop. 16) such that

$$A : B = X : Z,$$
$$C : D = Y : Z.$$

Then
$$(A : B) + (C : D)$$
$$= (X : Z) + (Y : Z)$$
$$= X + Y : Z.$$

4th stage. The process described in the last stage requires justification, because the *form* of the resulting ratio depends on the arbitrary magnitude Z. If the process is to be of any use, it is necessary to show that the *value* of the resulting ratio does not depend on the arbitrary magnitude Z. This will be accomplished when it is shown that if any other magnitude Z' be taken instead of Z, and the same process followed, the *value* found for the resulting ratio is the same.

Suppose, then, that
$$A : B = X' : Z',$$
and
$$C : D = Y' : Z'.$$

Then
$$(A : B) + (C : D) = X' + Y' : Z'.$$

Hence it is necessary to show that
$$X' + Y' : Z' = X + Y : Z.$$

Now
$$X' : Z' = A : B = X : Z,$$
$$\therefore\ X' : Z' = X : Z \qquad \text{(Prop. 2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also
$$Y' : Z' = C : D = Y : Z,$$
$$\therefore\ Y' : Z' = Y : Z \qquad \text{(Prop. 2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$
$$\therefore\ X' + Y' : Z' = X + Y : Z \text{ from (1) and (2) by Prop. 14.}$$

Hence the process described in the third stage is justified.

Art. 49. The next step is to prove the commutative and associative laws for the addition of ratios.

PROPOSITION 20.

Art. 50. To prove the commutative law for the addition of ratios, i.e.
$$(A : B) + (C : D) = (C : D) + (A : B).$$

Let
$$A : B = X : Z,$$
and
$$C : D = Y : Z.$$

Then
$$(A : B) + (C : D) = X + Y : Z$$
$$= Y + X : Z.$$

Also
$$(C : D) + (A : B) = Y + X : Z,$$
$$\therefore\ (A : B) + (C : D) = (C : D) + (A : B).$$

Or, denoting a ratio by a single Greek letter,
$$\alpha + \beta = \beta + \alpha.$$

PROPOSITION 21.

Art. 51. To prove the associative law for the addition of ratios, i.e.

$$[(A : B) + (C : D)] + (E : F) = (A : B) + [(C : D) + (E : F)].$$

Let

$$A : B = X : Z,$$
$$C : D = Y : Z,$$
$$E : F = U : Z.$$

$$\therefore\ [(A : B) + (C : D)] + (E : F) = [(X : Z) + (Y : Z)] + (U : Z)$$
$$= (X + Y : Z) + (U : Z)$$
$$= X + Y + U : Z,$$
$$(A : B) + [(C : D) + (E : F)] = (X : Z) + [(Y : Z) + (U : Z)]$$
$$= (X : Z) + (Y + U : Z)$$
$$= X + Y + U : Z.$$
$$\therefore\ [(A : B) + (C : D)] + (E : F) = (A : B) + [(C : D) + (E : F)].$$

Or, denoting a ratio by a single Greek letter,

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

PROPOSITION 22.

Art. 52. To prove that the sum of the measures of two ratios is equal to the measure of the single ratio which is the sum of the two ratios.

Let ρ, σ be the measures of $A : B$, $C : D$ respectively.

Take any arbitrary magnitude Z, and then take X, Y so that

$$A : B = X : Z.$$
$$C : D = Y : Z.$$

Then

$$(A : B) + (C : D) = X + Y : Z.$$

Since

$$A : B = \rho : 1,$$

and

$$C : D = \sigma : 1,$$
$$\therefore\ X : Z = \rho : 1,$$
$$Y : Z = \sigma : 1,$$
$$\therefore\ X + Y : Z = \rho + \sigma : 1. \quad \text{(Prop. 14.)}$$

Hence $\rho + \sigma$ is the measure of the single ratio, which is the sum of the ratios $A : B$ and $C : D$.

Therefore the measure of the single ratio which is the sum of $A : B$ and $C : D$ is equal to the sum of the measures of $A : B$ and $C : D$.

Hence the multiplicity of ratios is *measurable*.

The Compounding of Ratios.

Art. 53. 1st stage. The general idea at the root of the process of compounding ratios is this:—

When it is necessary to determine the relative magnitude of two magnitudes A and C of the same kind, it is permissible to make the comparison indirectly by taking another magnitude B of the same kind as A and C, and then comparing A with B, and B with C.

From this point of view the relative magnitude of A and C is considered to be determined by the relative magnitude of A and B, and the relative magnitude of B and C.

2nd stage. Euclid renders the above general idea quite precise by the following definition:—

The ratio of A to C is compounded of the ratio of A to B and the ratio of B to C.

(See the use made of the definition in the 23rd Proposition of the 6th Book, which gives a clearer view of the process than the 5th Definition of the 6th Book.)

3rd stage. To apply this definition to the compounding of any two ratios $P:Q$ and $T:U$ the following process is to be followed:—

Take any arbitrary magnitude A, and then take B and C (Prop. 16) so that

$$P:Q=A:B,$$
$$T:U=B:C.$$

Then the ratio compounded of $P:Q$ and $T:U$ is the ratio compounded of $A:B$ and $B:C$, and is therefore $A:C$.

4th stage. The process described in the last stage requires justification, because the *form* of the resulting ratio depends on the arbitrary magnitude A. If the process is to be of any use it is necessary to show that the *value* of the resulting ratio does not depend on the arbitrary magnitude A. This will be accomplished when it is shown that if any other magnitude A' be taken instead of A, and the same process followed, the *value* found for the resulting ratio is the same.

Suppose, then, that $$P:Q=A':B',$$
and $$T:U=B':C'.$$

Then the resulting ratio would be that compounded of $A':B'$ and $B':C'$, and would therefore be $A':C'$.

In order that this may agree with the former result it is necessary to show that $A:C=A':C'$.

Since $A : B = P : Q = A' : B'$,

and $B : C = T : U = B' : C'$,

it follows by Prop. 2 that $A : B = A' : B'$,

and $B : C = B' : C'$.

$\therefore A : C = A' : C'$ (Prop. 12).

Hence the process described in the third stage is justified.

Notation for the Compounding of Ratios.

Art. 54. The following notation is convenient.

Let $P : Q$ compounded with $T : U$ be written

$$(P : Q) \ast (T : U).$$

Consequently

$$(A : B) \ast (B : C) = (A : C).$$

Note on the Compounding of Ratios.

Art. 55. It is possible to compound two ratios by proceeding according to the following rule:—

Let the ratios to be compounded be $A : B$ and $C : D$.

Take the ratio of equality in the form $E : E$, where E is any magnitude of the same kind as C and D.

Take P and Q so that

$$E : C = A : P \qquad \text{.........(1)}.$$

$$E : D = B : Q \qquad \text{.........(2)}.$$

Then $P : Q$ is the ratio compounded of $A : B$ and $C : D$.

To prove this take R so that

$$C : D = B : R \qquad \text{.........(3)}.$$

Then $(A : B) \ast (C : D)$

$= (A : B) \ast (B : R)$

$= A : R$.

Hence it is necessary to show that

$$P : Q = A : R \qquad \text{.........(4)}.$$

From (3) by Prop. 3,

$$D : C = R : B \qquad \text{.........(5)}.$$

From (2) and (5) by Prop. 13,

$$E : C = R : Q \text{..(6).}$$

From (1) and (6) by Prop. 2,

$$A : P = R : Q :$$

∴ by Prop. 6, $A : R = P : Q$,

which is the result (4) to be proved.

This mode of compounding ratios is of interest on account of its connection with the extension of the idea of multiplication.

The unit ratio is taken to be the ratio of equality $E : E$.

One way of deriving from the unit ratio the ratio $C : D$ (which corresponds to the multiplier) is to change the antecedent of the unit ratio in the ratio $E : C$, and the consequent of the unit ratio in the ratio $E : D$.

Let these changes be performed on the ratio $A : B$ (which corresponds to the multiplicand). Then the antecedent becomes P by (1), and the consequent becomes Q by (2), so that the resulting ratio is $P : Q$ (which corresponds to the product).

(The above process contains an arbitrary element E, but its value does not affect the value of the resulting ratio, for $P : Q = A : R$, and R is determined by (3) into which E does not enter.)

PROPOSITION 23.

Art. 56. If $A : B = Q : R$,

and $C : D = S : T$,

to prove that $(A : B) * (C : D) = (Q : R) * (S : T)$.

Take $A : B = K : L$,

and $C : D = L : M$,

then $Q : R = K : L$ (Prop. 2),

and $S : T = L : M$ (Prop. 2),

$$\therefore (A : B) * (C : D) = (K : L) * (L : M) = K : M,$$

and

$$(Q : R) * (S : T) = (K : L) * (L : M) = K : M,$$

$$\therefore (A : B) * (C : D) = (Q : R) * (S : T).$$

Art. 57. The next step is to prove the commutative, associative and distributive laws for the Compounding of Ratios.

PROPOSITION 24.

Art. 58. To prove the Commutative Law for the Compounding of Ratios, i.e.

$$(A : B) \ast (C : D) = (C : D) \ast (A : B).$$

Take $$C : D = B : E \text{ ......(1)}$$

and $$A : B = D : F \text{ ......(2)}$$

$$\therefore\ (A : B) \ast (C : D) = (A : B) \ast (B : E)$$
$$= A : E,$$

and $$(C : D) \ast (A : B) = (C : D) \ast (D : F)$$
$$= C : F.$$

Hence it is necessary to prove that

$$A : E = C : F.$$

Now (2) is $$A : B = D : F,$$

and from (1) $$B : E = C : D,$$

$\therefore$ by Prop. 13 $$A : E = C : F,$$

$$\therefore\ (A : B) \ast (C : D) = (C : D) \ast (A : B).$$

Or, denoting a ratio by a single Greek letter,

$$\alpha \ast \beta = \beta \ast \alpha.$$

PROPOSITION 25.

Art. 59. To prove the Associative Law for the Compounding of Ratios, i.e.

$$[(A : B) \ast (C : D)] \ast (E : F)$$
$$= (A : B) \ast [(C : D) \ast (E : F)].$$

Take $$C : D = B : G \text{ ......(1).}$$

$$E : F = G : H \text{ ......(2).}$$

$$E : F = D : K \text{ ......(3),}$$

$$C : K = B : L \text{ ......(4).}$$

Then $$[(A : B) \ast (C : D)] \ast (E : F)$$
$$= [(A : B) \ast (B : G)] \ast (G : H)$$
$$= (A : G) \ast (G : H)$$
$$= A : H.$$

Also

$$(A : B) * [(C : D) * (E : F)]$$
$$= (A : B) * [(C : D) * (D : K)]$$
$$= (A : B) * (C : K)$$
$$= (A : B) * (B : L)$$
$$= A : L.$$

To prove the proposition it is necessary to show that

$$A : H = A : L,$$

and ∴ by Prop. 4 (ii) it is sufficient to show that

$$H = L.$$

From (2) and (3) by Prop. 2,

$$G : H = D : K \quad \text{...(5)}$$

From (1) and (4) by Props. 3 and 12,

$$D : K = G : L \quad \text{...(6).}$$

From (5) and (6) by Prop. 2,

$$G : H = G : L,$$
$$\therefore \quad H = L \quad \text{[Prop. 5 (ii)].}$$

Hence, denoting a ratio by a single Greek letter,

$$[\alpha * \beta] * \gamma = \alpha * [\beta * \gamma].$$

PROPOSITION 26.

Art. 60. To prove the Distributive Law for Ratios, i.e.

$$[(A : B) + (C : D)] * (E : F) = [(A : B) * (E : F)] + [(C : D) * (E : F)].$$

Take magnitudes P, Q, X, Y (Prop. 16) such that

$$A : B = P : X,$$
$$C : D = Q : X,$$
$$E : F = X : Y.$$

Then

$$[(A : B) + (C : D)] * (E : F)$$
$$= [(P : X) + (Q : X)] * (X : Y)$$
$$= [P + Q : X] * (X : Y)$$
$$= P + Q : Y,$$
$$[(A : B) * (E : F)] = (P : X) * (X : Y)$$
$$= P : Y,$$
$$[(C : D) * (E : F)] = (Q : X) * (X : Y)$$
$$= Q : Y;$$

$$\therefore \ [(A : B) \ast (E : F)] + [(C : D) \ast (E : F)]$$
$$= (P : Y) + (Q : Y)$$
$$= P + Q : Y$$
$$= [(A : B) + (C : D)] \ast (E : F).$$

Hence, denoting a ratio by a single Greek letter,

$$(\alpha + \beta) \ast \gamma = (\alpha \ast \gamma) + (\beta \ast \gamma).$$

Note to Art. 42, § 6. To explain why, on the hypothesis that no such magnitude as D, [which is such that the scale of D, C is the same as that of B, A,] exists, it is not possible to have a mode of division in which the lower class has no greatest magnitude, and the upper class has no least magnitude.

If this were the case, then a magnitude must exist which separates the two classes, but itself belongs to neither class.

Now if any magnitude D of the same kind as C be taken, only three alternatives are possible:

(i) the scale of D, C is below that of B, A;

(ii) the scale of D, C is the same as that of B, A;

(iii) the scale of D, C is above that of B, A.

If, therefore, a magnitude exist which belongs to neither class, let it be called D, and then the scale of D, C is the same as that of B, A. But this is contrary to the hypothesis made, viz.:—that no such magnitude exists.

XI. *A New Method in Combinatory Analysis, with application to Latin Squares and associated questions.*

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Introduction.

Euler in the *Verhandelingen uitgegeven door het Zeeuwsch Genootschap der Wetenschappen te Vlissingen*, vol. 9, 1782, has a paper entitled "Recherches sur une nouvelle espèce de Quarrés Magiques."

He commences as follows:—

"Une question fort curieuse, qui a exercé pendant quelque temps la sagacité de bien du monde, m'a engagé à faire les recherches suivantes, qui semblent ouvrir une nouvelle carrière dans l'Analyse, et en particulier dans la doctrine des combinaisons. Cette question rouloit sur une assemblée de 36 officiers de six différens grades et tirés de six Régimens différens, qu'il s'agissoit de ranger dans un quarré, de manière, que sur chaque ligne tant horizontale que verticale il se trouva six officiers tant de différens caractères que de Régimens différens. Or après toutes les peines qu'on s'est donné pour résoudre ce Problème, on a été obligé de reconnoître qu'un tel arrangement est absolument impossible, quoiqu'on ne puisse pas en donner de démonstration rigoureuse."

He denotes the six regiments by the Latin letters a, b, c, d, e, f and the six ranks or grades by the Greek letters $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, and remarks that the 'character' of an officer is determined by two letters, the one Latin and the other Greek, and that the problem consists in arranging the 36 combinations

$$\begin{array}{cccccc} a\alpha & a\beta & a\gamma & a\delta & a\epsilon & a\zeta \\ b\alpha & b\beta & b\gamma & b\delta & b\epsilon & b\zeta \\ c\alpha & c\beta & c\gamma & c\delta & c\epsilon & c\zeta \\ d\alpha & d\beta & d\gamma & d\delta & d\epsilon & d\zeta \\ e\alpha & e\beta & e\gamma & e\delta & e\epsilon & e\zeta \\ f\alpha & f\beta & f\gamma & f\delta & f\epsilon & f\zeta \end{array}$$

in a square in such a manner that every row and column contains the six Latin and the six Greek letters.

He finds no solution of this particular problem and gives his opinion that none can be obtained whenever the order of the square is of the form 2 mod 4.

In other cases as far as the order 9 he obtains solutions.

The first step is to arrange the Latin letters in a square so that no letter is missing either from any row or any column. He calls this a Latin Square, and in regard to their enumeration for a given order observes, § 148, p. 230:

"J'observe encore à cette occasion que le parfait dénombrement de tous les cas possibles de variations semblables seroit un objet digne de l'attention des Géomètres, d'autant plus que tous les principes connus dans la doctrine des combinaisons n'y sçauroient prêter le moindre secours."

And again, § 152, p. 234:

"J'avois observé ci-dessus, qu'un parfait dénombrement de toutes les variations possibles des quarrés latins seroit une question très importante, mais qui me paroissoit extrèmement difficile et presque impossible dès que le nombre n surpassoit 5. Pour approcher de cette énumération il faudroit commencer par cette question:

En combien de manières différentes, la première bande horizontale étant donnée, peut-on varier la seconde bande horizontale pour chaque nombre proposé n?"

He in fact gives a solution of the last-mentioned question. Many different ones are now in existence and, incidentally, a new one is given in this paper.

It is the well-known 'Problème des rencontres' which was first proposed by Montmort.

For the rest the paper is entirely concerned with the actual construction of what may be termed Graeco-Latin Squares, to which one is led by considering the problem of the officers above mentioned, and with their transformation so as to enable one to obtain many solutions from any one that has been arrived at.

Euler himself admits the unsatisfactory nature of his investigation. He remarks § 11, p. 94 "La formation des formules directrices est donc le premier et le principal objet dans ces recherches; mais je dois avoüer, que jusqu'ici je n'avois aucune méthode sûre qui puisse conduire à cette investigation. Il semble même qu'on doit se contenter d'une espèce de simple tâtonnement que je vais expliquer pour le quarré latin de 49 cases rapporté ci-dessus."

To explain the meaning of the phrase 'formules directrices' which will be referred to in the sequel, I take Euler's Graeco-Latin Square of order 7, writing with him the natural numbers instead of Latin and Greek letters and writing the latter as exponents to the former:

$$\begin{array}{ccccccc}
1^1 & 2^6 & 3^4 & 4^3 & 5^7 & 6^5 & 7^2 \\
2^2 & 3^7 & 1^5 & 5^4 & 4^1 & 7^6 & 6^3 \\
3^3 & 6^1 & 5^6 & 7^5 & 1^2 & 4^7 & 2^4 \\
4^4 & 5^2 & 6^7 & 1^6 & 7^3 & 2^1 & 3^5 \\
5^5 & 1^3 & 7^1 & 2^7 & 6^4 & 3^2 & 4^6 \\
6^6 & 7^4 & 4^2 & 3^1 & 2^5 & 5^3 & 1^7 \\
7^7 & 4^5 & 2^3 & 6^2 & 3^6 & 1^4 & 5^1
\end{array}$$

Consider the Latin Square as given by erasing the exponent numbers.

In order to find possible places for the exponent 1 we must select 7 different numbers, from the square, one being in each column and one in each row.

Thus we may select the set

$$1 \quad 6 \quad 7 \quad 3 \quad 4 \quad 2 \quad 5$$

and give each the exponent 1 as above.

This is called 'une formule directrice' for the exponent 1.

We may similarly find a formula for the exponent 2, viz.:—

$$2 \quad 5 \quad 4 \quad 6 \quad 1 \quad 3 \quad 7$$

and if we are successful for each of the 7 numbers and can fill in all the exponents clearly the Graeco-Latin Square will have been constructed. In this case if 'une formule directrice' be written for each number, so that they form a square, it is clear that it must be a Latin Square.

I pass now to the paper by Cayley, *Messenger of Mathematics*, vol. XIX. (1890), pp. 135—137. This is little more than a statement of the problem involved in the enumeration of Latin Squares. With Euler he reduces the number of squares by taking the top row and left-hand column in the same determinate order; say for the order 5,

$$\begin{array}{ccccc} a & b & c & d & e \\ b & d & e & a & c \\ c & e & d & b & a \\ d & a & e & c & b \\ e & c & b & a & d \end{array}$$

and he remarks that "if the number of such squares be $= N$, then obviously the whole number of squares that can be formed with the same n arrangements is $= N[n]^n$." This however is not so; the number is $N \, . \, n!(n-1)!$ the factor $n!$ appearing from permutation of columns and $(n-1)!$ from permutation of the lower $n-1$ rows. He speaks of the possible arrangements of the second line, the 'Problème des rencontres' and of the difficulty of proceeding a step further to the enumeration of the arrangements of the second and third lines. This problem, previous to the present paper, had never been solved, and to quote from Cayley (*loc. cit.*) "the difficulty of course increases for the next following lines." He further makes the valuable, if obvious, observation "when all the lines are filled up except the bottom line, the bottom line is completely determined." In the above quotation I have substituted 'bottom' for 'top,' as I read with Euler from top to bottom instead of with Cayley from bottom to top.

The 'Problème des ménages' involves to some extent the consideration of the second and third lines of the square and a fairly satisfactory solution has been obtained. (See post.)

For the rest nothing of value has been accomplished and the whole question awaits elucidation.

SECTION 1.

Art. 1. It occurred to me to attack the problem by the powerful methods of the calculus of symmetric functions, and I have been led to the complete analytical solution of the main and many allied questions. The enumerations are given in the form of coefficients in the developments of certain generating functions whose inner structures are seen to implicitly involve the solutions.

In particular I have obtained the following simple and elegant theorem.

If the symmetric function

$$\Sigma\alpha_1^{2^{n-1}}\alpha_2^{2^{n-2}}\ldots\ldots\alpha_{n-1}^2\alpha_n$$

be raised to the power n, so that

$$(\Sigma\alpha_1^{2^{n-1}}\alpha_2^{2^{n-2}}\ldots\ldots\alpha_{n-1}^2\alpha_n)^n=\ldots\ldots+K\Sigma\alpha_1^{2^n-1}\alpha_2^{2^n-1}\ldots\ldots\alpha_{n-1}^{2^n-1}\alpha_n^{2^n-1}+\ldots\ldots,$$

then K is the number of Latin Squares of order n and division by $n!(n-1)!$ is merely necessary to obtain the number of reduced Latin Squares.

I make the following references to papers by myself where the principles employed are set forth and employed in various interesting questions of combinatory analysis.

"Symmetric Functions and the Theory of Distributions" (*Proc. L. M. S.*, vol. XIX. pp. 220—256).

"A theorem in the calculus of linear partial differential equations" (*Q. M. J.*, vol. XXIV. pp. 246—250).

"Memoir on a new Theory of Symmetric Functions" (*Amer. Journ. Math.*, vol. XI. 1889, pp. 1—36).

"Second Memoir......" (*ibid.* vol. XII. 1890, pp. 61—102).

"Third Memoir......" (*ibid.* vol. XIII. 1891, pp. 193—234).

"Fourth Memoir......" (*ibid.* vol. XIV. 1892, pp. 15—32).

"A certain class of Generating Functions in the Theory of Numbers" (*Phil. Trans. R. S.*, vol. 185 A, 1894, pp. 111—160).

"The Algebra of Multi-Linear Partial Differential Operators" (*Proc. L. M. S.* vol. XIX.).

"The Multiplication of Symmetric Functions" (*Mess. of Math.*, New Series, No. 167, March 1885).

Art. 2. I reproduce, with slightly altered notation, the master theorem, given in the last but one quoted paper, which has special reference to the present investigation.

Let $u_1, u_2, \ldots u_s$ be any linear operators, whatever, in regard to the elements

$$p_1, p_2, p_3, \ldots,$$

and put

$$\Theta=\lambda_1u_1+\lambda_2u_2+\ldots+\lambda_su_s$$
$$=\Theta_1\partial_{p_1}+\Theta_2\partial_{p_2}+\Theta_3\partial_{p_3}+\ldots.$$

Further let

$$\phi_1,\ \phi_2, \dots \phi_m$$

be any m functions of $p_1, p_2, p_3, \dots$

and

$$\phi = \phi_1\phi_2\phi_3 \dots\dots \phi_m.$$

Then

$$\phi_t (p_1 + \Theta_1,\ \ p_2 + \Theta_2,\ \ p_3 + \Theta_3, \dots)$$

$$= \phi_t + \Theta\phi_t + \frac{(\Theta^2)}{2\,!}\phi_t + \frac{(\Theta^3)}{3\,!}\phi_t + \dots,$$

and

$$\phi + \Theta\phi + \frac{(\Theta^2)}{2\,!}\phi_t + \frac{\Theta^3}{3\,!}\phi_t + \dots$$

$$= \prod_{t=1}^{t=m}\left\{\phi_t + \Theta\phi_t + \frac{(\Theta^2)}{2\,!}\phi_t + \frac{(\Theta^3)^3}{3\,!}\phi_t + \dots\dots\right\},$$

that is

$$\phi + (\lambda_1 u_1 + \lambda_2 u_2 + \dots\dots + \lambda_s u_s)\,\phi + \frac{1}{2\,!}(\lambda_1 u_1 + \lambda_2 u_2 + \dots\dots + \lambda_s u_s)^2\,\phi + \dots\dots$$

$$= \prod_{t=1}^{t=m}\left\{\phi_t + (\lambda_1 u_1 + \lambda_2 u_2 + \dots\dots + \lambda_s u_s)\,\phi_t + \frac{1}{2\,!}(\lambda_1 u_1 + \lambda_2 u_2 + \dots\dots + \lambda_s u_s)^2\,\phi_t + \dots\dots\right\}.$$

We now compare the coefficients of

$$\lambda_1^{\chi_1}\lambda_2^{\chi_2} \dots\dots \lambda_s^{\chi_s},$$

on the two sides of the identity, and obtain the result

$$\frac{(u_1^{\chi_1}u_2^{\chi_2}\dots u_s^{\chi_s})\,\phi}{\chi_1\,!\,\chi_2\,!\dots\chi_s\,!} = \Sigma\Sigma\,\frac{(u_1^{\alpha_1}u_2^{\alpha_2}\dots u_s^{\alpha_s})\,\phi_1}{\alpha_1\,!\,\alpha_2\,!\dots\alpha_s\,!}\,\frac{(u_1^{\beta_1}u_2^{\beta_2}\dots u_s^{\beta_s})\,\phi_2}{\beta_1\,!\,\beta_2\,!\dots\beta_s\,!}\dots\frac{(u_1^{\mu_1}u_2^{\mu_2}\dots u_s^{\mu_s})\,\phi_m}{\mu_1\,!\,\mu_2\,!\dots\mu_s\,!},$$

where

$$\alpha_t + \beta_t + \dots + \mu_t = \chi_t,\ (t = 1,\ 2,\ 3, \dots s),$$

and the double summation is in regard to every positive integral solution of these s equations and to every permutation of

$$\phi_1,\ \phi_2,\ \phi_3 \dots \phi_m.$$

This important result is similar to the well-known theorem of Leibnitz in form, but in form only, for here such an operator product as

$$u_1^{\delta_1}u_2^{\delta_2}\dots u_s^{\delta_s}$$

does not mean $\Sigma\delta$ successive operations but the single operation of an operator of order $\Sigma\delta$ obtained by symbolic multiplication.

Art. 3. Putting s equal to unity we find

$$\frac{(u^{\chi})\,\phi}{\chi\,!} = \Sigma\Sigma\,\frac{(u^{\alpha})\,\phi_1}{\alpha\,!}\,\frac{(u^{\beta})\,\phi_2}{\beta\,!}\dots\frac{(u^{\mu})\,\phi_m}{\mu\,!},$$

and now putting

$$u = \partial_{p_1} + p_1\partial_{p_2} + p_2\partial_{p_3} + \dots$$

$$\frac{(u^{\delta})}{\delta\,!} = D_\delta,$$

$$D_\chi\phi = \Sigma\Sigma\,D_\alpha\phi_1 D_\beta\phi_2 \dots D_\mu\phi_m,$$

where

$$\alpha + \beta + \dots + \mu = \chi,$$

and the double summation has reference to every solution of this equation in positive integers (including zero) and to every permutation of $\phi_1, \phi_2, \ldots \phi_m$.

Art. 4. As a further particular case,

$$D_\chi \phi^m = \Sigma\Sigma D_\alpha \phi D_\beta \phi \ldots D_\mu \phi,$$

and we have a summation in regard to every partition of χ into μ parts, zero being counted as a part, and to every permutation of the parts of each partition.

The operation $$\Sigma D_\alpha \phi_1 D_\beta \phi_2 \ldots D_\mu \phi_m,$$

the summation being for every permutation of the parts $\alpha, \beta, \ldots \mu$, may be denoted by

$$D_{(\alpha\beta \ldots \mu)},$$

and thence we have the equivalence

$$D_\chi \equiv \Sigma D_{(\alpha\beta \ldots \mu)},$$

where the summation is for all partitions of χ.

It may happen that $(\alpha\beta \ldots \mu)$ is the only partition of χ, for which $D_{(\alpha\beta \ldots \mu)} \phi$ does not vanish, so that

$$D_\chi \equiv D_{(\alpha\beta \ldots \mu)}.$$

As will be seen the assignment of ϕ, so that this may be the case, is the key to the solution of the problem of the Latin Square.

Art. 5. Take the polynomial

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n),$$

and, with the object of investigating Latin Squares with the n elements

$$a_1, a_2, \ldots a_n,$$

consider the symmetric function

$$\Sigma \alpha_1^{a_1} \alpha_2^{a_2} \ldots \alpha_n^{a_n},$$

wherein $a_1, a_2, \ldots a_n$ are to be regarded as unspecified different integers.

Let $$a_1 + a_2 + \ldots + a_n = w,$$

and form the partial differential operator

$$D_w = \frac{1}{w!} (\partial_{p_1} + p_1 \partial_{p_2} + p_2 \partial_{p_3} + \ldots)^w,$$

the linear operator being raised to the power w by symbolic multiplication, so that D_w is an operator of order w.

The symmetric function $$\Sigma \alpha_1^{a_1} \alpha_2^{a_2} \ldots \alpha_n^{a_n}$$

may be conveniently symbolised, in the usual manner, by the partition notation

$$(a_1 a_2 \ldots a_n).$$

As is well known the operation of D is shewn by

$$D_{a_1}(a_1 a_2 \ldots a_n) = (a_2 \ldots a_n),$$

$$D_{a_s}(a_1 a_2 \ldots a_{s-1} a_s \ldots a_n) = (a_1 a_2 \ldots a_{s-1} a_{s+1} \ldots a_n),$$

$$D_{a_1}(a_1) = 1,$$

$$D_{a_1} D_{a_2} D_{a_3} \ldots D_{a_n} (a_1 a_2 a_3 \ldots a_n) = 1.$$

I now construct the symmetric function

$$(a_1 a_2 \dots a_n)^n,$$

and suppose it multiplied out until its equivalent representation as a sum of monomial symmetric functions is arrived at.

We have $$(a_1 a_2 \dots a_n)^n = \Sigma K (b_1 b_2 b_3 \dots),$$

and operating on both sides with

$$D_{b_1} D_{b_2} D_{b_3} \dots,$$

we find $$D_{b_1} D_{b_2} D_{b_3} \dots (a_1 a_2 \dots a_n)^n = K,$$

and we can calculate K by performing a definite series of operations.

If we operate with D_w only those symmetric functions, on the right-hand side, survive which include a part w in their partitions, and, in each of these, the number of times w occurs as a part is diminished by unity. We must consider the operation of D_w upon the left-hand side

$$(a_1 a_2 \dots a_n)^n.$$

First take for simplicity $n = 4$, and write

$$(a_1, a_2, a_3, a_4) = (a, b, c, d),$$

and $$w = a + b + c + d.$$

Let a, b, c, d be so assigned that

$$(abcd)$$

is the *only partition* of $w = a + b + c + d$ into four or fewer parts, repeated or not, drawn from the parts a, b, c, d.

Ex. gr. we may take $$(a, b, c, d) = (8, 4, 2, 1),$$

and it will be noticed that 15 has the single partition (8421) into four or fewer parts, repeated or not, drawn from the parts 8, 4, 2, 1.

Then $$D_w (abcd)^4 = D_{(abcd)} (abcd)^4 \text{ (see \textit{ante} Art. 4)},$$

and $$D_w (abcd)^4 = \Sigma D_{v_1} (abcd) D_{v_2} (abcd) D_{v_3} (abcd) D_{v_4} (abcd),$$

wherein $v_1 v_2 v_3 v_4$ is a permutation of the letters a, b, c, d, and the summation is for every such permutation.

(Cf. *Q. M. J.* No. 85, 1886, "The Law of Symmetry and other Theorems in Symmetric Functions," § 4.)

Art. 6. It is this valuable property of the operation D, when performed upon a product, that is the essential feature of this investigation. The right-hand side consists of $24 (= 4!)$ terms, each of which is

$$(bcd)(acd)(abd)(abc),$$

and we have the identity

$$D_w (abcd)^4 = 4!(bcd)(acd)(abd)(abc),$$

and clearly also

$$D_w (a_1 a_2 a_3 \dots a_{n-1} a_n)^n = n!(a_2 a_3 \dots a_n)(a_1 a_3 \dots a_n) \dots (a_1 a_2 \dots a_{n-1}).$$

As remarked above we may consider, in general, Latin Rectangles. If we consider a Rectangle of n columns and 1 row we obtain a trivial result, which however it is proper, for the orderly development of the subject, to notice. We are subject here to no condition. The whole number of rectangles is $n!$; of reduced rectangles

$$\frac{n!}{n!} = 1.$$

This we may take to be indicated by the above operation of D_w upon

$$(a_1a_2 \dots a_n)^n.$$

To give it analytical expression, in the identity

$$D_w(a_1a_2 \dots a_n)^n = n!(a_2a_3 \dots a_n) \dots (a_1a_2 \dots a_{n-1})$$

put
$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 1,$$

so that
$$[D_w(a_1a_2 \dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}$$
$$= n!\left(\frac{n!}{1!}\right)^n n_1,$$

where n_1 is the number of reduced Rectangles of n columns and 1 row.

I call to mind that the case is trivial, and $n_1 = 1$.

Thence
$$n_1 = \frac{[D_w(a_1a_2 \dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}}{n!\left(\frac{n!}{1!}\right)^n}.$$

Art. 7. I pass on to consider
$$D_w^2(abcd)^4$$
$$= 4!\,D_w(bcd)(acd)(abd)(abc)$$
$$= 4!\,\Sigma D_{v_1}(bcd)\,D_{v_2}(acd)\,D_{v_3}(abd)\,D_{v_4}(abc),$$

where $v_1v_2v_3v_4$ is any permutation of the letters a, b, c, d, and the summation is for all such permutations.

Now observe that certain terms out of the 24 vanish. $D_{v_1}(bcd)$ vanishes when $v_1 = a$, $D_{v_2}(acd)$ when $v_2 = b$, $D_{v_3}(abd)$ when $v_3 = c$, $D_{v_4}(abc)$ when $v_4 = d$. We are *only* concerned with those permutations

$$v_1v_2v_3v_4$$

of a, b, c, d which are competent to form a Latin Rectangle of two rows, the top row being a, b, c, d.

We have in fact the 'Problème des rencontres.'

$v_1v_2v_3v_4$ can only assume the 9 permutations

$$\begin{array}{cccc} b & c & d & a \\ b & d & a & c \\ b & a & d & c \\ c & d & a & b \\ c & d & b & a \\ c & a & d & b \\ d & a & b & c \\ d & c & a & b \\ d & c & b & a. \end{array}$$

9 being the solution of the corresponding 'Problème des rencontres,' and representing the number of Latin Rectangles in which the top row is *abcd* but in which the first letter in the second row is not necessarily *b*.

To obtain the number of reduced rectangles we must divide the number 9 by 3 $(= n-1)$, and so reach the number 3.

On the right-hand side of the identity

$$D_w^2 (abcd)^4$$
$$= 4!\,\Sigma D_{v_1}(bcd)\, D_{v_2}(acd)\, D_{v_3}(abd)\, D_{v_4}(abc)$$

we obtain, after operation, 9 terms; 3 for each reduced Latin Rectangle of two rows.

Let n_2 denote the number of reduced Latin Rectangles of n columns and two rows. On the right-hand side of the identity

$$D_w^2 (a_1 a_2 \dots a_n)^n$$
$$= n!\, D_{v_1}(a_2 a_3 \dots a_n)\, D_{v_2}(a_1 a_3 \dots a_n) \dots D_{v_n}(a_1 a_2 \dots a_{n-1})$$

we obtain after operation

$$(n-1)\, n_2 \text{ terms,}$$

corresponding to the Latin Rectangles, reduced as regards columns but unreduced as regards rows,—or

$$n!\,(n-1)\, n_2 \text{ terms,}$$

corresponding to the totality of unreduced Latin Rectangles.

We have

$$n_2 = \frac{\text{sum of coefficients of terms in the development of } D_w^2 (a_1 a_2 \dots a_n)^n}{n!\,(n-1)},$$

the development being that reached by the performance of D_w^2 in the manner indicated, and the numerator of the fraction necessarily representing the number of unreduced Latin Rectangles of n columns and two rows.

As before we can give n_2 an analytical expression.

Each of the $n!\,(n-1)\, n_2$ terms in $D_w^2 (a_1 a_2 \dots a_n)^n$ assumes the value

$$\left(\frac{n!}{2!}\right)^n$$

for $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$.

Hence
$$n_2 = \frac{[D_w^2 (a_1 a_2 \dots a_n)^n]_{\alpha_1 = \alpha_2 = \dots = \alpha_n = 1}}{n!\,(n-1)\left(\frac{n!}{2!}\right)^n},$$

which is a new solution of the much considered 'Problème des rencontres*.'

Art. 8. Observe that in the above process we have a series of operations corresponding to every unreduced Latin Rectangle. I pass now to the Latin Rectangle of three rows, a problem hitherto regarded as unassailable.

* Another solution has been given by the author, *Phil. Trans. R. S.* vol. 185 A, 1894.

Suppose

$$v_1\ v_2\ \dots\ v_n,$$
$$v_1'\ v_2'\ \dots\ v_n',$$
$$v_1''\ v_2''\ \dots\ v_n'',$$

to be such a rectangle.

In operating with D_w upon $(a_1a_2\dots a_n)^n$, we get one term corresponding to

$$D_{v_1} * D_{v_2} * \dots D_{v_n} *.$$

Operating upon *this term* with D_w we get one term corresponding to the double operation

$$D_{v_1} * D_{v_2} * \dots D_{v_n} *,$$
$$D_{v_1'} * D_{v_2'} * \dots D_{v_n'} *,$$

and again operating upon *this last term* with D_w we get one term corresponding to the treble operation

$$D_{v_1} * D_{v_2} * \dots D_{v_n} *,$$
$$D_{v_1'} * D_{v_2'} * \dots D_{v_n'} *,$$
$$D_{v_1''} * D_{v_2''} * \dots D_{v_n''} *;$$

that is to say, in performing D_w^3, we have a one-to-one correspondence between the terms involved and the unreduced Latin Rectangles of n columns and 3 rows. To obtain the reduced Latin Rectangles we have merely to divide the number of terms by

$$n!\,(n-1)\,(n-2).$$

Hence calling n_3 the number of reduced Rectangles, we have, by previous reasoning,

$$n_3 = \frac{[D_w^3\,(a_1a_2\dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}}{n!\,(n-1)\,(n-2)\left(\frac{n!}{3!}\right)^n}.$$

Art. 9. It is now easy to pass to the general case and to demonstrate the result

$$n_s = \frac{[D_w^s\,(a_1a_2\dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}}{n!\,\frac{(n-1)!}{(n-s)!}\left(\frac{n!}{s!}\right)^n}.$$

In the case of the Latin Square

$$s = n,$$

and we should be able to shew, after Cayley's remark, that

$$n_{n-1} = n_n.$$

Now

$$n_{n-1} = \frac{[D_w^{n-1}\,(a_1a_2\dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}}{n!\,(n-1)!\,n^n},$$

$$n_n = \frac{[D_w^n\,(a_1a_2\dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}}{n!\,(n-1)!},$$

and it will be shewn that these expressions have the same value.

After performance of D_w^{n-1} we obtain a number of terms of the nature

$$(c_1)(c_2)\dots(c_n),$$

and putting herein $\alpha_1=\alpha_2=\dots=\alpha_n=1$, we obtain a factor n^n.

On proceeding to perform D_w again, before putting $\alpha_1=\alpha_2=\dots=\alpha_n=1$, we obtain simply unity and the quantities $\alpha_1, \alpha_2, \dots \alpha_n$ have disappeared.

Hence

$$[D_w^{n-1}(a_1a_2\dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1}$$

$$=n^n[D_w^n(a_1a_2\dots a_n)^n]_{\alpha_1=\alpha_2=\dots=\alpha_n=1},$$

and thence $$n_{n-1}=n_n.$$

The expression for n_n may be simplified because

$$D_w^n(a_1a_2\dots a_n)^n$$

is an integer and does not involve the roots $\alpha_1, \alpha_2, \dots \alpha_n$.

$$\therefore\ n_n=\frac{D_w^n(a_1a_2\dots a_n)^n}{n!(n-1)!}.$$

The numerator of the fraction is the coefficient of the symmetric function

$$(w^n)$$

in the development of the power

$$(a_1a_2\dots a_n)^n,$$

w being equal to Σa.

Art. 10. In the symmetric functions the letters $a_1, a_2, \dots a_n$ are taken to represent different integers and so far their values have been unspecified. They must be appropriately chosen or the analysis will fail.

It is essential that the number

$$a_1+a_2+\dots+a_n$$

shall possess but a single partition, into n or fewer parts, drawn from the numbers $a_1, a_2, a_3, \dots a_n$, repetitions of parts permissible.

Thus of order 4 the simplest system is

$$(a_1, a_2, a_3, a_4)=(8, 4, 2, 1),$$

the number 15 possesses the single partition 8421, of four or fewer parts, drawn from the integers 8, 4, 2, 1, repetitions permitted.

The system 7421 would not do because 14 possesses the partition 7, 7.

In general, of order n, the simplest system is

$$(a_1, a_2, \dots a_{n-1}, a_n)=(2^{n-1}, 2^{n-2}, \dots 2, 1).$$

To perform the operations indicated we have to express the function

$$(a_1a_2\dots a_n)^n$$

in terms of the coefficients

$$p_1, p_2, \dots p_n,$$

and operate with

$$D_w = \frac{1}{w!}(\partial_{p_1} + p_1\partial_{p_2} + \dots + p_{n-1}\partial_{p_n})^w,$$

as many times successively as may be necessary. We then write $\binom{n}{s}$ for p_s and substitute in the formulæ.

The calculations will, no doubt, be laborious but that is here not to the point, as an enumeration problem may be considered to be solved when definite algebraical processes are set forth which lead to the solution.

In the case of the Latin Square we may write the result

$$(2^{n-1}2^{n-2}\dots 21)^n = \dots + n!\,(n-1)!\,n_n\,(\overline{2^n-1}^{\,n}) + \dots,$$

or in the form

$$(\Sigma\alpha_1^{2^{n-1}}\alpha_2^{2^{n-2}}\dots\alpha^2_{n-1}\alpha_n)^n = \dots + n!\,(n-1)!\,n_n\Sigma\alpha_1^{2^n-1}\alpha_2^{2^n-1}\dots\alpha_n^{2^n-1} + \dots,$$

$n!\,(n-1)!\,n_n$ and n_n enumerating respectively the unreduced and reduced Latin Squares of order n.

It will be noticed that $(2^{n-1}2^{n-2}\dots 21)$ is what I have elsewhere* called a perfect partition of the number 2^n-1; that is, from its parts can be composed, in one way only, the number 2^n-1 and every lower number.

SECTION 2.

Art. 11. I proceed to discuss the 'Problème des ménages' by the same method.

Lucas in his *Théorie des Nombres* thus enunciates the question:—

"Des femmes, en nombre n, sont rangées autour d'une table, dans un ordre determiné; on demande quel est le nombre des manières de placer leurs maris respectifs, de telle sorte qu'un homme soit placé entre deux femmes, sans se trouver à côté de la sienne?"

He then remarks that it is necessary to determine the number of 'permutations discordantes' with the two permutations

$$\begin{matrix} 1 & 2 & 3 & 4 \dots n-1, & n \\ 2 & 3 & 4 & 5 \dots \quad n, & 1. \end{matrix}$$

He remarks as follows:—

"Nous ne connaissons aucune solution simple de cette question, dont l'énoncé donne lieu à l'étude du nombre des permutations discordantes de deux permutations déjà discordantes et plus généralement, du nombre des permutations discordantes de deux permutations quelconques."

* "The theory of perfect partitions of numbers and the compositions of multipartite numbers." *Messenger of Mathematics*, Vol. 20, 1891, pp. 103—119.

He gives solutions due to M. Laisant and M. C. Moreau of which the most convenient is represented by the difference equation

$$(n-1)\lambda_{n+1} = (n^2-1)\lambda_n + (n+1)\lambda_{n-1} + 4(-1)^n$$

with the initial values

$$(\lambda_3, \ \lambda_4) = (1, \ 2).$$

The reader, who has mastered the preceding solution of the problem of the Latin Rectangle, will have no difficulty in applying the same method here.

Construct the symmetric function

$$(a_3a_4 \ldots a_n)(a_1a_4 \ldots a_n)(a_1a_2a_5 \ldots a_n) \ldots (a_2a_3 \ldots a_{n-1}),$$

where the sth factor from the left is deprived of the symbols a_s, a_{s+1} (a suffix when $>n$ being taken to the modulus n) by the operation of $D_{a_s}D_{a_{s+1}}$.

We now operate with D_w and obtain

$$\Sigma D_{v_1}(a_3a_4 \ldots a_n) D_{v_2}(a_1a_4 \ldots a_n) \ldots D_{v_n}(a_2a_3 \ldots a_{n-1}),$$

the summation being for every permutation

$$v_1v_2 \ldots v_n$$

of the letters $a_1a_2 \ldots a_n$.

The number of products that survive is precisely the number of *ménages* denoted by Lucas by the symbol λ_n. Each factor of each product contains $n-3$ symbols a in brackets and for

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$$

has the value

$$\frac{n\,!}{3\,!}.$$

Hence $$[D_w(a_3a_4 \ldots a_n)(a_1a_4 \ldots a_n) \ldots (a_2a_3 \ldots a_{n-1})]_{a_1=a_2=\ldots=a_n=1} = \lambda_n \left(\frac{n\,!}{3\,!}\right)^n,$$

or $$\lambda_n = \left(\frac{n\,!}{3\,!}\right)^{-n} [D_w(a_3a_4 \ldots a_n)(a_1a_4 \ldots a_n) \ldots (a_2a_3 \ldots a_{n-1})]_{a_1=a_2=\ldots=a_n=1}.$$

As before we may take in the calculation

$$(a_1, \ a_2, \ldots a_n) = (2^{n-1}, \ 2^{n-2}, \ldots 1).$$

Art. 12. Similarly we can find the number of permutations discordant with each of any two permutations whatever that are mutually discordant.

If these two permutations be

$$v_1v_2 \ldots v_n,$$
$$v_1'v_2' \ldots v_n',$$

we have merely to form a product in which the sth factor from the left is deprived of the symbols v_s, v_s' and proceed as before. We thus arrive at a similar but, of course, not an identical result.

Art. 13. It is equally easy to find the number of permutations discordant with each of two permutations which are not mutually discordant.

Let these permutations be

$$v_1 v_2 \dots v_s v_{s+1} \dots v_n$$

$$v_1 v_2 \dots v_s v'_{s+1} \dots v_n'.$$

We take a product in which the left-hand factor is without v_1, the next without v_2, &c....the sth without v_s, but beyond this the tth factor is without the two symbols v_t, v_t'.

Denoting this product by P we, by the usual method, reach the solution

$$\mu_n = \left(\frac{n\,!}{2\,!}\right)^{-s} \left(\frac{n\,!}{3\,!}\right)^{-(n-s)} [D_w P]_{a_1=a_2=\dots=a_n=1}.$$

Art. 14. On a similar principle we can enumerate the number of permutations discordant with *any number* of given permutations whatever.

Let

$$\begin{array}{l} v_1 \quad v_2 \quad v_3 \dots v_n, \\ v_1' \quad v_2' \quad v_3' \dots v_n', \\ v_1'' \quad v_2'' \quad v_3'' \dots v_n'', \\ \dots\dots\dots\dots\dots\dots \end{array}$$

be the m permutations and of the m letters v_s, v_s', v_s'', ...; let the different ones be u_s, u_s', u_s'', ... k_s in number.

Take for the sth factor $(a_1 a_2 \dots a_n)$,

deprived of the letters u_s, u_s', u_s'', ... ,

and form the resulting product P.

Proceeding as before we obtain the result

$$\nu_n = \left\{\frac{n\,!}{(k_1+1)\,!}\right\}^{-j_1} \left\{\frac{n\,!}{(k_2+1)\,!}\right\}^{-j_2} \dots \left\{\frac{n\,!}{(k_s+1)\,!}\right\}^{-j_s} \dots \times [D_w P]_{a_1=a_2=\dots=a_n=1},$$

where j_1, j_2, ... j_s, ... are numbers, at once ascertainable, and $\Sigma j = n$.

Art. 15. A more direct generalisation of the 'Problème des ménages' is obtained by imposing the condition that no husband is to have less than $2m$ persons between himself and his wife.

In the problem above considered $m = 1$. If $m = 2$ we must enumerate the permutations discordant with

$$\begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 \dots a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_2 & a_3 & a_4 & a_5 \dots a_{n-2} & a_{n-1} & a_n & a_1 \\ a_3 & a_4 & a_5 & a_6 \dots a_{n-1} & a_n & a_1 & a_2 \\ a_4 & a_5 & a_6 & a_7 \dots a_n & a_1 & a_2 & a_3 \end{array}$$

and we form the product

$$P_4 = (a_5 \dots a_n)(a_1 a_6 \dots a_n)(a_1 a_2 a_7 \dots a_n) \dots (a_4 \dots a_{n-1}),$$

the solution being given by

$$\text{number} = \left(\frac{n\,!}{5\,!}\right)^{-n} [D_w P_4]_{\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1},$$

and in general

$$\text{number} = \left(\frac{n\,!}{(2m+1)\,!}\right)^{-n} [D_w P_{2m}]_{\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1}.$$

SECTION 3.

Art. 16. The notion of a Latin Rectangle may be generalised. Instead of n different letters we may have s_1 of one kind, s_2 of a second, s_3 of a third, and so on. The letters may be, ex. gr.,

$$a_1^{s_1} a_2^{s_2} a_3^{s_3} \ldots\ldots a_k^{s_k},$$

where

$$\Sigma s = n.$$

To obtain a Latin Rectangle of t rows we take t permutations of the letters such that in no column does a_1 occur more than s_1 times, a_2 more than s_2, a_3 more than s_3, and so on. The reduced rectangles have the top row and left-hand column in the same assigned order and evidently we can obtain their number by dividing the number of unreduced rectangles by

$$\frac{n\,!}{s_1!\, s_2!\, s_3! \ldots s_k!} \times \frac{(n-1)\,!}{(s_1-1)!\, s_2!\, s_3! \ldots s_k!}$$

in the case when the rectangle is a square, and by factors of similar forms in the other particular cases.

Examples of such quasi-Latin Squares are

a	a	b	c	a	a	b	b	a	a	a	b
a	b	c	a	a	b	a	b	a	a	b	a
b	c	a	a	b	b	a	a	a	b	a	a
c	a	a	b	b	a	b	a	b	a	a	a.

Art. 17. We have Latin Squares and Rectangles associated with every partition of every number. The three, given above, correspond to the partitions 21^2, 2^2, 31; we have already, in the first part of the paper, considered the case *abcd* corresponding to 1^4 and there remains the case *aaaa*, of partition 4, which is trivial. Part of this theory is intimately connected with certain chessboard problems that might be proposed.

Take the unreduced Latin Squares on the letters

$$a\ a\ a\ a\ a\ a\ a\ b.$$

The enumeration gives the number of ways of placing 8 rooks on the board so that no one can take any of the others.

Similarly the enumeration connected with

$$a\ a\ a\ a\ a\ a\ b\ c$$

gives the number of ways of placing 16 rooks, 8 white and 8 black, on the board so that no rook can be taken by another of the same colour.

Like problems can be connected with other cases.

Art. 18. Let us consider the general question of the enumeration.

First take the simple case

$$a_1^{n-1}\,a_2,$$

where

$$(n-1)\,a_1 + a_2 = w.$$

Assuming a_1 and a_2 to be undetermined integers and remembering the law of operation of D_w, we have

$$D_w\,(a_1^{n-1}\,a_2)^n = n\,(a_1^{n-2}\,a_2)^{n-1}\,(a_1^{n-1}).$$

The coefficient n indicates that for unreduced rectangles there are n possible first rows, viz.:—the n permutations of

$$a_1^{n-1}\,a_2,$$

and

$$D_w^2(a_1^{n-1}\,a_2)^n = n\,(n-1)\,(a_1^{n-3}\,a_2)^{n-2}\,(a_1^{n-2})^2,$$

the coefficient $n(n-1)$ shewing that there are $n(n-1)$ possible pairs of two first rows in unreduced rectangles.

Also

$$D_w^s(a_1^{n-1}\,a_2)^n = \frac{n!}{(n-s)!}\,(a_1^{n-s-1}\,a_2)^{n-s}\,(a_1^{n-s})^s,$$

giving $\frac{n!}{(n-s)!}$ unreduced rectangles of s rows.

Hence

$$D_w^{n-1}\,(a_1^{n-1}\,a_2)^n = n!\,(a_2)\,(a_1)^{n-1}$$

$$D_w^n\,(a_1^{n-1}\,a_2)^n = n!$$

intimating (as is otherwise immediately evident) that the number of unreduced rectangles of $n-1$ rows or of squares is $n!$.

To enumerate the reduced rectangles observe that in Row 1 we have one place for a_2 instead of n places; in Row 2, $n-2$ places instead of $n-1$; in Row 3, $n-3$ instead of $n-2$, &c.

Therefore for the rectangle of s rows we have a divisor

$$\frac{n}{1}\cdot\frac{n-1}{n-2}\cdot\frac{n-2}{n-3}\cdots\cdots\frac{n-s+1}{n-s},$$

which is

$$\frac{n!}{(n-s)!}\cdot\frac{(n-s-1)!}{(n-2)!}.$$

Therefore the reduced rectangles, of s rows, are in number

$$\frac{(n-2)!}{(n-s-1)!}\qquad (s<n).$$

If $s = n$, the case of the square, the last fraction factor of the divisor must be omitted and we find the number

$$(n-2)!.$$

Suppose the symmetric functions to appertain to the quantities

$$\alpha_1,\ \alpha_2,\ \alpha\ .\ \ldots\ldots\ \alpha_n,$$

and in the identity

$$D_w^s (a_1^{n-1} a_2)^n = \frac{n!}{(n-s)!} (a_1^{n-s-1} a_2)^{n-s} (a_1^{n-s})^s,$$

suppose

$$\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 1.$$

Then
$$\frac{n!}{(n-s)!} = \left\{\frac{n!}{(n-s-1)!\,s!}\right\}^{-(n-s)} \left\{\frac{n!}{(n-s)!\,s!}\right\}^{-s} \times [D_w^s (a_1^{n-2} a_2)^n]_{\alpha_1=\alpha_2=\ldots=\alpha_n=1},$$

for

$$s \not> n-1.$$

When $s = n-1$ we have also the case of the Square as before remarked.

The right-hand side is therefore an analytical expression for the number of unreduced rectangles and we have merely to divide by $\frac{n!}{(n-s)!} \cdot \frac{(n-s-1)!}{(n-2)!}$ to obtain that of the reduced rectangles.

Art. 19. This simple case has been worked out to shew that the desired number can be obtained as the result of definite algebraical processes performed upon a certain symmetric function. In the actual working it is essential to select a_1 and a_2 in such wise that

$$a_1^{n-1} a_2$$

is the *only* partition of

$$w = (n-1) a_1 + a_2$$

into n or fewer parts drawn from the symbols a_1 and a_2 each any number of times repeated.

It will be found that the simplest system is

$$a_1 = 1,\ a_2 = n,$$

necessitating the consideration of the symmetric function

$$(n 1^{n-1})^n.$$

Art. 20. If we next proceed to enumerate the Latin Rectangles of

$$a_1^{n-2} a_2^2$$

we find that the Square enumeration is most easily expressed, those connected with Rectangles having complicated expressions.

The reason for this will be obvious from the results

$$D_w(a_1^{n-2}a_2^2)^n = \binom{n}{2}(a_1^{n-3}a_2^2)^{n-2}(a_1^{n-2}a_2)^2,$$

$$D_w^2(a_1^{n-2}a_2^2)^n = \binom{n}{2}\Big\{(a_1^{n-4}a_2^2)^{n-2}(a_1^{n-2}a_2^2)$$
$$+ 2(n-2)(a_1^{n-4}a_2^2)^{n-3}(a_1^{n-3}a_2)^2(a_1^{n-2})$$
$$+ \binom{n-2}{2}(a_1^{n-4}a_2^2)^{n-4}(a_1^{n-3}a_2)^4\Big\},$$

for it will be seen that the right-hand side of the identity just obtained, as containing terms of three different types, is not simply evaluated for unit values of the quantities

$$\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n,$$

and the terms will not have a single type until we reach D_w^{n-1} which is the case of the square.

Then every term is some permutation of

$$(a_1)^{n-2}(a_2)^2.$$

If K be the *whole* number of these terms, K is the number of unreduced squares, and then putting

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1,$$

we obtain

$$K = n^{-n}[D_w^{n-1}(a_1^{n-2}a_2^2)^n]_{\alpha_1=\alpha_2=\ldots=\alpha_n=1}$$
$$= D_w^n(a_1^{n-2}a_2^2)^n,$$

and dividing K by

$$\frac{n!}{(n-2)!\,2!} \times \frac{(n-1)!}{(n-3)!\,2!}$$

we obtain the number of reduced squares.

Art. 21. We get a precisely similar result for the enumeration of the squares derived from

$$a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k} \qquad \begin{pmatrix}\Sigma s = k \\ \Sigma sa = w\end{pmatrix},$$

viz. for the unreduced squares

$$K = n^{-n}[D_w^{n-1}(a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k})^n],$$

and then division by

$$\frac{n!}{s_1!\,s_2!\ldots s_k!} \times \frac{(n-1)!}{(s_1-1)!\,s_2!\ldots s_k!}$$

gives the number of reduced squares.

The choice of $a_1, a_2, \ldots a_k$ is determined by the circumstance that, for the validity of the process, $w = \Sigma sa$ must possess no partition of n into n or fewer parts drawn from the set

$$a_1, a_2, \ldots a_k,$$

each repeatable as many as n times ($n = \Sigma s$), except

$$(a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k}).$$

This condition is satisfied if

$$a_2 > s_1 a_1,$$
$$a_3 > s_1 a_1 + s_2 a_2,$$
$$\cdots\cdots\cdots\cdots$$
$$a_k > s_1 a_1 + s_2 a_2 + \ldots + s_{k-1} a_{k-1}.$$

Putting therefore $a_1 = 1$, we can take

$$a_1 = 1,$$
$$a_2 = s_1 + 1,$$
$$a_3 = (s_1 + 1)(s_2 + 1),$$
$$a_4 = (s_1 + 1)(s_2 + 1)(s_3 + 1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_k = (s_1 + 1)(s_2 + 1) \ldots (s_{k-1} + 1),$$

and the symmetric function to be considered is

$$\{1^{s_1}\, \overline{s_1 + 1}^{\,s_2}\, \overline{(s_1 + 1)(s_2 + 1)}^{\,s_3} \ldots\ldots \overline{(s_1 + 1)(s_2 + 1) \ldots (s_{k-1} + 1)}^{\,s_k}\}.$$

Observe that the partition, which here presents itself, is of necessity a perfect partition.

Ex. gr. To determine the Latin Squares on the base *aabb*, we take the function

$$(a^2 b^2)^4,$$

or

$$(3^2 1^2)^4,$$

$$(3^2 1^2) = p_2^2 p_4 - 2p_1 p_3 p_4 + 2p_4^2 - p_1 p_2 p_5 + 3p_3 p_5 + 5p_1^2 p_6 - 9p_2 p_6 - 5p_1 p_7 + 12p_8,$$

$$D_8 = \frac{1}{8!}\left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + p_2 \frac{d}{dp_3} + p_3 \frac{d}{dp_4} + p_4 \frac{d}{dp_5} + p_5 \frac{d}{dp_6} + p_6 \frac{d}{dp_7} + p_7 \frac{d}{dp_8}\right)^8,$$

and I find

$$D_8\,(3^2 1^2)^4 = 6\,(3^2 1)^2 (31^2)^2,$$
$$D_8^2\,(3^2 1^2)^4 = 6\,\{(3^2)^2 (1^2)^2 + 4\,(3^2)(31)^2 (1^2) + (31)^4\},$$
$$D_8^3\,(3^2 1^2)^4 = 90\,(3)^2 (1)^2,$$
$$D_8^4\,(3^2 1^2)^4 = 90.$$

Hence the number of unreduced Latin Squares is 90, and to obtain the reduced forms we divide by $\binom{4}{2}\binom{3}{1} = 18$ and obtain 5 for the number of reduced squares.

These are

a a b b	*a a b b*	*a a b b*
a a b b	*a b a b*	*a b b a*
b b a a	*b a b a*	*b b a a*
b b a a	*b b a a*	*b a a b*

a a b b	*a a b b*
a b a b	*a b b a*
b b a a	*b a a b*
b a b a	*b b a a*

SECTION 4.

Art. 22. Let us take into consideration the Græco-Latin Square of Euler (see Introduction).

Instead of Greek letters I find it more convenient to use accented Latin letters, so that for instance a Græco-Latin Square is

$$\begin{matrix} a^{a'} & b^{c'} & c^{b'}, \\ b^{b'} & c^{a'} & a^{c'}, \\ c^{c'} & a^{b'} & b^{a'}, \end{matrix} \quad \text{or more conveniently} \quad \begin{matrix} aa' & bc' & cb' \\ bb' & ca' & ac' \\ cc' & ab' & ba'. \end{matrix}$$

I remark that the Latin and accented Latin letters form, separately, Latin Squares, and that two other Latin Squares are obtainable,

(1) by taking the bases to the exponents a', b', c' in succession,

(2) by taking the exponents to the bases a, b, c in succession.

In order to apply to the question the method of this paper it is necessary to construct suitable operators and operands for use in the master operator theorem of § 1.

It is necessary to form symmetric functions of two systems of quantities

$$\alpha_1 \ \alpha_2 \ \dots \ \alpha_n,$$
$$\alpha_1' \ \alpha_2' \ \dots \ \alpha_n'.$$

Write
$$(1 + \alpha_1 x + \alpha_1' y)(1 + \alpha_2 x + \alpha_2' y) \dots (1 + \alpha_n x + \alpha_n' y)$$
$$= 1 + p_{10} x + p_{01} y + \dots + p_{ww'} x^w y^{w'} + \dots,$$
$$g_{10} = \Sigma p_{w-1,\, w'} \, \partial p_{ww'}; \quad g_{01} = \Sigma p_{w,\, w'-1} \, \partial p_{ww'},$$
$$G_{ww'} = \frac{1}{w!\, w'!} g_{10}{}^{w} g_{01}{}^{w'},$$

where $g_{10}{}^{w} g_{01}{}^{w'}$ denotes that the multiplication of operators is symbolic, or non-operational, as in the symbolic form of Taylor's Theorem.

The reader should refer to the author's paper "Memoir on Symmetric Functions of the Roots of Systems of Equations," *Phil. Trans. R. S.*, 181 A, 1890, § 3, p. 488 et seq.

Denote the symmetric function

$$\Sigma \alpha_1^{a_1} \alpha_1'^{a_1'} \alpha_2^{a_2} \alpha_2'^{a_2'} \alpha_3^{a_3} \alpha_3'^{a_3'} \dots$$

by
$$(\overline{a_1 a_1'} \ \overline{a_2 a_2'} \ \overline{a_3 a_3'} \dots),$$

and observe the results given (*loc. cit.*, p. 490),

$$G_{a_1 a_1'} (\overline{a_1 a_1'} \ \overline{a_2 a_2'} \ \overline{a_3 a_3'} \dots) = (\overline{a_2 a_2'} \ \overline{a_3 a_3'} \dots),$$
$$G_{a_1 a_1'} (\overline{a_1 a_1'}) = 1,$$
$$G_{a_1 a_1'} G_{a_2 a_2'} \dots G_{a_n a_n'} (\overline{a_1 a_1'} \ \overline{a_2 a_2'} \dots \overline{a_n a_n'}) = 1.$$

Also (see § 10, p. 516 et seq.) if

$$f_1,\ f_2,\ f_3, \dots f_m$$

denote, each, any symmetric functions and

$$f = f_1 f_2 f_3 \dots f_m,$$

$$G_{ww'} = \Sigma\Sigma\,(G_{\alpha_1\alpha_1'}f_1)(G_{\alpha_2\alpha_2'}f_2)\dots(G_{\alpha_s\alpha_s'}f_s)f_{s+1}\dots f_m.$$

where the double summation is for every partition

$$(\overline{\alpha_1\alpha_1'}\ \overline{\alpha_2\alpha_2'} \dots \overline{\alpha_s\alpha_s'})$$

of the bipartite number $\overline{ww'}$,

and for every permutation of the m suffixes of the functions $f_1, f_2, f_3, \dots f_m$.

We may denote the operation indicated by the single summation

$$\Sigma\,(G_{\alpha_1\alpha_1'}f_1)(G_{\alpha_2\alpha_2'}f_2)\dots(G_{\alpha_s\alpha_s'}f_s)f_{s+1}\dots f_m$$

by

$$G_{(\overline{\alpha_1\alpha_1'}\ \overline{\alpha_2\alpha_2'} \dots \overline{\alpha_s\alpha_s'})},$$

so that there is the operator equivalence

$$G_{ww'} = \Sigma G_{(\overline{\alpha_1\alpha_1'}\ \overline{\alpha_2\alpha_2'} \dots \overline{\alpha_s\alpha_s'})}$$

the summation having regard to every partition of the bipartite $\overline{ww'}$.

Let

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = w$$
$$\alpha_1' + \alpha_2' + \dots + \alpha_n' = w',$$

and suppose the integers

$$\alpha_1,\ \alpha_2, \dots \alpha_n$$
$$\alpha_1',\ \alpha_2', \dots \alpha_n'$$

so chosen that on the one hand w possesses the single partition $(\alpha_1\alpha_2 \dots \alpha_n)$ composed of n or fewer parts drawn from the parts $\alpha_1, \alpha_2, \dots \alpha_n$ repetitions permissible, and on the other hand w' possesses the single partition $(\alpha_1'\alpha_2' \dots \alpha_n')$ composed of n or fewer parts drawn from the parts $\alpha_1', \alpha_2', \dots \alpha_n'$ repetitions permissible. Then we have

$$G_{ww'} = \Sigma G_{(\overline{\alpha_{s_1}\alpha_{t_1}'}\ \overline{\alpha_{s_2}\alpha_{t_2}'} \dots \overline{\alpha_{s_n}\alpha_{t_n}'})}$$

where

$$s_1 s_2 \dots s_n,\quad t_1 t_2 \dots t_n$$

are some permutations of $1 \, . \, 2 \, . \, 3 \dots n$ respectively, and the summation is in regard to every association of a permutation

$$s_1 s_2 \dots s_n$$

with a permutation

$$t_1 t_2 \dots t_n.$$

Art. 23. First take

$$n = 3$$
$$w = \alpha_1 + \alpha_2 + \alpha_3$$
$$w' = \alpha_1' + \alpha_2' + \alpha_3',$$

and, as operand, the product

$$(\overline{\alpha_1\alpha_1'}\ \overline{\alpha_2\alpha_2'}\ \overline{\alpha_3\alpha_3'})\ (\overline{\alpha_1\alpha_2'}\ \overline{\alpha_2\alpha_3'}\ \overline{\alpha_3\alpha_1'})\ (\overline{\alpha_1\alpha_3'}\ \overline{\alpha_2\alpha_1'}\ \overline{\alpha_3\alpha_2'})$$

where a_1, a_2, a_3 are in the same order in each factor, and the dashed letters in successive factors, being written in successive lines, a Latin Square is formed, viz.:—

$$\begin{matrix} a_1' & a_2' & a_3' \\ a_2' & a_3' & a_1' \\ a_3' & a_1' & a_2' \end{matrix}$$

We find

$$G_{ww'}\left(a_1a_1'\ \overline{a_2a_2'}\ \overline{a_3}a_3'\right)\left(\overline{a_1a_2'}\ \overline{a_2}a_3'\ \overline{a_3a_1'}\right)\left(a_1\overline{a_3'}\ a_2a_1'\ a_3a_2'\right)$$

$$=\left(a_2a_2'\ a_3a_3'\right)\left(\overline{a_1a_2'}\ \overline{a_3a_1'}\right)\left(\overline{a_1a_3'}\ a_2a_1'\right)$$

$$+\left(a_1a_1'\ \overline{a_3a_3'}\right)\left(\overline{a_1a_2'}\ a_2a_3'\right)\left(\overline{a_2a_1'}\ \overline{a_3a_2'}\right)$$

$$+\left(\overline{a_1a_1'}\ \overline{a_2a_2'}\right)\left(a_2a_3'\ \overline{a_3a_1'}\right)\left(\overline{a_1a_3'}\ a_3a_2'\right)$$

three terms respectively derived from the partition operators

$$G_{(a_1a_1'\ \overline{a_2a_3'}\ \overline{a_3a_2'})}$$

$$G_{(a_2a_2'\ a_3a_1'\ \overline{a_1a_3'})}$$

$$G_{(a_3a_3'\ \overline{a_1a_2'}\ a_2a_1')}.$$

Operating again with $G_{ww'}$ we obtain

$$\left(\overline{a_3a_3'}\right)\left(\overline{a_1a_2'}\right)\left(\overline{a_2a_1'}\right)+\left(a_2a_2'\right)\left(a_3a_1'\right)\left(a_1a_3'\right)$$

$$+\left(\overline{a_1a_1'}\right)\left(\overline{a_2a_3'}\right)\left(\overline{a_3a_2'}\right)+\left(a_3a_3'\right)\left(a_1a_2'\right)\left(a_2a_1'\right)$$

$$+\left(a_2a_2'\right)\left(\overline{a_3a_1'}\right)\left(\overline{a_1a_3'}\right)+\left(\overline{a_1a_1'}\right)\left(a_2a_3'\right)\left(a_3a_2'\right)$$

terms respectively derived from the operators

$$G_{(a_2\overline{a_2'}\ \overline{a_3a_1'}\ a_1a_3')},\quad G_{(\overline{a_3a_3'}\ \overline{a_1a_2'}\ a_2a_1')}$$

$$G_{\overline{a_3a_3'}\ a_1\overline{a_2'}\ \overline{a_2a_1'}},\quad G_{(a_1\overline{a_1'}\ a_2a_3'\ a_3\overline{a_2'})}$$

$$G_{(\overline{a_1a_1'}\ \overline{a_2a_3'}\ \overline{a_3a_2'})},\quad G_{(\overline{a_2a_2'}\ a_3a_1'\ a_1a_3')}.$$

Operating again, on each term with the corresponding partition operator, we obtain the number 6.

We have obtained this number from the six combinations of operators

$$\left\{\begin{matrix} G_{(\overline{a_1a_1'}\ \overline{a_2a_3'}\ \overline{a_3a_2'})} \\ G_{(\overline{a_3a_3'}\ a_1\overline{a_2'}\ \overline{a_2a_1'})} \\ G_{(\overline{a_2a_2'}\ \overline{a_3a_1'}\ \overline{a_1a_3'})} \end{matrix}\right\}\left\{\begin{matrix} G_{(\overline{a_2a_2'}\ a_3a_1'\ a_1\overline{a_3'})} \\ G_{(\overline{a_1a_1'}\ a_2a_3'\ a_3a_2')} \\ G_{(\overline{a_3a_3'}\ a_1a_2'\ \overline{a_2a_1'})} \end{matrix}\right\}\left\{\begin{matrix} G_{(\overline{a_3a_3'}\ a_1a_2'\ a_2a_1')} \\ G_{(a_2\overline{a_2'}\ a_3a_1'\ a_1\overline{a_3'})} \\ G_{(\overline{a_1a_1'}\ a_2\overline{a_3'}\ \overline{a_3a_1'})} \end{matrix}\right\}$$

$$\left\{\begin{matrix} G_{(\overline{a_1a_1'}\ \overline{a_2a_3'}\ \overline{a_3a_2'})} \\ G_{(\overline{a_2a_2'}\ \overline{a_3a_1'}\ \overline{a_1a_3'})} \\ G_{(\overline{a_3a_3'}\ a_1\overline{a_2'}\ \overline{a_2a_1'})} \end{matrix}\right\}\left\{\begin{matrix} G_{(\overline{a_2a_2'}\ a_3a_1'\ \overline{a_1a_3'})} \\ G_{(\overline{a_3a_3'}\ a_1a_2'\ \overline{a_2a_1'})} \\ G_{(\overline{a_1a_1'}\ a_2\overline{a_3'}\ \overline{a_3a_2'})} \end{matrix}\right\}\left\{\begin{matrix} G_{(\overline{a_3a_3'}\ \overline{a_1a_2'}\ \overline{a_2}a_1')} \\ G_{(a_1\overline{a_1'}\ a_2\overline{a_3'}\ \overline{a_3a_2'})} \\ G_{(\overline{a_2a_2'}\ a_3a_1'\ \overline{a_1a_3'})} \end{matrix}\right\}$$

and to these correspond the six Græco-Latin Squares

$$\begin{array}{ccc} a_1a_1' & a_2a_3' & a_3a_2' \\ a_3a_3' & a_1a_2' & a_2a_1' \\ a_2a_2' & a_3a_1' & a_1a_3' \end{array} \qquad \begin{array}{ccc} a_2a_2' & a_3a_1' & a_1a_3' \\ a_1a_1' & a_2a_3' & a_3a_2' \\ a_3a_3' & a_1a_2' & a_2a_1' \end{array} \qquad \begin{array}{ccc} a_3a_3' & a_1a_2' & a_2a_1' \\ a_2a_2' & a_3a_1' & a_1a_3' \\ a_1a_1' & a_2a_3' & a_3a_2' \end{array}$$

$$\begin{array}{ccc} a_1a_1' & a_2a_3' & a_3a_2' \\ a_2a_2' & a_3a_1' & a_1a_3' \\ a_3a_3' & a_1a_2' & a_2a_1' \end{array} \qquad \begin{array}{ccc} a_2a_2' & a_3a_1' & a_1a_3' \\ a_3a_3' & a_1a_2' & a_2a_1' \\ a_1a_1' & a_2a_3' & a_3a_2' \end{array} \qquad \begin{array}{ccc} a_3a_2' & a_1a_2' & a_2a_1' \\ a_1a_1' & a_2a_3' & a_3a_2' \\ a_2a_2' & a_3a_1' & a_1a_3' \end{array}$$

By forming the operand, as above, we have insisted upon the left-hand column of the square involving only the three products

$$a_1a_1', \qquad a_2a_2', \qquad a_3a_3',$$

but by permuting a_1', a_2', a_3' we get an additional factor 6 and by permuting the 2nd and 3rd columns a further factor 2 ! we find that the unreduced number of Græco-Latin Squares of order 3 is

$$6 \times 6 \times 2 = 72.$$

If we insist upon the suffixes appearing in numerical order in the left-hand column for both undashed and dashed letters and also in the top row in the case of undashed letters, we obtain the reduced squares. In this instance there is but one, viz.

$$\begin{array}{ccc} a_1a_1' & a_2a_3' & a_3a_2' \\ a_2a_2' & a_3a_1' & a_1a_3' \\ a_3a_3' & a_1a_2' & a_2a_1'. \end{array}$$

In general the enumeration of the reduced squares is obtained by dividing the number of unreduced squares by

$$(n\,!)^2\,(n-1)\,!\,.$$

Above an operand was formed corresponding to the single reduced Latin Square,

$$\begin{array}{ccc} a_1' & a_2' & a_3' \\ a_2' & a_3' & a_1' \\ a_3' & a_1' & a_2' \end{array}$$

of order 3.

Operating with $G^3_{ww'}$, viz.:—three times successively with $G_{ww'}$, we obtained the number 6 and this has been shewn to give 3 ! times the number of reduced Græco-Latin Squares for, as remarked above, 3 ! is the number of ways in which the products a_1a_1', a_2a_2', a_3a_3' can be permuted.

Hence the number of reduced Græco-Latin Squares is

$$\frac{6}{3\,!} = 1.$$

In general we shall find that we must form an operand corresponding with each reduced Latin Square in the dashed letters, operate upon each, with $G_{ww'}$ n times successively, take the sum of the resulting numbers, and divide by $n!$.

The result will be the number of reduced Græco-Latin Squares of order n.

Art. 24. To elucidate the matter I will work out (not quite in full) the case of order 4 and deduce the reduced Græco-Latin Squares. There are four operands, since there are four reduced Latin Squares of order 4, viz. :—

$$\begin{array}{cccc|cccc|cccc|cccc}
a_1' & a_2' & a_3' & a_4' & a_1' & a_2' & a_3' & a_4' & a_1' & a_2' & a_3' & a_4' & a_1' & a_2' & a_3' & a_4' \\
a_2' & a_3' & a_4' & a_1' & a_2' & a_1' & a_4' & a_3' & a_2' & a_1' & a_4' & a_3' & a_2' & a_4' & a_1' & a_3' \\
a_3' & a_4' & a_1' & a_2' & a_3' & a_4' & a_1' & a_2' & a_3' & a_4' & a_2' & a_1' & a_3' & a_1' & a_4' & a_2' \\
a_4' & a_1' & a_2' & a_3' & a_4' & a_3' & a_2' & a_1' & a_4' & a_3' & a_1' & a_2' & a_4' & a_3' & a_2' & a_1'
\end{array}$$

These are

$$(A)\quad (\overline{a_1a_1'}\ \overline{a_2a_2'}\ \overline{a_3a_3'}\ \overline{a_4a_4'})(\overline{a_1a_2'}\ \overline{a_2a_3'}\ \overline{a_3a_4'}\ \overline{a_4a_1'})(\overline{a_1a_3'}\ \overline{a_2a_4'}\ \overline{a_3a_1'}\ \overline{a_4a_2'})(\overline{a_1a_4'}\ \overline{a_2a_1'}\ \overline{a_3a_2'}\ \overline{a_4a_3'}),$$

$$(B)\quad (\overline{a_1a_1'}\ \overline{a_2a_2'}\ \overline{a_3a_3'}\ \overline{a_4a_4'})(\overline{a_1a_2'}\ \overline{a_2a_1'}\ \overline{a_3a_4'}\ \overline{a_4a_3'})(\overline{a_1a_3'}\ \overline{a_2a_4'}\ \overline{a_3a_1'}\ \overline{a_4a_2'})(\overline{a_1a_4'}\ \overline{a_2a_3'}\ \overline{a_3a_2'}\ \overline{a_4a_1'}),$$

$$(C)\quad (\overline{a_1a_1'}\ \overline{a_2a_2'}\ \overline{a_3a_3'}\ \overline{a_4a_4'})(\overline{a_1a_2'}\ \overline{a_2a_1'}\ \overline{a_3a_4'}\ \overline{a_4a_3'})(\overline{a_1a_3'}\ \overline{a_2a_4'}\ \overline{a_3a_2'}\ \overline{a_4a_1'})(\overline{a_1a_4'}\ \overline{a_2a_3'}\ \overline{a_3a_1'}\ \overline{a_4a_2'}),$$

$$(D)\quad (\overline{a_1a_1'}\ \overline{a_2a_2'}\ \overline{a_3a_3'}\ \overline{a_4a_4'})(\overline{a_1a_2'}\ \overline{a_2a_4'}\ \overline{a_3a_1'}\ \overline{a_4a_3'})(\overline{a_1a_3'}\ \overline{a_2a_1'}\ \overline{a_3a_4'}\ \overline{a_4a_2'})(\overline{a_1a_4'}\ \overline{a_2a_3'}\ \overline{a_3a_2'}\ \overline{a_4a_1'}).$$

The operation of $D^4_{ww'}$ upon (A), (C) and (D) causes them to vanish and on (B) the result is 48; hence the number of reduced Græco-Latin Squares is

$$\frac{48}{4!} = 2.$$

These are

$$\begin{array}{cccc|cccc}
a_1a_1' & a_2a_4' & a_3a_2' & a_4a_3' & a_1a_1' & a_2a_3' & a_3a_4' & a_4a_2' \\
a_2a_2' & a_1a_3' & a_4a_1' & a_3a_4' & a_2a_2' & a_1a_4' & a_4a_3' & a_3a_1' \\
a_3a_3' & a_4a_2' & a_1a_4' & a_2a_1' & a_3a_3' & a_4a_1' & a_1a_2' & a_2a_4' \\
a_4a_4' & a_3a_1' & a_2a_3' & a_1a_2' & a_4a_4' & a_3a_2' & a_2a_1' & a_1a_3'
\end{array}$$

Observe that the undashed Latin Square is the same in both cases and that the second square is obtainable from the first by a cyclical interchange of the 2nd, 3rd and 4th columns of dashed letters. By the method we can, by regular process, determine the number of Græco-Latin Squares appertaining to any given Latin Square. The enumeration, however, in all but the simplest cases is so laborious as to be impracticable. And I do not see the way to prove that no Græco-Latin of the order 6, and generally of order $\equiv 2 \bmod 4$, exists.

SECTION 5.

Art. 25. It naturally occurs to one to seek other systems of operators and operands which lead to the solution of interesting problems. In the master theorem

$$\frac{(u_1^{\chi_1}u_2^{\chi_2}\dots u_s^{\chi_s})\,\phi}{\chi_1!\,\chi_2!\dots\chi_s!} = \Sigma\Sigma\frac{(u_1^{\alpha_1}u_2^{\alpha_2}\dots u_s^{\alpha_s})\,\phi_1}{\alpha_1!\,\alpha_2!\dots\alpha_s!}\frac{(u_1^{\beta_1}u_2^{\beta_2}\dots u_s^{\beta_s})\,\phi_2}{\beta_1!\,\beta_2!\dots\beta_s!}\dots\frac{(u_1^{\mu_1}u_2^{\mu_2}\dots u_s^{\mu_s})\,\phi_m}{\mu_1!\,\mu_2!\dots\mu_s!},$$

where $$\phi = \phi_1\phi_2\dots\phi_m,$$

put $$\phi = \theta^m, \quad u_t = \partial_{x_t},$$

$$\chi_1 = \chi_2 = \dots = \chi_s = 1,$$

and $$s = n.$$

Then

$$(\partial_{x_1}\partial_{x_2}\dots\partial_{x_n})\,\theta^n = \Sigma\Sigma\,(X_1)\,\theta\,(X_2)\,\theta\dots(X_k)\,\theta\,.\,\theta^{n-k},$$

where $$X_1X_2\dots X_k = \partial_{x_1}\partial_{x_2}\dots\partial_{x_n},$$

and as usual we must take every factorization of the operator and then distribute the operations upon the right-hand side in all possible ways.

Take $\theta = x_1x_2\dots x_n$ and putting $n=3$, we have

$$\begin{aligned}(\partial_{x_1}\partial_{x_2}\partial_{x_3})(x_1x_2x_3)^3 &= (\partial_{x_1}\partial_{x_2}\partial_{x_3})(x_1x_2x_3)\,.\,(x_1x_2x_3)\,.\,(x_1x_2x_3)\\ &+ (x_1x_2x_3)\,.\,(\partial_{x_1}\partial_{x_2}\partial_{x_3})(x_1x_2x_3)\,.\,(x_1x_2x_3)\\ &+ (x_1x_2x_3)\,.\,(x_1x_2x_3)\,.\,(\partial_{x_1}\partial_{x_2}\partial_{x_3})(x_1x_2x_3)\\ &+ (\partial_{x_1}\partial_{x_2})(x_1x_2x_3)\,.\,\partial_{x_3}(x_1x_2x_3)\,.\,(x_1x_2x_3)\\ &+ 5\text{ similar terms}\\ &+ (\partial_{x_2}\partial_{x_3})(x_1x_2x_3)\,.\,\partial_{x_1}(x_1x_2x_3)\,.\,(x_1x_2x_3)\\ &+ 5\text{ similar terms}\\ &+ \partial_{x_3}\partial_{x_1}(x_1x_2x_3)\,.\,\partial_{x_2}(x_1x_2x_3)\,.\,(x_1x_2x_3)\\ &+ 5\text{ similar terms}\\ &+ \partial_{x_1}(x_1x_2x_3)\,.\,\partial_{x_2}(x_1x_2x_3)\,.\,\partial_{x_3}(x_1x_2x_3)\\ &+ 5\text{ similar terms.}\end{aligned}$$

In all $27\,(=3^3)$ terms corresponding to the 27 permuted partitions of $x_1x_2x_3$ into exactly 3 parts, zero being reckoned as a part.

Selecting any term on the right-hand side, say

$$\partial_{x_1}\partial_{x_2}(x_1x_2x_3)\,.\,\partial_{x_3}(x_1x_2x_3)\,.\,(x_1x_2x_3)$$

we obtain $$(x_3)\,.\,(x_1x_2)\,.\,(x_1x_2x_3),$$

and if we were to proceed to perform the operation $\partial_{x_1}\partial_{x_2}\partial_{x_3}$ a second time, of the whole number of 27 operations, into which the operator is seen to break up, only a certain number will be effective in producing a non-zero term.

We are subject to the conditions

(1) 1st operator factor must not contain ∂_{x_1} or ∂_{x_2},

(2) 2nd factor must not contain ∂_{x_3}.

As one operation we can take

$$(x_3)\,.\,\partial_{x_1}\partial_{x_2}(x_1x_2)\,.\,\partial_{x_3}(x_1x_2x_3),$$

resulting in

$$(x_3)\,.\,(\cdot)\,(x_1x_2).$$

Again operating with $\partial_{x_1}\partial_{x_2}\partial_{x_3}$, we find that only one of the 27 operations can be performed, and we have

$$\partial_{x_3}(x_3)\,.\,(\cdot)\,.\,\partial_{x_1}\partial_{x_2}(x_1x_2),$$

resulting in

$$(\cdot)\,.\,(\cdot)\,.\,(\cdot).$$

Forming a square table of these operations we find

$\partial_{x_1}\partial_{x_2}$	∂_{x_3}	.
.	$\partial_{x_1}\partial_{x_2}$	∂_{x_3}
∂_{x_3}	.	$\partial_{x_1}\partial_{x_2}$.

and it will be seen that each of the three operators ∂_{x_1}, ∂_{x_2}, ∂_{x_3} occurs exactly once in each row and in each column.

This feature is a necessary result of the process.

Art. 26. We may symbolise the above taken successive differential operations by the scheme

12	3	
	12	3
3		12

and selecting the operations in any manner possible, so that an annihilating effect is not produced, we will obtain a Square of order 3, having the property that each of the three numbers 1, 2, 3 appears exactly once in each row and column without restriction in regard to the number of them that may appear in each compartment.

We have in fact the Latin Square freed from the condition that one letter must appear in each compartment. Hence it is seen that these squares are enumerated by the number of terms which survive the operations performed on the right-hand side of the identity after three successive operations of $\partial_{x_1}\partial_{x_2}\partial_{x_3}$.

Therefore the enumeration is given by

$$(\partial_{x_1}\partial_{x_2}\partial_{x_3})^3 (x_1^3 x_2^3 x_3^3) = (3!)^3.$$

In general for the order n the enumeration of these squares is given by

$$(\partial_{x_1}\partial_{x_2} \ldots \partial_{x_n})^n (x_1^n x_2^n \ldots x_n^n) = (n!)^n.$$

We may state the problem in the following way:—

"n^2 different towns form a square and there are n^2 inspectors, n of each of n different nationalities. Find the number of arrangements of the inspectors in the towns subject to the condition that one inspector of each nationality must be in each row or column of towns combined with the circumstance that no restriction is placed upon the number of inspectors that may be stationed in a particular town."

The result is, as shewn, $(n!)^n$.

Art. 27. We may also consider the operator

$$(\partial_{x_1}\partial_{x_2} \ldots \partial_{x_n})^m,$$

in conjunction with the operand

$$(x_1 x_2 \ldots x_n)^m,$$

where

$$m \gtrless n.$$

Thus, in particular, taking $(\partial_{x_1}\partial_{x_2})^3 (x_1 x_2)^3$, we find arrangements such as

1		2
	12	
2		1

which is a square of 3^2 compartments, and the numbers 1, 2 are arranged, in such manner, that each is contained once in each row and in each column.

The enumeration is given by

$$(\partial_{x_1}\partial_{x_2})^3 (x_1 x_2)^3 = (3!)^2,$$

and in general the result

$$(\partial_{x_1}\partial_{x_2} \ldots \partial_{x_n})^m (x_1 x_2 \ldots x_n)^m = (m!)^n$$

shews that, in a square of m^2 compartments, the n numbers 1, 2, 3, ... n can be arranged, in such a manner, that each is contained once in each row and in each column in exactly $(m!)^n$ ways.

Art. 28. It is very interesting to see that these results can also be obtained by means of the symmetric function operators employed in the body of the paper.

For take as operand $\{(a_1)(a_2)(a_3)\ldots(a_n)\}^n$,

where of course $(a_1) = \Sigma\alpha^{a_1}$, &c.;

and as operator $D^n_{a_1+a_2+a_3+\ldots+a_n} = D^n_w$,

where as usual w possesses the single partition $(a_1 a_2 \ldots a_n)$ into n or fewer parts drawn from $a_1, a_2, \ldots a_n$, repetitions permissible.

For simplicity take $n = 3$, and write

$$(a_1,\ a_2,\ a_3) = (a,\ b,\ c).$$

Then $\{(a)(b)(c)\}^3 = \{(a+b+c) + (a+b,\ c) + (a+c,\ b) + (b+c,\ a) + (abc)\}^3$,

and $D_{a+b+c} = D_{(a+b+c)} + D_{(a+b,\ c)} + D_{(a+c,\ b)} + D_{(b+c,\ a)} + D_{(abc)}$,

according to the notation explained above (Art. 4).

Now
$$D^3_{a+b+c}\{(a)(b)(c)\}^3 = D^3_{a+b+c}(a)^3(b)^3(c)^3$$
$$= D^3_{(abc)}(a)^3(b)^3(c)^3 = (3!)^3,$$

and, performing the developed operator upon

$$\{(a+b+c) + (a+b,\ c) + (a+c,\ b) + (b+c,\ a) + (abc)\}^3,$$

we have to consider the 125 terms of which the expanded power is composed.

One of these is $(a+b+c)(a+c,\ b)(abc)$,

and, performing D_{a+b+c}, we obtain two terms,

$$\begin{array}{ccc} \cdot & (a+c,\ b) & (abc) \\ +(a+b+c) & (b) & (ac), \end{array}$$

and now it is easy to see that

$$D^3_{a+b+c}(a+b+c)(a+c,\ b)(abc) = 0.$$

But selecting out of the 125 the term

$$(a+b,\ c)(a+c,\ b)(abc),$$

the operation of D_{a+b+c} produces

$$\begin{array}{ccc} (c) & (a+c,\ b) & (ab) \\ +(a+b,\ c) & (b) & (ac) \\ +(a+b) & (a+c) & (bc); \end{array}$$

the terms corresponding respectively to

$$\begin{array}{ccc} D_{a+b}* & \cdot & * \cdot D_c * \\ \cdot & * D_{a+c} & * \cdot D_b * \\ D_c & * D_b & * \quad D_a *, \end{array}$$

and, selecting the first of the terms produced,

$$D_c * D_b * D_a *$$

yields $(a+c)(b)$;

and now operating again, $* D_{a+c} * D_b *$

yields unity.

Hence we obtain one resulting term corresponding to the operator scheme

$a+b$		c
c	b	a
	$a+c$	b

or say

12		3
3	2	1
	13	2

which is a square having the desired property, viz.:—each row, as well as each column, contains each of the three numbers, without restriction in regard to the number of numbers appearing in a compartment.

And when we carry out the whole process we must arrive at

$$(3!)^3$$

such squares, each square typifying a succession of operations.

Art. 29. Hence we establish the general theorem, above enunciated, as the enumeration is given by

$$D^n_{a_1+a_2+\ldots+a_n}(a_1)^n(a_2)^n\ldots(a_n)^n.$$

which is

$$D^n_{(a_1a_2\ldots a_n)}(a_1^n)(a_2^n)\ldots(a_n^n)\times(n!)^n,$$

or

$$(n!)^n.$$

Clearly, the enlarged theorem corresponds to

$$D^m_{a_1+a_2+\ldots+a_n}(a_1)^m(a_2)^m\ldots(a_n)^m,$$

which gives

$$(m!)^n,$$

as before.

Art. 30. This interesting result shews that we may expect to meet with many pairs of operators and operands differing widely in character which conduct to the same theorem in combinatory analysis. I believe that the method of research, above set forth, is of considerable promise and worthy of the attention of mathematicians. It is probable that known theorems in combinatory analysis will lead conversely to theorems connected with operations which will prove both interesting and valuable.

XII. *On some differential equations in the theory of symmetrical algebra.* By Professor A. R. Forsyth.

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In a recent memoir*, I have discussed the theory of partial differential equations which are of order higher than the first and involve more than two independent variables. Most of the investigations apply, as indicated in the title of the memoir, to equations which are of the second order and involve three independent variables. The cause of this limitation was a desire to secure brevity in the formulæ; it is however evident from the course of the analysis (and there is a statement to this effect) that the investigations apply, *mutatis mutandis*, to equations which are of order m and involve n independent variables.

It is an inference from the theory of partial differential equations there given that the most general solution of an equation of order m in n variables involves m independent arbitrary functions and that each of these functions involves $n-1$ (or fewer) arguments. The arguments are shewn to satisfy an equation which, reproducing itself for all transformations of the independent variables, is called the characteristic invariant; and the form of this equation depends, in the first instance, only upon the aggregate of the derivatives of highest order that exist in the original differential equation. When the original equation is denoted by

$$F(\ldots,\ z_{m_1,\ m_2,\ \ldots,\ m_n},\ldots)=0.$$

(where

$$z_{m_1,\ m_2,\ \ldots,\ m_n}=\frac{\partial^m z}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}},$$

and $m_1+m_2+\ldots+m_n=m$ which is taken to be the order of the equation), then the characteristic invariant is

$$\Sigma\left\{\frac{\partial F}{\partial z_{m_1,\ m_2,\ \ldots,\ m_n}}\left(\frac{\partial u}{\partial x_1}\right)^{m_1}\left(\frac{\partial u}{\partial x_2}\right)^{m_2}\ldots\left(\frac{\partial u}{\partial x_n}\right)^{m_n}\right\}=0,$$

where the summation extends over all terms that arise through derivatives of F with regard to all the partial differential coefficients of z of order m which occur explicitly in F.

* "Memoir on the integration of partial differential equations of the second order in three independent variables when an intermediary integral does not exist in general," read before the Royal Society on 16 December, 1897.

The characteristic invariant is only one of a set of subsidiary equations, which can be obtained as follows. Denoting an argument of an arbitrary function by u, let the independent variables be changed from the set $x_1, x_2, \ldots, x_{n-1}, x_n$ to $x_1, x_2, \ldots, x_{n-1}, u$, differentiation with regard to the latter set being denoted by d and to the former by ∂ when the differential operator is expressed. To effect the change, we may consider x_n as a function of $x_1, \ldots, x_{n-1}, u$, and we write

$$\frac{dx_n}{dx_r} = p_r;$$

accordingly, we have

$$\frac{dz_{s_1, s_2, \ldots, s_n}}{dx_r} = z_{s_1, s_2, \ldots, 1+s_r, s_{r+1}, \ldots, s_n} + p_r z_{s_1, s_2, \ldots, s_{n-1}, 1+s_n},$$

for $r = 1, 2, \ldots, n-1$. By means of the aggregate of these relations, each of the derivatives of order m can be expressed in terms of $z_{0, 0, \ldots, 0, m}$ and of new derivatives with regard to $x_1, x_2, \ldots, x_{n-1}$ of differential coefficients of order $m-1$. Now assuming that the solution of the original equation is of the type known as free from partial quadratures, we have

$$z_{0, 0, \ldots, 0, m} = \left(\frac{1}{\frac{dx_n}{du}} \frac{d}{du}\right)^m z,$$

so that $z_{0, 0, \ldots, 0, m}$ involves derivatives of the arbitrary function with regard to u of order m higher than those which occur in the value of z, while the derivatives of order $m-1$ involve derivatives of the arbitrary function with regard to u only of order $m-1$ higher than those which occur in the value of z. Accordingly, when the transformed equation is arranged in powers of $z_{0, 0, \ldots, 0, m}$, in the form

$$z_{0, 0, \ldots, 0, m}^{\alpha} I + z_{0, 0, \ldots, 0, m}^{\alpha-1} I_1 + z_{0, 0, \ldots, 0, m}^{\alpha-2} I_2 + \ldots = 0,$$

we must have $$I = 0, \quad I_1 = 0, \quad I_2 = 0, \ldots\ldots,$$

the first of these being the characteristic invariant. This is an aggregate of subsidiary equations. If there be any integrable combination, it is proved to be of the nature of an intermediary integral of the original equation; and there may be as many of these as there are distinct values of u, or sets of values of u, determined through the characteristic invariant.

There are various classes of partial differential equations, discriminated according to the resolubility of the invariant into equations of lower degrees in the first derivatives of u. In particular, if the invariant is resoluble into m equations, each linear (and homogeneous) in the derivatives of u, then each of the m arbitrary functions involves $n-1$ arguments, being the $n-1$ functionally independent solutions of the corresponding linear equation. But if the resolubility of the invariant into linear equations is not of this complete character, there is a corresponding declension from the number of arguments in some or all of the arbitrary functions.

If for none of the arguments u there should exist an integrable combination, then we take the deduced equations

$$\frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0, \quad \dots\dots, \quad \frac{\partial F}{\partial x_n} = 0,$$

which must be satisfied identically by the solution of F. This is a set of n equations, each of order $m+1$ and linear in the derivatives of highest order. Each of them is treated by the preceding method as to the derivatives of order $m+1$ instead of order m: it appears that the characteristic invariant of each of them is the same for all, being the characteristic invariant of F: and each of them provides one other equation, so that, in addition to the characteristic invariant, there is an aggregate of n subsidiary equations involving new derivatives with regard to $x_1, x_2, \dots, x_{n-1}$ of the differential coefficients of z of order m. When an integrable combination, other than $F=0$, of the subsidiary equations exists, it leads to an integral equation of order m that can be associated with the original equation: and so for each value of u, or set of values of u, leading to an integrable combination.

If there should be no integrable combination of this set, then we take the deduced equations

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = 0 \dots (i, j = 1, 2, \dots, n).$$

and proceed from derivatives of order $m+2$ as before: the characteristic invariant persisting throughout. Of the corresponding new set of equations, integrable combinations other than

$$F=0, \quad \frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0, \quad \dots\dots, \quad \frac{\partial F}{\partial x_n} = 0$$

are required: if and when obtained, they lead to equations which can be associated with the set just indicated. And so on, for successive systems of deduced equations.

There is another method of proceeding which, dispensing temporarily with the subsidiary equations other than the characteristic invariant, proves effective (partially or wholly) in individual examples. It consists in effecting the actual transformation from

$$x_1, x_2, \dots, x_{n-1}, x_n \text{ to } x_1, x_2, \dots, x_{n-1}, u$$

upon the original equation: when the new form is obtained, it is regarded as a new equation; and it may happen that this new equation can be solved wholly or in part.

Examples of both methods of proceeding are given in the memoir referred to: the special example to which the latter method is applied being Laplace's equation. The purpose of the present paper is to indicate the application of the latter method to a set of equations, the most general solution of which can be obtained explicitly and completely. These are the equations

$$\left(\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \dots\dots + x_{n-1} \frac{\partial}{\partial x_n}\right)^m U = 0,$$

where the multiplication of the operator is symbolical and not operational: they are of importance in the theory of symmetrical algebra*, and there is accordingly reason for obtaining their most general solution.

SINGLE EQUATIONS.

1. As the representative equation is of order m and involves n independent variables, the most general solution must contain m arbitrary functions. Every argument of any one of these functions must satisfy the characteristic invariant of the equation, which is

$$\left(\frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} + x_2 \frac{\partial u}{\partial x_3} + \ldots\ldots + x_{n-1} \frac{\partial u}{\partial x_n}\right)^m = 0.$$

The invariant is resoluble, consisting in fact of the m-fold repetition of an equation of the first degree in the partial derivatives of u; hence the arguments of all the m arbitrary functions are the same, they are $n-1$ in number and, in value, they are any $n-1$ functionally independent solutions of the equation

$$\frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} + x_2 \frac{\partial u}{\partial x_3} + \ldots\ldots + x_{n-1} \frac{\partial u}{\partial x_n} = 0.$$

As the arbitrary functions involve only those $n-1$ quantities and as there is no limitation on the arbitrary character of the functions, it is sufficient for the purpose in view to choose as simple a set of $n-1$ functionally independent solutions as may be possible. These solutions are integrals of the subsidiary system

$$\frac{dx_1}{1} = \frac{dx_2}{x_1} = \frac{dx_3}{x_2} = \ldots\ldots = \frac{dx_n}{x_{n-1}},$$

which can be otherwise represented in the form

$$\frac{dx_s}{dx_1} = x_{s-1} \qquad (s = 2, 3, \ldots, n),$$

and therefore

$$\frac{d^{s-1}x_s}{dx_1^{s-1}} = x_1.$$

Hence the integrals are

$$x_2 = \frac{x_1^2}{2!} + u_2,$$

$$x_3 = \frac{x_1^3}{3!} + u_2 x_1 + u_3,$$

....

$$x_s = \frac{x_1^s}{s!} + \frac{x_1^{s-2}}{(s-2)!} u_2 + \frac{x_1^{s-3}}{(s-3)!} u_3 + \ldots\ldots + x_1 u_{s-1} + u_s,$$

* See MacMahon's memoirs "On a new theory of symmetric functions," *American Journal of Mathematics*, vol. XI (1889), pp. 1—36; *ib.*, vol. XII (1890), pp. 61—102; *ib.*, vol. XIII (1891), pp. 193—234; *ib.*, vol. XIV (1892), pp. 15—38.

for $s = 2, 3, \ldots, n$. When these $n-1$ equations are solved for $u_2, u_3, \ldots, u_n$, they give the explicit forms of the requisite $n-1$ arguments, as follows:

$$u_2 = \frac{1}{2\,!}(x_2', x_1, 1 \between 1, -x_1)^2,$$

$$u_3 = \frac{1}{3\,!}(x_3', x_2', x_1, 1 \between 1, -x_1)^3,$$

....

$$u_s = \frac{1}{s\,!}(x_s', x'_{s-1}, \ldots, x_2', x_1, 1 \between 1, -x_1)^s,$$

for $s = 2, 3, \ldots, n$: the quantities $x_2', x_3', \ldots, x_n'$ being defined by

$$r\,!\; x_r = x_r',$$

for $r = 2, 3, \ldots, n$. The arguments are therefore known quantities.

The similarity of form of the arguments to the leading coefficients of the (Hermite) covariants associated with the quantic

$$(1, x_1, x_2', \ldots, x_n' \between X, Y)^n$$

is complete.

2. Instead of pursuing the general theory, it proves to be more direct (owing to the special character of the equation) to proceed to the final solution by introducing $x_1, u_2, u_3, \ldots, u_n$ as the independent variables in place of $x_1, x_2, x_3, \ldots, x_n$. Denoting any function of the variables by P, and differentiations with regard to the new variables by $\frac{d}{dx_1}, \frac{d}{du_2}, \frac{d}{du_3}, \ldots\ldots, \frac{d}{du_n}$, then in order to obtain the transforming differential relations, we have

$$\frac{dP}{dx_1}dx_1 + \frac{dP}{du_2}du_2 + \frac{dP}{du_3}du_3 + \ldots\ldots + \frac{dP}{du_n}du_n$$

$$= dP$$

$$= \frac{\partial P}{\partial x_1}dx_1$$

$$+ \frac{\partial P}{\partial x_2}(x_1 dx_1 + du_2)$$

$$+ \frac{\partial P}{\partial x_3}(x_2 dx_1 + x_1 du_2 + du_3)$$

$$+ \frac{\partial P}{\partial x_4}\left(x_3 dx_1 + \frac{x_1^2}{2\,!}du_2 + x_1 du_3 + du_4\right)$$

$+$ ....

$$+ \frac{\partial P}{\partial x_s}\left\{x_{s-1}dx_1 + \frac{x_1^{s-2}}{(s-2)\,!}du_2 + \frac{x_1^{s-3}}{(s-3)\,!}du_3 + \ldots\ldots + x_1 du_{s-1} + du_s\right\}$$

$+$ ....

and therefore

$$\frac{d}{dx_1} = \frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4} + \ldots\ldots + x_{s-1}\frac{\partial}{\partial x_s} + \ldots\ldots + x_{n-1}\frac{\partial}{\partial x_n}:$$

$$\frac{d}{du_2} = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + \frac{x_1^2}{2\,!}\frac{\partial}{\partial x_4} + \ldots\ldots + \frac{x_1^{s-2}}{(s-2)!}\frac{\partial}{\partial x_s} + \ldots\ldots + \frac{x_1^{n-2}}{(n-2)!}\frac{\partial}{\partial x_n},$$

$$\frac{d}{du_3} = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + \ldots\ldots + \frac{x_1^{s-3}}{(s-3)!}\frac{\partial}{\partial x_s} + \ldots\ldots + \frac{x_1^{n-3}}{(n-3)!}\frac{\partial}{\partial x_n},$$

$$\frac{d}{du_4} = \frac{\partial}{\partial x_4} + \ldots\ldots + \frac{x_1^{s-4}}{(s-4)!}\frac{\partial}{\partial x_s} + \ldots\ldots + \frac{x_1^{n-4}}{(n-4)!}\frac{\partial}{\partial x_n},$$

....

The operators which, as will presently be seen, are required are, in addition to $\frac{d}{dx_1}$, the set

$$\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4} + x_3\frac{\partial}{\partial x_5} + \ldots$$

$$\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} + x_3\frac{\partial}{\partial x_6} + \ldots$$

$$\frac{\partial}{\partial x_4} + x_1\frac{\partial}{\partial x_5} + x_2\frac{\partial}{\partial x_6} + x_3\frac{\partial}{\partial x_7} + \ldots$$

....

say these are d_2, d_3, d_4, ... ; and denote $\frac{d}{dx_1}$ by d_1, so that the whole series is $d_1, d_2, d_3, d_4, \ldots, d_n$. By the above relations, we easily find

$$d_2 = \frac{d}{du_2} + u_2\frac{d}{du_4} + u_3\frac{d}{du_5} + u_4\frac{d}{du_6} + \ldots,$$

$$d_3 = \frac{d}{du_3} + u_2\frac{d}{du_5} + u_3\frac{d}{du_6} + u_4\frac{d}{du_7} + \ldots,$$

$$d_4 = \frac{d}{du_4} + u_2\frac{d}{du_6} + u_3\frac{d}{du_7} + u_4\frac{d}{du_8} + \ldots.$$

.... :

which are the expressions of the operators in terms of the transformed system of variables.

These operators possess the commutative property—the verification of the statement is easy—that

$$d_r d_s = d_s d_r,$$

when operating upon any variable quantity, for all combinations of r and s; consequently when the operators are combined operationally and not solely symbolically, they obey the laws of ordinary algebra.

3. In order to distinguish between symbolical and operational combinations of these differential operators, we shall use $\bar{d}$ to imply a purely symbolical expression for the operator and shall retain d to denote the fully operational expression. Since d_1 is the operator

$$\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \dots\dots + x_{n-1} \frac{\partial}{\partial x_n},$$

it follows that, with the notation just indicated, the original differential equation can be expressed in the form

$$\bar{d}_1{}^n U = 0.$$

It proves desirable to change the expression so that the operators which occur are operational and not solely symbolical; in other words, it is desirable to express $\bar{d}_1{}^n$ in terms of operational quantities that are not purely symbolical. The necessary expression is given by a particular case of a theorem in linear differential operators, due to MacMahon *: it is as follows: The relation

$$e^{y\bar{d}_1} = e^{yd_1 - \frac{1}{2}y^2 d_2 + \frac{1}{3}y^3 d_3 - \frac{1}{4}y^4 d_4 + \dots}$$

holds for any value of y, when expansion of both sides in powers of y is effected: on the left-hand side the powers of $\bar{d}_1$ are symbolical only, on the right-hand side the powers of the operators are operational.

Consequently, we have

$$\bar{d}_1 = d_1,$$

$$\frac{1}{2!}\bar{d}_1{}^2 = \frac{1}{2!}d_1{}^2 - \frac{1}{2}d_2,$$

$$\frac{1}{3!}\bar{d}_1{}^3 = \frac{1}{3!}d_1{}^3 - \frac{1}{2}d_1 d_2 + \frac{1}{3}d_3,$$

$$\vdots$$

$$\begin{aligned}\frac{1}{m!}\bar{d}_1{}^m = {} & \frac{1}{m!}d_1{}^m - \frac{1}{(m-2)!}d_1{}^{m-2}.\frac{1}{2}d_2 + \frac{1}{(m-3)!}d_1{}^{m-3}.\frac{1}{3}d_3 \\ & - \frac{1}{(m-4)!}d_1{}^{m-4}.\frac{1}{4}\left(d_4 - \frac{1}{2}d_2{}^2\right) \\ & + \frac{1}{(m-5)!}d_1{}^{m-5}\left\{\frac{1}{5}d_5 + \frac{1}{2!}\left(\frac{-2}{2.3}d_2 d_3\right)\right\} \\ & + \frac{1}{(m-6)!}d_1{}^{m-6}\left\{-\frac{1}{6}d_6 + \frac{1}{2!}\left(\frac{2}{2.4}d_2 d_4 + \frac{1}{3^2}d_3{}^2\right) + \frac{1}{3!}\frac{1}{2^3}d_2{}^3\right\} \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;\end{aligned}$$

* *Quarterly Journal of Mathematics*, vol. XXIV (1890), pp. 246—250. It should be added that, for simplicity of notation and printing in the present paper, I have reversed the relative signification of $\bar{d}_1{}^n$ and $d_1{}^n$ which is implied (but not explicitly used as regards the operators) in the paper referred to: see pp. 247, 248.

the coefficient of $\frac{1}{(m-s)!} d_1^{m-s}$ is

$$\Sigma \frac{1}{p_2!\, p_3!\, p_4! \dots} (-\tfrac{1}{2} d_2)^{p_2} (\tfrac{1}{3} d_3)^{p_3} (-\tfrac{1}{4} d_4)^{p_4} \dots\dots,$$

the summation extending over all the terms corresponding to integer solutions of the equation

$$2p_2 + 3p_3 + 4p_4 + \dots = s.$$

These results are most simply obtained on expanding e^{yd_1} and $e^{-\frac{1}{2}y^2 d_2 + \frac{1}{3}y^3 d_3 - \frac{1}{4}y^4 d_4 + \dots}$ separately, and then multiplying: for a reason that will soon appear, it is convenient to arrange the expression in powers of d_1.

4. The transformation having been now obtained explicitly, let it be written in the form

$$\bar{d}_1^m = d_1^m + d_1^{m-2}\Delta_2 + d_1^{m-3}\Delta_3 + d_1^{m-4}\Delta_4 + \dots;$$

consequently the differential equation is

$$(d_1^m + d_1^{m-2}\Delta_2 + d_1^{m-3}\Delta_3 + d_1^{m-4}\Delta_4 + \dots) U = 0.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ denote the roots of the equation

$$\xi^m + \xi^{m-2}\Delta_2 + \xi^{m-3}\Delta_3 + \xi^{m-4}\Delta_4 + \dots\dots = 0,$$

where it is easy to see that $\alpha_1, \alpha_2, \dots, \alpha_m$ are distinct from one another.

Also, let $A_1, A_2, \dots, A_m$ denote m independent arbitrary functions, each of them involving the $n-1$ arguments $u_2, u_3, \dots, u_n$ quite arbitrarily. Then the solution of the equation can be expressed in the form

$$U = \sum_{s=1}^{m} e^{x_1 \alpha_s} A_s:$$

a form which however is only symbolical and for any effective use of which moreover the solution of the foregoing algebraical equation of degree m would be needed. Such a necessity is superfluous: and the expression of the solution can be changed as follows.

5. Manifestly, we have

$$U = u_0 + x_1 u_1 + \frac{x_1^2}{2!} u_2 + \frac{x_1^3}{3!} u_3 + \frac{x_1^4}{4!} u_4 + \dots\dots,$$

where
$$u_\theta = \alpha_1^\theta A_1 + \alpha_2^\theta A_2 + \dots + \alpha_m^\theta A_m.$$

Replace the m independent arbitrary functions A by other m arbitrary functions

$$P_1, P_2, \dots, P_m,$$

defined by the relations

$$\left.\begin{array}{l} A_1 + A_2 + \ldots\ldots + A_m = P_1 \\ \alpha_1 A_1 + \alpha_2 A_2 + \ldots\ldots + \alpha_m A_m = P_2 \\ \alpha_1^2 A_1 + \alpha_2^2 A_2 + \ldots\ldots + \alpha_m^2 A_m = P_3 \\ \vdots \\ \alpha_1^{m-1} A_1 + \alpha_2^{m-1} A_2 + \ldots\ldots + \alpha_m^{m-1} A_m = P_m \end{array}\right\};$$

the m functions P are evidently independent of one another. Denoting

$$\left|\begin{array}{cccc} 1, & 1, & \ldots\ldots, & 1 \\ \alpha_1, & \alpha_2, & \ldots\ldots, & \alpha_m \\ \alpha_1^2, & \alpha_2^2, & \ldots\ldots, & \alpha_m^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{m-1}, & \alpha_2^{m-1}, & \ldots\ldots, & \alpha_m^{m-1} \end{array}\right|$$

by $\square$, we have

$$\square u_\theta = \alpha_1^\theta \square A_1 + \alpha_2^\theta \square A_2 + \ldots\ldots + \alpha_m^\theta \square A_m$$

$$= \left|\begin{array}{cccc} \alpha_1^\theta, & \alpha_2^\theta, & \ldots\ldots, & \alpha_m^\theta \\ \alpha_1, & \alpha_2, & \ldots\ldots, & \alpha_m \\ \alpha_1^2, & \alpha_2^2, & \ldots\ldots, & \alpha_m^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{m-1}, & \alpha_2^{m-1}, & \ldots\ldots, & \alpha_m^{m-1} \end{array}\right| P_1 + \left|\begin{array}{cccc} 1, & 1, & \ldots\ldots, & 1 \\ \alpha_1^\theta, & \alpha_2^\theta, & \ldots\ldots, & \alpha_m^\theta \\ \alpha_1^2, & \alpha_2^2, & \ldots\ldots, & \alpha_m^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{m-1}, & \alpha_2^{m-1}, & \ldots\ldots, & \alpha_m^{m-1} \end{array}\right| P_2 + \ldots\ldots,$$

where the (operator) coefficient of P_r differs from $\square$ by having $\alpha_1^\theta, \alpha_2^\theta, \ldots, \alpha_m^\theta$ in the r^{th} row instead of $\alpha_1^{r-1}, \alpha_2^{r-1}, \ldots, \alpha_m^{r-1}$.

Let H_κ denote the sum of the integral homogeneous functions of the roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ which are of weight κ: in particular,

$$H_1 = \alpha_1 + \alpha_2 + \ldots\ldots + \alpha_m = 0,$$

and in general, these functions are given by

$$(1 + z\Delta_1 + z^2\Delta_2 + z^3\Delta_3 + \ldots\ldots)(1 + zH_1 + z^2H_2 + z^3H_3 + \ldots\ldots) = 1,$$

on introducing a term $z\Delta_1$ which is zero. We thus have

$$0 = H_1 + \Delta_1,$$

$$0 = H_2 + H_1\Delta_1 + \Delta_2,$$

$$0 = H_3 + H_2\Delta_1 + H_1\Delta_2 + \Delta_3,$$

$$\vdots$$

and therefore $H_1\,(=0)$, H_2, H_3, ... are operators which can be regarded as known. Then by Jacobi's theorem on alternants*, we have

$$\frac{1}{\square}\begin{vmatrix} 1, & 1, & \ldots\ldots, & 1 \\ \alpha_1, & \alpha_2, & \ldots\ldots, & \alpha_m \\ \vdots & & & \\ \alpha_1^{r-2}, & \alpha_2^{r-2}, & \ldots\ldots, & \alpha_m^{r-2} \\ \alpha_1^{r}, & \alpha_2^{r}, & \ldots\ldots, & \alpha_m^{r} \\ \vdots & & & \\ \alpha_1^{m-1}, & \alpha_2^{m-1}, & \ldots\ldots, & \alpha_m^{m-1} \\ \alpha_1^{\theta}, & \alpha_2^{\theta}, & \ldots\ldots, & \alpha_m^{\theta} \end{vmatrix} = \begin{vmatrix} 1, & H_1, & H_2, & \ldots\ldots, & H_{r-2}, & H_r, & \ldots\ldots, & H_{m-1}, & H_\theta \\ 0, & 1, & H_1, & \ldots\ldots, & H_{r-3}, & H_{r-1}, & \ldots\ldots, & H_{m-2}, & H_{\theta-1} \\ 0, & 0, & 1, & \ldots\ldots, & H_{r-4}, & H_{r-2}, & \ldots\ldots, & H_{m-3}, & H_{\theta-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots\ldots, & 0, & 0, & \ldots\ldots, & H_1, & H_{\theta-m+2} \\ 0, & 0, & 0, & \ldots\ldots, & 0, & 0, & \ldots\ldots, & 1, & H_{\theta-m+1} \end{vmatrix}$$

$$= \begin{vmatrix} H_1, & H_2, & H_3, & \ldots\ldots, & H_{m-r}, & H_{\theta-r+1} \\ 1, & H_1, & H_2, & \ldots\ldots, & H_{m-r-1}, & H_{\theta-r} \\ 0, & 1, & H_1, & \ldots\ldots, & H_{m-r-2}, & H_{\theta-r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots\ldots, & H_1, & H_{\theta-m+2} \\ 0, & 0, & 0, & \ldots\ldots, & 1, & H_{\theta-m+1} \end{vmatrix}$$

the latter determinant being the value of the former because the first $r-1$ constituents in the diagonal of the former are each of them unity and all the constituents below the diagonal and belonging to the first $r-1$ columns are each of them zero. To evaluate the new determinant, multiply the second row by Δ_1, the third by Δ_2, the fourth by Δ_3, and so on down to the last row which is to be multiplied by Δ_{m-r}: and then add each of these rows to the first, replacing the first row by the new constituents thus obtained. These operations do not alter the value of the determinant. But now, in the first row, every constituent except the last is zero; and the last is

$$\begin{aligned} & H_{\theta-r+1} + \Delta_1 H_{\theta-r} + \Delta_2 H_{\theta-r-1} + \ldots\ldots + \Delta_{m-r} H_{\theta-m+1} \\ &= -(\Delta_{m-r+1} H_{\theta-m} + \Delta_{m-r+2} H_{\theta-m-1} + \ldots\ldots + \Delta_m H_{\theta-m-r+1}) \\ &= E_{r,\theta}, \end{aligned}$$

say: hence the determinant

$$= (-1)^{m-r} E_{r,\theta} \begin{vmatrix} 1, & H_1, & H_2, & \ldots, & H_{m-r-1} \\ 0, & 1, & H_1, & \ldots, & H_{m-r-2} \\ 0, & 0, & 1, & \ldots, & H_{m-r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots, & 1 \end{vmatrix}$$

$$= (-1)^{m-r} E_{r,\theta}.$$

* *Ges. Werke*, t. III, pp. 441—452; Scott's *Theory of Determinants*, p. 124.

But the coefficient of P_r in the expression for u_θ

$$= \frac{1}{\square} \begin{vmatrix} 1, & 1, & \ldots\ldots, & 1 \\ \alpha_1, & \alpha_2, & \ldots\ldots, & \alpha_m \\ \vdots & & & \\ \alpha_1^{r-2}, & \alpha_2^{r-2}, & \ldots\ldots, & \alpha_m^{r-2} \\ \alpha_1^{\theta}, & \alpha_2^{\theta}, & \ldots\ldots, & \alpha_m^{\theta} \\ \alpha_1^{r}, & \alpha_2^{r}, & \ldots\ldots, & \alpha_m^{r} \\ \vdots & & & \\ \alpha_1^{m-1}, & \alpha_2^{m-1}, & \ldots\ldots, & \alpha_m^{m-1} \end{vmatrix}$$

$$= \frac{(-1)^{m-r+2}}{\square} \begin{vmatrix} 1, & 1, & \ldots\ldots, & 1 \\ \alpha_1, & \alpha_2, & \ldots\ldots, & \alpha_m \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{r-2}, & \alpha_2^{r-2}, & \ldots\ldots, & \alpha_m^{r-2} \\ \alpha_1^{r}, & \alpha_2^{r}, & \ldots\ldots, & \alpha_m^{r} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{m-1}, & \alpha_2^{m-1}, & \ldots\ldots, & \alpha_m^{m-1} \\ \alpha_1^{\theta}, & \alpha_2^{\theta}, & \ldots\ldots, & \alpha_m^{\theta} \end{vmatrix}$$

$$= E_{r,\theta};$$

and therefore

$$u_\theta = E_{1,\theta} P_1 + E_{2,\theta} P_2 + E_{3,\theta} P_3 + \ldots + E_{m,\theta} P_m,$$

for values of θ greater than $m-1$. Accordingly, we have

$$U = \sum_{s=1}^{m} \left[\left\{ \frac{x_1^{s-1}}{(s-1)!} + \sum_{\theta=m}^{\infty} \frac{x_1^{\theta}}{\theta!} E_{s,\theta} \right\} P_s \right],$$

where $E_{s,\theta}$ is an operator defined as

$$E_{s,\theta} = H_{\theta-s+1} + \Delta_1 H_{\theta-s} + \Delta_2 H_{\theta-s-1} + \ldots + \Delta_{m-s} H_{\theta-m+1},$$

and the quantities P are m arbitrary independent functions of the $n-1$ arguments $u_2, u_3, \ldots, u_n$. This is *the most general solution of the equation*

$$\left(\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \ldots + x_{n-1} \frac{\partial}{\partial x_n} \right)^m U = 0,$$

where the multiplication of the operator is symbolical and not operational.

6. A few simple cases occur for the lowest values of m.

First, taking $m = 1$; we have

$$U = F(u_2, u_3, \ldots, u_n),$$

where F is an arbitrary function, as the most general solution of

$$\left(\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \ldots + x_{n-1} \frac{\partial}{\partial x_n} \right) U = 0.$$

Secondly, take $m = 2$. The subsidiary algebraical equation is

$$\xi^2 - d_2 = 0;$$

and accordingly we have

$$U = \left(1 + \frac{x_1^2}{2!} d_2 + \frac{x_1^4}{4!} d_2^2 + \frac{x_1^6}{6!} d_2^3 + \dots\right) \Phi(u_2, u_3, \dots, u_n)$$

$$+ \left(x_1 + \frac{x_1^3}{3!} d_2 + \frac{x_1^5}{5!} d_2^2 + \frac{x_1^7}{7!} d_2^3 + \dots\right) \Psi(u_2, u_3, \dots, u_n),$$

where Φ and Ψ are arbitrary, as the most general solution of

$$\left(\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \dots + x_{n-1} \frac{\partial}{\partial x_n}\right)^2 U = 0.$$

Thirdly, take $m = 3$. The transformed differential equation is

$$(d_1^3 - 3d_1 d_2 + 2d_3) U = 0,$$

and the subsidiary algebraical equation is

$$\xi^3 - 3\xi d_2 + 2d_3 = 0,$$

so that

$$\Delta_1 = 0, \quad \Delta_2 = -3d_2, \quad \Delta_3 = 2d_3.$$

Then H_p denoting the sum of the homogeneous products of p dimensions of the three roots, we have $H_p =$ coefficient of z^p in the expansion of

$$\frac{1}{1 - 3d_2 z^2 + 2d_3 z^3}$$

in ascending powers of z, that is,

$$H_p = \Sigma (3d_2)^\lambda (-2d_3)^\mu \frac{(\lambda + \mu)!}{\lambda!\, \mu!},$$

where the summation extends over all the terms that correspond to integer solutions of

$$2\lambda + 3\mu = p.$$

Then for the present case,

$$E_{1,\theta} = H_\theta + \Delta_1 H_{\theta-1} + \Delta_2 H_{\theta-2} = -\Delta_3 H_{\theta-3} = -2d_3 H_{\theta-3},$$
$$E_{2,\theta} = H_{\theta-1} + \Delta_1 H_{\theta-2} = H_{\theta-1},$$
$$E_{3,\theta} = H_{\theta-2};$$

and accordingly we have

$$U = \left\{1 - \left(\frac{x_1^3}{3!} H_0 + \frac{x_1^4}{4!} H_1 + \frac{x_1^5}{5!} H_2 + \dots\right) 2d_3\right\} P_1(u_2, u_3, \dots, u_n)$$

$$+ \left\{x_1 + \frac{x_1^3}{3!} H_2 + \frac{x_1^4}{4!} H_3 + \frac{x_1^5}{5!} H_4 + \dots\right\} P_2(u_2, u_3, \dots, u_n)$$

$$+ \left\{\frac{x_1^2}{2!} + \frac{x_1^3}{3!} H_1 + \frac{x_1^4}{4!} H_2 + \frac{x_1^5}{5!} H_3 + \dots\right\} P_3(u_2, u_3, \dots, u_n).$$

where P_1, P_2, P_3 are arbitrary, as the most general solution of

$$\left(\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \ldots + x_{n-1} \frac{\partial}{\partial x_n}\right)^3 U = 0.$$

7. In the form in which (§ 5) the solution has been obtained, it is easy to associate the result with Cauchy's existence-theorem*. Let it be proposed to obtain that solution of the differential equation which is such that, when $x_1 = 0$, the values of

$$U, \frac{\partial U}{\partial x_1}, \ldots\ldots, \frac{\partial^{m-1} U}{\partial x_1^{m-1}}$$

are respectively equal to

$$\phi_1(x_2, x_3, \ldots, x_n), \; \phi_2(x_2, x_3, \ldots, x_n), \; \ldots, \; \phi_m(x_2, x_3, \ldots, x_n).$$

On referring to § 1, it is at once evident that, when $x_1 = 0$, we have

$$u_s = \frac{1}{s!} x_s' = x_s.$$

Again, taking the general solution as given in § 5, we have, when $x_1 = 0$,

$$\frac{\partial^r U}{\partial x_1^r} = P_{r+1}(u_2, u_3, \ldots, u_n) \text{ when } x_1 = 0$$

$$= P_{r+1}(x_2, x_3, \ldots, x_n).$$

But the value should be $\phi_{r+1}(x_2, x_3, \ldots, x_n)$: so that

$$P_{r+1}(x_2, x_3, \ldots, x_n) = \phi_{r+1}(x_2, x_3, \ldots, x_n),$$

and therefore $\qquad P_{r+1}(u_2, u_3, \ldots, u_n) = \phi_{r+1}(u_2, u_3, \ldots, u_n),$

for all values of r. Hence all the arbitrary functions are determined in accordance with the assigned conditions.

8. In the preceding investigation, the multiplication of the operator has throughout been supposed symbolical and not operational. Corresponding results occur in the case of equations for which the multiplication of the operator is partially or wholly operational. The three kinds of cases that can arise can, in the notation previously adopted, be represented in the forms

$$\bar{d}_1^m U = 0,$$

$$d_1^\mu \bar{d}_1^{m-\mu} U = 0,$$

$$d_1^m U = 0.$$

In each instance, the most general solution involves m arbitrary functions independent of one another: and the functions involve the $n-1$ distinct arguments $u_2, u_3, \ldots, u_n$.

The solution of the first has been obtained. That of the second can be made to depend upon it; it is derivable from the solution of the first by noting that the transformed equation is

$$d_1^\mu (d_1^{m-\mu} + d_1^{m-\mu-2} \Delta_2' + d_1^{m-\mu-3} \Delta_3' + \ldots) U = 0.$$

* Jordan, *Cours d'Analyse*, t. III, p. 306.

so that in the final expression obtained above in § 5, we only need to replace the quantities H_p by their modified values: or what is the same thing, we replace H_p by H_p', the sum of the homogeneous products of p dimensions in the roots of

$$\xi^{m-\mu} + \xi^{m-\mu-2}\Delta_2' + \xi^{m-\mu-3}\Delta_3' + \dots = 0.$$

The operators $E_{s,\theta}$ for $s = 1, 2, \dots, \mu$ are each zero.

The third is the limiting case of the second. All the operators E are zero: and the solution is

$$\sum_{s=0}^{m-1} x_1^s \Phi_s (u_2, u_3, \dots, u_n),$$

where all the functions Φ are arbitrary.

9. The equation which has been discussed is one of a series that arise in the theory of symmetric algebra and it appears, at present, to be the most important of the series. There is one class of equations of a similar form; and their general solutions can be constructed in a similar fashion. They all can be represented in the form

$$\left(\frac{\partial}{\partial x_p} + x_1 \frac{\partial}{\partial x_{p+1}} + x_2 \frac{\partial}{\partial x_{p+2}} + \dots + x_{n-p} \frac{\partial}{\partial x_n}\right)^m U = 0,$$

where the multiplication of the operator is symbolical only and not operational; with the previous notation, this can be represented by

$$\bar{d}_p^m U = 0.$$

The case already discussed corresponds to $p = 1$; for the remaining cases, p has the values 2, 3, 4, ... We proceed to the integration of the equation, retaining a general value for p, so as to include all the cases: the march of the integration is similar to that in the earlier part of the paper.

10. The most general solution of the equation

$$\bar{d}_p^m U = 0$$

contains m independent arbitrary functions, each of $n-1$ arguments. These arguments satisfy the equation

$$\left(\frac{\partial u}{\partial x_p} + x_1 \frac{\partial u}{\partial x_{p+1}} + x_2 \frac{\partial u}{\partial x_{p+2}} + \dots + x_{n-p} \frac{\partial u}{\partial x_n}\right)^m = 0.$$

This equation being resoluble, as before, into the m-fold repetition of a linear equation in the derivatives of u, the $n-1$ arguments of each function are the same for all the m arbitrary functions; and they are any $n-1$ functionally independent solutions of the equation

$$\frac{\partial u}{\partial x_p} + x_1 \frac{\partial u}{\partial x_{p+1}} + x_2 \frac{\partial u}{\partial x_{p+2}} + \dots + x_{n-p} \frac{\partial u}{\partial x_n} = 0.$$

As they are the arguments of quite arbitrary functions, it is manifestly sufficient to obtain the $n-1$ solutions in the simplest forms possible. Accordingly, we require the simplest integrals of the equations

$$\frac{dx_r}{0} = \frac{dx_p}{1} = \frac{dx_{2p}}{x_p} = \frac{dx_{3p}}{x_{2p}} = \dots$$

$$= \frac{dx_{p+r}}{x_r} = \frac{dx_{2p+r}}{x_{p+r}} = \frac{dx_{3p+r}}{x_{2p+r}} = \dots$$

for $r = 1, 2, 3, \dots, p-1$. These integrals are given by

$$x_s = u_s, \qquad (s = 1, 2, \dots, p-1);$$

$$x_{2p} = \frac{x_p^2}{2!} + u_{2p},$$

$$x_{3p} = \frac{x_p^3}{3!} + x_p u_{2p} + u_{3p},$$

$$\vdots$$

$$x_{\kappa p} = \frac{x_p^\kappa}{\kappa!} + \frac{x_p^{\kappa-2}}{(\kappa-2)!} u_{2p} + \frac{x_p^{\kappa-3}}{(\kappa-3)!} u_{3p} + \dots + x_p u_{(\kappa-1)p} + u_{\kappa p},$$

for values of $\kappa = 2, 3, \dots$, up to the integral part of $\frac{n}{p}$: and

$$x_{p+r} = x_r x_p + u_{p+r},$$

$$x_{2p+r} = x_r \frac{x_p^2}{2!} + x_p u_{p+r} + u_{2p+r},$$

$$x_{3p+r} = x_r \frac{x_p^3}{3!} + \frac{x_p^2}{2!} u_{p+r} + x_p u_{2p+r} + u_{3p+r},$$

$$\vdots$$

$$x_{\kappa p+r} = x_r \frac{x_p^\kappa}{\kappa!} + \frac{x_p^{\kappa-1}}{(\kappa-1)!} u_{p+r} + \frac{x_p^{\kappa-2}}{(\kappa-2)!} u_{2p+r} + \dots + x_p u_{(\kappa-1)p+r} + u_{\kappa p+r},$$

for values of $\kappa = 2, 3, \dots$, up to the integral part of $\frac{n-r}{p}$: and for values $r = 1, 2, \dots, p-1$. The quantities u are the arguments of each of the m arbitrary functions: their values are easily proved to be

$$u_s = x_s, \qquad (s = 1, 2, \dots, p-1);$$

$$u_{\kappa p} = \frac{1}{\kappa!} (x'_{\kappa p}, x'_{(\kappa-1)p}, \dots, x'_{2p}, x_p, 1 \between 1, -x_p)^\kappa,$$

for the values 2, 3, ... of κ, the quantities $x'_{\theta p}$ being defined by the relation

$$x'_{\theta p} = \theta! \, x_{\theta p};$$

and
$$u_{\kappa p+r} = \frac{1}{\kappa!} (x'_{\kappa p+r}, x'_{(\kappa-1)p+r}, \dots, x'_{2p+r}, x_{p+r}, x_r \between 1, -x_p)^\kappa,$$

for the values 1, 2, ..., $p-1$ of r, for the values 1, 2, 3, ... of κ, the quantities $x'_{\theta p+r}$ being defined by the relation

$$x'_{\theta p+r} = \theta! \, x_{\theta p+r}.$$

11. The next step is the transformation of the variables from the set $x_1, x_2, \ldots, x_n$ to the set $u_1, u_2, \ldots, u_{p-1}, x_p, u_{p+1}, \ldots, u_n$. For this purpose, we denote derivatives with regard to the new set of variables by $\frac{d}{du_1}, \frac{d}{du_2}, \ldots$; so that, denoting any function by P, we have

$$\begin{aligned}
\sum_{s=1}^{p-1} \frac{dP}{du_s} du_s + \frac{dP}{dx_p} dx_p + \sum_\kappa \frac{dP}{du_{\kappa p}} du_{\kappa p} + \sum_{r=1}^{p-1} \sum_\kappa \frac{dP}{du_{\kappa p+r}} du_{\kappa p+r} \\
&= dP \\
&= \sum_{\theta=1}^{n} \frac{\partial P}{\partial x_\theta} dx_\theta \\
&= \sum_{s=1}^{p-1} \frac{\partial P}{\partial x_s} du_s \\
&\quad + \frac{\partial P}{\partial x_p} dx_p + \sum_\kappa \frac{\partial P}{\partial x_{\kappa p}} \left\{ x_{(\kappa-1)p} dx_p + \frac{x_p^{\kappa-2}}{(\kappa-2)!} du_{2p} + \frac{x_p^{\kappa-3}}{(\kappa-3)!} du_{3p} + \ldots \right\} \\
&\quad + \sum_{r=1}^{p-1} \sum_\kappa \frac{\partial P}{\partial x_{\kappa p+r}} \left\{ \frac{x_p^\kappa}{\kappa!} du_r + x_{(\kappa-1)p+r} dx_p + \frac{x_p^{\kappa-1}}{(\kappa-1)!} du_{p+r} + \frac{x_p^{\kappa-2}}{(\kappa-2)!} du_{2p+r} + \ldots \right\}.
\end{aligned}$$

Hence

$$\frac{d}{du_s} = \frac{\partial}{\partial x_s} + x_p \frac{\partial}{\partial x_{p+s}} + \frac{x_p^2}{2!} \frac{\partial}{\partial x_{2p+s}} + \frac{x_p^3}{3!} \frac{\partial}{\partial x_{3p+s}} + \ldots.$$

for the values $s = 1, 2, \ldots, p-1$; also

$$\begin{aligned}
\frac{d}{dx_p} &= \frac{\partial}{\partial x_p} + \sum_\kappa x_{(\kappa-1)p} \frac{\partial}{\partial x_{\kappa p}} + \sum_{r=1}^{p-1} \sum_\kappa x_{(\kappa-1)p+r} \frac{\partial}{\partial x_{\kappa p+r}} \\
&= \frac{\partial}{\partial x_p} + x_1 \frac{\partial}{\partial x_{p+1}} + x_2 \frac{\partial}{\partial x_{p+2}} + \ldots + x_{n-p} \frac{\partial}{\partial x_n} \\
&= d_p;
\end{aligned}$$

also the series

$$\left.\begin{aligned}
\frac{d}{du_{2p}} &= \frac{\partial}{\partial x_{2p}} + x_p \frac{\partial}{\partial x_{3p}} + \frac{x_p^2}{2!} \frac{\partial}{\partial x_{4p}} + \frac{x_p^3}{3!} \frac{\partial}{\partial x_{5p}} + \frac{x_p^4}{4!} \frac{\partial}{\partial x_{6p}} + \ldots \\
\frac{d}{du_{3p}} &= \frac{\partial}{\partial x_{3p}} + x_p \frac{\partial}{\partial x_{4p}} + \frac{x_p^2}{2!} \frac{\partial}{\partial x_{5p}} + \frac{x_p^3}{3!} \frac{\partial}{\partial x_{6p}} + \ldots \\
\frac{d}{du_{4p}} &= \frac{\partial}{\partial x_{4p}} + x_p \frac{\partial}{\partial x_{5p}} + \frac{x_p^2}{2!} \frac{\partial}{\partial x_{6p}} + \ldots \\
&\vdots
\end{aligned}\right\};$$

and the set

$$\left.\begin{aligned}
\frac{d}{du_{p+r}} &= \frac{\partial}{\partial x_{p+r}} + x_p \frac{\partial}{\partial x_{2p+r}} + \frac{x_p^2}{2!} \frac{\partial}{\partial x_{3p+r}} + \frac{x_p^3}{3!} \frac{\partial}{\partial x_{4p+r}} + \ldots \\
\frac{d}{du_{2p+r}} &= \frac{\partial}{\partial x_{2p+r}} + x_p \frac{\partial}{\partial x_{3p+r}} + \frac{x_p^2}{2!} \frac{\partial}{\partial x_{4p+r}} + \ldots \\
\frac{d}{du_{3p+r}} &= \frac{\partial}{\partial x_{3p+r}} + x_p \frac{\partial}{\partial x_{4p+r}} + \ldots \\
&\vdots
\end{aligned}\right\},$$

this set existing for the values $r = 1, 2, \ldots, p-1$.

From these, we have

$$\frac{d}{du_{2p}} + u_p \frac{d}{du_{3p}} + u_{2p} \frac{d}{du_{4p}} + \dots$$

$$= \frac{\partial}{\partial x_{2p}} + x_p \frac{\partial}{\partial x_{3p}} + x_{2p} \frac{\partial}{\partial x_{4p}} + x_{3p} \frac{\partial}{\partial x_{5p}} + \dots;$$

also

$$u_r \frac{d}{du_{2p+r}} + u_{p+r} \frac{d}{du_{3p+r}} + u_{2p+r} \frac{d}{du_{4p+r}} + \dots$$

$$= x_r \frac{\partial}{\partial x_{2p+r}} + x_{p+r} \frac{\partial}{\partial x_{3p+r}} + x_{2p+r} \frac{\partial}{\partial x_{4p+r}} + \dots,$$

for all the values $r = 1, 2, \dots, p-1$. Hence

$$\frac{d}{du_{2p}} + \sum_{s=1} u_s \frac{d}{du_{2p+s}} = \frac{\partial}{\partial x_{2p}} + \sum_{s=1} x_s \frac{\partial}{\partial x_{2p+s}}$$

$$= d_{2p}.$$

Similarly, we find

$$\frac{d}{du_{3p}} + \sum_{s=1} u_s \frac{d}{du_{3p+s}} = \frac{\partial}{\partial x_{3p}} + \sum_{s=1} x_s \frac{\partial}{\partial x_{3p+s}}$$

$$= d_{3p};$$

$$\frac{d}{du_{4p}} + \sum_{s=1} u_s \frac{d}{du_{4p+s}} = \frac{\partial}{\partial x_{4p}} + \sum_{s=1} x_s \frac{\partial}{\partial x_{4p+s}}$$

$$= d_{4p};$$

and so on.

The series of operators d_p, d_{2p}, d_{3p}, d_{4p}, ... are (as will be seen from the next section) the only operators required for our immediate purpose; the transformation from the old set of variables to the new set of variables has been effected for each of them*. The remaining operators do, however, occur in other connections: the necessary transformations are easily effected, and they lead to the result

$$\frac{d}{du_{p+r}} + u_{2p} \frac{d}{du_{3p+r}} + u_{3p} \frac{d}{du_{4p+r}} + \dots$$

$$+ \sum_{s=1}^{p-1} \left(u_s \frac{d}{du_{p+r+s}} + u_{p+s} \frac{d}{du_{2p+r+s}} + u_{2p+s} \frac{d}{du_{3p+r+s}} + \dots \right)$$

$$= \frac{\partial}{\partial x_{p+r}} + x_1 \frac{\partial}{\partial x_{p+r+1}} + x_2 \frac{\partial}{\partial x_{p+r+2}} + \dots$$

$$= d_{p+r}.$$

* The invariantive form of all the sets of operators, that occur in connection with the class of differential equations under consideration, can hardly fail to be remarked: their simplest (monomial) expressions are obtained in § 24.

where it will be noticed that the term $u_p \dfrac{d}{du_{2p+r}}$ is the one term which is absent from the transformed expression for d_{p+r}. As a matter of fact, this expression can be regarded as the expression for all the operators when values 1, 2, 3, ... are assigned to r: one term in each case being absent from the transformed expression.

12. We now proceed to change the equation

$$\bar{d}_p^m U = 0,$$

in which the repetition of the operator is symbolical and not operational, into an equation in which the operators are not solely symbolical. For this purpose, another particular case of MacMahon's theorem (quoted in § 3) is effective, as follows:—The relation

$$e^{y\bar{d}_p} = e^{yd_p - \frac{1}{2}y^2 d_{2p} + \frac{1}{3}y^3 d_{3p} - \frac{1}{4}y^4 d_{4p} + \dots}$$

holds for any value of y, when expansion of both sides in powers of y is effected: on the left-hand side, the powers of $\bar{d}_p$ are symbolical only; on the right-hand side, the powers of the operators are operational.

Consequently, we have

$$\bar{d}_p = d_p,$$

$$\frac{1}{2\,!}\bar{d}_p^2 = \frac{1}{2\,!}d_p^2 - \frac{1}{2}d_{2p},$$

$$\frac{1}{3\,!}\bar{d}_p^3 = \frac{1}{3\,!}d_p^3 - \frac{1}{2}d_p d_{2p} + \frac{1}{3}d_{3p},$$

$$\vdots$$

$$\frac{1}{m\,!}\bar{d}_p^m = \frac{1}{m\,!}d_p^m + \sum_{s=2}^{m}\frac{1}{(m-s)\,!}d_p^{m-s}\Theta_{s,p},$$

where $\Theta_{s,p}$ denotes

$$\Sigma\frac{1}{q_2!\;q_3!\;q_4!\;\dots}(-\tfrac{1}{2}d_{2p})^{q_2}(\tfrac{1}{3}d_{3p})^{q_3}(-\tfrac{1}{4}d_{4p})^{q_4}\dots,$$

in which the summation extends over all the terms corresponding to integer solutions of the equation

$$2q_2 + 3q_3 + 4q_4 + \dots = s.$$

It is also to be noted that, when the number n of original variables x is finite, the highest suffix that can occur in the symbol of an operator occurs in $d_{\kappa p}$, where κ is the integral part of $\dfrac{n}{p}$: but that when n is infinite in value, there is no such restriction upon the number of the operators that can occur. Manifestly,

$$\Theta_{2,p} = -\tfrac{1}{2}d_{2p},$$

$$\Theta_{3,p} = \tfrac{1}{3}d_{3p},$$

$$\Theta_{4,p} = -\tfrac{1}{4}(d_{4p} - \tfrac{1}{2}d_{2p}^2),$$

and so on. Writing

$$\Delta_{s,\,p} = \frac{m!}{(m-s)!}\,\Theta_{s,\,p},$$

the equation $\bar{d}_p^m U = 0$ is transformed to

$$(d_p^m + \Delta_{2,\,p}\, d_p^{m-2} + \Delta_{3,\,p}\, d_p^{m-3} + \ldots)\, U = 0.$$

13. The analysis that leads to the solution of this equation is, *mutatis mutandis*, precisely similar to that given in §§ 4, 5: the result for the present case is as follows.

Let $H_{\kappa,\,p}$ denote the sum of the integral homogeneous functions of weight κ in the m roots of the equation

$$\xi^m + \xi^{m-2}\Delta_{2,\,p} + \xi^{m-3}\Delta_{3,\,p} + \xi^{m-4}\Delta_{4,\,p} + \ldots = 0\,;$$

and let

$$E_{r,\,\theta,\,p} = H_{\theta-r+1,\,p} + \Delta_{2,\,p}\, H_{\theta-r-1,\,p} + \Delta_{3,\,p}\, H_{\theta-r-2,\,p} + \ldots + \Delta_{m-r,\,p}\, H_{\theta-m+1,\,p},$$

for $r = 1, 2, \ldots, m$ and for all values θ given by $m, m+1, \ldots$. Further, let $Q_1, Q_2, \ldots, Q_m$ denote m independent arbitrary functions of the arguments

$$u_1,\ u_2,\ \ldots,\ u_{p-1},\ u_{p+1},\ u_{p+2},\ \ldots,\ u_n.$$

Then *the most general solution of the equation*

$$\left(\frac{\partial}{\partial x_p} + x_1\frac{\partial}{\partial x_{p+1}} + x_2\frac{\partial}{\partial x_{p+2}} + \ldots + x_{n-p}\frac{\partial}{\partial x_n}\right)^m U = 0$$

is

$$U = \sum_{s=1}^{m}\left[\left\{\frac{x_p^{s-1}}{(s-1)!} + \sum_{\theta=m}^{\infty}\frac{x_p^{\theta}}{\theta!}\,E_{s,\,\theta,\,p}\right\}Q_s\right].$$

14. As in § 8 with the operator d_1, so here with the operator d_p, we may consider equations associated with the equation just solved given by

$$d_p^{\mu}\,\bar{d}_p^{m-\mu}\,U = 0,$$

$$d_p^m\,U = 0.$$

The solutions of these equations are derivable from those of the equations as given in § 8 in the same manner as the solution of

$$\bar{d}_p^m\,U = 0$$

has been derived from that of

$$\bar{d}_1^m\,U = 0.$$

15. In the preceding examples, the characteristic invariant is in every case an exact power of a single linear equation: with the result that the arbitrary functions, which occur in the respective solutions, are merely different arbitrary functions of the same set of arguments.

It is however easy to suggest other equations in which the characteristic invariant is in every case resoluble into equations each of which is linear but all of which are not the same. Thus, e.g., we might take

$$d_1 d_2 \ldots d_m U = 0,$$

the characteristic invariant of which is the product of m distinct linear equations: or

$$d_1^{m_1} \,.\, d_2^{m_2} \,.\, \ldots \,.\, d_p^{m_p} U = 0,$$

the characteristic invariant of which is the product of p distinct equations repeated m_1 times, m_2 times, ..., m_p times respectively.

Only a single example, as simple as possible, will be discussed: it will provide a sufficient indication of methods of solution.

16. Consider the equation

$$d_1 d_2 U = (d_1 d_2 - d_3) U = 0.$$

The arguments of the arbitrary functions, which are contained in the general solution satisfy the equation

$$(d_1\theta)(d_2\theta) = 0,$$

that is, they satisfy either

$$d_1\theta = 0$$

or

$$d_2\theta = 0.$$

As regards the equation $d_1\theta = 0$, we know that it possesses $n-1$ functionally independent solutions denoted by $u_2, u_3, \ldots, u_n$; and that when the independent variables are changed from the set $x_1, x_2, x_3, \ldots, x_n$ to $x_1, u_2, u_3, \ldots, u_n$, then

$$d_1 = \frac{d}{dx_1},$$

$$d_2 = \frac{d}{du_2} + u_2 \frac{d}{du_4} + u_3 \frac{d}{du_5} + \ldots\ldots,$$

$$d_3 = \frac{d}{du_3} + u_2 \frac{d}{du_5} + u_3 \frac{d}{du_6} + \ldots\ldots,$$

$$\ldots\ldots\ldots\ldots$$

Next, consider the equation

$$d_2\theta = 0,$$

in the first place when the variables are $u_2, u_3, \ldots, u_n$. It possesses $n-2$ functionally independent solutions, say $v_3, v_4, \ldots, v_n$, given by the expressions in § 10 when $p = 2$; and when the variables $u_2, u_3, \ldots, u_n$ are changed to the set $u_2, v_3, \ldots, v_n$, then

$$d_2 = \frac{D}{Du_2},$$

$$d_3 = \frac{D}{Dv_3} + v_3 \frac{D}{Dv_6} + v_4 \frac{D}{Dv_7} + v_5 \frac{D}{Dv_8} + \ldots.$$

where $\dfrac{D}{Dv}$ denotes derivation with regard to the last set of variables. The differential equation thus becomes

$$\frac{d}{dx_1}\left(\frac{DU}{du_2}\right) = d_3 U.$$

Let $\Theta(u_2, v_3, v_4, v_5, v_6, v_7, \ldots)$ be any arbitrary function of the arguments, say Θ; and denote by $\int\Phi du_2$, the value of U which is such that

$$\frac{DU}{Du_2} = \Phi,$$

Φ denoting any function of the arguments contained in Θ. Then the solution of the equation can be expressed in the form

$$U = \Theta + x_1 \int d_3\Theta du_2 + \frac{x_1^2}{2\,!}\iint d_3^2\,\Theta du_2 du_2 + \frac{x_1^3}{3\,!}\iiint d_3^3\,\Theta du_2 du_2 du_2 + \ldots.$$

Another form can be given to the solution. Take a series of variables

$$v_6 = \frac{1}{2}v_3^2 + w_6,$$

$$v_9 = \frac{1}{3\,!}v_3^3 + v_3 w_6 + w_9,$$

$$v_{12} = \frac{1}{4\,!}v_3^4 + \frac{v_3^2}{2\,!}w_6 + v_3 w_9 + w_{12},$$

$$\vdots$$

$$v_4 = w_4,$$

$$v_7 = v_3 w_4 + w_7,$$

$$v_{10} = \frac{v_3^2}{2!}w_4 + v_3 w_7 + w_{10},$$

$$\vdots$$

$$v_5 = w_5,$$

$$v_8 = v_3 w_5 + w_8,$$

$$v_{11} = \frac{v_3^2}{2!}w_5 + v_3 w_8 + w_{11},$$

$$\vdots$$

the explicit values of the variables w being given in terms of the variables v as in § 10. Then when the variables $v_3, v_4, \ldots, v_n$ are replaced by $v_3, w_4, \ldots, w_n$, the operator d_3 becomes simply $\frac{\delta}{\delta v_3}$, where $\frac{\delta}{\delta v_3}$ implies the partial differentiation when the variables are $x_1, u_2, v_3, w_4, w_5, \ldots, w_n$; and now the differential equation is

$$\frac{d}{dx_1}\left(\frac{DU}{Du_2}\right) = \frac{\delta U}{\delta v_3},$$

the quantity U being a function of the variables $x_1, u_2, v_3, w_4, \ldots, w_n$. All the derivations being partial, we may write the result in the form

$$\frac{\partial^2 U}{\partial x_1 \partial u_2} = \frac{\partial U}{\partial v_3}.$$

The form of solution already obtained is, of course, still effective, all that is necessary by way of change being that the function Θ it contains should be regarded as a function of $u_2, v_3, w_4, w_5, \ldots$: and then, in the integrations, d_3 is replaced by $\frac{\partial}{\partial v_3}$. The other form of solution indicated is as follows. Let F denote any arbitrary function of $x_1, u_2, w_4, w_5, \ldots, w_n$, the argument v_3 being omitted; then the solution of the equation can be expressed in the form

$$U = \left(1 + v_3 \frac{\partial^2}{\partial x_1 \partial u_2} + \frac{v_3^2}{2!} \frac{\partial^4}{\partial x_1^2 \partial u_2^2} + \frac{v_3^3}{3!} \frac{\partial^6}{\partial x_1^3 \partial u_2^3} + \ldots\right) F.$$

This is a solution deduced by using one of the argument-equations, viz.

$$d_1\theta = 0.$$

17. Turning now to the other of the argument-equations, viz. $d_2\theta = 0$, a corresponding investigation will be carried out, though this, indeed, is unnecessary, as we have dealt with the whole equation *ab initio* when once a form of argument was suggested. Before passing, however, to the other argument-equation, some remarks upon the solution should be noted.

First, the equation

$$\frac{\partial^2 U}{\partial x_1 \partial u_2} = \frac{\partial U}{\partial v_3}$$

can be solved (not in the most general way) by taking

$$U = e^{\alpha v_3} V,$$

where α is a constant and V is independent of v_3, so that we have

$$\frac{\partial^2 V}{\partial x_1 \partial u_2} = \alpha V:$$

here V is a function of x_1, u_2 and the variables w_4, w_5, ... which are parametric for the last form. But this equation is a special instance of Laplace's linear equation in which the invariants are equal: as they are constants, the Darboux-sequence is infinite and therefore the solution (which contains linearly one function of $x_1, w_4, w_5, \ldots$ and its derivatives, and linearly one function of $u_2, w_4, w_5, \ldots$ and its derivatives) is not expressible in finite terms. To compare the two solutions, we have $V = F$, provided the arbitrary function F of the first solution be limited so as to satisfy

$$\frac{\partial^2 F}{\partial x_1 \partial u_2} = \alpha F:$$

consequently the solution $e^{\alpha v_3} V$ is less general than the former.

Secondly, only one of the two argument-equations has been used. In one form of solution, the arguments of the arbitrary function are $u_2, v_3, \ldots$ which are solutions of $d_1\theta = 0$; in the other form, the arguments of the arbitrary function are $x_1, u_2, w_4, w_5, \ldots$ all of which save the first are solutions of $d_1\theta = 0$ and the first of which is a solution of $d_2\theta = 0$: that is to say, in both cases the arguments are solutions of

$$(d_1\theta)(d_2\theta) = 0,$$

in accordance with the general theorem. In each case, however, there has been obtained only one arbitrary function in the solution. This is not in real contradiction with the theorem that the general solution contains two arbitrary functions: for the theorem is proved to apply only to those equations whose integrals are expressible in finite terms and are free from partial quadratures, neither of which conditions is satisfied in the present case.

18. Reverting to the argument-equation

$$d_2\theta = \frac{\partial\theta}{\partial x_2} + x_1\frac{\partial\theta}{\partial x_3} + x_2\frac{\partial\theta}{\partial x_4} + \dots + x_{n-2}\frac{\partial\theta}{\partial x_n} = 0,$$

it possesses the $n-1$ functionally independent solutions $x_1, u_3, u_4, \dots, u_n$, where

$$\begin{aligned} x_4 &= \frac{1}{2\,!}x_2^2 + u_4, \\ x_6 &= \frac{1}{3\,!}x_2^3 + x_2u_4 + u_6, \\ x_8 &= \frac{1}{4\,!}x_2^4 + \frac{1}{2\,!}x_2^2u_4 + x_2u_6 + u_8, \\ &\vdots \\ x_3 &= x_2x_1 + u_3, \\ x_5 &= \frac{1}{2\,!}x_2^2x_1 + x_2u_3 + u_5, \\ x_7 &= \frac{1}{3\,!}x_2^3x_1 + \frac{1}{2\,!}x_2^2u_3 + x_2u_5 + u_7, \\ &\vdots\,. \end{aligned}$$

Proceeding as before, we make $x_1, x_2, u_3, u_4, \dots, u_n$ the new set of variables and, using the former notation, we find

$$\begin{aligned} \frac{d}{dx_1} &= \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + \frac{x_2^2}{2!}\frac{\partial}{\partial x_5} + \frac{x_2^3}{3!}\frac{\partial}{\partial x_7} + \dots, \\ \frac{d}{dx_2} &= \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4} + x_3\frac{\partial}{\partial x_5} + \dots \\ &= d_2, \\ \frac{d}{du_s} &= \frac{\partial}{\partial x_s} + x_2\frac{\partial}{\partial x_{s+2}} + \frac{x_2^2}{2!}\frac{\partial}{\partial x_{s+4}} + \frac{x_2^3}{3!}\frac{\partial}{\partial x_{s+6}} + \dots. \end{aligned}$$

for $s = 3, 4, 5, \dots$.

From these, we find

$$\frac{d}{dx_1} + u_4\frac{d}{du_5} + u_6\frac{d}{du_7} + \dots = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_5} + x_6\frac{\partial}{\partial x_7} + \dots$$

$$x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_6} + \dots$$

$$\begin{aligned} &= x_1\left(\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_4} + \frac{x_2^2}{2!}\frac{\partial}{\partial x_6} + \dots\right) \\ &\qquad + u_3\left(\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_6} + \frac{x_2^2}{2!}\frac{\partial}{\partial x_8} + \dots\right) + \dots \\ &= x_1\left(\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_4} + \frac{x_2^2}{2!}\frac{\partial}{\partial x_6} + \dots\right) + \sum_{s=1} u_{2s+1}\frac{d}{du_{2s+2}}, \end{aligned}$$

so that
$$d_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_5} + \dots$$
$$+ x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4} + \dots$$
$$= \frac{d}{dx_1} + \sum_{r=3} u_r \frac{d}{du_{r+1}} + x_1 \left(\frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_4} + \frac{x_2^2}{2!} \frac{\partial}{\partial x_6} + \dots \right).$$

Again
$$\frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_6} + x_6 \frac{\partial}{\partial x_8} + \dots - \left(\frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_4} + \frac{x_2^2}{2!} \frac{\partial}{\partial x_6} + \dots \right)$$
$$= u_4 \frac{\partial}{\partial x_6} + (x_2 u_4 + u_6) \frac{\partial}{\partial x_8} + \left(\frac{x_2^2}{2!} u_4 + x_2 u_6 + u_8 \right) \frac{\partial}{\partial x_{10}} + \dots$$
$$= u_4 \frac{d}{du_6} + u_6 \frac{d}{du_8} + \dots,$$

and also
$$x_1 \frac{d}{du_3} + u_3 \frac{d}{du_5} + u_5 \frac{d}{du_7} + \dots = x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_7} + \dots,$$

so that
$$d_2 - \left(\frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_4} + \frac{x_2^2}{2!} \frac{\partial}{\partial x_6} + \dots \right)$$
$$= x_1 \frac{d}{du_3} + u_3 \frac{d}{du_5} + u_4 \frac{d}{du_6} + u_5 \frac{d}{du_7} + u_6 \frac{d}{du_8} + \dots$$
$$= \Delta_2 ;$$

and therefore
$$d_1 = \frac{d}{dx_1} + \sum_{r=3} u_r \frac{d}{du_{r+1}} + x_1 (d_2 - \Delta_2).$$

Again,
$$\frac{d}{du_3} + u_4 \frac{d}{du_7} + u_6 \frac{d}{du_9} + \dots = \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_5} + x_4 \frac{\partial}{\partial x_7} + \dots,$$
$$x_1 \frac{d}{du_4} + u_3 \frac{d}{du_6} + u_5 \frac{d}{du_8} + \dots = x_1 \frac{\partial}{\partial x_4} + x_3 \frac{\partial}{\partial x_6} + x_5 \frac{\partial}{\partial x_8} + \dots ;$$

and therefore
$$\frac{d}{du_3} + x_1 \frac{d}{du_4} + u_3 \frac{d}{du_6} + u_4 \frac{d}{du_7} + u_5 \frac{d}{du_8} + \dots$$
$$= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + x_3 \frac{\partial}{\partial x_6} + \dots$$
$$= d_3.$$

Denoting by Δ_1 the operator
$$\frac{d}{dx_1} + u_3 \frac{d}{du_4} + u_4 \frac{d}{du_5} + \dots,$$
we have
$$d_1 = \Delta_1 + x_1 (d_2 - \Delta_2) ;$$
and accordingly the equation becomes
$$\{x_1 d_2^2 - (x_1 \Delta_2 - \Delta_1) d_2 - d_3\} U = 0,$$
where d_2 denotes $\dfrac{d}{dx_2}$.

19. A solution of this equation is obtainable in the form

$$U = V_0 + x_2 V_1 + \frac{x_2^2}{2!} V_2 + \frac{x_2^3}{3!} V_3 + \ldots,$$

where V_0, V_1 are arbitrary functions of $x_1, u_3, u_4, \ldots, u_n$, and the remaining coefficients V are expressible in terms of these two according to the law

$$x_1 V_{n+2} + (\Delta_1 - x_1 \Delta_2) V_{n+1} - d_3 V_n = 0,$$

for $n = 0, 1, 2, \ldots$.

This form of solution is, however, not so convenient as that which was given in the preceding investigation.

It seems paradoxical that, of the same equation, quite general solutions should exist which are so different in type that one of them involves a single arbitrary function and another of them involves a couple of independent arbitrary functions. The explanation is similar to that which explains the corresponding paradox in the case of the equation*

$$\frac{\partial^2 u}{\partial x^2} = \kappa \frac{\partial u}{\partial t}.$$

SIMULTANEOUS EQUATIONS.

20. Major MacMahon, who indeed originally proposed to me the solution of the equations

$$\bar{d}_p^m U = 0$$

by asking whether they could be treated according to the theory contained in the memoir to which a reference has already (§ 1) been made, tells me that the theory of symmetric functions indicates that solutions of the simultaneous equations

$$\bar{d}_1^{m_1} U = 0, \quad \bar{d}_1^{m_2} U = 0, \ \ldots\ldots$$

(any in number), exist.

A full investigation of the problem thus suggested seems difficult, not on account of the general theory of the equations, but on account of the elaborate analysis required in all cases except the very simplest. It is, of course, possible to deal with particular cases: and it also is possible to infer some of the characteristics of the solutions in the most general cases. All that will here be discussed is the solution of a couple of simultaneous equations of the above type: the extension to the case of more than two is obvious on a review of the analysis in § 26.

21. As an example, consider the simultaneous equations

$$\bar{d}_1^2 U = 0, \quad \bar{d}_1^3 U = 0.$$

* See my *Treatise on differential equations*, § 257.

The most general solution of the former has been shewn to be

$$U = \left(1 + \frac{x_1^2}{2!} d_2 + \frac{x_1^4}{4!} d_2^2 + \frac{x_1^6}{6!} d_2^3 + \ldots\right) \Phi$$

$$+ \left(x_1 + \frac{x_1^3}{3!} d_2 + \frac{x_1^5}{5!} d_2^2 + \frac{x_1^7}{7!} d_2^3 + \ldots\right) \Psi :$$

the solution of the second is of the same type, and therefore it is necessary to determine the limitations upon the arbitrary characters of Φ and Ψ in order that this quantity U may satisfy

$$\bar{d}_1^3 U = 0,$$

that is,

$$(d_1^3 - 3d_1 d_2 + 2d_3)\, U = 0.$$

Now

$$(d_1^3 - 3d_1 d_2 + 2d_3)\, U$$

$$= 2\left(1 + \frac{x_1^2}{2!} d_2 + \frac{x_1^4}{4!} d_2^2 + \ldots\right)(d_3\Phi - d_2\Psi)$$

$$+ 2\left(x_1 + \frac{x_1^3}{3!} d_2 + \frac{x_1^5}{5!} d_2^2 + \ldots\right)(d_3\Psi - d_2^2\Phi);$$

in order that the equation may be satisfied, this expression must vanish identically and therefore

$$\left.\begin{aligned} d_3\Phi &= d_2\Psi \\ d_2^2\Phi &= d_3\Psi \end{aligned}\right\},$$

shewing that both Φ and Ψ satisfy the equation

$$(d_2^3 - d_3^2)\, \Theta = 0,$$

which is of the third order. The operators d_2 and d_3 are

$$\frac{d}{du_2} + u_2 \frac{d}{du_4} + u_3 \frac{d}{du_5} + u_4 \frac{d}{du_6} + \ldots,$$

$$\frac{d}{du_3} + u_2 \frac{d}{du_5} + u_3 \frac{d}{du_6} + u_4 \frac{d}{du_7} + \ldots,$$

respectively; and Θ is some function of $u_2, u_3, \ldots, u_n$. The general theory of differential equations shews that every general solution Θ of the equation involves three independent arbitrary functions; that each of these arbitrary functions involves $n-2$ arguments; and that each of these arguments satisfies the equation

$$\left(\frac{\partial\theta}{\partial u_2} + u_2 \frac{\partial\theta}{\partial u_4} + u_3 \frac{\partial\theta}{\partial u_5} + u_4 \frac{\partial\theta}{\partial u_6} + \ldots\right)^3 = 0.$$

As this characteristic invariant is the triple repetition of a single equation, it follows that the arguments of the three arbitrary functions are the same for each of them; moreover, these arguments are (the simplest) $n-2$ functionally independent solutions of the equation

$$\frac{\partial\theta}{\partial u_2} + u_2 \frac{\partial\theta}{\partial u_4} + u_3 \frac{\partial\theta}{\partial u_5} + u_4 \frac{\partial\theta}{\partial u_6} + \ldots = 0.$$

As in §§ 10, 16, these can be taken to be $v_3, v_4, \ldots, v_n$, where

$$u_4 = \frac{1}{2!} u_2^2 + v_4,$$

$$u_6 = \frac{1}{3!} u_2^3 + u_2 v_4 + v_6,$$

$$u_8 = \frac{1}{4!} u_2^4 + \frac{1}{2!} u_2^2 v_4 + u_2 v_6 + v_8,$$

....

$$u_3 = v_3,$$

$$u_5 = v_3 u_2 + v_5,$$

$$u_7 = v_3 \frac{u_2^2}{2!} + u_2 v_5 + v_7,$$

$$u_9 = v_3 \frac{u_2^3}{3!} + \frac{u_2^2}{2!} v_5 + u_2 v_7 + v_9,$$

....

the explicit values of the variables v being

$$v_{2s} = \frac{1}{s!}(u'_{2s}, u'_{2s-2}, \ldots, u'_4, u_2, 1 \between 1, -u_2)^s,$$

$$v_{2s+3} = \frac{1}{s!}(u'_{2s+3}, u'_{2s+1}, \ldots, u'_7, u_5, u_3 \between 1, -u_2)^s,$$

where

$$u'_{2\kappa} = \kappa!\ u_{2\kappa}, \quad u'_{2\kappa+3} = \kappa!\ u_{2\kappa+3}.$$

To solve the equation for Θ, we change the variables from the set $u_2, u_3, u_4, \ldots, u_n$ to the set $u_2, v_3, v_4, \ldots, v_n$. We denote differentiations with regard to this new set by $\frac{D}{Du_2}, \frac{D}{Dv_3}, \frac{D}{Dv_4}, \ldots, \frac{D}{Dv_n}$; the modified expressions of the operators are obtained as in § 11 and, in particular, we find

$$\frac{D}{Du_2} = \frac{d}{du_2} + u_2 \frac{d}{du_4} + u_3 \frac{d}{du_5} + u_4 \frac{d}{du_6} + \ldots$$

$$= d_2,$$

$$\frac{D}{Dv_3} + v_3 \frac{D}{Dv_6} + v_4 \frac{D}{Dv_7} + v_5 \frac{D}{Dv_8} + \ldots$$

$$= \frac{d}{du_3} + u_2 \frac{d}{du_5} + u_3 \frac{d}{du_6} + u_4 \frac{d}{du_7} + \ldots$$

$$= d_3.$$

Accordingly, the equation for Θ, which is to be determined as a function of $u_2, v_3, v_4, v_5, \ldots, v_n$, now is

$$\frac{D^3\Theta}{Du_2^3} - d_3^2\Theta = 0,$$

where the operator d_3, expressed in terms of the variables $v_3, v_4, v_5, \ldots$ does not involve u_2.

22. Let F, G, H denote three independent arbitrary functions of the $n-2$ arguments $v_3, v_4, v_5, \ldots, v_n$; then the most general solution of the equation for Θ is

$$\Theta = \left(1 + \frac{u_2^3}{3\,!} d_3^2 + \frac{u_2^6}{6\,!} d_3^4 + \frac{u_2^9}{9\,!} d_3^6 + \ldots\right) F$$

$$+ \left(u_2 + \frac{u_2^4}{4\,!} d_3^2 + \frac{u_2^7}{7\,!} d_3^4 + \frac{u_2^{10}}{10\,!} d_3^6 + \ldots\right) G$$

$$+ \left(\frac{u_2^2}{2\,!} + \frac{u_2^5}{5\,!} d_3^2 + \frac{u_2^8}{8\,!} d_3^4 + \frac{u_2^{11}}{11\,!} d_3^6 + \ldots\right) H.$$

We have seen that both Φ and Ψ are solutions of the equation for Θ, subject to the relations

$$d_3\Phi = d_2\Psi, \quad d_2^2\Phi = d_3\Psi:$$

consequently, let the three arbitrary functions in Φ be denoted by F_1, G_1, H_1 and the three in Ψ by F_2, G_2, H_2. As the relation

$$d_3\Phi = d_2\Psi$$

must be satisfied identically, we find, on substituting the values of Φ and Ψ, that

$$d_3 F_1 = G_2, \quad d_3 G_1 = H_2, \quad d_3 H_1 = d_3^2 F_2;$$

and similarly from the relation

$$d_2^2\Phi = d_3\Psi,$$

we find that

$$H_1 = d_3 F_2, \quad d_3^2 F_1 = d_3 G_2, \quad d_3^2 G_1 = d_3 H_2,$$

all of which are satisfied by taking F_1, G_1, F_2 as three independent arbitrary functions, say L, M, N; and then

$$\left.\begin{array}{lll} F_1 = L\ , & G_1 = M\ , & F_2 = N \\ G_2 = d_3 L, & H_2 = d_3 M, & H_1 = d_3 N \end{array}\right\}.$$

When these are substituted, we possess the values of Φ and Ψ; and when these values of Φ and Ψ are inserted in the expression for U, we have the following result:—

The most general solution of the simultaneous equations

$$\bar{d}_1^2 U = 0, \quad \bar{d}_1^3 U = 0,$$

is given by

$$U = \sum_m \sum_p \left\{ \frac{x_1^{2m} u_2^{3p-m}}{2m!\,(3p-m)!} d_3^{2p} + \frac{x_1^{2m+1} u_2^{3p-m+1}}{(2m+1)!\,(3p-m+1)!} d_3^{2p+} \right\} L$$

$$+ \sum_m \sum_p \left\{ \frac{x_1^{2m} u_2^{3p-m+1}}{2m!\,(3p-m+1)!} d_3^{2p} + \frac{x_1^{2m+1} u_2^{3p-m+2}}{(2m+1)!\,(3p-m+2)!} d_3^{2p+1} \right\} M$$

$$+ \sum_m \sum_p \left\{ \frac{x_1^{2m} u_2^{3p-m+2}}{2m!\,(3p-m+2)!} d_3^{2p+1} + \frac{x_1^{2m+1} u_2^{3p-m}}{(2m+1)!\,(3p-m)!} d_3^{2p} \right\} N,$$

where L, M, N are three independent arbitrary functions of the $n-2$ arguments $v_3, v_4, \ldots, v_n$.

23. Another example in which the ultimate solution is differently obtained is as follows: required the most general simultaneous solution of

$$\bar{d}_1^3 U = 0, \quad \bar{d}_1^5 U = 0.$$

Now we have

$$\bar{d}_1^4 = d_1^3 - 3d_1 d_2 + 2d_3,$$

$$\bar{d}_1^5 = d_1^5 - 10d_1^3 d_2 + 20d_1^2 d_3 - 30d_1 (d_4 - \tfrac{1}{2} d_2^2) + 24d_5 - 20d_2 d_3$$

$$= (d_1^2 - 7d_2)(d_1^3 - 3d_1 d_2 + 2d_3)$$

$$+ 6 \{3d_1^2 d_3 - d_1 (5d_4 + d_2^2) + 4d_5 - d_2 d_3\};$$

therefore any solution of $\bar{d}_1^3 U = 0$, which satisfies $\bar{d}_1^5 U = 0$, satisfies also

$$RU = \{3d_1^2 d_3 - d_1 (5d_4 + d_2^2) + 4d_5 - d_2 d_3\} U = 0,$$

and any solution of $\bar{d}_1^3 U = 0$, which satisfies $RU = 0$, satisfies also $\bar{d}_1^5 U = 0$. Consequently, we may replace $\bar{d}_1^5 U = 0$ by $RU = 0$.

The general solution of $\bar{d}_1^3 U = 0$ is known, being given by

$$U = \left\{1 - 2 \sum_{p=3}^{\infty} \frac{x_1^p}{p!} H_{p-3} d_3\right\} P_1 (u_2, u_3, \ldots, u_n)$$

$$+ \left\{x_1 + \sum_{p=3}^{\infty} \frac{x_1^p}{p!} H_{p-1}\right\} P_2 (u_2, u_3, \ldots, u_n)$$

$$+ \left\{\frac{x_1^2}{2!} + \sum_{p=3}^{\infty} \frac{x_1^p}{p!} H_{p-2}\right\} P_3 (u_2, u_3, \ldots, u_n)$$

where
$$H_p = \Sigma (3d_2)^\lambda (-2d_3)^\mu \frac{(\lambda + \mu)!}{\lambda!\ \mu!},$$

the summation in H_p extending over all the terms that correspond to integer solutions of

$$2\lambda + 3\mu = p.$$

This solution must satisfy $RU = 0$ identically: consequently when the value is substituted, the coefficients of the various powers of x_1 must vanish. Writing

$$\Phi_p = H_{p-2} P_3 + H_{p-1} P_2 - 2d_3 H_{p-3} P_1,$$

we have

$$U = P_1 + x_1 P_2 + \frac{x_1^2}{2!} P_3 + \sum_{p=3}^{\infty} \frac{x_1^p}{p!} \Phi_p;$$

so that, substituting in RU and equating to zero the coefficients of the various powers of x_1, we find

$$3d_3 P_3 - D_4 P_2 + D_5 P_1 = 0,$$

$$3d_3 \Phi_3 - D_4 P_3 + D_5 P_2 = 0,$$

$$3d_3 \Phi_4 - D_4 \Phi_3 + D_5 P_3 = 0,$$

and
$$3d_3 \Phi_{p+2} - D_4 \Phi_{p+1} + D_5 \Phi_p = 0,$$

for $p = 3, 4, 5, \ldots$: here, D_4 and D_5 denote $5d_4 + d_2^2$ and $4d_5 - d_2 d_3$ respectively.

Now from the expression given above for Φ_p, and remembering that

$$H_m - 3d_2 H_{m-2} + 2d_3 H_{m-3} = 0,$$

we have

$$\Phi_m - 3d_2 \Phi_{m-2} + 2d_3 \Phi_{m-3} = 0:$$

consequently

$$3d_3\Phi_{p-2} - D_4\Phi_{p+1} + D_5\Phi_p = 3d_2(3d_3\Phi_p - D_4\Phi_{p-1} + D_5\Phi_{p-2}) - 2d_3(3d_3\Phi_{p-1} - D_4\Phi_{p-2} + D_5\Phi_{p-3}).$$

The difference-equation for Φ_p is therefore satisfied for p, if it is satisfied for $p-2$ and for $p-1$: and therefore it is satisfied for all values of p, if satisfied for $p=2$ (with the justifiable convention that $\Phi_2 = P_3$), $p=3$, $p=4$. Moreover, we have at once

$$3d_3\Phi_5 - D_4\Phi_4 + D_5\Phi_3 = H_3(3d_3P_3 - D_4P_2 + D_5P_1) + H_2(3d_3\Phi_3 - D_4P_3 + D_5P_2),$$

on using the relation $H_4 = H_2^2$; and

$$3d_3\Phi_6 - D_4\Phi_5 + D_5\Phi_4 = H_3(3d_3\Phi_3 - D_4P_3 + D_5P_2) + H_2(3d_3\Phi_4 - D_4\Phi_3 + D_5P_3),$$

in a similar manner. Consequently the first three equations are all that need be retained; when rearranged, these three equations are

$$\left.\begin{aligned} 0 &= 3d_3P_3 - D_4P_2 + D_5P_1 \\ 0 &= -D_4P_3 + (D_5 + 9d_2d_3)P_2 - 6d_3^2P_1 \\ 0 &= (D_5 + 9d_2d_3)P_3 - (3d_2D_4 + 6d_3^2)P_2 + 2d_3D_4P_1 \end{aligned}\right\}.$$

Let Δ denote the operator

$$2d_3D_4^3 - 3d_2D_4^2D_5 - 54d_2d_3^2D_4 - 18d_3^2D_4D_5 + D_5^3 + 18d_2d_3D_5^2 + 81d_2^2d_3^2D_5 + 108d_3^5;$$

then P_1, P_2, P_3 are each of them a solution of the equation

$$\Delta P = 0.$$

It may be added that, as indeed is to be expected, Δ is the eliminant of $\bar{d}_1^4$ and $\bar{d}_1^5$ when d_1 is eliminated between them: when expressed in terms of d_2, d_3, d_4, d_5, the value of Δ is

$$\begin{aligned} & 64d_5^3 \\ & + d_5^2(240d_3d_2) \\ & + d_5(-300d_4^2d_2 - 360d_4d_3^2 - 120d_4d_2^3 + 120d_3^2d_2^2 - 12d_2^5) \\ & + 250d_4^3d_3 + 225d_4^2d_3d_2^2 - 180d_4d_3^3d_2 + 60d_4d_3d_2^4 \\ & + 108d_3^5 - 100d_2^3d_3^3 + 5d_3d_2^6, \end{aligned}$$

where it will be noticed that, the sum of the positive coefficients being 1072, the sum of the negative coefficients is -1072.

24. The quantities determined as solutions of this equation are functions of $u_2, u_3, \ldots, u_n$ or, in order to facilitate the construction of an algorithm, say of $u_{21}, u_{31}, u_{41}, \ldots, u_{n1}$: the values of these in terms of the original variables being known.

Let there be now introduced a succession of sets of variables, denoted by

$$u_{32},\ u_{42},\ u_{52},\ u_{62},\ \ldots,\ u_{n2}:$$
$$u_{43},\ u_{53},\ u_{63},\ \ldots,\ u_{n3}:$$
$$u_{54},\ u_{64},\ \ldots,\ u_{n4}:$$
$$u_{65},\ \ldots,\ u_{n5}:$$
$$\ldots\ldots\ldots$$

and defined by the respective sets of equations

$$\left.\begin{aligned} u_{41} &= \frac{u_{21}^2}{2!} + u_{42}, & u_{31} &= u_{32}, \\ u_{61} &= \frac{u_{21}^3}{3!} + u_{21}u_{42} + u_{62}, & u_{51} &= u_{21}u_{32} + u_{52}, \\ u_{81} &= \frac{u_{21}^4}{4!} + \frac{u_{21}^2}{2!}u_{42} + u_{21}u_{62} + u_{82}, & u_{71} &= \frac{u_{21}^2}{2!}u_{32} + u_{21}u_{52} + u_{72}, \\ &\vdots \end{aligned}\right\},$$

$$\left.\begin{aligned} u_{62} &= \frac{u_{32}^2}{2!} + u_{63}, \\ u_{92} &= \frac{u_{32}^3}{3!} + u_{32}u_{63} + u_{93}, \\ &\vdots \\ u_{42} &= u_{43}, \\ u_{72} &= u_{32}u_{43} + u_{73}, \\ u_{10,\,2} &= \frac{u_{32}^2}{2!}u_{43} + u_{32}u_{73} + u_{10,\,3}, \\ &\vdots \\ u_{52} &= u_{53}, \\ u_{82} &= u_{32}u_{53} + u_{83}, \\ u_{11,\,2} &= \frac{u_{32}^2}{2!}u_{53} + u_{32}u_{83} + u_{11,\,3} \\ &\vdots \end{aligned}\right\},$$

and so on. Of the new sets, the first contains two groups of variables, the second three groups, the fourth three groups, and so on. At the respective stages, the sets of n independent variables are as follow*:

$$\begin{array}{llllllllll} x_1, & u_{21}, & u_{31}, & u_{41}, & u_{51}, & u_{61}, & u_{71}, & u_{81}, & \ldots, & u_{n1}: \\ & & | & & & & & & & \\ x_1, & u_{21}, & u_{32}, & u_{42}, & u_{52}, & u_{62}, & u_{72}, & u_{82}, & \ldots, & u_{n2}: \\ & & & | & | & & & & & \\ x_1, & u_{21}, & u_{32}, & u_{43}, & u_{53}, & u_{63}, & u_{73}, & u_{83}, & \ldots, & u_{n3}: \\ & & & & | & | & | & & & \\ x_1, & u_{21}, & u_{32}, & u_{43}, & u_{54}, & u_{64}, & u_{74}, & u_{84}, & \ldots, & u_{n4}: \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

* The vertical lines in the tableau shew the variables which are unchanged in passing from one set to the next.

The operators at the respective stages are as follow.

With the variables in the first line,

$$d_1 = \frac{d}{dx_1}:$$

and the remaining operators are multiple-termed.

With the variables in the second line,

$$d_1 = \frac{d}{dx_1}, \quad d_2 = \frac{d}{du_{21}}:$$

and the remaining operators are multiple-termed.

With the variables in the third line,

$$d_1 = \frac{d}{dx_1}, \quad d_2 = \frac{d}{du_{21}}, \quad d_3 = \frac{d}{du_{32}}:$$

and the remaining operators are multiple-termed; in each case being of the form in § 11. And so on.

If then, in any differential equation, the operator of highest suffix is d_s, it will be sufficient to effect the first s of the above transformations in order to be able to use the preceding simplified forms of $d_1, d_2, d_3, \ldots, d_s$.

25. In order to obtain solutions of $\Delta P = 0$ in § 23, where the highest suffix in the operators is 5, let it be taken in the form

$$d_5^3 P + d_5^2 \Delta_1 P + d_5 \Delta_2 P + \Delta_3 P = 0,$$

and suppose the variables changed to those in the fifth row in the preceding tableau: then as P is independent of x_1, it is a function of $u_{21}, u_{32}, u_{43}, u_{54}, u_{65}, u_{75}, \ldots, u_{n5}$. The operators d_2, d_3, d_4, d_5 that occur are

$$\frac{d}{du_{21}}, \quad \frac{d}{du_{32}}, \quad \frac{d}{du_{43}}, \quad \frac{d}{du_{54}},$$

respectively: and the quantities $u_{65}, u_{75}, \ldots, u_{n5}$ are independent of $u_{21}, u_{32}, u_{43}, u_{54}$ for the purposes of partial derivation.

By proceeding as in §§ 4, 13, we obtain the solution of $\Delta P = 0$ in the following form:

Let H_κ denote the sum of the integral homogeneous functions of weight κ in the roots of the equation

$$\xi^3 + \xi^2 \Delta_1 + \xi \Delta_2 + \Delta_3 = 0,$$

so that H_κ is a rational integral algebraical function of d_2, d_3, d_4. Further, let

$$E_{r,\theta} = H_{\theta-r+1} + \Delta_1 H_{\theta-r} + \ldots + \Delta_{3-r} H_{\theta-2},$$

(so that $E_{3,\theta}$ contains one term, $E_{2,\theta}$ contains two terms, and $E_{1,\theta}$ contains three terms). Finally, let A, B, C denote three independent arbitrary functions of $u_{21}, u_{32}, u_{43}, u_{65}, u_{75}, \ldots, u_{n5}$, that is, of the set of independent variables other than u_{54}; then the most general solution of the equation $\Delta P = 0$ is

$$P = \left\{1 + \sum_{\theta-3}^{\infty} \frac{u_{54}^p}{p!} E_{1,p}\right\} A$$

$$+ \left\{u_{54} + \sum_{\theta-3}^{\infty} \frac{u_{54}^p}{p!} E_{2,p}\right\} B$$

$$+ \left\{\frac{u_{54}^2}{2!} + \sum_{\theta-3}^{\infty} \frac{u_{54}^p}{p!} E_{3,p}\right\} C.$$

This is accordingly the form of each of the quantities P_1, P_2, P_3; let the functions for the three quantities be A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively.

Let Ψ_r denote

$$E_{13}A_r + E_{23}B_r + E_{33}C_r,$$

where

$$E_{13} = H_3 + \Delta_1 H_2 + \Delta_2 H_1 = -\Delta_3,$$

$$E_{23} = H_2 + \Delta_1 H_1, \qquad E_{33} = H_1.$$

Then when the values of P_1, P_2, P_3 are substituted in the three equations

$$0 = 3d_3P_3 \qquad - D_4P_2 \qquad + D_5P_1,$$

$$0 = -D_4P_3 \qquad + (D_5 + 9d_2d_3)\, P_2 \quad - 6d_3^2\, P_1,$$

$$0 = (D_5 + 9d_2d_3)\, P_3 - (3d_2D_4 + 6d_3^2)\, P_2 + 2d_3D_4P_1,$$

the resulting equations must be identically satisfied: that is, the coefficients of the various powers of u_{54} are zero. There thus arise three sets of equations, each singly infinite in number and similar in form to the set in § 23 for Φ_p: it appears, as in that investigation, that each set is satisfied in virtue of three equations; and the three sets of triplets are

$$\left.\begin{aligned} -4B_1 &= 3d_3A_3 - D_4A_2 - d_2d_3A_1 \\ -4C_1 &= 3d_3B_3 - D_4B_2 - d_2d_3B_1 \\ -4\Psi_1 &= 3d_3C_3 - D_4C_2 - d_2d_3C_1 \end{aligned}\right\},$$

$$\left.\begin{aligned} -4B_2 &= -D_4A_3 + 8d_2d_3A_2 - 6d_3^2\, A_1 \\ -4C_2 &= -D_4B_3 + 8d_2d_3B_2 - 6d_3^2\, B_1 \\ -4\Psi_2 &= -D_4C_3 + 8d_2d_3C_2 - 6d_3^2\, C_1 \end{aligned}\right\},$$

$$\left.\begin{aligned} -4B_3 &= 8d_2d_3A_3 - (3d_2D_4 + 6d_3^2)\, A_2 + 2d_3D_4A_1 \\ -4C_3 &= 8d_2d_3B_3 - (3d_2D_4 + 6d_3^2)\, B_2 + 2d_3D_4B_1 \\ -4\Psi_3 &= 8d_2d_3C_3 - (3d_2D_4 + 6d_3^2)\, C_2 + 2d_3D_4C_1 \end{aligned}\right\}.$$

The three first equations give B_1, B_2, B_3 in terms of A_1, A_2, A_3; the three second equations give C_1, C_2, C_3 in terms of B_1, B_2, B_3 and so in terms of A_1, A_2, A_3; and with these values, the three third equations are identically satisfied.

Accordingly we retain A_1, A_2, A_3 as the arbitrary functions. The values of P_1, P_2, P_3 are then known: when these are substituted in U, we have the most general solution of the equations

$$\bar{d}_1^3 U = 0, \quad \bar{d}_1^5 U = 0.$$

expressed in the form of a doubly-infinite series in powers of x_1 and of u_{54}, the coefficients of the various power-combinations being the appropriate derivatives of three independent arbitrary functions of $u_{21}, u_{32}, u_{43}, u_{65}, u_{75}, \ldots, u_{n5}$, derivation of the arbitrary functions taking place solely with regard to u_{21}, u_{32}, u_{43}.

26. The last special example has been worked out in full, because it is sufficiently significant of the march of the analysis required to solve the simultaneous equations

$$\bar{d}_1^q U = 0, \quad \bar{d}_1^p U = 0.$$

We assume $q > p$, $n > q$: if the latter be not justified, some modifications would be required, analogous to those in the last case if the number of variables were less than 5.

The most general solution of

$$\bar{d}_1^p U = 0,$$

is known; it contains p arbitrary functions P of the variables $u_{21}, u_{31}, \ldots, u_{n1}$. It must now satisfy the equation

$$\bar{d}_1^q U = 0.$$

The necessary conditions are that p homogeneous linear relations involving derivatives of the functions P may be satisfied. It follows immediately that each of the functions P satisfies the equation

$$\Delta P = 0,$$

where Δ is the resultant of $\bar{d}_1^p$, $\bar{d}_1^q$ and has the form

$$d_q^p + d_q^{p-1}\Theta_1 + \ldots = 0,$$

while $\Theta_1, \ldots$ involve the differential operators $d_2, d_3, \ldots, d_{q-1}$.

Transform the variables so that they are the qth line in the tableau of § 24: then the operators are

$$d_2 = \frac{d}{du_{21}}, \quad d_3 = \frac{d}{du_{32}}, \quad \ldots, \quad d_{q-1} = \frac{d}{du_{q-1,\,q-2}}, \quad d_q = \frac{d}{du_{q,q-1}}.$$

The solution of the equation

$$\Delta P = 0$$

is obtained as in the corresponding case of § 25: it is a series proceeding in powers of $u_{q,q-1}$, and the coefficients in the series are derivatives of p arbitrary functions of $u_{21}, u_{32}, \ldots, u_{q-1,\,q-2}, u_{q+1,\,q}, u_{q+2,\,q}, \ldots, u_{n,q}$, derivatives with regard to $u_{21}, u_{32}, \ldots, u_{q-1,q-2}$ alone occurring. This form, therefore, is characteristic of each of the p functions P that occur in U: selecting accordingly p functions Q for each of the functions P, there are linear relations among these functions Q and they are such that, out of the p^2 functions Q thus introduced, $p^2 - p$ of them can be expressed in terms of the remaining p which thus remain arbitrary functions. Let these be $Q_1, Q_2, \ldots, Q_p$.

Then each of the functions P is expressible as a series of powers of $u_{q,q-1}$, the coefficients being aggregates of derivatives (with regard to $u_{21}, u_{32}, \ldots, u_{q-1,q-2}$) of the p arbitrary functions Q which involve the arguments $u_{21}, u_{32}, \ldots, u_{q-1,q-2}, u_{q+1,q}, \ldots, u_{n,q}$. Finally, when these values of P are substituted in U, the result is the most general solution of

$$\bar{d}_1^q U = 0, \quad \bar{d}_1^p U = 0,$$

in the form of a doubly-infinite series of powers of x_1 and $u_{q,q-1}$, the coefficients being the appropriate aggregates of derivatives of the p arbitrary functions $Q_1, Q_2, \ldots, Q_p$.

The form of the solution is not unique: another form would be obtained were we to begin with the most general solution of $\bar{d}_1^q U = 0$; and we afterwards should require to transform the variables in Δ, which now would be

$$d_p^q + d_p^{q-1}\Theta_1' + \ldots,$$

only to those which occur in the pth-line in the tableau in § 24. As however both solutions are quite general, they can be changed into one another (and so also for other forms) in a manner similar to that which marks the corresponding transformation for the case already (§ 19) quoted.

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CONTENTS.

I. *Theorems relating to the Product of two Hypergeometric Series.* By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin.

[Received June 1897.]

1. The following theorem is stated without proof by Cayley (*Phil. Mag.* Nov. 1858, and *Collected Papers,* Vol. III. page 268), viz., writing as usual

$$F(\alpha,\ \beta;\ \gamma;\ x) = 1 + \frac{\alpha \,.\, \beta}{1 \,.\, \gamma} x + \frac{\alpha \,.\, \alpha+1 \,.\, \beta \,.\, \beta+1}{1 \,.\, 2 \,.\, \gamma \,.\, \gamma+1} x^2 + \ldots\ldots$$

then the product $\qquad F(\alpha,\ \beta;\ \gamma+\tfrac{1}{2};\ x) \,.\, F(\gamma-\alpha,\ \gamma-\beta;\ \gamma+\tfrac{1}{2};\ x)$

is connected with $\qquad (1-x)^{\alpha+\beta-\gamma} \,.\, F(2\alpha,\ 2\beta;\ 2\gamma;\ x)$

by a simple relation; for if the last-mentioned expression is put equal to

$$1 + Bx + Cx^2 + Dx^3 + \ldots,$$

then the product in question is equal to

$$1 + \frac{\gamma}{\gamma+\frac{1}{2}} \,.\, Bx + \frac{\gamma \,.\, \gamma+1}{\gamma+\frac{1}{2} \,.\, \gamma+\frac{3}{2}} \,.\, Cx^2 + \frac{\gamma \,.\, \gamma+1 \,.\, \gamma+2}{\gamma+\frac{1}{2} \,.\, \gamma+\frac{3}{2} \,.\, \gamma+\frac{5}{2}} \,.\, Dx^3 + \ldots.$$

The object of this paper is to establish the above and other similar theorems.

2. Having given any series $\qquad u \equiv \Sigma a_r x^r \qquad \ldots\ldots(1),$

if we form from it another $\qquad v \equiv \Sigma b_r x^r \qquad \ldots\ldots(2),$

by means of the relation $\qquad \dfrac{b_{r+1}}{b_r} = \dfrac{r+\theta}{r+\phi} \,.\, \dfrac{a_{r+1}}{a_r} \qquad \ldots\ldots(3),$

where θ and ϕ are constants we shall express the connection between v and u by writing

$$v = (\theta;\ \phi;\ u) \text{ or } v = \{\theta;\ \phi;\ u\} \qquad \ldots\ldots(4).$$

The method of proof pursued is to take the normal forms of the equations satisfied by two independent hypergeometric functions, obtain the linear differential equation satisfied by the product of the solutions of these, and investigate in what cases it can be identical with that satisfied by

$$y = (x-1)^{-\kappa} x^{-h} \{\theta;\ \phi;\ (1-x)^{-\sigma} z\} \qquad \ldots\ldots(5),$$

where z is a third hypergeometric function, all three functions having x as argument.

3. Considering any two equations of the types

$$(D^2 + I)\,\zeta = 0, \quad (D^2 + I')\,\zeta' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(6), (7)},$$

it is easily found that the product $\zeta\zeta'$ satisfies the equation

$$D\left[\frac{D^3y + 2PDy + yDP}{Q}\right] + Qy = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(8)},$$

where $I + I' = P$, $I - I' = Q$; D denoting differentiation with respect to x. (Compare Schafheitlin, *Pr.* (No. 99), *Sophien-Realgymn.* Berlin, 29 S. 4°.)

If (6), (7) are the normal forms of the equations satisfied by

$$\eta = F(\alpha,\ \beta;\ \gamma;\ x) \quad \text{and} \quad \eta' = F(\alpha',\ \beta';\ \gamma';\ x)$$

respectively, we have
$$I = \tfrac{1}{4}\left[\frac{1-\lambda^2}{x^2} + \frac{\lambda^2 - \mu^2 + \nu^2 - 1}{x(x-1)} + \frac{1-\nu^2}{(x-1)^2}\right],$$

where
$$\lambda = 1 - \gamma, \quad \mu = \alpha - \beta, \quad \nu = \gamma - \alpha - \beta,$$

with corresponding relations in case of the dashed letters.

The most general solution of (8) in this case then is

$$y = x^{\frac{1}{2}(\gamma+\gamma')}(1-x)^{\frac{1}{2}(\alpha+\beta-\gamma+\alpha'+\beta'-\gamma'+2)}\,\{A\eta_1\eta_1' + B\eta_1\eta_2' + C\eta_2\eta_1' + D\eta_2\eta_2'\} \;\ldots\ldots\text{(9)},$$

where η_1, η_2 are any independent solutions of the equation satisfied by η; η_1', η_2' are any independent solutions of that satisfied by η', and A, B, C, D are arbitrary constants.

4. We now proceed to find the equation whose solution is given by (5) where z is the most general solution of

$$x(1-x)\frac{d^2z}{dx^2} + (A - Bx)\frac{dz}{dx} - Cz = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(10)}.$$

Writing $(1-x)^{-\sigma}z = u$, we have

$$x^3D^2u + (B + 2\sigma)\,x^2Du + \{\sigma(\sigma-1) + \sigma B + C\}\,xu$$
$$- \{2x^2D^2u + (A + B + 2\sigma)\,xDu + (\sigma A + C)\,u\} + xD^2u + ADu = 0 \;\ldots\ldots\text{(11)}.$$

If a solution of this be $\qquad u = \Sigma a_r x^r,$

we obtain the relation

$$\{(r-1)(r-2) + (B + 2\sigma)(r-1) + \sigma(\sigma-1) + \sigma B + C\}\,a_{r-1}$$
$$- \{2r(r-1) + (A + B + 2\sigma)\,r + \sigma A + C\}\,a_r + (r+1)(r+A)\,a_{r+1} = 0 \ldots\ldots\text{(12)}.$$

Making the transformation indicated in (2), (3), and writing D' for $x\dfrac{d}{dx}$, we obtain the equation

$$x(D' + \theta + 1)(D' + \theta)\,\{D'(D'-1) + (B + 2\sigma)\,D' + \sigma(\sigma-1) + \sigma B + C\}\,v$$
$$- (D' + \theta)(D' + \phi - 1)\,\{2D'(D'-1) + (A + B + 2\sigma)\,D' + \sigma A + C\}\,v$$
$$+ \frac{1}{x}(D' + \phi - 1)(D' + \phi - 2)\,D'(D' + A - 1)\,v = 0 \ldots\ldots\ldots\ldots\text{(13)}.$$

On writing $y' = x^{-h}v$ the equation for y' may be obtained from (13) by changing D' into $D' + h$, and on writing

$$\left.\begin{aligned}
&A + 2h = A' \\
&B + 2\sigma + 2h = B' \\
&h(h + A - 1) = L \\
&2h(h-1) + h(A + B + 2\sigma) + \sigma A + C = M \\
&h(h-1) + h(B + 2\sigma) + \sigma(\sigma - 1) + \sigma B + C = N \\
&\theta + h + 1 = \theta' \\
&\phi + h + 1 = \phi'
\end{aligned}\right\} \text{......................................(14)}.$$

$$\left.\begin{aligned}
&2\phi' + A' = P_1 \\
&2\theta' + B' - 2\phi' - A' + 4 = Q_1 \\
&2(\phi' - 1)A' + \phi'(\phi' - 1) + L = P_2 \\
&2\phi'(\theta' - \phi' + 2) + (\theta' + \phi')(A' + B') - 4(\phi' - 1)A' + M - 2L = Q_2 \\
&(\theta' - \phi' + 2)(\theta' - \phi' + 1) + (\theta' - \phi' + 2)(B' - A') + L - M + N = R_2 \\
&(\phi' - 2)\{(\phi' - 1)A' + 2L\} = P_3 \\
&(\phi' - 1)\{\theta'(A' + B') - 2(\phi' - 2)A'\} + (\theta' + \phi' - 2)M - 4(\phi' - 2)L = Q_3 \\
&\theta'(\theta' - \phi' + 2)B' - (\theta' - \phi' + 2)(\phi' - 1)A' + 2(\phi' - 2)L - (\theta' + \phi' - 2)M + 2\theta' N = R_3 \\
&(\phi' - 2)(\phi' - 3)L = P_4 \\
&(\phi' - 2)\{(\theta' - 1)M - 2(\phi' - 3)L\} = Q_4 \\
&\theta'(\theta' - 1)N - (\theta' - 1)(\phi' - 2)M + (\phi' - 2)(\phi' - 3)L = R_4
\end{aligned}\right\} \text{...(15)},$$

and rearranging, this equation becomes

$$\left[D^4 + \left\{\frac{P_1}{x} + \frac{Q_1}{x-1}\right\}D^3 + \left\{\frac{P_2}{x^2} + \frac{Q_2}{x(x-1)} + \frac{R_2}{(x-1)^2}\right\}D^2 + \left\{\frac{P_3}{x^3} + \frac{Q_3}{x^2(x-1)} + \frac{R_3}{x(x-1)^2}\right\}D + \left\{\frac{P_4}{x^4} + \frac{Q_4}{x^3(x-1)} + \frac{R_4}{x^2(x-1)^2}\right\}\right]y' = 0 \text{......(16)}.$$

If we now write
$$y = (x-1)^{-\kappa}y',$$
we finally obtain as the equation for y,

$$\begin{aligned}
D^4y &+ \left\{\frac{P_1}{x} + \frac{Q_1 + 4\kappa}{x-1}\right\}D^3y + \left\{\frac{P_2}{x^2} + \frac{Q_2 + 3\kappa P_1}{x(x-1)} + \frac{R_2 + 3\kappa Q_1 + 6\kappa(\kappa-1)}{(x-1)^2}\right\}D^2y \\
&+ \left\{\frac{P_3}{x^3} + \frac{Q_3 + 2\kappa P_2}{x^2(x-1)} + \frac{R_3 + 2\kappa Q_2 + 3\kappa(\kappa-1)P_1}{x(x-1)^2} + \frac{2\kappa R_2 + 3\kappa(\kappa-1)Q_1 + 4\kappa(\kappa-1)(\kappa-2)}{(x-1)^3}\right\}Dy \\
&+ \left\{\frac{P_4}{x^4} + \frac{Q_4 + \kappa P_3}{x^3(x-1)} + \frac{R_4 + \kappa Q_3 + \kappa(\kappa-1)P_2}{x^2(x-1)^2} + \frac{\kappa R_3 + \kappa(\kappa-1)Q_2 + \kappa(\kappa-1)(\kappa-2)P_1}{x(x-1)^3}\right. \\
&\left. + \frac{\kappa(\kappa-1)R_2 + \kappa(\kappa-1)(\kappa-2)Q_1 + \kappa(\kappa-1)(\kappa-2)(\kappa-3)}{(x-1)^4}\right\}y = 0 \text{............................(17)}.
\end{aligned}$$

This then is an equation two independent solutions of which are given by (5). The other two will be considered later. (See Art. 8.)

5. Let us now examine whether the last equation can be identical with (8). If (8) be written in a form in which the coefficient of D^4y is unity, that of D^3y is $-\frac{1}{Q}\frac{dQ}{dx}$; hence it is evident that the equations cannot be identical unless Q be of the form $x^{-m}(x-1)^{-n}$, where m, n are some constants, positive or negative; but the most general value of P is of the form

$$P=\frac{a}{x^2}+\frac{b}{x(x-1)}+\frac{c}{(x-1)^2},$$

and of Q,

$$Q=\frac{a'}{x^2}+\frac{b'}{x(x-1)}+\frac{c'}{(x-1)^2};$$

hence there are six cases in which the above condition is satisfied, viz.

I. $m=1,\ n=2,\ a'=0,\ b'=-c'$, i.e. $\lambda^2=\lambda'^2,\ \mu^2=\mu'^2$;

II. $m=1,\ n=1,\ a'=0,\ c'=0,\quad \lambda^2=\lambda'^2,\ \nu^2=\nu'^2$;

III. $m=2,\ n=1,\ c'=0,\ a'=-b',\quad \mu^2=\mu'^2,\ \nu^2=\nu'^2$;

IV. $m=2,\ n=0,\ b'=0,\ c'=0,\quad \nu^2=\nu'^2,\ \lambda^2-\mu^2=\lambda'^2-\mu'^2$;

V. $m=0,\ n=2,\ a'=0,\ b'=0,\quad \lambda^2=\lambda'^2,\ \mu^2-\nu^2=\mu'^2-\nu'^2$;

VI. $m=2,\ n=2,\ c'=a',\ b'=-2a',\quad \mu^2=\mu'^2,\ \lambda^2-\nu^2=\lambda'^2-\nu'^2$.

6. In Case I., writing $Q=c'x^{-1}(x-1)^{-2}$, equation (8) can be written in the form

$$D^4y+\left\{\frac{1}{x}+\frac{2}{x-1}\right\}D^3y+2\left\{\frac{a}{x^2}+\frac{b}{x(x-1)}+\frac{c}{(x-1)^2}\right\}D^2y$$
$$+\left\{\frac{-4a}{x^3}+\frac{4a-b}{x^2(x-1)}+\frac{b+2c}{x(x-1)^2}-\frac{2c}{(x-1)^3}\right\}Dy$$
$$+\left\{\frac{4a}{x^4}+\frac{b-4a}{x^3(x-1)}+\frac{-b+c'^2}{x^2(x-1)^2}-\frac{2c+2c'^2}{x(x-1)^3}+\frac{2c+c'^2}{(x-1)^4}\right\}y=0 \quad\text{..........(18).}$$

Comparing (17) and (18) we obtain 14 equations connecting the 16 quantities

$$P_1,\ Q_1,\ P_2,\ Q_2,\ R_2,\ P_3,\ Q_3,\ R_3,\ P_4,\ Q_4,\ R_4,\ c',\ a,\ b,\ c,\ \kappa.$$

Eliminating the 12 first we obtain the equations

$$(2\kappa+1)\{c+\kappa(\kappa+1)\}=0 \quad\text{..................................(19),}$$

$$(2\kappa+1)\left\{(\kappa+1)(3\kappa-2)-b+2\frac{\kappa-1}{\kappa}c\right\}=0 \quad\text{......................(20),}$$

which are both satisfied by $\kappa = -\frac{1}{2}$,
and we thus obtain

$$\left.\begin{aligned} P_1 &= 1 \\ Q_1 &= 4 \\ P_2 &= 2a \\ Q_2 &= 2b + \tfrac{3}{2} \\ R_2 &= 2c + \tfrac{3}{2} \\ P_3 &= -4a \\ Q_3 &= 6a - b \\ R_3 &= 3b + 2c - \tfrac{3}{4} \\ P_4 &= 4a \\ Q_4 &= b - 6a \\ R_4 &= \frac{3a - 3b - c}{2} + \frac{3}{16} \end{aligned}\right\} \qquad (21),$$

and

$$c'^2 = \frac{3}{16} - \frac{c}{2} \qquad (22).$$

The alternatives to $\kappa = -\frac{1}{2}$ involve two relations instead of one among a, b, c, c', and thus cannot lead to cases of equal generality.

Using the above values in the 11 equations (15), connecting the 7 quantities θ', ϕ', A', B', L, M, N with a, b, c, they are found to be satisfied for any values of a, b, c, by

$$\theta' = \tfrac{1}{2},\ \ \phi' = 1,\ \ A' = -1,\ \ B' = 0,\ \ L = 2a,\ \ M = 2b + 4a,\ \ N = 2a + 2b + 2c - \tfrac{3}{4} \qquad (23),$$

and cannot hold in any other case of equal generality.

The relations
$$a' = 0,\ \ b' = -c',\ \ c'^2 = \frac{3}{16} - \frac{c}{2},$$
which characterize this case are equivalent to

$$\lambda = \pm\lambda',\quad \mu = \pm\mu',\quad \nu \pm \nu' = \pm 1 \qquad (24).$$

Consider first the case in which all these ambiguous signs are taken positive, that is to say, in which

$$\gamma = \gamma',\quad \alpha - \beta = \alpha' - \beta',\quad \alpha + \beta + \alpha' + \beta' = 2\gamma - 1 \qquad (25).$$

The values given in (23), when substituted in equations (14), lead to

$$\sigma \quad \pm \left(\frac{\nu - \nu'}{2}\right) = \pm(\nu - \tfrac{1}{2}) \qquad (26).$$

Here also we take the ambiguous sign positive and further obtain either

$$h = -\gamma \ \text{ or } \ h = \gamma - 2 \qquad (27).$$

Taking the former alternative we arrive at the results

$$\left.\begin{aligned} A &= 2\gamma - 1 \\ B &= 2\alpha + 2\beta - 1 \\ C &= 4\alpha\beta \\ \theta &= \gamma - \tfrac{1}{2} \\ \phi &= \gamma \end{aligned}\right\} \quad \text{.................................(28).}$$

It will be found that if in case of the ambiguous equations (24), (26), (27) we make any other choice of alternatives than that above, the theorems thereby deducible will be of exactly the same type as those which follow.

If $z = F(\alpha'', \beta''; \gamma''; x)$ be a solution of equation (10) the values given in (28) are equivalent to the equations

$$\alpha'' = 2\alpha, \quad \beta'' = 2\beta, \quad \gamma'' = 2\gamma - 1 \quad \text{............................(29).}$$

7. We now insert the values we have found in the solution given by (5) of the equation (17) which has been proved identical with (8) whose solution is given by (9), and on dividing both sides by $(1-x)^{\frac{1}{2}} x^{\gamma}$ we obtain the theorem that if z be *any* solution of the equation satisfied by $F(2\alpha, 2\beta; 2\gamma - 1; x)$ then $(\gamma - \frac{1}{2}; \gamma; z(1-x)^{\alpha+\beta+\frac{1}{2}-\gamma})$ is a linear function of the four independent functions which are the products of a solution of the equation satisfied by $F(\alpha, \beta; \gamma; x)$ and a solution of that satisfied by $F(\gamma - \frac{1}{2} - \alpha, \gamma - \frac{1}{2} - \beta; \gamma; x)$. If we denote the general solutions of the last-mentioned equations by η, η', respectively, and use the suffixes 1, 2, &c. to distinguish the particular solutions as in Forsyth's *Differential Equations*, a consideration of the general forms of z, η, η' shows that unless 2γ be an integer, positive or negative, if in this theorem we write $z = z_1 = z_2$, then the function of η, η' involved is

$$\eta_1\eta_1' = \eta_1\eta_2' = \eta_2\eta_1' = \eta_2\eta_2',$$

and the theorem thus gives the equations

$$\{\gamma - \tfrac{1}{2};\ \gamma;\ (1-x)^{\alpha+\beta+\frac{1}{2}-\gamma} F(2\alpha, 2\beta;\ 2\gamma - 1;\ x)\}$$

or

$$\{\gamma - \tfrac{1}{2};\ \gamma;\ (1-x)^{\gamma-\alpha-\beta-\frac{1}{2}} F(2\gamma - 2\alpha - 1,\ 2\gamma - 2\beta - 1;\ 2\gamma - 1;\ x)\}$$

$$= F(\alpha, \beta;\ \gamma;\ x)\, F(\gamma - \alpha - \tfrac{1}{2}, \gamma - \beta - \tfrac{1}{2};\ \gamma;\ x) \text{ ....(30)}$$

$$= (1-x)^{\alpha+\beta-\gamma+1} F(\alpha, \beta;\ \gamma;\ x)\, F(\alpha + \tfrac{1}{2}, \beta + \tfrac{1}{2};\ \gamma;\ x) \text{ ....(30')}$$

$$= (1-x)\, F(\gamma - \alpha, \gamma - \beta;\ \gamma;\ x)\, F(\alpha + \tfrac{1}{2}, \beta + \tfrac{1}{2};\ \gamma;\ x) \text{ ....(30'')}$$

$$= (1-x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta;\ \gamma;\ x)\, F(\gamma - \alpha - \tfrac{1}{2}, \gamma - \beta - \tfrac{1}{2};\ \gamma;\ x) \text{.........(30''').}$$

The type of the theorem is the same whether the first or the second form of the left-hand member be used.

If we take the first form, Cayley's theorem and equation (30) are identical, and if in the latter γ were changed into $\gamma + \frac{1}{2}$ they would be expressed by the same symbols.

If we write $z=z_3=z_4$, then the function of η, η' involved is

$$\eta_3\eta_3'=\eta_3\eta_4'=\eta_4\eta_3'=\eta_4\eta_4',$$

and the equations obtained are of exactly the same type as those above but expressed in different symbols.

It will also be found that we obtain equations of exactly the same type by choosing appropriate functions of z, η, η' which proceed in descending powers of x.

8. A question naturally arises as to the other two solutions of (17) which are not given by (5). Its most general solution is of the form

$$y=(x-1)^{-\kappa}x^{-h}v,$$

where v is given by (13). The relation (12) which exists among the coefficients when v is expanded in powers, shows that there are values of v which proceed in ascending powers of x of which the first terms are respectively

$$1,\quad x^{1-A},\quad x^{1-\phi},\quad x^{2-\phi};$$

of these the first two are, but the last two cannot be, given by (5).

If however, reversing the train of substitutions by which v is derived from z we write $u'=(\phi;\ \theta;\ v)$ where v is given by (13), the relation between successive coefficients in u' is not (12) but the result of multiplying equation (12) by

$$(\theta+r)(\theta+r-1)(\phi+r)(\phi+r-1),$$

this relation being obtained from (13) in a manner similar to that in which (13) is obtained from (12).

The relation thus obtained leads to a differential equation for u' identical not with (11) but with the result of performing on (11) the operation indicated by

$$(xD+\theta)(xD+\theta-1)(xD+\phi)(xD+\phi-1).$$

Such an equation is of the sixth order; four and only four of its solutions (combined linearly) are admissible from which to derive v by means of the relation $v=(\theta;\ \phi;\ u)$ for our theorem. This equation is equivalent to that obtained by equating the left-hand member of (11) not to zero but to

$$C_1x^{-\phi}+C_2x^{1-\phi}+C_3x^{-\theta}+C_4x^{1-\theta} \quad\text{.................................(31)},$$

C_1, C_2, C_3, C_4 being arbitrary constants. Changing in such an equation the value u' into z' by means of the relation $u'=(1-x)^{-\sigma}z'$ it becomes

$$[x(1-x)D^2+(A-Bx)D-C]\,z'=(C_1x^{-\phi}+C_2x^{1-\phi}+C_3x^{-\theta}+C_4x^{1-\theta})(1-x)^{\sigma-1}\ \text{...(32)}.$$

By expanding the right-hand member in powers of x its solution can be deduced from that of a series of equations of the type

$$[x(1-x)D^2+(A-Bx)D-C]\,z'=x^{\epsilon},$$

or

$$[x(1-x)D^2+(\gamma''-(\alpha''+\beta''+1)x)D-\alpha''\beta'']\,z'=x^{\epsilon}\ \text{.....................(33)}.$$

It may be easily established that a solution of this is

$$z' = \frac{x^{\epsilon+1}}{(\epsilon+\gamma'')(\epsilon+1)} + \frac{(\epsilon+\alpha''+1)(\epsilon+\beta''+1)}{(\epsilon+\gamma'')(\epsilon+\gamma''+1)(\epsilon+1)(\epsilon+2)} x^{\epsilon+2} + \ldots,$$

which we will write in the form,

$$z' = \frac{x^{\epsilon+1}}{(\epsilon+\gamma'')(\epsilon+1)} F(\alpha''+\epsilon+1,\ \beta''+\epsilon+1;\ \gamma''+\epsilon+1,\ \epsilon+2;\ x) \ldots\ldots\ldots\ldots(34).$$

It should be noted that the natural numbers do not here occur factorially in the denominators of the coefficients of the powers of x.

A solution of the equation

$$[x(1-x)D^2 + (\gamma'' - (\alpha''+\beta''+1)x)D - \alpha''\beta'']\, z' = x^{\epsilon}(1-x)^{\sigma-1} \ldots\ldots\ldots\ldots(35),$$

is therefore

$$z' = \frac{x^{\epsilon+1}}{(\epsilon+\gamma'')(\epsilon+1)} F(\alpha''+\epsilon+1,\ \beta''+\epsilon+1;\ \gamma''+\epsilon+1,\ \epsilon+2;\ x)$$

$$- \frac{(\sigma-1)\,x^{\epsilon+2}}{1\,.\,(\epsilon+\gamma''+1)(\epsilon+2)} F(\alpha''+\epsilon+2,\ \beta''+\epsilon+2;\ \gamma''+\epsilon+2,\ \epsilon+3;\ x)$$

$$+ \frac{(\sigma-1)(\sigma-2)\,x^{\epsilon+3}}{1\,.\,2\,.\,(\epsilon+\gamma''+2)(\epsilon+3)} F(\alpha''+\epsilon+3,\ \beta''+\epsilon+3;\ \gamma''+\epsilon+3,\ \epsilon+4;\ x) - \ldots\ldots(36),$$

the complementary function being the most general value of z, as given by (10).

On forming v by the relation

$$v = (\theta;\ \phi;\ (1-x)^{-\sigma} z') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37),$$

it appears that the leading term of v has the same index as the leading term of z' so that in order to obtain the expressions for v, the indices of whose leading terms are respectively $1-\phi$, $2-\phi$, we must (unless θ differs from ϕ by an integer which is not the case here) write $C_3 = C_4 = 0$.

The two values of z' to be inserted in (37) in order to obtain thereby the missing solutions of (13) may thus be obtained from (36) by giving ϵ the values $-\phi$, $1-\phi$.

Since

$$(C_1 x^{-\phi} + C_2 x^{1-\phi})(1-x)^{\sigma-1} = (C_1 + C_2)\, x^{-\phi}(1-x)^{\sigma-1} - C_2 x^{-\phi}(1-x)^{\sigma},$$

another solution may be obtained from (36) by giving ϵ the value $-\phi$, and increasing σ by unity.

9. The two linear relations which connect the two independent solutions of (17) last obtained with the other two solutions of (8) may of course be expressed in an infinite number of ways. We may specify the following

$$(\gamma-1)\Big(\tfrac{1}{2};\ 1;\ (1-x)^{\alpha+\beta+\frac{1}{2}-\gamma}\Big\{\frac{1}{\gamma-1} F(2\alpha-\gamma+1,\ 2\beta-\gamma+1;\ \gamma,\ 2-\gamma;\ x)$$

$$+ \frac{\alpha+\beta+\frac{1}{2}-\gamma}{1}\,\frac{x}{\gamma} F(2\alpha-\gamma+2,\ 2\beta-\gamma+2;\ \gamma+1,\ 3-\gamma;\ x)$$

$$+ \frac{(\alpha+\beta+\frac{1}{2}-\gamma)(\alpha+\beta+\frac{3}{2}-\gamma)}{1\,.\,2}\,\frac{x^2}{\gamma+1} F(2\alpha-\gamma+3,\ 2\beta-\gamma+3;\ \gamma+2,\ 4-\gamma;\ x) + \ldots\ldots\Big\}\Big)$$

$$= F(\alpha,\ \beta;\ \gamma;\ x)\,.\,F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 2-\gamma;\ x) \ldots\ldots\ldots\ldots\ldots\ldots(38).$$

The right-hand member of this equation and that of nearly all those which follow may be written in other forms, showing a power of $1-x$ as a factor.

By interchanging α and $\frac{1}{2}-\alpha$, β and $\frac{1}{2}-\beta$, γ and $2-\gamma$ we obtain another form for the left-hand member.

Another relation among the same solutions independent of the last may be obtained from it by interchanging α and $\gamma-\alpha-\frac{1}{2}$, β and $\gamma-\beta-\frac{1}{2}$ ....(39).

10. If Case II. of Art. 5 be investigated in a similar manner it will be found that similar theorems hold provided in addition to the preliminary conditions $\lambda = \pm\lambda'$, $\nu = \pm\nu'$, the relation $\mu \pm \mu' = \pm 1$ also holds, and that without loss of generality we may as in Case I. take all these ambiguous signs positive. These conditions are equivalent to

$$\beta' = \alpha - \tfrac{1}{2}, \quad \alpha' = \beta + \tfrac{1}{2}, \quad \gamma' = \gamma \quad \text{.................................(40).}$$

In this case we obtain the following values:

$$\left.\begin{aligned} &\sigma = \gamma-\alpha-\beta+\tfrac{1}{2} \ (\text{or } \sigma = \alpha+\beta-\gamma+\tfrac{1}{2}),\\ &h = -\gamma \qquad\qquad (\text{or } h = \gamma-2)\\ &\kappa = -1\\ &\theta = \gamma - \tfrac{1}{2}\\ &\phi = \gamma\\ &A = 2\gamma - 1\\ &B = 2\alpha + 2\beta\\ &C = 2\beta(2\alpha-1),\end{aligned}\right\} \text{.........................(41).}$$

the last three leading to $\quad \alpha'' = 2\alpha - 1, \quad \beta'' = 2\beta, \quad \gamma'' = 2\gamma.$

As in Case I. the alternatives rejected only lead to theorems of the same type as do those chosen.

The equations which correspond to (30)—(30‴) in this case may be written in the form

$$\{\gamma-\tfrac{1}{2};\ \gamma;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}} F(2\alpha-1, 2\beta;\ 2\gamma-1;\ x)\},$$

or

$$\{\gamma-\tfrac{1}{2};\ \gamma;\ (1-x)^{\gamma-\alpha-\beta-\frac{1}{2}} F(2\gamma-2\alpha, 2\gamma-2\beta-1;\ 2\gamma-1;\ x)\}$$

$$= (1-x)^{\alpha+\beta-\gamma} F(\alpha, \beta;\ \gamma;\ x) . F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \gamma;\ x) \text{ ....(42)}$$

$$= F(\alpha, \beta;\ \gamma;\ x) . F(\gamma-\alpha+\tfrac{1}{2}, \gamma-\beta-\tfrac{1}{2};\ \gamma;\ x) \text{ ....(42')}$$

$$= F(\gamma-\alpha, \gamma-\beta;\ \gamma;\ x) . F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \gamma;\ x) \text{ ....(42'')}$$

$$= (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta;\ \gamma;\ x) . F(\gamma-\alpha+\tfrac{1}{2}, \gamma-\beta-\tfrac{1}{2};\ \gamma;\ x) \text{ ......(42''').}$$

As in Case I. if we consider the product of those solutions of the equations satisfied by $F(\alpha, \beta;\ \gamma;\ x)$ and $F(\alpha-\frac{1}{2}, \beta+\frac{1}{2};\ \gamma;\ x)$ which, proceeding in ascending powers of x, begin with $x^{1-\gamma}$, the equations obtained are of exactly the same type as these, but expressed in different symbols.

Relations among the other two solutions of (8) and of (17) may, after dividing by $x^{1-\gamma}$, be written in the form

$$(1+2\beta-2\alpha)(\gamma-1)\left\{\tfrac{1}{2};\ 1;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\left(\frac{1}{\gamma-1} F(2\alpha-\gamma, 2\beta-\gamma+1;\ \gamma, 2-\gamma;\ x)\right.\right.$$

$$+ \frac{\alpha+\beta-\gamma-\frac{1}{2}}{1}\frac{x}{\gamma} F(2\alpha-\gamma+1, 2\beta-\gamma+2;\ \gamma+1, 3-\gamma;\ x)$$

$$\left.\left. + \frac{(\alpha+\beta-\gamma-\frac{1}{2})(\alpha+\beta-\gamma+\frac{1}{2})}{1 . 2}\frac{x^2}{\gamma+1} F(2\alpha-\gamma+2, 2\beta-\gamma+3;\ \gamma+2, 4-\gamma;\ x) + \dots\right)\right\}$$

$$= (2-2\alpha) F(\alpha, \beta;\ \gamma;\ x) . F(\tfrac{3}{2}-\alpha, \tfrac{1}{2}-\beta;\ 2-\gamma;\ x)$$

$$+ (2\beta-1) F(1-\alpha, 1-\beta;\ 2-\gamma;\ x) . F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \gamma;\ x) \text{.........(43).}$$

Another independent relation may be obtained by writing in the above

$$\alpha-\gamma+1 \text{ for } \alpha, \quad \beta-\gamma+1 \text{ for } \beta, \quad 2-\gamma \text{ for } \gamma,$$

and using an obvious alternative form of the right-hand member ....(44).

Another relation among the same solutions may be deduced from (43) by changing

$$\alpha \text{ into } \gamma-\alpha+\tfrac{1}{2} \text{ and } \beta \text{ into } \gamma-\beta-\tfrac{1}{2},$$

and using a similar alternative form of the right-hand member............................(45).

A fourth relation may be obtained from (43) by changing

$$\alpha \text{ into } \tfrac{3}{2}-\alpha, \quad \beta \text{ into } \tfrac{1}{2}-\beta, \quad \gamma \text{ into } 2-\gamma \text{.......................(46).}$$

The left-hand members of (45), (46) involve the alternative value of σ [see (41) and (36)] and the corresponding values of α'', β'', γ''.

By taking the solutions of equations (8) and (17) which proceed in ascending powers commencing with $x^{2-\gamma}$ we obtain a fifth relation which, after dividing by $x^{1-\gamma}$, may be written in the form

$$x\left\{\tfrac{3}{2};\ 2;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\left(\frac{1}{(2-\gamma)\gamma}F(2\alpha-\gamma+1,\ 2\beta-\gamma+2;\ \gamma+1,\ 3-\gamma;\ x)\right.\right.$$

$$+\frac{\alpha+\beta-\gamma+\frac{1}{2}}{(3-\gamma)(\gamma+1)1}xF(2\alpha-\gamma+2,\ 2\beta-\gamma+3;\ \gamma+2,\ 4-\gamma;\ x)$$

$$\left.\left.+\frac{(\alpha+\beta-\gamma+\frac{1}{2})(\alpha+\beta-\gamma+\frac{3}{2})}{(4-\gamma)(\gamma+2)1.2}x^2F(2\alpha-\gamma+3,\ 2\beta-\gamma+4;\ \gamma+3,\ 5-\gamma;\ x)+\dots\right)\right\}$$

$$=\frac{2}{(1+2\beta-2\alpha)(1-\gamma)}[F(\alpha,\ \beta;\ \gamma;\ x).F(\tfrac{3}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 2-\gamma;\ x)$$

$$-F(\alpha-\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma;\ x).F(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x)] \text{....................(47).}$$

As the right-hand member is unaltered by changing at the same time α into $1-\beta$, β into $1-\alpha$, and γ into $2-\gamma$, so also must the left.

11. If in Case II. we choose appropriate solutions proceeding in descending powers of x we arrive at Case III., in which similar theorems hold provided in addition to the preliminary conditions $\mu=\pm\mu'$, $\nu=\pm\nu'$, the further relation $\lambda\pm\lambda'=\pm1$ is satisfied.

This case may also be investigated independently in the same manner as the others.

The equations which in this case correspond to (30)—(30‴) may be written in the form

$$\{\gamma+\tfrac{1}{2};\ \gamma+1;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}F(2\alpha,\ 2\beta;\ 2\gamma;\ x)\},$$

or

$$\{\gamma+\tfrac{1}{2};\ \gamma+1;\ (1-x)^{\gamma-\alpha-\beta-\frac{1}{2}}F(2\gamma-2\alpha,\ 2\gamma-2\beta;\ 2\gamma;\ x)\}$$

$$=F(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ x).F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x) \text{..............................(48)}$$

$$=(1-x)^{\alpha+\beta-\gamma}F(\alpha,\ \beta;\ \gamma;\ x).F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x) \text{.......................(48')}$$

$$=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ x).F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta+\tfrac{1}{2};\ \gamma+1;\ x) \text{ ...(48'')}$$

$$=F(\alpha,\ \beta;\ \gamma;\ x).F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta+\tfrac{1}{2};\ \gamma+1;\ x) \text{ ....(48''').}$$

A relation among the other two solutions of (8) and of (17) may, on division by $x^{-\gamma}$, be written in the form

$$(\gamma-1)\left\{\tfrac{1}{2};\ 1;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\ \frac{1}{\gamma-1}F(2\alpha-\gamma,\ 2\beta-\gamma;\ \gamma,\ 1-\gamma;\ x)\right.$$

$$+\frac{\alpha+\beta-\gamma-\frac{1}{2}}{1}\frac{x}{\gamma}F(2\alpha-\gamma+1,\ 2\beta-\gamma+1;\ \gamma+1,\ 2-\gamma;\ x)$$

$$\left.+\frac{(\alpha+\beta-\gamma-\frac{1}{2})(\alpha+\beta-\gamma+\frac{1}{2})}{1\,.\,2}\frac{x^2}{\gamma+1}F(2\alpha-\gamma+2,\ 2\beta-\gamma+2;\ \gamma+2,\ 3-\gamma;\ x)+\ldots)\right\}$$

$$=F(\alpha,\ \beta;\ \gamma;\ x)\,.\,F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 1-\gamma;\ x)$$

$$+\frac{(\alpha-\frac{1}{2})(\beta-\frac{1}{2})}{\gamma(1-\gamma)}xF(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x)\,.\,F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x)\ldots(49).$$

Another relation among the same solutions may be obtained from this by changing

α into $\frac{1}{2}-\alpha$, β into $\frac{1}{2}-\beta$, γ into $1-\gamma$ ....(50).

A third relation among the same solutions may be obtained from (49) by changing

α into $\gamma-\alpha$, β into $\gamma-\beta$,

and using an obvious alternative form of the right-hand member ....(51).

A fourth relation may be obtained from (49) by changing

α into $\alpha+\frac{1}{2}-\gamma$, β into $\beta+\frac{1}{2}-\gamma$, γ into $1-\gamma$,

and using an obvious alternative form of the right-hand member ....(52).

A fifth relation on division by $x^{1-\gamma}$ may be written in the form

$$\gamma(1-\gamma)\left\{\tfrac{3}{2};\ 2;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\left(\frac{1}{\gamma(1-\gamma)}F(2\alpha-\gamma+1,\ 2\beta-\gamma+1;\ \gamma+1,\ 2-\gamma;\ x)\right.\right.$$

$$+\frac{\alpha+\beta-\gamma+\frac{1}{2}}{1}\frac{x}{(\gamma+1)(2-\gamma)}F(2\alpha-\gamma+2,\ 2\beta-\gamma+2;\ \gamma+2,\ 3-\gamma;\ x)$$

$$\left.\left.+\frac{(\alpha+\beta-\gamma+\frac{1}{2})(\alpha+\beta-\gamma+\frac{3}{2})}{1\,.\,2}\frac{x^2}{(\gamma+2)(3-\gamma)}F(2\alpha-\gamma+3,\ 2\beta-\gamma+3;\ \gamma+3,\ 4-\gamma;\ x)+\ldots\right)\right\}$$

$$=F(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x)\,.\,F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x)\ldots\ldots(53)$$

$$=F(\alpha-\gamma+1,\ \beta-\gamma+1;\ 2-\gamma;\ x)\,.\,F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta+\tfrac{1}{2};\ \gamma+1;\ x)\ \ldots\ldots(53').$$

By interchanging α and $\gamma-\alpha$, β and $\gamma-\beta$, which does not alter the value of the right-hand member, we obtain another form of the left-hand member.

12. Of the three relations which have been obtained in Case I. in equations (30)—(30‴), in Case II. in equations (42)—(42‴), in Case III. in equations (48)—(48‴), any two can be deduced from the remaining one by changing the constants and using the relations connecting three hypergeometric series whose constants differ by integers (Gauss, *Complete Works*, Vol. III. p. 133).

13. Cases IV., V., VI. of Art. 5, will be found on examination to lead to no theorems of equal generality with those above.

14. We next proceed to consider some special cases in which the relations we have obtained can be expressed more simply.

Suppose in Case I. we have $\gamma = \alpha + \beta + \frac{1}{2}$. Using this relation in equations (30)—(30‴) the left-hand member becomes a hypergeometric series of the third order and we obtain the relations

$$F(2\alpha, 2\beta, \alpha+\beta;\ 2\alpha+2\beta, \alpha+\beta+\tfrac{1}{2};\ x) = [F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x)]^2 \quad \text{.........(54)}$$
$$= (1-x)^{\frac{1}{2}} F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) . F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \quad \text{.........(54')}$$
$$= (1-x)[F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x)]^2 \quad \text{.........(54'')}.$$

The natural numbers *do* occur factorially in the denominators in this hypergeometric series of the third order and in others below.

Equation (38) now identical with (39) becomes

$$F(\alpha-\beta+\tfrac{1}{2}, \beta-\alpha+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$= F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) . F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x) \quad \text{..............(55)},$$

which can also be expressed in other ways by taking out a power of $1-x$ as a factor.

It may be shown more easily directly that the linear differential equation satisfied by the square of the hypergeometric series $F(\alpha, \beta;\ \gamma;\ x)$ becomes identical with that of a hypergeometric series of the third order if, and only if, $\gamma = \alpha + \beta + \frac{1}{2}$. This has been done by von Clausen, *Crelle's Journal*, Vol. III., where equations (54), (55) are given. On equating the appropriate third independent solutions of the two equations to each other the result is of the same type as (54)—(54″) expressed in different symbols.

If $\gamma = \alpha + \beta - \frac{1}{2}$ relations are obtained of the same type as (54)—(55) expressed in different symbols.

If in Case II. the same relation $\gamma = \alpha + \beta + \frac{1}{2}$ holds, equations (42)—(42‴) become, taking the second form of the left-hand member,

$$F(2\alpha, 2\beta+1, \alpha+\beta;\ 2\alpha+2\beta, \alpha+\beta+\tfrac{1}{2};\ x)$$
$$= F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) . F(\alpha, \beta+1;\ \alpha+\beta+\tfrac{1}{2};\ x) \quad \text{..........................(56)}$$
$$= F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) . F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \quad \text{...........(56')}.$$

Equations (45), (46) are now identical and reduce to the form

$$(1+2\beta-2\alpha) F(\beta-\alpha+\tfrac{3}{2}, \alpha-\beta+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$= (1-2\alpha) F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) . F(\tfrac{3}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$+ 2\beta F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) . F(1-\alpha, 1-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x) \quad \text{...............(57)},$$

in which the right-hand member may be written in a variety of forms.

Equation (47) may now be written in the form

$$F(\alpha-\beta-\tfrac{1}{2}, \beta-\alpha+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x) - 1$$
$$= \frac{(1+2\beta-2\alpha)}{2(2\alpha+2\beta-1)} [F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) . F(\tfrac{3}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$- F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) . F(1-\alpha, 1-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)] \quad \text{.........(58)}.$$

If instead of the relation $\gamma = \alpha + \beta + \frac{1}{2}$, $\gamma = \alpha + \beta - \frac{1}{2}$ holds, equations (43), (44) reduce to one of the same type as (57) expressed in different symbols, while (47) again reduces to one of the same type as (58).

In Case III. also if we suppose $\gamma = \alpha + \beta + \frac{1}{2}$ or $\gamma = \alpha + \beta - \frac{1}{2}$, the results already obtained may be expressed more simply and both suppositions lead to equations of the same type; we choose the latter. Equations (48)—(48‴) then become

$$F(2\alpha, 2\beta, \alpha+\beta;\ 2\alpha+2\beta-1, \alpha+\beta+\tfrac{1}{2};\ x)$$
$$= F(\alpha-\tfrac{1}{2}, \beta-\tfrac{1}{2};\ \alpha+\beta-\tfrac{1}{2};\ x)\,.\,F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \quad\ldots\ldots(59)$$
$$= F(\alpha, \beta;\ \alpha+\beta-\tfrac{1}{2};\ x)\,.\,F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \quad\ldots\ldots(59').$$

Equations (49) and (52) are now identical and reduce to

$$F(\alpha-\beta+\tfrac{1}{2}, \beta-\alpha+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta-\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$= F(\alpha, \beta;\ \alpha+\beta-\tfrac{1}{2};\ x)\,.\,F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$+ \frac{(\alpha-\frac{1}{2})(\beta-\frac{1}{2})}{(\alpha+\beta-\frac{1}{2})(\frac{3}{2}-\alpha-\beta)}\, xF(1-\alpha, 1-\beta;\ \tfrac{5}{2}-\alpha-\beta;\ x)\,.\,F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \ldots(60).$$

Equations (53), (53′) now become of the form

$$\left\{\tfrac{3}{2};\ 2;\ (1-x)^{-1}F(\alpha-\beta+\tfrac{1}{2}, \beta-\alpha+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{5}{2}-\alpha-\beta;\ x)\right\}$$
$$= F(1-\alpha, 1-\beta;\ \tfrac{5}{2}-\alpha-\beta;\ x)\,.\,F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \quad\ldots\ldots(61)$$
$$= F(\tfrac{3}{2}-\alpha, \tfrac{3}{2}-\beta;\ \tfrac{5}{2}-\alpha-\beta;\ x)\,.\,F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \quad\ldots\ldots(61').$$

The relations obtained also admit of some simplification in all three cases if γ and $\alpha+\beta$ differ by half any odd integer.

15. We have seen in the preceding article that if $\gamma = \alpha+\beta+\frac{1}{2}$ all the solutions of the linear differential equation satisfied by $[F(\alpha, \beta;\ \gamma;\ x)]^2$ are solutions of an equation of the type of (17); and, which is the same thing, that if $\gamma = \alpha+\beta-\frac{1}{2}$ all the solutions of the differential equation satisfied by $(1-x)[F(\alpha, \beta;\ \gamma;\ x)]^2$ are solutions of an equation of the same type; the theorems deducible in these cases being stated in (54)—(55).

The cases discussed include two others in which all the solutions of the equation satisfied by $(1-x)^a F[\alpha, \beta;\ \gamma;\ x]^2$ are solutions of (17), viz. if in Case II. we write $\beta+\frac{1}{2}=\alpha$, and if in Case III. we write $\gamma = \frac{1}{2}$ (or $\frac{3}{2}$); it seems unnecessary to give the forms which equations (42)—(53′) then assume.

The theorems obtained involving the square of a hypergeometric series require that either λ, μ, or ν should be $\pm\frac{1}{2}$.

It appears probable that in other cases than these three all the solutions of the equation satisfied by $(1-x)^a[F(\alpha, \beta;\ \gamma;\ x)]^2$ satisfy an equation like (17).

16. In connection with the preceding remark the writer has investigated in what cases the linear equation satisfied by $[F(\alpha, \beta;\ \gamma;\ x)]^2$ may have one or two solutions but not all three in the form of a hypergeometric series of the third order. The method pursued, (that of examining in what cases the successive coefficients of the

latter series can obey the law required in F^2), was very tedious; it is believed however that the following cases include all.

The existing relations in the various cases, after division in some instances by a power of x, may be written as follows:

Case I. $\nu=\gamma-\alpha-\beta=\frac{3}{2},\quad \lambda=1-\gamma=\pm\frac{1}{2}.$

$$\left[F\left(\frac{-1+\mu}{2},\ \frac{-1-\mu}{2};\ \tfrac{1}{2};\ x\right)\right]^2-\left(\frac{1-\mu^2}{\mu}\right)^2 x\left[F\left(\frac{\mu}{2},\ -\frac{\mu}{2};\ \tfrac{3}{2};\ x\right)\right]^2$$

$$=F(\mu-1,\ -\mu-1,\ 1-2\mu^2;\ \tfrac{1}{2},\ -2\mu^2;\ x)\ \ldots\ldots\ldots\ldots(62),$$

or

$$=F(\mu-1,\ -\mu-1;\ \tfrac{1}{2};\ x)+\frac{\mu^2-1}{\mu^2}xF(\mu,\ -\mu;\ \tfrac{3}{2};\ x)\ldots\ldots\ldots\ldots(62'),$$

$$F\left(\frac{-1+\mu}{2},\ \frac{-1-\mu}{2};\ \tfrac{1}{2};\ x\right).F\left(\frac{\mu}{2},\ -\frac{\mu}{2};\ \tfrac{3}{2};\ x\right)=F(\mu-\tfrac{1}{2},-\mu-\tfrac{1}{2},\tfrac{3}{2}-2\mu^2;\ \tfrac{3}{2},\ \tfrac{1}{2}-2\mu^2;\ x)\ldots(63)$$

$$=F(\mu-\tfrac{1}{2},\ -\mu-\tfrac{1}{2};\ \tfrac{3}{2};\ x)+\tfrac{1}{3}xF(\mu+\tfrac{1}{2},\ -\mu+\tfrac{1}{2};\ \tfrac{5}{2};\ x)\ldots\ldots\ldots\ldots(63'),$$

$$\left[F\left(\frac{-1+\mu}{2},\ \frac{-1-\mu}{2};\ \tfrac{1}{2};\ x\right)\right]^2+\left(\frac{1-\mu^2}{\mu}\right)^2 x\left[F\left(\frac{\mu}{2},\ -\frac{\mu}{2};\ \tfrac{3}{2};\ x\right)\right]^2$$

$$=F(-1,\ \mu^2-1;\ \mu^2;\ x)=1+\frac{1-\mu^2}{\mu^2}x\ \ldots\ldots\ldots\ldots(64).$$

Case II. $\nu=\gamma-\alpha-\beta=\frac{3}{2},\quad \mu=\alpha-\beta=\pm\frac{1}{2}.$

$$\left[F\left(-\frac{\lambda}{2},\ \frac{-\lambda-1}{2};\ 1-\lambda;\ x\right)\right]^2=F(-\lambda-1,\ -\lambda-\tfrac{1}{2},\ 2\lambda^2-\lambda;\ -2\lambda+1,\ 2\lambda^2-\lambda-1;\ x)\ldots(65)$$

$$=F(-\lambda-1,\ -\lambda-\tfrac{1}{2};\ -2\lambda+1;\ x)-\frac{\lambda+1}{2(1-\lambda)(1-2\lambda)}xF(-\lambda,\ -\lambda+\tfrac{1}{2};\ -2\lambda+2;\ x)\ldots(65').$$

Another relation of the same type obtainable by changing the sign of λ......(66).

And

$$F\left(-\frac{\lambda}{2},\ \frac{-\lambda-1}{2};\ 1-\lambda;\ x\right).F\left(\frac{\lambda}{2},\ \frac{\lambda-1}{2};\ 1+\lambda;\ x\right)=1+\frac{\lambda^2}{1-\lambda^2}x\ldots\ldots\ldots(67).$$

Case III. $\mu=\frac{3}{2},\quad \lambda=\pm\frac{1}{2}.$

$$F\left(-\frac{\nu}{2},\ \frac{-\nu+3}{2};\ \tfrac{3}{2};\ x\right).F\left(\frac{-\nu-1}{2},\ \frac{-\nu+2}{2};\ \tfrac{1}{2};\ x\right)$$

$$=F\left(-\nu-\tfrac{1}{2},\ -\nu+1,\ \frac{-2\nu^2+3\nu+3}{2\nu+2};\ \tfrac{3}{2},\ \frac{-2\nu^2+\nu+1}{2\nu+2};\ x\right)\ \ldots\ldots\ldots\ldots(68)$$

$$=F(-\nu-\tfrac{1}{2},\ -\nu+1;\ \tfrac{3}{2};\ x)-\frac{2\nu+2}{3}xF(-\nu+\tfrac{1}{2},\ -\nu+2;\ \tfrac{5}{2};\ x)\ \ldots\ldots\ldots\ldots(68'),$$

$$\left[F\left(\frac{-\nu-1}{2},\ \frac{-\nu+2}{2};\ \tfrac{1}{2};\ x\right)\right]^2+\left(\frac{\nu^2-1}{\nu}\right)^2 x\left[F\left(-\frac{\nu}{2},\ \frac{-\nu+3}{2};\ \tfrac{3}{2};\ x\right)\right]^2$$

$$=F\left(-\nu-1,\ -\nu+\tfrac{1}{2},\ \frac{-\nu^2+\nu+1}{\nu+1};\ \tfrac{1}{2},\ -\frac{\nu^2}{\nu+1};\ x\right)\ \ldots\ldots\ldots\ldots(69)$$

$$=F(-\nu-1,\ -\nu+\tfrac{1}{2};\ \tfrac{1}{2};\ x)+\frac{(\nu+1)^2(-2\nu+1)}{\nu^2}xF(-\nu,\ -\nu+\tfrac{3}{2};\ \tfrac{3}{2};\ x)\ldots(69'),$$

$$\left[F\left(\frac{-\nu-1}{2}, \frac{-\nu+2}{2}; \tfrac{1}{2}; x\right)\right]^2 - \left(\frac{\nu^2-1}{\nu}\right)^2 x \left[F\left(-\frac{\nu}{2}, \frac{-\nu+3}{2}; \tfrac{3}{2}; x\right)\right]^2$$

$$= F\left(-\nu-1, \frac{\nu^2-\nu+1}{-\nu+1}; \frac{\nu^2}{-\nu+1}; x\right) = (1-x)^\nu \left(1 - \frac{x}{\nu^2}\right) \quad \text{.................(70)}.$$

Case IV. $\mu = \tfrac{1}{2}$, $\lambda = \pm \tfrac{3}{2}$.

$$F\left(-\frac{\nu}{2}, \frac{-\nu-1}{2}; -\tfrac{1}{2}; x\right). F\left(\frac{2-\nu}{2}, \frac{3-\nu}{2}; \tfrac{5}{2}; x\right) = F\left(\tfrac{1}{2}-\nu, 1-\nu, \frac{\nu+3}{2\nu+2}; \tfrac{5}{2}, \frac{1-\nu}{2\nu+2}; x\right) \quad \text{...(71)}$$

$$= F(\tfrac{1}{2}-\nu, 1-\nu; \tfrac{5}{2}; x) + \frac{2(1+\nu)(1-2\nu)}{5} xF(\tfrac{3}{2}-\nu, 2-\nu; \tfrac{7}{2}; x) \quad \text{........(71')},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{-\nu-1}{2}; -\tfrac{1}{2}; x\right)\right]^2 + \left(\frac{\nu(\nu^2-1)}{3}\right)^2 x^3 \left[F\left(\frac{2-\nu}{2}, \frac{3-\nu}{2}; \tfrac{5}{2}; x\right)\right]^2$$

$$= F\left(-\nu-1, -\nu-\tfrac{1}{2}, -\frac{\nu}{\nu+1}; -\tfrac{1}{2}, \frac{-2\nu-1}{\nu+1}; x\right) \quad \text{..............................(72)}$$

$$= F(-\nu-1, -\nu-\tfrac{1}{2}; -\tfrac{1}{2}; x) + (\nu+1)^2 xF(-\nu, \tfrac{1}{2}-\nu; \tfrac{1}{2}; x) \quad \text{..............(72')},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{-\nu-1}{2}; -\tfrac{1}{2}; x\right)\right]^2 - \left(\frac{\nu(\nu^2-1)}{3}\right)^2 x^3 \left[F\left(\frac{2-\nu}{2}, \frac{3-\nu}{2}; \tfrac{5}{2}; x\right)\right]^2$$

$$= F\left(-\nu-1, -\frac{\nu}{1-\nu}; -\frac{1}{1-\nu}; x\right) = (1-x)^\nu (1-\nu^2 x) \quad \text{......(73)}.$$

Case V. $\mu = \tfrac{1}{2}$, $\lambda = \pm \tfrac{1}{2}$.

$$F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; \tfrac{1}{2}; x\right). F\left(\frac{1-\nu}{2}, \frac{2-\nu}{2}; \tfrac{3}{2}; x\right) = F(\tfrac{1}{2}-\nu, 1-\nu; \tfrac{3}{2}; x) \quad \text{...........(74)},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; \tfrac{1}{2}; x\right)\right]^2 + \nu^2 x \left[F\left(\frac{1-\nu}{2}, \frac{2-\nu}{2}; \tfrac{3}{2}; x\right)\right]^2 = F(-\nu, \tfrac{1}{2}-\nu; \tfrac{1}{2}; x) \quad \text{...(75)},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; \tfrac{1}{2}; x\right)\right]^2 - \nu^2 x \left[F\left(\frac{1-\nu}{2}, \frac{2-\nu}{2}; \tfrac{3}{2}; x\right)\right]^2 = (1-x)^\nu \quad \text{......(76)}.$$

In each of the foregoing five cases the equation satisfied by $[F(\alpha, \beta; \gamma; x)]^2$ has two independent solutions which are also solutions of that satisfied by a hypergeometric series of the third order and a third independent solution of the type $x^a(1-x)^b(1+cx)$.

If in Cases I., II. the left-hand member be transformed by means of the identity $F(\alpha, \beta; \gamma; x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; x)$, we obtain Case VI. $\nu = -\tfrac{3}{2}$, $\lambda = \pm\tfrac{1}{2}$, and Case VII. $\nu = -\tfrac{3}{2}$, $\mu = \pm\tfrac{1}{2}$. In each of these last cases the differential equation for the square of a hypergeometric series has one solution which is a solution of the equation satisfied by another hypergeometric series of the second order and is in fact of the form $x^a(1-x)^b(1+cx)$; two other independent solutions in these cases are the product of a power of $1-x$ and a solution of another hypergeometric series of the third order. It does not seem necessary to give the results in these cases.

II. *On the possibility of deducing magneto-optic phenomena from a direct modification of an electro-dynamic energy function.* By J. G. LEATHEM, M.A., Fellow of St John's College.

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INTRODUCTION.

1. THE method initiated by Maxwell for the explanation of the Faraday effect depended on the direct insertion of a magneto-optic term in the energy. This method was extended by FitzGerald[1] and others to the explanation of Kerr's effect, namely the modification introduced in the circumstances of optical reflexion by magnetisation of the reflector. A difficulty occurred however in satisfying all the interfacial conditions, which virtually shewed that such a scheme was not formally self-consistent. The origin of the discrepancy has been traced by Mr. Larmor[2] to omission to secure what may for shortness be called the electromotive incompressibility of the medium: in the ordinary problem of optical reflexion there is no tendency for this to be disturbed, but when Maxwell's magneto-optic energy terms are included the reaction against compression introduces what may be termed an electric pressure, which must appear in the equations. It was necessary to compare the modified scheme thus obtained with experimental knowledge: and the calculations given in this paper shew that in fact it does not represent the phenomena.

The paper is only a summary of the actual calculations; because since they were completed I have shewn[3] that the other rigorous theory formulated as an alternative by Mr. Larmor[4], which leads to a system of equations practically the same as those advanced on various hypotheses by FitzGerald, Goldhammer, Basset, Drude, and others, is in much more satisfactory agreement with experiment.

[1] G. F. FitzGerald, *Phil. Trans.* 1880.

[2] "Report on the Action of Magnetism on Light," *Brit. Assoc. Rep.* 1893.

[3] "On the magneto-optic phenomena of Iron, Nickel, and Cobalt," *Phil. Trans.* 1897. Considerable discussion has taken place in Wiedemann's *Annalen*, both before and subsequent to the publication of this memoir, on the question of the formal identity of the schemes of equations employed by these various writers. That this should have been possible is in itself a sufficient indication of the obscurity in which the fundamental principles of the subject had become enveloped. I had myself stated in the memoir that my system of equations was, as far as I could judge, formally the same as those of Goldhammer and Drude: but the theoretical principles from which they were derived, though bearing considerable resemblance to those of the former of these writers, seemed to me to be free from the empirical and tentative character of both. Indeed the relations of these theories to each other and to the one which I adopted had been fully indicated by Mr Larmor in 1893 in the Report above referred to.

[4] *Loc. cit.*

This brief history of the subject shews the desirability of also examining how far the former method of explanation agrees with the phenomena. The result is however what was to be expected by those who adhere to the more recent formulation of optical theory[1], which treats a material medium as free aether pervaded by discrete molecules involving in their constitution electrons considered as nuclei of intrinsic aethereal strain: on such a view a continuous energy function is not the starting point, and the influence of these discrete nuclei could hardly be expected to modify the propagation in the intervening aether in so fundamental a manner as an electromotive pressure would demand.

2. Mr. Larmor's modification of FitzGerald's scheme consists in the introduction of a new quantity λ, naturally suggested by the analysis, which may be interpreted as an irrotational or pressural wave propagated along the surface of separation of two different media. In the Report above referred to he obtains on this hypothesis the equations of propagation in a dielectric, and the conditions which must be satisfied at an interface between two non-conducting media. It is unnecessary to recapitulate here the results arrived at; but it may be well to mention that while in §§ 8 and 11, Mr. Larmor inadvertently states that λ must be continuous across a bounding surface, he has since pointed out that this is not the case, as the pressure is not λ but is the coefficient of $\delta\zeta$ in the variation of the action integral, and it is this pressure which must be continuous.

In the present paper it is proposed, by using the principle of Least Action and introducing a Dissipation Function, to obtain from the above hypotheses the differential equations of propagation in a *conducting* medium, and the boundary conditions which hold good at an interface between two such media. These will then be applied to the solution of the problem of the reflexion of light from a magnet; and the formulae so arrived at will be compared with the available experiments in this subject, with a view to testing to what extent the theory is capable of accounting for the observed phenomena.

NOTATION, AND ASSUMPTIONS.

3. The notation is the same as Maxwell's; and there are introduced quantities ξ, η, ζ defined by the relation

$$(\alpha, \beta, \gamma) = d/dt\,(\xi, \eta, \zeta).$$

For brevity we put

$$\alpha_0 \frac{d}{dx} + \beta_0 \frac{d}{dy} + \gamma_0 \frac{d}{dz} \equiv \frac{d}{d\theta},$$

$(\alpha_0, \beta_0, \gamma_0)$ being the intensity of the imposed magnetisation; this is slightly different from the definition of $d/d\theta$ given by FitzGerald, but the alteration is justified by the consideration that magneto-optic rotations are proportional, not to the magnetic force, but to the intensity of magnetisation of the material medium.

[1] Cf. Larmor, "A Dynamical Theory of the Electric and Luminiferous Medium," Part III. *Phil. Trans.* 1898.

The form of T', the magneto-optic part of the energy, given by FitzGerald (Larmor's Report, § 9) applies only to insulating media: it will be assumed that the corresponding expression for conducting media is

$$T' = C\iiint\left\{\frac{d\xi}{d\theta}\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\frac{d\eta}{d\theta}\frac{d}{dt}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\frac{d\zeta}{d\theta}\frac{d}{dt}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}dx\,dy\,dz.$$

This is one of several possible forms of T' which, for the particular case of a non-conducting medium, would be identical with that assumed by FitzGerald; it is of no importance which we choose, as they differ from one another only as to a complex factor in the magneto-optic constant C.

EQUATIONS OF PROPAGATION, AND BOUNDARY CONDITIONS.

4. The equations of propagation and boundary conditions are to be derived by the principle of Least Action from the energy functions T, T', and W; but for conducting media it is necessary to combine with these Rayleigh's Dissipation Function.

The Dissipation Function F is a homogeneous quadratic function of the velocities (which in the present instance are α, β, γ) representing half the rate at which energy is being dissipated.

In general if T be kinetic energy, V potential energy, and F the dissipation function, the Lagrangian equation of motion corresponding to a coordinate ψ is

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}+\frac{dF}{d\dot{\psi}}+\frac{dV}{d\psi}=0.$$

This equation cannot be arrived at by introducing F before variation into $\delta\int(T-V)\,dt$; but if we treat the energy and dissipation functions separately and afterwards piece together the results of the variations we shall get the desired result. For, neglecting terms at the time limits,

$$\delta\int(T-V)\,dt=-\Sigma\int\left\{\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}+\frac{dV}{d\psi}\right\}\delta\psi\,dt,$$

and

$$\delta\int F dt=+\Sigma\int\frac{dF}{d\dot{\psi}}\,\delta\dot{\psi}\,dt;$$

and so the Lagrangian equation is obtained by adding the coefficient of $\delta\dot{\psi}$ under the integral in $-\delta\int F dt$ to the coefficient of $\delta\psi$ under the integral in $+\delta\int(T-V)\,dt$.

5. In the present instance the coordinates ξ, η, ζ are not independent, being subject to the limitation $\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0$; and so, to get the conditions of the motion, we have to add the coefficient of $\delta\alpha$ under the integral in $-\delta\int F dt$ to the coefficient of $\delta\xi$ under the integral in

$$\delta\int(T+T'-W)\,dt+\int dt\,\delta\iiint\lambda\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz,$$

where the notation is now that of Larmor's Report, namely T and W representing the kinetic or electromagnetic and the static parts respectively of the energy of the medium, and T' the magneto-optic part. The introduction of λ is the characteristic feature of the theory.

6. F is given by the relation

$$2F = \iiint (Pp + Qq + Rr)\, dx\, dy\, dz,$$

and as

$$P = \frac{4\pi}{K} f \quad \text{and} \quad p = \frac{P}{\sigma} = \frac{4\pi}{\sigma K} f,$$

where σ is specific resistance, it follows that

$$F = \frac{8\pi^2}{\sigma K^2} \iiint (f^2 + g^2 + h^2)\, dx\, dy\, dz.$$

To express this in terms of α, β, γ, we notice that

$$4\pi\left(p + \frac{df}{dt}\right) = 4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

and

$$p = (4\pi/\sigma K) f,$$

so that

$$4\pi\left(\frac{4\pi}{\sigma K} + \frac{d}{dt}\right) f = \frac{d\gamma}{dy} - \frac{d\beta}{dz}.$$

In the case of light oscillations we assume all the variables proportional to $e^{\iota pt}$, where ι denotes $\sqrt{-1}$, and p is not to be confused with the x component of the conduction current; the above relation then becomes

$$f = \frac{1}{4\pi(4\pi/\sigma K + \iota p)} \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right),$$

and therefore

$$F = \frac{1}{2\sigma(4\pi/\sigma + \iota pK)^2} \iiint \left[\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)^2 + \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)^2 + \left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)^2\right] dx\, dy\, dz.$$

Varying, and integrating by parts in the usual way, we obtain

$$\delta F = \frac{1}{\sigma(4\pi/\sigma + \iota pK)^2} \iiint \left[\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)\left(\frac{d\delta\gamma}{dy} - \frac{d\delta\beta}{dz}\right) + \dots\right] dx\, dy\, dz$$

$$= \frac{1}{\sigma(4\pi/\sigma + \iota pK)^2} \left[\iint \left[\left\{n\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) - m\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)\right\} \delta\alpha + \text{two similar}\right] dS \right.$$

$$\left. - \iiint \left[\left\{\frac{d}{dz}\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) - \frac{d}{dy}\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)\right\} \delta\alpha + \text{two similar}\right] dx\, dy\, dz\right],$$

where l, m, n are the direction cosines of the outward normal to the element dS of the bounding surface, and the surface integral is taken over all bounding surfaces.

7. The value of T being

$$T=\frac{\mu}{8\pi}\iiint\left\{\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right\}dxdydz,$$

the corresponding variation is

$$\delta\int Tdt=\int dt\,\frac{\mu}{4\pi}\iiint\left\{\frac{d\xi}{dt}\frac{d\delta\xi}{dt}+\frac{d\eta}{dt}\frac{d\delta\eta}{dt}+\frac{d\zeta}{dt}\frac{d\delta\zeta}{dt}\right\}dxdydz$$
$$=-\int dt\,\frac{\mu}{4\pi}\iiint\left\{\frac{d^2\xi}{dt^2}\delta\xi+\frac{d^2\eta}{dt^2}\delta\eta+\frac{d^2\zeta}{dt^2}\delta\zeta\right\}dxdydz,$$

terms at the time limits being neglected.

8. From the expression for T' assumed in § 3 we derive

$$\delta\int T'dt=C\int dt\left[\iiint\left\{\frac{d\delta\xi}{d\theta}\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\frac{d\delta\eta}{d\theta}\frac{d}{dt}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\frac{d\delta\zeta}{d\theta}\frac{d}{dt}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}dxdydz\right.$$
$$\left.-\iiint\left\{\frac{d^2\xi}{d\theta dt}\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)+\frac{d^2\eta}{d\theta dt}\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)+\frac{d^2\zeta}{d\theta dt}\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\right\}dxdydz\right]$$
$$=C\int dt\left[\iint(\alpha_0 l+\beta_0 m+\gamma_0 n)\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\delta\xi dS+\text{two similar}\right.$$
$$+\iint\left(m\frac{d^2\zeta}{d\theta dt}-n\frac{d^2\eta}{d\theta dt}\right)\delta\xi dS+\text{two similar}$$
$$\left.-2\iiint\frac{d^2}{d\theta dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\delta\xi dxdydz-\text{two similar}\right].$$

Also $\delta\int dt\iiint\lambda\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz$

$$=\int dt\left[\iint\lambda(l\delta\xi+m\delta\eta+n\delta\zeta)dS-\iiint\left(\frac{d\lambda}{dx}\delta\xi+\frac{d\lambda}{dy}\delta\eta+\frac{d\lambda}{dz}\delta\zeta\right)dx\,dy\,dz\right].$$

9. The static part of the energy of the medium is given by

$$W=(2\pi/K)\iiint(f^2+g^2+h^2)\,dx\,dy\,dz$$
$$=\frac{1}{8\pi K(4\pi/\sigma K+\iota p)^2}\iiint\left[\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)^2+\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)^2+\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)^2\right]dx\,dy\,dz;$$

from which, noticing that $(\alpha,\beta,\gamma)=\iota p(\xi,\eta,\zeta)$, it follows that

$$\delta\int(-W)\,dt=\int dt\,\frac{p^2}{4\pi K(4\pi/\sigma K+\iota p)^2}\left[\iiint\left\{\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)\right.\right.$$
$$\left.\left.+\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)+\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\right\}dx\,dy\,dz\right]$$
$$=\int dt\,\frac{p^2}{4\pi K(4\pi/\sigma K+\iota p)^2}\left[\iint\left\{n\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-m\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}\delta\xi dS+\text{two similar}\right.$$
$$\left.-\iiint\left\{\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}\delta\xi dxdydz-\text{two similar}\right].$$

10. If now we bring together the results of the last four paragraphs in the manner specified in § 5, we obtain the following expression of the conditions of the motion:—

$$
\begin{aligned}
-\frac{\mu}{4\pi}\iiint\left(\frac{d^2\xi}{dt^2}\delta\xi+\frac{d^2\eta}{dt^2}\delta\eta+\frac{d^2\zeta}{dt^2}\delta\zeta\right)dx\,dy\,dz \\
+C\Bigg[\iint(\alpha_0 l+\beta_0 m+\gamma_0 n)\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\delta\xi\,dS+\text{two similar} \\
+\iint\left(m\frac{d^2\zeta}{d\theta dt}-n\frac{d^2\eta}{d\theta dt}\right)\delta\xi\,dS+\text{two similar} \\
-2\iiint\frac{d^2}{d\theta dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\delta\xi\,dx\,dy\,dz-\text{two similar}\Bigg] \\
+\frac{p^2}{4\pi K(4\pi/\sigma K+\iota p)^2}\Bigg[\iint\left\{n\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-m\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}\delta\xi\,dS+\text{two similar} \\
-\iiint\left\{\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}\delta\xi\,dx\,dy\,dz-\text{two similar}\Bigg] \\
+\iint\lambda\,(l\delta\xi+m\delta\eta+n\delta\zeta)\,dS-\iiint\left(\frac{d\lambda}{dx}\delta\xi+\frac{d\lambda}{dy}\delta\eta+\frac{d\lambda}{dz}\delta\zeta\right)dx\,dy\,dz \\
-\frac{1}{\sigma(4\pi/\sigma+\iota pK)^2}\Bigg[\iint\left\{n\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)-m\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}\delta\xi\,dS+\text{two similar} \\
-\iiint\left\{\frac{d}{dz}\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)-\frac{d}{dy}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}\delta\xi\,dx\,dy\,dz-\text{two similar}\Bigg] \\
=0.
\end{aligned}
$$

Taking together the volume integrals we get the bodily equations of propagation, of the type

$$
\mu\frac{d^2\xi}{dt^2}=\frac{\iota p}{4\pi/\sigma+\iota pK}\left[\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right]-8\pi C\frac{d^2}{d\theta dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)-4\pi\frac{d\lambda}{dx}\quad\ldots\text{(i)}.
$$

From the surface integrals we obtain the boundary conditions which hold good at an interface between two different media; in this case of course the integrals extend over both sides of the surface of separation. If, for the sake of simplification, we take the axis of z normal to the element of surface considered, we have $l=0$, $m=0$, $n=1$. Now $\delta\xi$ and $\delta\eta$ must be continuous across the interface, and therefore so also must be the expressions which are their coefficients in the surface integral: $\delta\zeta$ is not necessarily continuous, for reasons explained by Mr. Larmor in the footnote to § 11 of his Report; he has however pointed out to me that the supposition that it is continuous is perfectly allowable and involves no inconsistency. Thus we have analytically the alternative of supposing that both $\delta\zeta$ and its coefficient in the surface integral are continuous across the interface, or on the other hand of supposing that $\delta\zeta$ is discontinuous, and that therefore its coefficient vanishes at both sides of the bounding surface. Of these the former supposition seems the more natural, but the consequences of both will be investigated.

The boundary conditions are accordingly as follows:

(I) ξ and η continuous,

(II) $\frac{-\iota p}{4\pi/\sigma+\iota pK}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-4\pi C\frac{d^2\eta}{d\theta dt}+4\pi C\gamma_0\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)$ continuous,

(III) $\frac{\iota p}{4\pi/\sigma+\iota pK}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+4\pi C\frac{d^2\xi}{d\theta dt}+4\pi C\gamma_0\frac{d}{dt}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)$ continuous,

(IV) Either (1°) ζ continuous,

and $\lambda+C\gamma_0\frac{d}{dt}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)$ continuous,

or (2°) $\lambda+C\gamma_0\frac{d}{dt}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)=0$ at both sides of the boundary.

PLANE WAVES.

11. In dealing with the problem of the reflexion and refraction of plane waves at a plane surface of magnetised metal, it is convenient to take the reflecting surface as the plane $z=0$, and the plane of incidence as the plane $y=0$; the positive direction of the axis of z is from the metal into the air.

If we suppose that the expressions representing the optical circumstances in the incident wave depend on the exponential $e^{\iota(lx+mz+pt)}$, then those which represent the circumstances in a corresponding reflected or refracted wave must, in so far as they involve x and t, depend on the exponential $e^{\iota(lx+pt)}$. Hence the most general assumption that we can make about a reflected or refracted wave is that its rotational part depends on one or more exponentials of the type $e^{\iota(lx+m'z+pt)}$, and its condensational part on others of similar form, say $e^{\iota(lx+m''z+pt)}$. In fact, for such a wave

$$\left.\begin{aligned}\xi&=\Sigma Ae^{\iota(lx+m'z+pt)}+\Sigma d\phi/dx\\ \eta&=\Sigma Be^{\iota(lx+m'z+pt)}+\Sigma d\phi/dy\\ \zeta&=\Sigma(-l/m')Ae^{\iota(lx+m'z+pt)}+\Sigma d\phi/dz\\ \phi&=\mathfrak{A}e^{\iota(lx+m''z+pt)}\\ \lambda&=\Sigma Le^{\iota(lx+m''z+pt)}\end{aligned}\right\}\quad\text{(v)},$$

where A, B, $\mathfrak{A}$, and L are constants, real or complex.

The form of ζ has been so chosen that the rotational parts of ξ, η, ζ satisfy the condition

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0\quad\text{(vi)};$$

the substitution of their irrotational parts in this equation leads at once to the condition $l^2+m''^2=0$, which shews that there are only two possible values of m'', namely $m''=+\iota l$ and $m''=-\iota l$. The corresponding exponentials are $e^{-lz}e^{\iota(lx+pt)}$ and $e^{+lz}e^{\iota(lx+pt)}$, of which the former can occur only in a reflected, the latter only in a refracted wave, an amplitude which increases without limit in the direction of propagation being impossible.

The parts of (ξ, η, ζ) corresponding to different exponentials are really different waves, travelling each with its own velocity; the above condition shews that the velocity of propagation of either of the irrotational waves is infinitely great, since its square is equal to $p^2/(l^2 + m''^2)$.

12. In order to obtain further information about the various constants which occur in the assumed expressions for ξ, η, ζ, we substitute these values in the equations of propagation. In the case of the irrotational terms this leads to the relation

$$p^2 \mu \mathfrak{A} = 4\pi L \qquad\text{.........(vii).}$$

From the rotational terms we get, for each value of m', two relations which readily reduce to the form

$$\left.\begin{aligned} A\left\{\mu\iota p + \frac{l^2 + m'^2}{4\pi/\sigma + \iota pK}\right\} &= -8\pi C(\alpha_0 l + \gamma_0 m')\, m'B, \\ B\left\{\mu\iota p + \frac{l^2 + m'^2}{4\pi/\sigma + \iota pK}\right\} &= +8\pi C(\alpha_0 l + \gamma_0 m')\,\frac{l^2 + m'^2}{m'}\,A \end{aligned}\right\} \qquad\text{.........(viii).}$$

Eliminating from these the ratio B/A, we get

$$\mu\iota p + \frac{l^2 + m'^2}{4\pi/\sigma + \iota pK} = \pm\,\iota\,.\,8\pi C(\alpha_0 l + \gamma_0 m')\,(l^2 + m'^2)^{\frac{1}{2}} \qquad\text{.........(ix),}$$

an equation which determines m'. In this, to avoid ambiguity, we shall make the convention that $(l^2 + m'^2)^{\frac{1}{2}}$, or ω', as it may for brevity be called, is a complex whose imaginary part is negative. As the equation is a quartic, there are four possible values of m'; and if we neglect the second and higher powers of C, which for all media is an exceedingly small quantity, these four values are found to be of the form $+m_1$, $-m_1$, $+m_2$, $-m_2$, of which the two former correspond to the positive, the two latter to the negative sign on the right-hand side of the equation in m'. We complete the definition by the supposition that of the complexes $+m_1$ and $-m_1$ the former is that which has its imaginary part negative, and $+m_2$ is chosen in the same way; as a matter of fact it is found that m_1 and m_2 so defined have their real parts positive.

If (A_1, B_1), (A_1', B_1'), (A_2, B_2), (A_2', B_2') be the pairs of values of (A, B) corresponding to the roots $+m_1$, $-m_1$, $+m_2$, $-m_2$ respectively, and if we make the abbreviations

$$l^2 + m_1^2 \equiv \omega_1^2, \quad l^2 + m_2^2 \equiv \omega_2^2 \qquad\text{.........(x),}$$

either of the above relations between A and B readily yields the following

$$\left.\begin{aligned} A_1\iota\omega_1 &= -B_1 m_1, & A_1'\iota\omega_1 &= +B_1' m_1 \\ A_2\iota\omega_2 &= +B_2 m_2, & A_2'\iota\omega_2 &= -B_2' m_2 \end{aligned}\right\} \qquad\text{.........(xi).}$$

The consideration that a wave whose amplitude increases continually in the direction of propagation cannot occur, indicates that in the problem of reflection the reflected waves involve only those exponentials corresponding to $m' = -m_1$ and $m' = -m_2$, while the refracted waves involve only those corresponding to $m' = +m_1$ and $m' = +m_2$.

13. In the particular case when α_0 and γ_0 are zero, m_1 and m_2 are equal and their common value may be denoted by M; the corresponding common value of ω_1 and ω_2 may be called Ω. To evaluate these quantities we have only to change the right-hand side of equation (ix) to zero, when we find

$$\left.\begin{aligned}\Omega^2 &= -\mu\iota p\,(4\pi/\sigma + \iota pK),\\ M^2 &= -l^2 - \mu\iota p\,(4\pi/\sigma + \iota pK)\end{aligned}\right\} \qquad \text{(xii).}$$

whence of course

Returning to the general case, we see that m_1 and m_2 differ from M, and ω_1 and ω_2 differ from Ω, by quantities which have C for a factor and which are therefore small of the first order compared with M and Ω. If we neglect small quantities of the second order we may substitute Ω for ω' and M for m' on the right-hand side of equation (ix), which may therefore be written in the form

$$\frac{\omega'^2 - \Omega^2}{4\pi/\sigma + \iota pK} = \pm\,\iota\,.\,8\pi C\,(\alpha_0 l + \gamma_0 M)\,\Omega.$$

On introducing the abbreviation

$$\begin{aligned}\varpi &= -\,\iota\,.\,8\pi C\,(\alpha_0 l + \gamma_0 M)\,(4\pi/\sigma + \iota pK)/\Omega,\\ &= (8\pi\Omega/\mu p)\,C\,(\alpha_0 l + \gamma_0 M) \qquad \text{(xiii),}\end{aligned}$$

equation (ix) further reduces to

$$\omega'^2 - \Omega^2 = \mp\,\varpi\Omega^2 \qquad \text{(xiv).}$$

Hence we have $\quad \omega_1^2 = (1-\varpi)\,\Omega^2,\quad \omega_2^2 = (1+\varpi)\,\Omega^2 \qquad$ (xv),

and therefore also $\quad m_1^2 = M^2 - \varpi\Omega^2,\quad m_2^2 = M^2 + \varpi\Omega^2 \qquad$ (xvi);

and as ϖ is small, these lead to

$$\left.\begin{aligned}\omega_1 &= (1-\tfrac{1}{2}\varpi)\,\Omega, & \omega_2 &= (1+\tfrac{1}{2}\varpi)\,\Omega,\\ m_1 &= \left(1 - \frac{1}{2}\frac{\Omega^2}{M^2}\varpi\right)M, & m_2 &= \left(1 + \frac{1}{2}\frac{\Omega^2}{M^2}\varpi\right)M\end{aligned}\right\} \qquad \text{(xvii),}$$

expressions which will prove exceedingly useful in the subsequent analysis.

14. For a medium in which there is no magneto-optic effect the relations (xi) do not hold good. In fact, as for such a medium C is zero, the equations (viii) and (ix) all reduce to the same form, namely

$$\mu\iota p + \frac{l^2 + m'^2}{4\pi/\sigma + \iota pK} = 0,$$

and the ratio A/B is left quite arbitrary. If, as in the case of air, the medium is also a non-conductor, σ is infinitely great, and equation (ix) assumes the form

$$l^2 + m'^2 - p^2\mu K = 0,$$

which expresses the fact that, if V be the velocity of propagation of light, $V^2 = (\mu K)^{-1}$. There are two values of m' of the forms $+m$ and $-m$.

THE REFLEXION PROBLEM.

15. On passing now to a more detailed investigation of the problem of magnetic reflexion, the preceding paragraphs justify the representation of the optical circumstances *in the air* by the following expressions:—

$$\xi = A_0 e^{\iota(lx+mz+pt)} + A e^{\iota(lx-mz+pt)} + d\phi/dx,$$

$$\eta = B_0 e^{\iota(lx+mz+pt)} + B e^{\iota(lx-mz+pt)} + d\phi/dy,$$

$$\zeta = -(l/m)\,A_0 e^{\iota(lx+mz+pt)} + (l/m)\,A e^{\iota(lx-mz+pt)} + d\phi/dz,$$

$$\phi = \mathfrak{A} e^{-lz}\, e^{\iota(lx+pt)}, \qquad \lambda = (p^2/4\pi)\,\mathfrak{A} e^{-lz}\, e^{\iota(lx+pt)},$$

wherein A_0, B_0 represent the incident wave and A, B the reflected wave; the system of units being the electromagnetic, μ for air is equal to unity.

In the metal the refracted light may, in accordance with § 12, be represented by

$$\xi = A_1 e^{\iota(lx+m_1z+pt)} + A_2 e^{\iota(lx+m_2z+pt)} + d\phi/dx,$$

$$\eta = -\iota(\omega_1/m_1)\,A_1 e^{\iota(lx+m_1z+pt)} + \iota(\omega_2/m_2)\,A_2 e^{\iota(lx+m_2z+pt)} + d\phi/dy,$$

$$\zeta = -(l/m_1)\,A_1 e^{\iota(lx+m_1z+pt)} - (l/m_2)\,A_2 e^{\iota(lx+m_2z+pt)} + d\phi/dz,$$

$$\phi = \mathfrak{A}' e^{+lz}\, e^{\iota(lx+pt)}, \qquad \lambda = (p^2\mu/4\pi)\,\mathfrak{A}' e^{+lz}\, e^{\iota(lx+pt)}.$$

Getting rid of the ϕ's, the values of ξ, η, ζ, and λ in the two media may with advantage be rewritten as follows:—

In the air

$$\left.\begin{aligned}
\xi &= A_0 e^{\iota(lx+mz+pt)} + A e^{\iota(lx-mz+pt)} + \iota l\mathfrak{A} e^{-lz}\, e^{\iota(lx+pt)},\\
\eta &= B_0 e^{\iota(lx+mz+pt)} + B e^{\iota(lx-mz+pt)},\\
\zeta &= -(l/m)\,A_0 e^{\iota(lx+mz+pt)} + (l/m)\,A e^{\iota(lx-mz+pt)} - l\mathfrak{A} e^{-lz} e^{\iota(lx+pt)},\\
\lambda &= (p^2/4\pi)\,\mathfrak{A} e^{-lz}\, e^{\iota(lx+pt)}
\end{aligned}\right\}\quad\text{............(xviii).}$$

In the metal

$$\left.\begin{aligned}
\xi &= A_1 e^{\iota(lx+m_1z+pt)} + A_2 e^{\iota(lx+m_2z+pt)} + \iota l\mathfrak{A}' e^{lz} e^{\iota(lx+pt)},\\
\eta &= -\iota(\omega_1/m_1)\,A_1 e^{\iota(lx+m_1z+pt)} + \iota(\omega_2/m_2)\,A_2 e^{\iota(lx+m_2z+pt)},\\
\zeta &= -(l/m_1)\,A_1 e^{\iota(lx+m_1z+pt)} - (l/m_2)\,A_2 e^{\iota(lx+m_2z+pt)} + l\mathfrak{A}' e^{lz} e^{\iota(lx+pt)},\\
\lambda &= (p^2\mu/4\pi)\,\mathfrak{A}' e^{lz}\, e^{\iota(lx+pt)}
\end{aligned}\right\}\quad\text{.........(xix).}$$

15 *a*. Let us first consider the second of the alternative hypotheses referred to at the end of § 10, namely that which supposes ζ discontinuous.

The boundary condition (IV, 2°) must now be used, and from it it appears that, since in air C is zero, $\mathfrak{A}$ must also be zero. A slight simplification thus takes place in the expressions representing the optical circumstances in the air. Substituting these expressions in the boundary conditions we obtain:—

(I) From the continuity of ξ and η

$$A_0 + A = A_1 + A_2 + \iota l \mathfrak{A}' \quad\text{(xx)},$$

$$B_0 + B = -\iota(\omega_1/m_1)A_1 + \iota(\omega_2/m_2)A_2 \quad\text{(xxi)}.$$

(II) From condition (II)

$$-\frac{1}{K}\iota\frac{l^2+m^2}{m}(A_0 - A) = \frac{-\iota p}{4\pi/\sigma + \iota p K'}\iota\left\{\frac{l^2+m_1^2}{m_1}A_1 + \frac{l^2+m_2^2}{m_2}A_2\right\}$$
$$+ 4\pi C p\{(\alpha_0 l + \gamma_0 m_1)(-\iota\omega_1/m_1)A_1 + (\alpha_0 l + \gamma_0 m_2)(+\iota\omega_2/m_2)A_2\}$$
$$+ 4\pi C\gamma_0 \iota p(-\omega_1 A_1 + \omega_2 A_2) \quad\text{(xxii)}.$$

(III) From condition (III)

$$\frac{1}{K}(-\iota m)(B_0 - B) = \frac{\iota p}{4\pi/\sigma + \iota p K'}(-\omega_1 A_1 + \omega_2 A_2)$$
$$- 4\pi C p\{(\alpha_0 l + \gamma_0 m_1)A_1 + (\alpha_0 l + \gamma_0 m_2)A_2 + (\iota\alpha_0 l + \gamma_0 l)l\mathfrak{A}'\}$$
$$+ 4\pi C\gamma_0 \iota p\left\{\iota\frac{l^2+m_1^2}{m_1}A_1 + \iota\frac{l^2+m_2^2}{m_2}A_2\right\} \quad\text{(xxiii)}.$$

(IV) From condition (IV, 2°)

$$(p^2\mu/4\pi)\mathfrak{A}' - C\gamma_0 p l\{-\iota(\omega_1/m_1)A_1 + \iota(\omega_2/m_2)A_2\} = 0 \quad\text{(xxiv)}.$$

In these results the specific inductive capacity of the metal is denoted by K' to distinguish it from that of air. Equation (xxiv) shews that when γ_0 is zero so also is $\mathfrak{A}'$, so that when the reflexion is equatorial there is no condensational wave; it also shews that $\mathfrak{A}'$ is small of the first order compared with A_1 or A_2, and may therefore be omitted from equation (xxiii).

If we eliminate $\mathfrak{A}'$ from these five equations, and, neglecting small quantities of the second order, substitute in terms containing the factor C the first approximations Ω for ω_1 or ω_2 and M for m_1 or m_2, we obtain

$$A_0 + A = A_1 + A_2 + \frac{4\pi C\gamma_0}{p\mu}l^2\frac{\Omega}{M}(A_1 - A_2) \quad\text{(xxv)},$$

$$\iota(B_0 + B) = (\omega_1/m_1)A_1 - (\omega_2/m_2)A_2 \quad\text{(xxvi)},$$

$$\frac{1}{K}\frac{\omega^2}{m}(A_0 - A) = \frac{\iota p}{4\pi/\sigma + \iota p K'}\left\{\frac{\omega_1^2}{m_1}A_1 + \frac{\omega_2^2}{m_2}A_2\right\}$$
$$+ 4\pi C p\frac{\Omega}{M}(\alpha_0 l + 2\gamma_0 M)(A_1 - A_2) \quad\text{(xxvii)},$$

$$\frac{1}{K}m\iota(B_0 - B) = \frac{\iota p}{4\pi/\sigma + \iota p K'}\{\omega_1 A_1 - \omega_2 A_2\}$$
$$+ 4\pi C p(\alpha_0 l + \gamma_0 M)(A_1 + A_2) + 4\pi C p\gamma_0\frac{\Omega^2}{M}(A_1 + A_2) \quad\text{(xxviii)}.$$

On substituting the values of m_1, m_2, ω_1, ω_2 from § 13 and remembering result (xii), these become

$$\left.\begin{aligned} A_0 + A &= A_1 + A_2 + \frac{4\pi C\gamma_0}{p\mu} l^2 \frac{\Omega}{M}(A_1 - A_2), \\ \iota(B_0 + B) &= \frac{\Omega}{M}(A_1 - A_2) + \frac{4\pi C}{p\mu}\cdot\frac{l^2\Omega^2}{M^3}(\alpha_0 l + \gamma_0 M)(A_1 + A_2), \\ \frac{1}{K}\frac{\omega^2}{m}(A_0 - A) &= \frac{p^2\mu}{M}(A_1 + A_2) + 4\pi Cp\frac{\Omega}{M^3}(\alpha_0 l^3 + \gamma_0\Omega^2 M)(A_1 - A_2), \\ \frac{1}{K} m\iota(B_0 - B) &= \frac{p^2\mu}{\Omega}(A_1 - A_2) + 4\pi Cp\gamma_0\frac{\Omega^2}{M}(A_1 + A_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(\text{xxix}).$$

From the first and second of these equations we obtain, to first order of small quantities,

$$A_1 + A_2 = A_0 + A - \frac{4\pi C\gamma_0}{p\mu} l^2 \iota(B_0 + B),$$

$$\frac{\Omega}{M}(A_1 - A_2) = \iota(B_0 + B) - \frac{4\pi C}{p\mu}\cdot\frac{l^2\Omega^2}{M^3}(\alpha_0 l + \gamma_0 M)(A_0 + A);$$

and if we substitute these values in the second and third equations, and remember that $1/K = V^2 = p^2/\omega^2$, we get relations which readily reduce to

$$\left.\begin{aligned} \frac{1}{m}(A_0 - A) &= \frac{\mu}{M}(A_0 + A) + \frac{4\pi C}{pM^2}(\alpha_0 l^3 + \gamma_0 M^3)\,\iota(B_0 + B), \\ \frac{m}{\omega^2}\iota(B_0 - B) &= \mu\frac{M}{\Omega^2}\iota(B_0 + B) - \frac{4\pi C}{pM^2}(\alpha_0 l^3 - \gamma_0 M^3)(A_0 + A) \end{aligned}\right\} \quad \ldots\ldots\ \ldots\ldots(\text{xxx}).$$

Solving these for A and B we have

$$\left.\begin{aligned} A &= \left[\left(\frac{1}{m} - \frac{\mu}{M}\right)\left(\frac{m}{\omega^2} + \mu\frac{M}{\Omega^2}\right)A_0 - \frac{8\pi Cm}{pM^2\omega^2}(\alpha_0 l^3 + \gamma_0 M^3)\,\iota B_0\right] \Big/ \left(\frac{1}{m} + \frac{\mu}{M}\right)\left(\frac{m}{\omega^2} + \mu\frac{M}{\Omega^2}\right) \\ \iota B &= \left[\frac{8\pi C}{pM^2 m}(\alpha_0 l^3 - \gamma_0 M^3)A_0 + \left(\frac{1}{m} + \frac{\mu}{M}\right)\left(\frac{m}{\omega^2} - \mu\frac{M}{\Omega^2}\right)\iota B_0\right] \Big/ \left(\frac{1}{m} + \frac{\mu}{M}\right)\left(\frac{m}{\omega^2} + \mu\frac{M}{\Omega^2}\right) \end{aligned}\right\} \ldots(\text{xxxi}),$$

which, since A and B specify the reflected light, constitute the complete formal solution of the problem of metallic reflexion.

15 *b*. Turning now to the consideration of the first of the alternative hypotheses of § 10, namely that which supposes ζ continuous, we proceed exactly as before, save only that we use the boundary conditions (IV, 1°) instead of (IV, 2°). In this case, of course, $\mathfrak{A}$ is not zero.

Substituting the full expressions (xviii) and (xix) in the boundary conditions we obtain:—

(I) From the continuity of ξ, η, and ζ

$$A_0 + A + \iota l\mathfrak{A} = A_1 + A_2 + \iota l\mathfrak{A}',$$

$$B_0 + B = -\iota(\omega_1/m_1)A_1 + \iota(\omega_2/m_2)A_2,$$

$$-\frac{1}{m}(A_0 - A) - \mathfrak{A} = -\frac{1}{m_1}A_1 - \frac{1}{m_2}A_2 + \mathfrak{A}'.$$

(II) From condition (II)

$$-\frac{1}{K}\iota\frac{l^2+m^2}{m}(A_0-A)=\frac{-\iota p}{4\pi/\sigma+\iota pK'}\iota\left\{\frac{l^2+m_1^2}{m_1}A_1+\frac{l^2+m_2^2}{m_2}A_2\right\}$$

$$+4\pi Cp\{(\alpha_0 l+\gamma_0 m_1)(-\iota\omega_1/m_1)A_1+(\alpha_0 l+\gamma_0 m_2)(\iota\omega_2/m_2)A_2\}$$

$$+4\pi C\gamma_0\iota p(-\omega_1 A_1+\omega_2 A_2).$$

(III) From condition (III)

$$\frac{1}{K}(-\iota m)(B_0-B)=\frac{\iota p}{4\pi/\sigma+\iota pK'}(-\omega_1 A_1+\omega_2 A_2)$$

$$-4\pi Cp\{(\alpha_0 l+\gamma_0 m_1)A_1+(\alpha_0 l+\gamma_0 m_2)A_2+(\iota\alpha_0 l+\gamma_0 l)l\mathfrak{A}'\}$$

$$+4\pi C\gamma_0\iota p\left\{\iota\frac{l^2+m_1^2}{m_1}A_1+\iota\frac{l^2+m_2^2}{m_2}A_2\right\}.$$

(IV) From condition (IV, 1°)

$$(p^2/4\pi)\mathfrak{A}=(p^2\mu/4\pi)\mathfrak{A}'-C\gamma_0 pl\{-\iota(\omega_1/m_1)A_1+\iota(\omega_2/m_2)A_2\}.$$

If we substitute in these equations the values of m_1, m_2, ω_1, ω_2 from § 13, replace $(4\pi/\sigma+\iota pK')$ by $\iota\Omega^2/\mu p$ in virtue of (xii), and omit small quantities of the second and higher orders, we obtain :—

$$A_0+A+\iota l\mathfrak{A}=A_1+A_2+\iota l\mathfrak{A}',$$

$$\iota(B_0+B)=\frac{\Omega}{M}(A_1-A_2)+\frac{4\pi C}{p\mu}\frac{l^2\Omega^2}{M^3}(\alpha_0 l+\gamma_0 M)(A_1+A_2),$$

$$\frac{1}{m}(A_0-A)+\mathfrak{A}=\frac{1}{M}(A_1+A_2)+\frac{4\pi C}{p\mu}\frac{\Omega^3}{M^3}(\alpha_0 l+\gamma_0 M)(A_1-A_2),$$

$$\frac{1}{K}\frac{\omega^2}{m}(A_0-A)=\frac{p^2\mu}{M}(A_1+A_2)+4\pi Cp\frac{\Omega}{M^3}(\alpha_0 l^3+\gamma_0\Omega^2 M)(A_1-A_2),$$

$$\frac{1}{K}m\iota(B_0-B)=\frac{p^2\mu}{\Omega}(A_1-A_2)+4\pi Cp\gamma_0\frac{\Omega^2}{M}(A_1+A_2)+4\pi Cpl^2(\iota\alpha_0+\gamma_0)\mathfrak{A}',$$

$$\iota p^2\mathfrak{A}=\iota p^2\mu\mathfrak{A}'-4\pi Cp\gamma_0\frac{l\Omega}{M}(A_1-A_2).$$

Solving the second and third of these for (A_1+A_2) and (A_1-A_2), substituting the values so obtained in the others, and remembering that $1/K=p^2/\omega^2$, we get :—

$$(M-\iota l)\mathfrak{A}+\iota l\mathfrak{A}'=A_0+A-\frac{M}{m}(A_0-A)+\frac{4\pi C}{p\mu}\frac{\Omega^2}{M}(\alpha_0 l+\gamma_0 M)\iota(B_0+B),$$

$$p^2\mu\mathfrak{A}=-p^2\frac{\mu-1}{m}(A_0-A)+4\pi Cp\alpha_0 l\iota(B_0+B),$$

$$-4\pi Cp\frac{1}{M}(\alpha_0 l^3-\gamma_0 M^3)\mathfrak{A}+4\pi Cpl^2(\iota\alpha_0+\gamma_0)\mathfrak{A}'$$

$$=\frac{p^2 m}{\omega^2}\iota(B_0-B)-p^2\mu\frac{M}{\Omega^2}\iota(B_0+B)+4\pi Cp\frac{1}{M}(\alpha_0 l^3-\gamma_0 M^3)\frac{1}{m}(A_0-A),$$

$$\iota p^2\mathfrak{A}-\iota p^2\mu\mathfrak{A}'=-4\pi Cp\gamma_0 l\iota(B_0+B).$$

When we assume, as it is usual to do, that for magnetic forces alternating as rapidly as those in light waves, the magnetic permeability is unity, the form of these equations becomes simpler; and the elimination from them of $\mathfrak{A}$ and $\mathfrak{A}'$ which are now seen to be small of the first order, leads to:—

$$A_0 + A - \frac{M}{m}(A_0 - A) + \frac{4\pi C}{p}\frac{1}{M}(\alpha_0 l^3 + \gamma_0 M^3)\,\iota\,(B_0 + B) = 0,$$

$$\frac{m}{\omega^2}\,\iota\,(B_0 - B) - \frac{M}{\Omega^2}\,\iota\,(B_0 + B) + \frac{4\pi C}{p}\frac{1}{M}(\alpha_0 l^3 - \gamma_0 M^3)\frac{1}{m}(A_0 - A) = 0.$$

Solving for A and B, we get

$$A = \left[\left(\frac{1}{m} - \frac{1}{M}\right)\left(\frac{m}{\omega^2} + \frac{M}{\Omega^2}\right)A_0 - \frac{8\pi Cm}{pM^2\omega^2}(\alpha_0 l^3 + \gamma_0 M^3)\,\iota B_0\right] \Big/ \left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{m}{\omega^2} + \frac{M}{\Omega^2}\right),$$

$$\iota B = \left[\frac{8\pi C}{pM^2 m}(\alpha_0 l^3 - \gamma_0 M^3)A_0 + \left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{m}{\omega^2} - \frac{M}{\Omega^2}\right)\iota B_0\right] \Big/ \left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{m}{\omega^2} + \frac{M}{\Omega^2}\right).$$

Now the expressions here obtained for A and B are identical with those of equations (xxxi) when in the latter μ is, as usual, put equal to unity. Thus it appears that the alternative hypotheses as to boundary conditions discussed above lead to precisely the same results, and it is a matter of indifference which we adopt. The subsequent calculations apply equally well to the two views.

16. The value of Ω is determined by the consideration that $\Omega^2/\omega^2 = R^2 e^{2\iota\alpha}$, where $Re^{\iota\alpha}$ is the quasi refractive index of the metal*. The quantities R and α are connected by the relations

$$R^2 \cos 2\alpha = n^2(1 - k^2), \qquad R^2 \sin 2\alpha = -2n^2 k,$$

with Drude's optic constants, whose values for different metals are quoted in Thomson's *Recent Researches*†. The value of M is obtained from that of Ω by the relation $M^2 = \Omega^2 - l^2$, and it will be convenient to denote M/ω by the symbol $\mathfrak{M}$, so that

$$\mathfrak{M}^2 = R^2 e^{2\iota\alpha} - l^2/\omega^2 \qquad \text{(xxxii)}.$$

If i be the angle of incidence, and if we suppose the direction of the incident light to lie in the quadrant between the positive direction of the axis of x and the negative direction of the axis of z, then, ω and p being assumed essentially positive, we have

$$l = -\omega \sin i, \qquad m = +\omega \cos i \qquad \text{(xxxiii)}:$$

of course $p = V\omega$, and $\omega = 2\pi/\lambda$ where λ is the wave length of the light in air.

We shall also put $\mu = 1$, as it is usually taken for granted that for magnetic forces alternating as rapidly as those in light waves the magnetic permeability is unity.

* J. J. Thomson, *Recent Researches*, p. 419.

† Drude, *Wied. Ann.* XXXIX. p. 481. For the constants of Cobalt see Drude, *Wied. Ann.* XLVI. p. 407.

When these substitutions are made, equations (xxxi) assume the following form:—

$$\left.\begin{aligned} A &= \frac{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i) A_0 + 16\pi^2 C V^{-1} \lambda^{-1} (\cos^2 i / \mathfrak{M})(\alpha_0 \sin^3 i - \gamma_0 \mathfrak{M}^3) \iota B_0}{(\mathfrak{M} + \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)} \\ \iota B &= \frac{-16\pi^2 C V^{-1} \lambda^{-1} \mathfrak{M}^{-1} (\alpha_0 \sin^3 i + \gamma_0 \mathfrak{M}^3) A_0 - (\mathfrak{M} + \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} - \cos i) \iota B_0}{(\mathfrak{M} + \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)} \end{aligned}\right\} \quad \text{......(xxxv).}$$

It is to be noticed that these expressions contain only one undetermined constant, namely C; this we may assume complex, of the form $C_0 e^{\iota x}$, where x is defined as lying between -90° and $+90^\circ$, and C_0 may be positive or negative. The test of the theory consists in ascertaining whether it is possible, by attributing, for each metal, suitable values to C_0 and x, to derive from these formulae results in numerical agreement with those arrived at experimentally. Should the values of these constants indicated by the different experimental observations or series of observations prove to be the same, the theory may be regarded as offering a satisfactory account of the phenomena; but if the different series of observations point to considerably different values of the constants, it must be concluded that the theory is at fault.

THE KERR EXPERIMENTS.

17. In the original experiments on magnetic reflexion, namely those of Dr Kerr, the incident light was plane polarised; and observations were made, for various incidences, of the angle between the direction of the major axis of the ellipse of polarisation of the reflected light and that direction which it would have had if there had been no magnetisation. We may denote this angle by θ; it is the rotation required to bring the analyser from the position of extinction or greatest darkness before magnetisation of the mirror into the corresponding position after the magnetising current has been made, and it is to be reckoned positive when, as seen by the observer, it takes place in the direction contrary to that of the hands of a watch.

When the incident light is polarised either in or perpendicularly to the plane of incidence the theoretical value of θ, which in these particular cases we shall denote by θ_i and θ_p respectively, is very simply obtained from the formulae (xxxiv). For, as in the former case $B_0 = 0$, and in the latter $A_0 = 0$, we have

$$\begin{aligned} \theta_i &= \text{real part of } \{B \cos i / A\}_{B_0=0} \\ &= \text{real part of } \frac{16\pi^2 \iota C_0 e^{\iota x} V^{-1} \lambda^{-1} \mathfrak{M}^{-1} (\alpha_0 \sin^3 i + \gamma_0 \mathfrak{M}^3) \cos i}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)} \quad \text{..................(xxxv),} \end{aligned}$$

$$\begin{aligned} \theta_p &= -\text{real part of } \{A / B \cos i\}_{A_0=0} \\ &= \text{real part of } \frac{16\pi^2 \iota C_0 e^{\iota x} V^{-1} \lambda^{-1} (\cos i / \mathfrak{M})(\alpha_0 \sin^3 i - \gamma_0 \mathfrak{M}^3)}{(\mathfrak{M} + \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} - \cos i)} \quad \text{................(xxxvi).} \end{aligned}$$

When the reflexion is equatorial, so that $\gamma_0 = 0$, it was observed by Kerr that θ_p vanishes when the angle of incidence is about 75°, the mirror being of iron. Later experiments point to a rather greater value of the angle of incidence for which θ_p is zero, the results obtained by different observers being as follows:

Kerr	75°	Sissingh	80°
Righi	78° 54′	Kundt	80° to 82°
Drude	79°		

Now when θ_p is zero, the complex of which it is the real part must have its vector angle equal to an odd number of right angles; and from (xxxvi) we see that this vector angle is ϕ, where

$$\phi \equiv x + 90^\circ - \text{the sum of the vector angles of } \mathfrak{M},\ (\mathfrak{M} + \cos i),\ \text{and}\ (\mathfrak{M} R^{-2} e^{-2ia} - \cos i).$$

For any assigned value of i the values of these three vector angles may be calculated from relation (xxxii), utilising the tabulated values of the optical constants for iron; the calculations, though tedious, are quite straightforward. The following table shews values of ϕ obtained in this way:

Angle of Incidence	75°	78° 54′	80°
ϕ	$x + 80^\circ\ 18'$	$x + 98^\circ\ 1'$	$x + 103^\circ\ 36'$

Hence the theory agrees with Kerr's observation provided $x + 80^\circ\ 18' = 90^\circ$, or $x = +9^\circ\ 42'$; but according to Righi's result $x = -8^\circ\ 1'$, and according to Sissingh's $x = -13^\circ\ 36'$. Thus the uncertainty of the observations renders it impossible to draw from them any definite conclusion as to what value ought to be attributed to the constant x in the theory. We can, however, determine the sign of C_0; for all the observers agree that, for angles of incidence less than 75°, θ_p has the same sign as α_0*; but for such incidences $\cos\phi$ is positive, and accordingly C_0 must also be positive.

When the reflexion is polar, so that $\alpha_0 = 0$, the mirror still being of iron, it was observed by Kerr that θ_p has the sign opposite to that of γ_0 for all angles of incidence. In order that this should be in accordance with the theory it is necessary that, if

$$\phi' \equiv x + 90^\circ + \text{vector angle of } \mathfrak{M}^2 - \text{the sum of the vector angles of}$$
$$(\mathfrak{M} + \cos i)\ \text{and}\ (\mathfrak{M} R^{-2} e^{-2ia} - \cos i),$$

$C_0 \cos\phi'$ should be positive for all angles of incidence. Now the values of ϕ' lie between $x + 219^\circ\ 43'$ corresponding to $i = 0$, and $x + 342^\circ\ 40'$ corresponding to $i = 90^\circ$; so that either C_0 is positive and x between $50^\circ\ 17'$ and $107^\circ\ 20'$, or C_0 is negative and x between $-72^\circ\ 40'$ and $-129^\circ\ 43'$. This experiment was repeated by Kundt, who found that θ_p vanishes, changing sign, when i is about 82°; the corresponding value of ϕ' is $x + 312^\circ\ 37'$, and as the cosine of this is to vanish, either $x = +137^\circ\ 23'$ and C_0 is positive, or $x = -42^\circ\ 37'$ and C_0 is negative: as by definition x is numerically less than 90°, only the latter of these values is admissible.

* Kerr, *Phil. Mag.*, March 1878, p. 166.

With regard to θ_i, the experiments shew that in the case of polar reflexion it has always the sign opposite to that of γ_0, and that in the case of equatorial reflexion it has always the sign opposite to that of α_0. Hence if we define ϕ'' and ϕ''' by the relations

$$\phi'' \equiv x + 90^\circ + \text{vector angle of } \mathfrak{M}^2 - \text{the sum of the vector angles of } (\mathfrak{M} - \cos i) \text{ and } (\mathfrak{M}R^{-2}e^{-2\iota a} + \cos i),$$

$$\phi''' \equiv x + 90^\circ - \text{the sum of the vector angles of } \mathfrak{M}, (\mathfrak{M} - \cos i), \text{ and } (\mathfrak{M}R^{-2}e^{-2\iota a} + \cos i),$$

the theory requires that $C_0 \cos\phi''$ and $C_0 \cos\phi'''$ shall be negative for all angles of incidence. Now the values of ϕ'' lie between $x + 39^\circ\ 43'$ corresponding to $i = 0^\circ$, and $x - 17^\circ\ 20'$ corresponding to $i = 90^\circ$; and the values of ϕ''' lie between $x + 200^\circ\ 43'$ corresponding to $i = 0^\circ$, and $x + 148^\circ\ 44'$ corresponding to $i = 90^\circ$. And therefore, when we remember that x has by definition been restricted to be numerically less than 90°, it appears from the first condition that C_0 must be negative and x somewhere between $-72^\circ\ 40'$ and $+50^\circ\ 17'$; while the second condition indicates that C_0 must be positive, and x somewhere between $-58^\circ\ 44'$ and $+69^\circ\ 17'$.

Thus it appears that there are very serious discrepancies in the values of C_0 and x indicated by the four original Kerr experiments for iron.

THE EXPERIMENTS OF SISSINGH AND ZEEMAN.

18. A more decisive test of the present theory is obtained by comparing its results with the elaborate series of experiments which have been recently made by Drs. Sissingh, Zeeman, and Wind, at the laboratory of Leyden. These consist in observations of the amplitude (μ) and phase (m) of the "magneto-optic component" of light reflected from magnetised mirrors of iron, nickel, and cobalt, for various angles of incidence. The details of the definition of these quantities will be found in Sissingh's paper in the *Archives Néerlandaises**; μ is always reckoned on the supposition that the amplitude of the incident light is unity, and m is defined as retardation of phase calculated relatively to that component of the ordinary metallic reflexion which is polarised in the plane of incidence. The values of these quantities corresponding to the particular cases when the incident light is polarised in, or perpendicularly to the plane of incidence, are distinguished by the suffixes ($_i$) and ($_p$) respectively. It may readily be shewn that the components of the incident light in the directions defined by Sissingh as "principal directions," are in the present notation represented by $-A_0 \sec i$ and $-B_0$; while the corresponding principal components of the reflected light are $-A \sec i$ and $-B$.

19. When the incident light is polarised in the plane of incidence $B_0 = 0$; and in formulae (xxxiv) the incident ray is represented by $-A_0 \sec i$, the magneto-optic component of the reflected ray by $-B$, and the component relatively to which phase

* Sissingh, "Mesures relatives au phénomène de Kerr," *Archives Néerlandaises*, vol. XXVII.

is to be measured, by $-A \sec i$. Hence

$$360^\circ - m_i = \text{vector angle of } \{B \cos i / A\}_{B_0=0}$$

$$= \text{vector angle of } \frac{16\pi^2 C_0 e^{\iota x} \iota V^{-1} \lambda^{-1} \mathfrak{M}^{-1} (\alpha_0 \sin^3 i + \gamma_0 \mathfrak{M}^3) \cos i}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota\alpha} + \cos i)} \quad \text{............(xxxvii).}$$

When the incident light is polarised perpendicularly to the plane of incidence $A_0 = 0$, and the incident ray is represented by $-B_0$; the magneto-optic component of the reflected ray is $-A \sec i$, or rather that term of $-A \sec i$ that contains the factor B_0. The ray relatively to which phase is measured is represented by that term of $-A \sec i$ which contains the (vanishing) factor A_0. If A_0 be supposed to be only just not zero, then, since the incident ray is supposed to be plane polarised, B_0/A_0 is a real quantity. Hence we have

$$360^\circ - m_p = \text{vector angle of } \frac{16\pi^2 C_0 e^{\iota x} \iota V^{-1} \lambda^{-1} (\cos^2 i / \mathfrak{M}) (\alpha_0 \sin^3 i - \gamma_0 \mathfrak{M}^3)}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota\alpha} + \cos i)} \quad \text{...(xxxviii).}$$

From (xxxvii) and (xxxviii) we see at once that when the reflexion is equatorial, that is when $\gamma_0 = 0$,

$$m_i = m_p = m \text{ (say)};$$

this agrees with the observation of Sissingh who, from the results of his experiments on equatorial reflexion, came to the conclusion that for any given angle of incidence the magneto-optic component has the same amplitude and phase, whether the incident light be polarised in or perpendicularly to the plane of incidence.

We also see that when the reflexion is polar, that is when $\alpha_0 = 0$,

$$m_i = m_p \pm 180^\circ.$$

Now Zeeman*, as a result of experiments on polar reflexion, came to the conclusion that for all angles of incidence $m_i = m_p$: here therefore is a discrepancy.

20. When the reflexion is equatorial, we see from (xxxvii) that

$$360^\circ - m = \text{vector angle of } \frac{16\pi^2 C_0 e^{\iota x} \iota V^{-1} \lambda^{-1} \alpha_0 \sin^3 i \cos i}{\mathfrak{M} (\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota\alpha} + \cos i)}.$$

In determining m from this expression there is an ambiguity to the extent of 180°, for in defining m Sissingh requires that it shall not be altered when α_0 changes sign. Examining his paper, we see that his standard case corresponds to α_0 negative. We shall also assume C_0 negative; in what follows the consequences of the alternative assumption, viz. C_0 positive, are obtained by adding 180° to the calculated values of m. We now find

$m = 270^\circ - x +$ the sum of the vector angles of

$$\mathfrak{M}, \ (\mathfrak{M} - \cos i), \text{ and } (\mathfrak{M} R^{-2} e^{-2\iota\alpha} + \cos i) \text{.........(xxxix),}$$

and, to get m accurately for any particular angle of incidence, these three vector angles must be calculated from formula (xxxii), using the known values of R and α for the particular metal considered.

* Zeeman, "Mesures relatives au phénomène de Kerr," *Archives Néerlandaises*, vol. XXVII. p. 252.

The following table shews the results of Sissingh's observations on the phase for various angles of incidence, and the theoretical values of the phase for the same incidences, calculated from the present theory.

Equatorial Reflexion from Iron. Yellow Light. $\alpha_0 = -1400$ C.G.S.

Angle of incidence	Calculated value of m	Sissingh's observed value of m	Excess of m (observed) over m (calculated)
$86^\circ\ 0'$	$199^\circ\ 55' - x$	$209^\circ\ 26'$	$9^\circ\ 31' + x$
$82\ 30'$	$192\ 39' - x$	$204\ 22'$	$11\ 43' + x$
$76^\circ\ 30'$	$183^\circ\ 51' - x$	$194^\circ\ 49'$	$10^\circ\ 58' + x$
$71\ 25'$	$178\ 33' - x$	$190\ \ 3'$	$11\ 30' + x$
$61^\circ\ 30'$	$171^\circ\ 38' - x$	$181^\circ\ 49'$	$10^\circ\ 11' + x$
$51^\circ\ 22'$	$167^\circ\ \ 8' - x$	$179^\circ\ \ 0'$	$11^\circ\ 52' + x$
$36^\circ\ 10'$	$162^\circ\ 47' - x$	$174^\circ\ \ 9'$	$11^\circ\ 22' + x$

In order that there should be agreement of theory with experiment it is necessary that the value of x for iron should be about -11°; if this be so the agreement is extremely good.

21. When the reflexion is polar we see from (xxxviii) that

$$360^\circ - m_p = \text{vector angle of } \frac{-16\pi^2 C_0 e^{\iota x} \iota V^{-1}\lambda^{-1}\gamma_0 \cos^2 i\, \mathfrak{M}^2}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)}.$$

Taking γ_0 positive in the standard case, and still assuming C_0 negative, we find that $m_p = 270^\circ - x -$ vector angle of $\mathfrak{M}^2$

+ the sum of the vector angles of $(\mathfrak{M} - \cos i)$ and $(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)$......(xl);

and m_i differs from this by 180°.

Observations of the amplitude and phase of the magneto-optic component, in the case of polar reflexion from an iron mirror, have been made by Zeeman. An account of these experiments will be found in the paper which we have already referred to; he confines himself to one angle of incidence, viz. $i = 51^\circ\ 22'$. His result as regards phase compares with theory as follows—

Polar Reflexion from Iron. Yellow Light. $\gamma_0 = +850$ C.G.S.

Angle of incidence	Calculated value of m_i	Zeeman's observed value of m	Excess of m (observed) over m_i (calculated)
$51^\circ\ 22'$	$151^\circ\ 14' - x$	$229^\circ\ 55'$	$78^\circ\ 41' + x$

Thus the agreement of m_i with Zeeman's m requires that $x = -78^\circ\ 41'$. The same value of x would correspond to agreement between m_p and Zeeman's m if C_0 were assumed positive; both results are at variance with that of the preceding paragraph. Values of x numerically greater than 90° are excluded by the definition in § 16, and so, of course, need not be discussed.

22. In considering the amplitude of the magneto-optic component, it is to be noticed that, when the incident light is polarised in the plane of incidence, the incident ray is $-A_0 \sec i$ and the magneto-optic component is $-B$; when the incident light is polarised perpendicularly to the plane of incidence, the incident ray is $-B_0$ and the magneto-optic component is $-A \sec i$. Hence

$$\mu_i = \text{mod}\left(\frac{B\cos i}{A_0}\right)_{B_0=0}, \quad \mu_p = \text{mod}\left(\frac{A}{B_0 \cos i}\right)_{A_0=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{xli}).$$

Thus, for equatorial reflexion, we readily derive from (xxxiv)

$$\mu_i = \text{mod}\, \frac{16\pi^2 C_0 e^{\iota x} \iota V^{-1}\lambda^{-1}\alpha_0 \sin^3 i \cos i}{\mathfrak{M}(\mathfrak{M} + \cos i)(\mathfrak{M}R^{-2}e^{-2\iota a} + \cos i)},$$

$$\mu_p = \text{the same};$$

and therefore $\mu_i = \mu_p = \mu$ (say), which agrees with Sissingh's result.

If for brevity we put $16\pi^2 C_0 V^{-1}\lambda^{-1}\alpha_0 \equiv L$, then

$$\mu = L \,.\, \text{mod}\, \frac{\sin^3 i \cos i}{\mathfrak{M}(\mathfrak{M} + \cos i)(\mathfrak{M}R^{-2}e^{-2\iota a} + \cos i)},$$

and the latter factor may be calculated for any angle of incidence.

In the following table the values of μ derived from theory for various angles of incidence are compared with the values observed by Sissingh.

Equatorial Reflexion from Iron. Yellow Light. $\alpha_0 = -1400$ C.G.S.

Angle of incidence	Calculated value of $\log_{10}\mu - \log_{10}L$	Sissingh's observed value of $10^3 \times \mu$	$\left(\frac{\text{Calculated value of } \mu/L}{\text{Observed value of } \mu}\right)$
86° 0′	$\bar{2}$·1485	·284	49·57
82° 30′	$\bar{2}$·3438	·530	41·64
76° 30′	$\bar{2}$·4673	·715	41·02
71° 25′	$\bar{2}$·4932	·815	38·20
61° 30′	$\bar{2}$·4512	·820	34·46
51° 22′	$\bar{2}$·3228	·760	27·67
36° 10′	$\bar{3}$·9724	·630	14·90
24° 16′	$\bar{3}$·5853	·430	8·95
12° 0′	$\bar{4}$·6192	·260	1·60
6° 0′	$\bar{5}$·7233	·125	·423

In order that the theory should agree with experiment it is necessary that all the numbers in the last column should be equal. Obviously this is not the case; and their inequality is so pronounced, and depends in such a regular manner upon the angle of incidence, that it cannot possibly be attributed to accidental errors of observation. We must therefore conclude that here the theory is distinctly at variance with experiment.

23. For polar reflexion, we derive from (xli) and (xxxiv)

$$\mu_p = \text{mod}\,\frac{-16\pi^2 C_0 e^{\iota\chi}\iota V^{-1}\lambda^{-1}\mathfrak{M}^2\gamma_0 \cos i}{(\mathfrak{M} + \cos i)(\mathfrak{M}R^{-2}e^{-2\iota\alpha} + \cos i)}$$

$$= -\mu_i = \mu\,(\text{say}).$$

Comparing this with the amplitude in equatorial reflexion, we find

$$\frac{\mu\,(\text{equatorial})}{\mu\,(\text{polar})} = \text{mod}\left[\frac{(-\alpha_0)\sin^3 i}{\gamma_0\,\mathfrak{M}^3}\right].$$

If $-\alpha_0 = 1400$, $\gamma_0 = 850$, $i = 51^\circ\,22'$, the value of this ratio for iron, as calculated from theory, is ·0122. But the values ascribed to α_0, γ_0, and i correspond to the experiments of Sissingh and Zeeman; and the latter found experimentally

$$\frac{\mu\,(\text{Sissingh})}{\mu\,(\text{Zeeman})} = {\cdot}294,$$

so that here again there is a serious discrepancy between theory and experiment.

Nickel.

24. In the paper of Zeeman's already quoted there are given some measurements which he made upon polar reflexion from nickel. He also quotes experimental results of Kundt* and Drude†, which he expresses in a form similar to his own. These I have used to form the following tables, wherein the theoretical values of the phase and amplitude have in all cases been calculated for yellow light.

Equatorial Reflexion from Nickel. White Light.

Angle of incidence	Calculated value of m	Kundt's observed value of m_i	Excess of m_i (observed) over m (calculated)
30° 6′	$146^\circ\,11' - x$	176° 10′	$29^\circ\,59' + x$
40° 0′	$148^\circ\,50' - x$	115° 42′	$-33^\circ\;\;8' + x$
50° 0′	$152^\circ\,13' - x$	115° 14′	$-36^\circ\,59' + x$
61° 30′	$158^\circ\,17' - x$	127° 39′	$-30^\circ\,38' + x$
65° 18′	$161^\circ\;\;1' - x$	126° 42′	$-34^\circ\,19' + x$
75° 0′	$170^\circ\,45' - x$	130° 6′	$-40^\circ\,39' + x$

* *Wied. Ann.* vol. XXIII.

† *Wied. Ann.* vol. XLVI.

The value of x indicated by the figures in the last column is about $+35'$; this would give fairly good agreement except in the case of the first angle of incidence.

Equatorial Reflexion from Nickel. White Light.

Angle of incidence	Calculated value of m	Drude's observed value of m	Excess of m (observed) over m (calculated)
60°	$157° 19' - x$	131° 38′	$-25° 41' + x$
65	$160° 48' - x$	133° 57′	$-26° 51' + x$
75°	$170° 45' - x$	191° 41′	$20° 56' + x$
80°	$178° 17' - x$	171° 18′	$-6° 59' + x$

If the third angle of incidence, for which Drude's result differs widely from that of Kundt, be left out of account the mean value indicated for x is about $+17°$.

Equatorial Reflexion from Nickel.

Angle of incidence	Calculated value of $\log_{10}\mu - \log_{10}L$	Kundt's observed value of $10^3 \times \mu$	$\left(\frac{\text{Calculated value of } \mu/L}{\text{Observed value of } \mu}\right)$
30° 6′	$\bar{3}{\cdot}8199$	·21	31·46
40° 0′	$\bar{2}{\cdot}1366$	·77	17·79
50° 0′	$\bar{2}{\cdot}3527$	1·39	16·21
61° 30′	$\bar{2}{\cdot}5017$	·90	35·28
65° 18′	$\bar{2}{\cdot}5278$	·84	40·14
75° 0′	$\bar{2}{\cdot}5334$	·23	145·1

The inequality of the numbers in the last column shews that the theory does not here agree with experiment.

Polar Reflexion from Nickel. Yellow Light.

Angle of incidence	Calculated value of m_i	Zeeman's observed value of m	Excess of m (observed) over m_i (calculated)
50°	$160° 19' - x$	191° 40′	$32° 21' + x$

so that the value of x required for agreement is $-32° 21'$.

The experimental results used in the following table are due to Dr C. H. Wind*.

Polar Reflexion from Nickel. Yellow Light.

Angle of incidence	Calculated value of m_i	Observed value of m_i	Excess of m_i (observed) over m_i (calculated)
39° 4′	155° 35′ − x	14° 32′	− 141° 3′ + x
55° 0′	163° 1′ − x	17° 47′	− 145° 3′ + x
75° 0′	180° 30′ − x	32° 25′	− 148° 5′ + x

As the value of x here indicated, viz. about + 145°, is inadmissible, it appears that this set of experiments requires C_0 to be positive; the value of x then indicated is − 35°.

Cobalt.

25. Experiments made by Zeeman and by Drude on mirrors of cobalt are used in the following tables.

Polar Reflexion from Cobalt. White Light.

Angle of incidence	Calculated value of m_i	Zeeman's observed value of m	Excess of m (observed) over m_i (calculated)
45°	157° 55′ − x	200° 34′	42° 39′ + x
60°	165° 6′ − x	207° 40′	42° 34′ + x
73°	175° 54′ − x	217° 55′	42° 1′ + x

Here the value indicated for x is − 42° approximately.

Polar Reflexion from Cobalt.

Angle of incidence	Calculated value of m_i	Zeeman's observed value of m	Excess of m (observed) over m_i (calculated)
50°	159° 55′ − x	205° 9′	45° 14′ + x
60°	165° 6′ − x	212° 30′	47° 24′ + x
72°	174° 47′ − x	225° 51′	51° 4′ + x

The indicated value of x is about − 47°.

* *Communications from the Leiden Laboratory of Physics*, No. 9.

Equatorial Reflexion from Cobalt.

Angle of incidence	Calculated value of m	Drude's observed value of m	Excess of m (observed) over m (calculated)
35°	146° 49′ − x	102° 36′	− 44° 13′ + x
60°	155° 48′ − x	154° 33′	− 1° 15′ + x
75°	168° 29′ − x	167° 4′	− 1° 25′ + x
83°	182° 19′ − x	167° 3′	− 15° 16′ + x

No value of x will make the theory agree with this series of experiments; the mean of the indicated values is about + 15° 30′.

The experiments used in the following table are described in the *Communications from the Leiden Laboratory of Physics*, No. 5.

Polar Reflexion from Cobalt. White Light. $\gamma_0 = 430$ C.G.S.

Angle of incidence	Calculated value of $\log_{10}\mu_p - \log_{10} L'$	Zeeman's observed value of $10^3 \times \mu$	$\left(\dfrac{\text{Calculated value of } \mu_p/L'}{\text{Observed value of } \mu}\right)$
45°	·5559	1·58	2276
60°	·5349	1·50	2284
73°	·4690	1·17	2516

In this, L' is an abbreviation for $-16\pi^2 C_0 V^{-1}\lambda^{-1}\gamma_0$. The approximate equality of the numbers in the last column indicates a very good agreement of the theory with this set of experiments.

Conclusion.

26. On comparing with one another the results of the last six paragraphs it is readily seen that, while it is possible to assign to the x of any one of the metals considered such a value as will bring the theory into a more or less rough agreement with the experiments on equatorial reflexion, or again such a value as will bring about agreement with the experiments on polar reflexion, yet these two values of x are so widely separated from one another that they cannot be reconciled even by the utmost allowances for errors of observation. The results as regards amplitude moreover, in the cases of nickel and iron, shew that no two of the experiments can be accounted for

by the same value of C_0. The discrepancies between the theory and the Kerr experiments are also very noticeable, though their importance is perhaps not so great on account of the extreme delicacy which is required in these experiments. On the whole then, it is clear that the theory which we have been considering does not account for the observed facts. A confirmation of this conclusion is afforded by the absence of β_0 from the formulae (xxxiv), which signifies that, according to the theory, the component of magnetisation perpendicular to the plane of incidence produces no effect; but such an effect does exist, and has been measured by Zeeman*.

* *Communications from the Leiden Laboratory of Physics*, No. 29.

III. *On the solutions of the equation* $(\nabla^2+\kappa^2)\psi=0$ *in elliptic coordinates and their physical applications.* By R. C. Maclaurin, St John's College.

[*Received* and *read* 16 May, 1898.]

It is well known that the solution of a very large number of physical problems depends almost entirely on the successful treatment of the differential equation $(\nabla^2+\kappa^2)\psi=0$. The difficulty in any case is to obtain a solution in terms of coordinates that lend themselves readily to the symbolic expression of the "boundary conditions" of the problem. When the boundaries are either right circular cylinders or spheres all the analytical difficulties have been most successfully overcome, but comparatively little headway has been made with other forms of bounding surfaces.

The present paper deals with problems relating to elliptic cylinders and spheroids.

The two-dimensional problem seems first to have been attacked by Mathieu [Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique, *Journal de Liouville*, t. XIII., p. 137]. This was in 1868. In the following year H. Weber published a paper in the *Math. Annalen* (Bd. I.) dealing with the subject. Further references will be found in Heine [*Handbuch der Kugelfunctionen*, Bd. II., p. 208] and in a recent work by Pockels (1891) *Ueber die partielle Differentialgleichung* $\Delta u+\kappa^2 u=0$.

The three-dimensional problem is from an analytical point of view very similar to the one for two dimensions. It has been attacked by C. Niven in the *Phil. Trans.* 1880, in a paper on the "Conduction of Heat in ellipsoids."

The present essay will be found to contain very little in common with any of the above—except that the physical problem that occupied Prof. Niven in 1880 receives a brief mention here, although the method of treatment is quite different. Since this paper was written, my attention has been called to a short article by Lindemann, "Ueber die Differentialgleichung der Functionen des Elliptischen Cylinders" [*Math. Annalen*, Bd. 22, p. 117]. He uses independent variables practically the same as those of this essay (p. 43, et seq.) and obtains some of the results reached here, but is mainly occupied with proving some theorems about the *product* of two solutions of the differential equation.

In dealing with elliptic cylinders, we may define the position of any point by its distance z measured along the axis from some fixed normal section and by the semi-axes a and a' of the confocal ellipse and hyperbola that pass through the point. We may develope ψ (regarded as a function of z) in a Fourier series of the form $\Sigma A_n \cos(nz-\epsilon_n)$, where the coefficients A_n are functions of a and a'. Since

$$\frac{\partial^2}{\partial z^2} A_n \cos(nz-t_n) = -n^2 A_n \cos(nz-\epsilon_n),$$

we see that the equation $(V^2+\kappa^2)\psi=0$ reduces to $(V_1^2+\kappa'^2)\psi=0$, where $\kappa'^2=\kappa^2-n^2$ and $V_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. Thus practically the whole difficulty is reduced to finding a suitable solution of the equation $(V_1^2+\kappa^2)\psi=0$.

Now with the usual notation $ds^2=\frac{da^2}{h_1^2}+\frac{da'^2}{h_2^2}$,

we have
$$V_1^2\psi = h_1h_2\left[\frac{\partial}{\partial a}\left(\frac{h_1}{h_2}\frac{\partial\psi}{\partial a}\right)+\frac{\partial}{\partial a'}\left(\frac{h_2}{h_1}\frac{\partial\psi}{\partial a'}\right)\right],$$

and if we take $x=a/h$; $x'=a'/h$ where $2h$ is the distance between the foci of the confocal system we get

$$V_1^2\psi=\frac{1}{h^2(x^2-x'^2)}\left[(x^2-1)\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}-\left\{(x'^2-1)\frac{\partial^2\psi}{\partial x'^2}+x'\frac{\partial\psi}{\partial x'}\right\}\right].$$

If then $(V_1^2+\kappa^2)\psi=0$, we have, putting $h\kappa=\lambda$,

$$\lambda^2(x^2-x'^2)\psi=-\left[(x^2-1)\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}\right]+(x'^2-1)\frac{\partial^2\psi}{\partial x'^2}+x'\frac{\partial\psi}{\partial x'}.$$

$$(x^2-1)\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}+\lambda^2x^2\psi=(x'^2-1)\frac{\partial^2\psi}{\partial x'^2}+x'\frac{\partial\psi}{\partial x'}+\lambda^2x'^2\psi.$$

Now put $\psi=yy'$ where y is a function of x only and y' of x', and we get

$$\frac{1}{y}\left[(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\lambda^2x^2y\right]=\frac{1}{y'}\left[(x'^2-1)\frac{d^2y'}{dx'^2}+x'\frac{dy'}{dx'}+\lambda^2x'^2y'\right]$$

$= p^2$ say, where p^2 is some constant.

Hence we have
$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-p^2)y=0,$$

and a similar equation for y' in terms of x'.

We have $x=\frac{a}{h}=\frac{1}{e}$. Thus x is the reciprocal of the eccentricity of the ellipse and so is always *greater* than unity.

Also $x'=a'/h=\frac{1}{e'}$, and x', being the reciprocal of the eccentricity of the hyperbola, is always *less* than unity.

We see then that everything now depends on the solution of the equation

$$(x^2-1)\,y''+xy'+(\lambda^2x^2-p^2)\,y=0.$$

This equation has three critical points, $x=+1$, $x=-1$, $x=\infty$.

Hence we must endeavour to obtain suitable solutions for the three domains corresponding to these critical points.

To obtain a solution in the neighbourhood of $x=1$ we make the substitution $x=-z+1$ in the above equation, which becomes

$$z\,(z-2)\,y''+(-z+1)\,y'+[\lambda^2\,(z-1)^2-p^2]\,y=0.$$

If we write this in the normal form (i.e. with the coefficient of y'' unity) we see at once, by Fuch's Theorem, that its integrals are *regular* in the neighbourhood of $z=0$.

Hence y is of the form:—

$$y=a_0z^m+a_1z^{m+1}+\ldots\ldots+a_nz^{m+n}+\ldots\ldots,$$

there being at the most only a *finite* number of negative powers of z.

The *indicial equation* proves to be $m\,(2m-1)=0$, so that we have two series corresponding to $m=0$ and $m=\frac{1}{2}$.

Equating the coefficient of z^{m+n} to zero we get:—

$$-(m+n+1)(2m+2n+1)\,a_{n+1}+(\overline{m+n}^2+\lambda^2-p^2)\,a_n-2\lambda^2\,a_{n-1}+\lambda^2a_{n-2}=0.$$

Thus for the series corresponding to $m=0$, we have

$$-(n+1)(2n+1)\,a_{n+1}+(n^2+\lambda^2-p^2)\,a_n-2\lambda^2a_{n-1}+\lambda^2a_{n-2}=0\ldots\ldots\ldots\ldots(1),$$

and for that corresponding to $m=\frac{1}{2}$,

$$-(n+1)(2n+3)\,a_{n+1}+(\overline{n+\tfrac{1}{2}}^2+\lambda^2-p^2)\,a_n-2\lambda^2a_{n-1}+\lambda^2a_{n-2}=0\ldots\ldots\ldots(2).$$

Now consider the first series $(m=0)$ and put $v_{n+1}=a_{n+1}/a_n$.

Then we have $$v_{n+1}=+\frac{n^2+\lambda^2-p^2}{(n+1)\,(2n+1)}-\frac{2\lambda^2}{(n+1)\,(2n+1)\,v_n}+\frac{\lambda^2}{(n+1)\,(2n+1)\,v_nv_{n-1}}$$

$$=+\frac{1}{2}+\frac{An+B}{(n+1)\,(2n+1)}-\frac{2\lambda^2}{(n+1)\,(2n+1)\,v_n}+\frac{\lambda^2}{(n+1)\,(2n+1)\,v_nv_{n-1}}.$$

Thus when n *is very large, either* v_n is indefinitely small *or* v_n approaches the limit $+\frac{1}{2}$.

The series is therefore convergent if $|z|<2$.

It is easy to show that the series also converges if $|z|=2$.

For this purpose, put $z=+2z_1$, then we have

$$\phi(z)=\sum_0^\infty a_n z^n=\sum_0^\infty b_n z_1^n=\phi_1(z_1) \text{ say, where } b_n=(+2)^n a_n,$$

$$(n+1)(2n+1)\,b_{n+1}=2(n^2+\lambda^2-p^2)\,b_n-8\lambda^2(b_{n-1}-b_{n-2}),$$

$$b_{n-1}=\frac{2(n^2+\lambda^2-p^2)}{(n+1)(2n+1)}\,b_n-\frac{\delta\lambda^2}{(2n+1)(n+1)}(b_{n-1}-b_{n-2}).$$

Thus when n is very great $b_{n+1}=b_n$, so that $\dfrac{b_{n-1}-b_{n-2}}{(2n+1)(n+1)}$ is negligible and we may write $b_{n+1}=\dfrac{2(n^2+\lambda^2-p)}{(n+1)(2n+1)}\,b_n$.

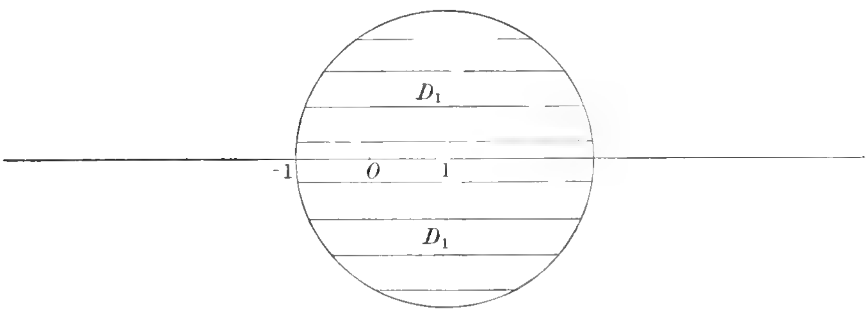
$\operatorname*{Lt}_{n=\infty} n\left(\dfrac{b_n}{b_{n-1}}-1\right)=\dfrac{3}{2}$, and as this is greater than unity it follows that the series $\sum_0^\infty b_n=\phi_1(1)$ is *convergent;* so that $\phi(+2)$ is convergent.

We can prove in exactly the same way that the other series (corresponding to $m=\frac{1}{2}$) is convergent if $|z|\not>2$.

We have thus obtained two solutions of our differential equation appropriate to the neighbourhood of the critical point $x=1$. These are:—

$$P=\phi(z)=\sum_0^\infty a_n z^n\;[a_n \text{ given by (1) p. 43, } a_0=1],$$

$$Q=z^{\frac{1}{2}}\psi(z)=\sum_0^\infty a_n z^{n+\frac{1}{2}}\;[a_n \text{ .... (2) ...... } a_0=1].$$



The '*domain*' of these functions P and Q is the interior of the circle, whose centre is $z=0$ and radius $=2$; i.e. the circle with centre at the critical point $x=1$, and passing through the next critical point $x=-1$. We shall call this the domain D_1.

$P = \phi(z) = \phi(1-x)$ is a 'uniform' function, returning to its original value, when the argument z traces out any closed contour. On the other hand

$$Q = z^{\frac{1}{2}}\,\psi(z) = (1-x)^{\frac{1}{2}}\,\psi(1-x)$$

is 'multiform'. $\psi(z)$ is uniform, but $z^{\frac{1}{2}}$ changes sign if z makes a tour round the pole $z=0$. If ρ is the modulus and θ the amplitude of z we may take

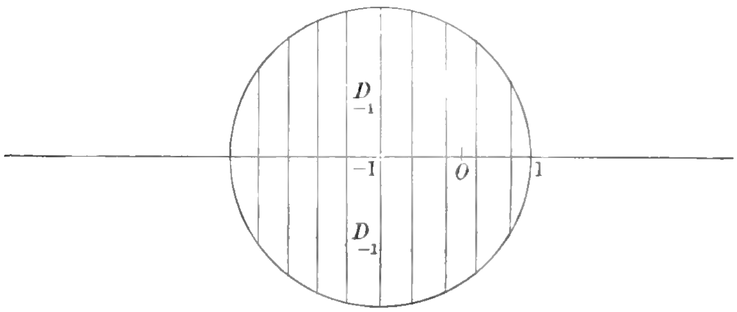
$$z^{\frac{1}{2}} = +\sqrt{\rho}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

Our fundamental equation is not changed if for x we write $-x$. Hence following out the same argument as above we shall get two solutions in the neighbourhood of the critical point $x=-1$, viz.:—

$$P' = \phi(1+x), \quad Q' = (1+x)^{\frac{1}{2}}\,\psi(1+x),$$

where ϕ and ψ are the functions already obtained.

The domain of these functions P' and Q' is the interior of the circle with centre at the critical point $x=-1$ and passing through the next critical point $x=1$. We shall refer to this as the domain D_{-1}.



We must now turn to the consideration of the integrals in the neighbourhood of the third critical point $x=\infty$. For this purpose we make the substitution $x=\frac{1}{x_1}$, a substitution which is simply and elegantly represented in a geometrical form by the aid of Neumann's sphere—in the well-known manner.

Our equation now becomes:—

$$x_1^4(1-x_1^2)\,y'' + x_1^3(1-2x_1^2)\,y' + (\lambda^2 - p^2x_1^2)\,y = 0.$$

The critical points of this equation are

$$x_1 = 0, \quad x_1 = \pm 1,$$

corresponding to $\quad x=\infty, \quad x=\pm 1.$

We have to consider the solutions in the neighbourhood of $x_1=0$.

Writing this equation in the normal form

$$y'' + p_1 y' + p_2 y = 0,$$

we see that $x_1 = 0$ is a pole of p_1 of order 1 and of p_2 of order 4. It follows from Fuch's theorem that the *integrals in the neighbourhood of* $x_1 = 0$ *are irregular.* We must not then expect quite the same simplicity in the treatment here as that which characterised the earlier work.

From the form of our equation we see that if y be expanded in powers of x_1 the coefficients of even and odd powers will be quite independent so that we may assume two solutions in the forms:—

$$y = x_1^s (a_0 + a_1 x_1^2 + \ldots + a_n x_1^{2n} + \ldots$$
$$+ a_{-1}/x_1^2 + \ldots + a_{-n}/x_1^{2n} + \ldots)$$
$$= x_1^s \sum_{n=-\infty}^{n=\infty} a_n x_1^{2n},$$

and
$$y = x_1^{s'} \sum_{n=-\infty}^{n=\infty} a_n' x_1^{2n+1}.$$

Take the first series $y = x_1^s \sum_{-\infty}^{\infty} a_n x_1^{2n}$ and substitute in the equation

$$x_1^4 (1 - x_1^2) y'' + x_1^3 (1 - 2x_1^2) y' + (\lambda^2 - p^2 x_1^2) y = 0.$$

We must have:—

$$x_1^4 (1 - x_1^2) \left[\frac{s(s-1) a_0}{x_1^2} + (s+2)(s+1) a_1 + (s+4)(s+3) a_2 x_1^2 + \ldots \right.$$
$$\left. + (s+2n)(s+2n-1) a_n x_1^{2n-2} + \ldots \right]$$
$$\left[+ \frac{(2-s)(3-s) a_{-1}}{x_1^4} + \frac{(4-s)(5-s) a_{-2}}{x_1^6} + \ldots + \frac{(2n-s)(2n+1-s) a_{-n}}{x_1^{2n+2}} + \ldots \right]$$
$$+ x_1^3 (1 - 2x_1^2) \left[\frac{s a_0}{x_1^2} + (s+2) a_1 x_1 + (s+4) a_2 x_1^3 + \ldots + (s+2n) a_n x_1^{2n-1} + \ldots \right.$$
$$\left. - \frac{(2-s) a_{-1}}{x_1^3} - \frac{(4-s) a_2}{x_1^5} - \ldots - \frac{(2n-s) a_{-2n}}{x_1^{2n+1}} \right]$$
$$+ (\lambda^2 - p^2 x_1^2) \left[a_0 + a_1 x_1^2 + \ldots + a_n x_1^{2n} + \ldots \right.$$
$$\left. + \frac{a_{-1}}{x_1^2} + \ldots + \frac{a_{-n}}{x_1^{2n}} + \ldots \right] \equiv 0.$$

Hence we must have:

$$\lambda^2 a_0 + [(2-s)^2 - p^2] a_{-1} - (3-s)(4-s) a_{-2} = 0,$$
$$\lambda^2 a_1 + [s^2 - p^2] a_0 - (2-s)(1-s) a_{-1} = 0,$$
$$\lambda^2 a_{n+1} + [2n + s^2 - p^2] a_n - (2n-1+s)(2n-2+s) a_{n-1} = 0.$$

The last equation holding also if for n we write $-n$.

Considering the ascending part of the series (for which n is +) we see at once from the last relation that in the general case (s unrestricted) the series a_n, $a_{n+1}\ldots$ will continually increase with n, so that the series formed in the way indicated on the last page will usually be divergent and so useless. But by properly choosing s it may be possible to make $a_n = 0$ for indefinitely large values of n, in which case as we shall prove our series will converge in the region $|x_1| \ngtr 1$. It is easy to see that if the ascending part of the series converges, the descending part will also converge.

For we have $(2n+1-s)(2n+2-s)\,a_{-(n+1)} - (2n-s^2-p^2)\,a_{-n} - \lambda^2 a_{-(n-1)} = 0$.

Let $a_{-n} = \dfrac{\lambda^{2n}(-1)^n c_n}{\Pi(2n-s)}$ where $\Pi(x)$ is Gauss' function $= \Gamma(x+1)$ and we get

$$\lambda^2 c_{n+1} + [2n - s^2 - p^2]\,c_n - (2n-s)(2n-s-1)\,c_{n-1} = 0.$$

Comparing this with the relation between a_{n+1}, a_n and a_{n-1}, we see that for very large values of n, the relations are practically identical. It is easy to see that c_n cannot be infinite for any *finite* value of n, hence it follows from what has just been said that for *large values of* n we may put $a_{-n} = \dfrac{\kappa\lambda^{2n}(-1)^n a_n}{\Pi(2n-s)}$ where κ is finite.

If then the ascending part of the series converges, the descending part will do so with great rapidity for *any finite value* of the argument.

We have said that c_n cannot be infinite for any *finite* value of n. For, putting

$$p^2 - 2n - s^2 = -\lambda^2 v_n,$$

$$(2n-s)(2n-s-1) = -\lambda^2 u_{n-1},$$

we have $c_{+1} + v_n c_n + u_{n-1} c_{n-1} = 0$. Suppose we make

$$c_1 = 0,$$

$$c_0 = 1,$$

then we have a system of equation to determine c_2, $c_3 \ldots c_n \ldots$

$$c_2 = -u_0,$$

$$c_3 + v_2 c_2 = 0,$$

$$c_4 + v_3 c_3 + u_2 c_2 = 0,$$

$$c_5 + v_4 c_4 + u_3 c_3 \qquad = 0,$$

and so on.

Solving we get

$$c_2 = -u_0,$$

$$c_3 = -\begin{vmatrix} -u_0 & 1 \\ 0 & v_2 \end{vmatrix} = u_0 v_2,$$

$$c_4 = \begin{matrix} -u_0 & 0 & 1 \\ 0 & 1 & u_2 \\ 0 & v_2 & u_2 \end{matrix},$$

and so on; the determinant in the denominator of c_n being $= (-1)^n$.

Thus as the denominators cannot vanish and the numerators cannot become infinite for finite values of n, we conclude that c_n is necessarily finite when n is so.

Returning now to p. 47, we have seen that s must be chosen so as to make $a_\infty = 0$, if the series is to converge.

The condition $a_\infty = 0$ is of course a necessary, but not a sufficient, condition for convergence. But it is easy to show that when this relation is satisfied the series *does* converge in the region

$$|x_1| < 1, \quad \{\text{i.e. } |x| > 1\}.$$

The series we are considering is

$$\Sigma u_n = x_1^s \Sigma a_n x_1^{2n},$$

where
$$\lambda^2 a_{n+1} + [(2n+s)^2 - p^2] a_n - (2n+s-1)(2n+s-2) a_{n+1} = 0.$$

For large values of n we have a_{n+1} very small, by hypothesis, and then the above relation gives

$$\text{Lt} \frac{a_n}{a_{n-1}} = \text{Lt} \frac{(2n+s-1)(2n+s-2)}{(2n+s)^2 - p^2} = 1\,;$$

$$\therefore \text{ Lt} \frac{u_n}{u_{n-1}} = x_1^2 \text{ [we are considering the } \textit{ascending} \text{ powers of the series].}$$

Hence the ascending series is convergent if $|x_1| < 1$, and divergent if $|x_1| > 1$.

When $|x_1| = 1$, we have $\text{Lt} \dfrac{u_n}{u_{n-1}} = 1$, so that the higher test $\text{Lt}\, n\left[\dfrac{u_n}{u_{n+1}} - 1\right]$ must be used. This limit is $\frac{3}{2}$, so that the series converges *on* the circle $|x_1| = 1$.

Thus we have proved that the *ascending* part of the series converges when

$$|x_1| \not> 1, \text{ i.e. } |x| \not< 1.$$

But by p. 47 if a_n is finite the *descending* part converges for *all* finite values of the argument. Of course when the ascending part converges a_n is finite, so that the *whole* series converges in any region in which the ascending part is convergent. Clearly also, if the ascending part *di*verges the series as a whole is divergent.

Summing up then, we find the necessary and sufficient condition for the convergence of the series we have obtained as a formal solution of our differential equation in the region $|x_1| \not> 1$ is that s should be a root of the equation $a_\infty = 0$.

For brevity, put

$$(2n+s)^2 - p^2 = \lambda^2 v_n\,; \quad (2n+s-1)(2n+s-2) = -\lambda^2 u_{n-1},$$

and we have
$$a_{n+1} + u_n a_n + u_{n+1} a_{n-1} = 0.$$

Some of the constants are necessarily arbitrary. Taking $a_0 = 1$, $a_1 = 0$, we have

$$a_1 = -v_0,$$

$$a_2 + v_1 a_1 = -u_0,$$

$$a_3 + v_2 a_2 + u_1 a_1 = 0, \quad \text{and so on.}$$

Solving, we get

$$a_1 = -v_0\,; \quad a_2 = \begin{vmatrix} v_0 & 1 \\ v_0 & v_1 \end{vmatrix}; \quad a_3 = -\begin{vmatrix} v_0 & 0 & 1 \\ u_0 & 1 & v_1 \\ 0 & v_2 & u_1 \end{vmatrix} = \begin{vmatrix} v_0 & 1 & 0 \\ u_0 & v_1 & 1 \\ 0 & u_1 & v_2 \end{vmatrix};$$

$$a_\infty = \begin{vmatrix} v_0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ u_0 & v_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & u_1 & v_2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & u_2 & v_3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & u_3 & v_4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & u_4 & v_5 & 1 & \dots \\ 0 & \dots \\ \vdots & \dots \end{vmatrix},$$

and s is determined as a root of $a_\infty = 0$.

We proceed to show how s may be developed in a series of ascending powers of λ^2.

Let $$a'_n = \lambda^{2n} a_n, \quad v_n = p^2 - (2n+s)^2,$$

then we have $$a'_{n+1} = v_n a'_n + \lambda^2 (s + 2n - 1)(s + 2n - 2)\, a'_{n-1}\,; \quad a'_1 = v_0\,;$$

$$a_2 = v_0 v_1 + \lambda^2 s(s+1) = v_0 v_1 \left[1 + \lambda^2 . \frac{s\,.\,s+1}{v_0 v_1}\right];$$

$$a'_3 = v_0 v_1 v_2 \left[1 + \lambda^2 . \left(\frac{s\,.\,s+1}{v_0\,.\,v_1} + \frac{s+2\,.\,s+3}{v_1\,.\,v_2}\right)\right],$$

and so on. The equation determining s is $a'_\infty = 0$.

It is clear that this equation is equivalent to

$$v_0 v_1 v_2 \dots v_\infty \left[1 + \lambda_0{}^2 S_1 + \lambda_0{}^4 S_2 + \lambda_0{}^6 S_3 \dots\right] = 0,$$

where $${}_0S_1 = \frac{s\,.\,s+1}{v_0\,.\,v_1} + \frac{(s+2)(s+3)}{v_1\,.\,v_2} + \frac{s+4\,.\,s+5}{v_2\,.\,v_3} \dots \text{ (mode of formation is obvious)},$$

${}_0S_2$ = sum of products of every *two* non-adjacent terms of the last series,

${}_0S_3$ = sum of products of every *three* non-adjacent terms of that series; and so on.

If $\lambda = 0$ we have $v_n = 0$ where n is zero or any positive integer, and this gives

$$s = \pm p - 2n.$$

To indicate the method of procedure let us obtain a few terms of the expansion of s, corresponding to $v_0 = 0$, as a first approximation.

We have $${}_0S_1 = \frac{s\,.\,s+1}{v_0\,.\,v_1} + {}_1S_1,$$

$${}_0S_2 = \frac{s\,.\,s+1}{v_0\,.\,v_1}\,{}_2S_1 + {}_1S_2\,; \quad {}_1S_2 = \frac{s+2\,.\,s+3}{v_1 v_2}\,{}_3S_1 + {}_2S_2,$$

$${}_0S_3 = \frac{s\,.\,s+1}{v_0\,.\,v_1}\,{}_2S_2 + {}_1S_3\,; \quad {}_1S_3 = \frac{s+2\,.\,s+3}{v_1 v_2}\,{}_3S_2 + {}_2S_3,$$

and so on.

Our equation to determine s is $v_0[1+\lambda^2{}_0S_1+\lambda^4{}_0S_2+\ldots\ldots]=0$,

i.e. $$v_0+\frac{s\,.\,s+1}{v_1}[\lambda^2+\lambda^4{}_2S_1+\lambda^6{}_2S_2+\ldots\ldots]+v_0\lambda^2[{}_1S_1+\lambda^2{}_1S_2+\ldots\ldots]=0.$$

First approximation:

$$v_0=0,\quad p^2-s^2=0,\quad s=p \text{ (taking + sign)}.$$

Second:

$$v_0+\lambda^2\,.\,\frac{s\,.\,s+1}{v_1}=0,$$

$$p^2-s^2-\lambda^2\,.\,\frac{p(p+1)}{4(p+1)}=0;\quad s=p-\frac{\lambda^2}{8}.$$

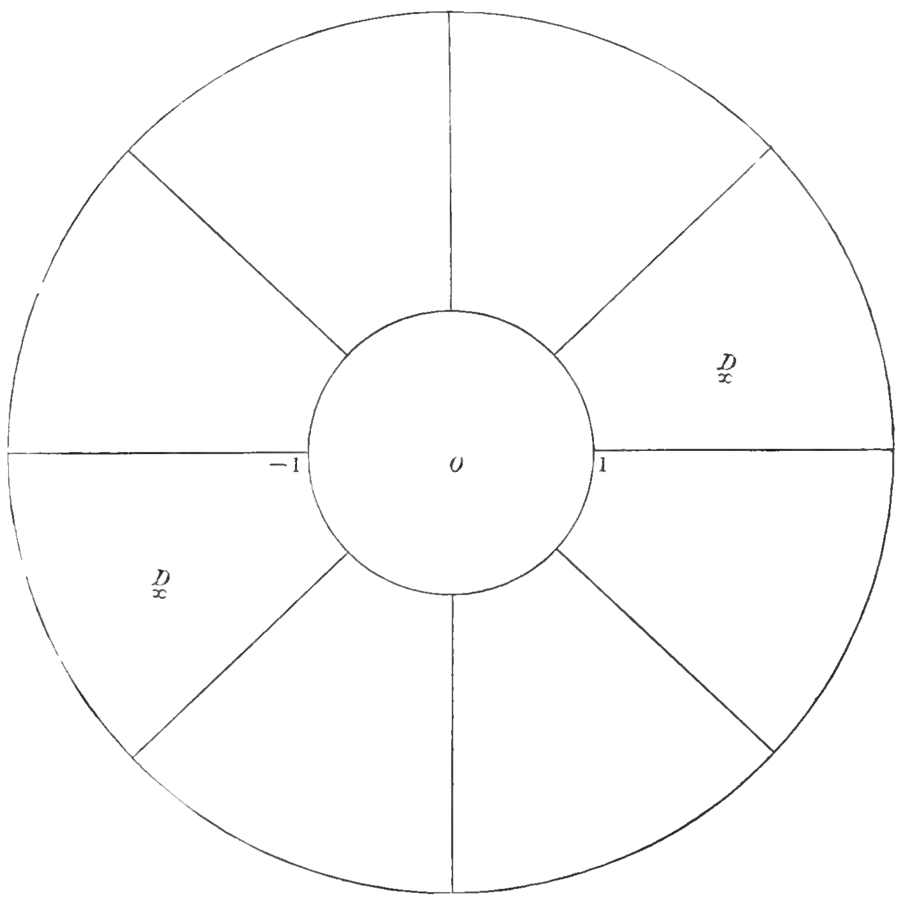
Third:

$$v_0+\frac{s(s+1)}{v_1}\lambda^2+\frac{s(s+1)}{v_1}\lambda^4{}_2S_1-\frac{s\,.\,s+1}{v_1}\lambda^4{}_1S_1=0,$$

$$v_0+\frac{s(s+1)}{v_1}\,.\,\lambda^2-\frac{s\,.\,s+1\,.\,s+2\,.\,s+3}{v_1^2v_2}\lambda^4=0,$$

$$s^2=p^2-\frac{p\lambda^2}{4}+\frac{4+7p-p^2}{128\,.\,(p+1)}\lambda^4,$$

and so on.



We have now shown how to obtain six different solutions of our fundamental equation:—

$$P = \phi(1-x); \text{ and } Q = (1-x)^{\frac{1}{2}} \psi(1-x),$$

which are applicable to the domain D_1,

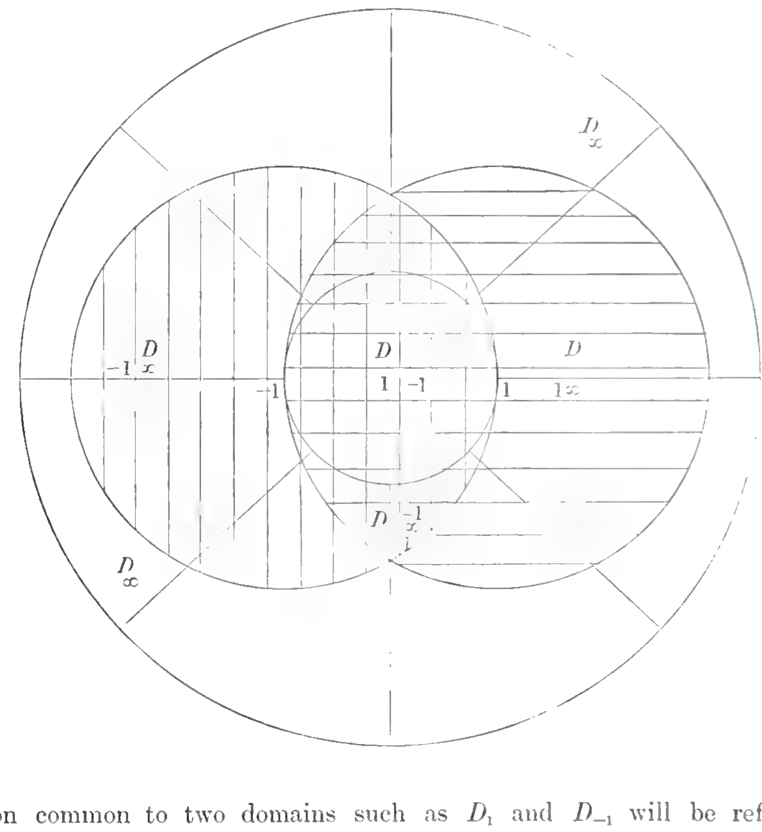
$$P' = \phi(1+x); \text{ and } Q' = (1+x)^{\frac{1}{2}} \psi(1+x) \ldots\ldots D_{-1},$$

$$P'' = x_1^s \Sigma a_n x_1^{2n} = x^{-s} \Sigma a_n x^{-2n} = x^{-s} \Phi(x),$$

and $$Q'' = x_1^{s'} \Sigma a_n' x_1^{2n+1} = x^{-s'} \Sigma a_n' x^{-2n-1} = x^{-s'} \Phi(x).$$

The last two series are convergent for all *finite* values of x such that $x > 1$. Thus the domain D_∞ is the ring bounded by the two circles $x = 1$, and $x = \infty$.

If we draw these various domains we see that they overlap.



The region common to two domains such as D_1 and D_{-1} will be referred to as the domain D_{1-1}.

In the domain D_{1-1} we have four solutions P, Q, P', Q', and there must consequently be a linear relation between any three of these.

We proceed to determine the values of the various constants in these linear relations.

In the domain D_{1-1} we have $P'=AP+BQ$,

i.e. $$\phi(1+x)=A\phi(1-x)+B(1-x)^{\frac{1}{2}}\psi(1-x).$$

Make x approach the point $x=1$ and we get $\phi(2)=A\phi(0)=A$.

[We have proved, p. 44, that $\phi(2)$ is finite.]

Next, keeping x real let it approach the point $x=-1$, so that the amplitude of $1-x$ is π, then we get
$$1=\phi(0)=A\phi(2)+Bi\sqrt{2}\,\psi(2).$$

Thus we have $A=\phi(2)$; $B=\dfrac{i}{\sqrt{2}}\cdot\dfrac{-1+\phi^2(2)}{\psi(2)}$.

Also we have in the same domain $Q'=A'P+B'Q$,

i.e. $$(1+x)^{\frac{1}{2}}\psi(1+x)=A'\phi(1-x)+B'(1-x)^{\frac{1}{2}}\psi(1-x).$$

Making $x=1$ we get $\sqrt{2}\,\psi(2)=A'\phi(0)=A'$.

And making x move along the real axis to $x=-1$,

we get $$0=A'\phi(2)+B'i\sqrt{2}\,\psi(2).$$

Hence we have $$A'=\sqrt{2}\,\psi(2);\quad B'=+i\phi(2).$$

We may note that $\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}=-i.$

We have then, $\begin{matrix} P'=AP+BQ \\ Q'=A'P+B'Q \end{matrix}$, from which we deduce
$$P=\frac{\begin{vmatrix} P' & B \\ Q' & B' \end{vmatrix}}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}=i(+BQ'-B'P'),$$

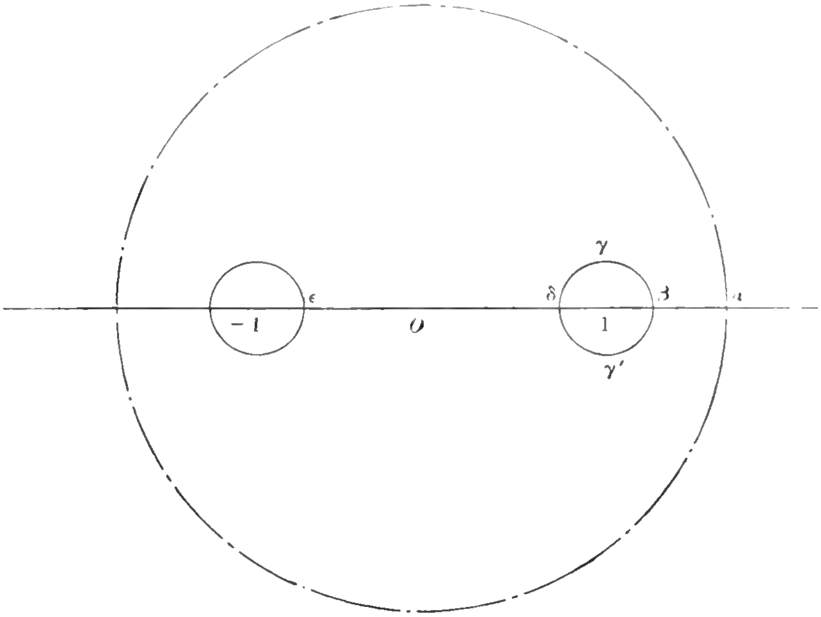
and $$Q=i(-AQ'+A'P').$$

These relations having been found we can determine with what solution of the differential equation we shall reach any point β (say) in D_1 or D_{-1} when we start with a definite solution from a point α in D_1 or D_{-1} and move along a given path to β, provided the path avoids the critical points 1, and -1.

As an example of this we shall work out a particular case that will afterwards be of service. Suppose we start at α on the real axis in the region D_1 and move along the dotted path in the figure so as to come back to α.

The only poles of the functions involved are the points 1 and -1, so that the effect of going along the dotted road is the same as going along the path $\alpha\beta\gamma\delta\epsilon$, round the loop enclosing -1, then back along $\epsilon\delta\gamma\beta\alpha$, then along $\alpha\beta\gamma\delta\gamma'\alpha$.

Suppose then we start at α with the solution P which is appropriate to D_1. As soon as we get to δ we can (if we choose) express P in the form $i(BQ' - B'P')$ for



we are now in the domain D_{1-1}. The functions $Q'P'$ hold throughout the domain D_{-1} and so are suitable for use when we wish to make a tour round -1. On making the circuit round -1, P' will be unaffected, while Q' will change sign. Thus we get back to ϵ with the solution $-i(BQ' + B'P')$. We are again in the domain D_{1-1} so that we can express P' and Q' in terms of P and Q by the help of the relations just obtained. We have, in fact,

$$\begin{aligned} -i(BQ' + B'P') &= -i[B(A'P + B'Q) + B'(AP + BQ)] \\ &= -i[(AB' + A'B)P + 2BB'Q]. \end{aligned}$$

P and Q hold all along the path $\epsilon\delta\gamma\beta\alpha$, so that we arrive at α with the solution

$$-i[(AB' + A'B)P + 2BB'.Q].$$

Starting now with this solution and going along $\alpha\beta\gamma\delta\gamma'\beta\alpha$ we note that P and Q hold all along the path, and that on making the circuit round 1, P is unaltered while Q changes sign. We conclude then that if we start from α with the solution P and go along the dotted path we shall return to α with the solution

$$-i[(AB' + A'B)P - 2BB'Q].$$

In exactly the same way we might show that if we had started with the solution Q we should have returned with $-i[(AB' + A'B)Q - 2AA'.P]$.

Next consider the various solutions that we have in the domain $D_{1\infty}$. They are:—

$$\left.\begin{aligned} P &= \phi(1-x);\ Q = (1-x)^{\frac{1}{2}}\psi(1-x) \\ P'' &= x^{-s}\Phi(x);\ Q'' = x^{-s'}\Psi(x) \end{aligned}\right\}.$$

We must therefore have relations of the form

$$x^{-s}\Phi(x) = P'' = \alpha P + \beta Q = \alpha\phi(1-x) + \beta(1-x)^{\frac{1}{2}}\psi(1-x),$$
$$x^{-s'}\Psi(x) = Q'' = \alpha' P + \beta' Q = \alpha'\phi(1-x) + \beta'(1-x)^{\frac{1}{2}}\psi(1-x).$$

The work on the last page will enable us to get a relation between α and β. For suppose we start with the function P'' from a point such as α on p. 53 in the domain $D_{1\infty}$ and describe a contour enclosing the points 1 and -1 but entirely confined to the domain D_∞ so that P'' holds throughout this path. We shall return to α with the function $(\cos 2s\pi - i\sin 2s\pi)P''$.

But in the region $D_{1\infty}$, $P''=\alpha P+\beta Q$, and from page 53 we see that if we start with the function $\alpha P+\beta Q$ and describe a path such as the one we have just followed with P'' we return to the starting point with the function

$$-i\alpha[(AB'+A'B)P-2BB'.Q]-i\beta[(AB'+A'B)Q-2AA'.P].$$

This then must be identical with $(\cos 2s\pi - i\sin 2s\pi)(\alpha P+\beta Q)$. Equating the coefficients of P and Q in these identities we get

$$\alpha(\cos 2s\pi - i\sin 2s\pi) = -i\alpha(AB'+A'B) + i.2AA'.\beta,$$
$$\beta(\cos 2s\pi - i\sin 2s\pi) = -i\beta(AB'+A'B) + i.2BB'.\alpha.$$

These two equations are really identical, they give us

$$\frac{\beta}{\alpha} = \frac{2iBB'}{\cos 2s\pi - i\sin 2s\pi + i(AB'+A'B)}.$$

Now we have $x^{-s}\Phi(x) = \alpha\phi(1-x) + \beta(1-x)^{\frac{1}{2}}\psi(x)$.

Putting $x=1$ we get $\Phi(1)=\alpha\phi(0)=\alpha$, so that we have

$$\alpha=\Phi(1);\ \beta = \frac{2iBB'.\Phi(1)}{\cos 2s\pi - i\sin 2s\pi + i(AB'+A'B)}.$$

Similarly we can determine α' and β'.

We have now completed the formal solution of our differential equation. It is an equation of the second order with three critical points, and we have obtained two solutions in the domain of each critical point and determined the constants in the linear relations that connect different solutions in a common domain. There is no finite region of the plane for which we have not obtained an appropriate solution. But for dealing with physical problems which it is the main object of this paper to attack, the solutions in the vicinity of the origin are not in a very convenient form. In the domain $|x|<1$ we want solutions expressed in powers of x. It is easy to build up such solutions by taking proper linear functions of $P.Q.P'.Q'$, but we may as well attack the problem directly.

Our equation is $(x^2-1)y''+xy'+(\lambda^2x^2-p^2)y=0$. Assuming a solution of the form $y=a_0x^m+a_1x^{m-1}+\dots$ we find, on equating the coefficient of the lowest power of x to zero, that we must have $m(m-1)=0$.

Thus we have two solutions of the forms:—

$$y = a_0 + a_1 x^2 + \ldots + a_n x^{2n} + \ldots$$

$$y = c_0 x + c_1 x^3 + \ldots + c_n x^{2n+1} + \ldots$$

The equations connecting the coefficients are:—

$$\begin{cases} -2a_1 - p^2 a_0 = 0\,; \; -12a_2 + (4 - p^2)\,a_1 + \lambda^2 a_0 = 0 \\ -(2n+1)(2n+2)\,a_{n+1} + (4n^2 - p^2)\,a_n + \lambda^2 a_{n-1} = 0 \end{cases},$$

$$\begin{cases} -6c_1 + (1 - p^2)\,c_0 = 0\,; \; -20c_2 + (3^2 - p^2)\,c_1 + \lambda^2 c_0 = 0 \\ -(2n+2)(2n+3)\,c_{n+1} + [(2n+1)^2 - p^2]\,c_n + \lambda^2 c_{n-1} = 0 \end{cases}.$$

[We may note here that if $\lambda = 0$ our equation reduces to $(x^2 - 1)\,y'' + xy' - p^2 y = 0$, which is satisfied by $y = A\cos p\theta + B\sin p\theta$ where $x = \cos\theta$; $|x| \ngtr 1$. For some purposes it is convenient to take $z = \lambda x$ as variable and then we have $(z^2 - \lambda^2)\,y'' + zy' + (z^2 - p^2)\,y = 0$, which, when $\lambda = 0$, reduces to Bessel's equation and gives $y = AJ_p(z) + BK_p(z)$.]

Let us examine the convergence of these series.

We have $\qquad -(2n+1)(2n+2)\,a_{n+1} + (4n^2 - p^2)\,a_n + \lambda^2 a_{n-1} = 0.$

Let $v_n = a_{n+1}/a_n$ and we get $v_n = \dfrac{4n^2 - p^2}{(2n+1)(2n+2)} + \dfrac{\lambda^2}{(2n+1)(2n+2)} \cdot \dfrac{1}{v_{n-1}}$

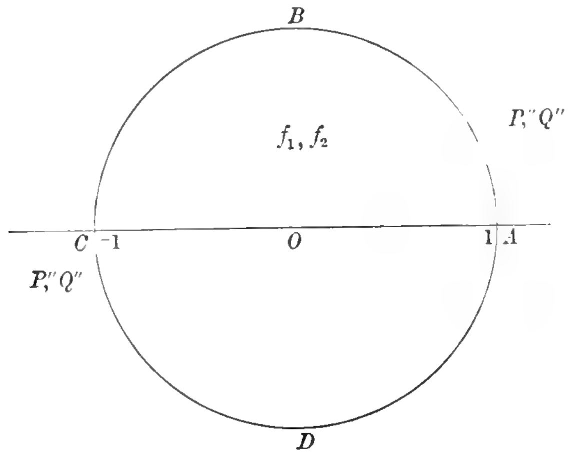
$$= 1 - \frac{6n + 2 + p^2}{(2n+1)(2n+2)} + \frac{\lambda^2}{(2n+1)(2+2)\,v_{n-1}}.$$

Thus *either* v_n is very small when n is large, in which case v_n approaches the limit $\dfrac{-\lambda^2}{(2n+1)(2n+2)}$, *or* v_n is not indefinitely small and approaches the limit 1. In the former case the series converges for *all* finite values of the argument. In the latter it converges when $|x| < 1$, and also, as we can easily show by proceeding as on p. 44, when $|x| = 1$. Thus in the most unfavourable case the series converges when $|x| \ngtr 1$ and this quite independently of the value of p. Exactly similar reasoning applies to the odd series $c_0 x + \ldots$. Denoting these solutions by $f(x)$ [two functions, one odd, and the other even] we have four solutions of our differential equation expressed as series of powers of x. Two solutions (f) are confined to the region $|x| < 1$, the other two (P'' and Q'') to the region $|x| \nless 1$. All these series hold *on* the circle $|x| = 1$.

If we start from A (see next page) and go with f_1 or f_2 along the real axis to C and back again to A, we must return to A with the *same* function with which we started if the functions involved are suitable for use in *physical problems* dealing with the *complete* cylinder.

For a tour round the cylinder must bring us back to the *same* physical conditions from which we started.

Now *on* the circle $|x| = 1$ we can express f_1 and f_2 as linear functions of P'' and Q''. And starting with f from A and going along $AOCOA$ must bring us to the same result as going along $ABCDA$. But along this latter path we can replace f by a linear function of P'' and Q''; hence it follows that, in dealing with physical problems concerning the *complete* cylinder, P'' and Q'' must be such that they return to their



original values when taken round the contour $ABCDA$. It is at once obvious that this requires *that s and s' should be either zero,* or *positive integers* (we can take them to be zero).

But if s and s' are zero, the condition for the convergence of the series P'' and Q'' will restrict us to a *particular set of values of p*, viz. the roots of $a_\infty=0$, [p. 48].

Now a few pages back, in dealing with the convergence of the f functions, we saw that there were *two* alternatives—v_n tends either to zero or unity in the limit. In the latter case the series is convergent in *a certain region* f whatever be the value of p. In the former the series is convergent for *all* finite values of the argument. If then we choose p as a root of $v_\infty=0$, we shall have series that are convergent for all portions of the x plane at a finite distance from the origin.

Now if $s=0$ the relations connecting the coefficients on p. 46, are the same as those that connect the coefficients of the f functions.

We see then that in this case the particular values of p to which we are confined for the convergence of our functions P'' and Q'' are identical with the roots of $v_\infty=0$. [To distinguish the odd and even series we shall refer to these as the roots of $a_\infty=0$ and $c_\infty=0$ respectively.] Thus the f functions are convergent for *all finite* values of the argument and our problem is to a certain extent simplified by the fact that we can use the same functions [an odd and even power series of x, respectively] for *all* finite values of x. However, this simplification is counterbalanced by the fact that the two solutions on p. 55, i.e. the odd and even series correspond to different values of p^2, so that we are obliged to complete the solution of our differential equation by the aid of new functions.

As the determination of the appropriate values of p^2 is important for the physical applications, we must consider this part of the problem in some detail and obtain some numerical results.

Considering the even series (the discussion of the odd series proceeds on exactly the same lines), we have seen that p^2 must be chosen so as to be a root of $a_\infty = 0$.

Putting $a_0 = 1$, the various coefficients are given by the equations

$$2a_1 = -p^2,$$

$$12a_2 = (4 - p^2)\,a_1 + \lambda^2,$$

$$(2n+1)(2n+2)\,a_{n+1} = (4n^2 - p^2)\,a_n + \lambda^2 a_{n-1}.$$

Before proceeding with the actual calculation of the roots of $a_\infty = 0$, we shall notice some points as to the position of the various roots of $a_n = 0$.

The equation $a_n = 0$ considered as an equation in p^2 is clearly of the nth degree. Its roots are all positive, for it is obvious that when p^2 is negative a_n is necessarily positive. For if this is true of a_n and a_{n-1}, then since

$$(2n+1)(2n+2)\,\lambda^2 a_{n+1} = (4n^2 - p^2)\,a_n + \lambda^2 a_{n-1},$$

it is true also of a_{n+1}. But it is clearly true of a_1 and a_2 so that, by induction, it must be true for a_n.

For some purposes it is rather more convenient to replace a_n by $(-1)^n a_n'$, so that we have:—

$$2a_1' = p^2;\quad 12a_2' = (p^2 - 4)\,a_1' + \lambda^2,$$

$$(2n+1)(2n+2)\,a'_{n+1} = (p^2 - 4n^2)\,a_n' + \lambda^2 a'_{n-1}.$$

It is now obvious that all the roots of $a'_{n+1} = 0$ are less than $4n^2$. For if we put $p^2 = 4n^2$ or any greater quantity $a_1' a_2' \ldots\ldots a'_{n+1}$ are all positive.

Again when $p^2 = (2n-2)^2$, $a_1' a_2' \ldots\ldots a_n'$ are all positive and

$$a'_{n+1} = \frac{-8n+4}{(2n+1)(2n+2)}\,a_n' + \frac{\lambda^2 a'_{n-1}}{(2n+1)(2n+2)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For large values of n the last term on the right is negligible compared with the first, and as a_n' is positive it follows that for large values of n, a'_{n+1} will be negative. [Our interest is centred mainly in the roots of $a_n = 0$ *when n is large*, and in what follows we shall suppose n large enough to make the right-hand side of (1) negative when a_n' is positive.]

We have seen then that a'_{n+1} is positive when $p^2 = (2n)^2$, and negative when

$$p^2 = (2n-2)^2.$$

Hence the equation $a'_{n+1} = 0$ has a root between $(2n-2)^2$ and $(2n)^2$.

We have seen that all the roots of $a_n'=0$ are less than $(2n-2)^2$. Let p_n^2 be any such root. Then the expressions

$$p_n^2-(2n)^2;\quad p_n^2-(2n+2)^2;\ \ldots\ldots$$

are all *negative*.

Also since $p^2=p_n^2$ satisfies $a_n'=0$ we have these relations:—

$$(2n+3)(2n+4)\,a'_{n+2}=[p_n-(2n+2)^2]\,a'_{n+1},$$

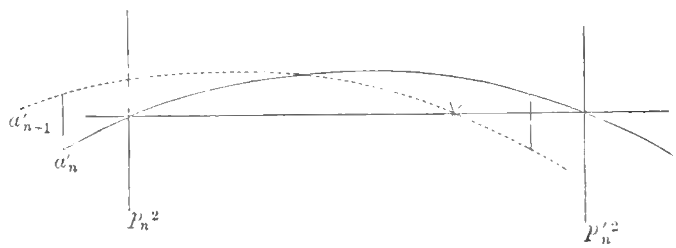
$$(2n+5)(2n+6)\,a'_{n+3}=[p_n-(2n+4)^2]\,a'_{n+2}+\lambda^2 a'_{n+1}.$$

etc. etc.

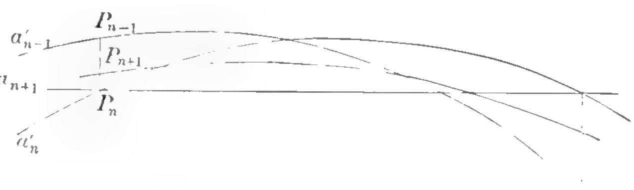
Thus when $p^2=p_n^2$ we see that a'_{n+2} is of opposite sign to a'_{n+1}; and a'_{n+3} is of opposite sign to a'_{n+2} and so on [n being large enough—see remark on last page]. If a'_{n-1} is positive then a'_{n+1} is positive, a'_{n+2} is negative, and so on.

Now let $p_n'^2$ be the *next* root of $a_n'=0$ greater than p_n^2.

Suppose the graph of a_n' is as in the figure, the dotted line representing a'_{n+1}. We have taken a_n' to be negative when p^2 is a little less than p_n^2. For this value



of p^2, a'_{n+1} must be of opposite sign to a_n' and so positive. Next consider a value of p^2 slightly less than $p_n'^2$. Here a_n' is positive and a'_{n+1} consequently negative. Thus a'_{n+1} has changed sign in the interval. Hence we see that a root of $a'_{n+1}=0$ lies between each pair of roots of $a_n'=0$. From what was said above this is true also of $a'_{n+2}=0$, $a'_{n+3}+0$; etc. Each of these equations has a root lying between any pair of roots of $a_n'=0$.



From the relation $(2n+1)(2n+2)a'_{n+1} = (p^2 - 4n^2)a_n' + \lambda^2 a'_{n-1}$ we see that when $a_n' = 0$, a'_{n+1} and a'_{n-1} have the same signs and that for large values of n, a'_{n+1} is then but a small fraction of a'_{n-1}.

Thus if in the figure P_nP_{n-1} is finite, P_nP_{n+1} is *very* small (for large values of n)—so that the roots of $a'_{n+1} = 0$ are very close to those of $a_n' = 0$.

We conclude then that as n increases the roots of $a'_{n+1} = 0$, approximate more and more closely to the roots of $a_n' = 0$ and so in the limit when $n = \infty$, the roots of $a_\infty = 0$ are *definite in position* and independent of n.

We shall now proceed to calculate some of the roots of $a_\infty = 0$ and $c_\infty = 0$. As on p. 49 we can express these quantities a_∞ and c_∞ as infinite determinants. But the second method there referred to is the practical one, i.e. we develope p^2 in a series of powers of λ^2, the series being rapidly convergent if λ^2 is not too large.

For brevity let $v_n = 4n^2 - p^2$; $a_n = \dfrac{u_n}{\lfloor 2n}$, then we get

$$u_{n+1} = u_n v_n + \lambda^2 . (2n-1)\, 2n . u_{n-1}.$$

Taking $u_0 = 1$ we have

$$u_1 = v_0,$$

$$u_2 = v_0 v_1 \left[1 + \lambda^2 . \frac{1.2}{v_0 . v_1}\right],$$

$$u_3 = v_0 v_1 v_2 \left[1 + \lambda^2 \left(\frac{1.2}{v_0 . v_1} + \frac{3.4}{v_1 . v_2}\right)\right],$$

$$u_4 = v_0 v_1 v_2 v_3 \left[1 + \lambda^2 \left(\frac{1.2}{v_0 . v_1} + \frac{3.4}{v_1 . v_2} + \frac{5.6}{v_2 . v_3}\right) + \lambda^4 . \frac{1.2.5.6}{v_0 v_1 v_2 v_3}\right],$$

and so on.

We see then that the equation $a_\infty = 0$ is equivalent to

$$v_0 v_1 v_2 \ldots v_\infty \left[1 + \lambda_0^2 S_1 + \lambda_0^4 S_2 + \lambda_0^6 S_3 + \ldots\right] = 0,$$

where
$$ {}_0S_1 = \frac{1.2}{v_0 . v_1} + \frac{3.4}{v_1 . v_2} + \frac{5.6}{v_2 . v_3} + \ldots,$$

${}_0S_2$ = sum of products of every *two* non-adjacent terms of the last series, and so on: as on p. 49.

If $\lambda = 0$ we have $v_0 v_1 \ldots v_\infty = 0$, giving $p^2 = (2n)^2$, where n is zero or any integer and leading to Bessel's functions $[J_0 J_1 \ldots]$.

We shall denote the values of p^2 that correspond to roots of $v_0 = 0$, $v_1 = 0$, etc., by p_0^2, p_2^2, etc., respectively.

p_0^2 *(corresponding to* $v_0=0$*).*

We have ${}_0S_1=\frac{1\,.\,2}{v_0\,.\,v_1}+\frac{3\,.\,4}{v_1\,.\,v_2}+\ldots$; ${}_1S_1=\frac{3\,.\,4}{v_1\,.\,v_2}+\ldots$ (the same as ${}_0S_1$ but beginning with second term) and similar notation for ${}_0S_2$, etc.

Clearly ${}_0S_1=\frac{1\,.\,2}{v_0\,.\,v_1}+{}_1S_1$; ${}_0S_2=\frac{1\,.\,2}{v_0\,.\,v_1}\,{}_2S_1+{}_1S_2$; ${}_0S_3=\frac{1\,.\,2}{v_0\,.\,v_1}\,{}_2S_2+{}_1S_3$ and so on.

$${}_1S_1=\frac{3\,.\,4}{v_1\,.\,v_2}\,{}_3S_1+{}_3S_2;\quad {}_1S_3=\frac{3\,.\,4}{v_1\,.\,v_2}\,{}_3S_2+{}_2S_3 \text{ and so on.}$$

So that our equation $v_0[1+\lambda^2{}_0S_1+\ldots]=0$ may be written

$$v_0+\frac{2}{v_1}(\lambda^2+\lambda^4{}_2S_1+\lambda^6{}_2S_2+\ldots)+v_0\lambda^2({}_1S_1+\lambda^2{}_1S_2+\ldots)=0.$$

First approximation: $\qquad v_0=0,\quad \therefore\ p_0^2=0.$

Second. $\qquad v_0+\frac{2}{v_1}\lambda^2=0.\quad \therefore\ v_0=-\frac{2\lambda^2}{v_1}=-\frac{\lambda^2}{2};\ p_0^2=\frac{\lambda^2}{2}.$

Third. $\qquad v_0+\frac{2\lambda^2}{v_1}+\frac{2\lambda^4}{v_1}\,.\,{}_2S_1-\frac{2\lambda^4}{v_1}\,.\,{}_1S_1=0,$

$$v_0+\frac{2\lambda^2}{v_1}-\frac{2\,.\,3\,.\,4}{v_1^2v_2}\lambda^4=0,$$

$$\therefore\ 0=-p_0^2+\frac{2\lambda^2}{4-\frac{\lambda^2}{2}}-\frac{24}{16\,.\,16}\lambda^4=-p_0^2+\frac{\lambda^2}{2}-\frac{1}{3^2}\lambda^4\ldots$$

$$\therefore\ p_0^2=\frac{\lambda^2}{2}-\frac{\lambda^4}{32}.$$

Fourth. $\qquad 0=v_0+\frac{2\lambda^2}{v_1}+\frac{2\lambda^4}{v_1}\,{}_2S_1+\frac{2\lambda^6}{v_1}\,{}_2S_2+\left(-\frac{2\lambda^4}{v_1}+\frac{2\,.\,3\,.\,4}{v_1^2v_2}\lambda^6\right)({}_1S_1+\lambda^2{}_1S_2)$

$$=v_0+\frac{2\lambda^2}{v_1}+\frac{2\lambda^4}{v_1}({}_2S_1-{}_1S_1)+\frac{2\lambda^6}{v_1}\left({}_2S_2-{}_1S_1+\frac{3\,.\,4}{v_1v_2}\,{}_1S_1\right)$$

$$=v_0+\frac{2\lambda^2}{v_1}-\frac{2\lambda^4}{v_1}\,.\,\frac{3\,.\,4}{v_1v_2}+\frac{2\lambda^6}{v_1}\,.\,\frac{3\,.\,4}{v_1v_2}\left(\frac{3\,.\,4}{v_1v_2}+\frac{5\,.\,6}{v_2v_3}\right)$$

$$=-p_0^2+\frac{2\lambda^2}{4-\frac{\lambda^2}{2}+\frac{\lambda^4}{16}}-\frac{2\lambda^4\,.\,12}{\left(4-\frac{\lambda^2}{2}\right)^2\left(16-\frac{\lambda^2}{2}\right)}+\frac{24\lambda^6}{16\,.\,16}\left(\frac{12}{4\,.\,16}+\frac{30}{16\,.\,36}\right)$$

$$=-p_0^2+\frac{\lambda^2}{2}-\frac{\lambda^4}{32}\ \text{(coefficient of } \lambda^6 \text{ vanish).}$$

Thus up to λ^8 the expansion of p_0^2 in terms of λ is

$$p_0^2=\frac{\lambda^2}{2}-\frac{\lambda^4}{32}.$$

p_2^2 *(corresponding to root $v_1 = 0$).*

Our equation is $\quad v_1 [1 + \lambda^2{}_0S_1 + \lambda^4{}_0S_2 + \lambda^6{}_0S_3 + \ldots] = 0. \quad (1).$

$$_0S_1 = \frac{1 \cdot 2}{v_0 \cdot v} + \frac{3 \cdot 4}{v_1 \cdot v_2} + {}_2S_1,$$

$$_0S_2 = \frac{1 \cdot 2}{v_0 \cdot v_1} {}_2S_1 + \frac{3 \cdot 4}{v_1 \cdot v_2} \cdot {}_3S_1 + {}_2S_2,$$

$$_0S_0 = \frac{1 \cdot 2}{v_0 \cdot v_1} \cdot {}_2S_2 + \frac{3 \cdot 4}{v_1 \cdot v_2} \cdot {}_3S_2 + {}_2S_3.$$

(1) is equivalent to $v_1 + \frac{1 \cdot 2}{v_0} \cdot \lambda^2 (1 + \lambda^2{}_2S_1 + \lambda^4{}_2S_2 + \ldots) + \frac{3 \cdot 4}{v_2} \cdot \lambda^2 (1 + \lambda^2{}_3S_1 + \lambda^4{}_3S_2)$

$$+ v_1\lambda^2 ({}_2S_1 + \lambda^2{}_3S_1 + \lambda^4{}_2S_3 + \ldots) = 0.$$

First approximation: $\quad v_1 = 0, \quad \therefore p_2^2 = 2^2 = 4.$

Second. $\quad v_1 + \lambda^2 \left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) = 0, \quad \therefore v_1 = -\frac{\lambda^2}{2}; \; p_2^2 = 4 + \frac{\lambda^2}{2}.$

Third. $\quad 0 = v_1 + \lambda^2 \left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) + \lambda^4 \left(\frac{1 \cdot 2}{v_0} \cdot {}_2S_1 + \frac{3 \cdot 4}{v_2} \cdot {}_3S_1 - \frac{1 \cdot 2}{v_0} \cdot {}_2S_1 - \frac{3 \cdot 4}{v_0} {}_2S_1\right)$

$$= v_1 + \lambda^2 \left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) + \lambda^4 \left[\frac{3 \cdot 4}{v_2} \cdot ({}_3S_1 - {}_2S_1)\right]$$

$$= v_1 + \lambda^2 \left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) - \lambda^4 \cdot \frac{3 \cdot 4}{v_2} \cdot \frac{5 \cdot 6}{v_2 v_3}.$$

Proceeding as before this gives $p_2^2 = 4 + \frac{\lambda^2}{2} + \frac{5\lambda^4}{12 \times 16}.$

Fourth. $\quad 0 = v_1 + \lambda^2 \left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) - \lambda^4 \cdot \frac{3 \cdot 4 \cdot 5 \cdot 6}{v_2{}^2 v_3} + \lambda^6 \cdot \frac{3 \cdot 4 \cdot 5 \cdot 6}{v_2{}^2 v_3} \left(\frac{5 \cdot 6}{v_2 v_3} + \frac{7 \cdot 8}{v_0 v_4}\right)$

$$= 4 - p_2^2 + \lambda^2 \left(\frac{-2}{4 + \frac{\lambda^2}{2} + \frac{5\lambda^4}{16 \cdot 12}} + \frac{12}{12 - \frac{\lambda^2}{2} - \frac{5\lambda^4}{16 \cdot 12}}\right)$$

$$- \lambda^4 \cdot \frac{3 \cdot 4 \cdot 5 \cdot 6}{\left(12 - \frac{\lambda^2}{2}\right)^2 \left(32 - \frac{\lambda^2}{2}\right)} + \lambda^6 \cdot \frac{3 \cdot 4 \cdot 5 \cdot 6}{12^2 \cdot 32} \left(\frac{30}{12 \cdot 32} + \frac{5 \cdot 6}{32 \cdot 60}\right).$$

Here again the coefficient of λ^6 vanishes and we have up to λ^8,

$$p_2^2 = 4 + \frac{\lambda^2}{2} + \frac{5\lambda^4}{12 \times 16}.$$

We shall turn next to the odd series. We have the general relation

$$(2n + 2)(2n + 3)\, c_{n+1} = [(2n + 1)^2 - p^2]\, c_n + \lambda^2 c_{n-1}.$$

In this case put $\quad v_n = (2n + 1)^2 - p^2; \quad c_n = \frac{u_n}{2n + 1},$

and we have $\quad u_{n+1} = v_n u_n + \lambda^2 \cdot (2n + 1)(2n)\, u_{n-1}.$

Taking $u_0=1$ we get
$$u_1 = v_0,$$
$$u_2 = v_0 v_1\left(1+\lambda^2 . \frac{2.3}{v_0 . v_1}\right),$$
$$u_3 = v_0 v_1 v_2\left[1+\lambda^2\left(\frac{2.3}{v_0 . v_1}+\frac{4.5}{v_1 . v_2}\right)\right],$$
and so on.

Just as before the equation to determine p^2 is
$$v_0 v_1 v_2 \ldots v_\infty\,[1+\lambda^2{}_0S_1+\lambda^4{}_0S_2+\ldots]=0,$$
where
$${}_0S_1=\frac{2.3}{v_0 . v_1}+\frac{4.5}{v_1 . v_2}+\frac{6.7}{v_2 . v_3}+\ldots$$

[Everything is exactly as before except that

2.3 is associated with $v_0 v_1$ instead of 1.2,

4.5 .... $v_1 v_2$ 3.4,

and so on.]

p_1^2 (*corresponding to root* $v_0=0$).

First approximation: $v_0=0$; $\therefore\ p_1^2=1$.

Second.
$$v_0+\lambda^2 . \frac{2.3}{v_1}=0;\quad \therefore\ v_0=-\frac{3}{4}\lambda^2;\quad p_1^2=1+\frac{3}{4}\lambda^2.$$

Third.
$$v_0=-\lambda^2 . \frac{2.3}{v_1}+\lambda^4 . \frac{2.3.4.5}{v_1^2 v_2}$$
$$=-\frac{6\lambda^2}{8-\frac{3}{4}\lambda^2}+\frac{2.3.4.5\lambda^4}{8^2 . 24}$$
$$=-\frac{3}{4}\lambda^2+\frac{\lambda^4}{128};\quad \therefore\ p_1^2=1+\frac{3}{4}\lambda^2-\frac{\lambda^4}{128}.$$

Fourth.
$$0=v_0+\frac{2.3}{v_1}\lambda^2-\frac{2.3.4.5}{v_1^2 v_2}\lambda^4+\frac{2.3.4.5}{v_1 v_2}\lambda^6\left(\frac{4.5}{v_1 v_2}+\frac{6.7}{v_2 v_3}\right)$$
$$=1-p_1^2+\frac{2.3\lambda^2}{8-\frac{3}{4}\lambda^2+\frac{\lambda^4}{128}}-\frac{2.3.4.5\lambda^4}{\left(8-\frac{3}{4}\lambda^2\right)^2\left(24-\frac{3}{4}\lambda^2\right)}$$
$$+\frac{2.3.4.5\lambda^6}{8.24}\left(\frac{4.5}{8.24}+\frac{6.7}{24.48}\right);$$
$$\therefore\ p_1^2=1+\frac{3}{4}\lambda^2-\frac{1}{128}\lambda^4-\frac{241}{12\times(32)^2}\lambda^6+\ldots.$$

p_3^2 *(root corresponding to $v_1 = 0$).*

Our equation is $$v_1 [1 + \lambda^2{}_0S_1 + \lambda^4{}_0S_2 + \ldots] = 0 \quad \ldots\ldots(1),$$

and $${}_0S_1 = \frac{2.3}{v_0.v_1} + \frac{4.5}{v_1.v_2} + \frac{6.7}{v_2.v} + \ldots = \frac{2.3}{v_0.v} + \frac{4.5}{v_1v_2} + {}_2S_1,$$

$${}_0S_2 = \frac{2.3}{v_0v_1}\,{}_2S_1 + \frac{4.5}{v_0.v_2}\,{}_3S_1 + {}_2S_2\,; \quad {}_0S_3 = \frac{2.3}{v_0v_1}\,{}_2S_2 + \frac{4.5}{v_1.v_2}\,{}_3S_2 + {}_2S_3.$$

Hence (1) becomes $$0 = v_1 + \frac{2.3}{v_0}\lambda^2 [1 + \lambda^2{}_2S_1 + \lambda^4{}_2S_2 + \ldots]$$

$$+ \frac{4.5}{v_2}\lambda^2 [1 + \lambda^2{}_3S_1 + \lambda^4{}_3S_2 + \ldots] + v_1\lambda^2 ({}_2S_1 + \lambda^2{}_2S_2 + \ldots).$$

First approximation: $v_1 = 0$; $\therefore\ p_3^2 = 9.$

Second. $$0 = v_1 + \lambda^2 \left(\frac{2.3}{v_0} + \frac{4.5}{v_2}\right)$$

$$= 9 - p_3^2 + \frac{\lambda^2}{2}; \quad \therefore\ p_3^2 = 9 + \frac{1}{2}\lambda^2.$$

Third. $$0 = v_1 + \lambda^2 \left(\frac{2.3}{v_0} + \frac{4.5}{v_2}\right) + \lambda^4 \left[\frac{2.3}{v_0}\,{}_2S_1 + \frac{4.5}{v_2}.{}_3S_1 - \frac{2.3}{v_0}.{}_2S_1 - \frac{4.5}{v_2}\,{}_2S_1\right]$$

$$= v_1 + \lambda^2 \left(\frac{2.3}{v_0} + \frac{4.5}{v_2}\right) - \lambda^4 . \frac{4.5.6.7}{v_2^2 v_3}$$

$$= 9 - p_3^2 + \lambda^2 \left[-\frac{6}{8 + \frac{\lambda^2}{2}} + \frac{20}{16 - \frac{\lambda^2}{2}} \right] - \frac{4.5.6.7}{16^2.40}\lambda^4;$$

$$\therefore\ p_3^2 = 9 + \frac{1}{2}\lambda^2 + \frac{17}{16 \times 32}\lambda^4.$$

Fourth. $$0 = v_1 + \lambda^2 \left(\frac{2.3}{v_0} + \frac{4.5}{v_2}\right) - \lambda^4 . \frac{4.5.6.7}{v_2^2 v_3} + \lambda^6 . \frac{4.5.6.7}{v_2 v_3}\left(\frac{6.7}{v_2 v_3} + \frac{8.9}{v_3 v_4}\right)$$

$$= 9 - p_3^2 + \lambda^2 \left[-\frac{6}{8 + \frac{\lambda^2}{2} + \frac{17}{16.32}\lambda^4} + \frac{20}{16 - \frac{\lambda^2}{2} - \frac{17}{16.32}\lambda^4} \right]$$

$$- \lambda^4 . \frac{4.5.6.7}{\left(16 - \frac{\lambda^2}{2}\right)^2 \left(40 - \frac{\lambda^2}{2}\right)} + \lambda^6 . \frac{4.5.6.7}{16^2.40}\left(\frac{6.7}{16.40} + \frac{8.9}{40.55}\right);$$

$$\therefore\ p_3^2 = 9 + \frac{1}{2}\lambda^4 + \frac{17}{32 \times 16}\lambda^4 + \frac{20287}{22 \times 16^3 \times 100}\lambda^6 + \ldots.$$

We have seen if we proceed to solve one fundamental equation in a series of ascending powers of $z\,(\equiv \lambda x)$ we get two separate series, one even and the other odd. But from what we have just done it is clear that these series correspond to *different values of* p^2; so that for any particular value of p^2 we have really only one solution.

We shall denote this solution by $f(z)$, using suffixes $f_0(z)$, $f_1(z)$, etc. to indicate the values of p^2 ($p_0^2 p_1^2 \ldots$) to be taken in the series. One equation is

$$(z^2-\lambda^2)y''+zy'+(z^2-p^2)y=0 \quad \ldots\ldots(1),$$

of which we have one solution $y=f(z)$. To get another solution assume $y=vf+w$. Substitute in (1) and we get

$$v[(z^2-\lambda^2)f''+zf'+(z^2-p^2)f]+f[(z^2-\lambda^2)v''+zv'] \\ +2(z^2-\lambda^2)v'f'+(z^2-\lambda^2)w''+zw'+(z^2-p^2)w=0,$$

Since f satisfies equation (1) the coefficient of v in the first line of the last equation vanishes.

Since v is quite at our disposal we may choose it so that

$$(z^2-\lambda^2)\frac{d^2v}{dz^2}+z\frac{dv}{d^2}=0,$$

which is satisfied by $\qquad v=\log(z+\sqrt{(z^2-\lambda^2})$.

Thus our problem is reduced to finding a *particular* solution (the simpler the better) of the equation

$$(z^2-\lambda^2)\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-p^2)w=-2(z^2-\lambda^2)\frac{dv}{dz}\frac{df}{dz}=-2\sqrt{z^2-\lambda^2}\frac{df}{dz} \quad \ldots\ldots(2).$$

Let $w=\sqrt{z^2-\lambda^2}\,.\,u$. Substitute in (2) and divide by $\sqrt{z^2-\lambda^2}$, and we get

$$(z^2-\lambda^2)\frac{d^2u}{dz^2}+3z\frac{du}{dz}+(z^2-p^2+1)u=-2\frac{df}{dz}.$$

We shall obtain u in the form of an ascending series of powers of z. The form of the solution will be different for the odd and even series $f_0 f_2 \ldots$ and $f_1 f_3 \ldots$.

Take first the *even* series for f.

$$f(z)=a_0+a_1z^2+\ldots+a_nz^{2n}+\ldots,$$

$$(z^2-\lambda^2)\frac{d^2u}{dz^2}+3z\frac{du}{dz}+(z^2-p^2+1)u=-2\frac{df}{dz}=-4a_1z+\ldots-4na_nz^{2n-1}+\ldots.$$

Clearly u must be an *odd* series.

Assume $\qquad u=A_0z+A_1z^3+\ldots+A_nz^{2n+1}+\ldots.$

$$(z^2-\lambda^2)[6A_1z+20A_2z^3+\ldots+(2n+1)(2n)A_nz^{2n-1}+\ldots] \\ +3z[A_0+3A_1z^2+\ldots+(2n+1)A_nz^{2n}+\ldots]+(z^2-p^2+1)[A_0z+A_1z^3+\ldots+A_nz^{2n+1}+\ldots] \\ \equiv -4a_1z+\ldots-4na_nz^{2n-1}.$$

Equating the coefficients of different powers of z, we get

$$-\ \ 6A_1\lambda^2+(4-p^2)A_0=-4a_1 \quad \ldots\ldots(1),$$

$$-20A_2\lambda^2+(16-p^2)A_1+A_0=-8a_2;$$

$$-(2n+2)(2n+3)A_{n+1}\lambda^2+[(2n+2)^2-p^2]A_n+A_{n-1}=-4(n+1)a_{n+1} \ldots(2),$$

which enable us to determine the coefficients successively.

As we require only a particular solution, we may give any values to A_0 or A_1 we choose, consistently with (1).

We must now examine the convergence of the series u.

In the first place we may prove, by proceeding as on p. 47, that A_n cannot be infinite for any *finite* value of n.

The relation connecting successive coefficients in $f(z)$ is

$$-(2n+1)(2n+2)\,a_{n+1}\lambda^2 + (4n^2 - p^2)\,a_n + a_{n-1} = 0 \quad\text{..........................(1).}$$

If $v_{n+1} = a_{n+1}/a_n$ we have

$$v_{n-1} = \frac{1}{\lambda^2} - \frac{6n+2+p^2}{(2n+1)(2n+2)\lambda^2} + \frac{1}{\lambda^2(2n+1)(2n+2)v_n}.$$

Hence *when n is large* we have

either $$v_n = -\frac{1}{(2n+1)(2n+2)} \quad\text{..(2),}$$

or v_n is not indefinitely small, and approximates more and more nearly to $\frac{1}{\lambda^2}$ as n increases.

But p^2 is chosen so as to make $a_\infty = 0$, hence we are confined to the first case (2).

Since then v_n approximates to $-\frac{1}{(2n+1)(2n+2)}$ and a_n to zero, we see that $-4(n+1)\,a_{n+1}$ is indefinitely small for very large values of n.

This being the case we see by comparing (1) above with (2) of p. 64, that when n is very large the coefficients of u are connected by the same relation as the coefficients of the convergent series f. Hence as the coefficients of u are finite for finite values of n, it follows that the series u is convergent in the same domain as the series f.

In exactly the same way we may proceed with the odd series for f,

$$f(z) = c_0 z + c_1 z^3 + \ldots + c_n z^{2n+1} + \ldots,$$

$$(z^2 - \lambda^2)\frac{d^2u}{dz^2} + 3z\frac{du}{dz} + (z^2 - p^2 + 1)\,u = -2\frac{df}{dz}$$

$$= -2c_0 - 6c_1 z^2 \ldots - 2(2n+1)\,c_n z^m \ldots .$$

In this case we take $u = C_0 + C_1 z^2 + \ldots + C_n z^{2n} + \ldots$

and get $$(z^2 - \lambda^2)\left[2C_1 + 12C_2 z^2 + \ldots + 2n(2n-1)\,C_n z^{2n-2} + \ldots\right]$$
$$+ 3z\left[2C_1 z + 4C_2 z^3 + \ldots + 2nC_n z^{2n-1} + \ldots\right] + (z^2 - p^2 + 1)\left[C_0 + C_1 z^2 + \ldots + C_n z^{2n} + \ldots\right]$$
$$\equiv -2c_0 - 6c_1 z^3 \ldots - 2(2n+1)\,c_n z^{2n} + \ldots .$$

This gives $$-2C_1\lambda^2 + (1 - p^2)\,C_0 = -2c_0$$
$$-12C_2\lambda^2 + (9 - p^2)\,C_1 + C_0 = -6c_1$$
$$-(2n+1)(2n+2)\,C_{n+1}\lambda^2 + \left[(2n+1)^2 - p^2\right]C_n + C_{n-1} = -2(2n+1)\,c_n,$$

from which, as before, the coefficients may be determined in succession, and the convergence of the series u established.

We have now obtained two independent solutions of our equation, viz.

$$y=f(z)\qquad\qquad(1)$$

$$y=f(z)\log(z+\sqrt{z^2-\lambda^2})+u\sqrt{z^2-\lambda^2}$$
$$=F(z)\qquad\qquad(2).$$

The complete integral is therefore $y=Af(z)+BF(z)$.

The two series are convergent for all *finite* values of the argument. We shall now proceed to obtain two other solutions which represent y 'asymptotically' (to use Poincaré's term)—i.e. we shall obtain two series which approximate more and more closely to solutions of our equation as the argument increases. These series will be very useful for numerical calculation when the argument is not small, and they will also help us to determine what linear function of f and F we must take if we are seeking a solution of our equation which is to vanish when $z=\infty$—a problem that confronts us in many physical applications of our analysis.

Our equation is $$(z^2-\lambda^2)y''+zy'+(z^2-p^2)y=0.$$

Let $y=ue^t$ where $t=iz$ and the equation above becomes

$$(t^2+\lambda^2)u''+(2t^2+t+2\lambda^2)u'+(t-p^2+\lambda^2)u=0.$$

Assuming a serial solution in descending powers of t of the form

$$u=t^m[a_0+a_1/t+a_2/t^2+\dots],$$

we find that we must have $2n+1=0$, therefore $m=-\frac{1}{2}$, and equating coefficients of the various powers of t to zero we get

$$2\frac{a_1}{a_0}=\frac{1}{4}+\lambda^2-p^2,$$

$$4\frac{a_2}{a_0}=\left(\frac{9}{4}+\lambda^2-p^2\right)\frac{a_1}{a_0}-\lambda^2,$$

$$2(n+1)a_{n+1}=\left[\frac{(2n+1)^2}{4}+\lambda^2-p^2\right]a_n-(2n-1)\lambda^2a_{n-1}+(2n-1)(2n+3)\lambda^2a_{n-2},$$

which enable us to determine the coefficients in succession in terms of a_0 which is of course arbitrary.

Now let us examine the convergence of the series just obtained.

Let $u_{n+1}=a_{n+1}/a_n$. Then we have

$$2u_{n+1}=\left[n+\frac{\lambda^2-p+1/4}{n+1}\right]-\lambda^2.\frac{2n-1}{n+1}\left[\frac{1}{u_n}-\frac{2n+3}{u_n.u_{n-1}}\right].$$

Hence *when n is large, either* u_n is large, in which case $u_{n+1}=\frac{n}{2}$ nearly, *or* u_n is small, in which case we have approximately when n is large

$$0=1-\frac{2\lambda^2}{n+1}\left[\frac{1}{u_n}-\frac{2n+3}{u_n.u_{n-1}}\right].$$

This relation cannot be satisfied by a *small* value of u_n, so that we conclude that when n is large u_n is large, in other words the series is *ultimately divergent*. But, for large values of the argument, the series *begins* by converging rapidly, and so if we stop before the terms begin to diverge the series obtained will be quite well adapted for purposes of calculation. Suppose, for example, that in any numerical problem we agree to neglect terms of the order $\frac{1}{10,000}$ and are dealing with values of the argument which make the nth term of the above series of the order $\frac{1}{10,000}$.

If we stop at the nth term of our series and substitute in the differential equation, the equation will not be *quite* satisfied; but instead of having zero on one side of our equation we shall have a few terms of the order $\frac{1}{10,000}$. In other words, the series we have taken (stopping at the nth term) satisfies our equation *approximately*, the error being of the order we agreed to neglect.

The solution thus obtained is

$$y = \frac{e^{iz}}{\sqrt{z}}\left[a_0 + \frac{a_1}{iz} + \frac{a_2}{(iz)^2} + \dots\right]$$

$$= \frac{e^{iz}}{\sqrt{z}}\left[a_0 - \frac{ia_1}{z} - \frac{a_2}{z^2} - \dots\right] = \phi^+(z) \text{ say.}$$

Changing the sign of i we get another solution, viz.:

$$y = \frac{e^{-iz}}{\sqrt{z}}\left[a_0 + \frac{ia}{z} - \frac{a_2}{z^2} - \dots\right] = \phi^-(z) \text{ say.}$$

The complete integral is $y = C\phi^+(z) + D\phi^-(z)$ where C and D are arbitrary constants.

By taking certain linear functions of ϕ^+ and ϕ^- we build up two solutions that will be of service afterwards. We shall denote these by $\chi^+(z)$ and $\chi^-(z)$, where

$$\chi^+(z) = \frac{\phi^+(z) + \phi^-(z)}{2} = \frac{\cos z}{\sqrt{z}}\left(a_0 - \frac{a_2}{z^2} + \dots\right) + \frac{\sin z}{\sqrt{z}}\left(\frac{a_1}{z} - \frac{a_3}{z^3} + \dots\right)$$

$$= \frac{1}{\sqrt{z}}\left[R\cos z + S\sin z\right],$$

where
$$R = a_0 - \frac{a_2}{z^2} + \dots; \quad S = \frac{a_1}{z} - \frac{a_3}{z^3} + \dots,$$

$$\therefore \chi^-(z) = i\,.\,\frac{\phi^-(z) - \phi^+(z)}{2} = \frac{1}{\sqrt{z}}\left[R\sin z - S\cos z\right].$$

These give $\phi^+ = \chi^+ + i\chi^-$; $\phi^- = \chi^+ - i\chi^-$,

$$\phi^+ + i\phi^- = (1+i)(\chi^+ + \chi^-) = \sqrt{2}e^{i\pi/4}(\chi^+ + \chi^-),$$

$$\phi^+ - i\phi^- = (1-i)(\chi^+ + \chi^-) = \sqrt{2}e^{-i\pi/4}(\chi^+ - \chi^-).$$

We proceed to consider the linear relations connecting the functions f, F, ϕ^+ and ϕ^-.

Let $f=\alpha\phi^+ + \beta\phi^-$ and suppose we are dealing with the *even* series f.

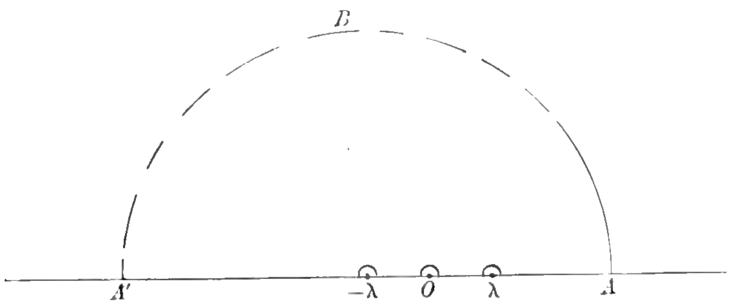
$f(z)$ is a *uniform* and continuous function of z, so that if we make a circuit round the origin we shall return to our starting point with the same value of f. This however is not the case with ϕ^+ and ϕ^- which are *multiform* functions. It follows then that to insure the continuity of f, the constants α and β in the above relation must be *discontinuous*.

If we put $z=r(\cos\theta+i\sin\theta)$, we shall find that α must change discontinuously as θ goes through an odd multiple of π and β as θ goes through an even multiple.

We have $e^{iz}=e^{-r\sin\theta}e^{ir\cos\theta}$, so that the modulus of e^{iz} is $e^{-r\sin\theta}$.

But $f=\alpha\phi^+ + \beta\phi^- = \alpha e^{-r\sin\theta}(\ldots)+\beta e^{r\sin\theta}(\ldots)$ and, when θ lies between 0 and π, $\sin\theta$ is positive, so that ϕ^- is (for large values of r to which we are confined when dealing with the functions ϕ) exceedingly large compared with ϕ^+, and so is far more important than ϕ^+. But when θ passes through π these relations are reversed. The term ϕ^+ is now the all important one. Hence as θ passes through π the constant α must change discontinuously to insure the continuity of f. There is no possibility of another discontinuity till we get to $\theta=2\pi$, ϕ^- must now become the important term so that β must change abruptly, and so on. We conclude then that α changes discontinuously as θ passes through an odd multiple of π, and β when θ goes through an even multiple.

We can obtain a relation between α and β by working on the same lines as on p. 52.



If we start with any of our functions at A (for which $\theta=0$) and go along the circle of very great radius ABA' to $A'(\theta=\pi)$, we must get to the same result as if we go along AOA' [avoiding the points λ, $O-\lambda$ by describing small semi-circles round them], for the space enclosed by these two paths contains no critical points of our functions.

Suppose then we take f along AOA' and its equivalent $\alpha\phi^+ + \beta\phi^-$ along the great circle.

Then f is valid all along its path and so is $\alpha\phi^+ + \beta\phi^-$, the functions ϕ being defined only for very large values of the argument.

Starting from A and going to A' we increase θ by π. f is even and so does not change sign. Hence we reach A' with f; but f at A' is not equal to $\alpha\phi^- + \beta\phi^-$ but to $\alpha'\phi^+ + \beta\phi^-$ (since α is discontinuous at $\theta = \pi$). For brevity we shall denote $f(z) = f(r \,.\, \overline{\cos\theta + i\sin\theta})$ by $f(\theta)$.

Thus we get $\alpha\phi^+(0) + \beta\phi^-(0) = f(0) = f(\pi)$

$$= \alpha'\phi^+(\pi) + \beta\phi^-(\pi)$$

$$= -i\alpha'\phi^-(0) - i\beta\phi^+(0).$$

Hence $$\alpha = -i\beta \text{ and } \beta = -i\alpha'.$$

We have found then that when $\pi > \theta > 0$, $f = \alpha(\phi^+ + i\phi^-)$, and in exactly the same way we see that for the *odd* function we have $f = \alpha_1(\phi^- - i\phi^-)$.

Now consider the other function

$$F = u\sqrt{z^2 - \lambda^2} + f\log(z + \sqrt{z^2 - \lambda^2}).$$

To make this definite we may take $\sqrt{z^2 - \lambda^2} = +\sqrt{\mu}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ where μ is the modulus and θ the amplitude of $z^2 - \lambda^2$; and for $\log(z + \sqrt{z^2 - \lambda^2})$ take its principal value.

Suppose first f is even, then u is odd (see p. 64).

Let $F = A\phi^+ + B\phi^-$ and carry out the process of last page. On going round $z = \lambda$, $\sqrt{z^2 - \lambda^2}$ becomes $i\sqrt{z^2 - \lambda^2}$ and so F becomes $iu\sqrt{z^2 - \lambda^2} + f\log(z + i\sqrt{z^2 - \lambda^2})$. Passing $z = 0$ z becomes negative, and u being odd becomes $-u$; while f is unaffected. Thus F has become $-iu\sqrt{z^2 - \lambda^2} + f\log(-z + i\sqrt{z^2 - \lambda^2})$, where of course z here means z.

Then going round $-\lambda$, F becomes $u\sqrt{z^2 - \lambda^2} + f\log(-z - \sqrt{z^2 + \lambda^2}) = F + \pi i f$, where F means the value with which we started.

Just as before A must be a discontinuous constant, changing as θ goes through an odd multiple of π and B as θ goes through an even multiple.

If A becomes A' when $\theta = \pi$ we then have :—

$$A'\phi^+(\pi) + B\phi^-(\pi) = F(0) + i\pi f(0)$$

$$= A\phi^+(0) + B\phi^-(0) + i\pi\alpha(\phi^+ + i\phi^-);$$

$$\therefore\ -iA'\phi^-(0) - iB\phi^+(0) = A\phi^+(0) + B\phi^-(0) + i\pi\alpha(\phi^+ + i\phi^-);$$

$$\therefore\ -iA' = B - \pi\alpha\,;\ -iB = A + \pi i\alpha,$$

and $$F = A(\phi^+ + i\phi^-) - \pi\alpha\phi^-,\ [\pi > \theta > 0].$$

Treating the *odd* series f in the same way we get

$$F = A'(\phi^+ - i\phi^-) + \pi\alpha_1\phi^-,\ [\pi > \theta > 0].$$

We have now only *two* undetermined constants, and these can be easily found by calculating the different series for *one* value of the argument. But for a large and important class of problems even this labour is not required. In these cases all that is wanted is the roots of $f(z)=0$ or $f'(z)=0$, or something of this form. The larger roots in these cases necessitate the use of the functions ϕ, but it is at once evident that the value of the constant α is not required. We shall now investigate the larger roots of the equation $f(z)=0$. Taking the even series $[f_0, f_2 \ldots]$ we see from last page that our equation is $\phi^+(z)+i\phi^-(z)=0$, or $\chi^+(z)+\chi^-(z)=0$,

or
$$R\cos\left(z-\frac{\pi}{4}\right)+S\sin\left(z-\frac{\pi}{4}\right)=0.$$

If
$$\tan\psi=S/R=\frac{\dfrac{a_1}{z}-\dfrac{a_3}{z^3}+\ldots}{a_0-\dfrac{a_2}{z^2}+\ldots} \quad \ldots\ldots\ldots\ldots (1),$$

then our equation becomes $\cos\left(z-\frac{\pi}{4}-\psi\right)=0$, and $z-\frac{\pi}{4}-\psi=(2m+1)\frac{\pi}{2}$ where m is an integer, so that

$$z=m\pi+\frac{3\pi}{4}+\psi \quad \ldots\ldots\ldots\ldots (2).$$

When z is *very large* we see from (1) that $\psi=0$, and thus the very large roots are given by $z=m\pi+\frac{3\pi}{4}$. ψ diminishes as z increases, so that the difference between two consecutive roots is more and more nearly equal to π as the sign of the root increases.

In the general case having got $\tan\psi$ from (1) in terms of z we use the expansion $\psi=\tan\psi-\frac{1}{3}\tan^3\psi+\frac{1}{5}\tan^5\psi\ldots$, substitute in (2) and proceed by successive approximations.

The treatment of the odd series $[f_1, f_3 \ldots]$ is of course precisely similar. Our equation is now $\phi^+(z)-i\phi^-(z)=0$, i.e. $\chi^+(z)-\chi^-(z)=0$, or

$$R\sin\left(z-\frac{\pi}{4}\right)-S\cos\left(z-\frac{\pi}{4}\right)=0,$$

$$\sin\left(z-\frac{\pi}{4}-\psi\right)=0, \quad \therefore\ z-\frac{\pi}{4}-\psi=m\pi,$$

where m is an integer, so that

$$z=m\pi+\frac{\pi}{4}+\psi.$$

In exactly the same way we may deal with the larger roots of $f'(z)=0$.

In most physical applications, the root of most importance is the *lowest* root; and it will sometimes happen that that root is too small to make the process just considered effective. In such a case there is no difficulty in finding the root directly from the *ascending* series, which converges rapidly (the argument being small).

We shall illustrate this by finding the lowest root of $f_0(z)=0$.

We have $p_0^2 = \frac{\lambda^2}{2} - \frac{\lambda^4}{32} - \ldots$ (up to λ^8) ... and using this value of p^2 we get

$$\frac{a_1}{a_0} = -\frac{1}{4} + \frac{\lambda^2}{64} + \ldots; \qquad \frac{a_2}{a_0} = \frac{1}{64} - \frac{\lambda^2}{12.64} + \ldots;$$

$$\frac{a_3}{a_0} = -\frac{1}{(6.8)^2} + \frac{7\lambda^2}{90 \times 64 \times 30} + \ldots, \qquad \frac{a_4}{a_0} = \frac{1}{(2.4.6.8)^2} + \ldots.$$

Hence
$$f_0(z) = 1 - z^2\left[\frac{1}{2^2} - \frac{\lambda^2}{64} + \ldots\right] + z^4\left[\frac{1}{(2.4)^2} - \frac{\lambda^2}{12.64} + \ldots\right]$$
$$- z^6\left[\frac{1}{(2.4.6)^2} - \frac{7\lambda^2}{96.64.30} + \ldots\right] + z^8\left[\frac{1}{(2.4.6.8)^2} + \ldots\right] + \ldots$$

Now $z = \lambda x = \kappa a$; $\lambda = \kappa a e$, so that $f_0(z)=0$ is equivalent to

$$0 = 1 - z^2 . \frac{1}{2^2} + z^4\left[\frac{1}{(2.4)^2} + \frac{e^2}{64}\right] - z^6\left[\frac{1}{(2.4.6)^2} + \frac{e^2}{12.64}\right]$$
$$+ z^8\left[\frac{1}{(2.4.6.8)^2} + \frac{7e^2}{96.64.30}\right] + \ldots \qquad \ldots\ldots\ldots\ldots(1).$$

The series on the right of (1) is rapidly convergent, and the terms are alternately positive and negative. If we stop at the nth term we get an algebraic equation which (for different values of e) can be solved by Horner's process with very little difficulty if n is not very great. The root thus obtained will not of course be exactly a root of $f_0(z)=0$, but it will be a close approximation if n is not too small. Also it is clear that the roots of the equation corresponding to n and $n+1$ will be the one greater and the other less than the root of $f_0(z)=0$. Thus by solving the equations corresponding to n and $n+1$ and taking z to lie between the two roots we shall get a close approximation to the real root.

Retaining the first five terms of our series and putting $y = \frac{1}{z^2}$ we have to solve:—

$$y^4 - {\cdot}25y^3 + y^2[{\cdot}015625 + {\cdot}015625e^2]$$
$$- y[{\cdot}000434 + {\cdot}001302e^2]$$
$$+ [{\cdot}000006781 + {\cdot}000038e^2] = 0.$$

For $e=0$, the equation becomes

$$y^4 - {\cdot}25y^3 + {\cdot}015625y^2 - {\cdot}000434y + {\cdot}000006781 = 0.$$

Attacking this by Horner's process we get

1	−2·5	1·5625	−·434	·0678 (·1728
	1	−1·5	·0625	−3715
	−1·5	·0625	−·3715	−3037
	1	−·5	−·4375	2178·75
	−·5	−·4375	−·809	−8582 500
	1	·5	1121·75	5957 556
	·5	6·25	311·25	−2624 944
	1	154	2542·75	
	1·5	160·25	2854 000	
	7	203	124 778	
	22	363·25	2978 778	
	7	252	126514	
	29	615·25	3105292	
	7	864		
	36	62389		
	7	868		
	430	63257		
	2			
	432			
	2			
	434			

The equation $y^3 - \cdot25y^2 + \cdot015625y - \cdot000434 = 0$ gives in the same way $y = \cdot1748$. Thus the real root must lie between ·1728 and ·1748. If we go to a higher order we find $y = \cdot1730$ as a very close approximation.

Since $y = \frac{1}{z^2}$ this makes $z = 2\cdot404$.

Treating the equation corresponding to other values of e in the same way we get these results:—

For

$$\begin{aligned} e &= 0, & z &= 2\cdot404\,; & e &= 0\cdot4, & z &= 2\cdot512.\\ e &= 0\cdot1, & z &= 2\cdot411\,; & e &= 0\cdot5, & z &= 2\cdot585.\\ e &= 0\cdot2, & z &= 2\cdot429\,; & e &= 0\cdot6, & z &= 2\cdot774.\\ e &= 0\cdot3, & z &= 2\cdot457\,; & e &= 0\cdot7, & z &= 3\cdot092.\end{aligned}$$

For larger values of e, we should take in more terms of our series to get a close approximation to the root, which is getting too large to make our approximation very accurate. For these larger roots it is better to use the *descending* series, and the method of finding the roots from them, explained on p. 70.

We shall proceed to consider briefly several physical problems that can be solved by the aid of the functions we have been considering.

Vibrations of elliptic membranes:

If T is the tension, ρ the density and w the small displacement normal to the plane of the membrane, the equation of motion is

$$\frac{d^2w}{dt^2} = \frac{T}{\rho} V_1{}^2 w = c^2 V_1{}^2 w \text{ where } T = \rho c^2.$$

Taking w to vary as $e^{i\kappa ct}$, we get $(V_1^2+\kappa^2)\,w=0$. Since w must be finite all over the membrane we are confined to the f functions—supposing the membrane to be a complete ellipse.

Thus we take $$w=\Sigma A_n f_n(\lambda x) f_n(\lambda x')\, e^{i\kappa ct}.$$

κ (on which the frequency depends) is determined by the boundary condition; which is $w=0$ when $x=x_0$.

This gives $f_n(\lambda x_0)=0$, or $f_n(\kappa a)=0$, where a is the semi-major axis of the ellipse.

The nodal system is composed of a series of confocal ellipses given by $f_n(\lambda x)=0$, and a series of confocal hyperbolas $f_n(\lambda x')=0$.

To determine the frequencies corresponding to the various fundamental modes of vibration we have to find the lowest roots of $f_0(z)=0$, $f_1(z)=0$, and so on, and we have already seen how to do this.

Taking the values found for these roots in the case of $f_0(z)=0$, we find that the ratios of the frequency of the fundamental note for $e=0$, $e=0{\cdot}1$, $e=0{\cdot}2$, $e=0{\cdot}3$, ... $e=0{\cdot}7$ to that for $e=0$ are 1; 1·003; 1·013; 1·022; 1·045; 1·075; 1·154; 1·286. Thus there is very little difference between the notes emitted in the different cases. The interval between the notes for $e=0$ and $e=0{\cdot}1$ is less than a *comma*; between $e=0$ and $e=0{\cdot}2$ just about a comma, between $e=0{\cdot}6$ and $e=0$ about a *minor third*; and between $e=0$ and $e=0{\cdot}7$ an interval between a minor third and a fourth. Of course the frequency rises as the eccentricity increases.

The vibrations of an elliptic plate can also be determined. If E be Young's modulus for the material, ρ the density, $2t$ the thickness, and μ Poisson's ratio, then if w is the displacement normal to the plane of the plate the dynamical equation is $\ddot{w}+c^4V^4w=0$, where $c^4=\dfrac{Et^2}{3\rho(1-\mu^2)}$. If as usual we take w to vary as e^{ipt} and put $\kappa^4=p^2/c^4$ our equation becomes $(V^4-\kappa^4)\,w=0$, or $(V^2+\kappa^2)(V^2-\kappa^2)\,w=0$. This can be solved in terms of our functions, and as w must be finite all over the plate we are confined to the f functions.

We may take then $w=Ae^{i^2\kappa^2c^2t}[f(z)+\mu'f(iz)][f(z')+\mu'f(iz^2)]$, where $z=\lambda x$; $z'=\lambda x'$.

The nodal system consists of the series of confocal ellipses given by $f(z)+\mu'f(iz)=0$, and the series of confocal hyperbolas given by $f(z')+\mu'f(iz')=0$.

If the plate is *clamped at the edge* ($x=x_0$), we must have w and $\dfrac{dw}{dz}=0$ at the edge,

$$\therefore\ f(z_0)+\mu'f(iz_0)=0,$$
$$f'(z_0)+i\mu'f(iz_0)=0.$$

Eliminating μ' we get the frequency equation $[z_0=\kappa a]$

$$\frac{f'(z_0)}{f(z_0)}=\frac{if'(iz_0)}{f(iz_0)}.$$

Vibration of air in elliptic cylinders.

If ψ is the velocity potential we have as usual $\frac{d^2\psi}{dt^2}=c^2\nabla_1{}^2\psi$, and taking ψ to vary as $e^{i\kappa ct}$ we get $(\nabla_1{}^2+\kappa^2)\psi=0$.

As ψ must be finite at all points in the cylinder we take

$$\psi=\Sigma A_n f_n(\lambda x) f_n(\lambda x')\, e^{i\kappa ct}.$$

Since there is no normal displacement at the boundary we have $\frac{\partial\psi}{\partial x}=0$ when $x=x_0$, so that corresponding to $\psi=A_n e^{i\kappa ct} f_n(\lambda x) f_n(\lambda x')$ we have the frequency equation $f_n'(\lambda x_0)=0$, or $f_n'(\kappa a)=0$. We have already explained (p. 70) how the roots of this equation are to be obtained from the *descending* series.

We shall write down the results for the lowest note of the modes whose velocity potentials are

$$\psi=Af_0(\lambda x) f_0(\lambda x^2)\, e^{i\kappa ct},$$

and

$$\psi=A_1 f_1(\lambda x) f_1(\lambda x^2)\, e^{i\kappa ct}.$$

Any other mode may of course be discussed by the same method.

Mode $\psi=Af_0(\lambda x) f_0(\lambda x')\, e^{i\kappa ct}$.

($c=331{,}00$ centimetres per second.)

e	κa	$a\times$ frequency	wave length/a	Ratio of frequency to that for $e=0$
0	3·832	20156	1·6400	1
0·1	3·8432	20221	1·6346	1·0033
0·2	3·8720	20366	1·6230	1·0103
0·3	3·9259	20650	1·6007	1·0245
0·4	4·0578	21346	1·5495	1·0591
0·5	4·1874	22025	1·5008	1·0927
0·6	4·3799	23036	1·4348	1·1430
0·7	4·7328	24895	1·3277	1·2351
0·8	5·2923	27835	1·1875	1·4132

We see from the above that there is an interval of rather less than a comma between the notes emitted in the case $e=0$ and in $e=0{\cdot}2$. If $e=0$ gives out the note c then $e=0{\cdot}8$ will sound F♯ very nearly.

In order that $e=0$ may omit the middle c of a piano (264 vibrations a second) its radius must be

$$a = 76{\cdot}35 \text{ centimetres} = 30{\cdot}061 \text{ inches.}$$

$$\text{Mode } \psi = A f_1(\lambda x) f_1(\lambda x') e^{i\kappa ct}.$$

[The lowest roots of $f_1'(\lambda x)$ is not very large and is best obtained by the method of p. 71.]

e	κa	$a \times$ frequency	wave length/a	Ratio of frequency to that for $e=0$
0	1·8410	9683	3·4135	1
0·1	1·8417	9687	3·4120	1·0005
0·2	1·8421	9690	3·4112	1·0007
0·3	1·8449	9705	3·4057	1·0024
0·4	1·8494	9728	3·3979	1·0047
0·5	1·8527	9744	3·3900	1·0063
0·6	1·8576	9772	3·3823	1·0093
0·7	1·8657	9813	3·3683	1·0135
0·8	1·8758	9867	3·3496	1·0191
0·9	1·8875	9929	3·3290	1·0255

This is the gravest of the normal modes. The influence of eccentricity on the frequency is exceedingly minute.

If the motion is not the same in every transverse plane we can still solve the problem as indicated on p. 42. Thus suppose, for example, the cylinder is bounded by rigid transverse walls at $z=0$ and $z=l$, then we take

$$\psi = \Sigma A_n \cos\frac{m\pi z}{l},$$

and get

$$(V_1^2 + \kappa'^2) A_n = 0,$$

where

$$\kappa'^2 = \kappa^2 - \frac{m^2\pi^2}{l^2},$$

so that we have finally

$$\psi = \Sigma B_n \,.\, f_n(\lambda' x) f_n(\lambda' x') \cos\frac{m\pi z}{l}\, e^{i\kappa ct},$$

where $\lambda' = h\kappa'$ and κ' is given above.

The *lowest* note corresponds to $m=1$, and

$$\kappa^2=\kappa'^2+\frac{\pi^2}{l^2},$$

or

$$(\kappa a)^2=(\kappa' a)^2+\frac{\pi^2 a^2}{l^2} \quad \text{..(1).}$$

The lowest values of $\kappa' a$ for the different cases have just been written down for the case $n=0$, and the corresponding frequencies are thus obtainable with the aid of (1).

Another problem that is readily solved by the help of our functions is the determinations of the *tidal waves in a cylindrical vessel* of elliptical boundary.

If ξ be the elevation of the free surface above the undisturbed level then it is well known that 'the equation of continuity' and the dynamical equations lead at once to $\frac{d^2\xi}{dt^2}=c^2V_1^2\xi$ where $c^2=gh$, (h being the depth).

Thus we have $\zeta=\Sigma A_n f_n(\lambda x) f_n(\lambda x') e^{i\kappa ct}$, the boundary equation being $\frac{d\zeta}{dn}=0$ or $f_n'(\lambda x_0)=0$. This is the frequency equation, giving the admissible values of κ, and the corresponding 'speeds' of the oscillations are then $\kappa c=\kappa\sqrt{gh}$.

Another hydrodynamical problem that naturally presents itself here is the consideration of certain possible forms of *steady vortex motion in an elliptic cylinder.* In steady motion in two dimensions the angular velocity (ω) is constant along each stream line. But if ψ be the stream function we have $V_1^2\psi=2\omega$, so that every possible form of steady motion is included in $V_1^2\psi=\chi(\psi)$ where $\chi(\psi)$ is an arbitrary function of ψ. If we put $\chi(\psi)=-\kappa^2\psi$ where κ is a constant we get our equation $(V_1^2+\kappa^2)\psi=0$, and as ψ must be finite at all points within the cylinder we have

$$\psi=\Sigma A_n f_n(\lambda x) f_n(\lambda x').$$

As ψ must be constant along the boundary, we get for the case corresponding to $\psi=Af_n(\lambda x)f_n(\lambda x')$, $f_n(\lambda x_0)=0$ or $f_n(\kappa a)=0$, the roots of which determine the admissible values of κ. We have already shown how to obtain these roots and have written down the lowest root for $n=0$, corresponding to various values of the eccentricity, (p. 72).

The *periods of vibration of electricity* in a cylindrical cavity of elliptic cross-section inside a conductor are readily obtained.

Since everything is independent of z (measured along the axis) we have, with usual notation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$; thus there is a stream function ψ, from which u and v are got by differentiation $u=\frac{\partial\psi}{\partial y}$; $v=-\frac{\partial\psi}{\partial x}$. By applying the circuital laws of Ampère and Faraday in the usual way we get $\frac{\partial^2\psi}{\partial t^2}=V^2(V_1^2\psi)$ in the dielectric, where V is the velocity of propagation of electro-dynamic action through the dielectric.

Putting $\psi \,.\, \propto e^{i\kappa Vt}$ we get $(V_1^2 + \kappa^2)\psi = 0$, and since ψ must be finite everywhere in the cavity, we thus get

$$\psi = \Sigma A_n f_n(\lambda x) f_n(\lambda x') e^{i\kappa Vt}.$$

Now the wave length will clearly be comparable in magnitude with the diameter of the cylinder, so that for ordinary sized cylinders the frequency is enormously high. But in such cases the currents are confined to a mere 'skin' on the surface of the conductor. Inside this skin (i.e. in the conductor) there is no E.M.F.; and the tangential E.M.F. is continuous on crossing the skin. Thus the tangential E.M.F. must vanish at the surface of the dielectric. Since the tangential E.M.F. vanishes, the tangential current also vanishes, therefore $\dfrac{\partial \psi}{\partial n} = 0$, at the boundary, or in our notation $\dfrac{\partial \psi}{\partial x} = 0$, when $x = x_0$.

Thus if $\psi = A f_n(\lambda x) f_n(\lambda x') e^{i\kappa Vt}$, the admissible values of κ $(\lambda = h\kappa)$ are given by the roots of $f_n'(\lambda x_0) = 0$. The roots of this are given (for $n = 0$ and $n = 1$) on pp. 74, 75: the wave lengths there determined apply also to the electrical problem.

If we are dealing with the problem of electric waves in the dielectric surrounding an elliptic cylinder then we must replace $f_n(\lambda x)$ above by $\phi_n^-(\lambda x)$ and take $\psi = \Sigma A_n \phi_n^-(\lambda x) f_n(\lambda x') e^{i\kappa Vt}$ in the dielectric. In this case the admissible values of κ are given by the roots the equation $\phi_n'^-(\kappa a) = 0$.

If we wish to estimate the *decay of magnetic force in a metal cylinder*, the lines of magnetic force being parallel to the axis we have (neglecting polarisation currents) $V_1^2 c = \dfrac{4\pi\mu}{\sigma}\dfrac{dc}{dt}$ or, if $c \,.\, \propto e^{-mt}$, $(V_1^2 + \kappa^2) c = 0$ where $\kappa^2 = \dfrac{4\pi\mu m}{\sigma}$.

The appropriate solution is $c = \Sigma A_n f_n(\lambda x) f_n(\lambda x') e^{-mt}$. Now we are neglecting polarisation currents so that c is constant in the dielectric. Taking this constant value as I, and noting that the tangential component of the magnetic force is continuous, we have at the boundary $(x = x_0)$

$$\Sigma A_n f_n(\lambda x_0) f_n(\lambda x') e^{-mt} = \mu I.$$

In the case of *free currents* $I = 0$ and the admissible values of κ are given by the roots of $f_n(\lambda x_0) = 0$, or $f_n(\kappa a) = 0$.

The 'modulus of decay' is $\dfrac{1}{m} = \dfrac{4\pi m}{\sigma\kappa^2} = \dfrac{4\pi\mu a^2}{\sigma(\kappa a)^2}$. The decay is slowest for the smallest values of (κa), the slowest of all corresponding to the least root of $f_0(\kappa a) = 0$. We shall write down the modulus for a copper rod semi-major axis 1 *cm*, for which $\sigma = 1600$; and for an iron rod of same size for which $\mu = 1000$, and $\sigma = 10^4$.

e	κa	Modulus: Copper (seconds)	Modulus: Iron (seconds)	Ratio to case $e=0$
0	2·404	·001359	·2174	1
0·1	2·411	·001351	·2162	0·9942
0·2	2·429	·001331	·2130	0·9793
0·3	2·457	·001301	·2082	0·9573
0·4	2·512	·001245	·1991	0·9158
0·5	2·585	·001175	·1881	0·8649
0·6	2·774	·001021	·1633	0·7510
0·7	3·092	·000803	·1284	0·5907

In dealing with the case where the *currents are longitudinal* and the magnetic force transversal, we may express everything in terms of R the E.M.F. along the axis. Adopting the usual notation we have:—

$$\frac{da}{dt}=-\frac{\partial R}{\partial y}; \quad \frac{db}{dt}=\frac{\partial R}{\partial x}; \quad c=0: \quad u=v=0:$$

$$4\pi\mu ip\omega=4\pi\mu\frac{dw}{dt}=V_1^2R.$$

In the cylinder we have $(V_1^2+\kappa_1^2)R_1=0$ where $\kappa_1^2=-\dfrac{4\pi\mu ip}{\sigma}$, and in the dielectric $(V_1^2+\kappa_0^2)R_0=0$ where $\kappa_0^2=\left(\dfrac{p}{V}\right)^2$.

Thus
$$R_1=\Sigma A_n f_n(\lambda_1 x) f_n(\lambda_1 x')e^{ipt},$$
$$R_0=\Sigma B_n \phi_n^-(\lambda_0 x) f_n(\lambda_0 x')e^{ipt}.$$

The E.M.F. parallel to the axis is continuous at the surface of the cylinder, $\therefore$ $R_1=R_0$ when $x=x_0$, $\therefore$ $A_n f_n(\lambda_1 x_0)=B_n\phi_n^-(\lambda_0 x_0)$. R is a 'stream function' for a and b, so that the magnetic induction parallel to the bounding ellipse is proportional to $\dfrac{\partial\psi}{\partial x}$.

But the tangential magnetic force is continuous; so that we get another boundary equation $\dfrac{A_n\lambda_1}{\mu}f_n'(\lambda_1 x_0)=B_n\lambda_0\phi_n'^-(\lambda_0 x_0)$.

Eliminating the ratio of A_n to B_n we get:—

$$\frac{\lambda_1 f_n'(\lambda_1 x_0)}{\mu f_n(\lambda_1 x_0)}=\frac{\lambda_0\phi_n'^-(\lambda_0 x_0)}{\phi_n^-(\lambda_0 x_0)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the currents will decay slowly p/V (and therefore λ_0) is very small.

Now if $\lambda=0$, the equation for $\phi_n^-(\lambda x)$ is $z^2\dfrac{dy^2}{dz^2}+z\dfrac{dy}{dz}+(z^2-n^2)\,y=0$.

Let $y=ue^t$ where $t=-iz$ and we get $t^2\dfrac{d^2u}{dt^2}+(2t^2+t)\dfrac{du}{dt}+(t-n^2)\,u=0$.

Solving this in an *ascending* series of the form $\dfrac{a_m}{t^m}+\dfrac{a_{m+1}}{t^{m-1}}+\ldots$, we get as the "indicial" equation $m^2-n^2=0$.

Thus we have $\phi_n^-(\lambda x)=\dfrac{ae^{-i\lambda x}}{(\lambda x)^n}$ approximately when λ is small.

Making use of this we find that (1), of p. 78, becomes

$$\frac{\lambda_1 f_n'(\lambda_1 x_0)}{\mu f_n(\lambda_1 x_0)}=-n\lambda_0,$$

or

$$\lambda_1 f_n'(\lambda_1 x_0)+n\lambda_0\mu f_n(\lambda_1 x_0)=0,$$

which determines the admissible values of p. If, as in case of iron, μ is large we have $f_n(\lambda_1 x_0)=0$, which has just been considered for the case $n=0$ on p. 78.

Section II. Spheroids.

We shall proceed now to the more interesting problem of finding solutions of the equation $(V^2+\kappa^2)\psi=0$ that will enable us to deal with a variety of questions relating to spheroids.

Generally if α_1, α_2, α_3 form a system of orthogonal coordinates so that the line element ds is given by $ds^2=\dfrac{d\alpha_1^2}{h_1^2}+\dfrac{d\alpha_2^2}{h_2^2}+\dfrac{d\alpha_3^2}{h_3^2}$, it is well known that

$$V^2\psi=h_1h_2h_3\left[\frac{\partial}{\partial\alpha_1}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial\alpha_1}\right)+\ldots+\ldots\right].$$

Also if α_1, α_2, α_3 satisfy $V^2\psi=0$, we have the simpler relation

$$V^2\psi=h_1^2\frac{\partial^2\psi}{\partial\alpha_1^2}+h_2^2\frac{\partial^2\psi}{\partial\alpha_2^2}+h_0^2\frac{\partial^2\psi}{\partial\alpha_0^2}.$$

We shall be dealing with prolate spheroids, and shall define the position of a point by the semi-axes a and a' of the ellipse and hyperbola confocal with some given one that pass through the point, and by the azimuth ϕ.

We may take $\alpha_1=-\frac{1}{2}\displaystyle\int_\epsilon^\infty\frac{d\lambda}{\sqrt{(a_0^2+\lambda)(b_0^2+\lambda)}}$ (with the usual notation).

$$=-\int_a^\infty\frac{da}{b^2},\ \text{so that } d\alpha_1=\frac{da}{b^2},$$

$$pdp=a\,da;\ h_1=\frac{d\alpha_1}{dn}=\frac{p}{a}\frac{\partial\alpha}{\partial a}=\frac{p}{ab^2}=\frac{1}{bD}=\frac{1}{b\sqrt{a^2-a'^2}}.$$

In this way we get $V^2\psi=\frac{1}{a^2-a'^2}\left[\frac{1}{b^2}\frac{\partial^2\psi}{\partial\alpha_1^2}+\frac{1}{b'^2}\frac{\partial^2\psi}{\partial\alpha_2^2}+\left(\frac{1}{1-\frac{a'^2}{h^2}}-\frac{1}{1-\frac{a^2}{h^2}}\right)\frac{\partial^2\psi}{\partial\phi^2}\right]$;

$$\text{i.e. } V^2\psi=\frac{1}{h^2(x^2-x'^2)}\left[-\frac{\partial}{\partial x}(1-x^2)\frac{\partial\psi}{\partial x}+\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'}+\left(\frac{1}{1-x'^2}-\frac{1}{1-x^2}\right)\frac{\partial^2\psi}{\partial\phi^2}\right],$$

where $x=a/h$ and $x'=a'/h$.

If we take ψ to vary as $\sin(n\phi+\gamma)$ we have $\frac{\partial^2\psi}{\partial\phi^2}=-n^2\psi$,

and hence if $(V^2+\kappa^2)\psi=0$ we get

$$h^2\kappa^2(x^2-x'^2)\psi=\frac{\partial}{\partial x}(1-x^2)\frac{\partial\psi}{\partial x}-\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'}+\left(\frac{1}{1-x'^2}-\frac{1}{1-x^2}\right)n^2\psi.$$

As before assume $\psi=yy'$ where y is a function of x only and y' of x' only. We then get (putting $h\kappa=\lambda$)

$$\frac{1}{y}\frac{d}{dx}(1-x^2)\frac{dy}{dx}-\frac{n^2}{1-x^2}-\lambda^2x^2=\frac{1}{y'}\frac{d}{dx'}(1-x'^2)\frac{dy'}{dx^2}-\frac{n^2}{1-x'^2}-\lambda^2x'^2$$

$=-p$ say (where p is an arbitrary constant).

Thus y and y' both satisfy an equation of the form

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)y-\frac{n^2}{1-x^2}y=0.$$

In case of symmetry about the axis we have $n=0$ and

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)y=0.$$

Consider $$(1-x^2)y''-2xy'+(p-\lambda^2x^2)y-\frac{n^2}{1-x^2}y=0 \;\ldots\ldots\ldots\ldots\ldots\;(1).$$

Let $y=(1-x^2)^{n/2}.z$. Substitute in (1) and divide by $(1-x)^{n/2}$.

This leads to the equation

$$(1-x^2)z''-2(n+1)xz'+[p-n(n+1)-\lambda^2x^2]z=0\;\ldots\ldots\ldots\;(2).$$

Everything now depends on the solution of this equation which we may therefore regard as the fundamental one of our problem. It corresponds to the equation of p. 43 in our former work.

This equation has the same critical points as that just referred to, viz. 1, -1, and ∞. Its solution in the neighbourhood of these critical points might be investigated on exactly the same lines as before. The work however would be so nearly identical with our earlier work that it is hardly worth repeating it here.

We shall proceed at once to obtain solutions in a form convenient for physical applications, i.e. in a series of powers of x.

Assuming $z = a_0 x^m + a_1 x^{m+1} + \ldots$ and substituting in (2) of last page we find that we must have $m(m-1)=0$. Thus we get two independent serial solutions, one even and the other odd.

Taking the *even* series first we have:—

$$z = a_0 + a_1 x^2 + \ldots + a_m x^{2m} + \ldots.$$

Substituting in the differential equation and equating coefficients of different powers to zero we get

$$2a_1 + p'a_0 = 0\,; \quad (2m+1)(2m+2)\,a_{m+1} + [p' - 2m(2m+n-1)]\,a_m - \lambda^2 a_{m-1} = 0,$$

where $p' = p - n(n+1)$.

If $v_{m+1} = a_{m+1}/a_m$ we find $v_{m+1} = 1 + \dfrac{4(n-1)m - 2 - p'}{(2m+2)(2m+1)} + \dfrac{\lambda^2}{(2m+2)(2m+1)\,v_m}$.

Thus when m *is large, either* v_m is small in which case v_m tends to the limit $-\dfrac{\lambda^2}{(2m+1)(2m+2)}$, *or* v_m is not small and then v_m approaches the limit 1, when m is very large. In the former case the ultimate convergence is the same as that of the series for $\cos\lambda x$, and the series is convergent for all *finite* values of the argument.

We must consider more particularly the other case which is more unfavourable to the convergence of our series. In the first place we note that the series converges for all values of the argument whose modulus is less than unity. It is important to examine the convergence in the extreme case when $|x| = 1$.

The series is $a_0 + a_1 + \ldots + a_m + \ldots$ to ∞,

where $$(2m+1)(2m+2)\,a_{m+1} = [2m(2m+2n+1) - p']\,a_m + \lambda^2 a_{m-1}.$$

As we have already seen $\operatorname*{Lt}_{m=\infty} \dfrac{a_{m-1}}{a_m} = 1$,

and when m is very large we have $\dfrac{a_{m+1}}{a_m} = \dfrac{2m(2m+2n+1) - p'}{(2m+1)(2m+2)}$ approx.;

$$\therefore\ m\left(\frac{a_m}{a_{m+1}} - 1\right) = m\,.\,\frac{(2m+1)(2m+2) - 2m(2m+2n+1) + p'}{2m(2m+2n+1) - p'}$$

$$= 1 - n \text{ in the limit when } m = \infty.$$

For all cases except $n=0$, this is less than 1, so that this test of convergence shows that all the series (corresponding to different values of n, except $n=0$) *diverge* when $x = 1$.

For the case of $n=0$ we must use a higher test of convergence.

We have in this case:—

$$(\log m) \times \left[m\left(\frac{a_m}{a_{m+1}} - 1\right) - 1\right] = (\log m)\,.\,\frac{p'.(m+1)}{4m^2 + 2m - p'}$$

$$= \frac{p'}{4}\,.\,\frac{\log m}{m} \text{ ultimately} = \frac{p'}{4}\,.\,\frac{1}{m} \text{ ultimately} = 0,$$

so that for this case also the series *diverges* when $x = 1$.

Now $x'=1$ on the axis of the spheroid, so that if we are dealing with physical problems that require the dependent variable to be expressed in terms of x and x' throughout a region from which the axis of the spheroid is *not* excluded, we cannot make use of the series just considered. For such problems we are confined to the first case mentioned on p. 81. In other words we must have $v_\infty=0$, and then the series we obtain converge for all finite values of the argument. As before this limits us to particular values of p.

The series just considered are the *even* series: but of course precisely similar results are obtained from a treatment of the *odd* series.

The cases that will prove most useful to us in the discussion of physical problems are those corresponding to $n=0$ and $n=1$, and we shall therefore consider these more in detail.

For the case of symmetry $n=0$ we have $\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)y=0$. It will be rather more convenient to solve this in powers of λx, and taking

$$y=a_0+a_1(\lambda x)^2+\ldots+a_n(\lambda x)^{2n}+\ldots$$

we get

$$2a_1\lambda^2+pa_0=0;\ 12a_2\lambda^2-(6-p)a_1-a_0=0,$$

$$(2n+1)(2n+2)a_{n+1}\lambda^2-[2n(2n+1)-p]a_n-a_{n-1}=0.$$

We have seen that for most physical applications p is confined to the roots of $a_\infty=0$. The determination of these roots proceeds exactly as on p. 60, et seq., the only difference is that in the present case we take $v_n=2n(2n+1)-p$ instead of $(2n)^2-p^2$. Thus we can use our former work to give us the values of p with comparatively very little trouble.

p_0 (*corresponding to* $v_0=0$), see p. 60.

First approximation: $v_0=v$, $\therefore p_0=0$.

Second.
$$v_0+\frac{1.2}{v_1}\lambda^2=0,\ \therefore v_0=-\frac{\lambda^2}{3};\ p_0=\frac{\lambda^2}{3}.$$

Third.
$$0=v_0+\frac{2\lambda^2}{v_1}-\frac{2.3.4}{v_1^2v_2}\lambda^4,$$

giving
$$p_0=\frac{\lambda^2}{3}-\frac{2}{135}\lambda^4.$$

Fourth.
$$0=v_0+\frac{2\lambda^2}{v_1}-\frac{2\lambda^4}{v_1}\cdot\frac{3.4}{v_1v_2}+\frac{2\lambda^6}{v_1}\cdot\frac{3.4}{v_1v_2}\left(\frac{3.4}{v_1v_2}+\frac{5.6}{v_2v_3}\right),$$

giving
$$p_0=\frac{\lambda^2}{3}-\frac{2}{135}\lambda^4+\frac{4}{3^5.5.7}\lambda^6,$$

Fifth.
$$0=v_0+\frac{2\lambda^2}{v_1}-\frac{2\lambda^4}{v_1}\cdot\frac{3.4}{v_1v_2}+\frac{2\lambda^6}{v_1}\left(\frac{3.4}{v_1v_2}\right)\left(\frac{3.4}{v_1v_2}+\frac{5.6}{v_1v_2}\right)$$
$$-\frac{2\lambda^8}{v_1}\cdot\frac{3.4}{v_1v_2}\left[\frac{3.4}{v_1v_2}\left(\frac{3.4}{v_1v_2}+\frac{5.6}{v_2v_3}\right)+\frac{5.6}{v_2v_3}\left(\frac{3.4}{v_1v_2}+\frac{5.6}{v_2v_3}+\frac{7.8}{v_3v_4}\right)\right].$$

Hence up to λ^{10} we have $p_0 = \dfrac{\lambda^2}{3} - \dfrac{2}{135}\lambda^4 + \dfrac{4}{3^5 \cdot 5 \cdot 7}\lambda^6 + \dfrac{182\lambda^8}{3^7 \cdot 5^3 \cdot 7^3} \dots$

p_2 *(corresponding to* $v_1 = 0$*)* see p. 61.

First approximation: $v_1 = 0, \quad \therefore p_2 = 6.$

Second.
$$v_1 + \lambda^2\left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) = 0; \quad p_2 = 6 + \frac{11}{21}\lambda^2.$$

Third.
$$v_1 + \lambda^2\left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) - \lambda^4\left(\frac{3 \cdot 4}{v_2} \cdot \frac{5 \cdot 6}{v_2 v_3}\right) = 0,$$
$$\therefore p_2 = 6 + \frac{11}{21}\lambda^2 + \frac{94}{(3 \times 7)^3}\lambda^4.$$

Fourth.
$$v_1 + \lambda^2\left(\frac{1 \cdot 2}{v_0} + \frac{3 \cdot 4}{v_2}\right) - \lambda^4\frac{3 \cdot 4 \cdot 5 \cdot 6}{v_2^2 v_3} + \lambda^6\frac{3 \cdot 4 \cdot 5 \cdot 6}{v_2^2 v_3}\left(\frac{5 \cdot 6}{v_2 v_3} + \frac{7 \cdot 8}{v_3 v_4}\right).$$

So that
$$p_2 = 6 + \frac{11}{21}\lambda^2 + \frac{94}{(21)^3}\lambda^4 - \frac{21388}{99 \times 7^5}\lambda^6 \dots$$

p_4 *(corresponding to* $v_2 = 0$*).*

First approximation: $v_2 = 0, \quad \therefore p_4 = 20.$

Second.
$$v_2 + \lambda^2\left(\frac{3 \cdot 4}{v_1} + \frac{5 \cdot 6}{v_3}\right) = 0, \quad \therefore p_4 = 20 + \frac{39}{77}\lambda^2.$$

Third.
$$v_2 + \lambda^2\left(\frac{3 \cdot 4}{v_1} + \frac{5 \cdot 6}{v_3}\right) - \lambda^4\left[\frac{1 \cdot 2 \cdot 3 \cdot 4}{v_0 v_1^2} + \frac{5 \cdot 6 \cdot 7 \cdot 8}{v_3^2 v_4}\right] = 0,$$
$$\therefore p_4 = 20 + \frac{39}{77}\lambda^2 + \frac{52644}{5 \cdot 7^3 \cdot 11^3 \cdot 13}\lambda^4 + \dots$$

Similarly with the odd series let
$$y = c_0\lambda x + c_1(\lambda x)^3 + \dots + c_n(\lambda x)^{2n+1} + \dots$$
and we get
$$(1 - x^2)\left[6c_1\lambda^3 x + 20c_2\lambda^5 x^3 + \dots + 2n(2n+1)c_n\lambda^{2n+1}x^{2n-1} + \dots\right]$$
$$- 2x\left[c_0\lambda + 3c_1\lambda^3 x^2 + 5c_2\lambda^5 x^4 + \dots + (2n+1)c_n\lambda^{2n+1}x^{2n} + \dots\right]$$
$$+ (p - \lambda^2 x^2)\left[c_0\lambda x + c_1(\lambda x)^3 + \dots + c_n(\lambda x)^{n+1} + \dots\right] = 0,$$
$$\therefore -6c_1\lambda^2 + (2 - p)c_0 = 0; \quad -20c_2\lambda^2 + (12 - p)c_1 + c_0 = 0$$
$$-(2n+2)(2n+3)\lambda^2 c_{n+1} + [(2n+1)(2n+2) - p]c_n + c_{n-1} = 0.$$

In this case we take $v_n = (2n+1)(2n+2) - p$ and proceed on exactly the same lines as before.

p_1 *(corresponding to root* $v_0 = 0$*).*

First approximation: $v_0 = 0, \quad \therefore p_1 = 2,$

Second.
$$v_0 + \lambda^2\frac{2 \cdot 3}{v_1} = 0, \quad \therefore p_1 = 2 + \frac{3}{5}\lambda^2.$$

Third. $$v_0+\lambda^2\frac{2.3}{v_1}-\lambda^4\frac{2.3.4.5}{v_1^2v_2}=0.$$

Thus $$p_1=2+\frac{3}{5}\lambda^2-\frac{12}{7\times 250}\lambda^4+\dots.$$

p_3 (*corresponding to* $v_1=0$).

First approximation: $v_1=0,\quad \therefore\ p_3=12.$

Second. $$v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)=0,\quad \therefore\ p_3=12+\frac{23}{45}\lambda^2.$$

Third. $$v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)-\lambda^4\frac{4.5.6.7}{v_2^2v_3}=0,$$

$$\therefore\ p_3=12+\frac{23}{45}\lambda^2+\frac{7229}{2.3^6.5^3.11}\lambda^4+\dots.$$

p_5 (*corresponding to* $v_2=0$).

First approximation: $v_2=0,\quad \therefore\ p_5=30.$

Second. $$v_2+\lambda^2\left(\frac{4.5}{v_1}+\frac{6.7}{v_3}\right)=0,\quad \therefore\ p_5=30+\frac{59}{117}\lambda^2.$$

Third. $$v_2+\lambda^2\left(\frac{4.5}{v_1}+\frac{6.7}{v_3}\right)-\lambda^4\left(\frac{2.3.4.5}{v_0v_1^2}+\frac{6.7.8.9}{v_3^2v_4}\right)=0,$$

giving $$p_5=30+\frac{59}{117}\lambda^2+\frac{63338}{3^5.5.7.13^3}\lambda^4+\dots.$$

Turning now from the case of symmetry ($n=0$) to the case $n=1$, our equation becomes $(1-x^2)\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(p-2-\lambda^2x^2)\,y=0.$

If $$y=a_0+a_1(\lambda x)^2+\dots+a_n(\lambda x)^{2n}+\dots$$

we have $$2a_1\lambda^2+(p-2)\,a_0=0;\quad 12a_2\lambda^2+(p-12)\,a_1-a_0=0,$$

$$-(2n+1)(2n+2)\,a_{n+1}\lambda^2+[(2n+1)(2n+2)-p]\,a_n+a_{n-1}=0,$$

and in the determination of p we proceed as before, the only difference being that now we have $v_n=(2n+1)(2n+2)-p$. In this way we get

$$p_0=2+\frac{1}{5}\lambda^2-\frac{4}{7.5^3}\lambda^4+\frac{8}{21\times 5^5}\lambda^6+\dots,$$

$$p_2=12+\frac{7}{15}\lambda^2+\frac{1064}{77.3^4.5^3}\lambda^4+\dots,$$

$$p_4=30+\frac{19}{39}\lambda^2+\frac{1108}{7.3^4.13^3}\lambda^4+\dots$$

For the *odd* series $\qquad y = c_0\lambda x + c_1(\lambda x)^3 + \ldots + c_n(\lambda x)^{2n+1} + \ldots,$

we have $\qquad 6c_1\lambda^2 + (p-6)\,c_0 = 0; \quad 20c_2\lambda^2 + (p-20)\,c_1 - c_0 = 0$

$$-(2n+2)(2n+3)\,c_{n+1}\lambda^2 + [(2n+2)(2n+3) - p]\,c_n + c_{n-1} = 0,$$

giving by the usual method

$$p_1 = 6 + \frac{3}{7}\lambda^2 - \frac{4}{1029}\lambda^4 + \frac{8}{33\,.\,7^5}\lambda^6 + \ldots,$$

$$p_3 = 20 + \frac{37}{77}\lambda^2 + \frac{7064}{7^3\,.\,11^3\,.\,13}\lambda^4 + \ldots,$$

$$p_5 = 42 + \frac{27}{55}\lambda^2 + \ldots.$$

Just as on p. 63 the odd and even series obtained as solutions of our differential equation correspond to different values of p and so are really solutions of different equations. But having obtained one solution of our equation there is no difficulty in finding another. In the case of symmetry our equation is

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx} + (p - \lambda^2x^2)\,y = 0.$$

Take $y = uf + w$ where f is the solution already found.

We then have

$$\frac{d}{dx}(1-x^2)\frac{dw}{dx} + (p-\lambda^2x^2)\,w + f\,.\,\frac{d}{dx}(1-x^2)\frac{du}{dx} + 2(1-x^2)\frac{du}{dx}\frac{df}{dx} = 0.$$

Choose u so that $\dfrac{d}{dx}(1-x^2)\dfrac{du}{dx} = 0$, which is satisfied by $u = \frac{1}{2}\log\dfrac{x+1}{x-1}$.

Our equation then reduces to $\dfrac{d}{dx}(1-x^2)\dfrac{dw}{dx} + (p - \lambda^2x^2)\,w = -2\dfrac{df}{dx}$, and we want a particular value of w (the simpler the better) to satisfy this.

If $f = a_0 + a_1x^2 + \ldots$ (even), we assume $w = A_0x + A_1x^3 + \ldots$ (odd).

Substituting in the differential equation and equating coefficients we get

$$6A_1 + (p-2)\,A_0 = -4a_1; \quad 20A_2 + (p-12)\,A_1 - \lambda^2A_0 = -8a_2,$$

$$(2n+2)(2n+3)\,A_{n+1} + [p - (2n+1)(2n+2)]\,A_n - \lambda^2A_{n-1} = -4(n+1)\,a_{n+1},$$

from which we can determine the coefficients, in succession A_0 (or A_1) being chosen at pleasure.

With the aid of these relations we can show just as on p. 65 that the series for w is convergent in the same domain as f.

Thus we have two independent solutions of our equation

$$y = f(x) \text{ and } y = \tfrac{1}{2}\log\frac{x+1}{x-1}\times f(x) + w = F(x),$$

the complete integral being $y = Af(x) + BF(x)$.

The series defining f is convergent for all finite values of the argument; but F becomes infinite when $x=\pm 1$, and so must not be used in regions where x can have either of these values.

We proceed to find some series that approach 'asymptotically' to solutions of our equation and that will prove useful for numerical calculation for large values of the argument.

Putting $z=i\lambda x$ and $y=ue^z$, the equation

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)\,y=0 \text{ becomes } (z^2+\lambda^2)\frac{d^2u}{dz^2}+2(z^2+z+\lambda^2)\frac{du}{dz}+(2z-p+\lambda^2)\,u=0.$$

Assume $u=\frac{b_0}{z}+\frac{b_1}{z^2}+\ldots\frac{b_n}{z^{n+1}}+\ldots$ and we get

$$(z^2+\lambda^2)\left[\frac{2b_0}{z^3}+\frac{6b_1}{z^4}+\frac{12b_2}{z^5}+\ldots+n(n+1)\frac{b_{n-1}}{z^{n+2}}+\ldots\right]$$

$$-2(z^2+z+\lambda^2)\left[\frac{b_0}{z^2}+\frac{2b_1}{z^3}+\ldots+\frac{b_{n-1}}{z^{n+1}}+\ldots\right]$$

$$+(2z-p+\lambda^2)\left[\frac{b_0}{z}+\frac{b_1}{z^2}+\ldots+\frac{b_n}{z^{n+1}}+\ldots\right]=0.$$

Hence $\qquad +2b_1+(p-\lambda^2)\,b_0=0;\quad 4b_2+(p-2-\lambda^2)\,b_1+\lambda^2b_0=0,$

$$2nb_n=[n(n-1)+\lambda^2-p]\,b_{n-1}-\lambda^2(n-1)\,[2b_{n-2}-\overline{n-2}\,b_{n-3}].$$

So that if $u_n=b_n/b_{n-1}$ we have

$$2u_n=n-1+\frac{\lambda^2-p}{n}-\lambda^2\frac{n-1}{n}\left[\frac{2}{u_{n-1}}-(n-2)\frac{1}{u_{n-1}u_{n-2}}\right].$$

Thus when n is large, either u_n is large and equal to $\frac{n-1}{2}$ approximately, or if u_n is small we have when n is large $0=1-\frac{\lambda^2}{n}\left[\frac{2}{u_{n-1}}-\frac{n-2}{u_n\,.\,u_{n-1}}\right]$, and as this cannot be satisfied by a small value of u_n, we see that u_n is ultimately large, so that the series is ultimately divergent. But as before it is useful for arithmetical purposes if we confine ourselves to the convergent part, the argument being large.

The solution thus obtained is (if $z=\lambda x$),

$$y=\frac{e^{iz}}{z}\left[b_0+\frac{b_1}{iz}+\frac{b_2}{(iz)^2}+\ldots\right]$$

$$=\frac{e^{iz}}{z}\left[b_0-i\frac{b_1}{z}-\frac{b_2}{z^2}+\ldots\right]=\phi^+(z) \text{ say.}$$

Changing the sign of i we get another solution

$$y=\frac{e^{-iz}}{z}\left[b_0+i\frac{b_1}{z}-\frac{b_2}{z^2}+\ldots\right]=\phi^-(z),$$

and the complete integral is $y=C\phi^+(z)+D\phi^-(z)$ where C and D are arbitrary constants.

As on p. 67 we shall take $\chi^+(z) = \dfrac{\phi^+(z) + \phi^-(z)}{2}$; $\chi^-(z) = i \cdot \dfrac{\phi^-(z) - \phi^+(z)}{2}$,

giving
$$\chi^+(z) = \frac{1}{z}[R\cos z + S\sin z],$$

$$\chi^-(z) = \frac{1}{z}[R\sin z - S\cos z],$$

where
$$R = b_0 - \frac{b_2}{z^2} + \ldots; \quad S = \frac{b_1}{z} - \frac{b_3}{z^3} + \ldots.$$

By proceeding very much as on p. 68 we can obtain some relations between the various solutions of our equation. Take $z = r(\cos\theta + i\sin\theta)$, obviously we have

$$\phi^+(-z) = -\phi^-(z); \quad \phi^-(-z) = -\phi^+(z) \quad \text{or} \quad \phi^+(\theta+\pi) = -\phi^-(\theta); \quad \phi^-(\theta+\pi) = -\phi^+(\theta).$$

Also we have $F(z) = \frac{1}{2}\log\dfrac{z+\lambda}{z-\lambda} f(z) + w(z)$ where $w(z)$ is odd if f is even and even if f is odd.

Taking f even, we have $F(-z) = \frac{1}{2}\log\dfrac{-z+\lambda}{-z-\lambda} f(-z) + w(-z)$

$$= -\left[\tfrac{1}{2}\log\frac{z+\lambda}{z-\lambda} f(z) + w(z)\right] = -F(z),$$

and if f is odd we get similarly $F(-z) = +F(z)$.

Then proceeding as on p. 69 we find these relations:—

if f is even,
$$f = \alpha'(\phi^+ - \phi^-) = \alpha\chi^-,$$
$$F = \beta'(\phi^+ + \phi^-) = \beta\chi^+,$$

and *if f is odd,*
$$f = \alpha_1'(\phi^+ + \phi^-) = \alpha_1\chi^+,$$
$$F = \beta_1'(\phi^+ - \phi^-) = \beta_1\chi^-.$$

The larger roots of $f(z)$ can be got from these results by the same process as on p. 70.

For the even series $(f_0 f_2 \ldots)$ we require the roots of $\chi^-(z) = 0$,

i.e.
$$R\sin z - S\cos z = 0.$$

As before, putting

$$\tan\psi = \frac{S}{R} = \frac{\dfrac{b_1}{z} - \dfrac{b_3}{z^3} + \ldots}{b_0 - \dfrac{b_2}{z^2} + \ldots},$$

our equation becomes $\tan(z - \psi) = 0$, so that $z = m\pi + \psi$ where m is an integer.

Similarly with the odd series; and with the roots of $f'(z) = 0$. The smaller roots may be got by Horner's method as already illustrated on p. 72 directly from the ascending series. For example, I find that the equation to determine the roots of $f_1'(z)=0$ is

$$0=1-\cdot 3z^2+(\cdot 017857+\cdot 0034285e^2)z^4$$
$$-(\cdot 00059524+\cdot 00031747e^2-\cdot 00003047e^4)z^6$$
$$+(\cdot 00006764+\cdot 0000111e^2-\cdot 0000010885e^4)z^8+\ldots,$$

and the lowest root for $e=0$, $e=0\cdot 1$; $e=0\cdot 2$; $e=0\cdot 3\ldots$, $e=0\cdot 9$ is 2·0815; 2·0825; 2·0848; 2·0865; 2·0902; 2·0961; 2·1035; 2·1124; 2·1233; 2·1364.

The treatment of the equation corresponding to $n=1$ instead of $n=0$ of our fundamental equation proceeds on quite similar lines. We have already seen that on putting $y=\sqrt{1-x^2}\,.\,y_1$, the equation becomes $(1-x^2)\frac{d^2y_1}{dx^2}-4x\frac{dy_1}{dx}+(p-2-\lambda^2x^2)y_1=0$ and we have obtained solutions of this in the form of *ascending* series of x. To obtain a solution corresponding to that on p. 86 we put $z=i\lambda x$ and $y_1=ue^z$, and the equation becomes

$$(z^2+\lambda^2)\frac{d^2u}{dz^2}+2\left[z^2+\lambda^2+2z\right]\frac{du}{dz}+(4z-p+2+\lambda^2)u=0.$$

Assume
$$u=\frac{b_0}{z^2}+\frac{b_1}{z^3}+\frac{b_2}{z^4}+\ldots+\frac{b_n}{z^{n+2}}+\ldots,$$

and we get $2b_1=(\lambda^2-p)b_0$; $4b_2=(\lambda^2+2-p)b_1-4\lambda^2b_0$

$$2(n+1)b_{n+1}=\left[\lambda^2+n(n+1)-p\right]b_n+(n+1)\lambda^2\left[nb_{n-2}-2b_{n-1}\right].$$

As before the series is ultimately divergent, but is none the less useful for purposes of calculation. The work on p. 87 is, with very slight and obvious modifications, applicable to these functions.

If y and y_1 are two solutions of our fundamental equation corresponding to different values of κ, then y and y_1 have a "*conjugate*" *property*, which will be of use to us.

We have
$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)y=0,$$
$$\frac{d}{dx}(1-x^2)\frac{dy_1}{dx}+(p_1-\lambda_1^2x^2)y_1=0.$$

Hence
$$\int_{-1}^{1}\left[p-p_1-(\lambda^2-\lambda_1^2)x^2\right]yy_1\,dx=\int_{-1}^{1}\left[y\frac{d}{dx}(1-x^2)\frac{dy_1}{dx}-y_1\frac{d}{dx}(1-x^2)\frac{dy}{dx}\right]dx$$
$$=\left[y(1-x^2)\frac{dy_1}{dx}-y_1(1-x^2)\frac{dy}{dx}\right]_{-1}^{1}=0.$$

By aid of this property we can determine the coefficients in the expansion of any function of x' in a series of terms like $f_n(x')$ by a process similar to that used when dealing with Legendre's or Bessel's functions.

In the case where $\lambda=\lambda_1$ but $p\neq p_1$ the conjugate property takes the simpler form $\int_{-1}^{1}yy_1\,dx=0.$

The expansion of the function $e^{i\lambda xx'}$ in terms of our functions is interesting in itself and leads to some results of importance.

If x_1 be the ordinary Cartesian coordinate of a point (the axis of x_1 being the axis of the spheroid) we have $x_1 = hxx'$ in our notation. Now $e^{i\kappa x_1}$ obviously satisfies $(\nabla^2 + \kappa^2)\psi = 0$, and is finite when $x = \pm 1$, so that it must be expressible in a series of f functions.

Assume then
$$e^{i\lambda xx'} \equiv \Sigma A_n f_n(\lambda x) f_n(\lambda x').$$

By p. 88 we have in this case
$$\int_{-1}^{1} f_n(\lambda x') f_m(\lambda x')\, dx' = 0.$$

Hence
$$\int_{-1}^{1} e^{i\lambda xx'} f_n(\lambda x')\, dx' \equiv A_n f_n(\lambda x) \int_{-1}^{1} \overline{f_n(\lambda x')}^2\, dx',$$
a relation which must hold for all values of λx, $(= z$ say).

Now
$$e^{i\lambda xx'} = e^{izx'} = 1 + izx' - \frac{z^2 x'^2}{\underline{|2}} - \frac{iz^3 x'^3}{\underline{|3}} + \ldots$$
and
$$f_n(z) = a_0 + a_1 z^2 + \ldots \text{ if } n \text{ is even,}$$
and
$$= c_0 z + c_1 z^3 + \ldots \text{ if } n \text{ is odd.}$$

Hence, if *n is even*,
$$\int_{-1}^{1} \left(1 + izx' - \frac{z^2 x'^2}{\underline{|2}} - \frac{iz^3 x'^3}{\underline{|3}} + \ldots\right) f_n(\lambda x')\, dx'$$
$$\equiv \left\{A_n \int_{-1}^{1} \overline{f_n(\lambda x')}|^2\, dx'\right\} (a_0 + a_1 z^2 + a_2 z^4 + \ldots).$$

Equating the coefficients of different powers of z we see that $\int_{-1}^{1} x'^{2r-1} f_n(\lambda x')\, dx' = 0$, as is otherwise evident since $f_n(\lambda x')$ is an *even* function.

We also get
$$\int_{-1}^{1} f_n(\lambda x')\, dx' = a_0 A_n \int_{-1}^{1} \overline{f_n(\lambda x')}^2\, dx',$$
which determines A_n, see p. 90,

and
$$-\frac{1}{2}\int_{-1}^{1} x'^2 f_n(\lambda x')\, dx' = \frac{a_1}{a_0}\int_{-1}^{1} f_n(\lambda x')\, dx',$$
$$\frac{1}{\underline{|4}}\int_{-1}^{1} x'^4 f_n(\lambda x')\, dx' = \frac{a_2}{a_0}\int_{-1}^{1} f_n(\lambda x')\, dx',$$
and so on.

Similarly if *n is odd* we have
$$\int_{-1}^{1} \left(1 + izx' - \frac{z^2 x'^2}{\underline{|2}} - \frac{iz^3 x'^3}{\underline{|3}} + \ldots\right) f_n(\lambda x')\, dx'$$
$$\equiv \left\{A_n \int_{-1}^{1} \overline{f_n(\lambda x')}^2\, dx'\right\} \left(c_0 z + c_1 z^3 + c_2 z^5 + \ldots\right).$$

This gives $\int_{-1}^{1} x'^{2m} f_n(\lambda x')\,dx' = 0$, as is obvious, since f_n is an *odd* function.

Also
$$i\int_{-1}^{1} x' f_n(\lambda x')\,dx' = c_0 A_n \int_{-1}^{1} \overline{f_n(\lambda x')}^2\,dx',$$

which determines A_n, see below.

And
$$-\frac{1}{\underline{|3}}\int_{-1}^{1} x'^3 f_n(\lambda x')\,dx' = \frac{c_1}{c_0}\int_{-1}^{1} x' f_n(\lambda x')\,dx',$$

$$\frac{1}{\underline{|5}}\int_{-1}^{1} x'^5 f_n(\lambda x')\,dx' = \frac{c_2}{c_0}\int_{-1}^{1} x' f_n(\lambda x')\,dx',$$

$$\frac{(-1)^m}{\underline{|2m+1}}\int_{-1}^{1} x'^{2m+1} f_n(\lambda x')\,dx' = \frac{c_m}{c_0}\int_{-1}^{1} x' f_n(\lambda x')\,dx'.$$

Now when n is even $f_n(\lambda x) = a_0 + \ldots a_1(\lambda x)^2 + a_2(\lambda x)^4 + \ldots,$

$$\therefore \int_{-1}^{1} \overline{f_n(\lambda x)}|^2\,dx = a_0\int_{-1}^{1} f_n(\lambda x)\,dx + a_1\lambda^2\int_{-1}^{1} x^2 f_n(\lambda x)\,dx + \ldots$$

$$= \frac{1}{a_0}\left[a_0^2 - a_1^2\lambda^2\,\underline{|2} + a_2^2\lambda^4\,\underline{|4}\ldots\right]\int_{-1}^{1} f_n(\lambda x)\,dx,$$

by p. 89.

Similarly, when n is odd, we have $f_n(\lambda x) = c_0\lambda x + c_1(\lambda x)^3 + \ldots,$

$$\therefore \int_{-1}^{1} \overline{f_n(\lambda x)}|^2\,dx = \frac{1}{c_0}\left[c_0^2\lambda - c_1^2\lambda^3\,\underline{|3} + c_2^2\lambda^5\,\underline{|5} + \ldots\right]\int_{-1}^{1} x f_n(\lambda x)\,dx,$$

by means of the relations above.

Since $\int_{-1}^{1} f_n(\lambda x)\,dx$ and $\int_{-1}^{1} x f_n(\lambda x)\,dx$ are easily determined by direct integration of the series, we can thus determine $\int_{-1}^{1} \overline{f_n(\lambda x)}|^2\,dx$.

The above gives us the value of the constant A_n of this and the last page. When n is even we have

$$A_n = \frac{\int_{-1}^{1} f_n(\lambda x)\,dx}{a_0\int_{-1}^{1} \overline{f_n(\lambda x)}|^2\,dx},$$

so that in this case $A_n = 1/(a_0^2 - a_1^2\lambda^2\,\underline{|2} + a_2^2\lambda^4\,\underline{|4} + \ldots),$

and similarly, when n is odd, we get

$$A_n = i/c_0^2\lambda^2 - c_1^2\lambda^3\,\underline{|3} + c_2^2\lambda^5\,\underline{|5} + \ldots).$$

We shall proceed to discuss some physical problems, involving the use of the functions now under consideration.

Vibrations of spheroidal sheets of air.

If c be the velocity of sound in air and ψ the velocity potential our characteristic equation is $\frac{\partial^2\psi}{\partial t^2} = c^2 \nabla^2 \psi$. Taking ψ to vary as $e^{i\kappa ct}$ we get $(\nabla^2 + \kappa^2)\psi = 0$. As we are confined to a thin layer we may regard x as a constant, and so our fundamental equation reduces to $\frac{d}{dx'}(1 - x'^2)\frac{d\psi}{dx'} + (\kappa^2 a^2 - \lambda^2 x'^2)\psi = 0$. This is the same form as before with $p = \kappa^2 a^2$.

If the layer is *complete* the series for ψ must be convergent for all possible values of x' (i.e. from $x' = -1$ to $x' = 1$, inclusive), and so we have $\psi = Af(x')$; $[F(x')$ becomes infinite at the poles $x' = \pm 1]$ and the necessity of convergence at the poles determines the hitherto arbitrary constant κ, or rather confines it to certain definite values. See p. 82.

If the layer of air instead of being complete is bounded by two parallels of latitude, then our solution is $\psi = Af_0(x') + Bf_1(x')$ where f_0 and f_1 are the odd and even series of p. 81. p is not now restricted by conditions of convergence, the series $f(x')$ being convergent (whatever p be) provided $x' < 1$, which is the case in the present problem.

At the bounding walls we must have $\frac{d\psi}{dx'} = 0$; so that if $x' = \alpha$, $x' = \beta$ are the boundaries, we have

$$Af_0'(\alpha) + Bf_1'(\alpha) = 0,$$
$$Af_0'(\beta) + Bf_1'(\beta) = 0.$$

Eliminating $A : B$ we get the frequency equation

$$\begin{vmatrix} f_0'(\alpha), & f_1'(\alpha) \\ f_0'(\beta), & f_1'(\beta) \end{vmatrix} = 0,$$

which determines the admissible values of κ.

If one of the boundaries (β) is the equator, then the frequency equation takes the simple form $f_0'(\alpha) = 0$.

Returning to the case of the complete spheroid we shall determine the frequency corresponding to some of the simpler modes of vibration.

Corresponding to p_0: (p. 82),

we have
$$\kappa^2 a^2 = p_0 = \frac{1}{3}\lambda^2 - \frac{2}{135}\lambda^4 + \frac{4}{3^5 . 5 . 7}\lambda^6 + \dots\dots(1),$$

and $\lambda = h\kappa = \kappa ae$, so that (1) becomes

$$\kappa^2 a^2 \left[1 - \frac{1}{3}e^2 + \frac{2}{135}\kappa^2 a^2 e^4 + \dots\right] = 0.$$

It follows from this that for our problem we must have $\kappa a = 0$. The expression in brackets [ ] cannot vanish for real values of κa, to which we are confined in the present case.

κ being zero, ψ is independent of the time—so that there is really no *vibration* at all, and this so called 'mode' is without physical interest.

Corresponding to p_1. This is the *slowest* periodic movement for the case of symmetry round the axis. The vibration is *un*symmetrical with respect to the ends of the spheroid.

We have
$$\kappa^2 a^2 = p_1 = 2 + \frac{3}{5}\lambda^2 - \frac{6}{875}\lambda^4 - \frac{4}{3\,.\,5^5\,.\,7}\lambda^6 + \dots$$
$$= 2 + \frac{3}{5}\kappa^2 a^2 e^2 - \frac{6}{875}\kappa^4 a^4 e^4 + \dots,$$
giving
$$\kappa^2 a^2 = 2 + 1\,.\,2e^2 + \cdot692e^4 + \cdot38e^6 + \dots.$$

e	κ^2a^2	$a\times$ frequency	wave length/a
0	2	7439	4·450
0·1	2·0121	7461	4·436
0·2	2·0491	7528	4·396
0·3	2·1136	7647	4·328
0·4	2·209	7818	4·234
0·5	2·3491	8061	4·106
0·6	2·5397	8383	3·948
0·7	2·7892	8786	3·767
0·8	3·1497	9335	3·546
0·9	3·6181	10,000	3·310

The ratios of the frequencies for $e=0·1 \dots e=0·9$ to that for the sphere $(e=0)$ are 1·003; 1·012; 1·028; 1·051; 1·083; 1·126; 1·181; 1·214; 1·330.

The interval between successive cases is rather more than a *comma*, that between $e=0·8$ and $e=0$ is rather more than a *minor third*, and between $e=0·9$ and $e=0$ is just about a *fourth*.

If then the sphere gives out the note c (or *Do*)

the spheroid $e=\frac{6}{10}$ „ „ „ „ d (or *Re*) very nearly,

„ „ $e=\frac{8}{10}$ „ „ „ „ e (or *Mi*) fairly nearly,

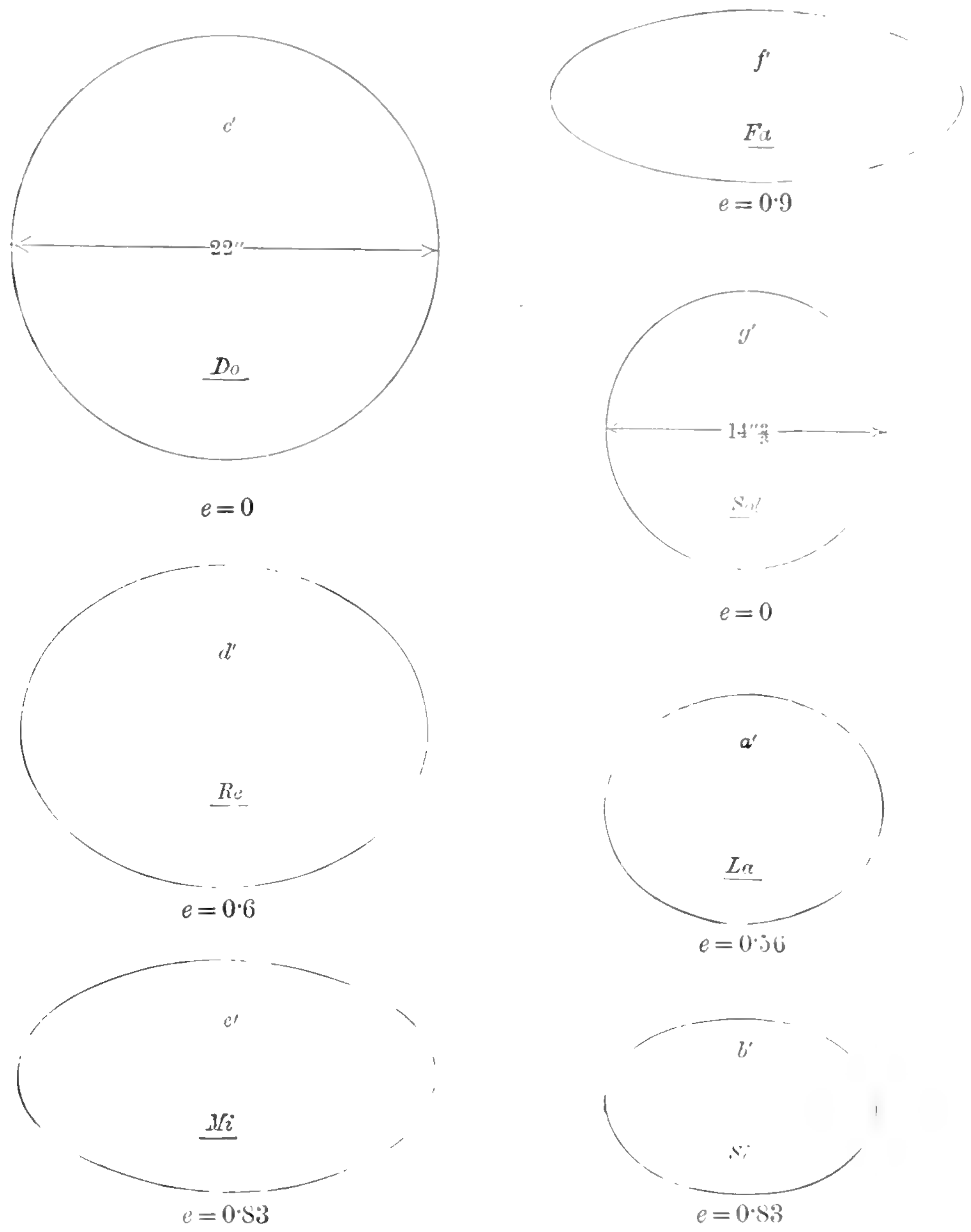
„ „ $e=\frac{9}{10}$ „ „ „ „ f (or *Fa*) very nearly.

For the sphere to sound the middle *c* of a piano (c'; frequency 264) it must have a radius $a = 28$ centimetres $= 11$ inches.

If the eccentricity is $\frac{5}{10}$ then $a = 31$ centimetres ($12\frac{1}{5}$ inches).

" " " $\frac{9}{10}$ " $a = 37$ " ($14\frac{3}{5}$ inches).

The accompanying rough sketch will illustrate to the eye the influence of size and eccentricity on the note emitted.



Corresponding to p_2 (the slowest *symmetrical* vibration),

$$\kappa^2a^2=p_2=6+\frac{11}{21}\lambda^2+\frac{94}{(21)^3}\lambda^4-\frac{21388}{99\times 7^5}\lambda^6\ldots \quad \text{(p. 83)}$$

$$=6+\frac{11}{21}\kappa^2a^2e^2+\frac{94}{(21)^3}\kappa^4a^4e^4-\frac{21388}{7^5\times 99}\kappa^6a^6e^6+\ldots,$$

so that $$\kappa^2a^2=6+3{\cdot}14e^2+2e^4-1{\cdot}4e^6+\ldots$$

e	κ^2a^2	$a\times$ frequency	wave length/a	Ratio of frequency to that for $e=0$
0	6	12883	2·569	1
0·1	6·0314	12918	2·563	1·003
0·2	6·1286	13020	2·544	1·010
0·3	6·2978	13201	2·507	1·024
0·4	6·5459	13461	2·459	1·045
0·5	6·9136	13830	2·393	1·073
0·6	7·3238	14234	2·325	1·105
0·7	7·8456	14734	2·247	1·143
0·8	8·4620	15301	2·163	1·186
0·9	9·3266	16063	2·061	1·247

The interval between the notes corresponding to $e=0$ and $e=\frac{2}{10}$ is just about a *comma*: between those for $e=0$ and $e=\frac{9}{10}$ about a *major third.*

We shall now turn to the case of *non*-symmetry about the axis. Taking ψ to vary as $\sin\phi$ (where ϕ is the azimuth) we have $n=1$ in the equation of p. 80.

Corresponding to p_0 (the gravest note we can get),

$$\kappa^2a^2=p_0=2+\frac{1}{5}\lambda^2-\frac{4}{5^3\,.\,7}\lambda^4+\frac{8}{21\,.\,5^5}\lambda^6\ldots$$

$$=2+\frac{1}{5}\kappa^2a^2e^2-\frac{4}{5^3\,.\,7}(\kappa a)^4e^4+\frac{8}{21\,.\,5^5}(\kappa a)^6e^6+\ldots,$$

giving $$\kappa^2a^2=2+0{\cdot}4e^2+0{\cdot}062e^4+0{\cdot}006e^6+\ldots.$$

The values are tabulated on the next page. The lowest note in the series (that for the sphere) is the same as that in the mode discussed on p. 92. But for the same eccentricity (except $e=0$) the notes in this mode are rather lower than in the mode on p. 92. In the present case if the sphere gives the note *Do* (*c*), then the spheroid $e=\frac{9}{10}$ gives out a note slightly *lower* than *Re* (*d*), while the note corresponding to $e = \frac{9}{10}$ in the mode of p. 92 is *Fa* (*f*). The spheroid $e = \frac{5}{10}$ gives in the mode of p. 92 about the same note as $e = \frac{9}{10}$ in the present mode.

e	$\kappa^2 a^2$	$a \times$ frequency	wave length/a
0	2	7439	4·450
0·1	2·004	7446	4·445
0·2	2·016	7468	4·432
0·3	2·037	7508	4·409
0·4	2·066	7561	4·377
0·5	2·104	7630	4·338
0·6	2·152	7716	4·289
0·7	2·211	7822	4·231
0·8	2·282	7944	4·167
0·9	2·367	8093	4·090

Corresponding to p_1

$$\kappa^2 a^2 = p_1 = 6 + \frac{3}{7}\lambda^2 - \frac{4}{1029}\lambda^4 + \frac{8}{33 \cdot 7^5}\lambda^6 + \dots$$

$$= 6 + \frac{3}{7}\kappa^2 a^2 e^2 - \frac{4}{1029}(\kappa a)^4 e^4 + \frac{8}{33 \cdot 7^5}(\kappa a)^6 e^6 + \dots,$$

so that

$$\kappa^2 a^2 = 6 + 2{\cdot}57 e^2 + 0{\cdot}96 e^4 + 0{\cdot}33 e^6 + \dots.$$

e	$\kappa^2 a^2$	$a \times$ frequency	wave length/a
0	6	12883	2·569
0·1	6·0257	12916	2·562
0·2	6·1032	12996	2·546
0·3	6·2393	13141	2·519
0·4	6·4375	13347	2·480
0·5	6·6067	13528	2·446
0·6	7·0632	13980	2·366
0·7	7·5226	14427	2·294
0·8	8·1242	14994	2·207
0·9	8·8902	15686	2·110

The notes of this mode are rather lower than in the corresponding cases on p. 94; $e=\frac{9}{10}$ of this mode nearly corresponding with $e=\frac{8}{10}$ of the former.

Vibrations in a hemi-spheroidal sheet.

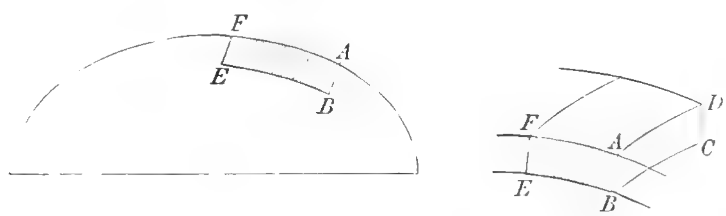
The symmetrical vibrations (corresponding to p_2, p_4 ...) make the equator *nodal*, so that we may at once adopt the above results to vibrations in a hemispheroid *closed at the edge.* The notes are given on p. 94.

The unsymmetrical type (corresponding to p_1, p_3 ...) make $\psi=0$ and therefore $\dot{\psi}=0$ at the equator ($x'=0$). Thus in these modes there is no pressure variation at the equator, so that there is a *loop* there. If we make the rough assumption usually adopted in the elementary treatment of open ends—viz. that there is a loop at the end, we can thus apply our results to the determination of the notes in a hemi-spheroidal layer with an *open end* (open at the equator). The notes are given on pp. 92 and 95.

The case discussed on last page ($n=1$) gives us the gravest note in a *quadrant of a spheroid,* closed at the sides and open at the equator.

A problem that is analytically closely analogous to the one just discussed is the investigation of the modes of vibration for the *electrical oscillations in a thin homœoidal layer of dielectric* between two conducting spheroidal surfaces. There are clearly two perfectly distinct types of oscillation. In one the wave surges backwards and forwards between the bounding surfaces, the *magnetic* force being normal to the ellipsoid and so the electric force tangential. In this type the wave length will be comparable with the thickness τ of the layer, and the frequency will be consequently excessively great. The longest wave will be that which crosses the layer and gets back again in a period, so that its wave length is 2τ and there will be harmonics of wave lengths τ, $\frac{2}{3}\tau$, etc. For the other type—which we are now to examine—the waves advance along the ellipsoidal surface, the electric force being normal to the ellipsoid and the magnetic force tangential.

The electric force R is normal to the spheroid. Let the line element on the spheroid be given by $ds^2=\dfrac{dx'^2}{h_1^2}+\dfrac{d\phi^2}{h_2^2}$.



Then, on applying the fundamental circuital laws of Faraday and Ampère to the elements $ABEF$ and $ABCD$, we get $\left\{\text{with help of relation } w=\dfrac{K}{4\pi}\dfrac{\partial R}{\partial t}\right\}$,

$$\mu K\frac{\partial^2 R}{\partial t^2}=\mu\,.\,h_1 h_2\left[\frac{\partial}{\partial x'}\left(\frac{h_1}{h_2}\cdot\frac{1}{\mu\tau}\frac{\partial(\tau R)}{\partial x'}\right)+\frac{\partial}{\partial\phi}\left(\frac{h_2}{h_1}\cdot\frac{1}{\mu\tau}\frac{\partial(\tau R)}{\partial\phi}\right)\right].$$

Owing to the symmetry about the axis τR is independent of ϕ and our equation becomes

$$\mu K\frac{\partial^2 R}{\partial t^2} = \mu h_1 h_2 \frac{\partial}{\partial x'}\left(\frac{h_1}{h_2}\cdot\frac{1}{\mu\tau}\frac{\partial \tau R}{\partial x'}\right)$$

$$= h_1 h_2 \frac{\partial}{\partial x'}\left(\frac{h_1}{h_2}\frac{1}{\tau}\frac{\partial(\tau R)}{\partial x'}\right) \quad\text{......................................(1),}$$

if μ is constant, as we shall suppose.

Now
$$h_1^2 = \frac{p'^2}{h^4 x'^2} = \frac{1-x'^2}{h^2(x^2-x'^2)}; \quad h_2^2 = \frac{1}{h^2(x^2-1)(1-x'^2)},$$

and for a homœoid $\tau = \alpha p$ where α is a constant, or $\tau = \alpha . \dfrac{hx\sqrt{x^2-1}}{\sqrt{x^2-x'^2}}$.

Substituting in (1) we get

$$\mu K\frac{\partial^2 R}{\partial t^2} = \frac{1}{h^2\sqrt{x^2-x'^2}}\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial}{\partial x'}\frac{R}{\sqrt{x^2-x'^2}};$$

or if
$$\psi = R/\sqrt{x^2-x'^2}, \quad \mu K h^2(x^2-x'^2)\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'}.$$

Putting $c = \dfrac{1}{\sqrt{\mu K}}$ so that c is the velocity of propagation and taking ψ to vary as $e^{i\kappa ct}$, we get

$$\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'} + \lambda^2(x^2-x'^2)\psi = 0,$$

where $\lambda = h\kappa$.

This is exactly the same equation as we had to deal with in considering the vibrations of spheroidal layers of air. The solution is of course the same as before: $p = \lambda^2 x^2 = \kappa^2 a^2$ being determined in the case of a *complete* spheroid by the condition of convergence at the poles. The only difference is that c, the velocity of propagation, is different for the two problems. The *wave lengths* are the same in both cases; and have been determined in some of the more important cases on p. 92 *et seq.*

If the surface is not complete, but is bounded by a parallel of latitude x', we must consider the condition at the edge. The layer of dielectric being supposed very thin, it is clear that the current (i) at the edge will be almost entirely tangential (along the parallel of latitude). Thus the boundary condition may be taken to be the vanishing of the component of the *magnetic* force along the parallel of latitude or $\dfrac{\partial R}{\partial x'} = 0$ at the edge.

Since $R=\sqrt{x^2-x'^2}\,.\,\psi$, the boundary condition becomes

$$(x^2-x'^2)\frac{\partial\psi}{\partial x'}-x'\psi=0.$$

At *the equator* $x'=0$, and the boundary equation is then $\frac{\partial\psi}{\partial x'}=0$, so that the conditions are exactly the same as in the sound problem with a *closed edge*. See p. 96.

Vibrations of air contained in a spheroidal envelope.

If ψ is the velocity potential our characteristic equation is $\frac{\partial^2\psi}{\partial t^2}=c^2V^2\psi$, to solve which we take ψ to vary as $e^{i\kappa ct}$ and proceed as on p. 79.

If we are dealing with a *complete* spheroidal surface we must confine ourselves to the f functions, and take

$$\psi=\Sigma A_n f_n(x) f_n(x')\, e^{i\kappa ct}.$$

At a rigid boundary $\frac{\partial\psi}{\partial x}=0$, so that the boundary condition is $f_n'(x_0)=0$, which determines the admissible values of κ.

We have already shown p. 87 how the roots of this equation can be obtained in any case. The gravest note will be the fundamental one of the type $y=f_1(x)f_1(x')\,e^{i\kappa ct}$, the values of κ being determined by roots of $f_1'(x_0)=0$.

On p. 88 we have given the lowest root of this equation for different values of the eccentricity of the bounding spheroid. We thus get these results:—

e	κa	$a\times$ frequency	wave length/a	Ratio of frequency to case $e=0$
0	2·0815	10950	3·0186	1
0·1	2·0825	10955	3·0172	1·0005
0·2	2·0848	10960	3·0158	1·0010
0·3	2·0865	10975	3·0117	1·0024
0·4	2·0902	10993	3·0061	1·0042
0·5	2·0961	11026	2·9978	1·0070
0·6	2·1035	11064	2·9876	1·0104
0·7	2·1124	11113	2·9745	1·0150
0·8	2·1233	11170	2·9594	1·020
0·9	2·1364	11240	2·9411	1.0264

The most striking thing about these results is the exceedingly small influence of moderate eccentricity on the frequency.

The interval between the fundamental notes for a sphere and a spheroid of eccentricity $\frac{6}{10}$ is less than a comma.

For the sphere to sound the middle c of a piano (c') its radius must be

$$a = 41{\cdot}487 \text{ centimetres} = 16\tfrac{1}{3} \text{ inches.}$$

If the eccentricity is $\frac{1}{2}$ then $a = 41{\cdot}764$ centimetres $= 16{\cdot}444$ inches.

" " $\frac{9}{10}$ " $a = 42{\cdot}57$ " $= 16{\cdot}761$ "

If we wish to find *the motion of the enclosed air due to a given normal motion of the bounding spheroid,* we have

$$\frac{\partial\psi}{\partial x} = \alpha\,.f(x'\kappa)\,e^{i\kappa ct} \text{ (say) when } x = x_0;$$

and so we assume $\psi = Af(x')f(x)\,e^{i\kappa ct}$, the boundary condition being $Af'(x_0) = \alpha$, $\therefore\ A = \dfrac{\alpha}{f'(x_0)}$ and our solution becomes $\psi = \alpha\dfrac{f(x)}{f'(x_0)}f(x')\,e^{i\kappa ct}$.

To determine the motion *of air contained between two confocal spheroids,* we must use two *independent* solutions of our fundamental equation, neither of which become infinite anywhere between the spheroids.

We may take $$\psi = [Af(x) + B\phi(x)]f(x')\,e^{i\kappa ct}.$$

When $x = x_1$ and $x = x_2$, we must have $\dfrac{\partial\psi}{\partial x} = 0$. Hence

$$Af'(x_1) + B\phi'(x_1) = 0, \quad Af'(x_2) + B\phi'(x_2) = 0.$$

Eliminating $A : B$ we get

$$\begin{vmatrix} f'(x_1), & \phi'(x_1) \\ f'(x_2), & \phi'(x_2) \end{vmatrix} = 0,$$

which is the frequency equation to determine κ.

On the communication of vibrations from a spheroid to the surrounding gas.

We shall suppose the disturbance due to a periodic normal motion of the surface $x = x_0$. If the normal displacement is $\phi(x')\,e^{i\kappa ct}$ we can expand $\phi(x')$ in a series of $f(x')$ functions, so that at the surface $x = x_0$ we have to deal with a boundary condition of the form $\dfrac{\partial\psi}{\partial x} = f(\lambda x')\,e^{i\kappa ct}$.

To represent the motion in the surrounding gas we want a solution appropriate to *divergent* waves; and so take $\psi = A\phi^-(z)f(z')\,e^{i\kappa ct}$, where $z = \lambda x$, $z' = \lambda x'$.

The boundary condition gives $1 = \lambda A\phi'^-(z_0)$ so that

$$\psi = \frac{\phi^-(z)}{\lambda\phi'^-(z_0)}f(z')\,e^{i\kappa ct}.$$

At a great distance from the spheroid z is very large, so that $\phi^-(z)=b_0\dfrac{e^{-iz}}{z}$ approximately.

Hence at a great distance from the spheroid we may take

$$\psi=\frac{b_0}{\lambda\phi'^-(z_0)}f(z')\frac{e^{i(\kappa ct-z)}}{z}$$

$$=\frac{b_0}{\lambda\phi'^-(z_0)}f(z')\frac{e^{i\kappa(ct-a)}}{\kappa a},$$

where a is the semi-major axis of the confocal ellipse through the point in question.

In estimating the energy emitted by the vibrating spheroid we may note that since there can be *on the whole* no accumulation of energy between two spheroidal surfaces, the energy transmitted across any of the confocal surfaces must be independent of z. Thus we may if we choose take the particular case when z is very great for estimating the transmission of energy across any one of the confocal spheroidal surfaces.

Now we have just seen that when z is very large we may take $\psi=Af(z')\dfrac{e^{i(\kappa ct-z)}}{z}$ where $A=b_0/\lambda\phi^{-1}(z_0)$.

The energy transmitted across a sphèroid z up to time t is

$$W=\int dt\iint-\rho\dot{\psi}\frac{\partial\psi}{\partial n}dS.$$

With the usual notation $pdp=ada=h^2xdx$; $p'dp'=h^2x'dx'$,

$$dS=2\pi y\,.\,dp'=2\pi\frac{bb'}{h}dp'=2\pi h\sqrt{(x^2-1)(1-x'^2)}\,.\,dp'.$$

But $$p=hx\sqrt{\frac{x^2-1}{x^2-x'^2}};\quad p'=hx'\sqrt{\frac{1-x'^2}{x^2-x'^2}};\quad\therefore\ \frac{p}{p'}=\frac{x}{x'}\sqrt{\frac{x^2-1}{1-x'^2}},$$

$$\frac{\partial\psi}{\partial n}dS=\frac{\partial\psi}{\partial p}dS=\frac{p}{h^2x}\frac{\partial\psi}{\partial x}2\pi h\sqrt{(x^2-1)(1-x'^2)}\,.\,\frac{h^2x'dx'}{p'}$$

$$=2\pi h(x^2-1)\frac{\partial\psi}{\partial x}dx'.$$

$$\therefore\quad W=\int dt\int_{-1}^{1}-\rho\dot{\psi}2\pi h(x^2-1)\frac{\partial\psi}{\partial x}dx'.$$

As already explained, we may take the expressions for $\dot{\psi}$ and $\dfrac{\partial\psi}{\partial x}$ corresponding to *large* values of x.

This gives $$\dot{\psi}=i\kappa cAf(z')\frac{e^{i(\kappa ct-z)}}{z},$$

$$\frac{\partial\psi}{\partial x}=\lambda\frac{\partial\psi}{\partial z}=-i\lambda\,.\,Af(z')\frac{e^{i(\kappa ct-z)}}{z},$$

when z is large.

Hence $$W=\int dt\int_{-1}^{1}\rho\kappa c\lambda\overline{A^2f(z')}|^2\frac{e^{2i(\kappa ct-z)}}{z^2}\,.\,2\pi h\left(\frac{z^2}{\lambda^2}-1\right)dx'.$$

If we integrate $\cos m\kappa(ct-a)$ {m an integer} with respect to t over any number of periods we get zero: while the integral of $\cos^2\kappa(ct-a)$ or $\sin^2\kappa(ct-a)$ is $\frac{1}{2}t$, where t is the time. Thus integrating over a long range of time—or rather over a range including a number of periods—we have

$$W=\tfrac{1}{2}\,.\,t\,.\,\rho\kappa c\lambda\,|A|^2\,2\pi h\left(\frac{1}{\lambda^2}-\frac{1}{z^2}\right)\int_{-1}^{1}\{f(\lambda x')\}^2\,dx',$$

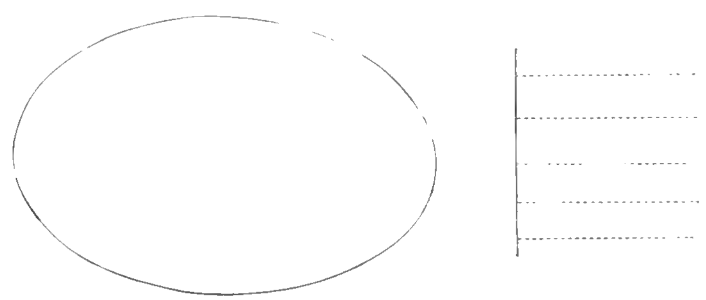
where $|A|$ is the modulus of $A=\text{mod.}\ \dfrac{b_0}{\lambda\phi^{-1}(z_0)}$.

Also since we are supposing z very large $\dfrac{1}{z^2}$ may be neglected and we get (remembering that $\lambda=h\kappa$)

$$W=\pi\rho c\,|A|^2\int_{-1}^{1}\{f(\lambda x')\}^2\,dx'.$$

Scattering of waves by an obstructing spheroid.

Suppose a series of plane waves with their fronts perpendicular to the axis of a



spheroid move towards the spheroid and are scattered by it. The velocity potential of the impinging wave is

$$\phi=e^{i\kappa(ct+x_1)}=e^{i\kappa ct}\,e^{i\lambda xx'},\ \text{see p. 89},$$
$$=e^{i\kappa ct}\,\Sigma A_nf_n(\lambda x)f_n(\lambda x'),$$

where the coefficients A_n are determined as on p. 90.

For the *scattered* wave we take $\psi=e^{i\kappa ct}\,\Sigma B_n\phi_n^{-}(\lambda x)f_n(\lambda x')$.

At the surface $x=x_0$ we must have

$$\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial x}=0,$$

$$\therefore\ A_nf_n'(\lambda x_0)+B_n\phi_n^{-1}(\lambda x_0)=0,$$

which determines B_n, so that the velocity potential of the scattered wave is known.

At a distance from the spheroid we may take $\phi_n^{-}(\lambda x)=b_0\dfrac{e^{-i\lambda x}}{\lambda x}$ giving

$$\psi=\frac{b_0e^{i(\kappa ct-\lambda x)}}{\lambda x}\,\Sigma B_nf_n(\lambda x').$$

We shall now turn to a brief discussion of the *electrical oscillations* over the surface of a *spheroid.*

Consider the case where the currents are in meridian planes through the axis of the spheroid and the magnetic force in parallels of latitude. If a, b, c denote the components of the magnetic induction parallel to rectangular fixed axes, c being along the axis of the spheroid, we have $c=0$, and

$$a=\sin\phi\,.\,\chi(xx'),$$
$$b=-\cos\phi\,.\,\chi(xx').$$

If as usual we take all variables proportional to e^{ipt} we have $(\nabla^2+\kappa^2)\,a=0$ in a dielectric, where $\kappa=p/V$, V being the velocity of propagation in the dielectric; and $(\nabla^2+\kappa'^2)\,a=0$ in a conductor, where $\kappa'^2=-4\pi\mu ip/\sigma$, the conductor being of specific resistance σ and magnetic permeability μ.

Since in the present case $a=\sin\phi\,.\,\chi(xx')$, we see that χ is a solution of the equation on p. 80 for the case $n=1$.

Thus *inside* the spheroid we have

$$a=A\sin\phi\,.\,f(\lambda'x)f(\lambda'x')\,e^{ipt},$$
$$b=-A\cos\phi\,.\,f(\lambda'x)f(\lambda'x')\,e^{ipt},$$
$$c=0.$$

And *outside* (in the dielectric)

$$a=B\sin\phi\,.\,\Phi^-(\lambda x)f(\lambda x')\,e^{ipt},$$
$$b=-B\cos\phi\,.\,\Phi^-(\lambda x)f(\lambda x')\,e^{ipt},$$
$$c=0.$$

Since the tangential magnetic force is continuous we have

$$\frac{A}{\mu}f(\lambda'x_0)=B\Phi^-(\lambda x_0),$$

$x=x_0$ being the bounding spheroid.

Another surface condition is got from the consideration that the electromotive intensity parallel to the spheroidal surface is continuous. Now the oscillations of the surface distribution of electricity are very rapid for ordinarily sized spheroids—the wave length being comparable with a diameter of the spheroid, and the velocity of propagation very great.

But we know that in the case of very rapid oscillations the disturbance is confined to a very thin 'skin'.

Inside this skin there is no E.M.F. [or there would be an electrical disturbance, which does *not* occur]. Hence for continuity of E.M.F. we see that the tangential E.M.F. vanishes at the *surface of the dielectric.*

To get an expression for the E.M.F. we make use of the circuital relation of Ampère, that the line integral of the magnetic force round a circuit $=4\pi\times$ current through the circuit.

Apply this to the ring-shaped circuit in the figure and we find, if H is the resultant magnetic force and C the intensity of the current,

$$4\pi C \,.\, 2\pi y \,.\, dp = \frac{d}{dx}(2\pi y \,.\, H)\, dx.$$

Now the E.M.F. parallel to C in the dielectric is $\frac{4\pi}{ipK}C$, and we have seen that this vanishes, so that our surface condition is

$$\frac{d}{dx}(yH) = 0,$$

when $x = x_0$, and
$$y = h\sqrt{(x^2-1)(1-x'^2)},$$

$$\therefore\ \frac{d}{dx_0}[\sqrt{x_0{}^2 - 1}\,\Phi^-(\lambda x_0)] = 0.$$

Now from p. 88 we see that

$$\phi^-(\lambda x) = \frac{\sqrt{x^2-1}}{\lambda^2 x^2}\left[b_0 - \frac{ib_1}{\lambda x} - \frac{b_2}{\lambda^2 x^2}\dots\right] e^{-i\lambda x},$$

$$\therefore\ \sqrt{x^2-1}\,\phi^-(\lambda x) = \frac{x^2-1}{\lambda^2 x^2}\left[b_0 + \frac{ib_1}{\lambda x} - \dots\right] e^{-i\lambda x},$$

so putting $z = \lambda x_0$, the frequency equation becomes

$$\frac{d}{dz}\cdot\left[\left(1-\frac{\lambda^2}{z^2}\right)\left(b_0 + \frac{ib_1}{z} - \dots\right) e^{-iz}\right] = 0.$$

In the case of the sphere, $\lambda = 0$ and the series for $\phi_n^-(\lambda x)$ are finite, so that the problem is very easy. The most important case corresponds to $n = 0$, giving $p_0 = 2$, $b_1 = -b_0$, b_2 and all other b's vanish, so that $\frac{d}{dz}\left(1 - \frac{i}{z}\right)e^{-iz} = 0$, $z^2 - iz - 1 = 0$, and

$$z = \lambda x = \kappa a = \frac{i+\sqrt{3}}{2}.$$

Thus $e^{ipt} = e^{i\kappa Vt} = e^{-\frac{Vt}{2a}} e^{i\frac{\sqrt{3}V}{2a}t}$. Thus the frequency is $\frac{V\sqrt{3}}{4\pi a}$ and the modulus of decay is $\frac{2a}{V}$, so that for ordinary sized spheres the vibrations are almost "dead beat."

For values of λ other than zero the question is not so easily answered. The modulus of z for the case of the sphere just considered is unity, and in the general case the series $b_0 + \frac{ib_1}{z} + \dots$ converges only for large values of the argument. For small values of the eccentricity we can get z from this series sufficiently accurately, as the coefficients b_2, b_3 ... will be very small; but for larger values of e it will be necessary to replace this descending series by the *ascending* ones f and F, in the manner indicated on p. 87.

If we wish to consider the oscillations between two confocal spheroids (conductors) we must take, in the dielectric between the conductors,

$$a = \sin\phi \,.\, f(\lambda x')\, e^{ipt}\, [C\chi^+(\lambda x) + D\chi^-(\lambda x)],$$

$$b = -\cos\phi \ldots;\ c = 0.$$

By the same argument as before we must have $\frac{d}{dx}(yH) = 0$ when $x = x_1$ and $x = x_2$ (the bounding surfaces),

$$\therefore\ \frac{d}{dx_1}\{[C\chi^+(\lambda x_1) + D\chi^-(\lambda x_1)]\sqrt{x_1^2-1}\} = 0;\quad \frac{d}{dx_2}\{[C\chi^+(\lambda x_2) + D\chi^-(\lambda x_2)]\sqrt{x_2^2-1}\} = 0,$$

and eliminating $C:D$ from these two equations we get an equation to determine the frequency and modulus of decay. The case of a thin homœoid has already been discussed on p. 96.

On the decay of electric currents in conducting spheroids.

First take the case when the lines of magnetic force are parallels of latitude. Then just as on p. 102 the tangential magnetic force in the conductor is $Af(\lambda' x)f(\lambda' x')e^{ipt}$, where $ip = -\frac{\sigma\kappa'^2}{4\pi\mu}$. This must be continuous at the surface of the conductor, and if we neglect displacement currents it is zero in the dielectric. Thus the admissible values of κ are given by $f(\lambda' x_0) = 0$, where $x = x_0$ is the surface of the conductor.

We have already shown how the roots of this equation are to be found. The tangential magnetic force is $\Sigma A_s f(\lambda_s' x) f(\lambda_s' x')\, e^{-\frac{\sigma\kappa_s'^2}{4\pi\mu}t}$, where the summation extends to the different values of κ_s given by the roots of $f_n(\lambda' x_0) = 0$. The constants A_s are determined in terms of their initial values by the usual process (see p. 88).

Next take the case when the currents are in parallels of latitude. If P, Q, R denote the components of E.M.F. (corresponding to a, b, c of last problem) we have:—

Inside the spheroid
$$P = A\sin\phi\, f(\lambda' x) f(\lambda' x')\, e^{ipt},$$
$$Q = -A\cos\phi \ldots;\ R = 0.$$

Outside the spheroid (in the dielectric)
$$P = B\sin\phi\Phi^-(\lambda x) f(\lambda x')\, e^{ipt},$$
$$Q = -B\cos\phi\ldots;\ R = 0.$$

The continuity of the E.M.F. parallel to the spheroidal surface gives
$$Af(\lambda' x_0) = B\Phi^-(\lambda x_0).$$

Another "boundary equation" is got by considering the fact that the magnetic induction tangential to a meridian is also continuous. By making use of the circuital relation of Faraday and proceeding just as on p. 103 we get in this way

$$\frac{A}{\mu}\frac{d}{dx_0}\left[\sqrt{x_0^2-1}\,f(\lambda' x_0)\right] = B\frac{d}{dx_0}\left[\sqrt{x_0^2-1}\,\Phi^-(\lambda x_0)\right].$$

Eliminating the ratio $A:B$ from the two boundary equations we get

$$\frac{\dfrac{d}{dx_0}\left[\sqrt{x_0^2-1}\,f(\lambda' x_0)\right]}{\mu f(\lambda' x_0)} = \frac{\dfrac{d}{dx_0}\left[\sqrt{x_0^2-1}\,\Phi^-(\lambda x_0)\right]}{\Phi^-(\lambda x_0)}.$$

This equation can be considerably simplified by noting that the currents will decay so slowly that p/V must be a very small quantity. Thus λ is in this case exceedingly small.

When $\lambda = 0$ the equation on p. 88 becomes

$$z^2\frac{d^2u}{dz^2} + 2(z^2+2z)\frac{du}{dz} + (4z - p + 2)u = 0,$$

while $p = n(n+1)$.

If we solve this in *ascending* series in the form $u = a_m z^{-m} + a_{m+1} z^{-m+1} + \ldots$, we have as the indicial equation $m^2 - 3m + 2 - n(n+1) = 0$, so that $m = n+2$, and we have $y_1 = \alpha \cdot \dfrac{e^{-i\lambda x}}{(\lambda x)^{n+2}}$ approximately when λ is very small.

Hence when λ is very small we have

$$\Phi_n^-(\lambda x) = \alpha\sqrt{x^2-1}\,\frac{e^{-i\lambda x}}{(\lambda x)^{n+2}},$$

and

$$\frac{\dfrac{d}{dx}\left[\sqrt{x^2-1}\,\Phi_n^-(\lambda x)\right]}{\Phi_n^-(\lambda x)} = -\frac{nx}{\sqrt{x^2-1}}.$$

So that the frequency equation becomes

$$(1+n\mu)x_0 f_n(\lambda' x_0) + \lambda'(x_0^2-1)f_n'(\lambda' x_0) = 0,$$

and when μ is very great this is practically equivalent to $f_n(\lambda' x_0) = 0$.

Various problems on the *conduction of heat* in spheroids may be discussed on the same lines as above.

For an isotropic solid of specific heat c, density ρ, and uniform conductivity κ_1, the equation of conduction is $\dfrac{\partial v}{\partial t} = \dfrac{\kappa_1}{c\rho}\nabla^2 v$, where v is the temperature.

Suppose we have a spheroid with its surface $x = x_0$ kept at uniform temperature v_0 and wish to ascertain how the heat diffuses into the interior. We take $v - v_0$ to vary as $e^{-\frac{\kappa_1}{c\rho}\kappa^2 t}$ and so get $(\nabla^2 + \kappa^2)(v - v_0) = 0$.

$$\therefore\ v - v_0 = \Sigma A_n f_n(\lambda x) f_n(\lambda x') e^{-\frac{\kappa_1}{c\rho}\kappa^2 t}.$$

which represents a temperature gradually approaching v_0 everywhere. The surface condition gives $f_n(\lambda x_0)=0$, and so determines the admissible values of κ.

Initially (when $t=0$) we have $v=v_0+\Sigma A_n f_n(\lambda x) f_n(\lambda x')$, and to determine the constants A_n we should require to expand the initial temperature in a series of this form. (See p. 88.)

For example suppose that initially $v=v_0+A f_0(\lambda x) f_0(\lambda x')$, then at time t we should have $v=v_0+A f_0(\lambda x) f_0(\lambda x') e^{-\frac{\kappa_1}{c\rho}\kappa^2 t}$. The 'modulus of decay' is $\frac{c\rho}{\kappa_1\kappa^2}=\frac{c\rho a^2}{\kappa_1 \,.\, (\kappa a)^2}$. The values of κa are the different roots of $f_0(\kappa a)=0$. The decay is slowest for the smallest root. Calculating this by the method of p. 87, and taking $\frac{\kappa_1}{c\rho}=1{\cdot}13$ for copper and $=0{\cdot}22$ for iron, we are led to these results:—

e	κa	Modulus/a^2 in seconds for *Copper*	Modulus/a^2 in seconds for *Iron*	Ratio to case $e=0$
0	3·1416	·08964	·4615	1
0·1	3·1538	·08853	·4547	0·9877
0·2	3·1902	·08696	·4467	0·9701
0·3	3·2528	·08381	·4295	0·9328
0·4	3·3476	·07898	·4057	0·8810
0·5	3·4785	·07319	·3760	0·8166

If the surface temperature is a given harmonic function of the time, say $v_0 e^{i\sigma t}$, we take v to vary as $e^{i\sigma t}$ and get $(\nabla^2+\kappa^2)v=0$, where $\kappa^2=-i\sigma\,.\,\frac{c\rho}{\kappa_1}$.

To represent waves of heat travelling inwards from the surface we take

$$v=\Sigma A_n \phi_n^+(\lambda x) f_n(\lambda x') e^{i\sigma t}.$$

The constants A_n are determined from the surface condition which makes

$$v_0=\Sigma A_n \phi_n^+(\lambda x_0) f_n(\lambda x').$$

Suppose the surface temperature is $v_0 f_0(\lambda x') e^{i\sigma t}$, where v_0 is a constant. [We know that when $\lambda=0$, $f_n(\lambda x')=P_n(x')$, so that for small values of λ, $f_0(\lambda x')$ differs very little from a constant.]

Then the temperature at any point of the spheroid is given by

$$v=v_0\frac{\phi_0^+(\lambda x)}{\phi_0^+(\lambda x_0)} f_0(\lambda x') e^{i\sigma t}.$$

We have $$\kappa^2 = -i\sigma\frac{c\rho}{\kappa_1} = -i\sigma m \text{ say},$$

$$\therefore \quad \kappa = \sqrt{\frac{m\sigma}{2}}(1-i); \qquad z = \lambda r = \kappa a = \sqrt{\frac{m\sigma}{2}}(1-i)\,a; \quad \lambda = \frac{a}{r}\sqrt{\frac{m\sigma}{2}}(1-i).$$

Now $$\Phi_0^+(\lambda x) = \Phi^+_0(z) = \frac{e^{iz}}{z}\left(1 - \frac{ib_1}{z} - \frac{b_2}{z^2}\ldots\right) = \frac{e^{iz}}{z}Re^{i\psi} \text{ say},$$

and $$\Phi_0^+(\lambda r_0) = \frac{e^{iz_0}}{z_0}R_0 e^{i\psi_0}.$$

So $$v = v_0\frac{z_0}{z}\cdot\frac{R}{R_0}e^{i[(z-z_0)+\psi-\psi_0+\sigma t]}f_0(\lambda x')$$

$$= v_0\frac{a_0 R}{aR_0}e^{-\sqrt{\frac{m\sigma}{2}}(a_0-a)}f_0(\lambda x')\,.\,e^{i\left[-\sqrt{\frac{m\sigma}{2}}(a_0-a)+\psi-\psi_0+\sigma t\right]}.$$

This represents a wave moving inwards from the surface with velocity

$$\sqrt{\frac{2\sigma}{m}} = \sqrt{\frac{2\sigma\kappa_1}{c\rho}}.$$

The phase is not the same throughout, the change at the surface a from that at a_0 being $\psi - \psi_0$. The amplitude also diminishes as we go inwards, its ratio at a to that at the surface a_0 being $\frac{a_0 R}{aR_0}\cdot e^{-\sqrt{\frac{m\sigma}{2}}(a_0-a)}$.

Making use of the relations connecting the coefficients b_0, $b_1\ldots$ on p. 86, and using the value of p_0 on p. 82 we easily get,

$$Re^{i\psi} = 1 - \left(\frac{1}{3a}\sqrt{\frac{m\sigma}{2}} - \frac{1}{12a^2}\right)a_0^2e^2 - \frac{(m\sigma)^{\frac{3}{2}}}{135\sqrt{2}\,.\,a}a_0^4e^4\ldots$$

$$- i\frac{a_0^2e^2}{3a}\sqrt{\frac{m\sigma}{2}}\left[1 - \left(m\sigma + \frac{8\sqrt{2m\sigma}}{a}\right)\frac{a_0^2e^2}{45} + \frac{2m^2\sigma^2a_0^4e^4}{3^4\,.\,5\,.\,7}\ldots\right],$$

and of course $R_0e^{i\psi_0}$ is got from this by writing a_0 for a.

Thus $$\tan\psi = -\frac{\frac{a_0^2e^2}{3a}\sqrt{\frac{m\sigma}{2}}\left[1 - \left(m\sigma + \frac{8\sqrt{2m\sigma}}{a}\right)\frac{a_0^2e^2}{45} + \frac{2m^2\sigma^2a_0^4e^4}{3^4\,.\,5\,.\,7}\ldots\right]}{1 - \left(\frac{1}{3a}\sqrt{\frac{m\sigma}{2}} - \frac{1}{12a^2}\right)a_0^2e^2 - \frac{(m\sigma)^{\frac{3}{2}}}{135\sqrt{2}\,.\,a}a_0^4e^4\ldots},$$

and $$R^2 = \left[1 - \left(\frac{1}{3a}\sqrt{\frac{m\sigma}{2}} - \frac{1}{12a^2}\right)a_0^2e^2 - \text{etc.}\right]^2 + \frac{a_0^4e^4}{9a^2}\frac{m\sigma}{2}[1 - (\quad)\ldots]^2.$$

The quantity $\frac{1}{m}$ is the "thermal diffusivity" of the substance. For copper we may take $\frac{1}{m} = 1{\cdot}13$ and for iron $\frac{1}{m} = 0{\cdot}22$. We have seen that the velocity with which the wave travels inwards along the axis is $\sqrt{\frac{2\sigma}{m}}$. If the period were ten minutes, this would give a velocity of ·02367 centimetres per second for copper and ·004609 for iron.

The following table gives the change of phase and amplitude on the spheroid $a = 20$, the surface $a_0 = 30$ being exposed to a fluctuating temperature, the period 10 minutes.

Eccentricity	R/R_0	Ratio of amplitude at $a=20$ to that at $a_0=30$	Difference of phase $(\psi_0 - \psi)$ between $a=20$ and $a_0=30$
0	1	0·91074	0
0·1	0·9999	0·91064	4′
0·2	0·99954	0·91032	14′
0·3	0·99826	0·90915	28′
0·4	0·99761	0·90856	42′
0·5	0·99529	0·90644	50′

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CONTENTS.

IV. *Certain Systems of Quadratic Complex Numbers.*

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1. The object of this paper is to discuss the Theory of quadratic complex numbers from the point of view indicated by Prof. Klein in his *Lectures on Mathematics* (1894), Lecture VIII. on "Ideal Numbers."

A quantity ϕ which is a root of an equation

$$x^n + a_1 x^{n-1} + \dots + a_n = 0,$$

the coefficients $a_1, a_2, \dots a_n$ being rational numbers, is called an "algebraic number." (See Weber's *Lehrbuch der Algebra*, Vol. II., Chapters 16 and foll.) In particular, if the degree of the equation of lowest degree satisfied by ϕ is 2, ϕ is "an algebraic number of the second order," or more briefly, a "quadratic number." I do not propose to discuss non-integral quadratic numbers, and I shall therefore speak of "quadratic numbers," meaning thereby "quadratic integers." [ϕ is an "integral algebraic number" when $a_1, a_2, \dots a_n$ are all integers.]

Every quadratic number is then a root of an equation

$$x^2 + a_1 x + a_2 = 0,$$

where a_1 and a_2 are integers. Solving this equation, we obtain

$$\phi = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}.$$

Let $a_1^2 - 4a_2 = e^2 d$, where d does not contain a square; then

$$\phi = \frac{-a_1 + e\sqrt{d}}{2}.$$

Note that $d \equiv 1$, 2, or 3 (mod. 4), for if d were $\equiv 0$ (mod. 4) it would contain a square, contrary to hypothesis.

There are now two cases to be considered:—

(1) a_1 even: then e is even, and $d \equiv 1$ or 2 or 3 (mod. 4), and therefore ϕ is of the form $x + y\sqrt{d}$, where x and y are integers.

(2) a_1 odd: then e is odd, and $d \equiv 1$ (mod. 4), and therefore ϕ is of the form

$$\frac{x + y\sqrt{d}}{2},$$

where x and y are odd integers; this may also be written

$$\phi = x' + y'\frac{1 + \sqrt{d}}{2},$$

x' and y' being integers, and y' being odd. (In the volume above cited, pp. 601 and foll., Weber discusses these systems of quadratic numbers and obtains the above results.)

I shall only consider the case d negative; the resulting quadratic numbers are "complex," in the usual sense of the term. And I shall call numbers of the forms $x + y\sqrt{d}$ and $x + y\dfrac{1+\sqrt{d}}{2}$ numbers of the first and second "type" respectively. It will be convenient to use the symbol θ in lieu of $\sqrt{d}$, or of $\dfrac{1+\sqrt{d}}{2}$, so that for any assigned value of d, and whichever type is being considered, the general form of the numbers of the system is

$$x + y\theta.$$

In the first type, $$\theta^2 - d = 0;$$

in the second, $$\theta^2 - \theta + \frac{1-d}{4} = 0.$$

By a "system," I mean the totality of numbers of the form $x + y\theta$ of a given type, for a given value of d.

2. The product of any two numbers of a system is a third number of the system; in fact, for a system of the first type,

$$(x + y\theta)(x' + y'\theta) = (xx' + dyy') + (xy' + x'y)\theta,$$

and for one of the second type,

$$(x + y\theta)(x' + y'\theta) = \left(xx' - \frac{1-d}{4}yy'\right) + (xy' + x'y + yy')\theta.$$

Each system is therefore, for multiplication, complete in itself.

When however the question of factorisation is considered, a system is not necessarily complete in itself. E.g.

$$6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

Thus in the system of the first type, given by $d = -5$, the number 6 can be decomposed in two distinct ways into prime factors. In face of this difficulty, Dedekind invented his theory of "ideals" (Supplement XI. to Dirichlet's *Zahlentheorie*); and Weber, in his *Algebra*, follows the same method. This theory, owing partly to its generality and partly to the novelty of the conceptions introduced, is difficult; Klein's treatment introduces clearness and simplicity. Gauss' Composition Theory in the Theory of Binary Quadratic Forms is in fact the key to the factorisation of quadratic numbers. The connection between the Theory of Binary Quadratic Forms and the Theory of Quadratic Numbers is due to the fact that every principal form of a given determinant is the product of two conjugate quadratic numbers.

When $D \equiv 0$ (mod. 4) $= -4d$, the principal form of determinant D* is $x^2 - \frac{D}{4} y^2$, which is equal to $(x + y\sqrt{-d})(x - y\sqrt{-d})$.

And when $D \equiv 1$ (mod. 4) $= 1 - 4\delta$, the principal form of determinant D is

$$x^2 + xy + \delta y^2,$$

which is equal to

$$\left(x + y\frac{1+\sqrt{D}}{2}\right)\left(x + y\frac{1-\sqrt{D}}{2}\right).$$

Since D is to be taken negative in this paper, d and δ will henceforth denote positive integers.

The product of a quadratic number and its conjugate will be called the "norm" of that quadratic number. The norm is evidently a real positive integer.

The following notation will be convenient:—

$a, b, c, \ldots$ denote quadratic numbers, $a', b', c', \ldots$ their respective conjugates, and $A, B, C, \ldots$ their respective norms, and

$$A = aa' = N(a) = N(a').$$

If then

$$a = bc,$$

it follows that

$$A = BC,$$

so that to every multiplication of quadratic numbers, corresponds a composition of quadratic forms.

Now Gauss' law of Composition asserts, that if f and f' are two quadratic forms of the same determinant, then the product of any two integers representable by f and f' respectively is representable by a definite form F (which according to circumstances may belong to the same class as either f or f', or may belong to a different class to either). If A is an integer representable by the principal form, and if B and C are factors of A representable by forms belonging to classes other than the principal class, and if $A = BC$ it is evident that there corresponds the factorisations $a = bc$ and $a' = b'c'$, where b and b' are the linear factors of B regarded as a quadratic form, and similarly c and c' of C.

* As suggested by Klein, I write a quadratic form $ax^2 + bxy + cy^2$: its determinant is $D = b^2 - 4ac$.

To complete a system of quadratic numbers, it is therefore necessary to introduce as "numbers" of the system the linear factors of the representative forms of each class for the given determinant: the numbers thus introduced will be called "secondary," in contradistinction to the "principal" numbers originally defined: they will be denoted by $x\lambda + y\mu$, where x and y are any real integers.

3. It is clear from what precedes that the properties of a system of quadratic numbers are very intimately connected with the number of classes of Quadratic Forms of the corresponding determinant D. Where there is only one such class, the corresponding system of quadratic numbers of the form $x + y\theta$ is complete, not only for multiplication but also for factorisation. The negative values of D for which this is the case are

$$-3,\ -4,\ -7,\ -8,\ -11,\ -12,\ -16,\ -19,\ -27,\ -28,\ -43,\ -67,\ -163$$

(see table in Gauss' *Werke*, Bd. II. p. 450).

The first two of these cases are well known; $D = -3$ gives the numbers $x + y\rho$, where ρ is a cube root of unity, and $D = -4$ gives the numbers $x + y\iota$, ι being a fourth root of unity.

As my object is to consider the character of the secondary numbers, in the form in which Klein has presented them, I shall set aside the systems of quadratic numbers for the values of D above given, and shall devote the remainder of this paper to the consideration of the case which comes next in simplicity, when there are *two* classes only of quadratic forms.

The negative values of D for which this is the case are given by $-D = 15$, 20, 24, 32, 35, 36, 40, 48, 51, 52, 60, 64, 72, 75, 88, 91, 99, 100, 112, 115, 123, 147, 148, 187, 232, 235, 267, 403, 427. (Gauss, *loc. cit.*)

When $D = -15$, the forms representing the two classes are

$$x^2 + xy + 4y^2,$$

and

$$2x^2 + xy + 2y^2.$$

The former is equal to

$$\left(x + y\frac{1 + \sqrt{-15}}{2}\right)\left(x + y\frac{1 - \sqrt{-15}}{2}\right);$$

the latter may be written

$$5\left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2,$$

and hence it is

$$\left(x\frac{\sqrt{5} + \sqrt{-3}}{2} + y\frac{\sqrt{5} - \sqrt{-3}}{2}\right)\left(x\frac{\sqrt{5} - \sqrt{-3}}{2} + y\frac{\sqrt{5} + \sqrt{-3}}{2}\right);$$

and therefore

$$\lambda = \frac{\sqrt{5} + \sqrt{-3}}{2}, \quad \mu = \frac{\sqrt{5} - \sqrt{-3}}{2}.$$

The object of expressing the linear factors in this particular form is to ensure that the product of two secondary numbers should be a number of the system: this is now the case; for

$$\lambda^2 = \left(\frac{\sqrt{5}+\sqrt{-3}}{2}\right)^2 = \frac{1+\sqrt{-15}}{2} = \theta,$$

$$\lambda\mu = \left(\frac{\sqrt{5}+\sqrt{-3}}{2}\right)\left(\frac{\sqrt{5}-\sqrt{-3}}{2}\right) = 2,$$

and
$$\mu^2 = \left(\frac{\sqrt{5}-\sqrt{-3}}{2}\right)^2 = \frac{1-\sqrt{-15}}{2} = 1 - \frac{1+\sqrt{-15}}{2} = 1-\theta.$$

Also we have
$$\lambda\theta = \lambda - 2\mu, \quad \mu\theta = 2\lambda.$$

If, however, we took

$$2x^2 + xy + 2y^2 = \left(\sqrt{2}x + \frac{1+\sqrt{-15}}{2\sqrt{2}}y\right)\left(\sqrt{2}x + \frac{1-\sqrt{-15}}{2\sqrt{2}}y\right),$$

we should find that neither of these factors behaves as an integral number; for

$$\left(\frac{1+\sqrt{-15}}{2\sqrt{2}}\right)^2 = \frac{-7+\sqrt{-15}}{4} = \tfrac{1}{2}(-4+\theta),$$

which is a *non-integral* quadratic number.

In the Lecture VIII. before referred to, Klein discusses generally this question of the proper factorisation to obtain the secondary numbers, and states that "it is always possible to bring about the important result that the product of any two complex numbers" of the system of the principal and secondary numbers "will again be a complex number of the system, so that the totality of these complex numbers forms, likewise, for multiplication a complete system."

When $D = -20$, the forms representing the two classes are

$$x^2 + 5y^2,$$

and
$$2x^2 + 2xy + 3y^2.$$

The principal numbers are given by $\quad x + y\sqrt{-5},$

the secondary numbers by $\quad x\sqrt{2} + y\dfrac{1+\sqrt{-5}}{\sqrt{2}}.$

Therefore in this case

$$\lambda^2 = 2, \quad \lambda\mu = 1+\theta, \quad \mu^2 = -2+\theta, \quad \lambda\theta = -\lambda + 2\mu, \quad \mu\theta = -3\lambda + \mu;$$

the number conjugate to μ is $\quad \dfrac{1-\sqrt{-5}}{\sqrt{2}},$

$$\text{i.e. } \lambda - \mu, \text{ and } N(\mu) = \mu(\lambda - \mu) = 3.$$

This furnishes a full explanation of the paradox $6 = 2 \times 3 = (1+\theta)(1-\theta)$ mentioned above (§ 2). Neither 2, 3, $1+\theta$, nor $1-\theta$ are primes, but

$$2 = \lambda^2, \qquad 3 = \mu(\lambda - \mu),$$
$$1 + \theta = \lambda\mu, \quad 1 - \theta = \lambda(\lambda - \mu).$$

In like manner the values of λ and μ may be determined for the other values of D given above; it is, however, unnecessary to do so here, as the general theory of any of these systems of quadratic numbers is to a large extent independent of the particular numerical value of D; where this is not the case, I shall confine myself to the system $D = -20$. It should be observed that in general secondary numbers are not quadratic numbers as defined in § 1: the latter are those which have in § 2 been given the name of principal quadratic numbers. In the case $D = -20$, the only exceptions to this statement are the numbers $x\lambda$, whose square is $2x^2$, and $x\lambda\theta = x(-\lambda + 2\mu)$, whose square is $10x^2$, x being a real integer.

4. A number whose norm is 1 is called a "unity"; a secondary number cannot be a unity, for 1 is representable only by forms of the principal class.

If $\alpha_0 + \alpha_1\theta$ is a unity of a system of the first type, then $\alpha_0^2 + d\alpha_1^2 = 1$.

Since d is positive, and >1, the only solutions of this are $\alpha_0 = \pm 1$, $\alpha_1 = 0$.

Similarly for systems of the second type, we get

$$\alpha_0^2 + \alpha_0\alpha_1 + \delta\alpha_1^2 = 1.$$

Since $\delta > 1$, this gives only $\alpha_0 = \pm 1$, $\alpha_1 = 0$.

Therefore the unities of the systems now being considered are simply $+1$, and -1, just as in the theory of real numbers.

There are three kinds of primes in any of the systems:

(i) principal primes, which will be denoted by p, and whose norms P are prime numbers in the ordinary sense;

(ii) secondary primes, denoted by q, whose norms Q are also ordinary prime numbers; and

(iii) real primes, denoted by r, which are ordinary prime numbers not representable by either form of the determinant D. The norm of r is r^2.

The primes, as thus defined, are evidently indivisible into actual factors belonging to their system. It will now be proved that they are primes in the full sense, so that any number can be expressed as the product of these primes in one and only one way.

Let m be a quadratic number, principal or secondary, M its norm, m being such that M is odd, M_1 a prime factor (in the ordinary sense) of M; and let $M = M_1M_2$. Then M being a norm is representable by a quadratic form of determinant D, corresponding to the given system of quadratic numbers. It follows from the ordinary Theory of Numbers that all the factors of M are also representable by forms of the same determinant.

Let H, H_1, and H_2 be forms respectively representing M, M_1, and M_2. Then the theory of Composition supplies the algebraical identity

$$H(X,\ Y) = H_1(x,\ y) \times H_2(x',\ y'),$$

where X and Y are lineo-linear functions of x, y, x', y'.

Each linear factor of $H_1(x, y)$ therefore divides $H(X, Y)$, and therefore must divide one or other of the linear factors of $H(X, Y)$.

Each linear factor of $H(X, Y)$ is therefore the product of a linear factor of $H_1(x, y)$ and a linear factor of $H_2(x', y')$; each side of this equation may then, if necessary, be multiplied by the numerical factor required to bring the linear factors of H, H_1, and H_2 to the correct forms (see § 3); lastly, if X, Y, x, y, x', y' be given the values for which the forms H, H_1, and H_2 respectively represent the integers M, M_1, and M_2, the algebraic equation becomes numerical, and gives the factorisation of m in the form $m_1 m_2$. If m_2 is not prime, it can be similarly treated.

Thus finally we obtain a unique expression for m as a product of prime factors. A method is given later for actually carrying out the process of factorisation. The proof above given is not applicable to the factorisation of the determinant itself in quadratic numbers; in most cases the determinant has a unique factorisation, but when $\frac{1}{4}D$ (D being even) or D (if D is odd) contains a square, this is not the case; e.g. $D = -36$: then $9 = 3^2 = -\theta^2$.

Nor is it applicable to the factorisation of 2, though as a matter of fact, in most cases no exception to the general law of unique factorisation arises in connection with 2. Besides the cases where $\frac{D}{4}$ is a multiple of 4 (a particular case of $\frac{D}{4}$ containing a square) there is only one exception, which arises in the case $D = -60$; the quadratic numbers are then of the forms $x + y\sqrt{-15}$, and $x\sqrt{3} + y\sqrt{-5}$, and so

$$\lambda^2 - \mu^2 = 3 + 5 = 8,$$

and therefore the number 8 can be factorised in two distinct ways, $2 \times 2 \times 2$, and $(\lambda + \mu)(\lambda - \mu)$. In these anomalous cases Klein's method breaks down; it fails to give the ultimate prime factors. Apparently these cases can only be dealt with by Dedekind's method. Putting aside those values of D with which we cannot deal, the following values of $-D$ remain to be studied: 15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, 427.

The systems arising from the latter values of D can be divided into three sets:

(i) Those in which $2 = \lambda^2$; these are given by $-D = 20$, 24, 40, 52, 88, 148, and 232.

(ii) That in which $2 = \lambda\mu$, being the case $D = -15$.

(iii) Those in which 2 is a prime; these are given by $-D = 35$, 51, 91, 115, 123, 187, 235, 267, 403, 427.

The numerical values of θ, λ, and μ, and of their norms, and the values of $\lambda\theta$, $\mu\theta$, etc., are given for these 18 values of D in Tables I. and II., at the end of the paper.

5. All the elementary theorems of the ordinary Theory of Numbers in regard to prime numbers and divisibility are true for the 18 systems to which I now confine myself, since for each of them the law of unique factorisation exists. I shall therefore freely adopt, without definition, the technical terms, such as "modulus," "residue," "congruence," of the ordinary Theory of Numbers, and the usual notation connected therewith.

The operation of multiplication can be applied to any numbers of a system, whether secondary or principal. It follows at once from the theory of composition, applied to the case of determinants with two classes, that the product of two principal numbers, or of two secondary numbers, is a principal number, and that the product of a principal and a secondary number is a secondary number. Examples of these laws are given in § 3, for the cases $D = -15$, and -20. On the other hand addition only operates between two numbers of the same class, either both principal or both secondary. For the quantity obtained by adding together a principal and a secondary number does not belong to and has no necessary connection with the system of principal and secondary numbers, and must therefore be considered as irrelevant to the present subject.

6. Residues.—In accordance with the principle above stated, a principal number cannot be congruent to a secondary number. Complete sets of principal residues and of secondary residues to a given modulus are required: the most convenient complete sets are given by the following formulae, which relate to the case $D = -20$; the method of obtaining them is the same for each system, but the actual results differ slightly from each other.

I. Principal modulus, $n = g(x + y\theta)$, where x is prime to y; let M be $x^2 + 5y^2$, the norm of $x + y\theta$.

(1) Principal residues:—A complete set is given by $s + t\theta$, where

$$s = 0,\ 1,\ \ldots\ldots\ (Mg - 1),$$

$$t = 0,\ 1,\ \ldots\ldots\ (g - 1).$$

(2) Secondary residues:—A complete set is given by $s\lambda + t\mu$, where

$$s = 0,\ 1,\ \ldots\ldots\ \left(\frac{M}{k}g - 1\right),$$

$$t = 0,\ 1,\ \ldots\ldots\ (kg - 1),$$

k being 1 or 2, according as $x + y$ is odd or even, i.e. according as M is odd or even.

II. Secondary modulus, $n = g(x\lambda + y\mu)$, where x is prime to y. $M = 2x^2 + 2xy + 3y^2$, the norm of $x\lambda + y\mu$.

(1) Principal residues:—A complete set is given by $s + t\theta$, where

$$s = 0,\ 1,\ \ldots\ldots\ (Mg - 1),$$
$$t = 0,\ 1,\ \ldots\ldots\ (g - 1).$$

(2) Secondary residues:—A complete set is given by $s\lambda + t\mu$, where

$$s = 0,\ 1,\ \ldots\ldots\ \left(\frac{M}{k}g - 1\right),$$
$$t = 0,\ 1,\ \ldots\ldots\ (kg - 1),$$

k being 1 or 2, according as y is odd or even, i.e. according as M is odd or even.

It should be noticed that in all cases, the number of residues in a complete set is Mg^2, which is the norm of the modulus.

The proof of I. (2) will be given as a specimen. If $s\lambda + t\mu$ is a multiple of n, then

$$s\lambda + t\mu = g(x + y\theta)(u\lambda + v\mu).$$

Now $\lambda\theta = -\lambda + 2\mu$, $\mu\theta = -3\lambda + \mu$, so equating coefficients of μ on each side, we get

$$t = g\{v(x + y) + 2uy\};$$

so, k being 1 or 2, according as $x + y$ is odd or even,

$$t \equiv 0 \ (\text{mod. } kg) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i}).$$

Also, $$(s\lambda + t\mu)(x - y\theta) = gM(u\lambda + v\mu),$$

therefore $$s(x + y) + t(3y) \equiv 0 \ (\text{mod. } gM),$$
$$s(-2y) + t(x - y) \equiv 0 \ (\text{mod. } gM).$$

Find ξ and η, so that $(x + y)\eta - 2y\xi = k$; then we get

$$ks + t(3y\eta + x\xi - y\xi) \equiv 0 \ (\text{mod. } gM).$$

Now $M = x^2 + 5y^2$, where x and y are not both even, for they are coprime: if one is odd, the other even, M is odd, and $k = 1$; if both x and y are odd, M is even, and $k = 2$; in either case $M \equiv 0$ (mod. k).

Therefore, dividing the last congruence throughout by k,

$$s + \frac{t}{k}(3y\eta + x\xi - y\xi) \equiv 0 \left(\text{mod. } g\frac{M}{k}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{ii}).$$

Congruences (i) and (ii) are the necessary and sufficient conditions that $s\lambda + t\mu$ should be a multiple of n.

No two numbers of the set $s\lambda + t\mu$, where

$$s = 0,\ 1,\ \ldots\ldots\ \left(\frac{M}{k}g - 1\right),$$
$$t = 0,\ 1,\ \ldots\ldots\ (kg - 1),$$

are therefore congruent for the modulus n, and every other secondary number is congruent to one of them: in other words, this set is complete.

In the important case of the modulus being prime, the results are simpler, and admit of being stated generally for all the systems.

(i) Modulus n, a principal or secondary prime, whose norm is N.

Principal residues:—$0, 1, 2, \ldots\ldots, N-1$.

Secondary residues:—$0, \lambda, 2\lambda, \ldots\ldots, (N-1)\lambda$, provided that n is not λ.

(ii) Modulus r, a real prime.

Principal residues are given by $s+t\theta$,

Secondary residues by $s\lambda+t\mu$,

where
$$s=0, 1, 2, \ldots\ldots, r-1,$$
$$t=0, 1, 2, \ldots\ldots, r-1.$$

7. Factorisation of numbers.—Each real prime factor of a real number can either be represented by one of the quadratic forms connected with the system, in which case it is the product of two conjugate prime factors, or it is a real prime of the system.

To factorise $n=x+y\theta$, or $x\lambda+y\mu$, where x and y are coprime. Express N, the norm of n, as the product of real prime factors $A, B, \ldots\ldots$ Each of these numbers is representable by one of the quadratic forms connected with the system; therefore $A=aa'$, $B=bb'$, ..., where a, a', ... are primes of the system.

By the method shewn in the previous section, calculate the residue of n (mod. a); if this residue is 0, a is a factor of n; if it is not 0, a' must be a factor of n. Similarly the other factors of n are determined.

Example, in the system $D=-20$.
$$55(61+2\theta).$$
$$5=-\theta^2,\ 11 \text{ is a prime.}$$

The norm of $61+2\theta$ is $61^2+5.2^2=3741=3.29.43$.
$$3=\mu(\lambda-\mu),$$
$$29=3^2+5.2^2=(3+2\theta)(3-2\theta),$$
$$43=2.4^2+2.4.1+3.1^2=(4\lambda+\mu)(5\lambda-\mu).$$

If
$$\theta\equiv x \pmod{3+2\theta},$$
$$(\theta-x)(3-2\theta)\equiv 0 \pmod{29};$$
hence
$$\begin{cases}3x-10\equiv 0 \pmod{29}\\ 2x+\ \ 3\equiv 0 \pmod{29},\end{cases}$$
and so
$$x\equiv 13 \pmod{29}.$$

Then
$$61+2\theta\equiv 61+2x\equiv 87\equiv 0 \pmod{3+2\theta}.$$

Similarly $61+2\theta$ is a multiple of $\lambda-\mu$, and of $4\lambda+\mu$.

Therefore the number $55(61+2\theta)$ is $-11\theta^2(\lambda-\mu)(4\lambda+\mu)(3+2\theta)$.

8. Congruences.—Precisely as in the ordinary Theory of Numbers, Lagrange's theorem as to congruences may be proved:—a congruence of the nth degree, the modulus being prime, cannot have more than n incongruent roots.

And if such a congruence appears to have more than n incongruent roots, it is an identical congruence.

The linear congruence

$$ex \equiv f \pmod{n},$$

n being any number, and e being prime to n, has one and only one root.

E and N being the respective norms of e and n, real integers ξ and η can be found such that

$$E\eta - N\xi = 1.$$

Then since

$$E\eta \equiv 1 \pmod{n},$$

$$ee'\eta f \equiv f \pmod{n},$$

where e' is the conjugate of e.

The solution is therefore $x \equiv e'\eta f \pmod{n}$.

9. Fermat's Theorem.—I. For powers of a principal number h, prime to the modulus n, which is a prime of any one of the three kinds.

Let $K_1, K_2, \ldots K_{N-1}$ denote a complete set (except 0) of principal residues to the modulus n. Then $hK_1, hK_2, \ldots hK_{N-1}$ is also a complete set.

Therefore

$$h^{N-1}(K_1K_2 \ldots K_{N-1}) \equiv K_1K_2 \ldots K_{N-1} \pmod{n},$$

and so

$$h^{N-1} \equiv 1 \pmod{n}.$$

II. For powers of a secondary number j, prime to the modulus n.

In any system, a secondary number j_0 can be found whose square is a real integer, and which is prime to n.

In systems of the first type, provided n is odd, j_0 may be taken to be λ, and then $j_0^2 = 2$.

In systems of the second type, $j_0 = \lambda + \mu$, or $\lambda - \mu$, one or other of which is prime to n.

Then $j_0^2 = f$, where f may be found for each system from Table I.

Then

$$jK_1, jK_2, \ldots jK_{N-1},$$

and

$$j_0K_1, j_0K_2, \ldots j_0K_{N-1},$$

are two complete sets of secondary residues to the modulus n.

And therefore

$$j^{N-1} \equiv j_0^{N-1} \pmod{n}.$$

For systems of the first type, we get

$$\begin{aligned} j^{N-1} &\equiv 2^{\frac{N-1}{2}} \pmod{n} \\ &\equiv (2/N) \pmod{n} \\ &\equiv (-1)^{\frac{N^2-1}{8}} \pmod{n}. \end{aligned}$$

For systems of the second type,

$$\begin{aligned} j^{N-1} &\equiv f^{\frac{N-1}{2}} \pmod{n} \\ &\equiv (f/N) \pmod{n}. \end{aligned}$$

When a particular system is specified, the expression of Fermat's theorem can be simplified. Thus for the system $D=-20$, the theory of generic characters shews that P (the norm of an odd principal prime p), being an odd integer representable by the principal form, satisfies the congruence

$$P \equiv 1 \pmod{4};$$

and that Q (the norm of an odd secondary prime q) satisfies $Q \equiv -1 \pmod{4}$.

Applying the general theorem that has just been proved

$$j^{N-1} \equiv (-1)^{\frac{N^2-1}{8}} \pmod{n},$$

first to the case of $n=p$, we obtain

$$j^{P-1} \equiv +1 \text{ or } -1 \pmod{p},$$

according as

$$P \equiv 1 \text{ or } 5 \pmod{8}.$$

And therefore

$$j^{P-1} \equiv (-1)^{\frac{P-1}{4}} \pmod{p}.$$

Similarly

$$j^{Q-1} \equiv (-1)^{\frac{Q+1}{4}} \pmod{q},$$

and

$$j^{R-1} \equiv 1 \pmod{r},$$

since

$$R = r^2 \equiv 1 \pmod{8}.$$

10. In the theory of real numbers, if a is an odd number

$$a^2 \equiv 1 \pmod{8}.$$

I propose to consider the analogue of this for quadratic numbers, but here and in the sequel, whenever the properties of 2 are in question, I shall confine the discussion to the seven systems mentioned in § 4, for which $2=\lambda^2$. In these cases $\theta=\sqrt{-d}$ where $d=\frac{-D}{4}$.

Numbers will be called odd, semi-even or even, according as they are prime to λ, a multiple of λ but not of 2, or a multiple of 2. As above h denotes a principal number, j a secondary number.

If h is odd, $h \equiv 1 \pmod{\lambda}$,

that is $h = 1 + \lambda k$;

therefore
$$h^2 = 1 + 2\lambda k + 2k^2 \equiv 1 \pmod{2}.$$

To proceed further, with higher powers of λ for moduli, it is necessary to distinguish the cases where d is odd, from those where it is even.

First, the case d odd; then $\theta = \sqrt{-d}$ is odd.

So, h being odd $\equiv 1$ or $\theta \pmod{2}$,

that is $h = 1 + 2k$, or $\theta + 2k$;

therefore $h^2 = 1 + 4k + 4k^2$, or $-d + 4\theta k + 4k^2$.

Now, if k is odd, so are θk and k^2; and if k is a multiple of λ, so are θk and k^2: in either case $k^2 + k$ and $k^2 + \theta k$ are multiples of λ.

And so $h^2 \equiv 1$ or $-d \pmod{4\lambda}$,

according as $h \equiv 1$ or $\theta \pmod{2}$.

Similarly when d is even, and, as before, h is odd,
$$h^2 \equiv 1 \text{ or } 1 - d + 2\theta \pmod{4\lambda},$$
according as $h \equiv 1$ or $1 + \theta \pmod{2}$.

Similar results hold for squares of odd secondary numbers. First, the case d odd.

Then
$$\mu = \frac{1 + \sqrt{-d}}{\sqrt{2}}, \text{ so } \mu^2 = \frac{1 - d}{2} + \theta.$$

As before we get $j^2 \equiv \mu^2 \pmod{4\lambda}$, if $j \equiv \mu \pmod{2}$, and $j^2 \equiv (\lambda - \mu)^2 \pmod{4\lambda}$, if $j \equiv \lambda - \mu \pmod{2}$.

And so
$$j^2 \equiv \frac{1 - d}{2} + \theta, \text{ or } \frac{1 - d}{2} - \theta \pmod{4\lambda},$$
according as $j \equiv \mu$ or $\lambda - \mu \pmod{2}$.

Secondly, in the case d even $\mu = \sqrt{-\frac{d}{2}}$,

and so
$$j^2 \equiv -\frac{d}{2} \text{ or } 2 - \frac{d}{2} - 2\theta \pmod{4\lambda},$$
according as $j \equiv \mu$ or $\lambda - \mu \pmod{2}$.

It is worthy of notice that the squares of odd numbers (principal or secondary) in any one of these quadratic systems can be congruent only to 4 out of the 16 odd residues to the modulus λ^5, i.e. 4λ; a result remarkably similar to the corresponding fact in the system of real numbers, viz. that all odd squares are congruent to 1 out of the 4 odd residues to the modulus 2^3, i.e. 8.

It will be convenient for future use to gather up in tabular form these results and some other similar ones, for the particular system $D = -20$.

	Residue of Number to mod. 2	Residue of Square to mod. 4	Residue of Square to mod. 4λ
Principal numbers	0	0	0 or 4
	$1+\theta$	2θ	$\pm 2\theta$
	1	1	1
	θ	-1	3
Secondary numbers	0	0	0 or 4
	λ	2	± 2
	μ	$2+\theta$	$-2+\theta$
	$\lambda+\mu$	$-2-\theta$	$-2-\theta$

11. Quadratic Congruences to a prime modulus.—Since the square of either a principal or a secondary number is a principal number, the general form of congruence to be considered is

$$x^2 \equiv h \pmod{n},$$

h being a principal number prime to n, and n an odd prime of any of the three kinds. According to circumstances, this congruence may have either principal solutions only, or secondary solutions only, or neither, or both.

(i) Principal solutions:—the necessary and sufficient condition for solubility is

$$h^{\frac{N-1}{2}} \equiv 1 \pmod{n}.$$

This condition is necessary, for

$$h^{\frac{N-1}{2}} \equiv x^{N-1} \equiv 1 \pmod{n}; \quad (\S\ 9,\ \text{I.})$$

and also sufficient, for taking the complete set of principal residues to the modulus n in the form

$$\pm K_1,\ \pm K_2,\ \ldots\ldots \pm K_{\frac{N-1}{2}},$$

and squaring each, we obtain $\frac{N-1}{2}$ different residues of squares of principal numbers: each of them satisfies $h^{\frac{N-1}{2}} \equiv 1 \pmod{n}$, a congruence (in h as unknown) which cannot have more than $\frac{N-1}{2}$ roots; therefore every value of h such that $h^{\frac{N-1}{2}} \equiv 1 \pmod{n}$ is congruent to the squares of two of the principal residues of n.

Further the $\frac{N-1}{2}$ residues which are not congruent to squares of principal numbers are the roots of

$$h^{\frac{N-1}{2}} \equiv -1 \pmod{n}.$$

The symbol (h/n) will be used to denote the least residue (either $+1$, or -1) of $h^{\frac{N-1}{2}} \pmod{n}$. This must be distinguished from the analogous symbol (A/B) in the ordinary Theory of Numbers. It should be noticed that (h/n) has no meaning, unless h is a principal number, *and* n is a prime number.

(ii) Secondary solutions of $x^2 \equiv h \pmod{n}$.

Then $$h^{\frac{N-1}{2}} \equiv x^{N-1} \equiv \pm 1 \pmod{n},$$

the ambiguous sign depending on the system of numbers and the value of n (see § 9, II.).

As before it may be proved that the condition $h^{\frac{N-1}{2}} \equiv \pm 1 \pmod{n}$, (as the case may be) is sufficient as well as necessary for the existence of secondary roots of the congruence.

For systems of the first type, this ambiguous sign may be expressed as $(-1)^{\frac{N^2-1}{8}}$ (§ 9, II.). Accordingly if $N \equiv \pm 1 \pmod{8}$, the congruence $x^2 \equiv h \pmod{n}$ is soluble both in principal and in secondary numbers, provided that $(h/n) = +1$; but it is soluble in neither, if $(h/n) = -1$. On the other hand, if $N \equiv \pm 3 \pmod{8}$ the congruence is soluble in principal numbers only if $(h/n) = +1$, and in secondary numbers only if $(h/n) = -1$.

12. The value of the symbol (h/n) can always be expressed in terms of the corresponding symbol in the ordinary theory of numbers, and so, when h and n are given, its actual value, $+1$ or -1, can easily be determined. The proof of this statement differs according as n is a real prime or not.

First, let n be a principal or secondary prime. Then (§ 6) a real number h_0 may be found congruent to $h \pmod{n}$. Hence $(h/n) \equiv h^{\frac{N-1}{2}} \equiv h_0^{\frac{N-1}{2}} \pmod{n}$; but

$$h_0^{\frac{N-1}{2}} \equiv (h_0/N) \pmod{N},$$

and so, since N is a multiple of n,

$$h_0^{\frac{N-1}{2}} \equiv (h_0/N) \pmod{n};$$

therefore $$(h/n) = (h_0/n) = (h_0/N).$$

Secondly, let $n=r$, a real prime.

Let $h=x+y\theta$, and its conjugate $h'=x+y\theta'$.

If the system is of the first type, $\theta=\sqrt{-d}$, and so

$$\theta^r=(-d)^{\frac{r-1}{2}}\theta\equiv(-d/r)\,\theta\ (\text{mod. } r).$$

Now $(-d/r)=-1$, for r is not representable by any form of determinant $-4d$; and $-\theta=\theta'$; therefore $$\theta^r\equiv\theta'\ (\text{mod. } r).$$

The same result holds for systems of the second type: then

$$2\theta=1+\sqrt{D},\quad 2\theta'=1-\sqrt{D}:$$

so $$2^r\theta^r=(1+\sqrt{D})^r\equiv1+D^{\frac{r-1}{2}}\sqrt{D}\ (\text{mod. } r)$$
$$\equiv1-\sqrt{D}\quad(\text{mod. } r)$$
$$\equiv2\theta'\quad(\text{mod. } r);$$

also, $$2^r\equiv2\quad(\text{mod. } r),$$

and hence $$\theta^r\equiv\theta'\quad(\text{mod. } r).$$

Then, in either case, $$h^r=(x+y\theta)^r\equiv x^r+y^r\theta^r\quad(\text{mod. } r).$$

But $$x^r\equiv x,\ \ y^r\equiv y,\ \text{ and }\ \theta^r\equiv\theta'\ (\text{mod. } r),$$

so $$h^r\equiv x+y\theta'\equiv h'\quad(\text{mod. } r);$$

therefore $$h^{r+1}\equiv hh'\equiv H\quad(\text{mod. } r).$$

And so finally, $$(h/r)\equiv h^{\frac{r^2-1}{2}}\equiv H^{\frac{r-1}{2}}\equiv(H/r)\ (\text{mod. } r).$$

It should be observed that, if both the numbers a and b in the symbol (a/b) are real, the symbol is still different from (a/b). For in that case,

$$(a/b)\equiv a^{\frac{b^2-1}{2}}\equiv(a^{b-1})^{\frac{b+1}{2}}\equiv1\ (\text{mod. } b),$$

while (a/b) of course may be either ±1.

It is evident that $$(h_1h_2h_3\ldots/n)=(h_1/n)\,(h_2/n)\,(h_3/n)\ldots.$$

13. Laws of Quadratic Reciprocity between two principal (including real) primes exist for all systems of principal quadratic numbers; these laws are analogous to, and indeed are deduced from, the Law of Quadratic Reciprocity in the ordinary Theory of Numbers. In determining these laws, systems of the first and second types must be separately considered, and the former will have to be subdivided according as

$$d=\frac{-D}{4}\equiv1\ (\text{mod. } 2),\text{ or } 2\ (\text{mod. } 4).$$

For the reason appearing in § 4, systems for which $d\equiv0$ (mod. 4) are omitted.

The following notation will be used, in discussing the laws of reciprocity for systems of the first type: $p = x + y\theta$, and $p' = x' + y'\theta$ denote two (non-conjugate) principal primes; $P = x^2 + dy^2$ and $P' = x'^2 + dy'^2$ are their respective norms.

Then $$PP' = (x^2 + dy^2)(x'^2 + dy'^2) = X^2 + dY^2,$$

where $$X = xx' + dyy', \quad Y = x'y - xy'.$$

Now we have identically

$$x(x' + y'\theta) = xx' + dyy' + y'\theta(x + y\theta),$$

that is $$xp' \equiv X \pmod{p},$$

and so $$(x/p)(p'/p) = (X/p).$$

Since x and X are real numbers, $(x/p) = (x/P)$, and $(X/p) = (X/P)$.

Therefore $$(p'/p) = (x/P)(X/P).$$

Similarly $$(p/p') = (x'/P')(X/P'),$$

and so $$(p/p')(p'/p) = (x/P)(x'/P')(X/PP').$$

This formula is true for all systems of the first type: in order to evaluate the right-hand side of the last equation, we must consider separately the cases when d is even or odd.

14. First, let d be odd. Then since $P = x^2 + dy^2$, either x is even and y odd, or vice-versa. If $p \equiv 1 \pmod{2}$, then $y \equiv 0 \pmod{2}$, and $P \equiv x^2 \equiv 1 \pmod{4}$; but if $p \equiv \theta \pmod{2}$, then $x \equiv 0 \pmod{2}$, and $P \equiv d \pmod{4}$.

In order to evaluate (x/P), three cases must be treated:

(i) x odd, and y even.

(ii) $x = 2\xi$, where ξ is odd, and y odd.

(iii) $x = 2^\mu\xi$, where ξ is odd, and $\mu > 1$, and y odd.

(i) Here $p \equiv 1 \pmod{2}$: then $(x/P) = (P/x)$, since $P \equiv 1 \pmod{4}$

$$= (x^2 + dy^2/x) = (dy^2/x) = (d/x)$$

$$= (-1)^{\frac{x-1}{2} \cdot \frac{d-1}{2}} (x/d).$$

(ii) $x = 2\xi$, where ξ is odd, and y odd.

Then $$P = 4\xi^2 + dy^2 \equiv 4 + d \pmod{8},$$

and so $$(x/P) = (2/P)(\xi/P) = (2/P)(-1)^{\frac{\xi-1}{2} \cdot \frac{P-1}{2}} (P/\xi).$$

Now $$(2/P) = (2/4 + d) = -(2/d),$$

and $$\frac{P-1}{2} \equiv \frac{d-1}{2} \pmod{2},$$

and $$(P/\xi) = (d/\xi).$$

Therefore
$$(x/P) = -(2/d)(-1)^{\frac{\xi-1}{2}\cdot\frac{d-1}{2}}(d/\xi)$$
$$= -(2/d)(\xi/d) = -(x/d).$$

(iii) $x = 2^{\mu}\xi$, where $\mu > 1$, and ξ is odd: also y is odd.

Then
$$P \equiv d \pmod{8},$$
therefore
$$(x/P) = (2/P)^{\mu}(\xi/P) = (2/P)^{\mu}(-1)^{\frac{\xi-1}{2}\cdot\frac{P-1}{2}}(P/\xi)$$
$$= (2/d)^{\mu}(-1)^{\frac{\xi-1}{2}\cdot\frac{d-1}{2}}(d/\xi)$$
$$= (2^{\mu}/d)(\xi/d) = (x/d).$$

Cases (ii) and (iii) can be summed up in one theorem: if x is even, then
$$(x/P) = (-1)^{\frac{x}{2}}(x/d).$$

15. While continuing to treat the case of d odd, we are now in a position to evaluate $(p/p')(p'/p)$. Since $PP' = X^2 + dY^2$, the results of the last paragraph give the value of (X/PP'), as well as of (x/P) and (x'/P').

Three cases must be separately considered:

(i) $p \equiv p' \equiv 1 \pmod{2}$;

(ii) $p \equiv 1 \pmod{2}$, $p' \equiv \theta \pmod{2}$;

(iii) $p \equiv p' \equiv \theta \pmod{2}$.

It will suffice to give the detailed working in the first of these cases only. In this x and x' are odd, y and y' are even. So $X = xx' + dyy' \equiv xx' \pmod{4}$, and therefore X is odd, and Y is even. Hence the result of § 14 (i) gives
$$(x/P) = (-1)^{\frac{x-1}{2}\cdot\frac{d-1}{2}}(x/d),$$
$$(x'/P') = (-1)^{\frac{x'-1}{2}\cdot\frac{d-1}{2}}(x'/d),$$
and
$$(X/PP') = (-1)^{\frac{X-1}{2}\cdot\frac{d-1}{2}}(X/d),$$
$$= (-1)^{\frac{xx'-1}{2}\cdot\frac{d-1}{2}}(xx'/d).$$

Therefore
$$(p/p')(p'/p) = (x/P)(x'/P')(X/PP')$$
$$= (-1)^{\frac{d-1}{2}\cdot\left(\frac{x-1+x'-1+xx'-1}{2}\right)}.$$

Now
$$x - 1 + x' - 1 + xx' - 1 = (x+1)(x'+1) - 4 \equiv 0 \pmod{4}.$$

And so the index of -1 on the right side of the last equation is even, and therefore, when $p \equiv p' \equiv 1 \pmod{2}$,
$$(p/p')(p'/p) = 1.$$

The result when $p \equiv 1 \pmod{2}$, $p' \equiv \theta \pmod{2}$ is similarly found to be
$$(p/p')(p'/p) = (-1)^{\frac{x-1}{2}\cdot\frac{d-1}{2}+\frac{y}{2}};$$

that is, if $d \equiv 1 \pmod{4}$, $\qquad (-1)^{\frac{y}{2}}$,

but if $d \equiv 3 \pmod{4}$, $\qquad (-1)^{\frac{x+y-1}{2}}$.

And when $p \equiv p' \equiv \theta \pmod{2}$, $\quad (p/p')\,(p'/p) = (-1)^{\frac{x+x'}{2} + \frac{d-1}{2} \cdot \frac{y+y'}{2}}$;

that is, if $d \equiv 1 \pmod{4}$, $\qquad (-1)^{\frac{x+x'}{2}}$,

but if $d \equiv 3 \pmod{4}$, $\qquad (-1)^{\frac{x+x'+y+y'}{2}}$.

Finally, the Laws of Quadratic Reciprocity for systems of the first type, for which $d \equiv 1 \pmod{4}$, can be summed in the following simple and beautiful shape:—

If either p or $p' \equiv 1 \pmod{2}$, then $(p/p')\,(p'/p) = (-1)^{\frac{yy'}{2}}$.

If both p and $p' \equiv \theta \pmod{2}$, then $(p/p')\,(p'/p) = (-1)^{\frac{x+x'}{2}}$.

This law holds for the systems $D = -20, -52, -148$ (for which $d = 5, 13, 37$ respectively); but it also holds for all other systems of principal quadratic numbers of the first type for which $d \equiv 1 \pmod{4}$, as the existence of secondary numbers is irrelevant to its proof.

In the case $d \equiv 3 \pmod{4}$ we similarly have:—

If either p or $p' \equiv 1 \pmod{2}$, $(p/p')\,(p'/p) = (-1)^{\frac{x(x-1)y' + x'(x'-1)y + yy'}{2}}$.

If both p and $p' \equiv \theta \pmod{2}$, $(p/p')\,(p'/p) = (-1)^{\frac{x+x'+y+y'}{2}}$.

16. The case $d \equiv 2 \pmod{4}$ remains to be treated, in order to complete the discussion as regards systems of the first type.

The procedure is similar to what has been given, so that there is no need to set out the working, which is somewhat lengthy, owing to the number of different cases.

If either p or $p' \equiv 1 \pmod{2}$, $\qquad (p/p')\,(p'/p) = (-1)^{\frac{(x-1)y' + (x'-1)y + yy'}{2}}$.

If both p and $p' \equiv 1 + \theta \pmod{2}$, $(p/p')\,(p'/p) = (-1)^{\frac{y+y'}{2}}$.

17. It is a remarkable fact that the laws of reciprocity between a principal prime and a real prime, or between two real primes, are the same as would be obtained from the law between two principal primes by making one or both of them become real primes. That is to say, if in the formulæ for $(p/p')\,(p'/p)$ we write $x' = r$, $y' = 0$, we obtain the true expression for $(p/r)\,(r/p)$.

For, in every system of the first type, $(p/r) = (P/r)$ and $(r/p) = (r/P)$, (§ 12); and so

$$(p/r)\,(r/p) = (P/r)\,(r/P) = (-1)^{\frac{P-1}{2}\frac{r-1}{2}}.$$

Now if $d \equiv 1 \pmod{4}$, $P \equiv 1 \pmod{4}$, (§ 14), and hence

$$(p/r)\,(r/p) = +1.$$

This is what is obtained from

$$(p/p')(p'/p) = (-1)^{\frac{yy'}{2}}$$

when we put r for x', 0 for y'; but it need hardly be said that this substitution of r for p' does not prove the result.

Next, in the cases $d \equiv 2$ or 3 (mod. 4), $P \equiv 1$ or 3 (mod. 4) according as $y \equiv 0$ or 1 (mod. 2): that is

$$\frac{P-1}{2} \equiv y \text{ (mod. 2)}.$$

And so we get $$(p/r)(r/p) = (-1)^{y\frac{r-1}{2}}.$$

And this is what we get from the formulæ for $d \equiv 3$ (mod. 4) (§ 15), and for $d \equiv 2$ (mod. 4) (§ 16) on putting r for x', 0 for y'.

In the case of two real primes, r and r', since $(r/r') = +1$, and $(r'/r) = +1$, we get $(r/r')(r'/r) = +1$, which agrees with the formulæ for $(p/p')(p'/p)$.

18. Similar but more complicated laws hold for systems of the second type. Here, if

$$p = x + y\theta, \qquad p' = x' + y'\theta,$$

then $$P = x^2 + xy + \delta y^2, \quad P' = x'^2 + x'y' + \delta y'^2.$$

Also, p_0' being the conjugate of p', $p_0' = x' + y'\theta' = x' + y' - y'\theta$;

and so $$pp_0' = X + Y\theta,$$

where $$X = xx' + xy' + \delta yy', \text{ and } Y = x'y - xy';$$

and so $$PP' = X^2 + XY + \delta Y^2.$$

The identities on which the theorem depends are

$$x(x' + y'\theta) = xx' + xy' + \delta yy' - y'\theta'(x + y\theta),$$

and $$(x' + y')(x + y\theta) = xx' + xy' + \delta yy' + y\theta(x' + y'\theta),$$

the latter being obtained from the former by changing x to $x' + y'$, x' to $x + y$, y to $-y'$, y' to $-y$, and θ to θ'.

These identities may be written

$$x \,.\, p' \equiv X \text{ (mod. } p),$$

and $$(x' + y') \,.\, p \equiv X \text{ (mod. } p'):$$

and therefore $$(p'/p)(p/p') = (x/P)(x' + y'/P')(X/PP').$$

Finally, we obtain the following results, in which for brevity I write

$$(p'/p)(p/p') = (-1)^M.$$

I. δ even: here $P \equiv x^2 + xy$ (mod. 2), and so x must be odd, and y even, if P is odd; and therefore $p \equiv p' \equiv 1$ (mod. 2). Then

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{x-1}{2} \cdot \frac{x'-1}{2}.$$

II. $\delta \equiv 1 \pmod{4}$. Here an odd prime p may be $\equiv 1$, or θ, or $1+\theta \pmod{2}$, so that there are six different cases:

(i) $p \equiv p' \equiv 1 \pmod{2}$,

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{x-1}{2} \cdot \frac{x'-1}{2};$$

(ii) $p \equiv 1,\ p' \equiv \theta \pmod{2}$,

$$M = \frac{x-1}{2} \cdot \frac{x'}{2} + \frac{y}{2} \cdot \frac{y'+1}{2};$$

(iii) $p \equiv p' \equiv \theta \pmod{2}$,

$$M = \frac{x+y+1}{2} \cdot \frac{x'+y'+1}{2} + \frac{y+1}{2} \cdot \frac{y'+1}{2};$$

(iv) $p \equiv 1,\ p' \equiv 1+\theta \pmod{2}$,

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'}{2} + \frac{y}{2} \cdot \frac{y'-1}{2};$$

(v) $p \equiv \theta,\ p' \equiv 1+\theta \pmod{2}$,

$$M = \frac{x+y+1}{2} \cdot \frac{x'+y'}{2} + \frac{x}{2} \cdot \frac{x'+1}{2};$$

(vi) $p \equiv p' \equiv 1+\theta \pmod{2}$,

$$M = \frac{x+1}{2} \cdot \frac{x'+1}{2} + \frac{y+1}{2} \cdot \frac{y'+1}{2} + 1.$$

III. $\delta \equiv 3 \pmod{4}$. As before, there are six different cases:

(i) $p \equiv p' \equiv 1 \pmod{2}$,

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{x-1}{2} \cdot \frac{x'-1}{2};$$

(ii) $p \equiv 1,\ p' \equiv \theta \pmod{2}$,

$$M = \frac{x-1}{2} \cdot \frac{x'+2}{2} + \frac{y}{2} \cdot \frac{y'+1}{2};$$

(iii) $p \equiv p' \equiv \theta \pmod{2}$,

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{y-1}{2} \cdot \frac{y'-1}{2};$$

(iv) $p \equiv 1,\ p' \equiv 1+\theta \pmod{2}$,

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'+2}{2} + \frac{y}{2} \cdot \frac{y'-1}{2};$$

(v) $p \equiv \theta,\ p' \equiv 1+\theta \pmod{2}$,

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'+2}{2} + \frac{x+2}{2} \cdot \frac{x'-1}{2};$$

(vi) $p \equiv p' \equiv 1+\theta \pmod{2}$,

$$M = \frac{x-1}{2} \cdot \frac{x'-1}{2} + \frac{y+1}{2} \cdot \frac{y'+1}{2} + 1.$$

It is remarkable that for all values of δ, when $p \equiv p' \equiv 1$ (mod. 2), the expression for M is the same.

As in the case of systems of the first type, it may be proved that these laws remain true when for p is written r, or (and) for p' is written r'.

19. Just as in the ordinary Theory of Numbers, the Legendrian symbol was generalised by Jacobi, so the analogous symbol (h/n) may be generalised. In § 11, the symbol was only defined to exist when n is a prime number.

Now let $n = n_1 n_2 n_3 \ldots$, where $n_1, n_2, n_3, \ldots$ are odd prime numbers, all prime to h: then we define (h/n) thus

$$(h/n) = (h/n_1)(h/n_2) \ldots .$$

It follows at once from this definition that

$$(h/m)(h/n) = (h/mn), \ m \text{ and } n \text{ being any odd numbers prime to } h;$$

$$\text{that } (h_1/n)(h_2/n) \ldots = (h_1 h_2 \ldots /n), \ h_1, h_2 \ldots \text{ being prime to the odd number } n;$$

$$\text{that if } h \equiv k \pmod{n}, \text{ then } (h/n) = (k/n);$$

and lastly, that if h_0 is real, n being as before any odd number, whose norm is N,

$$(h_0/n) = (h_0/N).$$

Now let p and p' denote any two odd principal numbers in one of the systems which are specially considered in this paper (§ 4); then the prime factors of p or p' are a certain number of odd principal or real primes, together with an even number of odd secondary primes.

Then the laws of reciprocity for the product of the generalised symbols (p/p') and (p'/p) are the same in form as those already proved when p and p' denote odd principal primes. The reason for this is simply that throughout §§ 13—18 no use is made of the supposition that p and p' are primes; all that is assumed about them is that they are odd principal numbers.

20. To complete the Laws of Reciprocity, we must evaluate $(\lambda q/p)(p/q)$ for systems of the first type, in which $\lambda^2 = 2$.

Let $p = x' + y'\theta$, $q = x\lambda + y\mu$. Then the following results may be proved:

I. For the systems $d = 5$, 13, 37, for each of which $d \equiv 5$ (mod. 8).

(i) $p \equiv 1$ (mod. 2), then $\quad (\lambda q/p)(p/q) = (-1)^{\frac{xy'}{2}} \phi;$

(ii) $p \equiv \theta$, $q \equiv \mu$ (mod. 2), then $\quad (\lambda q/p)(p/q) = (-1)^{1 + \frac{x+x'}{2}} \phi;$

(iii) $p \equiv \theta$, $q \equiv \lambda + \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{1 + \frac{x-y}{2}} \phi;$

where
$$\phi = (-1)^{\frac{1}{8}(x'+y'-1)(x'+y'-3)}.$$

II. For the systems $d=6$, and 22, for which $d\equiv 6 \pmod{8}$.

(i) $p\equiv 1$, $q\equiv\mu \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{\frac{x'-1}{2}}\phi$;

(ii) $p\equiv 1$, $q\equiv\lambda+\mu \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{\frac{y'}{2}}\phi$;

(iii) $p\equiv 1+\theta$, $q\equiv\mu \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{\frac{x+y+y'}{2}}\phi$;

(iv) $p\equiv 1+\theta$, $q\equiv\lambda+\mu \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{\frac{x-1}{2}}\phi$;

where
$$\phi=(-1)^{\frac{x^2-1}{8}}.$$

III. For the systems $d=10$, and 58, for which $d\equiv 2 \pmod{8}$.

(i) $p\equiv 1 \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{\frac{x(x'+y-1)}{2}}\phi$;

(ii) $p\equiv 1+\theta$, $q\equiv\mu \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{x+y+\frac{x'+y'-1}{2}}\phi$;

(iii) $p\equiv 1+\theta$, $q\equiv\lambda+\mu \pmod{2}$, then $\quad (\lambda q/p)\,(p/q)=(-1)^{\frac{x+x'}{2}}\phi$;

where
$$\phi=(-1)^{\frac{x^2-1}{8}}.$$

BINARY QUADRATIC FORMS.

21. In this branch of the subject the analogy with the ordinary Theory of Numbers is not so complete as in the earlier portions of this paper. As in the ordinary Theory, the quadratic form $ax^2+bxy+cy^2$ will be denoted by (a, b, c); a, b, c, x and y denote quadratic numbers. A binary quadratic form with quadratic numbers for coefficients will be called throughout the paper briefly a "form." The number m is represented by the form (a, b, c) when $m=ax^2+bxy+cy^2$, x and y being numbers: it will always be supposed that x is prime to y, and that a, b, and c have no common factor. The determinant of this form is $\Delta=b^2-4ac$.

Δ is therefore always a principal number.

As there is no distinction in the systems here considered between positive and negative numbers, there is nothing corresponding to the division of forms into definite and indefinite forms which occurs in the ordinary Theory.

Forms may be divided into sets, according to whether their coefficients and variables are principal or secondary numbers.

Remembering that a principal and a secondary number may not be added, the following kinds of forms exist:—

(1) a, b, and c principal numbers.

(2) a, b, and c secondary numbers; in this and the former case,

either (α) x and y principal numbers,
or (β) x and y secondary numbers.

(3) a and c principal, b secondary.

(4) a and c secondary, b principal; in this and in case (3)

either (α) x principal, y secondary,

or (β) x secondary, y principal.

These will be called the 1st, 2nd, 3rd and 4th *kinds* of forms.

22. A different classification of forms can be made according to the residue of Δ (mod. 4), since $\Delta \equiv b^2$ (mod. 4).

For the system $D = -20$, $d = 5$, the table in § 10 furnishes the following results:—

(1) $b \equiv 0$ (mod. 2), the form (a, b, c) being of any of the four kinds; then

$$\Delta \equiv 0 \text{ (mod. 4)}.$$

The form $\left(1,\ 0,\ -\frac{\Delta}{4}\right)$ is an example of this case.

(2) $b \equiv 1 + \theta$ (mod. 2), the form being of the first or fourth kinds; then

$$\Delta \equiv 2\theta \text{ (mod. 4)}.$$

(3) $b \equiv 1$ (mod. λ), the form being of the first or fourth kinds; then

$$\Delta \equiv \pm 1 \text{ (mod. 4)}.$$

(4) $b \equiv \lambda$ (mod. 2), the form being of the second or third kinds; then

$$\Delta \equiv 2 \text{ (mod. 4)}.$$

(5) $b \equiv \mu$ (mod. λ), the form being of the second or third kinds; then

$$\Delta \equiv 2 \pm \theta \text{ (mod. 4)}.$$

And conversely for any value of Δ satisfying one of the above congruences forms exist of the corresponding kinds; for $ac = \frac{1}{4}(b^2 - \Delta)$; the number on the right can always be split into the principal factors 1 and $\frac{1}{4}(b^2 - \Delta)$, and it is fairly evident that out of the infinity of possible values of b, $\frac{1}{4}(b^2 - \Delta)$ will often split into the product of two secondary factors.

23. Forms are said to be equivalent when the substitution

$$\begin{cases} x = \alpha x' + \beta y', \\ y = \gamma x' + \delta y', \text{ where } \alpha\delta - \beta\gamma = 1 \end{cases}$$

transforms one of them into the other.

Then the inverse substitution

$$\begin{cases} x' = \delta x - \beta y, \\ y' = -\gamma x + \alpha y \end{cases}$$

transforms the latter into the former, so that the relation of equivalence is a mutual one.

There are four "kinds" of such substitutions:

(i) α, β, γ, δ all principal; then x and y are both of the same kind, i.e. both principal, or both secondary.

(ii) α, β, γ, δ all secondary; x and y are both of the same kind.

(iii) α and δ principal, β and γ secondary; x and y are of different kinds.

(iv) α and δ secondary, β and γ principal; x and y are of different kinds.

The first and second kinds of substitutions alone are applicable to forms of the first or second kind; and the third and fourth kinds of substitutions alone to forms of the third or fourth kind.

If the form (a', b', c') is equivalent to (a, b, c), being obtained from the latter by the substitution $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, then

$$a' = a\alpha^2 + b\alpha\gamma + c\gamma^2,$$

$$b' = 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta,$$

$$c' = a\beta^2 + b\beta\delta + c\delta^2.$$

Therefore a form is necessarily transformed into another of the same kind.

Since $b'^2 - 4a'c' = (\alpha\delta - \beta\gamma)^2(b^2 - 4ac) = b^2 - 4ac$, equivalent forms have the same determinant.

Let $$S = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, \quad S' = \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix},$$

then the substitution arising from their composition in this order is

$$SS' = \begin{pmatrix}\alpha\alpha' + \beta\gamma' & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma' & \gamma\beta' + \delta\delta'\end{pmatrix}.$$

Now let σ_κ denote a substitution of the κth kind.

Then σ_1^2 and σ_2^2 are of the first kind,

$\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ are of the second kind.

Therefore the substitutions of the first and second kinds together form a group, of which those of the first kind form a sub-group. So also the substitutions of the third and fourth kinds form a group, of which those of the third kind form a sub-group. Other compositions of substitutions, e.g. $\sigma_1\sigma_3$, are impossible, for they would necessitate the addition of principal and secondary numbers.

We can now give a more precise definition of "equivalence"; it is an essential part of the notion of equivalence, that forms that are equivalent to the same form should be equivalent to one another.

In other words the set of substitutions, assumed to exist by the definition of "equivalence," must form a group.

There are accordingly two kinds of equivalence, corresponding to the two groups of substitutions applicable to a given form. A form of the first or second kind will be called "narrowly equivalent" to any form obtained from it by a substitution of the first kind, but "widely equivalent" to a form obtained from it by a substitution of either the first or the second kind. Similar definitions of the terms "narrow" and "wide" equivalence apply to forms of the third and fourth kinds.

Forms which are equivalent are said to belong to the same class, the class being narrow or wide, according to whether the equivalence is narrow or wide.

A wide class obviously contains the whole of a narrow class, if it contains a single form belonging to the narrow class.

Since all forms of a class are of the same kind, we can speak of the "kind" of the class.

24. There is, as in the ordinary Theory, a close connection between any class of forms and the set of numbers representable by any form of the class.

For, if $a' = a\alpha^2 + b\alpha\gamma + c\gamma^2$, α being prime to γ, and β and δ be chosen so that $\alpha\delta - \beta\gamma = 1$ (§ 8), then the substitution $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ converts the form (a, b, c) into an equivalent form (a', b', c').

Conversely, if (a', b', c') is equivalent to (a, b, c), then the extreme coefficients a' and c' are representable by (a, b, c). Hence the following theorems:—

The set of the extreme coefficients of all the forms of a narrow class of the first or second kind coincides with the set of all numbers representable by a form of the class, the variables x and y taking coprime principal values.

The set of the extreme coefficients of all the forms of a wide class of the first or second kind coincides with the set of all numbers representable by a form of the class, the variables taking all possible coprime values.

The set of first coefficients of all the forms of a narrow class of the third or fourth kind coincides with the set of all numbers representable by a form of the class, x taking principal and y taking secondary values; and the set of third coefficients of the same to the set of numbers representable by a form of the class, x taking secondary and y principal coprime values.

Lastly, the set of the extreme coefficients of all the forms of a wide class of the third and fourth kind coincides with the set of all numbers representable by a form of the class, the variables taking all possible coprime values.

If a' is an odd number representable by (a, b, c), and b', c' have the same meanings as before, then

$$b'^2 - 4a'c' = \Delta,$$

and so

$$b'^2 \equiv \Delta \pmod{a'}.$$

Therefore, if a' is representable by a form of the first or fourth kind (in each of which the middle coefficient is a principal number), a' must satisfy the condition $(\Delta/a')=1$.

And if a' is representable by a form of the second or third kind, the condition is $(\Delta/a')=1$, or -1, according to the value of a' and the system considered. (§ 11 (ii).)

Conversely, if $b'^2 \equiv \Delta \pmod{a'}$ is soluble, then writing $\dfrac{b'^2-\Delta}{a'}=c'$, the form (a', b', c') is of determinant Δ, and represents the number a'.

25. In the ordinary Theory of Numbers, there is an elementary proof of the finiteness of the number of classes for a given determinant, which depends on the method of Reduction. This method of proof does not apply to the systems of numbers now being studied.

It will be remembered that in the ordinary Theory the first step in the process is to apply to the form (a, b, c) the substitution $\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}$, thus producing the form (c, b', a'), where

$$b' = -b - 2c\delta,$$
$$a' = a + b\delta + c\delta^2;$$

and δ is then determined so that $|b'| \gtreqless c$.

In this way a reduced form (A, B, C) is obtained, such that

$$C \not< A \not< |B|.$$

Now, in any of the systems of numbers here considered the substitution $\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}$ is either of the first or fourth kind, according as δ is principal or secondary. In either case, it is not always possible to find a residue b' of $-b$ to mod. $2c$ such that $N(b') < N(2c)$, and the process of reduction therefore breaks down. If this were possible when b and c are any principal numbers, the Euclidean process for finding their greatest common factor would work, and there would be no need of secondary numbers to complete the laws of factorisation. And it is easy to prove the impossibility of always finding a residue b' such that $N(b') < N(2c)$, where either b or c, or both, are secondary numbers. Possibly Dirichlet's analytical method of determining the class-number for a given determinant in the ordinary Theory would apply to these systems of quadratic numbers.

GENERIC CHARACTERS.

26. Just as in the ordinary Theory of Numbers, the classes for a given determinant may be divided into genera. Since the results mainly depend on the residues of squares to moduli consisting of powers of 2 or its factors, I shall confine the remainder of this paper to the case $D=-20$, i.e. $d=5$; and for brevity I shall only consider forms of the first kind.

All the results of this nature are derived from the simple identity

$$4nn' = x^2 - \Delta y^2,$$

where

$$n = au^2 + buv + cv^2,$$
$$n' = au'^2 + bu'v' + cv'^2,$$
$$x = 2auu' + b(uv' + u'v) + 2cvv',$$
$$y = uv' - u'v,$$
$$\Delta = b^2 - 4ac.$$

Three cases will need separate consideration, according as b is even, semi-even, or odd, i.e. according as $\Delta \equiv 0,\ 2\theta$, or ± 1 (mod. 4). (§ 22.)

(1) When $\Delta \equiv 0 \,(\text{mod. } 4) = 4\delta$, the above identity may be divided by 4, giving

$$nn' = x^2 - \delta y^2,$$

x here meaning half the expression above given for x.

(2) When $\Delta \equiv 2\theta \,(\text{mod. } 4) = 2\delta$,

$$2nn' = x^2 - \delta y^2,$$

where

$$\lambda x = 2auu' + b(uv' + u'v) + 2cvv'.$$

(3) When $\Delta \equiv \pm 1$ (mod. 4),

$$4nn' = x^2 - \Delta y^2.$$

With regard to narrow classes, in any of these three cases, there exist quadratic characters precisely analogous to the quadratic characters in the ordinary theory.

For if t be any odd prime factor of Δ, we have $4nn' \equiv x^2 \,(\text{mod. } t)$, where x is principal (since the form is of the first kind, and the numbers u, u', v, v' are principal).

Therefore

$$(nn'/t) = +1,$$

that is

$$(n/t) = (n'/t).$$

27. There exist besides supplementary characters, which depend on theorems as to the residues of n and n' (being odd numbers) to moduli of the form λ^k.

These characters may be defined as follows: $n = n_0 + n_1\theta$ is an odd number whose norm is N; then

$$\psi(n) = (-1)^{n_1},$$
$$\chi(n) = (-1)^{\frac{N-1}{4}},$$
$$\omega(n) = (-1)^{\frac{1}{8}(n_0 + \psi n_1 - 1)(n_0 + \psi n_1 - 3)}.$$

Where no ambiguity is caused, I shall write ψ for $\psi(n)$, etc.

Therefore $\psi = +1$ or -1, according as $n \equiv 1$ or θ (mod. 2);

when $\chi = +1$, $\quad n \equiv \pm 1$, or $\pm(2 + \theta)$ (mod. 4),

when $\chi = -1$, $\quad n \equiv \pm(1 + 2\theta)$, or $\pm\theta$ (mod. 4).

And the following table is easily deduced from the definitions just given.

Residues of n to modulus λ^5, i.e. 4λ.		ψ	χ	ω
$+1,$	$+3$	$+$	$+$	$+$
$-1,$	-3	$+$	$+$	$-$
$1+2\theta,$	$-1+2\theta$	$+$	$-$	$+$
$-1-2\theta,$	$1-2\theta$	$+$	$-$	$-$
$2+\theta,$	$2-\theta$	$-$	$+$	$+$
$-2-\theta,$	$-2+\theta$	$-$	$+$	$-$
$-\theta,$	$4+\theta$	$-$	$-$	$+$
$\theta,$	$4-\theta$	$-$	$-$	$-$

From the definitions or from this table it may be verified that

$$\psi(nn') = \psi(n)\,\psi(n'), \quad \chi(nn') = \chi(n)\,\chi(n'), \quad \omega(nn') = \omega(n)\,\omega(n').$$

One specimen of the reasoning by which the existence of these supplementary characters is proved will be sufficient, and I shall then present a table shewing all the appropriate supplementary characters of narrow classes for the various values of Δ.

In the case $\Delta = 4\delta$, and $\delta \equiv \pm(1+\theta)$ (mod. 4λ).

Then $nn' \equiv x^2 \pm (1+\theta)\,y^2$ (mod. 4λ),

where x must be odd, but y may be odd, semi-even, or even, both x and y being principal; then (§ 10, table)

$$x^2 \equiv 1 \text{ or } 3 \pmod{4\lambda},$$

$$y^2 \equiv \pm 1,\ 2\theta \text{ or } 0 \pmod{4},$$

and so $nn' \equiv 2+\theta,\ -\theta,\ 4+\theta,\ 2-\theta,\ -1+2\theta,\ 1+2\theta,\ 1$ or 3 (mod. 4λ).

Therefore $\omega(nn') = +1,$

that is $\omega(n) = \omega(n').$

SUPPLEMENTARY CHARACTERS OF NARROW CLASSES.

I. $\Delta = 4\delta$.

Odd Residues of δ to mod. 4λ.	Characters.	Even Residues of δ to mod. 4λ.	Characters.
± 1	ψ	0	ψ, χ, ω
± 3	ψ	4	ψ, χ
$\pm (1+2\theta)$	ψ	± 2	ψ, χ
$\pm (1-2\theta)$	ψ	$\pm 2\theta$	ψ
$\pm \theta$	$\psi\chi$	$2+2\theta$	$\psi, \chi\omega$
$\pm (4+\theta)$	$\psi\chi$	$2-2\theta$	ψ, ω
$\pm (2+\theta)$	χ	$\pm (1+\theta)$	ω
$\pm (2-\theta)$	χ	$\pm (1-\theta)$	$\chi\omega$
		$\pm (3+\theta)$	$\psi\chi\omega$
		$\pm (3-\theta)$	$\psi\omega$

II. $\Delta = 2\delta \equiv 2\theta$ (mod. 4).

Residues of δ to mod. 4.	Characters.
$\pm \theta$	ψ
$\pm (2+\theta)$	none

III. $\Delta \equiv \pm 1$ (mod. 4); there are no supplementary characters in this case.

28. Two classes of a given determinant are said to belong to the same "genus," when all their generic characters have the same values.

Half the assignable genera of narrow classes of the first kind for a given determinant are impossible; this result is obtained (as in the ordinary Theory) by applying the Law of Quadratic Reciprocity to the equation $(\Delta/n) = +1$ (§ 24), n being here any odd number prime to Δ representable by some form of the first kind of determinant Δ.

For example, let $\Delta = 4ts^2$, s being principal and the largest square in Δ, and t also principal and $= t_1 t_2 \ldots$; and suppose $t \equiv \pm\theta$ (mod. 4). Then n being $n_0 + n_1\theta$, the law of reciprocity (§ 15) gives

$$(\Delta/n) = (t/n) = \epsilon\,(n/t),$$

where $\epsilon = (-1)^{\frac{n_0}{2}}$, if n_1 is odd, but $(-1)^{\frac{n_1}{2}}$, if n_1 is even.

That is, $\epsilon = +1$, if $n \equiv \pm 1$ or $\pm\theta$ (mod. 4),

and $\epsilon = -1$, if $n \equiv \pm(1+2\theta)$ or $\pm(2+\theta)$ (mod. 4).

Therefore $\epsilon = \psi\chi$; and $(n/t) = (n/t_1)(n/t_2) \ldots\ldots$,

so that the condition $(\Delta/n) = +1$

becomes $\psi\chi \cdot (n/t_1) \cdot (n/t_2) \ldots\ldots = +1$.

Similar applications of the law of reciprocity (§§ 15 and 20) to the numerous cases furnish the facts set out in the table below, which is arranged in a similar manner to Dirichlet's table in the ordinary Theory (Mathews, *Theory of Numbers*, Pt. I. p. 135), and to H. J. S. Smith's table in his paper "On Complex Binary Quadratic Forms" (in the system of numbers $x+y\sqrt{-1}$) (*Collected Papers*, Vol. I. p. 421). In the table s^2 denotes the largest square dividing δ in cases (I) and (II), or dividing Δ in case (III); $t_1, t_2 \ldots\ldots$ are the different odd prime factors of t which is itself odd; $s_1, s_2 \ldots\ldots$ are those odd prime factors of s which do not divide t; and I is the index of the highest power of λ contained in s. In each line is the complete set of characters for the corresponding value of Δ, those characters to the left of the vertical line being subject to the condition that their product is $+1$.

POSSIBLE GENERIC CHARACTERS FOR NARROW CLASSES.

I. $\Delta = 4\delta$.

(1) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm 1 \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm(2+\theta) \text{ (mod. 4)}, \end{cases}$

$I = 0$ or 1	$(n/t_1), \ldots\ldots$	$\psi, (n/s_1), \ldots\ldots$
$I = 2$	$(n/t_1), \ldots\ldots$	$\psi, \chi, (n/s_1), \ldots\ldots$
$I > 2$	$(n/t_1), \ldots\ldots$	$\psi, \chi, \omega, (n/s_1), \ldots\ldots$

(2) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm(1+2\theta) \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm\theta \text{ (mod. 4)}, \end{cases}$

$I = 0$ or 1	$\psi, (n/t_1), \ldots\ldots$	$(n/s_1), \ldots\ldots$
$I = 2$	$\psi, (n/t_1), \ldots\ldots$	$\chi, (n/s_1), \ldots\ldots$
$I > 2$	$\psi, (n/t_1), \ldots\ldots$	$\chi, \omega, (n/s_1), \ldots\ldots$

(3) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm\theta \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm(1+2\theta) \text{ (mod. 4)}, \end{cases}$

$I = 0$	$\psi\chi, (n/t_1), \ldots\ldots$	$(n/s_1), \ldots\ldots$
$I = 1$ or 2	$\psi, \chi, (n/t_1), \ldots\ldots$	$(n/s_1), \ldots\ldots$
$I > 2$	$\psi, \chi, (n/t_1), \ldots\ldots$	$\omega, (n/s_1), \ldots\ldots$

(4) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm(2+\theta) \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm 1 \text{ (mod. 4)}, \end{cases}$

$I = 0$	$\chi, (n/t_1), \ldots\ldots$	$(n/s_1), \ldots\ldots$
$I = 1$ or 2	$\chi, (n/t_1), \ldots\ldots$	$\psi, (n/s_1), \ldots\ldots$
$I > 2$	$\chi, (n/t_1), \ldots\ldots$	$\psi, \omega, (n/s_1), \ldots\ldots$

(5) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(1+\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(1-\theta) \pmod{4\lambda}, \end{cases}$

$I=0$	ω, (n/t_1), ...	(n/s_1), ...
$I=1$	ω, (n/t_1), ...	ψ, (n/s_1), ...
$I>1$	ω, (n/t_1), ...	ψ, χ, (n/s_1), ...

(6) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(1-\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(1+\theta) \pmod{4\lambda}, \end{cases}$

$I=0$	$\chi\omega$, (n/t_1), ...	(n/s_1), ...
$I=1$	$\chi\omega$, (n/t_1), ...	ψ, (n/s_1), ...
$I>1$	χ, ω, (n/t_1), ...	ψ, (n/s_1), ...

(7) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(3+\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(3-\theta) \pmod{4\lambda}, \end{cases}$

$I=0$	$\psi\chi\omega$, (n/t_1), ...	(n/s_1), ...
$I=1$	ψ, $\chi\omega$, (n/t_1), ...	(n/s_1), ...
$I>1$	ψ, χ, ω, (n/t_1), ...	(n/s_1), ...

(8) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(3-\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(3+\theta) \pmod{4\lambda}, \end{cases}$

$I=0$	$\psi\omega$, (n/t_1), ...	(n/s_1), ...
$I=1$	ψ, ω, (n/t_1), ...	(n/s_1), ...
$I>1$	ψ, ω, (n/t_1), ...	χ, (n/s_1), ...

II. $\Delta \equiv 2\theta \pmod{4} = 2\delta = 2ts^2$.

$\delta \equiv \pm\theta \pmod{4}$	ψ, (n/t_1), ...	(n/s_1), ...
$\delta \equiv \pm(2+\theta) \pmod{4}$	(n/t_1), ...	(n/s_1), ...

III. $\Delta \equiv \pm 1 \pmod{4} = ts^2$.

(n/t_1), ... | (n/s_1), ...

29. The generic characters of wide classes remain to be considered; we shall find that characters here occur of a kind which have no analogy in the ordinary Theory. Using the notation of § 26, if n is a number of the wide class represented by the form (a, b, c) of the first kind, u and v are either both principal or both secondary (§ 24). Similarly for u' and v'; and so x and y are both principal, or both secondary. Let t be any odd prime factor of Δ.

Then $$4nn' \equiv x^2 \pmod{t}.$$

If x is principal, this gives $$(nn'/t) = +1.$$

If x is secondary $$(nn'/t) \equiv x^{T-1} \equiv (-1)^{\frac{T^2-1}{8}} \pmod{t}. \quad (\S\ 9,\ \text{II.})$$

Now either $T \equiv \pm 1 \pmod{8}$, in which case $(nn'/t) = +1$, whether x is principal or secondary, and we get the quadratic character (n/t); or $T \equiv \pm 3 \pmod{8}$, in which case

(nn'/t) is $+1$ or -1, according as x is principal or secondary. Let t_1' and t_2' be two odd prime factors of Δ, both of whose norms are $\equiv \pm 3 \pmod{8}$; then it follows that $(nn'/t_1't_2') = +1$, whether x is principal or secondary; that is, $(n/t_1't_2') = (n'/t_1't_2')$; thus in the case of wide classes, there is a quadratic character corresponding to every pair of those odd prime factors of Δ, whose norms are $\equiv \pm 3 \pmod{8}$. These however are not all independent, but it is evident that if $t_1', t_2', \ldots\ldots t_r'$ are the prime factors of Δ of this kind, then $(n/t_1't_2'), (n/t_1't_3'), \ldots\ldots (n/t_1't_r')$ are all independent, and form a complete set, $r-1$ in number; for

$$(n/t_1't_l') \times (n/t_1't_m') = (n/t_l't_m').$$

For certain values of Δ the supplementary characters χ, ω, etc. occur; and there are also in some cases mixed characters, as $\psi\,(n/t_1')$, etc.

The latter arises when $\psi\,(nn') = +1$, and $(nn'/t_1') = +1$, if x is principal; but $\psi\,(nn') = -1$, $(nn'/t_1') = -1$, if x is secondary. Then $\psi\,(nn')\,(nn'/t_1') = +1$, in either case: and so $\psi\,(n) \,.\, (n/t_1') = \psi\,(n') \,.\, (n'/t_1')$.

I now set out in a table the supplementary and mixed characters of wide classes for the various values of Δ; it will be observed that they are identical with the corresponding results for narrow classes, except that some of the supplementary characters are associated with (n/t_1').

SUPPLEMENTARY AND MIXED CHARACTERS OF WIDE CLASSES.

I. $\Delta = 4\delta$.

Odd Residues of δ to mod. 4λ.	Characters.	Even Residues of δ to mod. 4λ.	Characters.
± 1	$\psi\,(n/t_1')$	0	$\psi\,(n/t_1')$, χ, $\psi\omega$
± 3	$\psi\,(n/t_1')$	4	$\psi\,(n/t_1')$, χ
$\pm(1+2\theta)$	$\psi\,(n/t_1')$	± 2	$\psi\,(n/t_1')$, χ
$\pm(1-2\theta)$	$\psi\,(n/t_1')$	$\pm 2\theta$	$\psi\,(n/t_1')$
$\pm\theta$	$\psi\chi\,(n/t_1')$	$2+2\theta$	$\psi\,(n/t_1')$, $\psi\chi\omega$
$\pm(4+\theta)$	$\psi\chi\,(n/t_1')$	$2-2\theta$	$\psi\,(n/t_1')$, $\psi\omega$
$\pm(2+\theta)$	χ	$\pm(1+\theta)$	$\omega\,(n/t_1')$
$\pm(2-\theta)$	χ	$\pm(1-\theta)$	$\chi\omega\,(n/t_1')$
		$\pm(3+\theta)$	$\psi\chi\omega$
		$\pm(3-\theta)$	$\psi\omega$

II. $\Delta = 2\delta \equiv 2\theta$ (mod. 4).

Residues of δ to mod. 4.	Characters.
$\pm\theta$	$\psi\,(n/t_1')$
$\pm(2+\theta)$	none

III. $\Delta \equiv \pm 1$ (mod. 4). No supplementary, or mixed characters.

30. In the following table, the notation is the same as in the corresponding table for narrow classes, the only difference being that the dashed letters t_1', t_2', ... s_1', ... denote prime factors of t and s whose norms are $\equiv \pm 3$ (mod. 8), the other factors being denoted by undashed letters: and r denotes the number of the former kind of factors of t. It is evident that if $T \equiv \pm 1$ (mod. 8), then r is even, and if $T \equiv \pm 3$ (mod. 8), then r is odd; and it is easy to prove that if

$$t \equiv \pm 1,\ \pm(2+\theta),\ \pm(\lambda+\mu),\ \text{or}\ \pm(2\lambda-\mu)\ (\text{mod. } 4),$$

then
$$T \equiv \pm 1\ (\text{mod. } 8);$$

while if
$$t \equiv \pm(1+2\theta),\ \pm\theta,\ \pm\mu,\ \text{or}\ \pm(\lambda-\mu)\ (\text{mod. } 4),$$

then
$$T \equiv \pm 3\ (\text{mod. } 8).$$

Bearing these results in mind as to r being even or odd, it will be seen that the product of the generic characters to the left of the vertical line is the same as in the previous table; this is of course necessary, since the product in question is a transformation of (Δ/n), and the latter expression is equal to $+1$, for either narrow or wide classes.

In the event of there being no factors of the form t_1' or s_1', or only one of them, the results given below need some modification; it is however easy to see in each such case what the complete set of characters is. For instance in the first line of the table, if there is no t_1', but there is s_1', ..., the characters are

$$(n/t_1),\ (n/t_2),\ \ldots\ldots \mid \psi\,.\,(n/s_1'),\ (n/s_1),\ \ldots\ldots\ (n/s_1's_2'),\ \ldots\ldots;$$

but if s_1' is also absent, then the mixed character containing ψ disappears, and the characters are

$$(n/t_1),\ (n/t_2),\ \ldots\ldots\ (n/s_1),\ \ldots\ldots.$$

POSSIBLE GENERIC CHARACTERS FOR WIDE CLASSES.

I. $\Delta = 4\delta$.

(1) $\delta = ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm 1 \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm (2+\theta) \pmod{4} \end{cases}$ r even.

$I = 0$ or 1	$(n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\psi\,(n/t_1'), (n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I = 2$	$(n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\psi\,(n/t_1'), \chi, (n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I > 2$	$(n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\psi\,(n/t_1'), \chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots$

(2) $\delta = ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm (1+2\theta) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm \theta \pmod{4} \end{cases}$ r odd.

$I = 0$ or 1	$\psi\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$(n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I = 2$	$\psi\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\chi, (n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I > 2$	$\psi\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots.$

(3) $\delta = ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm \theta \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm (1+2\theta) \pmod{4} \end{cases}$ r odd.

$I = 0$	$\psi\chi\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$(n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I = 1$ or 2	$\psi\chi\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\chi, (n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I > 2$	$\psi\chi\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots$

(4) $\delta = ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm (2+\theta) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm 1 \pmod{4} \end{cases}$ r even.

$I = 0$	$\chi, (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$(n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I = 1$ or 2	$\chi, (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\psi\,(n/t_1'), (n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I > 2$	$\chi, (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\psi\,(n/t_1'), \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots.$

(5) $\delta = \lambda ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm \mu \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm (\lambda - \mu) \pmod{4} \end{cases}$ r odd.

$I = 0$	$\omega\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$(n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I = 1$	$\omega\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots$
$I > 1$	$\omega\,(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$	$\chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots.$

(6) $\delta = \lambda ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm(\lambda-\mu) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm\mu \pmod{4} \end{cases}$ r odd.

$I=0$	$\chi\omega\,(n/t_1'),\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$(n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$I=1$	$\chi\omega\,(n/t_1'),\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$\psi\chi\omega,\ (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$I>1$	$\chi,\ \omega\,(n/t_1'),\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$\psi\omega,\ (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots.$

(7) $\delta = \lambda ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm(\lambda+\mu) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(2\lambda-\mu) \pmod{4} \end{cases}$ r even.

$I=0$	$\psi\chi\omega,\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$(n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$I=1$	$\psi\chi\omega,\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$\psi\,(n/t_1'),\ (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$I>1$	$\chi,\ \psi\omega,\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$\psi\,(n/t_1'),\ (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots.$

(8) $\delta = \lambda ts^2 \begin{cases} s \text{ principal, and } t \equiv \pm(2\lambda-\mu) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(\lambda+\mu) \pmod{4} \end{cases}$ r even.

$I=0$	$\psi\omega,\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$(n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$I=1$	$\psi\omega,\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$\psi\,(n/t_1'),\ (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$I>1$	$\psi\omega,\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$\psi\,(n/t_1'),\ \chi,\ (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots.$

II. $\Delta \equiv 2\theta \pmod{4} = 2\delta = 2ts^2$.

$\delta \equiv \pm\theta \pmod{4}$, r odd	$\psi\,(n/t_1'),\ (n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$(n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$
$\delta \equiv \pm(2+\theta) \pmod{4}$, r even	$(n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')$	$(n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots$

III. $\Delta \equiv \pm 1 \pmod{4} = ts^2$; r even.

$(n/t_1),\ \dots,\ (n/t_1't_2'),\ \dots,\ (n/t_1't_r')\quad (n/s_1),\ \dots,\ (n/t_1's_1'),\ \dots.$

31. As an illustration of the results obtained in reference to Binary Quadratic Forms, I now consider the case $\Delta = -4$. Let n be an odd principal number, representable by a form of the first kind having this determinant; then (§ 24), n_1 being any prime factor of n,

$$(-4/n_1) = +1.$$

Therefore $(-1/N_1) = +1$, that is $N_1 \equiv 1 \pmod{4}$. But the norm of any odd secondary number $\equiv 3 \pmod{4}$, so n_1 and therefore every prime factor of n is either a principal or a real prime. And conversely all such primes and all numbers composed of them are representable by some form of the first kind with this determinant.

I have searched for forms of the first kind in the same way as is done in the ordinary theory, and after eliminating all forms (within the limit of my search) narrowly equivalent to simpler ones, the following forms remain:—(1, 0, 1), $(\theta, 4, -\theta)$, $(\theta, -4, -\theta)$.

The form $(-1, 0, -1)$ is equivalent to $(1, 0, 1)$, the former being derived from the latter by the substitution

$$\begin{pmatrix} 2, & \theta \\ \theta, & -2 \end{pmatrix}.$$

First consider narrow classes; then the first line of the Table I. (§ 28) for $\Delta = 4\delta$ shews that there is one generic character ψ, which apparently can be either $+1$ or -1; this is an exception to the general rule that half the assignable genera are impossible, this exception being due to the fact that when $\Delta = -4$, there is no generic character (n/t_1), t_1 not existing in this case.

Both these genera in fact exist, the class represented by the form $(1, 0, 1)$ having the character $\psi = +1$, and the classes represented by $(\theta, \pm 4, -\theta)$ having the character $\psi = -1$; neither of the latter classes can therefore be narrowly equivalent to $(1, 0, 1)$.

Further, it is not difficult to prove that the forms $(\theta, 4, -\theta)$ and $(\theta, -4, -\theta)$ are non-equivalent (narrowly). I have accordingly proved the existence of at least three narrow classes of forms of the first kind; there may be more such, for, as pointed out in § 25, I am not aware of any method of ascertaining the number of classes of a given determinant.

Turning now to wide equivalence, the three forms above mentioned are all equivalent; for

$$\theta = \lambda^2 + \mu^2,$$

$$2 = \lambda\mu + \lambda(\lambda - \mu),$$

$$-\theta = \lambda^2 + (\lambda - \mu)^2,$$

and therefore the substitutions $\begin{pmatrix} \mu, & \lambda \\ \lambda, & \lambda - \mu \end{pmatrix}$ and $\begin{pmatrix} -\mu, & \lambda \\ \lambda, & -\lambda + \mu \end{pmatrix}$ respectively convert $(1, 0, 1)$ into $(\theta, 4, -\theta)$ and $(\theta, -4, -\theta)$. There is apparently therefore only one wide class; it has no generic character (§ 30).

TABLE I.

32. Systems of numbers of the first type; $D = -4d$, $\lambda = \sqrt{2}$.

$-D$	$\theta = \sqrt{-d}$	μ	$\lambda\mu$	μ^2	$\lambda\theta$	$\mu\theta$
20	$\sqrt{-5}$	$\frac{1+\sqrt{-5}}{\sqrt{2}}$	$1+\theta$	$-2+\theta$	$-\lambda+2\mu$	$-3\lambda+\mu$
24	$\sqrt{-6}$	$\sqrt{-3}$	θ	-3	2μ	-3λ
40	$\sqrt{-10}$	$\sqrt{-5}$	θ	-5	2μ	-5λ
52	$\sqrt{-13}$	$\frac{1+\sqrt{-13}}{\sqrt{2}}$	$1+\theta$	$-6+\theta$	$-\lambda+2\mu$	$-7\lambda+\mu$
88	$\sqrt{-22}$	$\sqrt{-11}$	θ	-11	2μ	-11λ
148	$\sqrt{-37}$	$\frac{1+\sqrt{-37}}{\sqrt{2}}$	$1+\theta$	$-18+\theta$	$-\lambda+2\mu$	$-19\lambda+\mu$
232	$\sqrt{-58}$	$\sqrt{-29}$	θ	-29	2μ	-29λ

TABLE II.

Systems of numbers of the second type; $D = -4\delta + 1$, $\theta = \frac{1+\sqrt{D}}{2}$, λ and μ are conjugate.

$-D$	$N(\theta)=\delta$	λ	λ^2	$\lambda\mu$	$\lambda\theta$	$\mu\theta$
15	4	$\frac{\sqrt{5}+\sqrt{-3}}{2}$	θ	2	$\lambda - 2\mu$	2λ
35	9	$\frac{\sqrt{7}+\sqrt{-5}}{2}$	θ	3	$\lambda - 3\mu$	3λ
51	13	$\frac{\sqrt{17}+\sqrt{-3}}{2}$	$3+\theta$	5	$4\lambda - 5\mu$	$5\lambda - 3\mu$
91	23	$\frac{\sqrt{13}+\sqrt{-7}}{2}$	$1+\theta$	5	$2\lambda - 5\mu$	$5\lambda - \mu$
115	29	$\frac{\sqrt{23}+\sqrt{-5}}{2}$	$4+\theta$	7	$5\lambda - 7\mu$	$7\lambda - 4\mu$
123	31	$\frac{\sqrt{41}+\sqrt{-3}}{2}$	$9+\theta$	11	$10\lambda - 11\mu$	$11\lambda - 9\mu$
187	47	$\frac{\sqrt{17}+\sqrt{-11}}{2}$	$1+\theta$	7	$2\lambda - 7\mu$	$7\lambda - \mu$
235	59	$\frac{\sqrt{47}+\sqrt{-5}}{2}$	$10+\theta$	13	$11\lambda - 13\mu$	$13\lambda - 10\mu$
267	67	$\frac{\sqrt{89}+\sqrt{-3}}{2}$	$21+\theta$	23	$22\lambda - 23\mu$	$23\lambda - 21\mu$
403	101	$\frac{\sqrt{31}+\sqrt{-13}}{2}$	$4+\theta$	11	$5\lambda - 11\mu$	$11\lambda - 4\mu$
427	107	$\frac{\sqrt{61}+\sqrt{-7}}{2}$	$13+\theta$	17	$14\lambda - 17\mu$	$17\lambda - 13\mu$

TABLE III.

33. The system $D=-20$; $\theta=\sqrt{-5}$.

Table of numbers, their prime factors and norms, in the order of magnitude of their norms.

Principal Numbers.			Secondary Numbers.		
Numbers.	Prime Factors.	Norms.	Numbers.	Prime Factors.	Norms.
1	—	1	λ	—	2
2	λ^2	4	μ	—	3
θ	—	5	$\lambda-\mu$	—	,,
$1+\theta$	$\lambda\mu$	6	$\lambda+\mu$	—	7
$1-\theta$	$\lambda(\lambda-\mu)$	,,	$2\lambda-\mu$	—	,,
3	$\mu(\lambda-\mu)$	9	2λ	λ^3	8
$2+\theta$	$-(\lambda-\mu)^2$	,,	$\lambda-2\mu$	$-\lambda\theta$	10
$2-\theta$	$-\mu^2$	,,	2μ	$\lambda^2\mu$	12
$3+\theta$	$\lambda(\lambda+\mu)$	14	$2\lambda-2\mu$	$\lambda^2(\lambda-\mu)$	,,
$3-\theta$	$\lambda(2\lambda-\mu)$	,,	$2\lambda+\mu$	$\theta(\lambda-\mu)$	15
4	λ^4	16	$3\lambda-\mu$	$-\theta\mu$	,,
2θ	$\lambda^2\theta$	20	3λ	$\lambda\mu(\lambda-\mu)$	18
$4+\theta$	$\mu(2\lambda-\mu)$	21	$\lambda+2\mu$	$-\lambda(\lambda-\mu)^2$	,,
$4-\theta$	$(\lambda-\mu)(\lambda+\mu)$	,,	$3\lambda-2\mu$	$-\lambda\mu^2$	,,
$1+2\theta$	$-(\lambda-\mu)(2\lambda-\mu)$	,,	$\lambda-3\mu$	—	23
$1-2\theta$	$-\mu(\lambda+\mu)$	,,	$2\lambda-3\mu$	—	,,
$2+2\theta$	$\lambda^3\mu$	24	3μ	$\mu^2(\lambda-\mu)$	27
$2-2\theta$	$\lambda^3(\lambda-\mu)$	,,	$3\lambda-3\mu$	$\mu(\lambda-\mu)^2$	,,
5	$-\theta^2$	25	$3\lambda+\mu$	$-\mu^3$	,,
			$4\lambda-\mu$	$-(\lambda-\mu)^3$	,,

V. *Partitions of Numbers whose Graphs possess Symmetry.*

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It will be remembered that in *Phil. Trans. R. S. of London*, Vol. 187, 1896 A. pp. 619—673, I undertook the extension to three dimensions of Sylvester's constructive theory of Partitions. In Sylvester's two-dimensional theory every partition of a unipartite number can be associated with a regular two-dimensional graph. In the present theory only a limited number of the partitions of multipartite numbers can be represented by regular graphs in three dimensions. But whereas Sylvester was only concerned with unipartite numbers, the three-dimensional theory has to do with multipartite numbers of unrestricted multiplicity. Though the partitions of such are not all involved the field is infinitely greater, and all which come within the purview of the regular graph are brought harmoniously together. If in this new theory we restrict ourselves to two dimensions but view the graphs from a three-dimensional standpoint, we obtain in general six interpretations of the graphs instead of two and multipartite numbers are brought under consideration as well as those which are unipartite. The enumeration of the three-dimensional graphs of given weight (number of nodes), the numbers of nodes along the axes being restricted not to exceed l, m, n respectively, was conjectured in Part I. but only established for some particular values of l, m, n.

For $m \geqq l$ it may be written

$$\frac{1-x^{n+1}}{1-x}\cdot\left(\frac{1-x^{n+2}}{1-x^2}\right)^2\cdots\left(\frac{1-x^{n+l-1}}{1-x^{l-1}}\right)^{l-1}$$

$$\times\left\{\frac{1-x^{n+l}}{1-x^l}\cdot\frac{1-x^{n+l+1}}{1-x^{l+1}}\cdots\frac{1-x^{n+m}}{1-x^m}\right\}^l$$

$$\times\left(\frac{1-x^{n+m+1}}{1-x^{m+1}}\right)^{l-1}\cdot\left(\frac{1-x^{n+m+2}}{1-x^{m+2}}\right)^{l-2}\cdots\frac{1-x^{n+l+m-1}}{1-x^{l+m-1}}.$$

The symmetry of this expression and its real nature are best shewn by a symbolic crystalline form.

Observing that it is composed of factors of the form $(1-x^s)^t$, where t may be positive or negative, put

$$1-x^s=\exp.(-u^s) \text{ in the case of every factor,}$$

and it will be found, after a few simplifications, to take the form

$$\exp.\frac{u}{(1-u)^2}(1-u^l)(1-u^m)(1-u^n).$$

In the two-dimensional theory the generating function

$$\frac{(1-x^{l+1})(1-x^{l+2})\dots(1-x^{l+m})}{(1-x)(1-x^2)\dots(1-x^m)}$$

has the symbolic crystalline form

$$\exp.\frac{u}{1-u}(1-u^l)(1-u^m),$$

whilst in one-dimensional theory

$$\frac{1-x^{l+1}}{1-x}$$

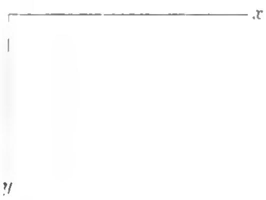
obviously leads to

$$\exp.u(1-u^l).$$

Hence we seem to have before us a system in κ dimensions associated with the crystalline form

$$\exp.\frac{u}{(1-u)^{\kappa-1}}(1-u^{l_1})(1-u^{l_2})\dots(1-u^{l_\kappa}).$$

In general a graph by rotations about the axes of x, y, and z



may assume six forms.

When these forms are identical the graph is said to be symmetrical or to possess xyz-symmetry.

Such ex. gr. is

When the six forms reduce to three the graph is said to be quasi-symmetrical. If it be such that each layer of nodes is symmetrical in two dimensions or, the same thing, is a Sylvester self-conjugate graph, it is said to possess xy-symmetry. Ex. gr.

Similarly the graph

possesses yz-symmetry, and by rotation about the y axis, or that of z, may be converted into one possessing xy or zx-symmetry.

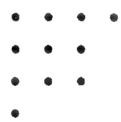
It is proposed to investigate generating functions for the enumeration of graphs possessing xy and xyz-symmetry, the former naturally including the latter.

Algebraic theorems will be evolved in the course of the work by the method initiated by Sylvester.

xy-SYMMETRICAL GRAPHS.

The self-conjugate Sylvester graphs which have i nodes along each axis can be formed by fitting into an angle of $2i-1$ nodes any number of angles of nodes, any angle containing an uneven number, less than $2i-1$, of nodes and no two angles possessing the same number of nodes. Ex. gr. for $i=7$ we have the angles

which by selection of the 1st, 3rd and 4th of the angles may be formed up into the graph



Hence, as Sylvester shewed, the generating function of such graphs is immediately seen to be

$$x^{2i-1}(1+x)(1+x^3)\dots(1+x^{2i-3}).$$

Each layer of the three-dimensional graph has this form, and if there be two layers at most we may construct a generating function

$$\Omega a_1a_2\dots a_i x^{2i-1}(1+a_1x)(1+a_1a_2x^3)\dots(1+a_1a_2\dots a_{i-1}x^{2i-3})$$
$$\times\left\{\left(1+\frac{x}{a_1}\right)\left(1+\frac{x^3}{a_1a_2}\right)\left(1+\frac{x^5}{a_1a_2a_3}\right)\dots \text{ ad inf.}\right\}$$

where Ω is a symbol of Cayley's which means that after multiplication all terms containing negative powers of a_1, a_2, $a_3\dots a_i$ are to be struck out and then each of these letters put equal to unity.

The first line of the expression following Ω is derived from

$$x^{2i-1}(1+x)(1+x^3)\dots(1+x^{2i-3})$$

by placing as coefficient to each x^{2s-1} the product $a_1a_2\dots a_s$.

The angles of the first or lower layer correspond to the powers of x in the first line, those of the second layer to the powers in the second line, and the operation of Ω is such as to prevent any combinations of the former and the latter which give rise to an irregular graph.

Summing this function from $i=1$ to $i=i$ and supposing its value unity when $i=0$ (a convention that is made only for convenience: no form exists for $i=0$) we obtain

$$\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots(1+a_1a_2\ldots a_ix^{2i-1})$$
$$\times\left\{\left(1+\frac{x}{a_1}\right)\left(1+\frac{x^3}{a_1a_2}\right)\left(1+\frac{x^5}{a_1a_2a_3}\right)\ldots\text{ ad inf.}\right\}$$

as the generating function which enumerates xy-symmetrical graphs of at most two layers, the number of nodes along an x or y axis being limited not to exceed i.

Further if i be infinite this becomes:—

$$\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots\ldots\text{ ad inf.}$$
$$\times\left(1+\frac{x}{a_1}\right)\left(1+\frac{x^3}{a_1a_2}\right)\left(1+\frac{x^5}{a_1a_2a_3}\right)\ldots\ldots\text{ ad inf.}$$

It is moreover clear that the generating function of xy-symmetrical graphs which have at most i nodes along each of the axes x, y and at most j nodes along the axis of z (i.e. which involve at most j layers) is:—

$$\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots\ldots(1+a_1a_2\ldots a_ix^{2i-1})$$
$$\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\ldots\ldots\text{ ad inf.}$$
$$\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\ldots\ldots\text{ ad inf.}$$
$$\times\left(1+\frac{d_1}{c_1}x\right)\left(1+\frac{d_1d_2}{c_1c_2}x^3\right)\left(1+\frac{d_1d_2d_3}{c_1c_2c_3}x^5\right)\ldots\ldots\text{ ad inf.}$$

....

j rows,

Ω operating in regard to all the symbols, a, b, c, d, &c. ...

If the graphs be unrestricted, as regards i, we put $i=\infty$; and, if they be totally unrestricted, we regard the tableau, upon which Ω operates, as possessing an unlimited number of rows and columns.

The generating function is crude. One, which only involves x, is ultimately to be desired. It should be possible, by algebraic processes, to perform the operation Ω and thus to pick out the terms of the product which constitute the reduced generating function. This appears to be a matter of considerable difficulty, and in order to determine the probable form of the reduced function I have examined many particular cases and attempted its construction. My conclusion is that, writing (s) to denote $1-x^s$, the reduced function is, in all probability, an algebraic fraction of which the numerator is

$$\begin{array}{ll}
(j+1)(j+3)(j+5) & \ldots\ldots(j+2i-1) \\
\times(2j+4)(2j+6)(2j+8) & \ldots\ldots(2j+4i-4) \\
\times(2j+8)(2j+10)(2j+12) & \ldots\ldots(2j+4i-8) \\
\times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\times(2j+4s)(2j+4s+2)(2j+4s+4) & \ldots\ldots(2j+4i-4s) \\
\times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}$$

wherein, if i be even, there are $\frac{1}{2}i$ rows the last of which is

$$(2j+2i);$$

and, if i be uneven, there are $\frac{1}{2}(i-1)$ rows the last of which is

$$(2j+2i-2)(2j+2i)(2j+2i+2);$$

and the denominator is obtained from the numerator by putting $j=0$, viz.:—it is

$$\begin{array}{ll}
(1)(3)(5) & \ldots\ldots(2i-1) \\
\times(4)(6)(8) & \ldots\ldots(4i-4) \\
\times(8)(10)(12) & \ldots\ldots(4i-8) \\
\times\ldots\ldots\ldots\ldots\ldots\ldots & \\
\times(4s)(4s+2)(4s+4) & \ldots\ldots(4i-4s) \\
\times\ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}$$

the last row being $(2i)$ or $(2i-2)(2i)(2i+2)$ according as i is even or uneven.

The proof of this formula, the truth of which seems unquestionable, is much to be desired.

When the number of layers of nodes is unrestricted we put $j=\infty$ and the numerator reduces to unity. When moreover both i and j are unrestricted in magnitude the reduced function becomes

$$\frac{1}{(1)(3)(5)(7)\ldots(4)(6)(8)^2(10)^2(12)^3(14)^3(16)^4(18)^4\ldots\ldots},$$

or as it may be also written

$$\frac{(1+x)(1+x^3)(1+x^5)(1+x^7)\ldots\ldots}{(2)(4)(6)^2(8)^2(10)^3(12)^3(14)^4(16)^4\ldots\ldots},$$

wherein the numerator denotes the generating function of Sylvester's unrestricted self-conjugate graphs in two dimensions.

Some particular cases are interesting.

By putting $j=1$ we should obtain Sylvester's result in two dimensions.

We find

$$\frac{(2)(6)(10)(14)\ldots(4i-2)}{(1)(3)(5)(7)\ldots(2i-1)},$$

which may be written

$$(1+x)(1+x^3)(1+x^5)(1+x^7)\ldots(1+x^{2i-1})$$

and is right.

When $j=2$, we find

$$\frac{(2i+1)}{(1)}\cdot\frac{(2i+4)(2i+6)\ldots(4i-2)(4i)}{(4)\quad(6)\quad\ldots(2i-2)(2i)},$$

or

$$\frac{(2i+2)(2i+4)\ldots(4i-2)(4i)}{(2)\quad(4)\quad\ldots(2i-2)(2i)}+x\frac{(2i+4)(2i+6)\ldots(4i-2)(4i)}{(2)\ (4)\ (6)\quad\ldots(2i-2)}.$$

For an even weight $2w$ we must take the coefficients of x^w in

$$\frac{(i+1)(i+2)\ldots(2i-1)(2i)}{(1)\quad(2)\quad\ldots\ (i-1)\ (i)},$$

and this is the generating function of two-dimensional graphs of weight w, not more than i nodes being allowed along either the x or y axis. Hence a correspondence between the at-most-two-layer xy-symmetrical graphs of weight $2w$ restricted as to the x and y axes by the number i and the graphs in two dimensions of weight w restricted as to the axes by the number i.

Ex. gr. for $w=4$, $i=3$ the correspondence is

1 1 1	1 1	1 1	2 2 1	2 2	1 1 1
1	1 1	1	2	2 2	1 1 1
		1	1		1 1

For an uneven weight $2w+1$ we take the coefficients of x^w in

$$\frac{(i+2)(i+3)\ldots(2i-1)(2i)}{(1)(2)(3)\ \ldots\ (i-1)},$$

and this is the generating function of two-dimensional graphs of weight w, not more than $i+1$ nodes being allowed along the x axis nor more than $i-1$ along the y axis.

The correspondence established is that between the at-most-two-layer xy-symmetrical graphs of weight $2w+1$ restricted as to each of the x and y axes by the number i and the graphs of two dimensions of weight w restricted as to the x axis by the number $i+1$ and as to the y axis by the number $i-1$.

Ex. gr. for $w=5$, $i=4$ we have the five to five correspondence

1 1 1 1 1	1 1 1 1	1 1 1	1 1 1	1 1	1 1 1 1	2 1 1 1	2 2 1 1	2 2 1	2 2 2
	1	1 1	1	1 1	1 1 1	1 1 1	2 1	2 1 1	2 1
			1	1	1 1 1	1 1	1	1 1	2
					1	1	1		

where i is indefinite the generating function becomes

$$\frac{1+x}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots\text{ ad inf.}}$$

This curious result shews that the number of at-most-two-layer xy-symmetrical graphs of weight w is equal to the whole number of partitions of $\frac{1}{2}w$ or of $\frac{1}{2}(w-1)$ according as w is even or uneven.

There is another solution of the problem that has been under consideration.

Instead of constructing a generating function from successive layers of nodes parallel to the plane of xy, we may build one up by first considering all the exterior angles of nodes; then those which become exterior when the former are removed; and so on. Thus if any graph were

$$\begin{array}{ccccc} 4 & 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 & \\ 2 & 2 & 1 & 1 & \\ 2 & 1 & 1 & & \\ 1 & & & & \end{array}$$

we first take

$$\begin{array}{ccccc} 4 & 3 & 2 & 2 & 1 \\ 3 & & & & \\ 2 & & & & \\ 2 & & & & \\ 1 & & & & \end{array}$$

as constructed by the superposition of

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ 1 & & & & \\ 1 & & & & \\ 1 & & & & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 1 & \end{array} \qquad 1\,;$$

then

$$\begin{array}{ccc} 2 & 2 & 1 \\ 2 & & \\ 1 & & \end{array} \quad \text{made up of} \quad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & & \\ 1 & & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 1 & \end{array}\ ;$$

then

$$\begin{array}{cc} 1 & 1 \\ 1 & \end{array}\ .$$

We are then led to the crude generating function

$$\Omega\,\frac{1}{(1-m)(1-mx)(1-max^3)(1-mabx^5)(1-mabcx^7)\ldots(1-mabc\ldots x^{2i-1})}$$

$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots\text{ ad inf.,}$$

$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots\text{ ad inf.}$$

$$\vdots$$

$$\text{ad inf.}$$

in the ascending expansion of which we must take the coefficient of m^j, Ω operating in regard to the letters

$$\begin{array}{lll} a, & b, & c, \ldots \\ a', & b', & c', \ldots \\ a'', & b'', & c'', \ldots \\ \ldots & \ldots & \ldots \end{array}$$

We have, therefore, the identity

$$\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots(1+a_1a_2\ldots a_ix^{2i-1})$$
$$\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\ldots \text{ ad inf.}$$
$$\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\ldots \text{ ad inf.}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$j \text{ rows}$$

$$= \text{Co } m^j\,\Omega\,\frac{1}{(1-m)(1-mx)(1-max^3)(1-mabx^5)(1-mabcx^7)\ldots(1-mabc\ldots x^{2i-1})}$$
$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots \text{ ad inf.}$$
$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots \text{ ad inf.}$$
$$\vdots$$
$$\text{ad inf.}$$

and, when j is unrestricted,

$$\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots(1+a_1a_2\ldots a_ix^{2i-1})$$
$$\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\ldots \text{ ad inf.}$$
$$\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\ldots \text{ ad inf.}$$
$$\vdots$$
$$\text{ad inf.}$$

$$= \Omega\,\frac{1}{(1-x)(1-ax^3)(1-abx^5)(1-abcx^7)\ldots(1-abc\ldots x^{2i-1})}$$
$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots \text{ ad inf.}$$
$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots \text{ ad inf.}$$
$$\vdots$$
$$\text{ad inf.,}$$

a remarkable result, which it would be difficult to establish algebraically.

As it is necessary in the sequel we will now determine the generating function which enumerates the xy-symmetrical graphs, limited as above, but subject to a new restriction, viz. each layer of nodes is to be formed by, at most, s plane angles of nodes.

The enumeration, it is easy to see, is given by the coefficients of $m^s x^w$ in the development of

$$\frac{1}{1-m}\,\Omega\,(1+ma_1x)(1+ma_1a_2x^3)(1+ma_1a_2a_3x^5)\dots(1+ma_1a_2\dots a_ix^{2i-1})$$

$$\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\dots\text{ ad inf.}$$

$$\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\dots\text{ ad inf.}$$

$$\vdots$$

$$j\text{ rows,}$$

and also by the coefficients of $m^j x^w$ in the development of

$$\frac{1}{1-m}\,\Omega\,\frac{1}{(1-mx)(1-max^3)(1-mabx^5)(1-mabcx^7)\dots(1-mabc\dots x^{2i-1})}$$

$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\dots\text{ ad inf.}$$

$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\dots\text{ ad inf.}$$

$$\vdots$$

$$s\text{ rows.}$$

Let the coefficients of m^s in the former of these generating functions be denoted by $F_{j,s}(x)$, and denoting the generating functions by A and B respectively, we have:—

$$A = 1 + mF_{j,1}(x) + m^2F_{j,2}(x) + \dots + m^sF_{j,s}(x) + \dots,$$

$$B = 1 + mF_{1,s}(x) + m^2F_{2,s}(x) + \dots + m^jF_{j,s}(x) + \dots.$$

Moreover for $j=\infty$, we have

$$A = 1 + mF_{\infty,1}(x) + m^2F_{\infty,2}(x) + m^3F_{\infty,3}(x) + \dots,$$

and for $s=\infty$,

$$B = 1 + mF_{1,\infty}(x) + m^2F_{2,\infty}(x) + m^3F_{3,\infty}(x) + \dots.$$

THE xyz-SYMMETRICAL GRAPHS.

Just as Sylvester dissected the xy-symmetrical graph in two dimensions into plane angles we may dissect the xyz-symmetrical graphs in three dimensions into solid angles. Each solid angle is in the shape of a symmetrical fragment of half of a hollow cube.

In each of the planes xy, yz, zx we find the same symmetrical two-dimensional graph. If this graph has i columns or rows the number of nodes which lie on one or other of the three axes is $1+3(i-1)$ or $3i-2$. In the plane of xy we can place plane angles of nodes so as to form a symmetrical graph in two dimensions. If w be the weight of the solid angle we have $w-3i+2$ nodes to dispose symmetrically in the three planes and this can be done in a number of ways which is given by the coefficients of

$$x^{\frac{1}{3}(w-3i+2)}$$

in
$$(1+x)(1+x^3)(1+x^5)\ldots(1+x^{2i-3}),$$
that is, by the coefficients of x^w in
$$Q_i = x^{3i-2}(1+x^3)(1+x^9)(1+x^{15})\ldots(1+x^{6i-9}),$$
which is therefore the generating function of the solid angles in question which have exactly i nodes along each axis. Observe that $i-1$ factors follow x^{3i-2}, and that, when it is convenient, we suppose the expression to have the value unity when $i=0$.

Hence the solid angles which possess i or fewer nodes along the axes are enumerated by

$$\begin{aligned}
&1\\
&+x\\
&+x^4(1+x^3)\\
&+x^7(1+x^3)(1+x^9)\\
&+\ldots\ldots\\
&+x^{3i-2}(1+x^3)(1+x^9)(1+x^{15})\ldots(1+x^{6i-9}).
\end{aligned}$$

Fitting solid angle graphs together when possible produces xyz-symmetrical graphs.

When $i=2$, $Q_2 = x^4(1+x^3)$, the two solid angles being

⊙ •   ⊙ ⊙
•   ⊙ •

of contents 4 and 7 respectively.

We cannot fit a solid angle into the first of these, for there is no node upon which it can rest. In the case of the second we can fit in the solid angle for which $i=1$, $Q_1 = x$ represented by a single node •, and thus form the symmetrical graph

⊙ ⊙
⊙ ⊙ of content 8.

Synthetically we form the generating function

$$\Omega x^4\left(1+ax^3\right)\left(1+\frac{1}{a}x\right) = x^4 + x^7 + x^8$$

of all symmetric graphs having $i=2$.

Observe that the construction of the factors, following the operator Ω, permits the association of

⊙ ⊙
⊙ • and •

and does not permit that of

⊙ •
• and •

Restricting ourselves to two solid angles when

$$i = 3,\quad Q_3 = x^7(1+x^3)(1+x^9)$$

we are similarly led to the construction of the generating function

$$\Omega x^7(1+ax^3)(1+abx^9)\times\left\{1+\frac{x}{a}+\frac{x^4}{ab}\left(1+\frac{x^3}{a}\right)\right\},$$

whence after expansion and operation we find

$$(x^7+x^{10}+x^{16}+x^{19})+(x^{11}+x^{17}+x^{20})+(x^{20}+x^{23})+x^{26},$$

and the correspondence is

x^7	$x^7 . ax^3$	$x^7 \times abx^9$	$x^7 . ax^3 . abx^9$
3 1 1 1 1	3 2 1 2 1 1	3 3 2 3 1 1 2 1	3 3 3 3 1 1 3 1 1

$x^7 . ax^3 . \frac{x}{a}$	$x^7 . abx^9 . \frac{x}{a}$	$x^7 . ax^3 . abx^9 . \frac{x}{a}$
3 2 1 2 2 1	3 3 2 3 2 1 2 1	3 3 3 3 2 1 3 1 1

$x^7 . abx^9 . \frac{x^4}{ab}$	$x^7 . ax^3 . abx^9 . \frac{x^4}{ab}$
3 3 2 3 3 2 2 2	3 3 3 3 3 2 3 2 1

$x^7 . ax^3 . abx^9 . \frac{x^4}{ab} . \frac{x^3}{a}$.
3 3 3 3 3 3 3 3 2

In the form which arises from the product $x^7 . abx^9 . \frac{x}{a}$, the largest solid angle is given by $x^7 . abx^9$; that is, x^7 gives the axial portion $\begin{matrix}3&1&1\\1&&\\1&&\end{matrix}$, x^9 yields $\begin{matrix}1&1\\1&\end{matrix}$ in each of the three planes, so that the resulting angle is $\begin{matrix}3&3&2\\3&1&1\\2&1&\end{matrix}$; the next largest solid angle is given by $\frac{x}{a}$ and this fits into the larger.

Again from $x^7 . ax^3 . abx^9 . \frac{x^4}{ab} . \frac{x^3}{a}$ we get first $\begin{matrix} 3 & 1 & 1 \\ 1 & & \\ 1 & & \end{matrix}$ from x^7, and then $\begin{matrix} 1 & 1 \\ 1 & \end{matrix}$ and 1, in each plane, from x^9 and x^3, yielding $\begin{matrix} 3 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{matrix}$ the outer solid angle; and $\frac{x^4}{ab} . \frac{x^3}{a}$ gives a solid angle, composed of $\begin{matrix} 2 & 1 \\ 1 & \end{matrix}$ and 1 fitting in each plane, viz.:— $\begin{matrix} 2 & 2 \\ 2 & 1 \end{matrix}$, and this fits into the larger solid angle yielding $\begin{matrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 2 \end{matrix}$.

It will be clear now that the generating function for symmetrical graphs having i nodes along each axis and formed of at most two solid angles is

$$\Omega x^{3i-2} (1 + a_1 x^3)(1 + a_1 a_2 x^9) \ldots (1 + a_1 a_2 \ldots a_{i-1} x^{6i-9})$$

$$\times \left\{ 1 + \frac{x}{a_1} + \frac{x^4}{a_1 a_2} \left(1 + \frac{x^3}{a_1}\right) + \frac{x^7}{a_1 a_2 a_3} \left(1 + \frac{x^3}{a_1}\right) \left(1 + \frac{x^9}{a_1 a_2}\right) + \ldots \text{ad inf.} \right\},$$

the general term in the series to infinity being

$$\frac{x^{3s-5}}{a_1 a_2 \ldots a_{s-1}} \left(1 + \frac{x^3}{a_1}\right) \left(1 + \frac{x^9}{a_1 a_2}\right) \ldots \left(1 + \frac{x^{6s-15}}{a_1 a_2 \ldots a_{s-2}}\right).$$

Summing this function, for values of i, it is found that the generating function, for the graphs composed of at most two solid angles and having at most i nodes along each axis, is

$$\Omega \{1 + x + x^4 (1 + a_1 x^3) + x^7 (1 + a_1 x^3)(1 + a_1 a_2 x^9) + \ldots$$
$$+ x^{3i-2} (1 + a_1 x^3)(1 + a_1 a_2 x^9) \ldots (1 + a_1 a_2 \ldots a_{i-1} x^{6i-9})\}$$

$$\times \left\{ 1 + \frac{x}{a_1} + \frac{x^4}{a_1 a_2} \left(1 + \frac{x^3}{a_1}\right) + \frac{x^7}{a_1 a_2 a_3} \left(1 + \frac{x^3}{a_1}\right) \left(1 + \frac{x^9}{a_1 a_2}\right) + \ldots \text{ad inf.} \right\}.$$

If the graphs are to be composed of at most two solid angles but to be otherwise unrestricted we obtain

$$\Omega \{1 + x + x^4 (1 + a_1 x^3) + x^7 (1 + a_1 x^3)(1 + a_1 a_2 x^9) + \ldots \text{ad inf.}\}$$

$$\times \left\{ 1 + \frac{x}{a_1} + \frac{x^4}{a_1 a_2} \left(1 + \frac{x^3}{a_1}\right) + \frac{x^7}{a_1 a_2 a_3} \left(1 + \frac{x^3}{a_1}\right) \left(1 + \frac{x^9}{a_1 a_2}\right) + \ldots \text{ad inf.} \right\}.$$

It is now easy to pass to the general case in which the composition is to be from at most s solid angles. The generating function is

$$\Omega \{1 + x + x^4 (1 + a_1 x^3) + x^7 (1 + a_1 x^3)(1 + a_1 a_2 x^9) + \ldots$$
$$+ x^{3i-2} (1 + a_1 x^3)(1 + a_1 a_2 x^9) \ldots (1 + a_1 a_2 \ldots a_{i-1} x^{6i-9})\}$$
$$\times \left\{1 + \frac{x}{a_1} + \frac{x^4}{a_1 a_2}\left(1 + \frac{b_1}{a_1} x^3\right) + \frac{x^7}{a_1 a_2 a_3}\left(1 + \frac{b_1}{a_1} x^3\right)\left(1 + \frac{b_1 b_2}{a_1 a_2} x^9\right) + \ldots \text{ ad inf.}\right\}$$
$$\times \left\{1 + \frac{x}{b_1} + \frac{x^4}{b_1 b_2}\left(1 + \frac{c_1}{b_1} x^3\right) + \frac{x^7}{b_1 b_2 b_3}\left(1 + \frac{c_1}{b_1} x^3\right)\left(1 + \frac{c_1 c_2}{b_1 b_2} x^9\right) + \ldots \text{ ad inf.}\right\}$$
$$\times \left\{1 + \frac{x}{c_1} + \frac{x^4}{c_1 c_2}\left(1 + \frac{d_1}{c_1} x^3\right) + \frac{x^7}{c_1 c_2 c_3}\left(1 + \frac{d_1}{c_1} x^3\right)\left(1 + \frac{d_1 d_2}{c_1 c_2} x^9\right) + \ldots \text{ ad inf.}\right\}$$
$$\times \ldots\ldots$$

s rows.

When the first row is also continued to infinity, and the number of rows is infinite, we have the crude form of generating function for xyz-symmetrical graphs quite unrestricted.

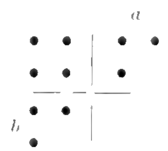
When $s = 1$ and $i = \infty$ it may be easily proved that the generating function may be written

$$1 + \frac{x}{1 - x^3} + \frac{x^7}{(1 - x^3)(1 - x^9)} + \frac{x^{19}}{(1 - x^3)(1 - x^9)(1 - x^{15})} + \ldots$$
$$+ \frac{x^{(k-1)^3 - (k-2)^3}}{(1 - x^3)(1 - x^9)(1 - x^{15}) \ldots (1 - x^{6k-9})} + \ldots$$

There is another mode of enumeration of xyz-symmetrical graphs which it is important to consider.

Durfee has shewn how to dissect a symmetrical graph in two dimensions into a square of nodes and two appendages lateral and subjacent.

Ex. gr. the graph



where this is a square of four nodes, a lateral appendage a and one which is subjacent b. This dissection leads to the expression of the generating function in the form of an infinite series of algebraic fractions. Sylvester further applied the same dissection to unsymmetrical graphs and derived algebraic identities of great interest.

In the case of three dimensions we also have a dissection of the same nature. This is not based upon the isolation of a cube of nodes as might at first appear.

If we take such a cube, for example,

2 2
2 2

we may, it is true, attach appropriate lateral, subjacent, and superjacent graphs and thus obtain an xyz-symmetrical graph; but a slight consideration shews that a large number of symmetrical graphs escape enumeration by this process. Ex. gr. the graph

$$\begin{array}{cc|c} 3 & 3 & 2 \\ 3 & 2 & 1 \\ \hline 2 & 1 & \end{array}$$

is based upon the cube in question, whereas the graph

$$\begin{array}{cc|c} 3 & 3 & 3 \\ 3 & 2 & 1 \\ \hline 3 & 1 & 1 \end{array}$$

is not based upon that or any other cube, yet it is without doubt symmetrical.

In the former of the two graphs observe that the appendages are

$$\text{lateral} \quad \begin{array}{c} 2 \\ 1 \end{array}$$

$$\text{subjacent} \quad 2\ 1$$

$$\text{superjacent} \quad \begin{array}{cc} 1 & 1 \\ 1 & \end{array}.$$

The fact is that symmetrical graphs are based also upon graphs other than those which are perfect cubes.

The whole series is formed as follows:—

We have, first, those based upon the cube 1, viz.

the base is 1.

Secondly, we have those based upon graphs such that there is a square of four nodes in each of the three planes of reference. These are of two kinds, viz.:—

$$\begin{array}{cc} 2 & 2 \\ 2 & 1 \end{array} \qquad \begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array}$$

where the nodes of the former are in the shape of the half of a hollow cube and the latter is obtained from the former by combining with it the cube 1.

Thirdly, we have four bases derived from the graph which has the shape of a half-hollow-cube of side 3; viz.:—

$$\begin{array}{ccc} 3 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 3 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 2 \end{array} \qquad \begin{array}{ccc} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{array}$$

where observe that the three latter bases are derived from the former by combination with the three bases previously constructed, viz.:—

$$1 \qquad \begin{array}{cc} 2 & 2 \\ 2 & 1 \end{array} \qquad \begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array}.$$

Similarly, of the fourth order, we have eight bases, viz.:—

$$\begin{array}{cccc} 4\,4\,4\,4 & 4\,4\,4\,4 & 4\,4\,4\,4 & 4\,4\,4\,4 \\ 4\,1\,1\,1 & 4\,2\,1\,1 & 4\,3\,3\,1 & 4\,3\,3\,1 \\ 4\,1\,1\,1 & 4\,1\,1\,1 & 4\,3\,2\,1 & 4\,3\,3\,1 \\ 4\,1\,1\,1 & 4\,1\,1\,1 & 4\,1\,1\,1 & 4\,1\,1\,1 \end{array}$$

$$\begin{array}{cccc} 4\,4\,4\,4 & 4\,4\,4\,4 & 4\,4\,4\,4 & 4\,4\,4\,4 \\ 4\,4\,4\,4 & 4\,4\,4\,4 & 4\,4\,4\,4 & 4\,4\,4\,4 \\ 4\,4\,2\,2 & 4\,4\,3\,2 & 4\,4\,4\,4 & 4\,4\,4\,4 \\ 4\,4\,2\,2 & 4\,4\,2\,2 & 4\,4\,4\,3 & 4\,4\,4\,4 \end{array}$$

the seven latter being derived from the former by combination with the seven forms previously constructed.

The way in which the bases are built up is now plain and we see that, of order n, we can construct 2^{n-1} bases of which $2^{n-1}-1$ are derived by combining with the half-hollow-square of order n, all the bases of lower orders in number,

$$1+2+2^2+\ldots+2^{n-2}=2^{n-1}-1.$$

As one illustration take the graph

$$\begin{array}{l|lll|llll} o & & & & & & & x \\ \hline & & & & & a & & \\ & 6 & 5 & 4 & 3 & 2 & 1 & \\ c & 5 & 2 & 1 & 1 & 1 & & \\ & 4 & 1 & 1 & 1 & & & \\ \cline{2-4} & 3 & 1 & 1 & & & & \\ b & 2 & 1 & & & & & \\ y & 1 & & & & & & \end{array}$$

the z axis being perpendicular to the plane of the paper.

The graph is built upon the base

$$\begin{array}{ccc} 3 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 1; \end{array}$$

a the lateral appendage being

$$\begin{array}{ccc} 3 & 2 & 1 \\ 1 & 1 & \\ 1; & & \end{array}$$

b the subjacent appendage being

$$\begin{array}{ccc} 3 & 1 & 1 \\ 2 & 1 & \\ 1: & & \end{array}$$

c the superjacent appendage being

$$\begin{array}{ccc} 3 & 2 & 1 \\ 2 & & \\ 1. & & \end{array}$$

From the lateral appendage we derive in succession the subjacent and superjacent appendages.

The rule is to face the origin and give the lateral graph right-handed rotations through 90° about the axes of z and y in succession. We thus derive the subjacent graph, and a repetition of the process upon the latter then gives the superjacent graph.

Thus starting with the lateral

$$\begin{array}{lll} 3 & 2 & 1 \\ 1 & 1 & \\ 1 & & \end{array} ,$$

the two rotations give in succession

$$\begin{array}{rrr} 1 & 1 & 3 \\ & 1 & 2 \\ & & 1 \end{array} \quad \text{and} \quad \begin{array}{lll} 3 & 1 & 1 \\ 2 & 1 & \\ 1 & & \end{array} ,$$

the latter being the subjacent, and operating similarly on the latter we obtain in succession

$$\begin{array}{rrr} 1 & 2 & 3 \\ & 1 & 1 \\ & & 1 \end{array} \quad \text{and} \quad \begin{array}{lll} 3 & 2 & 1 \\ 2 & & \\ 1 & & \end{array} ,$$

the last written graph being the superjacent.

As another example, if the lateral be

$$\begin{array}{lll} 2 & 2 & 1 \\ 1 & 1 & \end{array} ,$$

we obtain by operation

$$\begin{array}{rr} 1 & 2 \\ 1 & 2 \\ & 1 \end{array} \quad \text{and} \quad \begin{array}{ll} 2 & 1 \\ 2 & 1 \\ 1 & \end{array} ,$$

giving $\begin{array}{ll} 2 & 1 \\ 2 & 1 \\ 1 & \end{array}$ the subjacent; operating upon this

$$\begin{array}{rrr} 1 & 2 & 2 \\ & 1 & 1 \end{array} \quad \text{and} \quad \begin{array}{ll} 3 & 2 \\ 2 & \end{array}$$

giving $\begin{array}{ll} 3 & 2 \\ 2 & \end{array}$ the superjacent.

Compare the graph

$$\begin{array}{llllll} 5 & 4 & & 2 & 2 & 1 \\ 4 & 1 & & 1 & 1 & \\ 2 & 1 & & & & \\ 2 & 1 & & & & \\ 1 & & & & & \end{array}$$

upon the base

$$\begin{array}{ll} 2 & 2 \\ 2 & 1 \end{array} .$$

We have arrived at the point of shewing the construction of the bases and we have seen how to construct the graph, being given the base and the lateral; the base and the lateral completely determine the graph, and if they be of contents w_1, w_2 respectively the complete graph is of content $w_1 + 3w_2$. For a given base we have now to determine the possible forms of lateral appendage preparatory to attempting their enumeration.

Every line of numbers parallel to the axis of y in a symmetrical graph is of necessity a self-conjugate partition of a number, for otherwise more than one interpretation of the graph would be obtainable. Ex. gr. in the graph

5 4 2 2 1
4 1 1 1
2 1
2 1
1

5 4 2 2 1 is a self-conjugate partition of the number 14,
2 1 " " " " " 3,

the corresponding symmetrical two-dimensional graphs being

Hence this self-conjugate property appertains also to the lateral appendage, the lines of numbers being taken parallel to the axis of y, *not* parallel to the axis of x. The reverse would naturally be the case if we were considering the subjacent appendage. This property imposes a limitation upon the possible forms of lateral appendage.

Let w_1 be the content of the base, i_1 its order i.e. the number of nodes along an axis; also let w_2 and i_2 refer to the lateral.

Then for the complete symmetrical graph we have content $w_1 + 3w_2$ and order $i_1 + i_2$, or say, w, i referring to the complete graph,

$$w = w_1 + 3w_2, \quad i = i_1 + i_2.$$

For the base 1

$w_1 = 1$, $i_1 = 1$ the lateral must have the form

1 1 1 1 ...,

and the generating function for such laterals whose order does not exceed i_2 is

$$\frac{1 - x^{i_2+1}}{1 - x};$$

therefore if $F(x)$ denote the generating function of the associated symmetrical graphs

$$*\,\mathrm{Co}\, x^{w} F(x) = \mathrm{Co}\, x^{1+3w_2} F(x) = \mathrm{Co}\, x^{w_2} \frac{1-x^{i_2+1}}{1-x},$$

$$\therefore\ F(x) = x\frac{1-x^{3i_2+3}}{1-x^3},$$

the generating function of symmetrical graphs on the base 1.

Since $i_2 = i-1$, we may write this

$$F(x) = x\frac{1-x^{3i}}{1-x^3}.$$

For the base $\begin{matrix}2 & 2\\ 2 & 1\end{matrix}$ the lateral may involve $\begin{matrix}2\\1\end{matrix}$ and 1 but not $\begin{matrix}1\\1\end{matrix}$; hence its form must be

$$\begin{matrix}2 & 2 & 2 \ldots & 1 & 1 & 1 \ldots\\ 1 & 1 & 1 & & & \end{matrix}$$

If i_2 be unrestricted the lateral generating function is

$$\frac{1}{1-x\,.\,1-x^3};$$

otherwise we have to seek the coefficient of $m^{i_2}x^{w_2}$ in

$$\frac{1}{1-m\,.\,1-mx\,.\,1-mx^3};$$

and since $w_1 = 7,\ i_1 = 2,\ i_2 = i-2,$

we obtain the generating function of symmetrical graphs

$$\frac{x^7}{1-m\,.\,1-mx^3\,.\,1-mx^9},$$

in which we seek the coefficient of $m^{i-2}x^{w}$.

Similarly for the base $\begin{matrix}2 & 2\\ 2 & 2\end{matrix}$ since the lateral must be of the form $\begin{matrix}2 & 2\\ 2 & 2\end{matrix}\ldots\begin{matrix}2 & 2\\ 1 & 1\end{matrix}\ldots\begin{matrix}1 & 1\ldots\\ & \end{matrix}$ we are led to the generating function

$$\frac{x^8}{1-m\,.\,1-mx^3\,.\,1-mx^9\,.\,1-mx^{12}},$$

in which we seek the coefficient of $m^{i-2}x^{w}$.

If the base is at most of order 2 we may say that the enumeration of symmetrical graphs of content w and having at most i nodes along an axis is given by the coefficient of m^0x^w in

$$1 + \frac{m^{1-i}x}{1-m\,.\,1-mx^3} + \frac{m^{2-i}x^7}{1-m\,.\,1-mx^3\,.\,1-mx^9} + \frac{m^{2-i}x^8}{1-m\,.\,1-mx^3\,.\,1-mx^9\,.\,1-mx^{12}}.$$

* $\mathrm{Co}\,x^w F(x)$ denotes the coefficient of x^w in the expansion of $F(x)$.

If i_2 and therefore i be unrestricted this expression naturally becomes

$$1 + \frac{x}{1-x^3} + \frac{x^7}{1-x^3 \,.\, 1-x^9} + \frac{x^8}{1-x^3 \,.\, 1-x^9 \,.\, 1-x^{14}} \cdot$$

The question now arises as to the direct formation of the fractions appertaining to the bases of order i_1.

The form of the lateral appendage depends, as we have seen, upon the self-conjugate unipartite partition represented by the right-hand column or boundary of the base. We will call this partition the base-lateral. So far of the first four orders we have met with certain base-laterals, viz.:—

Order $i_1 =$	Base-lateral			
1	1			
2	2	2		
	1	2		
3	3	3	3	
	1	3	3	
	1	2	3	
4	4	4	4	4
	1	4	4	4
	1	2	4	4
	1	2	3	4

Of order i_1 there are i_1 different base-laterals; for consider the formation of the base of order n from those of inferior orders. Combination of the half-hollow-cube form of order i_1 with the bases of orders less than $i_1 - 1$ can only result in base-laterals identical with that of the half-hollow-cube base; and assuming that base-laterals of order $i_1 - 1$ are $i_1 - 1$ in number, it is plain that the combination referred to can only produce $i_1 - 1$ additional base-laterals. Hence, on the assumption made, the whole number of base-laterals of order i_1 is $1 + i_1 - 1 = i_1$. By induction the theorem is established.

The i_1 base-laterals of order i_1 are (writing them for convenience horizontally instead of vertically)

$$i_1 1^{i_1 - 1}, \quad i_1^2 2^{i_1 - 2}, \quad i_1^3 3^{i_1 - 3}, \;\ldots\ldots\; i_1^{i_1}.$$

We must discover the generating function of bases having a given base-lateral $i_1^s s^{i_1 - s}$.

The base-lateral in question is associated with $2^{i_1 - s - 1}$ different bases if $s < n$, while $i_1^{i_1}$ is associated with but a single base. Taking $s = 1$, the simplest base, having $i_1 1^{i_1 - 1}$ for base-lateral, is the half-hollow-graph of content $i_1^3 - (i_1 - 1)^3$. The remaining bases with this base-lateral are obtained by combination with the bases of the first $i_1 - 2$ orders.

Denote by $u_s - 1$ the generating function of the bases of the first s orders: then

$$u_{s+1} = \{1 + x^{(s+1)^3 - s^3}\}\, u_s,$$

or

$$u_{i_1-2} = \{1 + x^{(i_1-2)^3 - (i_1-3)^3}\}\, u_{i_1-3},$$

whence

$$u_{i_1-2} = (1+x)(1+x^{2^3-1^3})(1+x^{3^3-2^3})\dots\{1+x^{(i_1-2)^3-(i_1-3)^3}\}.$$

Therefore the generating function of bases, having the base-lateral $i_1 1^{i_1-1}$, is

$$x^{i_1^3-(i_1-1)^3}[(1+x)(1+x^{2^3-1^3})(1+x^{3^3-2^3})\dots\{1+x^{(i_1-2)^3-(i_1-3)^3}\}].$$

Next consider the bases having the base-lateral

$$i_1^2 2^{i_1-2}.$$

The simplest base of this nature is derived by combining the half-hollow-cube form of content $i_1^3-(i_1-1)^3$ with the similar form of content $(i_1-1)^3-(i_1-2)^3$ and thus it has the content $i_1^3-(i_1-2)^3$. With this we can again combine every base of the first i_1-3 orders without altering the base-lateral, which remains $i_1^2 2^{i_1-2}$. Hence the bases are enumerated by the generating function

$$x^{i_1^3-(i_1-2)^3}[(1+x)(1+x^{2^3-1^3})(1+x^{3^3-2^3})\dots\{1+x^{(i_1-3)^3-(i_1-4)^3}\}].$$

In general the simplest base with base-lateral $i_1^s s^{i_1-s}$ is obtained by combining the half-hollow-cube forms of orders $i_1,\ i_1-1,\ i_1-2, \dots i_1-s+1$, and thus has the content

$$i_1^3-(i_1-s)^3.$$

By reasoning before employed we arrive at the fact that the bases with base-lateral $i_1^s s^{i_1-s}$ are enumerated by

$$x^{i_1^3-(i_1-s)^3}[(1+x)(1+x^{2^3-1^3})(1+x^{3^3-2^3})\dots\{1+x^{(i_1-s-1)^3-(i_1-s-2)^3}\}].$$

In this expression, between the brackets [ ] there are i_1-s-1 factors; if $s=i_1-1$ or i_1 we take merely $x^{i_1^3-1^3}$ and $x^{i_1^3}$ respectively.

The next question is the ascertainment of the generating function which enumerates the lateral appendages that can be associated with the base-lateral $i_1^s s^{i_1-s}$. When $s=1$ this is easy because the lateral must be composed of columns

1	2	3 ...	i_1
	1	1	1
		1	1
			1
			⋮

not more than i_2 being taken.

The generating function is

$$\frac{1}{1-m\,.\,1-mx\,.\,1-mx^3\,.\,1-mx^5\,.\dots 1-mx^{2i_1-1}},$$

where we seek the coefficient of $m^{i_2}x^{w_2}$ or of $m^{i-i_1}x^{w_2}$ since $i=i_1+i_2$.

Finally for the symmetrical graphs constructed on bases whose base-laterals are $i_1 1^{i_1-1}$ we have the generating function

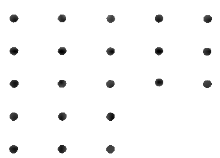
$$\frac{x^{i_1^3-(i_1-1)^3}[(1+x)(1+x^{2^3-1^3})(1+x^{3^3-2^3})\dots\{1+x^{(i_1-2)^3-(i_1-3)^3}\}]}{(1-m)(1-mx^3)(1-mx^9)\dots(1-mx^{6i_1-3})},$$

in the expansion of which we must seek the coefficient of $m^{i-i_1}x^{w}$.

When the base-lateral is $i_1^s s^{i_1-s}$ the matter is by no means so simple. The lateral appendage is composed of columns each of which is a self-conjugate partition of a number, and the possible forms of the columns are further limited by the form of the base-lateral. To explain take $i_1 = 5$, $s = 3$ so that the base-lateral (written horizontally) is

5 5 5 3 3.

This has a graph



formed of three plane angles. Any column of the lateral must have a graph which can be superposed; the condition for this is obviously that it must be composed of not more than three plane angles, the largest angle containing not more than 9 nodes. So with base-lateral $i_1^s s^{i_1-s}$ a lateral column must have a graph composed of not more than s plane angles, the largest angle containing not more than $2i_1 - 1$ nodes.

The complete lateral appendage constitutes a multipartite partition whose graph is symmetrical in two dimensions. We have therefore to enumerate the graphs of this nature, each layer being composed of at most s plane angles and no angle containing more than $2i_1 - 1$ nodes and the number of layers not exceeding $i - i_1$ or i_2.

The crude form of this generating function was found earlier in the paper to be

$$\Omega \frac{1}{(1-m)(1-mx)(1-max^3)(1-mabx^5)\dots(1-mabc\dots x^{2i_1-1})},$$

$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\dots \text{ ad inf.},$$

$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\dots \text{ ad inf.},$$

....

s rows,

in which we take the coefficient of $m^{i-i_1}x^{w_2}$.

We can now assert that the symmetrical graphs which appertain to the base-lateral $i_1^s s^{i_1-s}$, and have at most i nodes along an axis, are enumerated by the coefficient of $m^{i-i_1}x^w$ in

$$\Omega \frac{x^{i_1^3-(i_1-s)^3}[(1+x)(1+x^{2^3-1^3})(1+x^{3^3-2^3})\dots\{1+x^{(i_1-s-1)^3-(i_1-s-2)^3}\}]}{(1-m)(1-mx^3)(1-max^9)(1-mabx^{15})\dots(1-mabc\dots x^{6i_1-3})},$$

$$\left(1-\frac{x^3}{a}\right)\left(1-\frac{a'}{ab}x^9\right)\left(1-\frac{a'b'}{abc}x^{15}\right)\dots \text{ad inf.},$$

$$\left(1-\frac{x^3}{a'}\right)\left(1-\frac{a''}{a'b'}x^9\right)\left(1-\frac{a''b''}{a'b'c'}x^{15}\right)\dots \text{ad inf.},$$

....

s rows.

Denoting this expression by $S(m, x)$, we see that

$$\sum_{i_1=1}^{i_1=i}\sum_{s=1}^{s=i_1} S(m, x)\,.\,m^{i_1-i}$$

enumerates, by the coefficient of m^0x^w, the whole of the symmetrical graphs subject to the single restriction that more than i nodes are not to occur along an axis.

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CONTENTS.

VI. *On Divergent (or Semiconvergent) Hypergeometric Series.*

By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin.

[*Received* and *read* May 16, *Revised* September 1898.]

1. The series

$$1+\frac{\alpha_1\alpha_2\ldots\alpha_m}{1.\rho_1\rho_2\ldots\rho_n}x+\frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)\ldots\alpha_m(\alpha_m+1)}{1.2.\rho_1(\rho_1+1)\rho_2(\rho_2+1)\ldots\rho_n(\rho_n+1)}x^2+\ldots\ldots\ldots\ldots(1)$$

is convergent for all values of x if $m \ngtr n$ and convergent for values of x whose modulus is less than unity if $m=n+1$; in such cases if its sum be denoted by

$$F(\alpha_1, \alpha_2, \ldots \alpha_m; \rho_1, \rho_2, \ldots \rho_n; x)$$

the successive differential coefficients of this expression are represented numerically by the convergent series obtained by taking the corresponding differential coefficients of the terms of (1). The relation connecting the coefficients of two consecutive terms a_rx^r, $a_{r+1}x^{r+1}$, viz.:—

$$(1+r)(\rho_1+r)\ldots(\rho_n+r)a_{r+1}=(\alpha_1+r)(\alpha_2+r)\ldots(\alpha_m+r)a_r\ldots\ldots\ldots\ldots(2),$$

is equivalent to the differential equation

$$\left\{(\theta+\alpha_1)(\theta+\alpha_2)\ldots(\theta+\alpha_m)-\frac{1}{x}\theta(\theta+\rho_1-1)(\theta+\rho_2-1)\ldots(\theta+\rho_n-1)\right\}y=0\ \ldots(3),$$

in which θ stands for the operator xd/dx. The series (1) is therefore, when convergent, a solution of this equation. Relation (2) is satisfied by n other series, convergent if (1) be convergent, one of which is

$$x^{1-\rho_1}F(\alpha_1-\rho_1+1,\ \alpha_2-\rho_1+1,\ldots\alpha_m-\rho_1+1;\ 2-\rho_1,\ \rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\ x)\ldots(4),$$

the others being analogous. Each of these n series when convergent is therefore a solution of equation (3), and the $n+1$ series thus furnish the complete solution of this equation for all values of x if $m \ngtr n$, and for values of x whose modulus is less than unity if $m=n+1$. It is supposed that no two of the quantities $\alpha_1,\ldots\alpha_m,\ \rho_1,\ldots\rho_n,\ 1$ are equal, or differ by an integer.

Relation (2) is also satisfied by m series proceeding in descending powers of x, one of which is

$$x^{-\alpha_1}\left\{1+\frac{\alpha_1(\alpha_1-\rho_1+1)\ldots(\alpha_1-\rho_n+1)}{1.(\alpha_1-\alpha_2+1)\ldots(\alpha_1-\alpha_m+1)}(-)^{n-m+1}\frac{1}{x}\right.$$
$$\left.+\frac{\alpha_1(\alpha_1+1)(\alpha_1-\rho_1+1)(\alpha_1-\rho_1+2)\ldots(\alpha_1-\rho_n+1)(\alpha_1-\rho_n+2)}{1.2.(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2+2)\ldots(\alpha_1-\alpha_m+1)(\alpha_1-\alpha_m+2)}\frac{1}{x^2}+\ldots\right\}\ldots(5),$$

the others being analogous. These m series are all divergent when the former $n+1$ are convergent. The object of the present paper is to show that in such a case if the real part of $(-1)^{n-m+1}x$ is negative, then provided s exceeds a certain number independent of x, the sum of the first s terms of such a series as (6) differs by some quantity whose modulus is less than that of the next term from a certain linear function of the $n+1$ convergent series (1), (5), etc., and that whether the real part of $(-1)^{n-m+1}x$ is positive or negative, for any specified value of s, x can be taken so great that the sum of the first s terms differs from the same linear function of the convergent series by a quantity whose modulus is less than that of the next term multiplied by $1+\epsilon$, where ϵ is any assigned small positive quantity.

It may be remarked that the theorem stated for the case in which $(-)^{n-m+1}x$ has its real part negative *cannot* be true without *some* restriction on the argument or the modulus of $(-)^{n-m-1}x$. It has been pointed out by Hankel (*Math. Annal.* Vol. I.) that such a theorem cannot hold for a series proceeding in powers of u whose terms, after a certain one, are real positive and increasing as the three results to which it would lead on terminating the series successively before each of three consecutive positive terms, of which the first is less than the second and the second less than the third, involve an inconsistency. Hankel however appears to consider that it may hold for all other values of u; this is a mistake unless the function of u to which the semiconvergent series is "equal" (in the sense above) is discontinuous on crossing some curve other than the positive part of the axis of real quantities, as it is evident that, with this exception, if the theorem be true for all save real and positive values of u it must be true even for these.

The well-known semiconvergent expansions of $J_n(x)$, for example, *cannot* therefore hold for complex values in the sense that for each of the two divergent series occurring therein the error committed in stopping after the sth term, provided $s-n+\frac{1}{2}$ is positive, has a modulus less than that of the next term, without some restriction on the value of x; and in fact the demonstrations given by Lipschitz (*Crelle*, LVI.), Hankel (*loc. cit.*), and Gray and Mathews (*Treatise on Bessel Functions*) are invalid unless x be wholly real. Lipschitz, who discusses only the case in which n is zero, appears in fact to consider only real values of x. The fallacies involved in the proofs by Hankel and by Gray and Mathews will be noted presently. The magnitude of the error in case x is complex has been discussed by H. Weber (*Math. Annal.* Vol. XXXVII.), who has not however explicitly referred to the fallacies in question.

2. As a lemma to be used in establishing the theorem of the present paper I proceed to prove that if $(1-t)^{-a}$ be expanded in ascending powers of t, where the modulus of t may be greater than unity, then provided $a+s$ be positive the modulus of the error outstanding after s terms is, if the real part of t be negative, less than that of the next term, and if the real part of t be positive, and t not wholly real, less than that of a certain multiple (involving the argument of t) of the next term.

We have

$$\frac{1}{z-t}=\frac{1}{z}\left\{1+\frac{t}{z}+\frac{t^2}{z^2}+\dots\frac{t^{s-1}}{z^{s-1}}\right\}+\frac{t^s}{z^s(z-t)}.$$

Multiplying by $\dfrac{(1-z)^{-a}\,dz}{2\pi i}$, where $(1-z)^{-\alpha}$ is equal to unity at the origin, and integrating along the path $ABCDEFA$ in Fig. 1, $CDEFA$ being supposed a curve, every point of which is at an infinite distance from the origin, we obtain

$$(1-t)^{-a} = 1 + \alpha t + \dots \text{(to } s \text{ terms)} + \frac{t^s}{2\pi i}\int \frac{(1-z)^{-a}}{z^s(z-t)}\,dz,$$

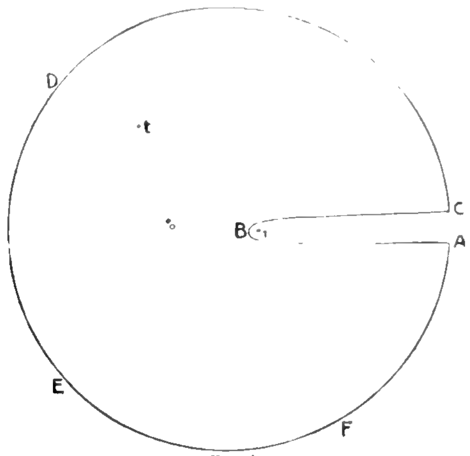

FIG. I

wherein the path of integration is the same and $(1-t)^{-\alpha}$ reduces to unity at the origin. If now $\alpha+s$ is positive the path $CDEFA$ contributes nothing to the integral on the right and the remainder after s terms, which we will denote by $R(\alpha, s)$, is equal to the integral taken along the path ABC only. This may be written, following Pochhammer (*Math. Annal.* XXXV.), in the form

$$\frac{-t^s}{2\pi i}\int_{\infty}^{(1)} \frac{(1-z)^{-a}}{z^s(z-t)}\,dz.$$

Suppose first that $1-\alpha$ is positive. We then have

$$R(\alpha, s) = \frac{t^s}{2\pi i}(e^{a\pi i} - e^{-a\pi i})\int_1^{\infty} \frac{(z-1)^{-a}}{z^s(z-t)}\,dz$$

$$= \frac{t^s}{\pi}\sin\alpha\pi \int_1^{\infty} \frac{(z-1)^{-a}}{z^s(z-t)}\,dz.$$

If the real part of t be negative

$$\text{mod.} \int_1^{\infty} \frac{(z-1)^{-a}}{z^s(z-t)}\,dz < \int_1^{\infty} \frac{(z-1)^{-a}}{z^{s+1}}\,dz < \frac{\Pi(-\alpha)\,\Pi(s+\alpha-1)}{\Pi(s)}.$$

Therefore since $\Pi(\alpha-1)\,\Pi(-\alpha) = \pi \operatorname{cosec} \alpha\pi$ we have

$$\text{mod. } R(\alpha, s) < \text{mod. } \frac{t^s\Pi(s+\alpha-1)}{\Pi(s)\Pi(\alpha-1)},$$

i.e. < modulus of first term omitted as stated.

If the real part of t be positive and $t = r(\cos\phi + i\sin\phi)$,

$$\text{mod.} \int_1^\infty \frac{(z-1)^{-\alpha}}{z^s(z-t)} dz < \frac{1}{\sin\phi} \int_1^\infty \frac{(z-1)^{-\alpha}}{z^{s+1}} dz,$$

and therefore

$$\text{mod. } R(\alpha, s) < t^s \frac{\Pi(s+\alpha-1)}{\Pi(s)\,\Pi(\alpha-1)} \frac{1}{\sin\phi}.$$

Next suppose that $1-\alpha$ is negative but that $n+1-\alpha$ is positive, n being a positive integer. By partial integration performed n times in succession we obtain the result

$$R(\alpha, s) = \frac{(-)^{n+1} t^s}{2\pi i.(1-\alpha)(2-\alpha)\ldots(n-\alpha)} \int_\infty^{(1)} dz.(1-z)^{n-\alpha} \frac{d^n}{dz^n}\left\{\frac{1}{z^s(z-t)}\right\}$$

$$= \frac{-\Pi(n)\, t^s}{2\pi i(1-\alpha)(2-\alpha)\ldots(n-\alpha)} \int_\infty^{(1)} (1-z)^{n-\alpha} \left\{\frac{1}{z^s(z-t)^{n+1}} + \frac{s}{1.z^{s+1}(z-t)^n} + \frac{s.(s+1)}{1.2.z^{s+2}(z-t)^{n-1}} + \ldots + \frac{s(s+1)\ldots(s+n-1)}{\Pi(n).z^{s+n}(z-t)}\right\} dz$$

$$= \frac{\Pi(n)\, t^s \sin(\alpha-n)\pi}{\pi.(1-\alpha)(2-\alpha)\ldots(n-\alpha)} \int_1^\infty (z-1)^{n-\alpha} \left\{\frac{1}{z^s(z-t)^{n+1}} + \frac{s}{1.z^{s+1}(z-t)^n} + \ldots + \frac{s(s+1)\ldots(s+n-1)}{\Pi(n)\,z^{s+n}(z-t)}\right\} dz \quad \ldots\ldots(6).$$

If now the real part of t be negative, this integral would be increased in absolute value by replacing every negative power of $z-t$ by the same power of z; if this were done the coefficient of $(z-1)^{n-\alpha} z^{-s-n-1}$ under the sign of integration would be

$$1 + \frac{s}{1} + \frac{s(s+1)}{1.2} + \ldots \frac{s(s+1)\ldots(s+n-1)}{1.2\ldots n},$$

which is equal to

$$\frac{(s+1)(s+2)\ldots(s+n)}{1.2\ldots n}.$$

Accordingly, since

$$\int_1^\infty (z-1)^{n-\alpha} z^{-s-n-1} dz = \frac{\Pi(n-\alpha)\,\Pi(s+\alpha-1)}{\Pi(s+n)},$$

we have

$$\text{mod. } R(\alpha, s) < \text{mod. } \frac{t^s \sin(\alpha-n)\pi.\,\Pi(s+\alpha-1)\,\Pi(-\alpha)}{\pi.\,\Pi(s)}$$

$$< \text{mod. } \frac{t^s\Pi(s+\alpha-1)}{\Pi(s)\,\Pi(\alpha-1)},$$

i.e. < modulus of next term as stated *.

If the real part of t be positive and $t = r(\cos\phi + i\sin\phi)$ the integral in (6) would be increased in absolute value by replacing every negative power of $z-t$ by the same power of $z\sin\phi$, and we evidently can deduce that

mod. $R(\alpha, s) <$ modulus of the next term multiplied by $\operatorname{cosec}^{n+1}\phi$.

* This result follows readily also from Lagrange's form of the remainder in Taylor's series, extended (as regards the modulus) to a function of a complex variable. See Darboux, *Liouville's Journal*, 1876.

3. In discussing the functions of Art. 1 it will be sufficient to consider the case in which $m = n$. The case in which $m = n + 1$ can be derived from this by making x and one of the ρ's infinite together, and any case in which $m < n$ by making one or more of the α's infinite and x infinitely small and of the proper order.

Omitting the case of $n = 0$, the simplest of such cases is that in which $n = 1$; this case will be considered somewhat fully. Equation (3) is now reduced to

$$(xD^2 - (x - \rho) D - \alpha) y = 0 \text{ .... } (7),$$

D denoting differentiation with respect to x. Its solutions in converging series have been fully discussed by Pochhammer (*Math. Annalen*, Vol. XXXVI.).

If the series $F(\alpha : \rho : x)$ be considered merely as the limiting form of that for

$$F(\alpha, \beta; \rho; u)$$

wherein x/β is written for u, and β is then made infinite, the limiting form of the equation

$$F(\alpha, \beta: \rho; u) = (1 - u)^{\rho-\alpha-\beta} F(\rho - \alpha, \rho - \beta; \rho; u)$$

shows that

$$F(\alpha : \rho : x) = e^x F(\rho - \alpha : \rho : -x) \text{ .... } (8),$$

and that

$$x^{1-\rho} F(\alpha - \rho + 1 : 2 - \rho : x) = x^{1-\rho} e^x F(1 - \alpha; 2 - \rho; -x) \text{ .... } (9),$$

while another particular integral suggests another divergent series, viz.

$$e^x x^{\alpha-\rho} \left\{ 1 + \frac{(1-\alpha)(\rho-\alpha)}{1} \frac{1}{x} + \frac{(1-\alpha)(2-\alpha)(\rho-\alpha)(\rho-\alpha+1)}{1 \,.\, 2} \frac{1}{x^2} + \dots \right\} \text{ ...... } (10),$$

besides that of the type (5) which now is

$$x^{-\alpha} \left\{ 1 - \frac{\alpha(\alpha-\rho+1)}{1} \frac{1}{x} + \frac{\alpha(\alpha+1)(\alpha-\rho+1)(\alpha-\rho+2)}{1 \,.\, 2} \frac{1}{x^2} + \dots \right\} \text{ .... } (11).$$

A particular case of relation (8), in which ρ is written $= 2\alpha = n$ and x is replaced by $2x$, is given by Glaisher (*Trans. Roy. Soc.*, 1881, Part 3, page 774).

Pochhammer shows that equation (7) is satisfied by the integral

$$\int e^u (u - x)^{-\alpha} u^{\alpha-\rho} du$$

taken along a path which starts from any part and returns to the same point, provided the path is such that the initial and final values of $e^u (u - x)^{-\alpha} u^{\alpha-\rho}$ differ by zero, and has considered the two solutions

$$\int_c^{(x, 0, x-, 0-)} e^u (u - x)^{-\alpha} u^{\alpha-\rho} du \text{ .... } (12),$$

$$\int_{-\infty}^{(x, 0)} e^u (u - x)^{-\alpha} u^{\alpha-\rho} du \text{ .... } (13).$$

In the former the path starts from any arbitrary point c (which may be taken in the finite line joining the points 0, x), and returns to the same point after having made a circuit round the points x, 0, in the positive direction, and then a circuit round the points x, 0 in the negative direction; this integral is a multiple of $x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x)$. In the latter the path starts from the point $-\infty$ and returns to the same point after making a circuit round the points x, 0, in the positive direction; this integral is a multiple of $F(\alpha;\rho;x)$.

The identities (8), (9) might easily be established from these integrals by making the substitution $u=x-v$.

In addition to (12), (13) the following integrals, not discussed by Pochhammer, also satisfy the necessary conditions and are therefore solutions of equation (7), viz.

$$\overline{\int}_{-\infty}^{(0)} e^u(u-x)^{-\alpha}u^{\alpha-\rho}du \quad\text{.......(14)},$$

$$\overline{\int}_{-\infty}^{(x)} e^u(u-x)^{-\alpha}u^{\alpha-\rho}du \quad\text{.......(15)}.$$

We take the path of u in (15), both in going and returning, to pass above or below the origin according as x is above or below the axis of real quantities.

Suppose in the first instance that the real part of x is positive.

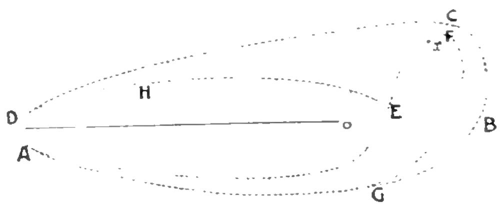

FIG. 2.

In Fig. 2 let A be a point near the negative part of the axis of real quantities and at a great distance from the origin. The path of u in (12) is equivalent to the paths $AGBCDA$, $AEFGA$, $ADHEA$, the value of the function $e^u(u-x)^{-\alpha}u^{\alpha-\rho}$ at the end of one part being taken as the value at the beginning of the next part. But for the path $AEFGA$ we may substitute $AEHDA$, $ADCBGA$. Thus for the whole path in (12) we may take $AGBCDA$, $AEHDA$, $ADCBGA$, $ADHEA$. And as the distance of A from the origin increases indefinitely, the first of these becomes the path in (13), the third this reversed, the second the path in (14), and the last that path reversed*.

In the integral (12) writing $(u-x)^{-\alpha}u^{\alpha-\rho}=x^{-\rho}\left(\frac{u}{x}-1\right)^{-\alpha}\left(\frac{u}{x}\right)^{\alpha-\rho}$, we will choose as Pochhammer virtually does those values of $\left(\frac{u}{x}-1\right)^{-\alpha}$ and of $\left(\frac{u}{x}\right)^{\alpha-\rho}$ which, when the path of u crosses the production of the line joining the points 0, x, for the first time at C, are wholly real and positive, and that value of $x^{-\rho}$ whose argument lies between $-\rho\pi/2$, and $+\rho\pi/2$. Pochhammer, *loc. cit.*, shows that this integral is

$$e^{\pi i(\alpha-\rho)}\mathfrak{C}(\alpha-\rho+1,\ 1-\alpha)\,x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x) \quad\text{.......(16)},$$

where $$\mathfrak{C}(a,b)=e^{-\pi i(a+b)}\overline{\int}_{c}^{(1,\,0,\,1-,\,0-)} v^{a-1}(1-v)^{b-1}dv \quad\text{.......(17)},$$

* On some grounds it would be more convenient to choose as the initial and final point one near the *positive* part of the axis of real quantities and at a great distance from the origin. I thought it advisable in the present case to follow Pochhammer.

c being an arbitrary point on the finite line joining the points 0, 1 and v^{a-1}, $(1-v)^{b-1}$ having there initially values which are real and positive. (See Pochhammer, "Zur Theorie der Euler'schen Integrale." *Math. Annal.* Vol. XXXV.)

But
$$\mathfrak{E}(a, b) = -4 \sin a\pi \sin b\pi \cdot \frac{\Pi(a-1)\,\Pi(b-1)}{\Pi(a+b-1)},$$

the Π functions having their extended meaning. The integral (12) is thus equal to

$$-4e^{\pi i(\alpha-\rho)} \sin(\alpha-\rho)\pi \sin(-\alpha\pi) \frac{\Pi(\alpha-\rho)\,\Pi(-\alpha)}{\Pi(1-\rho)} x^{1-\rho} F(\alpha-\rho+1\,;\; 2-\rho\,;\; x) \;\ldots(18).$$

Also of the four portions by which we have shown the path in (12) can be replaced, the first *AGBCDA* contributes to the integral as shown by Pochhammer,

$$\Gamma(1-\rho)\,F(\alpha\,;\;\rho\,;\;x) \text{ or } \frac{2\pi i}{\Pi(\rho-1)} F(\alpha\,;\;\rho\,;\;x) \;\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

and the third *ADCBGA* contributes this same multiplied by $-e^{2\pi i(\alpha-\rho)}$. These two together then contribute

$$\frac{4\pi e^{\pi i(\alpha-\rho)} \sin(\alpha-\rho)\pi}{\Pi(\rho-1)} F(\alpha;\;\rho;\;x) \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Again for the integral along the second portion *AEHDA*, bearing in mind the alteration of the argument, we have

$$e^{\pi i(\alpha-2\rho)} x^{-\alpha} \int_{-x}^{(0)} e^u (1-u/x)^{-\alpha} u^{\alpha-\rho}\,du \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

wherein at the point where the path of u crosses the line $0, x$, the values of $(1-u/x)^{-\alpha}$ and of $(u/x)^{\alpha-\rho}$ are real and positive and the argument of $x^{-\alpha}$ lies between $-\alpha\pi/2$ and $+\alpha\pi/2$.

Writing
$$(1-u/x)^{-\alpha} = 1 + \frac{\alpha}{1}\frac{u}{x} + \frac{\alpha(\alpha+1)}{1\,.\,2}\frac{u^2}{x^2} + \ldots \text{ (to } s \text{ terms)} + R_s,$$

and evaluating the several terms of the integral, (21) becomes

$$e^{\pi i(\alpha-2\rho)}\{e^{\pi i(\rho-\alpha)} - e^{\pi i(\alpha-\rho)}\}\,\Pi(\alpha-\rho)\,x^{-\alpha}$$
$$\left\{1 - \frac{\alpha(\alpha-\rho+1)}{1}\frac{1}{x} + \frac{\alpha(\alpha+1)(\alpha-\rho+1)(\alpha-\rho+2)}{1\,.\,2}\frac{1}{x^2} + \ldots \text{ to } s \text{ terms} + R_s'\right\} \ldots(22),$$

where
$$R_s' = \int_{-x}^{(0)} e^u u^{\alpha-\rho} R_s\,du \div \int_{-x}^{(0)} e^u u^{\alpha-\rho}\,du \ldots\ldots\ldots\ldots\ldots\ldots(23).$$

Now the origin of u is a multiple point of order s on R_s, or R_s in the neighbourhood of the origin of u is of order u^s, as may be seen by finding the limiting value of R_s/u^s, and therefore provided $\alpha-\rho+s+1$ is positive,

$$\int_{-\infty}^{(0)} e^u u^{\alpha-\rho} R_s\,du = -\left(e^{\pi i(\alpha-\rho+s)} - e^{-\pi i(\alpha-\rho+s)}\right) \int_{-\infty}^{0} e^u u^{\alpha-\rho} R_s\,du \ldots\ldots\ldots\ldots(24).$$

(the right-hand member involving a line integral and the argument of $e^u u^{\alpha-\rho} R_s$ being the same as that of R_s/u^s). And if $\alpha+s$ is positive the modulus of R_s is less than that of the $(s+1)$th term in the expansion of $(1-u/x)^{-\alpha}$, accordingly R_s' is less in absolute value than the $(s+1)$th term of the series in (22).

In this sense then provided $\alpha+s$, $\alpha-\rho+s+1$ and the real part of x are positive, s terms of the series represent the whole of the expression, including R_s', in brackets in (22), an expression which we will denote by $\phi(\alpha, \alpha-\rho+1; -1/x)$.

The fourth and last portion of the path in (12) contributes to (12) an expression equal to (22) multiplied by $-e^{2\pi i\rho}$.

Thus the second and last portions together contribute

$$4e^{\pi i(\alpha-\rho)} \sin\rho\pi \sin(\rho-\alpha)\pi \,.\, \Pi(\alpha-\rho)\, x^{-\alpha}\phi(\alpha, \alpha-\rho+1; -1/x) \;\ldots\ldots\ldots\ldots (25).$$

Equating (18) to the sum of (20) and (25) an equation is obtained which may be written in the form

$$\Pi(\alpha-1)\,\Pi(-\rho)\,F(\alpha; \rho; x)+\Pi(\alpha-\rho)\,\Pi(\rho-2)\,x^{1-\rho}F(\alpha-\rho+1; 2-\rho; x)$$
$$=\Pi(\alpha-1)\,\Pi(\alpha-\rho)\,x^{-\alpha}\phi(\alpha, \alpha-\rho+1; -1/x) \;\ldots\ldots\ldots\ldots (26),$$

the argument of x lying between $-\pi/2$ and $+\pi/2$, and the argument of every power x^m lying between $-m\pi/2$ and $+m\pi/2$.

Suppose next that the real part of x is negative and that the imaginary part is positive. Let us now take as the beginning and end of the paths of integration in (13), (14), (15) a point still at infinity, but whose argument instead of being π has some value between π and $3\pi/2$, say $5\pi/4$, and take values of the function under the sign of integration which are reconcilable with those formerly taken. These changes of path do not affect the values of the integrals. Proceeding as before, we have now to deal with $(1-u/x)^{-\alpha}$ where the argument of u is now $5\pi/4$ instead of π; this is of the form $(1-t)^{-\alpha}$ where the argument of t is somewhere between $\pi/4$ and $3\pi/4$ instead of between $\pi/2$ and $3\pi/2$; the integral on the right-hand side of (24) has now the same absolute value as

$$\int_{-\infty}^{0} e^{\frac{v(1+i)}{\sqrt{2}}}\left(v\frac{1+i}{\sqrt{2}\,.\,x}\right)^{\alpha-\rho} R_s\,\frac{1+i}{\sqrt{2}}\,dv,$$

where v is real and negative and R_s denotes the remainder after s terms of the expansion of $\left(1-\frac{v(1+i)}{x\sqrt{2}}\right)^{-\alpha}$. By Art. 2, R_s is certainly less in absolute value than the $(s+1)$th term of the expansion multiplied by $2^{\frac{\beta}{2}}$ where β is a positive integer such that $\beta-\alpha$ is positive, and as

$$\int_{-\infty}^{0} e^{\frac{v(1+i)}{\sqrt{2}}}\, v^m dv,$$

when $m+1$ is positive, is less in absolute value than $2^{\frac{m+1}{2}}\,\Pi(m)$, it follows that R_s' is less in absolute value than the $(s+1)$th term of the series in (22) multiplied by

$2^{\frac{\alpha+\beta-\rho+s+1}{2}}$ *. As there is no superior limit to the value of s for which this is true, and as, by taking x great enough, terms at the beginning may be made to outweigh as much as we please any finite number of succeeding terms, it follows that x may be taken so great that for any given value of s the error in taking s terms of the series in brackets in (22) instead of the whole expression will have a modulus less than that of the next term multiplied by $1+\epsilon$, where ϵ is any assigned small positive quantity.

If both the real and imaginary parts of x are negative the same result follows by taking for the point at infinity in (13), (14), (15), one whose argument lies between $\pi/2$ and π.

Equation (26) is thus true in the sense indicated when the real part of x is negative, the argument of every power x^m lying between $-m\pi$ and $-m\pi/2$ or between $+m\pi/2$ and $+m\pi$.

I proceed to obtain the relation connecting the divergent series (10) with the two convergent solutions of equation (7). If in (26) we change α into $1-\alpha$, ρ into $2-\rho$, x into $ye^{-\pi i}$, it becomes

$$\Pi(-\alpha)\,\Pi(\rho-2)\,F(1-\alpha;\ 2-\rho;\ -y)+\Pi(\rho-\alpha-1)\,\Pi(-\rho)\,e^{(1-\rho)\pi i}\,y^{\rho-1}F(\rho-\alpha;\ \rho;\ -y)$$
$$=\Pi(-\alpha)\,\Pi(\rho-\alpha-1)\,e^{(1-\alpha)\pi i}\,y^{\alpha-1}\phi(1-\alpha,\ \rho-\alpha;\ 1/y),$$

wherein the argument of every power y^m lies between 0 and $2m\pi$. After multiplication by e^y, bearing in mind equations (8), (9), and writing x instead of y, this may be written in the form

$$\Pi(\rho-\alpha-1)\,\Pi(-\rho)\,F(\alpha;\ \rho;\ x)+\Pi(-\alpha)\,\Pi(\rho-2)\,e^{(\rho-1)\pi i}\,x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x)$$
$$=\Pi(-\alpha)\,\Pi(\rho-\alpha-1)\,e^{(\rho-\alpha)\pi i}\,e^x x^{\alpha-\rho}\phi(1-\alpha,\ \rho-\alpha;\ 1/x)\ \ldots\ldots\ldots\ldots(27).$$

This is true in the sense that if the real part of x be negative the error in stopping the series on the right after s terms is less in absolute value than the next term, provided $s+1-\alpha$ and $s+\rho-\alpha$ are positive, and that whether the real part of x be positive or negative x can be taken so great that the error in stopping after s terms is less in absolute value than the next term multiplied by $1+\epsilon$, where ϵ has any assigned positive value and s has any given value.

The expression on the right of (27) is a multiple of (15) and may be obtained from that integral if x has its real part positive, by changing the starting point to some point at infinity whose real part is negative but which subtends with the origin an obtuse angle at the point x, and writing $u=x+v$.

Equation (27) may also be obtained directly by replacing the path of u in (12) by four other portions in a different manner. For instance instead of (12) we may write

$$-\int_c^{(0,\,x,\,0-,\,x-)} e^u(u-x)^{-\alpha}\,u^{\alpha-\rho}\,du,$$

(Pochhammer: "Ueber ein Integral mit doppeltem Umlauf," *Math. Annal.* Band XXXV.), and if the real part of x is negative and its imaginary part positive divide this path into

* This reasoning, in fact, proves that if the argument of x is $\pi\pm\gamma$, (γ acute), the multiplier is $[\operatorname{cosec}(\theta+\gamma)]^\beta(\sec\theta)^{\alpha-\rho+s+1}$ provided θ and $\theta+\gamma$ are acute; also α and $\alpha-\rho+1$ may be interchanged.

four others, that in (13) taken forwards, that in (15) taken forwards, that in (13) taken backwards, and that in (15) taken backwards. If the point x is differently situated a slight modification is necessary.

In the case which occurs in connection with Bessel Functions*, viz., that in which $\rho = 2\alpha$, Hankel (*loc. cit.*) considers the integral

$$\int_0^x e^{-xr} r^{n-\frac{1}{2}} (1 + ri)^{n-\frac{1}{2}} \, dr.$$

Although as Hankel proves, the remainder after s terms of $(1+ri)^{n-\frac{1}{2}}$ is less in absolute value than the next term (provided $n - s - \frac{1}{2}$ is negative), his inference that a similar statement holds for the integrals, is only valid when x is wholly real. For if u, v, w be complex functions of r which is wholly real, and if mod. $u <$ mod. v, we are not justified in inferring that mod. $\int_a^b uwdr <$ mod. $\int_a^b vwdr$.

Gray and Mathews (*Bessel Functions*, p. 69) apply Lagrange's form of the remainder in Taylor's series to the case of a complex function $\left(1 \pm \dfrac{\xi}{2ix}\right)^{n-\frac{1}{2}}$. Although this is a valid form of a superior limit to the modulus of the remainder, we are not justified in assuming that $\left(1 + \dfrac{\theta\xi}{2ix}\right)^{n-s-\frac{1}{2}}$ and $\left(1 - \dfrac{\theta'\xi}{2ix}\right)^{n-s-\frac{1}{2}}$ ($n - s - \frac{1}{2}$ being negative) are both less in absolute value than unity, where θ, θ', ξ are real, unless x be wholly real.

4. The theorem indicated by equation (26) is a particular case of the following which will be proved by induction:—

$$\frac{\Pi(\alpha_1 - 1)\,\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_2)\,\Pi(-\alpha_3)\ldots\Pi(-\alpha_n)} F(\alpha_1, \alpha_2, \ldots \alpha_n;\ \rho_1, \rho_2, \ldots \rho_n;\ x)$$

$$+ \frac{\Pi(\alpha_1 - \rho_1)\,\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_2 - 1)\,\Pi(\rho_1 - \alpha_3 - 1)\ldots\Pi(\rho_1 - \alpha_n - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_n - \rho_1 + 1;$$
$$2 - \rho_1, \rho_2 - \rho_1 + 1, \ldots \rho_n - \rho_1 + 1;\ x)$$

$+ (n-1)$ other terms analogous to the last

$$= \frac{\Pi(\alpha_1 - 1)\,\Pi(\alpha_1 - \rho_1)\ldots\Pi(\alpha_1 - \rho_n)}{\Pi(\alpha_1 - \alpha_2)\,\Pi(\alpha_1 - \alpha_3)\ldots\Pi(\alpha_1 - \alpha_n)} x^{-\alpha_1} F(\alpha_1, \alpha_1 - \rho_1 + 1, \ldots \alpha_1 - \rho_n + 1;\ \alpha_1 - \alpha_2 + 1, \ldots$$
$$\alpha_1 - \alpha_n + 1;\ -1/x) \quad \ldots\ldots(28),$$

wherein the argument of every power x^m lies between $-m\pi$ and $+m\pi$, the symbol of equality being interpreted in the sense that:—

(A) If the real part of x is positive, the error committed by stopping the series on the right-hand side after s terms is less in absolute value than the next term, provided s exceeds a certain number. Of the series on the left we may select the first so that $1, \rho_1, \ldots \rho_n$, are in ascending order as also $\alpha_2, \alpha_3, \ldots \alpha_n$. If a is any

* The semiconvergent series for $J_n(x)$ may be readily obtained by forming the equation satisfied by $e^x . I_n(x)$ and using the analogues of (26), (27).

fractional number let $[a]$ denote, if a is negative, zero, if a is positive the integer next higher than a. Then s is not to be less than the greater of the integers $[\alpha_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, $[\rho_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

(B) Whether the real part of x is positive or negative*, x can be taken so great that for any assigned value of s the error in stopping after s terms is less in absolute value than the next term multiplied by $1+\epsilon$, where ϵ is any assigned positive quantity however small.

It is to be noted that in the enunciation of (A) the additional factors occurring in the numerator and denominator of the second term omitted are all positive, and the argument of x is restricted to a range of π, including that value which makes all the terms omitted real and of alternate signs; *some* restriction on the argument or on the amplitude of x being, as remarked in Art. 1, not merely incidental to our method of proof but essentially necessary from the nature of the theorem.

5. We will first prove by induction that there is one solution of the differential equation satisfied by

$$F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_n;\ -x)$$

which can be written in the form $Ce^{-x}x^{\Sigma(\alpha-\rho)}$ where, as x increases indefinitely, having its real part positive, C tends to a fixed limit. (The minus sign has been inserted before the x as it is easier to reason about a negative quantity when we call it $-x$ than when we call it $+x$.) This is true also if the real part of x is negative but not required for the present purpose. Let us assume that this is true when there are n α's and n ρ's, and introduce another α and another ρ denoted simply by α and ρ. The differential equation for the new hypergeometric function is satisfied by

$$x^{1-\rho} \int_{\infty}^{(x)} (v-x)^{\rho-\alpha-1} v^{\alpha-1} \phi(v)\, dv,$$

where $\phi(v)$ is the solution of the old series referred to (see Pochhammer). Writing

$$\phi(v) = Ce^{-v} v^{\sum_1^n (\alpha_r - \rho_r)} = Ce^{-v} v^m$$

and making the substitution $v = x+u$, the above solution may be written in the form

$$e^{-x} x^{\alpha-\rho+m} \int_{x}^{(0)} Ce^{-u} u^{\rho-\alpha-1} (1+u/x)^{\alpha+m-1}\, du \quad \dots\dots\dots\dots\dots\dots\dots (29).$$

Let us assume in the first instance that $\rho - \alpha$ is positive; this is then a multiple (depending on the unspecified arguments) of the line integral

$$e^{-x} x^{\alpha-\rho+m} \int_0^{\infty} Ce^{-u} u^{\rho-\alpha-1} (1+u/x)^{\alpha+m-1}\, du.$$

If $(1+u/x)^{\alpha+m-1}$ be now expanded in powers of u/x the modulus of the remainder after a certain term will be less than that of the next term, and it is accordingly

* If the latter, it will also be shown that the error is less than a certain multiple of the next term (s restricted as before).

evident that the above may be written in the form $C'e^{-x}x^{\alpha-\rho+m}$ where, by increasing x sufficiently, C' can be made as nearly as we please equal to a certain constant.

Also in case $\rho-\alpha$ be negative, on integrating (29) by parts the integral may be written in the form

$$\frac{1}{\rho-\alpha}\left[Ce^{-u}u^{\rho-\alpha}(1+u/x)^{\alpha+m-1}\right]_{\infty}^{\infty}-\frac{1}{\rho-\alpha}\int_{\infty}^{(0)}e^{-u}u^{\rho-\alpha}(1+u/x)^{\alpha+m-2}\left\{\frac{\alpha+m-1}{x}C-C(1+u/x)\right.$$
$$\left.+(1+u/x)\,dC/du\right\}du.$$

The expression in the square brackets vanishes at the infinite limits, and by making x great enough dC/du may be made as small as we please by hypothesis, and the above integral, if x be large enough, can thus be made as nearly as we please equal to

$$-\frac{1}{\rho-\alpha}\int_{\infty}^{(0)}e^{-u}u^{\rho-\alpha}(1+u/x)^{\alpha+m-2}\left\{\frac{\alpha+m-1}{x}C-C(1+u/x)\right\}du,$$

from which it is evident that any limits to the value of $\rho-\alpha$ may be extended by unity, and therefore, for all values of $\rho-\alpha$, (29) may be written in the form

$$C'e^{-x}x^{\alpha-\rho+\sum\limits_{1}^{n}(\alpha_r-\rho_r)}$$

where C' tends to a fixed limit as x increases indefinitely.

6. The solution which does tend to this form is a multiple of

$$\frac{\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)}F(\alpha_1,\alpha_2,\ldots\alpha_n;\ \rho_1,\ldots\rho_n;\ -x)$$
$$+\sum_{r=1}^{r=n}\frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\ldots\Pi(\rho_r-\alpha_n-1)}x^{1-\rho_r}F(\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1;\,2-\rho_r,\rho_1-\rho_r+1,$$
$$\ldots\rho_n-\rho_r+1;\ -x),$$

wherein the argument of x^m lies between $-m\pi/2$ and $+m\pi/2$. This may be seen by making in the theorem indicated by equation (28) the arguments of x in succession $-\pi$ and $+\pi$ and subtracting the results after having multiplied one of them by $e^{2\pi\alpha i}$. It then appears that the above solution is one which, when in it x is made real and positive and sufficiently great, can be made less than a certain multiple of any specified term of the series

$$x^{-\alpha_1}F(\alpha_1,\ \alpha_1-\rho_1+1,\ldots;\ \alpha_1-\alpha_2+1,\ldots;\ +1/x),$$

and therefore must be the one in question, as from equation (28), assumed to hold for the above functions, no other solution can be of this order for infinite values of x which have their real part positive.

7. We require to evaluate the integral

$$\int_0^{\infty}x^mF(\alpha_1,\alpha_2,\ldots\alpha_n;\ \rho_1,\rho_2,\ldots\rho_n;\ -x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

when intelligible, that is when $m+1$ is positive and $m-\alpha_r+1$ is negative, where α_r is the algebraically least α. (When x is very great the hypergeometric series is of order $x^{-\alpha_r}$.)

The hypergeometric function is equal to

$$\frac{\Pi(\rho_n-1)}{\Pi(\alpha_n-1)\,\Pi(\rho_n-\alpha_n-1)}\int_0^x x^{1-\rho_n}(x-v)^{\rho_n-\alpha_n-1}v^{\alpha_n-1}F(\alpha_1,\dots\alpha_{n-1};\,\rho_1,\dots\rho_{n-1};\,-v)\,dv,$$

provided α_n and $\rho_n-\alpha_n$ are positive. Substituting this value in (30) and changing the order of integration, which can be shown to be legitimate since $m-\alpha_n+1$ is negative, (30) can be written in the form

$$\frac{\Pi(\rho_n-1)}{\Pi(\alpha_n-1)\,\Pi(\rho_n-\alpha_n-1)}\int_0^\infty v^{\alpha_n-1}F(\alpha_1,\dots\alpha_{n-1};\,\rho_1,\dots\rho_{n-1};\,-v)\,dv\int_v^\infty x^{m+1-\rho_n}(x-v)^{\rho_n-\alpha_n-1}dx;$$

but the x integral is equal to

$$\frac{\Pi(\alpha_n-m-2)\,\Pi(\rho_n-\alpha_n-1)}{\Pi(\rho_n-m-2)}\,v^{m+1-\alpha_n},$$

and accordingly (30) is equivalent to

$$\frac{\Pi(\rho_n-1)\,\Pi(\alpha_n-m-2)}{\Pi(\alpha_n-1)\,\Pi(\rho_n-m-2)}\int_0^\infty v^m\,F(\alpha_1,\dots\alpha_{n-1};\,\rho_1,\dots\rho_{n-1};\,-v)\,dv.$$

In the same way provided α_1 and $\rho_1-\alpha_1$ are positive

$$\int_0^\infty x^m\,F(\alpha_1;\,\rho_1;\,-x)\,dx=\frac{\Pi(\rho_1-1)\,\Pi(\alpha_1-m-2)}{\Pi(\alpha_1-1)\,\Pi(\rho_1-m-2)}\int_0^\infty x^m e^{-x}\,dx$$

$$=\frac{\Pi(\rho_1-1)\,\Pi(\alpha_1-m-2)}{\Pi(\alpha_1-1)\,\Pi(\rho_1-m-2)}\,\Pi(m).$$

Therefore by induction we finally obtain for (30) the value

$$\Pi(m)\,.\prod_{r=1}^{r=n}\frac{\Pi(\rho_r-1)\,\Pi(\alpha_r-m-2)}{\Pi(\alpha_r-1)\,\Pi(\rho_r-m-2)}\quad\dots\dots\dots\dots(31),$$

provided, in addition to the conditions necessary to make (30) intelligible, all such quantities as α_r and $\rho_r-\alpha_r$ are positive.

These latter conditions may however be removed. Since

$$F(\alpha_1+1,\,\alpha_2,\dots\alpha_n;\,\rho_1,\,\rho_2,\dots\rho_n;\,-x)-F(\alpha_1,\,\alpha_2,\dots\alpha_n;\,\rho_1,\,\rho_2,\dots\rho_n;\,-x)$$

$$=\frac{x}{\alpha_1}\frac{d}{dx}F(\alpha_1,\,\alpha_2,\dots\alpha_n;\,\rho_1,\,\rho_2,\dots\rho_n;\,-x),$$

by multiplying both sides by x^m and integrating we obtain

$$\int_0^\infty x^m\,F(\alpha_1+1,\,\alpha_2,\dots\alpha_n;\,\rho_1,\dots\rho_n;\,-x)\,dx-\int_0^\infty x^m\,F(\alpha_1,\dots\alpha_n;\,\rho_1,\dots\rho_n;\,-x)\,dx$$

$$=\left[\frac{x^{m+1}}{\alpha_1}F(\alpha_1,\dots\alpha_n;\,\rho_1,\dots\rho_n;\,-x)\right]_0^\infty-\frac{m+1}{\alpha_1}\int_0^\infty x^m\,F(\alpha_1,\dots\alpha_n;\,\rho_1,\dots\rho_n;\,-x)\,dx.$$

If the second integral in the left-hand member is intelligible, so also is the first; the expression in square brackets then vanishes at the limits and

$$\int_0^\infty x^m\,F(\alpha_1,\dots\alpha_n;\,\rho_1,\dots\rho_n;\,-x)\,dx=\frac{\alpha_1}{\alpha_1-m-1}\int_0^\infty x^m\,F(\alpha_1+1,\,\alpha_2,\dots\alpha_n;\,\rho_1,\dots\rho_n;\,-x)\,dx,$$

and accordingly any inferior limit imposed on any α may be extended by unity, the other α's and ρ's being kept unchanged, provided in (30) the integral remains intelligible.

Also from the equation

$$F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_n; -x) - F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_{n-1}, \rho_n - 1; -x)$$
$$= \frac{-x}{\rho_n - 1}\frac{d}{dx} F(\alpha_1, \dots \alpha_n; \rho_1, \dots \rho_n; -x) \quad \text{.........................(32)},$$

by multiplying both sides by x^m and integrating we obtain

$$\int_0^\infty x^m F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_n; -x)\,dx - \int_0^\infty x^m F(\alpha_1, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_{n-1}, \rho_n - 1; -x)\,dx$$
$$= \frac{-1}{\rho_n - 1}\left[x^{m+1} F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \dots \rho_n; -x)\right]_0^\infty$$
$$+ \frac{m+1}{\rho_n - 1}\int_0^\infty x^m F(\alpha_1, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_n; -x)\,dx \quad \text{......................(33)}.$$

If (30) is intelligible all the terms in this equation are finite and that in square brackets is zero at the limits and thus we obtain

$$\int_0^\infty x^m F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_{n-1}, \rho_n - 1; -x)\,dx$$
$$= \frac{\rho_n - m - 2}{\rho_n - 1}\int_0^\infty x^m F(\alpha_1, \dots \alpha_n; \rho_1, \dots \rho_n; -x)\,dx \quad \text{........(34)},$$

and accordingly any limit to the value of any ρ may be extended by unity, the other α's and ρ's being kept unchanged. The result stated as to the value of the integral (30) has thus been established *.

8. It is to be noted that the proofs given of the results of Arts. 6, 7 for functions of any order assume the truth of equation (28) for functions of the same order. We now proceed, assuming the results of Arts. 6, 7 and equation (28) for functions of any and the same order, to extend equation (28) to functions of the next higher order by the introduction of another ρ and another α. We do so in the present Article, taking the equation in sense (A) but subject to the restrictions that each of the quantities $\rho_2 - \alpha_2, \dots \rho_n - \alpha_n, \rho - \alpha, \alpha_1 + 1 - \rho_2, \dots \alpha_1 + 1 - \rho_n, \alpha_1 + 1 - \rho$, is positive. In Art. 9 we will extend the equation in the sense (B), and in Art. 10 the restrictions introduced in the present Article for the sense (A) will be removed.

As indicated by Pochhammer the equation satisfied by $F(\alpha, \alpha_1, \alpha_2, \dots \alpha_n; \rho, \rho_1, \dots \rho_n; +x)$ is satisfied by

$$x^{1-\rho}\int (v-x)^{\rho-\alpha-1} v^{\alpha-1} \phi(v)\,dv \quad \text{..............................(35)},$$

where $\phi(v)$ is any solution of that satisfied by $F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \dots \rho_n; +v)$ and the path of integration is a closed one such that the final value of $(v-x)^{\rho-\alpha-1} v^{\alpha-1}\phi(v)$ differs by zero from the initial one.

* This result may be generalized by omitting any number of α's, if the integral remains finite; write $x = y/\alpha$ and then make α infinite.

Let the path be one which makes a circuit round the point x in the positive direction, then round the origin in the positive direction, then round the point x in the negative direction and finally round the origin in the negative direction. Suppose in the first instance the real part of x to be positive.

Such a path is equivalent to the paths $ABCA$, $ADBA$, $ACBDA$ which may be replaced by the four portions $ABCA$, $ADBA$, $ACBA$, $ABDA$, (Fig. 3).

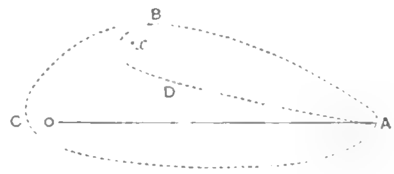

Fig. 3.

Let A be a point h on the axis of real quantities and let $\phi(v)$ be

$$CF(\alpha_1, \alpha_2, \ldots \alpha_n;\ \rho_1, \rho_2, \ldots \rho_n;\ +v)$$

$$+\sum_{r=1}^{r=n} C_r v^{1-\rho_r} F(\alpha_1-\rho_r+1, \ldots \alpha_n-\rho_r+1;\ 2-\rho_r, \rho_1-\rho_r+1, \ldots \rho_n-\rho_r+1;\ +v) \ldots\ldots\ldots (36),$$

those values being taken which make the initial arguments of every power of v zero at the point h (before multiplication by C, C_r), and make the initial argument of $(v-x)^{\rho-a-1}$ diminish indefinitely as h increases indefinitely.

On examining the values of the arguments at different points it will be seen that the second and fourth portions of the path together contribute to the integral (35) the expression

$$(e^{2\pi i(a-\rho)} - e^{2\pi i a})\, x^{1-\rho} \int_h^{(x)} C\,(v-x)^{\rho-a-1}\, v^{a-1}\, F(\alpha_1, \alpha_2, \ldots \alpha_n;\ \rho_1, \ldots \rho_n;\ +v)\, dv$$

$$+\sum_{r=1}^{r=n} (e^{2\pi i(a-\rho)} - e^{2\pi i(a-\rho_r)})\, x^{1-\rho} \int_h^{(x)} C_r (v-x)^{\rho-a-1}\, v^{a-\rho_r}\, F(\alpha_1-\rho_r+1, \ldots \alpha_n-\rho_r+1;\ 2-\rho_r, \ldots$$

$$\ldots \rho_n-\rho_r+1;\ +v)\, dv \ldots\ldots\ldots\ldots (37),$$

the initial arguments being taken as above.

If the differential coefficient of this expression with respect to h be written down it will be evident that in virtue of equation (28) assumed for the function of the $(n+1)$th order, i.e. that which satisfies the differential equation of the $(n+1)$th order, provided we take

$$\left.\begin{aligned} C\,(e^{2\pi i(a-\rho)} - e^{2\pi a i}) &= \frac{\Pi(\alpha_1-1)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_2)\,\Pi(-\alpha_3)\ldots\Pi(-\alpha_n)} \\ C_r(e^{2\pi i(a-\rho)} - e^{2\pi i(a-\rho_r)}) &= \frac{\Pi(\alpha_1-\rho_r)\,\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_2-1)\,\Pi(\rho_r-\alpha_3-1)\ldots\Pi(\rho_r-\alpha_n-1)} \end{aligned}\right\} \ldots\ldots\ldots (38),$$

this differential coefficient when h increases indefinitely will be of the order $h^{\rho-a_1-2}$; and the same is true also for a complex value of h provided the argument of h is kept between $-\pi$ and $+\pi$ and the value of the function to be integrated is reconcilable with that previously taken. Accordingly provided $\rho - \alpha_1 - 1$ is negative, a condition which we have for the present supposed satisfied, the expression (37) will remain finite when h increases indefinitely, and its value will be unaltered if for h we substitute any infinite limit whose argument lies between $-\pi$ and $+\pi$ and if the value of the function to be integrated is reconcilable with that already taken.

Now let h be increased indefinitely. We have also supposed for the present that $\rho - \alpha$ is positive; each symbol of integration in (37) may therefore be replaced by

$$(e^{2\pi i(\rho-\alpha)} - 1)\int_x^\infty$$

in which the infinite limit is on the production of the line joining the origin to the point x.

With the above values of the constants C, C_r, and assuming the theorem to hold for the functions of the $(n+1)$th order, the expression (37) may be written

$$(e^{2\pi i(\rho-\alpha)} - 1)\frac{\Pi(\alpha_1 - 1)\,\Pi(\alpha_1 - \rho_1)\ldots\Pi(\alpha_1 - \rho_n)}{\Pi(\alpha_1 - \alpha_2)\,\Pi(\alpha_1 - \alpha_3)\ldots\Pi(\alpha_1 - \alpha_n)}\int_x^\infty x^{1-\rho}(v - x)^{\rho-\alpha-1}v^{\alpha-\alpha_1-1}$$

$$F(\alpha_1, \alpha_1 - \rho_1 + 1, \ldots \alpha_1 - \rho_n + 1;\ \alpha_1 - \alpha_2 + 1, \alpha_1 - \alpha_3 + 1, \ldots \alpha_1 - \alpha_n + 1;\ -1/v)\,dv \ldots (39).$$

Expanding the divergent hypergeometric series in descending powers of v and integrating the terms successively we obtain

$$(e^{2\pi i(\rho-\alpha)} - 1)\frac{\Pi(\rho - \alpha - 1)\,\Pi(\alpha_1 - \rho)\,\Pi(\alpha_1 - 1)\,\Pi(\alpha_1 - \rho_1)\ldots\Pi(\alpha_1 - \rho_n)}{\Pi(\alpha_1 - \alpha)\,\Pi(\alpha_1 - \alpha_2)\ldots\Pi(\alpha_1 - \alpha_n)}x^{-\alpha_1}F(\alpha_1, \alpha_1 - \rho + 1, \ldots$$

$$\alpha_1 - \rho_n + 1;\ \alpha_1 - \alpha + 1, \alpha_1 - \alpha_2 + 1, \ldots \alpha_1 - \alpha_n + 1;\ -1/x) \ldots\ldots (39a),$$

the argument of $x^{-\alpha_1}$ lying between $-\alpha_1\pi/2$ and $+\alpha_1\pi/2$; and by expressing the error as an integral, the theorem being assumed for the function of the $(n+1)$th order, it appears that, provided $\alpha_1 + s$ and all the quantities of the type $\alpha_1 - \rho_r + 1 + s$ are positive, the error in stopping the divergent series at the sth term has a modulus less than that of the next term.

Returning to (35), the sum of the portions contributed by the first and third portions of the path is

$$(1 - e^{2\pi i(\alpha-\rho)})\int_h^{\overline{(x,0)}} x^{1-\rho}\,C\,(v - x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1, \ldots \alpha_n;\ \rho_1, \ldots \rho_n;\ +v)\,dv$$

$$+\sum_{r=1}^{r=n}(1 - e^{2\pi i(\alpha-\rho)})\int_h^{\overline{(x,0)}} C_r x^{1-\rho}(v - x)^{\rho-\alpha-1}v^{\alpha-\rho_r}$$

$$F(\alpha_1 - \rho_r + 1, \ldots \alpha_n - \rho_r + 1;\ 2 - \rho_r, \ldots \rho_n - \rho_r + 1;\ +v)\,dv \ldots\ldots (40),$$

the initial values of the arguments at the point h being the same as in (36). The differential coefficient of this with respect to h is of the order $h^{\rho-\alpha_1-2}$ owing to the particular values assigned to C, C_r, and accordingly (40) like (37) remains finite when h is increased indefinitely provided $\rho - \alpha_1 - 1$ is negative. We now expand $(v - x)^{\rho-\alpha-1}$ in ascending powers of x; the coefficient of $x^{1-\rho+m}$ in (40) is therefore

$$(1 - e^{2\pi i(\alpha-\rho)}) \frac{\Pi(\alpha - \rho + m)}{\Pi(\alpha - \rho)\,\Pi(m)} \int_h^{(0)} \Big[C v^{\rho-m-2} F(\alpha_1, \ldots \alpha_n;\ \rho_1, \ldots \rho_n;\ +v)$$

$$+ \sum_{r=1}^{r=n} C_r v^{\rho-m-1-\rho_r} F(\alpha_1 - \rho_r + 1, \ldots \alpha_n - \rho_r + 1;\ 2 - \rho_r, \ldots \rho_n - \rho_r + 1;\ +v) \Big] dv \quad \ldots\ldots(41),$$

wherein all the powers of v are initially real before multiplication by the complex coefficients C, C_r. The successive values of m are 0, 1, 2, &c.

We proceed to evaluate the integral in this when h is made infinite. By considering that, owing to the particular values assigned to C, C_r, its differential coefficient with respect to h is of the order $h^{\rho-m-\alpha_1-2}$, it appears that the integral remains finite when h increases indefinitely, (provided $\rho - m - \alpha_1 - 1$ is negative, which is certainly true if

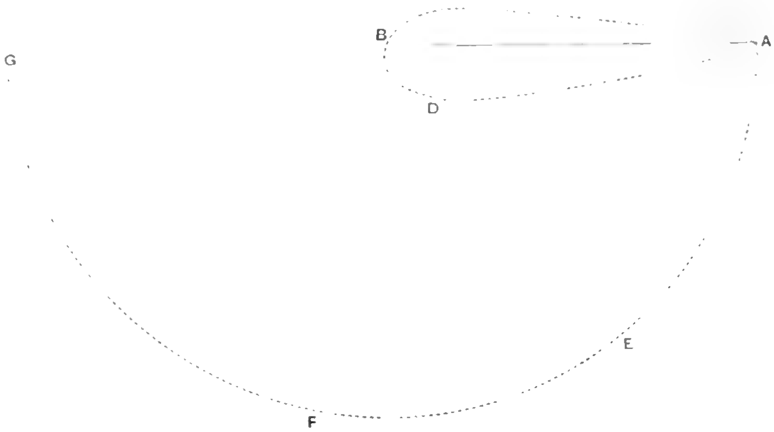

Fig. 4.

as already supposed $\rho - \alpha_1 - 1$ is negative), and as in the case of (37) this is true also for a complex value of h provided the argument of h is kept between $-\pi$ and $+\pi$ and the value of the function to be integrated is reconcilable with that previously taken. Accordingly by changing the argument of h to $-\pi$ as in the path $GFEABDAEFG$, (Fig. 4), the symbol of integration in (41) may be changed into $\int_{-\infty}^{(0)}$ where all the powers of v in the function to be integrated have zero argument (before multiplication by the complex coefficients C, C_r) at the point in which the path intersects the positive part of the axis of real quantities, and the initial and final limits of integration are not merely both negative and infinite but the same. (By changing the argument of h to $+\pi$ instead, we might obtain the symbol $\int_{-\infty}^{(0)}$ with different values for the arguments of the functions to be integrated; the evaluation of the integral would however lead to the same result as that to be presently deduced for the present case, as of course it should.)

Let us first suppose that $\rho - m - 1$ and all the quantities $\rho - m - \rho_r$ are positive; the integral can then be expressed as a line integral, and attending to the values of the arguments and of C, C_r, it is in fact

$$\int_0^\infty \left[e^{(\rho-2a)\pi i} \frac{\sin(\rho - m)\pi}{\sin \rho\pi} \cdot \frac{\Pi(\alpha_1 - 1)\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_2)\Pi(-\alpha_3)\dots\Pi(-\alpha_n)} u^{\rho-m-2} F(\alpha_1, \dots \alpha_n; \rho_1, \dots \rho_n; -u) \right.$$

$$+ \sum_{r=1}^{r=n} e^{(\rho+\rho_r-2a)\pi i} \frac{\sin(\rho - m - \rho_r)\pi}{\sin(\rho_r - \rho)\pi} \cdot \frac{\Pi(\alpha_1 - \rho_r)\Pi(\rho_r - 2)\Pi(\rho_r - \rho_1 - 1)\dots\Pi(\rho_r - \rho_n - 1)}{\Pi(\rho_r - \alpha_2 - 1)\dots\Pi(\rho_r - \alpha_n - 1)} u^{\rho-m-1-\rho_r}$$

$$\left. F(\alpha_1 - \rho_r + 1, \dots \alpha_n - \rho_r + 1; 2 - \rho_r, \rho_1 - \rho_r + 1, \dots \rho_n - \rho_r + 1; -u) \right] du.$$

Using the values given in (31) for each of the $n+1$ terms of this integral and making use of the relation $\Pi(n-1)\Pi(-n) = \pi \operatorname{cosec} n\pi$, the above may be written

$$\pi^2 \frac{\Pi(\alpha_1 - \rho + m)\Pi(\alpha_2 - \rho + m)\dots\Pi(\alpha_n - \rho + m)}{\Pi(\rho_1 - \rho + m)\Pi(\rho_2 - \rho + m)\dots\Pi(\rho_n - \rho + m)\Pi(1 - \rho + m)}$$

$$\left[-e^{(\rho-2a)\pi i} \frac{\sin \alpha_2\pi \sin \alpha_3\pi \dots \sin \alpha_n\pi}{\sin \rho\pi \sin \rho_1\pi \dots \sin \rho_n\pi} \right.$$

$$\left. + \sum_{r=1}^{r=n} e^{(\rho+\rho_r-2a)\pi i} \frac{\sin(\alpha_2 - \rho_r)\pi \sin(\alpha_3 - \rho_r)\pi \dots \sin(\alpha_n - \rho_r)\pi}{\sin \rho_r\pi \sin(\rho - \rho_r)\pi \sin(\rho_1 - \rho_r)\pi \dots \sin(\rho_n - \rho_r)\pi} \right] \quad \dots\dots\dots (42).$$

But the expression in square brackets is equivalent to

$$e^{(\rho-2a)\pi i} \cdot e^{\rho\pi i} \frac{\sin(\rho - \alpha_2)\pi \sin(\rho - \alpha_3)\pi \dots \sin(\rho - \alpha_n)\pi}{\sin \rho\pi \sin(\rho - \rho_1)\pi \sin(\rho - \rho_2)\pi \dots \sin(\rho - \rho_n)\pi},$$

for it is easily seen that this last may be written in the form

$$e^{(\rho-2a)\pi i} \left\{ \frac{A}{\sin \rho\pi} + \frac{A_1}{\sin(\rho - \rho_1)\pi} + \dots \right\},$$

where A, A_1, etc. are quantities independent of ρ and their evaluation in the usual way leads to the result stated.

The value of (41) is then

$$(e^{2\pi i(\rho-a)} - 1)\pi^2 \frac{\Pi(\alpha - \rho + m)}{\Pi(\alpha - \rho)\Pi(m)} \cdot \frac{\Pi(\alpha_1 - \rho + m)\Pi(\alpha_2 - \rho + m)\dots\Pi(\alpha_n - \rho + m)}{\Pi(\rho_1 - \rho + m)\Pi(\rho_2 - \rho + m)\dots\Pi(\rho_n - \rho + m)\Pi(1 - \rho + m)}$$

$$\times \frac{\sin(\rho - \alpha_2)\pi \sin(\rho - \alpha_3)\pi \dots \sin(\rho - \alpha_n)\pi}{\sin \rho\pi \sin(\rho - \rho_1)\pi \dots \sin(\rho - \rho_n)\pi} \quad \dots\dots\dots\dots\dots\dots\dots\dots (43).$$

The limit of the expression (41) when h is infinite still has the value (43) even if the additional conditions introduced in the process of evaluation (viz. that $\rho - m - 1$ and all the quantities such as $\rho - m - \rho_r$ should be positive) are not satisfied. For bearing in mind that

$$\frac{d}{dv} F(\alpha_1, \alpha_2, \dots \alpha_n; \rho_1, \rho_2, \dots \rho_n; v) = \frac{\alpha_1\alpha_2\dots\alpha_n}{\rho_1\rho_2\dots\rho_n} F(\alpha_1 + 1, \alpha_2 + 1, \dots \alpha_n + 1; \rho_1 + 1, \dots \rho_n + 1; v),$$

and that

$$\frac{d}{dv}\{v^{1-\rho_1}F(\alpha_1-\rho_1+1,\ldots\alpha_n-\rho_1+1;\ 2-\rho_1,\ \rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\ v)\}$$

$$=(1-\rho_1)\,v^{-\rho_1}F(\alpha_1-\rho_1+1,\ldots\alpha_n-\rho_1+1;\ 1-\rho_1,\ \rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\ v),$$

if we write the integral in (41) in the form

$$\int_h^{(0)} v^{\rho-m-2}\,\phi(v)\,dv$$

and integrate by parts, we obtain

$$\frac{1}{\rho-m-1}\Big[v^{\rho-m-1}\,\phi(v)\Big]_h^h-\frac{1}{\rho-m-1}\int_h^{(0)} v^{\rho-m-1}\,\phi'(v)\,dv.$$

But, attending to the values of C, C_r, given by (38), it appears that the difference between the two values of the expression in square brackets when v is equal to h is zero when h is infinite and that $-\phi'(v)$ only differs from $\phi(v)$ by having all the constants $\alpha_1,\ldots\alpha_n$, $\rho_1,\ldots\rho_n$ increased by unity, and accordingly we can increase all the quantities α, $\alpha_1,\ldots\alpha_n$, ρ, $\rho_1,\ldots\rho_n$ in the equation:—$\mathop{\mathrm{Lt.}}_{h=\infty}(41)=(43)$, keeping m unaltered.

In a similar manner we can show, by writing the integral in (41) in the form

$$\int_h^{(0)} v^{\rho-m-1-\rho_r}\,\psi(v)\,dv,$$

that we can increase ρ_r by unity, keeping the quantities α, $\alpha_1,\ldots\alpha_n$, ρ, $\rho_1,\ldots\rho_{r-1}$, $\rho_{r+1},\ldots\rho_n$, m unaltered, and still have the equation:—$\mathop{\mathrm{Lt.}}_{h=\infty}(41)=(43)$.

The initial value of m is zero and in that case (43) reduces to

$$-(e^{2\pi i(\rho-\alpha)}-1)\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\,\Pi(\rho-\rho_2-1)\ldots\Pi(\rho-\rho_n-1)\,\Pi(\alpha_1-\rho)}{\Pi(\rho-\alpha_2-1)\,\Pi(\rho-\alpha_3-1)\ldots\Pi(\rho-\alpha_n-1)},$$

and accordingly the expression (40) contributed by the first and third portions of the path is this multiple of

$$x^{1-\rho}F(\alpha-\rho+1,\ \alpha_1-\rho+1,\ \alpha_2-\rho+1,\ldots\alpha_n-\rho+1;$$
$$2-\rho,\ \rho_1-\rho+1,\ldots\rho_n-\rho+1;\ +x)\ldots\ldots\ldots\ldots(44),$$

wherein the argument of $x^{1-\rho}$ lies between $-\frac{\pi}{2}(1-\rho)$ and $+\frac{\pi}{2}(1-\rho)$.

Again returning to (35), (36),

$$\int_c^{(x,\,0,\,x-,\,0-)}(v-x)^{\rho-\alpha-1}\,v^{\alpha-1}\,F(\alpha_1,\ \alpha_2,\ldots\alpha_n;\ \rho_1,\ \rho_2,\ldots\rho_n;\ v)\,dv,$$

wherein the arguments have values reconcilable with those chosen, may be shown to be

$$-4e^{\pi i(\alpha-1)}\sin(\rho-\alpha-1)\pi\,\sin(\alpha-1)\pi\,\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha-1)}{\Pi(\rho-1)}\,x^{\rho-1}$$
$$F(\alpha,\ \alpha_1,\ldots\alpha_n;\ \rho,\ \rho_1,\ldots\rho_n;\ x)\ldots\ldots\ldots\ldots(45),$$

wherein the argument of $x^{\rho-1}$ lies between $-(\rho-1)\frac{\pi}{2}$ and $+(\rho-1)\frac{\pi}{2}$, and making use of the values of C, C_r, given by (38), the integral (35) may after some reduction be written in the form

$$(e^{2\pi i(\rho-\alpha)}-1)\left[\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha_1-1)\,\Pi(-\rho)\,\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha)\,\Pi(-\alpha_2)\,\Pi(-\alpha_3)\dots\Pi(-\alpha_n)}F(\alpha,\alpha_1,\dots\alpha_n;\ \rho,\rho_1,\dots\rho_n;\ x)\right.$$

$$+\sum_{r=1}^{r=n}\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha_1-\rho_r)\,\Pi(\rho_r-2)\,\Pi(\rho_r-\rho-1)\,\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha-1)\,\Pi(\rho_r-\alpha_2-1)\dots\Pi(\rho_r-\alpha_n-1)}x^{1-\rho_r}F(\alpha-\rho_r+1,\dots$$

$$\left.\alpha_n-\rho_r+1;\ 2-\rho_r,\ \rho_1-\rho_r+1,\dots\rho_n-\rho_r+1;\ x)\right]\ \dots\dots\dots\dots(46).$$

Equating (46) to the sum of (44) and (39a), and dividing by $(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)$, we obtain an equation of the same form as (28) and to be interpreted in the sense (A), but with an additional ρ and an additional α.

I do not see how to remove the restrictions imposed at the beginning of this Article without first showing that the theorem is true in the sense (B).

9. We now proceed in a different manner to extend the theorem in the sense (B).

The differential equation for $F(\alpha,\alpha_1,\dots\alpha_n;\ \rho,\rho_1,\dots\rho_n;\ +x)$ is satisfied by

$$\int_c^{(-x,\,0,\,-x-,\,0-)} x^{1-\rho}(v+x)^{\rho-\alpha-1}\,v^{\alpha-1}\,\phi(-v)\,dv\ \dots\dots\dots\dots(47),$$

where $\phi(-v)$ is any solution of that satisfied by

$$F(\alpha_1,\dots\alpha_n;\ \rho_1,\dots\rho_n;\ -v)\,dv.$$

(It would be more consistent with what has gone before to change the sign of v in the above. The introduction of the minus sign has however the advantage that the function of $-v$ with which we will be concerned does not involve i explicitly to so great an extent.)

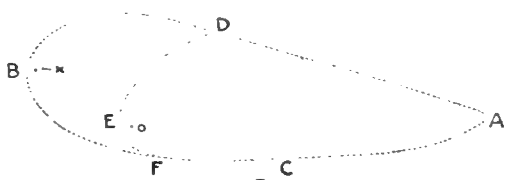

FIG 5

The above path is equivalent to the four paths $ABCA$, $ADEFA$, $ACBA$, $AFEDA$ (Fig. 5), A denoting a point h at a great distance on the positive part of the axis of real quantities.

It is assumed in the first instance that the real part of x is positive and the argument of $x^{1-\rho}$ is taken to be between $-(1-\rho)\pi/2$ and $+(1-\rho)\pi/2$. Let $\phi(-v)$ be

$$AF(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -v)$$

$$+\sum_{r=1}^{r=n} A_r v^{1-\rho_r} F(\alpha_1-\rho_r+1, \dots \alpha_n-\rho_r+1;\ 2-\rho_r,\ \rho_1-\rho_r+1, \dots \rho_n-\rho_r+1;\ -v),$$

those values being taken which make the initial arguments of every power of v zero at A (before multiplication by A, A_r) and make the initial argument of $(v+x)^{\rho-\alpha-1}$ diminish indefinitely as h increases indefinitely.

On examining the values of the arguments at different points it will be seen that the first and third portions of the path contribute to the integral (47)

$$(1-e^{2\pi\alpha i})\,x^{1-\rho}\int_h^{(-x,0)} A\,(v+x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -v)\,dv$$

$$+\sum_{r=1}^{r=n}(1-e^{2\pi i(\alpha-\rho_r)})\,x^{1-\rho}\int_h^{(-x,0)} A_r\,(v+x)^{\rho-\alpha-1}v^{\alpha-\rho_r}F(\alpha_1-\rho_r+1, \dots;\ 2-\rho_r, \dots \rho_n-\rho_r+1;\ -v)\,dv \dots(48),$$

the initial arguments being taken as above; and assuming the results stated to be true for the function of the $(n+1)$th order, if we take the differential coefficient of this with respect to h it will, by Art. 6, as h increases indefinitely, become of the order of a product of e^{-h} by a certain power of h and therefore diminish indefinitely, provided

$$\left.\begin{aligned}&[(1-e^{2\pi\alpha i})(e^{2\pi\rho i}-1)A=]\ 4e^{\pi(\alpha+\rho)i}\sin\alpha\pi\sin\rho\pi\,.\,A=\frac{\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_1)\dots\Pi(-\alpha_n)}\\ &[(1-e^{2\pi i(\alpha-\rho_r)})(e^{2\pi i(\rho-\rho_r)}-1)A_r=]\ 4e^{\pi(\alpha+\rho-2\rho_r)i}\sin(\alpha-\rho_r)\pi\sin(\rho-\rho_r)\pi\,.\,A_r\\ &\qquad=\frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\dots\Pi(\rho_r-\alpha_n-1)}\end{aligned}\right\}\dots\dots\dots(49),$$

and therefore with these values this integral will remain finite when h increases indefinitely.

In the expression (48) we now expand $(v+x)^{\rho-\alpha-1}$ in ascending powers of x; the coefficient of $x^{1-\rho+m}$ is

$$(-)^m\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\,\Pi(m)}\Bigg[(1-e^{2\pi\alpha i})\,A\int_h^{(-x,0)} v^{\rho-m-2}F(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -v)\,dv$$

$$+\sum_{r=1}^{r=n}(1-e^{2\pi i(\alpha-\rho_r)})\,A_r\int_h^{(-x,0)} v^{\rho-m-1-\rho_r}F(\alpha_1-\rho_r+1, \dots;\ 2-\rho_r, \dots \rho_n-\rho_r+1;\ -v)\,dv\Bigg]\dots(50),$$

wherein all the terms are initially real, and the successive values of m are 0, 1, 2, &c.

This too remains finite if h increases indefinitely for the same reason as (48).

We will first suppose that all the quantities $\rho-m-1$, $\rho-m-\rho_1, \dots\dots \rho-m-\rho_n$, are positive; the expression in square brackets can then be expressed as a line integral taken between the limits 0 and h, and when h increases indefinitely, using the value given by (31) for the integral (30), it is in fact

$$4e^{\pi(\alpha+\rho)i}\sin\alpha\pi\sin\rho\pi\,.\,A\,\Pi(\rho-m-2)\prod_{s=1}^{s=n}\frac{\Pi(\rho_s-1)\,\Pi(\alpha_s-\rho+m)}{\Pi(\alpha_s-1)\,\Pi(\rho_s-\rho+m)}$$

$$+\sum_{r=1}^{r=n}4e^{\pi i(\alpha+\rho-2\rho_r)}\sin(\alpha-\rho_r)\pi\sin(\rho-\rho_r)\pi\,.\,A_r\Pi(\rho-m-1-\rho_r)\prod_{s=1}^{s=n}\frac{\Pi(\rho_s-\rho_r)\,\Pi(\alpha_s-\rho+m)^*}{\Pi(\alpha_s-\rho_r)\,\Pi(\rho_s-\rho+m)}.$$

Attending to the values of A, A_r and making use of the relation

$$\Pi(n-1)\,\Pi(-n)=\Pi\operatorname{cosec} n\pi,$$

this may be written in the form

$$\frac{\pi}{\Pi(m-\rho+1)}\prod_{s=1}^{s=n}\frac{\Pi(\alpha_s-\rho+m)}{\Pi(\rho_s-\rho+m)}\left\{\frac{\sin\alpha_1\pi\sin\alpha_2\pi\ldots\sin\alpha_n\pi}{\sin\rho_1\pi\sin\rho_2\pi\ldots\sin\rho_n\pi\sin(m-\rho+2)\pi}\right.$$

$$\left.+\sum_{r=1}^{r=n}\frac{\sin(\alpha_1-\rho_r+1)\pi\sin(\alpha_2-\rho_r+1)\pi\ldots\sin(\alpha_n-\rho_r+1)\pi}{\sin(\rho_r-1)\pi\sin(\rho_1-\rho_r+1)\pi\ldots\sin(\rho_n-\rho_r+1)\pi\sin(m+1+\rho_r-\rho)\pi}\right\}.$$

But the expression in brackets is equal to

$$(-)^n\frac{\sin(\alpha_1+m-\rho+2)\pi\sin(\alpha_2+m-\rho+2)\pi\ldots\sin(\alpha_n+m-\rho+2)\pi}{\sin(m-\rho+2)\pi\sin(m+1+\rho_1-\rho)\pi\ldots\sin(m+1+\rho_n-\rho)\pi},$$

for it is readily seen that this last can be written in the form

$$\frac{B}{\sin(m-\rho+2)\pi}+\sum_{r=1}^{r=n}\frac{B_r}{\sin(m+1+\rho_r-\rho)\pi},$$

where B, B_r are quantities independent of ρ and their evaluation in the usual way leads to the result stated.

Accordingly (50) reduces to

$$(-)^m\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\,\Pi(m)}\frac{\Pi(\rho-m-2)\,\Pi(\rho-\rho_1-m-1)\ldots\Pi(\rho-\rho_n-m-1)}{\Pi(\rho-\alpha_1-m-1)\,\Pi(\rho-\alpha_2-m-1)\ldots\Pi(\rho-\alpha_n-m-1)}.$$

This result may then be extended to cases in which the conditions that $\rho-m-1$, $\rho-m-\rho_1$, ... $\rho-m-\rho_n$ should be positive are not satisfied, as is done in a parallel case in Art. 7.

Therefore the portion contributed by the first and third portions of the path is, when h is made infinite,

$$\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\ldots\Pi(\rho-\rho_n-1)}{\Pi(\rho-\alpha_1-1)\,\Pi(\rho-\alpha_2-1)\ldots\Pi(\rho-\alpha_n-1)}x^{1-\rho}F(\alpha-\rho+1,\,\alpha_1-\rho+1,\ldots\,\alpha_n-\rho+1;$$
$$2-\rho,\,\rho_1-\rho+1,\,\rho_2-\rho+1,\ldots\,\rho_n-\rho+1;\,+x)\;\ldots\ldots\ldots\ldots(50\,a).$$

* For $s=r$, ρ_s is to be replaced by unity.

Again, the second and fourth portions of the path contribute to the integral (47),

$$(e^{2\pi\rho i}-1)\,x^{1-\rho}\int_h^{(0)} A\,(v+x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1,\ldots\alpha_n;\ \rho_1,\rho_2,\ldots\rho_n;\ -v)\,dv$$

$$+\sum_{r=1}^{r=n}(e^{2\pi i(\rho-\rho_r)}-1)\,x^{1-\rho}\int_h^{(0)} A_r\,(v+x)^{\rho-\alpha-1}v^{\alpha-\rho_r}F(\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1;\ 2-\rho_r,\ldots\rho_n-\rho_r+1;\ -v)\,dv\ \ldots(51).$$

We now expand $(v+x)^{\rho-\alpha-1}$ in descending powers of x; the coefficient of $x^{-\alpha-m}$ in the above is

$$(-)^m\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\,\Pi(m)}\left[(e^{2\pi\rho i}-1)\,A\int_h^{(0)} v^{m+\alpha-1}F(\alpha_1\ldots\alpha_n;\ \rho_1\ldots\rho_n;\ -v)\,dv\right.$$

$$\left.+\sum_{r=1}^{r=n}(e^{2\pi i(\rho-\rho_r)}-1)\,A_r\int_h^{(0)} v^{m+\alpha-\rho_r}F(\alpha_1-\rho_r+1\ldots\alpha_n-\rho_r+1;\ 2-\rho_r\ldots\rho_n-\rho_r+1;\ -v)\,dv\right]\ldots(52).$$

Suppose at first that the quantities $m+\alpha$, $m+\alpha-\rho_1+1$, ... $m+\alpha-\rho_n+1$ are all positive; this then reduces to a line integral and the value of the expression in square brackets may be obtained from the value obtained for the expression in square brackets in (50) by changing m into $-m-1$, interchanging α and ρ and then multiplying by -1; accordingly the value of (52) is

$$(-)^{m+1}\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\Pi(m)}\cdot\frac{\Pi(m+\alpha-1)\,\Pi(m+\alpha-\rho_1)\ldots\Pi(m+\alpha-\rho_n)}{\Pi(m+\alpha-\alpha_1)\,\Pi(m+\alpha-\alpha_2)\ldots\Pi(m+\alpha-\alpha_n)}.$$

This value may as before be extended to the case in which the conditions that all the quantities $m+\alpha$, $m+\alpha-\rho_1+1$, ... $m+\alpha-\rho_n+1$, must be positive, do not hold. The successive values of m are 0, 1, 2 ... and accordingly we obtain for the part contributed to the integral (47) by the second and fourth portions of the path, the divergent series

$$-\frac{\Pi(\alpha-1)\,\Pi(\alpha-\rho_1)\,\Pi(\alpha-\rho_2)\ldots\Pi(\alpha-\rho_n)}{\Pi(\alpha-\alpha_1)\,\Pi(\alpha-\alpha_2)\ldots\Pi(\alpha-\alpha_n)}\,x^{-\alpha}F(\alpha,\alpha-\rho+1,\alpha-\rho_1+1,\ldots\alpha-\rho_n+1;$$

$$\alpha-\alpha_1+1,\alpha-\alpha_2+1,\ldots\alpha-\alpha_n+1;\ -1/x)\ldots(53).$$

As regards the remainder in this series after s terms, the remainder after s terms in the expansion of $(v+x)^{\rho-\alpha-1}$ has the origin of v for a multiple point of order s, and has, by Art. 2, a modulus less than that of the next term provided $\alpha-\rho+1+s$ is positive: and accordingly bearing in mind the order of $\phi(-v)$ in the neighbourhood of the origin of v, the remainder in (53) may be written in the form of a line integral

$$Cx^{-\alpha-s}\int_0^{\infty} pv^{\alpha-1+s}\phi(-v)\,dv,$$

provided, in addition, all the quantities $\alpha+s$, $\alpha-\rho+1+s$, $\alpha-\rho_1+1+s$, ... $\alpha-\rho_n+1+s$, are positive, C denoting the numerical factor, and p a quantity whose modulus is less than unity. We are not however justified in assuming that this integral would be increased numerically by replacing p by unity, and hence that the remainder in (53) is less in absolute value than the first term omitted; for it seems possible that $\phi(-v)$ may change sign between zero and infinity which would invalidate such reasoning*; (if this objection could be removed, this proof would establish the theorem in the sense (A) also). We will show however that if the inferior limit to s imposed by the conditions just laid down be raised by unity, the modulus of the remainder after s terms is less than that of a certain multiple of the next term. Denoting $\sum_1^n (\alpha_r - \rho_r)$ by σ, as v increases indefinitely $\phi(-v)$ becomes of order $e^{-v}v^{\sigma}$. Suppose that of values of v lying between zero and infinity v_0 is that which gives to $v^{\alpha-1+s}\phi(-v)$, which owing to the inferior limit of s being raised is now zero when v is zero, its numerically greatest value. Choosing any positive quantity γ less than unity, find a value v_1 of v so great that for all greater values $\phi(-v)$ lies between the limits $C'(1 \pm \gamma)e^{-v}v^{\sigma}$, C' being a constant which we could find if desired. The integral in the above remainder is therefore less than

$$v_1\phi(-v_0) + C'(1+\gamma)\int_{v_1}^{\infty} e^{-v}v^{\alpha-1+s+\sigma},$$

and therefore less than

$$v_1\phi(-v_0) + C'(1+\gamma)\,\Pi(\alpha - 1 + \sigma + s).$$

Therefore the remainder is less in absolute value than a certain multiple, independent of x, of the first term omitted.

If the real part of x is negative, the same may be proved, for a different multiple, depending on the argument but not on the modulus of x, in a manner similar to that in which the parallel case for the function $F(\alpha; \rho; x)$ was treated.

Since by taking x great enough, terms at the beginning may be made to outweigh as much as we please any finite number of those that come after, and since there is in the above no superior limit to s, it is evident that x may be taken so great that the error committed by stopping the series at any assigned term is less in absolute value than the next term multiplied by $1+\epsilon$, where ϵ is any assigned positive quantity.

Returning to (47)

$$\int_c^{\overline{(-x,0,-x-,0-)}} x^{1-\rho}(v+x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1,\ldots\alpha_n;\rho_1,\ldots\rho_n;-v)\,dv$$

with the values of the arguments reconcilable with those already chosen may be shown to be

$$-4e^{(\rho+\alpha)\pi i}\sin(\rho-\alpha)\pi\sin\alpha\pi\,.\,\frac{\Pi(\alpha-1)\,\Pi(\rho-\alpha-1)}{\Pi(\rho-1)}\,F(\alpha,\alpha_1,\ldots\alpha_n;\rho,\rho_1,\ldots\rho_n;+x),$$

and

$$\int_c^{\overline{(-x,0,-x-,0-)}} x^{1-\rho}(v+x)^{\rho-\alpha-1}v^{\alpha-\rho_r}F(\alpha_1-\rho_r+1,\ldots;2-\rho_r,\rho_1-\rho_r+1,\ldots;-v)\,dv$$

* As $\phi(-v)$ is an integral of the form of that discussed in Art. 5 it may be seen that it does not change sign for values of v between zero and $+\infty$ provided for all values of r from 2 to n, $\rho_r - \alpha_r$ is positive. The theorem might therefore be proved in this manner subject to these restrictions which may be removed as in Art. 10. Art. 8 (part of which had gone to press before this was noted) is therefore to a great extent unnecessary.

to be

$$-4e^{(\rho+\alpha-2\rho_r)\pi i}\sin(\rho-\alpha)\pi\sin(\alpha-\rho_r+1)\pi\,.\,\frac{\Pi(\alpha-\rho_r)\,\Pi(\rho-\alpha-1)}{\Pi(\rho-\rho_r-1)}\,x^{1-\rho_r}F(\alpha-\rho_r+1,$$

$$\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1\,;\,2-\rho_r,\rho-\rho_r+1,\ldots\rho_n-\rho_r+1\,;\,+x)$$

wherein the argument of $x^{1-\rho_r}$ lies between $-\pi(1-\rho_r)$ and $+\pi(1-\rho_r)$, and making use of the values of $\boldsymbol{A}$, $\boldsymbol{A}_r$, given by (49), the integral becomes

$$-\frac{1}{\Pi(\alpha-\rho)}\,\frac{\Pi(\alpha-1)\,\Pi(-\rho)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)}\,F(\alpha,\alpha_1,\ldots\alpha_n\,;\,\rho,\rho_1,\ldots\rho_n\,;\,x)$$

$$-\frac{1}{\Pi(\alpha-\rho)}\sum_1^n\frac{\Pi(\alpha-\rho_r)\,\Pi(\rho_r-2)\,\Pi(\rho_r-\rho-1)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\,\Pi(\rho_r-\alpha_2-1)\ldots\Pi(\rho_r-\alpha_n-1)}$$

$$x^{1-\rho_r}F(\alpha-\rho_r+1,\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1\,;\,2-\rho_r,\rho-\rho_r+1,\ldots\rho_n-\rho_r+1\,;\,+x)\ldots(54).$$

Equating this to the sum of (50a) and (53) and then multiplying by $\Pi(\alpha-\rho)$ we obtain a result similar to that indicated by equation (28) in the sense (B). It differs however from that obtained in the sense (A), in having α and α_1, ρ and ρ_1 interchanged.

10. Before proceeding to remove from equation (28) taken in the sense (A) the restrictions imposed in Art. 8 that certain quantities must be positive, we will first show that if $\alpha_1-\alpha_n$ is positive and if the theorem holds for the remainder after s terms of the function involving α_1, α_n, it holds also for the remainder after $(s+1)$ terms of the function involving α_1, α_n+1, the other α's and ρ's being unchanged.

If $\psi(\alpha_1,\alpha_2,\ldots\alpha_n\,;\,\rho_1,\rho_2,\ldots\rho_n\,;\,x)$ denote either any one of the $n+1$ series of the left-hand member of (28) including the constant multiplier, or the sum of the terms at the beginning of the divergent series on the right (including the constant multiplier and the factor $x^{-\alpha_1}$) up to and including the term involving a specified power of x, it is easily verified that

$$\psi(\alpha_1,\alpha_2,\ldots\alpha_{n-1},\alpha_n+1\,;\,\rho_1,\rho_2,\ldots\rho_n\,;\,x)$$

$$=(\alpha_1-\alpha_n)\,\psi(\alpha_1,\alpha_2,\ldots\alpha_{n-1},\alpha_n\,;\,\rho_1,\rho_2,\ldots\rho_n\,;\,x)$$

$$-\psi(\alpha_1+1,\alpha_2,\ldots\alpha_{n-1},\alpha_n\,;\,\rho_1,\rho_2,\ldots\rho_n\,;\,x).$$

If then $R(s+1)$, $R(s+1,\alpha_n+1)$, denote respectively the remainders after $(s+1)$ terms of the right-hand member of equation (28) for the function involving α_1, α_n, and for that involving α_1, α_n+1, and if $R(s,\alpha_1+1)$ denote the remainder after s terms for that involving α_1+1, α_n, we have

$$R(s+1,\alpha_n+1)=(\alpha_1-\alpha_n)\,R(s+1)-R(s,\alpha_1+1).$$

If then the theorem holds for $R(s,\alpha_n)$, a fortiori it holds for $R(s+1,\alpha_n)$, and it also holds for $R(s,\alpha_1+1)$, for $\sum_{r=2}^{r=n}[\alpha_r-\rho_r]$ is not increased by increasing α_1; we thus have

$$\text{mod. } R(s+1)<\text{mod. }\frac{\Pi(\alpha_1+s)\ldots\Pi(\alpha_1-\rho_n+1+s)}{\Pi(\alpha_1-\alpha_2+1+s)\ldots\Pi(\alpha_1-\alpha_n+1+s)}\,\frac{1}{\Pi(s+1)\,x^{\alpha_1+s+1}},$$

and mod. $R(s, \alpha_1+1) < (s+1)$ times the same, therefore provided $\alpha_1 - \alpha_n$ is positive we deduce that

$$\text{mod. } R(s+1,\ \alpha_n+1) < \text{mod. } \frac{\Pi(\alpha_1+s)\ldots\Pi(\alpha_1-\rho_n+1+s)}{\Pi(\alpha_1-\alpha_2+1+s)\ldots\Pi(\alpha_1-\alpha_{n-1}+1+s)\,\Pi(\alpha_1-\alpha_n+s)}\,\frac{1}{\Pi(s+1)\,x^{\alpha_1+s+1}},$$

which is the result stated; this reasoning holds for all values of s including zero.

We will next show that if the theorem holds for the remainder after s terms of the function involving α_1, it also holds for the remainder after $(s+1)$ terms of the function involving $\alpha_1 - 1$, the other α's and ρ's being unaltered.

If $\psi(\alpha_1)$ denote either the left-hand member of (28) or the sum of the terms at the beginning of the right up to and including the term involving a specified power of x, we have

$$x^{\alpha_1-2}\,\psi(\alpha_1) = \frac{d}{dx}\{x^{\alpha_1-1}\,\psi(\alpha_1-1)\} \quad\ldots\ldots\ldots\ldots(55).$$

If then we denote the difference between the left-hand member of (28) and s terms of the right-hand member by $R(s, \alpha_1, x)$, we have

$$x^{\alpha_1-2}\,R(s, \alpha_1, x) = \frac{d}{dx}\{x^{\alpha_1-1}\,R(s+1, \alpha_1-1, x)\} \quad\ldots\ldots\ldots\ldots(56),$$

and therefore

$$\left[x^{\alpha_1-1}\,R(s+1, \alpha_1-1, x)\right]_x^\infty = \int_x^\infty x^{\alpha_1-2}\,R(s, \alpha_1, x)\,dx \quad\ldots\ldots\ldots\ldots(57).$$

For all values of s including zero, $x^{\alpha_1-1}\,R(s+1, \alpha_1-1, x)$ vanishes when the modulus of x is infinite, whatever be its argument, since equation (28) is true in the sense (B) and since the first term omitted when multiplied by x^{α_1-1} has for its index α_1+s, the being negative of which is one of the conditions that the theorem should hold for $R(s, \alpha_1, x)$.

Accordingly

$$R(s+1, \alpha_1-1, x) = -\,x^{1-\alpha_1}\int_x^\infty x^{\alpha_1-2}R(s,\ \alpha_1,\ x)\,dx \quad\ldots\ldots\ldots\ldots(58).$$

By taking for the path of integration the production of the straight line joining the origin to the point x, it appears that either member of (58) would be increased in absolute value by replacing $R(s, \alpha_1, x)$ by the modulus of the first term omitted; but this change would replace the right-hand member by the modulus of the $(s+2)$th term in the series obtained from the right-hand member of (28) by diminishing α_1 by unity; therefore $R(s+1,\ \alpha_1-1,\ x)$ is less in absolute value than the next term: this reasoning holds for all values of s including zero.

We will now, having in fact proved the theorem for all values of s including zero subject to the conditions that for all values of r from 2 to n inclusive $\alpha_1 - \rho_r + 1$ and $\rho_r - \alpha_r$ should be positive, proceed to examine what restrictions should be placed on s if these conditions are violated.

We suppose that α_n is the greatest α and ρ_n the greatest ρ.

Suppose first that $\alpha_1 - \rho_n + 1$ is negative lying between $-a_n$ and $-a_n + 1$, and that $\rho_n - \alpha_n$ is negative lying between $-b_n$ and $-b_n + 1$. Then $\alpha_1 - \alpha_n + 1$ must also be negative lying between $-c_n$ and $-c_n + 1$ where c_n is either $a_n + b_n$ or $a_n + b_n - 1$. For the other values of r from 2 to $n-1$ let b_r denote $[\alpha_r - \rho_r]$, which for some values of r may be zero.

The theorem then applies for all values of s to the function involving

$$\alpha_1 + c_n,\ \alpha_2 - b_2,\ \ldots\ \alpha_n - b_n,\ 1,\ \rho_1,\ \rho_2,\ \ldots\ \rho_n,$$

since the necessary conditions are satisfied.

We may increase the value of $\alpha_n - b_n$ by unity b_n times in succession, keeping all the other α's and ρ's unaltered, provided at each such operation we increase the lowest value of s for which the theorem holds by unity, the condition for the validity of the last such process being that $\alpha_1 + c_n - (\alpha_n - 1)$ is to be positive; thus when we attain the value α_n, s has to be raised from zero to b_n. Then for each other value of r in turn we may increase in a similar manner the value of $\alpha_r - b_r$ by b_r, increasing the value of s at the same time by b_r also, the condition for the validity of this being that $\alpha_1 + c_n - (\alpha_r - 1)$ should be positive, which is true since α_n is the largest α. Thus when we attain the values $\alpha_2, \alpha_3, \ldots \alpha_n$, the lowest admissible value of s is $\sum_{r=2}^{r=n} b_r$. Finally we diminish the value of $\alpha_1 + c_n$ by unity c_n times in succession without altering the other α's or ρ's, at the same time increasing the value of s by c_n. Thus the lowest admissible value of s is $[\alpha_n - \alpha_1 - 1] + \sum_2^n [\alpha_r - \rho_r]$, as the enunciation states.

Next suppose that $\alpha_1 - \rho_n + 1$ is positive but that $\rho_n - \alpha_n$ is negative; we have now two sub-cases according as $\alpha_1 - \alpha_n + 1$ is negative or positive.

Taking first the former: as before the theorem applies for all values of s to the function involving

$$\alpha_1 + c_n,\ \alpha_2 - b_2,\ \ldots\ \alpha_n - b_n,\ 1,\ \rho_1,\ \rho_2,\ \ldots\ \rho_n,$$

and we proceed as before, with the result that when we attain the function involving

$$\alpha_1,\ \alpha_2,\ \ldots\ \alpha_n,\ 1,\ \rho_1,\ \ldots\ \rho_n,$$

the lowest admissible value of s is $[\alpha_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

Taking the latter sub-case, the theorem now applies for all values of s to the function involving

$$\alpha_1,\ \alpha_2 - b_2,\ \ldots\ \alpha_n - b_n,\ 1,\ \rho_1,\ \ldots\ \rho_n;$$

as before, for each value of r in turn we may increase the value of $\alpha_r - b_r$ by b_r, increasing the value of s by b_r also, this being legitimate since $\alpha_1 - \alpha_r + 1$ is positive; thus when we attain the values $\alpha_1, \alpha_2, \ldots \alpha_n, 1, \rho_1, \ldots \rho_n$, the lowest admissible value of s is $\sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

Next suppose that $\alpha_1 - \rho_n + 1$ is positive and $\rho_n - \alpha_n$ positive, but that for some values of r, $\rho_r - \alpha_r$ is negative. This is similar to the sub-case last considered and as in it, the lowest admissible value of s is $\sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, the term $[\alpha_n - \rho_n]$ being however zero.

Finally suppose that $\alpha_1 - \rho_n + 1$ is negative, and $\rho_n - \alpha_n$ positive. The theorem applies for all values of s to the function involving

$$\alpha_1 + [\rho_n - \alpha_1 - 1],\ \alpha_2 - b_2, \ldots \alpha_n,\ 1,\ \rho_1, \ldots \rho_n.$$

As before, for each value of r we may increase $\alpha_r - b_r$ by b_r, increasing s by b_r at the same time, this being legitimate since $\alpha_1 + [\rho_n - \alpha_1 - 1] - \alpha_r + 1$ is positive, ρ_n being greater than α_n and a fortiori than α_r. Finally we reduce the value of $\alpha_1 + [\rho_n - \alpha_1 - 1]$ to α_1 and thus when we attain the values $\alpha_1, \alpha_2, \ldots \alpha_n, 1, \rho_1, \ldots \rho_n$ the lowest admissible value of s is $[\rho_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, the term $[\alpha_n - \rho_n]$ being zero.

These several limits are all included by the statement that s is not to be less than the greater of the two integers, $[\rho_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, $[\alpha_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

11. We may in fact obtain a limit to the error even when the real part of x is negative. The reasoning of Art. 3 suffices to show that for the function $x^{-\alpha_1}\phi(\alpha_1, \alpha_1 - \rho_1 + 1;\ +1/x)$ if the argument of x be $\pm\gamma$, γ being $< \pi/2$, the modulus of the remainder is less than that of the next term divided by $(\sin(\theta+\gamma))^{[\alpha_1]} (\cos\theta)^{\alpha_1 - \rho_1 + s + 1}$, where θ and $\theta + \gamma$ are each less than $\pi/2$; and by changing in the integral (37) the point h to the point at infinity on the production of the line joining the points 0, x, we see that the same statement holds for the function of the $(n+1)$th order if for all values of r from 2 to n, $\alpha_1 - \rho_r + 1$ and $\rho_r - \alpha_r$ are positive. Also a reference to the method by which these restrictions are removed shows that in the most general case the index of $\sin(\theta+\gamma)$ may be replaced by the greater of the integers $[\alpha_1] + [\alpha_n - \alpha_1 - 1]$, $[\alpha_1] + [\rho_n - \alpha_1 - 1]$, while that of $\cos\theta$ is left unaltered in form, affected only by the increase in s, s being the number of terms taken and subject to the same restrictions as before. We must bear in mind that every ρ is greater than 1, ρ_n the greatest ρ, α_n the greatest α, and ρ_1 the ρ omitted from the sum $\sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

We may investigate the numerical value in the case of the semiconvergent series for the Bessel functions. In this case we may write $\alpha_1 = \frac{1}{2} - n$, $\rho_1 = 1 - 2n$. Hence $[\alpha_1]$ is 1, $\alpha_1 - \rho_1 + s + 1$ is $\frac{1}{2} + n + s$. The divisor is thus $\sin(\theta+\gamma)(\cos\theta)^{\frac{1}{2}+n+s}$; to make this as large as possible, θ should be nearly zero unless γ be very small, and we deduce that the error is less than the next term divided by $\sin\gamma$; if γ be very small, the greatest value is greater than when γ is zero, in which case it is $(\tfrac{1}{2}+n+s)^{\frac{1}{2}(\frac{1}{2}+n+s)}$. $(\tfrac{3}{2}+n+s)^{-\frac{1}{2}(\frac{1}{2}+n+s)}$; this tends to equality with $(\tfrac{1}{2}+n+s)^{-\frac{1}{2}}e^{-\frac{1}{2}}$, even for moderate values of s, and the error is thus less than the next term multiplied by a number which is nearly $(\tfrac{1}{2}+n+s)^{\frac{1}{2}}e^{\frac{1}{2}}$. The multiplier thus obtained when γ is zero is considerably larger than that given by Weber (*Math. Annal.* XXXVII.) for all arguments, which is about $s^{\frac{1}{2}}\pi^{-\frac{1}{2}}\cos n\pi$.

12. By reasoning similar to that by which it is shown that $\mathrm{Lt.}_{\alpha=\infty}(1-x/\alpha)^{\alpha}=e^{-x}$ we may show that by writing $x=y/h$, $\alpha=-h$ and increasing h indefinitely we can diminish the number of α's in equation (28) successively by unity; we thus obtain very general results. From the theorem that as τ increases indefinitely the ratio of $\Pi(\tau)$ to $e^{-\tau}\tau^{\tau}\sqrt{2\tau\pi}$ has unity for its limit it follows that if α and β are positive quantities and τ be increased indefinitely $\dfrac{\Pi(\tau-\alpha)}{\Pi(\tau-\beta)}\tau^{\alpha-\beta}$ has unity for its limit, and making use of this result we see that the general theorem may be written in the form that if $m \not> n$

$$\frac{\Pi(\alpha_1-1)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_2)\,\Pi(-\alpha_3)\ldots\Pi(-\alpha_m)}\,.\,F(\alpha_1,\alpha_2,\ldots\alpha_m;\ \rho_1,\rho_2,\ldots\rho_n;\ (-)^{n-m}x)$$

$$+\frac{\Pi(\alpha_1-\rho_1)\,\Pi(\rho_1-2)\,\Pi(\rho_1-\rho_2-1)\ldots\Pi(\rho_1-\rho_n-1)}{\Pi(\rho_1-\alpha_2-1)\,\Pi(\rho_1-\alpha_3-1)\ldots\Pi(\rho_1-\alpha_m-1)}\,x^{1-\rho_1}F(\alpha_1-\rho_1+1,\alpha_2-\rho_1+1,\ldots\alpha_m-\rho_1+1;$$
$$2-\rho_1,\rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\ (-)^{n-m}x),$$

$+(n-1)$ terms analogous to the last

$$=\frac{\Pi(\alpha_1-1)\,\Pi(\alpha_1-\rho_1)\ldots\Pi(\alpha_1-\rho_n)}{\Pi(\alpha_1-\alpha_2)\,\Pi(\alpha_1-\alpha_3)\ldots\Pi(\alpha_1-\alpha_m)}\,x^{-\alpha_1}F(\alpha_1,\alpha_1-\rho_1+1,\ldots\alpha_1-\rho_n+1;\ \alpha_1-\alpha_2+1,\ldots$$
$$\alpha_1-\alpha_m+1;\ -1/x)\ldots\ldots\ldots\ldots(59),$$

in the same senses as those in which equation (28) is true.

VII. *A semi-inverse method of solution of the equations of elasticity, and its application to certain cases of aeolotropic ellipsoids and cylinders.* By C. Chree, Sc.D., F.R.S.

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CONTENTS.

SECTION I.

Preliminary, and General Formulae.

§ 1. In two previous papers, here termed (A)* and (B)† for brevity, I developed a new method of treating the elastic solid equations for isotropic material, which led to a complete solution of the problem presented by an ellipsoid of uniform density ρ,

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \quad\quad (1),$$

acted on by bodily forces derivable from a potential

$$V = \tfrac{1}{2}(Px^2 + Qy^2 + Rz^2),$$

and by normal surface forces $Sx^2 + Ty^2 + Uz^2$.

Here x, y, z are ordinary Cartesian coordinates; while P, Q, R, S, T, U are constants, any of which may be zero.

* *Proc. Royal Soc.* Vol. LVIII. p. 39. † *Quarterly Journal*, Vol. XXVII. p. 338, 1895.

The bodily forces are such as the gravitational forces arising from the self-attraction of the ellipsoid, the tide generating influence of a distant body, or the 'centrifugal forces' due to rotation about a principal axis of the ellipsoid.

The present paper deals with the extension of the method to aeolotropic bodies.

§ 2. The most general kind of elastic solid possesses on the usual theory 21 independent elastic constants. The consideration of such material is laborious owing to the length of the expressions. I have thus considered in detail no case of greater complexity than that presented by material symmetrical with respect to three planes of elastic symmetry, coincident with the three principal planes of the ellipsoid.

In such material there are three principal Young's moduli E_1, E_2, E_3, relating to tractions parallel to the axes of x, y and z respectively. There are six corresponding Poisson's ratios η_{12}, η_{21}, etc. Here the first suffix gives the direction of the traction, the second that of the corresponding contraction. For instance, η_{12} is the ratio of the contraction parallel to the y-axis to the extension parallel to the x-axis, in the case of traction parallel to the latter axis.

Between the six Poisson's ratios there exist the three relations

$$\eta_{12}/E_1 = \eta_{21}/E_2 ; \quad \eta_{13}/E_1 = \eta_{31}/E_3 ; \quad \eta_{23}/E_2 = \eta_{32}/E_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

so that only three are really independent. There being nine independent elastic constants for this type of material, we shall take for our remaining three the principal slide coefficients n_1, n_2, n_3.

In the notation of Todhunter and Pearson's *History of Elasticity*, the strains are given in terms of the stresses by the following relations

$$\left.\begin{array}{lll} s_x = (\widehat{xx} - \eta_{12}\widehat{yy} - \eta_{13}\widehat{zz})/E_1 ; & s_y = (\widehat{yy} - \eta_{21}\widehat{xx} - \eta_{23}\widehat{zz})/E_2 ; & s_z = (\widehat{zz} - \eta_{31}\widehat{xx} - \eta_{32}\widehat{yy})/E_3 ; \\ \sigma_{yz} = \widehat{yz}/n_1 & ; \sigma_{zx} = \widehat{zx}/n_2 & ; \sigma_{xy} = \widehat{xy}/n_3 \end{array}\right\} \ldots\ldots (3).$$

A second and simpler type of material dealt with here is that which in addition to symmetry with respect to the planes of zx, xy and yz is completely symmetrical round the axis of z.

For such material we have

$$\left.\begin{array}{r} E_2 = E_1 = E', \\ E_3 = E, \\ \eta_{31} = \eta_{32} = \eta, \\ \eta_{12} = \eta_{21} = \eta', \\ \eta_{13} = \eta_{23} = \eta'', \\ \eta''/E' = \eta/E, \\ n_2 = n_1 = n, \\ n_3 = E' \div \{2(1+\eta')\} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

There are in this case only five independent elastic constants.

§ 3. For any kind of elastic material the stresses must satisfy the three body-stress equations

$$\left.\begin{aligned} \frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz} + \rho Px = 0, \\ \frac{d\widehat{xy}}{dx} + \frac{d\widehat{yy}}{dy} + \frac{d\widehat{yz}}{dz} + \rho Qy = 0, \\ \frac{d\widehat{xz}}{dx} + \frac{d\widehat{yz}}{dy} + \frac{d\widehat{zz}}{dz} + \rho Rz = 0 \end{aligned}\right\} \qquad \text{.......................(5)}$$

and the three surface equations

$$\left.\begin{aligned} (x/a^2)\,\widehat{xx} + (y/b^2)\,\widehat{xy} + (z/c^2)\,\widehat{xz} &= (x/a^2)(Sx^2 + Ty^2 + Uz^2), \\ (x/a^2)\,\widehat{xy} + (y/b^2)\,\widehat{yy} + (z/c^2)\,\widehat{yz} &= (y/b^2)(Sx^2 + Ty^2 + Uz^2), \\ (x/a^2)\,\widehat{xz} + (y/b^2)\,\widehat{yz} + (z/c^2)\,\widehat{zz} &= (z/c^2)(Sx^2 + Ty^2 + Uz^2) \end{aligned}\right\} \qquad \text{..................(6).}$$

From the six equations of compatibility* between the strains, viz. three of the type

$$\frac{d^2 s_x}{dy^2} + \frac{d^2 s_y}{dx^2} - \frac{d^2 \sigma_{xy}}{dx\,dy} = 0 \qquad \text{..................................(7).}$$

and three of the type

$$\frac{2d^2 s_x}{dy\,dz} + \frac{d^2 \sigma_{yz}}{dx^2} - \frac{d^2 \sigma_{zx}}{dx\,dy} - \frac{d^2 \sigma_{xy}}{dx\,dz} = 0 \qquad \text{..............................(8)}$$

we get six corresponding equations between the stresses.

These equations necessarily vary with the nature of the material. Thus for material symmetrical about the coordinate planes we have three equations of the type

$$\frac{1}{E_1}\frac{d^2}{dy^2}(\widehat{xx} - \eta_{12}\widehat{yy} - \eta_{13}\widehat{zz}) + \frac{1}{E_2}\frac{d^2}{dx^2}(\widehat{yy} - \eta_{21}\widehat{xx} - \eta_{23}\widehat{zz}) - \frac{1}{n_3}\frac{d^2}{dx\,dy}\widehat{xy} = 0 \qquad \text{......... (9).}$$

and three of the type

$$\frac{2}{E_1}\frac{d^2}{dy\,dz}(\widehat{xx} - \eta_{12}\widehat{yy} - \eta_{13}\widehat{zz}) + \frac{1}{n_1}\frac{d^2}{dx^2}\widehat{yz} - \frac{1}{n_2}\frac{d^2}{dx\,dy}\widehat{xz} - \frac{1}{n_3}\frac{d^2}{dx\,dz}\widehat{xy} = 0 \qquad \text{......... (10).}$$

In the present instance the equations of type (10) are identically satisfied and need not concern us further.

When the material is symmetrical round the z-axis we find in place of (9)

$$\left.\begin{aligned} \frac{d^2}{dy^2}\left(\frac{\widehat{xx} - \eta'\widehat{yy}}{E'} - \frac{\eta}{E}\widehat{zz}\right) + \frac{d^2}{dx^2}\left(\frac{\widehat{yy} - \eta'\widehat{xx}}{E'} - \frac{\eta}{E}\widehat{zz}\right) - \frac{2(1+\eta')}{E'}\frac{d^2}{dx\,dy}\widehat{xy} = 0, \\ \frac{d^2}{dz^2}\left(\frac{\widehat{yy} - \eta'\widehat{xx}}{E'} - \frac{\eta}{E}\widehat{zz}\right) + \frac{1}{E}\frac{d^2}{dy^2}(\widehat{zz} - \eta\widehat{xx} - \eta\widehat{yy}) - \frac{1}{n}\frac{d^2}{dy\,dz}\widehat{yz} = 0, \\ \frac{d^2}{dz^2}\left(\frac{\widehat{xx} - \eta'\widehat{yy}}{E'} - \frac{\eta}{E}\widehat{zz}\right) + \frac{1}{E}\frac{d^2}{dx^2}(\widehat{zz} - \eta\widehat{xx} - \eta\widehat{yy}) - \frac{1}{n}\frac{d^2}{dx\,dz}\widehat{xz} = 0 \end{aligned}\right\} \qquad \text{...... (11).}$$

* See Todhunter and Pearson's *History of Elasticity*, Vol. II. Part i. p. 74; or Love's *Treatise on Elasticity* Vol. I. p. 122.

§ 4. The greater complexity of (9) or (11), as compared to the corresponding equations for isotropy, does not affect the *type* of solution; and, as in the papers (A) and (B), we may assume

$$\left.\begin{aligned} \widehat{xx} &= A_0 + A_2x^2 + A_2'y^2 + A_2''z^2, \\ \widehat{yy} &= B_0 + B_2x^2 + B_2'y^2 + B_2''z^2, \\ \widehat{zz} &= C_0 + C_2x^2 + C_2'y^2 + C_2''z^2, \\ \widehat{yz} &= 2Lyz, \quad \widehat{xz} = 2Mxz, \quad \widehat{xy} = 2Nxy \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12).$$

Here $A_0, A_2 \ldots N$ are constants to be determined from the body-stress equations (5), the surface equations (6), and the equations of compatibility, the latter of which alone vary with the type of material.

Fortunately there is an immense economy of labour owing to my having in papers (A) and (B) expressed all the A, B, C constants in (12) in terms of the three L, M, N. In effecting this simplification I employed only (5) and (6), equations which, as pointed out above, apply to all kinds of elastic material. We are thus enabled at once to replace (12) by the following equations established in the two earlier papers:—

$$\left.\begin{aligned} \widehat{xx} = Sx^2 + Ty^2 + Uz^2 + a^2\Big[&(\tfrac{1}{2}P\rho + S)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\ &+ M\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + N\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right)\Big], \\ \widehat{yy} = Sx^2 + Ty^2 + Uz^2 + b^2\Big[&(\tfrac{1}{2}Q\rho + T)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\ &+ N\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) + L\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right)\Big], \\ \widehat{zz} = Sx^2 + Ty^2 + Uz^2 + c^2\Big[&(\tfrac{1}{2}R\rho + U)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\ &+ L\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) + M\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\Big], \\ \widehat{yz} = 2Lyz, \quad \widehat{xz} = 2Mxz, \quad \widehat{xy} &= 2Nxy \end{aligned}\right\} \quad \ldots\ldots(13).$$

§ 5. The results (13) apply to all kinds of elastic material, whether possessed of 2 or of 21 independent elastic constants; but the values of L, M, N vary of course with the material. The expressions for the strains corresponding to (13) vary. Thus, for material symmetrical with respect to the three coordinate planes, we find from (3)

$$\left.\begin{aligned} s_xE_1 = (1 - \eta_{12} - \eta_{13})(Sx^2 + Ty^2 + Uz^2) + \{&a^2(\tfrac{1}{2}P\rho + S) - \eta_{12}b^2(\tfrac{1}{2}Q\rho + T) \\ &- \eta_{13}c^2(\tfrac{1}{2}R\rho + U)\}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\ + (a^2M - \eta_{12}b^2L)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) &+ (a^2N - \eta_{13}c^2L)\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) \\ - (\eta_{12}b^2N + \eta_{13}c^2M)&\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right), \\ \sigma_{yz} = 2Lyz/n_1 & \end{aligned}\right\} \quad \ldots\ldots(14).$$

The expressions for the other four strains may be written down from symmetry.

If the material be symmetrical round the z-axis we have

$$
\left.
\begin{aligned}
s_x = {} & \left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)(Sx^2 + Ty^2 + Uz^2) + \left\{\frac{1}{E'}\left(a^2(\tfrac{1}{2}P\rho + S) - \eta' b^2(\tfrac{1}{2}Q\rho + T)\right)\right. \\
& \left. - \frac{\eta}{E}c^2(\tfrac{1}{2}R\rho + U)\right\}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\
& + \left(\frac{a^2M - \eta' b^2 L}{E'}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + \left(\frac{a^2N}{E'} - \frac{\eta c^2 L}{E}\right)\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) \\
& - \left(\frac{\eta'}{E'}b^2N + \frac{\eta}{E}c^2M\right)\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right), \\
s_y = {} & \text{expression obtained from that for } s_x \text{ by interchanging } a \text{ with } b,\ x \text{ with} \\
& y,\ L \text{ with } M,\ P \text{ with } Q, \text{ and } S \text{ with } T, \\
Es_z = {} & (1 - 2\eta)(Sx^2 + Ty^2 + Uz^2) + \{c^2(\tfrac{1}{2}R\rho + U) - \eta a^2(\tfrac{1}{2}P\rho + S) - \eta b^2(\tfrac{1}{2}Q\rho + T)\} \\
& \times\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\
& - \eta(a^2M + b^2L)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + (c^2L - \eta a^2 N)\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) \\
& + (c^2M - \eta b^2 N)\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right), \\
\sigma_{yz} = {} & 2Lyz/n, \quad \sigma_{xz} = 2Mxz/n, \quad \sigma_{xy} = 4(1 + \eta')Nxy/E'
\end{aligned}
\right\} \quad \ldots\ldots(15).
$$

§ 6. Results which depend only on the form of equations (13) are true irrespective of the nature of the elastic material. For instance* if S, T and U vanish, or there be no surface forces, the resultant stresses across parallel tangent planes at their points of contact with the system of confocals

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = \lambda$$

are all parallel, and their intensity varies as $1 - \lambda$.

§ 7. We have now to consider how L, M and N are to be determined. Substituting from (13) in the equations of compatibility, whether (9) or (11) as the case may be, we obtain three simple equations of the form

$$
\left.
\begin{aligned}
a_{11}L + a_{12}M + a_{13}N &= \varpi_1, \\
a_{12}L + a_{22}M + a_{23}N &= \varpi_2, \\
a_{13}L + a_{23}M + a_{33}N &= \varpi_3
\end{aligned}
\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),
$$

where $a_{11}, \ldots \varpi_1 \ldots$ are known functions of the elastic constants, the bodily and surface forces, and the semi-axes of the ellipsoid. Representing by Π_{11}, Π_{12}, &c. the minors of the determinant

$$
\Pi \equiv \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{12}, & a_{22}, & a_{23} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).
$$

* Cf. (A) § 2.

we have from (16)

$$\left.\begin{aligned} L &= (\varpi_1\Pi_{11} + \varpi_2\Pi_{12} + \varpi_3\Pi_{13}) \div \Pi, \\ M &= (\varpi_1\Pi_{12} + \varpi_2\Pi_{22} + \varpi_3\Pi_{23}) \div \Pi, \\ N &= (\varpi_1\Pi_{13} + \varpi_2\Pi_{23} + \varpi_3\Pi_{33}) \div \Pi \end{aligned}\right\} \quad \ldots\ldots(18).$$

§ 8. When the material is symmetrical with respect to the three principal planes of the ellipsoid

$$\left.\begin{aligned} a_{11} &= \frac{3b^4}{E_2} + \frac{3c^4}{E_3} + b^2c^2\left(\frac{1}{n_1} - \frac{2\eta_{23}}{E_2}\right), \\ a_{23} &= \frac{a^4}{E_1} - a^2b^2\frac{\eta_{12}}{E_1} - 3b^2c^2\frac{\eta_{23}}{E_2} - c^2a^2\frac{\eta_{31}}{E_3} \end{aligned}\right\} \quad \ldots\ldots(19),$$

and the other a's can be written down from symmetry, the relations (4) being borne in mind.

For the same kind of material

$$\left.\begin{aligned} \varpi_1 &= (\tfrac{1}{2}P\rho + S)\frac{a^2}{E_1}(\eta_{12}b^2 + \eta_{13}c^2) + \frac{b^2}{E_2}\left[\tfrac{1}{2}Q\rho(\eta_{23}c^2 - b^2) + T\left\{c^2(1-\eta_{31})\frac{E_2}{E_3} - b^2\right\}\right] \\ &\quad + \frac{c^2}{E_3}\left[\tfrac{1}{2}R\rho(\eta_{32}b^2 - c^2) + U\left\{b^2(1-\eta_{21})\frac{E_3}{E_2} - c^2\right\}\right], \\ \varpi_2 &= \frac{a^2}{E_1}\left[\tfrac{1}{2}P\rho(\eta_{13}c^2 - a^2) + S\left\{c^2(1-\eta_{32})\frac{E_1}{E_3} - a^2\right\}\right] + (\tfrac{1}{2}Q\rho + T)\frac{b^2}{E_2}(\eta_{23}c^2 + \eta_{21}a^2) \\ &\quad + \frac{c^2}{E_3}\left[\tfrac{1}{2}R\rho(\eta_{31}a^2 - c^2) + U\left\{a^2(1-\eta_{12})\frac{E_3}{E_1} - c^2\right\}\right], \\ \varpi_3 &= \frac{a^2}{E_1}\left[\tfrac{1}{2}P\rho(\eta_{12}b^2 - a^2) + S\left\{b^2(1-\eta_{23})\frac{E_1}{E_2} - a^2\right\}\right] \\ &\quad + \frac{b^2}{E_2}\left[\tfrac{1}{2}Q\rho(\eta_{21}a^2 - b^2) + T\left\{a^2(1-\eta_{13})\frac{E_2}{E_1} - b^2\right\}\right] + (\tfrac{1}{2}R\rho + U)\frac{c^2}{E_3}(\eta_{31}a^2 + \eta_{32}b^2) \end{aligned}\right\} \quad \ldots\ldots(20).$$

Under like conditions

$$\left.\begin{aligned} \Pi_{11} &= 8\frac{a^8}{E_1^2} + \frac{a^4b^4}{E_1E_2}(9 - \eta_{12}\eta_{21}) + \frac{a^4c^4}{E_1E_3}(9 - \eta_{13}\eta_{31}) + 9\frac{b^4c^4}{E_2E_3}(1 - \eta_{23}\eta_{32}) \\ &\quad + \frac{a^6b^2}{E_1}\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right) + \frac{a^6c^2}{E_1}\left(\frac{3}{n_2} - \frac{4\eta_{13}}{E_1}\right) + a^4b^2c^2\left\{\left(\frac{1}{n_2} - \frac{2\eta_{13}}{E_1}\right)\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1}\right) + \frac{6\eta_{23}}{E_1E_2} - \frac{2\eta_{12}\eta_{13}}{E_1^2}\right\} \\ &\quad + 3\frac{a^2b^4c^2}{E_2}\left(\frac{1}{n_2} - \frac{2\eta_{13}}{E_1} - \frac{2\eta_{12}\eta_{23}}{E_1}\right) + 3\frac{a^2b^2c^4}{E_3}\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right), \\ \Pi_{23} &= \frac{b^4c^4}{E_2}\left\{\frac{1}{E_3}(1 - 5\eta_{23}\eta_{32}) + \frac{3\eta_{23}}{n_1}\right\} - 3\frac{a^4b^4}{E_1E_2}(1 - \eta_{12}\eta_{21}) - 3\frac{a^4c^4}{E_1E_3}(1 - \eta_{13}\eta_{31}) \\ &\quad + 8\frac{b^2c^2}{E_2E_3}(\eta_{32}b^4 + \eta_{23}c^4) + \frac{a^2b^4c^2}{E_1}\left(\frac{2\eta_{13}}{E_2} + \frac{2\eta_{12}\eta_{23}}{E_2} + \frac{\eta_{12}}{n_1}\right) \\ &\quad + \frac{a^2b^2c^4}{E_1}\left(\frac{2\eta_{12}}{E_3} + \frac{2\eta_{13}\eta_{32}}{E_3} + \frac{\eta_{13}}{n_1}\right) + \frac{a^4b^2c^2}{E_1}\left(\frac{2\eta_{23}}{E_2} + \frac{10\eta_{12}\eta_{13}}{E_1} - \frac{1}{n_1}\right) \end{aligned}\right\} \quad \ldots(21).$$

The other four minors of the determinant (17) may be written down from symmetry.

In obtaining (20) and (21) free use has been made of the relations (2), by means of which various alternative forms can be obtained.

The full expression for Π is too long to write down. In practice, after determining the minors as above, one would determine Π from such an equation as

$$\Pi = a_{11}\Pi_{11} + a_{12}\Pi_{12} + a_{13}\Pi_{13} \qquad \ldots\ldots (22)$$

§ 9. When the material is symmetrical round the z-axis

$$\left.\begin{aligned}
a_{11} &= 3\left(\frac{b^4}{E'} + \frac{c^4}{E}\right) + b^2c^2\left(\frac{1}{n} - \frac{2\eta}{E}\right),\\
a_{22} &= 3\left(\frac{a^4}{E'} + \frac{c^4}{E}\right) + c^2a^2\left(\frac{1}{n} - \frac{2\eta}{E}\right),\\
a_{33} &= (3a^4 + 2a^2b^2 + 3b^4)/E',\\
a_{12} &= \frac{c^2}{E}\{c^2 - \eta(a^2 + b^2)\} - 3a^2b^2\frac{\eta'}{E'},\\
a_{13} &= \frac{b^2}{E'}(b^2 - \eta'a^2) - c^2\frac{\eta}{E}(3a^2 + b^2),\\
a_{23} &= \frac{a^2}{E'}(a^2 - \eta'b^2) - c^2\frac{\eta}{E}(a^2 + 3b^2)
\end{aligned}\right\} \qquad \ldots\ldots (23).$$

The corresponding ϖ's and Π's may be obtained from (20) and (21) by making use of (4); it would occupy space unduly to record them in full. In particular cases it is frequently simpler to employ the primitive equations (16) than to substitute their values for the ϖ's and Π's in (18).

My present object is rather to exemplify the utility of the method than to accumulate lengthy expressions, complete from a mathematical standpoint but indigestible by the ordinary physicist; I thus proceed to the consideration of some special cases.

SECTION II.

Sphere of Material symmetrical round an Axis.

§ 10. Let us first consider the effect of mutual gravitation of the material. We have

$$S = T = U = 0,$$
$$P = Q = R = -g\rho a,$$

where a is the radius of the sphere, g 'gravity' at its surface.

The equations (16) take the form

$$\left.\begin{aligned}
a_{11}L + a_{12}M + a_{13}N = a_{12}L + a_{22}M + a_{23}N &= \tfrac{1}{2}g\rho a^3\left(\frac{1-3\eta}{E} + \frac{1-\eta'}{E'}\right),\\
a_{13}L + a_{23}M + a_{33}N &= g\rho a^3\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)
\end{aligned}\right\} \qquad \ldots\ldots (24);$$

where a_{11}, &c. are obtained by putting $b = c = a$ in (23).

The formal solutions of (24) may conveniently be written

$$\left.\begin{aligned} L\Pi &= \tfrac{1}{2} g\rho a^3 \left[\left(\frac{1-3\eta}{E} + \frac{1-\eta'}{E'}\right)(\Pi_{11}+\Pi_{12}) + 2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\Pi_{13}\right], \\ M\Pi &= \tfrac{1}{2} g\rho a^3 \left[\left(\frac{1-3\eta}{E} + \frac{1-\eta'}{E'}\right)(\Pi_{12}+\Pi_{22}) + 2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\Pi_{23}\right], \\ N\Pi &= \tfrac{1}{2} g\rho a^3 \left[\left(\frac{1-3\eta}{E} + \frac{1-\eta'}{E'}\right)(\Pi_{13}+\Pi_{23}) + 2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\Pi_{33}\right] \end{aligned}\right\} \quad \ldots\ldots\ldots (25).$$

Here it is easily seen that

$$a_{22} = a_{11}, \quad a_{23} = a_{13},$$

$$\Pi_{22} = \Pi_{11}, \quad \Pi_{23} = \Pi_{13},$$

and so

$$M = L.$$

It is also easily verified that

$$\left.\begin{aligned} (\Pi_{11}+\Pi_{12}) \div (8/E') = \Pi_{13} \div \left(\frac{4\eta}{E} - \frac{1-\eta'}{E'}\right), \\ = \Pi \div \left[2a^4\left\{-\frac{16\eta^2}{E^2} + \frac{8(2-\eta-\eta\eta')}{EE'} + \frac{(1-\eta')(11+\eta')}{E'^2} + \frac{4}{nE'}\right\}\right], \\ = a^8\left\{\frac{2}{E} + \frac{3(1+\eta')}{E'} + \frac{1}{n}\right\} \end{aligned}\right\} \ldots (26).$$

Thus we can get rid of the common factor $a^8\left\{\frac{2}{E} + \frac{3(1+\eta')}{E'} + \frac{1}{n}\right\}$, which is an immense simplification.

In passing, the following simple way of obtaining the value of Π may be noted. We have

$$\Pi = a_{11}\Pi_{11} + a_{12}\Pi_{12} + a_{13}\Pi_{13}$$
$$= (a_{11} - a_{12})(\Pi_{11} - \Pi_{12}) + (a_{11}\Pi_{12} + a_{12}\Pi_{11} + a_{13}\Pi_{13}).$$

But

$$a_{11} = a_{22}, \text{ and } a_{13} = a_{23},$$

so

$$a_{11}\Pi_{12} + a_{12}\Pi_{11} + a_{13}\Pi_{13} = a_{12}\Pi_{11} + a_{22}\Pi_{12} + a_{23}\Pi_{13} = 0,$$

as being the value of a determinant of which two columns are identical.

Hence

$$\Pi = (a_{11} - a_{12})(\Pi_{11} - \Pi_{12}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27).$$

Returning to (25) we find

$$\left.\begin{aligned} L = M &= \frac{1}{2}\frac{g\rho}{a\Pi'}\left\{\frac{4(1-2\eta)}{EE'} + \left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\left(\frac{3+\eta'}{E'} + \frac{4\eta}{E}\right)\right\}, \\ N &= \frac{1}{2}\frac{g\rho}{a\Pi'}\left[\frac{3(1-2\eta)(1-\eta')}{EE'} + 2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\left(\frac{1-\eta'}{E'} + \frac{4\eta}{E} + \frac{1}{2n}\right)\right] \end{aligned}\right\} \ldots\ldots\ldots\ldots (28),$$

where

$$\Pi' \equiv \frac{-16\eta^2}{E^2} + \frac{8}{EE'}(2-\eta-\eta\eta') + \frac{(1-\eta')(11+\eta')}{E'^2} + \frac{4}{nE'} \ldots\ldots\ldots\ldots\ldots (29).$$

Employing these values of L, M and N, with

$$S = T = U = 0, \text{ and } P = Q = R = -g\rho a,$$

we have the stresses given by (13) and the strains by (15).

§ 11. It is often best to retain the general formulae (13) and (15) as long as possible, and only substitute for L, M and N in the final result. Suppose, for instance, we wish to find the change of length in radii along and perpendicular to the axis of material symmetry, due to the gravitational forces. Let δa_1 represent the change of radius along ox (and so perpendicular to the axis of symmetry), δa_3 the change along oz. Then, remembering that

$$M = L,$$

we have from (15)

$$\delta a_1 = \int_0^a s_x dx = -\frac{a}{3}\left\{\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)(g\rho a - 2La^2) - \frac{2N}{E'}a^2\right\},$$

$$\delta a_3 = \int_0^a s_z dz = -\frac{a}{3E}\{(1-2\eta)\, g\rho a - 4(L - \eta N)\, a^2\}.$$

After reduction I find

$$\delta a_1/a = -\frac{g\rho a}{\Pi'}\left[\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\left\{-\frac{4\eta^2}{E^2} + \frac{4-3\eta-\eta\eta'}{EE'} + \frac{2(1-\eta')}{E'^2} + \frac{1}{nE'}\right\} - (1-2\eta)\frac{1-\eta}{EE'^2}\right] \ldots (30).$$

$$\delta a_3/a = -\frac{1}{3}\frac{g\rho a}{E\Pi'}\left[(1-2\eta)\left\{-\frac{8\eta^2}{E^2} + \frac{2(4-5\eta-3\eta\eta')}{EE'} + \frac{(1-\eta')(11+\eta')}{E'^2} + \frac{4}{nE'}\right\}\right.$$
$$\left. - 2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\left(\frac{3-2\eta+\eta'+2\eta\eta'}{E'} - \frac{\eta}{n}\right)\right] \ldots\ldots\ldots\ldots (31);$$

where Π' is given by (29).

There is, as will presently appear, a reason for not multiplying up by the factors $1-2\eta$ and $\dfrac{1-\eta'}{E'} - \dfrac{\eta}{E}$.

§ 12. If we suppose E'/E very small, or that the stretching of a bar under given longitudinal traction is very much less when its axis is along than when it is perpendicular to the axis of material symmetry, we have as first approximations

$$\left.\begin{aligned} \delta a_1/a &= -g\rho a(1-\eta')\left\{2\frac{1-\eta'}{E'} + \frac{1}{n}\right\} \div (E'^2\Pi'), \\ \delta a_3/\delta a_1 &= \text{expression of order } E'/E \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

The change of length of the diameter along the axis of symmetry, which is here the direction of high resistance to extension, is thus relatively negligible.

If on the other hand E/E' be very small, we find as first approximations

$$\Pi' = -16\eta^2/E^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33),$$

$$\left.\begin{aligned} \delta a_1/a &= g\rho a\eta/(4E), \\ \delta a_3/a &= -\tfrac{1}{24}g\rho a\left\{\frac{4(1-2\eta)}{E} + \frac{1}{n}\right\} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

In this case δa_1 and δa_3 are in general of the same order of magnitude, only δa_1 is absolutely positive.

If while E and E' are of the same order of magnitude η vanishes, or traction parallel to the axis of material symmetry causes no lateral contraction in a bar, we have

$$\delta a_1/a = -g\rho a(1-\eta')\left\{\frac{3}{E} + \frac{2(1-\eta')}{E'} + \frac{1}{n}\right\} \div (\Pi' E'^2) \quad \text{.....................(35)},$$

$$\delta a_3/a = -g\rho a\left\{\frac{8}{E} + \frac{(1-\eta')(5-\eta')}{E'} + \frac{4}{n}\right\} \div (3\Pi' EE') \quad \text{.....................(36)},$$

with
$$\Pi' = \frac{1}{E'}\left\{\frac{16}{E} + \frac{(1-\eta')(11+\eta')}{E'} + \frac{4}{n}\right\} \quad \text{...............................(37)}.$$

Both (35) and (36) show contraction of diameters.

§ 13. If we apply our formulae to a sphere of the earth's mass the numerical results, when we attribute any ordinary values to the elastic constants, are inconsistent with the fundamental assumption of the ordinary mathematical theory of elasticity, according to which strains are small quantities whose squares are negligible.

In an isotropic "earth" consistency is attained only by supposing the material to be nearly incompressible, or $(1-2\eta)/E$ very nearly zero. This happens only if η be nearly ·5, unless E be enormously greater than for any known material. The former alternative is much the less improbable, because it implies that whilst resistance to change of volume is enormous—as it may well be under enormous pressure—resistance to change of shape need not be excessive.

When the material is not isotropic, but of the 5 elastic constant type symmetrical round an axis, the ordinary criterion for incompressibility is

$$2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right) + \frac{1-2\eta}{E} = 0 \quad \text{...............................(38)}.$$

This ensures that no change of volume will follow the application of *uniform* pressure however large; but it does not prevent change of volume under pressure which is not uniform. To provide against any change of volume two independent conditions must be satisfied, viz.

$$1 - 2\eta = 0 \text{...(39)},$$

$$\frac{1-\eta'}{E'} - \frac{\eta}{E} = 0 \text{.......................................(40)}.$$

If these hold simultaneously, of course (38) holds likewise. If (39) and (40) both hold, then (30) and (31) show at once that δa_1 and δa_3 absolutely vanish. If however (38) holds alone, or if one only of the two conditions (39) and (40) is satisfied, then neither δa_1 nor δa_3 vanishes.

§ 14. As the case when the material is only nearly incompressible is of special interest, it merits our attention.

Let us suppose then

$$\left.\begin{aligned} 1-2\eta &= p, \\ (1-\eta')E' - \eta E &= qE \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

where p and q are very small.

We easily reduce (30) and (31) to

$$\left.\begin{aligned} \delta a_1/a &= -\frac{g\rho a}{E\Pi'}\left\{q\left(\frac{3}{EE'} - \frac{3}{4E^2} + \frac{1}{nE'}\right) - \frac{1}{2}\frac{p}{EE'}\right\}, \\ \delta a_3/a &= -\frac{1}{3}\frac{g\rho a}{E\Pi'}\left\{p\left(\frac{6}{EE'} - \frac{3}{4E^2} + \frac{4}{nE'}\right) - 2\frac{q}{E}\left(\frac{4}{E'} - \frac{1}{E} - \frac{1}{2n}\right)\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (42);$$

where p^2, pq and q^2 are neglected, and

$$\Pi' = \frac{14}{EE'} - \frac{9}{4E^2} + \frac{4}{nE'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43).$$

If, in addition, the material be absolutely incompressible under uniform pressure,

$$p + 2q = 0,$$

and so

$$\left.\begin{aligned} \delta a_1/a &= \frac{g\rho a p}{2E\Pi'}\left(\frac{4}{EE'} - \frac{3}{4E^2} + \frac{1}{nE'}\right), \\ \delta a_3/a &= -\frac{g\rho a p}{3E\Pi'}\left(\frac{10}{EE'} - \frac{7}{4E^2} + \frac{4}{nE'} - \frac{1}{2nE}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (44).$$

Unless E and E' are widely different in magnitude, δa_1 has here the same sign as p, while δa_3 has the opposite sign.

These illustrations will, I hope, suffice to show how very varied are the possibilities in a gravitating elastic solid "earth."

Rotating Sphere of Material symmetrical round an Axis.

§ 15. The discussion of the influence of gravitation on an elastic solid earth naturally suggests that of rotation.

I have already* considered the influence of rotation on a spheroid of material symmetrical round the axis of rotation; and shall thus merely write down, for comparison with (30) and (31), the expressions for the changes in the equatorial and polar semi-axes of a sphere. These expressions may be obtained from the formulae (96) and (97) of the paper just quoted.

* *Camb. Phil. Trans.* Vol. xv. pp. 1—36.

Employing Π' as in (29), and denoting the angular velocity by ω, I find

$$\delta a_1/a = \frac{\omega^2\rho a^2(1-\eta')}{E'\Pi'}\left[\left\{-\frac{4\eta^2}{E^2}+\frac{4-3\eta-\eta\eta'}{EE'}+\frac{2(1-\eta')}{E'^2}+\frac{1}{nE'}\right\}+\frac{2\eta}{EE'}\right] \quad\ldots\ldots(45),$$

$$\delta a_3/a = -\frac{2\omega^2\rho a^2}{3E\Pi'}\left[\eta\left\{-\frac{8\eta^2}{E^2}+\frac{2(4-5\eta-3\eta\eta')}{EE'}+\frac{(1-\eta')(11+\eta')}{E'^2}+\frac{4}{nE'}\right\}\right.$$
$$\left.+\frac{1-\eta'}{E'}\left(\frac{3-2\eta+\eta'+2\eta\eta'}{E'}-\frac{\eta}{n}\right)\right] \quad\ldots\ldots(46).$$

I have manipulated these expressions so as to facilitate comparison with the corresponding results (30) and (31) for the influence of gravitation.

If in (45) and (46) we suppose E'/E very small, we have

$$\left.\begin{aligned}\delta a_1/a &= \frac{\omega^2\rho a^2(1-\eta')}{E'^2\Pi'}\left\{\frac{2(1-\eta')}{E'}+\frac{1}{n}\right\},\\ \delta a_3/\delta a_1 &= \text{expression of order } E'/E \text{ and so negligible}\end{aligned}\right\} \quad\ldots\ldots(47).$$

The similarity with the corresponding results (32) for gravitation is noteworthy.

If on the other hand E/E' be very small, we find, remembering that Π' is approximately equal to $-16\eta^2/E^2$,

$$\left.\begin{aligned}\delta a_3/a &= -\omega^2\rho a^2\eta/(3E),\\ \delta a_1/\delta a_3 &= \text{expression of order } E/E' \text{ and so negligible}\end{aligned}\right\} \quad\ldots\ldots(48).$$

If we suppose both (39) and (40) to hold, or the material to be absolutely incompressible, we find

$$\left.\begin{aligned}\delta a_1/a &= \frac{\omega^2\rho a^2}{2E\Pi'}\left(\frac{4}{EE'}-\frac{3}{4E^2}+\frac{1}{nE'}\right),\\ \delta a_3/a &= -\frac{\omega^2\rho a^2}{3E\Pi'}\left(\frac{10}{EE'}-\frac{7}{4E^2}+\frac{4}{nE'}-\frac{1}{2nE}\right)\end{aligned}\right\} \quad\ldots\ldots(49),$$

where Π' is given by (43).

In the case of rotation, unlike that of gravitation, a slight departure from incompressibility has very little effect; we may thus regard the results (49) as close approximations when the material is *slightly* compressible. In particular, if the material, though absolutely incompressible under uniform pressure, is slightly compressible under other circumstances, we find under combined gravitation and rotation from (44) and (49)

$$\left.\begin{aligned}\delta a_1/a &= \frac{g\rho a p+\omega^2\rho a^2}{2E\Pi'}\left(\frac{4}{EE'}-\frac{3}{4E^2}+\frac{1}{nE'}\right),\\ \delta a_3/a &= -\frac{g\rho a p+\omega^2\rho a^2}{3E\Pi'}\left(\frac{10}{EE'}-\frac{7}{4E^2}+\frac{4}{nE'}-\frac{1}{2nE}\right)\end{aligned}\right\} \quad\ldots\ldots(50).$$

So far as changes in the lengths of diameters are concerned, the gravitational and rotational influences are thus exactly parallel.

In the actual earth we have approximately

$$g\rho a/\omega^2\rho a^2 = 289,$$

and in such a case the gravitational and rotational effects on the diameters would be exactly equal if

$$p = 1/289 = \cdot 00346 \text{ approx.},$$

or

$$\eta = \cdot 4983 \text{ approx.}$$

If it were possible for η to equal $\cdot 5017$, the gravitational and rotational effects would in this case neutralise one another.

SECTION III.

Flat Ellipsoid.

§ 16. The next case considered is that of a very flat ellipsoid of material symmetrical with respect to the coordinate planes.

Supposing the axis of z taken along the short diameter $2c$, and retaining only terms independent of c, we have as first approximations, with our previous notation,

$$\left.\begin{aligned}
&a_{11} = 3b^4/E_2, \quad a_{22} = 3a^4/E_1, \quad a_{33} = 3(a^4/E_1) + 3(b^4/E_2) + a^2b^2\{(1/n_3) - 2(\eta_{12}/E_1)\}, \\
&a_{12} = -3a^2b^2\eta_{12}/E_1, \quad a_{13} = (b^4/E_2) - a^2b^2\eta_{12}/E_1, \quad a_{23} = a^4/E_1 - a^2b^2\eta_{12}/E_1, \\
&\Pi_{11} = \frac{8a^8}{E_1^2} + \frac{a^4b^4}{E_1E_2}(9 - \eta_{12}\eta_{21}) + \frac{a^6b^2}{E_1}\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right), \\
&\Pi_{22} = \frac{8b^8}{E_2^2} + \frac{a^4b^4}{E_1E_2}(9 - \eta_{12}\eta_{21}) + \frac{a^2b^6}{E_2}\left(\frac{3}{n_3} - \frac{4\eta_{21}}{E_2}\right), \\
&\Pi_{33} = 9a^4b^4(1 - \eta_{12}\eta_{21})/E_1E_2, \\
&\Pi_{12} = \frac{a^4b^4}{E_1}\left(\frac{1 - 5\eta_{12}\eta_{21}}{E_2} + \frac{3\eta_{12}}{n_3}\right) + \frac{8a^2b^2}{E_1E_2}(\eta_{21}a^4 + \eta_{12}b^4), \\
&\Pi_{13} = \Pi_{23} = -3a^4b^4(1 - \eta_{12}\eta_{21})/E_1E_2, \\
&\Pi = 3a^4b^4\frac{1 - \eta_{12}\eta_{21}}{E_1E_2}\left\{\frac{8a^4}{E_1} + \frac{8b^4}{E_2} + a^2b^2\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right)\right\}
\end{aligned}\right\} \quad \ldots\ldots\ldots (51).$$

Assuming

$$1 > \eta_{12}\eta_{21},$$

which can hardly fail to be universally true, Π is essentially positive.

Rotation about Short Axis.

§ 17. When the flat ellipsoid rotates with uniform angular velocity ω about its short axis, $P = Q = \omega^2$, $R = 0$, and the equations (16) take the form

$$a_{11}L + a_{12}M + a_{13}N = \tfrac{1}{2}\omega^2\rho b^2\{a^2(\eta_{12}/E_1) - b^2/E_2\},$$

$$a_{12}L + a_{22}M + a_{23}N = \tfrac{1}{2}\omega^2\rho a^2\{-a^2/E_1 + b^2(\eta_{21}/E_2)\},$$

$$a_{13}L + a_{23}M + a_{33}N = -\tfrac{1}{2}\omega^2\rho\left\{\frac{1}{E_1}a^2(a^2 - \eta_{12}b^2) + \frac{1}{E_2}b^2(b^2 - \eta_{21}a^2)\right\},$$

where a_{11}, &c. are given by (51).

Referring to the values found above for the determinant Π and its minors, we find on reduction, remembering (2),

$$\left.\begin{aligned} L = M &= -\frac{1}{2}\frac{\omega^2\rho}{\Pi'}\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right),\\ N &= -\frac{\omega^2\rho}{\Pi'}\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (52);$$

where

$$\Pi' = \frac{8a^4}{E_1}+\frac{8b^4}{E_2}+a^2b^2\left(\frac{3}{n_3}-\frac{4\eta_{12}}{E_1}\right) \quad\ldots\ldots\ldots\ldots (53).$$

We may reasonably regard L, M and N as essentially negative.

In our subsequent work the following result will be found useful,

$$3(L+M)+2N \equiv 2(3L+N) \equiv 2(3M+N) = -\omega^2\rho \quad\ldots\ldots\ldots\ldots (54).$$

Putting in (13)

$$S = T = U = R = 0, \quad P = Q = \omega^2,$$

we have

$$\widehat{xx}/a^2 = (\tfrac{1}{2}\omega^2\rho + M + N)(1 - x^2/a^2 - y^2/b^2) - (\tfrac{1}{2}\omega^2\rho + 3M + N)z^2/c^2 - 2Ny^2/b^2.$$

Having regard to (54) we see that the coefficient of z^2/c^2 vanishes, and deduce

$$\widehat{xx} = -2a^2L(1 - x^2/a^2 - y^2/b^2) - 2Ny^2a^2/b^2,$$

where L and N are given by (52) and (53).

Similarly we find

$$\widehat{zz} = 2Lc^2(1 - 2x^2/a^2 - 2y^2/b^2 - z^2/c^2).$$

But we have been treating terms of order c^2 as negligible and so may regard $\widehat{zz}$ as vanishing.

Again we have

$$\widehat{yz} = 2Lyz, \quad \widehat{zx} = 2Mzx;$$

or these two shearing stresses are of order c, and so though they are large compared to $\widehat{zz}$ we may neglect them for a first approximation. The complete stress system remaining may be written

$$\left.\begin{aligned} \widehat{xx} &= -2a^2L(1 - x^2/a^2 - y^2/b^2) - 2Na^2b^2y^2/b^4,\\ \widehat{yy} &= -2b^2L(1 - x^2/a^2 - y^2/b^2) - 2Na^2b^2x^2/a^4,\\ \widehat{xy} &= 2Nxy \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (55),$$

with L and N given as above by (52) and (53).

§ 18. If ψ be the inclination to the x-axis of one of the principal stress axes in planes parallel to xy, we have

$$\begin{aligned} \cot 2\psi &= \tfrac{1}{2}(\widehat{xx} - \widehat{yy})/\widehat{xy} \\ &= \cot 2\phi - (a^2 - b^2)L(1 - x^2/a^2 - y^2/b^2) \div (2Nxy) \quad\ldots\ldots\ldots\ldots (56); \end{aligned}$$

where ϕ is the inclination to the x-axis of the normal to the confocal

$$x^2/a^2 + y^2/b^2 = \lambda,$$

which passes through the point x, y, z.

This gives very readily the angles ψ and $\frac{\pi}{2}+\psi$ made with the x-axis by the two principal stress axes which lie in the plane parallel to xy. The third principal stress axis is always parallel to the z-axis.

Without any reference to the values of L and N we see from (56) that

$$\psi = \phi,$$

for all values of x and y if $\quad b^2 = a^2,$

and for all values of b/a if $\quad x^2/a^2 + y^2/b^2 = 1.$

We thus see that in a flat rotating spheroid, whatever be the relative values of the Young's moduli or Poisson's ratios, any perpendicular on the axis of rotation is a principal axis of stress at every point of its length.

Again for any shape of flat spheroid the principal stress axes at the rim in the central section $z=0$ coincide everywhere with the normal and tangent to the bounding ellipse.

The stress along the rim normal vanishes in accordance with the surface conditions, while the stress $\widehat{\pi}$ along the tangent is given by

$$\widehat{\pi} = p^2 \{(y^2/b^4)\,\widehat{xx} + (x^2/a^4)\,\widehat{yy} - 2\,(xy/a^2b^2)\,\widehat{xy}\},$$

where p is the perpendicular from the centre on the tangent at x, y.

Referring to (55), and remembering that

$$x^2/a^4 + y^2/b^4 = p^{-2},$$

we easily find

$$\widehat{\pi} = -\,2Na^2b^2\,p^2 \quad\text{...(57)};$$

or, writing in its value for N,

$$\widehat{\pi} = 2\omega^2\rho\,(a^2b^2/p^2)\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right) \div \left\{\frac{8a^4}{E_1}+\frac{8b^4}{E_2}+a^2b^2\left(\frac{3}{n_3}-\frac{4\eta_{12}}{E_1}\right)\right\} \quad\text{.........(58).}$$

The stress along the tangent to the rim in the central section is thus a traction, which varies inversely as the square of the perpendicular from the centre on the tangent.

§ 19. The strains which do not vanish are, as a first approximation,

$$\left.\begin{aligned} s_x &= -\,2\,(L/E_1)\,(a^2-\eta_{12}b^2)\,(1-x^2/a^2-y^2/b^2) - 2\,(N/E_1)\,(a^2y^2/b^2-\eta_{12}b^2x^2/a^2),\\ s_y &= -\,2\,(L/E_2)\,(b^2-\eta_{21}a^2)\,(1-x^2/a^2-y^2/b^2) - 2\,(N/E_2)\,(b^2x^2/a^2-\eta_{21}a^2y^2/b^2),\\ s_z &= 2\,(L/E_3)(\eta_{31}a^2+\eta_{32}b^2)(1-x^2/a^2-y^2/b^2)+2\,(N/E_3)(\eta_{31}a^2y^2/b^2+\eta_{32}b^2x^2/a^2),\\ \sigma_{xy} &= 2Nxy/n_3 \end{aligned}\right\} \quad\text{...(59);}$$

where L and N are given as before by (52) and (53).

To the present degree of approximation, the strains, like the stresses, do not vary with z; and at the rim in the central section $z = 0$ they depend on the constant N only.

Along the axis of rotation the strains are constants given by the simple expressions

$$\left.\begin{aligned} s_x &= -2(L/E_1)(a^2 - \eta_{12}b^2), \\ s_y &= -2(L/E_2)(b^2 - \eta_{21}a^2), \\ s_z &= 2(L/E_3)(\eta_{31}a^2 + \eta_{32}b^2) \end{aligned}\right\} \quad \ldots\ldots(60);$$

where L, as shown by (52), is a negative quantity.

An η in excess of 0·5 is at least highly exceptional, thus supposing a to be the longer semi-axis we may regard s_x at the axis as essentially positive, or a stretch. On the other hand s_y at the axis is positive or negative according as

$$b/a > \text{or} < \sqrt{\eta_{21}}.$$

For the changes in the lengths of the semi-axes we find from (59), by integration and substitution for L and N,

$$\left.\begin{aligned} \delta a/a &= \frac{2}{3}\frac{\omega^2\rho}{E_1\Pi'}\left[(a^2-\eta_{12}b^2)\left\{\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+a^2b^2\left(\frac{1}{n_3}-\frac{\eta_{12}}{E_1}\right)\right\}-(b^2-\eta_{21}a^2)\frac{\eta_{12}b^4}{E_2}\right], \\ \delta b/b &= \frac{2}{3}\frac{\omega^2\rho}{E_2\Pi'}\left[(b^2-\eta_{21}a^2)\left\{\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+a^2b^2\left(\frac{1}{n_3}-\frac{\eta_{12}}{E_1}\right)\right\}-(a^2-\eta_{12}b^2)\frac{\eta_{21}a^4}{E_1}\right], \\ \delta c/c &= -\frac{\omega^2\rho}{E_3\Pi'}(\eta_{31}a^2+\eta_{32}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right) \end{aligned}\right\} \quad \ldots(61),$$

where Π' is given by (53).

In passing, the following elegant relation may be noted

$$\frac{3(a^2\delta a/a - b^2\delta b/b)}{a^4/E_1 - b^4/E_2} + \frac{2\delta c/c}{(\eta_{31}a^2+\eta_{32}b^2)/E_3} = 0 \quad \ldots\ldots(62).$$

Regarding Π' as essentially positive, we see that $\delta c/c$ is invariably negative; or the short axis, about which the rotation occurs, necessarily shortens. The two perpendicular axes if similar in length in general both lengthen. If b, however, is much smaller than a it will usually shorten.

For instance, if $$b^2/a^2 = \eta_{21} \quad \ldots\ldots(63),$$

we have $$\delta b/b = -\tfrac{2}{3}\omega^2\rho a^2(\eta_{12}/E_1)(1-\eta_{12}\eta_{21}) \div (8 + 4\eta_{12}\eta_{21} + 3\eta_{21}E_1/n_3) \ldots\ldots(64).$$

The relations (60) and (57), it should be noticed, supply simple physical meanings to the constants L and N of the solution.

SECTION IV.

Thin Elliptic Disk rotating about the Perpendicular to its Plane through the Centre.

§ 20. In a previous paper* I have shown that in isotropic material the first approximations to the stresses and strains in a thin elliptic disk may be derived by applying the constant multiplier

$$\{4a^4 + (3+\eta)a^2b^2 + 4b^4\} \div (3a^4 + 2a^2b^2 + 3b^4)$$

to the values of the corresponding stresses and strains in a flat ellipsoid of the same (central) section πab, and equal axial thickness $2c$ or $2l$. A similar result holds when the material is of the more general type dealt with in the present paper, the constant multiplier being alone different. To find the suitable constant multiplier we may proceed as follows:

The mean values of the stresses, as I showed in an earlier paper†, are given by simple formulae of the type

$$\iiint \widehat{xx}\,dxdydz = \iiint Xx\,dxdydz + \iint Fx\,dS \quad\ldots\ldots\ldots\ldots\ldots\ldots(65):$$

where X is the x-component of the bodily forces per unit volume, and F the x-component of the surface forces per unit surface. The volume integrals extend throughout the entire volume, the surface integral over the whole surface of the solid.

In the present case we thus have

$$\iiint \widehat{xx}\,dxdydz = \omega^2\rho \iiint x^2\,dxdydz \quad\ldots\ldots\ldots\ldots\ldots\ldots(66).$$

Supposing C to denote the constant multiplier required for transformation from the flat ellipsoid to the thin disk, we find for the disk from (55)

$$\widehat{xx} = -2CLa^2(1 - x^2/a^2 - y^2/b^2) - 2CNa^2b^2y^2/b^4,$$

where L and N are given by (52) and (53).

Substituting for $\widehat{xx}$ in (66) and integrating, we find

$$-C \,.\, 2\pi abc \,.\, a^2(L + \tfrac{1}{2}N) = \omega^2\rho \,.\, 2\pi abc \,.\, a^2/4:$$

whence

$$C = -\omega^2\rho/(4L + 2N),$$

$$= (3L + N)/(2L + N) \text{ by (54).}$$

Referring to (52), we have at once

$$C = \left\{\frac{8a^4}{E_1} + \frac{8b^4}{E_2} + a^2b^2\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right)\right\} \div \left\{\frac{6a^4}{E_1} + \frac{6b^4}{E_2} + 2a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1}\right)\right\} \quad\ldots\ldots\ldots(67).$$

Thus, writing for shortness

$$\frac{6a^4}{E_1} + \frac{6b^4}{E_2} + 2a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1}\right) = W \quad\ldots\ldots\ldots\ldots\ldots\ldots(68),$$

* (A), p. 49.

† *Camb. Phil. Trans.* Vol. xv. equation (109), p. 336.

we have as first approximations to the stresses and strains in the thin rotating elliptic disk :—

$$\left.\begin{aligned}
\widehat{xx} &= \frac{\omega^2\rho a^2}{\Pi''}\left[\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\frac{y^2}{b^2}\right],\\
\widehat{yy} &= \frac{\omega^2\rho b^2}{\Pi''}\left[\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\frac{x^2}{a^2}\right],\\
\widehat{xy} &= -\frac{2\omega^2\rho}{\Pi''}\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)xy,\\
\widehat{zz} &= \widehat{xz} = \widehat{yz} = 0
\end{aligned}\right\} \quad \ldots\ldots(69);$$

$$\left.\begin{aligned}
s_x &= \frac{\omega^2\rho}{E_1\Pi''}\left[(a^2-\eta_{12}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\left.+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\left(\frac{a^2y^2}{b^2}-\eta_{12}\frac{b^2x^2}{a^2}\right)\right],\\
s_y &= \frac{\omega^2\rho}{E_2\Pi''}\left[(b^2-\eta_{21}a^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\left.+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\left(\frac{b^2x^2}{a^2}-\eta_{21}\frac{a^2y^2}{b^2}\right)\right],\\
s_z &= -\frac{\omega^2\rho}{E_3\Pi''}\left[(\eta_{31}a^2+\eta_{32}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\left.+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\left(\eta_{31}\frac{a^2y^2}{b^2}+\eta_{32}\frac{b^2x^2}{a^2}\right)\right],\\
\sigma_{xy} &= -\frac{2\omega^2\rho}{n_3\Pi''}\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)xy,\\
\sigma_{xz} &= \sigma_{yz} = 0
\end{aligned}\right\} \quad \ldots\ldots(70).$$

§ 21. The position of the principal stress axes in the disk is given by the same equation (56) as applies to the flat ellipsoid. Again, over the perimeter of the disk the normal stress component vanishes and the tangential component, in the plane of the cross-section, is given, cf. (58), by

$$\widehat{tt} = 2\omega^2\rho(a^2b^2/p^2)\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)\div\left\{\frac{6a^4}{E_1}+\frac{6b^4}{E_2}+2a^2b^2\left(\frac{1}{n_3}-\frac{2\eta_{12}}{E_1}\right)\right\} \quad \ldots\ldots\ldots\ldots(71),$$

where p is the perpendicular on the tangent from the centre of the ellipse.

The increments δa, δb and δl $(l=c)$ in the semi-axes of the ellipse and the axial semi-thickness are given by (61) when Π' is replaced by Π''. The relation (62) applies equally to the disk.

Employing δl as above, we find from the value of s_z that the displacement γ parallel to the axis of rotation is given by

$$\gamma = (\delta l/l)\,z\,(1-x^2/a_1^2-y^2/b_1^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72);$$

where

$$a_1^2 \equiv (\eta_{31}a^2 + \eta_{32}b^2)\left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right) \div \left\{\eta_{31}\left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right) + \eta_{32}\left(\frac{1}{n_3} + \frac{4\eta_{12}}{E_1}\right)b^4\right\},$$

$$b_1^2 = (\eta_{31}a^2 + \eta_{32}b^2)\left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right) \div \left\{\eta_{31}\left(\frac{1}{n_3} + \frac{4\eta_{12}}{E_1}\right)a^4 + \eta_{32}\left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right)\right\} \quad \ldots(73)$$

Since δl is negative, assuming Π'' positive, (72) shows very clearly how the originally plane cross-sections parallel to the faces of the disk become paraboloids whose concavities are directed away from the central section, and whose curvature increases with the distance z from that section.

The curvature at the centre of an originally plane section is greatest in the zy or in the zx plane according as

$$a_1^2 > \text{ or } < b_1^2.$$

The value of a_1^2/b_1^2 depends on the shape of the section as well as on the elastic properties of the medium. Assuming as before $a > b$, we easily find $a_1^2 > b_1^2$ if either

$$\eta_{32}/\eta_{31} = 1,$$

or

$$(a/b)^4 > \eta_{32}/\eta_{31} > 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74).$$

Thus the curvature is greatest in the plane containing the shorter axis of the ellipse if η_{32} and η_{31} are equal or if (74) holds.

Whilst the reduction in the thickness of the disk diminishes as we retire from the axis of rotation it remains a reduction right up to the rim. For it is obvious from (70) that γ_s, the value of γ over the curved surface, is given by

$$\gamma_s = -z(2\omega^2\rho/E_3\Pi'')\left(\frac{a^4}{E_1} + \frac{b^4}{E_2} - \frac{2a^2b^2\eta_{12}}{E_1}\right)\left(\eta_{31}\frac{a^2y^2}{b^2} + \eta_{32}\frac{b^2x^2}{a^2}\right) \quad \ldots\ldots\ldots\ldots\ldots (75),$$

and $\dfrac{a^4}{E_1} + \dfrac{b^4}{E_2} - \dfrac{2a^2b^2\eta_{12}}{E_1}$ can hardly fail to be positive.

It may be worth noticing that the reduction in the rim thickness is greatest at the ends of the minor axis or at the ends of the major axis according as

$$a^2\eta_{31} > \text{ or } < b^2\eta_{32}.$$

SECTION V.

Thin rotating Circular Disk of Material symmetrical round the Axis.

§ 22. When the disk is complete the approximate solution can be deduced from the results of the previous section by putting

$$b = a, \quad E_2 = E_1 = E', \quad \&c.$$

To include, however, the case of an annular disk, we must make a fresh start. In obtaining the following results I freely availed myself of a previous solution*, with

* 'On thin rotating isotropic disks,' *Camb. Phil. Soc. Proc.* Vol. VII. pp. 201—215, 1891.

which I foresaw that the present solution would agree in type. It will suffice to indicate generally the method of procedure.

Representing by $2l$ the thickness of the disk, by a and a' the radii of its outer and inner cylindrical boundaries, and employing cylindrical coordinates r, ϕ z, let us assume

$$\left.\begin{aligned} \widehat{rr} &= A\,(a^2 - r^2)(1 - a'^2/r^2) + B\,(l^2 - 3z^2), \\ \widehat{\phi\phi} &= C + Dr^2 + Fr^{-2} + B\,(l^2 - 3z^2), \\ \widehat{zz} &= \widehat{r\phi} = \widehat{rz} = \widehat{\phi z} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (76).$$

From the body-stress equation

$$\frac{d\widehat{rr}}{dr} + \frac{1}{r}(\widehat{rr} - \widehat{\phi\phi}) + \omega^2 \rho r = 0$$

we find

$$C = A\,(a^2 + a'^2), \quad D = \omega^2\rho - 3A, \quad F = Aa^2a'^2.$$

Only two of the equations of compatibility are not identically satisfied, and these two give us

$$A = \omega^2\rho\,(3 + \eta')/8,$$
$$B = \omega^2\rho\eta\,(1 + \eta')\,E' \div \{6\,(1 - \eta')\,E\}.$$

Substituting in (76) the values thus found for the constants, we have

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{8}\,\omega^2\rho\,(3 + \eta')(a^2 - r^2)\left(1 - \frac{a'^2}{r^2}\right) + \frac{1}{6}\,\frac{\omega^2\rho\eta\,(1 + \eta')\,E'}{(1 - \eta')\,E}\,(l^2 - 3z^2), \\ \widehat{\phi\phi} &= \frac{1}{8}\,\omega^2\rho\left\{(3 + \eta')\left(a^2 + a'^2 + \frac{a^2a'^2}{r^2}\right) - (1 + 3\eta')\,r^2\right\} + \frac{1}{6}\,\frac{\omega^2\rho\eta\,(1 + \eta')\,E'}{(1 - \eta')\,E}\,(l^2 - 3z^2), \\ \widehat{zz} &= \widehat{r\phi} = \widehat{rz} = \widehat{\phi z} = 0 \end{aligned}\right\} \quad \ldots\ldots (77).$$

The displacements (u along r, and w parallel to z) are easily obtained from the relations

$$u/r \equiv s_\phi = (\widehat{\phi\phi} - \eta'\,\widehat{rr})/E',$$
$$\frac{dw}{dz} \equiv s_z = -\,\eta\,(\widehat{rr} + \widehat{\phi\phi})/E\,;$$

whence we have, as w must vanish with z by symmetry,

$$\left.\begin{aligned} u &= \frac{1}{8}\,\frac{\omega^2\rho}{E'}\left\{(1 - \eta')(3 + \eta')\,r\,(a^2 + a'^2) - (1 - \eta'^2)\,r^3 + (1 + \eta')(3 + \eta')\,\frac{a^2a'^2}{r}\right\} \\ &\qquad + \frac{1}{6}\,\frac{\omega^2\rho\eta\,(1 + \eta')}{E}\,r\,(l^2 - 3z^2), \\ w &= -\frac{1}{4}\,\frac{\omega^2\rho\eta z}{E}\,\{(3 + \eta')(a^2 + a'^2) - 2\,(1 + \eta')\,r^2\} - \frac{1}{3}\,\frac{\omega^2\rho\eta^2\,(1 + \eta')\,E'}{(1 - \eta')\,E^2}\,z\,(l^2 - z^2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (78).$$

§ 23. If in (77) and (78) we put $a' = 0$ we obtain the correct values of the stresses and displacements in a complete disk of radius a.

It should be noticed, however, that the strains and stresses near the inner surface of a *nearly* complete disk (*i.e.* one in which a'/a is very small) are totally different from those at the same axial distance in a complete disk. This is due to the fact that even supposing $(a'/a)^2$ negligible, $(a'/r)^2$ approaches unity as r approaches a'.

The results supplied by (77) and (78) for a complete disk differ from those we should deduce from the formulae of Section IV., but only through containing the terms in $l^2 - 3z^2$ and $z(l^2 - z^2)$. Now we neglected terms of this order in Sections III. and IV., where terms in $(c/a)^2$—*i.e.* terms of order $(l/a)^2$—were omitted.

It is obvious in (77) and (78) that the mean values of $\widehat{rr}$, $\widehat{\phi\phi}$ and u taken between $z = -l$ and $z = +l$, or through the thickness of the disk, are unaffected by the presence of the terms in l^2 and z^2, and the same is true of the values of w over the faces of the disk.

§ 24. It may be well at this stage to make the status of our solution for the circular disk perfectly clear. It is not in general a *complete* solution of the mathematical equations. It satisfies indeed all the internal elastic solid equations, and all but one of the surface equations; but instead of making $\widehat{rr}$ identically zero at *every* point of both cylindrical surfaces, it only gives

$$\int_{-l}^{+l} \widehat{rr}\, dz = 0$$

over these surfaces. In other words, it only makes the statical *resultant* of the radial forces along a generator vanish. The solution is thus based on what Prof. Pearson terms the theory of equipollent systems of loading. According to the theory, which is very generally if not universally accepted, the error in such a solution is insensible, except in the immediate vicinity of the surfaces of small thickness—here the circular rims—where there is failure to satisfy the exact surface conditions. As the rim values of $\widehat{rr}$ in the present case are only of the order l^2 of small quantities, our solution is presumably an exceptionally favourable specimen of its class. Still it would not be legitimate to apply it without further investigation to the species of anchor ring which arises when $a - a'$ is comparable with l.

At first sight, it might appear better to have omitted the terms in l^2 and z^2 altogether; because in their absence $\widehat{rr}$ would vanish exactly over both rims. If, however, we omitted those terms, we should be unable to satisfy all the internal equations. Such a failure, in the absence of special knowledge, is much more serious than failure to satisfy a surface condition. For in dealing with internal equations we get, through differentiating, contributions of like magnitude from terms that are of widely different importance in the displacements and stresses. It is thus almost impossible to judge whether failure to satisfy an internal equation is trivial or absolutely fatal.

In the present case, while the terms in l^2 and z^2 serve mainly to save the proprieties and silence criticism, they fulfil a useful purpose in indicating the degree of approximation reached and the circumstances modifying it. For instance, the solution becomes absolutely exact if

$$\eta = 0, \text{ or } E'/E = 0;$$

and it is the more exact the smaller η or E'/E is.

On the other hand if E'/E be large the solution has a very limited application.

§ 25. When E and E' are of the same order of magnitude we may omit the terms in l^2 and z^2 in ordinary practical applications. When these terms are omitted I shall use the notation $\widehat{\overline{rr}}$, $\bar{u}$, &c. When the material is isotropic the values of $\bar{u}$, $\bar{w}$, &c., constitute what I have called elsewhere* the 'Maxwell solution,' as being the solution to which Maxwell's treatment of the problem would have led him in 1853 but for some small inaccuracies in his work.

It is noteworthy that $\widehat{\overline{rr}}$ and $\widehat{\overline{\phi\phi}}$ depend on no elastic constant other than η', while $\bar{u}$ is independent of η or E. Thus the stresses and radial displacement are exactly the same as in an isotropic material whose Young's modulus in E' and Poisson's ratio η'.

The longitudinal displacement $\bar{w}$ on the other hand depends on η and E, but even in its case the law of variation with the axial distance depends only on η'.

For the increments in the radii a and a', and in the semi-thickness at the two rims, we find

$$\left.\begin{aligned}(\overline{\delta a}/a) &= (\omega^2\rho/4E')\{(1-\eta')a^2 + (3+\eta')a'^2\},\\ (\overline{\delta a'}/a') &= (\omega^2\rho/4E')\{(3+\eta')a^2 + (1-\eta')a'^2\}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(79);$$

$$\left.\begin{aligned}(\delta l/l)_{r=a} &= -(\omega^2\rho\eta/4E)\{(1-\eta')a^2 + (3+\eta')a'^2\},\\ (\delta l/l)_{r=a'} &= -(\omega^2\rho\eta/4E)\{(3+\eta')a^2 + (1-\eta')a'^2\}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(80).$$

From these we deduce the following elegant relations

$$\left.\begin{aligned}(\delta l/l)_{r=a} &= -\eta(E'/E)(\overline{\delta a}/a) = -\eta''(\overline{\delta a}/a),\\ (\delta l/l)_{r=a'} &= -\eta(E'/E)(\overline{\delta a'}/a') = -\eta''(\overline{\delta a'}/a'),\\ (\delta l/l)_{r=a} + (\delta l/l)_{r=a'} &= -\omega^2\rho(\eta/E)(a^2+a'^2)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(81).$$

The arithmetic mean of the reductions in the thickness at the two rims of the disk is thus independent of η' or E'. The reduction in thickness is invariably greatest at the inner rim.

Originally plane sections parallel to the faces lie during rotation on paraboloids of revolution, the radius of curvature at whose vertices equals $E \div \{\omega^2\rho\eta(1+\eta')z\}$.

The curvature increases as we approach the faces $z = \pm l$. The general character of the phenomena is the same as when the material is isotropic (see *Camb. Phil. Soc. Proc.* Vol. VII. pp. 201—215).

* *Camb. Phil. Soc. Proc.* Vol. VII. p. 209.

SECTION VI.

ELONGATED ELLIPSOID, c/a AND c/b VERY LARGE.

§ 26. Retaining only the highest power of c in each case, we have for three-plane symmetry

$$
\left.
\begin{aligned}
&a_{11} = 3c^4/E_3, \quad a_{22} = 3c^4/E_3, \quad a_{33} = 3\,(a^4/E_1) + 3\,(b^4/E_2) + a^2b^2\,\{1/n_3 - 2\eta_{12}/E_1\},\\
&a_{12} = c^4/E_3, \quad a_{13} = -\,c^2\,(3a^2\eta_{31}/E_3 + b^2\eta_{32}/E_3), \quad a_{23} = -\,c^2\,(a^2\eta_{31} + 3b^2\eta_{32})/E_3,\\
&\Pi_{11} = (c^4/E_3)\left[\frac{a^4}{E_1}(9 - \eta_{13}\eta_{31}) + \frac{9b^4}{E_2}(1 - \eta_{23}\eta_{32}) + 3a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right)\right],\\
&\Pi_{22} = (c^4/E_3)\left[\frac{9a^4}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{b^4}{E_2}(9 - \eta_{23}\eta_{32}) + 3a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{21}}{E_2} - \frac{2\eta_{23}\eta_{31}}{E_2}\right)\right],\\
&\Pi_{33} = 8c^8/E_3^2,\\
&\Pi_{12} = (c^4/E_3)\left[-\frac{3a^4}{E_1}(1 - \eta_{13}\eta_{31}) - \frac{3b^4}{E_2}(1 - \eta_{23}\eta_{32}) + a^2b^2\left(\frac{10\eta_{31}\eta_{32}}{E_3} + \frac{2\eta_{12}}{E_1} - \frac{1}{n_3}\right)\right],\\
&\Pi_{13} = 8c^6a^2\eta_{13}/(E_1E_3),\\
&\Pi_{23} = 8c^6b^2\eta_{23}/(E_2E_3),\\
&\Pi = 8\,(c^8/E_3^2)\left[\frac{3a^4}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{3b^4}{E_2}(1 - \eta_{23}\eta_{32}) + a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right)\right]
\end{aligned}
\right\} \quad \ldots\ldots(82).
$$

ROTATION ABOUT THE LONG AXIS $2c$.

§ 27. The values of the ϖ's in equations (16) are as follows:

$$
\left.
\begin{aligned}
\varpi_1 = \varpi_2 &= \tfrac{1}{2}\omega^2\rho c^2\,(a^2\eta_{31} + b^2\eta_{32})/E_3,\\
\varpi_3 &= \tfrac{1}{2}\omega^2\rho\left\{\frac{a^2}{E_1}(\eta_{12}b^2 - a^2) + \frac{b^2}{E_2}(\eta_{21}a^2 - b^2)\right\}
\end{aligned}
\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(83).
$$

Substituting the above values of the Π's and ϖ's in the equations (18) we have the values of the constants L, M, N of the general solution. Thus for N we find

$$
N = \frac{\tfrac{1}{2}\omega^2\rho\left\{-\dfrac{a^4}{E_1}(1 - \eta_{13}\eta_{31}) - \dfrac{b^4}{E_2}(1 - \eta_{23}\eta_{32}) + \dfrac{2a^2b^2}{E_1}(\eta_{12} + \eta_{13}\eta_{32})\right\}}{\dfrac{3a^4}{E_1}(1 - \eta_{13}\eta_{31}) + \dfrac{3b^4}{E_2}(1 - \eta_{23}\eta_{32}) + a^2b^2\left(\dfrac{1}{n_3} - \dfrac{2\eta_{12}}{E_1} - \dfrac{2\eta_{13}\eta_{32}}{E_1}\right)} \quad \ldots\ldots\ldots\ldots(84).
$$

It is unnecessary to record the values of L and M as I have eliminated these quantities by aid of the following relations, which are not very difficult to verify :—

$$
\left.
\begin{aligned}
c^2\,(L - M) &= (\eta_{31}a^2 - \eta_{32}b^2)\,N,\\
c^2\,(L + M) &= (\eta_{31}a^2 + \eta_{32}b^2)\,(N + \tfrac{1}{4}\omega^2\rho)
\end{aligned}
\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(85);
$$

whence

$$
\left.
\begin{aligned}
c^2L &= \eta_{31}a^2N + \tfrac{1}{8}\omega^2\rho\,(\eta_{31}a^2 + \eta_{32}b^2),\\
c^2M &= \eta_{32}b^2N + \tfrac{1}{8}\omega^2\rho\,(\eta_{31}a^2 + \eta_{32}b^2)
\end{aligned}
\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(86).
$$

Retaining N (given explicitly by (84)) for brevity in the expressions for the stresses and strains, we have

$$\left.\begin{aligned}
\widehat{xx} &= a^2\left[(\tfrac{1}{2}\omega^2\rho + N)\left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right) - (\tfrac{1}{2}\omega^2\rho + 3N)\frac{y^2}{b^2}\right],\\
\widehat{yy} &= b^2\left[-(\tfrac{1}{2}\omega^2\rho + 3N)\frac{x^2}{a^2} + (\tfrac{1}{2}\omega^2\rho + N)\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right],\\
\widehat{zz} &= (\eta_{31}a^2 + \eta_{32}b^2)(\tfrac{1}{4}\omega^2\rho + N)\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2} - \frac{z^2}{c^2}\right) + (\eta_{31}a^2 - \eta_{32}b^2)N\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right),\\
\widehat{xy} &= 2Nxy,\\
&\widehat{yz} \text{ and } \widehat{zx} \text{ of order } a/c \text{ or } b/c, \text{ and so negligible}
\end{aligned}\right\}\quad\text{......(87);}$$

$$\left.\begin{aligned}
s_x = \tfrac{1}{2}(\omega^2\rho/E_1)&\left\{(a^2 - \eta_{12}b^2)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) - \tfrac{1}{2}\eta_{13}(\eta_{31}a^2 + \eta_{32}b^2)\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2} - \frac{z^2}{c^2}\right)\right\}\\
&+ (N/E_1)\left\{a^2(1 - \eta_{13}\eta_{31})\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) - b^2(\eta_{12} + \eta_{13}\eta_{32})\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right\},\\
s_y = \tfrac{1}{2}(\omega^2\rho/E_2)&\left\{(b^2 - \eta_{21}a^2)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) - \tfrac{1}{2}\eta_{23}(\eta_{31}a^2 + \eta_{32}b^2)\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2} - \frac{z^2}{c^2}\right)\right\}\\
&+ (N/E_2)\left\{b^2(1 - \eta_{23}\eta_{32})\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) - a^2(\eta_{21} + \eta_{23}\eta_{31})\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right)\right\},\\
s_z &= -\tfrac{1}{4}\omega^2\rho(\eta_{31}a^2 + \eta_{32}b^2)(1 - z^2/c^2)/E_3,\\
\sigma_{xy} &= 2Nxy/n_3,\\
\sigma_{yz} &\text{ and } \sigma_{zx} \text{ of order } a/c \text{ or } b/c \text{ and so negligible}
\end{aligned}\right\}\quad\text{...(88).}$$

§ 28. For the displacement parallel to the long axis we have

$$\gamma = -\tfrac{1}{4}\omega^2\rho(\eta_{31}a^2 + \eta_{32}b^2)z\left(1 - \frac{1}{3}\frac{z^2}{c^2}\right)\Big/E_3 \text{ ....(89).}$$

Sections perpendicular to the axis of rotation thus remain plane. The shortening of the long semi-axis is given by

$$\delta c/c = -\tfrac{1}{6}\omega^2\rho(\eta_{31}a^2 + \eta_{32}b^2)/E_3 \text{ ....(90).}$$

Using undashed letters as immediately above for the case of the long ellipsoid, and dashed letters for the case of the flat ellipsoid of Section III., the velocity of rotation, the material and the axes $2a$, $2b$ being the same in the two cases, we find from (90), (61) and (53)

$$(\delta c/c) \div (\delta c'/c') = 1 - \tfrac{1}{6}\left\{\frac{4a^4}{E_1} + \frac{4b^4}{E_2} + a^2b^2\left(\frac{3}{n_3} + \frac{4\eta_{12}}{E_1}\right)\right\} \div \left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right) \text{ ....(91).}$$

Thus the shortening per unit of length in the axis of rotation is less in the elongated than in the flat ellipsoid.

For the increments in the other principal semi-axes of the elongated ellipsoid we have

$$\left.\begin{aligned} \delta a/a &= \frac{1}{3}\frac{\omega^2\rho}{E_1}\{a^2 - \eta_{12}b^2 - \tfrac{1}{4}\eta_{13}(\eta_{31}a^2 + \eta_{32}b^2)\} + \frac{2}{3}\frac{N}{E_1}a^2(1 - \eta_{13}\eta_{31}), \\ \delta b/b &= \frac{1}{3}\frac{\omega^2\rho}{E_2}\{b^2 - \eta_{21}a^2 - \tfrac{1}{4}\eta_{23}(\eta_{31}a^2 + \eta_{32}b^2)\} + \frac{2}{3}\frac{N}{E_2}b^2(1 - \eta_{23}\eta_{32}) \end{aligned}\right\} \text{.........(92).}$$

The values obtained for $\delta a/a$ and $\delta b/b$ when its value (84) is substituted for N are somewhat lengthy even for a spheroid ($b = a$).

As, however, the influence of the elastic structure is very clearly exhibited in the case of a spheroid, I shall record the value obtained for the *difference* in the expansions of the two semi-axes taken along the directions of the two principal moduli E_1 and E_2. It is given by

$$\begin{aligned} \frac{\delta a_1 - \delta a_2}{\omega^2\rho a^3} &\left\{\frac{3}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{3}{E_2}(1 - \eta_{23}\eta_{32}) + \frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right\} \\ &= \frac{2}{3}\left(\frac{1}{E_1} - \frac{1}{E_2}\right)\left\{\frac{1}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{1}{E_2}(1 - \eta_{23}\eta_{32}) + \frac{1}{2n_3}\right\} \\ &+ \frac{1}{12}\cdot\frac{1}{E_3}(\eta_{31}^2 - \eta_{32}^2)\left\{\frac{1}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{1}{E_2}(1 - \eta_{23}\eta_{32}) - \frac{1}{n_3} - \frac{6\eta_{12}}{E_1} - \frac{6\eta_{13}\eta_{32}}{E_1}\right\} \text{.........(93).} \end{aligned}$$

By supposing equality first between E_1 and E_2, and secondly between η_{31} and η_{32}, we readily see how $\delta a_1 - \delta a_2$ depends on the difference between elastic moduli and on the difference between Poisson's ratios.

SECTION VII.

LONG ELLIPTIC CYLINDER ROTATING ABOUT ITS LONG AXIS.

§ 29. By a *long* elliptic cylinder is meant one whose length $2l$ is very large compared with the diameters $2a$, $2b$ of the cross-section. The solution for the elliptic cylinder—terms of order a/l or b/l being neglected—is obtained from that for the elongated ellipsoid by simply omitting all the terms in z^2. We thus have

$$\left.\begin{aligned} \widehat{xx} &= a^2\left[(\tfrac{1}{2}\omega^2\rho + N)\left(1 - \frac{x^2}{a^2}\right) - (\tfrac{1}{2}\omega^2\rho + 3N)\frac{y^2}{b^2}\right], \\ \widehat{yy} &= b^2\left[-(\tfrac{1}{2}\omega^2\rho + 3N)\frac{x^2}{a^2} + (\tfrac{1}{2}\omega^2\rho + N)\left(1 - \frac{y^2}{b^2}\right)\right], \\ \widehat{zz} &= (\eta_{31}a^2 + \eta_{32}b^2)(\tfrac{1}{4}\omega^2\rho + N)\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2}\right) + (\eta_{31}a^2 - \eta_{32}b^2)N\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \\ \widehat{xy} &= 2Nxy, \\ \widehat{xz} &= \widehat{yz} = 0 \end{aligned}\right\} \text{..............(94);}$$

$$\left.\begin{aligned}
s_x &= \frac{1}{2}\frac{\omega^2\rho}{E_1}\left\{(a^2-\eta_{12}b^2)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\tfrac{1}{2}\eta_{13}(\eta_{31}a^2+\eta_{32}b^2)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}\right)\right\}\\
&\quad+\frac{N}{E_1}\left\{a^2(1-\eta_{13}\eta_{31})\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}\right)-b^2(\eta_{12}+\eta_{13}\eta_{32})\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}\right)\right\},\\
s_y &= \frac{1}{2}\frac{\omega^2\rho}{E_2}\left\{(b^2-\eta_{21}a^2)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\tfrac{1}{2}\eta_{23}(\eta_{31}a^2+\eta_{32}b^2)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}\right)\right\}\\
&\quad+\frac{N}{E_2}\left\{b^2(1-\eta_{23}\eta_{32})\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}\right)-a^2(\eta_{21}+\eta_{23}\eta_{31})\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}\right)\right\},\\
s_z &= -\tfrac{1}{4}\omega^2\rho(\eta_{31}a^2+\eta_{32}b^2)/E_3,\\
\sigma_{xy} &= 2Nxy/n_3,\\
\sigma_{xz} &= \sigma_{yz} = 0
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(95);$$

where N is given by (84).

§ 30. The conclusion that the above solution applies to an elliptic cylinder may be justified as follows:

The terms containing z in equations (87) contribute nothing to the body-stress equations because $d\widehat{zz}/dz$, &c., are of the order of small quantities here neglected; thus the expressions (94) for the stresses will satisfy the body-stress equations. (This is easily verified of course directly.)

Again over the cylindrical surface

$$x^2/a^2 + y^2/b^2 = 1,$$

we have from (94)

$$\widehat{xx} = -2Na^2y^2/b^2,\quad \widehat{yy} = -2Nb^2x^2/a^2,\quad \widehat{xy} = 2Nxy \ldots\ldots\ldots\ldots\ldots\ldots\ldots (96);$$

whence

$$(x/a^2)\,\widehat{xx} + (y/b^2)\,\widehat{xy} = 0,$$

$$(x/a^2)\,\widehat{xy} + (y/b^2)\,\widehat{yy} = 0.$$

The equations over the *curved* surface are thus completely satisfied.

Over the terminal planes $z = \pm l$ the normal stress $\widehat{zz}$ does not vanish everywhere, as it strictly ought to do, but instead we have

$$\iint \widehat{zz}\,dxdy = 0.$$

Thus, according to the theory of equipollent systems of loading, the solution is satisfactory, except at points in the immediate vicinity of the terminal sections.

§ 31. The increments in the semi-axes a and b are given by the same formulae, viz. (92), as apply in the case of the elongated ellipsoid. The reduction in the half length l of the cylinder is given by

$$\delta l/l = -\tfrac{1}{4}\omega^2\rho(\eta_{31}a^2+\eta_{32}b^2)/E_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (97).$$

Comparing (97) with (90), supposing $l = c$, we see that the shortening in a long cylinder is greater than the corresponding axial shortening in an elongated ellipsoid in the ratio 3 : 2. This arises from the reduction in the strain s_z near the ends of the axis of rotation in the case of the long ellipsoid.

If undashed letters refer as above to a long cylinder, and dashed letters refer to the *axial* thickness of a thin disk of the same material and elliptic section rotating at the same speed, we find comparing (97) with (70) of Sect. IV.

$$(\delta l/l) \div (\delta l'/l') = 1 - \frac{1}{2}\left\{\frac{a^4}{E_1} + \frac{b^4}{E_2} + a^2b^2\left(\frac{1}{n_3} + \frac{2\eta_{12}}{E_1}\right)\right\} \div \left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right) \quad \text{.........(98).}$$

Thus the reduction per unit of length in the axis of rotation is invariably less in the long cylinder than in the thin disk.

As in the case of the disk, the tangential stress $\widehat{u}$ in the plane of the cross-section has a very simple form at the surface. For if p be the perpendicular from the central axis on the tangent plane at a point x, y on the surface, we easily find from (96)

$$\widehat{u} = -2Na^2b^2/p^2 \quad \text{...(99),}$$

N being given by (84). At least as a rule N is negative and $\widehat{u}$ a traction. The formula (99) differs from the corresponding result for a thin disk only in the value of N (cf. (71)).

We can easily attach a simple physical significance to N. Thus let $\widehat{u}_1$ and $\widehat{u}_2$ represent the minimum and maximum surface values of $\widehat{u}$, occurring respectively at the ends of the major and minor axes of the elliptic section, then

$$N = -\tfrac{1}{2}(\widehat{u}_1 + \widehat{u}_2) \div (a^2 + b^2) \quad \text{................................(100).}$$

SECTION VIII.

Long rotating Circular Cylinder of Material symmetrical round the Axis.

§ 32. When the cylinder is solid, the solution can be obtained by putting

$$b = a, \quad E_2 = E_1 = E', \quad \&c.,$$

in the results of last section. When the cylinder is hollow, an independent investigation is necessary.

In obtaining the following results I made use of the solution* published in 1892 for the case of isotropy, recognising that the type would remain unchanged. As the method adopted is practically identical with that applied in Section V. to the circular disk, I pass at once to the results. The origin has been taken at the mid-point of

* *Camb. Phil. Soc. Proc.* Vol. VII. pp. 283—305.

the cylinder's axis, and r, ϕ, z are ordinary cylindrical coordinates. The expressions for the stresses are as follows:

$$\left.\begin{aligned}\widehat{rr} &= \omega^2\rho\,\{3+\eta'-2\eta^2(E'/E)\}\,(a^2-r^2)\,(1-a'^2/r^2)\div\{8\,(1-\eta^2E'/E)\},\\ \widehat{\phi\phi} &= \omega^2\rho\,[\{3+\eta'-2\eta^2\,(E'/E)\}\,(a^2+a'^2+a^2a'^2/r^2)-(1+3\eta'+2\eta^2E'/E)\,r^2]\div\{8\,(1-\eta^2E'/E)\},\\ \widehat{zz} &= \omega^2\rho\,\eta\,(1+\eta')\,(a^2+a'^2-2r^2)\div\{4\,(1-\eta^2E'/E)\},\\ \widehat{r\phi} &= \widehat{rz}=\widehat{\phi z}=0\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots\ldots(101).$$

The displacements w parallel to the z-axis, and u along r, are given by

$$\left.\begin{aligned}w &= -\,\omega^2\rho\,\eta\,(a^2+a'^2)\,z\,/(2E),\\ u &= \frac{1}{8}\,\frac{\omega^2\rho}{1-\eta^2E'/E}\left[\left\{\frac{(1-\eta')(3+\eta')}{E'}-\frac{4\eta^2}{E}\right\}(a^2+a'^2)\,r-(1+\eta')\left(\frac{1-\eta'}{E'}-\frac{2\eta^2}{E}\right)r^3\right.\\ &\qquad\left.+\,\frac{1+\eta'}{E'}\,(3+\eta'-2\eta^2E'/E)\,a^2a'^2/r\right]\end{aligned}\right\}\ \ldots\ldots\ldots(102).$$

An alternative form for u, worth recording, is

$$u=\frac{1}{8}\,\frac{\omega^2\rho}{E'\,(1-\eta^2E'/E)}\,[(1-\eta')(3+\eta')(a^2+a'^2)\,r-(1-\eta'^2)\,r^3+(1+\eta')(3+\eta')\,a^2a'^2/r$$
$$-\,2\eta^2\,(E'/E)\,\{2\,(a^2+a'^2)\,r-(1+\eta')\,r^3+(1+\eta')\,a^2a'^2/r\}]\ldots\ldots(103).$$

I shall assume $1-\eta^2E'/E$ to be positive; if it could be zero the expressions for the stresses and displacements could become infinite.

§ 33. The solution, except when $\eta=0$, is dependent on the theory of equipollent systems of loading, in so far as we have to substitute for the exact surface equation

$$\widehat{zz}=0 \text{ over } z=\pm\,l,$$

$$\int_{a'}^{a} 2\pi r\,\widehat{zz}\,dr=0.$$

If we put $a'=0$ in (101), (102) and (103) we obtain the correct values of the stresses and displacements in a solid cylinder of radius a. The stresses and strains, however, near the inner surface of a *nearly* solid cylinder are, as in the case of the disk, totally different from those at the same axial distance in a wholly solid cylinder of the same external radius.

Comparing (101), (102) and (103) with (77) and (78), we see that when η or E'/E vanishes the formulae for the stresses and displacements in the long cylinder and thin disk become identical. This is true irrespective of the absolute values of η' or E'.

§ 34. The stress system (101) possesses several features of interest. The radial stress $\widehat{rr}$ is everywhere positive, or a traction, except at the surfaces, where it vanishes; it has its maximum value where

$$r=\sqrt{aa'}.$$

The orthogonal stress $\widehat{\phi\phi}$ is everywhere a traction. Its largest and smallest values occur respectively at the inner and outer surfaces. Distinguishing these values by the suffixes $_i$ and $_0$, we have

$$\left.\begin{aligned}\widehat{\phi\phi}_i &= \tfrac{1}{2}\omega^2\rho\,(a^2+a'^2)+\tfrac{1}{4}\omega^2\rho\,(1+\eta')\,(a^2-a'^2)\,(1-\eta^2E'/E),\\ \widehat{\phi\phi}_0 &= \tfrac{1}{2}\omega^2\rho\,(a^2+a'^2)-\tfrac{1}{4}\omega^2\rho\,(1+\eta')\,(a^2-a'^2)/(1-\eta^2E'/E)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(104).$$

This shows very clearly how $\widehat{\phi\phi}_i$ and $\widehat{\phi\phi}_0$ approach equality as the thickness of the cylinder wall diminishes.

The third principal stress $\widehat{zz}$, parallel to the axis of rotation, is a traction inside a pressure outside the cylindrical surface

$$r=\sqrt{\tfrac{1}{2}(a^2+a'^2)}.$$

The surface values of $\widehat{zz}$, using suffixes as above, are given by

$$\widehat{zz}_i=-\widehat{zz}_0=\tfrac{1}{4}\omega^2\rho\eta\,(1+\eta')\,(a^2-a'^2)/(1-\eta^2E'/E)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(105).$$

The numerical equality of $\widehat{zz}_i$ and $\widehat{zz}_0$ seems curious.

The following relation is also a neat one

$$\widehat{zz}_i-\widehat{zz}_0=\eta\,(\widehat{\phi\phi}_i-\widehat{\phi\phi}_0)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(106).$$

It somewhat reminds one of the results (81) established for the annular disk.

§ 35. Coming to the displacements, we see from (102) that the cross-sections—unlike those of the disk—remain plane. Further, if δl and $\delta l'$ denote the changes in the length of a hollow and a solid circular cylinder of equal length, the material, section, and velocity being the same, we have

$$\delta l/\delta l'=1+a'^2/a^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(107).$$

The influence of rotation on the length thus increases notably as the wall of the hollow cylinder becomes thinner. Comparing the first of equations (102) with the last of equations (81) we see that the change per unit length in the length of a long hollow cylinder is the exact arithmetic mean of the changes per unit thickness in the rim thicknesses of a thin disk of the same section and material rotating with equal velocity.

For the increments in the radii of the two surfaces of the long cylinder we find from (102)

$$\left.\begin{aligned}\delta a/a &= (\omega^2\rho/4E')\,\{(1-\eta')\,a^2+(3+\eta')\,a'^2\},\\ \delta a'/a' &= (\omega^2\rho/4E')\,\{(3+\eta')\,a^2+(1-\eta')\,a'^2\}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(108),$$

formulae in exact agreement with the corresponding results (79) for the annular disk.

A variety of interesting relationships exist amongst the different displacements. Thus if A represent the cross-section $\pi(a^2-a'^2)$, and t the wall thickness $a-a'$, we have from (102)

$$\left.\begin{aligned}(\delta a/a)+(\delta a'/a') &= \omega^2\rho\,(a^2+a'^2)/E',\\ -(dl/l)\div\{(\delta a/a)+(\delta a'/a')\} &= -\eta E'/(2E)=-\tfrac{1}{2}\eta'',\\ (\delta a'/a')-(\delta a/a) &= \omega^2\rho\,(a^2-a'^2)/(4n_3)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(109),$$

$$(\delta A/A)=\omega^2\rho\,(a^2+a'^2)\,(1-\eta')/(2E')\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(110),$$

$$(\delta t/t)=\omega^2\rho\,\{(a-a')^2-\eta'\,(a+a')^2\}/(4E')\ldots\ldots\ldots\ldots\ldots\ldots(111).$$

If the increments in a, a' and l could be measured, the relations (109) would give E', η'' and n_3 immediately.

From (110) we see that the area of the cross-section of the material is always increased by rotation; while (111) shows us that the cylinder wall becomes thicker or thinner according as

$$\eta' < \text{ or } > (a - a')^2/(a + a')^2 \qquad \text{.................................(112).}$$

For $\eta' = 1/4$ the thickness of the wall is unchanged when

$$a' = a/3 \qquad \text{...(113),}$$

and it is reduced when a'/a exceeds 1/3.

§ 36. The case when t/a is very small, or the cylinder wall very thin, merits separate attention. Of the stresses, $\widehat{rr}$ is then small compared to $\widehat{zz}$, while $\widehat{zz}$ in its turn is small compared to $\widehat{\phi\phi}$; and to a first approximation the sole stress left is

$$\widehat{\phi\phi} = \omega^2\rho a^2 \qquad \text{...(114).}$$

Under like conditions first approximations to the displacements are

$$\left.\begin{aligned} w &= -\,\omega^2\rho\eta a^2 z/E, \\ u &= \quad \omega^2\rho a^3/E' \end{aligned}\right\} \qquad \text{.....................................(115).}$$

VIII. *On the Change of a System of Independent Variables.*

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The main object of this paper is to establish a symbolical form for the result of changing a system of n independent variables in a partial differential coefficient. This form is a generalization of that obtained by Mr Leudesdorf* in the case of a single independent variable. The method adopted is the same as in a previous paper† in which I obtained another proof of Mr Leudesdorf's result. It consists in developing the formula given by Jacobi‡ for the reversion of series. An advantage of adopting this method is that the relation between the symbolical form and the fully developed form given by Sylvester§ and proved by Cayley‖ is readily perceived, so that from the symbolical form the developed form can be written down.

The first of the fundamental formulae of this paper is equation (10) of § 2. This formula may be developed in two ways. By the detailed work of §§ 1, 3 Sylvester's expanded form, just referred to, is deduced from it; in the succeeding sections it is developed into the symbolical formulae (21) and (22) of § 11, and (26), (27), (28), (29), (30) of §§ 14, 15. Of these formulae (29), or its equivalent (30), seems to be the most important. The crucial point in the establishment of these formulae is the proof of equations (18 a, b, c) in § 9. The somewhat complicated work of §§ 1, 3 is introduced for the sake of showing the connexion between the symbolical formulae and Sylvester's expanded form; it is not required in the subsequent developments. In § 17 a symbolical formula is obtained for the differentiation of implicit functions, and in § 18 this formula is applied to determine a solution of the general equation of infinite degree involving a dependent variable y and an independent variable x, when the equation has been deprived of the constant term.

The remainder of the paper is taken up with applications of the symbolical formulae (26)...(30). In several papers¶ published in the *Proceedings of the London Mathematical*

* "Second Paper on Change of the Independent Variable," *Proc. Lond. Math. Soc.* Vol. xviii.

† "Change of the Independent Variable in a Differential Coefficient," *Camb. Phil. Trans.* Vol. xvi.

‡ "De resolutione aequationum per series infinitas," *Crelle's Jour.* Vol. vi.; *Gesammelte Werke*, vi. pp. 26—61.

§ *Proc. Roy. Soc.* Vol. vii., 1855; and *Quar. Jour. Math.* Vol. i. 1857, with corrections.

‖ "Deuxième note sur une formule pour la réversion des séries," *Crelle*, Vol. liv.; *Collected Works*, Vol. iv. 234.

¶ "On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants," Vol. xviii. pp. 142—164.

"On Pure Ternary Reciprocants and Functions allied to them," Vol. xix. pp. 6—23.

"On Cyclicants, or Pure Ternary Reciprocants, and allied Functions," Vol. xix. pp. 377—405.

"On Projective Cyclic Concomitants, or Surface Differential Invariants," Vol. xx. pp. 131—160.

"On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables," Vol. xxii. pp. 79—104.

"On the Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups," Vol. xxix. p. 466.

Society, Prof. Elliott has considered the theory of Cyclicants and Reciprocants, mostly for the case of two independent variables. It is shown in this paper how most of Prof. Elliott's general theorems can be obtained for any number of independent variables as deductions from the general symbolical formulae. In § 19 the result of the change of variables by the general linear transformation is exhibited in formula (39), which may be regarded as fundamental in the theory of Pure Cyclicants. Prof. Elliott has also considered the conditions which must be satisfied by the functions which are the generalized forms of ordinary Reciprocants when there are two dependent and two independent variables. In §§ 29—31 these conditions are deduced from the general formulae of the paper for any number of variables, and it is also shown that Prof. Elliott's conditions are not independent and may be reduced in number.

For the sake of conciseness all the general results of the paper are worked out for the case of three independent variables, but care has been taken to use only such methods as would be applicable to any number of independent variables.

Summary of Contents.

§ 1. Let u, v, w, t be functions of three independent variables x, y, z; and let any differential coefficient

$$\frac{\partial^{p+q+r} u}{\partial x^p \partial y^q \partial z^r}$$

be denoted by u_{pqr}: let also the quantities

$$\frac{u_{pqr}}{p!\,q!\,r!},\quad \frac{v_{pqr}}{p!\,q!\,r!},\quad \frac{w_{pqr}}{p!\,q!\,r!},\quad \frac{t_{pqr}}{p!\,q!\,r!}$$

be denoted by a_{pqr}, b_{pqr}, c_{pqr}, d_{pqr}.

Further, let $D_x{}^p D_y{}^q D_z{}^r (tu^f v^g w^h)$ denote the result of suppressing all terms which explicitly contain t, u, v, w or first differential coefficients of u, v, w in

$$\frac{\partial^{p+q+r}}{\partial x^p \partial y^q \partial z^r} (tu^f v^g w^h).$$

The expanded form of this expression is easily obtained. Let ξ, η, ζ be quantities independent of x, y, z; let

$$U = a_{200}\xi^2 + a_{020}\eta^2 + a_{002}\zeta^2 + a_{110}\xi\eta + \ldots + a_{300}\xi^3 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

let V and W denote similar expressions with b's and c's instead of a's, and let

$$T = d_{100}\xi + d_{010}\eta + d_{001}\zeta + d_{200}\xi^2 + \ldots + d_{300}\xi^3 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Then

$$TU^f V^g W^h = \Sigma \frac{\xi^p \eta^q \zeta^r}{p!\,q!\,r!} D_\xi{}^p D_\eta{}^q D_\zeta{}^r (TU^f V^g W^h)$$

where, after the differentiations indicated by D_ξ, D_η, D_ζ, ξ, η, ζ are made to vanish.

But obviously, on this understanding,

$$D_\xi{}^p D_\eta{}^q D_\zeta{}^r (TU^f V^g W^h) = D_x{}^p D_y{}^q D_z{}^r (tu^f v^g w^h).$$

Therefore

$$\frac{D_x{}^p D_y{}^q D_z{}^r (tu^f v^g w^h)}{p!\,q!\,r!}$$

is equal to the coefficient of $\xi^p \eta^q \zeta^r$ in $TU^f V^g W^h$. Therefore by ordinary algebra, provided $p + q + r > 2f + 2g + 2h$,

$$\frac{D_x{}^p D_y{}^q D_z{}^r (tu^f v^g w^h)}{p!\,q!\,r!\,f!\,g!\,h!} = \Sigma d_{\lambda\mu\nu} \frac{1}{f_1!\,f_2!\ldots} (a_{\lambda_1\mu_1\nu_1})^{f_1} (a_{\lambda_2\mu_2\nu_2})^{f_2} \ldots$$
$$\frac{1}{g_1!\,g_2!\ldots} (b_{\lambda_1\mu_1\nu_1})^{g_1} (b_{\lambda_2\mu_2\nu_2})^{g_2} \ldots$$
$$\frac{1}{h_1!\,h_2!\ldots} (c_{\lambda_1\mu_1\nu_1})^{h_1} (c_{\lambda_2\mu_2\nu_2})^{h_2} \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

where summation extends to all zero and positive integral values of $f_1, f_2, \ldots g_1, g_2, \ldots h_1, h_2, \ldots \lambda, \mu, \nu, \lambda_1, \mu_1, \nu_1, \ldots$ which satisfy the conditions

$$\left.\begin{array}{ll} f_1+f_2+\ldots=f, \quad g_1+g_2+\ldots=g, \quad h_1+h_2+\ldots=h & \\ \lambda+\Sigma f_1\lambda_1+\Sigma g_1\lambda_1'+\Sigma h_1\lambda_1''=p, & \lambda+\mu+\nu \nless 1, \\ \mu+\Sigma f_1\mu_1+\Sigma g_1\mu_1'+\Sigma h_1\mu_1''=q, & \lambda_1+\mu_1+\nu_1 \nless 2, \\ \nu+\Sigma f_1\nu_1+\Sigma g_1\nu_1'+\Sigma h_1\nu_1''=r, & \lambda_1'+\mu_1'+\nu_1' \nless 2, \text{ \&c.} \end{array}\right\} \ldots\ldots(4).$$

Therefore, writing $[p, q, r;\ f, g, h;\ \lambda, \mu, \nu;\ \lambda_1, \mu_1, \nu_1;\ \&c.]$ for the typical term on the right hand of (3), we have

$$\frac{D_x{}^p D_y{}^q D_z{}^r (tu^f v^g w^h)}{p\,!\,q\,!\,r\,!\,f\,!\,g\,!\,h\,!}=\Sigma\,[p, q, r;\ f, g, h;\ \lambda, \mu, \nu;\ \lambda_1, \mu_1, \nu_1;\ \&c.] \ \ldots\ldots\ldots (5).$$

Again, on differentiating the equation

$$\frac{U^f}{f\,!}=\Sigma\frac{1}{f_1\,!\,f_2\,!\ldots}(a_{\lambda_1\mu_1\nu_1})^{f_1}(a_{\lambda_2\mu_2\nu_2})^{f_2}\ldots\xi^{\Sigma f_1\lambda_1}\eta^{\Sigma f_1\mu_1}\zeta^{\Sigma f_1\nu_1}$$

with respect to ξ, we find

$$\frac{U^{f-1}U_\xi}{(f-1)\,!}=\Sigma\,.\,\Sigma f_1\lambda_1\,.\,\frac{1}{f_1\,!\,f_2\,!\ldots}(a_{\lambda_1\mu_1\nu_1})^{f_1}\ldots\xi^{\Sigma f_1\lambda_1-1}\eta^{\Sigma f_1\mu_1}\zeta^{\Sigma f_1\nu_1}.$$

But

$$TU^{f-1}V^gW^hU_\xi=\Sigma D_x{}^p D_y{}^q D_z{}^r(tu^{f-1}v^g w^h u_x)\frac{\xi^p\eta^q\zeta^r}{p\,!\,q\,!\,r\,!};$$

therefore, by comparison of coefficients of powers of ξ, η, ζ,

$$\frac{D_x{}^{p-1}D_y{}^q D_z{}^r(tu^{f-1}v^g w^h u_x)}{(p-1)\,!\,q\,!\,r\,!\,(f-1)\,!\,g\,!\,h\,!}=\Sigma\,.\,\Sigma f_1\lambda_1\,.\,[p, q, r;\ f, g, h;\ \lambda, \mu, \nu;\ \lambda_1, \mu_1, \nu_1;\ \&c.]\ldots(6).$$

And similarly

$$\frac{D_x{}^{p-1}D_y{}^{q-1}D_z{}^r(tu^{f-1}v^{g-1}w^h u_x v_y)}{(p-1)\,!\,(q-1)\,!\,r\,!\,(f-1)\,!\,(g-1)\,!\,h\,!}=\Sigma\,.\,\Sigma f_1\lambda_1\,.\,\Sigma g_1\mu_1'\,[p, q, r;\ f, g, h;\ \lambda, \mu, \nu;\ \lambda_1, \mu_1, \nu_1;\ \&c.] \ \ldots\ldots\ldots(7),$$

$$\frac{D_x{}^{p-1}D_y{}^{q-1}D_z{}^{r-1}(tu^{f-1}v^{g-1}w^{h-1}u_x v_y w_z)}{(p-1)\,!\,(q-1)\,!\,(r-1)\,!\,(f-1)\,!\,(g-1)\,!\,(h-1)\,!}=\Sigma\,.\,\Sigma f_1\lambda_1\,.\,\Sigma g_1\mu_1'\,.\,\Sigma h_1\nu_1''$$
$$[p, q, r;\ f, g, h;\ \lambda, \mu, \nu;\ \lambda_1, \mu_1, \nu_1;\ \&c.]\ldots\ldots\ldots(8),$$

where the limits of summation are the same as in (4); and similar expressions can be written down with x, y, z interchanged.

It is obvious that, if $p+q+r \leqq 2(f+g+h)$,

$$D_x{}^p D_y{}^q D_z{}^r(tu^f v^g w^h)=0,$$

since the coefficient of $\xi^p\eta^q\zeta^r$ in $TU^fV^gW^h$ will be zero. For similar reasons

$$D_x{}^p D_y{}^q D_z{}^r(tu^f v^g w^h u_x)=0, \text{ if } p+q+r \leqq 2f+2g+2h+1,$$
$$D_x{}^p D_y{}^q D_z{}^r(tu^f v^g w^h u_x v_y)=0, \text{ if } p+q+r \leqq 2f+2g+2h+2, \text{ \&c.}$$

§ 2. We now proceed to establish the formula for the change of the independent variables in a partial differential coefficient. It is first necessary to state the theorem of Jacobi on which the method is based. Let υ, ν, ω be three quantities given in terms of three independent variables ξ, η, ζ by equations of the form

$$\upsilon = \xi + a_{200}\xi^2 + a_{020}\eta^2 + a_{002}\zeta^2 + a_{110}\xi\eta + \ldots + a_{300}\xi^3 + \ldots$$

$$= \xi + X, \text{ say;}$$

$$\nu = \eta + b_{200}\xi^2 + \ldots + b_{300}\xi^3 + \ldots = \eta + Y,$$

$$\omega = \zeta + c_{200}\xi^2 + \ldots + c_{300}\xi^3 + \ldots = \zeta + Z,$$

where the a's, b's and c's are any quantities independent of ξ, η, ζ. It is important to remark that the linear part of each of the expressions must consist of a single term. If these equations are solved for ξ, η, ζ in terms of υ, ν, ω one set of values will vanish when υ, ν, ω vanish, and can be expanded in series proceeding by integral powers of υ, ν, ω. Supposing that ξ, η, ζ have these values, we can then expand a general function $f(\xi, \eta, \zeta)$ in powers of υ, ν, ω, and Jacobi's theorem states that the coefficient of $\upsilon^l \nu^m \omega^n$ in the result is equal to the coefficient of $\xi^{-1}\eta^{-1}\zeta^{-1}$ in the expansion of

$$f(\xi, \eta, \zeta)\frac{\partial(\upsilon, \nu, \omega)}{\partial(\xi, \eta, \zeta)}\frac{1}{(\xi + X)^{l+1}(\eta + Y)^{m+1}(\zeta + Z)^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where the expansion is effected by first arranging $(\xi + X)^{-(l+1)}$, &c. in powers of X/ξ, Y/η, Z/ζ and then substituting for X, Y, Z and multiplying together the various terms.

Now let u, v, w, t be given functions of three independent variables x, y, z; and let it be required to change the variables from x, y, z to u, v, w and express $\partial^{l+m+n}t/\partial u^l \partial v^m \partial w^n$ in terms of differential coefficients of t, u, v, w with respect to x, y, z. To this end let x, y, z receive increments ξ, η, ζ, and let the consequent increments in u, v, w, t be υ, ν, ω, τ. The first differential coefficients of u, v, w, t will be denoted by special letters according to the following scheme

$$u_{100} = a, \quad u_{010} = a', \quad u_{001} = a'',$$

$$v_{100} = b, \quad v_{010} = b', \quad v_{001} = b'',$$

$$w_{100} = c, \quad w_{010} = c', \quad w_{001} = c'',$$

$$t_{100} = d, \quad t_{010} = d', \quad t_{001} = d''.$$

Then

$$\upsilon = a\xi + a'\eta + a''\zeta + U,$$

$$\nu = b\xi + b'\eta + b''\zeta + V,$$

$$\omega = c\xi + c'\eta + c''\zeta + W,$$

$$\tau = T,$$

where U, V, W, T have the same values as in the previous section.

But if t be regarded as a function of u, v, w, we have also

$$\tau = \Sigma \frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n} \frac{\upsilon^l \nu^m \omega^n}{l!\, m!\, n!},$$

and therefore $\partial^{l+m+n}t/\partial u^l \partial v^m \partial w^n$ may be determined as the coefficient of $\upsilon^l \nu^m \omega^n / l!\, m!\, n!$ in the expansion of τ in powers of υ, ν, ω.

To apply Jacobi's theorem we make the transformation

$$\xi' = a\xi + a'\eta + a''\zeta,$$

$$\eta' = b\xi + b'\eta + b''\zeta,$$

$$\zeta' = c\xi + c'\eta + c''\zeta;$$

so that

$$J\xi = A\xi' + B\eta' + C\zeta',$$

$$J\eta = A'\xi' + B'\eta' + C'\zeta',$$

$$J\zeta = A''\xi' + B''\eta' + C''\zeta',$$

where

$$J = \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix},$$

and A, B, C, ... are the first minors of J. We now have

$$\upsilon = \xi' + U', \quad \nu = \eta' + V', \quad \omega = \zeta' + W', \quad \tau = T',$$

where U', V', W', T' are the values of U, V, W, T in terms of ξ', η', ζ'.

To express these values take D_ξ, D_η, D_ζ as in § 1, and write

$$D_{\xi'} = \frac{1}{J}(AD_\xi + A'D_\eta + A''D_\zeta),$$

$$D_{\eta'} = \frac{1}{J}(BD_\xi + B'D_\eta + B''D_\zeta),$$

$$D_{\zeta'} = \frac{1}{J}(CD_\xi + C'D_\eta + C''D_\zeta),$$

and denote $D_{\xi'}{}^l D_{\eta'}{}^m D_{\zeta'}{}^n U'$, &c., by u'_{lmn}, &c.; and $u'_{lmn}/l!\, m!\, n!$ by a'_{lmn}, &c.

Then

$$U' = \Sigma a'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$

$$V' = \Sigma b'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$

$$W' = \Sigma c'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$

$$T' = \Sigma d'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$

where in U', V', W', $p + q + r \nless 2$, and in T', $p + q + r \nless 1$.

If also we take D_x, D_y, D_z as in § 1 and write

$$D_1 = \frac{1}{J}(AD_x + A'D_y + A''D_z),$$

$$D_2 = \frac{1}{J}(BD_x + B'D_y + B''D_z),$$

$$D_3 = \frac{1}{J}(CD_x + C'D_y + C''D_z),$$

then

$$D_{\xi'}{}^p D_{\eta'}{}^q D_{\zeta'}{}^r (T'U'^f V'^g W'^h) = D_1^p D_2^q D_3^r (tu^f v^g w^h),$$

provided that products of the operators D_1, D_2, D_3 are formed by mere algebraical multiplication; that is to say, in the product D_1D_2 it is supposed that D_1 does not operate on the coefficients of D_x, D_y, D_z which occur in D_2.

Therefore by Jacobi's formula

$$\frac{1}{l!\,m!\,n!}\frac{\partial^{l+m+n}t}{\partial u^l \partial v^m \partial w^n}$$

is equal to the coefficient of $\xi'^{-1}\eta'^{-1}\zeta'^{-1}$ in

$$T'\frac{\partial(u,\ v,\ w)}{\partial(\xi',\ \eta',\ \zeta')}\frac{1}{(\xi'+U')^{l+1}(\eta'+V')^{m+1}(\zeta'+W')^{n+1}},$$

or

$$\Sigma_{f=0,\ g=0,\ h=0}^{f=\infty,\ g=\infty,\ h=\infty}(-1)^{f+g+h}\frac{(l+f)!\,(m+g)!\,(n+h)!}{l!\,f!\,m!\,g!\,n!\,h!}$$

$$\begin{vmatrix} 1+\dfrac{\partial U'}{\partial \xi'}, & \dfrac{\partial V'}{\partial \xi'}, & \dfrac{\partial W'}{\partial \xi'} \\ \dfrac{\partial U'}{\partial \eta'}, & 1+\dfrac{\partial V'}{\partial \eta'}, & \dfrac{\partial W'}{\partial \eta'} \\ \dfrac{\partial U'}{\partial \zeta'}, & \dfrac{\partial V'}{\partial \zeta'}, & 1+\dfrac{\partial W'}{\partial \zeta'} \end{vmatrix}\frac{T'U'^f V'^g W'^h}{\xi'^{l+f+1}\eta'^{m+g+1}\zeta'^{n+h+1}};$$

that is, in

$$\Sigma_{f=0,\ g=0,\ h=0}^{f=\infty,\ g=\infty,\ h=\infty}(-1)^{f+g+h}\frac{(l+f)!\,(m+g)!\,(n+h)!}{l!\,f!\,m!\,g!\,n!\,h!}$$

$$.\,\Sigma D_1^p D_2^q D_3^r \begin{vmatrix} 1+u_1, & v_1, & w_1 \\ u_2, & 1+v_2, & w_2 \\ u_3, & v_3, & 1+w_3 \end{vmatrix} tu^f v^g w^h\,.\,\frac{\xi'^{p-l-f-1}\eta'^{q-m-g-1}\zeta'^{r-n-h-1}}{p!\,q!\,r!},$$

where u_1, v_1, w_1, &c. stand for D_1u, D_1v, D_1w, &c.

To obtain the term containing $\xi'^{-1}\eta'^{-1}\zeta'^{-1}$ we take $p = l + f$, $q = m + g$, $r = n + h$.

Hence

$$\frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n} = \Sigma_{f=0,\, g=0,\, h=0}^{f=\infty,\, g=\infty,\, h=\infty} (-1)^{f+g+h} D_1^{l+f} D_2^{m+g} D_3^{n+h} \begin{vmatrix} 1+u_1, & v_1, & w_1 \\ u_2, & 1+v_2, & w_2 \\ u_3, & v_3, & 1+w_3 \end{vmatrix} t \frac{u^f v^g w^h}{f!\,g!\,h!} \qquad \text{(10).}$$

Although in this expression summation extends to all values of f, g, h, it is obvious from the remark at the end of § 1 that terms for which $f+g+h \geq l+m+n$ will be zero. In particular, if $l=1$, $m=0$, $n=0$, the only term not zero is that for which $f=0$, $g=0$, $h=0$.

It is easily verified that

$$\begin{vmatrix} 1+u_1, & v_1, & w_1 \\ u_2, & 1+v_2, & w_2 \\ u_3, & v_3, & 1+w_3 \end{vmatrix} = \begin{vmatrix} A/J, & B/J, & C/J \\ A'/J, & B'/J, & C'/J \\ A''/J, & B''/J, & C''/J \end{vmatrix} \times \begin{vmatrix} a+u_x, & b+v_x, & c+w_x \\ a'+u_y, & b'+v_y, & c'+w_y \\ a''+u_z, & b''+v_z, & c''+w_z \end{vmatrix}$$

$$= \frac{1}{J} \begin{vmatrix} a+u_x, & b+v_x, & c+w_x \\ a'+u_y, & b'+v_y, & c'+w_y \\ a''+u_z, & b''+v_z, & c''+w_z \end{vmatrix} \qquad \text{(11),}$$

$$= \begin{vmatrix} u_x, & u_y, & u_z, & -1, & 0, & 0 \\ v_x, & v_y, & v_z, & 0, & -1, & 0 \\ w_x, & w_y, & w_z, & 0, & 0, & -1 \\ 1, & 0, & 0, & \frac{A}{J}, & \frac{B}{J}, & \frac{C}{J} \\ 0, & 1, & 0, & \frac{A'}{J}, & \frac{B'}{J}, & \frac{C'}{J} \\ 0, & 0, & 1, & \frac{A''}{J}, & \frac{B''}{J}, & \frac{C''}{J} \end{vmatrix} \qquad \text{(12).}$$

It must be understood that when the determinant (11) or (12) is substituted in the expression (10) the operators D_1, D_2, D_3 affect u_x, v_x, w_x, &c. but not a, b, c, &c., or A, B, C, &c.

§ 3. To obtain Sylvester's expanded form of the result we use the form (12) for the determinant and expand $D_1^{l+f} D_2^{m+g} D_3^{n+h}$ in powers of D_x, D_y, D_z. This operator is equal to

$$\Sigma \frac{(l+f)!}{p_1!\,p_2!\,p_3!} \frac{(m+g)!}{q_1!\,q_2!\,q_3!} \frac{(n+h)!}{r_1!\,r_2!\,r_3!}$$

$$\left(\frac{AD_x}{J}\right)^{p_1} \left(\frac{A'D_y}{J}\right)^{p_2} \left(\frac{A''D_z}{J}\right)^{p_3} \left(\frac{BD_x}{J}\right)^{q_1} \left(\frac{B'D_y}{J}\right)^{q_2} \left(\frac{B''D_z}{J}\right)^{q_3} \left(\frac{CD_x}{J}\right)^{r_1} \left(\frac{C'D_y}{J}\right)^{r_2} \left(\frac{C''D_z}{J}\right)^{r_3},$$

where summation extends to all zero or positive integral values of p_1, p_2, p_3, &c. for which

$$p_1+p_2+p_3=l+f,\quad q_1+q_2+q_3=m+g,\quad r_1+r_2+r_3=n+h.$$

We now re-arrange the grouping of the terms and transform (10) into

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n}=\Sigma_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h-3}D_1^{\,l-f-1}D_2^{\,m+g-1}D_3^{\,n+h-1}$$

$$\cdot\begin{vmatrix}(uD_x), & (uD_y), & (uD_z), & D_1, & 0, & 0\\ (vD_x), & (vD_y), & (vD_z), & 0, & D_2, & 0\\ (wD_x), & (wD_y), & (wD_z), & 0, & 0, & D_3\\ 1, & 0, & 0, & \frac{A}{J}, & \frac{B}{J}, & \frac{C}{J}\\ 0, & 1, & 0, & \frac{A'}{J}, & \frac{B'}{J}, & \frac{C'}{J}\\ 0, & 0, & 1, & \frac{A''}{J}, & \frac{B''}{J}, & \frac{C''}{J}\end{vmatrix}\, t\,\frac{u^f v^g w^h}{f!\,g!\,h!},$$

where (uD_x), (uD_y), (uD_z) are the same in effect as D_x, D_y, D_z but operate on u only, whilst (vD_x),... operate on v only, and (wD_x),... on w only.

We can now make use of the results of § 1 and obtain finally

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n}=\Sigma_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h-3}\frac{(l+f-1)!}{p_1!\,p_2!\,p_3!}\frac{(m+g-1)!}{q_1!\,q_2!\,q_3!}\frac{(n+h-1)!}{r_1!\,r_2!\,r_3!}$$

$$\times\Sigma(p_1+q_1+r_1-1)!\,(p_2+q_2+r_2-1)!\,(p_3+q_3+r_3-1)!$$

$$\times J^{-(l+m+n+f+g+h)}A^{p_1}A'^{p_2}A''^{p_3}B^{q_1}B'^{q_2}B''^{q_3}C^{r_1}C'^{r_2}C''^{r_3}$$

$$\times[(p_1+q_1+r_1),\,(p_2+q_2+r_2),\,(p_3+q_3+r_3);\,f,\,g,\,h;\,\lambda,\,\mu,\,\nu;\,\lambda_1,\,\mu_1,\,\nu_1;\,\&c.]$$

$$\times\begin{vmatrix}\Sigma f_1\lambda_1, & \Sigma f_1\mu_1, & \Sigma f_1\nu_1, & l+f, & 0, & 0\\ \Sigma g_1\lambda_1', & \Sigma g_1\mu_1', & \Sigma g_1\nu_1', & 0, & m+g, & 0\\ \Sigma h_1\lambda_1'', & \Sigma h_1\mu_1'', & \Sigma h_1\nu_1'', & 0, & 0, & n+h\\ p_1+q_1+r_1, & 0, & 0, & p_1, & q_1, & r_1\\ 0, & p_2+q_2+r_2, & 0, & p_2, & q_2, & r_2\\ 0, & 0, & p_3+q_3+r_3, & p_3, & q_3, & r_3\end{vmatrix}\quad\ldots\ldots\ldots\ldots(13),$$

where the limits of summation are given by (4).

A little consideration is needed to see the truth of the last result. It is obtained by regarding the determinant as expanded, then expanding the various terms by § 1 and grouping together all the terms which give rise to the particular term denoted by

$$[(p_1+q_1+r_1),\,(p_2+q_2+r_2),\,(p_3+q_3+r_3);\,f,\,g,\,h;\,\lambda,\,\mu,\,\nu;\,\lambda_1,\,\mu_1,\,\nu_1;\,\&c.].$$

The last result agrees with that given by Sylvester but differs in sign from that obtained by Cayley when the number of independent variables is odd.

In the case of certain terms, such as those which occur in the evaluation of $\frac{\partial^2 t}{\partial u^2}$, the coefficients as given by (13) take an indeterminate form, but as Sylvester explains, there is no difficulty in deducing the proper value.

§ 4. In order to obtain a symbolical expression for the result of the change of variables we return to equation (10), and by a re-arrangement of terms write it in the form

$$\frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n} = \Sigma_{f=0,\, g=0,\, h=0}^{f=\infty,\, g=\infty,\, h=\infty} (-1)^{f+g+h} D_1^{l+f-1} D_2^{m+g-1} D_3^{n+h-1}$$

$$\begin{vmatrix} D_1 - (uD_1), & -(vD_1), & -(wD_1) \\ -(uD_2), & D_2 - (vD_2), & -(wD_2) \\ -(uD_3), & -(vD_3), & D_3 - (wD_3) \end{vmatrix} t \frac{u^f v^g w^h}{f!\, g!\, h!} \quad \text{.................. (14),}$$

where, as in the previous section, (uD_1), (uD_2), (uD_3) are equivalent to D_1, D_2, D_3 but operate on u only. This re-arrangement is effected by grouping together terms homogeneous in D_1, D_2, D_3, (uD_1), (uD_2), ...

The operators required for the purpose of expressing the result symbolically will be considered in the following sections.

§ 5. Taking U, V, W, T as in § 1, let suffixes 1, 2, 3, 4 indicate that (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), ... are substituted in them for ξ, η, ζ; so that

$$U_1 = a_{200}\xi_1^2 + a_{020}\eta_1^2 + \dots$$

Let also U_{ξ_1}, U_{ξ_2}, ... denote $\frac{\partial}{\partial \xi_1} U_1$, $\frac{\partial}{\partial \xi_2} U_2$, ... Let $\{U_1 U_{\xi_1}\}$ denote an operator formed by replacing terms such as $\xi_1^p \eta_1^q \zeta_1^r$ in the product $U_1 U_{\xi_1}$ by $\frac{\partial}{\partial a_{pqr}}$. The particular brackets $\{\ \}$ will be used to indicate this operator and to distinguish it from a mere algebraical product. Similarly let $\{U_2 V_{\xi_2}\}$ be an operator formed from $U_2 V_{\xi_2}$ by replacing $\xi_2^p \eta_2^q \zeta_2^r$ by $\frac{\partial}{\partial b_{pqr}}$. Let also $\{U_3 W_{\xi_3}\}$ and $\{U_4 T_{\xi_4}\}$ be formed by replacing $\xi_3^p \eta_3^q \zeta_3^r$ by $\frac{\partial}{\partial c_{pqr}}$ and $\xi_4^p \eta_4^q \zeta_4^r$ by $\frac{\partial}{\partial d_{pqr}}$.

We shall also suppose eight similar operators $\{V_1 U_{\xi_1}\}$, $\{V_2 V_{\xi_2}\}$, ... to be formed in like manner, and twenty-four others by writing η and ζ for ξ.

Written at full length for a few terms the first operator is

$$\{U_1 U_{\xi_1}\} = 2a_{200}^2 \frac{\partial}{\partial a_{300}} + a_{020}a_{110} \frac{\partial}{\partial a_{030}} + a_{002}a_{101} \frac{\partial}{\partial a_{003}} + 3a_{200}a_{110} \frac{\partial}{\partial a_{210}} + (2a_{200}a_{020} + a_{110}^2) \frac{\partial}{\partial a_{120}}$$

$$+ 3a_{200}a_{101} \frac{\partial}{\partial a_{201}} + (2a_{200}a_{002} + a_{101}^2) \frac{\partial}{\partial a_{102}} + (a_{020}a_{101} + a_{011}a_{110}) \frac{\partial}{\partial a_{021}}$$

$$+ (a_{110}a_{002} + a_{011}a_{101}) \frac{\partial}{\partial a_{012}} + (2a_{200}a_{011} + 2a_{110}a_{101}) \frac{\partial}{\partial a_{111}}$$

$$+ \dots$$

Similarly

$$\{U_2 V_{\xi_2}\} = 2a_{200}b_{200}\frac{\partial}{\partial b_{300}} + a_{020}b_{110}\frac{\partial}{\partial b_{030}} + a_{002}b_{101}\frac{\partial}{\partial b_{003}} + (a_{200}b_{110} + 2a_{110}b_{200})\frac{\partial}{\partial b_{210}} + \dots$$

If it is desired to work with u_{pqr}, v_{pqr}, &c. instead of a_{pqr}, b_{pqr}, &c., the operators may be formed in similar fashion. Thus $\{U_1 U_{\xi_1}\}$ is formed from

$$\left(u_{200}\frac{\xi_1^2}{2\,!} + u_{020}\frac{\eta_1^2}{2\,!} + u_{002}\frac{\zeta_1^2}{2\,!} + u_{110}\xi_1\eta_1 + \dots + u_{300}\frac{\xi_1^3}{3\,!} + \dots\right)\left(u_{200}\xi_1 + u_{110}\eta_1 + u_{101}\zeta_1 + u_{300}\frac{\xi_1^2}{2\,!} + \dots\right)$$

by replacing $\xi_1^p\eta_1^q\zeta_1^r$ by $p!\,q!\,r!\,\dfrac{\partial}{\partial u_{pqr}}$.

And therefore the operators may be also expressed in the following manner

$$\{U_1 U_{\xi_1}\} = \Sigma\, D_x^p D_y^q D_z^r (uu_x)\frac{\partial}{\partial u_{pqr}},$$

$$\{U_2 V_{\xi_2}\} = \Sigma\, D_x^p D_y^q D_z^r (uv_x)\frac{\partial}{\partial v_{pqr}},$$

$$\{U_3 W_{\xi_3}\} = \Sigma\, D_x^p D_y^q D_z^r (uw_x)\frac{\partial}{\partial w_{pqr}},$$

$$\{U_4 T_{\xi_4}\} = \Sigma\, D_x^p D_y^q D_z^r (ut_x)\frac{\partial}{\partial t_{pqr}},$$

&c., &c.,

where summation may be supposed to extend to all positive integral values of p, q, r, though in the first three operators the coefficients of $\dfrac{\partial}{\partial u_{pqr}}$, $\dfrac{\partial}{\partial v_{pqr}}$, $\dfrac{\partial}{\partial w_{pqr}}$ will be zero if $p+q+r<3$, and in the fourth the coefficient of $\dfrac{\partial}{\partial t_{pqr}}$ will be zero if $p+q+r<2$.

The operators actually required will be nine formed by combinations of the above, viz.,

$$\{U, x\} = \{U_1 U_{\xi_1}\} + \{U_2 V_{\xi_2}\} + \{U_3 W_{\xi_3}\} + \{U_4 T_{\xi_4}\},$$

$$\{U, y\} = \{U_1 U_{\eta_1}\} + \{U_2 V_{\eta_2}\} + \{U_3 W_{\eta_3}\} + \{U_4 T_{\eta_4}\},$$

&c., &c.

§ 6. The first theorem to be established with regard to these operators $\{U, x\}$, $\{U, y\}, \dots$ is that they are all commutative with one another. But before proceeding to the proof of this theorem it is necessary to make a few preliminary remarks.

Let $F(\xi, \eta, \zeta)$ be any integral function consisting of terms $\xi^p\eta^q\zeta^r$ such that $p+q+r \nless 2$ and let $\{F(\xi_1, \eta_1, \zeta_1)\}$, $\{F(\xi_2, \eta_2, \zeta_2)\}, \dots$ be operators formed as in the preceding section.

It is then obvious that

$$\{F(\xi_1, \eta_1, \zeta_1)\} U_1 = F(\xi_1, \eta_1, \zeta_1),$$

$$\{F(\xi_1, \eta_1, \zeta_1)\} U_2 = F(\xi_2, \eta_2, \zeta_2),$$

$$\begin{aligned}\{F(\xi_1, \eta_1, \zeta_1)\} U_{\xi_1} &= \{F(\xi_1, \eta_1, \zeta_1)\} \frac{\partial}{\partial \xi_1} U_1 \\ &= \frac{\partial}{\partial \xi_1} \{F(\xi_1, \eta_1, \zeta_1)\} U_1 \\ &= \frac{\partial}{\partial \xi_1} F(\xi_1, \eta_1, \zeta_1),\end{aligned}$$

$$\{F(\xi_1, \eta_1, \zeta_1)\} U_{\xi_2} = \frac{\partial}{\partial \xi_2} F(\xi_2, \eta_2, \zeta_2),$$

$$\{F(\xi_1, \eta_1, \zeta_1)\} V_1 = 0.$$

A number of other similar results might be written down by interchanging suffixes and the quantities $U, V, W, T, \xi, \eta, \zeta$. Again

$$\begin{aligned}\{F(\xi_1, \eta_1, \zeta_1)\} U_1 U_{\xi_1} &= U_{\xi_1} \{F(\xi_1, \eta_1, \zeta_1)\} U_1 + U_1 \{F(\xi_1, \eta_1, \zeta_1)\} U_{\xi_1} \\ &= U_{\xi_1} F(\xi_1, \eta_1, \zeta_1) + U_1 \frac{\partial}{\partial \xi_1} F(\xi_1, \eta_1, \zeta_1);\end{aligned}$$

$$\begin{aligned}\{F(\xi_2, \eta_2, \zeta_2)\} U_1 V_{\xi_1} &= U_1 \{F(\xi_2, \eta_2, \zeta_2)\} V_{\xi_1} \\ &= U_1 \frac{\partial}{\partial \xi_1} F(\xi_1, \eta_1, \zeta_1).\end{aligned}$$

Many similar results could be obtained, but these will be sufficient to indicate the mode of procedure about to be adopted for forming the products of the operators

$$\{U, x\}, \{U, y\}, \ldots\ldots$$

§ 7. The essentially distinct cases to be considered are the products $\{U, x\}\{U, y\}$, $\{U, x\}\{V, x\}$, $\{U, x\}\{V, y\}$. We will take these cases in order.

Let
$$\{U, x\}\{U, y\} = \{U, x\} \cdot \{U, y\} + \{U, x\} * \{U, y\},$$

where the first term on the right is the result of algebraical multiplication, and the second is the result of operating with $\{U, x\}$ on the coefficients of $\{U, y\}$. It is only the second term that can possibly be unsymmetrical. We have

$$\begin{aligned}\{U, x\} * \{U, y\} &= [\{U_1 U_{\xi_1}\} + \{U_2 V_{\xi_2}\} + \{U_3 W_{\xi_3}\} + \{U_4 T_{\xi_4}\}] * [\{U_1 U_{\eta_1}\} + \{U_2 V_{\eta_2}\} + \{U_3 W_{\eta_3}\} + \{U_4 T_{\eta_4}\}] \\ &= \{U_1 U_{\xi_1}\} * \{U_1 U_{\eta_1}\} + \{U_1 U_{\xi_1}\} * \{U_2 V_{\eta_2}\} + \{U_1 U_{\xi_1}\} * \{U_3 W_{\eta_3}\} + \{U_1 U_{\xi_1}\} * \{U_4 T_{\eta_4}\} \\ &\quad + \{U_2 V_{\xi_2}\} * \{U_2 V_{\eta_2}\} + \{U_3 W_{\xi_3}\} * \{U_3 W_{\eta_3}\} + \{U_4 T_{\xi_4}\} * \{U_4 T_{\eta_4}\} \\ &= \{U_{\eta_1} U_1 U_{\xi_1}\} + \left\{U_1 \frac{\partial}{\partial \eta_1}(U_1 U_{\xi_1})\right\} + \{V_{\eta_2} U_2 U_{\xi_2}\} + \{W_{\eta_3} U_3 U_{\xi_3}\} + \{T_{\eta_4} U_4 U_{\xi_4}\} \\ &\quad + \left\{U_2 \frac{\partial}{\partial \eta_2}(U_2 V_{\xi_2})\right\} + \left\{U_3 \frac{\partial}{\partial \eta_3}(U_3 W_{\xi_3})\right\} + \left\{U_4 \frac{\partial}{\partial \eta_4}(U_4 T_{\xi_4})\right\}\end{aligned}$$

$$= \{U_1(U_1U_{\xi_1\eta_1} + 2U_{\xi_1}U_{\eta_1})\} + \{U_2(U_2V_{\xi_2\eta_2} + U_{\xi_2}V_{\eta_2} + U_{\eta_2}V_{\xi_2})\}$$

$$+ \{U_3(U_3W_{\xi_3\eta_3} + U_{\xi_3}W_{\eta_3} + U_{\eta_3}W_{\xi_3})\} + \{U_4(U_4T_{\xi_4\eta_4} + U_{\xi_4}T_{\eta_4} + U_{\eta_4}T_{\xi_4})$$

$$= \{U, y\} * \{U, x\},$$

from the symmetry of the expression with regard to ξ and η.

Again we have

$$\{U, x\} * \{V, x\} = [\{U_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} + \{U_3W_{\xi_3}\} + \{U_4T_{\xi_4}\}] * [\{V_1U_{\xi_1}\} + \{V_2V_{\xi_2}\} + \{V_3W_{\xi_3}\} + \{V_4T_{\xi_4}\}]$$

$$= \{U_1U_{\xi_1}\} * \{V_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} * \{V_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} * \{V_2V_{\xi_2}\} + \{U_2V_{\xi_2}\} * \{V_3W_{\xi_3}\}$$

$$+ \{U_2V_{\xi_2}\} * \{V_4T_{\xi_4}\} + \{U_3W_{\xi_3}\} * \{V_3W_{\xi_3}\} + \{U_4T_{\xi_4}\} * \{V_4T_{\xi_4}\}$$

$$= \left\{V_1\frac{\partial}{\partial\xi_1}(U_1U_{\xi_1})\right\} + \{U_{\xi_1}U_1V_{\xi_1}\} + \{V_{\xi_2}U_2V_{\xi_2}\} + \left\{V_2\frac{\partial}{\partial\xi_2}(U_2V_{\xi_2})\right\}$$

$$+ \{W_{\xi_3}U_3V_{\xi_3}\} + \{T_{\xi_4}U_4V_{\xi_4}\} + \left\{V_3\frac{\partial}{\partial\xi_3}(U_3W_{\xi_3})\right\} + \left\{V_4\frac{\partial}{\partial\xi_4}(U_4T_{\xi_4})\right\}$$

$$= \{V_1U_1U_{\xi_1\xi_1} + V_1U^2_{\xi_1} + U_1U_{\xi_1}V_{\xi_1}\} + \{V_2U_2V_{\xi_2\xi_2} + V_2U_{\xi_2}V_{\xi_2} + U_2V^2_{\xi_2}\}$$

$$+ \{U_3V_3W_{\xi_3\xi_3} + W_{\xi_3}(V_3U_{\xi_3} + U_3V_{\xi_3})\} + \{U_4V_4T_{\xi_4\xi_4} + T_{\xi_4}(U_4V_{\xi_4} + V_4U_{\xi_4})\}$$

$$= \{V, x\} * \{U, x\},$$

from the symmetry of the result with respect to U and V.

Finally

$$\{U, x\} * \{V, y\} = [\{U_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} + \{U_3W_{\xi_3}\} + \{U_4T_{\xi_4}\}] * [\{V_1U_{\eta_1}\} + \{V_2V_{\eta_2}\} + \{V_3W_{\eta_3}\} + \{V_4T_{\eta_4}\}]$$

$$= \{U_1U_{\xi_1}\} * \{V_1U_{\eta_1}\} + \{U_2V_{\xi_2}\} * \{V_1U_{\eta_1}\} + \{U_2V_{\xi_2}\} * \{V_2V_{\eta_2}\} + \{U_2V_{\xi_2}\} * \{V_3W_{\eta_3}\}$$

$$+ \{U_2V_{\xi_2}\} * \{V_4T_{\eta_4}\} + \{U_3W_{\xi_3}\} * \{V_3W_{\eta_3}\} + \{U_4T_{\xi_4}\} * \{V_4T_{\eta_4}\}$$

$$= \left\{V_1\frac{\partial}{\partial\eta_1}(U_1U_{\xi_1})\right\} + \{U_{\eta_1}U_1V_{\xi_1}\} + \{V_{\eta_2}U_2V_{\xi_2}\} + \left\{V_2\frac{\partial}{\partial\eta_2}(U_2V_{\xi_2})\right\} + \{W_{\eta_3}U_3V_{\xi_3}\}$$

$$+ \{T_{\eta_4}U_4V_{\xi_4}\} + \left\{V_3\frac{\partial}{\partial\eta_3}(U_3W_{\xi_3})\right\} + \left\{V_4\frac{\partial}{\partial\eta_4}(U_4T_{\xi_4})\right\}$$

$$= \{U_1V_1U_{\xi_1\eta_1} + V_1U_{\xi_1}U_{\eta_1} + U_1U_{\eta_1}V_{\xi_1}\} + \{U_2V_2V_{\xi_2\eta_2} + V_2U_{\eta_2}V_{\xi_2} + U_2V_{\eta_2}V_{\xi_2}\}$$

$$+ \{U_3V_3W_{\xi_3\eta_3} + W_{\xi_3}V_3U_{\eta_3} + W_{\eta_3}U_3V_{\xi_3}\} + \{U_4V_4T_{\xi_4\eta_4} + T_{\xi_4}V_4U_{\eta_4} + T_{\eta_4}U_4V_{\xi_4}\}.$$

This expression is unchanged when U, V and ξ, η are interchanged and likewise the suffixes 1, 2. Therefore

$$\{U, x\} * \{V, y\} = \{V, y\} * \{U, x\}.$$

By interchanges of U, V, W and x, y, z, the products of all pairs of the nine operators can be reduced to one or other of the three preceding. It has therefore been fully established that all the nine operators $\{U, x\}$, $\{U, y\}$, $\{U, z\}$, $\{V, x\}$, $\{V, y\}$, $\{V, z\}$, $\{W, x\}$, $\{W, y\}$, $\{W, z\}$ are commutative with one another.

§ 8. Now, taking U, V, W, T as in § 1 we have

$$\{U_1 U_{\xi_1}\} U^f = f U^{f-1} \{U_1 U_{\xi_1}\} U = f U^{f-1} . U U_\xi = U \frac{\partial}{\partial \xi} (U^f) \quad \text{(15 a)},$$

$$\{U_1 U_{\xi_1}\} \frac{\partial}{\partial \xi} U^f = \frac{\partial}{\partial \xi} \{U_1 U_{\xi_1}\} U^f = \frac{\partial}{\partial \xi} \left(U \frac{\partial}{\partial \xi} U^f \right) \quad \text{(15 b)},$$

$$\{U_1 U_{\xi_1}\} \frac{\partial}{\partial \eta} U^f = \frac{\partial}{\partial \eta} \{U_1 U_{\xi_1}\} U^f = \frac{\partial}{\partial \eta} \left(U \frac{\partial}{\partial \xi} U^f \right) \quad \text{(15 c)},$$

$$\{U_2 V_{\xi_2}\} V^g = g V^{g-1} \{U_2 V_{\xi_2}\} V = g V^{g-1} U V_\xi = U \frac{\partial}{\partial \xi} V^g \quad \text{(15 d)},$$

$$\{U_2 V_{\xi_2}\} \frac{\partial}{\partial \xi} V^g = \frac{\partial}{\partial \xi} \{U_2 V_{\xi_2}\} V^g = \frac{\partial}{\partial \xi} \left(U \frac{\partial}{\partial \xi} V^g \right) \quad \text{(15 e)},$$

$$\{U_2 V_{\xi_2}\} \frac{\partial}{\partial \eta} V^g = \frac{\partial}{\partial \eta} \{U_2 V_{\xi_2}\} V^g = \frac{\partial}{\partial \eta} \left(U \frac{\partial}{\partial \xi} V^g \right) \quad \text{(15 f)},$$

$$\{U_4 T_{\xi_4}\} T = U \frac{\partial T}{\partial \xi} \quad \text{(15 g)}.$$

By comparison of the coefficients of $\xi^p \eta^q \zeta^r$ on the two sides of these equations the following formulae are deduced:

$$\{U_1 U_{\xi_1}\} D_x{}^p D_y{}^q D_z{}^r u^f = D_x{}^p D_y{}^q D_z{}^r \left(u \frac{\partial}{\partial x} u^f \right) \quad \text{(16 a)},$$

$$\{U_1 U_{\xi_1}\} D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial x} u^f = D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial x} \left(u \frac{\partial}{\partial x} u^f \right) \quad \text{(16 b)},$$

$$\{U_1 U_{\xi_1}\} D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial y} u^f = D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial y} \left(u \frac{\partial}{\partial x} u^f \right) \quad \text{(16 c)},$$

$$\{U_2 V_{\xi_2}\} D_x{}^p D_y{}^q D_z{}^r v^g = D_x{}^p D_y{}^q D_z{}^r u \frac{\partial}{\partial x} v^g \quad \text{(16 d)},$$

$$\{U_2 V_{\xi_2}\} D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial x} v^g = D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial x} \left(u \frac{\partial}{\partial x} v^g \right) \quad \text{(16 e)},$$

$$\{U_2 V_{\xi_2}\} D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial y} v^g = D_x{}^p D_y{}^q D_z{}^r \frac{\partial}{\partial y} \left(u \frac{\partial}{\partial x} v^g \right) \quad \text{(16 f)},$$

$$\{U_4 T_{\xi_4}\} D_x{}^p D_y{}^q D_z{}^r t = D_x{}^p D_y{}^q D_z{}^r \left(u \frac{\partial t}{\partial x} \right) \quad \text{(16 g)}.$$

The form of the above results shows that they may be generalized by replacing $D_x{}^p D_y{}^q D_z{}^r$ by any function consisting of integral powers of D_x, D_y, D_z. These examples seem sufficient to show the effect of the operators. It will be noticed that the effect is to introduce a solitary u and to make certain alterations in the symbols of differentiation. The effects are perhaps best seen by examination of (16 e) and (16 f).

With a view to the application of these formulae to the result in § 2 it is convenient to re-write them in another form. Let (uD_x), as on previous occasions, represent D_x when operating on u only, and let $[uD_x]$ act only on the solitary u which is introduced into the last set of formulae.

With this notation (16 a) becomes, if $F(D)$ represents any function of D_x, D_y, D_z,

$$\{U_1U_{\xi_1}\}\ F(D)\,u^f = F(D)\,\{(uD_x)-[uD_x]\}\ u\,.\,u^f.$$

Moreover, since $\{U_1U_{\xi_1}\}$ does not operate on V, W or T, the equations (15 a, b, c) still hold if the functions operated on, viz. U^f, $\frac{\partial}{\partial\xi}U^f$, ..., are multiplied by powers of V, W, T and their differential coefficients. Thus from (15 b) we have, for example,

$$\{U_1U_{\xi_1}\}\,.\,\frac{\partial}{\partial\xi}U^f\,.\,V^gV_\xi W^hW_\eta T = \frac{\partial}{\partial\xi}\left(U\frac{\partial}{\partial\xi}U^f\right).\,V^gV_\xi W^hW_\eta T\,;$$

and corresponding to (16 a)

$$\{U_1U_{\xi_1}\}\,F(D)\,\phi(vD,\,wD,\,tD)\,u^fv^gw^ht = F(D)\,\phi(vD,\,wD,\,tD)\,\{(uD_x)-[uD_x]\}\ u\,.\,u^fv^gw^ht,$$

where $\phi(vD,\ wD,\ tD)$ represents a function of (vD_x), (vD_y), (vD_z), (wD_x), ...

Similarly

$$\{U_1U_{\eta_1}\}\,F(D)\,\phi(vD,\,wD,\,tD)\,u^fv^gw^ht = F(D)\,\phi(vD,\,wD,\,tD)\,\{(uD_y)-[uD_y]\}\ u\,.\,u^fv^gw^ht,$$

$$\{U_1U_{\zeta_1}\}\,F(D)\,\phi(vD,\,wD,\,tD)\,u^fv^gw^ht = F(D)\,\phi(vD,\,wD,\,tD)\,\{(uD_z)-[uD_z]\}\ u\,.\,u^fv^gw^ht,$$

and therefore

$$\left[\frac{A}{J}\{U_1U_{\xi_1}\}+\frac{A'}{J}\{U_1U_{\eta_1}\}+\frac{A''}{J}\{U_1U_{\zeta_1}\}\right]F(D)\,\phi(vD,\,wD,\,tD)\,u^fv^gw^ht$$
$$= F(D)\,\phi(vD,\,wD,\,tD)\,\{(uD_1)-[uD_1]\}\ u\,.\,u^fv^gw^ht\,.........(17\ a),$$

whilst similar results hold for D_2 and D_3.

Again, from formulae of which (16 d) is a type are deduced formulae exemplified by the following

$$\left[\frac{A}{J}\{U_2V_{\xi_2}\}+\frac{A'}{J}\{U_2V_{\eta_2}\}+\frac{A''}{J}\{U_2V_{\zeta_2}\}\right]F(D)\,\phi(uD,\,wD,\,tD)\,u^fv^gw^ht$$
$$= F(D)\,\phi(uD,\,wD,\,tD)\,(vD_1)\,u\,.\,u^fv^gw^ht\,.........(17\ b).$$

And, from formulae of which (16 e) and (16 f) are types are deduced others which are exemplified by

$$\left[\frac{A}{J}\{U_2V_{\xi_2}\}+\frac{A'}{J}\{U_2V_{\eta_2}\}+\frac{A''}{J}\{U_2V_{\zeta_2}\}\right]F(D)\,\phi(wD,\,tD)\,(vD_1)\,u^fv^gw^ht$$
$$= F(D)\,\phi(wD,\,tD)\,\{(vD_1)+[uD_1]\}\,(vD_1)\,u\,.\,u^fv^gw^ht\,...........(17\ c).$$

If in this last formula (vD_1) is replaced on the left by (vD_2), then on the right $\{(vD_1)+[uD_1]\}$ must be replaced by $\{(vD_2)+[uD_2]\}$; and if on the left A, A', A'' are replaced by B, B', B'', then on the right the second (vD_1) must be replaced by (vD_2).

§ 9. Return now to the expression (14) in § 4 and write, for brevity,

$$\begin{vmatrix} D_1-(uD_1), & -(vD_1), & -(wD_1) \\ -(uD_2), & D_2-(vD_2), & -(wD_2) \\ -(uD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix} = \Delta\,.$$

By adding the second and third columns to the first it is evident that

$$|\Delta| = \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix}.$$

By use of this form and consideration of the rules exemplified in the preceding formulae it then becomes evident from (17 a) that

$$\left[\frac{A}{J}\{U_1U_{\xi_1}\}+\frac{A'}{J}\{U_1U_{\eta_1}\}+\frac{A''}{J}\{U_1U_{\zeta_1}\}\right] D_1{}^lD_2{}^mD_3{}^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1{}^lD_2{}^mD_3{}^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix} \{(uD_1)-[uD_1]\}\,u\,.\,u^fv^gw^ht.$$

Also from (17 b) and (17 c)

$$\left[\frac{A}{J}\{U_2V_{\xi_2}\}+\frac{A'}{J}\{U_2V_{\eta_2}\}+\frac{A''}{J}\{U_2V_{\zeta_2}\}\right] D_1{}^lD_2{}^mD_3{}^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1{}^lD_2{}^mD_3{}^n \begin{vmatrix} (tD_1), & -(vD_1)-[uD_1], & -(wD_1) \\ (tD_2), & D_2-(vD_2)-[uD_2], & -(wD_2) \\ (tD_3), & -(vD_3)-[uD_3], & D_3-(wD_3) \end{vmatrix} (vD_1)\,u\,.\,u^fv^gw^ht.$$

Again by interchange of v and w in (17 b) and (17 c) it is obvious that

$$\left[\frac{A}{J}\{U_3W_{\xi_3}\}+\frac{A'}{J}\{U_3W_{\eta_3}\}+\frac{A''}{J}\{U_3W_{\zeta_3}\}\right] D_1{}^lD_2{}^mD_3{}^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1{}^lD_2{}^mD_3{}^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1)-[uD_1] \\ (tD_2), & D_2-(vD_2), & -(wD_2)-[uD_2] \\ (tD_3), & -(vD_3), & D_3-(wD_3)-[uD_3] \end{vmatrix} (wD_1)\,u\,.\,u^fv^gw^ht.$$

Similarly by interchange of v and t in (17 b) and (17 c) it follows that

$$\left[\frac{A}{J}\{U_4T_{\xi_4}\}+\frac{A'}{J}\{U_4T_{\eta_4}\}+\frac{A''}{J}\{U_4T_{\zeta_4}\}\right] D_1{}^lD_2{}^mD_3{}^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1{}^lD_2{}^mD_3{}^n \begin{vmatrix} (tD_1)+[uD_1], & -(vD_1), & -(wD_1) \\ (tD_2)+[uD_2], & D_2-(vD_2), & -(wD_2) \\ (tD_3)+[uD_3], & -(vD_3), & D_3-(wD_3) \end{vmatrix} (tD_1)\,u\,.\,u^fv^gw^ht.$$

Now add these four equations together. The operator on the left will become

$$\frac{A}{J}\{U, x\}+\frac{A'}{J}\{U, y\}+\frac{A''}{J}\{U, z\},$$

which it will be convenient to denote by $\{U, 1\}$.

On the right-hand side all the terms containing $[uD_1]$, $[uD_2]$ and $[uD_3]$ disappear. For the coefficient of $[uD_1]$ in the operator is easily seen to be

$$-D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix}$$

$$+D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix}$$

$$=0.$$

The coefficient of $[uD_2]$ is

$$D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_1), & -(vD_1), & -(wD_1) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix} = 0.$$

The coefficient of $[uD_3]$ is

$$D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_1), & -(vD_1), & -(wD_1) \end{vmatrix} = 0.$$

Hence

$$\{U, 1\}\, D_1^l D_2^m D_3^n \,|\, \Delta \,|\, u^f v^g w^h t = D_1^l D_2^m D_3^n \,|\, \Delta \;\; \{(uD_1)+(vD_1)+(wD_1)+(tD_1)\}\, u^{f+1} v^g w^h t$$
$$= D_1^{l+1} D_2^m D_3^n \,|\, \Delta \,|\, u^{f+1} v^g w^h t \quad \text{......................................(18 } a\text{)}.$$

We note that $\{U, 1\}$ may be written

$$\{U, 1\} = \frac{1}{J} \begin{vmatrix} \{U, x\}, & \{U, y\}, & \{U, z\} \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix}.$$

If we take similarly

$$\{V, 2\} = \frac{1}{J} \begin{vmatrix} a, & a', & a'' \\ \{V, x\}, & \{V, y\}, & \{V, z\} \\ c, & c', & c'' \end{vmatrix} = \frac{1}{J}[B\{V, x\} + B'\{V, y\} + B''\{V, z\}],$$

and

$$\{W, 3\} = \frac{1}{J} \begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ \{W, x\}, & \{W, y\}, & \{W, z\} \end{vmatrix} = \frac{1}{J}[C\{W, x\} + C'\{W, y\} + C''\{W, z\}],$$

we shall find

$$\{V, 2\}\, D_1^l D_2^m D_3^n \,|\, \Delta \,|\, u^f v^g w^h t = D_1^l D_2^{m+1} D_3^n \,|\, \Delta \,|\, u^f v^{g+1} w^h t \quad \text{..................(18 } b\text{)},$$

$$\{W, 3\}\, D_1^l D_2^m D_3^n \,|\, \Delta \,|\, u^f v^g w^h t = D_1^l D_2^m D_3^{n+1} \,|\, \Delta \,|\, u^f v^g w^{h+1} t \quad \text{..................(18 } c\text{)}.$$

We have therefore

$$D_1^{l+f-1}D_2^{m+g-1}D_3^{n+h-1}\ \Delta\ u^f v^g w^h t = \{U,\ 1\}\ D_1^{l+f-2}D_2^{m+g-1}D_3^{n+h-1}\ |\Delta|\ u^{f-1}v^g w^h t$$

$$= \{U,\ 1\}^f\ D_1^{l-1}D_2^{m+g-1}D_3^{n+h-1}\ |\Delta|\ v^g w^h t$$

$$= \{U,\ 1\}^f\ \{V,\ 2\}^g\ \{W,\ 3\}^h\ D_1^{l-1}D_2^{m-1}D_3^{n-1}\ |\Delta|\ t.$$

But $|\Delta|\ t = D_1D_2D_3t$, so that the expression becomes

$$\{U,\ 1\}^f\ \{V,\ 2\}^g\ \{W,\ 3\}^h\,.\,D_1^lD_2^mD_3^nt.$$

Therefore

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n} = \sum_{f=0,\ g=0,\ h=0}^{f=\infty,\ g=\infty,\ h=\infty} (-1)^{f+g+h}D_1^{l+f-1}D_2^{m+g-1}D_3^{n+h-1}\ |\Delta|\ \frac{u^fv^gw^ht}{f!\ g!\ h!}$$

$$= \Sigma\,(-1)^{f+g+h}\frac{\{U,\ 1\}^f}{f\,!}\ \frac{\{V,\ 2\}^g}{g\,!}\ \frac{\{W,\ 3\}^h}{h\,!}\,.\,D_1^lD_2^mD_3^nt$$

$$= e^{-\{U,\,1\}-\{V,\,2\}-\{W,\,3\}}\,.\,D_1^lD_2^mD_3^nt \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19\ a),$$

since the operators are commutative. The fact that the operators are commutative has been proved independently, but it is pretty obvious from the circumstance that the D_1, D_2, D_3 are commutative, so that the above reduction might have been effected in different orders.

If it is desired to bring the independent operators $\{U,\ x\}$, $\{U,\ y\}$, &c. into prominence the result may be written

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n} = e^{-\frac{A}{J}\{U,\ x\}}\,.\,e^{-\frac{A'}{J}\{U,\ y\}}\,.\,e^{-\frac{A''}{J}\{U,\ z\}}\,.\,e^{-\frac{B}{J}\{V,\ x\}}\,.\,e^{-\frac{B'}{J}\{V,\ y\}}\,.\,e^{-\frac{B''}{J}\{V,\ z\}}$$

$$.\,e^{-\frac{C}{J}\{W,\ x\}}\,.\,e^{-\frac{C'}{J}\{W,\ y\}}\,.\,e^{-\frac{C''}{J}\{W,\ z\}}\,.\,D_1^lD_2^mD_3^n\,t\ldots\ldots\ldots\ldots\ldots(19\ b).$$

If we write
$$D_{lmn} = \frac{1}{l\,!\ m\,!\ n\,!}\,\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n},$$

we have
$$D_{lmn} = e^{-\{U,\,1\}-\{V,\,2\}-\{W,\,3\}}\,.\,\frac{D_1^lD_2^mD_3^n}{l\,!\ m\,!\ n\,!}\,t\ldots\ldots\ldots\ldots(19\ c).$$

§ 10. It remains to express $\dfrac{D_1^lD_2^mD_3^nt}{l\,!\ m\,!\ n\,!}$ by means of operators acting upon d_{lmn}. Since $D_1^lD_2^mD_3^n$ is merely a linear transformation of $D_x^lD_y^mD_z^n$, the operators required for the purpose will be simply the ordinary operators of the theory of invariants. These operators we shall define as follows:—

$$\omega_{xy} = \{\xi T_\eta\} = \sum_{p=1,\ q=0,\ r=0} (q+1)\,d_{p-1,\ q+1,\ r}\,\frac{\partial}{\partial d_{pqr}},$$

$$\omega_{yx} = \{\eta T_\xi\} = \sum_{p=0,\ q=1,\ r=0} (p+1)\,d_{p+1,\ q-1,\ r}\,\frac{\partial}{\partial d_{pqr}},$$

$$\omega_{xz} = \{\xi T_\zeta\} = \sum_{p=1,\ q=0,\ r=0} (r+1)\,d_{p-1,\ q,\ r+1}\,\frac{\partial}{\partial d_{pqr}},$$

$$\omega_{zx} = \{\zeta T_\xi\} = \sum_{p=0,\, q=0,\, r=1} (p+1)\, d_{p+1,\, q,\, r-1} \frac{\partial}{\partial d_{pqr}},$$

$$\omega_{yz} = \{\eta T_\zeta\} = \sum_{p=0,\, q=1,\, r=0} (r+1)\, d_{p,\, q-1,\, r+1} \frac{\partial}{\partial d_{pqr}},$$

$$\omega_{zy} = \{\zeta T_\eta\} = \sum_{p=0,\, q=0,\, r=1} (q+1)\, d_{p,\, q+1,\, r-1} \frac{\partial}{\partial d_{pqr}}.$$

where T is as in (2), and the operators are formed by expanding ξT_η, &c., and replacing $\xi^p \eta^q \zeta^r$ by $\frac{\partial}{\partial d_{pqr}}$. The upper limits of p, q, r in the summations are all infinite.

These operators ω are identical with the operators Ω discussed in Elliott's *Algebra of Quantics*, Chap. XVI. Their properties are there obtained by forming the alternants, but as the formation is simplified by use of the symbolical method the process by this method is given here.

We remark that the operators $\partial/\partial\xi$ and $\{\xi T_\eta\}$ are independent and therefore, if T_ξ denotes an algebraical expression,

$$\{\xi T_\eta\}\, T_\xi = \{\xi T_\eta\} \frac{\partial}{\partial \xi} T = \frac{\partial}{\partial \xi} \{\xi T_\eta\}\, T = \frac{\partial}{\partial \xi} (\xi T_\eta).$$

Hence

$$\begin{aligned}
\omega_{xy}\omega_{yx} - \omega_{yx}\omega_{xy} &= \{\xi T_\eta\}\{\eta T_\xi\} - \{\eta T_\xi\}\{\xi T_\eta\} \\
&= \left\{\eta \frac{\partial}{\partial \xi}(\xi T_\eta)\right\} - \left\{\xi \frac{\partial}{\partial \eta}(\eta T_\xi)\right\} \\
&= \{\eta T_\eta\} - \{\xi T_\xi\} \quad \ldots\ldots\ldots\ldots\ldots (20\,a).
\end{aligned}$$

$$\begin{aligned}
\omega_{xy}\omega_{xz} - \omega_{xz}\omega_{xy} &= \{\xi T_\eta\}\{\xi T_\zeta\} - \{\xi T_\zeta\}\{\xi T_\eta\} \\
&= \left\{\xi \frac{\partial}{\partial \zeta}(\xi T_\eta)\right\} - \left\{\xi \frac{\partial}{\partial \eta}(\xi T_\zeta)\right\} \\
&= 0 \quad \ldots\ldots\ldots\ldots\ldots (20\,b),
\end{aligned}$$

$$\begin{aligned}
\omega_{xy}\omega_{zx} - \omega_{zx}\omega_{xy} &= \{\xi T_\eta\}\{\zeta T_\xi\} - \{\zeta T_\xi\}\{\xi T_\eta\} \\
&= \left\{\zeta \frac{\partial}{\partial \xi}(\xi T_\eta)\right\} - \left\{\xi \frac{\partial}{\partial \eta}(\zeta T_\xi)\right\} \\
&= \{\zeta T_\eta\} = \omega_{zy} \quad \ldots\ldots\ldots\ldots\ldots (20\,c).
\end{aligned}$$

Similarly

$$\omega_{xy}\omega_{yz} - \omega_{yz}\omega_{xy} = -\,\omega_{xz} \quad \ldots\ldots\ldots\ldots\ldots (20\,d),$$

$$\omega_{xy}\omega_{zy} - \omega_{zy}\omega_{xy} = 0 \quad \ldots\ldots\ldots\ldots\ldots (20\,e).$$

Equation (20 a) shows that if a function is annihilated both by ω_{xy} and ω_{yx} and is isobaric in first suffixes it must also be isobaric in second suffixes and the partial weights must be equal, and if the function is further annihilated by ω_{xz} and ω_{zx} it must also be isobaric in third suffixes and all the partial weights must be equal. Equations (20 b) and (20 e) show that any two ω's are commutative if they have the same letter for their first suffix, or the same letter for their second suffix. Equations (20 c) and (20 d) show that the ω's are not independent but that any one of them can be expressed as the alternant of two others, so that if a function is annihilated by the three operators ω_{yz}, ω_{zx}, ω_{xy}, it is also annihilated by all the others.

When the case of more than three independent variables is considered there will be pairs of ω's which have no common letter in their suffixes, and reference to the above proof of equations (20) shows that such pairs are commutative.

That the operators $\{U, x\}$, $\{V, x\}$, ... are not commutative with any of the ω's is easily seen by forming the alternants of typical pairs. Thus

$$\begin{aligned}
\{U, x\}\,\omega_{xy} - \omega_{xy}\,\{U, x\} &= \{U_4 T_{\xi_4}\}\,\{\xi_4 T_{\eta_4}\} - \{\xi_4 T_{\eta_4}\}\,\{U_4 T_{\xi_4}\} \\
&= \left\{\xi_4 \frac{\partial}{\partial \eta_4}(U_4 T_{\xi_4})\right\} - \left\{U_4 \frac{\partial}{\partial \xi_4}(\xi_4 T_{\eta_4})\right\} \\
&= \{\xi_4 U_{\eta_4} T_{\xi_4}\} - \{U_4 T_{\eta_4}\}, \\
\{U, x\}\,\omega_{yx} - \omega_{yx}\,\{U, x\} &= \{U_4 T_{\xi_4}\}\,\{\eta_4 T_{\xi_4}\} - \{\eta_4 T_{\xi_4}\}\,\{U_4 T_{\xi_4}\} \\
&= \left\{\eta_4 \frac{\partial}{\partial \xi_4}(U_4 T_{\xi_4})\right\} - \left\{U_4 \frac{\partial}{\partial \xi_4}(\eta_4 T_{\xi_4})\right\} \\
&= \{\eta_4 U_{\xi_4} T_{\xi_4}\}, \\
\{U, x\}\,\omega_{yz} - \omega_{yz}\,\{U, x\} &= \{U_4 T_{\xi_4}\}\,\{\eta_4 T_{\zeta_4}\} - \{\eta_4 T_{\zeta_4}\}\,\{U_4 T_{\xi_4}\} \\
&= \left\{\eta_4 \frac{\partial}{\partial \zeta_4}(U_4 T_{\xi_4})\right\} - \left\{U_4 \frac{\partial}{\partial \xi_4}(\eta_4 T_{\zeta_4})\right\} \\
&= \{\eta_4 U_{\zeta_4} T_{\xi_4}\}.
\end{aligned}$$

§ 11. We can now express

$$\frac{D_1^l D_2^m D_3^n}{l!\,m!\,n!}\,t$$

by means of these operators acting upon d_{lmn}.

We have, in the first place,

$$\omega_{xz} d_{lmn} = (n+1)\, d_{l-1,\, m,\, n+1},$$

$$\omega_{xz}{}^p\, d_{lmn} = \frac{(n+p)!}{n!}\, d_{l-p,\, m,\, n+p},$$

$$\omega_{xy}{}^q\, \omega_{xz}{}^p\, d_{lmn} = \frac{(n+p)!\,(m+q)!}{n!\,m!}\, d_{l-p-q,\, m+q,\, n+p}.$$

Therefore

$$\begin{aligned}
\frac{D_1^l D_y^m D_z^n t}{l!\,m!\,n!} &= \frac{1}{l!\,m!\,n!}\left(\frac{AD_x + A'D_y + A''D_z}{J}\right)^l D_y^m D_z^n t \\
&= \frac{1}{J^l}\frac{1}{l!\,m!\,n!}\Sigma\, \frac{l!}{p!\,q!\,l-p-q!}\, A^{l-p-q} A'^q A''^p\, D_x^{l-p-q}\, D_y^{m+q}\, D_z^{n+p}\, t
\end{aligned}$$

$$= \frac{1}{J^l} \frac{1}{m!\,n!} \Sigma \frac{1}{p!\,q!} A^{l-p-q} A'^q A''^p (m+q)!\,(n+p)!\, d_{l-p-q,\, m+q,\, n+p}$$

$$= \left(\frac{A}{J}\right)^l \Sigma \frac{1}{p!\,q!} \left(\frac{A'}{A}\right)^q \left(\frac{A''}{A}\right)^p \omega_{xy}{}^q\, \omega_{xz}{}^p\, d_{lmn}$$

$$= \left(\frac{A}{J}\right)^l e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}}\, d_{lmn},$$

since ω_{xy} and ω_{xz} are commutative.

To express $D_1{}^l D_2{}^m D_z{}^n t$ in a similar manner we proceed thus. We have

$$JD_1 = AD_x + A'D_y + A''D_z,$$

$$JD_2 = BD_x + B'D_y + B''D_z;$$

whence

$$AD_2 = BD_1 + c''D_y - c'D_z.$$

It may be noticed here that, if we were dealing with more than three independent variables, c'' and $-c'$ would be replaced by second minors of J.

We have therefore

$$\frac{D_1{}^l D_2{}^m D_z{}^n t}{l!\,m!\,n!} = \frac{A^{-m}}{l!\,m!\,n!} D_1{}^l (BD_1 + c''D_y - c'D_z)^m D_z{}^n t$$

$$= \frac{A^{-m}}{l!\,m!\,n!} \Sigma \frac{m!}{p!\,q!\,r!} B^p D_1{}^{l+p} (c''D_y)^q (-c'D_z)^r D_z{}^n t, \text{ where } p+q+r=m.$$

$$= \frac{A^{-m}}{l!\,n!} \Sigma \frac{1}{p!\,q!\,r!} B^p c''^q (-c')^r (l+p)!\,q!\,(n+r)! \left(\frac{A}{J}\right)^{l+p} e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}}\, d_{l+p,\, q,\, n+r}$$

$$= \left(\frac{A}{J}\right)^l \left(\frac{c''}{A}\right)^m e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}}\, \Sigma \frac{1}{p!\,r!} \left(\frac{AB}{Jc''}\right)^p \left(-\frac{c'}{c''}\right)^r \omega_{yx}{}^p\, \omega_{yz}{}^r\, d_{lmn}$$

$$= \left(\frac{A}{J}\right)^l \left(\frac{c''}{A}\right)^m e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}} \,.\, e^{\frac{AB}{Jc''}\omega_{yx} - \frac{c}{c''}\omega_{yz}} \,.\, d_{lmn}.$$

Finally, since $D_z = a''D_1 + b''D_2 + c''D_3$, we have

$$\frac{D_1{}^l D_2{}^m D_3{}^n t}{l!\,m!\,n!} = \frac{D_1{}^l D_2{}^m}{l!\,m!\,n!} c''^{-n} (D_z - a''D_1 - b''D_2)^n t$$

$$= \frac{c''^{-n}}{l!\,m!\,n!} \Sigma \frac{n!}{p!\,q!\,r!} (-a'')^q (-b'')^r D_1{}^{l+q} D_2{}^{m+r} D_z{}^p t, \text{ where } p+q+r=n,$$

$$= \frac{c''^{-n}}{l!\,m!} \Sigma \frac{(-a'')^q (-b'')^r}{p!\,q!\,r!} \left(\frac{A}{J}\right)^{l+q} \left(\frac{c''}{A}\right)^{m+r} e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}} e^{\frac{AB}{Jc''}\omega_{yx} - \frac{c'}{c''}\omega_{yz}} (l+q)!\,(m+r)!\,p!\, d_{l+q,\, m+r,\, p}$$

$$= \left(\frac{A}{J}\right)^l \left(\frac{c''}{A}\right)^m \frac{1}{c''^n} e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}} \,.\, e^{\frac{AB}{Jc''}\omega_{yx} - \frac{c'}{c''}\omega_{yz}}\, \Sigma \frac{1}{q!\,r!} \left(-\frac{a''A}{J}\right)^q \left(-\frac{b''c''}{A}\right)^r (\omega_{zx})^q (\omega_{zy})^r \,.\, d_{lmn}$$

$$= \frac{A^{l-m} c''^{m-n}}{J^l} e^{\frac{A'\omega_{xy} + A''\omega_{xz}}{A}} \,.\, e^{\frac{AB}{Jc''}\omega_{yx} - \frac{c'}{c''}\omega_{yz}} \,.\, e^{-\frac{a''A}{J}\omega_{zx} - \frac{b''c''}{A}\omega_{zy}} \,.\, d_{lmn}.$$

Now, for brevity, write

$$\omega_1 = \frac{A'}{A}\,\omega_{xy} + \frac{A''}{A}\,\omega_{xz},$$

$$\omega_2 = \frac{AB}{Jc''}\,\omega_{yx} - \frac{c'}{c''}\,\omega_{yz},$$

$$\omega_3 = -\frac{a''A}{J}\,\omega_{zx} - \frac{b''c''}{A}\,\omega_{zy},$$

$$\begin{aligned}\{U, V, W\} &= \{U, 1\} + \{V, 2\} + \{W, 3\} \\ &= \frac{1}{J}[A\{U, x\} + A'\{U, y\} + A''\{U, z\}] \\ &\quad + \frac{1}{J}[B\{V, x\} + B'\{V, y\} + B''\{V, z\}] \\ &\quad + \frac{1}{J}[C\{W, x\} + C'\{W, y\} + C''\{W, z\}].\end{aligned}$$

Then

$$\begin{aligned}D_{lmn} &= \frac{1}{l!\,m!\,n!}\,\frac{\partial^{l+m+n}\,t}{\partial u^l \partial v^m \partial w^n} \\ &= \frac{A^{l-m}\,c''^{m-n}}{J^l}\,e^{-\{U,\,V,\,W\}}\,.\,e^{\omega_1}\,.\,e^{\omega_2}\,.\,e^{\omega_3}\,.\,d_{lmn} \quad \text{..............(21).}\end{aligned}$$

Now if Ω denotes any linear operator which acts on two functions P and Q, we have

$$e^{\Omega}P\,.\,e^{\Omega}Q = e^{\Omega_1}\,.\,e^{\Omega_2}PQ,$$

where Ω_1, Ω_2 are equivalent to Ω but act respectively on P and Q alone. Therefore

$$\begin{aligned}e^{\Omega}P\,.\,e^{\Omega}Q &= e^{\Omega_1\,\cdot\,\Omega_2}PQ \\ &= e^{\Omega}PQ.\end{aligned}$$

By repeated applications of this principle we find that

$$D_{lmn}D_{l'm'n'} = \frac{A^{l+l'-m-m'}\,c''^{m+m'-n-n'}}{J^{l+l'}}\,e^{-\{U,\,V,\,W\}}\,.\,e^{\omega_1}\,.\,e^{\omega_2}\,.\,e^{\omega_3}\,.\,d_{lmn}d_{l'm'n'}.$$

And more generally if $F(d_{lmn},\ d_{l'm'n'},\ \ldots)$ represents any function isobaric in each set of suffixes being of weights p_1, p_2, p_3 in first, second and third suffixes,

$$F(D_{lmn},\ D_{l'm'n'},\ \ldots) = \frac{A^{p_1-p_2}\,c''^{p_2-p_3}}{J^{p_1}}\,e^{-\{U,\,V,\,W\}}\,e^{\omega_1}e^{\omega_2}e^{\omega_3}\,F(d_{lmn},\ d_{l'm'n'},\ \ldots) \quad \text{...... (22).}$$

§ 12. The asymmetry of that part of the operator which depends on the ω's is a consequence of their non-commutative character. By arranging the work a little differently nine different forms of the result could have been obtained. In the case of two independent variables the number of different forms will be four, and it will be convenient for some of the subsequent applications to have these four forms set out at length.

In modifying the work of § 11 for this case it is obvious by reference to the argument that only two ω's will be required, viz.

$$\omega_{xy} = \{\xi T_\eta\} = \sum_{p=1,\, q=0} (q+1)\, d_{p-1,\, q+1} \frac{\partial}{\partial d_{pq}},$$

$$\omega_{yx} = \{\eta T_\xi\} = \sum_{p=0,\, q=1} (p+1)\, d_{p+1,\, q-1} \frac{\partial}{\partial d_{pq}};$$

and their effect on d_{lm} is seen to be this:—

$$\omega_{xy}{}^p d_{lm} = \frac{(m+p)!}{m!} d_{l-p,\, m+p},$$

$$\omega_{yx}{}^p d_{lm} = \frac{(l+p)!}{l!} d_{l+p,\, m-p}.$$

Moreover, reference to the work shows that c'' must be replaced by unity, and that

$$J = \begin{vmatrix} a & a' \\ b & b' \end{vmatrix}.$$

Therefore

$$\frac{D_1{}^l D_2{}^m t}{l!\, m!} = \left(\frac{A}{J}\right)^l \left(\frac{1}{A}\right)^m e^{\frac{A'}{A}\omega_{xy}} e^{\frac{AB}{J}\omega_{yx}} d_{lm} \qquad \text{(23 } a\text{)}$$

We will next obtain the second form of the result. We have

$$D_1{}^l D_x{}^m t = \frac{1}{J^l}(AD_x + A'D_y)^l D_x{}^m t$$

$$= \frac{1}{J^l} \Sigma \frac{l!}{p!\, q!} A^p A'^q D_x{}^{m+p} D_y{}^q t, \text{ where } p + q = l.$$

Therefore

$$\frac{D_1{}^l D_x{}^m t}{l!\, m!} = \frac{1}{J^l} \Sigma \frac{(m+p)!}{m!\, p!} A^p A'^q d_{m+p,\, l-p}$$

$$= \left(\frac{A'}{J}\right)^l \Sigma \frac{1}{p!} \left(\frac{A}{A'}\right)^p \omega_{yx}{}^p d_{ml}$$

$$= \left(\frac{A'}{J}\right)^l e^{\frac{A}{A'}\omega_{yx}} d_{ml}.$$

Now

$$JD_1 = AD_x + A'D_y,$$

$$JD_2 = BD_x + B'D_y;$$

therefore, eliminating D_y,

$$A'D_2 = B'D_1 - D_x.$$

Hence

$$D_1{}^l D_2{}^m t = \frac{1}{A'^m}(B'D_1 - D_x)^m D_1{}^l t$$

$$= \frac{1}{A'^m}\Sigma \frac{m!}{p!\,q!} B'^p (-1)^q D_1{}^{l+p} D_x{}^q t$$

$$= \frac{1}{A'^m}\Sigma \frac{m!}{p!\,q!} B'^p (-1)^q (l+p)!\,q!\left(\frac{A'}{J}\right)^{l+p} e^{\frac{A}{A'}\omega_{yx}} d_{m-p,\ l+p}.$$

Therefore

$$\frac{D_1{}^l D_2{}^m t}{l!\,m!} = \left(\frac{A'}{J}\right)^l \left(-\frac{1}{A'}\right)^m e^{\frac{A}{A'}\omega_{yx}} \Sigma \frac{(l+p)!}{l!\,p!}\left(-\frac{A'B'}{J}\right)^p d_{m-p,\ l+p}$$

$$= \left(\frac{A'}{J}\right)^l \left(-\frac{1}{A'}\right)^m e^{\frac{A}{A'}\omega_{yx}} \Sigma \frac{1}{p!}\left(-\frac{A'B'}{J}\right)^p \omega_{xy}{}^p d_{ml}$$

$$= (-1)^m \frac{A'^{l-m}}{J^l} e^{\frac{A}{A'}\omega_{yx}} e^{-\frac{A'B'}{J}\omega_{xy}} d_{ml} \quad \text{.......... (23 } b).$$

Interchanging B, B' with A, A' and l with m, and writing $-J$ for J, we shall find similarly

$$\frac{D_1{}^l D_2{}^m t}{l!\,m!} = (-1)^l \left(\frac{B}{J}\right)^m \left(\frac{1}{B}\right)^l e^{\frac{B'}{B}\omega_{xy}} e^{-\frac{AB}{J}\omega_{yx}} d_{ml} \quad \text{.......... (23 } c)$$

$$= \left(\frac{B'}{J}\right)^m \left(\frac{1}{B'}\right)^l e^{\frac{B}{B'}\omega_{yx}} e^{\frac{A'B'}{J}\omega_{xy}} d_{lm} \quad \text{.......... (23 } d).$$

From these results it follows that, in the case of two independent variables, if F is isobaric and of weights p_1, p_2 in first and second suffixes,

$$F(D_{lm}, \ldots) = K\,.\,e^{-\{UV\}}\,.\,e^{\omega_1}\,.\,e^{\omega_2} F(d_{lm}, \ldots) \quad \text{.......... (24 } a),$$

where
$$\{UV\} = \frac{1}{J}[A\{U,\ x\} + A'\{U,\ y\} + B\{V,\ x\} + B'\{V,\ y\}] \quad \text{.......... (24 } b),$$

and the quantity K and the operators ω_1, ω_2 may have either of the two sets of values

$$K = \frac{A^{p_1-p_2}}{J^{p_1}},\quad \omega_1 = \frac{A'}{A}\omega_{xy},\quad \omega_2 = \frac{AB}{J}\omega_{yx} \quad \text{.......... (24 } c),$$

$$K = \frac{B'^{p_2-p_1}}{J^{p_2}},\quad \omega_1 = \frac{B}{B'}\omega_{yx},\quad \omega_2 = \frac{A'B'}{J}\omega_{xy} \quad \text{..........(24 } d).$$

And, as another form,

$$F(D_{lm}, \ldots) = Ke^{-\{UV\}}\,.\,e^{\omega_1}\,.\,e^{\omega_2} F(d_{ml}, \ldots) \quad \text{.......... (25 } a),$$

where K, ω_1, ω_2 may have either of the two sets of values

$$K = (-1)^{p_2} \frac{A'^{p_1-p_2}}{J^{p_1}}, \quad \omega_1 = \frac{A}{A'} \omega_{yx}, \quad \omega_2 = -\frac{A'B'}{J} \omega_{xy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25\ b),$$

$$K = (-1)^{p_1} \frac{B^{p_2-p_1}}{J^{p_2}}, \quad \omega_1 = \frac{B'}{B} \omega_{xy}, \quad \omega_2 = -\frac{AB}{J} \omega_{yx} \ldots\ldots\ldots\ldots\ldots\ldots (25\ c).$$

Here A, A', B, B' are the first minors of J, and therefore $A = b'$, $A' = -b$, $B = -a'$, $B' = a$.

§ 13. In the particular case when there is only one independent variable x which is transformed to u, we have $J = u_x$, $A = 1$,

$$\{U, V, W\} = \{U, x\} = \{U_1 U_{\xi_1}\} + \{U_2 T_{\xi_2}\};$$

there are no ω's, and we have, if F is an isobaric function of weight p,

$$F(D_l, D_l, \ldots) = \frac{1}{u_x{}^p} e^{-\{U, x\}} F(d_l, d_l, \ldots).$$

and

$$D_l = \frac{1}{l!} \frac{\partial^l t}{\partial u^l} = \frac{1}{u_x{}^l} e^{-\{U, x\}} \cdot d_l.$$

This form is not quite the same as that given by Mr Leudesdorf (*Proc. Lond. Math. Soc.*, Vol. XVIII.) and also established in my previous paper (*Trans. Camb. Phil. Soc.*, Vol. XVI.), but one formula can be deduced from the other by the method of the next section.

§ 14. Another form of the general result is often more useful than that stated in equation (21). It is obtained by exhibiting separately the terms containing first differential coefficients of t. For this purpose modified forms of the operators $\{U, x\}$, ... must be used; let $[U, x]$ denote the result of suppressing all terms in $\{U, x\}$ which contain d_{100}, d_{010}, d_{001}, so that $[U, x]$ may be formed in exactly the same way as $\{U, x\}$, except that in the process of formation the value of T used is

$$d_{200}\xi^2 + d_{020}\eta^2 + d_{002}\zeta^2 + d_{110}\xi\eta + \ldots$$

instead of that given in (2). Let $[V, x]$, $[W, x]$, $[U, y]$, ... represent similar modifications of $\{V, x\}$, $\{W, x\}$, $\{U, y\}$, ... Therefore

$$\{U, x\} = [U, x] + d_{100} [U_4],$$

$$\{U, y\} = [U, y] + d_{010} [U_4],$$

$$\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots$$

where $[U_4]$ is formed by replacing $\xi_4{}^p \eta_4{}^q \zeta_4{}^r$ in $a_2 \xi_4{}^2 + \ldots$ with $\dfrac{\partial}{\partial d_{pqr}}$.

The twelve operators $[U_4]$, $[V_4]$, $[W_4]$, $[U, x]$, $[U, y]$, ... are easily seen to be all commutative with one another. For, by § 7,

$$\{U, x\}\{V, y\} - \{V, y\}\{U, x\} = 0;$$

therefore

$$\{[U, x] + d_{100}[U_4]\}\{[V, y] + d_{010}[V_4]\} - \{[V, y] + d_{010}[V_4]\}\{[U, x] + d_{100}[U_4]\} = 0.$$

Hence, by selection of the coefficients of d_{100}, d_{010}, $d_{100}d_{010}$, it follows that

$$[U, x][V, y] - [V, y][U, x] = 0,$$
$$[U_4][V, y] - [V, y][U_4] = 0,$$
$$[U, x][V_4] - [V_4][U, x] = 0,$$
$$[U_4][V_4] - [V_4][U_4] = 0;$$

and in similar fashion it may be proved that the alternants of all other pairs of the operators are zero.

Now, by (19*c*),

$$D_{lmn} = e^{-\{U, V, W\}} \frac{D_1{}^l D_2{}^m D_3{}^n t}{l!\, m!\, n!}.$$

Therefore if $[U, V, W]$ is the modified form of $\{U, V, W\}$, so that

$$[U, V, W] = \frac{1}{J}(A[U, x] + A'[U, y] + A''[U, z])$$
$$+ \frac{1}{J}(B[V, x] + B'[V, y] + B''[V, z])$$
$$+ \frac{1}{J}(C[W, x] + C'[W, y] + C''[W, z]),$$

$$D_{lmn} = e^{-[U, V, W]} \,.\, e^{-\frac{1}{J}(Ad + A'd' + A''d'')[U_4]} \,.\, e^{-\frac{1}{J}(Bd + B'd' + B''d'')[V_4]} \,.\, e^{-\frac{1}{J}(Cd + C'd' + C''d'')[W_4]} \frac{D_1{}^l D_2{}^m D_3{}^n t}{l!\, m!\, n!}.$$

Now $D_1{}^l D_2{}^m D_3{}^n t$ is a linear function of d_{pqr}, ... Therefore the effect of $[U_4]$, $[V_4]$, $[W_4]$ operating on $D_1{}^l D_2{}^m D_3{}^n t$ is to change d_{pqr} into a_{pqr}, b_{pqr}, c_{pqr} and therefore to produce

$$D_1{}^l D_2{}^m D_3{}^n u, \quad D_1{}^l D_2{}^m D_3{}^n v, \quad D_1{}^l D_2{}^m D_3{}^n w;$$

whilst repeated operations by $[U_4]$, $[V_4]$, $[W_4]$ produce zero results.

Hence

$$D_{lmn} = e^{-[U, V, W]} \frac{1}{J}[J \,.\, D_1{}^l D_2{}^m D_3{}^n t - (Ad + A'd' + A''d'') D_1{}^l D_2{}^m D_3{}^n u$$
$$- (Bd + B'd' + B''d'') D_1{}^l D_2{}^m D_3{}^n v - (Cd + C'd' + C''d'') D_1{}^l D_2{}^m D_3{}^n w]$$

$$= \frac{1}{J} e^{-[U, V, W]} \begin{vmatrix} D_1{}^l D_2{}^m D_3{}^n t, & D_1{}^l D_2{}^m D_3{}^n u, & D_1{}^l D_2{}^m D_3{}^n v, & D_1{}^l D_2{}^m D_3{}^n w \\ d, & a, & b, & c \\ d', & a', & b', & c' \\ d'', & a'', & b'', & c'' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(26).$$

§ 15. Up to the present there has been no restriction on l, m, n except that they be not all zero; the last formula holds when $l+m+n=1$ on the understanding, assumed throughout, that $D_1^l D_2^m D_3^n u$, $D_1^l D_2^m D_3^n v$, $D_1^l D_2^m D_3^n w$ all vanish when $l+m+n=1$. But it is necessary to assume, in what follows, that $l+m+n>1$.

Corresponding to the operators ω of § 10 we introduce six operators Ω given by the equations

$$\Omega_{xy} = [\xi_1 U_{\eta_1}] + [\xi_2 V_{\eta_2}] + [\xi_3 W_{\eta_3}] + [\xi_4 T_{\eta_4}],$$
$$\Omega_{yx} = [\eta_1 U_{\xi_1}] + [\eta_2 V_{\xi_2}] + [\eta_3 W_{\xi_3}] + [\eta_4 T_{\xi_4}],$$
$$\Omega_{xz} = [\xi_1 U_{\zeta_1}] + [\xi_2 V_{\zeta_2}] + [\xi_3 W_{\zeta_3}] + [\xi_4 T_{\zeta_4}],$$
$$\Omega_{zx} = [\zeta_1 U_{\xi_1}] + [\zeta_2 V_{\xi_2}] + [\zeta_3 W_{\xi_3}] + [\zeta_4 T_{\xi_4}],$$
$$\Omega_{yz} = [\eta_1 U_{\zeta_1}] + [\eta_2 V_{\zeta_2}] + [\eta_3 W_{\zeta_3}] + [\eta_4 T_{\zeta_4}],$$
$$\Omega_{zy} = [\zeta_1 U_{\eta_1}] + [\zeta_2 V_{\eta_2}] + [\zeta_3 W_{\eta_3}] + [\zeta_4 T_{\eta_4}],$$

where
$$U_1 = a_{200}\xi_1^2 + \dots, \qquad V_2 = b_{200}\xi_2^2 + \dots,$$
$$W_3 = c_{200}\xi_3^2 + \dots, \qquad T_4 = d_{200}\xi_4^2 + \dots,$$
$$U_{\eta_1} = \frac{\partial}{\partial \eta_1} U_1, \qquad V_{\eta_2} = \frac{\partial}{\partial \eta_2} V_2, \dots,$$

and after expansion of the expressions $[\xi_1 U_{\eta_1}], \dots$, $\xi_1^p \eta_1^q \zeta_1^r$ is replaced by $\frac{\partial}{\partial a_{pqr}}$,

$$\xi_2^p \eta_2^q \zeta_2^r \text{ by } \frac{\partial}{\partial b_{pqr}}, \quad \xi_3^p \eta_3^q \zeta_3^r \text{ by } \frac{\partial}{\partial c_{pqr}}, \quad \xi_4^p \eta_4^q \zeta_4^r \text{ by } \frac{\partial}{\partial d_{pqr}}.$$

The four components of each operator are independent of one another and therefore commutative with one another; but as in § 10 the Ω's are not all commutative. In fact, applying the results of § 10 to corresponding pairs of the partial operators, we find the alternants of various pairs of Ω's to be

$$\Omega_{xy}\Omega_{yx} - \Omega_{yx}\Omega_{xy} = -[\xi_1 U_{\xi_1}] - [\xi_2 V_{\xi_2}] - [\xi_3 W_{\xi_3}] - [\xi_4 T_{\xi_4}] + [\eta_1 U_{\eta_1}] + [\eta_2 V_{\eta_2}] + [\eta_3 W_{\eta_3}] + [\eta_4 T_{\eta_4}],$$
$$\Omega_{xy}\Omega_{xz} - \Omega_{xz}\Omega_{xy} = 0,$$
$$\Omega_{xy}\Omega_{zx} - \Omega_{zx}\Omega_{xy} = \Omega_{zy},$$
$$\Omega_{xy}\Omega_{yz} - \Omega_{yz}\Omega_{xy} = -\Omega_{xz},$$
$$\Omega_{xy}\Omega_{zy} - \Omega_{zy}\Omega_{xy} = 0.$$

From these relations deductions can be made similar to those in § 10.

We next write

$$\Omega_1 = \frac{A'}{A}\Omega_{xy} + \frac{A''}{A}\Omega_{xz},$$
$$\Omega_2 = \frac{AB}{Jc''}\Omega_{yx} - \frac{c'}{c''}\Omega_{yz},$$
$$\Omega_3 = -\frac{a''A}{J}\Omega_{zx} - \frac{b''c''}{A}\Omega_{zy}.$$

Then
$$\frac{D_1^l D_2^m D_3^n t}{l!\ m!\ n!} = \frac{A^{l-m} c''^{m-n}}{J^l} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} d_{lmn},$$
$$\frac{D_1^l D_2^m D_3^n u}{l!\ m!\ n!} = \frac{A^{l-m} c''^{m-n}}{J^l} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} a_{lmn}, \ldots$$

Therefore
$$D_{lmn} = \frac{A^{l-m} c''^{m-n}}{J^{l-1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} \begin{matrix} d_{lmn} & a_{lmn} & b_{lmn} & c_{lmn} \\ d & a & b & c \\ d' & a' & b' & c' \\ d'' & a'' & b'' & c'' \end{matrix} \qquad \text{..............(27).}$$

Now write
$$J_1 = \begin{vmatrix} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \end{vmatrix} = Ad + A'd' + A''d'',$$
$$J_2 = \begin{vmatrix} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \end{vmatrix} = Bd + B'd' + B''d'',$$
$$J_3 = \begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix} = Cd + C'd' + C''d'';$$

therefore
$$D_{lmn} = \frac{A^{l-m} c''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} (Jd_{lmn} - J_1 a_{lmn} - J_2 b_{lmn} - J_3 c_{lmn}) \text{..............(28),}$$

and as in § 11, if F denotes a homogeneous function of degree i, which is also isobaric of partial weights p_1, p_2, p_3,
$$F(D_{lmn}, \ldots) = \frac{A^{p_1-p_2} c''^{p_2-p_3}}{J^{i+p_1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} F(Jd_{lmn} - J_1 a_{lmn} - J_2 b_{lmn} - J_3 c_{lmn}, \ldots) \quad \text{...(29),}$$

or using the operators $[U_4]$, $[V_4]$, $[W_4]$ defined in § 14
$$F(D_{lmn}, \ldots) = \frac{A^{p_1-p_2} c''^{p_2-p_3}}{J^{p_1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} e^{-\frac{J_1}{J}[U_4]} e^{-\frac{J_2}{J}[V_4]} e^{-\frac{J_3}{J}[W_4]} F(d_{lmn}, \ldots) \text{.........(30).}$$

This last form does not require F to be homogeneous, though it must be isobaric.

§ 16. If in (27) we put $t = x$, y, z in succession we obtain formulae for the interchange of the dependent and independent variables. Write
$$A_{lmn} = \frac{1}{l!\ m!\ n!} \frac{\partial^{l+m+n} x}{\partial u^l \partial v^m \partial w^n},$$
$$B_{lmn} = \frac{1}{l!\ m!\ n!} \frac{\partial^{l+m+n} y}{\partial u^l \partial v^m \partial w^n},$$
$$C_{lmn} = \frac{1}{l!\ m!\ n!} \frac{\partial^{l+m+n} z}{\partial u^l \partial v^m \partial w^n}.$$

Then, provided $l+m+n>1$,

$$A_{lmn}=-\frac{A^{l-m}c''^{m-n}}{J^{l+1}}e^{-[U,\,V,\,W]}\,.\,e^{\Omega_1}\,.\,e^{\Omega_2}\,.\,e^{\Omega_3}(Aa_{lmn}+Bb_{lmn}+Cc_{lmn})$$

$$=-\frac{A^{l-m}c''^{m-n}}{J^{l+1}}e^{-[U,\,V,\,W]}\,.\,e^{\Omega_1}\,.\,e^{\Omega_2}\,.\,e^{\Omega_3}\begin{vmatrix}a_{lmn} & b_{lmn} & c_{lmn}\\ a' & b' & c'\\ a'' & b'' & c''\end{vmatrix}\quad\text{......(31}a\text{)},$$

$$B_{lmn}=-\frac{A^{l-m}c''^{m-n}}{J^{l+1}}e^{-[U,\,V,\,W]}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}(A'a_{lmn}+B'b_{lmn}+C'c_{lmn})$$

$$=-\frac{A^{l-m}c''^{m-n}}{J^{l+1}}e^{-[U,\,V,\,W]}e^{\Omega_1}\,.\,e^{\Omega_2}\,.\,e^{\Omega_3}\begin{vmatrix}a & b & c\\ a_{lmn} & b_{lmn} & c_{lmn}\\ a'' & b'' & c''\end{vmatrix}\quad\text{.........(31}b\text{)},$$

$$C_{lmn}=-\frac{A^{l-m}c''^{m-n}}{J^{l+1}}e^{-[U,\,V,\,W]}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}(A''a_{lmn}+B''b_{lmn}+C''c_{lmn})$$

$$=-\frac{A^{l-m}c''^{m-n}}{J^{l+1}}e^{-[U,\,V,\,W]}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}\begin{vmatrix}a & b & c\\ a' & b' & c'\\ a_{lmn} & b_{lmn} & c_{lmn}\end{vmatrix}\quad\text{...........(31}c\text{)}.$$

And if $F(D_{lmn}, A_{lmn}, B_{lmn}, C_{lmn}, \ldots)$ is a function homogeneous of degree i in $A_{lmn}, B_{lmn}, C_{lmn}, D_{lmn}, \ldots$ and isobaric of weights p_1, p_2, p_3 in first, second and third suffixes, we have

$$F(D_{lmn}, A_{lmn}, B_{lmn}, C_{lmn}, \ldots)=(-1)^i\frac{A^{p_1-p_2}c''^{p_2-p_3}}{J^{i+p_1}}e^{-[U,\,V,\,W]}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}$$

$$.\,F(J_1a_{lmn}+J_2b_{lmn}+J_3c_{lmn}-Jd_{lmn},\ Aa_{lmn}+Bb_{lmn}+Cc_{lmn},$$

$$A'a_{lmn}+B'b_{lmn}+C'c_{lmn},\ A''a_{lmn}+B''b_{lmn}+C''c_{lmn}, \ldots)\quad\text{...........(32)}.$$

Since there are no d's occurring in equations (31a), (31b), (31c) the operators occurring in these equations, but not in (32), may be simplified by the omission of differential operators which affect only d's. Thus $[U_4T_{\xi_4}]$ may be omitted from $[U, x]$, $[\xi_4T_{\eta_4}]$ from $\Omega_{xy}, \ldots$.

It will be noted that $A_{lmn}, B_{lmn}, C_{lmn}$ are the coefficients of $\upsilon^l\nu^m\omega^n$ in the expansions of ξ, η, ζ when the series

$$\upsilon=a_{100}\xi+\ldots+a_{lmn}\xi^l\eta^m\zeta^n+\ldots,$$

$$\nu=b_{100}\xi+\ldots+b_{lmn}\xi^l\eta^m\zeta^n+\ldots,$$

$$\omega=c_{100}\xi+\ldots+c_{lmn}\xi^l\eta^m\zeta^n+\ldots,$$

are reversed, and ξ, η, ζ expanded in powers of υ, ν, ω.

§ 17. The formulae of § 15 may be adapted so as to give a symbolical form for the differential coefficient of an implicit function. The method is applicable to any number of variables, but for the sake of brevity the work will here be restricted to the case when there is only one dependent and one independent variable.

As in § 16, let

$$u = F(x, y), \quad v = G(x, y):$$

then formula (31*b*) shows how to determine B_{0m} or $\frac{1}{m!}\frac{\partial^m y}{\partial v^m}$; this differential coefficient is obtained on the assumption that u is constant, so that if we take $G(x, y) \equiv x$, we shall obtain $\frac{1}{m!}\frac{d^m y}{dx^m}$ on the assumption that x, y are connected by the equation $F(x, y) = \text{const.}$ As in § 16 let a_{pq} stand for

$$\frac{1}{p!\,q!}\frac{\partial^{p+q} F}{\partial x^p \partial y^q};$$

the b's of § 16 are in this case all zero except b_{10} which is equal to unity. Now we have

$$J = \begin{vmatrix} a_{10} & a_{01} \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} a & a' \\ 1 & 0 \end{vmatrix} = -a_{01}$$

so that $A = 0$, and therefore the forms of Ω_1, Ω_2 used in (27) are not applicable. Instead of these forms we may use forms similar to those given in (25*a*, *b*) and obtain

$$K = 1, \quad \Omega_1 = 0, \quad \Omega_2 = -\frac{a_{10}}{a_{01}}\Omega_{xy}.$$

We have therefore, if $m > 1$,

$$B_{0m} = \frac{1}{a_{01}} e^{-[U, V]} e^{-\frac{a_{10}}{a_{01}}\Omega_{xy}} \begin{vmatrix} a_{10} & 1 \\ a_{m0} & b_{m0} \end{vmatrix}.$$

Now, in general, when dealing with special values of the letters, it is necessary to carry out all the operations indicated and then substitute the special values. But in the present case, where all the b's involved in the operators are zero, it is allowable to suppress in the operators all terms which involve b's; for it is obvious from the form of the operators that they never diminish the degree of any function in b's, though they may increase the degree. It therefore follows that the terms which arise from the b-parts of the operators will all be zero. We therefore have

$$[U, V] = \frac{A'}{J}[U, y] = \frac{1}{a_{01}}[FF_\eta]$$

$$= \frac{1}{a_{01}}[(a_{20}\xi^2 + a_{11}\xi\eta + a_{02}\eta^2 + a_{30}\xi^3 + \ldots)(a_{11}\xi + 2a_{02}\eta + a_{21}\xi^2 + \ldots)],$$

$$\Omega_{xy} = [\xi F_\eta] = [\xi(a_{11}\xi + 2a_{02}\eta + a_{21}\xi^2 + \ldots)],$$

on the usual understanding that $\xi^p\eta^q$ is replaced by $\frac{\partial}{\partial a_{pq}}$. Hence finally the value of $\frac{d^m y}{dx^m}$ as found from the equation $F(x, y) = 0$ is given by

$$B_{0m} = \frac{1}{m!}\frac{d^m y}{dx^m} = -\frac{1}{a_{01}} e^{-\frac{1}{a_{01}}[FF_\eta]} \cdot e^{-\frac{a_{10}}{a_{01}}[\xi F_\eta]} \cdot a_{m0} \quad \text{......................(33).}$$

§ 18. The determination of the differential coefficients of implicit functions is equivalent to the solution of equations by series, so that the method of the last section leads to a symbolical form for the solution of a set of equations of infinite degree. It will be sufficient to illustrate the method by considering the case of a single equation,

$$0 = F(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + \dots;$$

it is required to determine that value of y which vanishes when x vanishes. The solution is

$$y = B_{01}x + B_{02}x^2 + \dots$$

where $B_{01} = -\frac{a_{10}}{a_{01}}$, and B_{0m} is given by (33). Now let P denote the terms of $F(x, y)$ which are independent of y; then the required solution of the equation $F(x, y) = 0$ may be written

$$y = -\frac{1}{a_{01}} e^{-\frac{1}{a_{01}}[FF_\eta]} \cdot e^{-\frac{a_{10}}{a_{01}}[\xi F_\eta]} P \quad \text{..............................(34),}$$

where the operators $[FF_\eta]$, $[\xi F_\eta]$ are the same as in the last section. For an equation of finite degree n it is necessary to suppose all the operations carried out, and then all the coefficients a_{pq} for which $p + q > n$ must be made zero.

If f denotes any rational integral function

$$f(y) = e^{-\frac{1}{a_{01}}[FF_\eta]} \cdot e^{-\frac{a_{10}}{a_{01}}[\xi F_\eta]} f\left(-\frac{P}{a_{01}}\right) \quad \text{..........................(34a).}$$

§ 19. As an illustration of the general methods established, we will employ them to effect the change when the variables are linearly transformed. Let the scheme of transformation be

$$X = \alpha x + \alpha' y + \alpha'' z + \alpha''' t,$$

$$Y = \beta x + \beta' y + \beta'' z + \beta''' t,$$

$$Z = \gamma x + \gamma' y + \gamma'' z + \gamma''' t,$$

$$T = \delta x + \delta' y + \delta'' z + \delta''' t,$$

and T being regarded as the dependent variables. Let

$$d_{lmn} = \frac{1}{l!\, m!\, n!} \frac{\partial^{l+m+n} t}{\partial x^l \partial y^m \partial z^n},$$

$$D_{lmn} = \frac{1}{l!\, m!\, n!} \frac{\partial^{l+m+n} T}{\partial X^l \partial Y^m \partial Z^n}.$$

It is required to express D_{lmn} in terms of $d_{lmn}, d_{l'm'n'}, \ldots$. In the formulae T, X, Y, Z are to be written for t, u, v, w so that $a, b, c, \ldots$ will denote first differential coefficients of T, not t, and therefore

$$a = \alpha + \alpha''' t_x, \quad b = \beta + \beta''' t_x, \ldots$$

Now equation (27) shows that by writing X, Y, Z, T for u, v, w, t the value of D_{lmn} can be found in terms of differential coefficients of X, Y, Z, T with respect to x, y, z and therefore expressed in terms of $t_{pqr}, \ldots$. But the operators $[U, x], \ldots$ which produce the expression can be replaced by others involving differential operators $\frac{\partial}{\partial t_{pqr}}, \ldots$.

For the operator $[U, x]$ or $[X, x]$ may be written

$$\Sigma \Delta_x^p \Delta_y^q \Delta_z^r (XX_x) \frac{\partial}{\partial X_{pqr}} + \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (XY_x) \frac{\partial}{\partial Y_{pqr}} + \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (XZ_x) \frac{\partial}{\partial Z_{pqr}}$$

$$+ \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (XT_x) \frac{\partial}{\partial T_{pqr}},$$

where $p+q+r>2$.

Now since all first differential coefficients are removed after operation with Δ's it is obvious that in $\Delta_x^p \Delta_y^q \Delta_z^r (XX_x), \ldots$ X, Y, Z, T may be replaced by $\alpha''' t, \beta''' t, \gamma''' t, \delta''' t$. Moreover, if $p+q+r \nless 2$,

$$X_{pqr} = \alpha''' t_{pqr}, \quad Y_{pqr} = \beta''' t_{pqr}, \quad Z_{pqr} = \gamma''' t_{pqr}, \quad T_{pqr} = \delta''' t_{pqr};$$

and therefore, for operations on a function of $X_{pqr}, Y_{pqr}, Z_{pqr}, T_{pqr}, \ldots$,

$$\frac{\partial}{\partial t_{pqr}} = \alpha''' \frac{\partial}{\partial X_{pqr}} + \beta''' \frac{\partial}{\partial Y_{pqr}} + \gamma''' \frac{\partial}{\partial Z_{pqr}} + \delta''' \frac{\partial}{\partial T_{pqr}}.$$

Also

$$\Sigma \Delta_x^p \Delta_y^q \Delta_z^r (XX_x) = \alpha'''^2 \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_x),$$

$$\Sigma \Delta_x^p \Delta_y^q \Delta_z^r (XY_x) = \alpha''' \beta''' \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_x), \text{ \&c.}$$

Therefore $[U, x]$ becomes

$$\alpha''' \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_x) \left[\alpha''' \frac{\partial}{\partial X_{pqr}} + \beta''' \frac{\partial}{\partial Y_{pqr}} + \gamma''' \frac{\partial}{\partial Z_{pqr}} + \delta''' \frac{\partial}{\partial T_{pqr}}\right]$$

$$= \alpha''' \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_x) \frac{\partial}{\partial t_{pqr}}.$$

Now denote the operators

$$\Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_x) \frac{\partial}{\partial t_{pqr}}, \quad \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_y) \frac{\partial}{\partial t_{pqr}}, \quad \Sigma \Delta_x^p \Delta_y^q \Delta_z^r (tt_z) \frac{\partial}{\partial t_{pqr}}$$

by V_1, V_2, V_3. When working with d_{pqr} instead of t_{pqr} it will be more convenient to form the operators V by writing

$$V_1 = [\tau \tau_\xi], \quad V_2 = [\tau \tau_\eta], \quad V_3 = [\tau \tau_\zeta],$$

where

$$\tau = d_{200} \xi^2 + d_{002} \eta^2 + \ldots$$

and after the algebraical multiplications $\xi^p \eta^q \zeta^r$ is replaced by $\frac{\partial}{\partial d_{pqr}}$.

We shall then find that $[U, x]$, $[U, y]$, $[U, z]$, $[V, x]$, ... become $\alpha''' V_1$, $\alpha''' V_2$, $\alpha''' V_3$, $\beta''' V_1$, ...; and it finally appears that $[U, V, W]$ becomes

$$\frac{A\alpha''' + B\beta''' + C\gamma'''}{J} V_1 + \frac{A'\alpha''' + B'\beta''' + C'\gamma'''}{J} V_2 + \frac{A''\alpha''' + B''\beta''' + C''\gamma'''}{J} V_3 = V, \text{ say,}$$

so that

$$V = \frac{1}{J}\begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ a & a' & a'' & \alpha''' \\ b & b' & b'' & \beta''' \\ c & c' & c'' & \gamma''' \end{vmatrix}$$

$$= \frac{1}{J}\begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ \alpha + \alpha''' t_x, & \alpha' + \alpha''' t_y, & \alpha'' + \alpha''' t_z, & \alpha''' \\ \beta + \beta''' t_x, & \beta' + \beta''' t_y, & \beta'' + \beta''' t_z, & \beta''' \\ \gamma + \gamma''' t_x, & \gamma' + \gamma''' t_y, & \gamma'' + \gamma''' t_z, & \gamma''' \end{vmatrix}$$

$$= \frac{1}{J}\begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ \alpha & \alpha' & \alpha'' & \alpha''' \\ \beta & \beta' & \beta'' & \beta''' \\ \gamma & \gamma' & \gamma'' & \gamma''' \end{vmatrix} \quad \text{...(35).}$$

When there are n independent variables the corresponding formula for V is

$$V = \frac{(-1)^{n-1}}{J}\begin{vmatrix} V_1, & V_2, \ldots V_n & , & 0 \\ \alpha \;, & \alpha' \;, \ldots \alpha^{(n-1)}, & & \alpha^{(n)} \\ \beta \;, & \beta' \;, \ldots \beta^{(n-1)}, & & \beta^{(n)} \\ \gamma \;, & \gamma' \;, \ldots \gamma^{(n-1)}, & & \gamma^{(n)} \\ \ldots\ldots & \ldots\ldots\ldots & & \end{vmatrix} \quad \text{...............................(36).}$$

The formula (26) then gives

$$D_{lmn} = \frac{A^{l-m}c''^{m-n}}{J^{l+1}} e^{-V}\begin{vmatrix} D_1{}^l D_2{}^m D_3{}^n T, & D_1{}^l D_2{}^m D_3{}^n X, & D_1{}^l D_2{}^m D_3{}^n Y, & D_1{}^l D_2{}^m D_3{}^n Z \\ T_x & X_x & Y_x & Z_x \\ T_y & X_y & Y_y & Z_y \\ T_z & X_z & Y_z & Z_z \end{vmatrix} \quad \text{......(37).}$$

Now, $l + m + n$ being greater than unity, the determinant becomes

$$\begin{vmatrix} \delta''' & \alpha''' & \beta''' & \gamma''' & D_1{}^l D_2{}^m D_3{}^n t \\ \delta + \delta''' t_x & \alpha + \alpha''' t_x & \beta + \beta''' t_x & \gamma + \gamma''' t_x & \\ \delta' + \delta''' t_y & \alpha' + \alpha''' t_y & \beta' + \beta''' t_y & \gamma' + \gamma''' t_y & \\ \delta'' + \delta''' t_z & \alpha'' + \alpha''' t_z & \beta'' + \beta''' t_z & \gamma'' + \gamma''' t_z & \end{vmatrix}$$

$$= \begin{vmatrix} \delta''' & \alpha''' & \beta''' & \gamma''' \\ \delta & \alpha & \beta & \gamma \\ \delta' & \alpha' & \beta' & \gamma' \\ \delta'' & \alpha'' & \beta'' & \gamma'' \end{vmatrix} D_1^l D_2^m D_3^n t$$

$$= \begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ \alpha'' & \beta'' & \gamma'' & \delta'' \\ \alpha''' & \beta''' & \gamma''' & \delta''' \end{vmatrix} D_1^l D_2^m D_3^n t$$

$$= M \,.\, D_1^l D_2^m D_3^n t, \text{ say,}$$

so that M is the modulus of the linear transformation.

To transform $D_1^l D_2^m D_3^n t$ we use the operators

$$\omega_{xy} = [\xi\tau_\eta], \qquad \omega_{xz} = [\xi\tau_\zeta],$$

$$\omega_{yx} = [\eta\tau_\xi], \qquad \omega_{yz} = [\eta\tau_\zeta],$$

$$\omega_{zx} = [\zeta\tau_\xi], \qquad \omega_{zy} = [\zeta\tau_\eta],$$

where $\tau = d_{200}\xi^2 + d_{020}\eta^2 + \dots$, and the operators are formed in the usual way by replacing $\xi^p\eta^q\zeta^r$ with $\dfrac{\partial}{\partial d_{pqr}}$. The properties of these ω's are precisely similar to those of the ω's investigated in § 10, and just as in previous cases we find that if we write

$$\Omega_1 = \frac{A'}{A}\,\omega_{xy} + \frac{A''}{A}\,\omega_{xz},$$

$$\Omega_2 = \frac{AB}{Jc''}\,\omega_{yx} - \frac{c'}{c''}\,\omega_{yz},$$

$$\Omega_3 = -\frac{a''A}{J}\,\omega_{zx} - \frac{b''c''}{A}\,\omega_{zy},$$

we shall have finally

$$D_{lmn} = M\,\frac{A^{l-m}c''^{m-n}}{J^{l+1}}\,e^{-V}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}\,d_{lmn} \qquad (38).$$

And if $F(d_{lmn}, d_{l'm'n'}, \dots)$ is a pure homogeneous function of degree i and isobaric of weights p_1, p_2, p_3 in first, second and third suffixes,

$$F(D_{lmn}, D_{l'm'n'}, \dots) = \frac{M^i}{J^{i+p_1}}\,A^{p_1-p_2}c''^{p_2-p_3}e^{-V}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}\,F(d_{lmn}, d_{l'm'n'}, \dots) \qquad (39).$$

The value of J is

$$\begin{vmatrix} \alpha + \alpha'''d, & \alpha' + \alpha'''d', & \alpha'' + \alpha'''d'' \\ \beta + \beta'''d, & \beta' + \beta'''d', & \beta'' + \beta'''d'' \\ \gamma + \gamma'''d, & \gamma' + \gamma'''d', & \gamma'' + \gamma'''d'' \end{vmatrix}$$

and A, B, C, ... are the minors of this determinant. It will be noticed that products of d, d', d'' do not appear in J, which is therefore a linear function of these quantities. In fact another form is

$$J = \begin{vmatrix} \alpha, & \alpha', & \alpha'', & \alpha''' \\ \beta, & \beta', & \beta'', & \beta''' \\ \gamma, & \gamma', & \gamma'', & \gamma''' \\ -d, & -d', & -d'', & 1 \end{vmatrix}$$

$$= \Delta''' - \Delta d - \Delta' d' - \Delta'' d'' \quad \text{.................................(40),}$$

where Δ, Δ', Δ'', Δ''' are the minors of δ, δ', δ'', δ''' in the determinant M.

§ 20. As in § 12 the part of the operator which depends on the ω's may be expressed in different forms.

For instance in the case of two independent variables we have

$$F(D_{lm}, \ldots) = Ke^{-V}e^{\Omega_1}e^{\Omega_2}F(d_{lm}, \ldots) \quad \text{............................(41),}$$

where

$$V = -\frac{1}{J}\begin{vmatrix} V_1 & V_2 & 0 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{vmatrix} \quad \text{............................(41}a\text{),}$$

and K, Ω_1, Ω_2 may have either of the two sets of values

$$K = \frac{M^i A^{p_1 - p_2}}{J^{p_1 + i}}, \quad \Omega_1 = \frac{A'}{A}\omega_{xy}, \quad \Omega_2 = \frac{AB}{J}\omega_{yx} \text{...................(41}b\text{);}$$

$$K = \frac{M^i B'^{p_2 - p_1}}{J^{p_2 + i}}, \quad \Omega_1 = \frac{B}{B'}\omega_{yx}, \quad \Omega_2 = \frac{A'B'}{J}\omega_{xy} \quad \text{................(41}c\text{).}$$

And, equally well,

$$F(D_{lm}, \ldots) = Ke^{-V}e^{\Omega_1}e^{\Omega_2}F(d_{ml}, \ldots) \quad \text{.........................(41}d\text{),}$$

where V is as before, and K, Ω_1, Ω_2 may have either of the two sets of values

$$K = (-1)^{p_2}\frac{M^i A'^{p_1 - p_2}}{J^{p_1 + i}}, \quad \Omega_1 = \frac{A}{A'}\omega_{yx}, \quad \Omega_2 = -\frac{A'B'}{J}\omega_{xy} \quad \text{..............(41}e\text{);}$$

$$K = (-1)^{p_1}\frac{M^i B^{p_2 - p_1}}{J^{p_2 + i}}, \quad \Omega_1 = \frac{B'}{B}\omega_{xy}, \quad \Omega_2 = -\frac{AB}{J}\omega_{yx} \quad \text{..............(41}f\text{).}$$

For example, suppose it is required to change the variables cyclically so that t, x, y are changed to x, y, t and x is the new dependent variable. Let x_{lm} stand for $\frac{1}{l!\,m!}\frac{\partial^{l+m}x}{\partial y^l \partial t^m}$.

The scheme of transformation is

$$X = 0\,.\,x + 1\,.\,y + 0\,.\,t,$$
$$Y = 0\,.\,x + 0\,.\,y + 1\,.\,t,$$
$$T = 1\,.\,x + 0\,.\,y + 0\,.\,t.$$

Therefore
$$J = \begin{vmatrix} 0 & 1 \\ t_x & t_y \end{vmatrix} = -t_x, \quad M = 1,$$

$$V = -\frac{1}{J}\begin{vmatrix} V_1 & V_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{V_1}{t_x}.$$

Here $B' = 0$ and therefore the form (41c) is not applicable, but the form (41b) gives

$$F(x_{lm}, \ldots) = \frac{t_y^{p_1 - p_2}}{(-t_x)^{p_1 + i}} e^{-\frac{V_1}{t_x}} e^{-\frac{t_x}{t_y}\omega_{xy}} e^{\frac{t_y}{t_x}\omega_{yx}} F(d_{lm}, \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots (42a);$$

and the forms (41e), (41f) give identical results, viz.,

$$F(x_{lm}, \ldots) = (-1)^{p_2}\frac{(-t_x)^{p_1 - p_2}}{(-t_x)^{i + p_1}} e^{-\frac{V_1}{t_x}} e^{-\frac{t_y}{t_x}\omega_{yx}} F(d_{ml}, \ldots)$$

$$= (-1)^i \frac{1}{t_x^{i + p_2}} e^{-\frac{V_1}{t_x}} e^{-\frac{t_y}{t_x}\omega_{yx}} F(d_{ml}, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42b).$$

If it is required to make the second cyclical change from t, x, y to y, t, x so that y is the dependent variable, let $y_{lm} = \dfrac{1}{l!\,m!}\dfrac{\partial^{l+m} y}{\partial t^l \partial x^m}$. Then the scheme of transformation is

$$X = 0 \,.\, x + 0 \,.\, y + 1 \,.\, t,$$
$$Y = 1 \,.\, x + 0 \,.\, y + 0 \,.\, t,$$
$$T = 0 \,.\, x + 1 \,.\, y + 0 \,.\, t.$$

Therefore
$$J = \begin{vmatrix} t_x & t_y \\ 1 & 0 \end{vmatrix} = -t_y,$$
$$M = 1,$$
$$V = -\frac{1}{J}\begin{vmatrix} V_1 & V_2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \frac{V_2}{t_y}.$$

Here $A = 0$, and the form (41b) is not applicable, but the other forms give

$$F(y_{lm}, \ldots) = \frac{(t_x)^{p_2 - p_1}}{(-t_y)^{p_2 + i}} e^{-\frac{V_2}{t_y}} e^{-\frac{t_y}{t_x}\omega_{yx}} e^{\frac{t_x}{t_y}\omega_{xy}} F(d_{lm}, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (43a)$$

and

$$F(y_{lm}, \ldots) = (-1)^{p_2}\frac{(-1)^{p_1 - p_2}}{(-t_y)^{i + p_1}} e^{-\frac{V_2}{t_y}} e^{-\frac{t_x}{t_y}\omega_{xy}} F(d_{ml}, \ldots)$$

$$= (-1)^i \frac{1}{t_y^{i + p_1}} e^{-\frac{V_2}{t_y}} e^{-\frac{t_x}{t_y}\omega_{xy}} F(d_{ml}, \ldots)$$

$$= (-1)^i \frac{1}{t_{01}^{i + p_1}} e^{-\frac{V_2}{t_{01}}} e^{-\frac{t_{10}}{t_{01}}\omega_{xy}} F(d_{ml}, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43b).$$

§ 21. The alternants of the operators ω with one another have been already examined in § 10; it remains to examine the alternants of the operators V combined with ω's. These alternants have, in the case of two independent variables, been given by Prof. Elliott (*Proc. Lond. Math. Soc.*, Vol. XIX. p. 9); but the proof is much simplified by making full use of the symbolical form for the operators. Only typical cases sufficient to establish the general results will be considered. We have

$$V_1 = [TT_\xi], \quad \omega_{xy} = [\xi T_\eta], \quad \omega_{yx} = [\eta T_\xi], \quad \omega_{yz} = [\eta T_\zeta],$$

where $T = d_{200}\xi^2 + \ldots$, and after multiplication $\xi^p\eta^q\zeta^r$ is replaced by $\dfrac{\partial}{\partial d_{pqr}}$.

Therefore
$$V_1V_2 - V_2V_1 = [TT_\xi][TT_\eta] - [TT_\eta][TT_\xi]$$
$$= \left[TT_\xi T_\eta + T\frac{\partial}{\partial\eta}(TT_\xi)\right] - \left[TT_\eta T_\xi + T\frac{\partial}{\partial\xi}(TT_\eta)\right]$$
$$= 0 \quad\ldots\ldots(44\,a),$$

$$V_1\omega_{xy} - \omega_{xy}V_1 = [TT_\xi][\xi T_\eta] - [\xi T_\eta][TT_\xi]$$
$$= \left[\xi\frac{\partial}{\partial\eta}(TT_\xi)\right] - \left[\xi T_\eta T_\xi + T\frac{\partial}{\partial\xi}(\xi T_\eta)\right]$$
$$= -[TT_\eta] = -V_2 \quad\ldots\ldots(44\,b),$$

$$V_1\omega_{yx} - \omega_{yx}V_1 = [TT_\xi][\eta T_\xi] - [\eta T_\xi][TT_\xi]$$
$$= \left[\eta\frac{\partial}{\partial\xi}(TT_\xi)\right] - \left[\eta T_\xi T_\xi + T\frac{\partial}{\partial\xi}(\eta T_\xi)\right]$$
$$= 0 \quad\ldots\ldots(44\,c),$$

$$V_1\omega_{yz} - \omega_{yz}V_1 = [TT_\xi][\eta T_\zeta] - [\eta T_\zeta][TT_\xi]$$
$$= \left[\eta\frac{\partial}{\partial\zeta}(TT_\xi)\right] - \left[\eta T_\zeta T_\xi + T\frac{\partial}{\partial\xi}(\eta T_\zeta)\right]$$
$$= 0 \quad\ldots\ldots(44\,d).$$

Similarly
$$V_1\omega_{xz} - \omega_{xz}V_1 = -V_3 \quad\ldots\ldots(44\,e),$$

and generally V_1 or $[TT_\xi]$ is commutative with all ω's except those which have x for the first suffix, whilst all the V's are commutative with one another.

§ 22. The applications to the theory of pure cyclicants are easily made. A cyclicant is defined as a function of differential coefficients which is unaltered when the dependent and independent variables are interchanged in any way whatever, save for the introduction of a factor which involves first differential coefficients only. The cyclicant is pure if it involves no first differential coefficients.

In the case of three independent variables, and the method will be perfectly general for any number, we shall show that if the function, supposed pure, is invariable save for the factor mentioned above, when t is interchanged with x and y, z are unaltered, and also unaltered when t is interchanged with y and z, x are unaltered, and also when t is interchanged with z and x, y are unaltered, then the function is invariable when any interchange whatever is made, and moreover the function is invariable when the general linear transformation is applied. We shall find that the necessary and sufficient conditions for the invariance of a homogeneous and isobaric function are that it be annihilated by the three operators V_1, V_2, V_3 and the six operators ω_{xy}, ω_{xz}, ω_{yx}, ω_{yz}, ω_{zx}, ω_{zy}. These conditions, though necessary and sufficient, are not independent. For it is evident, as in § 10, that annihilation by three ω's such as ω_{yz}, ω_{zx}, ω_{xy} will ensure annihilation by the remaining ω's, and it is proved in § 21, that annihilation by the ω's and V_1 will ensure annihilation by V_2 and V_3. Now annihilation by the ω's implies that the function is invariable when the independent variables only are linearly transformed, so that a pure function will be a cyclicant if it is unaltered by linear transformation of the independent variables and unaltered also by the interchange of the dependent and one independent variable.

In consequence of annihilation by the ω's any pure cyclicant will be an invariant of the system of quantics in ξ, η, ζ,

$$d_{200}\xi^2 + d_{020}\eta^2 + d_{002}\zeta^2 + d_{110}\xi\eta + d_{101}\xi\zeta + d_{011}\eta\zeta,$$

$$d_{300}\xi^3 + \ldots + d_{211}\xi^2\eta + \ldots + d_{111}\xi\eta\zeta,$$

$$d_{400}\xi^4 + \ldots$$

....

and conversely any invariant of these quantics which is annihilated by V_1 will be a pure cyclicant.

When the number of independent variables is n, there will be n operators of the V type and $n(n-1)$ operators of the ω type.

§ 23. To make the transformation by interchange of t and x the scheme will be

$$X = 0 . x + 0 . y + 0 . z + 1 . t,$$

$$Y = 0 . x + 1 . y + 0 . z + 0 . t,$$

$$Z = 0 . x + 0 . y + 1 . z + 0 . t,$$

$$T = 1 . x + 0 . y + 0 . z + 0 . t,$$

and the meaning of D_{lmn} will be $\dfrac{1}{l!\ m!\ n!}\dfrac{\partial^{l+m+n}x}{\partial t^l \partial y^m \partial z^n}$.

Here $M = -1$,

$$J = \begin{vmatrix} t_x & t_y & t_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = t_x,$$

$$A = 1,\quad A' = 0,\quad A'' = 0.\quad B = -t_y,\quad c' = 0,\quad a'' = t_z,\quad b'' = 0,\quad c'' = 1.$$

Therefore $$\Omega_1 = 0, \quad \Omega_2 = -\frac{t_y}{t_x}\omega_{yx}, \quad \Omega_3 = -\frac{t_z}{t_x}\omega_{zx},$$

and $$V = \begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = V_1.$$

Therefore if $F(d_{lmn}, d_{l'm'n'}, \ldots)$ is a pure homogeneous function of degree i and isobaric of weights p_1, p_2, p_3 in first, second and third suffixes,

$$F(D_{lmn}, D_{l'm'n'}, \ldots) = \frac{(-1)^i}{t_x^{i+p_1}} e^{-\frac{V_1}{t_x}} e^{-\frac{t_y}{t_x}\omega_{yx}} e^{-\frac{t_z}{t_x}\omega_{zx}} F(d_{lmn}, d_{l'm'n'}, \ldots).$$

The right-hand side can be arranged in powers of $\frac{1}{t_x}, \frac{t_y}{t_x}, \frac{t_z}{t_x}$, and since these are independent quantities it is obvious by observing the coefficients of their lowest powers that, in order that $F(d_{lmn}, \ldots)$ may be invariable, save for a factor, it is necessary that the function be annihilated by V_1, ω_{yx} and ω_{zx}. These conditions are obviously sufficient, and therefore if the conditions are satisfied we have

$$F(D_{lmn}, D_{l'm'n'}, \ldots) = \frac{(-1)^i}{t_x^{i+p_1}} F(d_{lmn}, d_{l'm'n'}, \ldots).$$

Similarly the necessary and sufficient conditions that F may be invariable when t is interchanged with y, and x, z are unaltered, are that F be annihilated by V_2, ω_{xy} and ω_{zy}; and, when t is interchanged with z and x, y are unaltered, the necessary and sufficient conditions for the permanence of F are that it be annihilated by V_3, ω_{xz} and ω_{yz}.

If F is annihilated by all the operators V and ω, equation (39) shows that F will be permanent in form, save for a factor, when any interchanges of variables are made or when both dependent and independent variables are changed by any linear transformation.

Since the annihilation of a function isobaric in first, second and third suffixes by the ω's implies that the three weights are equal, equation (39) shows that if F be a pure cyclicant the effect of the general linear transformation upon it is to transform it into

$$\frac{M^i}{J^{i+p}} F,$$

where i is the degree of F and p the weight in either set of suffixes.

In order that a function may remain permanent in form when the variables are changed by the general linear transformation it is therefore necessary that it be homogeneous and isobaric in each set of suffixes throughout.

§ 24. As another illustration we will give a proof of a theorem established by Prof. Elliott in a paper "On Pure Ternary Reciprocants, and Functions allied to them" (*Proc. Lond. Math. Soc.* Vol. XIX.). In this paper he considers two independent variables x, y with a dependent variable z. His operators V_1 and V_2 are the same as those considered in §§ 19, 20 and 21, with z written instead of t; his Ω_1 is ω_{yx} and his Ω_2 is ω_{xy}; and he writes z_{pq} for $\frac{1}{p!\,q!}\frac{\partial^{p+q}z}{\partial x^p\,\partial y^q}$. He considers a "reciprocantive covariant"

$$(P_0,\ P_1,\ P_2, \ldots P_m)(u,\ v)^m,$$

where u, v are any quantities and P_0, P_1, ... P_m are functions of z_{pq}, ... where $p+q \nless 2$, such that P_0 is a homogeneous and isobaric function annihilated by V_1, V_2 and ω_{yx}, and

$$\omega_{xy}P_0 = mP_1, \quad \omega_{xy}P_1 = (m-1)P_2, \ldots \omega_{xy}P_{m-1} = P_m, \quad \omega_{xy}P_m = 0.$$

In consequence of these conditions the function is a covariant of the emanants

$$(d_{20},\ d_{11},\ d_{02} \between u,\ v)^2, \quad (d_{30},\ d_{21},\ d_{12},\ d_{03} \between u,\ v)^3, \ldots$$

Therefore, if w_1, w_2 are the partial weights of P_0, $m = w_1 - w_2$ and the function is only altered by the factor $(-1)^{w_2}$ when u, v are interchanged. Hence if in P_r each quantity z_{rs} is replaced by z_{sr} the result is equal to $(-1)^{w_2}P_{m-r}$, and the quantities P therefore satisfy the conditions

$$\omega_{yx}P_m = mP_{m-1}, \quad \omega_{yx}P_{m-1} = (m-1)P_{m-2}, \ldots \omega_{yx}P_1 = P_0, \quad \omega_{yx}P_0 = 0.$$

Prof. Elliott shows also that all the P's are annihilated by V_1 and V_2; this property following from the relations

$$\omega_{yx}V_1 - V_1\omega_{yx} = 0, \qquad \omega_{xy}V_2 - V_2\omega_{xy} = 0,$$

$$\omega_{xy}V_1 - V_1\omega_{xy} = V_2, \quad \omega_{yx}V_2 - V_2\omega_{yx} = V_1.$$

See § 21.

Now let the variables be cyclically transformed from z, x, y to x, y, z so that x is the new dependent variable. Let x_{pq} denote $\frac{1}{p!\,q!}\frac{\partial^{p+q}x}{\partial y^p \partial z^q}$, and let $P_0(x)$ denote the result of substituting x_{pq}, ... for z_{pq}, ... in P_0.

Similarly when the variables are transformed from z, x, y to y, z, x so that y is the new dependent variable, let y_{pq} denote $\frac{1}{p!\,q!}\frac{\partial^{p+q}y}{\partial z^p \partial x^q}$, and let $P_m(y)$ denote the result of substituting y_{pq}, ... for z_{pq}, ... in P_m. Then Prof. Elliott's theorem states that*

$$(-1)^{i+w_2}\frac{P_0(x)}{x_{01}{}^{i+w_1}} = (-1)^{i+w_1}\frac{P_m(y)}{y_{10}{}^{i+w_1}} = (P_0,\ P_1,\ P_2, \ldots P_m)(-z_{01},\ z_{10})^m,$$

where i is the degree of P_0.

* Prof. Elliott gives different powers of (-1) in his statement of the theorem, but there is a slight error in his work which accounts for the difference.

The theorem follows at once from the results of § 20. Using equation (42 a), we have

$$P_0(x) = \frac{z_y^{w_1-w_1}}{(-z_x)^{i+w_1}} e^{-\frac{V_1}{z_x}} e^{-\frac{z_x}{z_y}\omega_{xy}} e^{\frac{z_y}{z_x}\omega_{yx}} P_0(z).$$

But ω_{yx} annihilates $P_0(z)$, and V_1 annihilates not only $P_0(z)$ but every function of the form $\omega_{xy}{}^r P_0(z)$; therefore

$$P_0(x) = \frac{z_y{}^m}{(-z_x)^{i+w_1}} e^{-\frac{z_x}{z_y}\omega_{xy}} P_0(z)$$

$$= \frac{z_y{}^m}{(-z_x)^{i+w_1}} \left[1 - \frac{z_x}{z_y}\omega_{xy} + \frac{1}{2!}\left(\frac{z_x}{z_y}\right)^2 \omega^2_{xy} - \dots\right] P_0(z)$$

$$= \frac{1}{(-z_x)^{i+w_1}} \left[P_0 z_y{}^m - mP_1 z_x z_y{}^{m-1} + \frac{m(m-1)}{1\,.\,2} P_2 z_x{}^2 z_y{}^{m-2} - \dots + (-1)^m P_m z_x{}^m\right]$$

$$= \frac{(-1)^m}{(-z_x)^{i+w_1}} (P_0, P_1, P_2, \dots P_m)(-z_y, z_x)^m$$

$$= \frac{(-1)^{i+w_2}}{z_{10}{}^{i+w_1}} (P_0, P_1, P_2, \dots P_m)(-z_{01}, z_{10})^m.$$

Again, the first and second partial weights of P_m are w_2 and w_1 respectively; therefore by equation (43 a)

$$P_m(y) = \frac{z_x^{w_1-w_2}}{(-z_y)^{i+w_1}} e^{-\frac{V_2}{z_y}} e^{-\frac{z_y}{z_x}\omega_{yx}} e^{\frac{z_x}{z_y}\omega_{xy}} P_m(z).$$

But ω_{xy} annihilates P_m, and V_2 annihilates not only P_m but every function of the form $\omega_{yx}{}^r P_m$; therefore

$$P_m(y) = \frac{z_x{}^m}{(-z_y)^{i+w_1}} e^{-\frac{z_y}{z_x}\omega_{yx}} P_m(z)$$

$$= \frac{z_x{}^m}{(-z_y)^{i+w_1}} \left[1 - \frac{z_y}{z_x}\omega_{yx} + \frac{1}{2!}\left(\frac{z_y}{z_x}\right)^2 \omega_{yx}{}^2 - \dots + \frac{(-1)^m}{m!}\left(\frac{z_y}{z_x}\right)^m \omega_{yx}{}^m\right] P_m$$

$$= \frac{1}{(-z_y)^{i+w_1}} \left[P_0(-z_y)^m + mP_1(-z_y)^{m-1} z_x + \frac{m(m-1)}{1\,.\,2} P_2(-z_y)^{m-2} z_x{}^2 + \dots + P_m z_x{}^m\right]$$

$$= \frac{1}{(-z_y)^{i+w_1}} (P_0, P_1, P_2, \dots P_m)(-z_y, z_x)^m.$$

The two results establish the theorem.

§ 25. It has been seen that when the variables are linearly transformed from t, x, y, z to T, X, Y, Z a pure function of differential coefficients will be unaltered in form, save for a factor, provided it is annihilated by the operators V_1, V_2, V_3 and ω_{xy}, ω_{xz}, ... It will be convenient temporarily to denote V_1, V_2, V_3 by V_x, V_y, V_z. It is evident that this permanence of form would be ensured if the transformed function expressed in terms of differential coefficients of T with respect to X, Y, Z

were annihilated by operators V_X, V_Y, V_Z, ω_{XY}, ω_{XZ}, ... formed with the quantities D_{pqr}, A_{pqr}, ... in the same way as V_x, V_y, V_z, ω_{xy}, ω_{xz}, ... are formed with d_{pqr}, a_{pqr}, ... It must be possible therefore to express the effect of these operators V_X, V_Y, V_Z, ω_{XY}, ω_{XZ}, ... on the transformed function by means of the original operators acting on the original function. It is now proposed to examine into the manner in which these operators V_X, ω_{XY}, ... can be so expressed.

The scheme of transformation is, as before,

$$X = \alpha x + \alpha' y + \alpha'' z + \alpha''' t,$$

$$Y = \beta x + \beta' y + \beta'' z + \beta''' t,$$

$$Z = \gamma x + \gamma' y + \gamma'' z + \gamma''' t,$$

$$T = \delta x + \delta' y + \delta'' z + \delta''' t,$$

and t, T are regarded as the respective dependent variables.

Suppose now that x, y, z receive increments ξ, η, ζ, and let the consequent increments in t, X, Y, Z, T be τ, ξ', η', ζ', τ', so that

$$\tau = d_{100}\xi + d_{010}\eta + d_{001}\zeta + d_{200}\xi^2 + d_{020}\eta^2 + \dots.$$

$$\tau' = D_{100}\xi' + D_{010}\eta' + D_{001}\zeta' + D_{200}\xi'^2 + D_{020}\eta'^2 + \dots.$$

Then $$V_x = V_1 = (\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta)(\tau_\xi - d_{100}).$$

$$\omega_{xy} = \xi(\tau_\eta - d_{010}),$$

$$\omega_{yx} = \eta(\tau_\xi - d_{100}), \text{ \&c.,}$$

where after expansion $\xi^p\eta^q\zeta^r$ is replaced by $\dfrac{\partial}{\partial d_{pqr}}$.

Therefore also

$$V_X = (\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')\left(\frac{\partial\tau'}{\partial\xi'} - D_{100}\right),$$

$$\omega_{XY} = \xi'\left(\frac{\partial\tau'}{\partial\eta'} - D_{010}\right),$$

$$\omega_{YX} = \eta'\left(\frac{\partial\tau'}{\partial\xi'} - D_{100}\right), \text{ \&c.,}$$

when $\xi'^p\eta'^q\zeta'^r$ is replaced after expansion by $\dfrac{\partial}{\partial D_{pqr}}$.

Now let $F(\xi', \eta', \zeta')$ be the symbolical expression for an operator obtained by expanding $F(\xi', \eta', \zeta')$ in powers of ξ', η', ζ' and replacing $\xi'^p\eta'^q\zeta'^r$ by $\dfrac{\partial}{\partial D_{pqr}}$, it being understood that F contains no term for which $p+q+r<2$. Then Prof. Elliott has proved* that the expression for this operator in terms of $\frac{\partial}{\partial d_{pqr}}$ is obtained by expanding a certain expression in powers of ξ, η, ζ and replacing $\xi^p\eta^q\zeta^r$ by $\frac{\partial}{\partial d_{pqr}}$. This expression is

$$\frac{1}{M} F(\xi', \eta', \zeta') \left(\Delta''' - \Delta \frac{\partial \tau}{\partial \xi} - \Delta' \frac{\partial \tau}{\partial \eta} - \Delta'' \frac{\partial \tau}{\partial \zeta}\right),$$

where
$$M = \begin{vmatrix} \alpha, & \alpha', & \alpha'', & \alpha''' \\ \beta, & \beta', & \beta'', & \beta''' \\ \gamma, & \gamma', & \gamma'', & \gamma''' \\ \delta, & \delta', & \delta'', & \delta''' \end{vmatrix}$$

and Δ, Δ', Δ'', Δ''' are the minors of δ, δ', δ'', δ''' in M.

The application of this rule to the operators considered here is simplified by use of formula (47) which we now proceed to establish. The rule itself may also be deduced from this formula, but Prof. Elliott adopts a different mode of proof.

We have, in consequence of the scheme of transformation,

$$\begin{aligned} \xi' &= \alpha\xi + \alpha'\eta + \alpha''\zeta + \alpha'''\tau, \\ \eta' &= \beta\xi + \beta'\eta + \beta''\zeta + \beta'''\tau, \\ \zeta' &= \gamma\xi + \gamma'\eta + \gamma''\zeta + \gamma'''\tau, \\ \tau' &= \delta\xi + \delta'\eta + \delta''\zeta + \delta'''\tau. \end{aligned} \quad \text{.................................}(45).$$

The simplest way of finding D_{100}, D_{010}, D_{001} is to determine them as the coefficients of ξ', η', ζ' when τ' is expressed in terms of these quantities.

Now, neglecting higher powers of ξ, η, ζ than the first, the last set of equations may be written

$$\begin{aligned} \xi' &= (\alpha + \alpha'''d_{100})\,\xi + (\alpha' + \alpha'''d_{010})\,\eta + (\alpha'' + \alpha'''d_{001})\,\zeta, \\ \eta' &= (\beta + \beta'''d_{100})\,\xi + (\beta' + \beta'''d_{010})\,\eta + (\beta'' + \beta'''d_{001})\,\zeta, \\ \zeta' &= (\gamma + \gamma'''d_{100})\,\xi + (\gamma' + \gamma'''d_{010})\,\eta + (\gamma'' + \gamma'''d_{001})\,\zeta, \\ \tau' &= (\delta + \delta'''d_{100})\,\xi + (\delta' + \delta'''d_{010})\,\eta + (\delta'' + \delta'''d_{001})\,\zeta. \end{aligned}$$

Therefore, eliminating ξ, η, ζ, we have

$$\begin{vmatrix} \alpha + \alpha'''d_{100}, & \alpha' + \alpha'''d_{010}, & \alpha'' + \alpha'''d_{001}, & \xi' \\ \beta + \beta'''d_{100}, & \beta' + \beta'''d_{010}, & \beta'' + \beta'''d_{001}, & \eta' \\ \gamma + \gamma'''d_{100}, & \gamma' + \gamma'''d_{010}, & \gamma'' + \gamma'''d_{001}, & \zeta' \\ \delta + \delta'''d_{100}, & \delta' + \delta'''d_{010}, & \delta'' + \delta'''d_{001}, & \tau' \end{vmatrix} = 0 \quad \text{............}(46).$$

* "The Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups." *Proc. Lond. Math. Soc.*, Vol. XXIX.

Write this equation in the form

$$\tau' = D_{100}\xi' + D_{010}\eta' + D_{001}\zeta',$$

and D_{100}, D_{010}, D_{001} are immediately determined.

As previously, write

$$J = \begin{vmatrix} \alpha + \alpha'''d_{100}, & \alpha' + \alpha'''d_{010}, & \alpha'' + \alpha'''d_{001} \\ \beta + \beta'''d_{100}, & \beta' + \beta'''d_{010}, & \beta'' + \beta'''d_{001} \\ \gamma + \gamma'''d_{100}, & \gamma' + \gamma'''d_{010}, & \gamma'' + \gamma'''d_{001} \end{vmatrix} = \frac{\partial (X, Y, Z)}{\partial (x, y, z)}.$$

The increments ξ, η, ζ, τ have only been temporarily assumed small for the purpose of finding D_{100}, D_{010}, D_{001}. Let them now be regarded as finite; then the determinant on the left of equation (46) will be the value of

$$J(\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta').$$

Multiply the first three columns by $-\xi$, $-\eta$, $-\zeta$, and add to the last. We then find, by means of equations (45),

$$J(\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')$$
$$= (\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta) \begin{vmatrix} \alpha + \alpha'''d_{100}, & \alpha' + \alpha'''d_{010}, & \alpha'' + \alpha'''d_{001}, & \alpha''' \\ \beta + \beta'''d_{100}, & \beta' + \beta'''d_{010}, & \beta'' + \beta'''d_{001}, & \beta''' \\ \gamma + \gamma'''d_{100}, & \gamma' + \gamma'''d_{010}, & \gamma'' + \gamma'''d_{001}, & \gamma''' \\ \delta + \delta'''d_{100}, & \delta' + \delta'''d_{010}, & \delta'' + \delta'''d_{001}, & \delta''' \end{vmatrix}$$

Multiply the last column by d_{100}, d_{010}, d_{001} and subtract from the first, second and third columns respectively. We then obtain the important equation

$$\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta' = \frac{M}{J}(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta) \quad \text{..............(47)}.$$

This theorem is the generalization of a statement by Prof. Elliott (*Proc. Lond. Math. Soc.*, Vol. XVIII., p. 147) made with reference to two independent variables when the linear transformation consists of a cyclical interchange.

§ 26. Now

$$V_X = (\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')\left(\frac{\partial \tau'}{\partial \xi'} - D_{100}\right).$$

Therefore the transformed expression for V_X is obtained from

$$\frac{1}{M}(\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')\left(\frac{\partial \tau'}{\partial \xi'} - D_{100}\right)\left(\Delta''' - \Delta\frac{\partial \tau}{\partial \xi} - \Delta'\frac{\partial \tau}{\partial \eta} - \Delta''\frac{\partial \tau}{\partial \zeta}\right),$$

by expressing it in terms of ξ, η, ζ. The first factor is transformed by equation (47) which gives

$$\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta' = \frac{M}{J}(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta).$$

Again, from this last equation

$$\frac{\partial \tau'}{\partial \xi'} - D_{100} = \frac{M}{J}\left\{\left(\frac{\partial \tau}{\partial \xi} - d_{100}\right)\frac{\partial \xi}{\partial \xi'} + \left(\frac{\partial \tau}{\partial \eta} - d_{010}\right)\frac{\partial \eta}{\partial \xi'} + \left(\frac{\partial \tau}{\partial \zeta} - d_{001}\right)\frac{\partial \zeta}{\partial \xi'}\right\},$$

where $\dfrac{\partial\xi}{\partial\xi'}$, $\dfrac{\partial\eta}{\partial\xi'}$, $\dfrac{\partial\zeta}{\partial\xi'}$ are to be obtained from equations (45). These equations give

$$1=\left(\alpha+\alpha'''\frac{\partial\tau}{\partial\xi}\right)\frac{\partial\xi}{\partial\xi'}+\left(\alpha'+\alpha'''\frac{\partial\tau}{\partial\eta}\right)\frac{\partial\eta}{\partial\xi'}+\left(\alpha''+\alpha'''\frac{\partial\tau}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\xi'},$$

$$0=\left(\beta+\beta'''\frac{\partial\tau}{\partial\xi}\right)\frac{\partial\xi}{\partial\xi'}+\left(\beta'+\beta'''\frac{\partial\tau}{\partial\eta}\right)\frac{\partial\eta}{\partial\xi'}+\left(\beta''+\beta'''\frac{\partial\tau}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\xi'},$$

$$0=\left(\gamma+\gamma'''\frac{\partial\tau}{\partial\xi}\right)\frac{\partial\xi}{\partial\xi'}+\left(\gamma'+\gamma'''\frac{\partial\tau}{\partial\eta}\right)\frac{\partial\eta}{\partial\xi'}+\left(\gamma''+\gamma'''\frac{\partial\tau}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\xi'}.$$

Now let

$$I=\begin{vmatrix}\alpha+\alpha'''\dfrac{\partial\tau}{\partial\xi}, & \alpha'+\alpha'''\dfrac{\partial\tau}{\partial\eta}, & \alpha''+\alpha'''\dfrac{\partial\tau}{\partial\zeta}\\ \beta+\beta'''\dfrac{\partial\tau}{\partial\xi}, & \beta'+\beta'''\dfrac{\partial\tau}{\partial\eta}, & \beta''+\beta'''\dfrac{\partial\tau}{\partial\zeta}\\ \gamma+\gamma'''\dfrac{\partial\tau}{\partial\xi}, & \gamma'+\gamma'''\dfrac{\partial\tau}{\partial\eta}, & \gamma''+\gamma'''\dfrac{\partial\tau}{\partial\zeta}\end{vmatrix}$$

$$=\Delta'''-\Delta\frac{\partial\tau}{\partial\xi}-\Delta'\frac{\partial\tau}{\partial\eta}-\Delta''\frac{\partial\tau}{\partial\zeta};$$

and let $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ denote the minors of the first row of I: therefore

$$I\frac{\partial\xi}{\partial\xi'}=\mathfrak{A},\quad I\frac{\partial\eta}{\partial\xi'}=\mathfrak{A}',\quad I\frac{\partial\zeta}{\partial\xi'}=\mathfrak{A}''.$$

Hence

$$I\left(\frac{\partial\tau'}{\partial\xi'}-D_{100}\right)=\frac{M}{J}\left\{\mathfrak{A}\left(\frac{\partial\tau}{\partial\xi}-d_{100}\right)+\mathfrak{A}'\left(\frac{\partial\tau}{\partial\eta}-d_{010}\right)+\mathfrak{A}''\left(\frac{\partial\tau}{\partial\zeta}-d_{001}\right)\right\}$$

$$=\frac{M}{J}\begin{vmatrix}\dfrac{\partial\tau}{\partial\xi}-d_{100}, & \dfrac{\partial\tau}{\partial\eta}-d_{010}, & \dfrac{\partial\tau}{\partial\zeta}-d_{001}\\ \beta+\beta'''\dfrac{\partial\tau}{\partial\xi}, & \beta'+\beta'''\dfrac{\partial\tau}{\partial\eta}, & \beta''+\beta'''\dfrac{\partial\tau}{\partial\zeta}\\ \gamma+\gamma'''\dfrac{\partial\tau}{\partial\xi}, & \gamma'+\gamma'''\dfrac{\partial\tau}{\partial\eta}, & \gamma''+\gamma'''\dfrac{\partial\tau}{\partial\zeta}\end{vmatrix}$$

$$=\frac{M}{J}\begin{vmatrix}\dfrac{\partial\tau}{\partial\xi}-d_{100}, & \dfrac{\partial\tau}{\partial\eta}-d_{010}, & \dfrac{\partial\tau}{\partial\zeta}-d_{001}\\ \beta+\beta'''d_{100}, & \beta'+\beta'''d_{010}, & \beta''+\beta'''d_{001}\\ \gamma+\gamma'''d_{100}, & \gamma'+\gamma'''d_{010}, & \gamma''+\gamma'''d_{001}\end{vmatrix}$$

$$=\frac{M}{J}\left\{A\left(\frac{\partial\tau}{\partial\xi}-d_{100}\right)+A'\left(\frac{\partial\tau}{\partial\eta}-d_{010}\right)+A''\left(\frac{\partial\tau}{\partial\zeta}-d_{001}\right)\right\}\quad\text{.....................(48)},$$

where A, A', A'',... are the minors of J.

The transformed expression for V_X is therefore

$$\frac{M}{J^2}(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta)\left\{A\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) + A'\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) + A''\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\right\}$$

$$= \frac{M}{J^2}(AV_x + AV_y + A''V_z),$$

or, in the previous notation,

$$\frac{M}{J^2}(AV_1 + A'V_2 + A''V_3) \quad \text{......................................(49).}$$

Similarly the transformations of V_Y and V_Z are

$$\frac{M}{J^2}(BV_1 + B'V_2 + B''V_3),\quad \frac{M}{J^2}(CV_1 + C'V_2 + C''V_3) \quad \text{..................(49}a\text{).}$$

Again $$\omega_{XY} = \xi'\left(\frac{\partial\tau'}{\partial\eta'} - D_{010}\right),$$

and the transformation of ω_{XY} is therefore

$$\frac{1}{M}\xi'\left(\frac{\partial\tau'}{\partial\eta'} - D_{010}\right)\left(\Delta''' - \Delta\frac{\partial\tau}{\partial\xi} - \Delta'\frac{\partial\tau}{\partial\eta} - \Delta''\frac{\partial\tau}{\partial\zeta}\right)$$

$$= \frac{1}{J}(\alpha\xi + \alpha'\eta + \alpha''\zeta + \alpha'''\tau)\left\{B\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) + B'\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) + B''\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\right\},$$

as in equation (48),

$$= \frac{1}{J}\{(\alpha + \alpha'''d_{100})\xi + (\alpha' + \alpha'''d_{010})\eta + (\alpha'' + \alpha'''d_{001})\zeta + \alpha'''(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta)\}$$

$$\left\{B\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) + B'\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) + B''\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\right\}.$$

Now let G_1, G_2, G_3 be operators defined by the equations

$$G_1 = \xi\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) = \Sigma p d_{pqr}\frac{\partial}{\partial d_{pqr}},$$

$$G_2 = \eta\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) = \Sigma q d_{pqr}\frac{\partial}{\partial d_{pqr}},$$

$$G_3 = \zeta\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right) = \Sigma r d_{pqr}\frac{\partial}{\partial d_{pqr}},$$

where in each case $p + q + r > 1$.

Then the transformation of ω_{XY} is

$$\frac{1}{J}\{(\alpha + \alpha'''d_{100})BG_1 + (\alpha' + \alpha'''d_{010})B'G_2 + (\alpha'' + \alpha'''d_{001})B''G_3\}$$

$$+ \frac{1}{J}\{(\alpha + \alpha'''d_{100})(B'\omega_{xy} + B''\omega_{xz}) + (\alpha' + \alpha'''d_{010})(B\omega_{yx} + B''\omega_{yz}) + (\alpha'' + \alpha'''d_{001})(B\omega_{zx} + B'\omega_{zy})\}$$

$$+ \frac{\alpha'''}{J}(BV_1 + B'V_2 + B''V_3) \quad \text{..(50).}$$

If the function F operated on is isobaric, and of weights w_1, w_2, w_3 in first, second and third suffixes, we have

$$G_1F = w_1F, \quad G_2F = w_2F, \quad G_3F = w_3F.$$

If further $w_1 = w_2 = w_3$, since

$$B(\alpha + \alpha''' d_{100}) + B'(\alpha' + \alpha''' d_{010}) + B''(\alpha'' + \alpha''' d_{001}) = 0,$$

we find for the transformation of ω_{XY}, a linear function of the operators V and ω, viz.

$$\frac{1}{J}\{(\alpha + \alpha''' d_{100})(B'\omega_{xy} + B''\omega_{xz}) + (\alpha' + \alpha''' d_{010})(B\omega_{yx} + B''\omega_{yz}) + (\alpha'' + \alpha''' d_{001})(B\omega_{zx} + B'\omega_{zy})$$

$$+ \frac{\alpha'''}{J}(BV_1 + B'V_2 + B''V_3) \qquad \text{(51)}.$$

The transformations of the remaining operators ω_{YX}, ω_{XZ}, ... can be written down at once from this last expression.

§ 27. For example, suppose a cyclical interchange is made from z, x, y with z for dependent variable to x, y, z with x for dependent variable. Using the symbolical notation, let

$$V_1 = (\zeta - z_{10}\xi - z_{01}\eta)(\zeta_\xi - z_{10}); \qquad V_2 = (\zeta - z_{10}\xi - z_{01}\eta)(\zeta_\eta - z_{01});$$

$$\omega_{xy} = \xi(\zeta_\eta - z_{01}); \qquad \omega_{yx} = \eta(\zeta_\xi - z_{10});$$

$$V_1' = (\xi - x_{10}\eta - x_{01}\zeta)(\xi_\eta - x_{10}); \qquad V_2' = (\xi - x_{10}\eta - x_{01}\zeta)(\xi_\zeta - x_{01});$$

$$\omega_{yz}' = \eta(\xi_\zeta - x_{01}); \qquad \omega_{zy}' = \zeta(\xi_\eta - x_{10}).$$

The scheme of transformation is

$$X = 0 \,.\, x + 1 \,.\, y + 0 \,.\, z,$$
$$Y = 0 \,.\, x + 0 \,.\, y + 1 \,.\, z,$$
$$Z = 1 \,.\, x + 0 \,.\, y + 0 \,.\, z.$$

so that $M = 1$, and

$$J = \begin{vmatrix} 0 & 1 \\ z_{10} & z_{01} \end{vmatrix} = -z_{10}.$$

The transformations of the operators are therefore respectively, if for simplicity we suppose them to act on pure isobaric functions with equal partial weights,

$$V_1' = \frac{1}{z_{10}^2}(z_{01}V_1 - z_{10}V_2) \qquad \text{(52}a\text{)},$$

$$V_2' = -\frac{1}{z_{10}^2}V_1 \qquad \text{(52}b\text{)},$$

$$\omega_{yz}' = \frac{1}{z_{10}}\omega_{yx} \qquad \text{(52}c\text{)},$$

$$\omega_{zy}' = -\frac{1}{z_{10}}(z_{01}V_1 - z_{10}V_2) - \frac{1}{z_{10}}\{z_{10}(-z_{10})\omega_{xy} + z_{01} \,.\, z_{01} \,.\, \omega_{yx}\}$$

$$= -\frac{1}{z_{10}}(z_{01}V_1 - z_{10}V_2) + \frac{1}{z_{10}}(z_{10}^2\omega_{xy} - z_{01}^2\omega_{yx}) \qquad \text{(52}d\text{)}.$$

§ 28. As an application of the last results we may employ them to prove two formulae which are fundamental in Prof. Elliott's paper "On the Linear Partial Differential Equations satisfied by Ternary Reciprocants," *Proc. Lond. Math. Soc.*, Vol. XVIII.

Let $F(z_{pq}, \ldots)$ be a pure function of the differential coefficients, and let μ be any number. It is required to evaluate

$$\frac{\partial}{\partial x_{01}}\left(\frac{F}{z_{10}{}^{\mu}}\right) \text{ and } \frac{\partial}{\partial x_{10}}\left(\frac{F}{z_{10}{}^{\mu}}\right),$$

where in the differentiation with respect to x_{01} it is assumed that $x_{10}, x_{20}, x_{11}, \ldots$ are regarded as constant, and, in the differentiation with respect to x_{10}, $x_{01}, x_{20}, x_{11}, \ldots$ are constant. We shall, for simplicity, assume F to be homogeneous of degree i and isobaric with equal partial weights w.

Now the change from x, yz to z, xy is the second cyclical interchange from x, yz; therefore by making suitable interchanges of letters in equation (43*b*) we have

$$F(z_{pq}, \ldots) = (-1)^i \frac{1}{x_{01}{}^{i+w}} e^{-\frac{V_2'}{x_{01}}} e^{-\frac{x_{10}}{x_{01}}\omega'_{yz}} F(x_{qp}, \ldots),$$

where V_2' and ω_{yz}' are the same operators as in the preceding section. It may be remarked that V_2' and ω_{yz}' are commutative by § 21. Therefore since $z_{10} = \dfrac{1}{x_{01}}$,

$$\frac{\partial}{\partial x_{01}}\left(\frac{F}{z_{10}{}^{\mu}}\right) = \left\{\frac{\mu - i - w}{x_{01}} + \frac{1}{x_{01}{}^2} V_2' + \frac{x_{10}}{x_{01}{}^2}\omega_{yz}'\right\} . (-1)^i x_{01}{}^{\mu-i-w} e^{-\frac{V_2'}{x_{01}}} e^{-\frac{x_{10}}{x_{01}}\omega_{yz}'} F(x_{qp}, \ldots)$$

$$= \left\{\frac{\mu - i - w}{z_{10}{}^{\mu-1}} - \frac{1}{z_{10}{}^{\mu}} V_1 - \frac{z_{01}}{z_{10}{}^{\mu}}\omega_{yx}\right\} F(z_{pq}, \ldots),$$

by (52*b*) and (52*c*).

Similarly

$$\frac{\partial}{\partial x_{10}}\left(\frac{F}{z_{10}{}^{\mu}}\right) = \frac{1}{z_{10}{}^{\mu}}\left(-\frac{1}{x_{01}}\right)\omega_{yz}' . (-1)^i \frac{1}{x_{01}{}^{i+w}} e^{-\frac{V_2'}{x_{01}}} e^{-\frac{x_{10}}{x_{01}}\omega_{yz}'} F(x_{qp}, \ldots)$$

$$= -\frac{1}{z_{10}{}^{\mu}}\omega_{yx} F(z_{pq}, \ldots),$$

by (52*c*).

These are Prof. Elliott's formulae.

§ 29. The theory of cyclicants is a generalization of the theory of ordinary reciprocants; in the case when there are two independent variables it plays a part which has the same reference to the theory of surfaces that the ordinary reciprocant has to the theory of plane curves. But the ordinary reciprocant may be looked at from another point of view. Regarding y as a function of x, let us suppose ξ, η to be corresponding increments of x and y: then writing a_n for $\frac{1}{n!}\frac{d^n y}{dx^n}$ and A_n for $\frac{1}{n!}\frac{d^n x}{dy^n}$, we have

$$\eta = a_1\xi + a_2\xi^2 + \dots,$$

$$\xi = A_1\eta + A_2\eta^2 + \dots$$

The second series is that obtained by reversion of the first, and a reciprocant may be looked upon as a function of the coefficients of a series which is unaltered in value, save for a factor involving a_1, when the series is reversed. From this point of view the generalized reciprocant may be defined in the following manner. Using the notation of § 16 let u, v, w be functions of x, y, z, and let a_{pqr} denote $\frac{1}{p!\,q!\,r!}\frac{\partial^{p+q+r}u}{\partial x^p \partial y^q \partial z^r}, \dots$ and let A_{pqr} denote $\frac{1}{p!\,q!\,r!}\frac{\partial^{p+q+r}x}{\partial u^p \partial v^q \partial w^r}$. Then $F(a_{pqr}, b_{pqr}, c_{pqr}, \dots)$ will be a reciprocant if

$$F(A_{pqr}, B_{pqr}, C_{pqr}, \dots) = \mu F(a_{pqr}, b_{pqr}, c_{pqr}, \dots),$$

where μ is a function of first differential coefficients only. The function will be called an n-ary reciprocant if there are n independent variables involved, and F will be a pure function if it is free from first differential coefficients. This kind of reciprocant may also be regarded as a function of the coefficients of series which is unaltered, save for a factor, when the series are reversed and the coefficients of the reversed series are substituted for those of the original series.

Sufficient conditions to ensure the permanence of such functions, when pure, are easily obtained from the results of § 16.

Reciprocants of the kind here considered have been discussed by Prof. Elliott* for the case of two independent variables. The conditions here obtained for n variables agree with those obtained by Prof. Elliott, who however does not examine into the question of the independence of his conditions.

Suppose $F(a_{lmn}, b_{lmn}, c_{lmn}, \dots)$ to be a homogeneous function of degree i and isobaric with equal partial weights w. Then

$$F(A_{lmn}, B_{lmn}, C_{lmn}, \dots) = \frac{(-1)^i}{J^{i+w}} e^{-(U, V, W)} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3}.$$

$$F(Aa_{lmn} + Bb_{lmn} + Cc_{lmn},\ A'a_{lmn} + B'b_{lmn} + C'c_{lmn},\ A''a_{lmn} + B''b_{lmn} + C''c_{lmn}, \dots) \dots(53).$$

The function will therefore be permanent in form if it is an invariant of the system of emanants

$$\left.\begin{array}{ll} a_{200}\xi^2 + a_{020}\eta^2 + a_{002}\zeta^2 + a_{110}\xi\eta + \dots, & a_{300}\xi^3 + \dots, \dots \\ b_{200}\xi^2 + \dots & , b_{300}\xi^3 + \dots, \dots \\ c_{200}\xi^2 + \dots & , c_{300}\xi^3 + \dots, \dots \end{array}\right\} \dots\dots\dots\dots(54).$$

* "On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables." *Proc. Lond. Math. Soc.*, Vol. XXII. pp. 79—104.

which remains an invariant when $\lambda u+\mu v+\nu w$, $\lambda' u+\mu' v+\nu' w$, $\lambda'' u+\mu'' v+\nu'' w$ are substituted for u, v, w, and which is further annihilated by the various operators $[U, x]$, $[V, x]$, $[W, x]$, $[U, y]$, ... The operators as defined in § 14 contain terms with differential operators involving t; such terms will of course be omitted here.

It is obvious that functions which satisfy the conditions just laid down will be unchanged or at most changed only in sign when u, v, w are interchanged; and that they will be unchanged or changed only in sign if first, second and third suffixes are interchanged. Such functions therefore, if homogeneous, will be of equal partial degrees in a_{pqr}, ..., b_{pqr}, ..., c_{pqr}, ...

When linear functions $\lambda u+\mu v+\nu w$, ... are substituted for u, v, w in a combinant the function is multiplied by the ith power of

$$\begin{vmatrix} \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \\ \lambda'', & \mu'', & \nu'' \end{vmatrix},$$

where i is equal to any one of the equal partial degrees of the combinant.

In the case of the reciprocants here considered the determinant is

$$\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix},$$

which is equal to J^2; and the determinant is equal to J^{n-1} when there are n independent variables.

Reference to equation (32) then shows that the factor for a reciprocant of equal partial degrees i and equal partial weights w is

$$\frac{(-1)^{ni}J^{(n-1)i}}{J^{ni-w}}=\frac{(-1)^{ni}}{J^{i+w}},$$

so that

$$F(A_{pqr}, B_{pqr}, C_{pqr}, \ldots)=\frac{(-1)^{ni}}{J^{i-w}}F(a_{pqr}, b_{pqr}, c_{pqr}, \ldots).$$

§ 30. One example of such reciprocants is easily seen to be the eliminant of the quadratic emanants just written down. For this is an invariant of the required type, and since it involves no differential coefficients a_{pqr} for which $p+q+r>2$, it is obviously annihilated by $[U, x]$, $[V, x]$, ...

This example for the case of two independent variables is given by Prof. Elliott. The eliminant in this case is

$$(a_{20}b_{11}-b_{20}a_{11})(a_{11}b_{02}-b_{11}a_{02})-(a_{20}b_{02}-b_{20}a_{02})^2.$$

The partial degree $i=2$, and the partial weight $w=4$; therefore by the last result of § 29 this expression is equal to

$$J^6 \{(A_{20}B_{11} - B_{20}A_{11})(A_{11}B_{02} - B_{11}A_{02}) - (A_{20}B_{02} - B_{20}A_{02})^2\},$$

where
$$J = ab' - a'b.$$

The invariant character of the function just considered corresponds to a simple theorem in the theory of the reversion of series. Let

$$v = a_{10}\xi + a_{01}\eta + a_{20}\xi^2 + a_{11}\xi\eta + a_{02}\eta^2 + \dots,$$

$$\nu = b_{10}\xi + b_{01}\eta + b_{20}\xi^2 + b_{11}\xi\eta + b_{02}\eta^2 + \dots,$$

and suppose that from these equations are deduced the series

$$\xi = A_{10}v + A_{11}\nu + A_{20}v^2 + A_{11}v\nu + A_{02}\nu^2 + \dots,$$

$$\eta = B_{10}v + B_{11}\nu + B_{20}v^2 + B_{11}v\nu + B_{02}\nu^2 + \dots.$$

The theorem then states that, if the quadratic terms in the original series have a common factor linear in ξ, η, the quadratic terms of the series obtained by reversion will have a quadratic factor linear in v, ν.

The theorem is easily proved independently. The property referred to is one unaltered by a linear transformation, and therefore we may take ξ for the common factor. The method of successive approximation then shows at once that the quadratic terms in the last two series must have a common factor.

§ 31. The conditions for pure reciprocants laid down in § 27 although sufficient are not independent. This statement can be proved by forming the alternants of various operators. If we assume F to be an invariant of the system of emanants (54) which remains unaltered save for a factor when $\lambda u + \mu v + \nu w$, $\lambda' u + \mu' v + \nu' w$, $\lambda'' u + \mu'' v + \nu'' w$ are substituted for u, v, w, then it can be shown that annihilation by one of the operators $[U, x]$ will ensure annihilation by all the others. In fact since F is invariant when $\lambda u + \mu v + \nu w$ is substituted for u, therefore F must be annihilated by the operators which in the usual symbolical notation will be denoted by $[V_1]$ and $[W_1]$, so that

$$[V_1] = [b_{200}\xi_1^2 + \dots] = b_{200}\frac{\partial}{\partial a_{200}} + \dots,$$

$$[W_1] = [c_{200}\xi_1^2 + \dots] = c_{200}\frac{\partial}{\partial a_{200}} + \dots.$$

Similarly F must be annihilated by $[U_2]$, $[W_2]$, $[U_3]$, $[V_3]$.

Now, in the present case, we have

$$[U, x] = [U_1 U_{\xi_1}] + [U_2 V_{\xi_2}] + [U_3 W_{\xi_3}];$$

therefore

$$[U, x][V_1] - [V_1][U, x] = [U_2 V_{\xi_2}] * [V_1] - [V_1] * [U_1 U_{\xi_1}] - [V_1] * [U_2 V_{\xi_2}] - [V_1] * [U_3 W_{\xi_3}].$$

And $$[V_1]\,U_1 = V_1, \quad [V_1]\,U_2 = V_2, \quad [V_1]\,U_{\xi_1} = \frac{\partial}{\partial \xi_1}[V_1]\,U_1 = \frac{\partial V_1}{\partial \xi_1}.$$

Therefore

$$\begin{aligned}
[U, x][V_1] - [V_1][U, x] &= [U_1 V_{\xi_1}] - [U_{\xi_1} V_1 + U_1 V_{\xi_1}] - [V_2 V_{\xi_2}] - [V_3 W_{\xi_3}] \\
&= -[V_1 U_{\xi_1}] - [V_2 V_{\xi_2}] - [V_3 W_{\xi_3}] \\
&= -[V, x] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55a).
\end{aligned}$$

Similarly

$$[U, x][W_1] - [W_1][U, x] = -[W, x] \ldots\ldots\ldots\ldots\ldots\ldots (55b);$$

and other equations can be written down with y, z in place of x, and with U, V, W interchanged.

Again $$\Omega_{xy} = [\xi_1 U_{\eta_1}] + [\xi_2 V_{\eta_2}] + [\xi_3 W_{\eta_3}];$$

therefore

$$\begin{aligned}
\Omega_{xy}[U, x] - [U, x]\,\Omega_{xy} &= [\xi_1 U_{\eta_1}] * [U_1 U_{\xi_1}] + [\xi_1 U_{\eta_1}] * [U_2 V_{\xi_2}] + [\xi_1 U_{\eta_1}] * [U_3 W_{\xi_3}] \\
&\quad + [\xi_2 V_{\eta_2}] * [V_2 V_{\xi_2}] + [\xi_3 W_{\eta_3}] * [U_3 W_{\xi_3}] \\
&\quad - [U_1 U_{\xi_1}] * [\xi_1 U_{\eta_1}] - [U_2 V_{\xi_2}] * [\xi_2 V_{\eta_2}] - [U_3 W_{\xi_3}] * [\xi_3 W_{\eta_3}] \\
&= \left[\xi_1 U_{\eta_1} U_{\xi_1} + U_1 \frac{\partial}{\partial \xi_1}(\xi_1 U_{\eta_1})\right] + [\xi_2 U_{\eta_2} V_{\xi_2}] + [\xi_3 U_{\eta_3} W_{\xi_3}] + \left[U_2 \frac{\partial}{\partial \xi_2}(\xi_2 V_{\eta_2})\right] \\
&\quad + \left[U_3 \frac{\partial}{\partial \xi_3}(\xi_3 W_{\eta_3})\right] - \left[\xi_1 \frac{\partial}{\partial \eta_1}(U_1 U_{\xi_1})\right] - \left[\xi_2 \frac{\partial}{\partial \eta_2}(U_2 V_{\xi_2})\right] - \left[\xi_3 \frac{\partial}{\partial \eta_3}(U_3 W_{\xi_3})\right] \\
&= [U_1 U_{\eta_1}] + [U_2 V_{\eta_2}] + [U_3 W_{\eta_3}] \\
&= [U, y] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55c).
\end{aligned}$$

Similarly

$$\Omega_{xz}[U, x] - [U, x]\,\Omega_{xz} = [U, z] \ldots\ldots\ldots\ldots\ldots\ldots (55d);$$

and other equations can also be obtained with V, W in place of U, or with x, y, z interchanged.

Equations (55a), (55b) show that any function annihilated by $[U, x]$, $[V_1]$ and $[W_1]$ will also be annihilated by $[V, x]$ and $[W, x]$; and then equations (55c) and (55d), with similar equations in which V, W are written for U, show that the function will also be annihilated by $[U, y]$, $[V, y]$, $[W, y]$, $[U, z]$, $[V, z]$, $[W, z]$.

Hence defining a combinant of the emanants (54) as an invariant which remains invariant when u, v, w are replaced by linear functions of u, v, w, we see that any combinant of the emanants which is annihilated by any one of the operators $[U, x]$, $[U, y]$, ... will be annihilated by all the others and will therefore be a reciprocant in the sense defined in § 29.

IX. *On Divergent Hypergeometric Series.* By Prof. W. M'F. Orr, M.A., Royal College of Science, Dublin.

*Addition**. [*Received* 3 April 1899.]

13. We have obtained the complete solution of equation (3) in divergent series only in the case in which $m = n + 1$. It has been shown by Stokes (*Camb. Phil. Soc. Proc.* Vol. VI.) that in any case in which $m < n + 1$, as x increases indefinitely, remaining real and positive, the ratio of

$$\frac{\Pi(\alpha_1 - 1)\,\Pi(\alpha_2 - 1)\ldots\Pi(\alpha_n - 1)}{\Pi(\rho_1 - 1)\,\Pi(\rho_2 - 1)\ldots\Pi(\rho_n - 1)} F(\alpha_1, \alpha_2, \ldots \alpha_m;\ \rho_1, \ldots \rho_n;\ x)$$

to

$$(n + 1 - m)^{-\frac{1}{2}}\,(2\pi y)^{\frac{m-n}{2}}\, y^{n-m+\Sigma\alpha-\Sigma\rho}\, e^{(n+1-m)y} \quad \ldots\ldots\ldots\ldots(60),$$

where $y^{n+1-m} = x$, has unity for its limit. The form of this expression, which admits for a complex x of $n + 1 - m$ independent values, suggests that $n + 1 - m$ (the missing number) further independent solutions may be obtained each as the product of one of the values of (60) by a divergent series proceeding in descending powers of y. The form of this series, even when no attention is paid to its arithmetical significance, is somewhat complicated even for the case in which $m = n$ and its complexity increases with every increase in the value of $n + 1 - m$. We will therefore content ourselves with establishing the forms towards which as x increases indefinitely the equations connecting the convergent and divergent functions tend. It may be convenient to use Lord Kelvin's symbol for approximate equality, viz. $\doteqdot$, to denote that under certain circumstances, obvious from the context, the ratio of the two expressions it connects may be made as nearly as desired equal to unity.

It may be readily proved by induction that when x is great

$$\frac{\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)} F(\alpha_1, \alpha_2, \ldots \alpha_n;\ \rho_1, \rho_2, \ldots \rho_n;\ -x)$$

$$+ \frac{\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_1 - 1)\,\Pi(\rho_1 - \alpha_2 - 1)\ldots\Pi(\rho_1 - \alpha_n - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_n - \rho_1 + 1;$$

$$2 - \rho_1, \rho_2 - \rho_1 + 1, \ldots \rho_n - \rho_1 + 1;\ -x)$$

$+ (n - 1)$ other terms analogous to the last

$$\doteqdot e^{-x}\, x^{\Sigma\alpha - \Sigma\rho} \quad \ldots\ldots\ldots\ldots(61).$$

* See *Trans. Camb. Phil. Soc.*, Vol. XVII. Part III. p. 171.

the argument of every power x^m lying between $-m\pi$ and $+m\pi$. (As limiting forms of this equation are required, it may be well to examine its genesis more fully.) In establishing this result we may follow the procedure of Art. 8; we consider

$$x^{1-\rho}\int (v-x)^{\rho-\alpha-1}\, v^{\alpha-1}\,\phi(-v)\,dv \quad\text{.............................(62)},$$

where $\phi(-v)$ is a solution of the equation satisfied by $F(\alpha_1, \ldots \alpha_n; \rho_1, \ldots \rho_n; -v)$ and the path of integration is the same as in Fig. 3.

Equations (38) are now to be replaced by

$$\left.\begin{aligned} C\,(e^{2\pi i(\alpha-\rho)} - e^{2\pi i\alpha}) &= \frac{\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\ldots\Pi(-\alpha_n)} \\ C_r\,(e^{2\pi i(\alpha-\rho)} - e^{2\pi i(\alpha-\rho_r)}) &= \frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\ldots\Pi(\rho_r-\alpha_n-1)} \end{aligned}\right\} \quad\text{......(63)};$$

hence if we assume equation (61) to hold for the function of the $(n+1)$th order, the integral remains finite for each of the four paths considered without any restriction on the values of the α's and ρ's. The expression which replaces (39) may be written in the form

$$\int_x^{(x)} x^{1-\rho}(v-x)^{\rho-\alpha-1}\,v^{\alpha-1}\,\psi(v)\quad\text{.................................(64)},$$

where as v increases indefinitely $\psi(v) \fallingdotseq e^{-v}\, v^{\sum_1^n(\alpha_r-\rho_r)}$ and initially the argument of $\psi(v)$ is zero. It may be shown as in Art. 5 that when x is increased indefinitely this tends to equality with

$$(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)\,e^{-x}\,x^{\alpha-\rho+\sum_1^n(\alpha_r-\rho_r)}\quad\text{..........................(65)}.$$

The expression which now replaces (41) may be shown to be equal to

$$-(e^{2\pi i(\rho-\alpha)}-1)\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\ldots\Pi(\rho-\rho_n-1)}{\Pi(\rho-\alpha_1-1)\,\Pi(\rho-\alpha_2-1)\ldots\Pi(\rho-\alpha_n-1)}\,x^{1-\rho}\,F(\alpha-\rho+1,$$
$$\alpha_1-\rho+1, \ldots \alpha_n-\rho+1;\, 2-\rho,\, \rho_1-\rho+1, \ldots \rho_n-\rho+1;\, -x)\quad\text{.........(66)}.$$

The value of the whole integral (62) is now, instead of (46),

$$(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)\left[\frac{\Pi(-\rho)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha)\,\Pi(-\alpha_1)\ldots\Pi(-\alpha_n)}\,F(\alpha, \ldots \alpha_n; \rho, \ldots \rho_n; -x)\right.$$
$$+\sum_{r=1}^{r=n}\frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha-1)\,\Pi(\rho_r-\alpha_1-1)\ldots\Pi(\rho_r-\alpha_n-1)}\,x^{1-\rho_r}\,F(\alpha-\rho_r+1, \ldots$$
$$\left.\alpha_n-\rho_r+1:\, 2-\rho_r,\, \rho-\rho_r+1, \ldots \rho_n-\rho_r+1;\, -x)\right]\quad\text{.........(67)}.$$

Equating (67) to the sum of (65) and (66) and dividing by $(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)$ an equation is obtained of the same type as (61) but with an additional α and an additional ρ. This equation, omitted through considerations of space, we will number (68).

Equation (61) holds even if the limits for the argument of x be extended to $-3\pi/2$

and $+3\pi/2$, for the difference between the two values of the left-hand member which are thus stated to be approximately equal to the same multiple of the right-hand member may be shown to be a linear function of the n solutions which when x is great are respectively of the orders $x^{-\alpha_1}, \ldots x^{-\alpha_n}$, and therefore when the real part of x is negative is very small compared with the right-hand member.

We now desire to find the limiting form of (68) when we write $\alpha = -h$, $x = y/h$ then make h increase indefinitely and finally suppose the modulus of y to be large. Multiplying the equation by $\Pi(h)$ the limiting form of the left-hand member, whatever be the modulus of y, is

$$\frac{\Pi(-\rho)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)}\,F(\alpha_1,\ldots\alpha_n:\ \rho,\ \rho_1,\ldots\rho_n:\ +y)$$

$$+\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\ldots\Pi(\rho-\rho_n-1)}{\Pi(\rho-\alpha_1-1)\ldots\Pi(\rho-\alpha_n-1)}\,y^{1-\rho}F(\alpha_1-\rho+1,\ldots\alpha_n-\rho+1;\ 2-\rho,$$

$$\rho_1-\rho+1,\ldots\rho_n-\rho+1:\ +y)$$

$+n$ other terms analogous to the last .... (69):

while the right-hand member, *i.e.* a certain multiple of (64), for any value of y becomes

$$\frac{\Pi(h)}{\Pi(\rho+h-1)}\int_{y/h}^{\infty}(y/h)^{1-\rho}(v-y/h)^{\rho+h-1}\,v^{-h-1}\,\psi(v),$$

or as h increases indefinitely

$$y^{1-\rho}\int_{\epsilon y}^{\infty}v^{\rho-2}e^{-y/v}\,\psi(v)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70),$$

where $\psi(v)$ when v is small is of the order of a power of v, and when v is great is approximately equal to $e^{-v}v^{\overset{n}{\underset{1}{\Sigma}}(\alpha_r-\rho_r)}$, provided the argument of v lies between $-3\pi/2$ and $+3\pi/2$, and ϵ is an indefinitely small positive quantity. It should be noted that the limitations placed on the argument of x in the integrals which have been expressed by divergent series were only imposed in order to make those series arithmetically intelligible in the sense of equation (28), but that while the integral forms are retained no such limitations are necessary. We may accordingly suppose that in (70) the limits of the argument of y are still further extended to -2π and $+2\pi$; for in evaluating

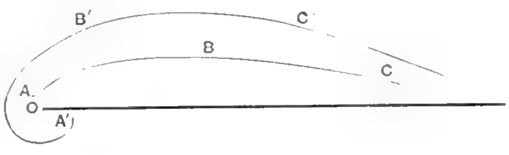

Fig. 6.

the integral when the argument of y lies between $-3\pi/2$ and -2π we may change the lower limit to a point whose argument is $-3\pi/2$ without altering its value, and so have all along the path of integration $\psi(v) = e^{-v}v^{\Sigma(\alpha_r-\rho_r)}$. As regards the path of

v a consideration of (37) of which it is a limiting form, and of Fig. 3, shows that if the argument of y lies between 0 and 2π the path must be such as ABC, or $A'B'C'$, (Fig. 6), while if the argument lies between 0 and -2π the path must be such as the image of this with respect to the axis of real quantities.

We can take y so great that $\psi(v)$ is as nearly equal as we please to $e^{-v}v^{\sum_1^{n}(\alpha_r-\rho_r)}$ for values of v for which $e^{-y/v}$ is as small as we please, and accordingly so that the value of (70) is as nearly as we please equal to that obtained by replacing $\psi(v)$ by this approximation. We would then have to consider an integral of the type

$$\int_{\epsilon y}^{\infty} e^{-v-y/v}\, v^p\, dv \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71).$$

This is a particular case of another with which we will have to deal, viz.:—

$$\int_{\epsilon y}^{\infty} e^{-sv-y^s v^{-s}}\,.\, v^p\, dv \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72),$$

s being a positive integer, and ϵ an indefinitely small positive quantity, the argument of v at the infinite limit being zero, and the argument of y lying between $-(s+1)\pi/s$ and $+(s+1)\pi/s$, the path of integration thus being permitted to make round the origin a number of revolutions determined by the initial argument. See Fig. 7, in

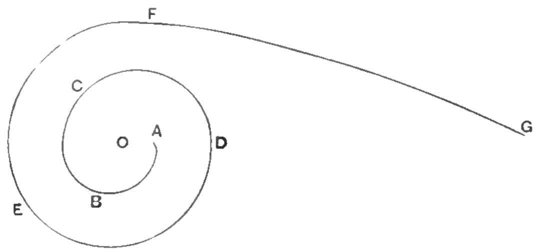

Fig. 7.

which $ABCDEFG$ represents a case in which the argument of y is positive, as we will at first suppose.

The value of $e^{-sv-y^s v^{-s}}$ is stationary for values of v given by the equation

$$1 - y^s v^{-s-1} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (73);$$

let v_1 be that root whose argument is $s/(s+1)$ times the argument of y, and thus lies between 0 and $+\pi$. It may be noted that if the point corresponding to any other solution of (73) lies in the region traversed by the path of integration in (72), the real part of v at any such point is very much greater algebraically than the real part of v_1, and therefore the modulus of $e^{-sv-y^s v^{-s}}$ very much less than that of $e^{-sv_1-y^s v_1^{-s}}$. We will now suppose the path of integration to pass through v_1, and consider separately the two portions of the path from v_1 to ∞ and from ϵy to v_1. Considering the former, let part of it be a straight line starting from v_1 in a direction whose argument is half that of v_1, a direction which makes an acute angle with the positive part of the axis of real quantities and an obtuse angle with the line joining v_1 to the origin.

Expanding v^{-s} in powers of $v - v_1$ we have for points on this line

$$sv + y^s v^{-s} = (s+1)\,v_1 + s\,(s+1)\,(v - v_1)^2/2v_1 + R,$$

where mod. $R < s\,(s+1)\,(s+2)\,(v - v_1)^3/6v_1^2$ (see Art. 2). We can thus take y so great that along this line R is less than any assigned quantity for a range such that throughout it the ratio of v to v_1 is as nearly as we please equal to unity, and at the end of it the term $s\,(s+1)\,(v - v_1)^2/2v_1$ is greater than any assigned quantity. If v_2 denote the value of v at the end of the range this may be done by increasing v_1 and making $v_2 - v_1$ vary as $v_1^{7/12}$ (say). Along such a range then $v^p\,e^{-sv-y^s v^{-s}}$ is as nearly as we please equal to

$$v_1^p\,e^{-(s+1)\,v_1}\,e^{-s\,(s+1)\,(v-v_1)^2/2v_1},$$

while at the end of it the final factor of the last expression is less than any assigned quantity, since $(v - v_1)^2/2v_1$ is real and positive.

The portion contributed to the integral (72) by this range is thus as nearly as we please equal to

$$v_1^p\,e^{-(s+1)\,v_1}\int_0^{\infty} e^{-s\,(s+1)\,(v-v_1)^2/2v_1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74),$$

or

$$\left(\frac{\pi v_1}{2s\,(s+1)}\right)^{\frac{1}{2}} v_1^p\,e^{-(s+1)\,v_1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75),$$

wherein the argument of $v_1^{p+1/2}$ is $p + 1/2$ times the argument of v_1.

We next proceed to show that the portion contributed to the integral by the path from v_2 to ∞ can at the same time be made less than any assigned quantity. Consider the expression

$$sv + y^s v^{-s} - \{sv_2 + y^s v_2^{-s} + s\,(v - v_2)\,(1 - c)\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (76),$$

where c is real, positive, and less than unity. This expression vanishes when $v = v_2$, and its real part is infinitely great and positive when, and only when, v is infinite and has its real part positive. It is therefore evident that a curve can be drawn from v_2 to ∞, in the negative direction round the origin, such that along it the real part of (76) increases continually. Everywhere along such a curve we would have

$$\text{mod. } e^{-sv-y^s v^{-s}} < \text{mod. } e^{-sv_2-y^s v_2^{-s}}\,e^{-s\,(v-v_2)\,(1-c)}.$$

The part contributed to the integral by this curve would be increased if each term were replaced by its modulus, but if the argument of v_1 is $\pi - \alpha$ this would replace

$$\int_{v_2}^{\infty} e^{-s\,(v-v_2)\,(1-c)}\,v^p dv$$

by something certainly less than

$$\Pi\,(p)\left\{s\,(1-c)\sin\frac{\alpha}{2}\right\}^{-p-1}$$

which is finite while the factor $e^{-sv_2-y^s v_2^{-s}}$ is less than any assigned quantity, therefore this part is less than any assigned quantity.

We next consider the portion of the integral contributed by the path from ϵy to v_1. By means of the substitutions

$$y^s v^{-s} = u s^{-1}, \quad s^{\frac{s-1}{s}} y = y_1^{\frac{1}{s}}$$

this portion can be made to depend on an integral similar to that just discussed, except that s and $1/s$ are interchanged, and can be shown to be also equal to (75) when y is very great.

Thus (72) tends to equality with twice (75); and a similar result is true when the argument of y is negative.

We now write in (75)

$$s = 1, \quad p = \rho - 2 + \sum_1^n (\alpha_r - \rho_r),$$

and accordingly when y is increased indefinitely (70) tends to equality with

$$\pi^{\frac{1}{2}} y^{\frac{1}{2}(\Sigma\alpha - \Sigma\rho + \frac{1}{2})} e^{-2y^{\frac{1}{2}}},$$

$\Sigma\rho$ now including ρ.

This result refers to the function of the $(n+2)$th order. If we now reduce the order by unity, omitting α_n and ρ, and change y into x, we have the equation

$$\frac{\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\ldots\Pi(-\alpha_{n-1})} F(\alpha_1, \ldots \alpha_{n-1};\ \rho_1, \ldots \rho_n:\ +x)$$

$$+ \frac{\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_1 - 1)\ldots\Pi(\rho_1 - \alpha_{n-1} - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_{n-1} - \rho_1 + 1;$$

$$2 - \rho_1,\ \rho_2 - \rho_1 + 1, \ldots \rho_n - \rho_1 + 1;\ +x)$$

$$+ (n-1) \text{ other terms analogous to the last}$$

$$= \pi^{\frac{1}{2}} x^{\frac{1}{2}(\Sigma\alpha - \Sigma\rho + \frac{1}{2})} e^{-2x^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(77)}.$$

As this relation holds while the argument of x ranges from -2π to $+2\pi$, it is equivalent to two independent relations among the functions considered.

It also holds even when the limits of the argument are extended to -3π and $+3\pi$, for the difference between the two values of the left-hand member which are thus stated to be approximately equal to the same multiple of the right-hand member may be shown to be small compared with either.

We next write in equation (68) $\alpha = -h$, $\alpha_n = -k$, $x = y/hk$. Let h and k increase indefinitely, and finally suppose the modulus of y large. On multiplying the equation by $\Pi(h)\,\Pi(k)$ the limiting form of the left-hand member, whatever be the modulus of y, is similar to (69) except that there is no α_n, while the right-hand member considered as a limiting form of (70) becomes for any value of y

$$\frac{\Pi(h)}{\Pi(\rho + h - 1)} \int_{y/h}^{\infty} (y/h)^{1-\rho} (v - y/h)^{\rho+h-1} v^{-h-1} \chi(v)\, dv \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(78)},$$

where $\chi(v)$ when v is small is of the order of a power of v, and when v is great is approximately equal to

$$\pi^{\frac{1}{2}} v^{\frac{1}{2}(\Sigma\alpha - \sum_1^n \rho_r + \frac{1}{2})} e^{-2v^{\frac{1}{2}}}$$

provided the argument of v lies between -3π and $+3\pi$. When h is increased indefinitely (78) becomes

$$y^{1-\rho} \int_{\epsilon y}^{\infty} v^{\rho-2} e^{-y/v} \chi(v)\, dv \qquad \text{(79)},$$

wherein, as may be seen by considering the integral of which this is a limiting form, the argument of every power of v at the upper limit is zero; the argument of y will be supposed to lie between -3π and $+3\pi$ in order that the approximate form for $\chi(v)$ may be applicable.

By the substitutions $v = v'^2$, $y = y'^2$, $\epsilon = \epsilon'^2$, (79) is reduced to the same type as (72), s having the value 2, and its evaluation thus leads to the result

$$2\pi \,.\, 3^{-\frac{1}{2}} y^{\frac{1}{3}(\Sigma\alpha - \Sigma\rho + 1)} e^{-3y^{\frac{1}{3}}} \qquad \text{(80)},$$

$\Sigma\rho$ now including ρ.

The resulting equation is thus obvious; it is established for values of y (or x) whose arguments lie between -3π and $+3\pi$, and thus gives three independent relations. As in the preceding case the limits of argument may be extended to -4π and $+4\pi$. This result is used in establishing the next case, and so on.

Thus we obtain the general result expressed by the following equation

$$\frac{\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_m)} F(\alpha_1, \alpha_2, \ldots \alpha_m;\ \rho_1, \rho_2, \ldots \rho_n;\ (-)^{n-m+1} x)$$

$$+ \frac{\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_1 - 1)\ldots\Pi(\rho_1 - \alpha_m - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_m - \rho_1 + 1 :$$

$$2 - \rho_1, \ldots \rho_n - \rho_1 + 1;\ (-)^{n-m+1} x)$$

$+ (n-1)$ other terms analogous to the last

$$\doteqdot (n+1-m)^{-\frac{1}{2}} (2\pi)^{\frac{n-m}{2}} v_1^{\frac{n-m}{2} + \Sigma\alpha - \Sigma\rho} e^{-(n+1-m)v_1} \qquad \text{(81)},$$

where $v_1^{n+1-m} = x$ and the argument of v_1 may have any value between $-\pi$ and $+\pi$. This equation is thus equivalent to $n+1-m$ independent equations.

I have verified that if, by means of the linear equations thus obtained, one of the convergent series be expressed in terms of the divergent functions, the result obtained for a real positive x agrees with that of Stokes (*loc. cit.*).

A complicated series may be obtained for the integral (72), by integrating by parts, writing $y^s = v_1^{s+1}$, $sv + v_1^{s+1} v^{-s} - (s+1) v_1 = u^2$, and expanding $v^{\rho+1}$ in powers of u by Lagrange's theorem.

If we denote $z^{s/2}(1+2z+3z^2+\dots sz^{s-1})^{-1/2}$ by $\phi(z)$ the result thus obtained is

$$\pi^{\frac{1}{2}}\,v_1^{p+\frac{1}{2}}\,e^{-(s+1)\,v_1}\left\{\left(\frac{2}{s(s+1)}\right)^{1/2}+\left[\frac{d^2}{dz^2}(z^p\{\phi(z)\}^3)\right]_{z=1}\frac{1}{4v_1}\right.$$
$$\left.+\left[\frac{d^4}{dz^4}(z^p\{\phi(z)\}^5)\right]_{z=1}\frac{1}{1\,.\,2\,.\,(4v_1)^2}+\dots\right\}\quad\dots\dots\dots\dots(82).$$

Thus if there are no α's and s ρ's the right-hand member of (81), in order that the equation should be exact, should be increased in the ratio of the above series to its first term, p being $(s-1)/2-\Sigma\rho$. If $s=1$ we may thus derive the semi-convergent series connected with the Bessel Functions. For other values the series is complicated.

If there are any α's the result is still more complicated.

In the case in which $m=n$, the result in case there is only one α and one ρ is given in equation (27).

If another α and ρ are now introduced it may be shown that to make equation (61) exact the right-hand member should be multiplied by the infinite series of divergent series:—

$$\phi(\rho_1-\alpha_2-\alpha_1+1,\ \rho_2-\alpha_2;\ -1/x)-\frac{(1-\alpha_1)(\rho_1-\alpha_1)}{1\,.\,x}\phi(\rho_1-\alpha_2-\alpha_1+2,\ \rho_2-\alpha_2;\ -1/x)$$
$$+\frac{(1-\alpha_1)(2-\alpha_1)(\rho_1-\alpha_1)(\rho_1-\alpha_1+1)}{1\,.\,2\,.\,x^2}\phi(\rho_1-\alpha_2-\alpha_1+3,\ \rho_2-\alpha_2;\ -1/x)+\dots$$

Another α and ρ make this series triply infinite, and so on.

If there is one α and two ρ's the limiting form of the above shows that the right-hand member of (77) should in that case be multiplied by

$$\phi\left(\rho_1-\rho_2-\alpha+3/2,\ \alpha+\rho_2-\rho_1-1/2;\ \frac{-1}{2v_1}\right)$$
$$-\frac{(1-\alpha)(\rho_1-\alpha)}{1\,.\,v_1}\phi\left(\rho_1-\rho_2-\alpha+5/2,\ \alpha+\rho_2-\rho_1-3/2;\ \frac{-1}{2v_1}\right)$$
$$+\frac{(1-\alpha)(2-\alpha)(\rho_1-\alpha)(\rho_1-\alpha+1)}{1\,.\,2\,.\,v_1^2}\phi\left(\rho_1-\rho_2-\alpha+7/2,\ \alpha+\rho_2-\rho_1-5/2;\ \frac{-1}{2v_1}\right),$$

where $v_1^2=x$. Each series here is a semi-convergent series connected with the Bessel functions. Another α and ρ make this series triply infinite, and so on.

It is to be noted that in the case in which $m=n+1$ no numerical connexion has been here established between the series which proceed in ascending and those which proceed in descending powers of x.