TRANSACTIONS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. VI.

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HISTORY

OF THE

SOCIETY.

IN the original charter of the Royal Society, it was provided that the collections of the Society should be deposited, if belonging to Natural History, in the Museum of the University, and if to Antiquities, in the Library of the Faculty of Advocates. Much inconvenience, however, could hardly fail to result from this arrangement, especially when the researches of the Society, having, as of late, been much turned to Geology, it became an object to collect together the specimens which served to illustrate the subjects under discussion, and to have them at hand when reference should be necessary.

IN a Museum arranged with a view to public lectures, (like that of the University), such an order as was required for this purpose could not easily be preserved; the Professor of Natural History must feel himself interrupted by the examinations which the Members of the Royal Society might wish to make ; and it would often be a point of delicacy, not to give him the trouble that such examinations would require.

These considerations induced the Society to apply for a new charter, under which its collections should remain in its own possession, so as to be at all times accessible to its Members.

As the interest of the two bodies just mentioned, might be somewhat affected by these alterations, the first step taken was to give them information of the intentions of the Society, and to request their concurrence in a measure of such manifest justice and utility. The Faculty of Advocates readily assented to this proposal; and the University, though at first in doubt whether it were not bound in duty to resist the alteration, on more mature deliberation, resolved to withdraw all opposition.

As it was not meant that the new charter should have any retrospect, the Huttonian Collection, with a great number of other articles, the property of the Society, still remain in the University Museum. The foundation of a new collection, in the Society's apartments, has been laid, by a cabinet presented by Mr Allan, containing specimens of the rocks round Edinburgh ; a collection by Colonel Imrie, illustrating the section of the Grampians which he has given in the 5th volume of the *Transactions* of the Society; and a collection of specimens from Sir George Mackenzie, illustrating the Mineralogy of Iceland.

The New Charter, which follows, hardly differs in any thing from that contained in the first volume of the *Transactions* of the Society, except in what respects the two restrictions that have just been mentioned.

CARTA

CARTA

NOVÆ ERECTIONIS

SOCIETATIS REGALIS EDINBURGI,

1808.

GEORGIUS TERTIUS, DEI *gratia*, *Britanniarum Rex, Fidei Defensor; omnibus probis hominibus ad quos præsentes literæ nostræ pervenerint, salutem: Quandoquidem Nos considerantes, quod petitio humilis nobis oblata fuerit a Regali Societate Edinburgi, et præfideli nostro et prædilecto consanguineo Henrico Duce de Buccleuch, ejusdem præside, in nomine et vice Societatis, et omnium ejusdem Sociorum; in qua petitione enarratur, quod per regiam nostram cartam, datam vigesimo nono die mensis Martii anno Domini millesimo septingentesimo et octogesimo tertio, Nobis benignè placuisset constituere, erigere et incorporare quosdam ibi nominatos in corpus politicum et corpo-*

 ratum,

ratum, nomine tituloque Regalis Societatis Edinburgi, *ad promovendas literas et scientiam utilem, cum facultatibus et privilegiis ibidem concessis, et speciatim, ut potens et capax sit tenendi proprietatem realem et personalem, causasque agendi et defendendi, Præsidem et Socios eligendi, canones ordinandi, et perpetuam successionem sub tali nomine habendi: quod, virtute prædictæ cartæ, Regalis Societas Edinburgi, ita creata, substituerit, suisque officiis a prima institutione, ritè functa sit: quod cartâ prædictâ ordinatum fuerit, cunctas res antiquas, tabulas publicas, librosque manuscriptos, quos acquisiverit Societas, in Bibliotheca Facultatis Juridicæ deponi; atque universas res ad historiam naturalem pertinentès, quasque Societas acquisiverit, in Musæo Academiæ Edinensis deponi: quod, ab hac constitutione incommodum haud parvum ortum fuerit; cùm Regalis Societas, nullum jus in Bibliothecarios Facultatis Juridicæ, nec in Custodes Musæi Academiæ Edinensis, habeat, nec horas eorum ministerii regulasve admissionis ad ea repositoria præscribere possit, nec Societati licitum sit congressus suos in eorum alterutro tenere; quæ cùm ita sint, hactenus Societati non licuit suas collectiones ita disponere, ut Sociorum aliorumve studio et disquisitioni aptè subjiciantur, undè et alia dona expectanda essent: Quod prædicta Societas, causâ hæc incommoda amovendi, nostraque bona proposita in hac institutione ad effectum perducendi, sapientiæ nostræ regiæ humiliter subjiciat, ut detur Societati jus collectiones suas cujuscunque generis uno in loco deponendi, quo sibi ordine placuerit, sub custodibus a Societate eligendis ejusque potestati subjectis; itaque ut cartam, cum privilegiis idoneis humilibus nostris petitoribus concedere dignemur; ut et in hac petitione oratum sit, ut Nobis benignè placeret de novo Cartam Nostram Regiam concedere dictæ Regali Societati Edinburgi, ejusque Sociis, qua iterum darentur jura, facultates, et privilegia, in carta regia per quam*

quam corpus istud creatum fuerat concessa, et qua insuper provideretur, uti nobis in regia nostra sapientia idoneum videatur, ut Societati potestas daretur collectiones suas anteà memoratas in uno ædificio deponendi, eis legibus, et eis ministris, qui Societati placerent, hosque sibi subjectos haberet: Et nos certiores facti hanc petitionem justam esse rationique consentaneam, et certis conditionibus et modis, in præsentibus expressis, concedi debere: Igitur, *constituimus, erigimus et incorporavimus, sicuti Nos regiâ nostrâ prærogativâ, et gratiâ speciali, pro Nobis notrisque regiis successoribus, per has præsentes, constituimus, erigimus, et incorporamus, prædictum Henricum Ducem de Buccleuch, Sociosque dictæ Regalis Societatis, atque alios qui postea eligentur Socii, in unum corpus politicum et corporatum, vel legalem incorporationem, nomine et titulo* Regalis Societatis Edinburgi, *ad promovendas literas et scientiam utilem, utque talis existens, et tali nomine, perpetuitatem habeat et successionem;* declarantes, *Quod dicta Societas capax sit capere, tenere, et frui proprietate reali seu personali, et petere, causas agere, defendere et respondere, et conveniri, in jus trahi, defendi et responderi, in omnibus seu ullis nostris Curiis Judicaturæ; et declarantes quod dictæ Societati fas sit, sigillo, tanquam Societatis sigillo, uti; dantes potestatem dictæ Societati, per majorem suffragiorum numerum eorum qui aderunt, eligendi Præsidem aliosque officiarios pro negotiorum administratione; necnon ordinandi canones, ad quos Socii sint eligendi et res Societatis sint administrandæ, conditionibus hujus cartæ sive donationis haud incongruentes, nec legibus et praxi nostri regni Scotiæ contrarios; et declarantes, quod hujusmodi canones sanciri nequeant, nisi ritè propositi fuerint in congressu habito saltem uno mensi ante illum congressum quo sanciendi sint: dantes etiam potestatem Societati ordinandi et administrandi collectiones re-*

rum

rum antiquarum, tabularum publicarum, librorum manuscriptorum, et rerum ad historiam naturalem pertinentium, quas Societas posteà acquisiverit, easque in Musæo et Bibliotheca, tali ordine et modo ut Societati placuerit, deponendi: SALVIS *tamen conditionibus, in hac nostra carta provisis; declarantes insuper hanc cartam nostram concessam esse sub his conditionibus sequentibus, videlicet, Quod jura, facultates, et privilegia, per præsentes in dictam Societatem collata, nullo modo detrahent de ullo jure dominii quod competit Academiæ Edinensi in collectiones antehac depositas in Musæo Academiæ, virtute cartæ nostræ Societati Regali datæ, prædicto vigesimo nono die mensis Martii millesimo septingentesimo et octogesimo tertio; antedicta Societate quantum in se est astricta, omne jus, ad collectiones antehac factas et in Musæo prædicto depositas, in dictam Academiam transferre; et quod Historiæ Naturalis Professori copia introitûs in Musæum et Bibliothecam Societatis Regalis detur æquè ac Sociis ipsius Societatis; et quod dictæ Societati non sit licitum constituere Professorem, prælectorem seu Doctorem Mineralogiæ, Geologiæ, aut Historiæ Naturalis, nec suis collectionibus uti ad talem institutionem promovendam, nisi quæ vel nunc sit, vel posthac fuerit, in Academia Edinensi.* —IN CUJUS REI TESTIMONIUM, *sigillum nostrum per Unionis Tractatum custodiend., et in Scotia vice et loco Magni Sigilli ejusdem utend., ordinat., præsentibus appendi mandavimus; Apud Aulam nostram apud St James's, vigesimo septimo die mensis Decembris anno Domini millesimo octingentesimo et octavo, regnique nostri anno quadragesimo nono.*

Per

Per signaturam manûs D. N. Regis supra script.

Written to the Seal, and registered the thirtieth day of August 1811.

JAMES DUNDAS, *Dep^t.*

Sealed at Edinburgh, the thirtieth of August, One thousand eight hundred and eleven years.

JAMES ROBERTSON,
Sub^t. £80 *Scots.*

THIS charter, as well as the former, having left the Society in possession of the power of making By-laws for the regulation of its affairs, it was proposed to revise the whole of those laws, and to make such alterations as, after the experience of thirty years, might appear to be necessary.

THE Society, therefore, having at several General Meetings taken this subject into consideration, after mature deliberation,

beration, and with due attention to the clause in the charter that respects the enactment of such laws, did, at a General Meeting, on the 23d of December 1811, sanction the Laws that follow, and declare them to be the rules by which the Society is to be governed, till all, or any of them are regularly repealed.

LAWS

LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH,

ENACTED 23d MAY 1811.

I.

THE ROYAL SOCIETY OF EDINBURGH shall be composed of Ordinary and Honorary Members.

II.

Every Ordinary Member, within three months after his election, shall pay as fees of admission Three Guineas, and shall further be bound to pay annually the sum of Two Guineas, into the hands of the Treasurer.

III.

Members shall be at liberty to compound for their annual subscription, each paying according to the value of an annuity on his life, determined as in the ordinary insurance on lives.

The power of raising the annual subscription shall remain with the Society.

IV.

Ordinary Members, not residing in Edinburgh, and not compounding for annual subscription, shall appoint some person residing in Edinburgh, by whom the payment of the said subscription is to be made, and shall signify the same to the Treasurer.

V.

Members failing to pay their subscriptions for three successive years, due application having been made to them by the Treasurer, shall cease to be Members of the Society, and the legal means for recovering such arrears shall be employed.

VI.

None but Ordinary Members are to bear any office in the Society, or to vote in the choice of Members or Office-bearers, nor to interfere in the patrimonial interest of the Society.

VII.

The number of Ordinary Members shall be unlimited.

VIII.

The Ordinary Members shall receive the volumes or parts of the Society's Transactions, when published, at the booksellers price, or the price at which they are sold to the trade. This regulation to continue in force for five years from the date of its enactment; and it is left to the Society then to consider, whether the volumes cannot be afforded *gratis* to the Members.

IX.

The Society having formerly admitted as Non-resident Members, gentlemen residing at such a distance from Edinburgh as to be unable regularly to attend the Meetings of the Society, with power to such Non-resident Members, when occasionally in Edinburgh, to be present at the Society Meetings, and to take a part in all their inquiries and proceedings, without being subjected to any contribution for defraying the expences of the Society; it is hereby provided, that the privileges of such Non-resident Members already elected shall remain as before; but no Ordinary Members shall be chosen in future under the title and with the privileges of Non-resident Members. The Members at present called Non-resident shall have an option of becoming Ordinary Members; if they decline this, they shall continue Non-resident as formerly.

X.

X.

The Honorary Members of the Society shall not be subject to the annual contributions. They shall be limited to Twenty-one, and shall consist of men distinguished for literature and science, not residing in Scotland.

XI.

Election of Members.

The election of Members, whether Ordinary or Honorary, shall be by ballot; it shall require the presence of Twenty-four Members at least to make a quorum, and the election shall be determined by the majority of votes.

XII.

The election of Members shall be made at one General Meeting annually, on the fourth Monday of January.

XIII.

No person shall be proposed as an Ordinary Member, without a recommendation presented by a Member of the Society, and subscribed by Three, to the purport mentioned below *; which recommendation shall be hung up in the Rooms of the Society, at least during Three Ordinary Meetings (of the Classes) previous to the day of election.

XIV.

In order to carry on with facility and success those improvements in science and literature, which are the objects of the institution, the Society shall be divided into two Classes, the Physical and the Literary Class; the former having for its department the sciences of Mathematics, Natural Philosophy, Chemistry, Medicine, Natural History, and what relates to the improvement of Arts and Manufactures; the latter having

* "A. B. a gentleman well skilled in many branches of Philosophy and Polite Learning, (Mathematics, Chemistry, Natural History, &c.) being to our knowledge desirous of becoming a Member of the Royal Society of Edinburgh, we whose names are subscribed, do recommend him as deserving of that honour, and as likely to prove an useful and valuable Member."

for its department the inquiries relative to Speculative Philosophy, Antiquities, Literature and Philology.

XV.

The Classes shall meet alternately on the first and third Mondays of every month, from November to June inclusive. It shall be competent, however, to bring matters of a Physical or Literary kind, before either Class of the Society indiscriminately. To facilitate this, one Minute-book shall be kept for both Classes; the Secretaries of the respective Classes either doing the duty alternately, or according to such agreement as they may find it convenient to make.

XVI.

The Society shall from time to time make a publication of its Transactions and Proceedings. For this purpose, the Council shall select and arrange the papers which they shall deem worthy of publication in the *Transactions* of the Society, and shall superintend the printing of the same.

The Transactions shall be published in Parts or *Fasciculi*, and the expence shall be defrayed by the Society.

XVII.

Office-bearers.

There shall be elected annually for conducting the publications and regulating the private business of the Society, a Council, consisting of a President; Two Vice-Presidents; a President for each Class of the Society; Six Counsellors for each Class; one Secretary for each; a Treasurer; a General Secretary; and a Keeper of the Museum and Library.

XVIII.

The election of the Office-bearers shall be on the fourth Monday of November.

XIX.

Four Counsellors, Two from each Class, shall go out annually. They are to be taken according to the order in which they presently stand on the list of the Council.

XX.

XX.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due. Treasurer.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the General Meeting in January, he shall present the accounts for the preceding year to be audited. At this Meeting the Treasurer shall also lay before the Society a list of all arrears due above twelve months, and the Society shall thereupon give such directions as they may find necessary for recovery thereof.

XXI.

At the General Meeting in November, a Committee of Three Members shall be chosen to audite the Treasurer's accounts, and give the necessary discharge of his intromissions.

The report of the examination and discharge shall be laid before the Society at the General Meeting in January, and inserted in the records.

XXII.

The General Secretary shall take down minutes of the proceedings of the General Meetings of the Society and of the Council, and shall enter them in two separate books. He shall keep a list of the Donations made to the Society, and take care that an account of such Donations be published in the Transactions of the Society. He shall, as directed by the Council, and with the assistance of the other Secretaries, superintend the publications of the Society. Secretary.

XXIII.

A Register shall be kept by the Secretary, in which copies shall be inserted of all the Papers read in the Society, or abstracts of those Papers, as the Authors shall prefer; no abstract or paper, however, to be published without the consent of the Author. It shall be understood, nevertheless, that a person choosing to read a paper, but not wishing to put

put it into the hands of the Secretary, shall be at liberty to withdraw it, if he has beforehand signified his intention of doing so.

For the above purpose, the Secretary shall be empowered to employ a Clerk, to be paid by the Society.

XXIV.

Another register shall be kept, in which the names of the Members shall be enrolled at their admission, with the date.

XXV.

A Seal shall be prepared and used, as the Seal of the Society.

XXVI.

The Librarian shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society ; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Members.

XXVII.

All articles of the above description shall be open to the inspection of the Members, at the Hall of the Society, at such times, and under such regulations, as the Council from time to time shall appoint.

PRESENTS made to the ROYAL SOCIETY OF EDINBURGH since the Year 1809.

The Sixth Volume of the Scriptores Logarithmici.—From Mr BARON MASERES.

Treatise on the Gout, by the late Dr HAMILTON of Lynn-Regis.—From the Author.

Traité de Mineralogie, par M. LE COMPTE DE BOURNON, 3 vols. 4to.—From the Author.

An Engraving, representing all the Mountains of the World, by R. Riddel, Esq; together with the History of Mountains, by Joseph Wilson, Esq; vols. 1st and 2d.—From Messrs RIDDEL and WILSON.

Recueil de Quelques Antiquités trouvées sur les Bords de la Mer Noire, par M. LEON DE WEXEL.—From the Author.

Nova Acta Petropolitana, tom. 14.—From the IMPERIAL ACADEMY of St Petersburgh.

Philosophical Essays, by THOMAS GORDON, Esq; 2 vols. 4to.—From the Author.

Transactions of the Linnean Society, vol. 8th and 9th.—From the LINNEAN SOCIETY.

Asiatic Researches, vol. 10th and 11th.—From the BENGAL SOCIETY.

Philosophical Transactions, for 1809, 1810, 1811.—From the ROYAL SOCIETY OF LONDON.

Memoirs of the American Academy, vols. 1st, 2d, and 3d.—From the AMERICAN ACADEMY.

Transactions of the American Philosophical Society, vol. 6th, part 2d.—From the AMERICAN PHILOSOPHICAL SOCIETY.

Observations on the Hydrargyria, by GEORGE ALLEY, M. D.—From the Author.

Transactions of the Geological Society of London, vol. 1st.—From the GEOLOGICAL SOCIETY.

Travels

Travels in Iceland, by Sir George Mackenzie, Baronet. Annals of Iceland, from 1796 to 1804. Manuscript copy of the Sturlinga Saga. History of Iceland during the 18th century. A compendium of Anatomy, translated into Icelandic, from the Works of Martinet. Pope's Essay on Man, in Icelandic verse.—From Sir George Mackenzie.

Essay on the Natural History of the Salt District in Cheshire, by Dr Holland.—From the Author.

Collection of Specimens, illustrating the Mineralogy of the Country round Edinburgh.—From Thomas Allan, Esq.

Collection of Specimens, illustrating the Section of the Grampians, at the beginning of this volume, with a descriptive Catalogue.—From Lieutenant-Colonel Imrie.

Model in Relief, representing the Granite Veins at the *Windy Shoulder* in Galloway.—From Sir James Hall, Baronet.

Collection of Specimens, illustrating the Mineralogy of Iceland.—From Sir George Mackenzie, Baronet.

I. *A* DESCRIPTION *of the* STRATA *which occur in aſcending from the* PLAINS *of* KINCARDINESHIRE *to the* SUMMIT *of* MOUNT BATTOC, *one of the moſt elevated points in the Eaſtern Diſtrict of the* GRAMPIAN MOUNTAINS. *By Lieutenant-Colonel* IMRIE, F. R. S. EDIN.

[*Read 5th March* 1804.]

THE moſt mountainous parts of Scotland are ſituated in its weſtern and north-weſt diſtricts. From thoſe parts of the country, ſeveral chains of mountains branch off, and continue their courſes in various directions, and to various extent. The moſt extended of thoſe chains is that of the Grampians. This chain takes its riſe from nearly about the centre of the above alpine diſtrict, and continues its courſe in a direction almoſt due eaſt, or perhaps a little to the ſouth of that point, until it diſappears in the German Ocean, betwixt the towns of Aberdeen and Stonehaven.

THIS chain, in its eaſtern diſtrict, conſiſts of three ranges, running nearly parallel to each other; the two lateral ranges being conſiderably lower than the central one. To the lateral mountains are attached a range of lower hills, that ſlope down into undulated grounds, which ſkirt the adjacent plains.

THE general ſhape of the individual mountains compoſing thoſe three ranges, is oblong, rounded, and ſometimes flattiſh on the tops; their length is always in the direction of the

 chain,

chain, that is to ſay, from weſt to eaſt: and I have obſerved, not unfrequently, that the weſtern ends of thoſe oblong mountains are more bulky than their eaſtern extremities, and that they ſlope and taper in ſome degree towards this quarter. Their general covering is that of a coarſe gravelly ſoil, produced by their own decompoſition; and the produce of this ſoil is heath. But upon ſome of the heights in the central range, I have found beds or layers of that ſpecies of turf called *Peat*, from fifteen to twenty feet in thickneſs, which repoſe upon the gravelly ſoil that there covers the native rock.

At this eaſtern part of the Grampians, where I am now about to endeavour to give a deſcription of the ſtratification, the mountains ſeldom ſhow any conſiderable extent of naked rock.

In their courſe to the eaſtward, as they approach the ſea, they begin to contract in breadth, and cover much leſs ſpace of country; and where they finiſh their courſe at the ſea, their height will ſcarcely entitle them to the appellation of hills: but although they become ſo diminutive in height and in breadth, yet the materials of which they are formed continue the ſame as thoſe which compoſe the ranges where they are in their greateſt altitude, and their exterior characters, as to form and figure, alſo continue the ſame.

Among the rivers which have their ſource in the Grampians, that of the *North Eſk* is not the firſt in rank as to ſize, nor is it the moſt diminutive. At a conſiderable diſtance from the plains in the interior of the mountains, a ſmall lake called *Loch Lee* is formed, in a rocky baſon, by a rivulet, and ſome ſprings and rills flowing from marſhy grounds. From this lake the North Eſk iſſues, not in a very conſiderable flow, but, being ſoon joined by other ſtreams and alpine torrents, it ſwells to a conſiderable ſize, and continues a courſe from this lake almoſt due eaſt, betwixt the central and ſouth lateral ranges of the mountains, for an extent of about ſeven miles: it then ſkirts Mount-Battoc, and being

ing there impeded, in its eaſtern direction, by ſome of the hills forming the baſis of that mountain, it then changes its courſe, almoſt at a right angle, and from thence flows in a due ſouth direction. In this laſt direction, it opens a way for itſelf through the ſouth lateral range, and enters the plains of Kincardine, and Forfar ſhires, where it immediately becomes the line of diviſion of thoſe two counties. It leaves thoſe plains by a hollow betwixt the two low hills of Garvoke and Pert, and after a courſe of nearly thirty miles from its ſource, it joins the ſea ſomewhat to the eaſtward of the town of Montroſe. It is in the bed of this river that I have examined the ſtrata of the Grampians of which I am now to give a deſcription. The ſection extends about ſix miles, from the horizontal grit or ſandſtone in the plain, to the granite of Mount Battoc, which is one of the mountains in the central range, and one of the higheſt of the chain in that part of the country. My direction, in this examination, is about due north, piercing through, almoſt at right angles, the ſtrata of the mountains, which are here nearly in a vertical poſition.

In this ſhort ſtretch of ſix miles, a great deal of matter highly intereſting to geology preſents itſelf. In it, we paſs from the ſecondary horizontal ſtrata of the neweſt formation, to the vertical, contorted, primary ſtrata of the oldeſt date, and terminate with granite, the primitive rock in the conception of many geologiſts. Thus, it embraces a complete range of the foſſil objects, which in this part of Scotland intervene between that which is deemed the oldeſt and what is accounted the moſt recent in point of formation. From the various ſtrata ſtanding in a poſition vertical, or nearly ſo, and the river North Eſk, cutting acroſs theſe ſtrata, at right angles, the ſucceſſion is uncommonly well exhibited to view, and a fair diſplay of the ſtructure of this country, and of the materials compoſing it to a great depth, is open to the attentive obſerver. In addition to this fine diſplay of the ſucceſſion of ſtrata, the arrangement of them will be be found to offer ſome very curious and important facts, particularly the gradual elevation, and the final perfect vertical poſition of the ſandſtone and puddingſtone, as well as the rather unuſual manner in which the ſecondary and the older ſtrata meet each other.

In the ſeries here to be deſcribed, the repeated occurrence of rocks of the *whin* and of the porphyry formation, reſpecting the origin of which opinions are ſo much divided, adds conſiderable intereſt; eſpecially when the form and ſituation in which they occur, and the condition of the contiguous rocks, are taken into conſideration.

In the account which I am now about to give, I ſhall endeavour to lay down a fair repreſentation of the facts as Nature preſents them, unbiaſſed by any of the prevailing theories of coſmogony. I ſhall avoid every geological diſcuſſion whatever, leaving it to others to draw thoſe concluſions, in relation to their own ſpeculations, which they ſhall imagine the facts to warrant.

In that part of the plains of Kincardineſhire from which I take my departure, the native rock conſiſts of Siliceous Grit or Sandſtone, which is here divided into an immenſe number of beds or layers, of various thickneſſes, from one inch to four feet, ſolid ſtone. In many places, gravel of various ſizes is found imbedded in this grit; which gravel conſiſts moſtly of water-worn quartz, and ſmall-grained granites. The colour of the general maſs of this grit is a dark-reddiſh brown, and in ſome few places it ſhows narrow lines and dots of a pearl-grey colour. The component parts of this grit conſiſt of ſmall particles of quartz, and ſtill more minute particles of ſilvery-luſtred mica: theſe owe their coheſion in maſs to a martial argillaceous cement, to which this rock alſo owes its colour. Thoſe lines and dots of pearl-grey colour, generally occur in the moſt ſolid and thickeſt beds

of

of the rock: they are formed of the ſame materials with the other parts of the ſtone; but into them the ferruginous ſtaining matter has not apparently been able to penetrate, and they derive their preſent greyiſh appearance from the natural colour of its particles of quartz, which are here *per ſe* of a bluiſh-white tint. This rock, in the plain, is perfectly horizontal in its poſition; but upon its approach towards the undulated grounds, which here form the loweſt baſis of the Grampians, it begins to riſe from its horizontal bed, and, gradually increaſing in its acclivity *towards the mountains*, it at laſt arrives at a poſition perfectly vertical.

For the firſt quarter of a mile from where this grit begins to leave its horizontal poſition, the riſe is very gradual; but after that diſtance, it becomes more rapid, and in a mile it gains its vertical poſition.

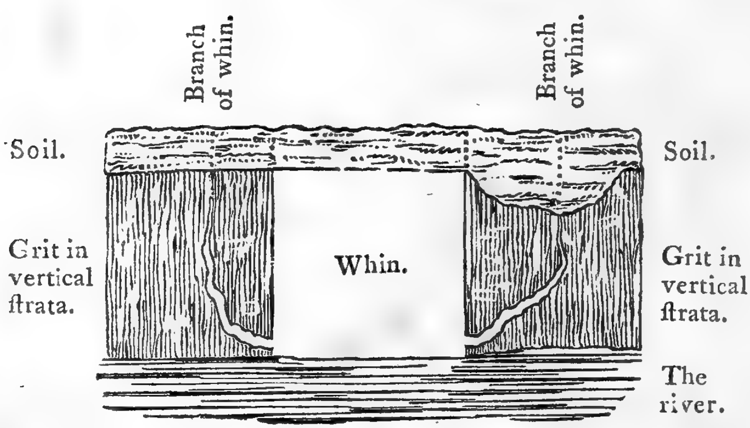
Where this grit or ſandſtone rock is in its moſt ſolid ſtate, and where its poſition is perfectly vertical, betwixt two beds or layers of it, there occurs a bed of Whinſtone forty feet broad.

The main body of this bed of whin interſects none of the layers of grit, but ſtands upright betwixt two of them, to both of which it is cloſely joined. The river, at this place, has, in its paſſage, worn down this bed of whin equally with that of the adjoining grit, and a perpendicular face of it can be examined upon each ſide of the river, from fifty to ſixty feet in height.

Upon examining the ſection of this bed, I found upon the eaſt ſide of the river two branches, which ſprung from the main body of the whin, nearly where the water of the river at preſent waſhes the baſe of its perpendicular ſurface. One of thoſe branches ſprings from the right ſide of the trunk, and the other ſprings from the left ſide: they at firſt diverge from the trunk as they aſcend, and where they puſh out laterally, they interſect the contiguous ſtrata, and penetrate them in a zig-zag manner; but at laſt, in a poſition betwixt two of the layers of the grit, they con-

tinue

tinue their direction upwards, decreasing in their diameters as they ascend, until they finish their course near to the superficial soil which here covers the rock. The grit contiguous upon both sides to the bed of whin, is considerably harder and more compact than it is in any other part of the stratification; and that angle of the grit which lies between the body of the whin and its branches, is more indurated than the strata of the grit upon each side.



This species of whin is not very compact in its texture. Its fracture is somewhat earthy, and is of a brownish-black colour; but it has a considerable degree of induration, and has some specks of lustre in it. Having passed this bed of whin, the grit continues in the same position as immediately before the whin occurred; but, soon after, the gravel, which I have mentioned to be in some places imbedded in the grit, increases in quantity, and at last the strata are formed of a rock composed entirely of that species of gravel, and which may be called Gravel-stone or Plum-pudding-rock. This aggregate constitutes a stratum four hundred yards thick. Its stretch is nearly from west to east, and it is vertical in its position. Its composition consists of quartz, porphyries, and some small-grained granites, all of which have evidently been rounded by attrition in water: they are

are of a vaſt variety of ſize, from that of a pea to the bulk of an oſtrich egg. Theſe are all firmly combined by an argillaceous ferruginous cement. In ſome parts of this gravel rock, are to be ſeen thin lines of a fine-grained grit, ſtretching through it from weſt to eaſt; it is by thoſe lines alone that the verticality and the ſtretch of this maſs is diſcoverable. Its general colour, in maſs, is that of a ferruginous red.

This plum-pudding rock is immediately followed by a ſucceſſion of ſtrata of fine-grained grit, in thin layers: it has a very conſiderable degree of induration, and is of a dark ferruginous brown colour. This deviates a little from the vertical poſition, and inclines to the ſouth: the ſtretch is from weſt to eaſt, and its extent towards the north is two hundred and ſixty yards. To this rock immediately ſucceeds a ſpecies of Porphyry, the principal maſs of which conſiſts of an indurated argil. Its colour is of a purple or lilac brown: its induration is very conſiderable, and its fracture is rough and earthy. The materials which are imbedded in its maſs, conſiſt of ſmall particles of quartz, felſpar, blackiſh-brown mica, and ſpecks of iron ochre; all of theſe are but thinly ſcattered. The ſpace in the courſe of the river occupied by this porphyry is two hundred and twenty yards: its ſtretch is nearly from weſt to eaſt, and it inclines in a ſmall degree to the ſouth. The rock which ſucceeds to this porphyry, and which is in contact with it, is difficult to deſcribe; and this difficulty ariſes from the great diſorder of the ſtratification, and the variety of materials compoſing it. The ſtrata of this bed do not ſucceed each other in a regular manner. Portions of them of various dimenſions lie together, but very variouſly diſpoſed: ſome are vertical, ſome horizontal, ſome dip to the ſouth, one only to the north, affording a ſolitary inſtance of a northern inclination of the ſtrata in this field of examination.

The materials of this maſs of confuſed ſtratification, are of very different deſcriptions. In one place, a quartzoſe ſtone

 abounds,

abounds, of a granular texture: it here, in general, resembles a fine-grained, highly indurated, and compact quartz sandstone: sometimes, however, it approaches to hornstone, and even sometimes to quartz in mass. Much of it has a white colour: the rest is tinted of an ochery brown, of different shades. In other places, the stratified matter consists of a stone of a laminated texture, with undulating lamellæ of a ferruginous tint, looking like an indurated shale; and various gradations of both kinds present themselves. This jumble is in thickness three hundred yards; and to it immediately succeeds a very narrow stratum of Argillite, which is of a greenish-grey colour, and very thinly lamellated.

This argillite is succeeded by a bed of Whin, thirty-three feet broad. This whin is of a dark blackish-brown colour, and is of a more compact texture, than the whin which I have described occurring in the grit, and is possessed of more induration: the materials of composition are nearly the same in both.

Its general stretch is nearly from west to east; but in this stretch, where it has been exposed to the eye by the river, it is somewhat curved, and presents its convex side to the mountains. To this bed of whin succeeds a narrow stratum of Argillite, perfectly similar to that which I have just now described upon its southern side. To this succeeds a seam of Limestone, six feet broad. This limestone is of a pale blue colour, and is much intersected by small veins of quartz trending through it in all directions.

In this limestone, I was unable to trace the remains of any animal or vegetable production. Its position is vertical, and it is immediately succeeded by another narrow stratum of argillite, thinly lamellated.

To this narrow stratum of argillite succeeds a bed of Whin, seventy-five feet broad. This whin is, in its texture, more compact; and its fracture displays a smoother surface than either

of

of the two former whins which I have had occaſion to mention. Its colour is of a dark-bluiſh black. In tracing, with my eye, its vertical cracks and fiſſures, I thought I could perceive a rude tendency to priſmatic forms. It is vertical in its poſition; and its ſtretch is from weſt to eaſt.

THIS bed of whin is ſucceeded by an Argillite of ſhivery texture, and confuſed ſtratification; but as it recedes from the whin, and approaches the mountains, it becomes regularly ſtratified. This ſtratum of ſlate is of great extended thickneſs; and it contains a vaſt variety of colour and of tint. The colours are, pale greyiſh-blue, yellowiſh-green, reddiſh-brown, purple and black, with a great variety of tints of all thoſe colours; but the predominant colours are the greyiſh-blue and the yellowiſh-green; of which two there are two ſorts; the one ſoft, and the other much indurated. The ſoft is thinly laminated, and frequently paſſes over into the highly indurated ſort, in which the appearance of the laminated texture is almoſt loſt.

IN this long ſucceſſion of argillite ſtrata, ſome ſubſtances occur that are heterogeneous to its rock, ſuch as jaſpers, limeſtone, &c.

THE jaſpers are in general of a blood-red colour, and are much veined with white quartz: they occur in large amorphous maſſes, and in neſts, of eliptic forms, of great variety of ſize. One of thoſe bodies of jaſper, in the eliptic form, has been cut through by the river, and is now to be ſeen in the face of the perpendicular rock, upon each ſide of the ſtream. Its ſize is thirty feet long, by ten broad: the points of its tranſverſe axis are ſharp; and it ſtands upright in the argillite. The maſſes of this matter which occur amorphous in the argillite, are of great magnitude. I have traced one of thoſe for thirty yards in extent. All of thoſe jaſpers are of great induration, and take a high poliſh. Both the amorphous and the eliptical formed maſſes are found imbedded, where the argillite is of a greeniſh-grey colour, thinly lamellated, of a ſilky luſtre, and ſaponaceous to the feel: it clings round thoſe maſſes in all their variety of direction, and of courſe its texture is there much twiſted. When the argillite ſtratification has extended its thickneſs to near three quarters of a mile, the limeſtone which I have mentioned above then occurs, in a bed of twelve feet thick. Its colour is bluiſh-black; and it is much pervaded by veins of quartz, and of calcareous ſpar; the laſt of thoſe are, in many places, of conſiderable breadth, and are of a pale fleſh colour. Where this limeſtone has been wrought, I obſerved it forked; that is to ſay, the bed is there ſplit or divided into two, by the intervention of an argillaceous body. Upon each ſide of this bed of limeſtone the argillite occurs of two colours. That which is next to, and in contact with the limeſtone, is black, of a ſhaly texture, ſoils the hand, and has veins of ferruginous-coloured quartz trending through it. The argillite which is more remote from the limeſtone is of a dark purple colour.

Immediately after this narrow bed of ſhale, the argillite reaſſumes its greeniſh-blue colour, and ſlaty texture, and becomes highly indurated: here ſome ſpecks of granulated quartz begin to appear, thinly ſcattered in its maſs, and, ſoon after, it is ſeen to paſs over into an aggregate rock, chiefly compoſed of grains of quartz, felſpar, and minute particles of mica. The particles of quartz and of felſpar ſeldom occur in this aggregate larger than the eight of an inch: theſe have very little the appearance of having ſuffered attrition: they are much mixed, and are frequently ſeen to take lineal directions; and in thoſe lines the particles of felſpar have frequently a compreſſed appearance, and an eye-like form. This rock, in maſs, has a greyiſh-blue colour: it is of great induration, and although lamellous or ſlaty in its texture, a croſs fracture is often more eaſily obtained than one with the lamellæ. Its croſs fracture is pretty even, but appears more granular than foliaceous. This rock occurs frequently in the

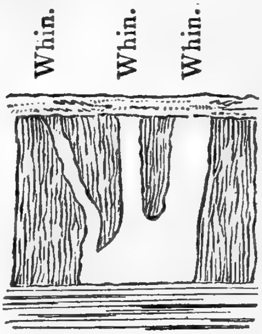
the diſtrict of blue clay ſlate, and may almoſt be ſaid to alternate with it. I have been perhaps more minute in the deſcription of this rock than it deſerved; but I have been ſo, becauſe doubts have ariſen relative to what name ought to be given to this aggregate. In all my geological reſearches, I have found this rock only twice; once, where I have here deſcribed it; and, again, near to Banff, on the Moray Frith. In both of thoſe ſituations, the aggregates are of the ſame compoſition, and ſimilar in poſition: they both lie among blue clay ſlate.

In this long alternation, two ſubſtances occur which are heterogeneous to the rocks among which they lie. The firſt of thoſe, is a bed of compact Felſpar, of great induration. This bed is ten feet broad: its ſtretch is nearly from weſt to eaſt: its poſition is vertical; and it ſtands between two of the layers of the blue clay ſlate. Its colour is of a reddiſh-brown, with a ſmall admixture of purple; and its general fracture is conchoidal, ſomewhat rough, but not earthy.

Not far diſtant from this bed, an appearance occurs worthy of ſome notice. Where the aggregate and the blue clay ſlate are alternating, a ſurface of conſiderable extent of the aggregate rock is expoſed to view, parallel to its ſtratification. This ſurface is regularly undulated in ſmall undulations, bearing a very ſtrong reſemblance to thoſe that may be ſeen upon the ſand of the ſea-beach, when recently left by the tide. After paſſing the bed of compact felſpar, the blue clay ſlate and aggregate rock again alternate; but here the blue clay ſlate predominates. Near to this, the ſecond ſubſtance heterogeneous to thoſe alternating rocks occurs. It is a bed of Whin, the form of which is ſomewhat ſingular. It conſiſts of a principal trunk, which the river, here cuts nearly at right angles. Upon the eaſt ſide of the river, this principal trunk is ſeen to ſplit into three branches; and thoſe three take an eaſtern direction, between the ſtrata of the aggregate rock and the blue clay ſlate, where thoſe two rocks are of great induration. The

breadth

breadth of this bed of whin is thirteen feet; and where it ſplits, its three branches are, ſix, four, and three feet in diameter. The trend or ſtretch of this bed is from weſt to eaſt; but upon the weſt ſide of the river, it curves ſomewhat to the ſouth-weſt. Its compoſition is nearly the ſame with the three other beds of whin which I have before mentioned. It is of a browniſh-black colour, and, when placed in certain directions, it ſhows ſpecks of luſtre. It is vertical in its poſition, has a great degree of induration, and its general fracture is roughly conchoidal.



UPON paſſing this bed of whin, the river ceaſes to be deeply imbedded in the rocks; but the aggregate rock and the clay ſlate ſtill continue to be ſeen for a ſhort diſtance, in a ſhelvy acclivity, where they are loſt to view in a long narrow plain, deeply covered with a bed of gravel, compoſed of the debris of the interior mountains. The river here flows over this bed of gravel for a conſiderable ſpace; and upon this narrow flat, we paſs through between two of the moſt elevated points in the ſouth lateral range of this part of the Grampians. Although the obtruſion of this maſs of gravel cuts off from inſpection the continuity of the laſt-mentioned rocks, yet the broken and abrupt ſides of the mountains, cloſe upon each hand, clearly points out, that this part of the ſouth lateral range is entirely compoſed of micaceous ſhiſtus. Here, we are deprived of the junction of the micaceous ſhiſtus with the two former rocks; and the loſs of all ſuch

ſuch junctions are always to be much regretted in mineralogical reſearch.

Having paſſed over this narrow plain, I advanced towards a ſecond range of hills, which here form the baſis of the central and higheſt chain. It is at this place where the river ſo ſuddenly changes its courſe from eaſt to ſouth, and where I was under the neceſſity of leaving its bed, to continue my northern direction towards Mount-Battoc. This, however, I was enabled to do to great advantage, by following up the deep cut bed of a winter torrent, which led me into the direction which I wiſhed to follow.

Upon entering the bed of this torrent, I found that the baſis of the hills here entirely conſiſted of micaceous ſhiſtus, much veined with quartz, and much twiſted in its texture. The ſtretch of this rock is here nearly from weſt to eaſt; and it has a ſoutherly dip of 45 degrees.

In paſſing through among thoſe hills towards the central range, I found in ſeveral of the beds of the torrents large blocks of reddiſh-brown porphyry, with ſcattered maſſes of micaceous ſhiſtus and granite.

In tracing up one of thoſe torrents, I ſaw the micaceous ſhiſtus rock and the porphyry both expoſed to view, near to each other; and, ſoon after, in the bed of the ſame torrent, I came to a caſcade which had laid bare both thoſe rocks at a point where they are in contact; and near thoſe a ſecond bed of porphyry made its appearance, in the front of a near hill. From my firſt view of thoſe, and from their relative poſitions, I was led to imagine, that they might here alternate in vertical poſition; but upon more minute inſpection, I found that the porphyry conſtituted vertical dikes, ſtretching nearly from ſouth to north; which courſe cuts the line of direction of the Grampians here almoſt at right angles: and, on the contrary, I found that the micaceous ſhiſtus which flanked thoſe dikes of porphyry, had a regular ſtretch

from

from weſt to eaſt, and a ſoutherly dip. To endeavour to have theſe appearances more fully explained to me, I directed my ſteps to the brow of that hill, where I had obſerved the rock laid bare; and in paſſing along the fronts of the hills from eaſt to weſt, I ſoon came to a dike of porphyry ſimilar to thoſe which I had immediately left. This dike is ſixty feet broad, ſtretching nearly from ſouth to north, and flanked upon both ſides by micaceous ſhiſtus, ſtretching and dipping as before deſcribed. In proceeding farther along the faces of thoſe hills, I found ſeveral other dikes of porphyry, of various breadths, and at various diſtances from each other; but all of them ſimilar in their lines of direction, and the micaceous ſhiſtus always interpoſing between them, through which they ſeemed to riſe. The porphyries of thoſe dikes are generally of a ferruginous colour, tending ſometimes to an orange-red, and of various tints of thoſe colours. They have great induration, are coarſe-grained, and produce a rough fracture. The particles of quartz which are ſcattered in their principal maſſes, are ſmall, amorphous, and are of a ferruginous colour. The particles of felſpar are of a light tint of the ſame colour, and are moſtly cryſtallized. The ſurface of thoſe dikes are in many places bare, and expoſed to the eye for long extents, in their lines of direction; and in all thoſe lines of direction which I have traced, I have never found any of them alter in their breadths, in their verticality, nor in their directions. Their ſurfaces, in general, conſiſt of oblong ſquare blocks, now looſe and unconnected with each other; and, in many places, the lines of fracture of thoſe blocks are ſo ſtraight, that one might almoſt ſuppoſe that they had been disjoined by the hand of art.

I HAVE often obſerved, in this diſtrict, and in other parts of the Grampians, that the looſe and outlying blocks of both granite and of porphyry, (which have not been worn down by at-

trition),

trition), confiſt, in general, of oblong ſquare ſhapes. This obſervation, when I firſt made it, led me to imagine, that thoſe rocks here were perhaps ſtratified. I have, however, as yet, not been able to trace real ſtratification of thoſe rocks in this diſtrict of the Grampians.

Upon ſome of the ſummits of thoſe hills which here form the baſis of the central range, I firſt diſcovered the granite in ſolid rock. In thoſe ſituations, the granite is only ſeen in patches, where the ſuperincumbent rocks have worn off it. Theſe ſuperincumbent rocks, which I here found in contact with the granite, are of two different compoſitions, and occur on the ſummits of different hills. The one of thoſe rocks, and the moſt prevalent one, is the micaceous ſhiſtus; the other is the granitelle, or a mixture of quartz and ſhorl. In ſome parts of this laſt-mentioned rock, I perceived a ſmall admixture of hornblende: where this appears in the compoſition, it perhaps ought to receive the appellation of granitine. In thoſe elevated ſituations, I found both of thoſe rocks, (eſpecially the micaceous ſhiſtus), in a ſtate of decompoſition, and faſt leaving the granite expoſed to the eye.

From thoſe appearances, it is to be inferred, that the interior of thoſe hills is compoſed of granite, which is but thinly coated by the ſuperincumbent rocks.

Upon leaving theſe hills, which, I have already ſaid, form the baſis of the central chain of the Grampians, I regretted very much, that all my endeavours proved abortive to trace out the whole extent, in line, of any one of thoſe dikes of porphyry which interſect their ſides. I conſtantly loſt them under peat or other ſoils, before I could trace them to their contact with the granite. It was my anxious wiſh to ſee how thoſe two rocks of porphyry and granite connected with each other at their junction.

In purſuing my reſearches towards the ſummit of Mount Batoc, I proceeded up the bed of a torrent, which, after heavy rains, daſhes down the immediate ſide of that mountain. In this bed, the blocks of micaceous ſchiſtus and of porphyry, (which I had ſeen ſo abundantly ſcattered among the hills that I had juſt left), totally diſappeared, and no outlyers of any kind were to be ſeen, excepting ſome granites, which were ſcattered in large maſſes; and in every part, where the torrent had carried off the ſuperincumbent ſoil, the granite was to be ſeen in ſolid rock.

In my progreſs towards the ſummit of this mountain, I fell in with a large face of the native granite rock expoſed to the eye. By the cracks in this face being in long-extended horizontal lines, it had at firſt the appearance of being ſtratified; but upon a nearer and more minute examination, I found that it was not ſtratified, and that the cracks which gave it that appearance were only ſuperficial.

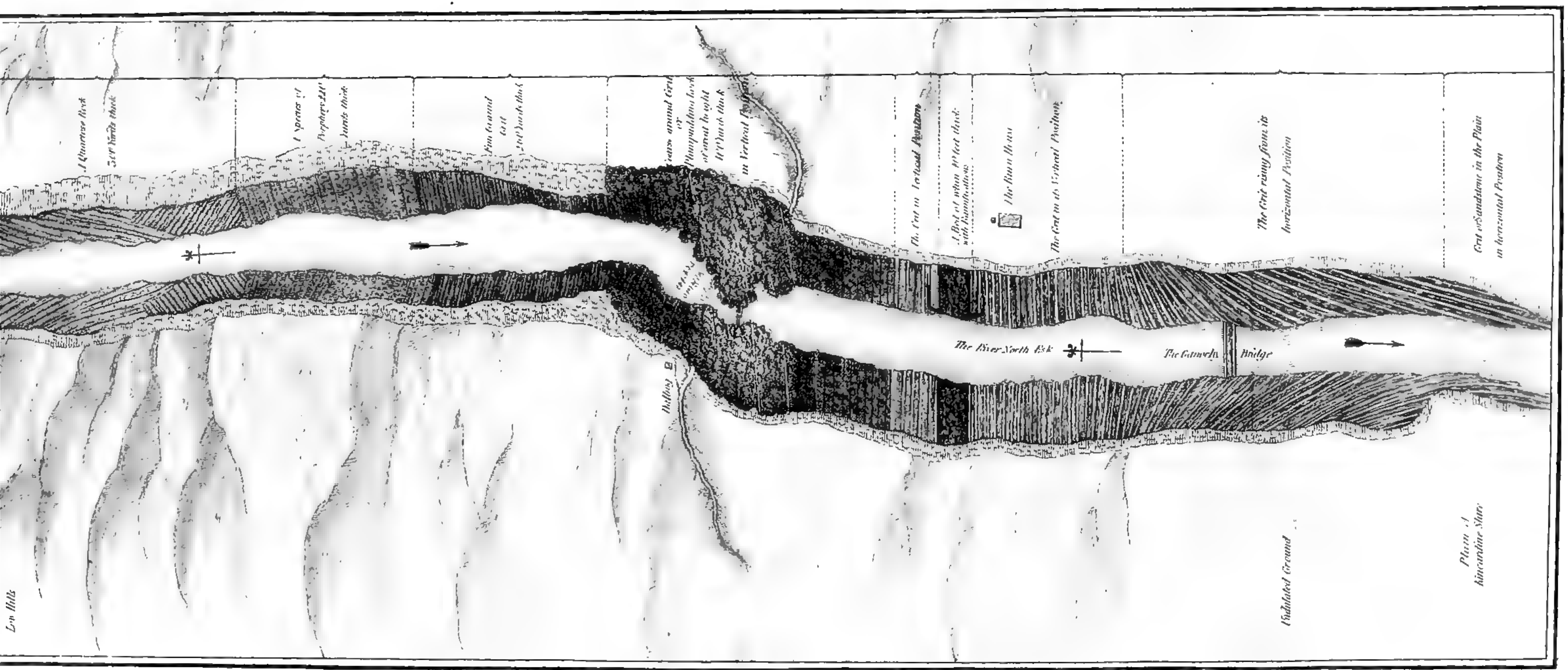
Around this face were ſcattered large blocks of granite, which were moſtly in oblong ſquare ſhapes.

Soon after paſſing this precipice, I gained the ſummit of the mountain, which, though not very highly elevated, is in this part of the chain the higheſt of the central range. It is about 3465 feet above the level of the ſea; and is entirely compoſed of a coarſe-grained granite, in which ſhorl ſometimes occurs; and its felſpar is very generally cryſtallized.

Having here finiſhed the extent of my intended inveſtigation, I beg to be permitted to add, that the line which I have here given the deſcription of, has been traced with much attention, and the true poſition of each foſſil has been moſt ſcrupulouſly attended to, and is correctly placed in the annexed plate.

I


The Grampians
Mount Battock in the Central Range, entirely composed of Granite
Micaceous Shistus
Hills forming the Bases of the Central Range
Micaceous Shistus
South Lateral range
The South Lateral Range entirely Composed of Micaceous Shistus
Plain
Plain
A Narrow Plain Covered with Gravel
Clay Slate Alternating with the aggregate rock
Clay Slate Alternating with the aggregate rock
Clay Slate and aggregate rock Alternating
Hills
Clay Slate passing into an aggregate Rock
Black Slate
Black Slate
Blue Clay Slate
Jasper in Eliptic forms
Clay Slate Shivery
Low Hills
A Quartzose Rock
A Species of
Porphery
Yards thick


The River North Esk
Bridge
Undulated Ground
Grit or Sandstone in the Plain
in horizontal Position
The Grit rising from its
horizontal Position
The Grit in its Vertical Position
The Burn House

I WISH that ſome more able pen than mine, would take up the further deſcription of this extended field of geology, ſo worthy of inveſtigation; but if none will come forward for that purpoſe, I may at ſome future period preſume to give to this Society more extended, and more general lines of deſcription of the Grampians, than that which I have now had the honour of ſubmitting to their examination.

II.

II. *A* GEOMETRICAL INVESTIGATION *of ſome curious and intereſting Properties of the* CIRCLE, *&c.* *By JAMES GLENIE, Eſq;* A. M. F. R. S. LOND. & EDIN.

[*Read April* 1. 1805.]

DEAR SIR, *Edinburgh, 22d March* 1805.

AS the following paper refers in a great meaſure to the *general theorems* publiſhed by your father, I now commit it to your care, and that of my friend Mr PLAYFAIR, Profeſſor of Natural Philoſophy. I wiſh it to be communicated to the Royal Society of Edinburgh, and, if approved of, to be inſerted in their Tranſactions as ſoon as poſſible. Indeed, I truſt, that even ſimple as it is, it will not be altogether unacceptable to that learned body.

I am,

Dear Sir,

Moſt ſincerely your, &c.

JAS GLENIE.

Dugald Stewart, Eſq;
Profeſſor of Moral Philoſophy.

THE

THAT truly elegant and inventive geometer the late Dr MATTHEW STEWART, publiſhed at Edinburgh, in 1746, without demonſtrations, a number of general theorems, of great uſe in the higher parts of mathematics, and much calculated for improving and extending geometry. Such of them as refer to the circle, and to regular figures inſcribed in, and circumſcribed about it, have not, as far as I can underſtand, been yet demonſtrated. Theſe, with an endleſs variety of other theorems, are derivable, as corollaries, from the following general though ſimple geometrical inveſtigation, that occurred to me fifteen years ago, and which, I ſuppoſe, has remained ſo long unknown and unattended to chiefly on account of its ſimplicity.

LET A, B, C, &c. (Pl. II. Fig. 1.) be any number of points in the circumference of a circle, and let that number be denoted by n. Let RA, RS, ST, &c. be tangents to the circle, in the points A, B, C, &c.; and let POQ be any diameter. Let Qc, Qd, Qf, &c. be perpendiculars from the point Q to the diameters paſſing through the points A, B, C, &c., and Pa, Pb, Pe, &c. perpendiculars from the point P to the ſame diameters.

THEN it is evident, that $\overline{PQ}^2 = \overline{AP}^2 + \overline{AQ}^2 = \overline{BP}^2 + \overline{BQ}^2 = \overline{CP}^2 + \overline{CQ}^2, =$ &c. Wherefore $\overline{PQ}^2 \times n = \overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 +$, &c. $+ \overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 +$, &c. But $\overline{AP}^2 = AG \times Aa = PQ \times Aa$, $\overline{BP}^2 = PQ \times Be$, $\overline{CP}^2 = PQ \times Cb$, &c.; and $\overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 +$, &c. $= PQ \times \overline{Ac + Bf + Cd +, \&c.}$ Now Aa, Be, Cb, &c. are reſpectively equal to perpendiculars drawn from P to the tangents RA, RS, ST, &c., as are Ac, Bf, Cd, &c. equal to perpendiculars drawn from Q to the ſame tangents. Conſequently the ſum of all the perpendiculars drawn from the points P and Q to lines touching the circle in the points, A, B, C, &c. is equal to $PQ \times n$, or a multiple of the diameter by n.

THE ſame may be proved othewiſe; for ſince $Oa = Oc$, $Aa = Gc$, $Aa + Ac =$ the diameter. In like manner, $Be + Bf =$ the diameter, and $Cb + Cd =$ diameter, &c.

IN

In the ſame way, it is demonſtrated, that if from any two points p, q, in the diameter PQ, equally diſtant from the centre O, perpendiculars be drawn to the lines touching the circle in the points A, B, C, &c. their ſum is equal to a multiple of the diameter by n.

But if from any two points V, W, in PQ produced, equally diſtant from the centre O, lines drawn perpendicular to any diameter B r, paſſing through any point of contact B, fall beyond its extremities B, r, the difference of the perpendiculars drawn from W, V, to the line touching the circle in B, is equal to the diameter, and ſo on.

So alſo, when perpendiculars from the points V, W in PQ, produced to the diameters paſſing through the points of contact A, B, C, &c. do not fall beyond the extremities of any of theſe diameters, perpendiculars from V and W to right lines touching the circle in the points A, B, C, &c. are taken together equal to a multiple of the diameter by the number of the ſaid points.

Cor. 1. Perpendiculars drawn from P and Q, or p and q, to lines touching the circle in the points A, B, C, &c. are together equal to a multiple of the radius by $2n$.

Cor. 2. The ſum of perpendiculars drawn from P, Q, or p, q, to the ſides of any regular figure circumſcribed about the circle, is equal to twice the ſum of perpendiculars drawn to the ſides of a regular figure of the ſame number of ſides circumſcribing the circle from any point within the ſame regular figure.

Cor. 3. $\frac{\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 +, \&c.}{d} =$ ſum of the perpendiculars drawn from P to right lines touching the circle in the points A, B, C, &c. d denoting the diameter.

Or a third proportional to the diameter and the chord AP, together with a third proportional to the diameter and the chord BP, together with a third proportional to the diameter and the chord

chord CP, &c. is equal to the ſum of the perpendiculars drawn from the point P to right lines touching the circle in the points A, B, C, &c.

AND $\frac{\overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 +, \&c.}{d}$ = ſum of perpendiculars drawn from Q to the ſame lines.

AGAIN, ſince by a well known property of the circle, $\overline{AP}^2 + \overline{AQ}^2 = \overline{BP}^2 + \overline{BQ}^2 = \overline{CP}^2 + \overline{CQ}^2 = \&c. = 2r^2 + 2\overline{Op}^2$, r denoting radius, the ſum of the ſquares of lines drawn from the points A, B, C, &c. to any two points p, q, in the diameter equally diſtant from the centre, is $= 2nr^2 + 2n \times \overline{Op}^2$ = a multiple of r^2, by twice the number of the points A, B, C, &c. together with the ſame multiple of the ſquare of Op or Oq.

IN like manner, $\overline{AV}^2 + \overline{AW}^2 + \overline{BV}^2 + \overline{BW}^2 + \overline{CV}^2 + \overline{CW}^2 +, \&c. = 2nr^2 + 2n \times \overline{OV}^2$ = a multiple of r^2 by twice the number of the points A, B, C, &c., together with the ſame multiple of $\overline{OV}^2$ or $\overline{OW}^2$.

AND ſince the ſquares of the chords AP, BP, CP, &c. are together equal to the ſum of the ſquares of the perpendiculars drawn from P to the right lines touching the circle in the points A, B, C, &c. together with the ſum of the ſquares of the perpendicular diſtances of P from the diameters paſſing through theſe points, the ſum of the ſquares of Ap, Bp, Cp, &c. is in like manner equal to the ſum of the ſquares of perpendiculars from p to theſe lines, together with the ſum of the ſquares of the perpendicular diſtances from p to the ſaid diameters.

IN like manner, $\overline{Aq}^2 + \overline{Bq}^2 + \overline{Cq}^2 +, \&c.$ = ſum of ſquares of perpendiculars from q to the lines touching the circle in A, B, C, &c. together with the ſum of the ſquares of the perpendicular diſtances of q from the diameters paſſing through A, B, C, &c.

WHEREFORE

WHEREFORE the ſquares of the perpendicular diſtances of either P or Q from diameters paſſing through the points of contact A, B, C, &c., are, taken together, equal to the exceſs of the rectangle under half the diameter PQ and the ſum of perpendiculars from P and Q to right lines touching the circle in the points A, B, C, &c. above half the ſum of the ſquares of ſaid perpendiculars $= nr^2 - rs$, (s being equal to the ſum of perpendiculars from O, as, in what follows, to right lines touching the circle, of which OQ is the diameter, in the points c, d, f, &c.). And the ſum of the ſquares of theſe perpendicular diſtances from both P and Q is $= 2nr^2 - 2rs$. This is alſo evident, from all angles in a ſemicircle being equal to right ones. For $\overline{AP}^2 + \overline{AQ}^2 + \overline{BP}^2 + \overline{BQ}^2 + \overline{CP}^2 + \overline{CQ}^2 +$ &c. $= n \times \overline{PQ}^2 = 4nr^2$; and $4nr^2 - 2nr^2 - 2rs = 2nr^2 - 2rs$.

CONSEQUENTLY, when the whole circle is divided into equal parts, in the points A, B, C, &c. $\overline{Ap}^2 + \overline{Bp}^2 + \overline{Cp}^2 +$ &c. $= \overline{Aq}^2 + \overline{Bq}^2 + \overline{Cq}^2 +$ &c. $= nr^2 + n \times \overline{Op}^2$; and $\overline{AV}^2 + \overline{BV}^2 + \overline{CV}^2 +$ &c. $= \overline{AW}^2 + \overline{BW}^2 + \overline{CW}^2 +$&c.$= nr^2 + n \times \overline{OV}^2$. For the ſum of perpendiculars drawn from p to the ſides of any regular figure circumſcribing the circle, is then equal to the ſum of the perpendiculars drawn from q to the ſides of the ſame figure. The ſame obſervation holds with regard to perpendiculars drawn from the points V, W.

FROM the foregoing general inveſtigation, when the circle is ſuppoſed to be equally divided in the points A, B, C, &c. Dr STEWART's firſt, ſecond, third, and eleventh theorems can be immediately derived.

I SHALL, however, proceed regularly with the inveſtigation; and, in the firſt place, take the ſquares of the perpendiculars from P and Q to the right lines touching the circle in the points A, B, C, &c. which perpendiculars are reſpectively equal to A a, A c; B f, B e; C d, C b; &c.

 Now

$$\text{Now } \overline{Ac}^2 + \overline{Aa}^2 = \overline{r + cO}^2 + \overline{r - cO}^2 = 2r^2 + 2 \times \overline{cO}^2$$

$$\overline{Bf}^2 + \overline{Be}^2 = \overline{r + Of}^2 + \overline{r - Of}^2 = 2r^2 + 2 \times \overline{Of}^2$$

$$\overline{Cd}^2 + \overline{Cb}^2 = \overline{r + Od}^2 + \overline{r - Od}^2 = 2r^2 + 2 \times \overline{Od}^2$$

&c. &c. &c.

Wherefore the ſum of the ſquares of perpendiculars from P, Q to lines touching the circle in the points A, B, C, &c. is $= 2n \times r^2 + 2 \times \overline{\overline{Oc}^2 + \overline{Of}^2 + \overline{Od}^2 + \&c.}$ But the points c, d, f, are in the circumference of a circle, of which the diameter is OQ or r, and by Cor. 3. the ſum of $\overline{Oc}^2 + \overline{Of}^2 + \overline{Od}^2 + \&c. = OQ \times$ into the ſum of perpendiculars drawn from O to lines touching the circle, of which OQ is the diameter, in the points c, d, f, &c. Call the ſum of theſe perpendiculars s. Then we have the ſum of the ſquares of perpendiculars drawn from P, Q to lines touching the circle APQ in the points A, B, C, &c. $= 2n \cdot r^2 + 2rs =$ (Cor. 3.) $\dfrac{\overline{AP}^4 + \overline{BP}^4 + \overline{CP}^4 + \&c. + \overline{AQ}^4 + \overline{BQ}^4 + \overline{CQ}^4 + \&c.}{d^2}$

When the circumference is divided into equal parts by the points A, B, C, &c. or the angles at O are equal, $s = \frac{n}{2} \times OQ$ or $\frac{n}{2} \times r$ and $2nr^2 + 2rs = 3nr^2$.

If a regular figure be inſcribed in the circle, having its angles at the points A, B, C, &c. or a regular figure be circumſcribed about the circle, having its ſides tangents to it in the points A, B, C, &c. we get from the general expreſſion $\dfrac{\overline{AP}^4 + \overline{BP}^4 + \overline{CP}^4 + \&c.}{r}$ or $\dfrac{\overline{AQ}^4 + \overline{BQ}^4 + \overline{CQ}^4 + \&c.}{r} = 4nr^3 + 4r^2 s = 4nr^3 + 2nr^3 = 6nr^3$, or third proportionals to radius, the chords drawn from either P or Q to the points A, B, C, &c. and the cubes of theſe chords equal, when taken together, to ſix times a multiple of the cube of radius by the number

ber of the fides of the infcribed or circumfcribed figure; or to fpeak algebraically, the fum of the fourth powers of the chords is equal to fix times a multiple of the fourth power of the femidiameter of the circle, by the number of the fides of the figure. This is Dr STEWART's 23d theorem.

IN like manner,

$$\overline{Ac}^3 + \overline{Aa}^3 = \overline{r+Oc}^3 + \overline{r-Oc}^3 = 2r^3 + 6r \times \overline{Oc}^2$$

$$\overline{Bf}^3 + \overline{Be}^3 = \overline{r+Of}^3 + \overline{r-Of}^3 = 2r^3 + 6r \times \overline{Of}^2$$

$$\overline{Cd}^3 + \overline{Cb}^3 = \overline{r+Od}^3 + \overline{r-Od}^3 = 2r^3 + 6r \times \overline{Od}^2$$

&c. &c. &c.

And the cubes of perpendiculars from P and Q to right lines touching the circle in the points A, B, C, &c. are taken together

$$= 2nr^3 + 6r \times \overline{\overline{Oc}^2 + \overline{Of}^2 + \overline{Od}^2 + \&c.} = \text{(by Corollary 3.)}$$

$$\frac{\overline{AP}^6 + \overline{BP}^6 + \overline{CP}^6 + \&c. + \overline{AQ}^6 + \overline{BQ}^6 + \overline{CQ}^6 + \&c.}{d^3}$$

BUT $\overline{Oc}^2 + \overline{Of}^2 + \overline{Od}^2 + \&c. = \frac{nr^2}{2}$, when the circumference is equally divided in the points A, B, C, &c. or when a regular figure is circumfcribed about the circle, with its fides touching the fame in faid points. Wherefore the cubes of perpendiculars from P and Q to the fides of a regular figure of a greater number of fides than three circumfcribed about the circle, are taken together $= 5nr^3$. This is Dr STEWART's 19th theorem.

AND if a regular figure of a greater number of fides than three be infcribed in the circle, having its angles in the points A, B, C, &c. third proportionals to the cube of the diameter and the cubes of chords drawn from P and Q to the points A, B, C, &c. will, taken together, be equal to $5nr^3$; or third proportionals to the cube of the diameter and chords drawn from either P or Q to the faid angular points, will taken together, be $= \frac{5nr^3}{2}$;

or,

or, to ſpeak algebraically, the ſum of the ſixth power of chords drawn from either P or Q to the ſaid points, will be equal to twenty times a multiple of the ſixth power of radius, by the number of the ſides of the inſcribed figure.

In like manner,

$$\frac{\overline{Ac}^4+\overline{Aa}^4}{r}=\frac{\overline{r+Oc}^4+\overline{r-Oc}^4}{r}=2r^3+12r\times\overline{Oc}^2+\frac{2\times\overline{Oc}^4}{r};$$

$$\frac{\overline{Bf}^4+\overline{Be}^4}{r}=\frac{\overline{r+Of}^4+\overline{r-Of}^4}{r}=2r^3+12r\times\overline{Of}^2+\frac{2\times\overline{Of}^4}{r}$$

$$\frac{\overline{Cd}^4+\overline{Cb}^4}{r}=\frac{\overline{r+Od}^4+\overline{r-Od}^4}{r}=2r^3+12r\times\overline{Od}^2+\frac{2\times\overline{Od}^4}{r}$$

&c. &c. &c.

And third proportionals to radius, perpendiculars from P and Q to right lines touching the circle in the points A, B, C, &c. and the cubes of ſaid perpendiculars are, taken together, equal to

$$2nr^3+12r\times\overline{\overline{Oc}^2+\overline{Of}^2+\overline{Od}^2+\&c.}+2\times\frac{\overline{Oc}^4+\overline{Of}^4+\overline{Od}^4+\&c.}{r}$$

$$=\text{(by Cor. 3.)}\ \frac{\overline{AP}^8+\overline{BP}^8+\overline{CP}^8+\&c.}{d^4\times r}+\frac{\overline{AQ}^8+\overline{BQ}^8+\overline{CQ}^8+\&c.}{d^4\times r}$$

But $\overline{Oc}^2+\overline{Of}^2+\overline{Od}^2+\&c.=\frac{nr^2}{2}$ when the circumference is equally divided in the points A, B, C, &c. or when a regular figure is circumſcribed about the circle, with its ſides touching the ſame in ſaid points, and $12r\times\frac{nr^2}{2}=6nr^3$. Alſo $2\times\frac{\overline{Oc}^4+\overline{Of}^4+\overline{Od}^4+\&c.}{r}$ is then $=\frac{3nr^3}{4}$. Wherefore theſe third proportionals are taken together equal to $8nr^3+3\frac{nr^3}{4}=\frac{35nr^3}{4}$; and four times their aggregate is equal to $35nr^3$. Or,

to

to ſpeak algebraically, eight times the ſum of the fourth powers of perpendiculars from either P or Q to the ſides of a regular figure of a greater number of ſides than four circumſcribed about the circle, and touching it in the points A, B, C, &c. are equal to thirty-five times the multiple of the fourth power of radius by the number of the ſides of the figure. This is Dr STEWART's 25th theorem.

AND if a regular figure of a greater number of ſides than four be inſcribed in the circle, having its angles in the points A, B, C, &c. the eighth powers of the chords drawn from either P or Q to the points A, B, C, &c. (to ſpeak algebraically) is equal to $70 n r^8 = n \times \frac{1.3.5.7}{1.2.3.4}. 2^4. r^8$ = ſeventy times a multiple of the eighth power of radius by the number of the ſides of the figure.

IN like manner,

$$\frac{\overline{r+Oc}^5 + \overline{r-Oc}^5}{r^2} = 2r^3 + 20r \times \overline{Oc}^2 + \frac{10.\overline{Oc}^4}{r},$$

$$\frac{\overline{r+Of}^5 + \overline{r-Of}^5}{r^2} = 2r^3 + 20r \times \overline{Of}^2 + 10 \times \frac{\overline{Of}^4}{r},$$

$$\frac{\overline{r+Od}^5 + \overline{r-Od}^5}{r^2} = 2r^3 + 20r \times \overline{Od}^2 + 10 \times \frac{\overline{Od}^4}{r}$$

&c. &c.

WHEREFORE $\frac{\overline{r+Oc}^5 + \overline{r-Oc}^5}{r^2} + \frac{\overline{r+Of}^5 + \overline{r-Of}^5}{r^2} + \frac{\overline{r+Od}^5 + \overline{r-Od}^5}{r^2}$ + &c. $= 2nr^3 + 10nr^3 + \frac{15nr^3}{4} = \frac{63nr^3}{4}$ equal (by Cor. 3.) to $\frac{\overline{AP}^{10} + \overline{BP}^{10} + \overline{CP}^{10} + \&c. + \overline{AQ}^{10} + \overline{BQ}^{10} + \overline{CQ}^{10} + \&c.}{d^5 r^2}$, when the circle is equally divided in the points A, B, C, &c.

AND

And generally when m is any integer whatfoever, we have

$$\frac{\overline{r+Oc}^{m}+\overline{r-Oc}^{m}}{r^{m-3}}+\frac{\overline{r+Of}^{m}+\overline{r-Of}^{m}}{r^{m-3}}+\frac{\overline{r+Od}^{m}+\overline{r-Od}^{m}}{r^{m-3}}$$

$+$ &c. equal to $2nr^3+\frac{m}{1}.\frac{m-1}{1}.r\times\overline{\overline{Oc}^2+\overline{Of}^2+\overline{Od}^2+\&c.}+$

$$\frac{m}{1}.\frac{m-1}{1}.\frac{m-2}{3}.\frac{m-3}{4}\times\frac{\overline{Oc}^4+\overline{Of}^4+\overline{Od}^4+\&c.}{r}+\frac{m}{1}.\frac{m-1}{1}.\frac{m-2}{3}.$$

$$\frac{m-3}{4}.\frac{m-4}{5}.\frac{m-5}{6}\times\frac{\overline{Oc}^6+\overline{Of}^6+\overline{Od}^6+\&c.}{r^3}+\&c.=(\text{Cor. }3.)$$

$$\frac{\overline{AP}^{2m}+\overline{BP}^{2m}+\overline{CP}^{2m}+\&c.}{d^m r^{m-3}}+\frac{\overline{AQ}^{2m}+\overline{BQ}^{2m}+\overline{CQ}^{2m}+\&c.}{d^m r^{m-3}};$$

which, when the circle is equally divided in the points A, B, C, &c. by the circumfcription or infcription of a regular figure, coincides with the 36th and 38th of Dr Stewart's general theorems.

And univerfally if m have to l any ratio whatfoever,

$$\frac{\overline{r+Oc}^{\frac{m}{l}}+\overline{r-Oc}^{\frac{m}{l}}}{r^{\frac{m}{l}-3}}+\frac{\overline{r+Of}^{\frac{m}{l}}+\overline{r-Of}^{\frac{m}{l}}}{r^{\frac{m}{l}-3}}+\frac{\overline{r+Od}^{\frac{m}{l}}+\overline{r-Od}^{\frac{m}{l}}}{r^{\frac{m}{l}-3}}+\&c.$$

is $=2nr^3+\frac{m}{l}.\frac{m-l}{l}.r\times\overline{\overline{Oc}^2+\overline{Of}^2+\overline{Od}^2+\&c.}+\frac{m}{l}.\frac{m-l}{l}.\frac{m-2l}{3l}.$

$$\frac{m-3l}{4l}\times\frac{\overline{Oc}^4+\overline{Of}^4+\overline{Od}^4+\&c.}{r}+\frac{m}{l}.\frac{m-l}{l}.\frac{m-2l}{3l}.\frac{m-3l}{4l}.\frac{m-4l}{5l}$$

$$\frac{m-5l}{6l}\times\frac{\overline{Oc}^6+\overline{Of}^6+\overline{Od}^6\times\&c.}{r^3}+\&c.\ \&c.$$

This laft theorem, or expreffion, is more general than any of Dr Stewart's theorems, and will furnifh an endlefs number of new and curious infinite feries, with their fummations. It may alfo be extended to the chords AP, BP, &c. and expreffed in terms of them. And as to the truth of the binomial

nomial and refidual theorems, when m has to l the ratio of any two homogeneous magnitudes whatfoever, I muft refer the reader to my general demonftration of both in Baron MASERES's *Scriptores Logarithmici*, vol. 5. and to fome of the geometrical formulæ in my *Univerfal Comparifon*.

IN like manner, if pg, ph, pi, &c. be perpendiculars refpectively to BO, CO, AO, &c. we have $\overline{r+Oi}^2 + \overline{r-Oi}^2 + \overline{r+Og}^2 + \overline{r-Og}^2 + \overline{r+Oh}^2 + \overline{r-Oh}^2 + \&c. = 2nr^2 + 2\times\overline{\overline{Oi}^2 + \overline{Og}^2 + \overline{Oh}^2 + \&c.} = 2nr^2 + n.\overline{Op}^2$, when the circle is equally divided in the points A, B, C, &c. or when a regular figure is circumfcribed about it, with its fides touching it in thefe points. This is Dr STEWART's third theorem, of which he gives a demonftration of confiderable length.

IN like manner,

$$\overline{r+Oi}^3 + \overline{r-Oi}^3 + \overline{r+Og}^3 + \overline{r-Og}^3 + \overline{r+Oh}^3 + \overline{r-Oh}^3$$

$+ \&c.$ are equal to $2nr^3 + 6r\times\overline{\overline{Oi}^2 + \overline{Og}^2 + \overline{Oh}^2 + \&c.} = 2nr^3 + 3r\times\overline{Op}^2$, when the circle is equally divided in the points, A, B, C, &c. or when a regular figure circumfcribing it touches it in thefe points. This is Dr STEWART's 20th theorem.

IN like manner,

$$\frac{\overline{r+Oi}^4 + \overline{r-Oi}^4}{r} + \frac{\overline{r+Og}^4 + \overline{r-Og}^4}{r} + \frac{\overline{r+Oh}^4 + \overline{r-Oh}^4}{r}$$

$+ \&c.$ is equal to $2nr^3 + 12r\times\overline{\overline{Oi}^2 + \overline{Og}^2 + \overline{Oh}^2 + \&c.} + 2\times$

$$\frac{\overline{Oi}^4 + \overline{Og}^4 + \overline{Oh}^4 + \&c.}{r} = \text{to } 2nr^3 + 6r\times\overline{Op}^2 \times n + \frac{3n.\overline{Op}^4}{4r},$$

when the circle is equally divided in the points A, B, C, &c. or when a regular figure, circumfcribing it, touches it in thefe points. And a multiple of this by four, or eight times the aggregate of third

third proportionals to r, the perpendiculars from either p or q to the sides of the regular circumscribing figure, and the cubes of these perpendiculars is equal to $8nr^3 + 24nr \times \overline{Op}^2 + 3n \times \frac{\overline{Op}^4}{r}$; or, speaking algebraically, eight times the sum of the fourth powers of perpendiculars from either p or q are equal to $8nr^4$, together with 24 times a multiple by n of the fourth power of the line whose square is equal to $r \times Op$, together with thrice a multiple by n of $\overline{Op}^4$. This is Dr STEWART's 26th theorem.

IN like manner,

$$\frac{\overline{r+Oi}^5 + \overline{r-Oi}^5}{r^2} + \frac{\overline{r+Og}^5 + \overline{r-Og}^5}{r^2} + \frac{\overline{r+Ob}^5 + \overline{r-Ob}^5}{r^2} + \&c.$$

are $= 2nr^3 + 20r \times \overline{\overline{Oi}^2 + \overline{Og}^2 + \overline{Ob}^2 + \&c.} + 10 \times \frac{\overline{Oi}^4 + \overline{Og}^4 + \overline{Ob}^4 + \&c.}{r}$

$= 2nr^3 + 10nr \times \overline{Op}^2 + \frac{15n.\overline{Op}^4}{4r}$ when the circle is equally divided in the points A, B, C, &c. or when a regular figure circumscribing it touches it in these points.

AND generally when m is any integer whatsoever, $\frac{\overline{r+Oi}^m + \overline{r-Oi}^m}{r^{m-3}} + \frac{\overline{r+Og}^m + \overline{r-Og}^m}{r^{m-3}} + \frac{\overline{r+Ob}^m + \overline{r-Ob}^m}{r^{m-3}}$ + &c. is $= 2nr^3 + \frac{m}{1}.\frac{m-1}{1}.r \times \overline{\overline{Oi}^2 + \overline{Og}^2 + \overline{Ob}^2 + \&c.} + \frac{m}{1}.\frac{m-1}{1}.\frac{m-2}{3}.\frac{m-2}{4} \times \frac{\overline{Oi}^4 + \overline{Og}^4 + \overline{Ob}^4 + \&c.}{r} + \frac{m}{1}.\frac{m-1}{1}.\frac{m-2}{3}.\frac{m-3}{4}.\frac{m-4}{5}.\frac{m-5}{6} \times$ into $\frac{\overline{Oi}^6 + \overline{Og}^5 + \overline{Ob}^6 + \&c.}{r}$ + &c. &c.; which, when the circle is equally divided in the points A, B, C, &c. or when

when a regular figure circumſcribing it touches it in ſaid points, gives Dr STEWART's 37th theorem, ſince the ſame reaſoning and mode of demonſtration holds good in regard to half the amount of this expreſſion, whether the points p and q be in PQ, or in PQ produced.

AND univerſally if m have to l any ratio whatſoever,

$$\frac{\overline{r+Oi}^{\frac{m}{l}}+\overline{r-Oi}^{\frac{m}{l}}}{r^{\frac{m}{l}-3}}+\frac{\overline{r+Og}^{\frac{m}{l}}+\overline{r-Og}^{\frac{m}{l}}}{r^{\frac{m}{l}-3}}+\frac{\overline{r+Ob}^{\frac{m}{l}}+\overline{r-Ob}^{\frac{m}{l}}}{r^{\frac{m}{l}-3}}+\&c.$$

$$=2nr^3+\frac{m}{l}\cdot\frac{m-l}{l}\cdot r\times\overline{\overline{Oi}^2+\overline{Og}^2+\overline{Ob}^2+\&c.}+\frac{m}{l}\cdot\frac{m-l}{l}\cdot\frac{m-2l}{3l}\cdot$$

$$\frac{m-3l}{4l}\cdot\times\frac{\overline{Oi}^4+\overline{Og}^4+\overline{Ob}^4+\&c.}{r}+\frac{m}{l}\cdot\frac{m-l}{l}\cdot\frac{m-2l}{3l}\cdot\frac{m-3l}{4l}\cdot\frac{m-4l}{5l}\cdot$$

$$\frac{m-5l}{6l}\cdot\times\text{ into }\frac{\overline{Oi}^6+\overline{Og}^6+\overline{Ob}^6+\&c.}{r^3}+\&c.\ \&c.$$

THIS laſt theorem or expreſſion is more general than any of Dr STEWART's theorems, and from it may eaſily be derived an endleſs number of new and curious infinite ſeries, with their ſummations.

IT is almoſt needleſs for me to obſerve, that beſides theſe theorems of Dr MATTHEW STEWART, an unlimited number of other theorems, reſpecting figures both regular and irregular, circumſcribing and inſcribed in the circle, may eaſily be derived from the foregoing inveſtigation, as well as a great number of geometrical infinite ſeries, with their ſummations. And as to theorems reſpecting given points, right lines and figures either regular or irregular, given by poſition, and right lines interſecting each other either in one point or in different points in angles either equal or unequal, that are deducible from it, they are innumerable.

 Now,

Now, let a circle (Fig. 2.) be divided into an uneven number of equal parts, by the points A, B, C, D, E, &c. and let PQ be any diameter; from P let Pa, Pb, Pc, Pd, Pk, &c. be drawn perpendicular to the diameters passing through the points A, B, C, &c. and from Q let Qe, Qf, Qg, Qh, Qi, &c. be perpendicular to the same diameters.

Then it is evident, that Aa, Ae are respectively equal to perpendiculars drawn from P, Q, to a tangent to the circle in the point A; and since $Oa = Oe$, their sum $Aa + Ae = \overline{r-Oa} + \overline{r+Oa}$. In like manner, the sum of the perpendiculars from P, Q to the tangent at B is $= \overline{r-Oc} + \overline{r+Oc}$, to the tangent at C is $= \overline{r-Ok} + \overline{r+Ok}$, to the tangent at D is $= \overline{r+Ob} + \overline{r-Ob}$, and to the tangent at E is $= \overline{r+Od} + \overline{r-Od}$. But $r-Oa + r-Oc + r-Ok + r+Ob + r+Od = r+Oa + r+Oc + r+Ok + r-Ob + r-Od$; $2 \times \overline{Ob+Od} = 2 \times \overline{Oa+Oc+Ok}$ and $Ob + Od = Oa + Oc + Ok$, and since $\overline{r-Oa}^2 + \overline{r-Oc}^2 + \overline{r-Ok}^2 + \overline{r+Ob}^2 + \overline{r+Od}^2 = \overline{r+Oa}^2 + \overline{r-Oc}^2 + \overline{r+Ok}^2 + \overline{r-Ob}^2 + \overline{r-Od}^2$, we have this equation $4 \times \overline{r \times Ob + r \times Od} = 4 \times \overline{r \times Oa + r \times Oc + r \times Ok}$, or $Ob + Od = Oa + Oc + Ok$.

But if from a point in the circumference of a circle, perpendiculars be drawn to the alternate sides of a regular figure of an even number of sides circumscribing the circle, or, which comes to the same thing, beginning with any one side, perpendiculars be drawn to the 1st, 3d, 5th, 7th, &c. sides, the sum of these perpendiculars, the sum of their squares, the sum of their cubes, &c. to the sum of their $\overline{\frac{n}{2}-1}^{\text{th}}$ or $\frac{n-2}{2}^{\text{th}}$ powers, is respectively equal to the sum of the perpendiculars drawn from the same

point

point to the other ſides, viz. the 2d, 4th, 6th, 8th, &c. the ſum of their ſquares, the ſum of their cubes, &c. to the ſum of their $\frac{n-2}{2}$ powers, but not in powers above $\frac{n-2}{2}$ (*n* being the number of the ſides).

THUS for inſtance, if a regular hexagon circumſcribe a circle, and from any point in the circumference perpendiculars be drawn to the alternate ſides, that is, to the ſides of an equilateral triangle circumſcribing it, the ſum of theſe perpendiculars, and the ſum of their ſquares, are reſpectively equal to the ſum of the perpendiculars drawn to the other three ſides, and the ſum of their ſquares. For the ſum of the perpendiculars to the three ſides of an equilateral triangle, is equal to half the ſum of the perpendiculars to the ſides of the hexagon, and the ſum of their ſquares in the one, equal to half the ſum of their ſquares in the other. But this does not hold in regard to the ſum of their cubes, as the ſum of the cubes of perpendiculars to the ſides of the triangle is not invariable.

IN like manner, if perpendiculars be drawn from a point in the circumference to any four ſides of a regular circumſcribing octagon, taking them alternately, that is, to the ſides of a circumſcribing ſquare, their ſum, the ſum of their ſquares, and the ſum of their cubes, are reſpectively equal to the ſum of perpendiculars to the other four ſides, the ſum of their ſquares and the ſum of their cubes. But this does not hold in regard of the ſum of their fourth powers, which to the ſides of a ſquare are not invariable.

IN like manner, the ſum of perpendiculars to the alternate ſides of a regular circumſcribing decagon, that is, to the ſides of a pentagon, the ſum of their ſquares, the ſum of their cubes, and the ſum of their fourth powers, are reſpectively equal to the sum, the sum of the squares, the sum of the cubes, and the sum of the fourth powers of perpendiculars to the other five sides. But this equality does not hold in the fifth powers, which to the sides of a pentagon are not invariable. For $\frac{10-2}{2} = 4$. And so on.

N. B. The same holds true if the perpendiculars be drawn from any point within the figure for odd powers, and either within or without, in even ones.

But as it was observed in the preceding page, that the equality between the sum of the powers of perpendiculars, drawn from any point in the circumference of a circle, to the alternate sides of any regular figure of an even number of sides, and the sum of the powers of perpendiculars drawn from the same point to the other sides, existed only to the $\frac{n-2}{2}^{th}$ power; so the equality between the sum of the powers of perpendiculars drawn from the extremities P and Q of any diameter to the sides of a regular figure of an odd number of sides circumscribing the circle, and the sum of perpendiculars from either of these, or any point in the circumference, to the sides of a regular circumscribing figure of double the number of sides, exists only to the $\overline{n-2}^{th}$ power.

A wide field is here opened for the geometrical solution of both determinate and indeterminate problems.

For instance, having two equal right lines given, to cut one into two parts, and the other into three, so that the sum of the squares on the two parts, into which the one is cut, shall be equal

equal to the ſum of the ſquares on the three parts, into which the other is cut.

SOLUTION.

WITH radius equal to one-third part of either of the given lines deſcribe a circle. If a regular hexagon circumſcribe it, perpendiculars drawn from the point where any ſide of the hexagon touches the circle, to the other five ſides, are reſpectively equal to the parts into which the two given equal right lines are required to be divided. Calling the ſide, from a point in which the perpendiculars are drawn, the 1ſt, the perpendiculars drawn to the 3d and 5th are the parts, into which one of the two equal given right lines is cut, and thoſe drawn to the 2d, 4th, and 6th ſides, the three parts into which the other given line is cut.

N. B. If the perpendiculars be drawn from any point in the circumference, that is not one of the points of contact, three of them taken alternately, are together equal to the other three, and equal to either of the given lines, and the ſum of their ſquares equal to the ſum of the ſquares of the other three. And if they be drawn from a point in the circumference equally diſtant from two points of contact, the 1ſt = the 6th, the 2d = the 5th, and 3d = the 4th.

AGAIN, let it be required to divide each of two equal given right lines into four unequal parts, ſo that none of the parts of the one ſhall be equal to any of the parts of the other, but the ſum of the ſquares of the parts of the one ſhall be equal to the ſum of the ſquares of the parts of the other, and alſo the ſum of the cubes of the parts of the one equal to the ſum of the cubes of the parts of the other.

SOLUTION.

Solution.

With a fourth part of either of the equal given right lines as radius deſcribe a circle. If a regular decagon circumſcribe the circle, and from any point in the circumference, that is neither one of the points, where the ſides of the figure touch the circle, nor at an equal diſtance between the points of contact, perpendiculars be drawn to the ſides of the octagon, theſe taken alternately are the parts into which the given right lines are required to be divided.

If the point coincide with one of the points of contact, one of the given lines is cut into three parts, and the other into four.

If the point be equally diſtant from two points of contact, the 1ſt perpendicular is = the 8th, the 2d = 7th, the 3d = 6th, and the 4th = 5th. $\frac{3-2}{2} = \frac{6}{2} = 3$ the higheſt power.

With ſuch problems one might proceed without end.

Since (fig. 1.) $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 +$ &c. $\overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 +$ &c. are equal to the ſquares of lines drawn to P and Q from the angles of a regular inſcribed figure of the ſame number of ſides with the irregular circumſcribing figure, or from the points where the ſides of a regular circumſcribing figure touch the circle, it is evident, that the ſum of the ſquares of perpendiculars drawn from P and Q to the ſides of any circumſcribing figure, regular or irregular, of a given number n of ſides, together with the ſquares of the perpendicular diſtances of P and Q from the diameters paſſing through the points of contact A, B, C, &c. viz. $\overline{Pa}^2 + \overline{Pb}^2 + \overline{Pe}^2 +$ &c. $+ \overline{Qc}^2 + \overline{Qd}^2 + \overline{Qf}^2 +$ &c. $= 2 \times \overline{\overline{Pa}^2 + \overline{Bb}^2 + \overline{Pe}^2 + \text{\&c.}}$ is an invariable quantity. For $\overline{Pa}^2 + \overline{aO}^2 = \overline{Pb}^2 + \overline{bO}^2 = \overline{Pe}^2 + \overline{eO}^2 = \overline{PO}^2$ whether the angles

angles at O formed by the diameters paſſing through the points of contact A, B, C, &c. be equal or unequal, or whether the circumference of the circle, of which the diameter is PO, be equally or unequally divided in the points *a*, *b*, *e*, &c.; and the ſum of perpendiculars from P and O, to the ſides of a circumſcribing figure, touching this circle in the points *a*, *b*, *e*, &c. is the ſame whether the figure be regular or irregular.

OTHERWISE,

$Aa + Ac = 2r : 4r^2 = \overline{Aa}^2 + \overline{Ac}^2$ $(\overline{aG}^2) + 2Aa \times aG$ $(2 \times \overline{Pa}^2)$. In like manner, $4r^2 = \overline{Cb}^2 + \overline{Cd}^2$ $(\overline{bb}^2) + 2Cb \times bb$ $(2 \times \overline{Pb}^2)$ and ſo on: $4nr^2 = \overline{Aa}^2 + \overline{Ac}^2 + \overline{Cb}^2 + \overline{Cd}^2 + \overline{Be}^2 + \overline{Bf}^2 +$ &c. $+ 2 \times \overline{Pa}^2 + \overline{Pb}^2 + \overline{Pe}^2 +$ &c. whether the angles AOB, BOC, &c. at O be equal or unequal.

IN like manner, the ſum of the cubes of perpendiculars drawn from P and Q to the ſides of any circumſcribing figure, regular or irregular, of a given number *n* of ſides, together with thrice the ſolid, which has the ſquares of perpendiculars from P and Q to the diameters paſſing through A, B, C, &c. for its baſe, and *r* for its altitude, is an invariable quantity; that is, $2nr^3 + 6r \times \overline{\overline{Oa}^2 + \overline{Ob}^2 + \overline{Oe}^2 + \&c. + \overline{Pa}^2 + \overline{Pb}^2 + \overline{Pe}^2 + \&c.}$ is an invariable quantity, being $= 2nr^3 + 6r \times n \times \overline{PO}^2 = 8nr^3 = n \times \overline{2r}^3$. For $\overline{Oa}^2 + \overline{Ob}^2 + \overline{Oe}^2 +$ &c. $+ \overline{Pa}^2 + \overline{Pb}^2 + \overline{Pe}^2 +$ &c. $= nr^2$. Conſequently, when AB, BC, &c. are unequal among themſelves, the ſum of the ſquares, cubes, &c. of perpendiculars drawn from P and Q to lines touching the circle in the points A, B, C, &c. is not invariable.

$\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 +$ &c. $+ \overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 +$ &c. being $= 4nr^2$, whether AB, AC, &c. be equal or unequal, is invariable.

variable. But their 4th, 6th, 8th, &c. are not invariable, when AB, BC, &c. are unequal among themſelves.

It is manifeſt, that when AB, BC, &c. are equal among themſelves, whatever be the number of the points A, B, C, &c. or whatever be the part of the circumference they take in or extend to, the ſum of the ſquares, cubes, &c. of perpendiculars drawn from P and Q to lines touching the circle in theſe points, is the ſame with the ſum of the ſquares, cubes, &c. of perpendiculars drawn from P and Q to the ſides of a regular circumſcribing figure having the ſame number of ſides as there are points, A, B, C, &c. Thus, if the number of the points be five, and theſe be comprehended in a ſemicircle, a quadrant, or any ſector, the ſum of the ſquares, cubes, and fourth powers of perpendiculars drawn from P and Q to lines touching the circle in theſe points, is the ſame with the ſum of the ſquares, cubes, and fourth powers of perpendiculars drawn from P and Q to the ſides of a regular pentagon circumſcribing the circle. And ſo on.

Perpendiculars drawn from P, (fig. 2.), one extremity of the diameter PQ, to the ſides of the figure of an uneven or odd number of ſides circumſcribing the circle, and touching it in the points A, B, C, D, E, &c. are reſpectively equal to perpendiculars drawn from Q, the other extremity of the diameter, to the ſides of a circumſcribing figure of double the number of ſides, which touch the circle in the points H, I, K, L, G, &c. or $Hg = Bc$, $Ge = Aa$, $Lb = Ed$, $Kf = Db$, $Ii = Ck$, &c.; and perpendiculars drawn from Q to the ſides of the figure of an odd number of ſides, are reſpectively equal to perpendiculars drawn from P to the ſides of a figure of double the number of ſides, which touch the circle in the points H, I, K, L, G, &c. or $Df = Kb$, $Ci = Ik$, $Bg = Hc$, $Ae = Ga$, $Eb = Ld$, &c. Wherefore, the ſum of the m powers of perpendiculars, drawn from P and Q to the ſides of any circumſcribing figure of an odd number of ſides, is equal to half the ſum of the m powers of of perpendiculars drawn from P and Q to the ſides of a circumſcribing figure of double the number of ſides.

Thus, if the figure be a pentagon, we get

$$\overline{r+Oa}^3+\overline{r-Oa}^3 = 2r^3+6r\times\overline{Oa}^2$$

$$\overline{r+Oc}^3+\overline{r-Oc}^3 = 2r^3+6r\times\overline{Oc}^2$$

$$\overline{r+Ok}^3+\overline{r-Ok}^3 = 2r^3+6r\times\overline{Ok}^2$$

$$\overline{r+Ob}^3+\overline{r-Ob}^3 = 2r^3+6r\times\overline{Ob}^2$$

$$\overline{r+Od}^3+\overline{r-Od}^3 = 2r^3+6r\times\overline{Od}^2$$

$$\text{Sum} = 10r^3+6r\times\overline{Oa}^2+\overline{Oc}^2+\overline{Ok}^2+\overline{Ob}^2+\overline{Od}^2=25r^3=10\times\frac{5r^3}{2}$$

When the diameter PQ biſects the arcs BK, DH, or is perpendicular to one of the diameters paſſing through a point of contact, O k, O i vaniſh, and it is then demonſtrated exactly in the ſame way as in figures of an even number of ſides, that the ſum of the cubes of perpendiculars drawn from either P or Q is $\frac{5n}{2}\times r^3$, and conſequently that the ſum of the cubes of thoſe drawn from P, is equal to the ſum of thoſe drawn from Q. But let the figure be a pentagon, and let the diameter AG be perpendicular to any ſide in the point of contact A. Draw C m, B n perpendicular to AG. Then G m is equal to each of the perpendiculars drawn from G to the ſides touching the circle in the points C and D; and A m to each of the perpendiculars drawn from A to the ſame ſides; G n is equal to each of the perpendiculars drawn from G to the ſides touching the circle in the points B, E, and A n, to each of the perpendiculars drawn from A to the ſame ſides. Wherefore $2\,Gm + 2\,Gn + GA\,(2r) = 2\,An + 2\,Am$, or $r - Om + r + On + r = An + Am = r - On$

$+ r + Om$, and $r = 2Om - 2On$. Therefore $Om = \frac{r + 2On}{2}$, $r - Om = \frac{r - 2On}{2} = Gm$, and $r + Om = \frac{3r + 2On}{2} = Am$.

Consequently we have $2 \times \overline{r + Om}^3 = \frac{27r^3 + 54r^2 \times On + 36r \times \overline{On}^2 + 8 \times On^3}{4}$, and $2 \times \overline{r - On}^3 = 2r^3 - 6r^2 \times On + 6r \times \overline{On}^2 - 2 \times \overline{On}^3$, and these added together give $\frac{35r^3 + 30r^2 \times On + 60r \times \overline{On}^2}{4}$. In like manner, $2 \times \overline{r - Om}^3 = \frac{r^3 - 6r^2 \times On + 12r \times \overline{On}^2 - 8 \times \overline{On}^3}{4}$, and $2 \times \overline{r + On}^3 = 2r^3 + 6r^2 \times On + 6r \times \overline{On}^2 + 2 \times \overline{On}^3$, and these added together give $\frac{9r^3 + 18r^2 \times On + 36r \times \overline{On}^2}{4}$, to which, if $\overline{GA}^3 = 8r^3$, the cube of the perpendicular to the side touching the circle in the point A, be added, we get $\frac{41r^3 + 18r^2 \times On + 36r \times \overline{On}^2}{4}$.

But the sum of the cubes of perpendiculars, drawn from A and G to the sides of the pentagon is $25r^3$, as has been demonstrated, when PQ coincides with AG. Wherefore $\frac{76r^3 + 48r^2 \times On + 96r \times \overline{On}^2}{4}$ or $19r^3 + 12r^2 \times On + 24r \times \overline{On}^2 = 25r^3$ and $4r \times \overline{On}^2 + 2r^2 \times On = r^3$. Now, if for this value of r^3, there be substituted r^3 in $\frac{41r^3 + 18r^2 \times On + 36r \times \overline{On}^2}{4}$,

we

we get $\frac{41r^3+9r^3}{4}=\frac{25r^3}{2}$; and if r^3 be ſubſtituted for its equal in $\frac{35r^3+30r^2\times On+60r\times\overline{On}^2}{4}$, we get $\frac{35r^3+15r^3}{4}=\frac{25r^3}{2}$. Wherefore the ſum of the cubes of perpendiculars drawn from the point G to the ſides of the pentagon, is equal to the ſum of the cubes of perpendiculars drawn from the point A to the ſame.

Since $2\times\overline{r+Om}^3+2\times\overline{r-On}^3=4r^3+3r^2\times\overline{2Om-2On}$ $+3r\times\overline{2\times\overline{Om}^2+2\times\overline{On}^2}+2\times\overline{Om}^3-2\times\overline{On}^3=\frac{25r^3}{2}$, and $\overline{2\times\overline{Om}^2+\overline{On}^2}=\frac{3r^2}{2}$, we have $3r^2\times\overline{2\times Om-2\times On}+$ $2\times\overline{\overline{Om}^3-\overline{On}^3}=4r^3$. But $2Om-2On=r$; therefore $2\times\overline{\overline{Om}^3-\overline{On}^3}=r^3$, or $\overline{Om}^3-\overline{On}^3=\frac{r^3}{2}$.

If P, inſtead of biſecting the arc BK, be any point between B and K, the ſum of the cubes of perpendiculars drawn from it to the ſides of the circumſcribing pentagon, is equal to the ſum of the cubes of perpendiculars drawn from Q to the ſame. For ſince $Oc+Oa+Ok=Ob+Od$ and $Oc-Ob$, $Od-Oa$ and Ok, begin together, and become maxima together, $Oc-Ob$ has to Ok a given ratio. Let that be the ratio of m to 1. Then $Oc-Ob=m\times Ok$, and $Od-Oa=Oc-Ob+Ok=$ $\overline{m+1}\times Ok$. $Oc=Ob+m\times Ok$, $Oa=Od-\overline{m+1}\times Ok$. $\overline{Oc}^3=\overline{Ob}^3+3m\times\overline{Ob}^2\times Ok+3m^2\times Ob\times\overline{Ok}^2+m^3\times$ $\overline{Ok}^3$. $\overline{Oa}^3=\overline{Od}^3-3\times\overline{m+1}\times\overline{Od}^2\times Ok+3\times\overline{m+1}\times Od\times$ $\overline{Ok}^2$

$\overline{Ok}^2 - \overline{Ok}^3$. Wherefore $\overline{Oc}^3 + \overline{Oa}^3 + \overline{Ok}^3 = \overline{Ob}^3 + \overline{Od}^3 + 3\,m \times \overline{Ob}^2 \times Ok - 3 \times \overline{m+1} \times \overline{Od}^2 \times Ok + 3\,m^2 \times Ob \times \overline{Ok}^2 + 3 \times \overline{m+1}^2 \times Od \times \overline{Ok}^2 + \overline{Ok}^3$. Now, let this be $= \overline{Ob}^3 + \overline{Od}^3 \pm V^3$. Then $3\,m \times \overline{Ob}^2 - 3 \times \overline{m+1} \times \overline{Od}^2 + 3\,m^2 \times Ob \times Ok + 3 \times \overline{m+1}^2 \times O\,d \times O\,k + \overline{Ok}^2 = \pm \frac{V^3}{Ok}$, and $m \times \overline{Ob}^2 = \overline{m+1} \times \overline{Od}^2 \pm \frac{V^3}{3}$, when $Ok = 0$. But $m \times \overline{Ob}^2 = \overline{m+1} \times \overline{Od}^2$; therefore $V = 0$. For when $O\,k = 0$, $O\,b$ is the ſine of 72°, and $O\,d$ the ſine of 36°. When $O\,k$ is a maximum, it is the ſine of 18°, $O\,c$ is $= r$, $O\,b$ is the coſine of 36°, and $O\,c - O\,b$ the verſed ſine of 36°. Wherefore, $m+1 : m = \overline{Ob}^2 : \overline{Od}^2 =$ verſed ſine of 36° + ſine of 18° : verſed ſine of 36°.

Let BD (Pl. II. fig. 3) = BH = ſide of an inſcribed pentagon; biſect BD in F, and draw OFC, AC, BC and DH. Then, ſince the angle FOB is 36°, CF is the verſed ſine of 36°, OG is the ſine of 18°. But ſince the triangles CFB, DGO, are ſimilar OG : CF = DG : FB = DH : DB, and OG + CF : CF = DH + DB : DB = $\overline{DH}^2 : \overline{DB}^2 = \overline{DG}^2 : \overline{FB}^2 =$ ſquare of the ſine of 72° : ſquare of the ſine of 36°. For when DH is cut in extreme and mean ratio, the greater part is equal to the ſide of the pentagon.

DH is cut in extreme and mean ratio in the point L, and LH = BD; the triangle CDP is ſimilar to the triangle DOB; and the triangle MDN to the triangle BOC.

This demonſtration, however, was unneceſſary. For if the ſum of the cubes of perpendiculars drawn from P to the ſides of the pentagon, be equal to the ſum of the cubes of perpendiculars drawn

drawn from Q to the ſame, both when O k is $= o$, and when it is a maximum, this equality muſt exiſt whatever be the magnitude of O k between theſe limits.

And in a ſimilar manner is it demonſtrated, that the ſum of the cubes of perpendiculars drawn from P to the ſides of any other regular figure of an odd number of ſides circumſcribing the circle, is equal to the ſum of the cubes of perpendiculars drawn from Q to the ſame. For if O m, &c. or ſuch parts of perpendiculars drawn from A to the ſides of any regular circumſcribing figure of an odd number, n, of ſides as lie between O and G, be denoted by A, B, C, &c.; and O n, &c. or ſuch parts of perpendiculars drawn from G to the ſame as lie between O and A, be denoted by a, b, c, &c. $A + B + C +$ &c, to $\frac{n-1}{4}$ terms, $-a-b-$ &c. to $\frac{n-1}{4}$ terms, $= \frac{r}{2}$, if $n-1$ be a multiple of 2 by an even number. Alſo $A^2+B^2+C^2+$ &c. to $\frac{n-1}{4}$ terms, $+a^2+b^2+c^2+$ &c. to $\frac{n-1}{4}$ terms, $= \frac{\overline{n-2} \times r^2}{4}$; and $A^3+B^3+C^3+$ &c. to $\frac{n-1}{4}$ terms, $-a^3-b^3-c^3-$ &c. to $\frac{n-1}{4}$ terms, $= \frac{r^3}{2}$. But if $n-1$ be a multiple of 2 by an odd number, $A + B + C +$ &c. to $\frac{n+1}{4}$ terms, $-a-b-$ &c. to $\frac{n-3}{4}$ terms, $= \frac{r}{2}$; $A^2 + B^2 + C^2 +$ &c. to $\frac{n+1}{4}$ terms, $+a^2+b^2+$ &c. to $\frac{n-3}{4}$ terms $= \frac{n-2}{4} \times r^2$, and $A^3 + B^3 + C^3 +$ &c. to $\frac{n+1}{4}$

terms

terms, $-a^3-b^3-$&c. to $\frac{n-3}{4}$ terms, $=\frac{r^3}{2}$. Thus, in the heptagon $A+B-a=\frac{r}{2}$, $A^2+B^2+a^2=\frac{5r^2}{4}$; and $A^3+B^3-a^3=\frac{r^3}{2}$. In the enneagon $A+B-a-b=\frac{r}{2}$, $A^2+B^2+a^2+b^2=\frac{7r^2}{4}$ and $A^3+B^3-a^3-b^3=\frac{r^3}{2}$, and so on. But in the enneagon, A is the cosine of 20°, or the sine of 70°, B is the sine of 30° $=\frac{r}{2}$, a is the sine of 10°, and b the sine of 50°. Wherefore the sine of 70° = sine of 10° + the sine of 50°; the square of the sine of 70° + square of the sine of 10° + square of the sine of 50° $=\frac{3r^2}{2}$; and the cube of the sine of 70° — cube of the sine of 10° — cube of the sine of 50° $=\frac{3r^3}{8}$, viz. thrice the cube of half the radius. And so on.

$$\overline{r-Oc}^4=r^4-4r^3\times Oc+6r^2\times\overline{Oc}^2-4r\times\overline{Oc}^3+\overline{Oc}^4,$$
$$\overline{r-Oa}^4=r^4-4r^3\times Oa+6r^2\times\overline{Oa}^2-4r\times\overline{Oa}^3+\overline{Oa}^4,$$
$$\overline{r-Ok}^4=r^4-4r^3\times Ok+6r^2\times\overline{Ok}^2-4r\times\overline{Ok}^3+\overline{Ok}^4,$$
$$\overline{r+Ob}^4=r^4+4r^3\times Ob+6r^2\times\overline{Ob}^2+4r\times\overline{Ob}^3+\overline{Ob}^4,$$
$$\overline{r+Od}^4=r^4+4r^3\times Od+6r^2\times\overline{Od}^2+4r\times\overline{Od}^3+\overline{Od}^4,$$
$$\text{Sum}=5r^4+6r^2\times\overline{\overline{Oc}^2+\overline{Oa}^2+\overline{Ok}^2+\overline{Ob}^2+\overline{Od}^2}+\overline{Oc}^4+\overline{Oa}^4+\overline{Ob}^4+\overline{Od}^4.$$

BUT

BUT ſince $5r$ ($=$ ſum of the perpendiculars drawn from P or Q to the ſides of the pentagon) $= \frac{\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 + \overline{EP}^2}{2r}$ $= \frac{\overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 + \overline{DQ}^2 + \overline{EQ}^2}{2r}$; and ſince the ſum of the ſquares of perpendiculars drawn from P to the ſides of the pentagon, has been demonſtrated equal to the ſum of the ſquares of perpendiculars from Q to the ſame, $\frac{\overline{AP}^4 + \overline{BP}^4 + \overline{CP}^4 + \overline{DP}^4 + \overline{EP}^4}{4r^2}$ $= \frac{\overline{AQ}^4 + \overline{BQ}^4 + \overline{CQ}^4 + \overline{DQ}^4 + \overline{EQ}^4}{4r^2} = \frac{15r^2}{2}$. And as the circumference of the circle, which has PO for its diameter, is divided into five equal parts, in the points a, b, c, d, k, $\overline{Oc}^2 + \overline{Oa}^2 + \overline{Ok}^2 + \overline{Ob}^2 + \overline{Od}^2 = r \times \frac{5 \times OP}{2} = \frac{5r^2}{2}$; and $\overline{Oc}^4 + \overline{Oa}^4 + \overline{Ok}^4 + \overline{Ob}^4 + \overline{Od}^4 = 4 \times \frac{\overline{OP}^2}{4} \times \frac{15}{2} \times \frac{\overline{OP}^2}{4} = \frac{15r^4}{8}$. Wherefore,

$$5r^4 + 6r^2 \times \overline{\overline{Oc}^2 + \overline{Oa}^2 + \overline{Ok}^2 + \overline{Ob}^2 + \overline{Od}^2} + \overline{Oc}^4 + \overline{Oa}^4 + \overline{Ok}^4 + \overline{Ob}^4 + \overline{Od}^4 = \frac{40r^4}{8} + \frac{120r^4}{8} + \frac{15r^4}{8} = \frac{175}{8} \times r^4 = \frac{35 \times 5}{8} \times r^4$$

$=$ half the ſum of the fourth powers of perpendiculars drawn from both P and Q to the ſides of the pentagon.

IN the ſame way is it demonſtrated, that the ſum of the fourth powers of perpendiculars drawn from P to the ſides of any regular circumſcribing figure of an odd number, n, of ſides, is equal to the ſum of the fourth powers of perpendiculars drawn from

Q

Q to the fame, $= \frac{35n}{8} \times r^4 = n \times \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \times r^4$. When P coincides with B, Oc is $= r$, and $A^4 + B^4 + C^4 +$ &c. to $\frac{n-1}{4}$ terms, $+ a^4 + b^4 + c^4 +$ &c. to $\frac{n-1}{4}$ terms, (when $n - 1$ is a multiple of 2 by an even number) is $= \frac{3n-8}{16} \times r^4$; and $A^4 + B^4 + C^4 +$ &c. to $\frac{n+1}{4}$ terms, $+ a^4 + b^4 +$ &c. to $\frac{n-3}{4}$ terms, (when $n - 1$ is a multiple of 2 by an odd number) is $= \frac{3n-8}{16} \times r^4$.

In like manner is it demonftrated, that the fum of the fifth powers of perpendiculars drawn from P to the fides of any regular circumfcribing figure of a greater number n of fides than 5, is equal to the fum of the fifth powers of perpendiculars drawn from Q to the fame; and that $A^5 + B^5 + C^5 +$ &c. to $\frac{n-1}{4}$ terms, $- a^5 - b^5 - c^5 -$ &c. to $\frac{n-1}{4}$ terms, (when $n - 1$ is a multiple of 2 by an even number) is $= \frac{r^5}{2}$, and $A^5 + B^5 + C^5 +$ &c. to $\frac{n+1}{4}$ terms $- a^5 - b^5 -$ &c. to $\frac{n-3}{4}$ terms, (when $n - 1$ is a multiple of 2 by an odd number) is $= \frac{r^5}{2}$.

A

A is the cosine of $\frac{180°}{n}$, B is the cosine of $3 \times \frac{180°}{n}$, C is the cosine of $5 \times \frac{180°}{n}$, and so on; and when $n - 1$ is a multiple of 2 by an even number, the last of the terms A, B, C, &c. is the cosine of $\frac{n-3}{2} \times \frac{180°}{n}$, and a, the first of the terms a, b, c, &c. is the cosine of $\frac{n+1}{2} \times \frac{180°}{n}$; b, the second, is the cosine of $\frac{n+5}{2} \times \frac{180°}{n}$; c, the third, is the cosine of $\frac{n+9}{2} \times \frac{180°}{n}$; and the last of the terms a, b, c, &c. is the cosine of $\frac{2n-4}{2} \times \frac{180°}{n}$, or of $\overline{n-2} \times \frac{180°}{n}$. But when $n - 1$ is a multiple of 2 by an odd number, the last of the terms A, B, C, &c. is the cosine of $\frac{n-1}{2} \times \frac{180°}{n}$; the first of the terms a, b, &c. is the cosine of $\frac{n+3}{2} \times \frac{180°}{n}$ and the last of the terms a, b, &c. is the cosine of $\overline{n-2} \times \frac{180°}{n}$. Universally, if m be any odd number less than n, we have $A^m + B^m + C^m +$ &c. to $\frac{n-1}{4}$ terms, $- a^m - b^m - c^m -$ &c. to $\frac{n-1}{4}$ terms, $= \frac{r^m}{2}$, when $n - 1$ is a multiple of 2 by an even number; and $A^m + B^m + C^m +$ &c., to $\frac{n+1}{4}$ terms, $- a^m - b^m -$ &c. to $\frac{n-3}{4}$ terms, $= \frac{r^m}{2}$, when

$n - 1$

$n-1$ is a multiple of 2 by an odd number. Thus, in the enneagon, or figure of nine sides, A is the cosine of 20° or the sine of 70°, B is the cosine of 60° or the sine of 30°, a is the cosine of 100° or the sine of 10°, and b is the cosine of 140° or the sine of 50, and $A^m - a^m - b^m = \frac{2^{m-1}-1}{2^m} \times r^m$. And if p be any even number less than n, $A^p + B^p + C^p +$ &c. to $\frac{n-1}{4}$ terms or $\frac{n+1}{4}$ terms $+ a^p + b^p + c^p +$ &c. to $\frac{n-1}{4}$ terms, or $\frac{n-3}{4}$ terms, (according as $n-1$ is a multiple of 2 by an even or odd number) is $= n \times \frac{1.3.5.7\ldots p-1}{1.2.3.4\ldots\frac{p}{2}} \times \frac{r^p}{2^{\frac{p+2}{2}}} - \frac{r^p}{2}$. Consequently, $\overline{A^p + a^p + A^m - a^m} + \overline{B^p + b^p + B^m - b^m} +$ &c. to $\frac{n-1}{4}$ terms, is $= n \times \frac{1.3.5.7\ldots p-1}{1.2.3.4\ldots\frac{p}{2}} \times \frac{r^p}{2^{\frac{p+2}{2}}} - \frac{r^p}{2} + \frac{r^m}{2} =$

$$\frac{\frac{p+2}{2}\cdot\frac{p+6}{2}\cdot\frac{p+10}{2}\ldots p-1}{2\cdot 4\cdot 6\ldots\frac{p}{2}} \times \frac{r^p}{2^{\frac{p+2}{2}}} - \frac{r^p}{2} + \frac{r^m}{2}$$

(when $n-1$ is a multiple of 2 by an even number) $=$

$$\frac{\frac{p+2}{2}\cdot\frac{p+6}{2}\cdot\frac{p+10}{2}\ldots \text{to } \frac{2p-2}{2}}{2\cdot 4\cdot 6\ldots \text{to } \frac{p}{2}} \times \frac{1}{2^{\frac{p+2}{2}}},$$

when $r=1$; and when $n-1$ is a multiple of 2 by an odd number $\overline{A^p + A^m} + \overline{B^p + B^m} + \overline{C^p + C^m} +$ &c. to $\frac{n+1}{4}$ terms, $+ \overline{a^p - a^m} + \overline{b^p - b^m}$

$\overline{b^p - b^m} + \overline{c^p - c^m} + \&c.$ to $\frac{n-3}{4}$ terms, is equal to $n \times$

$\frac{1.3.5.7 \ldots \text{ to } p-1}{1.2.3.4 \ldots \text{ to } \frac{p}{2}} \times \frac{r^p}{2^{\frac{p+2}{2}}} - \frac{r^p}{2} + \frac{r^m}{2}$, which when $r = 1$

is $= n \times \dfrac{\frac{p+2}{2} . \frac{p+6}{2} . \frac{p+10}{2} \ldots \text{ to } \frac{2p-2}{2}}{2 \;.\; 4 \;.\; 6 \;\ldots \text{ to } \frac{p}{2}} \times \frac{1}{2^{\frac{p+2}{2}}} = n \times$

$\dfrac{\overline{p+2} . \overline{p+6} \ldots \text{ to } \overline{2p-2}}{2 \;.\; 4 \ldots \text{ to } \frac{p}{2}} \times \frac{1}{2^{\frac{p+4}{2}}}$. Hence the ſummation of

an endleſs variety of ſeries, of which the terms are powers of the ſines and coſines of angles; and though they do not conſiſt of an infinite number of terms, they may conſiſt of any number of terms whatever, ſince $\frac{n-1}{4}$ may be equal to any given number as well as $\frac{n+1}{4}$ and $\frac{n-3}{4}$. The expreſſion $n \times \frac{1.3.5.7 \ldots \text{ to } p-1}{1.2.3.4 \ldots \text{ to } \frac{p}{2}}$

$\times \frac{1}{2^{\frac{p+2}{2}}}$ is $= n \times \dfrac{\frac{p+2}{2} . \frac{p+6}{2} \ldots \text{ to } \frac{2p-2}{2}}{2 \;.\; 4 \ldots \text{ to } \frac{p}{2}} \times \frac{1}{2^{\frac{p+2}{2}}}$, when

$\frac{p}{2}$ is equal to an even number, or p is a multiple of 2 by an even number. But when $\frac{p}{2}$ is an odd number, it is equal to $n \times$

$\dfrac{\frac{p+4}{2} . \frac{p+8}{2} \ldots \text{ to } \frac{2p-2}{2}}{2 \;.\; 4 \ldots \text{ to } \frac{p-2}{2}} \times \frac{1}{2^{\frac{p+2}{2}}}$, which muſt then be uſed.

 THE

THE sums of these series, however, vary with the variations in the magnitude of r. For when $r = 2, 3, 4,$ &c. $-\frac{rp}{2} + \frac{r^m}{2}$ does not vanish, and $\frac{r^p}{2^{\frac{p+2}{2}}}$ becomes respectively $2^{\frac{p}{2}-1}$, $\frac{3^p}{2^{\frac{p+2}{2}}}$, $\frac{4^p}{2^{\frac{p+2}{2}}}$, &c.

$\overline{A^m - a^m} + \overline{B^m - b^m} + \overline{C^m - c^m} +$ &c. to $\frac{n-1}{4}$ terms $= \frac{r^m}{2}$, when $n-1$ is a multiple of 2 by an even number, and the sum of the series is constant or invariable when m is given, let the number of the terms $\frac{n-1}{4}$ be great or small. This series, when n is infinitely great, or $n-1$ has to 4 a ratio greater than any given or assignable ratio, may be considered as infinite.

$$\frac{\overline{A^m - a^m} + \overline{B^m - b^m} + \overline{C^m - c^m} + \text{\&c. to } \frac{n-1}{4} \text{ terms}}{\overline{A-a} + \overline{B-b} + \overline{C-c} + \text{\&c. to } \frac{n-1}{4} \text{ terms}} = \frac{r^m}{2} \div \frac{r}{2} = r^{m-1}.$$

$\overline{\overline{A^m - a^m} + \overline{B^m - b^m} + \text{\&c. to } \frac{n-1}{4} \text{ terms}}$, $\times \overline{A-a} + \overline{B-b} +$ &c. to $\frac{n-1}{4}$ terms, $= \frac{r^{m+1}}{4}$. $\overline{A^m + A - a^m + a} + \overline{B^m + B - b^m + b} +$ &c. to $\frac{n-1}{4}$ terms $= \frac{r^m + r}{2} = 1$, when $r = 1$.

$\overline{A^m - A - a^m - a} + \overline{B^m - B - b^m - b} +$ &c. to $\frac{n-1}{4}$ terms, $= \frac{r^m - r}{2} = 0$, when $r = 1$.

It

It may not be unacceptable to geometers to ſee the foregoing concluſions in regard to regular figures circumſcribed about and inſcribed in a circle, derived by making uſe of one point only, inſtead of two, either in or not in the circumference, which is eaſily effected in the following manner.

Let the ſides of any regular figure of an even number of ſides touch the circle BRETCQLS (Pl. II. fig. 4.) in the points B, R, E, T, C, Q, L, S, and let DN, DH, DM, DV, be perpendiculars from the point D to the diameters joining the points of contact; and from the points of contact let chords be drawn to any point A in the circumference.

If GE, or the radius of the circle, be denoted by r, and A a, A b, A c, A d, be perpendiculars to the diameters joining the points of contact, a C, a B, T b, S b, L c, E c, Q d, d R, are reſpectively equal to the perpendiculars from the point A to the ſides of the figure, and are alſo reſpectively equal to $\frac{\overline{AC}^2}{2r}$, $\frac{\overline{AB}^2}{2r}$, $\frac{\overline{AT}^2}{2r}$, $\frac{\overline{AS}^2}{2r}$, $\frac{\overline{AL}^2}{2r}$, $\frac{\overline{AE}^2}{2r}$, $\frac{\overline{AQ}^2}{2r}$, $\frac{\overline{AR}^2}{2r}$. But if N denote the number of ſides of the figure, the ſum of the perpendiculars is $= N \times r$. Wherefore $\overline{AC}^2 + \overline{AB}^2 + \overline{AT}^2 +$ &c. $= 2 N \times r^2$. This is Prop. 4. Dr Stewart's Theor.

Again, the ſum of the ſquares of the two perpendiculars from A, parallel to BC, or $\overline{Ba}^2 + \overline{aC}^2 = 2r^2 + 2 \times \overline{Ga}^2$; and the ſquares of the two perpendiculars from A parallel to LE, or $\overline{Ec}^2 + \overline{cL}^2 = 2r^2 + 2 \times \overline{Gc}^2$; $\overline{Tb}^2 + \overline{Sb}^2 = 2r^2 + 2 \times \overline{Gb}^2$; alſo $\overline{Rd}^2 + \overline{dQ}^2 = 2r^2 + 2 \times \overline{Gd}^2$. Wherefore the ſum of the ſquares of the perpendiculars drawn from the point A to the ſides

sides of the figure, is $= N \times r^2 + 2 \times \overline{\overline{Ga}^2 + \overline{Gb}^2 + \overline{Gc}^2 + \overline{Gd}^2}$.

But since the angles G *a* A, G *b* A, G *c* A, G *d* A, are right ones, a circle passes through the points A, *a*, *b*, G, *c*, *d*, having GA for its diameter. And because the angles CGT, QGC, LGQ, are equal, this circle is equally divided by the points *a*, *b*, *c*, *d*. Consequently the squares of the chords drawn from these points to the point A, are together $= N \times \frac{\overline{GA}^2}{4}$; that is, $\overline{Ac}^2$ $(\overline{Ga}^2)$ $+ \overline{Aa}^2$ $(\overline{Gc}^2)$ $+ \overline{Ad}^2$ $(\overline{Gb}^2)$ $+ \overline{Ab}^2$ $(\overline{Gd}^2)$ $= N \times \frac{\overline{GA}^2}{4} = N \times \frac{r^2}{4}$. Wherefore the sum of the squares of the perpendiculars drawn from A to the sides of the figure, is $= N \times r^2 + 2N \times \frac{r^2}{4} = N \times \frac{3r^2}{2}$. But the sum of the squares of these perpendiculars is $= \frac{\overline{AC}^4}{4r^2} + \frac{\overline{AB}^4}{4r^2} + \frac{\overline{AT}^4}{4r^2} + \frac{\overline{AS}^4}{4r^2} + \frac{\overline{AL}^4}{4r^2} + \frac{\overline{AE}^4}{4r^2} + \frac{\overline{AQ}^4}{4r^2} + \frac{\overline{AR}^4}{4r^2}$. Therefore $\overline{AC}^4 + \overline{AB}^4 +$, &c. $= N \times 6r^4 = N \times 2r^2 \times 3r^2 = \overline{\overline{AC}^2 + \overline{AB}^2 +}$, &c. $\times 3r^2$. Whence this proposition:

If a circle be divided into any even number of parts, and from the points of division chords be drawn to any point in the circumference, the sum of the fourth powers of these chords is equal to the sum of their squares, multiplied by thrice the square of radius.

Wherefore

Wherefore $\overline{Aa}^4 + \overline{Ab}^4 + \overline{Ac}^4 + \overline{Ad}^4 = \overline{\overline{Aa}^2 + \overline{Ab}^2 + \overline{Ac}^2 + \overline{Ad}^2}$ $\times \frac{\overline{3GA}^2}{4} = N \times \frac{3r^4}{16}$; and $\overline{DM}^4 + \overline{DV}^4 + \overline{DH}^4 + \overline{DN}^4$ $= \overline{\overline{DM}^2 + \overline{DV}^2 + \overline{DH}^2 + \overline{DN}^2} \times \frac{3 \times \overline{DG}^2}{4} = \frac{3 \times N}{16} \times \overline{DG}^4 =$ $\frac{6N}{2} \times \frac{\overline{DG}^4}{16}$.

Now, it is evident, that perpendiculars drawn from the point D to the fides of the figures, are refpectively $r + DM$, $r - DM$, $r + DV$, $r - DV$, $r + DH$, $r - DH$, $r + DN$, $r - DN$.

But $\overline{r + DM}^2 + \overline{r - DM}^2 = 2r^2 + 2 \times \overline{DM}^2$,

$\overline{r + DV}^2 + \overline{r - DV}^2 = 2r^2 + 2 \times \overline{DV}^2$,

$\overline{r + DH}^2 + \overline{r - DH}^2 = 2r^2 + 2 \times \overline{DH}^2$,

$\overline{r + DN}^2 + \overline{r - DN}^2 = 2r^2 + 2 \times \overline{DN}^2$.

Wherefore the fum of their fquares is equal to $N \times r^2 +$ $2 \times \overline{\overline{DM}^2 + \overline{DV}^2 + \overline{DH}^2 + \overline{DN}^2} = N \times r^2 + N \times \frac{\overline{GD}^2}{2}$, fince the circle which paffes through the points D, M, V, G, H, N, is equally divided by the points M, V, H, N. This is Prop. 5. of Dr Stewart's Theorems.

$\overline{r + DM}^3 + \overline{r - DM}^3 = 2r^3 + 6r \times \overline{DM}^2$,

$\overline{r + DV}^3 + \overline{r - DV}^3 = 2r^3 + 6r \times \overline{DV}^2$,

$\overline{r + DH}^3 + \overline{r - DH}^3 = 2r^3 + 6r \times \overline{DH}^2$,

$\overline{r + DN}^3 + \overline{r - DN}^3 = 2r^3 + 6r \times \overline{DN}^2$.

Wherefore,

WHEREFORE, the ſum of the cubes of the perpendiculars, drawn from the point D to the ſides of the figure, is $= N \times r^3 + 6r \times \overline{DM^2 + DV^2 + DH^2 + DN^2} = N \times r^3 + 6r \times \overline{\frac{DG^2}{4}}$.

This is Prop. 23. STEWART's Theor. When DG $= r$, the ſum of the cubes of the perpendiculars is $= N \times \frac{5}{2} \times r^3 = N \times \frac{1.3.5}{1.2.3}. r^3$. This is Prop. 22. Dr STEWART's Theor.

WHEN DG $= r$, or D coincides with A, the ſum of the cubes of the perpendiculars is equal to $\frac{\overline{AC}^6}{8r^3} + \frac{\overline{AB}^6}{8r^3} + \frac{\overline{AT}^5}{8r^3} +$, &c.; and, conſequently, we get $\overline{AC}^6 + \overline{AB}^5 + \overline{AT}^5 +$, &c. $= \frac{5.8}{2} \times N \times r^6 = N \times 20 r^6 = N \times \frac{1.3.5}{1.2.3}..2^3 r^6 = N \times 20 \times r^6 = N \times 10 r^4 \times \overline{AC^2 + AB^2 + AT^2} +$, &c.

IF, therefore, the circumference of a circle be divided into an even number of equal parts, and from the points of diviſion chords be drawn to any point in the circumference, the ſum of the ſixth powers of theſe chords is equal to the ſum of their ſquares, multiplied by ten times the fourth power of radius.

$$\overline{r + DM}^4 + \overline{r - DM}^4 = 2r^4 + 12r^2 \times \overline{DM}^2 + 2 \times \overline{DM}^4,$$

$$\overline{r + DV}^4 + \overline{r - DV}^4 = 2r^4 + 12r^2 \times \overline{DV}^2 + 2 \times \overline{DV}^4,$$

$$\overline{r + DH}^4 + \overline{r - DH}^4 = 2r^4 + 12r^2 \times \overline{DH}^2 + 2 \times \overline{DH}^4,$$

$$\overline{r + DN}^4 + \overline{r - DN}^4 = 2r^4 + 12r^2 \times \overline{DN}^2 + 2 \times \overline{DN}^4.$$

WHEREFORE

Wherefore, the ſum of the fourth powers of the perpendiculars drawn from the point D to the ſides of the figure, is $= \text{N} \times r^4 + \text{N} \times 3r^2 \times \overline{\text{GD}}^2 + \text{N} \times \frac{3}{8} \times \overline{\text{GD}}^4$; and eight times this ſum $= \text{N} \times \overline{8r^4 + 24r^2 . \overline{\text{GD}}^2 + 3 \cdot \overline{\text{GD}}^4}$. This is Prop. 29. of Dr Stewart's Theorems.

When GD $= r$, the ſum of theſe fourth powers is $\text{N} \times \overline{4r^4 + \frac{3}{8}r^4} = \text{N} \times \frac{35}{8}r^4 = \text{N} \times \frac{1.3.5.7}{1.2.3.4} \times r^4$, which is Prop. 28. of Dr Stewart's Theorems.

And ſince $\frac{\overline{\text{AC}}^8}{2^4 r^4} + \frac{\overline{\text{AB}}^8}{2^4 r^4} + \frac{\overline{\text{AT}}^8}{2^4 r^4} +$, &c. $= \text{N} \times \frac{35 r^4}{8}$, we get $\overline{\text{AC}}^8 + \overline{\text{AB}}^8 + \overline{\text{AT}}^8 +$, &c. $= \text{N} \times \frac{35}{8} . 2^4 r^8 = \text{N} \times \frac{1.3.5.7}{1.2.3.4} \times 2^4 r^8$.

And by proceeding in this way, (the law of continuation being evident), we get Propoſitions 39, 40, 41, &c. of Dr Stewart's Theorems, ſince the powers of DM, DV, DH, DN, &c. however high, may always be expreſſed by thoſe of DG and r. The ſame reaſoning holds in all even powers, when the point D is without the figure, by taking the powers of DH, $+ r$, &c. when DH, &c. is greater than than r, inſtead of the powers of $r \pm$ DH, &c.

Let any regular figure of an odd number of ſides, (Pl. III. Fig. 5.), circumſcribe the circle, and touch it in the points B, E, C, Q, L; and from any point D, let perpendiculars DP, DR, DS, DO, DT, be drawn to the ſides of the figure; and DF, DM, DN, DH, DV, perpendiculars to the diameters paſſing through the points of contact:

Then, if radius be denoted by r, it is evident, that DP is $= r - GN$, $DR = r - GF$, $DS = r + GH$, $DO = r + GM$, and $DT = r + GV$; and calling N the number of the ſides of the figure, the ſum of the ſquares of theſe lines is $N \times r^2 + 2r \times \overline{GH + GM + GV - GN - GF} + \overline{GH}^2 + \overline{GM}^2 + \overline{GV}^2 + \overline{GN}^2 + \overline{GF}^2$. But ſince the angles HGN, NGM, MGF, FGV, are equal, and the angles at H, N, M, F, V, right ones, a circle, having its diameter $= GD$, paſſes through the points G, H, N, D, M, F, V, and its circumference is divided into equal parts at the points H, N, M, F, V. Wherefore $\overline{GH}^2 + \overline{GN}^2 + \overline{GM}^2 + \overline{GF}^2 + \overline{GV}^2 = 2 \times N \times \frac{\overline{GD}^2}{4} = N \times \frac{\overline{GD}^2}{2}$. But $\overline{DP}^2 + \overline{DR}^2 + \overline{DS}^2 + \overline{DO}^2 + \overline{DT}^2 = N \times r^2 + N \times \frac{\overline{GD}^2}{2}$. (Stewart's Theor. Prop. 5.). Therefore $2r \times \overline{GH + GM + GV - GN - GF} = 0$, or $GN + GF = GH + GM + GV$. Whence this propoſition: If, from any point, perpendiculars be drawn to the ſides of any regular figure of an odd number of ſides, circumſcribing a circle, the ſum of the parts by which thoſe perpendiculars, which are greater than radius, exceed it, is equal to the ſum of thoſe parts by which the perpendiculars, which are leſs than radius, fall ſhort of it. And this propoſition is alſo true with regard to any regular figure, of which the number of its ſides is a multiple of any odd number by 2, ſince the perpendiculars DF, DM, DN, DH, DV, &c. are the ſame both in number and magnitude, in any regular figure of an odd number of ſides, and a regular figure of double the number of ſides. Conſequently, in a hexagon, one of the three perpendiculars drawn from any point D to the diameters joining the oppoſite points of contact, is

equal

equal to the sum of the other two, and so on; and if, in the hexagon, the point D be taken in one of the three diameters, the perpendiculars drawn from it to the other two are equal.

AGAIN, $\overline{DP}^3 + \overline{DR}^3 + \overline{DS}^3 + \overline{DO}^3 + \overline{DT}^3 = N \times r^3 + 3r^2 \times \overline{GH + GM + GV - GN - GF} + 3r \times \overline{\overline{GH}^2 + \overline{GM}^2 + \overline{GV}^2 + \overline{GN}^2 + \overline{GF}^2} + \overline{GH}^3 + \overline{GM}^3 + \overline{GV}^3 - \overline{GN}^3 - \overline{GF}^3 = N \times r^3 + N \times 3r \cdot \frac{\overline{GD}^2}{2} + \overline{GH}^3 + \overline{GM}^3 + \overline{GV}^3 - \overline{GN}^3 - \overline{GF}^3$. But since $N \times r^3 + N \times 3r \times \frac{\overline{GD}^2}{2} = \overline{DP}^3 + \overline{DR}^3 + \overline{DS}^3 + \overline{DO}^3 + \overline{DT}^3$ $\overline{GH}^3 + \overline{GM}^3 + \overline{GV}^3 = \overline{GN}^3 + \overline{GF}^3$.

IF D be in a line perpendicular from G the centre, to a diameter drawn from any point of contact L, the odd chord GV vanishes, (V coinciding with G), and GN = GM, GH = GF; and the expression for the sum of the cubes of the perpendiculars, drawn from D to the sides of the circumscribing figure, is simply $N \times r^3 + N \times 3r \times \frac{\overline{GD}^2}{2}$

IF the figure circumscribing the circle be a pentagon, a line drawn from G, bisecting the angle QG*d* nearer to G. is perpendicular to LG; also, if D be in the line G*d*, the point M coincides with D, GN = GF, GH = GV, and GM coincides with GD, and twice the cube on GF or GN is equal to the three cubes on GD, GH, GV, or to the cube on GD with twice the cube on GH or GV; and the difference of the cubes on GF, GV, or on GN, GH, is then equal to half the cube on GD, or $2\,\overline{GF}^3 - 2\,\overline{GV}^3 = \overline{GD}^3$.

Hence an eaſy ſolution of this problem.

Having two equal right lines given, it is required to cut one of them into two parts, and the other into three parts; ſo that the cubes on the two parts, into which the one of theſe lines is cut, ſhall, together, be equal to the cubes on the three parts, into which the other is cut, taken together.

Hence, alſo, an eaſy conſtruction for this problem: On a given right line, to conſtitute a triangle, ſuch that twice the difference of the cubes on the other two ſides, ſhall be equal to the cube on the given line.

Let AC be the given line, (Pl. III. Fig. 6.). With A as radius, deſcribe an arc AB. Take the angle ACB = 36°. Draw AG perpendicular to CB, and join AB. From A and C as centres, deſcribe arcs with the radii $\frac{AB}{2}$, and CG, interſecting in the point F. Then CFA is the triangle required; and $2 \cdot \overline{CF}^3 - 2 \cdot \overline{AF}^3 = \overline{CA}^3$.

Demonstration.

Since the angle ACB is 36°, AB is the ſide of a decagon inſcribed in the circle, which has AC for its radius; and CG is the perpendicular to the ſide of an inſcribed pentagon. But it is well known, that CG is $= \frac{AC + AB}{2}$, and $\overline{AC}^2 = \overline{AB}^2 + AC \times AB$. Conſequently $3\,\overline{AC}^3 = 3\,AC^2 \times AB + 3\,AC \times \overline{AB}^2$; add $\overline{AC}^3$ to both, and we have $4\,\overline{AC}^3 = \overline{AC}^3 + 3\,\overline{AC}^2 \times AB +$

3 AC

$3\,AC \times AB^2$, and $\overline{AC}^3 = \frac{\overline{AC}^3 + 3\overline{AC}^2 \times AB + 3\,AC \times \overline{AB}^2}{4} =$

$\frac{AC + \overline{AB}^3 - \overline{AB}^3}{8} \times 2 = 2 \times \overline{CG}^3 - \overline{AF}^3$. Thus, in any circle, the cube of radius is equal to twice the difference between the cubes on the perpendicular to the ſide of the inſcribed pentagon, and half the ſide of the inſcribed decagon.

PROPOSITION. Let any regular figure of an odd number of ſides, be circumſcribed about a circle, and let (n) be any odd number, leſs than the number of the ſides of the figure; and from any point within the figure let perpendiculars be drawn to the ſides of the circumſcribing figure; then the ſum of the (n) powers of the parts by which thoſe perpendiculars, which are greater than radius, exceed it, is equal to the ſum of the (n) powers of thoſe parts by which the perpendiculars, which are leſs than radius, fall ſhort of it.

HENCE theſe problems.

HAVING two equal given right lines, to cut one of them into two parts, and the other into three, ſo that the cubes on the two parts, into which one of them is cut, ſhall, together, be equal to the cubes on the three parts, into which the other is cut, taken together.

AND having two equal right lines given, to cut one of them into ſeven parts, and the other into eight, ſo that the cubes, the 5th powers, the 7th, 9th, 11th and 13th powers, of the ſeven parts, into which the one is cut, ſhall, together, be reſpectively equal to the cubes, the 5th, the 7th, the 9th, the 11th, and the 13th powers, of the eight parts, into which the other is cut.

THE firſt of theſe two problems is effected by a pentagon, inſcribed in a circle; and the ſecond, by a quindecagon inſcribed.

IF

If V be as much on the other ſide of the centre G, towards L, as it is towards C, the lines GN, GM, exchange their values or magnitudes, as alſo do the lines GH, GF; and the perpendiculars to the ſides of the circumſcribing figure then become $r - GM$, $r - GH$, $r + GN$, $r + GF$, $r - GV$; and the ſum of their cubes $N \times r^3 + N \times 3r . \frac{\overline{GD}^2}{2} + \overline{GN}^3 + \overline{GF}^3 - \overline{GM}^3 - \overline{GH}^3 - \overline{GV}^3$; which added to $N \times r^3 + N \times 3r . \frac{\overline{GD}^2}{2} + \overline{GM}^3$ $\overline{GH}^3 + \overline{GV}^3 - \overline{GN}^3 - \overline{GF}^3$, the ſum of their cubes before found, and the aggregate divided by 2, gives $N \times r^3 + N \times 3r .$ $\frac{\overline{GD}^2}{2}$, the ſum of their cubes, when D is in the line drawn from the centre G perpendicular to LG.

Let a circle, (Pl. III. Fig. 7.), be deſcribed on BC, with the centre G, and let BF be a ſquare on the diameter BC; draw EGD from E, through the centre G, to meet the circle in D, and join D F.

Then, ſince $BG \times CS$, or $CG \times CS = \overline{GS}^2$, GC is cut in extreme and mean proportion in the point S, and GS is the ſide of a regular decagon, inſcribed in the circle. And ſince the perpendicular from G to the ſide of a regular inſcribed pentagon, is $= \frac{BG + GS}{2}$, BS is twice that perpendicular. But $\frac{\overline{BG + GS}^3}{2}$, or $\frac{\overline{r + GS}^3}{8} - \frac{\overline{GS}^3}{8} = \frac{r^3}{2}$. Conſequently $\overline{BS}^3 - \overline{GS}^3$, or $\overline{r + GS}^3 - \overline{GS}^3 = 4r^3$. Therefore $3r^3 = 3r^2 \times GS + 3r \times \overline{GS}^2$, and

r^3

$r^3 = r^2 \times GS + r \times \overline{GS}^2$. But BS is cut in G, in the ſame manner as GC is cut in S. Wherefore, if another circle be deſcribed, with BS as radius, and a line be drawn from one of the angles of a ſquare, deſcribed on the diameter, through the centre, to meet the circumference in a point, and if this point, and the other oppoſite angle of the ſquare be joined, $\overline{2r + GS}^3 - r^3$ will in like manner be $= 4 \times \overline{r + GS}^3$, or $4 \times \overline{BS}^3$, and $7r^3 + 12r^2. GS + 6r. \overline{GS}^2 + \overline{GS}^3 = 4r^3 + 12r^2. GS + 12r. \overline{GS}^2 + 4. \overline{GS}^3$. Therefore $3r^3 = 6r. \overline{GS}^2 + 3\overline{GS}^3$, and $r^3 = 2r. \overline{GS}^2 + \overline{GS}^3 = r^2. GS + r. \overline{GS}^2$. Therefore $2r. GS + \overline{GS}^2 = r^2 + r. GS$, and $\overline{GS}^2 + r. GS = r^2$, and $\overline{GS}^3 = r^2. GS - r. \overline{GS}^2$.

If, therefore, from any point in the circumference of the circle BDC, perpendiculars be drawn to the ſides of any regular figure circumſcribed about it, the ſum of their cubes being $= N \times \frac{5}{2}. r^3$, (calling N the number of the ſides of the figure), is $= N \times 5r. \overline{GS}^2 + N \times \frac{5}{2}. \overline{GS}^3$; and twice the ſum of the cubes of theſe perpendiculars is $N \times 5. \overline{GS}^3 + N \times 10r. \overline{GS}^2$; that is, equal to five times a multiple by the number of the ſides of the figure of the cube on the ſide of an inſcribed regular decagon, and ten times a multiple, by the ſame number of the ſolid, which has the ſquare of the ſide of the inſcribed decagon for its baſe, and radius for its altitude; and if the perpendiculars be drawn from any point P, within the circumſcribed figure, that is, not in the circumference of the circle,

circle, twice the ſum of their cubes will be equal to $2\,N \times \overline{\overline{GS}^3 + 2r.\overline{GS}^2} + 2\,N \times 6r \times \frac{\overline{GP}^2}{4}$; that is, equal to twice a multiple by the number of the ſides of the figure of the cube on the ſide of the inſcribed decagon, together with four times a multiple, by the ſame number of the ſolid which has the ſquare of the ſide of the decagon for its baſe, and r for its altitude, together with thrice a multiple by the ſame number of the ſolid, which has the ſquare of GP for its baſe, and r for its altitude.

In like manner, may the ſixth powers of lines drawn from the angles of any regular inſcribed figure of a greater number of ſides than three, to any point either in, or not in the circumference, be expreſſed in terms of the ſide of an inſcribed decagon, ſince their ſum is a multiple of the ſum of the cubes of the perpendiculars, to the ſides of the circumſcribing figure, by $8\,r^3$.

Again, ſince $r + GS : r :: r : GS :: GS : r - GS$, we have $2r + GS : r + GS :: r + GS : r :: r : GS :: GS : r - GS$.

Wherefore $\overline{3r + 2\,GS}^3 - \overline{r + GS}^3 = 4 \times \overline{2r + GS}^3$, or $26r^3 + 51r^2.GS + 33r.\overline{GS}^2 + 7\overline{GS}^3 = 32r^3 + 48r^2.GS + 24r.\overline{GS}^2 + 4\overline{GS}^3$, or $3r^2.GS + 9r.\overline{GS}^2 + 3\overline{GS}^3 = 6r^3$, or $r^2 \cdot GS + 3r \cdot \overline{GS}^2 + \overline{GS}^3 = 2r^3$.

Wherefore, ſince four times the ſum of the cubes of the perpendiculars drawn from any point in the circumference of the circle to the ſides of any regular circumſcribing figure, is $N \times 5 \times 2r^3$; four times the ſum of theſe cubes is $= N \times \overline{5r^2.GS + 15r.\overline{GS}^2 + 5\overline{GS}^3} = 5\,N \times \overline{r^2.GS + 3r.\overline{GS}^2 + \overline{GS}^3}$; that

that is, equal to five times a multiple, by the number of the ſides of the figure of the cube on the ſide of the inſcribed decagon, together with fifteen times a multiple, by the ſame number, of the ſolid, which has the ſquare on the ſide of the inſcribed decagon as its baſe, and r for its altitude, together with five times a multiple, by the ſame number of the ſolid, which has r^2 for its baſe, and the ſide of the decagon for its altitude.

LET the circumference of a circle be divided into any number n of equal parts, and from any point in the circumference let chords be drawn to the points of diviſion, and let $3m$ be any number leſs than n, the ſum of the $2m$ powers of the lines which have reſpectively to $2r$ the diameter, the ratios which the cubes of the chords have reſpectively to $8r^3$, the cube of the diameter, is equal to $n \times \frac{1.3.5.7 \ldots 6m-1}{1.2.3.4 \ldots 3m} \times \frac{r^{2m}}{2^m}$.

LET the chords be denoted by A, B, C, D, &c. to n terms; and let $8r^3 : A^3 = 2r : a$, $8r^3 : B^3 = 2r : b$, $8r^3 : C^3 = 2r : c$, $8r^3 : D^3 = 2r : d$, &c. Then $a = \frac{A^3}{4r^2}$, and $a^{2m} = \frac{A^{6m}}{2^{4m} r^{4m}}$ $b^{2m} = \frac{B^{6m}}{2^{4m} r^{4m}}$, &c.; and $a^{2m} + b^{2m} +$ &c. to n terms, is $= \frac{A^{6m}}{2^{4m} r^{4m}} + \frac{B^{6m}}{2^{4m} r^{4m}} +$ &c. to n terms. If $p = 3m$, we have $a^{2m} + b^{2m} +$ &c. $= \frac{A^{2p}}{2^{p+m} r^{p+m}} + \frac{B^{2p}}{2^{p+m} r^{p+m}} +$ &c. But the ſum of the $2p$ powers of the chords A, B, &c. is $n \times \frac{1.3.5.7 \ldots 2^{p-1}}{1.2.3.4 \ldots p} 2^p r^{2p}$.

 Therefore

Therefore $a^{2m} + b^{2m} + \&c. = n \times \frac{1.3.5.7..2^{p-1}}{1.2.3.4..p} \times \frac{r^{p-m}}{2^m} =$

$n \times \frac{1.3.5.7..6^{m-1}}{1.2.3.4..3^m} \times \frac{r^{2m}}{2^m}$.

If $m = 1$, $a^2 + b^2 + \&c. = n \times \frac{1.3.5}{1.2.3} \times \frac{r^2}{2} = \frac{5nr^2}{4}$; and the diameter, (or $2r$) $\times \overline{a^2 + b^2 + c^2 + \&c.}$ = sum of the cubes of perpendiculars drawn from any point in the circumference, to the sides of a regular circumscribing polygon of n number of sides, and $a^2 + b^2 + \&c.$ is to the sum of the squares of these perpendiculars as 5 to 6; and if the perpendiculars to the sides of the polygon corresponding to the chords A, B, C, D, &c. and drawn from the same point in the circumference that these chords are drawn from, be denoted by P, Q, R, S, &c. $a + b + c + \&c. = \frac{A \times P}{2r} + \frac{B \times Q}{2r} + \frac{C \times R}{2r} + \frac{D \times S}{2r} + \&c.$ $2^m r^m \times \overline{a^{2m} + b^{2m} + c^{2m} + \&c.} = P^{3m} + Q^{3m} + R^{3m} + \&c.$ = the sum of the $3m$ powers of these perpendiculars, $= n \times \frac{1.3.5.7...6^{m-1}}{1.2.3.4...3^m} \times r^{3m}$.

Theorem Φ. From any point C, (Pl. III. Fig. 8.), let the chord CA be drawn; let GAF be a tangent to the circle at A; and let AD be perpendicular to the diameter BC, and CF, BG to GF. The right line which has to BC ($2r$), the ratio of $\overline{AC}^3$ to $\overline{BC}^3$, or the triplicate ratio of the chord of the arc AC, to the diameter, is $\frac{AC \times CF}{BC}$, or $\frac{AC \times CD}{BC}$ = a fourth proportional to the diameter, the chord and the perpendicular drawn from one extremity of the

the chord to the tangent to the circle at the other extremity, or a fourth proportional to the diameter, the chord and versed sine of the arc AC.

For, the angle CAF = the angle ABC = the angle CAD. Therefore CD = CF, and AD = AF. But CD = $\frac{\overline{AC}^2}{BC}$. Consequently $\frac{AC \times CD}{BC} = \frac{\overline{AC}^3}{\overline{BC}^2}$, which has to BC the ratio of $\overline{AC}^3$ to $\overline{BC}^3$. Q. E. D.

Cor. 1. BD = perpendicular BG; GF = the chord AE of double the arc AC = twice the sine of the arc AC.

Cor. 2. $\frac{AB \times BG}{BC}$, or $\frac{AB \times BD}{BC}$, has to BC, the ratio of $\overline{AB}^3$ to $\overline{BC}^3$.

Cor. 3. $\overline{CF}^3 = \frac{\overline{AC}^6}{\overline{BC}^3}$, $\overline{BG}^3 = \frac{\overline{AB}^6}{\overline{BC}^3}$, $\overline{CF}^3 = \frac{\overline{AC}^2 \times \overline{CD}^2}{\overline{BC}^2} \times BC = \frac{\overline{AC}^2 \times \overline{CF}^2 \times BC}{\overline{BC}^2}$, and $\overline{BG}^3 = \frac{\overline{AB}^2 \times \overline{BD}^2}{\overline{BC}^2} \times BC = \frac{\overline{AB}^2 \times \overline{BG}^2}{\overline{BC}^2} \times BC$; and the lines, which have to BC the ratios of $\overline{AC}^3 : \overline{BC}^3$; and $\overline{AB}^3 : \overline{BC}^3$ are to each other as $AC \times CD$ to $AB \times BD$, or as $AC \times CF : AB \times BG$.

SEE Fig. 1. and Theorem Φ. Since the part of the tangent at the point A, that would be intercepted between perpendiculars drawn to it from P and Q, is equal to 2 P a, or 2 Q c, the part of the tangent at the point B, that would be intercepted between perpendiculars drawn to it from P and Q, is = 2 P e, or 2 Q f; and the part of the tangent at C, that would be intercepted between perpendiculars drawn to it from P and Q, is = 2 P b, or 2 Q d, we have (when AB, BC, &c. are equal, or when the diameters paſſing through A, B, C, &c. make equal angles with one another at the centre O) the ſum of the ſquares of theſe parts of the tangents, (calling n the number of the points of contact), $= n \times \frac{1}{1} . 2 r^2$; the ſum of their fourth powers $= n \times \frac{1.3}{1.2} \times 2^2 r^4$; and the ſum of the $2m$ powers of theſe parts (m being any integer leſs than n) $= n \times \frac{1.3.5. \dots 2m-1}{1.2.3. \dots m} \times 2_m r^{2m}$ (r being the radius OP or OQ) = the ſum of the $2m$ powers of the chords drawn from either P or Q, at right angles to the diameters paſſing through A, B, C, &c. = the ſum of the $2m$ powers of chords, drawn to any point in the circumference from the angles of a regular inſcribed figure of n number of ſides, or from the points where a regular inſcribed figure of n number of ſides, touches the circle, = the ſum of the $2m$ powers of perpendiculars, drawn from P or Q to n number of right lines paſſing through Q or P, and interſecting each other at equal angles. And the ſum of the $2m$ powers of the halves of theſe parts of the tangents, or of the parts intercepted between the points of contact and perpendiculars drawn from either P or Q to the ſides of the equal ſided figure circumſcribing the circle, or ſegment, is $= n \times \frac{1.3.5. \dots 2m-1}{1.2.3. \dots m\, 2^m} \times r^{2m}$ = the ſum of the $2m$ powers of the

the ſines of the angles formed at the centre O, by OP or OQ; and the diameters paſſing through the points of contact to the radius OP or OQ; that is, $= \overline{Pa}^{2m} + \overline{Pe}^{2m} + \overline{Pb}^{2m} +$, &c. or $\overline{Qc}^{2m} + \overline{Qf}^{2m} + \overline{Qd}^{2m} +$, &c. = the ſum of the $2m$ powers of perpendiculars drawn from any point in the circumference of a circle deſcribed from P as a centre, with PO as radius to n number of right lines, interſecting each other in P, and making all the angles equal, = the ſum of the m powers of the rectangles $Aa \times aG$, $Be \times er$, $Cb \times bh$, &c.; or of the rectangles $Gc \times cA$, $rf \times fB$, $hd \times dC$, &c. when the regular figure circumſcribing the circle has an odd number of ſides; but equal to twice the ſum of the $2m$ powers of ſaid ſines, or to twice the ſum of the m powers of ſaid rectangles, when the regular figure circumſcribing the circle has an even number of ſides, ſince the number of the diameters drawn through the oppoſite points of contact, and making equal angles with each other, at their interſection in the centre O, is only half the number of the points of contact or ſides of the figure. But theſe rectangles are reſpectively equal to $\overline{r - Oa} \times \overline{r + Oa}$, $\overline{r - Oe} \times \overline{r + Oe}$, $\overline{r - Ob} \times \overline{r + Ob}$, &c. or $r^2 - \overline{Oa}^2$, $r^2 - \overline{Oe}^2$, $r^2 - \overline{Ob}^2$, &c.; and the ſum of their m^{th} powers is $n r^{2m} - \frac{m}{1} \cdot r^{2m-2} \times \overline{\overline{Oa}^2 + \overline{Oe}^2 + \overline{Ob}^2 +, \&c.\text{ to } n \text{ terms}}$, $+ \frac{m}{1} \cdot \frac{m-1}{2} \cdot r^{2m-4} \times \overline{\overline{Oa}^4 + \overline{Oe}^4 + \overline{Ob}^4 +, \&c.\text{ to } n \text{ terms}} +$, &c. &c. $+ \overline{Oa}^{2m} + \overline{Oe}^{2m} + \overline{Ob}^{2m} +$, &c. to ($n$) terms, or to $- \overline{Oa}^{2m} - \overline{Oe}^{2m} - \overline{Ob}^{2m} -$, &c. to ($n$) terms, according as m is even or odd.

III.

PLATE II.

Fig. 1.

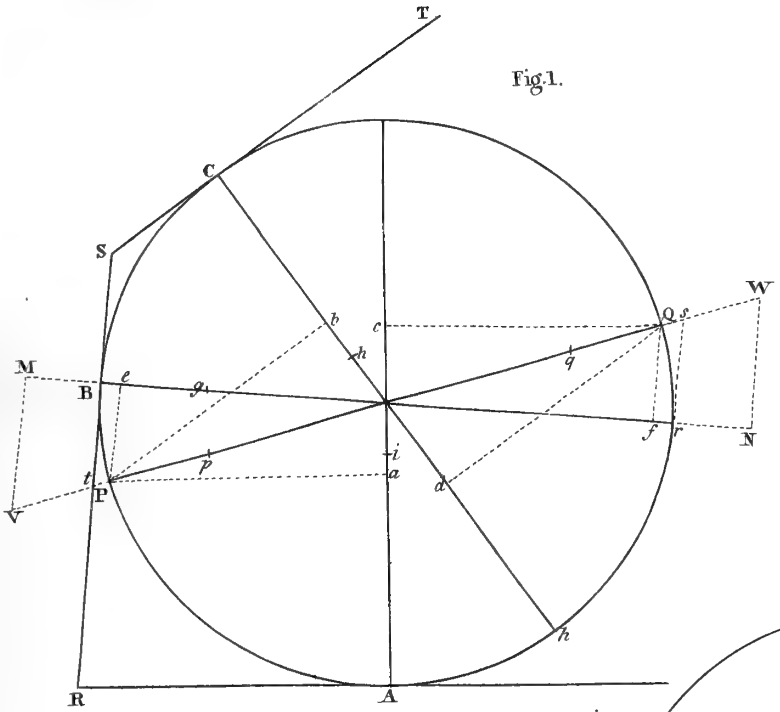


Fig. 3.

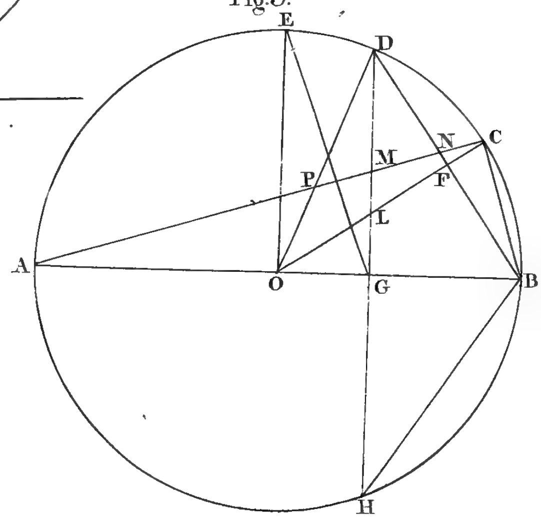


Fig. 2.

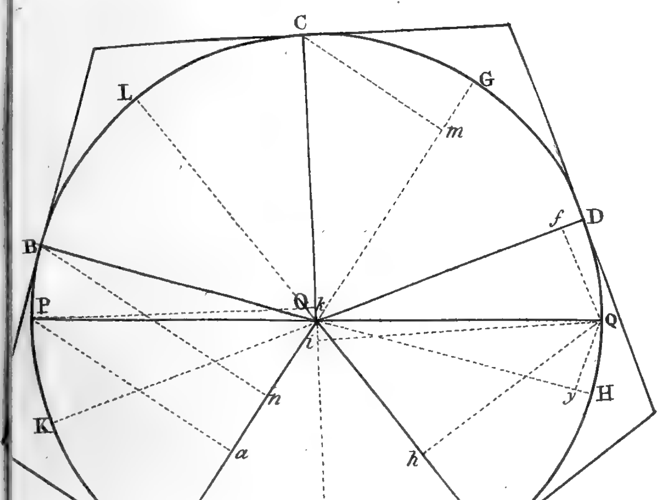


Fig. 4.

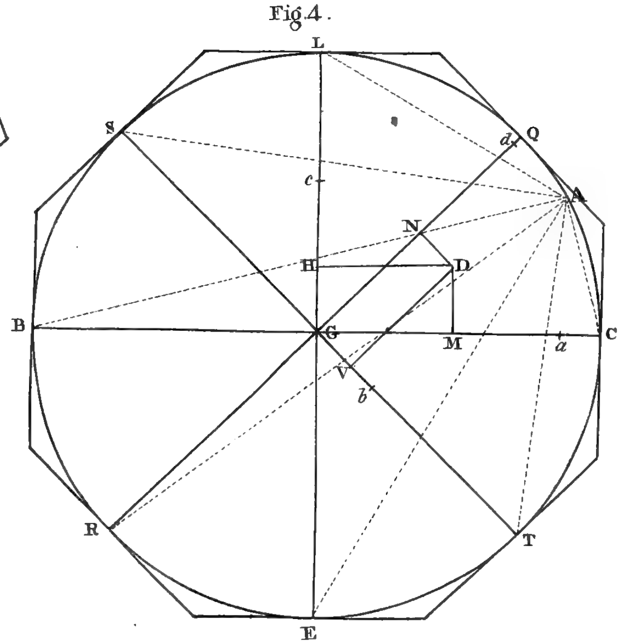


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Fig. 5.

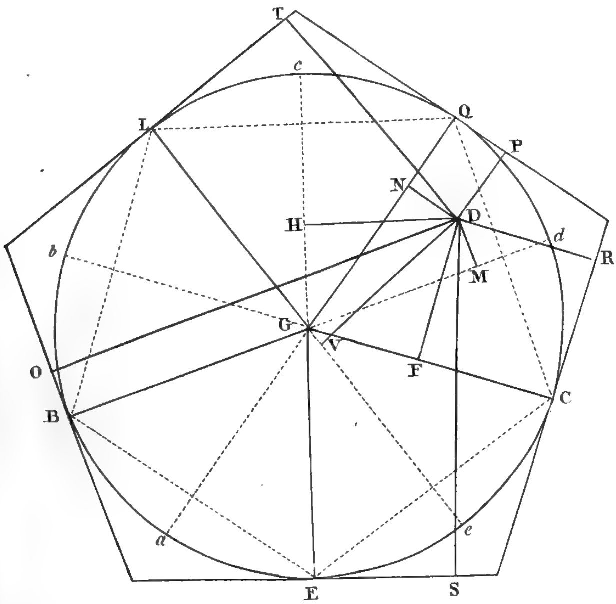


Fig. 6.

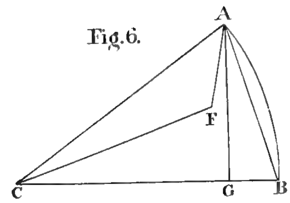


Fig. 7.

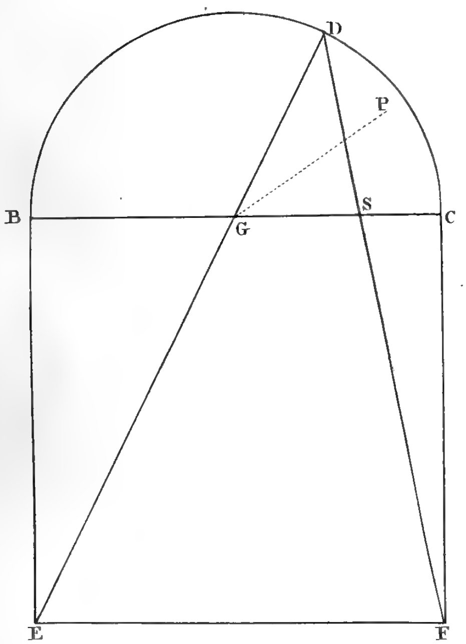
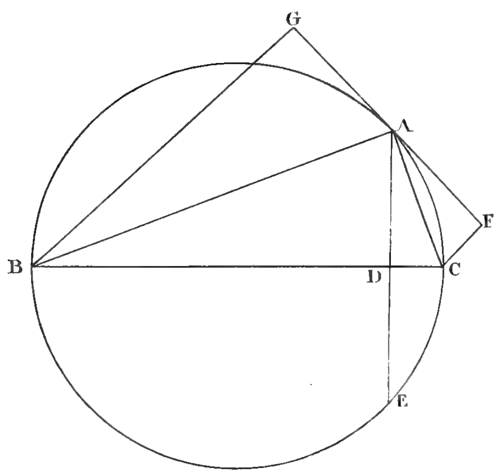


Fig. 8.



D. Lizars sculpt.

III. Account *of a* Series *of* Experiments, *ſhewing the* Effects *of* Compression *in modifying the* Action *of* Heat. *By Sir JAMES HALL, Bart.* F.R.S. Edin.

[*Read June* 3. 1805.]

I.

Ancient Revolutions of the Mineral Kingdom.—Vain attempts to explain them.—Dependence of Geology on Chemiſtry.—Importance of the Carbonate of Lime.—Dr Black's *diſcovery of Carbonic Acid, ſubverted the former theories depending on Fire, but gave birth to that of Dr* Hutton.*—Progreſs of the Author's Ideas with regard to that Theory.—Experiments with Heat and Compreſſion, ſuggeſted to Dr* Hutton *in* 1790.*—Undertaken by the Author in* 1798.*—Speculations on which his hopes of ſucceſs were founded.*

WHOEVER has attended to the ſtructure of Rocks and Mountains, muſt be convinced, that our Globe has not always exiſted in its preſent ſtate; but that every part of its maſs, ſo far at leaſt as our obſervations reach, has been agitated and ſubverted by the moſt violent revolutions.

Facts leading to ſuch ſtriking concluſions, however imperfectly obſerved, could not fail to awaken curioſity, and give riſe to a deſire of tracing the hiſtory, and of inveſtigating the cauſes, of ſuch ſtupendous events; and various attempts were made in this way, but with little ſucceſs; for while diſcoveries of of the utmoſt importance and accuracy were made in Aſtronomy and Natural Philoſophy, the ſyſtems produced by the Geologiſts were ſo fanciful and puerile, as ſcarcely to deſerve a ſerious refutation.

One principal cauſe of this failure, ſeems to have lain in the very imperfect ſtate of Chemiſtry, which has only of late years begun to deſerve the name of a ſcience. While Chemiſtry was in its infancy, it was impoſſible that Geology ſhould make any progreſs; ſince ſeveral of the moſt important circumſtances to be accounted for by this latter ſcience, are admitted on all hands to depend upon principles of the former. The conſolidation of looſe ſand into ſtrata of ſolid rock; the cryſtalline arrangement of ſubſtances accompanying thoſe ſtrata, and blended with them in various modes, are circumſtances of a chemical nature, which all thoſe who have attempted to frame theories of the earth have endeavoured by chemical reaſonings to reconcile to their hypotheſes.

Fire and *Water*, the only agents in nature by which ſtony ſubſtances are produced, under our obſervation, were employed by contending ſects of geologiſts, to explain all the phenomena of the mineral kingdom.

But the known properties of Water, are quite repugnant to the belief of its univerſal influence, ſince a very great proportion of the ſubſtances under conſideration are inſoluble, or nearly ſo, in that fluid; and ſince, if they were all extremely ſoluble, the quantity of water which is known to exiſt, or that could poſſibly exiſt in our planet, would be far too ſmall to accompliſh the office aſſigned to it in the Neptunian theory *. On the other hand, the known properties of Fire are no leſs inadequate to the purpoſe; for, various ſubſtances which frequently occur in the mineral kingdom, ſeem, by their preſence, to preclude

* *Illuſtrations of the Huttonian Theory*, by Mr Profeſſor Playfair, 430.

clude its ſuppoſed agency; ſince experiment ſhews, that, in our fires, they are totally changed or deſtroyed.

UNDER ſuch circumſtances, the advocates of either element were enabled, very ſucceſsfully, to refute the opinions of their adverſaries, though they could but feebly defend their own: and, owing perhaps to this mutual power of attack, and for want of any alternative to which the opinions of men could lean, both ſyſtems maintained a certain degree of credit; and writers on geology indulged themſelves, with a ſort of impunity, in a ſtyle of unphiloſophical reaſoning, which would not have been tolerated in other ſciences.

OF all mineral ſubſtances, the *Carbonate of Lime* is unqueſtionably the moſt important in a general view. As limeſtone or marble, it conſtitutes a very conſiderable part of the ſolid maſs of many countries; and, in the form of veins and nodules of ſpar, pervades every ſpecies of ſtone. Its hiſtory is thus interwoven in ſuch a manner with that of the mineral kingdom at large, that the fate of any geological theory muſt very much depend upon its ſucceſsful application to the various conditions of this ſubſtance. But, till Dr BLACK, by his diſcovery of Carbonic Acid, explained the chemical nature of the carbonate, no rational theory could be formed, of the chemical revolutions which it has undoubtedly undergone.

THIS diſcovery was, in the firſt inſtance, hoſtile to the ſuppoſed action of fire; for the decompoſition of limeſtone by fire in every common kiln being thus proved, it ſeemed abſurd to aſcribe to that ſame agent the formation of limeſtone, or of any maſs containing it.

THE contemplation of this difficulty led Dr HUTTON to view the action of fire in a manner peculiar to himſelf, and thus to form a geological theory, by which, in my opinion, he has furniſhed the world with the true ſolution of one of the moſt interesting problems that has ever engaged the attention of men of science.

He supposed,

I. That Heat has acted, at some remote period, on all rocks.

II. That during the action of heat, all these rocks (even such as now appear at the surface) lay covered by a superincumbent mass, of great weight and strength.

III. That in consequence of the combined action of Heat and Pressure, effects were produced different from those of heat on common occasions; in particular, that the carbonate of lime was reduced to a state of fusion, more or less complete, without any calcination.

The essential and characteristic principle of his theory is thus comprised in the word *Compression;* and by one bold hypothesis, founded on this principle, he undertook to meet all the objections to the action of fire, and to account for those circumstances in which minerals are found to differ from the usual products of our furnaces.

This system, however, involves so many suppositions, apparently in contradiction to common experience, which meet us on the very threshold, that most men have hitherto been deterred from the investigation of its principles, and only a few individuals have justly appreciated its merits. It was long before I belonged to the latter class; for I must own, that, on reading Dr Hutton's first geological publication, I was induced to reject his system entirely, and should probably have continued still to do so, with the great majority of the world, but for my habits of intimacy with the author; the vivacity and perspicuity of whose conversation, formed a striking contrast to the obscurity

ſcurity of his writings. I was induced by that charm, and by the numerous original facts which his ſyſtem had led him to obſerve, to liſten to his arguments, in favour of opinions which I then looked upon as viſionary. I thus derived from his converſation, the ſame advantage which the world has lately done from the publication of Mr PLAYFAIR's *Illuſtrations*; and, experienced the ſame influence which is now exerted by that work, on the minds of our moſt eminent men of ſcience.

AFTER three years of almoſt daily warfare with Dr HUTTON, on the ſubject of his theory, I began to view his fundamental principles with leſs and leſs repugnance. There is a period, I believe, in all ſcientific inveſtigations, when the conjectures of genius ceaſe to appear extravagant; and when we balance the fertility of a principle, in explaining the phenomena of nature, againſt its improbability as an hypotheſis: The partial view which we then obtain of truth, is perhaps the moſt attractive of any, and moſt powerfully ſtimulates the exertions of an active mind. The miſt which obſcured ſome objects diſſipates by degreee, and allows them to appear in their true colours; at the ſame time, a diſtant proſpect opens to our view, of ſcenes unſuſpected before.

ENTERING now ſeriouſly into the train of reaſoning followed by Dr HUTTON, I conceived that the chemical effects aſcribed by him to compreſſion, ought, in the firſt place, to be inveſtigated; for, unleſs ſome good reaſon were given us for believing that heat would be modified by preſſure, in the manner alleged, it would avail us little to know that they had acted together. He reſted his belief of this influence on analogy; and on the ſatisfactory ſolution of all the phenomena, furniſhed by this ſuppoſition. It occurred to me, however, that this principle was ſuſceptible of being eſtabliſhed in a direct manner by experiment, and I urged him to make the attempt; but he always rejected this propoſal, on account of the immenſity of the natural agents, whoſe operations he ſuppoſed to lie far beyond the reach of our imitation; and he ſeemed to imagine, that any ſuch attempt muſt undoubtedly fail, and thus throw diſcredit on opinions already ſufficiently eſtabliſhed, as he conceived, on other principles. I was far, however, from being convinced by theſe arguments; for, without being able to prove that any artificial compreſſion to which we could expoſe the carbonate, would effectually prevent its calcination in our fires, I maintained, that we had as little proof of the contrary, and that the application of a moderate force might poſſibly perform all that was hypothetically aſſumed in the Huttonian Theory. On the other hand, I conſidered myſelf as bound, in practice, to pay deference to his opinion, in a field which he had already ſo nobly occupied, and abſtained, during the remainder of his life, from the proſecution of ſome experiments with compreſſion, which I had begun in 1790.

In 1798, I reſumed the ſubject with eagerneſs, being ſtill of opinion, that the chemical law which forms the baſis of the Huttonian Theory, ought, in the firſt place, to be inveſtigated experimentally; all my ſubſequent reflections and obſervations having tended to confirm my idea of the importance of this purſuit, without in any degree rendering me more apprehenſive as to the reſult.

In the arrangement of the following paper, I ſhall firſt confine myſelf to the inveſtigation of the chemical effects of Heat and Compreſſion, reſerving to the concluding part, the application of my reſults to Geology. I ſhall, then, appeal to the volcanoes, and ſhall endeavour to vindicate the laws of action aſſumed in the Huttonian Theory, by ſhewing, that lavas, previous to their eruptions, are ſubject to ſimilar laws; and that the volcanoes, by their ſubterranean and ſubmarine exertions,

tions, muſt produce, in our times, reſults ſimilar to thoſe aſcribed, in that Theory, to the former action of fire.

In comparing the Huttonian operations with thoſe of the volcanoes, I ſhall avail myſelf of ſome facts, brought to light in the courſe of the following inveſtigations, by which a preciſe limit is aſſigned to the intenſity of the heat, and to the force of compreſſion, required to fulfil the conditions of Dr Hutton's hypotheſis: For, according to him, the power of thoſe agents was very great, but quite indefinite; it was therefore impoſſible to compare their ſuppoſed effects in any preciſe manner with the phenomena of nature.

My attention was almoſt excluſively confined to the Carbonate of Lime, about which I reaſoned as follows: The carbonic acid, when uncombined with any other ſubſtance, exiſts naturally in a gaſeous form, at the common temperature of our atmoſphere; but when in union with lime, its volatility is repreſſed, in that ſame temperature, by the chemical force of the earthy ſubſtance, which retains it in a ſolid form. When the temperature is raiſed to a full red-heat, the acid acquires a volatility by which that force is overcome, it eſcapes from the lime, and aſſumes its gaſeous form. It is evident, that were the attractive force of the lime increaſed, or the volatility of the acid diminiſhed by any means, the compound would be enabled to bear a higher heat without decompoſition, than it can in the preſent ſtate of things. Now, preſſure muſt produce an effect of this kind; for when a mechanical force oppoſes the expanſion of the acid, its volatility muſt, to a certain degree, be diminiſhed. Under preſſure, then, the carbonate may be expected to remain unchanged in a heat, by which, in the open air, it would have been calcined. But experiment alone can teach us, what compreſſing force is requiſite to enable it to reſiſt any given elevation of temperature; and what is to be the reſult of ſuch an operation. Some of the compounds of lime with acids

are

are fusible, others refractory; the carbonate, when constrained by pressure to endure a proper heat, may be as fusible as the muriate.

One circumstance, derived from the Huttonian Theory, induced me to hope, that the carbonate was easily fusible, and indicated a precise point, under which that fusion ought to be expected. Nothing is more common than to meet with nodules of calcareous spar inclosed in whinstone; and we suppose, according to the Huttonian Theory, that the whin and the spar had been liquid together; the two fluids keeping separate, like oil and water. It is natural, at the junction of these two, to look for indications of their relative fusibilities; and we find, accordingly, that the termination of the spar is generally globular and smooth; which seems to prove, that, when the whin became solid, the spar was still in a liquid state; for had the spar congealed first, the tendency which it shews, on all occasions of freedom, to shoot out into prominent crystals, would have made it dart into the liquid whin, according to the peculiar forms of its crystallization; as has happened with the various substances contained in whin, much more refractory than itself, namely, augite, felspar, &c.; all of which having congealed in the liquid whin, have assumed their peculiar forms with perfect regularity. From this I concluded, that when the whin congealed, which must have happened about 28° or 30° of Wedgwood, the spar was still liquid. I therefore expected, if I could compel the carbonate to bear a heat of 28° without decomposition, that it would enter into fusion. The sequel will shew, that this conjecture was not without foundation.

I shall now enter upon the description of those experiments, the result of which I had the honour to lay before this Society on the 30th of August last (1804); fully aware how difficult it is, in giving an account of above five hundred experiments, all tending to one point, but differing much from each other in vari-

ous

ous particulars, to ſteer between the oppoſite faults of prolixity and barrenneſs. My object ſhall be to deſcribe, as ſhortly as poſſible, all the methods followed, ſo as to enable any chemiſt to repeat the experiments; and to dwell particularly on ſuch circumſtances only, as ſeem to lead to concluſions of importance.

The reſult being already known, I conſider the account I am about to give of the execution of theſe experiments, as addreſſed to thoſe who take a particular intereſt in the progreſs of chemical operations: in the eyes of ſuch gentlemen, I truſt, that none of the details into which I muſt enter, will appear ſuperfluous.

II.

Principle of execution upon which the following Experiments were conducted.—Experiments with Gun-Barrels filled with baked Clay, and welded at the muzzle.—Method with the Fuſible Metal.—Remarkable effects of its expanſion.—Neceſſity of introducing Air.—Reſults obtained.

When I firſt undertook to make experiments with heat acting under compreſſion, I employed myſelf in contriving various devices of ſcrews, of bolts, and of lids, ſo adjuſted, I hoped, as to confine all elaſtic ſubſtances; and perhaps ſome of them might have anſwered. But I laid aſide all ſuch devices, in favour of one which occurred to me in January 1798; which, by its ſimplicity, was of eaſy application in all caſes, and accompliſhed all that could be done by any device, ſince it ſecured perfect ſtrength and tightneſs to the utmoſt that the veſſels employed could bear, whether formed of metallic or earthy ſubſtance. The device depends upon the

the following general view: If we take a hollow tube or barrel (AD, fig. 1.) closed at one end, and open at the other, of one foot or more in length; it is evident, that by introducing one end into a furnace, we can apply to it as great heat as art can produce, while the other end is kept cool, or, if necessary, exposed to extreme cold. If, then, the substance which we mean to subject to the combined action of heat and pressure, be introduced into the breech or closed end of the barrel (CD), and if the middle part be filled with some refractory substance, leaving a small empty space at the muzzle (AB), we can apply heat to the muzzle, while the breech containing the subject of experiment, is kept cool, and thus close the barrel by any of the numerous modes which heat affords, from the welding of iron to the melting of sealing-wax. Things being then reversed, and the breech put into the furnace, a heat of any required intensity may be applied to the subject of experiment, now in a state of constraint.

My first application of this scheme was carried on with a common gun-barrel, cut off at the touch-hole, and welded very strongly at the breech by means of a plug of iron. Into it I introduced the carbonate, previously rammed into a cartridge of paper or pasteboard, in order to protect it from the iron, by which, in some former trials, the subject of experiment had been contaminated throughout during the action of heat. I then rammed the rest of the barrel full of pounded clay, previously baked in a strong heat, and I had the muzzle closed like the breech, by a plug of iron welded upon it in a common forge; the rest of the barrel being kept cold during this operation, by means of wet cloths. The breech of the barrel was then introduced horizontally into a common muffle, heated to about 25° of Wedgwood. To the muzzle a rope was fixed, in such a manner, that the barrel could be withdrawn with-

out

out danger from an explosion *. I likewise, about this time, closed the muzzle of the barrel, by means of a plug, fixed by solder only; which method had this peculiar advantage, that I could shut and open the barrel, without having recourse to a workman. In these trials, though many barrels yielded to the expansive force, others resisted it, and afforded some results that were in the highest degree encouraging, and even satisfactory, could they have been obtained with certainty on repetition of the process. In many of them, chalk, or common limestone previously pulverised, was agglutinated into a stony mass, which required a smart blow of a hammer to break it, and felt under the knife like a common limestone; at the same time, the substance, when thrown into nitric acid, dissolved entirely with violent effervescence.

In one of these experiments, owing to the action of heat on the cartridge of paper, the baked clay, which had been used to fill the barrel, was stained black throughout, to the distance of two-thirds of the length of the barrel from its breech. This circumstance is of importance, by shewing, that though all is tight at the muzzle, a protrusion may take place along the barrel, greatly to the detriment of complete

* On one occasion, the importance of this precaution was strongly felt. Having inadvertently introduced a considerable quantity of moisture into a welded barrel, an explosion took place, before the heat had risen to redness, by which, part of the barrel was spread out to a flat plate, and the furnace was blown to pieces. Dr Kennedy, who happened to be present on this occasion, observed, that notwithstanding this accident, the time might come when we should employ water in these experiments to assist the force of compression. I have since made great use of this valuable suggestion: but he scarcely lived, alas! to see its application; for my first success in this way, took place during his last illness.—I have been exposed to no risk in any other experiment with iron barrels; matters being so arranged, that the strain against them has only commenced in a red heat, in which the metal has been so far softened, as to yield by laceration like a piece of leather.

plete compreſſion: and, at the ſame time, it illuſtrates what has happened occaſionally in nature, where the bituminous matter ſeems to have been driven by ſuperior local heat, from one part of a coaly bed, though retained in others, under the ſame compreſſion. The bitumen ſo driven off being found, in other caſes, to pervade and tinge beds of ſlate and of ſandſtone.

I WAS employed in this purſuit in ſpring 1800, when an event of importance interrupted my experiments for about a year. But I reſumed them in March 1801, with many new plans of execution, and with conſiderable addition to my apparatus.

IN the courſe of my firſt trials, the following mode of execution had occurred to me, which I now began to put in practice. It is well known to chemiſts, that a certain compoſition of different metals *, produces a ſubſtance ſo fuſible, as to melt in the heat of boiling-water. I conceived that great advantage, both in point of accuracy and diſpatch, might be gained in theſe experiments, by ſubſtituting this metal for the baked clay above mentioned: That after introducing the carbonate into the breech of the barrel, the fuſible metal, in a liquid ſtate, might be poured in, ſo as to fill the barrel to its brim: That when the metal had cooled and become ſolid, the breech might, as before, be introduced into a muffle, and expoſed to any required heat, while the muzzle was carefully kept cold. In this manner, no part of the fuſible metal being melted, but what lay at the breech, the reſt, continuing in a ſolid ſtate, would effectually confine the carbonic acid: That after the action of ſtrong heat had ceaſed, and after all had been allowed to cool completely, the fuſible metal might be removed entirely from the barrel, by means of a heat little above that of boiling water, and far too low to occaſion any decompoſition of the

* Eight parts of biſmuth, five of lead, and three of tin.

the carbonate by calcination, though acting upon it in freedom; and then, that the subject of experiment might, as before, be taken out of the barrel.

THIS scheme, with various modifications and additions, which practice has suggested, forms the basis of most of the following methods.

IN the first trial, a striking phenomenon occurred, which gave rise to the most important of these modifications. Having filled a gun-barrel with the fusible metal, without any carbonate; and having placed the breech in a muffle, I was surprised to see, as the heat approached to redness, the liquid metal exuding through the iron in innumerable minute drops, dispersed all round the barrel. As the heat advanced, this exudation increased, till at last the metal flowed out in continued streams, and the barrel was quite destroyed. On several occasions of the same kind, the fusible metal, being forced through some very minute aperture in the barrel, spouted from it to the distance of several yards, depositing upon any substance opposed to the stream, a beautiful assemblage of fine wire, exactly in the form of wool. I immediately understood, that the phenomenon was produced by the superior expansion of the liquid over the solid metal, in consequence of which, the fusible metal was driven through the iron as water was driven through silver * by mechanical percussion in the Florentine experiment. It occurred to me, that this might be prevented by confining along with the fusible metal a small quantity of air, which, by yielding a little to the expansion of the liquid, would save the barrel. This remedy

* *Essays of Natural Experiments made in the Academie del Cimento*, translated by WALLER, London, 1684, page 117. The same in MUSSCHENBROEK's Latin translation, Lugd. Bat. 1731, p. 63.

medy was found to anſwer completely, and was applied, in all the experiments made at this time *.

I NOW propoſed, in order to keep the carbonate clean, to incloſe it in a ſmall veſſel; and to obviate the difficulty of removing the reſult at the concluſion of the experiment, I further propoſed to connect that veſſel with an iron ramrod, longer than the barrel, by which it could be introduced or withdrawn at pleaſure.

A SMALL tube of glaſs †, or of Raumur's porcelain, about a quarter of an inch in diameter, and one or two inches in length, (fig. 2. A) was half filled with pounded carbonate of lime, rammed as hard as poſſible; the other half of the tube being

* I found it a matter of much difficulty to aſcertain the proper quantity of air which ought to be thus incloſed. When the quantity was too great, the reſult was injured by diminution of elaſticity, as I ſhall have occaſion fully to ſhew hereafter. When too ſmall, or when, by any accident, the whole of this included air was allowed to eſcape, the barrel was deſtroyed.

I hoped to aſcertain the bulk of air neceſſary to give liberty to the expanſion of the liquid metal, by meaſuring the actual quantity expelled by known heats from an open barrel filled with it. But I was ſurpriſed to find, that the quantity thus diſcharged, exceeded in bulk that of the air which, in the ſame heats, I had confined along with the carbonate and fuſible metal in many ſucceſsful experiments. As the expanſion of the liquid does not ſeem capable of ſenſible diminution by an oppoſing force, this fact can only be accounted for by a diſtention of the barrel. In theſe experiments, then, the expanſive force of the carbonic acid, of the included air, and of the fuſible metal, acted in combination againſt the barrel, and were yielded to in part by the diſtention of the barrel, and by the condenſation of the included air. My object was to increaſe the force of this mutual action, by diminiſhing the quantity of air, and by other devices to be mentioned hereafter. Where ſo many forces were concerned, the laws of whoſe variations were unknown, much preciſion could not be expected, nor is it wonderful, that in attempting to carry the compreſſing force to the utmoſt, I ſhould have deſtroyed barrels innumerable.

† I have ſince conſtantly uſed tubes of common porcelain, finding glaſs much too fuſible for this purpoſe.

being filled with pounded ſilex, or with whatever occurred as moſt likely to prevent the intruſion of the fuſible metal in its liquid and penetrating ſtate. This tube ſo filled, was placed in a frame or cradle of iron (*d f k h*, figs. 3, 4, 5, and 6,) fixed to the end (*m*) of a ram-rod (*m n*). The cradle was from ſix to three inches in length, and as much in diameter as a gun-barrel would admit with eaſe. It was compoſed of two circular plates of iron, (*d e f g* and *h i k l*, ſeen edgewiſe in the figures,) placed at right-angles to the ram-rod, one of theſe plates (*d e f h*) being fixed to it by the centre (*m*). Theſe plates were connected together by four ribs or flattened wires of iron (*d h*, *e i*, *f k*, and *g l*,) which formed the cradle into which the tube (A), containing the carbonate, was introduced by thruſting the adjacent ribs aſunder. Along with the tube juſt mentioned, was introduced another tube (B), of iron or porcelain, filled only with air. Likewiſe, in the cradle, a pyrometer * piece (C) was placed in contact with (A) the tube containing the carbonate. Theſe articles generally occupied the

* The pyrometer-pieces uſed in theſe experiments were made under my own eye. Neceſſity compelled me to undertake this laborious and difficult work, in which I have already ſo far ſucceeded as to obtain a ſet of pieces, which, though far from complete, anſwer my purpoſe tolerably well. I had lately an opportunity of comparing my ſet with that of Mr Wedgwood, at various temperatures, in furnaces of great ſize and ſteadineſs. The reſult has proved, that my pieces agree as well with each other as his, though with my ſet each temperature is indicated by a different degree of the ſcale. I have thus been enabled to conſtruct a table, by which my obſervations have been corrected, ſo that the temperatures mentioned in this paper are ſuch as would have been indicated by Mr Wedgwood's pieces. By Mr Wedgwood's pieces, I mean thoſe of the only ſet which has been ſold to the public, and by which the melting heat of pure ſilver is indicated at the 22d degree. I am well aware, that the late Mr Wedgwood, in his Table of Fuſibilities, has ſtated that fuſion as taking place at the 28th degree; but I am convinced that his obſervations muſt have been made with ſome ſet different from that which was afterwards ſold.

the whole cradle; when any ſpace remained, it was filled up by a piece of chalk dreſſed for the purpoſe. (Fig. 4. repreſents the cradle filled, as juſt deſcribed).

THINGS being thus prepared, the gun-barrel, placed erect with its muzzle upwards, was half filled with the liquid fuſible metal. The cradle was then introduced into the barrel, and plunged to the bottom of the liquid, ſo that the carbonate was placed very near the breech, (as repreſented in fig. 5, the fuſible metal ſtanding at *o*). The air-tube (B) being placed ſo as to enter the liquid with its muzzle downwards, retained great part of the air it originally contained, though ſome of it might be driven off by the heat, ſo as to eſcape through the liquid. The metal being now allowed to cool, and to fix round the cradle and ramrod, the air remaining in the air-tube was effectually confined, and all was held faſt. The barrel being then filled to the brim with fuſible metal, the apparatus was ready for the application of heat to the breech, (as ſhewn in fig. 6.)

IN the experiments made at this time, I uſed a ſquare brick furnace (figs. 7 and 8), having a muffle (*r s*) traverſing it horizontally and open at both ends. This muffle being ſupported in the middle by a very ſlender prop, was expoſed to fire from below, as well as all round. The barrel was placed in the muffle, with its breech in the hotteſt part, and the end next the muzzle projecting beyond the furnace, and ſurrounded with cloths which were drenched with water from time to time. (This arrangement is ſhewn in fig. 7). In this ſituation, the fuſible metal ſurrounding the cradle being melted, the air contained in the air-tube would of courſe ſeek the higheſt poſition, and its firſt place in the air-tube would be occupied by fuſible metal. (In fig. 6., the new poſition of the air is ſhewn at *p q*).

AT

At the conclufion of the experiment, the metal was generally removed by placing the barrel in the tranfverfe muffle, with its muzzle pointing a little downwards, and fo that the heat was applied firft to the muzzle, and then to the reft of the barrel in fucceffion. (This operation is fhewn in fig. 8). In fome of the firft of thefe experiments, I loofened the cradle, by plunging the barrel into heated brine, or a ftrong folution of muriate of lime; which laft bears a temperature of 250° of Fahrenheit before it boils. For this purpofe, I ufed a pan three inches in diameter, and three feet deep, having a flat bafon at top to receive the liquid when it boiled over. The method anfwered, but was troublefome, and I laid it afide. I have had occafion, lately, however, to refume it in fome experiments in which it was of confequence to open the barrel with the leaft poffible heat *.

By thefe methods I made a great number of experiments, with refults that were highly interefting in that ftage of the bufinefs, though their importance is fo much diminifhed by the fubfequent progrefs of the inveftigation, that I think it proper to mention but very few of them.

On the 31ft of March 1801, I rammed forty grains of pounded chalk into a tube of green bottle-glafs, and placed it in the cradle as above defcribed. A pyrometer in the muffle along with the barrel indicated 33°. The barrel was expofed to heat during feventeen or eighteen minutes. On withdrawing the cradle, the carbonate was found in one folid mafs, which had vifibly fhrunk in bulk, the fpace thus left within the tube being accurately

* In many of the following experiments, lead was ufed in place of the fufible metal, and often with fuccefs; but I loft many good refults in this way: for the heat required to liquefy the lead, approaches fo near to rednefs, that it is difficult to difengage the cradle without applying a temperature by which the carbonate is injured. I have found it anfwer well, to furround the cradle and a few inches of the rod, with fufible metal, and to fill the reft of the barrel with lead.

accurately filled with metal, which plated the carbonate all over without penetrating it in the leaſt, ſo that the metal was eaſily removed. The weight was reduced from forty to thirty-ſix grains. The ſubſtance was very hard, and reſiſted the knife better than any reſult of the kind previouſly obtained; its fracture was cryſtalline, bearing a reſemblance to white ſaline marble; and its thin edges had a decided ſemitranſparency, a circumſtance firſt obſerved in this reſult.

On the 3d of March of the ſame year, I made a ſimilar experiment, in which a pyrometer-piece was placed within the barrel, and another in the muffle; they agreed in indicating 23°. The inner tube, which was of Reaumur's porcelain, contained eighty grains of pounded chalk. The carbonate was found, after the experiment, to have loſt $3\frac{1}{2}$ grains. A thin rim, leſs than the 20th of an inch in thickneſs, of whitiſh matter, appeared on the outſide of the maſs. In other reſpects, the carbonate was in a very perfect ſtate; it was of a yellowiſh colour, and had a decided ſemitranſparency and ſaline fracture. But what renders this reſult of the greateſt value, is, that on breaking the maſs, a ſpace of more than the tenth of an inch ſquare, was found to be completely cryſtallized, having acquired the rhomboidal fracture of calcareous ſpar. It was white and opaque, and preſented to the view three ſets of parallel plates which are ſeen under three different angles. This ſubſtance, owing to partial calcination and ſubſequent abſorption of moiſture, had loſt all appearance of its remarkable properties in ſome weeks after its production; but this appearance has ſince been reſtored, by a freſh fracture, and the ſpecimen is now well preſerved by being hermetically incloſed.

III.

III.

Experiments made in Tubes of Porcelain.—Tubes of Wedgwood's Ware. —Methods uſed to confine the Carbonic Acid, and to cloſe the Pores of the Porcelain in a Horizontal Apparatus.—Tubes made with a view to theſe Experiments.—The Vertical Apparatus adopted.—View of Reſults obtained, both in Iron and Porcelain.—The Formation of Limeſtone and Marble.—Inquiry into the Cauſe of the partial Calcinations. —Tubes of Porcelain weighed previous to breaking.—Experiments with Porcelain Tubes proved to be limited.

WHILE I was carrying on the above-mentioned experiments, I was occaſionally occupied with another ſet, in tubes of porcelain. So much, indeed, was I prepoſſeſſed in favour of this laſt mode, that I laid gun-barrels aſide, and adhered to it during more than a year. The methods followed with this ſubſtance, differ widely from thoſe already deſcribed, though founded on the ſame general principles.

I PROCURED from Mr WEDGWOOD's manufactory at Etruria, in Staffordſhire, a ſet of tubes for this purpoſe, formed of the ſame ſubſtance with the white mortars, in common uſe, made there. Theſe tubes were fourteen inches long, with a bore of half an inch diameter, and thickneſs of 0.2; being cloſed at one end (figs. 9, 10, 11, 12, 13.)

I PROPOSED to ram the carbonate of lime into the breech (Fig. 9. A); then filling the tube to within a ſmall diſtance of its muzzle with pounded flint (B), to fill that remainder (C) with common borax of the ſhops (borat of ſoda) previouſly reduced to glaſs, and then pounded; to apply heat to the muzzle alone, ſo as to convert that borax into ſolid glaſs; then, reverſing the operation, to keep the muzzle cold, and apply the requiſite heat to the carbonate lodged in the breech.

I THUS expected to confine the carbonic acid; but the attempt was attended with confiderable difficulty, and has led to the employment of various devices, which I fhall now fhortly enumerate, as they occurred in the courfe of practice. The fimple application of the principle was found infufficient, from two caufes: Firft, The carbonic acid being driven from the breech of the tube, towards the muzzle, among the pores of the pounded filex, efcaped from the compreffing force, by lodging itfelf in cavities which were comparatively cold: Secondly, The glafs of borax, on cooling, was always found to crack very much, fo that its tightnefs could not be depended on.

To obviate both thefe inconveniences at once, it occurred to me, in addition to the firft arrangement, to place fome borax (fig. 10. C) fo near the breech of the tube, as to undergo heat along with the carbonate (A); but interpofing between this borax and the carbonate, a ftratum of filex (B), in order to prevent contamination. I trufted that the borax in a liquid or vifcid ftate, being thruft outwards by the expanfion of the carbonic acid, would prefs againft the filex beyond it (D), and totally prevent the elaftic fubftances from efcaping out of the tube, or even from wandering into its cold parts.

IN fome refpects, this plan anfwered to expectation. The glafs of borax, which can never be obtained when cold, without innumerable cracks, unites into one continued vifcid mafs in the loweft red-heat; and as the ftrefs in thefe experiments, begins only with rednefs, the borax being heated at the fame time with the carbonate, becomes united and impervious, as foon as its action is neceffary. Many good refults were accordingly obtained in this way. But I found, in practice, that as the heat rofe, the borax began to enter into too thin fufion, and was often loft among the pores of the filex, the fpace in which it had lain being found empty on breaking the tube. It was therefore

therefore found neceſſary to oppoſe ſomething more ſubſtantial and compact, to the thin and penetrating quality of pure borax.

In ſearching for ſome ſuch ſubſtance, a curious property of bottle-glaſs occurred accidentally. Some of this glaſs, in powder, having been introduced into a muffle at the temperature of about 20° of Wedgwood; the powder, in the ſpace of about a minute, entered into a ſtate of viſcid agglutination, like that of honey, and in about a minute more, (the heat always continuing unchanged,) conſolidated into a firm and compact maſs of *Reaumur's porcelain* *. It now appeared, that by placing this ſubſtance immediately behind the borax, the penetrating quality of this laſt might be effectually reſtrained; for, Reaumur's porcelain has the double advantage of being refractory, and of not cracking by change of temperature. I found, however, that in the act of conſolidation, the pounded bottle-glaſs ſhrunk, ſo as to leave an opening between its maſs and the tube, through which the borax, and, along with it, the carbonic acid, was found to eſcape. But the object in view was obtained by means of a mixture of pounded bottle-glaſs, and pounded flint, in equal parts. This compound ſtill agglutinates, not indeed into a maſs ſo hard as Reaumur's porcelain, but ſufficiently ſo for the purpoſe; and this being done without any ſenſible contraction, an effectual barrier was oppoſed to the borax; (this arrangement is ſhewn in fig. 11.); and thus the method of cloſing the tubes was rendered ſo complete, as ſeldom to fail in practice †. A ſtill further refinement upon this method

* In the ſame temperature, a maſs of the glaſs of equal bulk would undergo the ſame change; but it would occupy an hour.

† A ſubſtance equally efficacious in reſtraining the penetrating quality of borax, was diſcovered by another accident. It conſiſts of a mixture of borax and common ſand, by which a ſubſtance is formed, which, in heat, aſſumes the ſtate of a very tough paſte, and becomes hard and compact on cooling.

thod was found to be of advantage. A ſecond ſeries of powders, like that already deſcribed, was introduced towards the muzzle, (as ſhewn in fig. 12.). During the firſt period of the experiment, this laſt-mentioned ſeries was expoſed to heat, with all the outward half of the tube (*a b*); by this means, a ſolid maſs was produced, which remained cold and firm during the ſubſequent action of heat upon the carbonate.

I soon found, that notwithſtanding all the above-mentioned precautions, the carbonic acid made its eſcape, and that it pervaded the ſubſtance of the Wedgwood tubes, where no flaw could be traced. It occurred to me, that this defect might be remedied, were borax, in its thin and penetrating ſtate of fuſion, applied to the inſide of the tube; and that the pores of the porcelain might thus be cloſed, as thoſe of leather are cloſed by oil, in an air-pump. In this view, I rammed the carbonate into a ſmall tube, and ſurrounded it with pounded glaſs of borax, which, as ſoon as the heat was applied, ſpread on the inſide of the large tube, and effectually cloſed its pores. In this manner, many good experiments were made with barrels lying horizontally in common muffles, (the arrangement juſt deſcribed being repreſented in fig. 13.)

I was thus enabled to carry on experiments with this porcelain, to the utmoſt that its ſtrength would bear. But I was not ſatisfied with the force ſo exerted; and, hoping to obtain tubes of a ſuperior quality, I ſpent much time in experiments with various porcelain compoſitions. In this, I ſo far ſucceeded, as to produce tubes by which the carbonic acid was in a great meaſure retained without any internal glaze. The beſt material I found for this purpoſe, was the pure porcelain-clay of Cornwall, or a compoſition in the proportion of two of this clay to one of what the potters call *Corniſh-ſtone*, which I believe to be a granite in a ſtate of decompoſition. Theſe tubes were ſeven or eight inches long, with a bore

tapering

tapering from 1 inch to 0.6. Their thickneſs was about 0.3 at the breech, and tapered towards the muzzle to the thinneſs of a wafer.

I NOW adopted a new mode of operation, placing the tube vertically, and not horizontally, as before. By obſerving the thin ſtate of borax whilſt in fuſion, I was convinced, that it ought to be treated as a complete liquid, which being ſupported in the courſe of the experiment from below, would ſecure perfect tightneſs, and obviate the failure which often happened in the horizontal poſition, from the falling of the borax to the lower ſide.

IN this view, (fig. 16.), I filled the breech in the manner deſcribed above, and introduced into the muzzle ſome borax (C) ſupported at the middle of the tube by a quantity of ſilex mixed with bottle-glaſs (B). I placed the tube, ſo prepared, with its breech plunged into a crucible filled with ſand (E), and its muzzle pointing upwards. It was now my object to apply heat to the muzzle-half, whilſt the other remained cold. In that view, I conſtructed a furnace (fig. 14. and 15.), having a muffle placed vertically (*c d*), ſurrounded on all ſides with fire (*e e*), and open both above (at *c*), and below (at *d*). The crucible juſt mentioned, with its tube, being then placed on a ſupport directly below the vertical muffle, (as repreſented in fig. 14. at F), it was raiſed, ſo that the half of the tube next the muzzle was introduced into the fire. In conſequence of this, the borax was ſeen from above to melt, and run down in the tube, the air contained in the powder eſcaping in the form of bubbles, till at laſt the borax ſtood with a clear and ſteady ſurface like that of water. Some of this ſalt being thrown in from above, by means of a tube of glaſs, the liquid ſurface was raiſed nearly to the muzzle, and, after all had been allowed to become cold, the poſition of the tube was reverſed; the muzzle being now plun-

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ged into the ſand, (as in fig. 17.), and the breech introduced into the muffle. In ſeveral experiments, I found it anſwer well, to occupy great part of the ſpace next the muzzle, with a rod of ſand and clay previouſly baked, (fig. 19. K K), which was either introduced at firſt, along with the pounded borax, or, being made red hot, was plunged into it when in a liquid ſtate. In many caſes I aſſiſted the compactneſs of the tube by means of an internal glaze of borax; the carbonate being placed in a ſmall tube, (as ſhewn in fig. 18.)

These devices anſwered the end propoſed. Three-fourths of the tube next the muzzle was found completely filled with a maſs, having a concave termination at both ends, (*f* and *g* figs. 17. 18. 19.), ſhewing that it had ſtood as a liquid in the two oppoſite poſitions in which heat had been applied to it. So great a degree of tightneſs indeed was obtained in this way, that I found myſelf ſubjected to an unforeſeen ſource of failure. A number of the tubes failed, not by exploſion, but by the formation of a minute longitudinal fiſſure at the breech, through which the borax and carbonic acid eſcaped. I ſaw that this aroſe from the expanſion of the borax when in a liquid ſtate, as happened with the fuſible metal in the experiments with iron-barrels; for, the crevice here formed, indicated the exertion of ſome force acting very powerfully, and to a very ſmall diſtance. Accordingly, this ſource of failure was remedied by the introduction of a very ſmall air-tube. This, however, was uſed only in a few experiments.

In the courſe of the years 1801, 1802, and 1803, I made a number of experiments, by the various methods above deſcribed, amounting, together with thoſe made in gun-barrels, to one hundred and fifty-ſix. In an operation ſo new, and in which the apparatus was ſtrained to the utmoſt of its power, conſtant ſucceſs could not be expected, and in fact many experiments failed, wholly or partially. The reſults, however, upon the

the whole, were ſatisfactory, ſince they ſeemed to eſtabliſh ſome of the eſſential points of this inquiry.

These experiments prove, that, by mechanical conſtraint, the carbonate of lime can be made to undergo ſtrong heat, without calcination, and to retain almoſt the whole of its carbonic acid, which, in an open fire, at the ſame temperature, would have been entirely driven off: and that, in theſe circumſtances, heat produces ſome of the identical effects aſcribed to it in the Huttonian Theory.

By this joint action of heat and preſſure, the carbonate of lime which had been introduced in the ſtate of the fineſt powder, is agglutinated into a firm maſs, poſſeſſing a degree of hardneſs, compactneſs, and ſpecific gravity *, nearly approaching to theſe qualities in a ſound limeſtone; and ſome of the reſults, by their ſaline fracture, by their ſemitranſparency, and their ſuſceptibility of poliſh, deſerve the name of marble.

The ſame trials have been made with all calcareous ſubſtances; with chalk, common limeſtone, marble, ſpar, and the ſhells of fiſh. All have ſhewn the ſame general property, with ſome varieties as to temperature. Thus, I found, that, in the ſame circumſtances, chalk was more ſuſceptible of agglutination than ſpar; the latter requiring a heat two degrees higher than the former, to bring it to the ſame pitch of agglutination.

The chalk uſed in my firſt experiments, always aſſumed the character of a yellow marble, owing probably to ſome ſlight contamination of iron. When a ſolid piece of chalk, whoſe bulk had been previouſly meaſured in the gage of Wedgwood's pyrometer was ſubmitted to heat under compreſſion, its contraction was remarkable, proving the approach of the particles during their conſolidation; on theſe occaſions, it was found to

* See Appendix.

to ſhrink three times more than the pyrometer-pieces in the ſame temperature. It loſt, too, almoſt entirely, its power of imbibing water, and acquired a great additional ſpecific gravity. On ſeveral occaſions, I obſerved, that maſſes of chalk, which, before the experiment, had ſhewn one uniform character of whiteneſs, aſſumed a ſtratified appearance, indicated by a ſeries of parallel layers of a brown colour. This circumſtance may hereafter throw light on the geological hiſtory of this extraordinary ſubſtance.

I HAVE ſaid, that, by mechanical conſtraint, almoſt the whole of the carbonic acid was retained. And, in truth, at this period, ſome loſs of weight had been experienced in all the experiments, both with iron and porcelain. But even this circumſtance is valuable, by exhibiting the influence of the carbonic acid, as varied by its quantity.

WHEN the loſs exceeded 10 or 15 *per cent* *. of the weight of the carbonate, the reſult was always of a friable texture, and without any ſtony character; when leſs than 2 or 3 *per cent*. it was conſidered as good, and poſſeſſed the properties of a natural carbonate. In the intermediate caſes, when the loſs amounted, for inſtance, to 6 or 8 *per cent*., the reſult was ſometimes excellent at firſt, the ſubſtance bearing every appearance of ſoundneſs, and often poſſeſſing a high character of cryſtallization; but it was unable to reſiſt the action of the air; and, by attracting carbonic acid or moiſture, or both, crumbled to duſt more or leſs rapidly, according to circumſtances. This ſeems to prove, that the carbonate of lime, though not fully ſaturated with carbonic acid, may poſſeſs the properties of limeſtone; and perhaps a difference of this

* I have found, that, in open fire, the entire loſs ſuſtained by the carbonate varies in different kinds from 42 to 45.5 *per cent*.

this kind may exiſt among natural carbonates, give riſe to their different degrees of durability.

I HAVE obſerved, in many caſes, that the calcination has reached only to a certain depth into the maſs ; the internal part remaining in a ſtate of complete carbonate, and, in general, of a very fine quality. The partial calcination ſeems thus to take place in two different modes. By one, a ſmall proportion of carbonic acid is taken from each particle of carbonate ; by the other, a portion of the carbonate is quite calcined, while the reſt is left entire. Perhaps one reſult is the effect of a feeble calcining cauſe, acting during a long time, and the other of a ſtrong cauſe, acting for a ſhort time.

SOME of the reſults which ſeemed the moſt perfect when firſt produced, have been ſubject to decay, owing to partial calcination. It happened, in ſome degree, to the beautiful ſpecimen produced on the 3d of March 1801, though a freſh fracture has reſtored it.

A SPECIMEN, too, of marble, formed from pounded ſpar, on 15th May 1801, was ſo complete as to deceive the workman employed to poliſh it, who declared, that, were the ſubſtance a little whiter, the quarry from which it was taken would be of great value, if it lay within reach of a market. Yet, in a few weeks after its formation, it fell to duſt.

NUMBERLESS ſpecimens, however, have been obtained, which reſiſt the air, and retain their poliſh as well as any marble. Some of them continue in a perfect ſtate, though they have been kept without any precaution during four or five years. That ſet, in particular, remain perfectly entire, which were ſhewn laſt year in this Society, though ſome of them were made in 1799, ſome in 1801 and 1802, and though the firſt eleven were long ſoaked in water, in the trials made of their ſpecific gravity.

A CURIOUS circumſtance occurred in one of theſe experiments, which may hereafter lead to important conſequences. Some ruſt of iron had accidentally found its way into the tube: 10 grains of carbonate were uſed, and a heat of 28° was applied. The tube had no flaw; but there was a certainty that the carbonic acid had eſcaped through its pores. When broken, the place of the carbonate was found occupied, partly by a black ſlaggy matter, and partly by ſphericles of various ſizes, from that of a ſmall pea downwards, of a white ſubſtance, which proved to be quicklime; the ſphericles being interſperſed through the ſlag, as ſpar and agates appear in whinſtone. The ſlag had certainly been produced by a mixture of the iron with the ſubſtance of the tube; and the ſpherical form of the quicklime ſeems to ſhew, that the carbonate had been in fuſion along with the ſlag, and that they had ſeparated on the eſcape of the carbonic acid.

THE ſubject was carried thus far in 1803, when I ſhould probably have publiſhed my experiments, had I not been induced to proſecute the inquiry by certain indications, and accidental reſults, of a nature too irregular and uncertain to meet the public eye, but which convinced me, that it was poſſible to eſtabliſh, by experiment, the truth of all that was hypothetically aſſumed in the Huttonian Theory.

THE principal object was now to accompliſh the entire fuſion of the carbonate, and to obtain ſpar as the reſult of that fuſion, in imitation of what we conceive to have taken place in nature.

IT was likewiſe important to acquire the power of retaining all the carbonic acid of the carbonate, both on account of the fact itſelf, and on account of its conſequences; the reſult being viſibly improved by every approach towards complete ſaturation. I therefore became anxious to inveſtigate the cauſe of the partial calcinations which had always taken place, to

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a greater or a lefs degree, in all thefe experiments. The queftion naturally fuggefts itfelf, What has become of the carbonic acid, feparated in thefe partial calcinations from the earthy bafis? Has it penetrated the veffel, and efcaped entirely, or has it been retained within it in a gafeous, but highly compreffed ftate? It occurred to me, that this queftion might be eafily refolved, by weighing the veffel before and after the action of heat upon the carbonate.

With iron, a conftant and inappreciable fource of irregularity exifted in the oxidation of the barrel. But with porcelain the thing was eafy; and I put it in practice in all my experiments with this material, which were made after the queftion had occurred to me. The tube was weighed as foon as its muzzle was clofed, and again, after the breech had been expofed to the fire; taking care, in both cafes, to allow all to cool. In every cafe, I found fome lofs of weight, proving, that even in the beft experiments, the tubes were penetrated to a certain degree. I next wifhed to try if any of the carbonic acid feparated, remained within the tube in a gafeous form; and in that view, I wrapt the tube, which had juft been weighed, in a fheet of paper, and placed it, fo furrounded, on the fcale of the balance. As foon as its weight was afcertained, I broke the tube by a fmart blow, and then replaced upon the fcale the paper containing all the fragments. In thofe experiments, in which entire calcination had taken place, the weight was found not to be changed, for all the carbonic acid had already efcaped during the action of heat. But in the good refults, I always found that a lofs of weight was the confequence of breaking the tube.

These facts prove, that both caufes of calcination had operated in the porcelain tubes; that, in the cafes of fmall lofs, part of the carbonic acid had efcaped through the veffel, and that part had been retained within it. I had in view methods by which the laſt could be counteracted; but I ſaw no remedy for the firſt. I began, therefore, to deſpair of ultimate ſucceſs with tubes of porcelain *.

ANOTHER circumſtance confirmed me in this opinion. I found it impracticable to apply a heat above 27° to theſe tubes, when charged as above with carbonate, without deſtroying them, either by exploſion, by the formation of a minute rent, or by the actual ſwelling of the tube. Sometimes this ſwelling took place to the amount of doubling the internal diameter, and yet the porcelain held tight, the carbonate ſuſtaining but a very ſmall loſs. This ductility of the porcelain in a low heat is a curious fact, and ſhews what a range of temperature is embraced by the gradual tranſition of ſome ſubſtances from a ſolid to a liquid ſtate: For the ſame porcelain, which is thus ſuſceptible of being ſtretched out without breaking in a heat of 27°, ſtands the heat of 152°, without injury, when expoſed to no violence, the angles of its fracture remaining ſharp and entire.

IV.

* I am neverthelеſs of opinion, that, in ſome ſituations, experiments with compreſſion may be carried on with great eaſe and advantage in ſuch tubes. I allude to the ſituation of the geologiſts of France and Germany, who may eaſily procure, from their own manufactories, tubes of a quality far ſuperior to any thing made for ſale in this country.

IV.

Experiments in Gun-Barrels resumed.—The Vertical Apparatus applied to them.—Barrels bored in solid Bars.—Old Sable Iron.—Fusion of the Carbonate of Lime.—Its action on Porcelain.—Additional apparatus required in consequence of that action.—Good results; in particular, four experiments, illustrating the theory of Internal Calcination, *and shewing the efficacy of the Carbonic Acid as a Flux.*

Since I found that, with porcelain tubes, I could neither confine the carbonic acid entirely, nor expose the carbonate in them to strong heats; I at last determined to lay them aside, and return to barrels of iron, with which I had formerly obtained some good results, favoured, perhaps, by some accidental circumstances.

On the 12th of February 1803, I began a series of experiments with gun-barrels, resuming my former method of working with the fusible metal, and with lead; but altering the position of the barrel from horizontal to vertical; the breech being placed upwards during the action of heat on the carbonate. This very simple improvement has been productive of advantages no less remarkable, than in the case of the tubes of porcelain. In this new position, the included air, quitting the air-tube on the fusion of the metal, and rising to the breech, is exposed to the greatest heat of the furnace, and must therefore react with its greatest force; whereas, in the horizontal position, that air might go as far back as the fusion of the metal reached, where its elasticity would be much feebler. The same disposition enabled me to keep the muzzle of the barrel plunged, during the action of heat, in a vessel filled with water; which contributed

contributed very much both to the convenience and ſafety of theſe experiments.

In this view, making uſe of the brick-furnace with the vertical muffle, already deſcribed in page 93. I ordered a pit (*a a a*, fig. 20.) to be excavated under it, for the purpoſe of receiving a water-veſſel. This veſſel (repreſented ſeparately, fig. 21.) was made of caſt iron; it was three inches in diameter, and three feet deep; and had a pipe (*d e*) ſtriking off from it at right angles, four or five inches below its rim, communicating with a cup (*e f*) at the diſtance of about two feet. The main veſſel being placed in the pit (*a a*) directly below the vertical muffle, and the cup ſtanding clear of the furnace, water poured into the cup flowed into the veſſel, and could thus conveniently be made to ſtand at any level. (The whole arrangement is repreſented in fig. 20.) The muzzle of the barrel (*g*) being plunged into the water, and its breech (*b*) reaching up into the muffle, as far as was found convenient, its poſition was ſecured by an iron chain (*g f*). The heat communicated downwards generally kept the ſurface of the water (at *c*) in a ſtate of ebullition; the waſte thus occaſioned being ſupplied by means of the cup, into which, if neceſſary, a conſtant ſtream could be made to flow.

As formerly, I rammed the carbonate into a tube of porcelain, and placed it in a cradle of iron, along with an air-tube and a pyrometer; the cradle being fixed to a rod of iron, which rod I now judged proper to make as large as the barrel would admit, in order to exclude as much of the fuſible metal as poſſible; for the expanſion of the liquid metal being in proportion to the quantity heated, the more that quantity could be reduced, the leſs riſk there was of deſtroying the barrels.

In the courſe of practice, a ſimple mode occurred of removing the metal and withdrawing the cradle: it conſiſted in placing

cing the barrel with its muzzle downwards, ſo as to keep the breech above the furnace and cold, while its muzzle was expoſed to ſtrong heat in the muffle. In this manner, the metal was diſcharged from the muzzle, and the poſition of the barrel being lowered by degrees, the whole metal was removed in ſucceſſion, till at laſt the cradle and its contents became entirely looſe. As the metal was delivered, it was received in a crucible, filled with water, ſtanding on a plate of iron placed over the pit, which had been uſed, during the firſt ſtage of the experiment, to contain the water-veſſel. It was found to be of ſervice, eſpecially where lead was uſed, to give much more heat to the muzzle than ſimply what was required to liquefy the metal it contained; for when this was not done, the muzzle growing cold as the breech was heating, ſome of the metal delivered from the breech was congealed at the muzzle, ſo as to ſtop the paſſage.

According to this method, many experiments were made in gun-barrels, by which ſome very material ſteps were gained in the inveſtigation.

On the 24th February, I made an experiment with ſpar and chalk; the ſpar being placed neareſt to the breech of the barrel, and expoſed to the greateſt heat, ſome baked clay intervening between the carbonates. On opening the barrel, a long-continued hiſſing noiſe was heard. The ſpar was in a ſtate of entire calcination; the chalk, though crumbling at the outſide, was uncommonly hard and firm in the heart. The temperature had riſen to 32°.

In this experiment, we have the firſt clear example, in iron barrels, of what I call *Internal Calcination;* that is to ſay, where the carbonic acid ſeparated from the earthy baſis, has been accumulated in cavities within the barrel. For, ſubſequently to the action of ſtrong heat, the barrel had been completely cooled; the air therefore introduced by means of the air-tube, muſt

have

have reſumed its original bulk, and by itſelf could have no tendency to ruſh out; the heat employed to open the barrel being barely ſufficient to ſoften the metal. Since, then, the opening of the barrel was accompanied by the diſcharge of elaſtic matter in great abundance, it is evident, that this muſt have proceeded from ſomething ſuperadded to the air originally included, which could be nothing but the carbonic acid of the carbonate. It follows, that the calcination had been, in part at leaſt, internal; the ſeparation of the acid from the earthy matter being complete where the heat was ſtrongeſt, and only partial where the intenſity was leſs.

The chemical principles ſtated in a former part of this paper, authoriſed us to expect a reſult of this kind. As heat, by increaſing the volatility of the acid, tended to ſeparate it from the earth, we had reaſon to expect, that, under the ſame compreſſion, but in different temperatures, one portion of the carbonate might be calcined, and another not: And that the leaſt heated of the two, would be leaſt expoſed to a change not only from want of heat, but likewiſe in conſequence of the calcination of the other maſs; for the carbonic acid diſengaged by the calcination of the hotteſt of the two, muſt have added to the elaſticity of the confined elaſtic fluid, ſo as to produce an increaſe of compreſſion. By this means, the calcination of the coldeſt of the two might be altogether prevented, and that of the hotteſt might be hindered from making any further advancement. This reaſoning ſeemed to explain the partial calcinations which had frequently occurred where there was no proof of leakage; and it opened ſome new practical views in theſe experiments, of which I availed myſelf without loſs of time. If the internal calcination of one part of an incloſed maſs, promotes the compreſſion of other maſſes included along with it, I conceived that we might forward our views very much by placing a ſmall quantity of carbonate,

nate, carefully weighed, in the ſame barrel with a large quantity of that ſubſtance; and by arranging matters ſo that the ſmall fiducial part ſhould undergo a moderate heat, while a ſtronger heat, capable of producing internal calcination, ſhould be applied to the reſt of the carbonate. In this manner, I made many experiments, and obtained reſults which ſeemed to confirm this reaſoning, and which were often very ſatisfactory, though the heat did not always exert its greateſt force where I intended it to do ſo.

On the 28th of February, I introduced ſome carbonate, accurately weighed, into a ſmall porcelain tube, placed within a larger one, the reſt of the large tube being filled with pounded chalk; theſe carbonates, together with ſome pieces of chalk, placed along with the large tube in the cradle, weighing in all 195.7 grains. On opening the barrel, air ruſhed out with a long-continued hiſſing noiſe. The contents of the little tube were loſt by the intruſion of ſome borax which had been introduced over the ſilex, in order to exclude the fuſible metal. But the reſt of the carbonate, contained in the large tube, came out in a fine ſtate, being porous and frothy throughout; ſparkling every where with facettes, the angular form of which was diſtinguiſhable in ſome of the cavities by help of a lens: in ſome parts the ſubſtance exhibited the rounding of fuſion; in many it was in a high degree tranſparent. It was yellow towards the lower end, and at the other almoſt colourleſs. At the upper end, the carbonate ſeemed to have united with the tube, and at the places of contact to have ſpread upon it; the union having the appearance of a mutual action. The general maſs of carbonate efferveſced in acid violently, but the thin ſtratum immediately contiguous to the tube, feebly, if at all.

On the 3d of March, I introduced into a very clean tube of porcelain 36.8 of chalk. The tube was placed in the upper part of the cradle, the remaining ſpace being filled with two pieces of chalk, cut for the purpoſe; the uppermoſt of theſe being excavated, ſo as to anſwer the purpoſe of an air-tube. The pieces thus added, were computed to weigh about 300 grains. There was no pyrometer uſed; but the heat was gueſſed to be about 30°. After the barrel had ſtood during a few minutes in its delivering poſition, the whole lead, with the rod and cradle, were thrown out with a ſmart report, and with conſiderable force. The lowermoſt piece of chalk had ſcarcely been acted upon by heat. The upper part of the other piece was in a ſtate of marble, with ſome remarkable facettes. The carbonate, in the little tube, had ſhrunk very much during the firſt action of heat, and had begun to ſink upon itſelf, by a further advancement towards liquefaction. The maſs was divided into ſeveral cylinders, lying confuſedly upon each other; this diviſion ariſing from the manner in which the pounded chalk was rammed into the tube in ſucceſſive portions. In ſeveral places, particularly at the top, the carbonate was very porous, and full of decided air-holes, which could not have been formed but in a ſoft ſubſtance; the globular form and ſhining ſurface of all theſe cavities, clearly indicating fuſion. The ſubſtance was ſemitranſparent; in ſome places yellow, and in ſome colourleſs. When broken, the ſolid parts ſhewed a ſaline fracture, compoſed of innumerable facettes. The carbonate adhered, from end to end, to the tube, and incorporated with it, ſo as to render it impoſſible to aſcertain what loſs had been ſuſtained. In general, the line of contact was of a brown colour; yet there was no room for ſuſpecting the preſence of any foreign matter, except, perhaps, from the iron-rod which was uſed in ramming down the chalk. But, in ſubſequent experiments, I have obſerved the ſame brown or black colour at the union of the carbonate with the porcelain tubes, where the powder had been purpoſely rammed with a piece of

wood;

wood; ſo that this colour, which has occurred in almoſt every ſimilar caſe, remains to be accounted for. The carbonate effervеſced violently with acid; the ſubſtance in contact with the tube, doing ſo, however, more feebly than in the heart, leaving a copious depoſite of white ſandy matter, which is doubtleſs a part of the tube, taken up by the carbonate in fuſion.

On the 24th of March, I made a ſimilar experiment, in a ſtout gun-barrel, and took ſome care, after the application of heat, to cool the barrel ſlowly, with a view to cryſtallization. The whole maſs was found in a fine ſtate, and untouched by the lead; having a ſemitranſparent and ſaline ſtructure, with various facettes. In one part, I found the moſt decided cryſtallization I had obtained, though of a ſmall ſize: owing to its tranſparency it was not eaſily viſible, till the light was made to reflect from the cryſtalline ſurface, which then produced a dazzle, very obſervable by the naked eye: when examined by means of a lens, it was ſeen to be compoſed of ſeveral plates, broken irregularly in the fracture of the ſpecimen, all of which are parallel to each other, and reflect under the ſame angle, ſo as to unite in producing the dazzle. This ſtructure was obſervable equally well in both parts of the broken ſpecimen. In a former experiment, as large a facette was obtained in a piece of ſolid chalk; but this reſult was of more conſequence, as having been produced from chalk previouſly pounded.

The foregoing experiments proved the ſuperior efficacy of iron veſſels over thoſe of porcelain, even where the thickneſs was not great; and I perſevered in making a great many experiments with gun-barrels, by which I occaſionally obtained very fine reſults: but I was at laſt convinced, that their thickneſs was not ſufficient to enſure regular and ſteady ſucceſs: For this purpoſe, it appeared proper to employ veſſels of ſuch ſtrength, as to bear a greater expanſive force than was juſt neceſſary; ſince, occaſionally, (owing to our ignorance of the relation between the various forces of expanſion, affinity, tenacity, &c.), much more ſtrain has been given to the veſſels than was requiſite. In ſuch caſes, barrels have been deſtroyed, which, as the reſults have proved, had acted with ſufficient ſtrength during the firſt ſtages of the experiments, though they had been unable to reſiſt the ſubſequent overſtrain. Thus, my ſucceſs with gun-barrels, depended on the good fortune of having uſed a force no more than ſufficient, to conſtrain the carbonic acid, and enable it to act as a flux on the lime. I therefore determined to have recourſe to iron barrels of much greater ſtrength, and tried various modes of conſtruction.

I HAD ſome barrels executed by wrapping a thick plate of iron round a mandrel, as is practiſed in the formation of gun-barrels; and likewiſe by bringing the two flat ſides together, ſo as to unite them by welding. Theſe attempts, however, failed. I next thought of procuring bars of iron, and of having a cavity bored out of the ſolid, ſo as to form a barrel. In this manner I ſucceeded well. The firſt barrel I tried in this way was of ſmall bore, only half an inch: Its performance was highly ſatisfactory, and ſuch as to convince me, that the mode now adopted was the beſt of any that I had tried. Owing to the ſmallneſs of the bore, a pyrometer could not be uſed internally, but was placed upon the breech of the barrel, as it ſtood in the vertical muffle. In this poſition, it was evidently expoſed to a much leſs heat than the fiducial part of the apparatus, which was always placed, as nearly as could be gueſſed, at the point of greateſt heat.

ON the 4th of April, an experiment was made in this way with ſome ſpar; the pyrometer on the breech giving 33°. The ſpar came out clean, and free from any contamination, adhering to the inſide of the porcelain tube: it was very much ſhrunk, ſtill retaining a cylindrical form, though bent by partial adheſions. Its ſurface bore ſcarcely any remains of the impreſſion taken by the

the powder, on ramming it into the tube: it had, to the naked eye, the roughneſs and ſemitranſparency of the pith of a ruſh ſtripped of its outer ſkin. By the lens, this ſame ſurface was ſeen to be glazed all over, though irregularly, ſhewing here and there ſome air-holes. In fracture, it was ſemitranſparent, more vitreous than cryſtalline, though having a few facettes: the maſs, was ſeemingly formed of a congeries of parts, in themſelves quite tranſparent: and, at the thin edges, ſmall pieces were viſible of perfect tranſparency. Theſe muſt have been produced in the fire; for the ſpar had been ground with water, and paſſed through ſieves, the ſame with the fineſt of thoſe uſed at Etruria, as deſcribed by Mr Wedgwood, in his paper on the conſtruction of his Pyrometer.

With the ſame barrel I obtained many intereſting reſults, giving as ſtrong proofs of fuſion as in any former experiments; with this remarkable difference, that, in theſe laſt, the ſubſtance was compact, with little or no trace of frothing. In the gun-barrels where fuſion had taken place, there had always been a loſs of 4 or 5 *per cent.*, connected, probably, with the frothing. In theſe experiments, for a reaſon ſoon to be ſtated, the circumſtance of weight could not be obſerved; but appearances led me to ſuppoſe, that here the loſs had been ſmall, if any.

On the 6th of April, I made another experiment with the ſquare barrel, whoſe thickneſs was now much reduced by ſucceſſive ſcales, produced by oxidation, and in which a ſmall rent began to appear externally, which did not, however, penetrate to the bore. The heat roſe high, a pyrometer on the breech of the barrel giving 37°. On removing the metals, the cradle was found to be fixed, and was broken in the attempts made to withdraw it. The rent was much widened externally: but it was evident, that the barrel had not been laid open, for part of the carbonate was in a ſtate of ſaline

marble;

marble; another was hard and white, without any ſaline grains, and ſcarcely effervefced in acid. It was probably quicklime, formed by internal calcination, but in a ſtate that has not occurred in any other experiment.

The workman whom I employed to take out the remains of the cradle, had cut off a piece from the breech of the barrel, three or four inches in length. As I was examining the crack which was ſeen in this piece, I was ſurpriſed to ſee the inſide of the barrel lined with a ſet of tranſparent and well-defined cryſtals, of ſmall ſize, yet viſible by the naked eye. They lay together in ſome places, ſo as to cover the ſurface of the iron with a tranſparent coat; in others they were detached, and ſcattered over the ſurface. Unfortunately, the quantity of this ſubſtance was too ſmall to admit of much chemical examination; but I immediately aſcertained, that it did not in the leaſt effervеſce in acid, nor did it ſeem to diſſolve in it. The cryſtals were in general tranſparent and colourleſs, though a few of them were tinged ſeemingly with iron. Their form was very well defined, being flat, with oblique angles, and bearing a ſtrong reſemblance to the cryſtals of the Lamellated Stylbite of Haüy. Though made above two years ago, they ſtill retain their form and tranſparency unchanged. Whatever this ſubſtance may be, its appearance, in this experiment, is in the higheſt degree intereſting, as it ſeems to afford an example of the mode in which Dr Hutton ſuppoſes many internal cavities to have been lined, by the ſublimation of ſubſtances in a ſtate of vapour; or, held in ſolution, by matters in a gaſeous form. For, as the cryſtals adhered to a part of the barrel, which muſt have been occupied by air during the action of heat, it ſeems next to certain that they were produced by ſublimation.

The very powerful effects produced by this laſt barrel, the ſize of which (reduced, indeed, by repeated oxidation) was not above

above an inch ſquare, made me very anxious to obtain barrels of the ſame ſubſtance, which being made of greater ſize, ought to afford reſults of extreme intereſt. I found upon inquiry, that this barrel was not made of Swediſh iron, as I at firſt ſuppoſed, but of what is known by the name of *Old Sable*, from the figure of a Sable ſtamped upon the bars; that being the armorial badge of the place in Siberia where this iron is made *.

A WORKMAN explained to me ſome of the properties of different kinds of irons, moſt intereſting in my preſent purſuit; and he illuſtrated what he ſaid by actual trial. All iron, when expoſed to a certain heat, cruſhes and crumbles under the hammer; but the temperature in which this happens, varies with every different ſpecies. Thus, as he ſhewed me, caſt iron cruſhes in a dull-red heat, or perhaps about 15° of WEDGWOOD; ſteel, in a heat perhaps of 30°; Swediſh iron, in a bright white heat, perhaps of 50° or 60°; old ſable, itſelf, likewiſe yields, but in a much higher heat, perhaps of 100°. I merely gueſſed at theſe temperatures; but I am certain of this, that in a heat ſimilar to that in which Swediſh iron crumbled under the hammer, the old ſable withſtood a ſtrong blow, and ſeemed to poſſeſs conſiderable firmneſs. It is from a knowledge of this quality, that the blackſmith, when he firſt takes his iron from the forge, and lays it on the anvil, begins by very gentle blows, till the temperature has ſunk to the degree in which the iron can bear the hammer. I obſerved, as the ſtrong heat of the forge acted on the Swediſh iron, that it began to boil at the ſurface, clearly indicating the diſcharge of ſome gaſeous matter; whereas, the old ſable, in the ſame circumſtances, acquired the ſhining ſurface of a liquid, and melted away without any efferveſcence. I procured, at this time, a conſiderable number

* I WAS favoured with this account of it by the late Profeſſor ROBISON.

number of bars of that iron, which fully anſwered my expectations.

By the experiments laſt mentioned, a very important point was gained in this inveſtigation; the complete fuſibility of the carbonate under preſſure being thereby eſtabliſhed. But from this very circumſtance, a neceſſity aroſe of adding ſome new devices to thoſe already deſcribed: for the carbonate, in fuſion, ſpreading itſelf on the inſide of the tube containing it, and the two uniting firmly together, ſo as to be quite inſeparable, it was impoſſible, after the experiment, to aſcertain the weight of the carbonate by any method previouſly uſed. I therefore determined in future to adopt the following arrangement.

A small tube of porcelain (*i k*, fig. 23.) was weighed by means of a counterpoiſe of ſand, or granulated tin; then the carbonate was firmly rammed into the tube, and the whole weighed again: thus the weight of the carbonate, previous to the experiment, was aſcertained. After the experiment, the tube, with its contents, was again weighed; and the variation of weight obtained, independently of any mutual action that had taken place between the tube and the carbonate. The balance which I uſed, turned, in a conſtant and ſteady manner, with one hundredth of a grain. When pounded chalk was rammed into this tube, I generally left part of it free, and in that ſpace laid a ſmall piece of lump-chalk (*i*), dreſſed to a cylinder, with the ends cut flat and ſmooth, and I uſually cut a letter on each end, the more effectually to obſerve the effects produced by heat upon the chalk; the weight of this piece of chalk being always eſtimated along with that of the powder contained in the tube. In ſome experiments, I placed a cover of porcelain on the muzzle of the little tube, (this cover being weighed along with it), in order to provide againſt the caſe of ebullition:

ebullition: But as that did not often occur, I ſeldom took the trouble of this laſt precaution.

It was now of conſequence to protect the tube, thus prepared, from being touched during the experiment, by any ſubſtance, above all, by the carbonate of lime, which might adhere to it, and thus confound the appreciation by weight. This was provided for as follows: The ſmall tube (Fig. 23. *i k*), with its pounded carbonate (*k*), and its cylinder of lump-chalk (*i*,), was dropt into a large tube of porcelain (*p k*, Fig. 24.). Upon this a fragment of porcelain (*l*), of ſuch a ſize as not to fall in between the tubes, was laid. Then a cylinder of chalk (*m*) was dreſſed, ſo as nearly to fit and fill up the inſide of the large tube, one end of it being rudely cut into the form of a cone. This maſs being then introduced, with its cylindrical end downwards, was made to preſs upon the fragment of porcelain (*l*). I then dropped into the ſpace (*n*), between the conical part of this maſs and the tube, a ſet of fragments of chalk, of a ſize beyond what could poſſibly fall between the cylindrical part and the tube, and preſſed them down with a blunt tool, by which the chalk being at the ſame time cruſhed and rammed into the angle, was forced into a maſs of ſome ſolidity, which effectually prevented any thing from paſſing between the large maſs of chalk and the tube. In practice, I have found this method always to anſwer, when done with care. I covered the chalk, thus rammed, with a ſtratum of pounded flint (*o*), and that again with pounded chalk (*p*) firmly rammed. In this manner, I filled the whole of the large tube with alternate layers of ſilex and chalk; the muzzle being always occupied with chalk, which was eaſily preſſed into a maſs of tolerable firmneſs, and, ſuffering no change in very low heats, excluded the fuſible metal in the firſt ſtages of the experiment.

The large tube, thus filled, was placed in the cradle, ſometimes with the muzzle upwards, and ſometimes the reverſe. I

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have frequently altered my views as to that part of the arrangement, each mode poſſeſſing peculiar advantages and diſadvantages. With the muzzle upwards, (as ſhewn in fig. 24. and 25.) the beſt ſecurity is afforded againſt the intruſion of the fuſible metal; becauſe the air, quitting the air-tube in the working poſition, occupies the upper part of the barrel; and the fuſible metal ſtands as a liquid (at *q*, fig. 25.) below the muzzle of the tube, ſo that all communication is cut off, between the liquid metal and the inſide of the tube. On the other hand, by this arrangement, the ſmall tube, which is the fiducial part of the apparatus, is placed at a conſiderable diſtance from the breech of the barrel, ſo as either to undergo leſs heat than the upper part, or to render it neceſſary that the barrel be thruſt high into the muffle.

With the muzzle of the large tube downwards, the inner tube is placed (as ſhewn in fig. 22.), ſo as ſtill to have its muzzle upwards, and in contact with the breech of the large tube. This has the advantage of placing the ſmall tube near to the breech of the barrel: and though there is here leſs ſecurity againſt the intruſion of liquid metal, I have found that a point of little conſequence; ſince, when the experiment is a good one, and that the carbonic acid has been well confined, the intruſion ſeldom takes place in any poſition. In whichever of the two oppoſite poſitions the large tube was placed, a pyrometer was always introduced, ſo as to lie as near as poſſible to the ſmall tube. Thus, in the firſt-mentioned poſition, the pyrometer was placed immediately below the large tube, and, in the other poſition, above it; ſo that, in both caſes, it was ſeparated from the carbonate by the thickneſs only of the two tubes.

Much room was unavoidably occupied by this method, which neceſſarily obliged me to uſe ſmall quantities of carbonate,

bonate, the ſubject of experiment ſeldom weighing more than 10 or 12 grains, and in others far leſs *.

On the 11th of April 1803, with a barrel of old ſable iron having a bore of 0.75 of an inch, I made an experiment in which all theſe arrangements were put in practice. The large tube contained two ſmall ones; one filled with ſpar, and the other with chalk. I conceived that the heat had riſen to 33°, or ſomewhat higher. On melting the metals, the cradle was thrown out with conſiderable violence. The pyrometer, which, in this experiment, had been placed within the barrel, to my aſtoniſhment, indicated 64°. Yet all was ſound. The two little tubes came out quite clean and uncontaminated. The ſpar had loſt 17.0 *per cent.*: The chalk 10.7 *per cent.*: The ſpar was half ſunk down, and run againſt the ſide of the little tube: Its ſurface was ſhining, its texture ſpongy, and it was compoſed of a tranſparent and jelly-like ſubſtance: The chalk was entirely in a ſtate of froth. This experiment extends our power of action, by ſhewing, that compreſſion, to a conſiderable degree, can be carried on in ſo great a heat as 64°. It ſeems likewiſe to prove, that, in ſome of the late experiments with the ſquare barrel, the heat had been much higher than was ſuppoſed at the time, from the indication of the pyrometer placed on the breech of the barrel; and that in ſome of them, particularly in the laſt, it muſt have riſen at leaſt as high as in the preſent experiment.

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* I meaſured the capacity of the air-tubes by means of granulated tin, acting as a fine and equal ſand. By comparing the weight of this tin with an equal bulk of water, I found that a cubic inch of it weighed 1330.6 grains, and that each grain of it correſponded to 0.00075 of a cubic inch. From theſe data I was able, with tolerable accuracy, to gage a tube by weighing the tin required to fill it.

On the 21ſt of April 1805, a ſimilar experiment was made with a new barrel, bored in a ſquare bar of old ſable, of about two and a half inch in diameter, having its angles merely rounded; the inner tube being filled with chalk. The heat was maintained during ſeveral hours, and the furnace allowed to burn out during the night. The barrel had the appearance of ſoundneſs, but the metals came off quietly, and the carbonate was entirely calcined, the pyrometer indicating 63°. On examination, and after beating off the ſmooth and even ſcale of oxide peculiar to the old ſable, the barrel was found to have yielded in its peculiar manner; that is, by the opening of the longitudinal fibres. This experiment, notwithſtanding the failure of the barrel, was one of the moſt intereſting I had made, ſince it afforded proof of complete fuſion. The carbonate had boiled over the lips of the little tube, ſtanding, as juſt deſcribed, with its mouth upwards, and had run down to within half an inch of its lower end: moſt of the ſubſtance was in a frothy ſtate, with large round cavities, and a ſhining ſurface; in other parts, it was interſperſed with angular maſſes, which have evidently been ſurrounded by a liquid in which they floated. It was harder, I thought, than marble; giving no efferveſcence, and not turning red like quicklime in nitric acid, which ſeemed to have no effect upon it in the lump. It was probably a compound of quicklime with the ſubſtance of the tube.

With the ſame barrel repaired, and with others like it, many ſimilar experiments were made at this time with great ſucceſs; but to mention them in detail, would amount nearly to a repetition of what has been ſaid. I ſhall take notice of only four of them, which, when compared together, throw much light on the theory of theſe operations, and likewiſe ſeem to eſtabliſh a very important principle in geology. Theſe four

four experiments differ from each other only in the heat employed, and in the quantity of air introduced.

The first of these experiments was made on the 27th of April 1803, in one of the large barrels of old sable, with all the above-mentioned arrangements. The heat had risen, contrary to my intention, to 78° and 79°. The tubes came out uncontaminated with fusible metal, and every thing bore the appearance of soundness. The contents of the little tube, consisting of pounded chalk, and of a small piece of lump-chalk, came out clean, and quite loose, not having adhered to the inside of the tube in the smallest degree. There was a loss of 41 *per cent.*, and the calcination seemed to be complete; the substance, when thrown into nitric acid, turning red, without effervescence at first, though, after lying a few minutes, some bubbles appeared. According to the method followed in all these experiments, and lately described at length, (and shewn in fig. 24. & 25.), the large tube was filled over the small one, with various masses of chalk, some in lump, and some rammed into it in powder; and in the cradle there lay some pieces of chalk, filling up the space, so that in the cradle there was a continued chain of carbonate of four or five inches in length. The substance was found to be less and less calcined, the more it was removed from the breech of the barrel, where the heat was greatest. A small piece of chalk, placed at the distance of half an inch from the small tube, had some saline substance in the heart, surrounded and intermixed with quicklime, distinguished by its dull white. In nitric acid, this substance became red, but effervesced pretty briskly; the effervescence continuing till the whole was dissolved. The next portion of chalk, was in a firm state of limestone; and a lump of chalk in the cradle, was equal in perfection to any marble I have obtained by compression: the two last-mentioned pieces of chalk effervescing with violence in the acid, and shewing

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no redneſs when thrown into it. Theſe facts clearly prove, that the calcination of the contents of the ſmall tube had been internal, owing to the violent heat which had ſeparated its acid from the moſt heated part of the carbonate, according to the theory already ſtated. The ſoundneſs of the barrel was proved by the complete ſtate of thoſe carbonates which lay in leſs heated parts. The air-tube in this experiment had a capacity of 0.29, nearly one-third of a cubic inch.

The ſecond of theſe experiments was made on the 29th of April, in the ſame barrel with the laſt, after it had afforded ſome good reſults. The air-tube was reduced to one-third of its former bulk, that is, to one-tenth of a cubic inch. The heat roſe to 60°. The barrel was covered externally with a black ſpongy ſubſtance, the conſtant indication of failure, and a ſmall drop of white metal made its appearance. The cradle was removed without any exploſion or hiſſing. The carbonates were entirely calcined. The barrel had yielded, but had reſiſted well at firſt; for, the contents of the little tube were found in a complete ſtate of froth, and running with the porcelain.

The third experiment was made on the 30th of April, in another ſimilar barrel. Every circumſtance was the ſame as in the two laſt experiments, only that the air-tube was now reduced to half its laſt bulk, that is, to one-twentieth of a cubic inch. A pyrometer was placed at each end of the large tube. The uppermoſt gave 41°, the other only 15°. The contents of the inner tube had loſt 16 *per cent.*, and were reduced to a moſt beautiful ſtate of froth, not very much injured by the internal calcination, and indicating a thinner ſtate of fuſion than had appeared.

The fourth experiment was made on the 2d of May, like the reſt in all reſpects, with a ſtill ſmaller air-tube, of 0.0318, being leſs than one-thirtieth of a cubic inch. The upper pyrometer

rometer gave 25°, and the under one 16° : The loweſt maſſes of carbonate were ſcarcely affected by the heat : The contents of the little tube had loſt 2.9 *per cent.* ; both the lump and the pounded chalk were in a fine ſaline ſtate, and, in ſeveral places had run and ſpread upon the inſide of the tube, which I had not expected to ſee in ſuch a low heat. On the upper ſurface of the chalk rammed into the little tube, which, after its introduction had been wiped ſmooth, were a ſet of white cryſtals, with ſhining facettes, large enough to be diſtinguiſhed by the naked eye, and ſeeming to riſe out of the maſs of carbonate. I likewiſe obſerved, that the ſolid maſs on which theſe cryſtals ſtood, was uncommonly tranſparent.

In theſe four experiments, the bulk of the included air was ſucceſſively diminiſhed, and by that means its elaſticity increaſed. The conſequence was, that in the firſt experiment, where that elaſticity was the leaſt, the carbonic acid was allowed to ſeparate from the lime, in an early ſtage of the riſing heat, lower than the fuſing point of the carbonate, and complete internal calcination was effected. In the ſecond experiment, the elaſtic force being much greater, calcination was prevented, till the heat roſe ſo high as to occaſion the entire fuſion of the carbonate, and its action on the tube, before the carbonic acid was ſet at liberty by the failure of the barrel. In the third experiment, with ſtill greater elaſtic force, the carbonate was partly conſtrained, and its fuſion accompliſhed, in a heat between 41° and 15°. In the laſt experiment, where the force was ſtrongeſt of all, the carbonate was almoſt completely protected from decompoſition by heat, in conſequence of which it cryſtallized and acted on the tube, in a temperature between 25° and 16°. On the other hand, the efficacy of the carbonic acid as a flux on the lime, and in enabling the carbonate to act as a flux on other bodies, was clearly evinced ; ſince the firſt experiment

periment proved, that quicklime by itself, could neither be melted, nor act upon porcelain, even in the violent heat of 79°; whereas, in the last experiment where the carbonic acid was retained, both of these effects took place in a very low temperature.

V.

Experiments in which Water was employed to increase the Elasticity of the included Air.—Cases of complete Compression.—General Observations.—Some Experiments affording interesting results; in particular, shewing a mutual action between Silex and the Carbonate of Lime.

FINDING that such benefit arose from the increase of elasticity given to the included air in the last-mentioned experiments, by the diminution of its quantity; it now occurred to me, that a suggestion formerly made by Dr KENNEDY, of using water to assist the compressing force, might be followed with advantage: That while sufficient room was allowed for the expansion of the liquid metal, a reacting force of any required amount, might thus be applied to the carbonate. In this view, I adopted the following mode, which, though attended with considerable difficulty in execution, I have often practised with success. The weight of water required to be introduced into the barrel was added to a small piece of chalk or baked clay, previously weighed. The piece was then dropped into a tube of porcelain of about an inch in depth, and covered with pounded chalk, which was firmly rammed upon it. The tube was then placed in the cradle along with the subject of experiment, and the whole was plunged into the fusible metal, previously poured into the barrel, and heated so as merely to render it liquid. The metal being thus suddenly cooled, the

the tube was encafed in a folid mafs, before the heat had reached the included moifture. The difficulty was to catch the fufible metal at the proper temperature; for when it was fo hot as not to fix in a few feconds, by the contact of the cradle and its contents, the water was heard to bubble through the metal and efcape. I overcame this difficulty, however, by firft heating the breech of the barrel, (containing a fufficient quantity of fufible metal), almoft to rednefs, and then fetting it into a veffel full of water, till the temperature had funk to the proper pitch, which I knew to be the cafe when the hiffing noife produced in the water by the heated barrel ceafed; the cradle, during the laft ftage of this operation, being held clofe to the muzzle of the barrel, and ready to be thruft into it.

On the 2d of May, I made my firft experiment in this way, ufing the fame air-tube as in the laft experiment, which was equal in capacity to one-thirtieth of a cubic inch. Half a grain of water was introduced in the manner juft defcribed. The barrel, after an hour of red-heat, was let down by a rope and pulley, which I took care to ufe in all experiments, in which there was any appearance of danger. All was found. The metals rufhed out fmartly, and a flafh of flame accompanied the difcharge. The upper pyrometer gave 24°, and the lower one 14°. The contents of the inner tube had loft lefs than 1 *per cent.*, ftrictly 0.84. The carbonate was in a ftate of good limeftone; but the heat had been too feeble: The lower part of the chalk in the little tube was not agglutinated: The chalk round the fragment of pipe-ftalk (ufed to introduce the water), which had been more heated than the pyrometer, and the fmall rod, which had moulded itfelf in the boll of the ftalk, were in a ftate of marble.

On the 4th of May, I made an experiment like the laft, but with the addition of 1.05 grains water. After application of heat, the

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fire was allowed to burn out till the barrel was black. The metal was difcharged irregularly. Towards the end, the inflammable air produced, burnt at the muzzle, with a lambent flame, during fome time, arifing doubtlefs from hydrogen gas, more or lefs pure, produced by the decompofition of the water. The upper pyrometer indicated 36°, and the lower one 19°. The chalk which lay in the outer part of the large tube was in a ftate of marble. The inner tube was united to the outer one, by a ftar of fufed matter, black at the edges, and fpreading all round, furrounding one of the fragments of porcelain which had fallen by accident in between the tubes. The inner tube, with the ftarry matter adhering to it, but without the coated fragment, feemed to have fuftained a lofs of 12 *per cent.*, on the original carbonate introduced. But, the fubftance furrounding the fragment being inappreciable, it was impoffible to learn what lofs had been really fuftained. Examining the little tube, I found its edges clean, no boiling over having taken place. The top of the fmall lump of chalk had funk much. When the little tube was broken, its contents gave proof of fufion in fome parts, and in others, of the neareft approach to it. A ftrong action of ebullition had taken place all round, at the contact of the tube with the carbonate: in the heart, the fubftance had a tranfparent granular texture, with little or no cryftallization. The fmall piece of lump-chalk was united and blended with the rammed powder, fo that they could fcarcely be diftinguifhed. In the lower part of the carbonate, where the heat muft have been weaker, the rod had acted more feebly on the tube, and was detached from it: here the fubftance was firm, and was highly marked in the fracture with cryftalline facettes. Wherever the carbonate touched the tube, the two fubftances exhibited, in their mixture, much greater proofs of fufion than could be found in the pure carbonate. At one place, a ftream of this compound had penetrated a rent in the

the inner tube, which it had filled completely, conftituting a real vein, like thofe of the mineral kingdom: which is ftill diftinctly to be feen in the fpecimen. It had then fpread itfelf upon the outfide of the inner tube, to the extent of half an inch in diameter, and had enveloped the fragment of porcelain already mentioned. When pieces of the compound were thrown into nitric acid, fome effervefced, and fome not.

I REPEATED this experiment on the fame day, with two grains of water. The furnace being previoufly hot; I continued the fire during one half-hour with the muffle open, and another with a cover upon it. I then let the barrel down by means of the pulley. The appearance of a large longitudinal rent, made me at firft conceive that the experiment was loft, and the barrel deftroyed: The barrel was vifibly fwelled, and in fwelling had burft the cruft of fmooth oxide with which it was furrounded; at the fame time, no exudation of metal had happened, and all was found. The metals were thrown out with more fuddennefs and violence than in any former experiment, but the rod remained in its place, being fecured by a cord. The upper pyrometer gave 27°, the lower 23°. The contents of the inner tube had loft 1.5 *per cent.* The upper end of the little lump of chalk, was rounded and glazed by fufion; and the letter which I have been in the habit of cutting on thefe fmall pieces, in order to trace the degree of action upon them, was thus quite obliterated. On the lower end of the fame lump, the letter is ftill vifible. Both the lump and the rammed chalk were in a good femitranfparent ftate, fhining a little in the fracture, but with no good facettes, and no where appearing to have acted on the tube. This laft circumftance is of confequence, fince it feems to fhew, that this very remarkable action of heat, under compreffion, was performed without the affiftance of the fubftance of the tube, by which, in many other

 experiments,

experiments, a confiderable additional fufibility has been communicated to the carbonate.

These experiments, and many others made about the fame time, with the fame fuccefs, clearly prove the efficacy of water in affifting the compreffion; and refults approaching to thefe in quality, obtained, in fome cafes, by means of a very fmall air-tube, fhew that the influence of water on this occafion has been merely mechanical.

During the following fummer and autumn 1803, I was occupied with a different branch of this fubject, which I fhall foon have occafion to mention.

In the early part of laft year, 1804, I again refumed the fort of experiments lately defcribed, having in view principally to accomplifh abfolute compreffion, in complete imitation of the natural procefs. In this purfuit, I did not confine myfelf to water, but made ufe of various other volatile fubftances, in order to affift compreffion; namely, carbonate of ammonia, nitrate of ammonia, gunpowder, and paper impregnated with nitre. With thefe I obtained fome good refults, but none fuch as to induce me to prefer any of thefe compreffors to water. Indeed, I am convinced, that water is fuperior to them all. I found, in feveral experiments, made with a fimple air-tube, without any artificial compreffor, in which a very low red-heat had been applied, that the carbonate loft one or one and a half *per cent*. Now, as this muft have happened in a temperature fcarcely capable of inflaming gunpowder, it is clear, that fuch lofs would not have been prevented by its prefence: whereas water, beginning far below rednefs to affume a gafeous form, will effectually refift any calcination, in low as well as in high heats. And as the quantity of water can very eafily be regulated by weight, its employment for this purpofe feems liable to no objection.

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On the 2d of January 1804, I made an experiment with marble and chalk, with the addition of 1.1 grain of water. I aimed at a low heat, and the pyrometer, though a little broken, ſeemed clearly to indicate 22°. Unluckily, the muzzle of the large tube, which was cloſed as uſual with chalk, was placed uppermoſt, and expoſed to the ſtrongeſt heat. I found it rounded by fuſion, and in a frothy ſtate. The little tube came out very clean, and was ſo nearly of the ſame weight as when put in, that its contents had loſt but 0.074 *per cent.* of the weight of the original carbonate. The marble was but feebly agglutinated, but the chalk was in a ſtate of firm limeſtone, though it muſt have undergone a heat under 22°, or that of melting ſilver. This experiment is certainly a moſt remarkable one, ſince a heat has been applied, in which the chalk has been changed to hard limeſtone, with a loſs leſs than the 1000dth par of its weight, (exactly $\frac{1}{1351}$) ; while, under the ſame circumſtances of preſſure, though probably with more heat, ſome of the ſame ſubſtance had been brought to fuſion. What loſs of weight this fuſed part ſuſtained, cannot be known.

On the 4th of January, a ſimilar experiment was made, likewiſe with 1.1 grain of water. The diſcharge of the metal was accompanied with a flaſh of flame. The pyrometer indicated 26°. The little tube came out quite clean. Its contents had been reduced from 14.53 to 14.46, difference 0.07 grains, being 0.47 *per cent.* on the original carbonate, leſs than one two-hundredth part of the original weight, (exactly $\frac{1}{212}$). The chalk was in a ſtate of firm ſaline marble, but with no unuſual qualities.

These two laſt experiments are rendered ſtill more intereſting, by another ſet which I made ſoon after, which ſhewed, that one eſſential precaution in a point of ſuch nicety had been neglected, in not previouſly drying the carbonate. In ſeveral trials made in the latter end of the ſame month,

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I found, that chalk expoſed to a heat above that of boiling water, but quite ſhort of redneſs, loſt 0.34 *per cent.*; and in another ſimilar trial, 0.46 *per cent.* Now, this loſs of weight equals within 0.01 *per cent.* the loſs in the laſt-mentioned experiment, that being 0.47; and far ſurpaſſes that of the laſt but one, which was but 0.074. There is good reaſon, therefore, to believe, that had the carbonate, in theſe two laſt experiments, been previouſly dried, it would have been found during compreſſion to have undergone no loſs.

THE reſult of many of the experiments lately mentioned, ſeems fully to explain the perplexing diſcordance between my experiments with porcelain tubes, and thoſe made in barrels of iron. With the procelain tubes, I never could ſucceed in a heat above 28°, or even quite up to it; yet the reſults were often excellent. Whereas, the iron-barrels have currently ſtood firm in heats of 41° or 51°, and have reached even to 70° or 80° without injury. At the ſame time, the reſults, even in thoſe high heats, were often inferior, in point of fuſion, to thoſe obtained by low heats in porcelain. The reaſon of this now plainly appears. In the iron-barrels it has always been conſidered as neceſſary to uſe an air-tube, in conſequence of which, ſome of the carbonic acid has been ſeparated from the earthy baſis by internal calcination: what carbonic acid remained, has been more forcibly attracted, according to M. BERTHOLLET's principle, and, of courſe, more eaſily compreſſed, than when of quantity ſufficient to ſaturate the lime: but, owing to the diminiſhed quantity of the acid, the compound has become leſs fuſible than in the natural ſtate, and, of courſe, has undergone a higher heat with leſs effect. The introduction of water, by furniſhing a reacting force, has produced a ſtate of things ſimilar to that in the porcelain tubes; the carbonate ſuſtaining little or no loſs of weight,

weight, and the compound retaining its fuſibility in low heats *.

In the early part of 1804, ſome experiments were made with barrels, which I wiſhed to try, with a view to another ſeries of experiments. The reſults were too intereſting to be paſſed over; for, though the carbonic acid in them was far from being completely conſtrained, they afforded ſome of the fineſt examples I had obtained, of the fuſion of the carbonate, and of its union with ſilex.

On the 13th of February, an experiment was made with pounded oyſter-ſhell, in a heat of 33°, without any water being introduced to aſſiſt compreſſion. The loſs was apparently of 12 *per cent.* The ſubſtance of the ſhell had evidently been in viſcid fuſion: it was porous, ſemitranſparent, ſhining in ſurface and fracture; in moſt parts with the gloſs of fuſion, in many others with facettes of cryſtallization. The little tube had been ſet with its muzzle upwards; over it, as uſual, lay a fragment of porcelain, and on that a round maſs of chalk. At the contact of the porcelain and the chalk, they had run together, and the chalk had been evidently in a very ſoft ſtate; for, reſting with its weight on the porcelain, this laſt had been preſſed into the ſubſtance of the chalk, deeper than its own breadth, a rim of chalk being viſible without the ſurface of the porcelain; juſt as when the round end of a knife is preſſed upon

* The retentive power here aſcribed to the procelain tubes, ſeems not to accord with what was formerly mentioned, of the carbonic acid having been driven through the ſubſtance of the tube. But the loſs by this means has probably been ſo ſmall, that the native properties of the carbonate have not been ſenſibly changed. Or, perhaps, this penetrability may not be ſo univerſal as I have been induced to think, by having met with it in all the caſes which I tried. In this doubt, I ſtrenuouſly recommend a further examination of this ſubject to gentlemen who have eaſy acceſs to ſuch procelains as that of Dreſden or of Seve.

upon a piece of ſoft butter. The carbonate had ſpread very much on the inſide of the tube, and had riſen round its lip, as ſome ſalts riſe from their ſolution in water. In this manner, a ſmall quantity of the carbonate had reached the outer tube, and had adhered to it. The black colour frequently mentioned as accompanying the union of the carbonates with the porcelain, is here very remarkable.

On the 26th of February, I made an experiment, in which the carbonate was not weighed, and no foreign ſubſtance was introduced to aſſiſt the compreſſion. The temperature was 46°. The pyrometer had been affected by the contact of a piece of chalk, with which it had united; and ſome of the carbonate muſt have penetrated the ſubſtance of the pyrometer, ſince this laſt had viſibly yielded to preſſure, as appeared by a ſwelling near the contact. I obſerved in theſe experiments, that the carbonate had a powerful action on the tubes of Corniſh clay, more than on the pounded ſilex. Perhaps it has a peculiar affinity for argil, and this may lead to important conſequences. The chalk had viſibly firſt ſhrunk upon itſelf, ſo as to be detached from the ſides, and had then begun to run by ſucceſſive portions, ſo as ſtill to leave a pillar in the middle, very irregularly worn away; indicating a ſucceſſive liquefaction, like that of ice, not the yielding of a maſs ſoftening all at once.

On the 28th of February, I made an experiment with oyſter-ſhell unweighed, finely ground, and paſſed through the cloſeſt ſieves. The pyrometer gave 40°. The piece of chalk below it had been ſo ſoft, as to ſink to the depth of half an inch into the mouth of the iron air-tube, taking its impreſſion completely. A ſmall part of this lump was contaminated with iron, but the reſt was in a fine ſtate. The tube had a rent in it, through which the carbonate, united with the matter of the tube, had flowed in two or three places. The ſhell

ſhell had ſhrunk upon itſelf, ſo as to ſtand detached from the ſides, and bore very ſtrong marks of fuſion. The external ſurface was quite ſmooth, and ſhining like an enamel. The internal part conſiſted of a mixture of large bubbles and ſolid parts: the inſide of the bubbles had a luſtre much ſuperior to that of the outſide, and equal to that of glaſs. The general maſs was ſemitranſparent; but ſmall parts were viſible by the lens, which were completely tranſparent and colourleſs. In ſeveral places this ſmooth ſurface had cryſtallized, ſo as to preſent brilliant facettes, ſteadily ſhining in certain aſpects. I obſerved one of theſe facettes on the inſide of an air-bubble, in which it interrupted the ſpherical form as if the little ſphere had been preſſed inwards at that ſpot, by the contact of a plane ſurface. In ſome chalk near the mouth of the large tube, which lay upon a ſtratum of ſilex, another very intereſting circumſtance occurred. Connected with its lower end, a ſubſtance was viſible, which had undoubtedly reſulted from the union of the carbonate with the ſilex. This ſubſtance was white and ſemitranſparent, and bore the appearance of chalcedony. The maſs of chalk having attached itſelf to that above it, had ſhrunk upwards, leaving an interval between it and the ſilex, and carrying ſome of the compound up with it. From thence this laſt had been in the act of dropping in a viſcid ſtate of fuſion, as evidently appeared when the ſpecimen was entire; having a ſtalactite and ſtalagmite correſponding accurately to each other. Unluckily I broke off the ſtalactite, but the ſtalagmite continues entire, in the form of a little cone. This new ſubſtance effervefced in acid, but not briſkly. I watched its entire ſolution; a ſet of light clouds remained undiſſolved, and probably ſome jelly was formed; for I obſerved, that a ſeries of air-bubbles remained in the form of the fragment, and moved together without any viſible connection; thus ſeeming to indicate a chemical union be-

 tween

tween the ſilex and the carbonate. The ſhell, fuſed in the experiment, diſſolved entirely in the acid, with violent effervescence.

In the three laſt experiments, and in ſeveral others made at the ſame time, the carbonate had not been weighed; but no water being introduced to aſſiſt the compreſſion, it is probable there was much loſs by internal calcination; and owing doubtleſs to this, the carbonates have crumbled almoſt entirely to duſt, while the compounds which they had formed with ſilex remain entire.

On the 13th of March, I made a ſimilar experiment, in which, beſides ſome pounded oyſter-ſhell, I introduced a mixture of chalk, with 10 *per cent.* of ſilex intermixed, and ground together in a mortar with water, in a ſtate of cream, and then well dried. The contents of the tube when opened, were diſcharged with ſuch violence, that the tube was broken to pieces; but I found a lump of chalk, then in a ſtate of white marble, welded to the compound; which laſt, in its fracture, ſhewed that irregular black colour, interſperſed roughly through a cryſtalline maſs, that belongs to the alpine marbles, particularly to the kind called at Rome *Cipolline*. It was very hard and firm; I think unuſually ſo. It effervesced conſtantly to the laſt atom, in diluted nitric acid, but much more ſluggiſhly than the marble made of pure chalk. A cloudineſs appeared pervading all the liquid. When the effervescence was over, a ſeries of bubbles continued during the whole day in the acid, without any diſpoſition to burſt, or riſe to the ſurface. After ſtanding all next day and night, they maintained their ſtation; and the ſolution being ſtirred, was found to be entirely agglutinated into a tranſparent jelly, breaking with ſharp angles. This experiment affords a direct and poſitive proof of a chemical union having taken place between the carbonate and ſilex.

VI.

VI.

Experiments made in Platina,—with Spar,—with Shells,—and with Carbonate of Lime of undoubted purity.

SINCE I had the honour of laying before this Society a ſhort ſketch of the foregoing experiments, on the 30th of Auguſt laſt (1804), many chemiſts and mineralogiſts of eminence have favoured me with ſome obſervations on the ſubject, and have ſuggeſted doubts which I am anxious to remove. It has been ſuggeſted, that the fuſibility of the carbonates may have been the conſequence of a mixture of other ſubſtances, either originally exiſting in the natural carbonate, or added to it by the contact of the porcelain tube.

WITH regard to the firſt of theſe ſurmiſes, I beg leave to obſerve, that, granting this cauſe of fuſion to have been the real one, a material point, perhaps all that is ſtrictly neceſſary in order to maintain this part of the Huttonian Theory, was nevertheleſs gained. For, granting that our carbonates were impure, and that their impurity rendered them fuſible, ſtill the ſame is true of almoſt every natural carbonate; ſo that our experimenats were, in that reſpect, conformable to nature. And as to the other ſurmiſe, it has been ſhewn, by comparing together a varied ſeries of experiments, that the mutual action between the lime and the porcelain was occaſioned entirely by the preſence of the carbonic acid, ſince, when it was abſent, no action of this kind took place. The fuſion of our carbonates cannot, therefore, be aſcribed to the porcelain.

BEING convinced, however, by many obſervations, that the fuſibility of the carbonate did not depend upon impurity, I have exerted myſelf to remove, by freſh experiments, every doubt that has ariſen on the ſubject. In order to guard againſt natural impurities, I have applied to ſuch of my friends as have turned their attention to chemical analyſis, (a branch of the ſcience to which I have never attended,) to furniſh me with carbonate of lime of undoubted purity. To obviate the contamination ariſing from the contact of the porcelain tubes, I determined to confine the ſubject of experiment in ſome ſubſtance which had no diſpoſition to unite with the carbonate. I firſt tried charcoal, but found it very troubleſome, owing to its irregular abſorption of water and air.

I THEN turned my thoughts to the conſtruction of tubes or cups of platina for that purpoſe. Being unable readily to procure proper ſolid veſſels of this ſubſtance, I made uſe of thin laminated plates, formed into cups. My firſt method was, to fold the plate exactly as we do blotting-paper to form a filter (Fig. 26.); this produced a cup capable of holding the thinneſt liquid; and being covered with a lid, formed of a ſimilar thin plate, bent at the edges, ſo as to overlap conſiderably (Fig. 28.), the carbonate it contained was ſecured on all ſides from the contact of the porcelain tube within which it was placed. Another convenient device likewiſe occurred: I wrapt a piece of the plate of platina round a cylinder, ſo as to form a tube, each end of which was cloſed by a cover like that juſt deſcribed (Fig. 27. and 29). (In figure 26. and 27. theſe cups are repreſented upon a large ſcale, and in 28. and 29. nearly of their actual ſize). This laſt conſtruction had the advantage of containing eight or nine grains of carbonate, whereas the other would only hold about a grain and a half. On the other hand, it was not fit to retain a thin liquid; but, in moſt caſes, that circumſtance was of no conſequence; and I foreſaw that the carbonates could not

thus

thus eſcape without proving the main point under conſideration, namely, their fuſion.

The reſt of the apparatus was arranged in all reſpects as formerly deſcribed, the ſame precautions being taken to defend the platina veſſel as had been uſed with the inner tubes of porcelain.

In this manner I have made a number of experiments during this ſpring and ſummer, the reſult of which is highly ſatisfactory. They prove, in the firſt place, the propriety of the obſervations which led to this trial, by ſhewing, that the pure carbonate, thus defended from any contamination, is decidedly more refractory than chalk; ſince, in many experiments, the chalk has been reduced to a ſtate of marble, while the pure carbonate, confined in the platina veſſel, has been but very feebly acted upon, having only acquired the induration of a ſandſtone.

In other experiments, however, I have been more ſucceſsful, having obtained ſome reſults, worthy, I think, of the attention of this Society, and which I ſhall now ſubmit to their inſpection. The ſpecimens are all incloſed, for ſafety, in glaſs tubes, and ſupported on little ſtands of wax, (fig. 31, 32, 33.). The ſpecimens have, in general, been removed from the cup or tube of platina in which they were formed, theſe devices having the advantage of ſecuring both the veſſel and its contents, by enabling us to unwrap the folds without violence; whereas, in a ſolid cup or tube, it would have been difficult, after the experiment, to avoid the deſtruction either of the veſſel or its contents, or both.

April 16. 1805.—An experiment was made with pure calcareous ſpar from St Gothard, remarkably tranſparent, and having a ſtrong double refraction. A temperature of 40° was applied; but owing to ſome accident, the weight was not known. The conical cup came out clean and entire, filled

not

not quite to the brim with a yellowiſh-grey ſubſtance, having a ſhining ſurface, with longitudinal ſtreaks, as we ſometimes ſee on glaſs. This ſurface was here and there interrupted by little white tufts or protuberances, diſpoſed irregularly. On the ledge of the cup, formed by the ends of the folded platina, were ſeveral globular drops like minute pearls, viſible to the naked eye, the number of which amounted to ſixteen. Theſe ſeem to have been formed by the entire fuſion of what carbonate happened to lie on the ledge, or had been entangled amongſt the extremities of the folds, drawing itſelf together, and uniting in drops; as we ſee when any ſubſtance melts under the blowpipe. This reſult is preſerved entire, without deranging the tube. I am ſorry to find that it has begun to fall to decay, in conſequence, no doubt, of too great a loſs of its carbonic acid. But the globules do not ſeem as yet to have ſuffered any injury.

April 25.—The ſame ſpar was uſed, with two grains of water, and a heat of 33°. I have reaſon to ſuſpect, however, that, in this and ſeveral other experiments made at this time, the metal into which the cradle was plunged, on firſt introduction into the barrel, had been too hot, ſo as to drive off the water. There was a loſs of 6.4 *per cent.* The reſult lay in the cup without any appearance of frothing or ſwelling. The ſurface was of a clean white, but rough, having in one corner a ſpace ſhining like glaſs. The cup being unwrapt, the ſubſtance was obtained ſound and entire: where it had moulded itſelf on the platina, it had a ſmall degree of luſtre, with the irregular ſemitranſparency of ſaline marble: when broken, it preſerved that character more completely than in any reſult hitherto obtained; the fracture being very irregular and angular, and ſhining with facettes in various directions. I much regret that this beautiful ſpecimen no longer exiſts, having crumbled entirely to pieces, notwithſtanding all the care I took to incloſe it with glaſs and wax.

April 26.—An experiment was made with ſome carbonate of lime, purified by my friend Sir George Mackenzie. Two grains of water were introduced, but were loſt, I ſuſpect, as in the laſt caſe. The heat applied was 32°. The loſs of weight was 10.6 *per cent.* Yet, though made but one day after the laſt-mentioned ſpecimen, it remains as freſh and entire as at firſt, and promiſes to continue unchanged. The external ſurface, as ſeen on removing the lid of the conical cup, was found to ſhine all over like glaſs, except round the edges, which were fringed with a ſeries of white and rough ſphericles, one ſet of which advanced, at one ſpot, near to the centre. The ſhining ſurface was compoſed of planes, which formed obtuſe angles together, and had their ſurface ſtriated; the ſtriæ bearing every appearance of a cryſtalline arrangement. When freed from the cup, as before, the ſubſtance moulded on the platina was found to have aſſumed a fine pearly ſurface. Some large air-bubbles appeared, which had adhered to the cup, and were laid open by its removal, whoſe internal ſurface had a beautiful luſtre, and was full of ſtriæ like the outward ſurface. The maſs is remarkable for ſemitranſparency, as ſeen particularly where the air-bubbles diminiſh its thickneſs: a ſmall part of the maſs being broken at one end, ſhews an internal ſaline ſtructure.

April 29.—A cup of platina was filled with ſeveral large pieces of a periwinkle* ſhell, the ſharp point of the ſpiral being made to ſtand upright in the cup, (fig. 30.). A heat of 30° was applied, and no water was introduced. The carbonate loſt no leſs than 16 *per cent.* The ſhell, particularly [illegible] the

* Turbo terebra, Lin.

the ſharp end of the periwinkle, retained its original ſhape in a great meaſure, ſo as to be quite diſcernible; but the whole was glazed over with a truly vitreous luſtre. This glaze covered, at one place, a fragment of the ſhell which had been originally looſe, and had welded the two together. All the angles are rounded by this vitrifaction; the ſpace between the entire ſhell and the fragment being filled, and the angles of their meeting rounded, with this ſhining ſubſtance. The colour is a pale blue, contraſted, in the ſame little glaſs, with a natural piece of periwinkle, which is of a reddiſh-yellow. One of the fragments had adhered to the lid, and had been converted into a complete drop, of the ſize of a muſtard-ſeed. It is fixed on the wax (at *b*), along with the other ſpecimens of the experiment (fig. 32.). This reſult ſhews, as yet, no ſign of decay, notwithſtanding ſo great a loſs of weight.

THE laſt experiment was repeated on the ſame day, and prepared in the ſame manner, with large fragments of ſhell, and the point of the periwinkle ſtanding up in the cup. A heat of 34° was applied; a loſs took place of 13° *per cent.* All the original form had diſappeared, the carbonate lying in the cup as a complete liquid, with a concave ſurface, which did not ſhine, but was ſtudded all over with the white ſphericles or tufts, like thoſe ſeen in the former reſults, without any ſpace between them. When detached from the cup, the ſurface moulded on the platina, was white and pearly, with a ſlight gloſs. The maſs was quite ſolid; no veſtige whatever appearing, of the original form of the fragments, (fig. 33.). A ſmall piece, broken off near the apex of the cone, ſhewed the internal ſtructure to be quite ſaline. In the act of arranging the ſpecimen on its ſtand, another piece came off in a new direction, which preſented to view the moſt perfect cryſtalline arrangement: the ſhining plane extended acroſs the whole ſpecimen, and was more than the tenth of an inch in all directions. This fracture, likewiſe,

likewiſe, ſhewed the entire internal ſolidity of the maſs. Unfortunately, this ſpecimen has ſuffered much by the ſame decay to which all of them are ſubject which have loſt any conſiderable weight. The part next the outward ſurface alone remains entire. I have never been able to explain, in a ſatisfactory manner, this difference of durability; the laſt-mentioned reſult having loſt more in proportion to its weight than this.

About the beginning of June, I received from Mr Hatchett ſome pure carbonate of lime, which he was ſo good as to prepare, with a view to my experiments; and I have been conſtantly employed with it till within theſe few days.

My firſt experiments with this ſubſtance were peculiarly unfortunate, and it ſeemed to be leſs eaſily acted upon than any ſubſtance of the kind I had tried. Its extreme purity, no doubt, contributed much to this, though another circumſtance had likewiſe had ſome effect. The powder, owing to a cryſtallization which had taken place on its precipitation, was very coarſe, and little ſuſceptible of cloſe ramming; the particles, therefore, had leſs advantage than when a fine powder is uſed, in acting upon each other, and I did not chooſe to run any riſk of contamination, by reducing the ſubſtance to a finer powder. Whatever be the cauſe, it is certain, that in many experiments in which the chalk was changed to marble, this ſubſtance remained in a looſe and brittle ſtate, though conſiſting generally of clear and ſhining particles. I at laſt, however, ſucceeded in obtaining ſome very good reſults with this carbonate.

In an experiment made with it on the 18th of June, in a ſtrong heat, I obtained a very firm maſs with a ſaline fracture, moulded in ſeveral places on the platina, which was now uſed in the cylindrical form. On the 23d, in a ſimilar experiment, the barrel failed, and the ſubject of experiment was found in an entire ſtate of froth, proving its former fluidity.

 On

On the 25th, in a ſimilar experiment, a heat of 64° was applied without any water within the barrel. The platina tube, (having been contaminated in a former experiment with ſome fuſible metal), melted, and the carbonate retaining its cylindrical ſhape, had fallen through it, ſo as to touch the piece of porcelain which had been placed next to the platina tube. At the point of contact, the two had run together, as a hot iron runs when touched by ſulphur. The carbonate itſelf was very tranſparent, reſembling a piece of ſnow in the act of melting.

On the 26th of June, I made an experiment with this carbonate, which afforded a beautiful reſult. One grain of water was introduced with great care; yet there was a loſs of 6.5 *per cent.*, and the reſult has fallen to decay. The pyrometer indicated 43°. On the outſide of the platina cylinder, and on one of the lids, were ſeen a ſet of globules, like pearls, as once before obtained, denoting perfect fuſion. When the upper lid was removed, the ſubſtance was found to have ſunk almoſt out of ſight, and had aſſumed a form not eaſily deſcribed. (I have endeavoured to repreſent it in fig. 31. by an ideal ſection of the platina-tube and its contents, made through the axis of the cylinder). The powder, firſt ſhrinking upon itſelf in the act of agglutination, had formed a cylindrical rod, a remnant of which (*a b c*) ſtood up in the middle of the tube. By the continued action of heat, the ſummit of the rod (at *a*) had been rounded in fuſion, and the maſs being now ſoftened, had ſunk by its weight, and ſpread below, ſo as to mould itſelf in the tube, and fill its lower part completely (*d f g e*). At the ſame time, the viſcid fluid adhering to the ſides (at *e* and *d*), while the middle part was ſinking, had been in part left behind, and in part drawn out into a thin but tapering ſhape, united by a curved ſurface (at *b* and *c*) to the middle rod. When the platina tube was unwrapt, the thin edges (at *e* and *d*) were preſerved all round, and in a ſtate ftate of beautiful femitranfparency. (I have attempted to reprefent the entire fpecimen, as it ftood on its cone of wax, in fig. 34.). The carbonate, where moulded on the platina, had a clean pearly whitenefs, with a faline appearance externally, and in the fun, fhone with facettes. Its furface was interrupted by a few fcattered air-bubbles, which had lain againft the tube. The intervening fubftance was unufually compact and hard under the knife. The whole furface (*e b a c d*, fig. 31.), and the infide of the air-bubbles, had a vitreous luftre. Thus, every thing denoted a ftate of vifcid fluidity, like that of honey.

THESE laft experiments feem to obviate every doubt that remained with refpect to the fufibility of the pureft carbonate, without the affiftance of any foreign fubftance.

VII.

Meafurement of the Force required to conftrain the Carbonic Acid.—Apparatus with the Muzzle of the Barrel upwards, and the weight acting by a long Lever.—Apparatus with the Muzzle downwards.—Apparatus with Weight acting directly on the barrel.—Comparifon of various refults.

IN order to determine, within certain limits at leaft, what force had been exerted in the foregoing experiments, and what was neceffary to enfure their fuccefs, I made a number of experiments, in a mode nearly allied to that followed by Count RUMFORD, in meafuring the explofive force of gunpowder.

I BEGAN to ufe the following fimple apparatus in June 1803. I took one of the barrels, made as above defcribed, for the purpofe of compreffion, having a bore of 0.75 of an

 inch,

inch *, and dreſſed its muzzle to a ſharp edge. To this barrel was firmly ſcrewed a collar of iron (*a a*, fig. 36.) placed at a diſtance of about three inches from the muzzle, having two ſtrong bars (*b b*) projecting at right angles to the barrel, and dreſſed ſquare. The barrel, thus prepared, was introduced, with its breech downwards, into the vertical muffle (fig. 35.); its length being ſo adjuſted, that its breech ſhould be placed in the ſtrongeſt heat; the two projecting bars above deſcribed, reſting on two other bars (*c c*, fig. 35.) laid upon the furnace to receive them; one upon each ſide of the muffle. Into the barrel, ſo placed, was introduced a cradle, containing carbonate, with all the arrangements formerly mentioned; the rod connected with it being of ſuch length, as juſt to lie within the muzzle of the barrel. The liquid metal was then poured in till it filled the barrel, and ſtood at the muzzle with a convex ſurface; a cylinder of iron, of about an inch in diameter, and half an inch thick, was laid on the muzzle (fig. 35. and 37.), and to it a compreſſing weight was inſtantly applied. This was firſt done by the preſſure of a bar of iron (*d e*, fig. 35.), three feet in length, introduced looſely into a hole (*d*), made for the purpoſe in the wall againſt which the furnace ſtood; the diſtance between this hole and the barrel being one foot. A weight was then ſuſpended at the extremity of the bar (*e*), and thus a compreſſing force was applied, equal to three times that weight. In the courſe of practice, a cylinder of lead was ſubſtituted for that of iron, and a piece of leather was placed between it and the muzzle of the barrel, which laſt being dreſſed to a pretty ſharp edge, made an impreſſion in the lead: to aſſiſt this effect, one ſmart blow of a hammer was ſtruck upon the bar, directly over the barrel, as ſoon as the weight had been hung on.

It

* This was the ſize of barrel uſed in all the following experiments, where the fact is not otherwiſe expreſſed.

It was essential, in this mode of operation, that the whole of the metal should continue in a liquid state during the action of heat; but when I was satisfied as to its intensity and duration, I congealed the metal, either by extinguishing the furnace entirely, or by pouring water on the barrel. As soon as the heat began to act, drops of metal were seen to force themselves between the barrel and the leather, following each other with more or less rapidity, according to circumstances. In some experiments, there was little exudation; but few of them were entirely free from it. To save the metal thus extruded, I placed a black-lead crucible, having its bottom perforated, round the barrel, and luted close to it, (fig. 37.); some sand being laid in this crucible, the metal was collected on its surface. On some occasions, a sound of ebullition was heard during the action of heat; but this was a certain sign of failure.

The results of the most important of these experiments, have been reduced to a common standard in the second table placed in the Appendix; to which reference is made by the following numbers.

No. 1.—On the 16th of June 1803, I made an experiment with these arrangements. I had tried to use a weight of 30 lb. producing a pressure of 90 lb., but I found this not sufficient. I then hung on a weight of 1 cwt., or 112 lb.; by which a compressing force was applied of 3 cwt. or 336 lb. Very little metal was seen to escape, and no sound of ebullition was heard. The chalk in the body of the large tube was reduced to quicklime; but what lay in the inner tube was pretty firm, and effervesced to the last. One or two facettes, of good appearance, were likewise found. The contents of the small tube had lost but 2.6 *per cent.*; but there was a small visible intrusion of metal, and the result, by its appearance, indicated a greater loss. I considered this, however, as one point gained; that being the

first

firſt tolerable compreſſion accompliſhed by a determinate force. The pyrometer indicated 22°.

This experiment was repeated the ſame day, when a ſtill ſmaller quantity of metal eſcaped at the muzzle; but the barrel had given way below, in the manner of thoſe that have yielded for want of ſufficient air. Even this reſult was ſatisfactory, by ſhewing that a mechanical power, capable of forcing ſome of the barrels, could now be commanded. The carbonate in the little tube had loſt 20 *per cent.;* but part of it was in a hard and firm ſtate, efferveſcing to the laſt.

No. 2.—On the 21ſt June, I made an experiment with another barrel, with the ſame circumſtances. I had left an empty ſpace in the large tube, and had intended to introduce its muzzle downwards, meaning that ſpace to anſwer as an air-tube; but it was inverted by miſtake, and the tube entering with its muzzle upwards, the empty ſpace had of courſe filled with metal, and thus the experiment was made without any included air. There was no pyrometer uſed; but the heat was gueſſed to be about 25° where the ſubject of experiment lay. The barrel, when opened, was found full of metal, and the cradle being laid flat on the table, a conſiderable quantity of metal ran from it, which had undoubtedly been lodged in the vacuity of the large tube. When cold, I found that vacuity ſtill empty, with a plating of metal. The tube was very clean to appearance, and, when ſhaken, its contents were heard to rattle. Above the little tube, and the cylinder of chalk, I had put ſome borax and ſand, with a little pure borax in the middle, and chalk over it. The metal had not penetrated beyond the borax and ſand, by a good fortune peculiar to this experiment; the intruſion of metal in this mode of execution, being extremely troubleſome. The button of chalk, was found in a ſtate of clean white carbonate, and pretty hard, but without tranſparency. The little tube

tube was perfectly clean. Its weight with its contents, feemed to have fuffered no change from what it had been when firft introduced. Attending, however, to the balance with fcrupulous nicety, a fmall preponderance did appear on the fide of the weight. This was done away by the addition of the hundredth of a grain to the fcale in which the carbonate lay, and an addition of another hundredth produced in it a decided preponderance. Perhaps, had the tube, before its introduction, been examined with the fame care, as great a difference might have been detected; and it feems as if there had been no lofs, at leaft not more than one hundredth of a grain, which on 10.95 grains, amounts to 0.0912, fay 0.1 *per cent.* The carbonate was loofe in the little tube, and fell out by fhaking. It had a yellow colour, and compact appearance, with a ftony hardnefs under the knife, and a ftony fracture; but with very flight facettes, and little or no tranfparency. In fome parts of the fpecimen, a whitifh colour feemed to indicate partial calcination. On examining the fracture, I perceived, with the magnifier, a fmall globule of metal, not vifible to the naked eye, quite infulated and fingle. Poffibly the fubftance may have contained others of the fame fort, which may have compenfated for a fmall lofs, but there could not be many fuch, from the general clean appearance of the whole. In the fracture, I faw here and there fmall round holes, feeming, though imperfectly, to indicate a beginning of ebullition.

I MADE a number of experiments in the fame manner, that is to fay, with the muzzle of the barrel upwards, in fome of which I obtained very fatisfactory refults; but it was only by chance that the fubftance efcaped the contamination of the fufible metal; which induced me to think of another mode of applying the compreffing weight with the muzzle of the barrel downwards, by which I expected to repeat, with a determinate weight, all the experiments formerly

made

made in barrels clofed by congealed metal; and that, by making ufe of an air-tube, the air, rifing to the breech, would fecure the contents of the tube from any contamination. In this view, the barrel was introduced from below into the muffle with its breech upwards, and retained in that pofition by means of a hook fixed to the furnace, till the collar was made to prefs up againft the grate, by an iron lever, loaded with a weight, and refting on a fupport placed in front. In fome experiments made in this way, the refult was obtained very clean, as had been expected; but the force had been too feeble, and when it was increafed, the furnace yielded upwards by the mechanical ftrain.

I FOUND it therefore neceffary to ufe a frame of iron, (as in fig. 38.; the frame being reprefented feparately in fig. 39.), by which the brick-work was relieved from the mechanical ftrain. This frame confifted of two bars *(a b* and *f e*, figs. 38. and 39.), fixed into the wall, (at *a* and *f*,) paffing horizontally under the furnace, one on each fide of the muffle, turning downwards at the front, (in *b* and *e*), and meeting at the ground, with a flat bar *(c d)* uniting the whole. In this manner, a kind of ftirrup (*b c d e*) was formed in front of the furnace, upon the crofs bar *(c d)* of which a block of wood (*b b*, fig. 38.), was placed, fupporting an edge of iron, upon which the lever refted; the working end of the lever *(g)* acting upwards. A ftrain was exerted, by means of the barrel and its collar, againft the horizontal bars, *(a b* and *f e)*, which was effectually refifted by the wall (at *a* and *f)* at one end of thefe bars, and by the upright bars (*c b* and *d e)* at the other end. In this manner the whole ftrain was fuftained by the frame, and the furnace ftood without injury.

THE iron bar, at its working end, was formed into the fhape of a cup, (at *g*), and half filled with lead, the fmooth furface of which, was applied to the muzzle of the barrel. The lever, too, was lengthened, by joining to the bar of iron, a beam of wood, making

making the whole ten feet in length. In this manner, a preſſure upwards was applied to the barrel, equal to the weight of 10 cwt.

In the former method, in which the barrel ſtood with its muzzle upwards, the weight was applied while the metal was liquid. In this caſe, it was neceſſary to let it previouſly congeal, otherwiſe the contents would have run out in placing the barrel in the muffle, and to allow the liquefaction eſſential to theſe trials, to be produced by the propagation of heat from the muffle downwards. This method required, therefore, in every caſe, the uſe of an air-tube; for without it, the heat acting upon the breech, while the metal at the muzzle was ſtill cold, would infallibly have deſtroyed the barrel. A great number of theſe experiments failed, with very conſiderable waſte of the fuſible metal, which, on theſe occaſions was nearly all loſt. But a few of them ſucceeded, and afforded very ſatisfactory reſults, which I ſhall now mention.

In November 1803, ſome good experiments were made in this way, all with a bore of 0.75, and a preſſure of 10 cwt.

No. 3.—On the 19th, a good limeſtone was obtained in an experiment made in a temperature of 21°, with a loſs of only 1.1 *per cent.*

No. 4.—On the 22d, in a ſimilar experiment, there was little exudation by the muzzle. The pyrometer gave 31°. The carbonate was in a porous, and almoſt frothy ſtate.

No. 5.—In a ſecond experiment, made the ſame day, the heat roſe to 37° or 41°. The ſubſtance bore ſtrong marks of fuſion, the upper part having ſpread on the little tube: the whole was very much ſhrunk, and run againſt one ſide. The maſs ſparkling and white, and in a very good ſtate.

No. 6.—On the 25th, an experiment was made with chalk, and ſome fragments of ſnail ſhell, with about half a grain of water. The heat had riſen to near 51° or 49°. The barrel had been held tight by the beam, but was rent and a little ſwelled at the breech. The rent was wide, and ſuch as has always appeared in the ſtrongeſt barrels when they failed. The carbonate was quite calcined, it had boiled over the little tube, and was entirely in a frothy ſtate, with large and diſtinctly rounded air-holes. The fragments of ſhell which had occupied the upper part of the little tube, had loſt every trace of their original ſhape in the act of ebullition and fuſion.

No. 7.—On the 26th a ſimilar experiment was made, in which the barrel was thrown open, in ſpite of this powerful compreſſing force, with a report like that of a gun, (as I was told, not having been preſent), and the bar was found in a ſtate of ſtrong vibration. The carbonate was calcined, and ſomewhat frothy, the heart of one piece of chalk uſed was in a ſtate of ſaline marble.

It now occurred to me to work with a compreſſing force, and no air-tube, truſting, as happened accidentally in one caſe, that the expanſion of the liquid would clear itſelf by gentle exudation, without injury to the carbonate. In this mode, it was neceſſary, for reaſons lately ſtated, to place the muzzle upwards. Various trials made thus, at this time, afforded no remarkable reſults. But I reſumed the method, with the following alteration in the application of the weight, on the 27th of April 1804.

I conceived that ſome inconvenience might ariſe from the mode of employing the weight in the former experiments. In them it had been applied at the end of the bar, and its effect propagated along it, ſo as to preſs againſt the barrel at its other extremity. It occurred to me, that the propagation of motion in this way, requiring ſome ſenſible time, a conſiderable quantity of carbonic acid might eſcape by a ſudden eruption, before that propagation had taken effect. I therefore thought, that more effectual work might be done, by placing

placing a heavy maſs, (fig. 40.), ſo as to act directly and ſimply upon the muzzle of the barrel; this maſs being guided and commanded by means of a powerful lever, *(a b)*. For this purpoſe, I procured an iron roller, weighing 3 cwt. 7 lb., and ſuſpended it over the furnace, to the end of a beam of wood, reſting on a ſupport near the furnace, with a long arm guided by a rope *(c c)* and pulley *(d)*, by which the weight could be raiſed or let down at pleaſure.

WITH this apparatus I made ſome tolerable experiments; but I found the weight too light to afford certain and ſteady reſults of the beſt quality. I therefore procured at the foundry a large maſs of iron *(f)*, intended, I believe, for driving piles, and which, after allowing for the counterpoiſe of the beam, gave a direct preſſure of 8.1 cwt.; and I could, at pleaſure, diminiſh the compreſſing force, by placing a bucket (*e*) at the extremity of the lever, into which I introduced weights, whoſe effect on the ultimate great maſs, was known by trial. Many barrels failed in theſe trials: at laſt, I obtained one of ſmall bore, inch 0,54, which gave two good reſults on the 22d of June 1804.

No. 8.—WISHING to aſcertain the leaſt compreſſing force by which the carbonate could be effectually conſtrained in melting heats, I firſt obſerved every thing ſtanding firm in a heat of above 20°; I then gradually threw weights into the bucket, till the compreſſing force was reduced to 2 cwt. Till then, things continued ſteady; but, on the preſſure being ſtill further diminiſhed, metal began to ooze out at the muzzle, with increaſing rapidity. When the preſſure was reduced to 1½ cwt. air ruſhed out with a hiſſing noiſe. I then ſtopped the experiment, by pouring water on the barrel. The piece of chalk had loſt 12 *per cent*. It was white and ſoft on the outſide, but firm and good in the heart.

No. 9.—An experiment was made with chalk, in a little tube; to this, one grain of water was added. I had intended to work with 4 cwt. only; but the barrel was no ſooner placed, than an exudation of metal began at the muzzle, owing, doubtleſs, to the elaſticity of the water. I immediately increaſed the preſſure to 8.1 cwt. by removing the weight from the bucket, when the exudation inſtantly ceaſed. I continued the fire for three quarters of an hour, during which time no exudation happened; then all came out remarkably clean, with ſcarcely any contamination of metal. The loſs amounted to 2.58 *per cent.* The ſubſtance was tolerably indurated, but had not acquired the character of a complete ſtone.

In theſe two laſt experiments, the bore being ſmall, a pyrometer could not be admitted.

On the 5th of July 1804, I made three very ſatisfactory experiments of this kind, in a barrel with the large bore of 0.75 of an inch.

No. 10.—was made with a compreſſing force of only 3 cwt. A ſmall eruption at the muzzle being obſerved, water was thrown on the barrel: the pyrometer gave 21°: the chalk was in a firm ſtate of limeſtone.

No. 11.—with 4 cwt. The barrel ſtood without any eruption or exudation, till the heat roſe to 25°. There was a loſs of 3.6 *per cent.*: the reſult was ſuperior, in hardneſs and tranſparency, to the laſt, having ſomewhat of a ſaline fracture.

No. 12.—with 5 cwt. The reſult, with a loſs of 2.4 *per cent.*, was of a quality ſuperior to any of thoſe lately obtained.

These experiments appear to anſwer the end propoſed, of aſcertaining the leaſt preſſure, and loweſt heat, in which limeſtone can be formed. The reſults, with various barrels of different ſizes, agree tolerably, and tend to confirm each other. The table ſhews, when we compare numbers 1, 2, 8, 10, 11, 12, That a preſſure of 52 atmoſpheres, or 1700 feet of ſea, is capable

capable of forming a limeſtone in a proper heat: That under 86 atmoſpheres, anſwering nearly to 3000 feet, or about half a mile, a complete marble may be formed: and laſtly, That with a preſſure of 173 atmoſpheres, or 5700 feet, that is, little more than one mile of ſea, the carbonate of lime is made to undergo complete fuſion, and to act powerfully on other earths.

VIII.

Formation of Coal.—Accidental occurrence which led me to undertake theſe Experiments.—Reſults extracted from a former publication.—Explanation of ſome difficulties that have been ſuggeſted.—The Fibres of Wood in ſome caſes obliterated, and in ſome preſerved under compreſſion.—Reſemblance which theſe Reſults bear to a ſeries of Natural Subſtances deſcribed by Mr HATCHETT.—*Theſe reſults ſeem to throw light on the hiſtory of Surturbrand.*

As I intend, on ſome future occaſion, to reſume my experiments with inflammable ſubſtances, which I look upon as far from complete, I ſhall add but a few obſervations to what I have already laid before this Society, in the ſketch I had the honour to read in this place on the 30th of Auguſt laſt.

THE following incidental occurrence led me to enter upon this ſubject rather prematurely, ſince I had determined firſt to ſatisfy myſelf with regard to the carbonate of lime.

OBSERVING, in many of the laſt-mentioned claſs of experiments, that the elaſtic matters made their eſcape between the muzzle of the barrel and the cylinder of lead, I was in the habit, as mentioned above, of placing a piece of leather between the lead and the barrel; in which poſition, the heat to which the leather was expoſed, was neceſſarily below that of melting lead.

lead. In an experiment, made on the 28th November 1803, in order to aſcertain the power of the machinery, and the quantity of metal driven out by the expanſion of the liquid, there being nothing in the barrel but metal, I obſerved, as ſoon as the compreſſing apparatus was removed, (which on this occaſion was done while the lower part of the barrel was at its full heat, and the barrel ſtanding brim full of liquid metal,) that all the leather which lay on the outſide of the circular muzzle of the barrel, remained, being only a little browned and crumpled by the heat to which it had been expoſed. What leather lay within the circle, had diſappeared; and, on the ſurface of the liquid metal, which ſtood up to the lip of the barrel, I ſaw large drops, of a ſhining black liquid, which, on cooling, fixed into a criſp black ſubſtance, with a ſhining fracture, exactly like pitch or pure coal. It burned, though not with flame. While hot, it ſmelt decidedly of volatile alkali. The important circumſtance here, is the different manner in which the heat has acted on the leather, without and within the rim of the barrel. The only difference conſiſted in compreſſion, to which, therefore, the difference of effect muſt be aſcribed: by its force, the volatile matter of the leather which eſcaped from the outward parts, had within the rim, been conſtrained to remain united to the reſt of the compoſition, upon which it had acted as a flux, and the whole together had entered into a liquid ſtate, in a very low heat. Had the preſſure been continued till all was cool, theſe ſubſtances muſt have been retained, producing a real coal.

On the 24th April 1803, a piece of leather uſed in a ſimilar manner, (the compreſſing force being continued, however, till all was cold,) was changed to a ſubſtance like glue, owing doubtleſs to compreſſion, in a heat under that of melting lead.

These obſervations led me to make a ſeries of experiments with animal and vegetable ſubſtances, and with coal; the

the result of which I have already laid before the Society. I shall now repeat that communication, as printed in NICHOLSON's *Journal* for October last (1804).

"I HAVE likewise made some experiments with coal, treated in the same manner as the carbonate of lime: but I have found it much less tractable; for the bitumen, when heat is applied to it, tends to escape by its simple elasticity, whereas the carbonic acid in marble, is in part retained by the chemical force of quicklime. I succeeded, however, in constraining the bituminous matter of the coal, to a certain degree, in red heats, so as to bring the substance into a complete fusion, and to retain its faculty of burning with flame. But, I could not accomplish this in heats capable of agglutinating the carbonate; for I have found, where I rammed them successively into the same tube, and where the vessel has withstood the expansive force, that the carbonate has been agglutinated into a good limestone, but that the coal has lost about half its weight, together with its power of giving flame when burnt, remaining in a very compact state, with a shining fracture. Although this experiment has not afforded the desired result, it answers another purpose admirably well. It is known, that where a bed of coal is crossed by a dike of whinstone, the coal is found in a peculiar state in the immediate neighbourhood of the whin: the substance in such places being incapable of giving flame, it is distinguished by the name of *blind coal*. Dr HUTTON has explained this fact, by supposing that the bituminous matter of the coal, has been driven by the local heat of whin, into places of less intensity, where it would probably be retained by distillation. Yet the whole must have been carried on under the action of a pressure capable of constraining the carbonic acid of the calcareous spar, which occurs frequently in such rocks. In the last-mentioned experiment, we have a perfect representation

tation of the natural fact; ſince the coal has loſt its petroleum, while the chalk in contact with it has retained its carbonic acid.

" I HAVE made ſome experiments of the ſame kind, with vegetable and animal ſubſtances. I found their volatility much greater than that of coal, and I was compelled, with them, to work in heats below redneſs; for, even in the loweſt red-heat, they were apt to deſtroy the apparatus. The animal ſubſtance I commonly uſed was horn, and the vegetable, ſaw-duſt of fir. The horn was incomparably the moſt fuſible and volatile of the two. In a very ſlight heat, it was converted into a yellow red ſubſtance, like oil, which penetrated the clay tubes through and through. In theſe experiments, I therefore made uſe of tubes of glaſs. It was only after a conſiderable portion of the ſubſtance had been ſeparated from the maſs, that the remainder aſſumed the clear black peculiar to coal. In this way I obtained coal, both from ſaw-duſt and from horn, which yielded a bright flame in burning.

" THE mixture of the two produced a ſubſtance having exactly the ſmell of ſoot or coal-tar. I am therefore ſtrongly inclined to believe, that animal ſubſtance, as well as vegetable, has contributed towards the formation of our bituminous ſtrata. This ſeems to confirm an opinion, advanced by Mr KEIR, which has been mentioned to me ſince I made this experiment. I conceive, that the coal which now remains in the world, is but a ſmall portion of the organic matter originally depoſited: the moſt volatile parts have been driven off by the action of heat, before the temperature had riſen high enough to bring the ſurrounding ſubſtance into fuſion, ſo as to confine the elaſtic fluids, and ſubject them to compreſſion.

" IN ſeveral of theſe experiments, I found that, when the preſſure was not great, when equal, for inſtance, only to 80 atmoſpheres, that the horn employed was diſſipated entirely, the

glaſs

glaſs tube which had contained it being left almoſt clean: yet undoubtedly, if expoſed to heat without compreſſion, and protected from the contact of the atmoſphere, the horn would leave a cinder or coak behind it, of matter wholly devoid of volatility. Here, then, it would ſeem as if the moderate preſſure, by keeping the elements of the ſubſtance together, had promoted the general volatility, without being ſtrong enough to reſiſt that expanſive force, and thus, that the whole had eſcaped. This reſult, which I ſhould certainly not have foreſeen in theory, may perhaps account for the abſence of coal in ſituations where its preſence might be expected on principles of general analogy."

SINCE this publication, a very natural queſtion has been put to me. When the inflammable ſubſtance has loſt weight, or when the whole has been diſſipated, in theſe experiments, what has become of the matter thus driven off?

I MUST own, that to anſwer this queſtion with perfect confidence, more experiments are required. But, in the courſe of practice, two circumſtances have occurred as likely, in moſt caſes, to have occaſioned the loſs alluded to. I found in theſe experiments, particularly with horn, that the chalk, both in powder and in lump, which was uſed to fill vacuities in the tubes, and to fix them in the cradle, was ſtrongly impregnated with an oily or bituminous matter, giving to the ſubſtance the qualities of a ſtinkſtone. I conceive, that the moſt volatile part of the horn has been conveyed to the chalk, partly in a ſtate of vapour, and partly by boiling over the lips of the glaſs tube; the whole having been evidently in a ſtate of very thin fluidity. Having, in ſome caſes, found the tube, which had been introduced full of horn, entirely empty after the experiment, I was induced, as above ſtated, to conceive, that, under preſſure, it had acquired a greater general volatility than it had in freedom; and I find that, in the open fire, horn yields a charcoal equal to 20 *per cent.* of the original weight. But more experiments muſt be made on this ſubject.

Another cauſe of the loſs of weight, lay undoubtedly in the exceſs of heat employed in moſt of them, to remove the cradle from the barrel. With inflammable ſubſtances, no air-tube was uſed, and the heats being low, the air lodged in interſtices had been ſufficient to ſecure the barrels from deſtruction, by the expanſion of the liquid metal. In this view, likewiſe, I often uſed lead, whoſe expanſion in ſuch low heats, I expected to be leſs than that of the fuſible metal. And the lead requiring to melt it, a heat very near to that of redneſs, the ſubject of experiment was thus, on removing the cradle, expoſed in freedom to a temperature which was comparatively high. But, obſerving that a great loſs was thus occaſioned, I returned to the uſe of the fuſible metal, together with my former method of melting it, by plunging the barrel, when removed from the furnace, into a ſolution of muriate of lime, by which it could only receive a heat of 250° of Fahrenheit.

The effect was remarkable, in the few experiments tried in this way. The horn did not, as in the other experiments, change to a hard black ſubſtance, but acquired a ſemifluid and viſcid conſiſtency, with a yellow-red colour, and a very offenſive ſmell. This ſhews, that the ſubſtances which here occaſioned both the colour and ſmell of the reſults, had been driven off in the other experiments, by the too great heat applied to the ſubſtance, when free from compreſſion.

I found that the organization of animal ſubſtance was entirely obliterated by a ſlight action of heat, but that a ſtronger heat was required to perform the entire fuſion of vegetable matter. This, however, was accompliſhed; and in ſeveral experiments, pieces of wood were changed to a jet-black and inflammable ſubſtance, generally very porous, in which no

trace

trace could be discovered of the original organization. In others, the vegetable fibres were still visible, and are forced asunder by large and shining air-bubbles.

SINCE the publication of the sketch of my experiments, I have had the pleasure to read Mr HATCHETT's very interesting account of various natural substances, nearly allied to coal; and I could not help being struck with the resemblance which my results bear to them, through all their varieties, as brought into view by that able chemist; that resemblance affording a presumption, that the changes which, with true scientific modesty, he ascribes to an unknown cause, may have resulted from various heats acting under pressure of various force. The substance to which he has given the name of *Retinasphaltum*, seems to agree very nearly with what I have obtained from animal substance, when the barrel was opened by means of low heat. And the specimen of wood entering into fusion, but still retaining the form of its fibres, seems very similar to the intermediate substance of Bovey-coal and *Surturbrand*, which Mr HATCHETT has assimilated to each other. It is well known, that the surturbrand of Iceland, consists of the stems of large trees, flattened to thin plates, by some operation of nature hitherto unexplained. But the last-mentioned experiment seems to afford a plausible solution of this puzzling phenomenon.

IN all parts of the globe, we find proofs of slips, and various relative motions, having taken place amongst great masses of rock, whilst they were soft in a certain degree, and which have left unequivocal traces behind them, both in the derangement of the beds of strata, and in a smooth and shining surface, called *slickenside*, produced by the direct friction of one mass on another. During the action of subterranean heat, were a single stratum to occur, containing trees intermixed with animal substances, shell-fish, &c. these trees would be reduced, to a soft and unctuous state, similar to that of the piece of wood in the laſt-mentioned experiment, whilſt the ſubſtance of the contiguous ſtrata retained a conſiderable degree of firmneſs. In this ſtate of things, the ſtratum juſt mentioned, would very naturally become the ſcene of a ſlip, occaſioned by the unequal preſſure of the ſurrounding maſſes. By ſuch a ſliding motion, accompanied by great compreſſion, a tree would be flattened, as any ſubſtance is ground in a mortar, by the combination of a lateral and direct force. At the ſame time, the ſhells along with the trees, would be flattened, like thoſe deſcribed by BERGMAN; while thoſe of the ſame ſpecies in the neighbouring limeſtone-rock, being protected by its inferior fuſibility, would retain their natural ſhape.

IX.

Application of the foregoing reſults to Geology.—The fire employed in the Huttonian Theory is a modification of that of the Volcanoes.—This modification muſt take place in a lava previous to its eruption.—An Internal Lava is capable of melting Limeſtone.—The effects of Volcanic Fire on ſubſtances in a ſubterranean and ſubmarine ſituation, are the ſame as thoſe aſcribed to Fire in the Huttonian Theory.—Our Strata were once in a ſimilar ſituation, and then underwent the action of fire.—All the conditions of the Huttonian Theory being thus combined, the formation of all Rocks may be accounted for in a ſatisfactory manner.—Concluſion.

HAVING inveſtigated, by means of the foregoing experiments, ſome of the chemical ſuppoſitions involved in the Huttonian Theory, and having endeavoured to aſſign a determinate limit to the power of the agents employed; I ſhall now apply theſe reſults to Geology, and inquire how far the events ſuppoſed

ſuppoſed anciently to have taken place, accord with the exiſting ſtate of our globe.

THE moſt powerful and eſſential agent of the Huttonian Theory, is Fire, which I have always looked upon as the ſame with that of volcanoes, modified by circumſtances which muſt, to a certain degree, take place in every lava previous to its eruption.

THE original ſource of internal fire is involved in great obſcurity; and no ſufficient reaſon occurs to me for deciding whether it proceeds by emanation from ſome vaſt central reſervoir, or is generated by the local operation of ſome chemical proceſs. Nor is there any neceſſity for ſuch a deciſion: all we need to know is, that internal fire exiſts, which no one can doubt, who believes in the eruptions of Mount Veſuvius. To require that a man ſhould account for the generation of internal fire, before he is allowed to employ it in geology, is no leſs abſurd than it would be to prevent him from reaſoning about the conſtruction of a teleſcope, till he could explain the nature of the ſun, or account for the generation of light *. But while we remain in ſuſpenſe as to the prime cauſe of this tremendous agent, many circumſtances of importance with regard to it, may fairly become the ſubjects of obſervation and diſcuſſion.

SOME authors (I conceive through ignorance of the facts) have alleged, that the fire of Ætna and Veſuvius is merely ſuperficial. But the depth of its action is ſufficiently proved, by the great diſtance to which the eruptive percuſſions are felt, and ſtill more, by the ſubſtances thrown out uninjured by ſome eruptions

* THIS topic, however, has of late been much urged againſt us, and an unfair advantage has been taken of what Mr PLAYFAIR has ſaid upon it. What he gave as mere conjecture on a ſubject of collateral importance, has been argued upon as the baſis and fundamental doctrine of the ſyſtem.

eruptions of Mount Vesuvius. Some of these, as marble and gypsum, are incapable in freedom of resisting the action of fire. We have likewise granite, schistus, gneiss, and stones of every known class, besides many which have never, on any other occasion, been found at the surface of our globe. The circumstance of these substances having been thrown out, unaffected by the fire, proves, that it has proceeded from a source, not only as deep, but deeper, than their native beds; and as they exhibit specimens of every class of minerals, the formation of which we pretend to explain, we need inquire no further into the depth of the Vesuvian fire, which has thus been proved to reach below the range of our speculations.

VOLCANIC fire is subject to perpetual and irregular variations of intensity, and to sudden and violent renewal, after long periods of absolute cessation. These variations and intermissions, are likewise essential attributes of fire as employed by Dr HUTTON; for some geological scenes prove, that the indurating cause has acted repeatedly on the same substance, and that, during the intervals of that action, it had ceased entirely. This circumstance affords a complete answer to an argument lately urged against the Huttonian Theory, founded on the waste of heat which must have taken place, as it is alleged, through the surface. For if, after absolute cessation, a power of renewal exists in nature, the idea of waste by continuance is quite inapplicable.

THE external phenomena of volcanoes are sufficiently well known; but our subject leads us to inquire into their internal actions. This we are enabled to do by means of the foregoing experiments, in so far as the carbonate of lime is concerned.

SOME experiments which I formerly * laid before this Society and the public, combined with those mentioned in this paper,

* *Edinburgh Transactions*, Vol. V. Part I. p. 60—66.

paper, prove, that the feebleſt exertions of volcanic fire, are of ſufficient intenſity to perform the agglutination, and even the entire fuſion, of the carbonate of lime, when its carbonic acid is effectually confined by preſſure; for though lava, after its fuſion, may be made, in our experiments, to congeal into a glaſs, in a temperature of 16° or 18° of WEDGWOOD, in which temperature the carbonate would ſcarcely be affected; it muſt be obſerved, that a ſimilar congelation is not to be looked for in nature; for the maſs, even of the ſmalleſt ſtream of lava, is too great to admit of ſuch rapid cooling. And, in fact, the external part of a lava is not vitreous, but conſiſts of a ſubſtance which, as my experiments have proved, muſt have been congealed in a heat of melting ſilver, that is, in 22° of WEDGWOOD; while its internal parts bear a character indicating that they congealed in 27° or 28° of the ſame ſcale. It follows, that no part of the lava, while it remained liquid, can have been leſs hot than 22° of WEDGWOOD. Now, this happens to be a heat, in which I have accompliſhed the entire fuſion of the carbonate of lime, under preſſure. We muſt therefore conclude, that the heat of a running lava is always of ſufficient intenſity to perform the fuſion of limeſtone.

IN every active volcano, a communication muſt exiſt between the ſummit of the mountain and the unexplored region, far below its baſe, where the lava has been melted, and whence it has been propelled upwards; the liquid lava riſing through this internal channel, ſo as to fill the crater to the brim, and flow over it. On this occaſion, the ſides of the mountain muſt undergo a violent hydroſtatical preſſure outwards, to which they often yield by the formation of a vaſt rent, through which the lava is diſcharged in a lateral eruption, and flows in a continued ſtream ſometimes during months. On Ætna moſt of the eruptions are ſo performed; few lavas flowing from the ſummit, but generally breaking out laterally, at very elevated ſtations.

At

At the place of delivery, a quantity of gaſeous matter is propelled violently upwards, and, along with it, ſome liquid lava; which laſt, falling back again in a ſpongy ſtate, produces one of thoſe conical hills which we ſee in great number on the vaſt ſides of Mount Ætna, each indicating the diſcharge of a particular eruption. At the ſame time, a jet of flame and ſmoke iſſues from the main crater, proving the internal communication between it and the lava; this diſcharge from the ſummit generally continuing, in a greater or a leſs degree, during the intervals between eruptions. (Fig. 41. repreſents an ideal ſection of Mount Ætna; *a b* is the direct channel, and *b c* is a lateral branch).

Let us now attend to the ſtate of the lava within the mountain, during the courſe of the eruption; and let us ſuppoſe, that a fragment of limeſtone, torn from ſome ſtratum below, has been included in the fluid lava, and carried up with it. By the laws of hydroſtatics, as each portion of this fluid ſuſtains preſſure in proportion to its perpendicular diſtance below the point of diſcharge, that preſſure muſt increaſe with the depth. The ſpecific gravity of ſolid and compact lava is nearly 2.8; and its weight, when in a liquid ſtate, is probably little different. The table ſhews, that the carbonic acid of limeſtone cannot be conſtrained in heat by a preſſure leſs than that of 1708 feet of ſea, which correſponds nearly to 600 feet of liquid lava. As ſoon, then, as our calcareous maſs roſe to within 600 feet of the ſurface, its carbonic acid would quit the lime, and, aſſuming a gaſeous form, would add to the eruptive efferveſcence. And this change would commonly begin in much greater depths, in conſequence of the bubbles of carbonic acid, and other ſubſtances in a gaſeous form, which, riſing with the lava, and through it, would greatly diminiſh the weight of the column, and would render its preſſure on any particular ſpot extremely variable. With all theſe irregularities, however, and interruptions, the preſſure

preſſure would in all caſes, eſpecially where the depth was conſiderable, far ſurpaſs what it would have been under an equal depth of water. Where the depth of the ſtream, below its point of delivery, amounted, then, to 1708 feet, the preſſure, if the heat was not of exceſſive intenſity, would be more than ſufficient to conſtrain the carbonic acid, and our limeſtone would ſuffer no calcination, but would enter into fuſion; and if the eruption ceaſed at that moment, would cryſtallize in cooling along with the lava, and become a nodule of calcareous ſpar. The maſs of lava, containing this nodule, would then conſtitute a real whinſtone, and would belong to the kind called *amygdaloid*. In greater depths ſtill, the preſſure would be proportionally increaſed, till ſulphur, and even water, might be conſtrained; and the carbonate of lime would continue undecompoſed in the higheſt heats.

If, while the lava was in a liquid ſtate, during the eruption or previous to it, a new rent (*de*, fig. 41.), formed in the ſolid country below the volcano, was met by our ſtream (at *d*), it is obvious that the lava would flow into the aperture with great rapidity, and fill it to the minuteſt extremity, there being no air to impede the progreſs of the liquid. In this manner, a ſtream of lava might be led from below to approach the bottom of the ſea (*ff*), and to come in contact with a bed of looſe ſhells (*gg*), lying on that bottom, but covered with beds of clay, interſtratified, as uſually occurs, with beds of ſand, and other beds of ſhells. The firſt effect of heat would be to drive off the moiſture of the loweſt ſhell-bed, in a ſtate of vapour, which, riſing till it got beyond the reach of the heat, would be condenſed into water, producing a ſlight motion of ebullition, like that of a veſſel of water, when it begins to boil, and when it is ſaid to ſimmer. The beds of clay and ſand might thus undergo ſome heaving and partial derangement, but would ſtill poſſeſs the power of ſtopping, or of very much impeding,

peding, the descent of water from the sea above; so that the water which had been driven from the shells at the bottom, would not return to them, or would return but slowly; and they would be exposed dry to the action of heat *.

In this case, one of two things would inevitably happen. Either the carbonic acid of the shells would be driven off by the heat, producing an incondensable elastic fluid, which, heaving up or penetrating the superincumbent beds, would force its way to the surface of the sea, and produce a submarine eruption, as has happened at Santorini and elsewhere; or the volatility of the carbonic acid would be repressed by the weight of the superincumbent water (*k k*), and the shell-bed, being softened or fused by the action of heat, would be converted into a stratum of limestone.

The foregoing experiments enable us to decide in any particular case, which of these two events must take place, when the heat of the lava and the depth of the sea are known.

The table shews, that under a sea no deeper than 1708 feet, near one-third of a mile, a limestone would be formed by proper heat; and that, in a depth of little more than one mile, it would enter into entire fusion. Now, the common soundings of mariners extend to 200 fathoms, or 1200 feet. Lord Mulgrave † found bottom at 4680 feet, or nearly nine-tenths of a mile; and Captain Ellis let down a sea-gage to the depth of 5346 feet ‡. It thus appears, that

* This situation of things, is similar to what happens when small-coal is moistened, in order to make it cake. The dust, drenched with water, is laid upon the fire, and remains long wet, while the heat below suffers little or no abatement.

† *Voyage towards the North Pole*, p. 142.

‡ *Philosophical Transactions*, 1751, p. 212.

that at the bottom of a ſea, which would be ſounded by a line much leſs than double of the uſual length, and leſs than half the depth of that ſounded by Lord MULGRAVE, limeſtone might be formed by heat; and that, at the depth reached by Captain ELLIS, the entire fuſion would be accompliſhed, if the bed of ſhells were touched by a lava at the extremity of its courſe, when its heat was loweſt. Were the heat of the lava greater, a greater depth of ſea would, of courſe, be requiſite to conſtrain the carbonic acid effectually; and future experiments may determine what depth is required to co-operate with any given temperature. It is enough for our preſent purpoſe to have ſhewn, that the reſult is poſſible in any caſe, and to have circumſcribed the neceſſary force of theſe agents within moderate limits. At the ſame time it muſt be obſerved, that we have been far from ſtretching the known facts; for when we compare the ſmall extent of ſea in which any ſoundings can be found, with that of the vaſt unfathomed ocean, it is obvious, that in aſſuming a depth of one mile or two, we fall very ſhort of the medium. M. DE LA PLACE, reaſoning from the phenomena of the tides, ſtates it as highly probable that this medium is not leſs than eleven Engliſh miles *.

IF a great part or the whole of the ſuperincumbent maſs conſiſted, not of water, but of ſand or clay, then the depth requiſite to produce theſe effects would be leſſened, in the inverſe ratio of the ſpecific gravity. If the above-mentioned occurrence took place under a maſs compoſed of ſtone firmly bound together by ſome previous operation of nature, the power of the ſuperincumbent maſs, in oppoſing the eſcape of

* "On peut donc regarder au moins comme très probable, que la profondeur "moyenne de la mer n'eſt pas au-deſſous de quatre lieues." DE LA PLACE, *Hiſt. de l'Acad. Roy. des Sciences*, année 1776.

carbonic acid, would be very much increafed by that union and by the ftiffnefs or tenacity of the fubftance. We have feen numberlefs examples of this power in the courfe of thefe experiments, in which barrels, both of iron and porcelain, whofe thicknefs did not exceed one-fourth of an inch, have exerted a force fuperior to the mere weight of a mile of fea. Without fuppofing that the fubftance of a rock could in any cafe act with the fame advantage as that of a uniform and connected barrel; it feems obvious that a fimilar power muft, in many cafes, have been exerted to a certain degree.

We know of many calcareous maffes which, at this moment, are expofed to a preffure more than fufficient to accomplifh their entire fufion. The mountain of Saleve, near Geneva, is 500 French fathoms, or nearly 3250 Englifh feet, in height, from its bafe to its fummit. Its mafs confifts of beds, lying nearly horizontal, of limeftone filled with fhells. Independently, then, of the tenacity of the mafs, and taking into account its mere weight, the loweft bed of this mountain, muft, at this moment, fuftain a preffure of 3250 feet of limeftone, the fpecific gravity of which is about 2.65. This preffure, therefore, is equal to that of 8612 feet of water, being nearly a mile and a half of fea, which is much more than adequate, as we have fhewn, to accomplifh the entire fufion of the carbonate, on the application of proper heat. Now, were an emanation from a volcano, to rife up under Saleve, and to penetrate upwards to its bafe, and ftop there; the limeftone to which the lava approached, would inevitably be foftened, without being calcined, and, as the heat retired, would cryftallize into a faline marble.

Some other circumftances, relating to this fubject, are very deferving of notice, and enable us ftill further to compare the ancient and modern operations of fire.

It

It appears, at firſt ſight, that a lava having once penetrated the ſide of a mountain, all ſubſequent lavas ſhould continue, as water would infallibly do, to flow through the ſame aperture. But there is a material difference in the two caſes. As ſoon as the lava has ceaſed to flow, and the heat has begun to abate, the crevice through which the lava had been paſſing, remains filled with a ſubſtance, which ſoon agglutinates into a maſs, far harder and firmer than the mountain itſelf. This maſs, lying in a crooked bed, and being firmly welded to the ſides of the crevice, muſt oppoſe a moſt powerful reſiſtance to any ſtream tending to purſue the ſame courſe. The injury done to the mountain by the formation of the rent, will thus be much more than repaired; and in a ſubſequent eruption, the lava muſt force its way through another part of the mountain or through ſome part of the adjoining country. The action of heat from below, ſeems in moſt caſes to have kept a channel open through the axis of the mountain, as appears by the ſmoke and flame which is habitually diſcharged at the ſummit during intervals of calm. On many occaſions, however, this ſpiracle ſeems to have been entirely cloſed by the conſolidation of the lava, ſo as to ſuppreſs all emiſſion. This happened to Veſuvius during the middle ages. All appearance of fire had ceaſed for five hundred years, and the crater was covered with a foreſt of ancient oaks, when the volcano opened with freſh vigour in the ſixteenth century.

The eruptive force, capable of overcoming ſuch an obſtacle, muſt be tremendous indeed, and ſeems in ſome caſes to have blown the volcano itſelf almoſt to pieces. It is impoſſible to ſee the Mountain of Somma, which, in the form of a creſcent, embraces Mount Veſuvius, without being convinced that it is a fragment of a large volcano, nearly concentric

with

with the present inner cone, which, in some great eruption, had been destroyed all but this fragment. In our own times, an event of no small magnitude has taken place on the same spot; the inner cone of Vesuvius having undergone so great a change during the eruption in 1794, that it now bears no resemblance to what it was when I saw it in 1785.

The general or partial stagnation of the internal lavas at the close of each eruption seems, then, to render it necessary, that in every new discharge, the lava should begin by making a violent laceration. And this is probably the cause of those tremendous earthquakes which precede all great eruptions, and which cease as soon as the lava has found a vent. It seems but reasonable to ascribe like effects to like causes, and to believe that the earthquakes which frequently desolate countries not externally volcanic, likewise indicate the protrusion from below of matter in liquid fusion, penetrating the mass of rock.

The injection of a whinstone-dike into a frail mass of shale and sandstone, must have produced the same effects upon it that the lava has just been stated to produce on the loose beds of volcanic scoria. One stream of liquid whin, having flowed into such an assemblage, must have given it great additional weight and strength: so that a second stream coming like the first, would be opposed by a mass, the laceration of which would produce an earthquake, if it were overcome; or by which, if it resisted, the liquid matter would be compelled to penetrate some weaker mass, perhaps at a great distance from the first. The internal fire being thus compelled perpetually to change the scene of its action, its influence might be carried to an indefinite extent: So that the intermittance in point of time, as well as the versatility in point of place, already remarked as common to the Huttonian and Volcanic fires, are accounted for on our principles.

ples. And it thus appears, that whinſtone poſſeſſes all the properties which we are led by theory to aſcribe to an internal lava.

THIS conection is curiouſly illuſtrated by an intermediate caſe between the reſults of external and internal fire, diſplayed in an actual ſection of the ancient part of Veſuvius, which occurs in the Mountain of Somma mentioned above. I formerly deſcribed this ſcene in my paper on Whinſtone and Lava; and I muſt beg leave once more to preſs it upon the notice of the public, as affording to future travellers a moſt intereſting field of geological inquiry.

THE ſection is ſeen in the bare vertical cliff, ſeveral hundred feet in height, which Somma preſents to the view from the little valley, in form of a creſcent, which lies between Somma and the interior cone of Veſuvius, called the *Atrio del Cavallo*. (Fig. 42. repreſents this ſcene, done from the recollection of what I ſaw in 1785. *abc* is the interior cone of Veſuvius; *dfg* the mountain of Somma; and *cde* the *Atrio del Cavallo*). By means of this cliff (*fd* in figure 42. and which is repreſented ſeparately in fig. 44.), we ſee the internal ſtructure of the mountain, compoſed of thick beds (*kk*) of looſe ſcoria, which have fallen in ſhowers; between which thin but firm ſtreams (*mm*) of lava are interpoſed, which have flowed down the outward conical ſides of the mountain. (Fig. 43. is an ideal ſection of Veſuvius and Somma, through the axis of the cones, ſhewing the manner in which the beds of ſcoria and of lava lie upon each other; the extremities of which beds are ſeen edgewiſe in the cliff at *mm* and *kk*, fig. 42, 43, and 44.).

THIS aſſemblage of ſcoria and lava is traverſed abruptly and vertically, by ſtreams of ſolid lava (*nn*, fig. 44.), reaching from top to bottom of the cliff. Theſe laſt I conceive to have flowed in rents of the ancient mountain, which rents had acted

as

as pipes through which the lavas of the lateral eruptions were conveyed to the open air. This ſcene preſents to the view of an attentive obſerver, a real ſpecimen of thoſe internal ſtreams which we have juſt been conſidering in ſpeculation, and they may exhibit circumſtances deciſive of the opinions here advanced. For, if one of theſe ſtreams had formerly been connected with a lateral eruption, diſcharged at more than 600 feet above the *Atrio del Cavallo*, it might poſſibly contain the carbonate of lime. But could we ſuppoſe that depth to extend to 1708 feet, the interference of air-bubbles, and the action of a ſtronger heat than was merely required for the fuſion of the carbonate, might have been overcome.

Perhaps the height of Veſuvius has never been great enough for this purpoſe. But could we ſuppoſe Ætna to be cleft in two, and its ſtructure diſplayed, as that of Veſuvius has juſt been deſcribed, there can be no doubt that internal ſtreams of lava would be laid open, in which the preſſure muſt have far exceeded the force required to conſtrain the carbonic acid of limeſtone; ſince that mountain occaſionally delivers lavas from its ſummit, placed 10.954 feet above the level of the Mediterranean *, which waſhes its baſe. I recollect having ſeen, in ſome parts of Ætna, vaſt chaſms and crags, formed by volcanic revolutions, in which vertical ſtreams of lava, ſimilar to thoſe of Somma, were apparent. But my attention not having been turned to that object till many years afterwards, I have only now to recommend the inveſtigation of this intereſting point to future travellers.

What has been ſaid of the heat conveyed by internal volcanic ſtreams, applies equally to that deeper and more general heat by which the lavas themſelves are melted and propelled upwards.

* *Phil. Tranſ.* 1777, p. 595.

upwards. That they have been really ſo propelled, from a great internal maſs of matter, in liquid fuſion, ſeems to admit of no doubt, to whatever cauſe we aſcribe the heat of volcanoes. It is no leſs obvious, that the temperature of that liquid muſt be of far greater intenſity than the lavas, flowing from it, can retain when they reach the ſurface. Independently of any actual eruption, the body of heat contained in this vaſt maſs of liquid, muſt diffuſe itſelf through the ſurrounding ſubſtances, the intenſity of the heat being diminiſhed by ſlow gradations, in proportion to the diſtance to which it penetrates. When, by means of this progreſſive diffuſion, the heat has reached an aſſemblage of looſe marine depoſites, ſubject to the preſſure of a great ſuperincumbent weight, the whole muſt be agglutinated into a maſs, the ſolidity of which will vary with the chemical compoſition of the ſubſtance, and with the degree of heat to which each particular ſpot has thus been expoſed. At the ſame time, analogy leads us to ſuppoſe, that this deep and extenſive heat muſt be ſubject to viciſſitudes and intermiſſions, like the external phenomena of volcanoes. We have endeavoured to explain ſome of theſe irregularities, and a ſimilar reaſoning may be extended to the preſent caſe. Having ſhewn, that ſmall internal ſtreams of lava tend ſucceſſively to pervade every weak part of a volcanic mountain, we are led to conceive, that the great maſſes of heated matter juſt mentioned, will be ſucceſſively directed to different parts of the earth; ſo that every looſe aſſemblage of matter, lying in a ſubmarine and ſubterranean ſituation, will, in its turn, be affected by the indurating cauſe; and the influence of internal volcanic heat will thus be circumſcribed within no limits but thoſe of the globe itſelf.

A SERIES of undoubted facts prove, that all our ſtrata once lay in a ſituation ſimilar in all reſpects to that in which the marine depoſites juſt mentioned have been ſuppoſed to lie.

THE inhabitant of an unbroken plain, or of a country formed of horizontal ſtrata, whoſe obſervations have been confi-

ned

ned to his native ſpot, can form no idea of thoſe truths, which at every ſtep in an alpine diſtrict force themſelves on the mind of a geological obſerver. Unfortunately for the progreſs of geology, both London and Paris, are placed in countries of little intereſt; and thoſe ſcenes by which the principles of this ſcience are brought into view in the moſt ſtriking manner, are unknown to many perſons beſt capable of appreciating their value. The moſt important, and at the ſame time, the moſt aſtoniſhing truth which we learn by any geological obſervations, is, that rocks and mountains now placed at an elevation of more than two miles above the level of the ſea, muſt at one period have lain at its bottom. This is undoubtedly true of thoſe ſtrata of limeſtone which contain ſhells; and the ſame concluſion muſt be extended to the circumjacent ſtrata. The imagination ſtruggles againſt the admiſſion of ſo violent a poſition; but muſt yield to the force of unqueſtionable evidence; and it is proved by the example of the moſt eminent and cautious obſervers, that the concluſion is inevitable *.

Another queſtion here occurs, which has been well treated by Mr Playfair. Has the ſea retreated from the mountains? or have they riſen out of the ſea? He has ſhewn, that the balance of probability is incomparably in favour of the latter ſuppoſition; ſince, in order to maintain the former, we muſt diſpoſe of an enormous maſs of ſea, whoſe depth is ſeveral miles, and whoſe baſe is greater than the ſurface of the whole ſea. Whereas the elevation of a continent out of a ſea like ours, would not change its level above a few feet; and even were a great derangement thus occaſioned,

* Saussure, *Voyages dans les Alpes*, tom. ii. p. 99.—104.

fioned, the water would eafily find its level without the af-fiftance of any extraordinary fuppofition. The elevation of the land, too, is evinced by what has occafionally happened in volcanic regions, and affords a complete folution of the contortion and erection of ftrata, which are almoft univerfally admitted to have once lain in a plane and horizontal pofition.

WHATEVER opinion be adopted as to the mode in which the land and the water have been feparated, no one doubts of the ancient fubmarine fituation of the ftrata.

AN important feries of facts proves, that they were likewife fubterranean. Every thing indicates that a great quantity of matter has been removed from what now conftitutes the furface of our globe, and enormous depofites of loofe fragments, evidently detached from maffes fimilar to our common rock, evince the action of fome very powerful agent of deftruction. Analogy too, leads us to believe, that all the primary rocks have once been covered with fecondary; yet, in vaft diftricts, no fecondary rock appears. In fhort, geologifts feem to agree in admitting the general pofition, that very great changes of this kind have taken place in the folid furface of the globe, however much they may differ as to their amount, and as to their caufes.

DR HUTTON afcribed thefe changes to the action, during very long time, of thofe agents, which at this day continue flowly to corrode the furface of the earth; frofts, rains, the ordinary floods of rivers, &c. which he conceives to have acted always with the fame force, and no more. But to this opinion I could never fubfcribe, having early adopted that of SAUSSURE, in which he is joined by many of the continental geologifts. My conviction was founded upon the infpection of thofe facts in the neighbourhood of Geneva, which he has adduced in fupport of his opinion. I was then convinced, and I ſtill believe, that vaſt torrents, of depth ſufficient to overtop our mountains, have ſwept along the ſurface of the earth, excavating vallies, undermining mountains, and carrying away whatever was unable to reſiſt ſuch powerful corroſion. If ſuch agents have been at work in the Alps, it is difficult to conceive that our countries ſhould have been ſpared. I made it therefore my buſineſs to ſearch for traces of ſimilar operations here. I was not long in diſcovering ſuch in great abundance; and, with the help of ſeveral of my friends, I have traced the indications of vaſt torrents in this neighbourhood, as obvious as thoſe I formerly ſaw on Saleve and Jura. Since I announced my opinion on this ſubject, in a note ſubjoined to my paper on Whinſtone and Lava, publiſhed in the fifth volume of the *Tranſactions* of this Society, I have met with many confirmations of theſe views. The moſt important of theſe are derived from the teſtimony of my friend Lord SELKIRK, who has lately met with a ſeries of ſimilar facts in North America.

IT would be difficult to compute the effects of ſuch an agent; but if, by means of it, or of any other cauſe, the whole maſs of ſecondary ſtrata, in great tracts of country, has been removed from above the primary, the weight of that maſs alone muſt have been ſufficient to fulfil all the conditions of the Huttonian Theory, without having recourſe to the preſſure of the ſea. But when the two preſſures were combined, how great muſt have been their united ſtrength!

WE are authoriſed to ſuppoſe, that the materials of our ſtrata, in this ſituation, underwent the action of fire. For volcanoes have burnt long before the earlieſt times recorded in hiſtory, as appears by the magnitude of ſome volcanic mountains; and it can ſcarcely be doubted, that their fire has acted without any material ceſſation ever ſince the ſurface of our globe acquired its

preſent

present form. In extending that fame influence to periods of ftill higher antiquity, when our ftrata lay at the bottom of the fea, we do no more than afcribe permanence to the exifting laws of nature.

The combination of heat and compreffion refulting from thefe circumftances, carries us to the full extent of the Huttonian Theory, and enables us, upon its principles, to account for the igneous formation of all rocks from loofe marine depofites.

The fand would thus be changed to fandftone; the fhells to limeftone; and the animal and vegetable fubftances to coal.

Other beds, confifting of a mixture of various fubftances, would be ftill more affected by the fame heat. Such as contained iron, carbonate of lime, and alkali, together with a mixture of various earths, would enter into thin fufion, and, penetrating through every crevice that occurred, would, in fome cafes, reach what was then the furface of the earth, and conftitute lava: in other cafes, it would congeal in the internal rents, and conftitute porphyry, bafalt, greenftone, or any other of that numerous clafs of fubftances, which we comprehend under the name of *whinftone*. At the fame time, beds of fimilar quality, but of compofition fomewhat lefs fufible, would enter into a ftate of vifcidity, fuch as many bodies pafs through in their progrefs towards fufion. In this ftate, the particles, though far from poffeffing the fame freedom as in a liquid, are fufceptible of cryftalline arrangement *; and the fubftance

* This ftate of vifcidity, with its numberlefs modifications, is deferving of great attention, fince it affords a folution of fome of the moft important geological queftions. The mechanical power exerted by fome fubftances, in the act of affuming a cryftalline form, is well known. I have feen a fet of large and broad cryftals

ſubſtance, which, in this ſluggiſh ſtate, would be little diſpoſed to move, being confined in its original ſituation by contiguous beds of more refractory matter, would cryſtallize, without undergoing any change of place, and conſtitute one of thoſe beds of whinſtone, which frequently occur interſtratified with ſandſtone and limeſtone.

In other caſes where the heat was more intenſe, the beds of ſand, approaching more nearly to a ſtate of fuſion, would acquire ſuch tenacity and toughneſs, as to allow themſelves to be bent and contorted, without laceration or fracture, by the influence of local motions, and might aſſume the ſhape and character of primary ſchiſtus: the limeſtone would be highly cryſtallized, and would become marble, or, entering into thin fuſion, would penetrate the minuteſt rents in the form of calcareous ſpar. Laſtly, when the heat was higher ſtill, the ſand itſelf would be entirely melted, and might be converted, by the ſubſequent effects of ſlow cooling, into granite, ſienite, &c.; in ſome caſes, retaining traces of its original ſtratification, and conſtituting gneiſs and ſtratified granite; in others, flowing into the crevices, and forming veins of perfect granite.

In conſequence of the action of heat, upon ſo great a quantity of matter, thus brought into a fluid or ſemifluid ſtate, and in which, notwithſtanding the great preſſure, ſome ſubſtances would be volatilized, a powerful heaving of the ſuperincumbent maſs muſt have taken place; which, by repeated efforts, ſucceeding

cryſtals of ice, like the blade of a knife, formed in a maſs of clay, of ſuch ſtiffneſs, that it had juſt been uſed to make cups for chemical purpoſes. In many of my former experiments, I found that a fragment of glaſs made from whinſtone or lava, when placed in a muffle heated to the melting point of ſilver, aſſumed a cryſtalline arrangement, and underwent a complete change of character. During this change, it became ſoft, ſo as to yield to the touch of an iron rod; yet retained ſuch ſtiffneſs, that, lying untouched in the muffle, it preſerved its ſhape entirely; the ſharp angles of its fracture not being in the leaſt blunted.

ceeding each other from below, would at laſt elevate the ſtrata into their preſent ſituation.

The Huttonian Theory embraces ſo wide a field, and comprehends the laws of ſo many powerful agents, exerting their influence in circumſtances and in combinations hitherto untried, that many of its branches muſt ſtill remain in an unfiniſhed ſtate, and may long be expoſed to partial and plauſible objections, after we are ſatisfied with regard to its fundamental doctrines. In the mean time I truſt, that the object of our purſuit has been accompliſhed, in a ſatisfactory manner, by the fuſion of limeſtone under preſſure. This ſingle reſult affords, I conceive, a ſtrong preſumption in favour of the ſolution which Dr Hutton has advanced of all the geological phenomena; for, the truth of the moſt doubtful principle which he has aſſumed, has thus been eſtabliſhed by direct experiment.

APPEN-

APPENDIX.

No. I.

SPECIFIC GRAVITY OF SOME OF THE FOREGOING RESULTS.

AS many of the artificial limeſtones and marbles produced in theſe experiments, were poſſeſſed of great hardneſs and compactneſs, and as they had viſibly undergone a great diminution of bulk, and felt heavy in the hand, it ſeemed to me an object of ſome conſequence to aſcertain their ſpecific gravity, compared with each other, and with the original ſubſtances from which they were formed. As the original was commonly a maſs of chalk in the lump, which, on being plunged into water, begins to abſorb it rapidly, and continues to do ſo during a long time, ſo as to vary the weight at every inſtant, it was impoſſible, till the abſorption was complete, to obtain any certain reſult; and to allow for the weight thus gained, required the application of a method different from that uſually employed in eſtimating ſpecific gravity.

IN the common method, the ſubſtance is firſt weighed in air, and then in water; the difference indicating the weight of water diſplaced, and being conſidered as that of a quantity of water equal in bulk to the ſolid body. But as chalk, when ſaturated with water, is heavier, by about one-fourth, than when dry, it is evident, that its apparent weight, in water, muſt be increaſed, and the apparent loſs of weight diminiſhed exactly to that amount. To have a juſt eſtimate, then, of the quantity of water diſplaced by the ſolid body, the apparent loſs of weight muſt be increaſed, by adding the abſorption to it.

TWO diſtinct methods of taking ſpecific gravity thus preſent themſelves, which it is of importance to keep ſeparate, as each of them is applicable to a particular claſs of ſubjects.

ONE of theſe methods, conſiſts in comparing a cubic inch of a ſubſtance in its dry ſtate, allowing its pores to have their ſhare in conſtituting its bulk, with a cubic inch of water.

THE other depends upon comparing a cubic inch of the ſolid matter of which the ſubſtance is compoſed, independently of vacuities, and ſuppoſing the whole reduced to perfect ſolidity, with a cubic inch of water.

THUS, were an architect to compute the efficacy of a given bulk of earth, intended to load an abutment, which earth was dry, and ſhould always remain ſo, he would undoubtedly follow the firſt of theſe modes: Whereas, were a farmer to compare the ſpecific gravity of the ſame earth with that of any other ſoil, in an agricultural point of view, he would uſe the ſecond mode, which is involved in that laid down by Mr DAVY.

AS our object is to compare the ſpecific denſity of theſe reſults, and to aſcertain to what amount the particles have approached

proached each other, it ſeems quite evident that the firſt mode is ſuited to our purpoſe. This will appear moſt diſtinctly, by inſpection of the following Table, which has been conſtructed ſo as to include both.

TABLE OF SPECIFIC GRAVITIES.

I.	II. Weight in air, dry.	III. Weight in water.	IV. Weight in air, wet.	V. Difference between Columns II. & III.	VI. Difference between Columns II. & IV. or abſorption.	VII. Abſorption *per cent.*	VIII. Sum of Columns V. and VI.	IX. Specific gravity by common mode.	X. Specific gravity by new mode.
1.	125.90	77.55	135.65	47.35	9.75	7.74	57.10	2.604	2.204
2.	9.94	6.13	9.99	3.81	0.05	0.50	3.86	2.609	2.575
3.	15.98	9.70	16.02	6.28	0.04	0.25	6.32	2.544	2.528
4.	5.47	3.33	5.48	2.14	0.01	0.18	2.15	2.556	2.544
5.	18.04	10.14	18.06	7.90	0.02	0.11	7.92	2.283	2.277
6.	6.48	3.74	7.10	2.74	0.62	9.56	3.36	2.365	1.928
7.	10.32	5.97	10.36	4.35	0.04	0.39	4.39	2.372	2.350
8.	54.57	31.30	55.23	23.27	0.66	1.21	23.93	2.345	2.280
9.	72.27	41.10	76.13	31.17	3.86	5.34	35.03	2.318	2.063
10.	37.75	21.15	38.30	16.60	0.55	1.45	17.15	2.274	2.201
11.	21.21	12.55	21.26	8.66	0.05	0.24	8.71	2.449	2.435
12. Marble.	18.59	11.56	18.61	7.03	0.02	0.18	7.05	2.644	2.636
13. Chalk.	504.15	302.40	623.20	201.75	119.05	23.61	320.80	2.498	1.571
14. Average Chalk.	444.30	264.35	550.80	179.95	106.50	23.97	286.45	2.469	1.551
15. rammed Powder.	283.97	—	—	—	—	—	198.65	—	1.429

EXPLANATION.

Column I. contains the number affixed to each of the ſpecimens, whoſe properties are expreſſed in the table.

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THE first eleven are the ſame with thoſe uſed in the paper read in this Society on the 30th of Auguſt 1804, and publiſhed in NICHOLSON'S *Journal* for October following, and which refer to the ſame ſpecimens. No. 12. Is a ſpecimen of yellow marble, bearing a ſtrong reſemblance to No. 3. No. 13. A ſpecimen of chalk. No. 14. Shews the average of three trials with chalk. No. 15. Some pounded chalk, rammed in the manner followed in theſe experiments. In order to aſcertain its ſpecific gravity, I rammed the powder into a glaſs-tube, previouſly weighed; then, after weighing the whole, I removed the chalk, and filled the ſame tube with water. I thus aſcertained, in a direct manner, the weight of the ſubſtance, as ſtated in Column II., and that of an equal bulk of water, ſtated in Column VIII.

COLUMN II. Weight of the ſubſtance, dry in air, after expoſure, during ſeveral hours to a heat of 212° of FAHRENHEIT.

COLUMN III. Its weight in water, after lying long in the liquid, ſo as to perform its full abſorption; and all air-bubbles being carefully removed.

COLUMN IV. Weight in air, wet. The looſe external moiſture being removed by the touch of a dry cloth; but no time being allowed for evaporation.

COLUMN V. Difference between Columns II. and III., or apparent weight of water diſplaced.

COLUMN VI. Difference between Columns II. and IV., or the abſorption

COLUMN VII. Abſorption reduced to a *per centage* of the dry ſubſtance.

COLUMN VIII. Sum of Columns V. and VI., or the real weight of water diſplaced by the body.

COLUMN IX. Specific gravity, by the common mode, reſulting from the diviſion of Column II. by Column V.

COLUMN X. Specific gravity, in the new mode, reſulting from the diviſion of Column II. by Column VIII.

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THE ſpecific gravities aſcertained by the new mode, and expreſſed in Column X. correſpond very well to the idea which is formed of their comparative denſities, from other circumſtances, their hardneſs, compact appearance, ſuſceptibility of poliſh, and weight in the hand.

THE caſe is widely different, when we attend to the reſults of the common method contained in Column IX. Here the ſpecific gravity of chalk is rated at 2.498, which exceeds conſiderably that of a majority of the reſults tried. Thus, it would appear, by this method, that chalk has become lighter by the experiment, in defiance of our ſenſes, which evince an increaſe of denſity.

THIS ſingular reſult ariſes, I conceive, from this, that, in our ſpecimens, the faculty of abſorption has been much more decreaſed than the poroſity. Thus, if a piece of crude chalk, whoſe ſpecific gravity had previouſly been aſcertained by the common mode, and then well dried in a heat of 212°, were dipped in varniſh, which would penetrate a little way into its ſurface; and, the varniſh having hardened, the chalk were weighed in water, it is evident, that the apparent loſs of weight would now be greater by 23.61 *per cent.* of the dry weight, than it had been when the unvarniſhed chalk was weighed in water; becauſe the varniſh, cloſing the ſuperficial pores, would quite prevent the abſorption, while it added but little to the weight of the maſs, and made no change on the bulk. In computing, then, the ſpecific gravity, by means of this laſt reſult, the chalk would appear very much lighter than at firſt, though its denſity had, in fact, been increaſed by means of the varniſh.

A SIMILAR effect ſeems to have been produced in ſome of theſe reſults, by the agglutination or partial fuſion of part of the ſubſtance, by which ſome of the pores have been ſhut out from the water.

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THIS view derives ſome confirmation from an inſpection of Columns VI. and VII.; the firſt of which expreſſes the abſorption; and the ſecond, that reſult, reduced to a *per centage* of the original weight. It there appears, that whereas chalk abſorbs 23.97 *per cent.*, ſome of our reſults abſorb only 0.5, or ſo low as 0.11 *per cent.* So that the power of abſorption has been reduced from about one-fourth, to leſs than the five hundredth of the weight.

I HAVE meaſured the diminution of bulk in many caſes, particularly in that of No. 11. The chalk, when crude, ran to the 75th degree of WEDGWOOD's gage, and ſhrunk ſo much during the experiment, that it ran to the 161ſt.; the difference amountting to 86 degrees. Now, I find, that WEDGWOOD's gage tapers in breadth, from 0.5 at zero of the ſcale, to 0.3 at the 240th degree. Hence, we have for one degree 0.000833. Conſequently, the width, at the 75th degree, amounts to 0.437525; and at the 161ſt, to 0.365887. Theſe numbers, denoting the linear meaſure of the crude chalk, and of its reſult under heat and compreſſion, are as 100 to 83.8; or, in ſolid bulk, as 100 to 57.5. Computing the denſities from this ſource, they are as 1 to 1.73. The ſpecific gravities in the Table, of the chalk, and of this reſult, are as 1.551 : 2.435; that is, as 1 to 1.57. Theſe concluſions do not correſpond very exactly; but the chalk employed in this experiment, was not one of thoſe employed in determining average ſpecific gravity in the Table; and other circumſtances may have contributed to produce irregularity. Comparing this chalk with reſult ſecond, we have 1.551 : 2.575 ſo 1 : 1.6602.

TABLE

No. II.

TABLE,

CONTAINING THE REDUCTION OF THE FORCES MENTIONED IN CHAP. VII. TO A COMMON STANDARD.

I. Number of experiment referred to in Chap. VII.	II. Bore, in decimals of an inch.	III. Preſſure in hundred weights.	IV. Temperature by WEDGWOOD's pyrometer.	V. Depth of ſea in feet.	VI Ditto in miles.	VII. Preſſure, expreſſed in atmoſpheres
1	0.75	3	22	1708 05	0.3235	51.87
2	0.75	3	25	1708.05	0.3235	51.87
3	0.75	10	20	5693.52	1.0783	172.92
4	0.75	10	31	5693 52	1.0783	172.92
5	0.75	10	41	5693.52	1.0783	172.92
6	0.75	10	51	5693.52	1.0783	172.92
7	0.75	10	—	5693.52	1.0783	172.92
8	0.54	2	—	2196.57	0.4160	66.71
9	0.54	{ 4	—	4393.14	0.8320	133.43
		{ 8.1	—	8896.12	1.6848	270.19
10	0.75	3	21	1708.05	0.3235	51.87
11	0.75	4	25	2277.41	0.4313	69.70
12	0.75	5	—	2846.76	0.5396	86.46

EXPLANATION.

EXPLANATION.

COLUMN I. contains the number of the experiment, as referred to in the text. Column II. The bore of the barrel uſed, in decimals of an inch. Column III. The abſolute force applied to the barrel, in hundred-weights. Column IV. The temperature, in WEDGWOOD's ſcale. Column V. The depth of ſea at which a force of compreſſion would be exerted equal to that ſuſtained by the carbonate in each experiment, expreſſed in feet. Column VI. The ſame in miles. Column VII. Compreſſing force, expreſſed in atmoſpheres.

BOTH Tables were computed ſeparately, by a friend, Mr J. JARDINE, and myſelf.

THE following data were employed.

AREA of a circle of which the diameter is unity, 0.785398.

WEIGHT of a cubic foot of diſtilled water, according to Profeſſor ROBISON, 998.74 ounces avoirdupois.

MEAN ſpecific gravity of ſea-water, according to BLADH, 1.0272.

MEAN heighth of the barometer at the level of the ſea 29.91196 Engliſh inches, according to LAPLACE.

SPECIFIC gravity of mercury, according to CAVENDISH and BRISSON, 13.568.

EXPLANATION.

COLUMN I. contains the number of the experiment, as referred to in the text. Column II. The bore of the barrel used, in decimals of an inch. Column III. The absolute force applied to the barrel, in hundred-weights. Column IV. The temperature, in Wedgwood's scale. Column V. The depth of sea at which a force of compression would be exerted equal to that sustained by the carbonate in each experiment expressed in feet. Column VI. The same in miles. Column VII. Compressing force, expressed in atmospheres.

Both Tables were computed separately, by a friend, Mr J. Jardine, and myself.

The following data were employed.

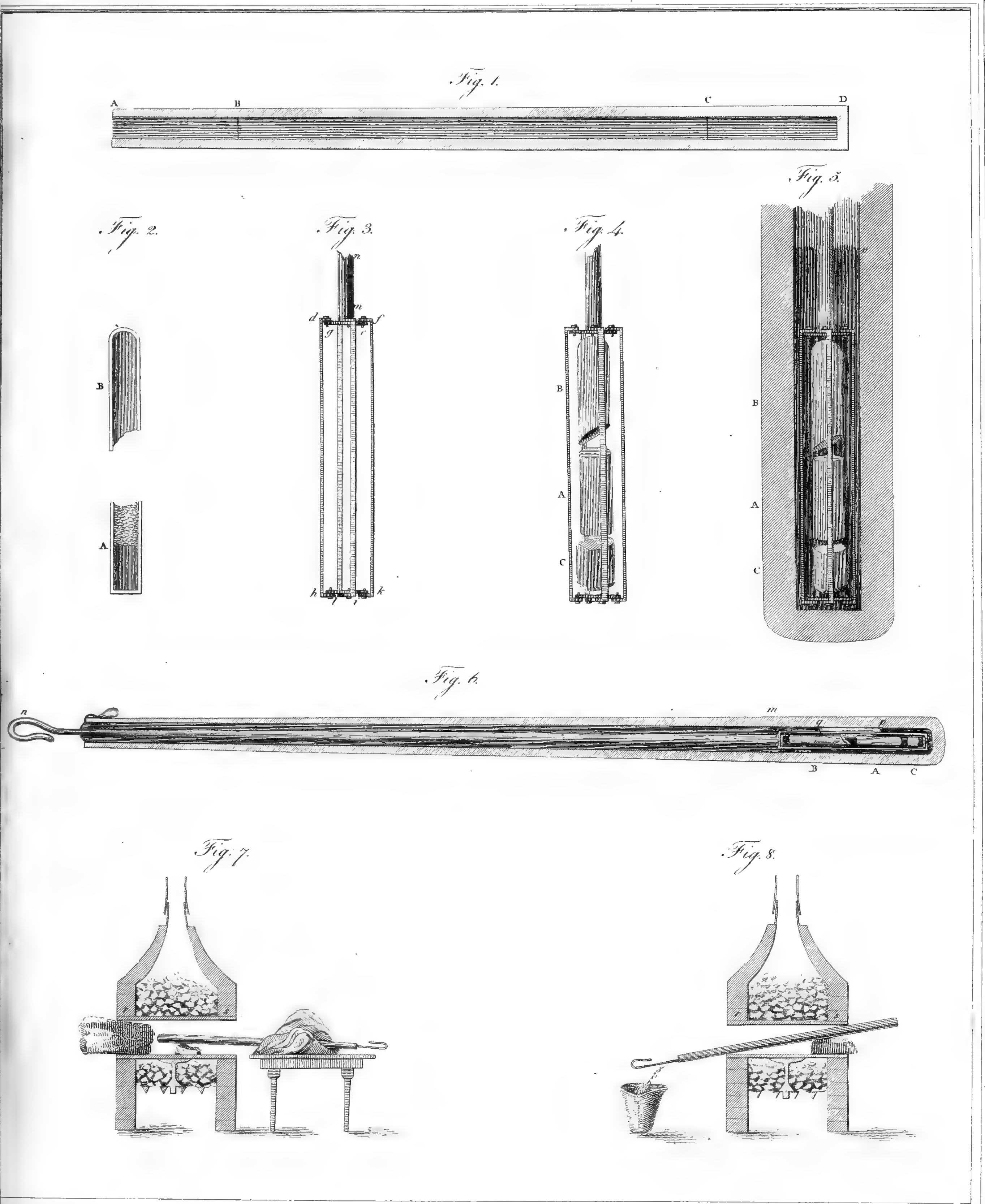
Area of a circle of which the diameter is unity, 0.785398.

Weight of a cubic foot of distilled water, according to Professor Robison, 998.74 ounces avoirdupois.

Mean specific gravity of sea-water, according to Black, 1.0272.

Mean height of the barometer at the level of the sea, 29.8196 English inches, according to Laplace.

Specific gravity of mercury, according to Cavendish and Brisson, 13.568.



...J. H. del.

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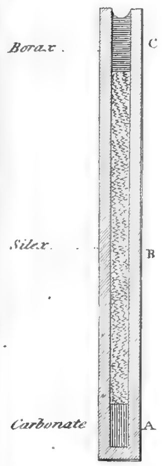
Fig. 9.



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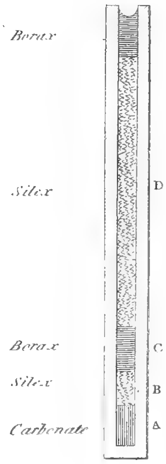
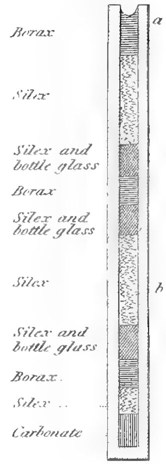


Fig. 11.

Fig. 12.



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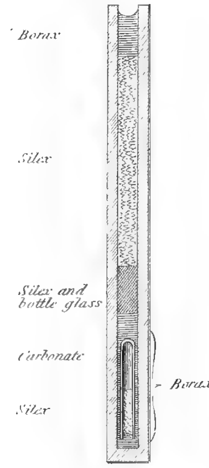
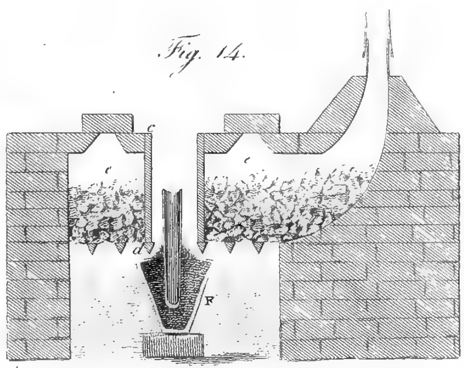


Fig. 14.



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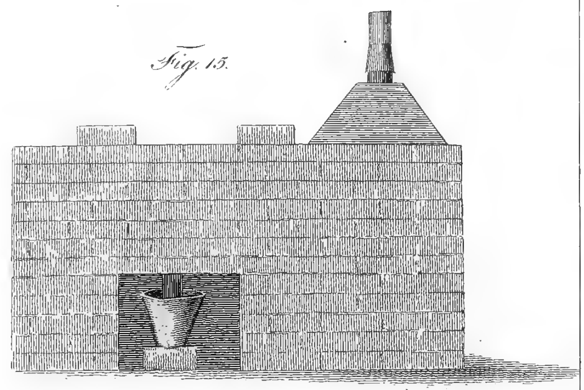

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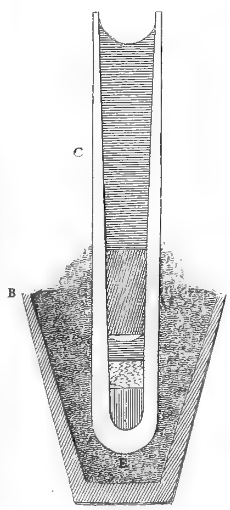

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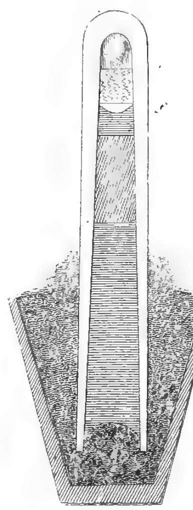


Fig. 18.

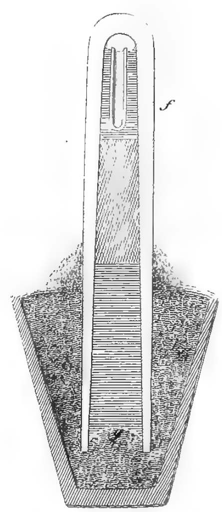
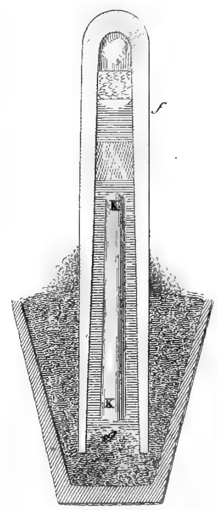
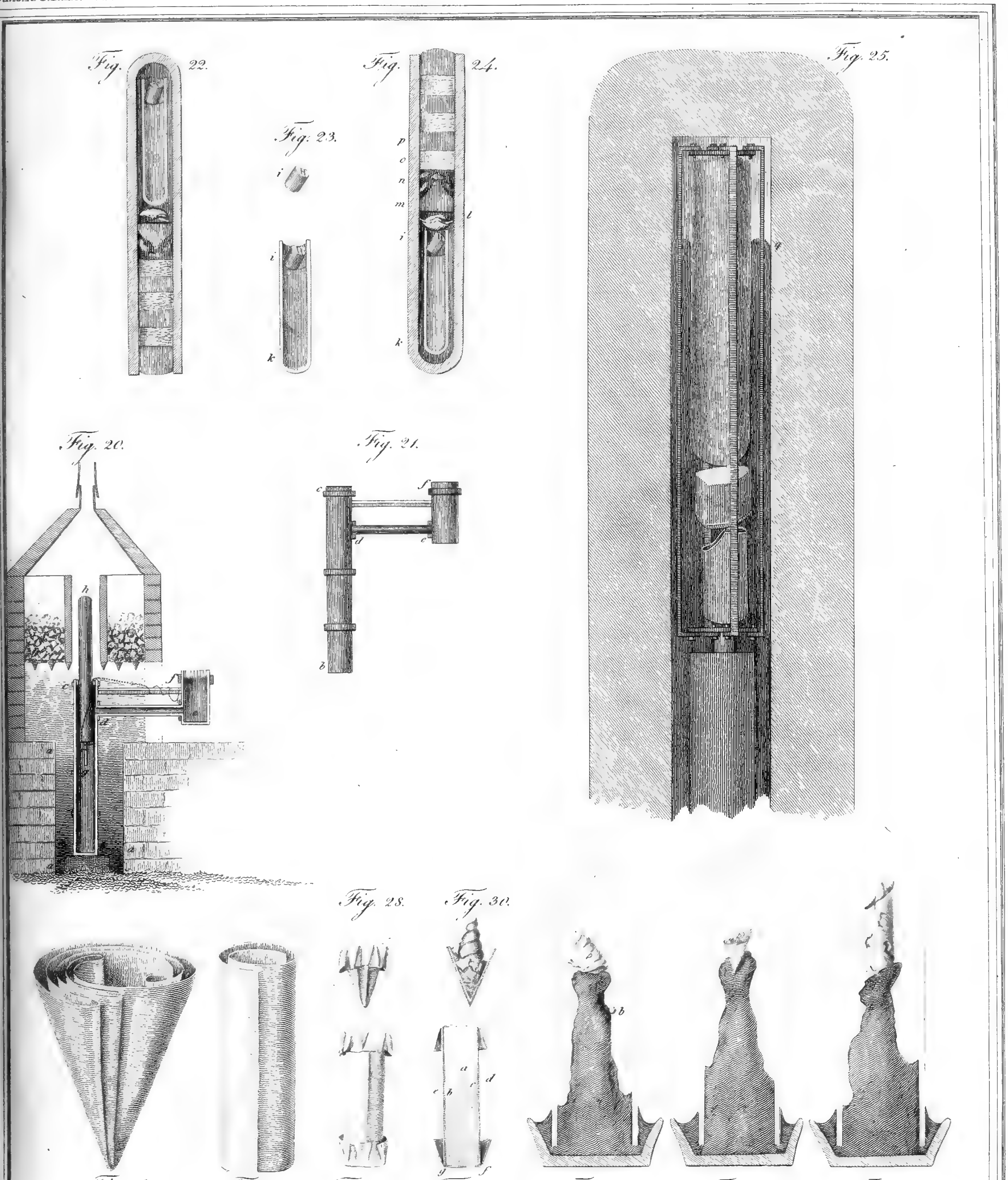


Fig. 19.



Sir J. Hall del.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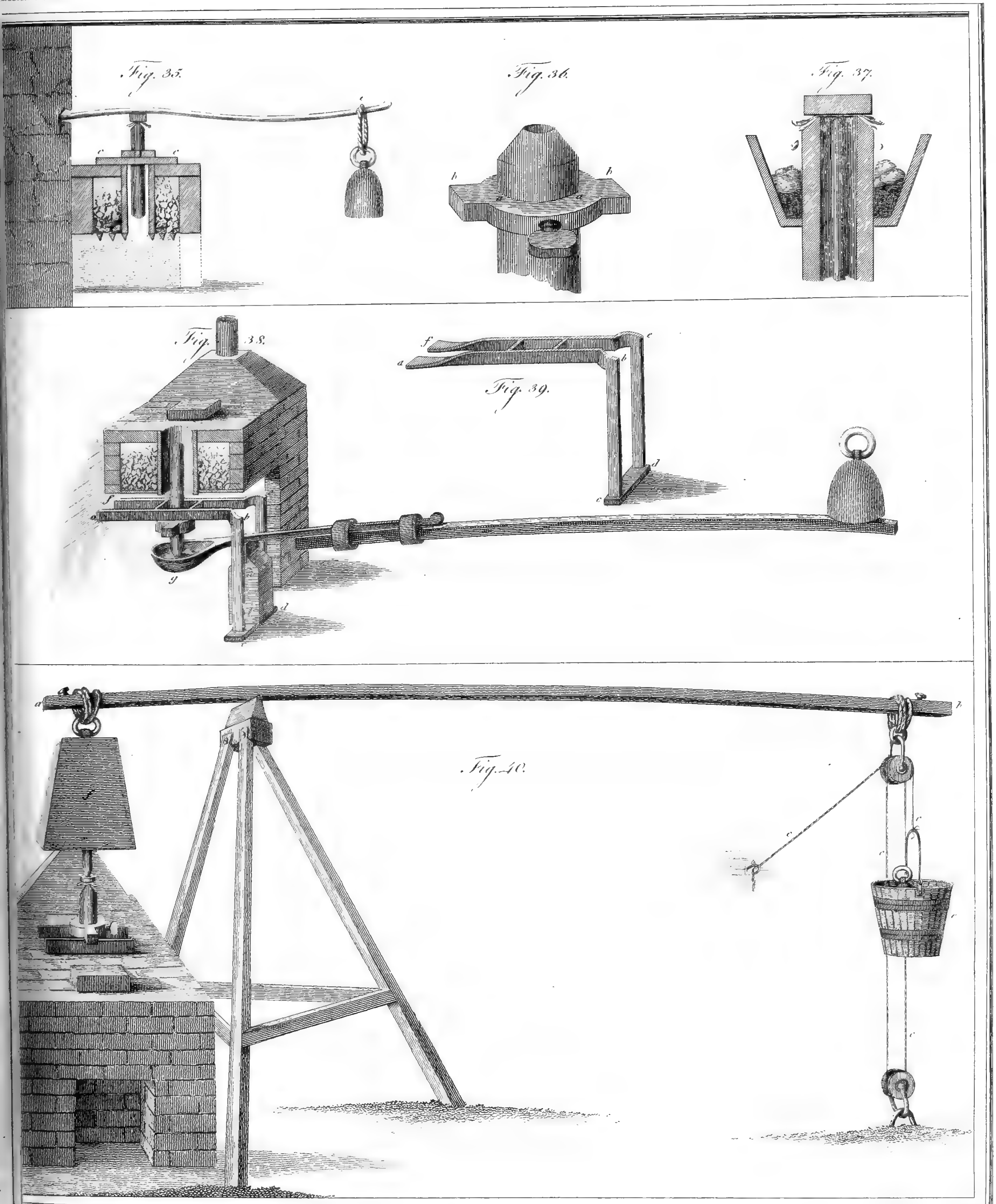
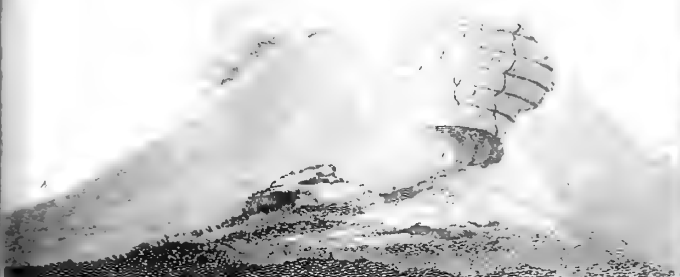


J.H. del. ... sculp.

Fig. 41.

Fig. 42.



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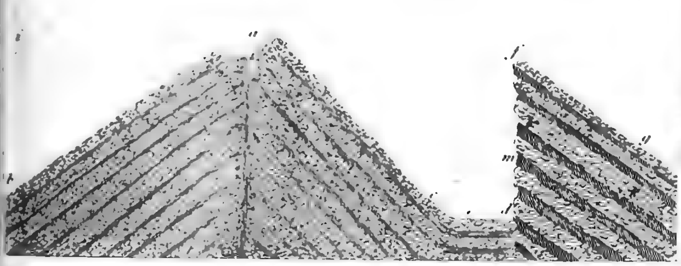
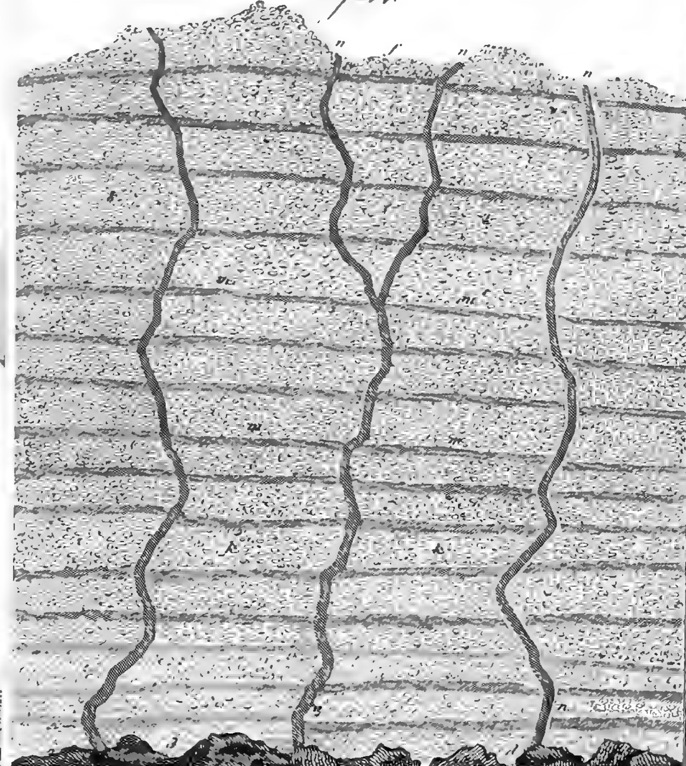


Fig. 44.



IV. *Of the* Solids *of* Greatest Attraction, *or those which, among all the* Solids *that have certain Properties, Attract with the greatest Force in a given Direction. By* John Playfair, F. R. S. *Lond. and Edin. and Professor of Natural Philosophy in the University of Edinburgh.*

[*Read 5th January* 1807.]

THE inveſtigations which I have at preſent the honour of ſubmitting to the Royal Society, were ſuggeſted by the experiments which have been made of late years concerning the gravitation of terreſtrial bodies, firſt, by Dr Maskelyne, on the Attraction of Mountains, and afterwards by Mr Cavendish, on the Attraction of Leaden Balls.

In reflecting on theſe experiments, a queſtion naturally enough occurred, what figure ought a given maſs of matter to have, in order that it may attract a particle in a given direction, with the greateſt force poſſible? This ſeemed an inquiry not of mere curioſity, but one that might be of uſe in the further proſecution of ſuch experiments as are now referred to. On conſidering the queſtion more nearly, I ſoon found, though it belongs to a claſs of problems of conſiderable difficulty, which the Calculus Variationum is uſually employed to reſolve, that it nevertheleſs admits of an eaſy ſolution, and one leading to reſults of remarkable ſimplicity, ſuch as may intereſt

Mathematicians by that circumſtance, as well as by their connection with experimental inquiries.

In the problem thus propoſed, no condition was joined to that of the greateſt attraction, but that of the quantity of homogeneous matter being given. This is the moſt general ſtate of the problem. It is evident, however, that other conditions may be combined with the two preceding; it may be required that the body ſhall have a certain figure, conical, for example, cylindric, &c. and the problem, under ſuch reſtrictions, may be ſtill more readily applicable to experiments than in its moſt general form.

Though the queſtion, thus limited, belongs to the common method of Maxima and Minima, it leads to inveſtigations that are in reality conſiderably more difficult than when it is propoſed in its utmoſt generality.

Among the following inveſtigations, there are alſo ſome that have a particular reference to the experiments on Schehallien. A few years ago, an attempt was made by Lord Webb Seymour and myſelf, toward ſuch a ſurvey of the rocks which compoſe that mountain, as might afford a tolerable eſtimate of their ſpecific gravity, and thereby ſerve to correct the concluſions, deduced from Dr Maskelyne's obſervations, concerning the mean denſity of the earth. The account of this ſurvey, and of the concluſions ariſing from it, belongs naturally to the Society under whoſe direction the original experiment was made; what is offered here, is an inveſtigation of ſome of the theorems employed in obtaining thoſe concluſions. When a new element, the heterogenity of the maſs, or the unequal diſtribution of denſity in the mountain, was to be introduced into the calculations, the ingenious methods employed by Dr Hutton could not always be purſued. The propoſitions that relate to the attraction of a half, or quarter cylinder, on a particle placed in its axis, are intended to remedy this inconveni-

ence,

ence, and will probably be found of ufe in all inquiries concerning the difturbance of the direction of the plumb-line by inequalities, whether in the figure or denfity of the exterior cruft of the globe.

THE firft of the problems here refolved, has been treated of by BOSCOVICH; and his folution is mentioned in the catalogue of his works, as publifhed in the memoirs of a philofophical fociety at Pifa. I have never, however, been able to procure a fight of thefe memoirs, nor to obtain any account of the folution juft mentioned, and therefore am fenfible of hazarding a good deal, when I treat of a fubject that has paffed through the hands of fo able a mathematician, without knowing the conclufions which he has come to, or the principles which he has employed in his inveftigation. In fuch circumftances, if my refult is juft, I cannot reafonably expect it to be new; and I fhould, indeed, be much alarmed to be told, that it has not been anticipated. The other problems contained in this paper, as far as I know, have never been confidered.

I.

To find the folid into which a mafs of homogeneous matter muft be formed, in order to attract a particle given in pofition, with the greateft force poffible, in a given direction.

LET A (Fig. 1. Pl. 6.) be the particle given in pofition, AB the direction in which it is to be attracted; and ACBH a fection of the folid required, by a plane paffing through AB.

SINCE the attraction of the folid is a maximum, by hypothefis, any fmall variation in the figure of the folid, provided the quantity of matter remain the fame, will not change the attraction in the direction AB. If, therefore, a fmall portion of matter be taken from any point C, in the fuperficies of the folid, and placed at D, another point in the fame fuperficies, there will be no variation produced in the force which the ſolid exerts on the particle A, in the direction AB.

THE curve ACB, therefore, is the locus of all the points in which a body being placed, will attract the particle A in the direction AB, with the ſame force.

THIS condition is ſufficient to determine the nature of the curve ABC. From C, any point in that curve, draw CE perpendicular to AB; then if a maſs of matter placed at C be called m^3, $\frac{m^3}{AC^2}$ will be the attraction of that maſs on A, in the direction AC, and $\frac{m^3 \times AE}{AC^3}$ will be its attraction in the direction AB. As this is conſtant, it will be equal to $\frac{m^3}{AB^2}$, and therefore $AB^2 \times AE = AC^3$.

ALL the ſections of the required ſolid, therefore, by planes paſſing through AB, have this property, that $AC^3 = AB^2 \times AE$; and as this equation is ſufficient to determine the nature of the curve to which it belongs, therefore all the ſections of the ſolid, by planes that paſs through AB, are ſimilar and equal curves; and the ſolid of conſequence may be conceived to be generated by the revolution of ACB, any one of theſe curves, about AB as an axis.

THE ſolid ſo generated may be called the *Solid of greateſt Attraction*; and the line ACB, the *Curve of equal Attraction*.

II.

TO find the equation between the co-ordinates of ACB, the curve of equal attraction.

FROM

From C (Fig. 1.) draw CE perpendicular to AB; let $AB=a$, $AE=x$, $EC=y$. We have found $AB^2 \times AE = AC^3$, that is, $a^2 x = (x^2+y^2)^{\frac{3}{2}}$, or $a^4 x^2 = (x^2+y^2)^3$, which is an equation to a line of the 6th order.

To have y in terms of x, $x^2+y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}}$, $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, and $y = x^{\frac{1}{3}} \sqrt{a^{\frac{4}{3}} - x^{\frac{4}{3}}}$.

Hence $y=0$, both when $x=0$, and when $x=a$. Alſo if x be ſuppoſed greater than a, y is impoſſible. No part of the curve, therefore, lies beyond B.

The parts of the curve on oppoſite ſides of the line AB, are ſimilar and equal, becauſe the poſitive and negative values of y are equal. There is alſo another part of the curve on the ſide of A, oppoſite to B, ſimilar and equal to ACB; for the values of y are the ſame whether x be poſitive or negative.

III.

The curve may eaſily be conſtructed without having recourſe to the value of y juſt obtained.

Let $AB=a$, (Fig. 1.) $AC=z$, and the angle $BAC=\varphi$. Then $AE = AC \times \text{coſ}\,\varphi = z \,\text{coſ}\,\varphi$, and ſo $a^2 z \,\text{coſ}\,\varphi = z^3$, or $a^2 \text{coſ}\,\varphi = z^2$; hence $z = a\sqrt{\text{coſ}\,\varphi}$.

From this formula a value of AC or z may be found, if φ or the angle BAC be given; and if it be required to find z in numbers, it may be conveniently calculated from this expreſſion. A geometrical conſtruction may alſo be eaſily derived from it. For if with the radius AB, a circle BFH be deſcribed from the centre A; if AC be produced to meet the circumfe-

rence

ence in F, and if FG be drawn at right angles to AB, then $\frac{AG}{AB} = \text{cof}\,\varphi$, and fo $z = a \times \sqrt{\frac{AG}{AB}} = \sqrt{AB \times AG} = AC$.

THEREFORE, if from the centre A, with the diftance AB, a circle BFH be defcribed, and if a circle be alfo defcribed on the diameter AB, as AKB, then drawing any line AF from A, meeting the circle BFH in F, and from F letting fall FG perpendicular on AB, interfecting the femicircle AKB in K; if AK be joined, and AC made equal to AK, the point C is in the curve.

FOR $AK = \sqrt{AB \times AG}$, from the nature of the femicircle, and therefore $AC = \sqrt{AB \times AG}$, which has been fhewn to be a property of the curve. In this way, any number of points of the curve may be determined; and the *Solid of greateft attraction* will be defcribed, as already explained, by the revolution of this curve about the axis AB.

IV.

To find the area of the curve ACB.

1. LET ACE, AFG (Fig. 2.) be two radii, indefinitely near to one another, meeting the curve ACB in C and F, and the circle, defcribed with the radius AB, in E and G. Let $AC = z$ as before, the angle $BAC = \varphi$, and $AB = a$. Then $GE = a\dot{\varphi}$, and the area $AGE = \frac{1}{2} a^2 \dot{\varphi}$, and fince $AE^2 : AC^2 :: \text{Sect. } AEG : \text{Sect. } ACF$, the fector $ACF = \frac{1}{2} z^2 \dot{\varphi}$. But $z^2 = a^2 \text{cof}\,\varphi$, (§ III.), whence the fector ACF, or the fluxion of the area $ABC = \frac{1}{2} a^2 \dot{\varphi} \text{cof}\,\varphi$, and confequently the area $ABC = \frac{1}{2} a^2 \text{fin}\,\varphi$, to which no conftant quantity need be added, becaufe it vanifhes when $\varphi = 0$, or when the area ABC vanifhes.

THE

The whole area of the curve, therefore, is $\frac{1}{2}a^2$, or $\frac{1}{2}AB^2$; for when ϕ is a right angle sin$\phi = 1$. Hence the area of the curve on both fides of AB is equal to the fquare of AB.

2. The value of x, when y is a maximum, is eafily found. For when y, and therefore y^2 is a maximum, $\frac{2}{3}a^{\frac{4}{3}}x^{-\frac{1}{3}} = 2x$, or $3x^{\frac{4}{3}} = a^{\frac{4}{3}}$, that is $x = \frac{a}{3^{\frac{3}{4}}} = \frac{a}{\sqrt[4]{27}}$.

Hence, calling b the value of y when a maximum,

$$b^2 = a^{\frac{4}{3}} \times \frac{a^{\frac{2}{3}}}{27^{\frac{1}{6}}} - \frac{a^2}{27^{\frac{1}{2}}} = a^2\left(\frac{27^{\frac{1}{3}} - 1}{27^{\frac{1}{2}}}\right) = \frac{2a^2}{\sqrt{27}}, \text{ and fo } b = a\frac{\sqrt{2}}{\sqrt[4]{27}},$$

and therefore $a : b :: \sqrt[4]{27} : \sqrt{2}$, or as $11 : 7$ nearly.

3. It is material to obferve, that the radius of curvature at A is infinite. For fince $y^2 = a^{\frac{4}{3}}x^{\frac{2}{3}} - x^2$, $\frac{y^2}{x} = \frac{a^{\frac{4}{3}}}{x^{\frac{1}{3}}} - x$. But when x is very fmall, or y indefinitely near to A, $\frac{y^2}{x}$ becomes the diameter of the circle having the fame curvature with ACB at A, and when x vanifhes, this value of $\frac{y^2}{x}$, or $\frac{a^{\frac{4}{3}}}{x^{\frac{1}{3}}} - x$, becomes infinite, becaufe of the divifor $x^{\frac{1}{3}}$ being in that cafe $= 0$. The diameter, therefore, and the radius of curvature at A are infinite. In other words, no circle, having its centre in AB produced, and paffing through A, can be defcribed with fo great a radius, but that, at the point A, it will be within the curve of equal attraction.

The

The solid of greatest attraction, then, at the extremity of its axis, where the attracted particle is placed, is exceedingly flat, approaching more nearly to a plane than the superficies of any sphere can do, however great its radius.

4. To find the radius of curvature at B, the other extremity of the axis, since $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, if we divide by $a - x$, we have $\frac{y^2}{a-x} = \frac{a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2}{a-x}$. But at B, when $a - x$, or the abscissa reckoned from B vanishes, $\frac{y^2}{a-x}$ is the diameter of the circle having the same curvature with ACB in B. But when $a - x = 0$, or $a = x$, both the numerator and denominator of the fraction $\frac{a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2}{a-x}$ vanish, so that its ultimate value does not appear. To remove this difficulty, let $a - x = z$, or $x = a - z$, then we have $y^2 = a^{\frac{4}{3}} (a-z)^{\frac{2}{3}} - (a-z)^2$. But when z is extremely small, its powers, higher than the first, may be rejected; and therefore $(a-z)^{\frac{2}{3}} = a^{\frac{2}{3}} \left(1 - \frac{z}{a}\right)^{\frac{2}{3}} = a^{\frac{2}{3}} (1 - \frac{2z}{3a}$, &c.) Therefore the equation to the curve becomes in this case, $y^2 = a^{\frac{4}{3}} \times a^{\frac{2}{3}} \left(1 - \frac{2z}{3a}\right) - a^2 + 2az = a^2 - \frac{2}{3}az - a^2 + 2az = \frac{4}{3}az$.

HENCE

HENCE $\frac{y^2}{2z}$, or the radius of curvature at B $= \frac{2}{3}a$. The curve, therefore, at B falls wholly without the circle BKA, defcribed on the diameter AB, as its radius of curvature is greater. This is alfo evident from the conftruction.

V.

To find the force with which the folid above defined attracts the particle A in the direction AB.

LET b (Fig. 2.) be a point indefinitely near to B, and let the curve A c b be defcribed fimilar to ACB. Through C draw C c D perpendicular to AB, and fuppofe the figure thus conftructed to revolve about AB; then each of the curves ACB, A c b will generate a folid of greateft attraction; and the excefs of the one of thefe folids above the other, will be an indefinitely thin fhell, the attraction of which is the variation of the attraction of the folid ACB, when it changes into A c b.

AGAIN, by the line DC, when it revolves along with the reft of the figure about AB, a circle will be defcribed; and by the part C c, a circular ring, on which, if we fuppofe a folid of indefinitely fmall altitude to be conftituted, it will make the element of the folid fhell AC c. Now the attraction exerted by this circular ring upon A, will be the fame as if all the matter of it were united in the point C, and the fame, therefore, as if it were all united in B.

BUT the circular ring generated by C c, is $= \pi(DC^2 - Dc^2)$ $= 2\pi DC \times Cc$. Now $2 DC \times Cc$ is the variation of y^2, or DC^2, while DC paffes into D c, and the curve BCA into the curve bcA; that is $2 DC \times Cc$ is the fluxion of y^2, or of $a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, taken on the ſuppoſition that x is conſtant and a variable, viz. $\frac{4}{3} a^{\frac{1}{3}} \dot{a} \times x^{\frac{2}{3}}$. Therefore the ſpace generated by Cc = $\frac{4\pi}{3} a^{\frac{1}{3}} x^{\frac{2}{3}} \dot{a}$.

If this expreſſion be multiplied by $\dot{x}$, we have the element of the ſhell $= \frac{4\pi}{3} a^{\frac{1}{3}} x^{\frac{2}{3}} \dot{a} \dot{x}$.

In order to have the ſolidity of the ſhell ACBbc, the above expreſſion muſt be integrated relatively to x, that is, ſuppoſing only x variable, and it is then $\frac{3}{5} \times \frac{4\pi}{3} a^{\frac{1}{3}} x^{\frac{5}{3}} \dot{a} + C$. But C=o, becauſe the fluent vaniſhes when x vaniſhes, therefore the portion of the ſhell ACc = $\frac{4}{5} x^{\frac{5}{3}} a^{\frac{1}{3}} \dot{a}$, and when $x=a$, the whole ſhell $= \frac{4\pi}{5} a^2 \dot{a}$.

Now, if the whole quantity of matter in the ſhell were united at B, its attractive force exerted on A, would be the ſame with that of the ſhell; therefore the whole force of the ſhell $= \frac{4\pi}{5} \dot{a}$. The ſame is true for every other indefinitely thin ſhell into which the ſolid may be ſuppoſed to be divided; and therefore the whole attraction of the ſolid is equal to $\int \frac{4\pi}{5} \dot{a}$, ſuppoſing a variable, that is $= \frac{4\pi}{5} a$.

Hence

HENCE we may compare the attraction of this solid with that of a sphere of which the axis is AB, for the attraction of that sphere $= \frac{\pi}{6} a^3 \times \frac{4}{a^2} = \frac{2\pi}{3} a$. The attraction of the solid ADBH, (Fig. 1.) is, therefore, to that of the sphere on the same axis as $\frac{4\pi}{5} a$ to $\frac{2\pi}{3} a$, or as 6 to 5.

VI.

To find the content of the solid ADBH, we need only integrate the fluxionary expression for the content of the shell, viz. $\frac{4\pi}{5} a^2 \dot{a}$. We have then $\frac{4\pi}{15} a^3 =$ the content of the solid ADBH. Since the solidity of the sphere on the axis a is $= \frac{\pi}{6} a^3$, the content of the solid ADBH is to that of the sphere on the same axis as $\frac{4\pi}{15} a^3$ to $\frac{\pi}{6} a^3$; that is, as $\frac{4}{15}$ to $\frac{1}{6}$, or as 8 to 5.

VII.

LASTLY, To compare the attraction of this solid with the attraction of a sphere of equal bulk, let $m^3 =$ any given mass of matter formed into the solid ADBH; then for determining AB, we have this equation, $\frac{4\pi}{15} a^3 = m^3$, and $a = m \sqrt[3]{\frac{15}{4\pi}}$; and there-

 fore

fore alſo the attraction of the ſolid, (which is $\frac{4\pi}{5}a$) $= \frac{4\pi}{5} m \sqrt[3]{\frac{15}{4\pi}}$

$$= m \left(\frac{4 \cdot 5^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot \pi^{\frac{2}{3}}}{5 \cdot 4^{\frac{1}{3}}}\right) = m \left(\frac{4^{\frac{2}{3}} \cdot 3^{\frac{1}{3}} \cdot \pi^{\frac{2}{3}}}{5^{\frac{2}{3}}}\right) = m \sqrt[3]{\frac{48\pi^2}{25}}.$$

Again, if m^3 be formed into a ſphere, the radius of that ſphere $= m \sqrt[3]{\frac{3}{4\pi}}$, and the attraction of it on a particle at its

ſurface $= \dfrac{m^3}{m^2 \left(\frac{3}{4\pi}\right)^{\frac{2}{3}}} = m \dfrac{(16\pi^2)^{\frac{1}{3}}}{9^{\frac{1}{3}}}$.

Hence the attraction of the ſolid ADBH, is to that of a ſphere equal to it, as $m \left(\frac{48}{25}\pi^2\right)^{\frac{1}{3}}$ to $m \left(\frac{16\pi^2}{9}\right)^{\frac{1}{3}}$; that is, as $(27)^{\frac{1}{3}}$ to $(25)^{\frac{1}{3}}$, or as 3 to the cube-root of 25.

The ratio of 3 to $\sqrt[3]{25}$, is nearly that of 3 to $3 - \frac{2}{27}$, or of 81 to 79; and this is therefore alſo nearly equal to the ratio of the attraction of the ſolid ADBH to that of a ſphere of equal magnitude.

VIII.

It has been ſuppoſed in the preceding inveſtigation, that the particle on which the ſolid of greateſt attraction exerts its force is in contact with that ſolid. Let it now be ſuppoſed, that the diſtance between the ſolid and the particle is given; the ſolid being

being on one ſide of a plane, and the particle at a given diſtance from the ſame plane on the oppoſite ſide. The maſs of matter which is to compoſe the ſolid being given, it is required to conſtruct the ſolid.

LET the particle to be attracted be at A (Fig. 3.), from A draw AA′ perpendicular to the given plane, and let EF be any ſtraight line in that plane, drawn through the point A′; it is evident that the axis of the ſolid required muſt be in AA′ produced. Let B be the vertex of the ſolid, then it will be demonſtrated as has been done above, that this ſolid is generated by the revolution of the curve of *equal attraction*, (that of which the equation is $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$), about the axis of which one extremity is at A, and of which the length muſt be found from the quantity of matter in the ſolid.

THE ſolid required, then, is a ſegment of the ſolid of greateſt attraction, having B for its vertex, and a circle, of which A′E or A′F is the radius, for its baſe.

To find the ſolid content of ſuch a ſegment, CD being $= y$, and AC $= x$, we have $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, and $\pi y^2 \dot{x} = \pi a^{\frac{4}{3}} x^{\frac{2}{3}} \dot{x} - \pi x^2 \dot{x}$ = the cylinder which is the element of the ſolid ſegment.

THEREFORE $\int \pi y^2 \dot{x}$, or the ſolid ſegment intercepted between B and D muſt be $\frac{3}{5} \pi a^{\frac{4}{3}} x^{\frac{5}{3}} - \frac{1}{3} \pi x^3 + C$. This muſt vaniſh when $x = a$, or when C comes to B, and therefore $C = -\frac{4\pi}{15} a^3$.

$-\frac{4\pi}{15}a^3$. The ſegment, therefore, intercepted between B and C, the line AC being x, is $\frac{4\pi}{15}a^3 - \frac{3\pi}{5}a^{\frac{4}{3}}x^{\frac{5}{3}} + \frac{\pi}{3}x^3$.

This alſo gives $\frac{4\pi}{15}a^3$, for the content of the whole ſolid, when $x = 0$, the ſame value that was found by another method at § VI.

Now, if we ſuppoſe x to be $=$ AA′, and to be given $= b$, the ſolid content of the ſegment becomes $\frac{4\pi}{15}a^3 - \frac{3}{5}\pi a^{\frac{4}{3}}b^{\frac{5}{3}} + \frac{\pi}{3}b^3$, which muſt be made equal to the given ſolidity which we ſhall ſuppoſe $= m^3$, and from this equation a, which is yet unknown, is to be determined. If, then, for $a^{\frac{1}{3}}$ we put u, we have $\pi\left(\frac{4}{15}u^9 - \frac{3}{5}b^{\frac{5}{3}}u^4 + \frac{1}{3}b^3\right) = m^3$, or $\frac{4}{15}u^9 - \frac{3}{5}b^{\frac{5}{3}}u^4 = \frac{m^3}{\pi} - \frac{1}{3}b^3$ and $u^9 - \frac{9}{4}b^{\frac{5}{3}}u^4 = \frac{15m^3}{4\pi} - \frac{15}{12}b^3$.

The ſimpleſt way of reſolving this equation, would be by the rule of falſe poſition. In ſome particular caſes, it may be reſolved more eaſily; thus, if $\frac{15m^3}{\pi} - \frac{15}{12}b^3 = 0$, $u^9 - \frac{9}{4}b^{\frac{5}{3}}u^4 = 0$, and $u^5 = \frac{9}{4}b^{\frac{5}{3}}$, that is $a^{\frac{5}{3}} = \frac{9}{4}b^{\frac{5}{3}}$ or $a = b \times \left(\frac{9}{4}\right)^{\frac{3}{5}} = b\sqrt[5]{\frac{729}{64}}$.

IX.

IX.

1. If it be required to find the equation to the ſuperficies of the ſolid of greateſt attraction, and alſo to the ſections of it parallel to any plane paſſing through the axis; this can readily be done by help of what has been demonſtrated above.

Let AHB (Fig. 4.) be a ſection of the ſolid, by a plane through AB its axis. Let G be any point in the ſuperficies of the ſolid, GF a perpendicular from G on the plane AHB, and FE a perpendicular from F on the axis. Let AE $= x$, EF $= z$, FG $= v$, then x, z, and v are the three co-ordinates by which the ſuperficies is to be defined. Let AB $= a$, EH $= y$, then, from the nature of the curve AHB, $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$. But becauſe the plane GEH is at right angles to AB, G and H are in the circumference of a circle of which E is the centre; ſo that GE $=$ EH $= y$. Therefore $EF^2 + FG^2 = EH^2$, that is, $z^2 + v^2 = y^2$, and by ſubſtitution for y^2 in the former equation, $z^2 + v^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, or $\left(x^2 + z^2 + v^2\right)^3 = a^4 x^2$, which is the equation to the ſuperficies of the ſolid of greateſt attraction.

2. If we ſuppoſe EF, that is z, to be given $= b$, and the ſolid to be cut by a plane through FG and CD, (CD being parallel to AB), making on the ſurface of the ſolid the ſection DGC; and if AK be drawn at right angles to AB, meeting DC in K, then we have, by writing b for z in either the preceding equations, $b^2 + v^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, and $v^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2 - b^2$ for the equation of the curve DGC, the co-ordinates being GF and FK, becauſe FK is equal to AE or x.

This

THIS equation alſo belongs to a curve of equal attraction; the plane in which that curve is being parallel to AB, the line in which the attraction is eſtimated, and diſtant from it by the ſpace b.

INSTEAD of reckoning the abſciſſa from K, it may be made to begin at C. If AL or CK $= h$, then the value of h is determined from the equation $b^2 = a^{\frac{4}{3}} h^{\frac{2}{3}} - h^2$, and if $x = h + u$, u being put for CF, $v^2 = a^{\frac{4}{3}} (h + u)^{\frac{2}{3}} - (h + u)^2 - a^{\frac{4}{3}} h^{\frac{2}{3}} + h^2$, or $v^2 + (h + u)^2 + b^2 = a^{\frac{4}{3}} (h + u)^{\frac{2}{3}}$, or $(v^2 + (h + u)^2 + b^2)^3 = a^4 (h + u)^2$.

WHEN b is equal to the maximum value of the ordinate EH, (IV. 2.) the curve CGD goes away into a point; and if b be ſuppoſed greater than this, the equation to the curve is impoſſible.

X.

THE ſolid of greateſt attraction may be found, and its properties inveſtigated, in the way that has now been exemplified, whatever be the law of the attracting force. It will be ſufficient, in any caſe, to find the equation of the generating curve, or the curve of equal attraction.

THUS, if the attraction which the particle C (Fig. 1.) exerts on the given particle at A, be inverſely as the m power of the diſtance, or as $\frac{1}{AC^m}$, then the attraction in the direction AE will be $\frac{AE}{AC^{m+1}}$, and if we make this $= \frac{1}{AB^m}$, we have $\frac{AE}{AC^{m+1}}$

$$= \frac{1}{AB^m},$$

$= \frac{1}{AB^m}$, or making AE $= x$, EC $= y$, and AB $= a$, as before,

$\frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} = \frac{1}{a^m}$, or $a^m x = (x^2+y^2)^{\frac{m+1}{2}}$, and $x^2 + y^2 =$

$a^{\frac{2m}{m+1}} x^{\frac{2}{m+1}}$, or $y^2 = a^{\frac{2m}{m+1}} x^{\frac{2}{m+1}} - x^2$.

If $m = 1$, or $m + 1 = 2$, this equation becomes $y^2 = ax - x^2$, being that of a circle of which the diameter is AB. If, therefore, the attracting force were inversely as the distance, the solid of greatest attraction would be a sphere.

If the force be inversely as the cube of the distance, or $m = 3$, and $m + 1 = 4$, the equation is $y^2 = a^{\frac{3}{2}} x^{\frac{1}{2}} - x^2$, which belongs to a line of the 4th order.

If $m = 4$, and $m + 1 = 5$, the equation is $y^2 = a^{\frac{8}{5}} x^{\frac{2}{5}} - x^2$; which belongs to a line of the 10th order.

In general, if m be an even number, the order of the curve is $\overline{m+1} \times 2$; but if m be an odd number, it is $m + 1$ simply.

XI.

In the same manner that the solid of greatest attraction has been found, may a great *class* of similar problems be resolved. Whenever the property that is to exist in its greatest or least degree, belongs to all the points of a plane figure, or to all the points of a solid, given in magnitude, the question is reduced to the determination of the locus of a certain equation, just as in the preceding example.

 Let

Let it, for inftance, be required to find a folid given in magnitude, fuch, that from all the points in it, ftraight lines being drawn to any affigned number of given points, the fum of the fquares of all the lines fo drawn fhall be a *minimum*. It will be found, by reafoning as in the cafe of the folid of greateft attraction, that the fuperficies bounding the required folid muft be fuch that the fum of the fquares of the lines drawn from any point in it, to all the given points, muft be always of the fame magnitude. Now, the fum of the fquares of the lines drawn from any point to all the given points, may be fhewn by plane geometry to be equal to the fquare of the line drawn to the centre of gravity of thefe given points, multiplied by the number of points, together with a given fpace. The line, therefore, drawn from any point in the required fuperficies to the centre of gravity of the given points, is given in magnitude, and therefore the fuperficies is that of a fphere, having for its centre the centre of gravity of the given points.

The magnitude of the fphere is next determined from the condition, that its folidity is given.

In general, if x, y, and z, are three rectangular co-ordinates that determine the pofition of any point of a folid given in magnitude, and if the value of a certain function Q of x, y and z, be computed for each point of the folid, and if the fum of all thefe values of Q added together, be a maximum or a minimum, the folid is bounded by a fuperficies in which the function Q is every where of the fame magnitude. That is, if the triple integral $\int \dot{x} \int \dot{y} \int Q \dot{z}$ be the greateft or leaft poffible, the fuperficies bounding the folid is fuch that $Q = A$, a conftant quantity.

The fame holds of plane figures; the propofition is then fimpler, as there are only two co-ordinates, fo that $\int \dot{x} \int Q \dot{y}$ is the

the quantity that is to be a maximum or a minimum, and the line bounding the figure is defined by the equation Q = A.

All the queſtions, therefore, which come under this deſcription, though they belong to an order of problems, which requires in general the application of one of the moſt refined inventions of the New Geometry, the *Calculus Variationum*, form a particular diviſion admitting of reſolution by much ſimpler means, and directly reducible to the conſtruction of loci.

In theſe problems alſo, the ſynthetical demonſtration will be found extremely ſimple. In the inſtance of the ſolid of greateſt attraction this holds remarkably. Thus, it is obvious, that (Fig. 1.) any particle of matter placed without the curve ACBH, will attract the particle at A in the direction AB, leſs than any of the particles in that curve, and that any particle of matter within the curve, will attract the particle at A more than any particle in the curve, and more, *à fortiori*, than any particle without the curve. The ſame is true of the whole ſuperficies of the ſolid. Now, if the figure of the ſolid be any how changed, while its quantity of matter remains the ſame, as much matter muſt be expelled from within the ſurface, at ſome one place C, as is accumulated without the ſurface at ſome other point H. But the action of any quantity of matter within the ſuperficies ACBH on A, is greater than the action of the ſame without the ſuperficies ACBH. The ſolid ACBH, therefore, by any change of its figure, muſt loſe more attraction than it gains; that is, its attraction is diminiſhed by every ſuch change, and therefore it is itſelf the ſolid of greateſt attraction. Q. E. D.

XII.

THE preceding theorems relate to the ſolids, which, of all ſolids whatſoever of a given content, have the greateſt attraction in a given direction. It may be intereſting alſo to know, among bodies of a given kind, and a given ſolid content, for example, among cones, cylinders, or parallelepipeds, given in magnitude, which has the greateſt attractive power, in the direction of a certain ſtraight line. We ſhall begin with the cone.

LET ABC (Fig. 5.) be a cone of which the axis is AD, required to find the angle BAC, when the force which the cone exerts, in the direction AD, on the particle A at its vertex, is greater than that which any other cone of the ſame ſolid content, can exert in the direction of its axis, on a particle at its vertex.

IT is known, if π be the ſemicircumference of the circle of which the radius is 1, that is, if $\pi = 3.14159$, &c. that the attraction of the cone ABC, on the particle A, in the direction AD, is $= 2\pi \times \left(AD - \frac{AD^2}{AB}\right)$. (SIMPSON's Fluxions, vol. ii. Art. 377.)

LET $AD = x$, $AB = z$, the ſolid content of the cone $= m^3$, and its attraction $= A$.

THEN $A = 2\pi \left(x - \frac{x^2}{z}\right)$, and $\pi x (z^2 - x^2) = 3m^3$.

THE quantity $x - \frac{x^2}{z}$, is to be a maximum, and therefore,

$$\dot{x} - \frac{2xz\dot{x} - x^2\dot{z}}{z^2} = 0, \text{ or } z^2 - 2xz + x^2 \cdot \frac{\dot{z}}{\dot{x}} = 0.$$

AGAIN,

AGAIN, from the equation $\pi x(z^2 - x^2) = 3m^3$, we have $2xz\dot{z} + z^2\dot{x} - 3x^2\dot{x} = 0$, and $\frac{\dot{z}}{\dot{x}} = \frac{3x^2 - z^2}{2xz}$, and by substituting this value of $\frac{\dot{z}}{\dot{x}}$ in the former equation, we have

$$z^2 - \frac{5}{2}xz + \frac{3}{2}\cdot\frac{x^3}{z} = 0.$$

As this equation is homogeneous, if we make $\frac{x}{z} = u$, we will obtain an equation involving u only, and therefore determining the ratio of z to x, or of AB to AD. Substituting, accordingly, uz for x in the last equation, we have

$$z^2 - \frac{5}{2}uz^2 + \frac{3}{2}u^3z^2 = 0, \text{ and } 1 - \frac{5}{2}u + \frac{3}{2}u^3 = 0.$$

THIS equation is obviously divisible by $u - 1$, and when so divided, gives $\frac{3}{2}u^2 + \frac{3}{2}u - 1 = 0$, or $u^2 + u = \frac{2}{3}$, whence

$$u = -\frac{1}{2} \pm \sqrt{\frac{11}{12}}.$$

THIS is the value of $\frac{x}{z}$, and as $\frac{x}{z}$ must be less than unity, because AB is greater than AD, the negative value of u, or $-\frac{1}{2} - \sqrt{\frac{11}{12}}$, is excluded; so that $u = -\frac{1}{2} + \sqrt{\frac{11}{12}} = .45761$ nearly.

Now $u = \frac{AD}{AB}$ = the cosine of the angle BAD, or half the angle of the cone; therefore that angle $= 62°. 46'$ nearly.

As

As the tangent of 62°.46′ is not far from being double of the radius, therefore the cone of greateſt attraction has the radius of its baſe nearly double of its altitude.

To compare the attraction of this cone with that of a ſphere containing the ſame quantity of matter, we muſt expreſs the attraction in terms of u, the ratio of x to z, which has now been found.

Because $\pi x\,(z^2 - x^2) = 3m^3$, and $z = \frac{x}{u}$, $\pi x\left(\frac{x^2}{u^2} - x^2\right) = \pi x^3\left(\frac{1-u^2}{u^2}\right) = 3m^3$, and $x = m\,.\sqrt[3]{\frac{3u^2}{\pi(1-u^2)}}$.

Now, we have $A = 2\pi\left(x - \frac{x^2}{z}\right)$, and ſince $\frac{x}{z} = u$, $\frac{x^2}{z} = m\,u\sqrt[3]{\frac{3u^2}{\pi(1-u^2)}}$, and $A = 2\pi\left(m.\sqrt[3]{\frac{3u^2}{\pi(1-u^2)}} - mu.\sqrt[3]{\frac{3u^2}{\pi(1-u^2)}}\right) = 2\pi m.(1-u)\sqrt[3]{\frac{3u^2}{\pi(1-u^2)}}$; wherefore, $A^3 = 8\pi^3 m^3 (1-u)^3 \times \frac{3u^2}{\pi(1-u^2)} = 24\pi^2 m^3 . \frac{u^2(1-u)^2}{1+u}$.

But if A' be the attraction of a ſphere of which the maſs is m^3, on a particle at its ſurface, $A' = m\sqrt[3]{\frac{16\pi}{9}}$, and $A'^3 = m^3.\frac{16\pi^2}{9}$. Therefore $A^3 : A'^3 :: \frac{24u^2(1-u)^2}{1+u} \quad \frac{16}{9} :: \frac{27u^2(1-u)^2}{2(1+u)} : 1$; and conſequently $A : A' :: 3\sqrt[3]{\frac{u^2(1-u)^2}{2(1+u)}} : 1$.

If, in this expreſſion, we ſubſtitute .45761 for u, we ſhall have $A : A' :: .82941 : 1$, ſo that the attraction of the cone,

when

when a maximum is about $\frac{4}{5}$ of the attraction of a ſphere of equal ſolidity.

XIII.

Of all the cylinders given in maſs, or quantity of matter, to find that which ſhall attract a particle, at the extremity of its axis, with the greateſt force.

Let DF (Fig. 6.) be a cylinder of which the axis is AB, if AC be drawn, the attraction of the cylinder on the particle A is $2\pi \times (AB + BC - AC)$ *, and we have therefore to find when $AB + BC - AC$ is a maximum, ſuppoſing $AB . BC^2$ to be equal to a given ſolid.

Let $AB = x$, $BC = y$, then $AC = \sqrt{x^2 + y^2}$, and the quantity that is to be a maximum is $x + y - \sqrt{x^2 + y^2}$. We have therefore $\dot{x} + \dot{y} - \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} = 0$, and $(\dot{x} + \dot{y})\ (x^2 + y^2)^{\frac{1}{2}} = x\dot{x} + y\dot{y}$, or $\left(1 + \frac{\dot{y}}{\dot{x}}\right)(x^2 + y^2)^{\frac{1}{2}} = x + y \cdot \frac{\dot{y}}{\dot{x}}$.

But ſince $\pi x y^2 = m^3$, $2xy\dot{y} + y^2\dot{x} = 0$, or $2x\dot{y} = -y\dot{x}$, and $\frac{\dot{y}}{\dot{x}} = -\frac{y}{2x}$.

Therefore

* Princip. Lib. I. Prop. 91. Alſo Simpson's Fluxions, vol. II. § 379. In the former, the conſtant multiplier 2π is omitted, as it is in ſome other of the theorems relating to the attraction of bodies. This requires to be particularly attended to, when theſe propoſitions are to be employed for comparing the attraction of ſolids of different ſpecies.

THEREFORE $\left(1 - \frac{y}{2x}\right) \cdot (x^2 + y^2)^{\frac{1}{2}} = x - y \cdot \frac{y}{2x}$, or $(2x - y)\ (x^2 + y^2)^{\frac{1}{2}} = 2x^2 - y^2$.

As this equation is homogeneous, if we make $\frac{y}{x} = u$, or $y = ux$, both x and y may be exterminated. For we have by ſubſtituting ux for y, $(2x - ux)\ (x^2 + u^2 x^2)^{\frac{1}{2}} = 2x^2 - u^2 x^2$, or $(2x^2 - ux^2)\cdot(1 + u^2)^{\frac{1}{2}} = 2x^2 - u^2 x^2$, and dividing by x^2, $(2 - u)\cdot(1 + u^2)^{\frac{1}{2}} = 2 - u^2$; whence ſquaring both ſides, $(4 - 4u + u^2)\ (1 + u^2) = 4 - 4u^2 + u^4$.

FROM this, by multiplying and reducing, we get $4u^2 - 9u = -4$, or $u^2 - \frac{9}{4}u = -1$; and $u = \frac{9 \pm \sqrt{17}}{8}$.

2. THE two values of u in this formula create an ambiguity which cannot be removed without ſome farther inveſtigation. If A be the attraction of the cylinder, then $A = 2\pi(x + y - \sqrt{x^2 + y^2})$, into which expreſſion, if we introduce u, and exterminate both x and y, by help of the equations $\pi x y^2 = m^3$, and $\frac{y}{x} = u$, we get $A = 2\pi^{\frac{2}{3}} m \frac{1 + u - \sqrt{1 + u^2}}{u^{\frac{2}{3}}}$.

NOTWITHSTANDING the radical ſign in this formula, there is but one value of A, correſponding to each value of u, as the poſitive root of $\sqrt{1 - u^2}$ is not applicable to the phyſical problem.

blem. This is evident, becaufe the attraction muft vanifh both when $y = 0$, and when $x = 0$; that is, both when u is nothing, and when it is infinite. This can only happen when $\sqrt{1+u^2}$ is negative.

FARTHER, the value of A is always pofitive (as it ought to be), $1+u$ being greater than $\sqrt{1+u^2}$, becaufe it is the fquare-root of $1+2u+u^2$.

3. PERHAPS the relation between A and u will be beft conceived, by fuppofing A to be the ordinate of a curve in which the abfciffæ are reprefented by the fucceffive values of u. Thus, if OP (Fig. 7.) $= u$, and PM = A, the locus of M is a curve of the figure OMM′, which interfects the axis at O, and has the ordinate PM a maximum, when OP $= \frac{9-\sqrt{17}}{8}$; beyond PM′ the curve has a point M′ of contrary flexure, where it becomes convex toward the axis OR, and afterwards approaches OR continually. It has alfo another branch $mm'n$, correfponding to the affirmative values of $\sqrt{1+u^2}$, which has the perpendicular OQ for an affymptote; and has the ordinate P′m' a minimum, when $u = \frac{9+\sqrt{17}}{8}$. After paffing the point where P′m' is a minimum, this branch of the curve recedes continually from the axis OR. Befides thefe, there are other two branches of the fame curve, on the oppofite fide of OQ, anfwering to the negative values of u. It is, however, only the firft-mentioned of thefe four branches that is connected with the mechanical queftion confidered here.

THE attraction is a maximum, therefore, when $u = \frac{9-\sqrt{17}}{8}$, that is, when y is to x, or the radius of the base of the cylinder, to its altitude, as $9-\sqrt{17}$ to 8, or as 5 to 8 nearly. Therefore also the diameter of the base is to the altitude, when the attraction of the cylinder is greatest, as $9-\sqrt{17}$ to 4, or as 5 to 4 nearly.

5. THE attraction of the cylinder, when a maximum is now to be compared with that of a sphere of equal solid content.

FIRST, to compute the quantity $\frac{1+u-\sqrt{1+u^2}}{u^{\frac{2}{3}}}$, when $u = \frac{9-\sqrt{17}}{8} = .6096$, since $u^2 = .37161$, $1+u^2 = 1.37161$, and $\sqrt{1+u^2} = 1.17116$; so that $1+u-\sqrt{1+u^2} = .43844$.

ALSO because $u^2 = .37161$, $u^{\frac{2}{3}} = .718945$; and therefore $\frac{1+u-\sqrt{1+u^2}}{u^{\frac{2}{3}}} = \frac{4384}{7189}$. Therefore $A = 2\pi^{\frac{2}{3}} m . \frac{1+u-\sqrt{1+u^2}}{u^{\frac{2}{3}}}$ $= 2\pi^{\frac{2}{3}} m \times \frac{4384}{7189}$.

NOW, if A′ be the attraction of a sphere of the solidity m, $A' = \pi^{\frac{2}{3}} m \times \left(\frac{16}{9}\right)^{\frac{1}{3}}$, and $A : A' :: \frac{2 \times 4384}{7189} : \left(\frac{16}{9}\right)^{\frac{1}{3}} ::$ $\frac{8768}{7189} : 1.2114$,

$\frac{8768}{7189}$: 1.2114, or as 1218 to 1211.4; ſo that the attraction of the cylinder, even when its form is moſt advantageous, does not exceed that of a ſphere, of the ſame ſolid content, by more than a hundred and eighty-third part.

6. In a note on one of the letters of G. L. Le Sage, publiſhed by M. Prevost of Geneva *, the following theorem is given concerning the attraction of a cylinder and a ſphere: If a cylinder be circumſcribed about a ſphere, the particle placed in the extremity of the axis of the cylinder, or at the point of contact of the ſphere, and the baſe of the cylinder, is attracted equally by the ſphere, and by that portion of the cylinder which has for its altitude two-thirds of the diameter of the ſphere, and of which the ſolidity is therefore juſt equal to that of the ſphere.

We may inveſtigate this theorem, by ſeeking the altitude of ſuch a part of the circumſcribing cylinder as ſhall have the ſame attraction with the ſphere at the point of contact. If r be the radius of the ſphere, the attraction at any point of its ſurface, is $\frac{4\pi r}{3}$; and if x be the altitude of the cylinder, and the radius of its baſe r, then its attraction on a particle at the extremity of its axis is $2\pi(x+r-\sqrt{x^2+r^2})$. Since theſe attractions are ſuppoſed equal, $2\pi(x+r-\sqrt{x^2+r^2})=\frac{4\pi r}{3}$, and $x+r-\sqrt{x^2+r^2}=\frac{2r}{3}$, whence $\frac{2rx}{3}=\frac{8r^2}{9}$, and $x=\frac{4r}{3}$.

 The

* Notice de la vie de G. L. Le Sage de Genève, par P. Prevost, p. 391.

The altitude of the cylinder is therefore $\frac{4}{3}$ of the radius, or $\frac{2}{3}$ of the diameter of the ſphere, which is Le Sage's Theorem.

This cylinder is alſo known to be equal in ſolidity to the ſphere; but its attraction is not greater than that of the latter, becauſe the proportion of its altitude to the diameter of its baſe is not that which gives the greateſt attraction. Its altitude is to the diameter of its baſe, as $\frac{4}{3}r$ to $2r$, or 4 to 6; in order to have the greateſt effect, it muſt be as 4 to 5 nearly, (§ 3.).

Notwithstanding, therefore, that the form of the one of theſe cylinders is conſiderably different from that of the other, their attractions are very nearly equal; the one of them being the ſame with that of the ſphere, and the other greater than it by about the 183d part. On each ſide of the form which gives the maximum of attraction, there may be great variations of figure, without much change in the attracting force. A ſimilar property belongs to all quantities near their greateſt or leaſt ſtate, but ſeems to hold eſpecially in what regards the attraction of bodies.

XIV.

In conſidering the attraction of the Mountain Shehallien, in ſuch a manner as to make a due allowance for the heterogeneity of the maſs, it became neceſſary to determine the attraction of a half cylinder, or of any ſector of a cylinder, on a point ſituated in its axis, in a given direction, at right angles

to

to that axis. The ſolution of this problem is much connected with the experimental inquiries concerning the attraction of mountains, and affords examples of maxima of the kind that form the principal object of this paper. The following lemma is neceſſary to the ſolution.

LET the quadrilateral DG (Fig. 8.) be the indefinitely ſmall baſe of a column DH, which has everywhere the ſame ſection, and is perpendicular to its baſe DG.

LET A be a point at a given diſtance from D, in the plane DG; it is required to find the force with which the column DH attracts a particle at A, in the direction AD.

LET the diſtance AD $= r$, the angle DAE $= \varphi$, DE (ſuppoſed variable) $= y$, and let EF be a ſection of the ſolid parallel, and equal to the baſe DG; and let the area of DG $= m^2$.

THE element of the ſolid DF is $m^2 \dot{y}$; and ſince DE, or $y = r \tan \varphi$, $\dot{y} = r\, \dot{\overline{\tan \varphi}} = r . \frac{\dot{\varphi}}{\text{coſ} \varphi^2}$, ſo that the element of the ſolid $= m^2 r . \frac{\dot{\varphi}}{\text{coſ} \varphi^2}$.

THIS quantity divided by AE^2, that is, ſince AE : AD : : 1 : coſ φ, by $\frac{r^2}{\text{coſ} \varphi^2}$, gives the element of the attraction in the direction AE equal to $\frac{m^2 r \dot{\varphi}}{\text{coſ} \varphi^2} \times \frac{\text{coſ} \varphi^2}{r^2} = \frac{m^2 \dot{\varphi}}{r}$. To reduce this to the direction AD, it muſt be multiplied into the coſine of the angle DAE or φ, ſo that the element of the attraction of the column in the direction AD is $\frac{m^2}{r} \dot{\varphi} \text{ coſ} \varphi$, and the attraction itſelf $=$ $\frac{m^2}{r} \int \dot{\varphi} \text{ coſ} \varphi = \frac{m^2}{r} \text{ ſin } \varphi$.

WHEN

When ϕ becomes equal to the whole angle ſubtended by the column, the total attraction is equal to the area of the baſe divided by the diſtance, and multiplied by the ſine of the angle of elevation of the column.

If the angle of elevation be 30°, the attraction of the column is juſt half the attraction it would have, ſuppoſing it extended to an infinite height.

In this inveſtigation, m^2 is ſuppoſed an infiniteſimal; but if it be of a finite magnitude, provided it be ſmall, this theorem will afford a ſufficient approximation to the attraction of the column, ſuppoſing the diſtance AD to be meaſured from the centre of gravity of the baſe, and the angle ϕ to be that which is ſubtended by the axis of the column, or by its perpendicular height above the baſe.

XV.

Let the ſemicircle CBG (Fig. 9.), having the centre A, be the baſe of a half cylinder ſtanding perpendicular to the horizon, AB a line in the plane of the baſe, biſecting the ſemicircle, and repreſenting the direction of the meridian; it is required to find the force with which the cylinder attracts a particle at A, in the direction AB, ſuppoſing the radius of the baſe, and the altitude of the cylinder to be given.

Let DF be an indefinitely ſmall quadrilateral, contained between two arches of circles deſcribed from the centre A, and two radii drawn to A; and let a column ſtand on it of the ſame height with the half cylinder, of which the baſe is the ſemicircle CBG. Let $z =$ the angle BAD, the azimuth of D; $v =$ the vertical angle ſubtended by the column on DF; $a =$ the

the height of that column, or of the cylinder, AD $= x$, AB, the radius of the baſe, $= r$.

By the laſt propoſition, the column ſtanding on DF, exerts on A an attraction in the direction AD, which is $= \frac{Dd \times Df}{AD} \times \sin v$.

Now $Dd = \dot{x}$, $Df = x\dot{z}$, and $Dd \times Df = x\dot{z}\dot{x}$. Therefore the attraction in the direction AD is $\frac{x\dot{x}\dot{z}}{x} \times \sin v = \dot{x}\dot{z}\sin v$, and reduced to the direction AB, it is $\dot{x}\dot{z}\sin v \times \cos z$.

This is the element of the attraction of the cylindric ſhell or ring, of which the radius is AD or x, and the thickneſs $\dot{x}$; and therefore integrated on the ſuppoſition that z only is variable, and x and v conſtant, it gives $\dot{x}\sin v \int \dot{z}\cos z = \dot{x}\sin v \times \sin z$ for the attraction of the ſhell. When $z = 90$, and $\sin z = 1$, we have the attraction of a quadrant of the ſhell $= \dot{x}\sin v$, and therefore that of the whole ſemicircle $= 2\dot{x}\sin v$.

Next, if x be made variable, and conſequently v, we have $2\int \dot{x}\sin v$ for the attraction of the ſemi-cylinder.

Now the angle v would have a for its ſine if the radius were $\sqrt{a^2+x^2}$, and ſo $\sin v = \frac{a}{\sqrt{a^2+x^2}}$; wherefore the above expreſſion is $\int \frac{2a\dot{x}}{\sqrt{a^2+x^2}} = 2a\,L(x+\sqrt{a^2+x^2}) + C$; and as this muſt vaniſh when $x = 0$, $2aLa + C = 0$, and $C = -2aLa$, ſo that the

the fluent is $2aL\frac{x+\sqrt{a^2+x^2}}{a}$, which, when $x=r$, gives the attraction of the ſemi-cylinder $=2aL\frac{r+\sqrt{a^2+r^2}}{a}$.

THIS expreſſion is very ſimple, and very convenient in calculation. It is probably needleſs to remark, that the logarithms meant are the hyperbolic.

XVI.

LET it be required to find the figure of a ſemi-cylinder given in magnitude, which ſhall attract a particle ſituated in the centre of its baſe with the greateſt force poſſible, in the direction of a line biſecting the baſe.

THE attraction of the cylinder, as juſt demonſtrated, is $2aL\frac{r+\sqrt{r^2+a^2}}{a}$; and becauſe the ſolid is ſuppoſed to be given in magnitude, we may put $ar^2=m^3$, or $a=\frac{m^3}{r^2}$; ſo that the formula above becomes $\frac{2m^3}{r^2}L\frac{r+\sqrt{r^2+\frac{m^6}{r^4}}}{\frac{m^3}{r^2}}=\frac{2m^3}{r^2}L\frac{r^3+\sqrt{r^6+m^6}}{m^3}$.

Now we may ſuppoſe $m=1$, and then the attraction of the cylinder $=\frac{2}{r^2}L(r^3+\sqrt{r^6+1})$.

THIS

THIS formula vaniſhes whether r be ſuppoſed infinitely great or infinitely ſmall, and, therefore, there muſt be ſome magnitude of r in which its value will be the greateſt poſſible.

IF r is very ſmall in reſpect of 1, $\sqrt{1+r^6} = 1+\frac{r^6}{2}$, and ſo $r^3+\sqrt{1+r^6} = 1+r^3+\frac{r^6}{2}$, or ſimply $= 1+r^3$. But L $(1+r^3)$, if r is very ſmall in reſpect of 1, is r^3; and therefore the ultimate value of the formula, when r is infinitely ſmall, is $\frac{2}{r^2} \times r^3 = 2r$, which is alſo infinitely ſmall.

AGAIN, let r be infinitely great; then $\sqrt{r^6+1} = r^3$, and ſo the formula is $\frac{2}{r^2}$ L . $2r^3$, or $\frac{2 \times 3}{r^2}$ L . $2r$. But the logarithm of an infinitely great quantity r, is an infinite of an order incomparably leſs than r, as is known from the nature of logarithms, (GREG. FONTANÆ Diſquiſitiones Phyſ. Math. de Infinito Logarithmico, Theor. 4.); ſo that $\frac{6}{r^2}$ L $2r$ is leſs than $\frac{6r}{r^2}$, or than $\frac{6}{r}$. But $\frac{6}{r}$ is infinitely ſmall, r being infinitely great, and therefore, when the radius of the cylinder becomes infinitely great, its ſolid content remaining the ſame, its attraction is leſs even than an infiniteſimal of the firſt order.

THE determination of the maximum, by the ordinary method, leads to an exponential equation of conſiderable difficulty, if an accurate ſolution is required. It is, however, eaſily found by trial, that, when the function $\frac{1}{r^2} L(r^3 + \sqrt{1 + r^6})$ is a maximum, r is nearly $= \frac{6}{5}$. Therefore, becaufe $a = \frac{1}{r^2} = \frac{25}{36}$, r is nearly to a as $\frac{6}{5}$ to $\frac{25}{36}$, or as 216 to 125; and this of confequence, is, nearly, the ratio of the radius of the bafe, to the altitude of the half-cylinder, when its attraction, eftimated according to the hypothefis of the problem, is the greateft poffible.

XVII.

To determine the oblate fpheroid of a given folidity which fhall attract a particle at its pole with the greateft force.

Let there be an oblate fpheroid generated by the revolution of the ellipfis ADBE (Pl. 7. Fig. 10.), about the conjugate axis AB, and let F be the focus; then if AF be drawn, and the arch CG defcribed from the centre A, the force with which the fpheroid draws a particle at A, in the direction AC, is $\frac{4\pi . AC . CD^2}{CF^3}$ (CF—CG *). (Maclaurin's Fluxions, § 650).

Let this force $= F$, $AC = a$, $CD = b$, the angle $CAF = \varphi$; then $CF = a \tan\varphi$, and $F = \frac{4\pi a b^2}{a^3 \tan\varphi^3} (\tan\varphi - \varphi)\, a = \frac{4\pi b^2}{a} . \frac{\tan\varphi - \varphi}{\tan\varphi^3}$.

Now if m^3 be the folidity of the fpheroid, fince that folidity is two-thirds of the cylinder, having CD for the radius of its bafe,

* The multiplier 2π, omitted by Maclaurin, is reftored as above, § XIII.

base, and AB for its altitude; therefore $m^3 = \frac{2}{3} \times \pi b^2 \times 2a$ $= \frac{4}{3} \pi a b^2$; so that $b^2 = \frac{m^3}{\frac{4}{3}\pi a} = \frac{3m^3}{4\pi a}$, and $\frac{b^2}{a} = \frac{3m^3}{4\pi a^2}$.

But because AF : AC :: 1 : cos φ, or $b : a :: 1 : \cos\varphi$, $b^2 = \frac{a^2}{\cos\varphi^2}$, and $\frac{b^2}{a} = \frac{a}{\cos\varphi^2}$.

Now since $b^2 = \frac{a^2}{\cos\varphi^2}$, and also $b^2 = \frac{3m^3}{4\pi a}$, we have $\frac{a^2}{\cos\varphi^2} = \frac{3m^3}{4\pi a}$, and $a^3 = \frac{3m^3}{4\pi} . \cos\varphi^2$, or if $\frac{3m^3}{4\pi} = n^3$, $a^3 = n^3 . \cos\varphi^2$, and $a = n \cos\varphi^{\frac{2}{3}}$.

Hence, as $\frac{b^2}{a} = \frac{a}{\cos\varphi^2}$, $\frac{b^2}{a} = \frac{n\cos\varphi^{\frac{2}{3}}}{\cos\varphi^2} = \frac{n}{\cos\varphi^{\frac{4}{3}}}$.

By substituting this value of $\frac{b^2}{a}$ in the value of F, we have F $= \frac{4\pi n}{\cos\varphi^{\frac{4}{3}}} . \frac{\tan\varphi - \varphi}{\tan\varphi^3}$, and because $\tan\varphi^3 = \frac{\sin\varphi^3}{\cos\varphi^3}$, F $=$

$$\frac{2\pi n . (\tan\varphi - \varphi) . \cos\varphi^3}{\sin\varphi^3 \times \cos\varphi^{\frac{4}{3}}} = \frac{2\pi n (\tan\varphi - \varphi) \cos\varphi^{\frac{5}{3}}}{\sin\varphi^3} =$$

$$2\pi n (\tan\varphi - \varphi) . \cos\varphi^{\frac{5}{3}} . \sin\varphi^{-3}.$$

Now when the product of any number of factors is a maximum, if the fluxion of each factor be divided by the factor it-

 self,

ſelf, the ſum of the quotients is equal to nothing. Therefore

$$\frac{\frac{\dot{\varphi}}{\text{coſ}\,\varphi^2} - \dot{\varphi}}{\tan\varphi - \varphi} + \frac{5\,\text{coſ}\,\varphi^{\frac{2}{3}} \,.\, \dot{\overline{\text{coſ}\,\varphi}}}{3\,\text{coſ}\,\varphi^{\frac{5}{3}}} - \frac{3\,\text{ſin}\,\varphi^{-4} \,.\, \dot{\overline{\text{ſin}\,\varphi}}}{\text{ſin}\,\varphi^{-3}} = 0, \text{ or}$$

$$\frac{\dot{\varphi}\,(1 - \text{coſ}\,\varphi^2)}{\text{coſ}\,\varphi^2\,(\tan\varphi - \varphi)} - \frac{5\,\dot{\varphi}\,\text{ſin}\,\varphi}{3\,\text{coſ}\,\varphi} - \frac{3\,\dot{\varphi}\,\text{coſ}\,\varphi}{\text{ſin}\,\varphi} = 0,$$

$$\text{and } \frac{\text{ſin}\,\varphi^2}{\text{coſ}\,\varphi^2\,(\tan\varphi - \varphi)} - \frac{5\,\text{ſin}\,\varphi}{3\,\text{coſ}\,\varphi} - \frac{3\,\text{coſ}\,\varphi}{\text{ſin}\,\varphi} = 0;$$

$$\frac{\text{ſin}\,\varphi}{\text{coſ}\,\varphi\,(\tan\varphi - \varphi)} = \frac{5}{3} + 3\,\frac{\text{coſ}\,\varphi^2}{\text{ſin}\,\varphi^2}, \text{ and } \frac{\tan\varphi}{\tan\varphi - \varphi} = \frac{5}{3} + 3\cot\varphi^2.$$

HENCE $\frac{3\tan\varphi}{5 + 9\cot\varphi^2} = \tan\varphi - \varphi$, and $\varphi = \tan\varphi -$

$$\frac{3\tan\varphi}{5 + 9\,.\cot\varphi^2} = \frac{5\tan\varphi + 9\tan\varphi\,.\cot\varphi^2 - 3\tan\varphi}{5 + 9\cot\varphi^2} = \frac{2\tan\varphi + 9\cot\varphi}{5 + 9\cot\varphi^2}.$$

LET $\tan\varphi = t$, then $\varphi = \frac{2t + \frac{9}{t}}{5 + \frac{9}{t^2}} = \frac{2t^3 + 9t}{5t^2 + 9} =$

$\frac{t\,(9 + 2t^2)}{9 + 5t^2}$; which, therefore, is the value of φ when F is a maximum.

THE

THE value of φ, now found, is remarkable for being a near approximation to any arch of which t is the tangent, provided that arch do not exceed 45°. The leſs the arch is, the more near is the approximation; but the expreſſion can only be conſidered as accurate when $\varphi = 0$.

THIS will be made evident by comparing the fraction $\frac{t(9+2t^2)}{9+5t^2}$ with the ſeries, that gives the arch in terms of the tangent t, viz. $\varphi = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} +$, &c. The fraction $\frac{t(9+2t^2)}{9+5t^2} = t - \frac{t^3}{3} + \frac{5t^5}{27} - \frac{5^2 t^7}{3 \cdot 9^2} +$, &c. The two firſt terms of theſe ſeries agree; and in the third terms, the difference is inconſiderable, while t is leſs than unity; but the agreement is never entire, unleſs $t = 0$, when both ſeries vaniſh.

THE attraction, therefore, or the gravitation at the pole of an oblate ſpheroid, is not a maximum, until the eccentricity of the generating ellipſis vaniſh, and the ſpheroid paſs into a ſphere.

FROM the circumſtance of the value of φ above found, agreeing nearly with an indefinite number of arches, we muſt conclude, that when a ſphere paſſes into an oblate ſpheroid, its attraction varies at firſt exceeding ſlowly, and continues to do ſo till its oblateneſs, or the eccentricity of the generating ellipſis, become very great. This may be ſhewn, by taking the value of F, and ſubſtituting in it that of φ, in terms of tan φ.

WE have $F = \frac{4\pi n}{\cos\varphi^{\frac{4}{3}}} \cdot \frac{\tan\varphi - \varphi}{\tan\varphi^3}$; and ſince $\varphi = \tan\varphi -$

$\frac{\overline{\tan\varphi}^3}{3}$

$\frac{\tan\varphi^3}{3}+\frac{\tan\varphi^5}{5}-$, &c. $\tan\varphi-\varphi=\frac{\tan\varphi^3}{3}-\frac{\tan\varphi^5}{5}+$, &c. and

$$F=\frac{4\pi n}{\text{cof}\,\varphi^{\frac{4}{3}}}\left(\frac{\tan\varphi^3}{3}-\frac{\tan\varphi^5}{5}+\frac{\tan\varphi^7}{7}\right)\frac{1}{\tan\varphi^3}=$$

$\frac{4\pi n}{\text{cof}\,\varphi^{\frac{4}{3}}}\left(\frac{1}{3}-\frac{\tan\varphi^2}{5}+\frac{\tan\varphi^4}{7}\right)$. When $\varphi=0$, $F=\frac{4\pi n}{3}$;

and fince $n=m\,.\sqrt[3]{\frac{3}{4\pi}}$, $F=\frac{4\pi m}{3}\sqrt[3]{\frac{3}{4\pi}}=m\sqrt[3]{\frac{64\pi^3\cdot 3}{27\cdot 4\pi}}=$

$m\sqrt[3]{\frac{16\pi^2}{9}}$, which is the attraction at the furface of a fphere of the folidity m^3, as was already fhewn. This laft is the conclufion we had to expect, the fpheroid, when it ceafes to have any oblatenefs, becoming of neceffity a fphere.

It is evident alfo, that the variations of φ will but little affect the magnitude of F, while φ and $\tan\varphi$ are fmall, as the leaft power of $\tan\varphi$ that enters into the value of F is the fquare.

For, inftead of $\text{cof}\,.\,\varphi^{-\frac{4}{3}}$, we may, when φ is very fmall, write $1+\frac{2}{3}\tan\varphi^2$; fo that $F=4\pi n\left(1+\frac{2}{3}\tan\varphi^2\right)\left(\frac{1}{3}-\frac{\tan\varphi^2}{5}+\frac{\tan\varphi^4}{7}-\text{, \&c.}\right)$.

If the oblatenefs of a fpheroid diminifh, while its quantity of matter remains the fame, its attraction will increafe till the oblatenefs vanifh, and the fpheroid become a fphere, when the attraction at its poles, as we have feen, becomes a maximum. If the polar axis continue to increafe, the fpheroid

roid becomes oblong, and the attraction at the poles again diminishes. This we may safely conclude from the law of continuity, though the oblong spheroid has not been immediately considered.

XVIII.

To find the force with which a particle of matter is attracted by a parallelepiped, in a direction perpendicular to any of its sides.

FIRST, let EM (Fig. 11.), be a parallelepiped, having the thickness CE indefinitely small, A, a particle situated anywhere without it, and AB a perpendicular to the plane CDMN. The attraction in the direction AB is to be determined.

LET the solid EM be divided into columns perpendicular to the plane NE, having indefinitely small rectangular bases, and let CG be one of those columns.

IF the angle CAB, the azimuth of this column relatively to AB, be called z, CAD, its angle of elevation from A, e, and m^2, the area of the little rectangle CF; then, as has been already shewn, the attraction of the column CG, in the direction AC, is $\frac{m^2}{AC} \cdot \sin e$; and that same attraction, reduced to the direction AB, is $\frac{m^2}{AC} \cdot \sin e \cdot \cos z$. This is the element of the attraction of the solid, and if we call that attraction f, $f = \frac{m^2}{AC} \cdot \sin e \cdot \cos z$.

Now, if $AB = a$, because $1 : \cos z :: AC : AB$, $AC = \frac{a}{\cos z}$; so that $f = \frac{m^2}{a} \cdot \sin e \cdot \cos z^2$.

BUT

But BC $= a \,.\, \tan z$; and therefore KC, the fluxion of BC, is $= a \,.\, \frac{\dot{z}}{\text{cos} z^2}$; if, then, CE $= n$, $m^2 =$ CE $\times$ CK $= n\, a \,.\, \frac{\dot{z}}{\text{cos} z^2}$, and substituting this for m^2, we get $\dot{f} = n \dot{z} \,.\, \text{sin}\, e$.

Next, to express sin e, in terms of z, if we make E $=$ BAL, the angle subtended by the vertical columns, when it is greatest, or the inclination of the plane ADM, to the plane ACN, then we may consider the angle CAD, as measured by the side of a right angled spherical triangle, of which the other side is $90 - z$, and E the angle, adjacent to that side, and therefore $\tan e = \text{sin}\,(90 - z) \,.\, \tan E = \text{cos}\, z \,.\, \tan E$. But $\tan E =$ $\tan \,.\, BAL = \frac{BL}{BA} = \frac{b}{a}$, supposing BL, or CD $= b$.

Therefore $\tan e = \frac{b}{a} \text{cos}\, z$, or $\frac{\text{sin}\, e}{\text{cos}\, e} = \frac{b}{a} \,.\, \text{cos}\, z$.

Hence $\frac{\text{sin}\, e^2}{\text{cos} e^2} = \frac{b^2}{a^2} \,.\, \text{cos}\, z^2$, or $\frac{\text{sin}\, e^2}{1 - \text{sin}\, e^2} = \frac{b^2}{a^2} \,.\, \text{cos}\, z^2$, and $\text{sin}\, e^2 = \frac{b^2}{a^2} \,.\, \text{cos}\, z^2 - \text{sin}\, e^2 \,.\, \frac{b^2}{a^2} \text{cos}\, z^2$; and therefore $\text{sin}\, e =$

$$\frac{\frac{b}{a} \,.\, \text{cos}\, z}{\sqrt{1 + \frac{b^2}{a^2} \text{cos}\, z^2}}.$$

If this value of sin e be substituted for it, we have

$$\dot{f} = n \dot{z} \,.\, \text{sin}\, e = \frac{b\, n\, \dot{z}\, \text{cos}\, z}{a \sqrt{1 + \frac{b^2}{a^2} \text{cos}\, z^2}}.$$

Let

Let $u = \text{fin}\, z$, then $\dot{u} = \dot{z} \,\text{cof}\, z$, and $\text{cof}\, z^2 = 1 - u^2$; wherefore, again, by fubftitution, $\dot{f} = \dfrac{b\,n\,\dot{u}}{a\sqrt{1 + \frac{b^2}{a^2}(1 - u^2)}} =$

$\dfrac{b\,n\,\dot{u}}{\sqrt{a^2 + b^2 - b^2\,u^2}}$. Let $a^2 + b^2$, or $AL^2 = c^2$, then $\dot{f} =$

$$\frac{b\,n\,\dot{u}}{\sqrt{c^2 - b^2\,u^2}} = \frac{b\,n\,\dot{u}}{c\sqrt{1 - \frac{b^2}{c^2}u^2}}.$$

If, therefore, ϕ be fuch an arch, that $\dfrac{b\,u}{c} = \text{fin}\,\phi$, $\dfrac{b\,\dot{u}}{c} =$

$\dot{\phi}\,.\,\text{cof}\,\phi$, and $\sqrt{1 - \frac{b^2}{c^2}u^2} = \text{cof}\,\phi$, then $\dfrac{b\,n\,\dot{u}}{c\sqrt{1 - \frac{b^2}{c^2}u^2}} =$

$\dfrac{n\,\dot{\phi}\,\text{cof}\,\phi}{\text{cof}\,\phi} = n\,\dot{\phi}$. Thus, $\dot{f} = n\,\dot{\phi}$, and $f = n\,\phi + B$, B being a conftant quantity.

Now, fince $\text{fin}\,\phi = \dfrac{b\,u}{c} = \dfrac{b}{c}\,\text{fin}\, z$, ϕ is nothing when z is nothing; and as f may be fuppofed to begin when z begins, we have likewife $B = 0$; and, $f = n\,\phi = n$ multiplied into an arch, the fine of which is to the fine of z, in the given ratio of c to b. Or f is fuch that $= \text{fin}\,\dfrac{f}{n} = \dfrac{b}{c} \times \dfrac{BC}{AC} = \dfrac{BL}{AL} \times \dfrac{BC}{AC}$.

HENCE this rule, multiply the ſine of the greateſt elevation, into the ſine of the greateſt azimuth of the ſolid; the arch of which this is the ſine, multiplied into the thickneſs of the ſolid, is equal to its attraction in the direction of the perpendicular from the point attracted.

THE heighth and the length of the parallelepiped, are, therefore, ſimilarly involved in the expreſſion of the force, as they ought evidently to be from the nature of the thing.

XIX.

THIS theorem leads directly to the determination of the attraction of a pyramid, having a rectangular baſe, on a particle at its vertex. For if we conſider EM (Fig. 11.) as a ſlice of a pyramid parallel to its baſe, A being the vertex, then the ſlice behind EM ſubtending the ſame angles that it does, will have its force of attraction $= n' \phi$, n' being its thickneſs, and ſo of all the reſt; and, therefore, the ſum of all theſe attractions, if p denote the whole height of the ſolid, or the perpendicular from A on its baſe, will be $p \phi$. But as $n \phi$ is only the attraction of the part HB, it muſt be doubled to give the attraction of the whole ſolid EM, which is, therefore, $2 n \phi$; and this muſt again be doubled, to give the attraction of the part which is on the ſide of AB, oppoſite to EM; thus the element of the attraction of the pyramid is $4 n \phi$, and the whole attraction correſponding to the depth p, is $4 p \phi$.

IF the ſolid is the fruſtum of a pyramid whoſe depth is p', and vertex A, the angle ϕ being determined as before, the attraction on A is $4 p' \phi$.

IF

If we ſuppoſe BC and BL to be equal, and therefore the angle BAL = the angle BAC, calling either of them η, then ſin φ = ſin η^2, by what has been already ſhewn; and from this equation, as η is ſuppoſed to be given, φ is determined.

This expreſſion for the attraction of an iſoſceles pyramid, having a rectangular baſe, may be of uſe in many computations concerning the attraction of bodies.

If the ſolidity of the pyramid be given, from the equations $f = 4p\varphi$, and ſin φ = ſin η^2, we may determine η, and p, that is, the form of the pyramid when f is a maximum.

Let the ſolidity of the pyramid $= m^3$, then p, being the altitude of the pyramid, and η half the angle at the vertex $p \tan \eta =$ half the ſide of the baſe, (which is a ſquare), and therefore the area of the baſe $= 4p^2 \tan \eta^2$, and the ſolidity of the pyramid $\frac{4}{3}p^3 \tan \eta^2$; ſo that $\frac{4}{3}p^3 \tan \eta^2 = m^3$.

Now $\tan \eta^2 = \frac{\text{ſin } \eta^2}{\text{coſ } \eta^2}$, and ſin φ = ſin η^2, alſo $1 - \text{ſin } \varphi = 1 - \text{ſin } \eta^2 = \text{coſ } \eta^2$, therefore $\tan \eta^2 = \frac{\text{ſin } \varphi}{1 - \text{ſin } \varphi}$; ſo that $m^3 = \frac{4}{3}p^3 \cdot \frac{\text{ſin } \varphi}{1 - \text{ſin } \varphi}$, and $p^3 = \frac{3}{4}m^3 \cdot \frac{1 - \text{ſin } \varphi}{\text{ſin } \varphi}$, or $p = m\sqrt[3]{\frac{3(1 - \text{ſin } \varphi)}{4 \text{ ſin } \varphi}}$; we have, therefore, f, that is $4p\varphi = 4m\varphi\sqrt[3]{\frac{3 \cdot (1 - \text{ſin } \varphi)}{4 \text{ ſin } \varphi}}$. This laſt is, therefore, a maximum by hypothesis; and, consequently, its cube, or $64\, m^3 \varphi^3 \times \frac{3(1 - \sin\varphi)}{4 \sin\varphi}$, or omitting the constant multipliers, $\varphi^3 . \frac{1 - \sin\varphi}{\sin\varphi}$ must be a maximum.

If we take the fluxion of each of these multipliers, and divide it by the multiplier itself, and put the sum equal to nothing, we shall have, $\frac{3\dot{\varphi}}{\varphi} - \frac{\dot{\varphi}\cos\varphi}{1 - \sin\varphi} - \frac{\dot{\varphi}\cos\varphi}{\sin\varphi} = 0$, or $\frac{3}{\varphi} =$

$$\frac{\cos\varphi}{1 - \sin\varphi} + \frac{\cos\varphi}{\sin\varphi} = \frac{\cos\varphi . \sin\varphi + \cos\varphi - \cos\varphi . \sin\varphi}{\sin\varphi(1 - \sin\varphi)} =$$

$\frac{\cos\varphi}{\sin\varphi(1 - \sin\varphi)}$, and inverting these fractions $\frac{\varphi}{3} =$

$$\frac{\sin\varphi(1 - \sin\varphi)}{\cos\varphi} = \tan\varphi(1 - \sin\varphi), \text{ or } \varphi = 3\tan\varphi(1 - \sin\varphi).$$

The solution of this transcendental equation may easily be obtained, by approximation, from the trigonometric tables, if we consider that $1 - \sin\varphi$ is the coversed sine of φ. Thus taking the logarithms, we have $L\varphi = L.3 + L.\tan\varphi + L.\text{coverf.}\,\varphi$. From which, by trial, it will soon be discovered, that φ is nearly equal to an arch of $48°$. To obtain a more exact value of φ, let $\varphi = \text{arc}\,(48° + \beta)$, β being a number of minutes to be determined. Because arc . $48° = .8377580$, and arc $(48° + \beta) = .8377580 + .0002909\,\beta$, therefore log. arc $(48° + \beta) = 9.9231186 + .0001506\,\beta$.

In

In the ſame manner,

$$L\tan(48^\circ + \beta) = 0.0455626 + .0002540\,\beta,$$

$$\text{and } L\,.\,\text{coverſ.}(48^\circ + \beta) = 9.4096883 - .0003292\,\beta$$

$$L\,3 = 0.4771213$$

$$\text{Sum} = 9.9323722 - .0000752\,\beta$$

$$\text{Subtract Log arc}(48^\circ + \beta) = 9.9231186 + .0001506\,\beta$$

$$\text{Remainder} = .0092536 - .0002258\,\beta = 0.$$

Whence, $\beta = \frac{92536}{2258} = 41'$ nearly.

A second approximation will give a correction $= - 20''$, ſo that $\phi = \text{arc}\,.\,48^\circ.\,40'\,\frac{2}{3}$; and ſince ſin ϕ = ſin η^2, ſin η = $\sqrt{\text{ſin}\,\phi}$, ſo that $\eta = 76^\circ.\,30'$, and $2\,\eta$, or the whole angle of the pyramid $= 153^\circ$.

An iſoſecles pyramid, therefore, with a ſquare baſe, will attract a particle at its vertex with greateſt force, when the inclination of the oppoſite planes to one another is an angle of 153°.

XX.

To return to the attraction of the parallelepiped, it may be remarked, that the theorem concerning this attraction already inveſtigated, § xviii. though it applies only to the caſe when the parallelepiped is indefinitely thin, leads, nevertheleſs, to ſome very general concluſions. It was ſhewn, that the attraction which the ſolid EL (Fig. 11.) exerts on the particle A, in the direction AB, is $n\,.\,\phi$, ϕ being an arch, ſuch that ſin ϕ = ſin BAC × ſin BAL = ſin z . ſin E; and, therefore, if B be the centre of a rectangle, of which the breadth is 2 BC, and the height 2 BL, the attraction of that plane, or of the thin solid, having that plane for its base, and n, for its thickness, is $4\,n\,.\,\phi$. Now, ϕ, which is thus proportional to the attraction of the plane, is also proportional to the spherical surface, or the angular space, subtended by the plane at the centre A.

For suppose PSQ (Fig. 12.) and OQ to be two quadrants of great circles of a sphere, cutting one another at right angles in Q; let QS = E, and QR = z. Through S, and O the pole of PSQ, draw the great circle OST, and through P and R, the great circle PTR, intersecting OS in T. The spherical quadrilateral SQRT, is that which the rectangle CL (Fig. 11.) would subtend, if the sphere had its centre at A, if the point Q was in the line AB, and the circle PQ, in the vertical plane ABL.

Now, in the spherical triangle PST, right angled at S, cos T = cos PS × sin SPT = sin QS × sin QR = sin E × sin z. But this is also the value of sin ϕ, and therefore ϕ is the complement of the angle T, or $\phi = 90 - T$.

But the area of the triangle PQR, in which both Q and R are right angles, is equal to the rectangle under the arch QR, which measures the angle QPR, and the radius of the sphere. Also the area SPT = arc . (S + T + P — 180°) r; that is, because S is a right angle, = arc . (T + P — 90) × r = arc . (T + QR — 90) × r; and taking this away from the triangle PQR, there remains the area QSTR = arc . (QR — T — QR + 90°) × r = (90 — T) r = $\phi \times r$. The arch ϕ, therefore, multiplied into the radius, is equal to the spherical quadrilateral QSTR, subtended by the rectangle BD.

This proposition is evidently applicable to all rectangles whatsoever. For when the point B, where the perpendicular from A meets the plane of the rectangle, falls anywhere, as in Fig. 15. then it may be shewn of each of the four rectangles BD, BD, BM, BM′, BD′, which make up the whole rectangle DM′, that its attraction in the direction AB is expounded by the area of the ſpherical quadrilateral ſubtended by it, and, therefore, that the attraction of the whole rectangle MD′, is expounded by the ſum of theſe ſpherical quadrilaterals, that is, by the whole quadrilateral ſubtended by MD′. In the ſame manner, if the perpendicular from the attracted particle, were to meet the plane without the rectangle MD′, the difference between the ſpherical quadrilaterals ſubtended by MC and M′C, would give the quadrilateral, ſubtended by the rectangle MD′, for the value of the attraction of that rectangle.

Therefore, in general, *if a particle* A, *gravitate to a rectangular plane, or to a solid indefinitely thin, contained between two parallel rectangular planes, its gravitation, in the line perpendicular to those planes, will be equal to the thickness of the solid, multiplied into the area of the spherical quadrilateral subtended by either of those planes at the centre* A.

The ſame may be extended to all planes, by whatever figure they be bounded, as they may all be reſolved into rectangles of indefinitely ſmall breadth, and having their lengths parallel to a ſtraight line given in poſition.

The gravitation of a point toward any plane, in a line perpendicular to it, is, therefore, equal to *n*, a quantity that expreſſes the intenſity of the attraction, multiplied into the area of the ſpherical figure, or, as it may be called, the angular ſpace ſubtended by the given plane.

Thus, in the caſe of a triangular plane, where the angles ſubtended at A, by the ſides of the triangle, are *a*, *b* and *c*; ſince Euler has demonſtrated * that the area of the ſpherical triangle contained by theſe arches, is equal to the rectangle under

* Nov. Acta Petrop. 1792, p. 47.

der the radius, and an arch Δ, ſuch that $\text{coſ}\, \frac{1}{2}\Delta = \frac{1 + \text{coſ}\, a + \text{coſ}\, b + \text{coſ}\, c}{4\, \text{coſ}\, \frac{1}{2} a \,.\, \text{coſ}\, \frac{1}{2} b \,.\, \text{coſ}\, \frac{1}{2} c}$; if Δ be computed, the attraction $= n \,.\, \Delta$.

In the caſe of a circular plane, our general propoſition agrees with what Sir Isaac Newton has demonſtrated. If CFD (Fig. 13.) be a circle, BA a line perpendicular to the plane of it from its centre B; A, a particle anywhere in that line; the force with which A is attracted, in the direction AB, is $2\pi\left(1 - \frac{AB}{AD}\right)$ *, in which the multiplier 2π is ſupplied, being left out in the inveſtigation referred to, where a quantity only proportional to the attraction is required. Now $\frac{AB}{AD}$ is the coſine of the angle BAD, and, therefore, $1 - \frac{AB}{AD}$ is its verſed ſine; and, therefore, if the arch GEK be deſcribed from the centre A, with the radius 1, and if the ſine GH, and the chord EG be drawn, HE is the verſed ſine of BAD, and the attraction $= 2\pi\,\text{EH}$. But $2 \,.\, \text{EH} = \text{EG}^2$, becauſe 2 is the diameter of the circle GEK; therefore the attraction $= \pi \,.\, \text{EG}^2 =$ the area of the circle of which EG is the radius, or the ſpherical ſurface included by the cone, which has A for its vertex, and the circle CFD for its baſe.

XXI.

* Princip. Lib. i. Prop. 90.

XXI.

From the general propofition, that the attraction of any plane figure, whatever its boundary may be, in a line perpendicular to the plane, is at any diftance proportional to the angular fpace, or to the area of the fpherical figure which the plane figure fubtends at that diftance, we can eafily deduce a demonftration of this other propofition, that whatever be the figure of any body, its attraction will decreafe in a ratio that approaches continually nearer to the inverfe ratio of the fquares of the diftances, as the diftances themfelves are greater. In other words, the inverfe ratio of the fquares of the diftances, is the limit to which the law by which the attraction decreafes, continually approaches as the diftances increafe, and with which it may be faid to coincide when the diftances are infinitely great.

This propofition, which we ufually take for granted, without any other proof, I believe, then, fome indiftinct perception of what is required by the law of continuity, may be rigoroufly demonftrated from the principle juft eftablifhed.

Let B (Fig. 14.) be a body of any figure whatfoever, A a particle fituated at a diftance from B vaftly greater than any of the dimenfions of B, fo that B may fubtend a very fmall angle at A; from C, a point in the interior of the body, fuppofe its centre of gravity, let a ftraight line be drawn to A, and let A′ be another point, more remote from B than A is, where a particle of matter is alfo placed.

The directions in which A and A′ gravitate to B, as they muft tend to fome point within B, muft either coincide with AC, or make a very fmall angle with it, which will be always the lefs, the greater the diftance.

Let the body B be cut by two planes, at right angles to AC, and indefinitely near to one another, ſo as to contain between them a ſlice or thin ſection of the body, to which A and A′ may be conſidered as gravitating, nearly in the direction of the line AG perpendicular to that ſection.

The gravitation of A, therefore, to the aforeſaid ſection, will be to that of A′ to the ſame, as the angular ſpace ſubtended by that ſection at A, to the angular ſpace ſubtended by it at A′. But theſe angular ſpaces, when the diſtances are great, are inverſely as the ſquares of thoſe diſtances, and therefore, alſo, the gravitation of A toward the ſection, will be to that of A′, inverſely as the ſquares of the diſtances of A and A′ from the ſection. Now theſe diſtances may be accounted equal to CA and CA′, from which they can differ very little, wherever the ſection is made.

The gravitations of A and A′ toward the ſaid ſection, are, therefore, as $\frac{1}{AC^2}$ to $\frac{1}{A'C^2}$. And the ſame may be proved of the gravitation to all the other ſections, or laminæ, into which the body can be divided by planes perpendicular to AC; therefore the ſums of all theſe gravitations, that is, the whole gravitations of A to B, and of A′ to B, will be in that ſame ratio, that is, as $\frac{1}{AC^2}$ to $\frac{1}{A'C^2}$, or inverſely as the ſquares of the diſtances from C. Q. E. D.

It is evident, that the greater the diſtances AC, A′C are, the nearer is this propoſition to the truth, as the quantities rejected in the demonſtration, become leſs in reſpect of the reſt, in the ſame proportion that AC and A′C increaſe.

It is here aſſumed, that the angular ſpace ſubtended by the ſame plane figure, is inverſely as the ſquare of the diſtance. This

This propofition may be proved to be rigoroufly true, if we confider the inverfe ratio of the fquares of the diftances, as a limit to which the other ratio conftantly converges.

It is a propofition alfo ufually laid down in optics, where the *visible space* fubtended by a furface, is the fame with what we have here called the *angular space* fubtended by it, or the portion of a fpherical fuperficies that would be cut off by a line paffing through the centre of the fphere, and revolving round the boundary of the figure. The centre of the fphere is fuppofed to coincide with the eye of the obferver, or with the place of the particle attracted, and its radius is fuppofed to be unity.

The propofitions that have been juft now demonftrated concerning the attraction of a thin plate contained between parallel planes, have an immediate application to fuch inquiries concerning the attraction of bodies, as were lately made by Mr Cavendish.

In fome of the experiments inftituted by that ingenious and profound philofopher, it became neceffary to determine the attraction of the fides of a wooden cafe, of the form of a parallelepiped, on a body placed anywhere within it. (Philofophical Tranfactions, 1798, p. 523.). The attraction in the direction perpendicular to the fide, was what occafioned the greateft difficulty, and Mr Cavendish had recourfe to two infinite feries, in order to determine the quantity of that attraction. The determination of it, from the preceding theorems, is eafier and more accurate.

Let MD′ (Fig. 15.) reprefent a thin rectangular plate, A, a particle attracted by it, AB a perpendicular on the plane MD′, NBC, LBL′, two lines drawn through B parallel to the fides of the rectangle MD′. Let AC, AL, AN, AL′, be drawn.

Then, if we find φ ſuch that ſin $\varphi = \frac{BL}{AL} \times \frac{BC}{AC}$, the attraction of the rectangle CL is $n \cdot \varphi$, n denoting the thickneſs of the plate.

So alſo, if ſin $\varphi' = \frac{BL}{AL} \times \frac{BN}{AN}$, the attraction of LN is $= n \cdot \varphi'$.

If ſin $\varphi'' = \frac{BN}{AN} \times \frac{BL'}{AL'}$, the attraction of NL′ is $= n \cdot \varphi''$.

Lastly, If ſin $\varphi''' = \frac{BL'}{AL'} \times \frac{BC}{AC}$, the attraction of L′C $= n \cdot \varphi'''$.

Thus the whole effect of the plane MD′, or $f = n\,(\varphi + \varphi' + \varphi'' + \varphi''')$.

We may either ſuppoſe φ, φ' &c. defined as above, or by the following equations, where η, η', η'', &c. denote the angles ſubtended by the ſides of the rectangles that meet in B, beginning with BC, and going round by L, N and L′ to C.

$$\text{ſin}\,\varphi = \text{ſin}\,\eta \,.\, \text{ſin}\,\eta'$$
$$\text{ſin}\,\varphi' = \text{ſin}\,\eta' \,.\, \text{ſin}\,\eta''$$
$$\text{ſin}\,\varphi'' = \text{ſin}\,\eta'' \,.\, \text{ſin}\,\eta'''$$
$$\text{ſin}\,\varphi''' = \text{ſin}\,\eta''' \,.\, \text{ſin}\,\eta.$$

If the computation is to be made by the natural ſines, it will be better to uſe the following formulæ :

$$\text{ſin}\,\varphi = \frac{1}{2}\,\text{coſ}\,(\eta - \eta') - \frac{1}{2}\,\text{coſ}\,(\eta + \eta')$$

$$\text{ſin}\,\varphi' = \frac{1}{2}\,\text{coſ}\,(\eta' - \eta'') - \frac{1}{2}\,\text{coſ}\,(\eta' + \eta'')$$

ſin φ''

$$\text{ſin}\,\varphi'' = \frac{1}{2}\,\text{coſ}\,(\eta'' - \eta''') - \frac{1}{2}\,\text{coſ}\,(\eta'' + \eta''')$$

$$\text{ſin}\,\varphi''' = \frac{1}{2}\,\text{coſ}\,(\eta''' - \eta\ ) - \frac{1}{2}\,\text{coſ}\,(\eta''' + \eta\ ).$$

BY either of theſe methods, the determination of the attraction is reduced to a very ſimple trigonometrical calculation.

XXII.

THE preceding theorems will alſo ſerve to determine the attraction of a parallelepiped, of any given dimenſions, in the direction perpendicular to its ſides.

LET BF (Fig. 16.) be a parallelepiped, and A, a point in BK, the interſection of two of its ſides, where a particle of matter is ſuppoſed to be placed; it is required to find the attraction in the direction AB.

THOUGH the placing of A in one of the interſections of the planes, ſeems to limit the inquiry, it has in reality no ſuch effect; for wherever A be with reſpect to the parallelepiped, by drawing from it a perpendicular to the oppoſite plane of the ſolid, and making planes to paſs through this perpendicular, the whole may be divided into four parallelepipeds, each having AB for an interſection of two of its planes; and being, therefore, related to the given particle, in the ſame way that the parallelepiped BF is to A.

LET GH be any ſection of the ſolid parallel to EC, and let it repreſent a plate of indefinitely ſmall thickneſs.

LET AB′ $= x$, B′*b*, the thickneſs of the plate $= \dot{x}$. Then φ being ſo determined, that ſin $\varphi =$ ſin B′AH × ſin B′AG, the attraction of the plate GH is $\varphi\,\dot{x}$, which, therefore, is the element

ment of the attraction of the solid. If that attraction $=$ F, then $F = \int \varphi \dot{x}$. But $\int \varphi \dot{x} = \varphi x - \int x \dot{\varphi}$; and the determination of F depends, therefore, on the integration of $x \dot{\varphi}$.

Now $\dot{\varphi} \cos \varphi = \dot{\overline{\sin \varphi}}$, and, therefore, $x \dot{\varphi} = \frac{x \dot{\overline{\sin \varphi}}}{\cos \varphi}$.

If $B'G = b$, and $B'H = \beta$, then $\sin B'AG = \frac{b}{AG} = \frac{b}{\sqrt{b^2 + x^2}}$, and $\sin B'AH = \frac{\beta}{AH} = \frac{\beta}{\sqrt{\beta^2 + x^2}}$; so that $\sin \varphi = \frac{b}{\sqrt{b^2 + x^2}} \times \frac{\beta}{\sqrt{\beta^2 + x^2}}$, and $\overline{\sin \varphi}^2 = \frac{b^2 \beta^2}{(b^2 + x^2)(\beta^2 + x^2)}$.

Hence, $\cos \varphi^2 = 1 - \sin \varphi^2 = 1 - \frac{b^2 \beta^2}{(b^2 + x^2)(\beta^2 + x^2)} = \frac{x^2 (b^2 + \beta^2 + x^2)}{(b^2 + x^2)(\beta^2 + x^2)}$, and $\cos \varphi = \frac{x \sqrt{b^2 + \beta^2 + x^2}}{\sqrt{(b^2 + x^2)(\beta^2 + x^2)}}$.

Again, because $\sin \varphi = \frac{b}{\sqrt{b^2 + x^2}} \cdot \frac{\beta}{\sqrt{\beta^2 + x^2}}$, $\dot{\overline{\sin \varphi}} = \frac{-b x \dot{x}}{(b^2 + x^2)^{\frac{3}{2}}} \times \frac{\beta}{(\beta^2 + x^2)^{\frac{1}{2}}} - \frac{\beta x \dot{x}}{(\beta^2 + x^2)^{\frac{3}{2}}} \times \frac{b}{(b^2 + x^2)^{\frac{1}{2}}}$.

Hence $\frac{\dot{\overline{\sin \varphi}}}{\cos \varphi}$ or $\dot{\varphi} =$

$$\left(\frac{-b \beta x \dot{x}}{(b^2 + x^2)^{\frac{3}{2}} \times (\beta^2 + x^2)^{\frac{1}{2}}} - \frac{b \beta x \dot{x}}{(\beta^2 + x^2)^{\frac{3}{2}} \times (b^2 + x^2)^{\frac{1}{2}}}\right)$$

$\times$

$$\times \frac{(b^2 + x^2)^{\frac{1}{2}} (\beta^2 + x^2)^{\frac{1}{2}}}{x \sqrt{b^2 + \beta^2 + x^2}} = - \frac{b \beta \dot{x}}{(b^2 + x^2)(c^2 + x^2)^{\frac{1}{2}}} -$$

$\frac{b \beta \dot{x}}{(\beta^2 + x^2)(c^2 + x^2)^{\frac{1}{2}}}$, c^2 being put for $b^2 + \beta^2$.

THEREFORE $x \dot{\varphi} = - \frac{b \beta x \dot{x}}{(b^2 + x^2)(c^2 + x^2)^{\frac{1}{2}}} -$

$\frac{b \beta x \dot{x}}{(\beta^2 + x^2)(c^2 + x^2)^{\frac{1}{2}}}$.

Now, $\int \frac{- b \beta x \dot{x}}{(b^2 + x^2)(c^2 + x^2)^{\frac{1}{2}}} = b \,\text{Log}\, \frac{\beta + \sqrt{c^2 + x^2}}{\sqrt{b^2 + x^2}} + C$;

(Harmonia Menfurarum, Form. IX.); and $\int \frac{- b \beta x \dot{x}}{(b^2 + x^2)(c^2 + x^2)^{\frac{1}{2}}} =$

$\beta \,\text{Log}\, \frac{b + \sqrt{c^2 + x^2}}{\sqrt{\beta^2 + x^2}} + C$.

THEREFORE $\int x \dot{\varphi} =$

$b \,\text{Log}\, \frac{\beta + \sqrt{c^2 + x^2}}{\sqrt{b^2 + x^2}} + \beta \,\text{Log}\, \frac{b + \sqrt{c^2 + x^2}}{\sqrt{\beta^2 + x^2}} + C$, and $\int \varphi \dot{x} =$

$\varphi x - b \,\text{Log}\, \frac{\beta + \sqrt{c^2 + x^2}}{\sqrt{b^2 + x^2}} - \beta \,\text{Log}\, \frac{b + \sqrt{c^2 + x^2}}{\sqrt{\beta^2 + x^2}} - C$.

IF, then, we determine C, fo that the fluent may begin at K, and end at B; if, alfo, we make η the value of φ, that correfponds to AB or a; and η', the value of it that correfponds to AK

AK or a', we have the whole attraction of the ſolid, or F =

$$\eta a - \eta' a' - b \text{ Log} \frac{\beta + \sqrt{c^2 + a^2}}{\sqrt{b^2 + a^2}} \times \frac{\sqrt{b^2 + a'^2}}{\beta + \sqrt{c^2 + a'^2}}$$

$$- \beta \text{ Log} \frac{b + \sqrt{c^2 + a^2}}{\sqrt{\beta^2 + a^2}} \times \frac{\sqrt{\beta^2 + a'^2}}{b + \sqrt{c^2 + a'^2}}.$$

If, in this value of F, we invert the ratios, in order to make the logarithms affirmative, and write like quantities, one under the other, we have $F = \eta a - \eta' a'$

$$+ b \text{ Log} \frac{\beta + \sqrt{c^2 + a'^2}}{\beta + \sqrt{c^2 + a^2}} \times \frac{\sqrt{b^2 + a^2}}{\sqrt{b^2 + a'^2}}$$

$$+ \beta \text{ Log} \frac{b + \sqrt{c^2 + a'^2}}{b + \sqrt{c^2 + a^2}} \times \frac{\sqrt{\beta^2 + a^2}}{\sqrt{\beta^2 + a'^2}}.$$

The firſt two terms of this expreſſion deſerve particular attention, as η is an arch, ſuch that $\text{ſin } \eta = \text{ſin BAE} \times \text{ſin BAC}$; therefore, by what has been before demonſtrated, η is the meaſure of the angular ſpace ſubtended at A by the rectangle BD. The firſt term in the value of F, therefore, is the product of the diſtance AB, into the angular ſpace ſubtended by the rectangle BD. In like manner, the ſecond term, or $\eta' a'$, is the product of the diſtance AK, into the angular ſpace ſubtended by the rectangle KF.

The relation of the quantities expreſſing the ratios, in the two logarithmic terms, will be beſt conceived by ſubſtituting for the algebraic quantities the lines that correſpond to them in the diagram. Becauſe $c^2 = b^2 + \beta^2 = EB^2 +$ BC²

$BC^2 = EC^2$, therefore $c = EC$ or BD. So alſo, $c^2 + a^2 = BD^2 + BA^2 = AD^2$, becauſe ABD is a right angle, &c. Thus,

$$F = \eta a - \eta' a' + BE . \text{Log} \frac{(AF + FN) AE}{(AD + DE) (AN} +$$

$$BC . \text{Log} \frac{(AF + FM) AC}{(AD + DC) AM}.$$

THIS expreſſion for the attraction of a parallelepiped, though conſiderably complex, is ſymmetrical in ſo remarkable a degree, that it will probably be found much more manageable, in inveſtigation, than might at firſt be ſuppoſed. That it ſhould be ſomewhat complex, was to be expected, as the want of continuity in the ſurface by which a ſolid is bounded, cannot but introduce a great variety of relations into the expreſſion of its attractive force. The farther ſimplification, however, of this theorem, and the application of it to other problems, are ſubjects on which the limits of the preſent paper will not permit us to enter. The determinations of certain *maxima* depend on it, ſimilar to thoſe already inveſtigated. It points at the method of finding the figure, which a fluid, whether elaſtic or unelaſtic, would aſſume, if it ſurrounded a cubical or priſmatic body by which it was attracted. It gives ſome hopes of being able to determine generally the attraction of ſolids bounded by any planes whatever; ſo that it may, ſome time or other, be of uſe in the Theory of Chryſtallization, if, indeed, that theory ſhall ever be placed on its true baſis, and founded, not on an hypotheſis purely Geometrical, or in ſome meaſure arbitrary, but on the known Principles of Dynamicks.

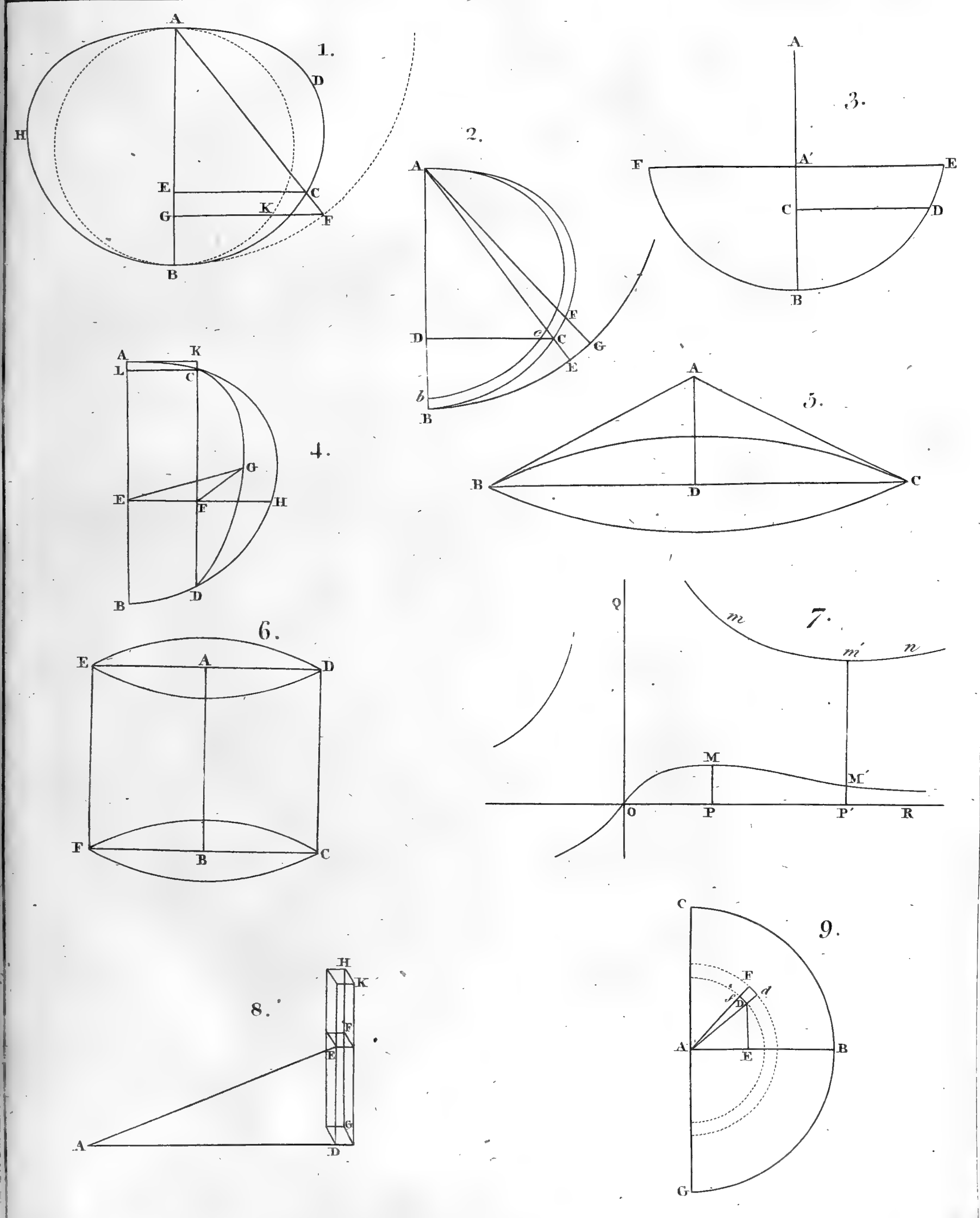
BC = EC, therefore c = AC or FD. So also, $c^2 + a^2 =$ $BD^2 + BA^2 = AD^2$, and ABD is a right angle, &c. Thus

$$T = pa - pa \cdot BC \cdot Log. \frac{(AE + EN)(AE}{(AD - DE)(AN} + \cdot$$

$$BC \cdot Log. \frac{(HF + CF)(CG}{(HD + DG)(DF} \cdot$$

This expression for the attraction of a parallelepiped, though considerably complex, is symmetrical in so remarkable a degree, that it will probably be found much more manageable in investigation, than might at first be supposed. That it should be somewhat complex, was to be expected, as the want of continuity in the surface by which a solid is bounded, cannot but introduce a great variety of relations into the expression of its attractive force. The farther simplification, however, of this theorem, and the application of it to other problems, are subjects on which the limits of the present paper will not permit us to enter. The determinations of certain maxima depend on it, similar to those already investigated. It points at the method of finding the figure, which a fluid, whether elastic or unelastic, would assume, if it surrounded a cubical or prismatic body by which it was attracted. It gives some hope of being able to determine generally the attraction of solids bounded by any planes whatever; so that it may, some time or other, be of use in the Theory of Chrystallization, if, indeed, that theory shall ever be placed on its true basis, and founded, not on an hypothesis purely Geometrical, or in some measure arbitrary, but on the known Principles of Dynamicks.

PLATE VI.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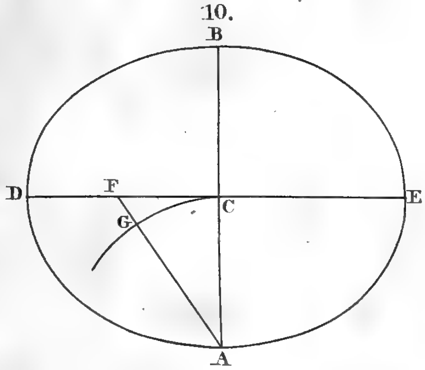

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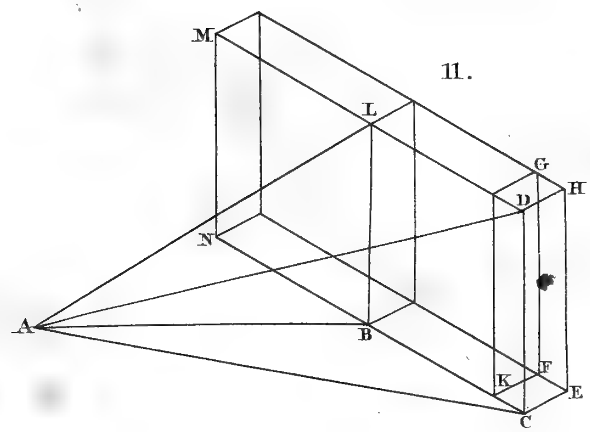

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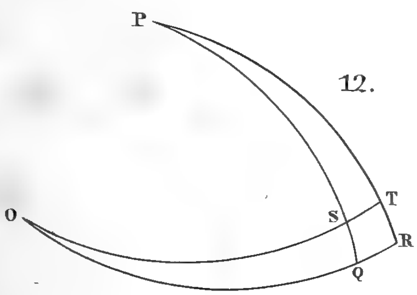

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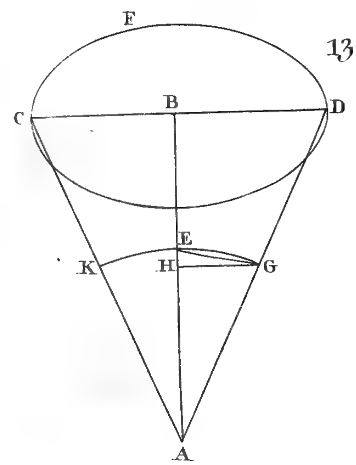

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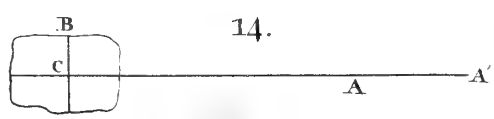

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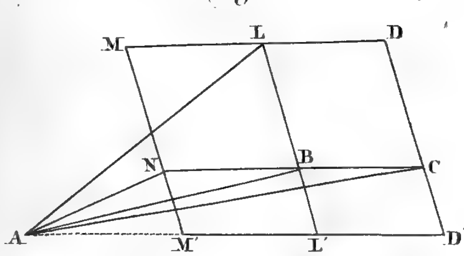

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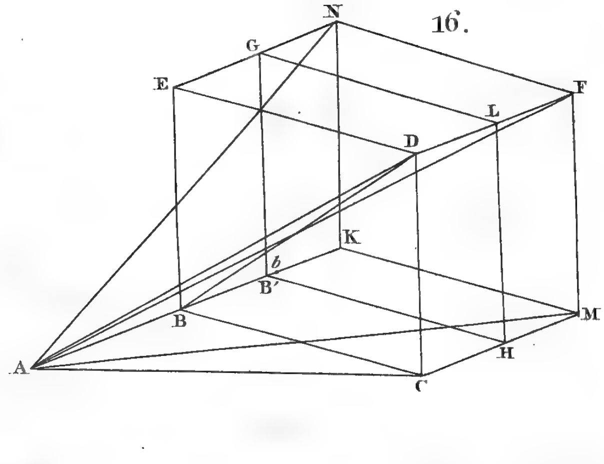

V. *An Account of a very extraordinary Effect of* REFRACTION, *observed at Ramsgate, by the Reverend* S. VINCE, A. M. F. R. S. *Plumian Professor of Astronomy and Experimental Philosophy at Cambridge. Communicated by* PATRICK WILSON, *Esq;* F. R. S. EDIN.

[*Read 5th January* 1807.]

THE phenomenon about to be defcribed, was feen on Auguft 6. 1806, about feven in the evening; the air being very ftill, and a little hazy. The tops of the four turrets of Dover Caftle ufually appear above the hill, lying between Ramfgate and Dover; but, at the above-ftated time, not only the tops were vifible, but the whole of the Caftle, appearing as if it were fituated on the fide of the hill next to Ramfgate, and rifing as much above the hill as ufual.

LET AB (Plate VIII. Fig. 1.) reprefent the termination of the hill; *v*, *x*, *w*, *y*, the tops of the four turrets of the Caftle, as they ufually appear. But, at the time above mentioned, befides thus feeing the turrets, the whole Caftle *m n r s* was vifible, and appeared as if it had been brought over and placed on the Ramfgate fide of the hill, as reprefented in the figure. This phenomenon was fo very fingular and unexpected, that, at firft fight, I thought it to be fome illufion; but, upon continuing my obfer-

 vation,

vation, I was ſatisfied that it was a real image of the Caſtle. Upon this I gave the teleſcope to a perſon preſent, who, upon attentive examination, ſaw alſo a very clear image of the Caſtle, exactly as I had deſcribed it. He continued to obſerve it for about twenty minutes, during which time the appearance remained preciſely the ſame; but rain coming on, we were prevented from making any further obſervations. Between us and the land, from which the hill riſes, there was about ſix miles of ſea, and from thence to the top of the hill about the ſame diſtance, and we were about ſeventy feet above the ſurface of the water.

The hill itſelf did not appear through the image, which, it might have been expected to do. The image of the Caſtle appeared very ſtrong, and well defined; and although the rays from the hill behind it, muſt undoubtedly have come to the eye, yet ſo it was, that the ſtrength of the image of the Caſtle ſo far obſcured the back-ground, that it made no ſenſible impreſſion upon us. Our attention was of courſe principally directed to the image of the Caſtle; but if the hill behind had been at all viſible, it could not have eſcaped our obſervation, as we continued to look at it for a conſiderable time with a good teleſcope.

A phenomenon of this kind I do not remember to have ſeen deſcribed; and it muſt have been a very extraordinary ſtate of the air to have produced it. It is manifeſt, that a ray of light coming from the top of the hill, muſt have come to the eye in a curve lying between the two curves deſcribed by the rays coming from the top and bottom of the Caſtle, in order to produce the effect.

Let AB (Plate VIII. Fig. 2.) repreſent the Caſtle, EC the Cliff (at Ramſgate), BTD the Hill, DC the Sea, E the place of the ſpectator, T the top of the hill, A*yv*E a ray of light coming from the top of the Caſtle to the ſpectator,

B*xw*E

B *x w* E a ray coming from the bottom, and T *x z* E a ray coming from the top of the hill, falling upon the eye at E, in a direction between thofe of the other two rays; then it is manifeft, that fuch a difpofition of the rays will produce the obferved appearance. To effect this, there muft have been a very quick variation of the denfity of the air which lay between the two curves *y v* E, *x w* E, fo as to increafe the curvature of the ray T *x z* E, after it cuts B *w* E in *x*, by which means, the ray T *x z* E, might fall between the other two rays. The phenomenon cannot be otherwife accounted for. As there are not, that I know of, any records of a phenomenon of this nature, the conftitution of the air muft have been fuch as but very rarely happens, or fuch an appearance would before have been taken notice of.

THE phenomena which I faw at the fame place, and which I defcribed in the Philofophical Tranfactions of the Royal Society for the year 1798, I explained upon the fame principle, that of a quick variation of denfity; and this was afterwards confirmed by fome very ingenious experiments made by Dr WOLLASTON. Perhaps this phenomenon may afterwards be fubjected to an experimental illuftration.

VI.

[illegible] ray coming from the bottom, and T [illegible] E a ray [illegible] falling upon the eye at E [illegible] between the other two rays; then it is [illegible], that such a disposition of the rays will produce the observed appearance. To effect this, there must have been a very quick variation of the density of the air which lay between the two curves [illegible] E, [illegible] E, so as to increase the curvature of the ray T [illegible] E, where it cuts [illegible] E in [illegible], by which means the ray T [illegible] E, might fall between the other two rays. The phenomenon cannot be otherwise accounted for. As there are not, that I know of, any records of a phenomenon of this nature, [illegible] of the air must have been such as but [illegible] such an appearance would before have been taken notice of. [illegible]

The phenomena which I saw at the same place, and which I described in the Philosophical Transactions of the Royal Society for the year 1798, I explained upon the same principle, that of a quick variation of density; and this was afterwards confirmed by some very ingenious experiments made by Dr Wollaston. Perhaps this phenomenon may afterwards be subjected to an experimental illustration.

VI.

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Fig. 1.

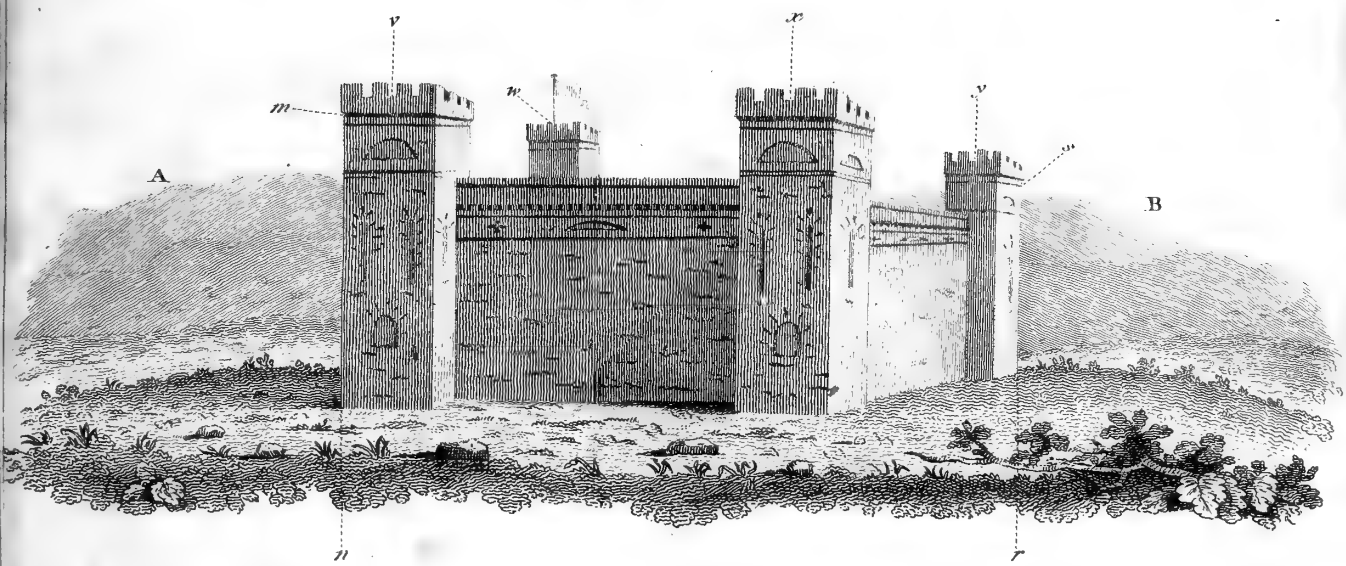
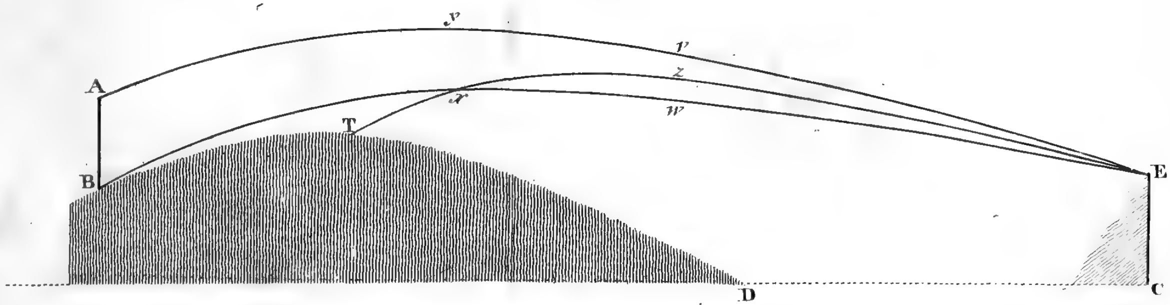


Fig. 2.



D. Lizars sculp^t

VI. *Some Account of the Large Snake* ALEA-AZAGUR, (BOA CONSTRICTOR *of* LINNÆUS), *found in the Province of Tipperah. Communicated by* MR JAMES RUSSELL. *Extracted from the Memorandum Book of* JOHN CORSE SCOTT, *Esq.*

[*Read 28th April* 1807].

February 1. 1787.

A LARGE ſnake of this ſpecies was brought to Comillah. It meaſured 15 feet 3 inches in length, and 18 inches in circumference about the middle. This meaſurement, however, varied conſiderably by the wreathings and contortions it made, in order to free itſelf from confinement.

THE œſophagus, from the mouth to the pylorus, or bottom of the ſtomach, meaſured altogether 9 feet 3 inches, and was ſo wide as to take in a man's head with eaſe. The ſtomach was eaſily diſtinguiſhed by the thickneſs of its coats, or the number of rugæ on its internal ſurface. But there was no contraction at the cardia or entrance of the ſtomach. The outlet or pylorus, however, was ſo narrow as hardly to admit two fingers.

THE head of the ſnake was ſmall in proportion to its body. And I was curious to obſerve the mechaniſm of the jaw, by which it can ſo eaſily take into its mouth any ſubſtance as large as the thickeſt part of its body.

THE

The lower jaw consists of two bones, connected anteriorly by skin and ligaments, which admit of considerable distension, so that the anterior ends can be separated an inch from each other. The posterior extremity, or condyle of each lower jaw-bone, is likewise connected to the head in such a manner, as to allow of considerable separation. The two bones which compose the upper jaw, are capable only of a very small degree of separation at the symphisis or anterior part.

This singular degree of laxity in the structure of the articulations, permits of a degree of distension which is incompatible with the firmness requisite to perform the function of mastication.

July 7. 1790.

A snake of the allea species was brought in, of a very uncommon thickness in proportion to its length, which induced me to open it. A very large guana was extracted from the gullet and stomach; for the animal was gorged to the throat. The guana, from the nose to the tip of the tail, measured 4 feet 3 inches, and in circumference round the belly 1 foot 6 inches; and yet the snake, after the guana was taken out, measured only 8 feet 6 inches in length.

The circumference of this snake is not given; but if it bore the same proportion to its length that it did in the former snake, it would be nearly 10 inches. In this instance, therefore, the snake had swallowed an animal of greater magnitude than itself almost in the proportion of 9 to 5.

On the 16th of the same month another snake was brought in, having nearly the same appearance as the last, but still more distended. It was opened while yet alive, and an entire fawn of one year old extracted. The fawn measured 1 foot 8 inches round the belly; and the extreme length of the snake was only 9 feet 3 inches.

April

April 5. 1791.

A SNAKE of the ſame ſpecies was brought to Comillah and opened, from which a fawn was taken ſtill larger than the one juſt mentioned; but the ſnake was 10 feet 6 inches in length.

IT is the general opinion, that ſnakes break the bones of their prey before they ſwallow it, if the animal be of any conſiderable ſize. This, however, I am diſpoſed to doubt, as in none of the above inſtances had the animal ſuffered ſuch oſſifraction, if I may be allowed the expreſſion. The mechaniſm of the jaws, and the width of the gullet above deſcribed, render ſuch violence unneceſſary.

THE animal is ſwallowed very gradually, being firſt, I ſuſpect, well lubricated with ſlime, with which this kind of large ſnake appears abundantly provided.

THESE circumſtances may undoubtedly be deemed rather fabulous by thoſe who have never ſeen nor examined large ſnakes. But they are facts not to be denied, and are well authenticated by every one who has had opportunities of ſeeing and opening ſuch ſnakes.

DURING Mr LECKIE's reſidence at Comillah, I have learned from undoubted authority, that a ſnake of the above mentioned ſpecies was found dead, with the horns of a large deer ſticking in his throat, ſuppoſed to be the cauſe of his death. The ſnake and the horns were both brought to Comillah in this ſituation; but in a putrid ſtate. The ſnake meaſured above 17 feet in length; and the bones of it were afterwards ſent to Mr CHARLES COLLINSON of Banleak.

VII. *Chemical Analysis of a* Black Sand, *from the River Dee in Aberdeenshire; and of a* Copper Ore, *from Arthrey in Stirlingshire.* *By* Thomas Thomson, M. D. *Lecturer on Chemistry, Edinburgh.*

[*Read* 18*th May* 1807.]

THE ſpecimen which formed the ſubject of the firſt of the following *analyses*, was brought from the banks of the river Dee, about ſeven years ago, by my friend Mr James Mill, who at that time reſided in Aberdeenſhire. By him I was informed, that conſiderable quantities of it are found in different parts of the bed of that river,—that it is called by the inhabitants *iron-sand*,—and that they uſe it for ſanding newly written paper. I tried ſome experiments in the year 1800, in order to aſcertain its nature; but was too little ſkilled at that time, both in mineralogy and practical chemiſtry, to manage an analyſis of any conſiderable difficulty.

The black powder is mixed with a good many ſmall whitiſh, reddiſh, and browniſh grains, which, when examined by means of a glaſs, prove to be pieces of quartz, felſpar, and mica. From this it would appear, that the ſand of the river Dee conſiſts chiefly of the detritus of granite or gneiſs.

When a magnet is paſſed over the ſand, ſome of the black grains adhere to it, and are by this means eaſily obtained ſepa-

 rate.

rate. But after all that can be attracted by the magnet is removed, the greater part of the black powder ſtill remains. This reſidue is indeed attracted by a powerful magnet, but ſo very feebly, that it is not poſſible by means of it to ſeparate it from the grains of ſand with which it is mixed. Thus we learn, that the black matter conſiſts of two diſtinct ſubſtances; one of which is powerfully attracted by the magnet, the other not. As this ſecond ſubſtance was obviouſly ſpecifically heavier than the grains of ſand with which it was mixed, I placed a quantity of the powder on an inclined plane, and by expoſing it cautiouſly, and repeatedly, to a jet of water, I ſucceeded in waſhing away moſt of the grains of ſand, and thus obtained it in a ſtate of tolerable purity.

THE firſt of theſe minerals we may call *iron-sand*, and the ſecond *iserine*, as they belong to mineral ſpecies which oryctognoſts have diſtinguiſhed by theſe names.

I. IRON-SAND.

THE iron-ſand is much ſmaller in quantity than the iſerine, and does not exceed one-fourth of the mixture at moſt. Its colour is iron-black. It is in very ſmall angular grains, commonly pretty ſharp-edged, and ſometimes having the ſhape of imperfect octahedrons. The ſurface is rough; the luſtre is feebly glimmering and metallic; the fracture, from the ſmallneſs of the grains, could not be accurately aſcertained, but it ſeemed to be conchoidal. Opake, ſemihard, brittle, eaſily reduced to powder. Powder has a greyiſh-black colour; powerfully attracted by the magnet; ſpecific gravity 4.765.

1. As acids were not found to act upon this mineral, 100 grains of it were reduced to a fine powder, mixed with twice its

its weight of carbonate of potash, and exposed for two hours to a red heat, in a porcelain crucible. The mass, being softened in water, was digested in muriatic acid. By repeating this process twice, the whole was dissolved in muriatic acid, except a brownish-white matter, which being dried in the open air, weighed $19\frac{1}{2}$ grains.

2. THE muriatic acid solution, which had a deep yellowish-brown colour, was concentrated almost to dryness, and then diluted with water. It assumed a milky appearance; but nothing was precipitated. Being boiled for some time, and then set aside, a curdy-like matter fell. It was of a milk-white colour, weighed, when dry, 7 grains, and possessed the properties of oxide of titanium.

3. THE residual liquid being supersaturated with ammonia, a dark reddish-brown matter precipitated, which being separated by the filter, dried, drenched in oil, and heated to redness, assumed the appearance of a black matter, strongly attracted by the magnet. It weighed 93.7 grains, and was oxide of iron.

4. THE 19.5 grains of residual powder, being mixed with four times its weight of carbonate of soda, and exposed for two hours to a red heat, in a platinum crucible, and afterwards heated with muriatic acid, was all dissolved, except about a grain of blackish matter, which was set aside.

5. THE muriatic solution being concentrated by evaporation, a little white matter was separated. It weighed $\frac{1}{4}$th of a grain, and possessed the characters of oxide of titanium.

6. WHEN

6. When evaporated to dryneſs, and rediſſolved in water, a white powder remained, which proved to be ſilica, and which, after being heated to redneſs, weighed one grain.

7. The watery ſolution being ſuperſaturated with potaſh, and boiled for a few minutes, was thrown upon a filter, to ſeparate a reddiſh-brown matter, which had been precipitated. The clear liquid which paſſed through the filter, was mixed with a ſolution of ſal ammoniac. A ſoft white matter ſlowly ſubſided. It was alumina, and, after being heated to redneſs, weighed half a grain.

8. The brown-coloured matter which had been precipitated by the potaſh, when dried upon the ſteam-bath, weighed 20.2 grains. It diſſolved with effervescence in muriatic acid. The ſolution had the appearance of the yolk of an egg. When boiled for ſome time, and then diluted with water, it became white, and let fall a curdy precipitate, which weighed, when dry, 4.6 grains, and poſſeſſed the properties of oxide of titanium.

9. The reſidual liquor being mixed with an exceſs of ammonia, let fall a brown matter, which, after being dried, drenched in oil, and heated to redneſs, weighed 6 grains. It was ſtrongly attracted by the magnet, but was of too light a colour to be pure oxide of iron. I therefore diſſolved it in muriatic acid, and placed it on the ſand-bath, in a porcelain capſule. When very much concentrated by evaporation, ſmall white needles began to make their appearance in it. The addition of hot-water made them diſappear; but they were again formed when the liquor became ſufficiently concentrated. Theſe cryſtals, when ſeparated, weighed 1.3 grains, and proved, on examination, to be white oxide of arſenic. During the ſolution of

of the 6 grains in muriatic acid, a portion of black matter ſeparated. It weighed 0.2 grains, and was totally diſſipated before the blow-pipe in a white ſmoke. Hence, it muſt have been arſenic. Theſe 1.5 gr. are equivalent to rather more than 1 grain of metallic arſenic. Thus, it appears, that the 6 grains contained 1 grain of arſenic, which explains the whiteneſs of their colour. The reſt was iron. It can ſcarcely be doubted, that the proportion of arſenic preſent was originally greater. Some of it muſt have been driven off when the iron oxide was heated with oil.

10. The inſoluble reſidue, (No. 4.), was with great difficulty diſſolved in ſulphuric acid. When the ſolution was mixed with ammonia, a white powder fell, which weighed 0.8 grains. It was accidentally loſt, before I examined its properties. But I have no doubt, from its appearance, that it was oxide of titanium.

11. Thus, from the 100 grains of iron-ſand, the following conſtituents have been extracted by analyſis:

Black oxide of iron, -	98.70
White oxide of titanium,	12.65
Arſenic, - -	1.00
Silica and alumina, -	1.50
Total,	113.85

Here there is an exceſs of nearly 14 grains, owing, without doubt, to the combination of oxygen with the iron and the titanium during the analyſis.

Had the iron in the ore been in the metallic ſtate, the exceſs of weight, inſtead of 14, could not have been leſs than 30. For the black oxide is known to be a compound of 100 metal and

and 37 oxygen. Hence, I think, it follows, that the iron in our ore muſt have been in the ſtate of an oxide, and that it muſt have contained leſs oxygen than black oxide of iron. A good many trials, both on iron-ſand, and on ſome of the other magnetic ores of iron, induce me to conclude, that the iron in moſt of them is combined with between 17 and 18 *per cent.* of oxygen. This compound, hitherto almoſt overlooked by chemiſts, I conſider as the real protoxide of iron. THENARD has lately demonſtrated, the exiſtence of an oxide intermediate between the black and the red; ſo that we are now acquainted with four oxides of this metal. But the protoxide, I preſume, does not combine with acids like the others. Analogy leads us to preſume the exiſtence of a fifth oxide, between the green and the red.

As to the titanium, it is impoſſible to know what increaſe of weight it has ſuſtained, becauſe we are neither acquainted with it in the metallic ſtate, nor know how much oxygen its different oxides contain. It is highly improbable, that, in iron-ſand, the titanium is in the metallic ſtate, if it be made out that the iron is in that of an oxide. The experiments of VAUQUELIN and HECHT, compared with thoſe of KLAPROTH, have taught us that there are three oxides of titanium, namely, the blue, the red, and the white. From an experiment of VAUQUELIN and HECHT, and from ſome of my own, I am diſpoſed to conſider theſe oxides as compoſed of the following proportions of metal and oxygen:

	METAL.	OXYGEN.
1. Blue,	100	16
2. Red,	100	33
3. White,	100	49

I find, that when the white oxide of titanium is reduced to the ſtate of red oxide, it loſes one-fourth of its weight; and that red

red oxide, when raifed to the ftate of white oxide, increafes exactly one-third of its weight. It was the knowledge of thefe facts, that led me to the preceding numbers. And I think they may be ufed, till fome more direct experiment lead us to precife conclufions.

RED oxide being the only ftate in which this metal has yet occurred feparate, we may conclude that it combines, in this ftate, with metallic oxides, and that the titanium in iron-fand, is moft probably in this ftate. But white oxide, diminifhed by one-fourth, gives us the equivalent quantity of red oxide. On that fuppofition, the titanium prefent, before the analyfis, in the 100 grains of ore, weighed 9.5 grains.

THE appearance of the arfenic furprifed me a good deal, as it was altogether unexpected. I am difpofed to afcribe it to fome particles of arfenic pyrites which might have been accidentally prefent. This conjecture will appear the more probable, when we reflect, that arfenic pyrites very frequently accompanies iron-fand. Before the microfcope, the iron-fand appears to contain fome white fhining particles, which, probably, are arfenic pyrites.

THE fmall quantity of filica and alumina, I afcribe, without hefitation, to grains of quartz and felfpar, which had adhered to the iron-fand, and been analyfed along with it. Some fuch grains were actually obferved and feparated. But others, probably, efcaped detection.

12. IF thefe fuppofitions be admitted as well founded, the iron-fand was compofed of

Protoxide of iron,	85.3
Red oxide of titanium,	9.5
Arfenic, - -	1.0
Silica and alumina, -	1.5
Lofs, - -	2.7
	100.0

The

The loſs will not appear exceſſive, if we conſider, that a portion of the arſenic muſt have been ſublimed, before the preſence of that metal was ſuſpected.

Upon the whole, I think we may conſider the ſpecimen of iron-ſand examined, as compoſed of 9 parts protoxide of iron, and 1 of red oxide of titanium. The preſence of titanium in this ore had been already detected by Lampadius, though, as I have not ſeen his analyſis, I cannot ſay in what proportion.

II. ISERINE.

The colour of this ore is iron-black, with a ſhade of brown. It conſiſts of ſmall angular grains, rather larger than thoſe of the iron-ſand, but very ſimilar to them in their appearance. Their edges are blunt; they are ſmoother, and have a ſtronger glimmering luſtre than thoſe of the iron-ſand. Luſtre ſemimetallic, inclining to metallic. The fracture could not be diſtinctly obſerved, but it ſeemed to be conchoidal; at leaſt nothing reſembling a foliated fracture could be perceived. Opake, ſemihard, brittle, eaſily reduced to powder; colour of the powder unaltered; ſpecific gravity 4.491 *; ſcarcely attracted by the magnet.

1. A hundred grains of the powdered ore were mixed with ſix times their weight of carbonate of ſoda, and expoſed for two hours to a red heat, in a platinum crucible. The maſs obtained being ſoftened with water, diſſolved completely in muriatic acid. When the ſolution was concentrated, it aſſumed the appearance

* If, as the following analyſis would lead us to expect, the ſpecimen examined was a mixture of four parts iſerine, and one part quartz and felſpar, the ſpecific gravity of pure iſerine ſhould be 4.964.

pearance of the yolk of an egg. It was boiled, diluted with water, and ſet aſide for ſome time. A white matter gradually depoſited, which, when dried on the ſteam-bath, weighed 53 grains, and poſſeſſed the properties of oxide of titanium.

2. THE liquid thus freed from titanium, was evaporated to dryneſs, and the reſidue rediſſolved in water, acidulated with muriatic acid. A white powder remained, which, after being heated to redneſs, weighed 16.8 grains, and poſſeſſed the properties of ſilica.

3. THE ſolution was precipitated by ammonia, and the brown matter which had ſeparated, boiled for ſome time in liquid potaſh. The whole was then thrown on a filter, to ſeparate the undiſſolved part, and the liquid which came through, was mixed with a ſolution of ſal ammoniac. A white powder fell, which, after being heated to redneſs, weighed 3.2 grains. It was alumina.

4. THE brown ſubſtance collected on the filter, was dried, drenched in oil, and heated to redneſs. It was ſtrongly attracted by the magnet, and weighed 52 grains.

5. IT was digeſted in diluted ſulphuric acid; but not being rapidly acted upon, a quantity of muriatic acid was added, and the digeſtion continued. The whole ſlowly diſſolved, except a blackiſh matter, which became white when expoſed to a red heat, and, as far as I could judge from its properties, was oxide of titanium, ſlightly contaminated with iron. It weighed 1.8 grains.

6. THE acid ſolution being concentrated by gentle evaporation, a number of ſmall yellowiſh-coloured needles made their ap-

pearance in it. By repeated evaporations, all the cryſtals that would form were ſeparated. They weighed 6 grains. I rediſſolved them in water, and added ſome ammonia to the ſolution. A fine yellow powder fell, which I ſoon recogniſed to be oxide of uranium. It weighed 4.2 grains.

7. THUS it appears, that the 52 grains (No. 4.), attracted by the magnet, contained 46 grains of iron, and 6 grains of uranium and titanium.

8. THE following are the ſubſtances ſeparated from 100 grains of iſerine, by the preceding analyſis:

Oxide of titanium,	54.8
Oxide of iron, -	46.0
Oxide of uranium,	4.2
Silica, - -	16.8
Alumina, -	3.2
Total,	125.0

Here is an exceſs of no leſs than 25 grains, to be accounted for by oxygen, which muſt have united to the three metals during the proceſs. As to the ſilica and alumina, there can be little heſitation in aſcribing them to grains of ſand, which had been mixed with the ore. The pure iſerine, in all probability, was compoſed of iron, titanium, and uranium. If we ſuppoſe that each of theſe metals exiſted in the ſtate of protoxide, we muſt diminiſh the titanium by one-fourth, the iron by one-ſeventh nearly, and the uranium, according to BUCHOLZ's experiments, by one-fifth. This would give us,

Titanium,

Titanium, -	41.1
Iron, - -	39.4
Uranium, -	3.4
Silica and alumina,	20.0
	103.9

Here, then, is ſtill an exceſs of nearly 4 *per cent.* But this I am diſpoſed to aſcribe to the oxides of titanium and uranium, having been only dried upon the ſteam-bath. Upon the whole, it appears, that, in the ſpecimens of iſerine analyſed, the proportions of titanium and iron were nearly equal, and that the uranium did not exceed 4 *per cent.* The appearance of uranium ſurpriſed me a good deal. I perceive, however, that it has already been detected in this ore, from an analyſis publiſhed by Profeſſor JAMESON, in the ſecond volume of his Mineralogy, which, I underſtand, was made by LAMPADIUS. The ſpecimen examined by LAMPADIUS yielded very nearly 60 parts of titanium, 30 of iron, and 10 of uranium. Whereas, in mine, if the foreign matter be removed, there was obtained, very nearly, - -

48	titanium,
48	iron,
4	uranium,
100	

But, there can be no doubt, that the iſerine which I analyſed was ſtill contaminated with a good deal of iron-ſand; for it was impoſſible to remove the whole.

13*th*

Analysis of the Grey Copper Ore, *from Airthrey.*

The copper mine of Airthrey, near Stirling, conſiſts of a thin vein, which runs through the weſt corner of the Ochils. It has been twice wrought, by two different companies. But, in both caſes, was abandoned, after a few years trial. I went to it ſome years ago, and examined the ore, at the requeſt of one of the proprietors. The ſpecimens which were employed for the ſubſequent analyſis, were the pureſt that I could ſelect, out of a conſiderable quantity. I was told, however, that from the lower level, which was at that time full of water, much richer ore had been extracted. But, afterwards, when the lower level was freed from its water, I went down to it myſelf, and found the ore preciſely of the ſame kind as in the upper, with this difference, that it was more mixed with calcareous ſpar, and perhaps, on that account, more eaſily ſmelted.

The veinſtones in the Airthrey mine are ſulphate of barytes, and carbonate of lime, and with theſe the ore is almoſt always more or leſs mixed.

The colour is at firſt light ſteel-grey; but the ſurface ſoon tarniſhes, and becomes of a dark dull leaden-grey, and in ſome places aſſumes a beautiful tempered ſteel tarniſh. Maſſive and diſſeminated. In ſome ſpecimens, it exhibits the appearance of imperfect cryſtals. Internal ſurface ſhining and metallic; but, by expoſure, it ſoon becomes dull. Fracture ſmall-grained, inclining to even. Fragments indeterminate, and rather blunt-edged. Semihard, the degree being almoſt the ſame as that of calcareous ſpar; for theſe two minerals reciprocally ſcratch each

each other. Streak ſimilar, opake, brittle, eaſily frangible; ſpecific gravity 4.878.

1. To free the ore as completely as poſſible from foreign matter, it was reduced to a coarſe powder, and carefully picked. It was then digeſted in diluted muriatic acid, which diſſolved a quantity of carbonate of lime, amounting to 13 *per cent.* of the original weight of the ore.

2. Thus purified, it was dried on the ſteam-bath, and 100 grains of it were reduced to a fine powder, and digeſted in diluted nitric acid, till every thing ſoluble in that menſtruum was taken up. The reſidue was digeſted in the ſame manner, in muriatic acid; and when that acid ceaſed to act, the reſidue was treated with nitro-muriatic acid till no farther ſolution could be produced. The inſoluble matter was of a white colour; it weighed 6.9 grains, and was almoſt entirely ſulphate of barytes. No traces of ſulphate of lead, nor of oxide of antimony, could be detected in it by the blow-pipe.

3. The three acid ſolutions being mixed together, no cloudineſs appeared, nor was any change produced; a proof that the ore contained no ſilver.

4. The ſolution being evaporated nearly to dryneſs, was diluted with water, and precipitated by muriate of barytes. By this means, the ſulphuric and arſenic acids, which had been formed during the long-continued action of the nitric acid on the ore, and the preſence of which had been indicated by reagents, were thrown down; for nitrate of lead, added to the reſidual liquid, occaſioned no precipitate; a proof that no arſenic acid was preſent.

5. The

5. The liquid, thus freed from arſenic acid, was mixed with an exceſs of ammonia. It aſſumed a deep blue colour, while a brown matter precipitated. It was ſeparated by the filter, and being dried, drenched in oil, and heated to redneſs, it was totally attracted by the magnet. It weighed 45.5 grains, and was iron.

6. The ammoniacal liquid was neutraliſed by ſulphuric acid, and the copper thrown down by means of an iron plate. It weighed 17.2 grains.

7. To aſcertain the quantity of ſulphur and arſenic, 100 grains of the purified ore, in the ſtate of a fine powder, were put into the bottom of a coated glaſs-tube, and expoſed for two hours to a red heat. When the whole was cold, and the bottom of the tube cut off, the ore was found in a round ſolid maſs, having the metallic luſtre, a conchoidal fracture, and the colour and appearance of *variegated copper-ore*. It had loſt 16 grains of its weight.

8. The upper part of the tube was coated with a yellowiſh-brown ſubſtance, like melted ſulphur. It weighed 12.6 grains. Thus, there was a loſs of 3.4 grains. As the tube was long, this loſs can ſcarcely be aſcribed to ſulphur driven off. I rather conſider it as water. For towards the beginning of the proceſs, drops of water were very perceptible in the tube. Whether this water was a conſtituent of the ore, or derived from the previous digeſtion in muriatic acid, cannot be determined.

9. When the 12.6 grains of yellowiſh brown matter detached from the tube, were digeſted in hot potaſh-ley, the whole was diſſolved, except a fine blackiſh powder, which weighed

weighed 1 grain, and was arſenic. The diſſolved portion I conſidered as ſulphur.

10. The potaſh ſolution, being mixed with nitric acid, 4 grains of ſulphur fell. The remaining 7.6 grains muſt have been converted into ſulphuric acid, by the action of the nitric acid. Accordingly, muriate of barytes occaſioned a copious precipitate.

11. The 84 grains of roaſted ore being reduced to a fine powder, mixed with half their weight of pounded charcoal, and roaſted a ſecond time in a glaſs-tube, one grain of ſulphur ſublimed. But the tube breaking before the roaſting had been continued long enough, the proceſs was completed in a crucible. The roaſted ore weighed 70 grains.

12. From the preceding analyſis, we learn that the conſtituents of the Airthrey ore, are as follows:

Iron, -	45.5
Copper, -	17.2
Arſenic, -	14.0
Sulphur, -	12.6
Water, -	3.4
Foreign bodies,	6.9
	99.6
Loſs, -	.4
	100.0

If we ſuppoſe the water and the earthy reſidue to be only accidentally preſent, then the only eſſential conſtituents are the firſt four, and the ore would be a compound of

Iron,	51.0
Copper,	19.2
Arſenic,	15.7
Sulphur,	14.1
	100.0

If we compare this analyſis with ſeveral analyſes of grey copper ore, lately publiſhed by KLAPROTH, we ſhall find, that the conſtituents are the ſame in both; but the proportions of the two firſt ingredients are very nearly reverſed. KLAPROTH obtained from 0.4 to 0.5 of copper, and from 0.22 to 0.27 of iron. This renders it obvious, that the two ores were not in the ſame ſtate. I have little doubt, that the difference, however, is merely apparent, and that it aroſe, altogether, from a quantity of iron pyrites, and perhaps alſo of arſenic pyrites, which I could not ſeparate from the grey copper ore which I examined. Both of theſe minerals could be diſtinctly ſeen in many of the ſpecimens, intimately mixed with the grey copper; and I have no doubt that the ſame mixture exiſted, even in thoſe ſpecimens which were ſelected as pureſt. The difference in the proportions of copper and arſenic, obtained by KLAPROTH * in his various analyſes, is ſo conſiderable, as to lead to a ſuſpicion, that even his ſpecimens, in all probability, contained a mixture of foreign matter.

* GEHLEN's Jour. vol. v. p. 9. 11. 13.

VIII.

VIII. *New Series for the* Quadrature *of the* Conic Sections, *and the* Computation *of* Logarithms.

By William Wallace, *one of the Professors of Mathematics in the Royal Military College at Great Marlow, and* F. R. S. Edin.

[*Read 27th June* 1808.]

1. THE Quadrature of the Conic Sections, and the Computation of Logarithms, are problems of conſiderable importance, not only in the elements of Mathematics, but alſo in the higher branches of that ſcience. On this account, every ſucceſsful attempt to ſimplify their reſolution, as well as any new formulæ which may be found applicable to that purpoſe, muſt always be intereſting, and muſt in ſome meaſure contribute to the improvement of mathematical knowledge.

2. The object of this Paper, is to give ſolutions of theſe problems, which ſhall be at once ſimple and elementary, without employing the fluxional or other equivalent calculus; and it is preſumed, that thoſe which follow, will be found to partake ſo much of both theſe properties, that they may even admit of being incorporated with the elements of Geometry and Analyſis. Beſides, the formulæ which reſult from the inveſtigations, are, as far as I know, entirely new, while each is appli-

cable to every poſſible caſe of the problem to be reſolved. Now this laſt circumſtance is the more remarkable, as it generally happens, that a ſeries which applies very well to the quadrature of a curve within certain limits, is quite inapplicable beyond them.

3. Although, in a general way, this Paper may be ſaid to treat of the quadrature of the Conic Sections, yet there is one of them, namely, the Parabola, which I ſhall not at all notice; becauſe, although its area may be found in a way analogous to that which is here employed in the caſe of the other two, yet the formula which would thence reſult, muſt, from its nature, be the ſame as would be found by any other mode of proceeding.

As the quadratures of the ellipſe, and any hyperbola may be deduced from thoſe of the circle and equilateral hyperbola, I ſhall, in the following Paper, treat only of the two laſt; and as the quadrature of a ſector of a circle, and the rectification of its bounding arch, are reducible the one to the other, it is a matter of indifference which of theſe we conſider. I ſhall, however, confine myſelf to the latter.

4. In treating of logarithms, I might, after the example of the earlier writers on this ſubject, deduce the formulæ for their computation from thoſe which we ſhall find for the quadrature of the equilateral hyperbola. I prefer, however, treating this ſubject in a manner purely analytical, without adverting at all to the hyperbola, being of opinion, that every branch of mathematics ought, as much as poſſible, to be deduced from its own peculiar principles; and therefore, that it would be contrary to good method, to have recourſe to the properties of geometrical figure, when treating of a ſubject entirely arithmetical.

5. To

5. To proceed now in the inveftigation of the different feries, for the rectification of an arch of a circle, let A denote any arch, the radius being fuppofed unity. Then, from the arithmetic of fines, we have

$$\frac{1}{\tan A} = \frac{1}{2\tan\frac{1}{2}A} - \frac{1}{2}\tan\frac{1}{2}A.$$

In this formula let each term of the feries of arches

$$a, \frac{1}{2}a, \frac{1}{4}a, \frac{1}{8}a \ldots\ldots \frac{1}{2^{n-2}}a, \frac{1}{2^{n-1}}a,$$

(which is a geometrical progreffion, having the number of its terms n, and its common ratio $\frac{1}{2}$,) be fucceffively fubftituted for A, and let the refults be multiplied by the terms of the correfponding feries of fractions

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\ldots \frac{1}{2^{n-2}}, \frac{1}{2^{n-1}};$$

then we fhall obtain the following feries of equations:

$$\frac{1}{\tan a} = \frac{1}{2\tan\frac{1}{2}a} - \frac{1}{2}\tan\frac{1}{2}a,$$

$$\frac{1}{2\tan\frac{1}{2}a} = \frac{1}{4\tan\frac{1}{4}a} - \frac{1}{4}\tan\frac{1}{4}a,$$

$$\frac{1}{4\tan\frac{1}{4}a} = \frac{1}{8\tan\frac{1}{8}a} - \frac{1}{8}\tan\frac{1}{8}a,$$

$$\frac{1}{8\tan\frac{1}{8}a} = \frac{1}{16\tan\frac{1}{16}a} - \frac{1}{16}\tan\frac{1}{16}a,$$

. . . .

$$\frac{1}{2^{n-2}\tan\frac{a}{2^{n-2}}} = \frac{1}{2^{n-1}\tan\frac{a}{2^{n-1}}} - \frac{1}{2^{n-1}}\tan\frac{a}{2^{n-1}},$$

$$\frac{1}{2^{n-1}\tan\frac{a}{2^{n-1}}} = \frac{1}{2^{n}\tan\frac{a}{2^{n}}} - \frac{1}{2^{n}}\tan\frac{a}{2^{n}}.$$

LET

Let the ſums of the correſponding ſides of theſe equations be taken, and obſerving that the ſeries

$$\frac{1}{2\tan\frac{1}{2}a}+\frac{1}{4\tan\frac{1}{4}a}+\frac{1}{8\tan\frac{1}{8}a}\ldots+\frac{1}{2^{n-1}\tan\frac{a}{2^{n-1}}}$$

is found in each ſum, let it be rejected from both; and the reſult will be

$$\frac{1}{\tan a}=\begin{cases}\dfrac{1}{2^n\tan\dfrac{a}{2^n}}\\ -\left(\frac{1}{2}\tan\frac{1}{2}a+\frac{1}{4}\tan\frac{1}{4}a+\frac{1}{8}\tan\frac{1}{8}a+\frac{1}{16}\tan\frac{1}{16}a\ldots\right.\\ \qquad\left.+\frac{1}{2^n}\tan\frac{a}{2^n}\right),\end{cases}$$

the number of terms of the ſeries in the parentheſis being n, and hence we have

$$\frac{1}{2^n\tan\frac{a}{2^n}}=\frac{1}{\tan a}+\frac{1}{2}\tan\frac{1}{2}a+\frac{1}{4}\tan\frac{1}{4}a+\frac{1}{8}\tan\frac{1}{8}a+$$
$$\frac{1}{16}\tan\frac{1}{16}a\ldots+\frac{1}{2^n}\tan\frac{a}{2^n}.$$

6. Now, $2\tan\frac{1}{2}a$ is the perimeter of a figure formed by drawing tangents at the ends of the arch a, and producing them till they meet; and $4\tan\frac{1}{4}a$ is the perimeter of a figure formed by biſecting the arch a, and drawing tangents at its extremities and at the point of biſection, producing each two adjoining tangents till they meet; and in general $2^n\tan\frac{a}{2^n}$ is the perimeter of a figure formed in the ſame way, by dividing the arch

arch a into 2^{n-1} equal parts, and drawing tangents at the points of diviſion, and the extremities of the arch. Therefore, denoting the perimeter of the figure thus conſtructed by P, we have

$$\frac{1}{P} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a$$

$$+ \frac{1}{16}\tan\frac{1}{16}a \ldots + \frac{1}{2^n}\tan\frac{a}{2^n};$$

and this is true, whatever be the number of terms in the ſeries

$$\frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a \ldots\ldots + \frac{1}{2^n}\tan\frac{a}{2^n}\ *.$$

7. Now ſuppoſing n the number of terms in the ſeries, to increaſe, then 2^{n-1}, the number of equal parts into which the arch is conceived to be divided, will alſo increaſe, and may become greater than any aſſignable number. But it is a principle admitted in the elements of geometry, that an arch being divided, and a polygon deſcribed about it in the manner ſpecified in article 6., the perimeter of the polygon will continually approach to the circular arch, and will at laſt differ from it by leſs than any given quantity. Therefore, if we ſuppoſe n indefinitely great, ſo that the ſeries may go on *ad infinitum*, then, inſtead of P in the formula of the laſt article, we may ſubſtitute its limit, namely, the arch a, and thus we ſhall have

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a +$$

$$\frac{1}{16}\tan\frac{1}{16}a +, \text{ \&c.}$$

Thus

* We may here obſerve, that this formula may be conſidered as the analytic expreſſion of a general theorem (which is not inelegant) relating to regular figures deſcribed about any arch of a circle; and others analogous to it will occur in the following inveſtigations.

Thus we have the circular arch, or rather its reciprocal (from which the arch itſelf is eaſily found), expreſſed by a ſeries of a very ſimple form; and this is the firſt formula which I propoſed to give for the rectification of the circle.

8. We now proceed to inquire what is the degree of convergency of this ſeries. In the firſt place, it appears, that the numeral co-efficients of the terms are each one-half of that which goes before it. Again, A being any arch of a circle, we have by a theorem in the elements of geometry, ſec A : 1 : : tan A — tan $\frac{1}{2}$ A : tan $\frac{1}{2}$ A; therefore, 1 + ſec A : 1 : : tan A : tan $\frac{1}{2}$ A, and hence $\tan\frac{1}{2}A = \frac{\tan A}{1+\sec A}$. But as ſec A is greater than 1, therefore 1 + ſec A muſt be greater than 2, and conſequently $\frac{\tan A}{1+\sec A}$ leſs than $\frac{\tan A}{2}$; hence it follows, that tan $\frac{1}{2}$ A muſt be leſs than $\frac{1}{2}$ tan A. Thus it appears, that *a* being any arch leſs than a quadrant, the tangent of any one of the ſeries of arches $\frac{1}{2}a$, $\frac{1}{4}a$, $\frac{1}{8}a$, &c. is leſs than half the tangent of the arch before it. By combining the rate of convergency of the tangents with that of their numeral co-efficients, it appears, that each term of the ſeries, after the ſecond, is leſs than one-fourth of the term before it; and this is one limit to the rate of convergency of the ſeries.

9. Again, to find another limit, let us reſume the formula $\tan\frac{1}{2}A = \frac{\tan A}{1+\sec A}$, from which it follows, that $\frac{\tan\frac{1}{2}A}{\tan A} = \frac{1}{1+\sec A}$, and ſimilarly, that $\frac{\tan\frac{1}{4}A}{\tan\frac{1}{2}A} = \frac{1}{1+\sec\frac{1}{2}A}$. But

ſince

fince fec $\frac{1}{2}$ A < fec A, and confequently $\frac{1}{1+\sec\frac{1}{2}A} > \frac{1}{1+\sec A}$,

therefore $\frac{\tan\frac{1}{4}A}{\tan\frac{1}{2}A} > \frac{\tan\frac{1}{2}A}{\tan A}$, and $\tan\frac{1}{4}A > \frac{\tan\frac{1}{2}A}{\tan A} \times \tan\frac{1}{2}A$.

In this expreffion, let $\frac{1}{2}a$, $\frac{1}{4}a$, $\frac{1}{8}a$, &c. be fubftituted for A, and let the refults be divided by 8, 16, 32, &c.; then we get

$$\tfrac{1}{8}\tan\tfrac{1}{8}a > \frac{\frac{1}{4}\tan\frac{1}{4}a}{\frac{1}{2}\tan\frac{1}{2}a} \times \tfrac{1}{4}\tan\tfrac{1}{4}a,$$

$$\tfrac{1}{16}\tan\tfrac{1}{16}a > \frac{\frac{1}{8}\tan\frac{1}{8}a}{\frac{1}{4}\tan\frac{1}{4}a} \times \tfrac{1}{8}\tan\tfrac{1}{8}a,$$

&c.

from which it appears, that in the feries, $\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a + \frac{1}{16}\tan\frac{1}{16}a +$, &c.

each term after the third (that is, after $\frac{1}{4}\tan\frac{1}{4}a$), is greater than a third proportional to the two terms immediately before it, taken in their order; and this is another limit to the rate of convergency of the feries.

10. THE limits which we have found to the rate of convergency of the feries, enable us alfo to affign limits to the fum of all the terms after any given term. Let the feries be put under this form,

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a \;\ldots + T_{(m)} + T_{(m+1)} + T_{(m+2)} +, \text{\&c.}$$

 where

where $T_{(m)}$, $T_{(m+1)}$, $T_{(m+2)}$, &c. denote the terms whofe places in the feries are expreffed by the numbers m, $m+1$, $m+2$, &c. Then, becaufe

$$T_{(m+2)} < \frac{1}{4} T_{(m+1)},$$

$$T_{(m+3)} < \frac{1}{4} T_{(m+2)},$$

$$T_{(m+4)} < \frac{1}{4} T_{(m+3)},$$

&c.

We have

$$T_{(m+2)} + T_{(m+3)} + T_{(m+4)} +, \&c. < \frac{1}{4} \Big(T_{(m+1)} + T_{(m+2)} + T_{(m+3)} +, \&c.\Big)$$

That is, putting S for $T_{(m+2)} + T_{(m+3)} + T_{(m+4)} +$, &c. or for the fum of all the terms after $T_{(m+1)}$,

$S < \frac{1}{4}\left(T_{(m+1)} + S\right)$, and hence $\frac{3}{4} S < \frac{1}{4} T_{(m+1)}$, & $S < \frac{1}{3} T_{(m+1)}$.

Thus it appears, that the fum of all the terms of the feries following any term after the firft, is lefs than the third part of that term.

11. AGAIN, from what has been faid in Article 9., we have $T_{(m+2)} > \frac{T_{(m+1)}}{T_{(m)}} T_{(m+1)}$, and therefore $\frac{T_{(m+2)}}{T_{(m+1)}} > \frac{T_{(m+1)}}{T_{(m)}}$, and fimilarly $\frac{T_{(m+3)}}{T_{(m+2)}} > \frac{T_{(m+2)}}{T_{(m+1)}}$, and $\frac{T_{(m+4)}}{T_{(m+3)}} > \frac{T_{(m+3)}}{T_{(m+2)}}$,

and

and ſo on. From which it follows, that

$$T_{(m+2)} > \frac{T_{(m+1)}}{T_{(m)}} T_{(m+1)},$$

$$T_{(m+3)} > \frac{T_{(m+1)}}{T_{(m)}} T_{(m+2)},$$

$$T_{(m+4)} > \frac{T_{(m+1)}}{T_{(m)}} T_{(m+3)},$$

&c.

HENCE, taking the ſum of the quantities on each ſide of the ſign $>$, and putting S for

$$T_{(m+2)} + T_{(m+3)} + T_{(m+4)} +, \text{ \&c.}$$

we get

$$S > \frac{T_{(m+1)}}{T_{(m)}} \left(T_{(m+1)} + S\right).$$

Therefore $S - \frac{T_{(m+1)}}{T_{(m)}} S > \frac{T^2_{(m+1)}}{T_{(m)}}$, and conſequently by reduction,

$$S > \frac{T_{(m+1)}}{T_{(m)} - T_{(m+1)}} T_{(m+1)};$$

from which it appears, that the ſum of all the terms following any aſſigned term after the third, is greater than a third proportional to the difference of the two terms immediately before it and the latter of the two. But ſince this limit will not differ much from the former, which is $\frac{1}{3} T_{(m+1)}$, it may be more conveniently expreſſed thus,

$$S > \frac{1}{3} T_{(m+1)} - \frac{T_{(m)} - 4 T_{(m+1)}}{3\left(T_{(m)} - T_{(m+1)}\right)} T_{(m+1)};$$

 which

which formula, by reduction, will be found to be the very same as the other.

12. THE result, then, of the whole investigation, may be briefly stated as follows: Let a denote any arch of a circle of which the radius is unity, then shall

$$\frac{1}{a} = \left\{ \begin{array}{l} \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a + \frac{1}{16}\tan\frac{1}{16}a \ldots \\ \quad + T_{(m)} + T_{(m+1)} + S; \end{array} \right.$$

where $T_{(m)}$ and $T_{(m+1)}$ denote any two succeeding terms of the series $\frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a +$, &c., their places in it being expressed by the numbers m and $m+1$; and where S is put for the sum of all the remaining terms; and the limits of S are the two quantities

$\frac{1}{3}T_{(m+1)}$ and $\frac{1}{3}T_{(m+1)} - \frac{\left(T_{(m)} - 4\,T_{(m+1)}\right)T_{(m+1)}}{3\left(T_{(m)} - T_{(m+1)}\right)}$, that is, S is less than the former, but greater than the latter. The expressions $\tan\frac{1}{2}a$, $\tan\frac{1}{4}a$, $\tan\frac{1}{8}a$, &c. are easily deduced from $\tan a$, and from one another, by a well-known formula in the arithmetic of sines, which may be expressed thus,

$$\tan\frac{1}{2}A = \sqrt{\frac{1}{\tan^2 A} + 1} - \frac{1}{\tan A}.$$

13. I NOW proceed to the investigation of a second formula for the rectification of the circle; and for this purpose resume the

the equation $\frac{1}{\tan A} = \frac{1}{2\tan\frac{1}{2}A} - \frac{1}{2}\tan\frac{1}{2}A$, which, by taking the ſquare of each ſide, is transformed to

$$\frac{1}{\tan^2 A} = \frac{1}{4\tan^2\frac{1}{2}A} + \frac{1}{4}\tan^2\frac{1}{2}A - \frac{1}{2}.$$

In this formula, let each term of the ſeries of arches

$$a,\ \frac{1}{2}a,\ \frac{1}{4}a,\ \frac{1}{8}a \ldots \frac{a}{2^{n-1}},$$

of which the number of terms is n, be ſubſtituted ſucceſſively for a, and let the reſults be multiplied by the correſponding terms of the ſeries of fractions,

$$1,\ \frac{1}{4},\ \frac{1}{4^2},\ \frac{1}{4^3} \ldots \frac{1}{4^{n-2}},\ \frac{1}{4^{n-1}};$$

thus there will be formed the ſeries of equations

$$\frac{1}{\tan^2 a} = \frac{1}{2^2\tan^2\frac{1}{2}a} + \frac{1}{4}\tan^2\frac{1}{2}a - \frac{1}{2},$$

$$\frac{1}{2^2\tan^2\frac{1}{2}a} = \frac{1}{4^2\tan^2\frac{1}{4}a} + \frac{1}{4^2}\tan^2\frac{1}{4}a - \frac{1}{2.4},$$

$$\frac{1}{4^2\tan^2\frac{1}{4}a} = \frac{1}{8^2\tan^2\frac{1}{8}a} + \frac{1}{4^3}\tan^2\frac{1}{8}a - \frac{1}{2.4^2},$$

$$\frac{1}{8^2\tan^2\frac{1}{8}a} = \frac{1}{16^2\tan^2\frac{1}{16}a} + \frac{1}{4^4}\tan^2\frac{1}{16}a - \frac{1}{2.4^3},$$

. . . .

$$\frac{1}{2^{2n-4}\tan\frac{a}{2^{n-2}}} = \frac{1}{2^{2n-2}\tan^2\frac{a}{2^{n-1}}} + \frac{1}{4^{n-1}}\tan^2\frac{a}{2^{n-1}} - \frac{1}{2.4^{n-2}},$$

$$\frac{1}{2^{2n-2}\tan\frac{a}{2^{n-1}}} = \frac{1}{2^{2n}\tan^2\frac{a}{2^n}} + \frac{1}{4^n}\tan^2\frac{a}{2^n} - \frac{1}{2.4^{n-1}}.$$

Let

Let the ſum of theſe equations be taken, as in the inveſtigation of the firſt formula, and obſerving that the ſeries

$$\frac{1}{2^2\tan^2\frac{1}{2}a}+\frac{1}{4^2\tan^2\frac{1}{4}a}+\frac{1}{8^2\tan^2\frac{1}{8}a}\ldots+\frac{1}{2^{2n-2}\tan^2\frac{a}{2^{n-1}}}$$

is found on both ſides of the reſulting equation, let it be rejected from both; then we obtain

$$\tan^2 a=\begin{cases}\dfrac{1}{2^{2n}\tan^2\dfrac{a}{2^n}}\\ +\dfrac{1}{4}\tan^2\dfrac{1}{2}a+\dfrac{1}{4^2}\tan^2\dfrac{1}{4}a+\dfrac{1}{4^3}\tan^2\dfrac{1}{8}a+\dfrac{1}{4^4}\tan^2\dfrac{1}{16}a\ldots\\ +\dfrac{1}{4^n}\tan^2\dfrac{a}{2^n}-\left(\dfrac{1}{2}+\dfrac{1}{2.4}+\dfrac{1}{2.4^2}+\dfrac{1}{2.4^3}\ldots+\dfrac{1}{2.4^{n-1}}\right).\end{cases}$$

Now it appears, that one part of this expreſſion, viz.

$$\frac{1}{2}+\frac{1}{2.4}+\frac{1}{2.4^2}+\frac{1}{2.4^3}\ldots+\frac{1}{2.4^{n-1}},$$

is a geometrical ſeries, the firſt term of which is $\frac{1}{2}$, the laſt term $\frac{1}{2.4^{n-1}}$, and common ratio $\frac{1}{4}$; therefore its ſum is $\frac{2}{3}\left(1-\frac{1}{4^n}\right)$.

Alſo, ſince $2^n\tan\frac{a}{2^n}$ is the expreſſion for the perimeter of a polygon, formed by dividing the arch a into 2^{n-1} equal parts, by drawing tangents at the points of diviſion, and producing the

the adjacent tangents until they meet, (Art. 6.); therefore $2^{2n}\tan^2\frac{a}{2^n}$ will be the ſquare of that perimeter. Let the perimeter itſelf be denoted by P, then, ſubſtituting P^2 in the equation inſtead of $2^{2n}\tan\frac{a}{2^n}$, and $\frac{2}{3}\left(1-\frac{1}{4^n}\right)$ inſtead of the ſeries to which it is equivalent, and bringing $\frac{1}{P^2}$ to one ſide, we get

$$\frac{1}{P^2}=\begin{cases}\dfrac{1}{\tan^2 a}+\dfrac{2}{3}\left(1-\dfrac{1}{4^n}\right)\\ -\left(\dfrac{1}{4}\tan^2\dfrac{1}{2}a+\dfrac{1}{4^2}\tan^2\dfrac{1}{4}a+\dfrac{1}{4^3}\tan^2\dfrac{1}{8}a+\dfrac{1}{4^4}\tan^2\dfrac{1}{16}a\ldots\right.\\ \left.+\dfrac{1}{4^n}\tan^2\dfrac{a}{2^n}\right).\end{cases}$$

15. THIS is true, whatever be the value of n, the number of terms of the ſeries in the parentheſis. Let us now conceive the ſeries to be continued indefinitely, then, as upon this hypotheſis, n may be conſidered as indefinitely great, $\frac{1}{4^n}$ will become leſs than any aſſignable quantity, and therefore $\frac{2}{3}\left(1-\frac{1}{4^n}\right)$ will become ſimply $\frac{2}{3}$; moreover, P will in this caſe become a, (Art. 7.), and P^2 will become a^2. Thus, upon the whole, we ſhall

ſhall have

$$\frac{1}{a^2} = \begin{cases} \dfrac{1}{\tan^2 a} + \dfrac{2}{3} \\ -\left(\dfrac{1}{4}\tan^2\dfrac{1}{2}a + \dfrac{1}{4^2}\tan^2\dfrac{1}{4}a + \dfrac{1}{4^3}\tan^2\dfrac{1}{8}a + \dfrac{1}{4^4}\tan^2\dfrac{1}{16}a \right. \\ \left. +, \&c.\right), \end{cases}$$

and this may be conſidered as a ſecond formula for the rectification of any arch of a circle; for the proceſs by which an arch is found from the ſquare of its reciprocal is ſo ſimple, that the latter being known, the former may alſo be regarded as known.

16. INSTEAD of expreſſing the ſquare of the reciprocal of the arch in this manner, by the ſquares of the tangents of its ſub-multiples, we may expreſs it otherwiſe by the ſquares of their ſecants. For ſince $\tan^2\frac{1}{2}a = \sec^2\frac{1}{2}a - 1$, and $\tan^2\frac{1}{4}a = \sec^2\frac{1}{4}a - 1$, and ſo on, therefore the ſeries

$$\frac{1}{4}\tan^2\frac{1}{2}a + \frac{1}{4^2}\tan^2\frac{1}{4}a + \frac{1}{4^3}\tan^2\frac{1}{8}a + \frac{1}{4^4}\tan^2\frac{1}{16}a +, \&c.$$

is equivalent to

$$\frac{1}{4}\sec^2\frac{1}{2}a + \frac{1}{4^2}\sec^2\frac{1}{4}a + \frac{1}{4^3}\sec^2\frac{1}{8}a + \frac{1}{4^4}\sec^2\frac{1}{16}a +, \&c.$$

$$-\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} +, \&c.\right),$$

But the latter part of this expreſſion is evidently an infinite geometrical

geometrical ſeries, of which the firſt term is $\frac{1}{4}$, and ratio $\frac{1}{4}$; therefore its ſum will be $\frac{1}{3}$; hence, by ſubſtitution, and putting $\frac{1}{\sin^2 a}$ for $\frac{1}{\tan^2 a} + 1$, we have

$$\frac{1}{a^2} = \frac{1}{\sin^2 a} - \left(\frac{1}{4}\sec^2 \frac{1}{2}a + \frac{1}{4^2}\sec^2 \frac{1}{4}a + \frac{1}{4^3}\sec^2 \frac{1}{8}a + \frac{1}{4^4}\sec^2 \frac{1}{16}a \right.$$

$$\left. +, \&c.\right),$$

which is the ſeries to be inveſtigated.

17. THERE is, however, another form, under which the ſeries brought out in Article 15. may be given, and which I conſider as the beſt adapted of any to the actual calculation of the length of an arch. This transformation will be effected, if, in the well-known formula, $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$, inſtead of A, we ſubſtitute ſucceſſively a, $\frac{1}{2}a$, $\frac{1}{4}a$, &c.; we ſhall then obtain a ſeries of fractions of the form $\frac{1 - \cos \frac{1}{n}a}{1 + \cos \frac{1}{n}a}$ which being ſubſtituted inſtead of their equivalents in the formula

$$\frac{1}{a^2} = \begin{cases} \frac{1}{\tan^2 a} + \frac{2}{3} \\ -\left(\frac{1}{4}\tan^2 \frac{1}{2}a + \frac{1}{4^2}\tan^2 \frac{1}{4}a + \frac{1}{4^3}\tan^2 \frac{1}{8}a + \frac{1}{4^4}\tan^2 \frac{1}{16}a \right. \\ \left. +, \&c.\right), \end{cases}$$

it

it is changed to

$$\frac{1}{a^2} = \begin{cases} \frac{1+\cos 2a}{1-\cos 2a} + \frac{2}{3} \\ -\left(\frac{1}{4}\,\frac{1-\cos a}{1+\cos a} + \frac{1}{4^2}\,\frac{1-\cos\frac{1}{2}a}{1+\cos\frac{1}{2}a} + \frac{1}{4^3}\,\frac{1-\cos\frac{1}{4}a}{1+\cos\frac{1}{4}a}\right. \\ \left. + \frac{1}{4^4}\,\frac{1-\cos\frac{1}{8}a}{1+\cos\frac{1}{8}a} +, \&c.\right); \end{cases}$$

or, putting $\frac{1}{2}a$ instead of a, and $\frac{1}{4}a$ instead of $\frac{1}{2}a$, and so on, in order that the formula may contain only the cosines of the arch and its sub-multiples, and dividing the whole expression by 4,

$$\frac{1}{a^2} = \begin{cases} \frac{1}{4}\,\frac{1+\cos a}{1-\cos a} + \frac{1}{6} \\ -\left(\frac{1}{4^2}\,\frac{1-\cos\frac{1}{2}a}{1+\cos\frac{1}{2}a} + \frac{1}{4^3}\,\frac{1-\cos\frac{1}{4}a}{1+\cos\frac{1}{4}a} + \frac{1}{4^4}\,\frac{1-\cos\frac{1}{8}a}{1+\cos\frac{1}{8}a}\right. \\ \left. + \frac{1}{4^5}\,\frac{1-\cos\frac{1}{16}a}{1+\cos\frac{1}{16}a} +, \&c.\right) \end{cases}$$

and this is the second series which I proposed to investigate, reduced to its most convenient form.

18. WE may determine two limits to the rate of convergency of the series just now found, in the same manner as we have found the limits of that of our first series; and, indeed, the reasoning employed in the one case is immediately applicable to the other. For if the first series, which is

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a +, \&c.$$

be

be put under this form

$$\frac{1}{a} = \frac{1}{\tan a} + T(1) + T(2) + T(3) + T(4) +, \&c.$$

where $T(1)$ is put for $\frac{1}{2}\tan\frac{1}{2}a$, and $T(2)$ for $\frac{1}{4}\tan\frac{1}{4}a$, and $T(3)$ for $\frac{1}{8}\tan\frac{1}{8}a$, &c., then, as the formula given at the conclufion of the laft article, becomes by fubftituting $\tan^2\frac{1}{4}a$ for $\frac{1-\operatorname{cof}\frac{1}{2}a}{1+\operatorname{cof}\frac{1}{2}a}$, and $\tan^2\frac{1}{8}a$ for $\frac{1-\operatorname{cof}\frac{1}{4}a}{1+\operatorname{cof}\frac{1}{4}a}$, and fo on,

$$\frac{1}{a^2} = \begin{cases} \frac{1}{4}\,\frac{1+\operatorname{cof} a}{1-\operatorname{cof} a} + \frac{1}{6} \\ -\left(\frac{1}{4^2}\tan^2\frac{1}{4}a + \frac{1}{4^3}\tan^2\frac{1}{8}a + \frac{1}{4^4}\tan^2\frac{1}{16}a +, \&c.\right) \end{cases}$$

it may be otherwife expreffed thus,

$$\frac{1}{a^2} = \begin{cases} \frac{1}{4}\,\frac{1+\operatorname{cof} a}{1-\operatorname{cof} a} + \frac{1}{6} \\ -\left(T^2(2) + T^2(3) + T^2(4) + T^2(5) +, \&c.\right), \end{cases}$$

where it is to be obferved, that the fymbols $T(2)$, $T(3)$, $T(4)$, denote the very fame quantities in both feries.

Now, as we have found (Art. 8. and 9.), that each term of the feries of quantities $T(2)$, $T(3)$, $T(4)$, &c. is lefs than $\frac{1}{4}$ of the term immediately before, but greater than a third proportional to the two terms immediately before it, taken in their order, it

 is

manifeft, that each term of the feries in our fecond formula muft be lefs than $\frac{1}{16}$ of the term before it, but greater than a third proportional to the two terms immediately preceding it; and thefe are the limits to the rates of convergency of our fecond feries.

19. We may alfo affign limits to the fum of all terms, after any propofed term: for putting it under this form

$$\frac{1}{a^2} = \begin{cases} \frac{1}{4}\,\frac{1+\operatorname{cof} a}{1-\operatorname{cof} a} + \frac{1}{6} \\ -\left(T_{(1)} + T_{(2)} \ldots + T_{(n)} + T_{(n+1)} + T_{(n+2)} +, \&c.\right), \end{cases}$$

where $T_{(1)}$, $T_{(2)}$, ... $T_{(n)}$, &c. now denote merely the terms of the feries taken in their order, then becaufe

$$T_{(m+2)} < \frac{1}{16}\,T_{(m+1)},$$

$$T_{(m+3)} < \frac{1}{16}\,T_{(m+2)},$$

$$T_{(m+4)} < \frac{1}{16}\,T_{(m+3)},$$

&c.

Therefore,

$$T_{(m+2)} + T_{(m+3)} + T_{(m+4)} +, \&c. < \frac{1}{16}\Big(T_{(m+1)} + T_{(m+2)} + T_{(m+3)} +, \&c.\Big)$$

That

That is, putting S for $T(m+2) + T(m+3) + T(m+4) +$, &c.

$$S < \frac{1}{16}\left\{T(m+1) + S\right\}, \text{ and hence } S < \frac{1}{15}T(m+1).$$

Thus it appears, that the ſum of all the terms following any term, is leſs than $\frac{1}{15}$ of that term.

20. As to the other limit, it muſt be the ſame as the like limit of our firſt ſeries, on account of their having the ſame limit to their correſponding rates of convergency. That is, putting S to denote as above, then

$$S > \frac{T(m+1)}{T(m) - T(m+1)} T(m+1);$$

$$\text{or } S > \frac{1}{15} T(m+1) - \frac{T(m) - 16\,T(m+1)}{15\,(T(m) - T(m+1))} T(m+1).$$

21. It yet remains for us to conſider how the ſeries of quantities $\frac{1 - \text{cof } a}{1 + \text{cof } a}$, $\frac{1 - \text{cof } \frac{1}{2} a}{1 + \text{cof } \frac{1}{2} a}$, &c. are to be found. Now this may be done, either by computing the coſines of the ſeries of arches a, $\frac{1}{2} a$, $\frac{1}{4} a$, $\frac{1}{8} a$, &c. one from another by means of the formula $\text{cof } \frac{1}{2} A = \sqrt{\frac{1 + \text{cof } A}{2}}$, and thence computing the ſeries of fractions $\frac{1 - \text{cof } \frac{1}{2} a}{1 + \text{cof } \frac{1}{2} a}$, $\frac{1 - \text{cof } \frac{1}{4} a}{1 + \text{cof } \frac{1}{4} a}$, &c. Or we may compute each fraction at once from that which precedes it, by a formula which may be thus inveſtigated:

Put

Put $\frac{1-\text{cos}\,A}{1+\text{cos}\,A}=t$, and $\frac{1-\text{cos}\,\frac{1}{2}A}{1+\text{cos}\,\frac{1}{2}A}=t'$, then $\text{cos}\,A=\frac{1-t}{1+t}$,

and $\frac{1+\text{cos}\,A}{2}=\frac{1}{1+t}$; alſo $\text{cos}\,\frac{1}{2}A=\frac{1-t'}{1+t'}$, now $\text{cos}\,\frac{1}{2}A$

$=\sqrt{\frac{1+\text{cos}\,A}{2}}$, therefore $\frac{1-t'}{1+t'}=\frac{1}{\sqrt{1+t}}$, and hence

$$t'=\frac{\sqrt{1+t}-1}{\sqrt{1+t}+1}.$$

22. Upon the whole, then, the reſult of the inveſtigation of the ſecond ſeries may be ſtated briefly as follows. Let a denote any arch of a circle, its radius being unity, then

$$\frac{1}{a^2}=\left\{\begin{array}{l}\frac{1}{4}\frac{1+\text{cos}\,a}{1-\text{cos}\,a}+\frac{1}{6}\\ -\left\{\begin{array}{l}\frac{1}{4^2}\frac{1-\text{cos}\,\frac{1}{2}a}{1+\text{cos}\,\frac{1}{2}a}+\frac{1}{4^3}\frac{1-\text{cos}\,\frac{1}{4}a}{1+\text{cos}\,\frac{1}{4}a}+\frac{1}{4^4}\frac{1-\text{cos}\,\frac{1}{8}a}{1+\text{cos}\,\frac{1}{8}a}\ldots\&c.\\ \qquad +T_{(m)}+T_{(m+1)}+S\end{array}\right\}\end{array}\right.$$

where $T_{(m)}$ and $T_{(m+1)}$ denote any two ſucceſſive terms of the ſeries in the parentheſis, and S denotes the ſum of all the following terms; and here S will always be between the limits

$$\frac{1}{15}T_{(m)}, \text{and } \frac{1}{15}T_{(m+1)}-\frac{(T_{(m)}-16\,T_{(m+1)})\,T_{(m+1)}}{15\,(T_{(m)}-T_{(m+1)})},$$

that is, it will be leſs than the former, but greater than the latter quantity.

The ſeries of coſines are to be deduced one from another by means of the formula

$$\text{cos}\,\frac{1}{2}A=\sqrt{\frac{1+\text{cos}\,A}{2}}.$$

Or,

Or, compute the ſeries of quantities t, t', t'', t''', &c., one from another by means of the formulæ

$$t = \frac{1 - \cos a}{1 + \cos a}, \quad t' = \frac{\sqrt{1+t} - 1}{\sqrt{1+t} + 1}, \quad t'' = \frac{\sqrt{1+t'} - 1}{\sqrt{1+t'} + 1}, \quad \&c.$$

Then will

$$\frac{1}{a^2} = \begin{cases} \frac{1}{4}\frac{1 + \cos a}{1 - \cos a} + \frac{1}{6} \\ -\left(\frac{1}{4^2}t' + \frac{1}{4^3}t'' + \frac{1}{4^4}t''' + \frac{1}{4^5}t^{\text{IV}} \ldots + T(m) + T(m+1) + S\right), \end{cases}$$

where $T(m)$, $T(m+1)$, and S denote the ſame as before.

I CONSIDER this ſecond ſeries for the rectification of the circle (under either of its forms), as preferable to the other ſeries given at Article 12. for two reaſons; firſt, becauſe of its greater rate of convergency, and, next, becauſe the quantities $\cos a$, $\cos \frac{1}{2} a$, $\cos \frac{1}{4} a$, &c., alſo the quantities t, t', t'', &c. can be deduced more eaſily one from another than the ſeries of tangents $\tan a$, $\tan \frac{1}{2} a$, $\tan \frac{1}{4} a$, $\tan \frac{1}{8} a$, &c.

23. THAT we may inveſtigate another ſeries, let us reſume the formula $\frac{1}{\tan A} = \frac{1}{2 \tan \frac{1}{2} A} - \frac{1}{2} \tan \frac{1}{2} A$; the cubes of both ſides of the equation being now taken, the reſult is

$$\frac{1}{\tan^3 A} = \frac{1}{8 \tan^3 \frac{1}{2} A} - \frac{3}{8}\left\{\frac{1}{\tan \frac{1}{2} A} - \tan \frac{1}{2} A\right\} - \frac{1}{8}\tan^3 \frac{1}{2} A.$$

To

To the fides of this equation let the correfponding fides of the equation

$$\frac{1}{\tan A} = \frac{1}{2}\left\{\frac{1}{\tan\frac{1}{2}A} - \tan\frac{1}{2}A\right\}$$

be added; then we get

$$\frac{1}{\tan^3 A} + \frac{1}{\tan A} = \frac{1}{8}\left(\frac{1}{\tan^3\frac{1}{2}A} + \frac{1}{\tan\frac{1}{2}A}\right) - \frac{1}{8}\left(\tan\frac{1}{2}A + \tan^3\frac{1}{2}A\right).$$

Now $\frac{1}{\tan^3 A} + \frac{1}{\tan A} = \frac{1+\tan^2 A}{\tan^3 A} = \frac{\sec^2 A}{\tan^3 A}$, and fimilarly,

$\frac{1}{\tan^3\frac{1}{2}A} + \frac{1}{\tan\frac{1}{2}A} = \frac{\sec^2\frac{1}{2}A}{\tan^3\frac{1}{2}A}$, alfo $\tan\frac{1}{2}A + \tan^3\frac{1}{2}A =$

$\tan\frac{1}{2}A\left(1+\tan^2\frac{1}{2}A\right) = \tan\frac{1}{2}A\sec^2\frac{1}{2}A$: therefore, by fubfti-

tuting, we get

$$\frac{\sec^2\frac{1}{2}A}{\tan^3\frac{1}{2}A} = \frac{\sec^2\frac{1}{2}A}{8\tan^3\frac{1}{2}A} - \frac{1}{8}\tan\frac{1}{2}A\sec^2\frac{1}{2}A.$$

24. From this formula, by fubftituting $a, \frac{1}{2}a, \frac{1}{4}a \ldots \frac{a}{2^{n-1}}$ for A, and multiplying the fucceffive refults by the fractions $1, \frac{1}{8}, \frac{1}{8^2} \ldots \frac{1}{8^{n-1}}$, we deduce the following feries of equations, the number of which is n:

$\frac{\sec^2 a}{\tan^3 a}$

$$\frac{\sec^2 a}{\tan^3 a} = \frac{\sec^2 \frac{1}{2} a}{2^3 \tan^3 \frac{1}{2} a} - \frac{1}{8} \tan \frac{1}{2} a \sec^2 \frac{1}{2} a,$$

$$\frac{\sec^2 \frac{1}{4} a}{2^3 \tan^3 \frac{1}{2} a} = \frac{\sec^2 \frac{1}{8} a}{4^3 \tan^3 \frac{1}{4} a} - \frac{1}{8^2} \tan \frac{1}{4} a \sec^2 \frac{1}{4} a,$$

$$\frac{\sec^2 \frac{1}{4} a}{4^3 \tan^3 \frac{1}{4} a} = \frac{\sec^2 \frac{1}{8} a}{8 \tan \frac{1}{8} a} - \frac{1}{8^3} \tan \frac{1}{8} a \sec^2 \frac{1}{8} a,$$

. . . .

$$\frac{\sec^2 \frac{a}{2^{n-2}}}{2^{3(n-1)} \tan^3 \frac{a}{2^{n-2}}} = \frac{\sec^2 \frac{a}{2^{n-1}}}{2^{3(n-1)} \tan^3 \frac{a}{2^{n-1}}} - \frac{1}{8^{n-1}} \tan \frac{a}{2^{n-1}} \sec^2 \frac{a}{2^{n-1}},$$

$$\frac{\sec^2 \frac{a}{2^{n-1}}}{2^{3(n-1)} \tan^3 \frac{a}{2^{n-1}}} = \frac{\sec^2 \frac{a}{2^n}}{2^{3n} \tan^3 \frac{a}{2^n}} - \frac{1}{8^n} \tan \frac{a}{2^n} \sec^2 \frac{a}{2^n}.$$

LET the ſums of the correſponding ſides of theſe equations be now taken, and rejecting from both the common ſeries

$$\frac{\sec^2 \frac{1}{2} a}{2^3 \tan^3 \frac{1}{2} a} + \frac{\sec^2 \frac{1}{4} a}{2^3 \tan^3 \frac{1}{4} a} \ldots + \frac{\sec^2 \frac{a}{2^{n-1}}}{2^{3(n-1)} \tan^3 \frac{a}{2^{n-1}}},$$

the reſult will be

$$\frac{\sec^2 a}{\tan^3 a} = \left\{ \begin{array}{l} \dfrac{\sec^2 \frac{a}{2^n}}{3^{3n} \tan^3 \frac{a}{2^n}} \\ - \left\{ \begin{array}{l} \frac{1}{8} \tan \frac{1}{2} a \sec^2 \frac{1}{2} a + \frac{1}{8^2} \tan \frac{1}{4} a \sec^2 \frac{1}{4} a \\ + \frac{1}{8^3} \tan \frac{1}{8} a \sec^2 \frac{1}{8} a \ldots + \frac{1}{8^n} \tan \frac{a}{2^n} \sec^2 \frac{a}{2^n} \end{array} \right\} \end{array} \right.$$

 and

and here the number of terms composing the series in the parenthesis is n.

Let us now conceive the series to go on *ad infinitum*, so that n may be considered as indefinitely great, then it is manifest, that $\sec^2 \frac{a}{2^n}$ will become equal to rad^2; now $2n \tan \frac{a}{2^n}$ will become a, (Art. 6. and 7.) therefore $2^{3n} \tan^3 \frac{a}{2^n}$ will become a^3; hence, substituting $\frac{1}{a^3}$ for $\dfrac{\sec^2 \frac{a}{2^n}}{2^{3n} \tan^3 \frac{a}{2^n}}$ in our equation, and transposing, we get at last

$$\frac{1}{a^3} = \left\{ \begin{array}{l} \dfrac{\sec^2 a}{\tan^3 a} + \dfrac{1}{8} \tan \dfrac{1}{2} a \sec^2 \dfrac{1}{2} a + \dfrac{1}{8^2} \tan \dfrac{1}{4} a \sec^2 \dfrac{1}{4} a \\ + \dfrac{1}{8^3} \tan \dfrac{1}{8} a \sec^2 \dfrac{1}{8} a +, \&c. \end{array} \right.$$

and this is the third series which I proposed to investigate for the rectification of an arch of a circle.

25. The series we have just now found, is evidently of a very simple form; it also converges pretty fast, each term being less than the 16th of that which precedes it. As, however, to apply it to actual calculation, it will be necessary to extract the cube root of a number, which is an operation of considerable labour when the root is to be found to several figures, perhaps, considered as a practical rule, this third formula is inferior to the two former. But if, on the other hand, we regard it merely as an elegant analytical theorem, it does not seem less deserving of notice than either of them.

26. The

26. The mode of reaſoning by which we have found ſeries expreſſing the three firſt powers of the reciprocal of an arch, will apply equally to any higher power, but the ſeries will become more and more complex as we proceed, beſides requiring in their application the extraction of high roots. In the caſe of the fourth power, however, the ſeries is ſufficiently ſimple, and converges faſter than any we have yet inveſtigated, while, at the ſame time, in its application we have only extractions of the ſquare root. On theſe accounts, I ſhall here give its inveſtigation.

Resuming the expreſſion $\frac{1}{\tan A} = \frac{1}{2\tan\frac{1}{2}A} - \frac{1}{2}\tan\frac{1}{2}A$;

let the fourth power, and alſo the ſquare of each ſide of the equation be taken, the reſult will be

$$\frac{1}{\tan^4 A} = \frac{1}{16\tan^4\frac{1}{2}A} - \frac{1}{4\tan^2\frac{1}{2}A} + \frac{3}{8} - \frac{1}{4}\tan^2\frac{1}{2}A + \frac{1}{16}\tan^4\frac{1}{2}A,$$

$$\frac{1}{\tan^2 A} = \frac{1}{4\tan^2\frac{1}{2}A} - \frac{1}{2} + \frac{1}{4}\tan^2\frac{1}{2}A.$$

Let the firſt of theſe equations be multiplied by 4, and the ſecond by 3, and let the reſults be added; then, reducing the fractions to a common denominator, we get

$$\frac{3+4\tan^2 A}{\tan^4 A} = \frac{1}{16}\,\frac{3+4\tan^2\frac{1}{2}A}{\tan^4\frac{1}{2}A} - \frac{14}{16} + \frac{1}{16}\left\{4\tan^2\tfrac{1}{2}A + 3\tan^4\tfrac{1}{2}A\right\}.$$

Let us, for the ſake of brevity, expreſs the complex quantity $\frac{3+4\tan^2 A}{\tan^4 A}$ by the ſymbol fA, (which is not to be underſtood as the product of two quantities f and A, but as a character denoting a particular function of the arch A;) and, ſimilarly, let $\frac{3+4\tan^2\frac{1}{2}A}{\tan^4 A}$ be denoted by $f\,\frac{1}{2}A$, and ſo on. Alſo let the other complex expreffion $4\tan^2\frac{1}{2}A + 3\tan^4\frac{1}{2}A$ be denoted by $F\frac{1}{2}A$; and if there were others like it, that is, which only differed by having $\frac{1}{4}A$, $\frac{1}{8}A$, &c. inftead of $\frac{1}{2}A$, they would be denoted by $F\frac{1}{4}A$, $F\frac{1}{8}A$, &c.; then our laft equation will ftand thus,

$$fA = \frac{1}{16}f\tfrac{1}{2}A - \frac{14}{16} + \frac{1}{16}F\tfrac{1}{2}A,$$

and fimilarly, putting $\frac{1}{2}A$, $\frac{1}{4}A$, $\frac{1}{8}A$, &c. fucceffively for A, and multiplying the refults by the feries of fractions $\frac{1}{16}$, $\frac{1}{16^2}$, $\frac{1}{16^3}$, &c.

$$\frac{1}{16}f\tfrac{1}{2}A = \frac{1}{16^2}f\tfrac{1}{4}A - \frac{14}{16^2} + \frac{1}{16^2}F\tfrac{1}{4}A,$$

$$\frac{1}{16}f\tfrac{1}{4}A = \frac{1}{16^3}f\tfrac{1}{8}A - \frac{14}{16^3} + \frac{1}{16^3}F\tfrac{1}{8}A,$$

&c.

By continuing this feries of equations to n terms, and then taking their fum, and rejecting what is common to each fide of the refult, exactly as in the inveftigations of the three preceding formulæ, we fhall get

$$fA = \frac{1}{2^{4n}}f\frac{A}{2^n} - 14\left\{\frac{1}{16} + \frac{1}{16^2} + \frac{1}{16^3} \ldots + \frac{1}{16^n}\right\}$$

$$+ \frac{1}{16}F\tfrac{1}{2}A + \frac{1}{16^2}F\tfrac{1}{4}A + \frac{1}{16^3}F\tfrac{1}{8}A \ldots + \frac{1}{16^n}F\frac{A}{2^n},$$

and this equation holds true, n being any whole pofitive number whatever.

27. Let

27. Let us now, however, ſuppoſe n indefinitely great, then the quantity $\frac{1}{2^{4n}} f \frac{A}{2^n}$, or $\frac{3+4\tan^2\frac{A}{2^n}}{\left(2^n \tan\frac{A}{2^n}\right)^4}$, becomes ſimply $\frac{3}{A^4}$, becauſe $\tan\frac{A}{2^n}$, and conſequently $4\tan^2\frac{A}{2^n}$, vaniſhes, and $2^n \tan\frac{A}{2^n}$ becomes A, as we have already had occaſion to obſerve (Art. 6. and 7.). Alſo the geometrical ſeries

$$\frac{1}{16} + \frac{1}{16^2} + \frac{1}{16^3} +, \&c.$$

having the number of its terms indefinitely great, and their common ratio $\frac{1}{16}$, will be $\frac{1}{15}$. Therefore, by ſubſtitution and tranſpoſition we have

$$\frac{3}{A^4} = fA + \frac{14}{15} - \left\{\frac{1}{16}F\tfrac{1}{2}A + \frac{1}{16^2}F\tfrac{1}{4}A + \frac{1}{16^3}F\tfrac{1}{8}A +, \&c.\right\}$$

or, ſubſtituting for fA, and $F\frac{1}{2}A$, &c. the quantities which theſe ſymbols expreſs,

$$\frac{3}{A^4} = \begin{cases} \dfrac{3+4\tan^2 A}{\tan^4 A} + \dfrac{14}{15} \\ -\left\{\dfrac{1}{16}(4\tan^2\tfrac{1}{2}A + 3\tan^4\tfrac{1}{2}A) + \dfrac{1}{16^2}(4\tan^2\tfrac{1}{4}A + 3\tan^4\tfrac{1}{4}A)\right. \\ \left. + \dfrac{1}{16^3}(4\tan^2\tfrac{1}{8}A + 3\tan^4\tfrac{1}{8}A) +, \&c.\right\} \end{cases}$$

and this is one form of the ſeries which we propoſed to inveſtigate.

28. This

28. THIS series, however, admits of being expressed under another form, better adapted to calculation, and to effect this transformation, let us begin with the term $\frac{3+4\tan^2 A}{\tan^4 A}$. In this quantity let $\frac{1-\cos 2A}{1+\cos 2A}$ be substituted for $\tan^2 A$; it then becomes, after proper reduction, $\frac{7+6\cos 2A-\cos^2 2A}{1-2\cos 2A+\cos^2 2A}$. Again, in this expression let $\frac{1+\cos 4A}{2}$ be substituted for $\cos^2 2A$, we then get

$$\frac{3+4\tan^2 A}{\tan^4 A}=\frac{13-\cos 4A+12\cos 2A}{3+\cos 4A-4\cos 2A}.$$

THE remaining terms of the series, which are similar to one another, and of the form $4\tan^2\frac{1}{2n}A+3\tan^4\frac{1}{2n}A$, admit of a like transformation; for by substituting $\frac{1-\cos\frac{1}{n}A}{1+\cos\frac{1}{n}A}$ for $\tan^2\frac{1}{2n}A$, and again $\frac{1+\cos\frac{2}{n}A}{2}$ for $\cos^2\frac{1}{n}A$ in the result, we get

$$4\tan^2\frac{1}{2n}A+3\tan^4\frac{1}{2n}A=\frac{13-\cos\frac{2}{n}A-12\cos\frac{1}{n}A}{3+\cos\frac{2}{n}A+4\cos\frac{1}{n}A}.$$

By

By substituting these transformed expressions in the series, it becomes

$$\frac{3}{A^4} = \frac{13 - \cos 4A + 12\cos 2A}{3 + \cos 4A - 4\cos 2A} + \frac{14}{15}$$

$$- \left\{ \frac{1}{16}\,\frac{13 - \cos 2A - 12\cos A}{3 + \cos 2A + 4\cos A} + \frac{1}{16^2}\,\frac{13 - \cos A - 12\cos\frac{1}{2}A}{3 + \cos A + 4\cos\frac{1}{2}A} \right.$$

$$\left. + \frac{1}{16^3}\,\frac{13 - \cos\frac{1}{2}A - 12\cos\frac{1}{4}A}{3 + \cos\frac{1}{2}A + 4\cos\frac{1}{4}A} +, \text{\&c.} \right\}$$

FINALLY, let $\frac{1}{4}a$ be now substituted for A, and $\frac{1}{8}a$ for $\frac{1}{2}$ A, and so on, and let the result be divided by 3×16; then we have

$$\frac{1}{a^4} = \frac{1}{3.16^2}\,\frac{13 - \cos a + 12\cos\frac{1}{2}a}{3 + \cos a - 4\cos\frac{1}{2}a} + \frac{7}{8.8.9.10}$$

$$- \left\{ \frac{1}{3.16^3}\,\frac{13 - \cos\frac{1}{2}a - 12\cos\frac{1}{4}a}{3 + \cos\frac{1}{2}a + 4\cos\frac{1}{4}a} + \frac{1}{3.16^4}\,\frac{13 - \cos\frac{1}{4}a - 12\cos\frac{1}{8}a}{3 + \cos\frac{1}{4}a + 4\cos\frac{1}{8}a} \right.$$

$$\left. + \frac{1}{3.16^5}\,\frac{13 - \cos\frac{1}{8}a - 12\cos\frac{1}{16}a}{3 + \cos\frac{1}{8}a + 4\cos\frac{1}{16}a} +, \text{\&c.} \right\},$$

which is our fourth general series for the rectification of an arch; and its rate of convergence is very considerable, for each term is less than $\frac{1}{64}$th of the term before it. The series, however, approaches continually to a geometrical progression, of which the common ratio is $\frac{1}{64}$.

29. THE preceding formulæ, as well as innumerable others, which may in like manner be deduced from the expression $\tan A = \frac{1}{2\tan\frac{1}{2}A} - \frac{1}{2}\tan\frac{1}{2}A$, all agree in expressing a power of the reciprocal of an arch by an infinite series, the terms of which

which are like functions of a series of arches, formed from the arch to be rectified, and from one another, by continual bisection. We may, however, in the very same way investigate the formulæ which shall express the reciprocal of the arch and its powers by like functions of arches deduced from one another by trisection, or any other section whatever. The most simple formula, which expresses an arch by functions deduced from it by trisection, may be investigated as follows:

30. FROM the known expression $\tan A = \frac{3\tan\frac{1}{3}A - \tan^3\frac{1}{3}A}{1 - 3\tan^2\frac{1}{3}A}$,

we readily get

$$\frac{1}{\tan A} = \frac{1}{3\tan\frac{1}{3}A} - \frac{8}{3}\,\frac{\tan\frac{1}{3}A}{3 - \tan^2\frac{1}{3}A},$$

and hence, by substituting a, $\frac{1}{3}a$, $\frac{1}{9}a$, &c. successively for A, and multiplying by the terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. we derive the following equations,

$$\frac{1}{\tan a} = \frac{1}{3\tan\frac{1}{3}a} - \frac{8}{3}\,\frac{\tan\frac{1}{3}a}{3 - \tan^2\frac{1}{3}a},$$

$$\frac{1}{3\tan\frac{1}{3}a} = \frac{1}{9\tan\frac{1}{9}a} - \frac{8}{3^2}\,\frac{\tan\frac{1}{9}a}{3 - \tan^2\frac{1}{9}a},$$

$$\frac{1}{9\tan\frac{1}{9}a} = \frac{1}{27\tan\frac{1}{27}a} - \frac{8}{3^3}\,\frac{\tan\frac{1}{27}a}{3 - \tan^2\frac{1}{27}a}.$$

&c.

Now conceive this series of equations to be continued, till the number of equations be n, and their sum to be taken, and the quantities common to each side of the result rejected, as in the investigations of the other formulæ; then we shall have

$\frac{1}{\tan a}$

$$\frac{1}{\tan a} = \frac{1}{3^n \tan \frac{a}{3^n}} - \left\{ \frac{8}{3}\,\frac{\tan \frac{1}{3} a}{3 - \tan^2 \frac{1}{3} a} + \frac{8}{3^2}\,\frac{\tan \frac{1}{9} a}{3 - \tan^2 \frac{1}{9} a} \right.$$

$$\left. + \frac{8}{3^3}\,\frac{\tan \frac{1}{27} a}{3 - \tan^2 \frac{1}{27} a} \ldots\ldots + \frac{8}{3^n}\,\frac{\tan \frac{a}{3^n}}{3 - \tan^2 \frac{a}{3^n}} \right\}.$$

And this is true, n being any number whatever. Now, if we confider that $3^n \tan \frac{a}{3^n}$ expreffes the fum of the fides of a figure formed by dividing the arch into 3^n equal parts, and drawing tangents at the points of divifion, whofe orders, reckoned from one end of the arch, are indicated by even numbers, (that end itfelf being reckoned one of them), and producing each to meet thofe adjoining to it, and the laft to meet a radius of the circle produced through the other end of the arch, it will be obvious, that n being fuppofed to increafe indefinitely, the expreffion $3^n \tan \frac{a}{3^n}$ will have for its limit the arch a, and in this cafe the feries will go on *ad infinitum*. Thus we fhall have

$$\frac{1}{\tan a} = \frac{1}{a} - \left\{ \frac{8}{3}\,\frac{\tan \frac{1}{3} a}{3 - \tan^2 \frac{1}{3} a} + \frac{8}{3^2}\,\frac{\tan \frac{1}{9} a}{3 - \tan^2 \frac{1}{9} a} + \frac{8}{3}\,\frac{\tan \frac{1}{27} a}{3 - \tan^2 \frac{1}{27} a} +, \&c. \right\},$$

and by tranfpofition,

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{8}{3}\,\frac{\tan \frac{1}{3} a}{3 - \tan^2 \frac{1}{3} a} + \frac{8}{3^2}\,\frac{\tan \frac{1}{9} a}{3 - \tan^2 \frac{1}{9} a} + \frac{8}{3^3}\,\frac{\tan \frac{1}{27} a}{3 - \tan^2 \frac{1}{27} a} +, \&c.$$

and this is the feries which I propofed to inveftigate.

31. THE ſeries we have juſt now found, may be preſented under various forms. Thus, by conſidering that

$$\frac{1}{\tan A} = \frac{\operatorname{cof} A}{\operatorname{fin} A} = \frac{2 \operatorname{fin} A \operatorname{cof} A}{2 \operatorname{fin}^2 A} = \frac{\operatorname{fin} 2 A}{1 - \operatorname{cof} 2 A},$$

and that

$$\frac{\tan A}{3 - \tan^2 A} = \frac{\frac{\operatorname{fin} A}{\operatorname{cof} A}}{3 - \frac{\operatorname{fin}^2 A}{\operatorname{cof}^2 A}} = \frac{1}{2} \frac{2 \operatorname{fin} A \operatorname{cof} A}{4 \operatorname{cof}^2 A - 1} = \frac{1}{2} \frac{\operatorname{fin} 2 A}{1 + 2 \operatorname{cof} 2 A}$$

it will appear that by due ſubſtitution the ſeries may be otherwiſe expreſſed as follows:

$$\frac{1}{a} = \frac{1}{2} \frac{\operatorname{fin} a}{1 - \operatorname{cof} a} + \frac{2}{3} \frac{\operatorname{fin} \frac{1}{3} a}{1 + 2 \operatorname{cof} \frac{1}{3} a} + \frac{2}{3^2} \frac{\operatorname{fin} \frac{1}{9} a}{1 + 2 \operatorname{cof} \frac{1}{9} a} + \frac{2}{3^3} \frac{\operatorname{fin} \frac{1}{27} a}{1 + 2 \operatorname{cof} \frac{1}{27} a} + \&c.$$

And other forms might be given to it, but they would all converge with the ſame quickneſs, and each term would be leſs than $\frac{1}{9}$th of the term before it. The ſeries, however, under whatever form it be given, and all others which like it require for their application the triſection of an arch, are, when compared with thoſe we formerly inveſtigated, of little uſe as practical rules; becauſe it is well known that to determine the ſine, or other ſuch function of an arch from a function of its triple, is a problem which produces a cubic equation of a form which does not admit of being reſolved otherwiſe than by trials, or by infinite ſeries, both of which proceſſes are ſufficiently laborious, and only to be employed where the object in view cannot be attained by eaſier means.

32. As from the different ſeries we have found for the rectification of an arch of a circle, the ſpirit of our method muſt be ſufficiently obvious, I ſhall not inveſtigate any others at preſent.

ſent. Before leaving this part of our ſubject, however, it may be proper to obſerve, that the ſecond ſeries may be deduced from the firſt, and the third from the ſecond, and ſo on with reſpect to innumerable others of the ſame kind, by the fluxional or differential calculus.

For reſuming the firſt ſeries

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a +, \&c.$$

if we take the fluxion of each term, conſidering a as a variable quantity, we have

$$\frac{-da}{a^2} = \frac{-da\,\text{ſec}^2 a}{\tan^2 a} + \frac{1}{4}da\,\text{ſec}^2\frac{1}{2}a + \frac{1}{4^2}da\,\text{ſec}^2\frac{1}{4}a + \frac{1}{4^3}da\,\text{ſec}^2\frac{1}{8}a +, \&c.$$

and hence, changing the ſigns, and rejecting da from each term, and putting $1+\tan^2\frac{1}{n}a$ for $\text{ſec}^2\frac{1}{n}a$, we find

$$\frac{1}{a^2} = \left\{\begin{array}{l} \frac{1}{\tan^2 a} + 1 - \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} +, \&c.\right) \\ -\left(\frac{1}{4}\tan^2\frac{1}{2}a + \frac{1}{4^2}\tan^2\frac{1}{4}a + \frac{1}{4^3}\tan^2\frac{1}{8}a +, \&c.\right) \end{array}\right.$$

In this expreſſion, inſtead of the numeral ſeries $\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+$, &c. (which is a geometrical progreſſion having its common ratio $\frac{1}{4}$) ſubſtitute its value, viz. $\frac{1}{3}$, and the reſult is

$$\frac{1}{a^2} = \frac{1}{\tan^2 a} + \frac{2}{3} - \left\{\frac{1}{4}\tan^2\frac{1}{2}a + \frac{1}{4^2}\tan^2\frac{1}{4}a + \frac{1}{4^3}\tan^2\frac{1}{4}a +, \&c.\right\}$$

which is identical with the formula found at Art. 15.

 From

From this ſeries, by a like mode of proceeding, we may deduce our third ſeries, and thence, again, our fourth, and ſo on: but this mode of inveſtigation, although very ſimple, is certainly leſs elementary than that which we have followed. And it muſt be kept in mind, that one principal object of this paper is to employ only the firſt principles of geometry and analyſis in treating of the ſubjects announced in its title.

33. By a mode of deduction differing but little from that employed in the laſt article, we may even derive our firſt ſeries from a known formula, the invention of which is attributed to Euler. It is this,

$$a = \text{ſin}\, a\ \text{ſec}\,\frac{1}{2}a\ \text{ſec}\,\frac{1}{4}a\ \text{ſec}\,\frac{1}{8}a +, \&c. *$$

From this expreſſion, by the theory of logarithms, we get

$$\log a = \log \text{ſin}\, a + \log \text{ſec}\,\frac{1}{2}a + \log \text{ſec}\,\frac{1}{4}a + \log \text{ſec}\,\frac{1}{8}a +, \&c.$$

we have now only to take the fluxions of all the terms, and reject $d\,a$, which is found in each, and the reſult is

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a +, \&c.$$

which is the ſeries in queſtion.

34. I now proceed to the inveſtigation of formulæ for the quadrature of the hyperbola, and as the principles from which they

* This formula, although very elegant as an analytical transformation, does not ſeem to admit of being applied with advantage to the rectification of an arch, on account of the great number of factors of the product which would be required to give a reſult tolerably correct.

they are to be deduced are in effect the fame as we have had occafion to employ when treating of the circle, it will be proper to ufe the fame form of reafoning, and the fame mode of notation, in the one cafe as in the other.

THEREFORE, in the equilateral hyperbola ABB′, of which C is the centre, (Plate IX. Fig. 1.), and CA the femitranfverfe axis; let CB be drawn to any point B of the curve, and BD perpendicular to CA; then, in imitation of the notation commonly ufed in the arithmetic of fines, which we have followed in the former part of this paper, we fhall confider the co-ordinates CD, DB, as functions of the hyperbolic fector ACB, and putting S to denote its area, we fhall denote the abfciffa CD by ab S, and the ordinate BD by ord S.

DRAW AE touching the curve at its vertex, and meeting CB in E; then, from fimilar triangles, we have $AE = \frac{DB}{CD} \times CA$; therefore fuppofing the femitranfverfe axis AC to be unity, $AE = \frac{\text{ord } S}{\text{ab } S}$. Now this expreffion for the tangent correfponding to a hyperbolic fector S, being analogous to $\frac{\text{fin } A}{\text{cof } A}$, the expreffion for the tangent of an angle A, we may fimilarly denote AE by the abbreviation tan S. In like manner, if CB′ be drawn to a point B′ of the curve, bifecting the fector ACB, and meeting AE in E′, and B′D′ be drawn perpendicular to CA; then, as the fector ACB′ will be $\frac{1}{2}$S, it follows, that $CD' = \text{ab } \frac{1}{2} S$, $B'D' = \text{ord } \frac{1}{2} S$, and $AE' = \tan \frac{1}{2} S$; and fo on.

35. FROM

35. From the nature of the hyperbola, we have

$$\text{ab S} = \text{ab}^2 \frac{1}{2}\text{S} + \text{ord}^2 \frac{1}{2}\text{S}, \quad \text{ord S} = 2\,\text{ab}\frac{1}{2}\text{S ord}\frac{1}{2}\text{S};$$

therefore, by diviſion,

$$\frac{\text{ab S}}{\text{ord S}} = \frac{\text{ab}\frac{1}{2}\text{S}}{2\,\text{ord}\frac{1}{2}\text{S}} + \frac{1}{2}\frac{\text{ord}\frac{1}{2}\text{S}}{\text{ab}\frac{1}{2}\text{S}},$$

that is, $$\frac{1}{\tan S} = \frac{1}{2\tan\frac{1}{2}S} + \frac{1}{2}\tan\frac{1}{2}S.$$

This laſt formula expreſſes a property of the hyperbola perfectly analogous to that of the circle (Art. 5.), from which we have deduced our firſt four ſeries for the rectification of an arch. Therefore ſimilarly, putting s, $\frac{1}{2}s$, $\frac{1}{4}s$, &c. ſucceſſively inſtead of S, and multiplying by the ſeries of numbers 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. we have as in that article

$$\frac{1}{\tan s} = \frac{1}{2\tan\frac{1}{2}s} + \frac{1}{2}\tan\frac{1}{2}s,$$

$$\frac{1}{2\tan\frac{1}{2}s} = \frac{1}{4\tan\frac{1}{4}s} + \frac{1}{4}\tan\frac{1}{4}s,$$

$$\frac{1}{4\tan\frac{1}{4}s} = \frac{1}{8\tan\frac{1}{8}s} + \frac{1}{8}\tan\frac{1}{8}s,$$

&c.

This ſeries of equations being ſuppoſed continued until their number be n, by proceeding exactly as in Art. 5. when treating of the circle, we obtain

$$\frac{1}{2^n \tan\frac{s}{2^n}} =$$

$$\frac{1}{2^n \tan\frac{s}{2^n}} = \left\{ \begin{array}{l} \frac{1}{\tan s} - \left(\frac{1}{2}\tan\frac{1}{2}s + \frac{1}{4}\tan\frac{1}{4}s + \frac{1}{8}\tan\frac{1}{8}s + \frac{1}{16}\tan\frac{1}{16}s \ldots \right. \\ \qquad \left. + \frac{1}{2^n}\tan\frac{s}{2^n}\right). \end{array} \right.$$

36. Let us now ſuppoſe the hyperbolic ſector ACB to be divided into 2^n equal parts, by lines drawn from the centre to the points 1, 2, 3, 4, ... 7 in the curve, and tangents to be drawn at the extremities of the hyperbolic arch AB, and at the alternate intermediate points of diviſion 2, 4, 6, &c. ſo as to form the polygon AFF′F″F‴BC. Then, by a known property of the hyperbola, the triangles ACF, FC 2, 2 CF′, F′C 4, . . . F‴CB are all equal, and as their number is 2^n, the whole polygon bounded by the tangents, and by the ſtraight lines AC, CB will be equal to the triangle ACF taken 2^n times. But the area of this triangle is $\frac{1}{2}AC \times AF = \frac{1}{2}\tan\frac{s}{2^n}$ (becauſe $AF = \tan\frac{s}{2^n}$), therefore $2^n \tan\frac{s}{2^n}$ expreſſes twice the area of the polygon AFF′F″F‴BC. Let Q denote this area, then, ſubſtituting $\frac{1}{2}Q$ for $2^n \tan\frac{s}{2^n}$, and multiplying all the terms of the ſeries by 2, we have

$$\frac{1}{Q} = \left\{ \begin{array}{l} \frac{2}{\tan s} - \left(\tan\frac{1}{2}s + \frac{1}{2}\tan\frac{1}{4}s + \frac{1}{4}\tan\frac{1}{8}s + \frac{1}{8}\tan\frac{1}{16}s \ldots \right. \\ \qquad \left. + \frac{1}{2^n}\tan\frac{s}{2^n}\right). \end{array} \right.$$

Now

Now, the rectilineal ſpace Q is evidently leſs than the hyperbolic ſector s; but n may be conceived ſo great that the difference between Q and s ſhall be leſs than any aſſignable ſpace, as it is eaſy to demonſtrate upon principles ſtrictly geometrical; therefore, if we ſuppoſe n indefinitely great, then Q becomes s; and as, upon this hypotheſis, the ſeries goes on *ad infinitum*, we have

$$\frac{1}{s} = \frac{2}{\tan s} - \left(\tan \frac{1}{2} s + \frac{1}{2} \tan \frac{1}{4} s + \frac{1}{4} \tan \frac{1}{8} s + \frac{1}{8} \tan \frac{1}{16} s +, \&c.\right)$$

which is our firſt ſeries for the quadrature of an hyperbolic ſector. And as $\frac{1}{\tan S} = \frac{1}{2 \tan \frac{1}{2} S} + \frac{1}{2} \tan \frac{1}{2} S$, by reſolving this equation in reſpect of $\tan \frac{1}{2} S$, we get the formula

$$\tan \frac{1}{2} S = \frac{1}{\tan S} - \sqrt{\frac{1}{\tan^2 S} - 1},$$

by which the ſeries of quantities $\tan \frac{1}{2} s$, $\tan \frac{1}{4} s$, &c. may be deduced from $\tan s = \frac{\text{ord } s}{\text{ab } s}$, and from one another.

37. THIS expreſſion for an hyperbolic ſector is perfectly ſimilar in its form to that given in Art. 7. for an arch of a circle. It may, however, be transformed into another better adapted to calculation, by means of a property of the hyperbola to which there is no correſponding property of the circle, or at leaſt none that can be expreſſed without employing the ſign $\sqrt{-1}$. The property alluded to may be deduced from the known

known formulæ

$$2\,\text{ab}\,\frac{s}{n} = (\text{ab}\,s + \text{ord}\,s)^{\frac{1}{n}} + (\text{ab}\,s - \text{ord}\,s)^{\frac{1}{n}},$$

$$2\,\text{ord}\,\frac{s}{n} = (\text{ab}\,s + \text{ord}\,s)^{\frac{1}{n}} - (\text{ab}\,s - \text{ord}\,s)^{\frac{1}{n}},$$

by proceeding as follows. Let each ſide of the latter of theſe equations be divided by the correſponding ſide of the former, the reſult is

$$\frac{\text{ord}\,\frac{s}{n}}{\text{ab}\,\frac{s}{n}} = \frac{(\text{ab}\,s + \text{ord}\,s)^{\frac{1}{n}} - (\text{ab}\,s - \text{ord}\,s)^{\frac{1}{n}}}{(\text{ab}\,s + \text{ord}\,s)^{\frac{1}{n}} + (\text{ab}\,s - \text{ord}\,s)^{\frac{1}{n}}},$$

which expreſſion is equivalent to this other one,

$$\frac{\text{ord}\,\frac{s}{n}}{\text{ab}\,\frac{s}{n}} = \frac{\left(\frac{\text{ab}\,s + \text{ord}\,s}{\text{ab}\,s - \text{ord}\,s}\right)^{\frac{1}{n}} - 1}{\left(\frac{\text{ab}\,s + \text{ord}\,s}{\text{ab}\,s - \text{ord}\,s}\right)^{\frac{1}{n}} + 1}.$$

Let us now put p for the fraction $\frac{\text{ab}\,s + \text{ord}\,s}{\text{ab}\,s - \text{ord}\,s}$, then, remarking that $\frac{\text{ord}\,\frac{s}{n}}{\text{ab}\,\frac{s}{n}} = \tan\frac{s}{n}$, we have

$$\tan\frac{s}{n} = \frac{p^{\frac{1}{n}} - 1}{p^{\frac{1}{n}} + 1},$$

an equation which expreſſes the property we propoſed to inveſtigate.

38. We have now only to ſuppoſe n in this formula to have theſe values, 2, 4, 8, &c. ſucceſſively, and to ſubſtitute inſtead of the terms of our ſeries

$$\frac{1}{s} = \frac{2}{\tan s} - \left(\tan\frac{1}{2}s + \frac{1}{2}\tan\frac{1}{4}s + \frac{1}{4}\tan\frac{1}{8}s +, \&c.\right),$$

their values as given by the formula, putting alſo $\frac{\text{ord}\, s}{\text{ab}\, s}$ inſtead of tan s, and the ſeries becomes

$$\frac{1}{s} = \frac{2\,\text{ab}\, s}{\text{ord}\, s} - \left\{\frac{p^{\frac{1}{2}} - 1}{p^{\frac{1}{2}} + 1} + \frac{1}{2}\,\frac{p^{\frac{1}{4}} - 1}{p^{\frac{1}{4}} + 1} + \frac{1}{4}\,\frac{p^{\frac{1}{8}} - 1}{p^{\frac{1}{8}} + 1} +, \&c.\right\}$$

and this is the new form under which we propoſed to exhibit it.

39. Let us now inquire what are the limits of the rate of convergency of this ſeries; and in doing this, it will be moſt convenient to refer to the firſt of its two forms. Now, from the formula $\frac{1}{\tan S} = \frac{1}{2\tan\frac{1}{2}S} - \frac{1}{2}\tan\frac{1}{2}S$, we get $\tan\frac{1}{2}S = \frac{1}{2}\tan S\,(1 + \tan^2\frac{1}{2}S)$. But $1 + \tan^2\frac{1}{2}S > 1$, and therefore $\frac{1}{2}\tan S\,(1 + \tan^2\frac{1}{2}S) > \frac{1}{2}\tan S$, hence it follows, that $\tan\frac{1}{2}S > \frac{1}{2}\tan S$. Thus it appears, that each term of the ſe-

ries

ries of quantities $\tan\frac{1}{4}s$, $\tan\frac{1}{8}s$, &c. is greater than half the term before it; and as these, multiplied by the fractions $\frac{1}{2}$, $\frac{1}{4}$, &c. respectively constitute the terms of the series, each term of the series, under either of its forms, is greater than one-fourth of the term before it.

40. AGAIN, from the formula $\tan\frac{1}{2}S = \frac{1}{2}\tan S\,(1+\tan^2\frac{1}{2}S)$ we find $\frac{2\tan\frac{1}{2}S}{\tan S} = 1+\tan^2\frac{1}{2}S$, and similarly, $\frac{2\tan\frac{1}{4}S}{\tan\frac{1}{2}S} = 1+\tan^2\frac{1}{4}S$. But from the nature of the hyperbola $1+\tan^2\frac{1}{4}S < 1+\tan^2\frac{1}{2}S$; therefore $\frac{2\tan\frac{1}{4}S}{\tan\frac{1}{2}S} < \frac{2\tan\frac{1}{2}S}{\tan S}$, and hence $\tan\frac{1}{4}S < \frac{\tan^2\frac{1}{2}S}{\tan S}$. Therefore, putting $\frac{1}{2n}s$ instead of S, and multiplying by $\frac{1}{4n}$, we have

$$\frac{1}{4n}\tan\frac{1}{8n}s < \frac{\frac{1}{2n}\tan\frac{1}{4n}s}{\frac{1}{n}\tan\frac{1}{2n}s} \times \frac{1}{2n}\tan\frac{1}{4n}s,$$

from which it appears, that each term of the series, following the second, is less than a third proportional to the two terms immediately before it. So that, upon the whole, it appears, that the limits of the rate of convergency of our first series for an hyperbolic sector, are the same as those of our first for an arch of a circle, (fee Art. 8. and 9.), only the greater limit in the one cafe correfponds to the leffer limit in the other, and *vice versa*.

41. We might now, from thefe limits to the rate of convergency, determine two limits to the fum of all the terms of the feries following any given term, by the mode of inveftigation employed at Art. 10. and Art. 11. in the cafe of the circle; but the refult in both cafes would be found to be the fame, with the difference of the fign $<$ for $>$, and $>$ for $<$; that is, we would find the fum of all the terms following any term of the feries, to be greater than one-third of that term, but lefs than a third proportional to the difference between the two terms immediately before it and the latter of the two.

42. Upon the whole, then, our firft formula, for the quadrature of an hyperbolic fector, may be expreffed as follows.

Let s denote the area of the fector, and put p for $\frac{\text{ab } s + \text{ord } s}{\text{ab } s - \text{ord } s}$.

Then,

$$\frac{1}{s} = \frac{2 \text{ abs } s}{\text{ord } s} - \left\{ \frac{p^{\frac{1}{2}} - 1}{p^{\frac{1}{2}} + 1} + \frac{1}{2} \frac{p^{\frac{1}{4}} - 1}{p^{\frac{1}{4}} + 1} + \frac{1}{4} \frac{p^{\frac{1}{8}} - 1}{p^{\frac{1}{8}} + 1} + \frac{1}{8} \frac{p^{\frac{1}{16}} - 1}{p^{\frac{1}{16}} + 1} \dots + T_{(m)} + T_{(m+1)} + R \right\}$$

where $T_{(m)}$ and $T_{(m+1)}$ denote any two fucceeding terms of the feries, and R the fum of all the following terms *. And here

* The fame feries may alfo be put under another form, which it may not be improper to notice briefly, on account of the facility with which the terms may be

here

$$R > \frac{1}{3} T_{(m+1)}$$

$$\text{but } R < \frac{T_{(m+1)}}{T_{(m)} - T_{(m+1)}} \times T_{(m+1)}.$$

As theſe limits to R differ but little when the terms $T_{(m)}$, $T_{(m+1)}$ are conſiderably advanced in the ſeries, the latter may be expreſſed more conveniently for calculation thus

$$R < \frac{1}{3} T_{(m+1)} + \frac{\left(4 T_{(m+1)} - T_{(m)}\right) T_{(m+1)}}{3\left(T_{(m)} - T_{(m+1)}\right)}.$$

43. Let us next inveſtigate a ſeries for the quadrature of the hyperbola, which may be analogous to our ſecond ſeries for the rectification of the circle. For this purpoſe, proceeding as at Art. 13. we reſume the formula $\frac{1}{\tan S} = \frac{1}{2 \tan \frac{1}{2} S} + \frac{1}{2} \tan \frac{1}{2} S$, and taking the ſquare of each ſide of the equation, get

$$\frac{1}{\tan^2 S} = \frac{1}{4 \tan^2 \frac{1}{2} S} + \frac{1}{4} \tan^2 \frac{1}{2} S + \frac{1}{2}.$$

Inſtead

be deduced one from another by the help of the common trigonometrical tables. It is this,

$$\frac{1}{s} = \frac{2\,\text{ab}\,s}{\text{ord}\,s} - (\text{ſin}\,\alpha' + \frac{1}{2}\,\text{ſin}\,\alpha'' + \frac{1}{4}\,\text{ſin}\,\alpha''' + \frac{1}{8}\,\text{ſin}\,\alpha^{\text{iv}} \ldots + T_{(m)} + T_{(m+1)} + R).$$

The arches α', α'', α''', α^{iv}, &c. are to be deduced one from another as follows. Take α ſuch that $\text{ſin}\,\alpha = \frac{\text{ord}\,s}{\text{ab}\,s}$, then, $\text{ſin}\,\alpha' = \tan \frac{1}{2} \alpha$, $\text{ſin}\,\alpha'' = \tan \frac{1}{2} \alpha'$, $\text{ſin}\,\alpha''' = \tan \frac{1}{2} \alpha''$, $\text{ſin}\,\alpha^{\text{iv}} = \tan \frac{1}{2} \alpha'''$, &c. The ſymbols $T_{(m)}$, $T_{(m+1)}$ and R, denote the ſame things as in the other form of the ſeries.

Inftead of S, we now fubftitute in this expreffion s, $\frac{1}{2}s$, $\frac{1}{4}s$, $\frac{1}{8}s$, &c. fucceffively, and multiply the refults by the terms of the feries 1, $\frac{1}{4}$, $\frac{1}{4^2}$, $\frac{1}{4^3}$, &c. fo as to form the following feries of equations, the number of which is n.

$$\frac{1}{\tan^2 s} = \frac{1}{2^2 \tan^2 \frac{1}{2}s} + \frac{1}{4}\tan^2 \frac{1}{2}s + \frac{1}{2}$$

$$\frac{1}{2^2 \tan^2 \frac{1}{2}s} = \frac{1}{4^2 \tan^2 \frac{1}{4}s} + \frac{1}{4^2}\tan^2 \frac{1}{4}s + \frac{1}{2.4}$$

$$\frac{1}{4^2 \tan^2 \frac{1}{4}s} = \frac{1}{8^2 \tan^2 \frac{1}{8}s} + \frac{1}{4^3}\tan^2 \frac{1}{8}s + \frac{1}{2.4^2}$$

$$\frac{1}{8^2 \tan^2 \frac{1}{8}s} = \frac{1}{16^2 \tan^2 \frac{1}{16}s} + \frac{1}{4^4}\tan^2 \frac{1}{16}s + \frac{1}{2.4^3},$$

&c.

From thefe, by proceeding in all refpects as in the article above quoted, that is, by adding, and rejecting what is common to each fide of the fum, we get

$$\frac{1}{\tan^2 \frac{1}{2}s} = \begin{cases} \dfrac{1}{2^{2n}\tan^2 \dfrac{s}{2^n}} \\ +\frac{1}{4}\tan^2 \frac{1}{2}s + \frac{1}{4^2}\tan^2 \frac{1}{4}s + \frac{1}{4^3}\tan^2 \frac{1}{8}s \ldots + \frac{1}{4^n}\tan^2 \frac{1}{2^n}s \\ +\frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4^2} \ldots + \frac{1}{2.4^{n-1}}. \end{cases}$$

Now,

Now, as we have found (Art. 36.) that $2^n \tan \frac{a}{2^n}$ expresses twice the area of the polygon AFF′F″F‴ (Plate IX.), the numerical value of which we have there denoted by Q, it follows, that $2^{2n} \tan^2 \frac{s}{2^n} = \frac{1}{4} Q^2$. Moreover, the geometrical series $\frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4^2} \ldots . + \frac{1}{2.4^{n-1}}$ is equivalent to $\frac{2}{3}\left(1 - \frac{1}{4^n}\right)$, therefore, by substitution and transposition, we get

$$\frac{1}{4Q^2}\left\{\begin{array}{l}\frac{1}{\tan^2 s} - \frac{2}{3}\left(1 - \frac{1}{4^n}\right) \\ -\left(\frac{1}{4}\tan^2\frac{1}{2}s + \frac{1}{4^2}\tan^2\frac{1}{4}s + \frac{1}{4^3}\tan^2\frac{1}{8}s \ldots . + \frac{1}{4^n}\tan^2\frac{s}{2^n}\right).\end{array}\right.$$

44. Let us now conceive n to be indefinitely great, then, as upon this hypothesis, Q becomes s, and $\frac{2}{3}\left(1 - \frac{1}{4^n}\right)$ becomes simply $\frac{2}{3}$, and the series whose terms were n in number, now goes on *ad infinitum*, we have at last, after multiplying the whole expression by 4,

$$\frac{1}{Q^2} = \left\{\begin{array}{l}\frac{4}{\tan^2 s} - \frac{8}{3} \\ -\left(\tan^2\frac{1}{2}s + \frac{1}{4}\tan^2\frac{1}{4}s + \frac{1}{4^2}\tan^2\frac{1}{8}s + \frac{1}{4^3}\tan^2\frac{1}{16}s +, \&c.\right).\end{array}\right.$$

And this is one form of the series to be investigated.

45. The

45. THE ſame ſeries, however, may be given under another form, better adapted to calculation. For ſince, by the nature of the hyperbola

$$\text{ab}^2\,S + \text{ord}^2\,S = \text{ab}\,2\,S, \text{ and } \text{ab}^2\,S - \text{ord}^2\,S = 1,$$

therefore, taking the ſum and difference of the correſponding ſides of theſe equations, we get

$$2\,\text{ab}^2\,S = \text{ab}\,2\,S + 1,\ 2\,\text{ord}^2\,S = \text{ab}\,2\,S - 1;$$

and hence, by dividing the latter of theſe equations by the former, and putting tan S inſtead of $\frac{\text{ord}\,S}{\text{ab}\,S}$, we find

$$\tan S = \frac{\text{ab}\,2\,S - 1}{\text{ab}\,2\,S + 1}.$$

FROM this formula, by ſubſtituting s, $\frac{1}{2}s$, $\frac{1}{4}s$, &c. inſtead of S, we obtain expreſſions for $\tan^2 s$, $\tan^2 \frac{1}{2}s$, $\tan^2 \frac{1}{4}s$, &c. Theſe being ſubſtituted in the ſeries, and afterwards s put inſtead of $2s$, $\frac{1}{2}s$ inſtead of s, $\frac{1}{4}s$ inſtead of $\frac{1}{2}s$, &c. (ſo as to produce a reſult involving only the abſciſſæ correſponding to the ſector s, and its ſub-multiples); and, finally, the whole being divided by 4, we ſhall get

$$\frac{1}{s^2} = \left\{ \begin{array}{l} \dfrac{\text{ab}\,s + 1}{\text{ab}\,s - 1} - \dfrac{2}{3} \\ - \left(\dfrac{1}{4}\,\dfrac{\text{ab}\,\frac{1}{2}s - 1}{\text{ab}\,\frac{1}{2}s + 1} + \dfrac{1}{4^2}\,\dfrac{\text{ab}\,\frac{1}{4}s - 1}{\text{ab}\,\frac{1}{4}s + 1} + \dfrac{1}{4^3}\,\dfrac{\text{ab}\,\frac{1}{8}s - 1}{\text{ab}\,\frac{1}{8}s + 1} +, \&c. \right) \end{array} \right.$$

and this expreſſion is analogous to our ſecond ſeries for an arch of a circle, as given at Art. 17.

46. WE may now inveſtigate what are the limits to the rate of convergency of this ſeries, as alſo the limits to the ſum of all its terms following any aſſigned term. With reſpect to the firſt of

of theſe inquiries, it appears, that the terms of the ſeries, under its firſt form, (Art. 43) are exactly the ſquares of the corresponding terms of the former ſeries, under its firſt form (Art. 36.), ſo that the one being written thus,

$$\frac{1}{s}=P-(T_{(1)}+T_{(2)}\ldots+T_{(m)}+T_{(m+1)}+T_{(m+2)}+,\&c.)$$

the other will be

$$\frac{1}{s^2}=P'-(T^2_{(1)}+T^2_{(2)}\ldots+T^2_{(m)}+T^2_{(m+1)}+T^2_{(m+2)}+,\&c.),$$

and here P and P′ are put for the parts of the two expreſſions which do not follow the law of the remaining terms, but $T_{(1)}$, $T_{(2)}$, &c. denote the ſame quantities in both. Now, as each term in the former ſeries has been proved to be greater than one-fourth of the term immediately before it (Art. 39), each term of the latter muſt be greater than one-ſixteenth of the term immediately before it; and this is one limit to the rate of convergency.

Again, as it has been proved (Art. 40.), that in the firſt ſeries $T_{(n+2)} < \frac{T^2_{(n+1)}}{T_{(n)}}$, therefore, ſquaring, we have $T^2_{(n+2)} < \frac{T^4_{(n+1)}}{T^2_{(n)}}$. Now this quantity is a third proportional to $T^2_{(n)}$ and $T^2_{(n+1)}$; hence it follows, that the greater limit of the rate of convergency in the two ſeries is the very ſame; that is, each term is leſs than a third proportional to the two terms immediately before it.

As theſe limits to the rate of convergency differ from thoſe of our ſecond ſeries for an arch of a circle (Art. 18.), only by the leſſer limit in the one caſe correſponding to the greater in the other, and the contrary, it is ſufficiently evident,

 that

that by proceeding, as in the caſe of the circle, to determine limits to the ſum of all the terms following any aſſigned term, we would obtain an analogous reſult, namely, that the ſum of all the terms following any aſſigned term is greater than $\frac{1}{15}$th of that term, but leſs than a third proportional to the difference of the two terms immediately before it, and the latter of the two.

47. It now only remains to be conſidered, how the numerical values of the terms of the ſeries are to be found. Now, this may evidently be done by computing the values of the quantities ab $\frac{1}{2}$ s, ab $\frac{1}{4}$ s, ab $\frac{1}{8}$ s, &c. from the abſciſſa correſponding to the whole ſector, and from one another by the known formula

$$\text{ab}\,\tfrac{1}{2}\,\text{S} = \sqrt{\frac{\text{ab S} + 1}{2}}$$

and thence the values of the quanties $\frac{\text{ab}\,\frac{1}{2}\,s - 1}{\text{ab}\,\frac{1}{2}\,s + 1}$, $\frac{\text{ab}\,\frac{1}{4}\,s - 1}{\text{ab}\,\frac{1}{4}\,s + 1}$, &c.

Or we may deduce each of theſe from that which precedes it, by a formula analogous to that found at Art. 21. in the caſe of the circle, and which may be inveſtigated as follows. Let $\frac{\text{ab S} - 1}{\text{ab S} + 1} = t$, and $\frac{\text{ab}\,\frac{1}{2}\,\text{S} - 1}{\text{ab}\,\frac{1}{2}\,\text{S} + 1} = t'$, then we have ab $\text{S} = \frac{1+t}{1-t}$, and $\frac{\text{ab S} + 1}{2} = \frac{1}{1-t}$; we have alſo ab $\frac{1}{2}$ $\text{S} = \frac{1+t'}{1-t'}$; and ſince by the nature of the hyperbola ab $\frac{1}{2}$ $\text{S} = \sqrt{\frac{\text{ab S} + 1}{2}}$;

therefore

therefore $\frac{1}{\sqrt{1-t}} = \frac{1+t'}{1-t'}$, and hence

$$t' = \frac{1-\sqrt{1-t}}{1+\sqrt{1-t}},$$

which is the formula required.

48. THE refult of the whole inveftigation of this fecond feries, for the area of an hyperbolic fector, may now be collected into one point of view, as follows.

PUTTING s for the area of the fector, let its correfponding abfciffa be denoted by the abbreviated expreffion ab s; alfo let the abfciffæ correfponding to the other fectors which are its fub-multiples be denoted fimilarly.

COMPUTE the feries of quantities ab $\frac{1}{2}s$, ab $\frac{1}{4}s$, ab $\frac{1}{8}s$, &c. from ab s, and one another, by the formula

$$\text{ab}\,\tfrac{1}{2}S = \sqrt{\frac{\text{ab}\,S+1}{2}}$$

Then fhall

$$\frac{1}{s^2} = \left\{ \begin{array}{l} \dfrac{\text{ab}\,s+1}{\text{ab}\,s-1} - \dfrac{2}{3} \\ - \left\{ \begin{array}{c} \dfrac{1}{4}\,\dfrac{\text{ab}\,\frac{1}{2}s-1}{\text{ab}\,\frac{1}{2}s+1} + \dfrac{1}{4^2}\,\dfrac{\text{ab}\,\frac{1}{4}s-1}{\text{ab}\,\frac{1}{4}s+1} + \dfrac{1}{4^3}\,\dfrac{\text{ab}\,\frac{1}{8}s-1}{\text{ab}\,\frac{1}{8}s+1} \\ \ldots\ldots + T_{(m)} + T_{(m+1)} + R \end{array} \right\}, \end{array} \right.$$

where R denotes the fum of all the terms following the term $T_{(m+1)}$, and this fum is always contained between the limits

$$\frac{1}{15}T_{(m+1)}, \text{ and } \frac{T^2_{(m+1)}}{T_{(m)} - T_{(m+1)}},$$

being

being greater than the former, but leſs than the latter. This laſt limit may alſo be otherwiſe expreſſed thus,

$$\frac{1}{15} T_{(m+1)} + \frac{(16\, T_{(m+1)} - T_{(m)})\; T_{(m+1)}}{15\,(T_{(m)} - T_{(m+1)})}.$$

Or compute the ſeries of quantities t, t', t'', &c. one from another by theſe formulæ

$$t = \frac{\text{ab}\, s - 1}{\text{ab}\, s + 1},\; t' = \frac{1 - \sqrt{1 - t}}{1 + \sqrt{1 + t}},\; t'' = \frac{1 - \sqrt{1 - t'}}{1 + \sqrt{1 + t'}}, \text{ \&c.}$$

Then ſhall

$$\frac{1}{s^2} = \frac{\text{ab}\, s + 1}{\text{ab}\, s - 1} - \frac{2}{3} - \left(\frac{1}{4} t' + \frac{1}{4^2} t'' + \frac{1}{4^3} t''' \ldots + T_{(m)} + T_{(m+1)} + R\right),$$

the ſymbols $T_{(m)}$, $T_{(m+1)}$, and R, being put to denote the ſame as before.

49. We might now inveſtigate other ſeries for the quadrature of an hyperbolic ſector, ſimilar to the third and fourth ſeries we have found for the rectification of an arch of a circle; but this inquiry would extend the Paper to too great a length. For this reaſon, and alſo becauſe the manner of proceeding in the one caſe is exactly the ſame as has been followed in the other, it ſeems unneceſſary, in the caſe of the hyperbola, to extend our inquiries farther. I ſhall therefore now proceed to the third and laſt object propoſed in this Paper, namely, the inveſtigation of formulæ for the calculation of logarithms, beginning with a few remarks that may ſerve to connect theſe formulæ with the common theory.

50. It is uſually ſhewn by writers on this ſubject, that all numbers whatever are conſidered as equal, or nearly equal, to one

one or other of the terms of a geometrical ſeries whoſe firſt term is unity and common ratio, a number very nearly equal to unity, but a little greater; and any quantities proportional to the exponents of the terms of the ſeries, are the logarithms of the numbers to which the terms are equal.

LOGARITHMS, then, being not abſolute but relative quantities, we may aſſume any number whatever as that whoſe logarithm is unity; but a particular number being once choſen, the logarithms of all other numbers are thereby fixed.

HENCE it follows, that there may be different ſyſtems, according as unity is made the logarithm of one or another number; the logarithms of two given numbers, however, will always have the ſame ratio to each other in every ſyſtem whatever; theſe properties which are commonly known, are mentioned here only for the ſake of what is to follow, as we have already premiſed.

51. TAKING this view of the theory of logarithms as the foundation of our inveſtigations,

LET us put r for the common ratio of the geometrical ſeries,
x for any number or term of the ſeries,
b for the number whoſe logarithm is unity,
y for the exponent of that power of r which is equal to x,
m for the exponent of the power of r which is equal to b.

Then we have $x = r^y$, and $b = r^m$, and becauſe by the nature of logarithms $\log x : \log b :: y : m$, therefore $\log x = \frac{y}{m} \times \log b$;

but by hypotheſis $\log b = 1$, therefore $\log x = \frac{y}{m}$.

52. LET

52. LET v denote any number greater than unity, and p and n any two whole poſitive numbers; then, by a known formula

$$v^p - 1 = \frac{v - 1}{v}\left\{v + v^2 + v^3 + v^4 \ldots + v^p\right\},$$

$$v^n - 1 = \frac{v - 1}{v}\left\{v + v^2 + v^3 + v^4 \ldots + v^n\right\};$$

therefore, dividing each ſide of the firſt of theſe equations by the correſponding ſide of the ſecond, we get

$$\frac{v^p - 1}{v^n - 1} = \frac{v + v^2 + v^3 + v^4 \ldots + v^p}{v + v^2 + v^3 + v^4 \ldots + v^n},$$

Now, v being by hypotheſis greater than unity, the fraction on the right hand ſide of this equation is leſs than this other fraction

$$\frac{v^p + v^p + v^p + v^p \ldots + v^p \text{ (to } p \text{ terms)}}{1 + 1 + 1 + 1 \ldots + 1 \text{ (to } n \text{ terms)}} = \frac{p\, v^p}{n},$$

becauſe it has manifeſtly a leſs numerator, and at the ſame time a greater denominator. The ſame fraction is, however, greater than this fraction

$$\frac{1 + 1 + 1 + 1 \ldots + 1 \text{ (to } p \text{ terms)}}{v^n + v^n + v^n + v^n \ldots + v^n \text{ (to } n \text{ terms)}} = \frac{p}{n\, v^n},$$

becauſe it has a greater numerator, and a leſs denominator. Therefore,

$$\frac{v^p - 1}{v^n - 1} < \frac{p\, v^p}{n}, \quad \frac{v^p - 1}{v^n - 1} > \frac{p}{n\, v^n},$$

and hence, dividing the firſt of theſe expreſſions by v^p, and multiplying the ſecond by v^n,

$$\frac{p}{n} > \frac{v^p - 1}{v^p\,(v^n - 1)}, \quad \frac{p}{n} < \frac{v^n\,(v^p - 1)}{v^n - 1}. \quad (\alpha).$$

53. PUTTING

53. Putting v and p to denote, as in laſt article, it is manifeſt that the ſeries

$$1+v^{\frac{1}{p}}+v^{\frac{2}{p}}+v^{\frac{3}{p}}\ldots+v^{\frac{p-1}{p}}$$

is greater than this other ſeries

$$1+1+1+1\ldots+1 \text{ (to } p \text{ terms)} = p,$$

but leſs than this ſeries

$$v+v+v+v\ldots+v \text{ (to } p \text{ terms)} = p\,v;$$

but by a known formula, the ſum of the firſt of theſe three ſeries is

$$\frac{v-1}{v^{\frac{1}{p}}-1}, \text{ therefore,}$$

$$\frac{v-1}{v^{\frac{1}{p}}-1} > p, \quad \frac{v-1}{v^{\frac{1}{p}}-1} < p\,v,$$

and hence it follows, that

$$\left.\begin{aligned} v^{\frac{1}{p}} &< 1+\frac{v-1}{p}, \quad p\left(v^{\frac{1}{p}}-1\right) < v-1, \\ v^{\frac{1}{p}} &> 1+\frac{v-1}{p\,v}, \quad p\left(v^{\frac{1}{p}}-1\right) > 1-\frac{1}{v}. \end{aligned}\right\} (\beta).$$

54. Let us now recur to the ſymbols r, x, b, y and m, whoſe values are aſſigned in Art. 51. and let us aſſume $y=p$, and $r=v^n$; then, from the two expreſſions (α) in Art. 52, we have

$\frac{y}{n}$ >

$$\frac{y}{n} > \frac{1}{r^{\frac{y}{n}}} \cdot \frac{r^{\frac{y}{n}} - 1}{r - 1}, \quad \frac{y}{n} < r \cdot \frac{r^{\frac{y}{n}} - 1}{r - 1};$$

and hence, multiplying by n, and dividing by m,

$$\frac{y}{m} > \frac{1}{r^{\frac{y}{n}}} \cdot \frac{n(r^{\frac{y}{n}} - 1)}{m(r - 1)}, \quad \frac{y}{m} < r \cdot \frac{n(r^{\frac{y}{n}} - 1)}{m(r - 1)}.$$

But $r^y = x$, and $r^m = b$, (Art. 51.), from which it follows, that $r^{\frac{y}{n}} = x^{\frac{1}{n}}$, and $r = b^{\frac{1}{m}}$; moreover, $\frac{y}{m} = \log x$; therefore, ſubſtituting, we get

$$\log x > \frac{1}{x^{\frac{1}{n}}} \cdot \frac{n(x^{\frac{1}{n}} - 1)}{m(b^{\frac{1}{m}} - 1)}, \quad \log x < b^{\frac{1}{m}} \cdot \frac{n(x^{\frac{1}{n}} - 1)}{m(b^{\frac{1}{m}} - 1)},$$

and in theſe expreſſions n denotes any whole poſitive number whatever.

55. By ſubtracting the leſſer of theſe limits to the logarithm of x from the greater, we find their difference to be

$$\frac{1}{x^{\frac{1}{n}}} \frac{n(x^{\frac{1}{n}} - 1)}{m(b^{\frac{1}{m}} - 1)} \times (b^{\frac{1}{m}} x^{\frac{1}{n}} - 1).$$

Now

Now we have found, that one factor of this expression, viz.

$$\frac{1}{x^{\frac{1}{n}}} \frac{n(x^{\frac{1}{n}}-1)}{m(b^{\frac{1}{m}}-1)}$$ cannot exceed the logarithm of x; with respect to the other factor $b^{\frac{1}{m}} x^{\frac{1}{n}} - 1$, since it appears from the first of the four formulæ (β), (Art. 53.), that $b^{\frac{1}{m}} < 1 + \frac{b-1}{m}$, and $x^{\frac{1}{n}} < 1 + \frac{x-1}{n}$; therefore, multiplying $b^{\frac{1}{m}} x^{\frac{1}{n}} < \left(1+\frac{b-1}{m}\right)\left(1+\frac{x-1}{m}\right)$, and hence

$$b^{\frac{1}{m}} x^{\frac{1}{n}} - 1 < \frac{b-1}{m} + \frac{x-1}{n} + \frac{(b-1)\ (x-1)}{mn}.$$

Now as we may conceive m and n to be as great as we please, it is evident that this quantity, which exceeds the factor $b^{\frac{1}{m}} x^{\frac{1}{n}} - 1$, may be smaller than any assignable quantity; therefore the product of the two factors, or the difference between the limits to the value of $\log x$, may, by taking m and n sufficiently great, be less than any assignable quantity.

Upon the whole, then, it appears, that the logarithm of x is a limit to which the two quantities

$$\frac{1}{x^{\frac{1}{n}}} \frac{n(x^{\frac{1}{n}}-1)}{m(b^{\frac{1}{m}}-1)},\quad b^{\frac{1}{m}} \frac{n(x^{\frac{1}{n}}-1)}{m(b^{\frac{1}{m}}-1)}$$

 continually

continually approach when m and n are conceived to increaſe indefinitely, and to which each at laſt comes nearer than by any aſſignable difference, juſt as a circle is the limit to all the polygons which can be inſcribed in it, or deſcribed about it.

56. THE two expreſſions $n(x^{\frac{1}{n}} - 1)$, $m(b^{\frac{1}{m}} - 1)$, which enter into theſe limits to the logarithm of x, and which are evidently functions of the ſame kind, have each a finite magnitude even when m or n is conſidered as greater than any aſſignable number; for ſince when v is greater than unity, and p any whole poſitive number, we have

$$p(v^{\frac{1}{p}} - 1) < v - 1, \quad p(v^{\frac{1}{p}} - 1) > 1 - \frac{1}{v}. \quad \text{(Art. 53.)}$$

Therefore, ſuppoſing x and b both greater than 1, (which may always be done in the theory of logarithms), the expreſſion $n(x^{\frac{1}{n}} - 1)$ is neceſſarily contained between the limits $x - 1$ and $1 - \frac{1}{x}$; and in like manner, $m(b^{\frac{1}{m}} - 1)$ is between $b - 1$ and $1 - \frac{1}{b}$.

57. As the expreſſion $m(b^{\frac{1}{m}} - 1)$ depends entirely upon the value of b, the number whoſe logarithm is aſſumed $= 1$, (and which is ſometimes called the *basis* of the ſyſtem), the limit to which it approaches when m increaſes indefinitely will be a conſtant quantity in a given ſyſtem; but the limit to which

which $n(x^{\frac{1}{n}}-1)$ approaches, when n is conceived to be indefinitely increafed, will be variable, as it depends upon the particular value of the number x.

Let us therefore denote the limit of $m(b^{\frac{1}{m}}-1)$, or that of $\frac{m(b^{\frac{1}{m}}-1)}{b^{\frac{1}{m}}}$, (for they are evidently the fame), by B, and then in the fyftem whofe bafis is b, the logarithm of x will be the limit of either of thefe two expreffions

$$\frac{1}{x^{\frac{1}{n}}}\,\frac{n(x^{\frac{1}{n}}-1)}{B},\ \frac{n(x^{\frac{1}{n}}-1)}{B},$$

when n is conceived to increafe indefinitely, or to fpeak briefly, $\log x=\frac{n(x^{\frac{1}{n}}-1)}{B}$, when n is indefinitely great.

The conftant multiplier $\frac{1}{B}$ is what writers on the fubject of logarithms have denominated the *modulus* of the fyftem. As in Napier's fyftem it is unity, we have, n being indefinitely great, Nap. $\log x=n(x^{\frac{1}{n}}-1)$, and fince in any fyftem whatever $B=m(b^{\frac{1}{m}}-1)$, or $B=n(b^{\frac{1}{n}}-1)$, for we may put m or n indifcriminately;

indiſcriminately; therefore $B = \text{Nap. log } b$, and conſequently

$$\log x \text{ (to baſis } b) = \frac{\text{Nap. log } x}{\text{Nap. log } b},$$

as is commonly known.

58. SINCE, therefore, the logarithms of any propoſed ſyſtem may be deduced from thoſe of NAPIER's ſyſtem, I ſhall throughout the reſt of this Paper attend only to the formula

$$\text{Nap. log } x = n\,(x^{\frac{1}{n}} - 1),\ n \text{ being indefinitely great.}$$

LET us then, agreeably to the mode of proceeding employed in the former part of this paper, aſſume the identical equation

$$\frac{X^2+1}{X^2-1} = \frac{X+1}{2\,(X-1)} + \frac{1}{2}\,\frac{X-1}{X+1}.$$

In this expreſſion let $x^{\frac{1}{2}}$, $x^{\frac{1}{4}}$, $x^{\frac{1}{8}}$, $x^{\frac{1}{16}}$, &c. be ſubſtituted ſucceſſively for X, and let the reſults be multiplied by the correſponding terms of the ſeries, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. Thus there will be formed a ſeries of equations, which, putting n for their number, and m for 2^n, may ſtand as follows:

$\frac{x+1}{x-1}$

$$\frac{x+1}{x-1} = \frac{x^{\frac{1}{2}}+1}{2\,(x^{\frac{1}{2}}-1)} + \frac{1}{2}\,\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1},$$

$$\frac{x^{\frac{1}{2}}+1}{2\,(x^{\frac{1}{2}}-1)} = \frac{x^{\frac{1}{4}}+1}{4\,(x^{\frac{1}{4}}-1)} + \frac{1}{4}\,\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1},$$

$$\frac{x^{\frac{1}{4}}+1}{4\,(x^{\frac{1}{4}}+1)} = \frac{x^{\frac{1}{8}}+1}{8\,(x^{\frac{1}{8}}-1)} + \frac{1}{8}\,\frac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1},$$

$$\frac{x^{\frac{1}{8}}-1}{8\,(x^{\frac{1}{8}}+1)} = \frac{x^{\frac{1}{16}}+1}{16\,(x^{\frac{1}{16}}-1} + \frac{1}{16}\,\frac{x^{\frac{1}{16}}-1}{x^{\frac{1}{16}}+1},$$

. . . .

$$\frac{2\,(x^{\frac{2}{m}}+1)}{m\,(x^{\frac{2}{m}}-1)} = \frac{x^{\frac{1}{m}}+1}{m\,(x^{\frac{1}{m}}-1)} + \frac{1}{2^{n}}\,\frac{x^{\frac{1}{m}}-1}{x^{\frac{1}{m}}+1},$$

Let the fum of thefe equations be now taken, and the quantities found on both fides of the refult rejected, then, after

ter transposing, we get

$$\frac{x^{\frac{1}{m}}+1}{m\,(x^{\frac{1}{m}}-1)} = \begin{cases} \dfrac{x+1}{x-1} \\ -\left(\dfrac{1}{2}\dfrac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1} + \dfrac{1}{4}\dfrac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1} + \dfrac{1}{8}\dfrac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1} + \dfrac{1}{16}\dfrac{x^{\frac{1}{16}}-1}{x^{\frac{1}{16}}+1} \ldots \right. \\ \left. + \dfrac{1}{2^n}\dfrac{x^{\frac{1}{m}}-1}{x^{\frac{1}{m}}+1}\right). \end{cases}$$

THIS equation, which is identical, holds true whatever be the value of n. Let us now, however, suppose, that n is indefinitely great, then the series will go on *ad infinitum*, and $m = 2^n$ will become indefinitely great; but this being the case, $x^{\frac{1}{m}} + 1$, which is always less than $2 + \frac{x-1}{m}$, (Art. 53.), will become simply 2; and $m\,(x^{\frac{1}{m}} - 1)$ will become Nap. log x, (Art. 57.); therefore substituting these limits, and dividing by 2, we have

$$\frac{1}{\log x} = \begin{cases} \dfrac{1}{2}\dfrac{x+1}{x-1} \\ -\left(\dfrac{1}{4}\dfrac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1} + \dfrac{1}{8}\dfrac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1} + \dfrac{1}{16}\dfrac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1} + \dfrac{1}{32}\dfrac{x^{\frac{1}{16}}-1}{x^{\frac{1}{16}}+1} +, \right. \\ \left. \text{\&c.}\right) \end{cases}$$

and

and this is the firſt ſeries which I propoſe to inveſtigate for the calculation of logarithms.

59. THE ſeries juſt now found agreeing exactly in its form with our firſt ſeries for an hyperbolic ſector, (Art. 40.), as it ought to do, will of courſe have the ſame limits to the rate of its convergency, and to the ſum of all its terms, following any propoſed term. As the latter of theſe have been deduced from the former, in the caſe of the hyperbola, by a proceſs purely analytical, and the ſame as we have followed in treating of the rectification of the circle, it is not neceſſary to repeat their inveſtigation in this place. The limits to the rate of convergency, however, having been made to depend partly upon the nature of the curve, it may be proper, in the preſent inquiry, to deduce them entirely from the analytical formula which has been made the baſis of the inveſtigation.

LET any three ſucceſſive terms of the ſeries of quantities $\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1}$, $\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1}$, &c. be denoted by t, t' and t''; then it is evident from the formula, (Art. 58.), that the relation of theſe quantities to one another will be expreſſed by the equations

$$\frac{2}{t}=\frac{1}{t'}+t', \quad \frac{2}{t'}=\frac{1}{t''}+t''.$$

From the firſt of theſe we get $2t'=t(1+t'^2)$, now each of the quantities t, t', &c. being evidently leſs than unity, it follows, that $1+t'^2<2$, but >1, and therefore that $2t'<2t$, and $t'<t$; alſo that $2t'>t$, and $t'>\frac{1}{2}t$. Hence it appears,

in

in the firſt place, that each term of our ſeries, taking its co-efficient into account, is greater than one-fourth of the term before it.

AGAIN, becauſe $\frac{t'}{t} = \frac{1}{2}(1+t'^2)$; and, ſimilarly, $\frac{t''}{t'} = \frac{1}{2}(1+t''^2)$,

and it having been proved that $t' < t$, ſo that ſimilarly, $t'' < t'$,

therefore $\frac{1}{2}(1+t''^2) < \frac{1}{2}(1+t'^2)$, and conſequently $\frac{t''}{t'} < \frac{t'}{t}$, and

$t'' < \frac{t'^2}{t}$. Thus it appears, that each of the quantities t'', &c. is

leſs than a third proportional to the two immediately before it, and the ſame muſt alſo be true of the terms of the ſeries.

60. UPON the whole, then, our firſt ſeries for the calculation of logarithms may be expreſſed as follows:

$$\frac{1}{\log x} = \frac{1}{2}\,\frac{x+1}{x-1} - \left(\frac{1}{4}\,\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1} + \frac{1}{8}\,\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1} + \frac{1}{16}\,\frac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1} \ldots\right.$$

$$\left. + T_{(m)} + T_{(m+1)} + R\right);$$

and here, as in the former ſeries, $T_{(m)}$ and $T_{(m+1)}$ denote any two ſucceeding terms, and R is a quantity greater than $\frac{1}{3}T_{(m+1)}$, but leſs than a third proportional to $T_{(m)} - T_{(m+1)}$ and $T_{m+1)}$; or it is leſs than

$$\frac{1}{3}T_{(m+1)} + \frac{4\,T_{(m+1)} - T_{(m)}}{3\,(T_{(m)} - T_{(m+1)})}\,T_{(m+1)}.$$

61. THAT

61. THAT we may inveſtigate a ſecond ſeries, we muſt take the ſquare of the formula, (Art. 58.), which will be

$$\left(\frac{X^2+1}{X^2-1}\right)^2 = \frac{(X+1)^2}{4(X-1)^2} + \frac{1}{4}\left(\frac{X-1}{X+1}\right)^2 + \frac{1}{2}.$$

From this expreſſion, proceeding exactly as in Art. 58. we form the following ſeries of equations,

$$\frac{(x+1)^2}{(x-1)^2} = \frac{(x^{\frac{1}{2}}+1)^2}{4(x^{\frac{1}{2}}-1)^2} + \frac{1}{4}\left(\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1}\right)^2 + \frac{1}{2},$$

$$\frac{(x^{\frac{1}{2}}+1)^2}{4(x^{\frac{1}{2}}-1)^2} = \frac{(x^{\frac{1}{4}}+1)^2}{4^2(x^{\frac{1}{4}}-1)^2} + \frac{1}{4^2}\left(\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1}\right)^2 + \frac{1}{2.4},$$

$$\frac{(x^{\frac{1}{4}}+1)^2}{4^2(x^{\frac{1}{4}}-1)^2} = \frac{(x^{\frac{1}{8}}+1)^2}{4^3(x^{\frac{1}{8}}-1)^2} + \frac{1}{4^3}\left(\frac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1}\right)^2 + \frac{1}{2.4^2},$$

. . . .

$$\frac{4(x^{\frac{2}{m}}+1)^2}{m^2(x^{\frac{2}{m}}-1)^2} = \frac{(x^{\frac{1}{m}}+1)^2}{m^2(x^{\frac{1}{m}}-1)^2} + \frac{1}{4^n}\left(\frac{x^{\frac{1}{m}}-1}{x^{\frac{1}{m}}+1}\right)^2 + \frac{1}{2.4^{n-1}},$$

Here n denotes the number of equations, and m is put for 2^n. Let the ſum of the correſponding ſides of theſe equations be taken, and the quantities common to each rejected, as uſual, and

the

the refult, after tranfpofition, will be

$$\frac{(x^{\frac{1}{m}}+1)^2}{m^2(x^{\frac{1}{m}}-1)^2}=\left\{\begin{array}{l}\left(\frac{x+1}{x-1}\right)^2-\left(\frac{1}{2}+\frac{1}{2.4}+\frac{1}{2.4^2}+\frac{1}{2.4^3}\cdots\cdots+\frac{1}{2.4^{n-1}}\right)\\ -\left\{\frac{1}{4}\left(\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1}\right)^2+\frac{1}{4^2}\left(\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1}\right)+\frac{1}{4^3}\left(\frac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1}\right)^2\cdots\right.\\ \left.+\frac{1}{4^n}\left(\frac{x^{\frac{1}{m}}-1}{x^{\frac{1}{m}}+1}\right)^2\right\}\end{array}\right.$$

This equation holds true when n is any whole pofitive number whatever. But if we fuppofe it indefinitely great, then the two feries will go on *ad infinitum*, and the limit of the numerical feries will be $\frac{2}{3}$, alfo the limit of $(x^{\frac{1}{m}}+1)^2$ will be 4, and the limit of $m^2(x^{\frac{1}{m}}-1)^2$ will be $\log^2 x$, (Art. 57.); therefore, fubftituting thefe limits, and alfo putting $\frac{4x}{(x-1)^2}+\frac{1}{3}$ for $\left(\frac{x+1}{x-1}\right)^2-\frac{2}{3}$, and dividing the whole expreffion by 4, we get

$$\frac{1}{\log^2 x}=$$

$$\frac{1}{\log^2 x} = \left\{ \begin{array}{l} \dfrac{x}{(x-1)^2} + \dfrac{1}{12} \\ \\ - \left\{ \dfrac{1}{4^2}\left(\dfrac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{2}} + 1}\right)^2 + \dfrac{1}{4^3}\left(\dfrac{x^{\frac{1}{4}} - 1}{x^{\frac{1}{4}} + 1}\right)^2 + \dfrac{1}{4^4}\left(\dfrac{x^{\frac{1}{8}} - 1}{x^{\frac{1}{8}} + 1}\right)^2 + , \right. \\ \\ \left. \qquad \&c. \right\} \end{array} \right.$$

and thus we have obtained a ſecond ſeries for the logarithm of a number, which, by putting t, t', &c. inſtead of the fractions $\frac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{2}} + 1}$, $\frac{x^{\frac{1}{4}} - 1}{x^{\frac{1}{4}} + 1}$, &c. and remarking that the relation which the quantities t, t', &c. have to one another is identical with that of the quantities tan s, tan $\frac{1}{2}s$, &c. (Art. 35.), it will appear to be the ſame as our ſecond ſeries for the area of an hyperbolic ſector, (Art. 43.). Of courſe it will have the ſame limits to the rate of its convergency, and to the ſum of all its terms following any given term. Now theſe have been found without any reference to the geometrical properties of the curve, therefore it is not neceſſary to repeat their inveſtigation.

62. We muſt now transform our ſeries upon principles purely analytical, ſo as to ſuit it to calculation. And, in the firſt place, becauſe $\left(\frac{x-1}{x+1}\right)^2 = \frac{x^2 - 2x + 1}{x^2 + 2x + 1} = \frac{\frac{1}{2}(x + \frac{1}{x}) - 1}{\frac{1}{2}(x + \frac{1}{x}) + 1}$, if

we

we put $\frac{1}{2}\left(x+\frac{1}{x}\right)=X$, it follows, that $\left(\frac{x-1}{x+1}\right)^2=\frac{X-1}{X+1}$.

In like manner, putting $\frac{1}{2}\left(x^{\frac{1}{2}}+\frac{1}{x^{\frac{1}{2}}}\right)=X'$, and $\frac{1}{2}\left(x^{\frac{1}{4}}-\frac{1}{x^{\frac{1}{4}}}\right)$

$=X''$, we have $\left(\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1}\right)^2=\frac{X'-1}{X'+1}$, and $\left(\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1}\right)^2=\frac{X''-1}{X''+1}$,

&c. Again, becauſe $X'=\frac{1}{2}\left(x^{\frac{1}{2}}+\frac{1}{x^{\frac{1}{2}}}\right)$, therefore $2X'^2=$

$\frac{1}{2}\left(x+\frac{1}{x}\right)+1$, but $\frac{1}{2}\left(x+\frac{1}{x}\right)=X$, therefore $2X'^2=X+1$

and $X'=\sqrt{\frac{X+1}{2}}$. In like manner, it will appear, that

$X''=\sqrt{\frac{X'+1}{2}}$, &c.

63. From the preceding inveſtigation it appears, upon the whole, that our ſecond ſeries for the calculation of a logarithm may be expreſſed as follows.

Putting x for any number, let a ſeries of quantities X, X', X'', X''', &c. be found ſuch that

$$X=\frac{1}{2}\left\{x+\frac{1}{x}\right\},\ X'=\sqrt{\frac{X+1}{2}},\ X''=\sqrt{\frac{X'+1}{2}},\ \&c.$$

Then will

$$\frac{1}{\log^2 x}=$$

$$\frac{1}{\log^2 x} = \begin{cases} \dfrac{x}{(x-1)^2} + \dfrac{1}{12} \\ -\left\{\dfrac{1}{4^2}\,\dfrac{X'-1}{X'+1} + \dfrac{1}{4^3}\,\dfrac{X''-1}{X''+1} + \dfrac{1}{4^4}\,\dfrac{X'''-1}{X'''+1} \cdots \right. \\ \left. + T_{(m)} + T_{(m+1)} + R\right\}; \end{cases}$$

and here $T_{(m)}$, $T_{(m+1)}$, are put for any two ſucceſſive terms of the ſeries, and R for the ſum of all the following terms: And in every caſe R is greater than $\frac{1}{15}\,T_{(m+1)}$, but leſs than

$$\frac{1}{15}\,T_{(m+1)} + \frac{16\,T_{(m+1)} - T_{(m)}}{15\,(T_{(m)} - T_{(m+1)})}\,T_{(m+1)}.$$

64. FROM the analogy of the two formulæ from which we have deduced the ſeries for the rectification of an arch of a circle, and for the calculation of logarithms, it is eaſy to infer that there will be correſponding ſeries for the reſolution of each of theſe problems. And as the two preceding ſeries for a logarithm have been inveſtigated in the very ſame way as the firſt two ſeries for an arch of a circle, ſo, by proceeding exactly as in the inveſtigation of the third and fourth ſeries for the circle, we may obtain a third and fourth ſeries for a logarithm. The mode of deduction, then, being the ſame in both caſes, and alſo ſufficiently evident, I ſhall ſimply ſtate the reſult of the inveſtigation of a ſeries for logarithms which is analogous to our fourth ſeries for an arch of a circle, (Art. 28.).

LET x be any number, and X, X′, X″, X‴, &c. a ſeries of quantities formed from x, and one another, as ſpecified in the beginning of the laſt article. Then

$$\frac{1}{\log^4 x} =$$

$$\frac{1}{\log^4 x} = \left\{ \begin{array}{l} \frac{x(x^2+4x+1)}{6(x-1)^4} - \frac{1}{8.9.10} \\ + \frac{1}{3.16^2}\,\frac{X+12X'-13}{X+4X'+3} + \frac{1}{3.16^3}\,\frac{X'+12X''-13}{X'+4X''+3} \\ + \frac{1}{3.16^4}\,\frac{X''+12X'''+13}{X''+4X'''+3} +, \&c. \end{array} \right.$$

The terms of this ſeries approach continually to thoſe of a geometrical ſeries, of which the common ratio is $\frac{1}{64}$: and hence it follows, that the ſum of all the terms after any aſſign-ed term, approaches the nearer to $\frac{1}{63}$ of that term, according as it is more advanced in the ſeries.

65. Besides the foregoing, our method furniſhes yet another kind of expreſſion for the logarithm of a number, namely, a product conſiſting of an infinite number of factors, which approach continually to unity. Such an expreſſion may be inveſtigated as follows. From the identical equation

$$X - 1 = (X^{\frac{1}{2}} - 1)(X^{\frac{1}{2}} + 1).$$

Let there be formed the ſeries of equations

$$x - 1 = 2(x^{\frac{1}{2}} - 1)\frac{x^{\frac{1}{2}} + 1}{2},$$

$$2(x^{\frac{1}{2}} - 1) = 4(x^{\frac{1}{4}} - 1)\frac{x^{\frac{1}{4}} + 1}{2},$$

$$4(x^{\frac{1}{4}} - 1) = 8(x^{\frac{1}{8}} - 1)\frac{x^{\frac{1}{8}} + 1}{2},$$

. . . .

$$\frac{m}{2}(x^{\frac{2}{m}} - 1) = m(x^{\frac{1}{m}} - 1)\frac{x^{\frac{1}{m}} + 1}{2},$$

here

here m is put for any integer power of 2. Let the product of the corresponding sides of these equations be now taken, and the common factors rejected, and the result will be

$$x - 1 = m\left(x^{\frac{1}{m}} - 1\right) \frac{x^{\frac{1}{2}} + 1}{2} \cdot \frac{x^{\frac{1}{4}} + 1}{2} \cdot \frac{x^{\frac{1}{8}} + 1}{2} \cdots \frac{x^{\frac{1}{m}} + 1}{2},$$

and hence

$$m\left(x^{\frac{1}{m}} - 1\right) = (x - 1) \frac{2}{x^{\frac{1}{2}} + 1} \cdot \frac{2}{x^{\frac{1}{4}} + 1} \cdot \frac{2}{x^{\frac{1}{8}} + 1} \cdots \frac{2}{x^{\frac{1}{m}} + 1}.$$

This equation holds true, m being any power of 2 whatever.

LET us, however, conceive it indefinitely great. Then the number of factors will become infinite, and $m\left(x^{\frac{1}{m}} - 1\right)$ will become Nap. log x (Art. 57.). Therefore,

$$\text{Nap. log } x = (x - 1) \frac{2}{x^{\frac{1}{2}} + 1} \cdot \frac{2}{x^{\frac{1}{4}} + 1} \cdot \frac{2}{x^{\frac{1}{8}} + 1} \cdot \frac{2}{x^{\frac{1}{16}} + 1}, \text{ \&c.}$$

ad infinitum.

THE product of any finite number of these factors being always a function of this form $m\left(x^{\frac{1}{m}} - 1\right)$ will of course be greater than log x, (Art. 54.). However, the function $\frac{1}{x^{\frac{1}{m}}} m\left(x^{\frac{1}{m}} - 1\right)$ or $m\left(1 - \frac{1}{x^{\frac{1}{m}}}\right)$, being in like manner expanded into an infi-

nite

nite product, we get from it

$$\log x = \left(1 - \frac{1}{x}\right) \frac{2}{\frac{1}{x^{\frac{1}{2}}}+1} \cdot \frac{2}{\frac{1}{x^{\frac{1}{4}}}+1} \cdot \frac{2}{\frac{1}{x^{\frac{1}{8}}}+1} \cdot \frac{2}{\frac{1}{x^{\frac{1}{16}}}+1}, \text{ \&c.}$$

ad infinitum.

and the product of any finite number of factors of this expression will always be less than log x.

THESE formulæ, which are analogous to that given by EULER for an arch of a circle, (see Art. 33.), are not inelegant, considered as analytical transformations. It does not seem, however, that without some analytical artifice, they can be applied with advantage to the actual calculation of logarithms, by reason of the great labour which would be necessary to obtain a result tolerably accurate.

66. I SHALL now conclude this Paper, with some examples of the application of the formulæ to the computation of the length of one-fourth the circumference of a circle whose radius is unity, (which is the extreme and the most unfavourable case), and to the computation of a logarithm; as also of the *modulus* of the common system of logarithms, which is the reciprocal of NAPIER's logarithm of 10.

EXAMPLE

EXAMPLE I. The length of an arch of 90°, computed to 12 places of decimals, by means of the first series, (Art. 12.). Here $a = 90°$.

$\frac{1}{\tan a} = \cot a = 0$

$\tan \frac{1}{2} a = 1$

$\tan \frac{1}{4} a = 0.414213562373 1$

$\tan \frac{1}{8} a = 0.198912367379 6$

$\tan \frac{1}{16} a = 0.098491403357 1$

$\tan \frac{1}{32} a = 0.049126849769 4$

$\tan \frac{1}{64} a = 0.024548622108 9$

$\tan \frac{1}{128} a = 0.012272462379.$

$\tan \frac{1}{256} a = 0.006136000157.$

$\tan \frac{1}{512} a = 0.003067971201.$

$\tan \frac{1}{1024} a = 0.00153398194..$

$\tan \frac{1}{2048} a = 0.00076699054..$

$\frac{1}{2} \tan \frac{1}{2} a = .500\ 000\ 000\ 000\ 0$

$\frac{1}{4} \tan \frac{1}{4} a = .103\ 553\ 390\ 593\ 3$

$\frac{1}{8} \tan \frac{1}{8} a = .024\ 864\ 045\ 922\ 5$

$\frac{1}{16} \tan \frac{1}{16} a = .006\ 155\ 712\ 709\ 8$

$\frac{1}{32} \tan \frac{1}{32} a = .001\ 535\ 214\ 055\ 3$

$\frac{1}{64} \tan \frac{1}{64} a = .000\ 383\ 572\ 220\ 5$

$\frac{1}{128} \tan \frac{1}{128} a = .000\ 095\ 878\ 612\ 3$

$\frac{1}{256} \tan \frac{1}{256} a = .000\ 023\ 968\ 750\ 6$

$\frac{1}{512} \tan \frac{1}{512} a = .000\ 005\ 992\ 131\ 3$

$\frac{1}{1024} \tan \frac{1}{1024} a = .000\ 001\ 498\ 029\ 3$

$\frac{1}{2048} \tan \frac{1}{2048} a = .000\ 000\ 374\ 507\ 1$

$\left.\begin{array}{l} S < .0000001248357 \\ S > .0000001248356 \end{array}\right\}$ Hence $S = .000\ 000\ 124\ 835\ 7$

$\frac{1}{a} = \overline{.636\ 619\ 772\ 367\ 7}$

Arch of 90°, or $a = 1.570\ 796\ 326\ 795.$

EXAMPLE II. The length of an arch of 90°, computed by the fecond feries, (Art. 22.).

$\text{cof}\, a = 0$ $\text{cof}\,\frac{1}{16} a = 0.99518472667\ldots$

$\text{cof}\,\frac{1}{2} a = 0.70710678 11865$ $\text{cof}\,\frac{1}{32} a = 0.99879545621\ldots$

$\text{cof}\,\frac{1}{4} a = 0.9238795325113$ $\text{cof}\,\frac{1}{64} a = 0.9996988187\ldots$

$\text{cof}\,\frac{1}{8} a = 0.980785280403.$

Amount of pofitive terms, $\frac{1}{4}\,\frac{1+\text{cof}\,a}{1-\text{cof}\,a} + \frac{1}{8} = \frac{5}{12} = .416\ 666\ 666\ 666\ 7$

$$\frac{1}{4^2}\,\frac{1-\text{cof}\,\frac{1}{2}a}{1+\text{cof}\,\frac{1}{2}a} = .010\ 723\ 304\ 703\ 4$$

$$\frac{1}{4^3}\,\frac{1-\text{cof}\,\frac{1}{4}a}{1+\text{cof}\,\frac{1}{4}a} = .000\ 618\ 220\ 779\ 6$$

$$\frac{1}{4^4}\,\frac{1-\text{cof}\,\frac{1}{8}a}{1+\text{cof}\,\frac{1}{8}a} = .000\ 037\ 892\ 799\ 0$$

$$\frac{1}{4^5}\,\frac{1-\text{cof}\,\frac{1}{16}a}{1+\text{cof}\,\frac{1}{16}a} = .000\ 002\ 356\ 882\ 2$$

$$\frac{1}{4^6}\,\frac{1-\text{cof}\,\frac{1}{32}a}{1+\text{cof}\,\frac{1}{32}a} = .000\ 000\ 147\ 127\ 6$$

$$\frac{1}{4^7}\,\frac{1-\text{cof}\,\frac{1}{64}a}{1+\text{cof}\,\frac{1}{64}a} = .000\ 000\ 009\ 192\ 7$$

$S < .000\ 000\ 000\ 612\ 8$, $S > .000\ 000\ 000\ 612\ 6$ } Hence $S = .000\ 000\ 000\ 612\ 7$

Amount of negative terms, $.011\ 381\ 932\ 097\ 2$

Difference between the pofitive and negative terms, } or $\frac{1}{a^2} = .405\ 284\ 734\ 569\ 5$

$$\frac{1}{a} = .636\ 619\ 772\ 367\ 7$$

Arch of 90°, or $a = 1.570\ 796\ 326\ 795.$

EXAMPLE

EXAMPLE III. The length of an arch of 90°, calculated from the fourth ſeries, (Art. 28.).

$\text{coſ}\, a = 0$ $\qquad$ $\text{coſ}\tfrac{1}{8} a = 0.980\,785\,280\ldots$

$\text{coſ}\tfrac{1}{2} a = 0.707\,106\,781\,186\,5$ $\qquad$ $\text{coſ}\tfrac{1}{16} a = 0.995\,184\,7\ldots$

$\text{coſ}\tfrac{1}{4} a = 0.923\,879\,532\,51\ldots$ $\qquad$ $\text{coſ}\tfrac{1}{32} a = 0.998\,80\ldots$

$$\frac{1}{3.16^2}\,\frac{13 - \text{coſ}\,a + 12\,\text{coſ}\tfrac{1}{2} a}{3 + \text{coſ}\,a - 4\,\text{coſ}\tfrac{1}{2} a} = .163\,053\,902\,010\,8$$

$$\frac{7}{8.8.9.10} = .001\,215\,277\,777\,8$$

Amount of poſitive terms, $\quad .164\,269\,179\,788\,6$

$$\frac{1}{3.16^3}\,\frac{13 - \text{coſ}\tfrac{1}{2} a - 12\,\text{coſ}\tfrac{1}{4} a}{3 + \text{coſ}\tfrac{1}{2} a + 4\,\text{coſ}\tfrac{1}{4} a} = .000\,013\,261\,796\,5$$

$$\frac{1}{3.16^4}\,\frac{13 - \text{coſ}\tfrac{1}{4} a - 12\,\text{coſ}\tfrac{1}{8} a}{3 + \text{coſ}\tfrac{1}{4} a + 4\,\text{coſ}\tfrac{1}{8} a} = .000\,000\,198\,794\,2$$

$$\frac{1}{3.16^5}\,\frac{13 - \text{coſ}\tfrac{1}{8} a - 12\,\text{coſ}\tfrac{1}{16} a}{3 + \text{coſ}\tfrac{1}{8} a + 4\,\text{coſ}\tfrac{1}{16} a} = .000\,000\,003\,074\,4$$

$$\frac{1}{3.16^6}\,\frac{13 - \text{coſ}\tfrac{1}{16} a - 12\,\text{coſ}\tfrac{1}{32} a}{3 + \text{coſ}\tfrac{1}{16} a + 4\,\text{coſ}\tfrac{1}{32} a} = .000\,000\,000\,047\,8$$

Each of the remaining terms, being nearly $\tfrac{1}{64}$th of the term before it, their ſum will be nearly $\tfrac{1}{63}$ of the laſt term, or $\quad .000\,000\,000\,000\,8$

Amount of negative terms, $\quad .000\,013\,463\,713\,7$

Difference between the poſitive and negative terms, or $\quad \frac{1}{a^4} = .164\,255\,716\,074\,9$

$$\frac{1}{a^2} = .405\,284\,734\,569\,3$$

$$\frac{1}{a} = .636\,619\,772\,367\,6$$

Arch of 90°, or $a = 1.570\,796\,326\,795.$

EXAMPLE IV. The reciprocal of NAPIER's logarithm of 10, (which is the *modulus* of the common system), calculated by the second series for logarithms. (See Art. 63.).

$x = 10$, and hence $X = 5.05$	$X^{IV} = 1.010\ 373\ 154\ 20\ldots$
$X' = 1.739\ 252\ 713\ 092\ 7$	$X^{V} = 1.002\ 589\ 934\ 6\ldots$
$X'' = 1.170\ 310\ 367\ 614\ 6$	$X^{VI} = 1.000\ 647\ 274\ \ldots$
$X''' = 1.041\ 707\ 820\ 748.$	$X^{VII} = 1.000\ 161\ 805\ \ldots$

$$\frac{x}{(x-1)^2} = .123\ 456\ 790\ 123\ 5$$

$$\tfrac{1}{12} = .083\ 333\ 333\ 333\ 3$$

Sum of positive terms, $.206\ 790\ 123\ 456\ 8$

$$\frac{1}{4^2}\,\frac{X'-1}{X'+1} = .016\ 867\ 116\ 475\ 8$$

$$\frac{1}{4^3}\,\frac{X''-1}{X''+1} = .001\ 226\ 137\ 760\ 6$$

$$\frac{1}{4^4}\,\frac{X'''-1}{X'''+1} = .000\ 079\ 796\ 518\ 0$$

$$\frac{1}{4^5}\,\frac{X^{IV}-1}{X^{IV}+1} = .000\ 005\ 038\ 882\ 6$$

$$\frac{1}{4^6}\,\frac{X^{V}-1}{X^{V}+1} = .000\ 000\ 315\ 745\ 2$$

$$\frac{1}{4^7}\,\frac{X^{VI}-1}{X^{VI}+1} = .000\ 000\ 019\ 746\ 9$$

$$\frac{1}{4^8}\,\frac{X^{VII}-1}{X^{VII}+1} = .000\ 000\ 001\ 234\ 4$$

$$\left.\begin{array}{l} R > .0000000000822,9 \\ R < .0000000000823,1 \end{array}\right\}\quad R = .000\ 000\ 000\ 082\ 3$$

Sum of negative terms, $.018\ 178\ 426\ 445\ 8$

Difference of the positive and negative terms, $\dfrac{1}{\log^2 10} = .188\ 611\ 697\ 011\ 0$

$$\frac{1}{\log 10} = .434\ 294\ 481\ 903.$$

EXAMPLE

EXAMPLE V. NAPIER's logarithm of 10, calculated by the third series for logarithms. (See Art. 64.).

$x = 10,\quad X = 5.05$

$X' = 1.7392527130 93.$

$X'' = 1.1703103676 1..$

$X''' = 1.0417078207...$

$X^{IV} = 1.01037315.....$

$X^{V} = 1.0025899......$

$$\frac{x(x^2+4x+1)}{6(x-1)^4} = .035\ 817\ 710\ 714\ 8$$

$$\frac{1}{3.16^2}\ \frac{X+12X'-13}{X+4X'+3} = .001\ 121\ 093\ 421\ 4$$

$$\frac{1}{3.16^3}\ \frac{X'+12X''-13}{X'+4X''+3} = .000\ 024\ 041\ 122\ 9$$

$$\frac{1}{3.16^4}\ \frac{X''+12X'''-13}{X''+4X'''+3} = .000\ 000\ 409\ 239\ 4$$

$$\frac{1}{3.16^5}\ \frac{X'''+12X^{IV}-13}{X'''+4X^{IV}+3} = .000\ 000\ 006\ 535\ 7$$

$$\frac{1}{3.16^6}\ \frac{X^{IV}+12X^{V}-13}{X^{IV}+4X^{V}+3} = .000\ 000\ 000\ 102\ 7$$

$\frac{1}{63}$ of last term = sum of the remaining terms nearly, .000 000 000 001 6

From sum of positive terms, = .036 963 261 138 5

Subtract $\frac{1}{8.9.10}$ = .001 388 888 888 9

There remains $\frac{1}{\log^4 10}$ = .035 574 372 249 6

$$\frac{1}{\log^2 10} = .188\ 611\ 697\ 011\ 3$$

$$\frac{1}{\log 10} = .434\ 294\ 481\ 903\ .$$

EXAMPLE

Example VI. To ſhew that the ſeries inveſtigated in this Paper are applicable in every caſe, whether the number whoſe logarithm is required be large or ſmall, let it be required to calculate the common logarithm of the large prime number 1243 to ſeven decimal places, by the ſecond ſeries, (Art. 63.).

$x = 1243$ $\qquad X''' = 1.42356148$

$X = 621.50040225$ $\qquad X^{IV} = 1.10080913$

$X' = 17.64228446$ $\qquad X^{V} = 1.0248925.$

$X'' = 3.05305457$ $\qquad X^{VI} = 1.006204..$

$$\frac{x}{(x-1)^2} = .000\,805\,801$$

$$\tfrac{1}{12} = .083\,333\,333$$

Sum of poſitive terms, $.084\,139\,134$

$$\frac{1}{4^2}\,\frac{X'-1}{X'+1} = .055\,794\,813$$

$$\frac{1}{4^3}\,\frac{X''-1}{X''+1} = .007\,914\,766$$

$$\frac{1}{4^4}\,\frac{X'''-1}{X'''+1} = .000\,682\,688$$

$$\frac{1}{4^5}\,\frac{X^{IV}-1}{X^{IV}+1} = .000\,046\,861$$

$$\frac{1}{4^6}\,\frac{X^{V}-1}{X^{V}+1} = .000\,003\,001$$

$$\frac{1}{4^7}\,\frac{X^{VI}-1}{X^{VI}+1} = .000\,000\,189$$

$$\left.\begin{matrix} R > .000000012{,}6 \\ R < .000000012{,}7 \end{matrix}\right\} \quad R = .000\,000\,013$$

Sum of negative terms, $.064\,442\,331$

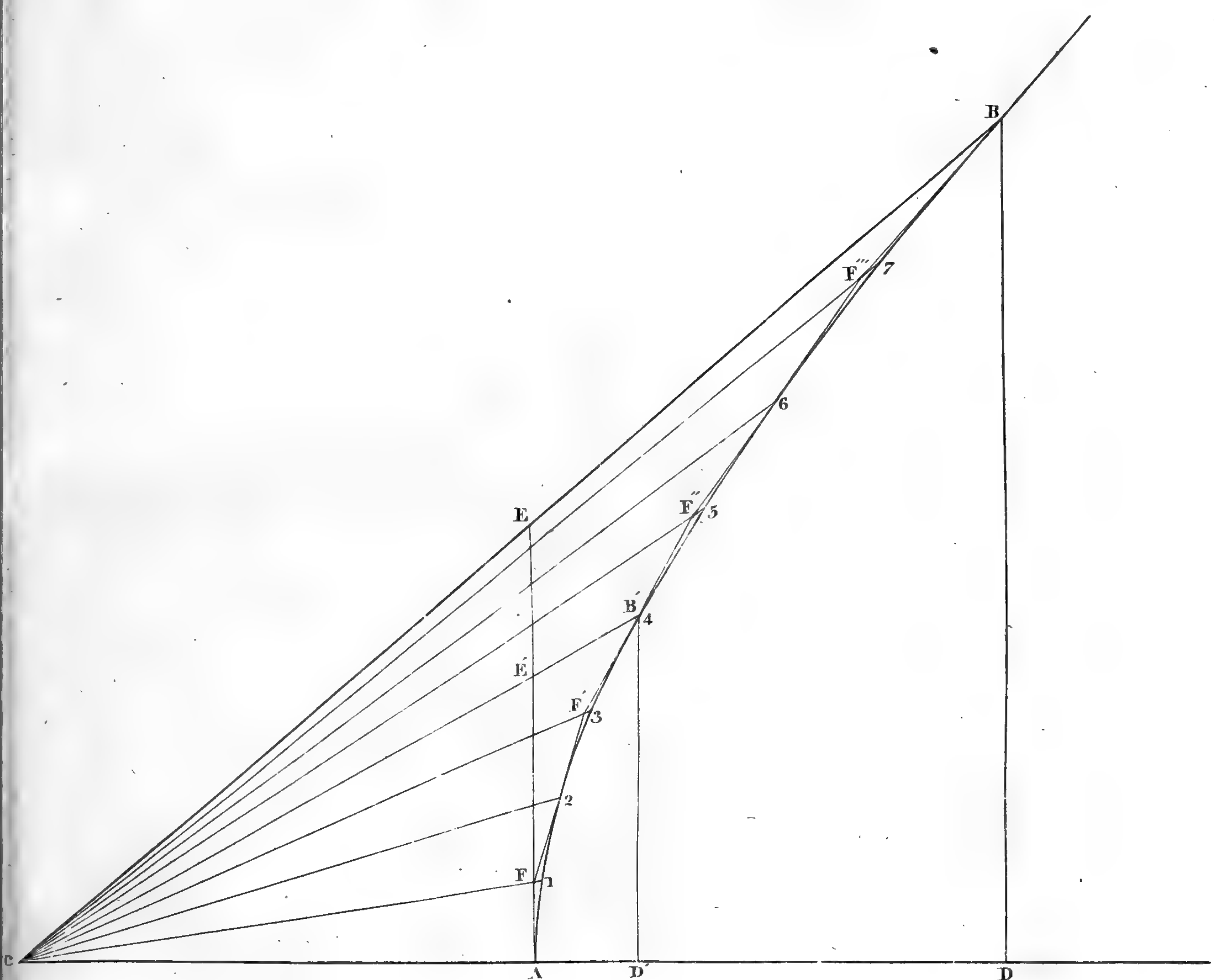
$$\frac{1}{\text{Nap. log}^2 x} = .019\,696\,803$$

$$\frac{1}{\text{Nap. log } x} = .140\,345\,300$$

$$\text{Common log of } 1243 = \frac{.434294482}{.140345300} = 3.094\,471\,1..$$

No. IX.

PLATE IX.



IX. Remarks *on a* Mineral *from* Greenland, *supposed to be* Crystallised Gadolinite. *By* Thomas Allan, Esq. F. R. S. Ed.

[*Read* 21*st November* 1808].

AMONG a parcel of minerals which I procured laſt ſpring, there are ſpecimens of two very rare foſſils; one of them, the Cryolite, the other I believe a variety of the Gadolinite. The former, is accurately deſcribed in the different mineralogical works, and I have little to add to the information contained in them. But the Gadolinite appears to be very imperfectly known, and has never yet been deſcribed as a cryſtalliſed foſſil.

The minerals in queſtion were found on board a Daniſh prize, captured on her paſſage from Iceland to Copenhagen, and were ſold with the reſt of her cargo at Leith. On examination, I was ſurpriſed to find they correſponded ſo little with the foſſils which are uſually brought from that iſland, and conſequently endeavoured to trace from the ſhip's papers, any particulars that might lead to the knowledge of their geographic origin. All I could learn was, that they were ſent from Davis' Straits by a Miſſionary.

I consider this limited information, however, ſufficient to fix on the coaſt of Greenland as the place from whence they had

had been brought; the only Cryolite known in Europe having been ſent by a Miſſionary from Greenland to Copenhagen.

THE Gadolinite, from its extreme ſcarcity, is a mineral to be found in very few cabinets; and when this collection fell into my hands, was one of thoſe I knew only by deſcription. I was led to ſuſpect that ſome of the minerals in this parcel belonged to that ſpecies, by obſerving, imbedded in a piece of granite, ſome ſmall ſhapeleſs maſſes, whoſe external characters appeared to correſpond entirely with thoſe aſſigned to the gadolinite; but on reference to the mineralogical works which treat of this ſtone, I found more difficulty than could have been ſuppoſed in aſcertaining whether they did ſo or not. The inveſtigation, however, furniſhed a ſtrong proof of the ſuperiority of chemical teſt over external character; for although the ſhape, luſtre, fracture, and geognoſtic relations, left me ſcarcely any room to doubt, yet on applying the blow-pipe and acids, it was quite evident, that the ſtone I firſt tried could not be gadolinite. I examined with great care the reſt of the parcel, and picked out ſeveral, which, though very different, reſembled in various reſpects the one that originally attracted attention; and with a view to ſatisfy myſelf, I ſent duplicates to a friend in London, from whom I learnt, that one of thoſe which I ſuppoſed to be gadolinite was certainly that mineral. Notwithſtanding the very reſpectable authority I had obtained, to which I was inclined to pay the utmoſt deference, it was not till after minute and repeated inveſtigations that I found myſelf diſpoſed to ſubmit to it; the phyſical characters of the ſpecimen in queſtion differed ſo very widely from thoſe I was taught to expect.

IT is more than twenty years ſince the gadolinite was firſt obſerved by M. ARRHENIUS, in an old quarry at Roſlagie, near Ytterby in Sweden. It was deſcribed by Mr GEYER, and by him conſidered as a black zeolite.

IN

In 1794, M. Gadolin analysed it, and found that it contained 38 *per cent.* of an unknown earth, whose properties approached to alumine in some respects, and to calcareous earth in others; but that it essentially differed from both, as well as from every other known earth.

In 1797 M. Ekeberg repeated the analysis of M. Gadolin, and obtained $47\frac{1}{2}$ *per cent.* of the new earth. This increase of quantity he attributed to the greater purity of the specimens he submitted to experiment, and in consequence of having confirmed the discovery of Gadolin, he called the stone after him, and gave the name of Yttria to the earth.

Analyses by Vauquelin and Klaproth have since appeared. The quantity of yttria observed by the former amounted only to 35 *per cent.*; but the latter states $59\frac{3}{4}$ *per cent.*

The small portions of this mineral, which, from its rarity, it is natural to conclude were at the disposal of these celebrated chemists, may in some measure account for the diversity of their results; but it is likewise by no means impossible, that the mineral itself may have varied in the proportions of its chemical ingredients.

The difference which we find in the mineralogical descriptions of this fossil, hitherto only found in one spot, is much more difficult to account for. If the information I have otherwise obtained be correct, of which I have not the slightest doubt, we cannot help attributing a certain degree of carelessness to some of the authors, particularly the French writers, who have such opportunities at command *, of investigating every point relative to natural history. The great veneration

* Lucas notes the Gadolinite as one of the minerals in the collection at the *Jardin de Plantes.*

they entertain for the talents and accuracy of the celebrated HAÜY, may induce them to think his obſervations require no concurring teſtimony ; and, on the other hand, the pupils of the German School, conſider no mineral deſerving a place in their ſyſtem, till it has been examined and claſſed by their illuſtrious maſter, whoſe authority will be handed down by them with equal reſpect to poſterity.

IT is unneceſſary to occupy the time of the Society, in giving a comparative view of the different deſcriptions of the Gadolinite. I ſhall only notice a few prominent features.

IT is deſcribed by every one of the authors, as poſſeſſing a ſpecific gravity of upwards of 4, and as acting powerfully upon the magnet. This laſt character is noticed by Profeſſor JAMESON, in the firſt account he gives of the gadolinite ; but in the ſecond it is omitted, along with ſome others. KLAPROTH takes no notice of its magnetic power, but ſtates the ſpecific gravity at 4.237.

THE French writers deſcribe the colour as black and reddiſh black. The German as raven or greeniſh black. Theſe variations, with ſeveral others which may be obſerved on referring to the different authors, ſhew that ſome incorrectneſs muſt exiſt. But the moſt remarkable of all is, that the gadolinite, if *ever* magnetic, is not always ſo ; for the ſpecimens in the poſſeſſion of the COUNT DE BOURNON are not, nor, as he informs me, are any that he has ever ſeen. It is therefore reaſonable to conclude, that magnetiſm in the gadolinite may depend on accidental cauſes.

THE following is the deſcription of the foſſil, which I ſuppoſe to be that ſubſtance in a cryſtalliſed ſtate ; although nothing ſhort of analyſis can afford indiſputable teſtimony of the identity of any mineral ſo little known.

SPECIFIC

Specific Gravity, 3.4802. The ſpecimen weighed 1136.39 grains. Its ſurface is a little decompoſed, and it has alſo ſome minute particles of felſpar intermixed with it; both of which would affect the reſult in ſome degree; but neither were of ſuch amount as to do ſo in any conſiderable degree.

Hardness: ſufficient to reſiſt ſteel, and ſcratch glaſs, but not quartz.

Lustre: ſhining, approaching to reſinous.

Fracture: uneven, verging to flat conchoidal.

Colour: pitch black which I conſider velvet black with a ſhade of brown; when pounded, of a greeniſh grey colour.

Figure: it occurs cryſtalliſed. The ſimpleſt figure, and perhaps the primitive form, is a rhomboidal priſm, whoſe planes meet under angles of 120° and 60°. In ſome of the ſpecimens, the acute angle is replaced by one face, in others by two, thereby forming ſix and eight ſided priſms. All the ſpecimens I poſſeſs are only fragments of cryſtals, none of which retain any part of a termination. They occur imbedded in felſpar, probably granite.

Chemical Characters: before the blow-pipe froths up, and melts but only partially, leaving a brown ſcoria; with borax it melts into a black glaſs. When pounded, and heated in diluted nitric acid, it tinges the liquid of a ſtraw colour; and, ſome time after cooling, gelatinates.

The principal diſtinguiſhing character of the gadolinite, is its forming a jelly with acid, a character belonging to few other minerals. The Mezotype Lazulite, Apophilite, Ædelite, and Oxide of Zinc, ſo far as I know, alone poſſeſs the ſame quality; and it cannot eaſily be miſtaken for any of them.

It has not the ſmalleſt attraction for the magnet; it does not decrepitate and diſperſe when expoſed to the blow-pipe; it is not in any ſhape tranſparent.

The Swediſh foſſil occurs in roundiſh amorphous maſſes, imbedded and diſſeminated in a granitic rock, having the external ſurfaces covered with a ſlight whitiſh coating, perhaps from the attachment of micaceous particles. There is no ſuch appearance on the ſurface of the cryſtalliſed gadolinite.

The ſituation which this mineral ſhould hold in the ſyſtem has been a matter of difficulty among mineralogiſts. Haüy has placed it in the claſs of Earthy Foſſils, immediately after his Anataſe and Dioptaſe,—rather an unfortunate ſituation, both theſe having been recogniſed as ores of known metals, titanium and copper, ſince the publication of his admirable treatiſe.

Werner, on account of its weight, has claſſed it among the metals; and from its natural alliances, and chemical compoſition, has given it a place among the irons *. If weight entitled it to be claſſed among the metals, ſeveral other minerals have an equal claim to the ſame ſituation. Of its natural alliances we know very little, farther than that the Swediſh diſtrict where it is found abounds in iron; and as to its chemical compoſition, if 17½ *per cent.* of iron be ſufficient to counterbalance 59¾ths of a new earth, it would be right to arrange it accordingly. The analyſes of ſo many chemiſts of known celebrity, are certainly ſufficient to juſtify the conſtitution of a new ſpecies for its reception. Werner, however, may feel himſelf licenſed in this arrangement, as he does not conſider it neceſſary that a mineral compound ſhall preſerve the characters of its components; but that any of the components may give to the compoſition characters ſufficiently marked, to determine its relations. It is upon this diſtinction that he founds the difference between the predominant and characteriſtic principles †.

The

* Jameson, vol. ii. p. 613.
† Brochant, vol. i. p. 44.

The arrangement of Brongniart appears much more judicious; he has placed it at the commencement of the Earthy Minerals, and aſſigns as a reaſon, that it is unique in its compoſition; and if placed in any other ſituation, it would interrupt the ſeries, either in reſpect to its compoſition or external characters.

Of the Cryolite I have very little to obſerve, in addition to the deſcriptions given in the different mineralogical works. The ſpecific gravity I found to be 2.961; Haüy ſtates it at 2.949. Among the various maſſes I examined, there was no trace of cryſtallization, farther than the cleavage, which is threefold, and nearly at right angles. The maſſes broke in two directions, (which may be ſuppoſed the ſides of the priſm), with great facility, leaving a very ſmooth ſurface; but the tranſverſe cleavage was more difficult, and by no means ſo ſmooth. Several of the ſpecimens being mixed with galena, pyrites, and cryſtals of ſparry iron-ore, it would appear that the cryolite is a vein-ſtone; but I was not ſo fortunate as to find any of it attached to a rock ſpecimen, ſo as to throw light on its geognoſtic relations.

X. *On the Progress of Heat when communicated to Spherical Bodies from their Centres.* *By* JOHN PLAYFAIR, F. R. S. LOND. *Sec*r. R. S. EDIN. *and Professor of Natural Philosophy in the University of Edinburgh.*

[*Read March* 6. 1809.]

1. AN argument againſt the hypotheſis of central heat has been ſtated by an ingenious author as carrying with it the evidence of demonſtration.

"THE eſſential and characteriſtic property of the power "producing heat, is its tendency to exiſt every where in a ſtate "of equilibrium, and it cannot hence be preſerved without loſs "or without diffuſion, in an accumulated ſtate. In the theory "of HUTTON, the exiſtence of an intenſe local heat, acting for "a long period of time, is aſſumed. But it is impoſſible to pro-"cure caloric in an inſulated ſtate. Waving every objection "to its production, and ſuppoſing it to be generated to any ex-"tent, it cannot be continued, but muſt be propagated to the "contiguous matter. If a heat, therefore, exiſted in the cen-"tral region of the earth, it muſt be diffuſed over the whole "maſs; nor can any arrangement effectually counteract this "diffuſion. It may take place ſlowly, but it muſt always con-"tinue progreſſive, and muſt be utterly ſubverſive of that ſy-"ſtem of indefinitely renewed operations which is repre-

"sented as the grand excellence of the Huttonian Theory *." " Again, he obſerves, in giving what he ſays appears to him a demonſtration of the fallacy of the firſt principles of the Huttonian Syſtem, " it will not be diſputed, that the tendency of " caloric is to diffuſe itſelf over matter, till a common tempe- " rature is eſtabliſhed. Nor will it probably be denied, that a " power conſtantly diffuſing itſelf from the centre of any maſs " of matter, cannot remain for an indefinite time locally accu- " mulated in that maſs, but muſt at length become equal or " nearly ſo over the whole †."

2. I muſt confeſs, notwithſtanding the reſpect I entertain for the acuteneſs and accuracy of the author of this reaſoning, that it does not appear to me to poſſeſs the force which he aſcribes to it; nor to be conſiſtent with many facts that fall every day under our obſervation. A fire ſoon heats a room to a certain degree, and though kept up ever ſo long, if its intenſity, and all other circumſtances remain the ſame, the heat continues very unequally diſtributed through the room; but the temperature of every part continues invariable. If a bar of iron has one end of it thruſt into the fire, the other end will not in any length of time become red-hot; but the whole bar will quickly come into ſuch a ſtate, that every point will have a fixed temperature, lower as it is farther from the fire, but remaining invariable while the condition of the fire, and of the medium that ſurrounds the bar, continues the ſame. The reaſon indeed is plain: the equilibrium of heat is not ſo much a primary law in the diſtribution of that fluid, as the limitation of another law which is general and ultimate, conſiſting in the tendency of heat to paſs with a greater or a leſs velocity, according to circumſtances, from bodies where the temperature is

* MURRAY's *Syſtem of Chemistry*, vol. iii. Appendix, p. 49.

† Page 51.

is higher, to thoſe where it is lower, or from thoſe which contain more heat, according to the indication of the thermometer, to thoſe which contain leſs. It is of this general tendency, that the equilibrium or uniform diſtribution of heat is a conſequence,—but a conſequence only contingent, requiring the preſence of another condition, which may be wanting, and actually is wanting, in many inſtances. This condition is no other, than that the quantity of heat in the ſyſtem ſhould be given, and ſhould not admit of continual increaſe from one quarter, nor diminution from another. When ſuch increaſe and diminution take place, what is uſually called "the equilibrium of heat" no longer exiſts. Thus, if we expoſe a thermometer to the ſun's rays, it immediately riſes, and continues to ſtand above the temperature of the ſurrounding air. The way in which this happens is perfectly underſtood: the mercury in the thermometer receives more heat from the ſolar rays than the air does; it begins therefore to riſe as ſoon as thoſe rays fall on it; at the ſame time, it gives out a portion of its heat to the air, and always the more, the higher it riſes. It continues to riſe, therefore, till the heat which it gives out every inſtant to the air, be equal to that which it receives every inſtant from the ſolar rays. When this happens, its temperature becomes ſtationary; the momentary increment and decrement of the heat are the ſame, and the total, of courſe, continues conſtant. The thermometer, therefore, in ſuch circumſtances, never acquires the temperature of the ſurrounding air; and the only equilibrium of the heat, is that which ſubſiſts between the increments and the decrements juſt mentioned: theſe indeed are, ſtrictly ſpeaking, *in equilibrio*, as they accurately balance one another. This ſpecies of equilibrium, however, is quite different from what is implied in the uniform diffuſion of heat.

3. In order to ſtate the argument more generally, let A, B, C, D, &c. be a ſeries of contiguous bodies; or let them be parts of the ſame body; and let us ſuppoſe that A receives, from ſome cauſe, into the nature of which we are not here to inquire, a conſtant and uniform ſupply of heat. It is plain, that heat will flow continually from A to B, from B to C, &c.; and in order that this may take place, A muſt be hotter than B, B than C, and ſo on; ſo that no uniform diſtribution of heat can ever take place. The ſtate, however, to which the ſyſtem will tend, and at which, after a certain time, it muſt arrive, is one in which the momentary increaſe of the heat of each body is juſt equal to its momentary decreaſe; ſo that the temperature of each individual body becomes fixed, all theſe temperatures together forming a ſeries decreaſing from A downwards. To be convinced that this is the ſtate which the ſyſtem muſt aſſume, ſuppoſe any body D, by ſome means or other, to get more heat than that which is required to make the portion of heat which it receives every moment from C, juſt equal to that which it gives out every moment to E; as its exceſs of temperature above E is increaſed, it will give out more heat to E, and as the exceſs of the temperature of C above that of D is diminiſhed, D will receive leſs heat from C; therefore, for both reaſons, D muſt become colder, and there will be no ſtop to the reduction of its temperature, till the increments and decrements become equal as before.

4. If, therefore, heat be communicated to a ſolid maſs, like the earth, from ſome ſource or reſervoir in its interior, it muſt go off from the centre on all ſides, toward the circumference. On arriving at the circumference, if it were hindered from proceeding farther, and if ſpace or vacuity preſented to heat an impenetrable barrier, then an accumulation of it at the ſurface, and at laſt a uniform diſtribution of it through the whole maſs, would inevitably be the conſequence. But if heat

heat may be loſt and diſſipated in the boundleſs fields of vacuity, or of ether, which ſurround the earth, no ſuch equilibrium can be eſtabliſhed. The temperature of the earth will then continue to augment only, till the heat which iſſues from it every moment into the ſurrounding medium, become equal to the increaſe which it receives every moment from the ſuppoſed central reſervoir. When this happens, the temperature at the ſuperficies can undergo no farther change, and a ſimilar effect muſt take place with reſpect to every one of the ſpherical and concentric ſtrata into which we may conceive the ſolid maſs of the globe to be divided. Each of theſe muſt in time come to a temperature, at which it will give out as much heat to the contiguous ſtratum on the outſide, as it receives from the contiguous ſtratum on the inſide; and, when this happens, its temperature will remain invariable.

5. THAT we may trace this progreſs with more accuracy, let us ſuppoſe a ſpherical body to be heated from a ſource of heat at its centre; and let h, h', h'', be the temperatures at the ſurfaces of two contiguous and concentric ſtrata, the diſtances from the centre being x, x', x''; and let it alſo be ſuppoſed, that the thickneſs of each of the ſtrata, to wit, $x'—x$, and $x''—x'$, is very ſmall.

THEN ſuppoſing the body to be homogeneous, the quantity of heat that flows from the inner ſtratum into the outward, in a given time, will be proportional to the exceſs of its temperature above that of the outward ſtratum multiplied into its quantity of matter, that is, to $(h - h')(x'^3 - x^3)$.

6. IN

6. IN the ſame manner, the heat which goes off from the ſecond ſtratum in the ſame time, is proportional to $(h'-h'')$ $(x''^3-x'^3)$; and theſe two quantities, when the temperature of the ſecond ſtratum becomes conſtant, muſt be equal to one another, or $(h-h')(x'^3-x^3)=(h'-h'')(x''^3-x'^3)$.

BUT becauſe $h-h'$, and $x'-x$ are indefinitely ſmall, $h-h'=\dot{h}$, and $x'^3-x^3=3x^2\dot{x}$; therefore $\dot{h}\times 3x^2\dot{x}=$ a given quantity; which quantity, ſince $\dot{x}$ is given, we may repreſent by $a^2\dot{x}^2$; ſo that $\dot{h}=\frac{a^2\dot{x}^2}{3x^2\dot{x}}=\frac{a^2\dot{x}}{3x^2}$, or, becauſe $\dot{h}$ is negative in reſpect of $\dot{x}$, being a deccrement, while the latter is an increment, $\dot{h}=-\frac{a^2\dot{x}}{3x^2}$, and therefore $h=C+\frac{a^2}{3x}$.

7. To determine the conſtant quantity C, let us ſuppoſe that the temperature at the ſurface of the internal *nucleus* of ignited matter is $=$ H, and $r=$ radius of that nucleus. Then, in the particular caſe, when $x=r$ and $h=H$, the preceding equation gives $H=C+\frac{a^2}{3r}$; ſo that $C=H-\frac{a^2}{3r}$, and conſequently $h=H-\frac{a^2}{3r}+\frac{a^2}{3x}$; or $h=H+\frac{a^2}{3}\left(\frac{1}{x}-\frac{1}{r}\right)$.

8. IT is evident, from this formula, that for every value of x there is a determinate value of h, or that for every diſtance from the centre there is a fixed temperature, which, after a certain time, muſt be acquired, and will remain invariable as long

long as the intenſity and magnitude of the central fire continue the ſame.

9. It remains for us to determine the value of a^2, which, though conſtant, is not yet given, or known from obſervation.

At the ſurface of the globe we may ſuppoſe the mean temperature to be known: let T be that temperature, and let R = the radius of the globe. Then, when $x = R$, $h = T$, and by ſubſtituting in the general formula, we have $T = H + \frac{a^2}{3}\left(\frac{1}{R} - \frac{1}{r}\right)$,

and $a^2 = \frac{3(T - H)}{\frac{1}{R} - \frac{1}{r}} = \frac{3Rr(H - T)}{R - r}$.

$$\text{Thus } h = H + \frac{Rr(H - T)}{R - r}\left(\frac{1}{x} - \frac{1}{r}\right)$$

$$= H - \frac{Rr(H - T)}{R - r}\left(\frac{1}{r} - \frac{1}{x}\right).$$

Hence alſo by reduction

$$h = \frac{RT - rH}{R - r} + \frac{Rr(H - T)}{x(R - r)},$$

$$\text{or } h = \frac{1}{R - r}(RT - rH) + \frac{Rr(H - T)}{x}.$$

From this equation, it is evident, that $h - \frac{RT - rH}{R - r}$, or the exceſs of the temperature at any diſtance x from the centre, above a certain given temperature, is inverſely as x. But the conſtruction of the hyperbola which is the locus of the preceding

preceding equation, will exhibit the relation between the temperature and the diſtance, in the way of all others leaſt ſubject to miſapprehenſion.

Let the circle (Plate X. fig. 3.) deſcribed with the radius A B, repreſent the globe of the earth; and the circle deſcribed with the radius A H an ignited maſs at the centre. Let H K, perpendicular to A B, be the temperature at H, the ſurface of the ignited maſs; and let F D be the temperature at any point whatever, in the interior of the earth, B M repreſenting that at the ſurface. Then A B being $=$ R in the preceding equation, A H $= r$, H K $=$ H, B M $=$ T; A F $= x$, and F D $= h$, theſe two laſt being variable quantities; ſince

$\left(h - \frac{RT - rH}{R - r}\right) x = \frac{Rr(H - T)}{R - r}$ we have, (taking A E

$= \frac{RT - rH}{R - r}$, and drawing E L parallel to A B, meeting H K

in N, and F D in O,) $OD \times OE = \frac{BA . AH (HK - BM)}{BH}$,

which is a given quantity.

Therefore D is in a rectangular hyperbola, of which the centre is E, the aſſymptotes E G and E L, and the rectangle of the coordinates, equal to $BA . AH \times \frac{HK - BM}{BH}$, or, which amounts

to the ſame, to K N . N E.

It is evident from this, that if the ſphere were indefinitely extended, the temperature at the point B and all other things remaining the ſame, the temperature at its ſuperfices would not

be leſs than A E, or than the quantity $\frac{RT - rH}{R - r}$.

The

THE quantity AE, or $\frac{RT - rH}{R - r}$, is ſuppoſed here to be ſubtracted; if RT be leſs than r H, it will change its ſign, and muſt be taken on the other ſide of the centre A.

10. THE reſults of theſe deductions may be eaſily repreſented numerically, and reduced into tables, for any particular values that may be aſſigned to the conſtant quantities. Thus, if the radius of the globe, or $R = 100$, that of the ignited nucleus or $r = 1$; the temperature of the nucleus, or $H = 1000$, and T the temperature at the ſurface $= 60$, the formula becomes $h = 50.505 + \frac{949.494}{x}$.

Values of x	Values of h
10	145°.454
20	98 .423
30	82 .599
40	74 .686
50	69 .938
60	66 .330
70	63 .926
80	62 .361
90	61 .055
100	60.

11. OTHER

11. OTHER things remaining as before, if we now make $r = 10$, then $h = - 44.444 + \frac{10444}{x}$

x	h
20	477°.556
30	303 .556
40	226 .556
50	164 .556
60	145 .556
70	104 .556
80	85 .056
90	64 .556
100	60 .000

12. IF $R = 10$, $r = 1$, $H = 10000$, and $T = 60$,

$$h = - 1044.44 + \frac{11044.44}{x}$$

Values of x	Values of h
1	10000°.00
2	4477 .70
3	2637 .04
4	1716 .67
5	1164 .44
6	796 .30
7	533 .33
8	346 .16
9	182 .72
10	60 .00

13. THE

13. 1. The general conclufions which refult from all this are, that when we fuppofe an ignited nucleus of a given magnitude, and a given intenfity of heat, there is in the fphere to which it communicates heat a fixed temperature for each particular ftratum, or for each fpherical fhell, at a given diftance from the centre; and that a great intenfity of heat in the interior, is compatible with a very moderate temperature at the furface.

2. However great the fphere may be, the heat at its furface cannot be lefs than a given quantity; R, r, H and T remaining the fame. It muft be obferved, that though R is put for the radius of the globe; it fignifies in fact nothing, but the diftance at which the temperature is T, as r does the diftance at which the temperature is H.

Therefore were the fphere indefinitely extended, the temperature at its fuperficies would not be lefs than the quantity $\frac{RT - rH}{R - r}$, that is, not lefs than 50.5 in the firft of the preceding examples, than -44.4 in the fecond, or -1044.4 in the third.

14. In all this the fphere is fuppofed homogeneous; but if it be otherwife, and vary in denfity, in the capacity of the parts for heat, or in their power to conduct heat, providing it do fo as any function of the diftance from the centre, the calculus may be inftituted as above. For example, let the denfity be fuppofed to vary as $\frac{b}{b + x}$, then we have as before $(b - b')\,(x'^3 - x^3)\,\frac{b}{b + x}$ for the momentary increment of heat in a ftratum placed at the diftance x from the centre,

 or

or $\dot{h} \times 3 x^2 \dot{x} \times \frac{b}{b+x}$ = to a given quantity, or to $a^2 \dot{x}^2$, and therefore $\dot{h} = -\frac{a^2 (b+x) \dot{x}}{3 b x^2} = -\frac{a^2 \dot{x}}{3 x^2} - \frac{a^2 \dot{x}}{b x}$. Hence $h =$ $C + \frac{a^2}{3x} - \frac{a^2}{b}$ Log x. Suppoſe that when $x = r$, the radius of the heated nucleus, $h = H$; then $H =$

$$C + \frac{a^2}{3r} - \frac{a^2}{b} \text{Log } r, \text{ and } C =$$

$$H - \frac{a^2}{3r} + \frac{a^2}{b} \text{Log } r; \text{ therefore } h =$$

$$H - \frac{a^2}{3r} + \frac{a^2}{3x} + \frac{a^2}{b} \text{Log } \frac{r}{x}.$$

In this expreſſion a^2 will be determined, if the temperature at any other diſtance R from the centre is known. Let this be T; then by ſubſtitution we have

$$T = H - \frac{a^2}{3r} + \frac{a^2}{3R} + \frac{a^2}{b} \text{Log } \frac{r}{R},$$

$$\text{and } a^2 = \frac{T - H}{\frac{1}{3R} - \frac{1}{3r} + \frac{1}{b} \text{Log } \frac{r}{R}}.$$

$$\text{Hence } h = H + \left(\frac{T - H}{\frac{1}{3R} - \frac{1}{3r} + \frac{1}{b} \text{Log } \frac{r}{R}} \right) \times$$

$$\left(-\frac{1}{3r} + \frac{1}{3x} + \frac{1}{b} \text{Log } \frac{r}{x} \right).$$

15. THIS

15. This is given merely as an example of the method of conducting the calculus when the variation of the denfity is taken into account, and not becaufe there is reafon to believe that the law which that variation actually follows, is the fame that has now been hypothetically affumed.

16. The principle on which we have proceeded, applies not only to folids, fuch as we fuppofe the interior of the earth, but it applies alfo to fluids like the atmofphere, provided they are fuppofed to have reached a fteady temperature. The propagation of heat through fluids is indeed carried on by a law very different from that which takes place with refpect to folids; it is not by the motion of heat, but by the motion of the parts of the fluid itfelf. Yet, when we are feeking only the mean refult, we may fuppofe the heat to be fo diffufed, that it does not accumulate in any particular ftratum, but is limited by the equality of the momentary increments and decrements of temperature which that ftratum receives. This is conformable to experience; for we know that a conftancy, not of temperature, but of difference between the temperature of each point in the atmofphere and on the furface, actually takes place. Thus, near the furface, an elevation of 280 feet produces, in this country, a diminution of one degree. The ftrata of our atmofphere, however, differ in their capacity of heat, or in the quantity of heat contained in a given fpace, at a given temperature. Concerning the law which the change of capacity follows, we have no certain information to guide us; and we have no refource, therefore, but to affume a hypothetical law, agreeing with fuch facts as are known, and, after deducing the refults of this law, to compare them with the obfervations made on the temperature of the air, at different heights above the furface of the earth.

17. Let

17. Let us then ſuppoſe, that the ſtrata of the atmoſphere have a capacity for heat, which increaſes as the air becomes rarer, ſo as to be proportional to $m b^{-\frac{r}{x}}$, x denoting, as before, the diſtance from the centre of the earth, r the radius of the earth, m and b determinate, but unknown quantities, ſuch that $m b^{-1}$ or $\frac{m}{b}$, expreſſes the capacity of air for heat, when of its ordinary denſity, at the ſurface of the earth. The formula thus aſſumed, agrees with the extreme caſes; for, when $x = r$, the capacity of heat $= \frac{m}{b}$, a finite quantity; when x increaſes, $\frac{r}{x}$ diminiſhes, and ſo alſo does $b^{\frac{r}{x}}$, if b is greater than unity, and therefore $\frac{m}{b^{\frac{r}{x}}}$ increaſes continually. It does not, however, increaſe beyond a certain limit, for when x is infinite $\frac{m}{b^{\frac{r}{x}}}$ becomes $\frac{m}{1}$, or m.

18. Hence,

18. Hence, by reaſoning as in § 6. the momentary increment of the temperature, or ſenſible heat, of any ſtratum, is as $-\frac{a^2 \dot{x}}{3x^2}$ directly, and its capacity for heat, or $m\,b^{-\frac{r}{x}}$ inverſely,

that is, $\dot{h} = -\frac{a^2 \dot{x}}{3x^2} \times \frac{b^{\frac{r}{x}}}{m} = -\frac{a^2\, b^{\frac{r}{x}}\, \dot{x}}{3\,m\,x^2}$.

Let $\frac{r}{x} = y$, then $-\frac{r\,\dot{x}}{x^2} = \dot{y}$, ſo that $-\frac{a^2\,\dot{x}}{3mx^2} = \frac{a^2\,\dot{y}}{3\,m\,r}$, and therefore $\dot{h} = \frac{a^2\; b^y}{3\,m\,r}\dot{y}$. Hence $h = C + \frac{a^2}{3mr\,\text{Log}\,b}\,b^y = C + \frac{a^2}{3mr\,\text{Log}\,b}\,b^{\frac{r}{x}}$.

19. To determine C, if T be the temperature of the air at the ſurface, when $x = r$, $T = C + \frac{a^2\,b}{3mr\,\text{Log}\,b}$, and $C = T - \frac{a^2\,b}{3mr\,\text{Log}\,b}$.

Hence $h = T - \frac{a^2\,b}{3mr\,\text{Log}\,b} + \frac{a^2\,b^{\frac{r}{x}}}{3\,mr\,\text{Log}\,b} =$

$$T - \frac{a^2\,(b - b^{\frac{r}{x}})}{3\,m\,r\,\text{Log}\,b}.$$

This formula, when $x = r$ gives $h = T$, and when x is infinite, it gives $h = T - \frac{a^2\,(b - 1)}{3\,m\,r\,\text{Log}\,b}$. In all intermediate

caſes,

cafes, as x is greater than r, $b^{\frac{r}{x}}$ is lefs than b, (b being a number greater than 1) and therefore $b - b^{\frac{r}{x}}$ is pofitive, fo that h is lefs than T, as it ought to be.

20. We may obtain an approximate value of this formula, without exponential quantities, that will apply to all the cafes in which x and r differ but little in refpect of r, that is, in all the cafes to which our obfervations on the atmofphere can poffibly extend.

If, in the term $b^{\frac{r}{x}}$ we write $r + z$ for x, z being the height of any ftratum of air above the furface of the earth, we have $b^{\frac{r}{x}} = b^{\frac{r}{r+z}}$.

21. But, from the nature of exponentials, we know $b^{\frac{r}{x}} =$

$$1 + \frac{r}{x} \text{Log } b + \frac{r^2 (\text{Log } b)^2}{2\, x^2} + \frac{r^3 (\text{Log } b)^3}{2.3\, x^3}, \text{ \&c.} =$$

$$1 + \frac{r}{r+z} \text{Log } b + \frac{r^2 (\text{Log } b)^2}{2\,(r+z)^2} +, \text{ \&c.}$$

Now $\frac{r}{r+z} = 1 - \frac{z}{r} + \frac{z^2}{r^2} -$, &c. And if we leave out the higher powers of z, we have nearly

$\frac{r}{r+z}$

$$\frac{r}{r+z} = 1 - \frac{z}{r}$$

$$\frac{r^2}{(r+z)^2} = 1 - \frac{2z}{r}$$

$$\frac{r^3}{(r+z)^3} = 1 - \frac{3z}{r}, \text{ \&c.}$$

THEREFORE, by ſubſtitution, we have $b^{\frac{r}{r+z}} =$

$$1 + \left(1 - \frac{z}{r}\right) \text{Log}\, b + \left(1 - \frac{2z}{r}\right) \frac{(\text{Log}\, b)^2}{2} +, \text{ \&c.} =$$

$$\left\{ \begin{array}{l} 1 + \text{Log}\, b + \frac{(\text{Log}\, b)^2}{2} + \frac{(\text{Log}\, b)^3}{2 \cdot 3} +, \text{ \&c.} \\ - \frac{z}{r} \text{Log}\, b - \frac{z}{r} (\text{Log}\, b)^2 - \frac{z}{r} \frac{(\text{Log}\, b)^3}{2}, \text{ \&c.} \end{array} \right\}$$

Now, from the nature of exponentials,

$$b = 1 + \text{Log}\, b + \frac{\text{Log}\, b^2}{2} + \frac{\text{Log}\, b^3}{2 \cdot 3} +, \text{ \&c.}$$

And $\frac{z}{r} \text{Log}\, b + \frac{z}{r} (\text{Log}\, b)^2 + \frac{z}{r} \frac{(\text{Log}\, b)^3}{2}$, &c.

$$= \frac{z}{r} \text{Log}\, b \left(1 + \text{Log}\, b + \frac{(\text{Log}\, b)^2}{2} +, \text{ \&c.}\right) =$$

$\frac{z}{r} b \,\text{Log}\, b$; therefore when z is very ſmall, $b^{\frac{r}{r+z}} =$

$b - \frac{z\,b}{r}$ Log b, and therefore (§ 19.), $a^2 \frac{\left(b - b^{\frac{r}{r+z}}\right)}{3\,m\,r \,\text{Log}\, b} =$

$$a^2 \frac{\left(b - b + \frac{b\,z\,\text{Log}\,b}{r}\right)}{3\,m\,r\,\text{Log}\,b} = \frac{a^2\,b\,z}{3\,m\,r^2};$$ hence when z is very

ſmall, $h = \text{T} - \frac{a^2\,b\,z}{3\,m\,r^2}$.

22. Therefore when z, or the height above the ſurface is ſmall, h diminiſhes in the ſame proportion that the height increaſes, which is conformable to experience.

In our climate, when $z = 280$ feet, $\frac{a^2\,b \times 280}{3\,m\,r^2} = 1°$; ſo that the co-efficient $\frac{a^2\,b}{3\,m\,r^2} = \frac{1}{280}$, and therefore

$$h = \text{T} - \frac{z}{280}.$$

When the conſtant quantities are thus determined, the formula agrees nearly with obſervation. In the rule for barometrical meaſurements, it is implied, that the heat of the atmoſphere decreaſes uniformly; but the rate for each particular caſe is determined by actual obſervation, or by thermometers obſerved at the top and bottom of the height to be meaſured.

XI.

XI. *Experiments on Allanite, a new Mineral from Greenland.* *By* THOMAS THOMSON, M. D. F. R. S. E. *Fellow of the Imperial Chirurgo-Medical Academy of Petersburgh.*

[*Read Nov.* 5. 1810.]

ABOUT three years ago, a Danifh veffel * was brought into Leith as a prize. Among other articles, fhe contained a fmall collection of minerals, which were purchafed by THOMAS ALLAN, Efq; and Colonel IMRIE, both members of this Society. The country from which thefe minerals had been brought was not known for certain; but as the collection abounded in Cryolite, it was conjectured, with very confiderable probability, that they had been collected in Greenland.

AMONG the remarkable minerals in this collection, there was one, which, from its correfpondence with Gadolinite, as defcribed in the different mineralogical works, particularly attracted the attention of Mr ALLAN. Confirmed in the idea of its being a variety of that mineral, by the opinion of Count BOURNON, added to fome experiments made by Dr WOLLASTON, he was induced to give the defcription which has fince been publifhed in a preceding part of the prefent volume.

ABOUT a year ago, Mr ALLAN, who has greatly diftinguifhed himfelf by his ardent zeal for the progrefs of mineralogy in all

 its

* DER FRUHLING, Captain JACOB KETELSON, captured, on her passage from Iceland to Copenhagen.

its branches, favoured me with ſome ſpecimens of this curious mineral, and requeſted me to examine its compoſition,—a requeſt which I agreed to with pleaſure, becauſe I expected to obtain from it a quantity of *yttria*, an earth which I had been long anxious to examine, but had not been able to procure a ſufficient quantity of the Swediſh Gadolinite for my purpoſe. The object of this paper, is to communicate the reſult of my experiments to the Royal Society,—experiments which cannot appear with ſuch propriety any where as in their Tranſactions, as they already contain a paper by Mr ALLAN on the mineral in queſtion.

I. DESCRIPTION.

I AM fortunately enabled to give a fuller and more accurate deſcription of this mineral than that which formerly appeared, Mr ALLAN having, ſince that time, diſcovered an additional quantity of it, among which, he not only found freſher and better characteriſed fragments, but alſo ſome entire cryſtals. In its compoſition, it approaches moſt nearly to Cerite, but it differs from it ſo much in its external characters, that it muſt be conſidered as a diſtinct ſpecies. I have therefore taken the liberty to give it the name of *Allanite*, in honour of Mr ALLAN, to whom we are in reality indebted for the diſcovery of its peculiar nature.

ALLANITE occurs maſſive and diſſeminated, in irregular maſſes, mixed with black mica and felſpar; alſo cryſtalliſed; the varieties obſerved are,

1. A four-ſided oblique priſm, meaſuring 117° and 63°.
2. A ſix-ſided priſm, acuminated with pyramids of four ſides, ſet on the two adjoining oppoſite planes. Theſe laſt are ſo minute as to be incapable of meaſurement. But, as nearly as the eye can determine, the form reſembles *Fig.* 1.; the priſm of which has two right angles, and four meaſuring 135°.

3. A

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Fig. 1.

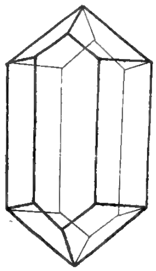
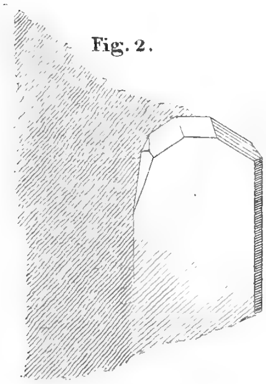


Fig. 2.



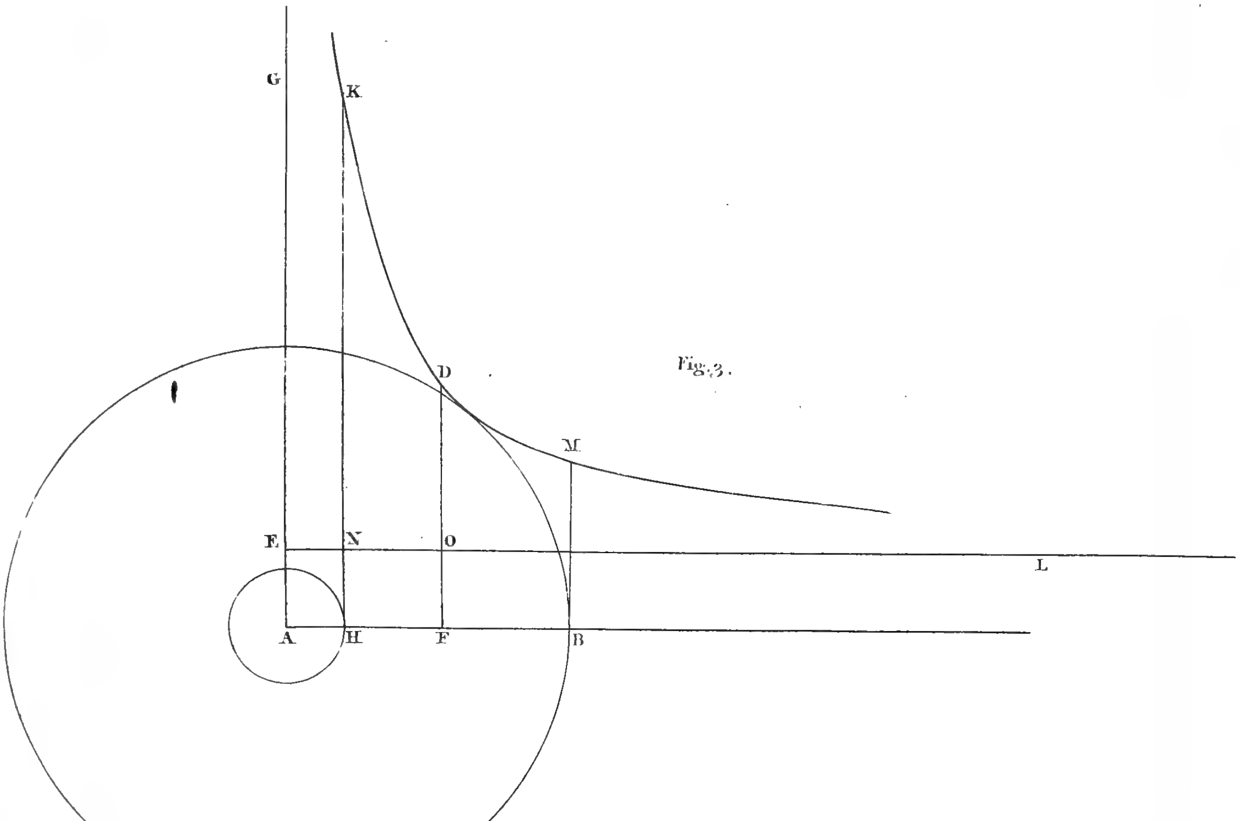
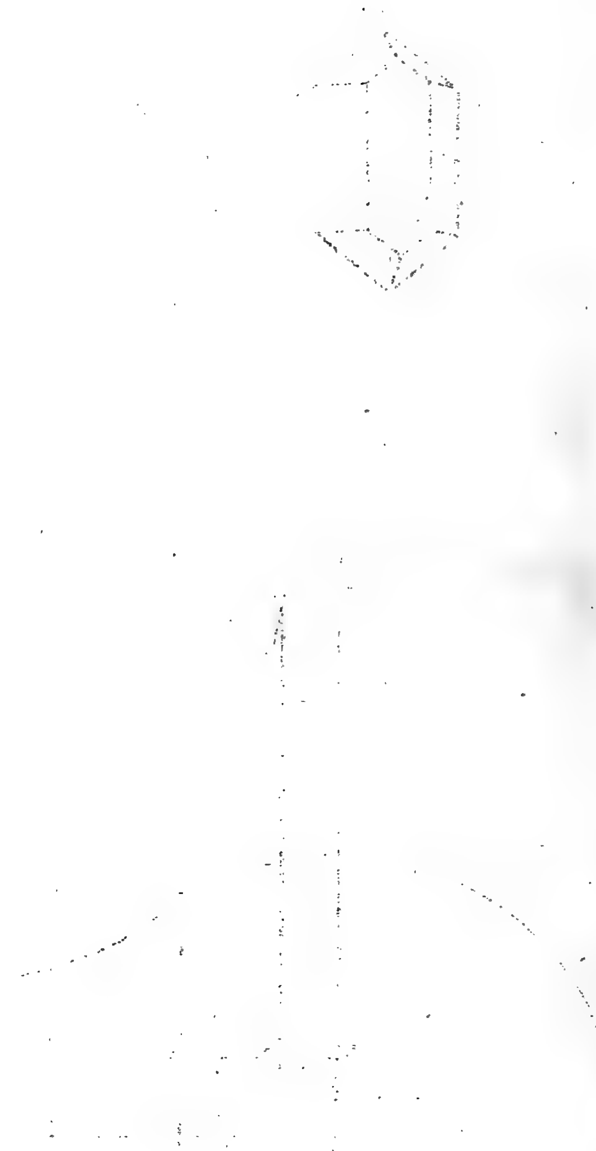


Fig. 3.

D. Lizars Sculpt.



3. A flat prifm, with the acute angle of 63° replaced by one plane, and terminated by an acumination, having three principal facettes fet on the larger lateral planes, with which the centre one meafures 125° and 55°. Of this fpecimen, an engraving is given in the annexed Plate, *Fig.* 2.

SPECIFIC gravity, according to my experiments, 3.533. The fpecimen appears to be nearly, though not abfolutely, pure. This fubftance, however, is fo very much mixed with mica, that no reliance can be placed on any of the trials which have been made. Count BOURNON, furprifed at the low fpecific gravity noted by Mr ALLAN, which was 3.480, broke down one of the fpecimens which had been fent him, in order to procure the fubftance in the pureft ftate poffible, and the refult of four experiments was as follows, 4.001

3.797

3.654

3.119

In a fubfequent experiment of Mr ALLAN's, he found it 3.665. From thefe it appears, that the fubftance is not in a pure ftate. Its colour is fo entirely the fame with the mica, with which it is accompanied, that it is only by mechanical attrition that they can be feparated.

COLOUR, brownifh-black.

EXTERNAL luftre, dull; internal, fhining and refinous, flightly inclining to metallic.

FRACTURE, fmall conchoidal.

FRAGMENTS, indeterminate, fharp-edged.

OPAKE.

SEMI-HARD in a high degree. Does not fcratch quartz nor felfpar, but fcratches hornblende and crown-glafs.

BRITTLE.

EASILY

Easily frangible.

Powder, dark greenifh-grey.

Before the blow-pipe it froths, and melts imperfectly into a brown fcoria.

Gelatinises in nitric acid. In a ftrong red heat it lofes 3.98 *per cent.* of its weight.

II. Experiments to ascertain its composition.

My firft experiments were made, on the fuppofition that the mineral was a variety of gadolinite, and were pretty much in the ftyle of thofe previoufly made on that fubftance by Ekeberg, Klaproth, and Vauquelin.

1. 100 grains of the mineral, previoufly reduced to a fine powder in an agate mortar, were digefted repeatedly on a fand bath in muriatic acid, till the liquid ceafed to have any action on it. The undiffolved refidue was filica, mixed with fome fragments of mica. When heated to rednefs, it weighed 33.4 grains.

2. The muriatic acid folution was evaporated almoft to drynefs, to get rid of the excefs of acid, diffolved in a large quantity of water, mixed with a confiderable excefs of carbonate of ammonia, and boiled for a few minutes. By this treatment, the whole contents of the mineral were precipitated in the ftate of a yellowifh powder, which was feparated by the filtre, and boiled, while ftill moift, in potafh-ley. A fmall portion of it only was diffolved. The potafh-ley was feparated from the undiffolved portion by the filtre, and mixed with a folution of fal ammoniac, by means of which a white powder precipitated from it. This white matter being heated to rednefs, weighed 7.9 grains. It was digefted in fulphuric acid, but 3.76 grains refufed to diffolve. This portion poffeffed the properties of filica. The diffolved portion being mixed with a few drops of fulphate

ſulphate of potaſh, ſhot into cryſtals of alum. It was therefore alumina, and amounted to 4.14 grains.

3. The yellow matter which refuſed to diſſolve in the pot-aſh-ley, was mixed with nitric acid. An efferveſcence took place, but the liquid remained muddy, till it was expoſed to heat, when a clear reddiſh-brown ſolution was effected. This ſolution was evaporated to dryneſs, and kept for a few minutes in the temperature of about 400°, to peroxidize the iron, and render it inſoluble. A ſufficient quantity of water was then poured on it, and digeſted on it for half-an-hour, on the ſand-bath. The whole was then thrown upon a filtre. The dark red matter which remained on the filtre, was drenched in oil, and heated to redneſs, in a covered crucible. It was then black, and attracted by the magnet; but had not exactly the appearance of oxide of iron. It weighed 42.4 grains.

4. The liquid which paſſed through the filtre, had not the ſweet taſte which I expected, but a ſlightly bitter one, ſimilar to a weak ſolution of nitrate of lime. Hence it was clear, that no yttria was preſent, as there ought to have been, had the mineral contained that earth. This liquid being mixed with carbonate of ammonia, a white powder precipitated, which, after being dried in a red heat, weighed 17 grains. It diſſolved in acids with efferveſcence; the ſolution was precipitated white by oxalate of ammonia, but not by pure ammonia. When diſſolved in ſulphuric acid, and evaporated to dryneſs, a light white matter remained, taſteleſs, and hardly ſoluble in water. Theſe properties indicate carbonate of lime. Now, 17 grains of carbonate of lime are equivalent to about 9.23 grains of lime.

5. From

5. FROM the preceding analyfis, fuppofing it accurate, it followed, that the mineral was compofed of

Silica, - - -	37.16
Lime, - - -	9.23
Alumina, - - -	4.14
Oxide of iron, - - -	42.40
Volatile matter, - -	3.98
	96.91
Lofs, - - -	3.09
	100.00

But the appearance of the fuppofed oxide of iron, induced me to fufpect, that it did not confift wholly of that metal. I thought it even conceivable, that the yttria which the mineral contained, might have been rendered infoluble by the application of too much heat, and might have been concealed by the iron with which it was mixed. A number of experiments, which it is needlefs to fpecify, foon convinced me, that, befides iron, there was likewife another fubftance prefent, which poffeffed properties different from any that I had been in the habit of examining. It poffeffed one property at leaft in common with yttria; its folution in acids had a fweet tafte; but few of its other properties had any refemblance to thofe which the chemifts to whom we are indebted for our knowledge of yttria, have particularifed. But as I had never myfelf made any experiments on yttria, I was rather at a lofs what conclufion to draw. From this uncertainty, I was relieved by Mr ALLAN, who had the goodnefs to give me a fmall fragment of gadolinite, which had been received directly from Mr EKEBERG. From this I extracted about 10 grains of yttria; and upon comparing its properties with thofe of the fubftance in queftion, I found

found them quite different. Convinced by thefe experiments, that the mineral contained no yttria, but that one of its conftituents was a fubftance with which I was ftill unacquainted, I had recourfe to the following mode of analyfis, in order to obtain this fubftance in a pure ftate.

III. Analysis of allanite.

1. 100 grains of the mineral, previoufly reduced to a fine powder, were digefted in hot nitric acid till nothing more could be diffolved. The undiffolved refidue, which was filica, mixed with fome fcales of mica, weighed, after being heated to rednefs, 35.4 grains.

2. The nitric acid folution was tranfparent, and of a light-brown colour. When ftrongly concentrated by evaporation, to get rid of the excefs of acid, and fet afide in an open capfule, it concreted into a whitifh folid matter, confifting chiefly of foft cryftals, nearly colourlefs, having only a flight tinge of yellow. Thefe cryftals being left expofed to the air, became gradually moift, but did not fpeedily deliquefce. The whole was therefore diffolved in water, and the excefs of acid, which was ftill prefent, carefully neutralifed with ammonia. By this treatment, the folution acquired a much deeper brown colour; but ftill continued tranfparent. Succinate of ammonia was then dropped in with caution. A copious reddifh-brown precipitate fell, which being wafhed, dried, and heated to rednefs in a covered crucible, weighed 25.4 grains. It poffeffed all the characters of black oxide of iron. For it was attracted by the magnet, completely foluble in muriatic acid, and the folution was not precipitated by oxalate of ammonia.

3. The liquid being ftill of a brown colour, I conceived it not to be completely free from iron. On this account, an ad-

ditional quantity of ſuccinate of ammonia was adde. And we precipitate fell; but inſtead of the dark reddiſh-brown colour, which characterizes ſuccinate of iron, it had a beautiful fleſh-red colour, which it retained after being dried in the open air. When heated to redneſs in a covered crucible, it became black, and had ſome reſemblance to gunpowder. It weighed 7.2 grains.

4. THIS ſubſtance attracted my peculiar attention, in conſequence of its appearance. I found it to poſſeſs the following characters:

a. IT was taſteleſs, and not in the leaſt attracted by the magnet, except a few atoms, which were eaſily ſeparated from the reſt.

b. IT was inſoluble in water, and not ſenſibly acted on when boiled in ſulphuric, nitric, muriatic, or nitro-muriatic acid.

c. BEFORE the blow-pipe it melted with borax and microcoſmic ſalt, and formed with both a colourleſs bead. With carbonate of ſoda it formed a dark-red opake bead.

d. WHEN heated to redneſs with potaſh, and digeſted in water, ſnuff-coloured flocks remained undiſſolved, which gradually ſubſided to the bottom. The liquid being ſeparated, and examined, was found to contain nothing but potaſh. When muriatic acid was poured upon the ſnuff-coloured flocks, a ſlight efferveſcence took place, and when heat was applied, the whole diſſolved. The ſolution was tranſparent, and of a yellow colour, with a ſlight tint of green. When evaporated to dryneſs, to get rid of the exceſs of acid, a beautiful yellow matter gradually ſeparated. Water boiled upon this matter diſſolved the whole. The taſte of the ſolution was aſtringent, with a ſlight metallic flavour, by no means unpleaſant, and no ſweetneſs was perceptible.

e. A

e. A PORTION of the black powder being expoſed to a red heat for an hour, in an open crucible, became reddiſh-brown, and loſt ſomewhat of its weight. In this altered ſtate, it was ſoluble by means of heat, though with difficulty, both in nitric and ſulphuric acids. The ſolutions had a reddiſh-brown colour, a ſlight metallic aſtringent taſte, but no ſweetneſs.

f. THE ſolution of this matter in nitric and muriatic acid, when examined by re-agents, exhibited the following phenomena:

(1.) With pruſſiate of potaſh, it threw down a white precipitate in flocks. It ſoon ſubſided; readily diſſolved in nitric acid; the ſolution was green.

(2.) Pruſſiate of mercury. A light yellow precipitate, ſoluble in nitric acid.

(3.) Infuſion of nut galls. No change.

(4.) Gallic acid. No change.

(5.) Oxalate of ammonia. No change.

(6.) Tartrate of potaſh. No change.

(7.) Phoſphate of ſoda. No change.

(8.) Hydro-ſulphuret of ammonia. Copious black flocks. Liquor remains tranſparent.

(9.) Arſeniate of potaſh. A white precipitate.

(10.) Potaſh. - - -	Copious yellow-coloured flocks; readily diſſolved in nitric acid.
(11.) Carbonate of ſoda. - -	
(12.) Carbonate of ammonia.	

(13.) Succinate of ammonia. A white precipitate.

(14.) Benzoate of potaſh. A white precipitate.

(15.) A plate of zinc being put into the ſolution in muriatic acid, became black, and threw down a black powder, which was inſoluble in ſulphuric, nitric, muriatic, nitro-muriatic, acetic, and phoſphoric acids, in every temperature, whether theſe acids were concentrated or diluted.

(16.) A plate of tin put into the nitric ſolution, occaſioned no change.

(17.) A portion being incloſed in a charcoal crucible, and expoſed for an hour to the heat of a forge, was not reduced to a metallic button, nor could any trace of it be detected when the crucible was examined.

These properties were all that the ſmall quantity of the matter in my poſſeſſion enabled me to aſcertain. They unequivocally point out a metallic oxide. Upon comparing them with the properties of all the metallic oxides known, none will be found with which this matter exactly agrees. Cerium is the metal, the oxides of which approach the neareſt. The colour is nearly the ſame, and both are precipitated white by pruſſiate of potaſh, ſuccinate of ammonia, and benzoate of potaſh. But, in other reſpects, the two ſubſtances differ entirely. Oxide of cerium is precipitated white by oxalate of ammonia and tartrate of potaſh; our oxide is not precipitated at all: Oxide of cerium is precipitated white by hydro-ſulphuret of ammonia; while our oxide is precipitated black: Oxide of cerium is not precipitated by zinc, while our oxide is thrown down black. There are other differences between the two, but thoſe which I have juſt mentioned are the moſt ſtriking.

These properties induced me to conſider the ſubſtance which I had obtained from the Greenland mineral as the oxide of a metal hitherto unknown; and I propoſed to diſtinguiſh it by the name of *Junonium*.

In the experiments above detailed, I had expended almoſt all the oxide of *Junonium* which I had in my poſſeſſion, taking it for granted that I could eaſily procure more of it from the Greenland

land mineral. But, ſoon after, I was informed by Dr WOLLASTON, to whom I had ſent a ſpecimen of the mineral, that he had not been able to obtain any of my ſuppoſed Junonium in his trials. This induced me to repeat the analyſis no leſs than three times, and in neither caſe was I able to procure any more of the ſubſtance which I have deſcribed above. Thus, it has been out of my power, to verify the preceding details, and to put the exiſtence of a new metal in the mineral beyond doubt. At the ſame time, I may be allowed to ſay, that the above experiments were made with every poſſible attention on my part, and moſt of them were repeated, at leaſt a dozen times. I have no doubt myſelf of their accuracy ; but think that the exiſtence of a new metal can hardly be admitted, without ſtronger proofs than the ſolitary analyſis which I have performed.

5. THE liquid, thus freed from iron and junonium, was ſuperſaturated with pure ammonia. A greyiſh-white gelatinous matter precipitated. It was ſeparated by the filtre, and became gradually darker coloured when drying. This matter, after being expoſed to a red heat, weighed about 38 grains. When boiled in potaſh-ley, 4.1 grains were diſſolved, of a ſubſtance which, ſeparated in the uſual way, exhibited the properties of alumina.

6. THE remaining 33.9 grains were again diſſolved in muriatic acid, and precipitated by pure ammonia. The precipitate was ſeparated by the filtre, and allowed to dry ſpontaneouſly in the open air. It aſſumed an appearance very much reſembling gum-arabic, being ſemi-tranſparent, and of a brown colour. When dried upon the ſand-bath, it became very dark-brown, broke with a vitreous fracture, and ſtill retained a ſmall degree of tranſparency. It was taſteleſs, felt gritty between the teeth, and was eaſily reduced to powder. It efferveſced in ſulphuric, nitric, muriatic, and acetic acids, and a ſolution of it

was

was effected in each by means of heat, though not without considerable difficulty. The solutions had an austere, and slightly sweetish taste. When examined by re-agents, they exhibited the following properties:

(1.) Prussiate of potash. A white precipitate.
(2.) Oxalate of ammonia. A white precipitate.
(3.) Tartrate of potash. A white precipitate.
(4.) Hydrosulphuret of potash. A white precipitate.
(5.) Phosphate of soda. A white precipitate.
(6.) Arseniate of potash. A white precipitate.
(7.) Potash and its carbonate. A white precipitate.
(8.) Carbonate of ammonia. A white precipitate.
(9.) Ammonia. A white gelatinous precipitate.
(10.) A plate of zinc. No change.

These properties indicated Oxide of Cerium. I was therefore disposed to consider the substance which I had obtained as oxide of cerium. But on perusing the accounts of that substance, given by the celebrated chemists to whose labours we are indebted for our knowledge of it, there were several circumstances of ambiguity which occurred. My powder was dissolved in acids with much greater difficulty than appeared to be the case with oxide of cerium. The colour of my oxide, when obtained from oxalate, by exposing it to a red heat, was much lighter, and more inclined to yellow, than the oxide of cerium.

In this uncertainty, Dr Wollaston, to whom I communicated my difficulties, offered to send me down a specimen of the mineral called *cerite*, that I might extract from it real oxide of cerium, and compare my oxide with it. This offer I thankfully

fully accepted *; and upon comparing the properties of my oxide with thofe of oxide of cerium extracted from *cerite*, I was fully fatisfied that they were identical. The more difficult folubility of mine, was owing to the method I had employed to procure it, and to the ftrong heat to which I had fubjected it; whereas the oxide of cerium from cerite had been examined in the ftate of carbonate.

7. In the many experiments made upon this powder, and upon oxide of cerium from cerite, I repeated every thing that had been eftablifhed by Berzelius and Hisinger, Klaproth and Vauquelin, and had an opportunity of obferving many particulars which they have not noticed. It may be worth while, therefore, without repeating the details of thefe chemifts, to mention a few circumftances, which will be found ufeful in examining this hitherto fcarce oxide.

a. The precipitate occafioned by oxalate of ammonia is at firft in white flocks, not unlike that of muriate of filver, but it foon affumes a pulverulent form. It diffolves readily in nitric acid, without the affiftance of heat. The fame remark applies to the precipitate thrown down by tartrate of potafh. But tartrate of cerium is much more foluble in acids than the oxalate.

b. The

* The fpecimen of cerite which I analifed, was fo much mixed with actinolite, that the ftatement of the refults which I obtained cannot be of much importance. The fpecific gravity of the fpecimen was 4.149. I found it compofed as follows:

A white powder, left by muriatic acid, and prefumed to be filica,	47.3
Red oxide of cerium, - - - -	44.
Iron, - - - -	4.
Volatile matter, - - -	3.
Lofs, - - -	1.7
	100.0

b. THE ſolution of cerium in acetic acid is precipitated grey by infuſion of nut-galls. Cerium is precipitated likewiſe by the ſame re-agent from other acids, provided the ſolution contain no exceſs of acid. This fact was firſt obſerved by Dr WOLLASTON, who communicated it to me laſt ſummer. I immediately repeated his experiments with ſucceſs.

c. CERIUM is not precipitated from its ſolution in acids by a plate of zinc. In ſome caſes, indeed, I have obtained a yellowiſh-red powder, which was thrown down very ſlowly. But it proved, on examination, to conſiſt almoſt entirely of red oxide of iron, and of courſe only appeared when the ſolution of cerium was contaminated with iron.

d. THE ſolutions of cerium in acids have an aſtringent taſte, with a perceptible ſweetneſs, which, however, is different from the ſweetneſs which ſome of the ſolutions of iron in acids poſſeſs.

e. THE muriate and ſulphate of cerium readily cryſtalliſe; but I could not ſucceed in obtaining cryſtals of nitrate of cerium.

f. THE beſt way of obtaining pure oxide of cerium, is to precipitate the ſolution by oxalate of ammonia, waſh the precipitate well, and expoſe it to a red heat. The powder obtained by this proceſs is always red; but it varies very much in its ſhade, and its beauty, according to circumſtances. This powder always contains carbonic acid.

g. I CONSIDER the following as the eſſential characters of cerium. The ſolution has a ſweet aſtringent taſte: It is precipitated white by pruſſiate of potaſh, oxalate of ammonia, tartrate of potaſh, carbonate of potaſh, carbonate of ammonia, ſuccinate of ammonia, benzoate of potaſh, and hydroſulphuret of ammonia: The precipitates are re-diſſolved by nitric or muriatic acids:

acids: Ammonia throws it down in gelatinous flocks: Zinc does not precipitate it at all.

b. The white oxide of cerium, mentioned by Hisinger and Berzelius, and defcribed by Vauquelin, did not prefent itfelf to me in any of my experiments; unlefs the white flocks precipitated by ammonia from the original folution be confidered as white oxide. They became brown on drying, and when heated to rednefs, were certainly converted into red oxide.

As cerium, as well as iron, is precipitated by fuccinate of ammonia, the preceding method of feparating the two from each other was not unexceptionable. Accordingly, in fome fubfequent analyfes, I feparated the cerium by means of oxalate of ammonia, before I precipitated the iron. I found that the proportions obtained by the analyfis above defcribed, were fo near accuracy that no material alteration is neceffary.

8. The liquid, thus freed from iron, alumina, and cerium, was mixed with carbonate of foda. It precipitated a quantity of carbonate of lime, which amounted, as before, to about 17 grains, indicating 9.2 grains of lime.

From the preceding analyfis, which was repeated no lefs than three times, a different method being employed in each, the conftituents of allanite are as follows:

Silica, - - -	35.4
Lime, - - -	9.2
Alumina, - -	4.1
Oxide of iron, - -	25.4
Oxide of cerium, - -	33.9
Volatile matter, - -	4.
	112.0

I omit the 7 grains of junonium, becaufe I only detected it in one fpecimen of allanite. The excefs of weight in the preceding numbers, is to be afcribed chiefly to the carbonic acid combined with the oxide of cerium, from which it was not completely freed by a red heat. I have reafon to believe, too, that the proportion of iron is not quite fo much as 25.5 grains. For, in another analyfis, I obtained only 18 grains, and in a third 20 grains. Some of the cerium was perhaps precipitated along with it in the preceding analyfis, and thus its weight was apparently increafed.

XII. *A Chemical Analysis of Sodalite, a new Mineral from Greenland.* By Thomas Thomson, M. D. F. R. S. E. *Fellow of the Imperial Chirurgo-Medical Academy of Petersburgh.*

[*Read Nov.* 5. 1810.]

THE mineral to which I have given the name of *Sodalite*, was alſo put into my hands by Mr Allan. In the Greenland collection which he purchaſed, there were ſeveral ſpecimens of a rock, obviouſly primitive. In the compoſition of theſe, the ſubſtance of which I am about to treat, formed a conſtituent, and, at firſt appearance, was taken for felſpar, to which it bears a very ſtriking reſemblance.

This rock is compoſed of no leſs than five different foſſils, namely, garnet, hornblende, augite, and two others, which form the paſte of the maſs. Theſe are evidently different minerals; but in ſome ſpecimens, are ſo intimately blended, that it required the ſkill of Count Bournon to make the diſcrimination, and aſcertain their real nature. Even this diſtinguiſhed mineralogiſt was at firſt deceived by the external aſpect, and conſidered the paſte as common lamellated felſpar, of a greeniſh colour. But a peculiarity which preſented itſelf to Mr Allan, in one of the minerals, induced him to call the attention of Count Bournon more particularly to its conſtruction.

On a cloſer examination of the mineral, M. de Bournon found that ſome ſmall fragments, which he had detached, prefented rectangular prifms, terminated by planes, meafuring, with the fides of the prifm, 110° and 70° or nearly fo,—a form which belongs to a rare mineral, known by the name of Sahlite, from Sweden. He further obferved, intermixed along with this, another material; and after fome trouble, fucceeded in detaching a mafs, prefenting a regular rhomboidal dodecahedron. It was to this form that Mr ALLAN had previoufly requefted his attention.

SOME time before this inveftigation, M. de BOURNON had examined a mineral from Sweden, of a lamellated ftructure, and a greenifh colour, which, he found, indicated the fame form. From this circumftance, together with fome external refemblance, which ftruck him, he was induced to conclude, that our mineral was a variety of that fubftance.

To that fubftance the name of Swedifh *natrolite* had been given, in confequence of the inveftigation of Dr WOLLASTON, who found that it contained a large proportion of foda.

THERE are few minerals, however, that are fo totally diftinct in their external characters as the natrolite of KLAPROTH, and the fubftance we are now treating of. The mineral examined by KLAPROTH occurs at Roegan *, on the Lake of Conftance, in porphyry-flate, coating the fides of veins and cavities in a mamellated form, the texture of which is compact, fibrous, and radiated; the colour pale yellow, in fome places paffing into white, and marked with brown zones. Hitherto it had never been found in a ftate fufficiently perfect to afford any indications of form. Lately, however, M. de BOURNON was fo fortunate as to procure fome of it, prefenting very delicate needleform cryftals, which, by means of a ftrong magnifier, he was able to afcertain, prefented flat rectangular prifms, terminated by planes, which, he thought, might form angles of 60°

* IT has been obferved alfo by Profeffor JAMESON, in the flœtz-trap rocks behind Burntifland.

60° and 120 with the ſides of the priſm. With this, neither our mineral nor the Swediſh can have any connection, farther than ſome analogy which may exiſt in their compoſition.

CONCERNING the Swediſh mineral, I have not been able to obtain much ſatisfactory information. There is a ſpecimen of it in Mr ALLAN's cabinet, which he received directly from Sweden, ſent by a gentleman who had juſt before been in London, and was well acquainted with the collections of that city, from which it is inferred, that the ſpecimen in queſtion is the ſame as that examined by Count BOURNON and Dr WOLLASTON.

WERNER has lately admitted into his ſyſtem a new mineral ſpecies, which he diſtinguiſhes by the name of *Fettstein*. Of this I have ſeen two deſcriptions; one by HAÜY, in his *Tableau Comparatif*, publiſhed laſt year; and another by Count DUNIN BORKOWSKI, publiſhed in the 69th volume of the *Journal de Physique*, and tranſlated in *Nicholson's Journal*, (vol. 26. p. 384). The ſpecimen, called *Swedish Natrolite*, in Mr ALLAN's poſſeſſion, agrees with theſe deſcriptions in every particular, excepting that its ſpecific gravity is a little higher. BORKOWSKI ſtates the ſpecific gravity of *fettstein* at 2.563; HAÜY at 2.6138; while I found the ſpecific gravity of Mr ALLAN's ſpecimen to be 2.779, and, when in ſmall fragments, to be as high as 2.790. This very near agreement in the properties of the Swediſh natrolite, with the characters of the fettſtein, leads me to ſuppoſe it the ſubſtance to which WERNER has given that name. This opinion is ſtrengthened, by a fact mentioned by HAÜY, that fettſtein had been at firſt conſidered as a variety of *Wernerite*. For the ſpecimen ſent to Mr ALLAN, under the name of *Compact Wernerite*, is obviouſly the very ſame with the ſuppoſed natrolite of Sweden. Now, if this identity be admitted, it will follow, that our mineral conſtitutes a ſpecies apart. It bears, indeed, a conſiderable reſemblance to it; but neither the cryſtalline form, nor the conſtituents of fettſtein,

as

as ſtated by HAUY, are ſimilar to thoſe of the mineral to which I have given the name of Sodalite. The conſtituents of fettſtein, as aſcertained by VAUQUELIN, are as follows:

Silica, - - -	44.00
Alumina, - -	34.00
Oxide of iron, - -	4.00
Lime, - - -	0.12
Potaſh and ſoda, - -	16.50
Loſs, - - -	1.38
	100.00

II. DESCRIPTION OF SODALITE.

SODALITE, as has been already mentioned, occurs in a primitive rock, mixed with ſahlite, augite *, hornblende, and garnet †.

IT occurs maſſive; and cryſtalliſed, in rhomboidal dodecahedrons, which, in ſome caſes, are lengthened, forming ſix-ſided priſms, terminated by trihedral pyramids.

ITS colour is intermediate between celandine and mountain green, varying in intenſity in different ſpecimens. In ſome caſes it ſeems intimately mixed with particles of ſahlite, which doubtleſs modify the colour.

EXTERNAL luſtre glimmering, internal ſhining, in one direction vitreous, in another reſinous.

FRACTURE foliated, with at leaſt a double cleavage; croſs fracture conchoidal.

FRAGMENTS indeterminate; uſually ſharp-edged.

TRANSLUCENT.

* THIS ſituation of the augite deſerves attention. Hitherto it has been, with a few exceptions, found only in flœtz trap rocks.

† THE particular colour and appearance of this garnet, ſhews that the rock came from Greenland: For ſimilar garnet has never been obſerved, except in ſpecimens from Greenland.

Translucent.

Hardness equal to that of felſpar. Iron ſcratches it with difficulty.

Brittle.

Easily frangible.

Specific gravity, at the temperature of 60°, 2.378. The ſpecimen was not abſolutely free from ſahlite.

When heated to redneſs, does not decrepitate, nor fall to powder, but becomes dark-grey, and aſſumes very nearly the appearance of the Swediſh natrolite of Mr Allan, which I conſider as fettſtein. If any particles of ſahlite be mixed with it, they become very conſpicuous, by acquiring a white colour, and the opacity and appearance of chalk. The loſs of weight was 2.1 *per cent.* I was not able to melt it before the blow-pipe.

II. Chemical analysis.

1. A hundred grains of the mineral, reduced to a fine powder, were mixed with 200 grains of pure ſoda, and expoſed for an hour to a ſtrong red heat, in a platinum crucible. The mixture melted, and aſſumed, when cold, a beautiful graſs-green colour. When ſoftened with water, the portion adhering to the ſides of the crucible acquired a fine browniſh-yellow. Nitric acid being poured upon it, a complete ſolution was obtained.

2. Suspecting, from the appearance which the fuſed maſs aſſumed, that it might contain chromium, I neutraliſed the ſolution, as nearly as poſſible, with ammonia, and then poured into it a recently prepared nitrate of mercury. A white precipitate fell, which being dried, and expoſed to a heat rather under redneſs, was all diſſipated, except a ſmall portion of grey matter,

matter, not weighing quite 0.1 grain. This matter was inſoluble in acids; but became white. With potaſh it fuſed into a colourleſs glaſs. Hence I conſider it as ſilica. This experiment ſhews that no chromium was preſent. I was at a loſs to account for the precipitate thrown down by the nitrate of mercury. But Mr ALLAN having ſhown me a letter from EKEBERG, in which he mentions, that he had detected muriatic acid in ſodalite, it was eaſy to ſee that the whole precipitate was calomel. The white powder weighed 26 grains, indicating, according to the analyſis of CHENEVIX, about 3 grains of muriatic acid.

3. THE ſolution, thus freed from muriatic acid, being concentrated by evaporation, gelatiniſed. It was evaporated nearly to dryneſs; the dry maſs, digeſted in hot water acidulated with nitric acid, and poured upon the filter. The powder retained upon the filter was waſhed, dried, and heated to redneſs. It weighed 37.2 grains, and was ſilica.

4. THE liquor which had paſſed through the filter, was ſuperſaturated with carbonate of potaſh, and the copious white precipitate which fell, collected by the filter, and boiled while yet moiſt in potaſh-ley. The bulk diminiſhed greatly, and the undiſſolved portion aſſumed a black colour, owing to ſome oxide of mercury with which it was contaminated.

5. THE potaſh-ley being paſſed through the filter, to free it from the undiſſolved matter, was mixed with a ſufficient quantity of ſal-ammoniac. A copious white precipitate fell, which being collected, waſhed, dried, and heated to redneſs, weighed 27.7 grains. This powder being digeſted in ſulphuric acid, diſſolved, except 0.22 grain of ſilica. Sulphate of potaſh being added, and the ſolution ſet aſide, it yielded alum cryſtals to the very laſt drop. Hence the 27.48 grains of diſſolved powder were alumina.

6. THE

6. The black refidue which the potafh-ley had not taken up, was diffolved in diluted fulphuric acid. The folution being evaporated to drynefs, and the refidue digefted in hot water, a white foft powder remained, which, heated to rednefs, weighed 3.6 grains, and was fulphate of lime, equivalent to about 2 grains of lime.

7. The liquid from which the fulphate of lime was feparated, being exactly neutralifed by ammonia, fuccinate of ammonia was dropped in; a brownifh-red precipitate fell, which, being heated to rednefs in a covered crucible, weighed 1 grain, and was black oxide of iron.

8. The refidual liquor being now examined by different reagents, nothing farther could be precipitated from it.

9. The liquid (No. 4.) from which the alumina, lime, and iron had been feparated by carbonate of potafh, being boiled for fome time, let fall a fmall quantity of yellow-coloured matter. This matter being digefted in diluted fulphuric acid, partly diffolved with effervefcence; but a portion remained undiffolved, weighing 1 grain. It was infoluble in acids, and with potafh melted into a colourlefs glafs. It was therefore filica. The fulphuric acid folution being evaporated to drynefs, left a refidue, which poffeffed the properties of fulphate of lime, and which weighed 1.2 grains, equivalent to about 0.7 grains of lime.

10. The conftituents obtained by the preceding analyfis being obvioufly defective, it remained to examine whether the mineral, according to the conjecture of Bournon, contained an alkali. For this purpofe, 100 grains of it, reduced to a fine powder, and mixed with 500 grains of nitrate of barytes, were expofed for an hour to a red heat, in a porcelain crucible. The fufed mafs was foftened with water, and treated with muriatic acid. The whole diffolved, except 25 grains of a white pow-

der, which proved on examination to be filica. The muriatic acid folution was mixed with fulphuric acid, evaporated to dryness; the refidue, digefted in hot water, and filtered, to feparate the fulphate of barytes. The liquid was now mixed with an excefs of carbonate of ammonia, boiled for an inftant or two, and then filtered, to feparate the earth and iron precipitated by the ammonia. The liquid was evaporated to drynefs, and the dry mafs obtained expofed to a red heat in a filver crucible. The refidue was diffolved in water, and expofed in the open air to fpontaneous evaporation. The whole gradually fhot into regular cryftals of fulphate of foda. This falt being expofed to a ftrong red heat, weighed 50 grains, indicating, according to BERTHOLLET's late analyfis, 23.5 grains of pure foda. It deferves to be mentioned, that during this procefs, the filver crucible was acted on, and a fmall portion of it was afterwards found among the fulphate of foda. This portion was feparated before the fulphate of foda was weighed.

THE preceding analyfis gives us the conftituents of fodalite as follows:

Silica, - - -	38.52
Alumina, - -	27.48
Lime, - - -	2.70
Oxide of iron, - -	1.00
Soda, - - -	23.50
Muriatic acid, - -	3.00
Volatile matter, - -	2.10
Lofs, - - -	1.70
	100.00

Mr

Mr Allan ſent a ſpecimen of this mineral to Mr Ekeberg, who analiſed it in the courſe of laſt ſummer. The conſtituents which he obtained, as he ſtates them in a letter to Mr Allan, are as follows:

Silica, - - -	36.
Alumina, - - -	32.
Soda, - - -	25.
Muriatic acid, - -	6.75
Oxide of iron, - -	0.25
	100.00

This reſult does not differ much from mine. The quantity of muriatic acid is much greater than mine. The lime and the volatile matter which I obtained, eſcaped his notice altogether. If we were to add them to the alumina, it would make the two analyſes almoſt the ſame. No mineral has hitherto been found containing nearly ſo much *soda* as this. Hence the reaſon of the name by which I have diſtinguiſhed it.

XIII. *Demonstration of the Fundamental Property of the Lever.* *By* DAVID BREWSTER, LL. D. F. R. S. EDIN.

[*Read December* 3. 1810.]

IT is a fingular fact in the hiftory of fcience, that, after all the attempts of the moft eminent modern mathematicians, to obtain a fimple and fatisfactory demonftration of the fundamental property of the lever, the folution of this problem given by ARCHIMEDES, fhould ftill be confidered as the moft legitimate and elementary. GALILEO, HUYGENS, DE LA HIRE, Sir ISAAC NEWTON, MACLAURIN, LANDEN, and HAMILTON, have directed their attention to this important part of mechanics; but their demonftrations are in general either tedious and abftrufe, or founded on affumptions too arbitrary to be recognifed as a proper bafis for mathematical reafoning. Even the demonftration given by ARCHIMEDES is not free from objections, and is applicable only to the lever, confidered as a phyfical body. GALILEO, though his demonftration is fuperior in point of fimplicity to that of ARCHIMEDES, reforts to the inelegant contrivance, of fufpending a folid prifm from a mathematical lever, and of dividing the prifm into two unequal parts, which act as the power and the weight. The demonftration given by HUYGENS, affumes as an axiom, that a given weight removed

removed from the fulcrum, has a greater tendency to turn the lever round its centre of motion, and is, besides, applicable only to a commensurable proportion of the arms. The foundation of Sir Isaac Newton's demonstration is still more inadmissible. He assumes, that if a given power act in any direction upon a lever, and if lines be drawn from the fulcrum to the line of direction, the mechanical effort of the power will be the same when it is applied to the extremity of any of these lines; but it is obvious, that this axiom is as difficult to be proved as the property of the lever itself. M. De la Hire has given a demonstration which is remarkable for its want of elegance. He employs the *reductio ad absurdum*, and thus deduces the proposition from the case where the arms are commensurable. The demonstration given by Maclaurin has been highly praised; but if it does not involve a *petitio principii*, it has at least the radical defect, of extending only to a commensurable proportion of the arms. The solutions of Landen and Hamilton are peculiarly long and complicated, and resemble more the demonstration of some of the abstrusest points of mechanics, than of one of its simplest and most elementary truths.

In attempting to give a new demonstration of the fundamental property of the lever, which shall be at the same time simple and legitimate, we shall assume only one principle, which has been universally admitted as axiomatic, namely, *that equal and opposite forces, acting at the extremities of the equal arms of a lever, and at equal angles to these arms, will be in equilibrio.* With the aid of this axiom, the fundamental property of the lever may be established by the three following propositions.

In Prop. I. the property is deduced in a very simple manner, when the arms of the lever are commensurable.

In

IN PROP. II., which is totally independent of the firſt, the demonſtration is general, and extends to any proportion between the arms.

IN PROP. III. the property is eſtabliſhed, when the forces act in an oblique direction, and when the lever is either rectilineal, angular, or curvilineal. In the demonſtrations which have generally been given of this laſt propoſition, the oblique force has been reſolved into two, one of which is directed to the fulcrum, while the other is perpendicular to that direction. It is then aſſumed, *that the force directed to the fulcrum has no tendency to disturb the equilibrium, even though it acts at the extremity of a bent arm;* and hence it is eaſy to demonſtrate, that the remaining force is proportional to the perpendicular drawn from the fulcrum to the line of direction in which the original force was applied. As the principle thus aſſumed, however, is totally inadmiſſible as an intuitive truth, we have attempted to demonſtrate the propoſition without its aſſiſtance.

PROP. I.—*If one arm of a straight lever is any multiple of the other, a force acting at the extremity of the one will be in equilibrio with a force acting at the extremity of the other, when these forces are reciprocally proportioned to the length of the arms to which they are applied.*

LET AB (PLATE XI. fig. 1.) be a lever ſupported on the two fulcra F, *f*, ſo that A*f* = *f*F = FB. Then, if two equal weights C, D, of 1 pound each, be ſuſpended from the extremities A, B, they will be in equilibrio, ſince they act at the end of equal arms A*f*, BF; and each of the fulcra *f*, F, will ſupport an equal part of the whole weight, or 1 pound. Let the fulcrum *f* be now removed, and let a weight E, of 1 pound, act upwards at the point *f*; the equilibrium will ſtill continue; but the weight E, of 1 pound, acting upwards at *f*, is equivalent to a weight G of 1 pound, acting downwards at B. Remove, therefore, the weight

E,

E, and ſuſpend the weight G from B; then, ſince the equilibrium is ſtill preſerved after theſe two ſubſtitutions, we have a weight C, of one pound, acting at the extremity of the arm AF, in equilibrio with the weights D and G, which together make two pounds, acting at the extremity of the arm FB. But FA is to FB as 2 is to 1; therefore an equilibrium takes place, when the weights are reciprocally proportional to the arms, in the particular caſe when the arms are as 2 to 1. By making F*f* ſucceſſively double, triple, &c. of FB, it may in like manner be ſhewn, that, in theſe caſes, the propoſition holds true.

Lemma.

If any weight BC c b, (fig. 2. No. 1.), of uniform shape and density, is placed on a lever A φ, whose fulcrum is φ, it has the same tendency to turn the lever round φ, as if it were suspended from a point G, so taken that b G = G *c.*

If a weight W, of the ſame magnitude with BC, acts upwards at the point G, it will be in equilibrium with the weight BC, and will therefore deſtroy the tendency of that weight to turn the lever round φ. But the weight W, acting upwards at the point G, has the ſame power to turn the lever round φ, as an equal weight *w*, acting downwards at G. Conſequently the tendency of the weight BC to turn the lever round φ, is the ſame as the tendency of an equal weight *w*, acting downwards at G.

Prop. II.

If two forces applied to a lever, and acting at right angles to it, have the same tendency to turn the lever round its centre of motion, they are reciprocally proportional to the distances of the points at which they are applied from the centre of motion.

Let A φ *d*, (fig. 2. No. 2.) be a lever whoſe fulcrum is φ, and let it be loaded with a weight BD *d b* of uniform ſhape and denſity.

ſity. Then by the lemma, this weight has the ſame tendency to turn the lever round, as if it were ſuſpended from the point n, ſo taken that $bn = dn$. Make $\varphi c = \varphi d$, and let the weight BDdb be divided at the points C and F, by the lines Cc, Fφ. The weights CFφc, DFφd, being in equilibrio, by the axiom, have no tendency to turn the lever round φ, conſequently the remaining weight BCcd, has the ſame tendency to turn the lever round φ as the whole weight BDdb. Hence if $bm = cm$, the weight BCcb acting at the point m, will have the ſame tendency to turn the lever round φ, as the weight BDdb acting at n. Now BDdb : BC$cb = bd : bc = nd : mc$; and ſince $bc = bd - cd$, we have $mc = \frac{1}{2}bd - \frac{1}{2}cd = nd - \frac{1}{2}cd = n\varphi$, and $nd = n\varphi + \frac{1}{2}cd = mc + \frac{1}{2}cd = m\varphi$. Conſequently,

$$BDdb : BCcb = m\varphi : n\varphi.$$

LEMMA.

Two equal forces acting at the ſame point of the arm of a lever, and in directions which form equal angles with a perpendicular drawn through that point of the arm, will have equal tendencies to turn the lever round its centre of motion.

LET AB (fig. 3.) be a lever with equal arms AF, FB. Through the points A, B, draw AD, BE, perpendicular to AB, and AP, Ap, BW, Bw, forming equal angles with the lines AD, BE. Produce PA to M. Then, equal forces acting in the directions AP, Bw, will be in equilibrio. But a force M equal to P, and acting in the direction AM, will counteract the force P, acting in the direction AB, or will have the ſame tendency to turn the lever round F; and the force W, acting in the direction BW, will have the ſame tendency to turn the lever round F as the

the force M: Consequently the force W will have the same tendency to turn the lever round F as the force w; and this will hold true, whether the arms AF, FB, are straight or curvilineal, provided that they are both of the same form.

PROP. III.—*If a force acts in different directions at the same point in the arm of a lever, its tendency to turn the lever round its centre of motion, will be proportional to the perpendiculars let fall from that centre on the lines of direction in which the force is applied.*

LET AB, (fig. 4.) be the lever, and let the two equal forces BM, B m, act upon it at the point B, in the direction of the lines BM, B m. Draw BN, B n, respectively equal to BM, B m, and forming the same angles with the line PB ω perpendicular to AB. To BM, B m, BN, B n, produced, draw the perpendiculars AY, A y, AX, A x. Now, the side AX = AY, and A x = A y, on account of the equality of the triangles ABX, ABY; and if M l, M λ, be drawn perpendicular to B ω, the triangles ABY, BM l, will be similar, and also the triangles AB y, B $m\lambda$: Hence we obtain

$$AB : AY = BM : Bl, \text{ and}$$
$$AB : Ay = BM : B\lambda$$

Therefore, *ex æquo*, $AY : Ay = Bl : B\lambda$.

Complete the parallelograms BM o N, B $m\omega n$, and B l, B λ will be respectively one-half of the diagonals B o, B ω.

Now let two equal forces BM, BN, act in these directions upon the lever at B, their joint force will be represented by the diagonal B o, and consequently one of the forces BM will be

Fig. 1.

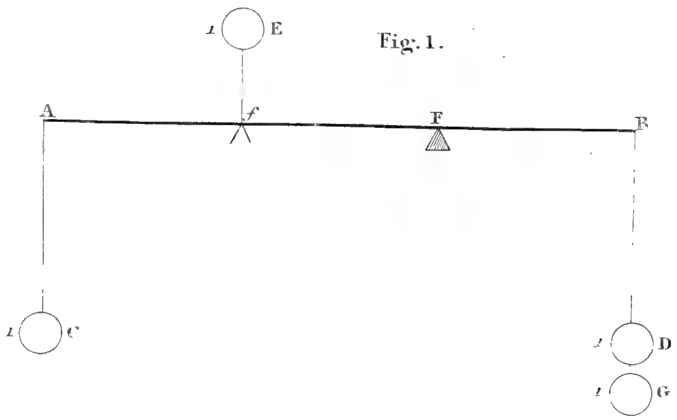


Fig. 2. No. 1.

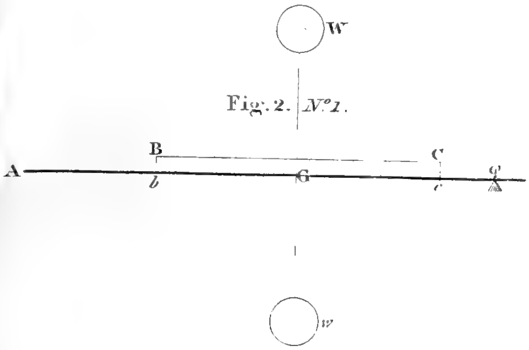


Fig. 2. No. 2.

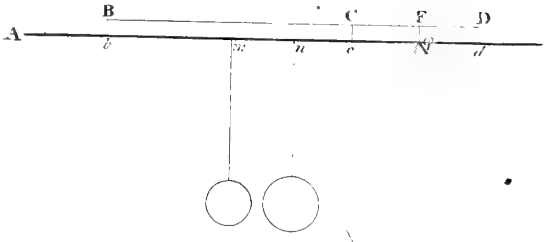


Fig. 3.

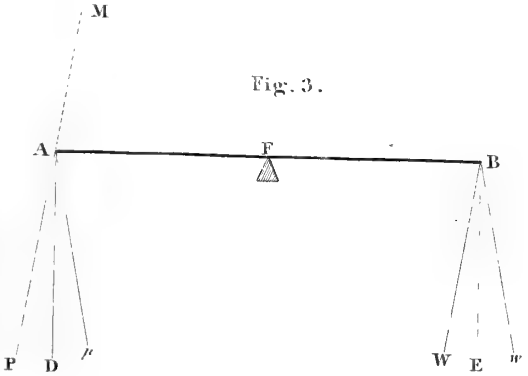
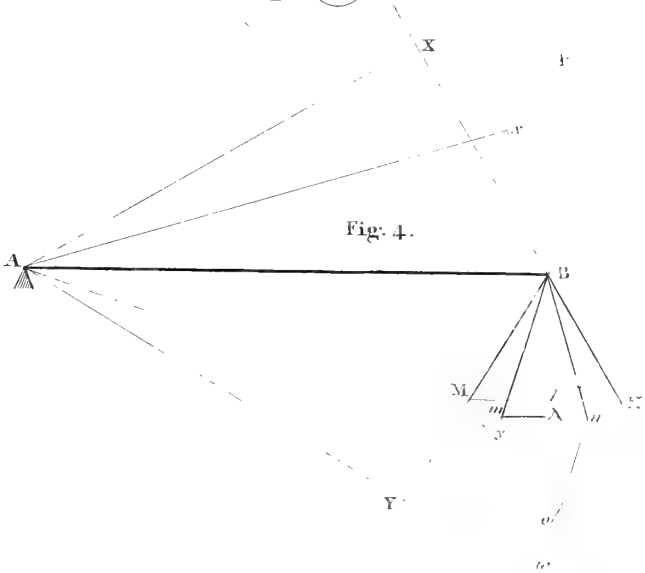


Fig. 4.



be reprefented by $Bl = \frac{1}{2} Bo$. In the fame manner, if the two equal forces Bm, Bn, act upon the lever at B, their joint force will be reprefented by $B\omega$, and one of them, Bm, will be reprefented by $B\lambda = \frac{1}{2} B\omega$. Confequently the power of the two forces BM, Bm, to turn the lever round its centre of motion, is reprefented by Bl, $B\lambda$, refpectively; that is, the force BM is to the force Bm as Bl is to $B\lambda$; that is, as AY is to Ay, the perpendiculars let fall upon the lines of their direction.

XIV. *On the Rocks in the vicinity of Edinburgh.* By THOMAS ALLAN, Esq; F. R. S. EDIN.

[*Read March* 4. 1811.]

ALTHOUGH ſcience has only within theſe few years acknowledged the importance of Geology, the eagerneſs with which it has been cultivated, affords ſufficient proof of the intereſt it is capable of creating. Of this we have a recent example in the laborious undertaking of Sir GEORGE MACKENZIE and his friends, who, notwithſtanding all the dangers preſented by a voyage through the moſt tempeſtuous ocean, and the deprivations to which they were expoſed, in a journey through a country deſtitute of the ſlighteſt trace to guide the route of the traveller, were not deterred from exploring the inhoſpitable ſhores of Iceland. Theſe, and other travellers, have extended our knowledge of various diſtricts on the ſurface of the globe; but we have ſtill to lament the extreme imperfection of the ſcience, which, as yet, has aſſumed no decided character or form. This appears principally owing to the want of ſome ſimple method, grounded on clear and intelligible principles:

ples; perhaps alſo, to its having always been the object of thoſe who have treated the ſubject, to accommodate their obſervations to a particular theory; and when this is the caſe, it is obvious, that the mind cannot refuſe itſelf the ſatisfaction, of dwelling with comparative enthuſiaſm on facts which appear favourable to the adopted ſyſtem; while others of a different tendency, are either reluctantly, and therefore ſuperficially conſidered, or what is yet worſe, even ſtudiouſly avoided.

In the preſent ſtate of our knowledge, to diveſt geology of theory, would be to deprive it of all its intereſt. We muſt not deſpair, however, that by the multiplication of particular facts, and the expoſition of others, with which we are ſtill unacquainted, a ſyſtem of geology may yet be formed, founded excluſively on the phenomena of nature, or at leaſt on reaſoning much leſs hypothetical than is now required.

The moſt obvious means of attaining this object, ſeems to be a careful, minute, and candid examination of every circumſtance which appears to convey an explanation of itſelf, without reference to any theory; and from theſe we may ultimately hope to obtain ſome data, equally certain and comprehenſive.

It is with this view, that I have always formed my collections of geological ſpecimens; and although it will appear, that the arguments I have deduced are favourable to one ſet of opinions, yet I can aſſert with confidence, that the diſtrict which it is the object of the preſent paper to examine, has been faithfully explored, and, I hope, candidly deſcribed.

It is generally admitted, that no city in Europe is more favourably ſituated than the metropolis of Scotland, for the ſtudy and purſuit of geology: even the ground which it occupies, when laid open for the erection of buildings, has occaſionally preſented ſome very intereſting phenomena. The hills in the immediate neighbourhood, always at command, afford a never-failing ſource of reſearch; and in the ſurrounding country, a greater variety of foſſils is to be met with, than almoſt in any ſpace of the ſame extent.

The importance of a complete acquaintance with the phenomena which ſurround this city, cannot therefore, I think, be conſidered of a trivial nature. Indeed, by the number of ingenious works already before the public, it may be thought that the ſubject is exhauſted. But this is an error I am very deſirous to combat, not only becauſe in my own experience I have found it to be one, but becauſe, as ſcience advances, our habits of inveſtigation improve, phenomena become more familiar, we learn to trace and to ſeize not only the objects we are in purſuit of, but alſo to detect others, which our leſs practiſed eye had originally paſſed over unnoticed.

We all think ourſelves perfectly acquainted with the rock, on which our Caſtle ſtands. But I ſuſpect there are many members of this Society, who will be ſurpriſed to learn, that ſandſtone occurs near its ſummit, and alſo at its baſe. Saliſbury

lisbury Craig and Arthur's Seat appear perfectly familiar to us; there are phenomena belonging to both, however, of which, I have no doubt, many are yet ignorant. That any circumstance of an interesting nature, should remain unobserved, can only be accounted for, by its being taken for granted, that these conspicuous objects, having already undergone much critical examination, nothing farther remains to be noticed. This is an opinion, which I shall prove in the sequel, to be without foundation.

Arthur's Seat and *Salisbury Craig*, are naturally the objects, which first attract the attention of the geological traveller, on his arrival in Edinburgh; and to these places he is generally conducted by some one of our *amateurs*, when the favourite theory is introduced, and each corroborative fact dwelt upon, with all the usual keenness of theoretic discussion. This was the ground which, in all probability, first suggested the Theory of HUTTON; and it was perhaps here, that his comprehensive mind originally laid the foundation, of the structure which he afterwards so successfully reared. But that theory, in itself so beautiful, and in many points so perfect, I am very far from embracing entirely. I am very far, indeed, from following him through his formation and consolidation of strata, or the transportation and arrangement of the materials, of which they are composed. There are other circumstances also, which, though totally irreconcilable with any other hypothesis, are yet but imperfectly explained by his. I particularly allude to the singular contortions, exhibited in what are termed *Transition* strata, so finely exemplified on the coast of Berwickshire. I wish to carry my inductions, just as far as facts will bear them out. It is therefore, only in the regions of unstratified rocks, or in their immediate vicinity, that I have as yet, been able to

discover

difcover a language, which, if ftudied with due attention, cannot fail, I think, to become intelligible, and carry conviction to thofe, who choofe to reafon impartially on the fubject.

In the writings of Dr Hutton, we do not meet with defcriptions of particular diftricts, his object being rather to eftablifh a general theory, by the particular facts which thefe diftricts afforded.

We cannot, therefore, look to him for a mineralogical account of the neighbourhood of Edinburgh; and we have to regret, that no other geologift has yet undertaken that tafk.

In a fhort notice, in the Appendix of a work on another county, by Profeffor Jameson, this vicinity is mentioned as principally belonging, to what is termed the *Coal Formation* by Werner, which, according to the fyftem of that celebrated naturalift, forms part of the *Flœtz* rocks.

To render thefe terms intelligible to the general reader, it is neceffary to give fome explanation, as, without a confiderable knowledge of the fyftem to which they exclufively belong, they muft be totally incomprehenfible.

Werner is the only perfon, who has attempted a regular arrangement of rocks; an arduous undertaking, which I have no doubt he has accomplifhed, with all the accuracy the fubject was fufceptible of, and fo far as the country he examined allowed *.

But it appears very evident, that the facts he met with were fuch, that, in confequence of the hypothefis he had previoufly thought proper to adopt, it became neceffary to invent a theo-

 ry

* Links from other quarters, having been fubfequently added to his formation-fuites, by his pupils.

ry capable of embracing all the phenomena, which the conſtruction of his ſyſtematic arrangement led him to obſerve. A peculiar language was therefore indiſpenſable; and as this language has been conſtructed with ſo much regard to his theory, unleſs that is underſtood and adopted, his terms become uſeleſs.

By a *formation* is meant, any ſeries or ſuite of rocks which uſually occur together; hence the Coal Formation is compoſed of

1. Sandſtone,	6. Limeſtone,
2. Coarſe Conglomerate,	7. Marl,
3. Slate-clay,	8. Clay-ironſtone,
4. Bituminous Shale,	9. Porphyritic Stone,
5. Indurated Clay,	10. Greenſtone *,

with which the Coal occurs in numerous beds, varying extremely in thickneſs. Theſe, however, never all occur together, and it is no detriment to the Coal Formation ſuite, even if Coal itſelf ſhould not be found among them.

Again, the term *Flœtz* is given to all the formations, contained between the tranſition and alluvial rocks, and implies that they are diſtinguiſhed by their frequent occurrence in beds, which are much more nearly horizontal, than the primitive and tranſition

* Greenſtone is a literal tranſlation from the German; it is an extremely improper name; but as we have no other by which we can distinguish this variety of trap, we muſt uſe it till a more appropriate is found, even at the expence of ſuch language as *red and blue greenſtones*. In the mean time, it muſt be underſtood merely as an arbitrary term.

tranſition rocks. If directly tranſlated, the word ſignifies *flat*, and may be correctly deſcriptive of the diſtricts originally examined by WERNER; but as this conſtruction will not apply univerſally to this claſs of rocks, and as it is particularly at variance with thoſe belonging to it in this country, it would be better to follow the example of Profeſſor DAVY, and uſe the term *parallel rocks*, which is much leſs liable to objection.

THE Huttonian Theory has no language peculiar to itſelf, having nothing to deſcribe, that cannot be done in the uſual phraſeology of any country. This, by the zealous admirers of that doctrine, may no doubt be lamented, as depriving it of an apparent ſyſtematic arrangement, to which the oppoſite theory is ſo deeply indebted.

IN forming a collection from the rocks in the neighbourhood of Edinburgh, the circumſtances above narrated, induced me to begin with thoſe of Saliſbury Craig and its vicinity. The collection I have now the honour of preſenting to the Society, I began ſome years ago: it is only part of a ſeries, which, as completed, I hope may be found worthy of a place in their cabinet. I conſider it of very great importance, that every geological paper, ſhould be accompanied with ſpecimens, in order that if the former be found deſerving of publication in your *Transactions*, thoſe who peruſe the deſcription may know, that the ſpecimens referred to, are to be ſeen in the repoſitories of this eſtabliſhment.

Salisbury Craig is ſituated on the north ſide of Arthur's Seat, againſt which its ſouthern extremity reſts: from this it extends, in a northern direction, and rounds towards the eaſt, ſo as to form the ſegment of a circle, about half a mile in length It is ſurmounted by a magnificent façade, which is loweſt at the extreme points; towards the middle, the perpendicular rock may be from 80 to 90 feet high. From the baſe of this precipice, a ſloping bank, covered with debris, reaches to the valley below, altogether forming an elevation of nearly 400 feet. From the upper edge of it, a regularly inclined plane, ſlopes gently, on the oppoſite ſide, at an angle of about 15°, in a north-eaſt direction, and forms the left bank of the valley, called the Hunters Bog. On the right of this valley, the rocks again riſe rapidly, affording indications of two or three ſeparate façades. Theſe are not characterized in the diſtinct manner of Saliſbury Craig, but are ſurmounted by a ſurface, which, though a little rounded, preſents an inclination correſponding with that of the Craig, in a very ſtriking manner.

From the baſe of Saliſbury Craig, or rather from the baſe of the debris by which it is encircled, towards the ſouthern extremity, the ground again riſes, and preſents an inclined plane, ſimilar to its own, but of leſs magnitude. This is known by the name of St Leonard's Hill.

Hence it appears, that there are three ſimilarly inclined planes or terraces, of which Saliſbury Craig forms the intermediate one, each of them having a different elevation. From this ſtructure we may eaſily conceive the origin of the Swediſh word *Trap*, which has been employed as a generic term, for the

the claſs of rocks to which this appearance may generally be attributed *.

If we imagine a vertical plane, to paſs from St Leonard's Hill in an E. N. E. direction, which ſhall cut Saliſbury Craig, and continue through the right bank of the Hunters Bog, we ſhall find the rocks diſpoſed in the following manner:

St Leonard's Hill.

Sandſtone.
Porphyritic Greenſtone.
Sandſtone.

Salisbury Craig.

Sandſtone.
Greenſtone.
Sandſtone.

Hunters' Bog.

Greenſtone.
Sandſtone.
Porphyritic Greenſtone.
Trap-Tuff.
Baſalt.

The

* One of the greateſt difficulties which geology as well as mineralogy has laboured under, is the multitude of ſynonymous terms which have been applied to every individual foſſil. Trap has ſuffered from this diſadvantage, perhaps more than any other variety of rocks; as above noticed, that name is derived from the ſimilarity to the ſteps of a ſtair, which may generally be traced in the outline of a country, in which this rock abounds; and as it has been employed as a generic term by mineralogiſts throughout Europe, I think it proper to use it, to the excluſion of *whinſtone*, the name it bears in the writings of Dr Hutton;

a

The two laſt of theſe are not comprehended in the Coal Formation ſuite; they are conſidered as members of another formation, denominated the Neweſt Flœtz-Trap.

THE upper ſandſtone of St Leonard's Hill, and the lower ſandſtone of Saliſbury Craig, are, ſo far as we know, continuous; but as theſe, ſuppoſing the lines of the ſtrata to be projected, would form a bed of 450 feet thick, it is poſſible alternations of greenſtone may occur in it. Above, I have only mentioned ſuch as are viſible.

THOSE on the right of the Hunters' Bog, are not ſo diſtinctly expoſed as the reſt; but the foſſils are all found in the order I have ſtated. Occaſionally ſmall ſeams of reddiſh-brown coloured ſlaty clay, and clay-ironſtone occur, principally intermixed with the ſandſtone; but they are ſo thin, and ſo unconnected, that they can ſcarely be conſidered as ſtrata.

THE ſeries of ſpecimens I am now about to deſcribe, are thoſe of St Leonard's Hill and Saliſbury Craig.

No. 1. is a ſpecimen of the Sandſtone of St Leonard's Hill; it is of a reddiſh-white colour, and extremely coarſe-grained. It was taken from the middle of the quarry, and preſents a ſpecies of conglomerate, the fragments of ſandſtone being agglutinated by a dark-red ferruginous paſte.

No. 2. from the ſame quarry, is more compact, and preſents a ſtreaked appearance, correſponding with the direction of the ſtratum. There is a conſiderable degree of irregularity to be obſerved, in tracing the line of junction at St Leonard's Hill. In ſome places, two or three folds of the ſtrata are cut off abruptly

a name which, though perfectly underſtood in this country, is not received abroad, and ought therefore to be relinquiſhed.

abruptly at each end by the greenſtone; in another, that ſubſtance ſinks ſuddenly as it were into a gap in the ſtrata, and being loſt in rubbiſh, has ſomewhat the appearance of a dike. Beyond this a double horizontal wedge of greenſtone, with the ends turned downwards, appears among the ſtrata; and a little farther, towards the north, a roundiſh maſs of the ſame ſubſtance alſo occurs; this has very much the appearance of an included fragment, but the decompoſition of the ſandſtone has juſt begun to expoſe its connection with the rock above.

On the ſandſtone, Porphyritic Greenſtone (No. 3.) reſts. The colour of this is reddiſh-brown; the texture is fine-grained; and it contains ſmall ſpecks of fleſh-coloured calcareous ſpar. It is traverſed in various places by veins of Hematitic Iron-ore (No. 4.) accompanied with ſulphate of barytes. Theſe two ſpecimens have very much the character of ſome varieties of porphyry-ſlate, and on breaking one maſs, I obſerved a tendency to a ſlaty arrangement. In different places of this quarry, the greenſtone aſſumes a variety of appearances (No. 5. and 6.), ſome of which might be attributed to decompoſition. I do not conceive, however, that any external cauſe has ever had much effect upon this rock, although in ſome places it has entirely loſt its luſtre, (No. 7.), and might be miſtaken for trap-tuff, were it not for the ſhape of the cryſtals.

Above this, the rock graduates into a highly cryſtalline Porphyritic ſtone, (No. 8.) the paſte of which is of a browniſh-grey colour, very cloſe-grained, with an uneven ſplintery fracture, containing both cryſtals of felſpar and hornblende.

In this quarry there are ſeveral inſtances of *ſlikenſides*, one of which is rather remarkable, it occurs in an inclined rent in the ſandſtone: the traces of the ſlip, (No. 9.), are horizontal,

tal, and extremely well defined; but immediately over it, in the greenſtone, the appearance of the ſlip is not continued. Some indications of a ſlip appear a little to the right of it.

In a part of the Greenſtone which is conſiderably decompoſed, a vein, ſtretching horizontally, of a dark-green fibrous ſubſtance occurs, (No. 10.); it is ſoft, and has a ſhining ſatiny luſtre, like aſbeſtus. I have not anywhere in this vicinity met with any ſimilar ſubſtance.

We now proceed to Saliſbury Craig, where the circumſtances I ſhall principally notice, are,

1. The texture of the greenſtone rock, with the foſſils it contains.
2. The vein of greenſtone by which the Craig is interſected.
3. The included maſs of ſandſtone which occurs in the greenſtone; and,
4. The indurations and interruptions of the ſtrata.

No. 11. is a ſpecimen of the greenſtone taken from the lower edge of the bed, at the great quarry, where it touches the ſandſtone; the point of contact being marked by a ſmall remaining fragment of the latter, at which the grain of the ſtone is much finer than at the other extremity. The colour is iron-grey, with ſmall ſpecks of calcareous matter interſperſed.

Nos. 12, 13, & 14. are different gradations of texture, taken in a vertical line, from the edge towards the centre, where the ſtone is always moſt perfectly cryſtalliſed; from hence it again declines in grain towards the upper ſurface, where we find it in the ſame earthy and uncryſtalliſed ſtate (No. 15.) obſerved at the bottom. In the laſt ſpecimen, there is a ſmall detached fragment

fragment of the ſtratified matter imbedded in the greenſtone, a circumſtance connected with a very important claſs of facts.

No. 16. This ſpecimen of greenſtone is remarkable, as exhibiting a variety of colours; theſe are not blended, but diſtinctly divided from each other. The colours are iron-grey, light-grey, dark-red, and brick-red.

No. 17. This ſpecimen is a ſtrong example of the impropriety of the name which it bears; it is a greenſtone, decidedly of a red colour. The ſingular penetration of ferruginous matter, which is exhibited in various parts of this rock, is not eaſily accounted for; but ſuppoſing it to have been once in a ſtate of fuſion, it may have obtained this ſuperabundance of iron by abſorption, as the adjoining ſtrata frequently abound in that mineral.

In various parts of the Craig, veins of a peculiar nature may be obſerved; they are compoſed preciſely of the ſame ingredients as the rock, and are diſtinguiſhable only by the red colour of the felſpar, (No. 18). Theſe are termed *contemporaneous veins*, or *veins of secretion;* they are deeply waved, and generally follow the direction of the bed. Some of them preſent a very bright brick-red colour, (No. 19.), mixed with ſpecks of calcareous ſpar.

Nos. 20, 21. in theſe ſpecimens, are ſmall globules of a black earthy ſubſtance, which I am at a loſs to name. I ſhould have conſidered it Amphibole, but for the next ſpecimen, (No. 22.), in which the ſame ſubſtance appears to occur in irregular fragments.

No. 23. Analcime with cryſtalliſed Calcareous Spar. I before noticed, that it was in the heart of the bed where the ſubſtance of the greenſtone preſented the cryſtalline texture in

 the

the higheſt perfection. The occurrence of the analcime is connected with the ſame fact. I have never been able to find it on Saliſbury Craig, excepting at one period, when an entire ſection of the bed was quarried off, and about the middle of this the analcime occurred.

No. 24. with ſulphate of barytes, with calcareous ſparry iron-ore.

No. 25. part of a very irregular vein. Its ſides are formed of calcareous ſparry iron-ore, which is followed by a coating of hematitic iron. Here the regular ſtratification, as it is called, of the vein ends, and calcedony, firſt ſemitranſparent, then opake, and common calcareous ſpar, occupy the reſt.

No. 26. calcareous ſparry iron-ore cryſtalliſed, with ſome tranſparent cryſtals of quartz.

No. 27. large cryſtals of calcareous ſpar, with cryſtalliſed and radiated tufts of quartz.

No. 28. red oxide of iron, with a vein of calcareous ſparry iron-ore.

No. 29. green coloured quartz, with a coating of cryſtalliſed quartz.

No. 30. cryſtalliſed quartz, with amethyſt.

Such are the minerals which occur on Saliſbury Craig. Some of them are rare, and others to be found only when the rock is working in particular places.

The next circumſtance I have to notice, is the vein of greenſtone *. It occurs a little to the north of the ſpot, to which

* The term *dyke* has been very generally applied to *veins* of this deſcription, and I am not ſatisfied that it is the leaſt proper of the two; as there certainly is a marked diſtinction between veins compoſed of rocks, and what we generally

which the cart-road, along the bafe of the rock extends, a few feet beyond a gap, known by the name of the *Cat's Nick*.

I do not think this vein attracted the attention of geologifts in any particular manner, prior to 1805. It certainly was obferved long before that period, but was not known to extend through the bed of greenftone, till Sir James Hall and myfelf noticed, that after cutting the fandftone, it continued its courfe uninterrupted to the top. This obfervation contributed very much to increafe our curiofity, and a man was employed to clear away the foil and rubbifh, which had accumulated on the furface. A confiderable portion of the rock was foon laid open, below the point from which it was at that time vifible. Nothing, however, of much intereft, was by this means difcovered. The dike, after bending a little to one fide, continued its courfe downwards.

The fpace which this dike occupies, may be from fix to eight feet wide; its width varies a little in fome parts, and thefe variations are apparently increafed, if the fection which is obferved be not at right angles with the walls. That portion embraced by the ftrata, which we found principally covered with debris, was very much decompofed, prefenting on the furface a certain degree of nodular exfoliation, of a rufty-

 brown

ly underftand by mineral veins. The firft are formed of one uniform rock, compofed in all their parts of the fame conftituents, and differing only in pofition, from the beds thefe materials more ufually form; while the latter, though fometimes formed only of one fubftance, fuch as quartz or calcareous fpar, are generally compofed of a feries of foffils, arranged in lines parallel to the walls. No fuch appearance ever prevails in rock veins, or conftituting mountain maffes; therefore, in ufing the term *vein*, when applied to greenftone, granite, or the like, it muft be underftood as a generic term, of which these latter, fpecify the variety.

brown colour. On breaking into the rock, it exhibited (No. 31.) * the concentric lines ſo common in decompoſing greenſtone; and beyond this, the ſtone preſented a degree of freſhneſs, with a very coarſe grain of a peculiarly light aſh-grey colour, and a very dull earthy texture, (No. 32.) Between this portion of the vein and that embraced by the greenſtone, there is a very remarkable difference, the latter being of the uſual iron-grey colour, and otherwiſe perfectly characteriſtic. Before it leaves the ſandſtone ſtrata, it ſeems to contain an unuſually large proportion of calcareous matter. This may have aided the decompoſition, together with the moiſture retained by the debris, ſo lately removed from its ſurface, and which has left it in a ſtate eaſily affected by the weather. Since I commenced writing this paper, I made an excurſion to the ſpot, and was greatly ſurpriſed to obſerve the devaſtation of laſt winter.

Before the vein riſes above the level of the ſtrata, a portion of it, ſtill more decompoſed than the reſt, of a dark-purple colour, branches off, and embraces a wedge-ſhaped maſs of the ſandſtone (No. 33. and 34.) indurated in a very high degree. Juſt at the top of this indurated maſs, the whole dike changes its colour, and, I may alſo ſay, its conſiſtence. It here preſents a light-greyiſh aſpect, deeply ſtained, with red ferruginous marks, of a dull earthy texture, an even fracture, and a tolerably fine grain, (No. 35.) That portion correſponding with, and immediately over the included ſandſtone, I found much coarſer in the grain, (No. 36.), and in a more decompoſed ſtate;

* Correſponding numbers will be found in the annexed engraving, which will explain more fully the relative poſition of the ſpecimens.

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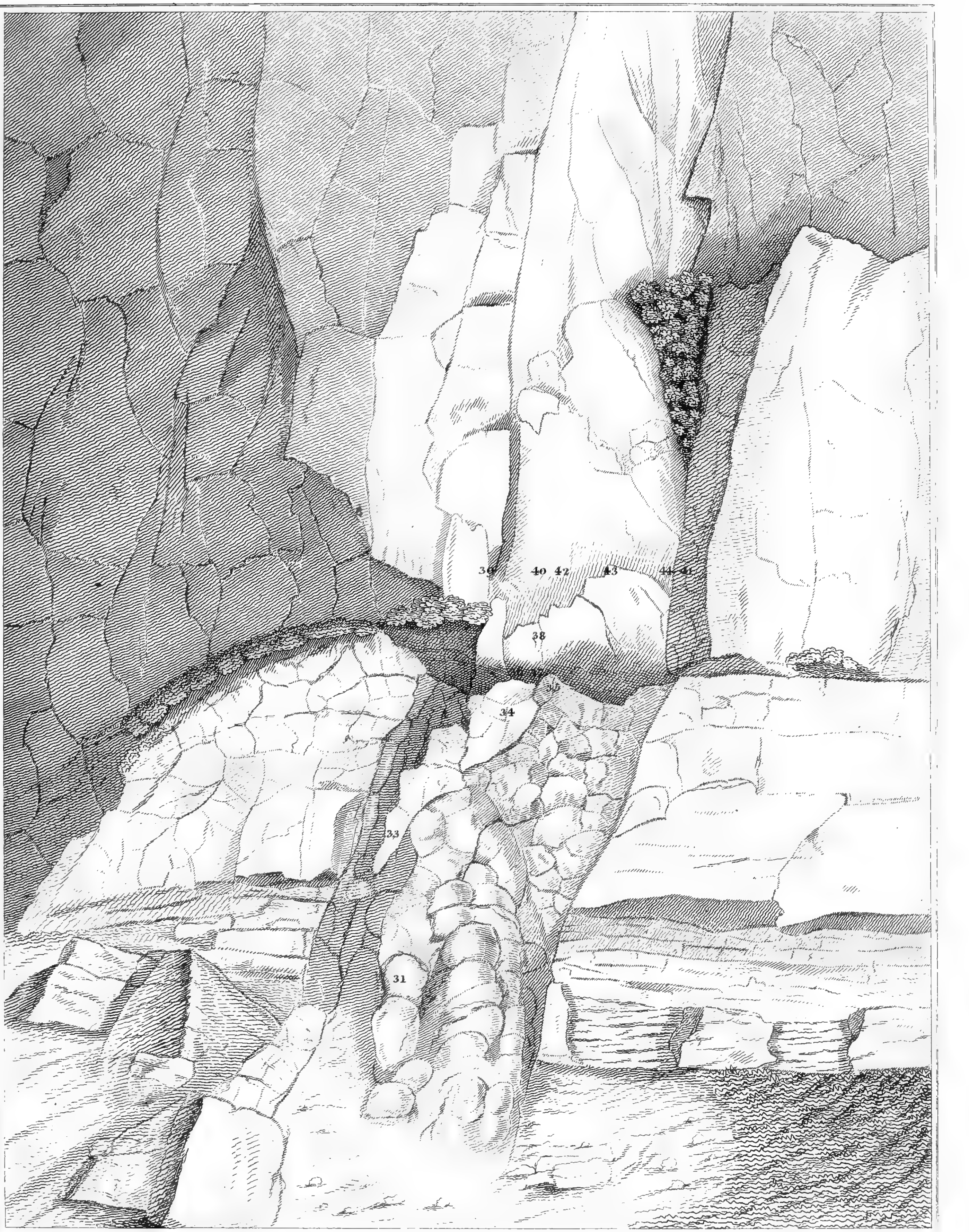

ſtate; while it differed from No. 37., the ſtone on the ſides, which were perfectly ſimilar to each other in compoſition.

Tracing the friable purple-coloured portion upwards, I found it gradually became harder, and, of a ſudden, change to a fine-grained blue-coloured greenſtone; and the part correſponding with the included maſs, alter to a hard coarſe-grained rock, (No. 38.) I ſoon obſerved, that this coarſe-grained maſs, which is about ten inches thick, continued upwards, maintaining an uniform dimenſion and poſition, in reſpect to the walls of the vein, as high as the eye could trace it in the rock, thus dividing it into two portions; that on the left ſide being about eighteen inches wide, while the other is about five feet.

On comparing the texture of the included ſtripe, with that of the walls on each ſide, (No. 39. left ſide; No. 40. included ſtripe; No. 41. right ſide,) taken in a horizontal line, about ſix feet above the ſtrata, I found as cloſe a reſemblance as it is poſſible to conceive; they are all coarſe-grained, and highly cryſtalliſed. This ſimilarity is not more remarkable, than the difference between the ſubſtance of the vein and the included maſs. Specimens taken from the junction of theſe, mark this in a ſtriking manner. No. 42. is from the left ſide of the right portion of the vein, to which the fine-grained part belongs. No. 43. is from the middle of this portion; and No. 44. from the ſide next the right wall. Theſe were alſo taken in a horizontal line, and exhibit the ſame gradation of grain noticed as exiſting in the great bed. Even in the narrow portion of eighteen inches, on the left ſide, this circumſtance is quite viſible; but the ſpecimens taken from the other are highly illuſtrative of the fact.

I.

I HAVE had an opportunity of examining many veins of greenſtone; but I know of none more intereſting in a geological point of view than this.

I THINK it can ſcarcely be doubted, that the ſame effort which ſeparated the included portion of ſandſtone, cleft the correſponding ſtripe of greenſtone from the great bed. This, as well as the gradation of grain, everywhere obſervable in beds and veins of trap, are remarks, in my opinion, of conſiderable value to the Huttonian hypotheſis. On a former occaſion, when I had the honour of ſubmitting ſome remarks on the north of Ireland to the Society, I took an opportunity of dwelling particularly on the laſt circumſtance. Like the charring of coal, when that ſubſtance is found in contact with whin, as has been ably remarked by Profeſſor PLAYFAIR, " few facts in the hiſtory of foſſils ſo directly aſ" ſimilates the operations of the mineral regions with thoſe " which take place on the ſurface of the earth *." This gradation of texture has a ſtrong analogy to many accidental facts obſervable in furnaces, of glaſshouſes and the like, and ſtill more ſo to thoſe experiments made expreſsly for the purpoſe of aſcertaining the effects of ſlow cooling, by Sir JAMES HALL and others. One additional argument for the igneous origin of theſe veins, has been added by the obſervations of Sir GEORGE MACKENZIE and his friends, in Iceland, in perfect correſpondence with the above fact. He there found many veins of this ſubſtance, coated on the ſides with a glaſſy covering, exactly ſimilar to melted greenſtone, when rapidly cooled.

I SHOULD expect the ſame circumſtance would be met with in veins of porphyry and granite; but I have not been able to extend

* Illuſtrations of the Huttonian Theory, § 68.

tend my obſervations ſo widely, as to embrace the facts reſpecting theſe rocks. One remark I ſhall, however, hazard in this place, reſpecting an eſſential difference between veins of granite and thoſe of greenſtone. The former ſeem to be of ſimultaneous formation with the great body of that rock, to which they may generally be traced, and, ſo far as I have hitherto obſerved, are never found to cut it. Veins of greenſtone, on the other hand, I have never ſeen connected with the great beds of that ſubſtance; they traverſe theſe juſt as they do every other kind of rock, and conſequently are in all inſtances of a poſterior formation. I am aware, that theſe ideas are very much at variance with certain received opinions. I therefore wiſh to be underſtood as ſpeaking ſolely upon my own experience.

I HAVE now to mention the well-known included maſs of ſandſtone. Along the edge of the ſtrata, a number of inſtances occur on Saliſbury Craig, affording the moſt unequivocal marks of diſturbance; but it preſents only one example, of a maſs totally enveloped in the ſubſtance of the greenſtone *.

THIS ſpot has been the ſcene of much controverſy, between contending geologiſts. While the Huttonian conſiders it as a moſt incontrovertible proof of violence and of heat, the Wernerian contends, that there is nothing in the leaſt extraordinary in the appearance, and aſſerts, that the ſuperficies of the apparently included maſs, is no more than the ſection of ſome part of the ſtratum, which, if traced, would be found to connect with the reſt; that it had been enveloped in the fluid menſtruum of the greenſtone, when in this elevated poſition; and that the rock being

* Since this paper was ſent to preſs, others have been obſerved in different parts of the rock.

ing cut in a certain direction, a ſection having the appearance of an inſulated maſs, would of courſe be expoſed to view. There is no doubt that ſuch a circumſtance is perfectly poſſible; but, in the preſent inſtance, this explanation will not be found at all applicable. In every other caſe, where the ſtrata appear diſplaced, they are not torn from the reſt, nor has the greenſtone inſinuated itſelf, except as a wedge, ſupporting the lifted maſſes. The included maſs is of a light greeniſh-grey colour, in ſhape quadrangular, and, when minutely examined, will be found ſhivered into numerous diſtinct fragments, with veins of greenſtone running through it in every direction. It partly retains its original ſtratified texture (No. 45.) although indurated in a very high degree, and is ſo firmly welded to the greenſtone, that it is no difficult matter to obtain ſpecimens (No. 46.) of the conjoined rocks; one ſmall ſpecimen (No. 47.) in the collection, is twice interſected by that ſubſtance. It, therefore, has no reſemblance whatever to thoſe pieces of ſtrata, which are only in part detached, and which, if cut in a tranſverſe direction, would, in all probability, exhibit an inſulated ſection. That ſection, however, would not diſplay the broken and diſtorted appearance deſcribed above, at leaſt if we may be allowed to judge by the integrity of the longitudinal ſections, of which there are ſo many examples in this vicinity. Beſides, the colour of the included maſs is totally different from that of any of the ſtrata near it, which are here of a deep red (No. 48.), and at this particular ſpot are remarkable for their apparent derangement. I therefore conclude, that there is every reaſon to conſider this, as a fragment detached from ſome other part of the ſandſtone, and left ſuſpended in its preſent ſituation, when the greenſtone aſſumed a ſolid conſiſtence, as was originally conjectured by Dr HUTTON.

I

I NOW come, as propofed, to that divifion of the fubject which relates to indurations. By *induration* is meant, a greater degree of compactnefs, obfervable in particular parts of ftratified rocks, than is ufual throughout their mafs. One part of a bed may be harder than another, confequently more indurated. But the induration here alluded to, is that which is fuppofed to have been effected, by an alteration in the denfity of the ftone, in confequence of the action of heat.

THESE phenomena are of a very ftriking nature, and were firft brought into notice by Dr HUTTON; in them, he found evidence, to him perfectly conclufive, of the igneous formation of whin, and, with that ingenuity and perfeverance which characterife the whole of his works, he did not fail to generalife his obfervations, and to place the facts, firft noticed in this fpot, in fuch a light, as to render them effentially ufeful to his theory.

THE anxiety which the difciples of the Wernerian fchool have always evinced, to undervalue the merit of this obfervation, is a fure mark of the eftimation in which they hold it; and it is, therefore, very properly confidered by the fupporters of the oppofite doctrine, as one of their ftrongeft holds. In the following lift, are comprehended moft of the varieties, which this indurated fandftone prefents on Salifbury Craig.

No. 49. is a junction fpecimen *, taken near the fouthern extremity of the Craig; here the greenftone is of the deep red tinge noticed at No. 17.

* By *junction fpecimen* is meant, a fpecimen which exhibits the greenftone and the fandftone conjoined.

No. 50. is another ſpecimen of the ſame kind; the greenſtone is here of the uſual colour, and the line of junction moſt admirably defined. This was taken from the great quarry. The next, (No. 51.), is a ſpecimen of the ſandſtone in its ſuppoſed unaltered ſtate. Nos. 49, and 50. are both from the lower junction. No. 52. is from the upper edge, taken about halfway between the higheſt part of the Craig and Holyroodhouſe. Here the ſandſtone preſents a facetted appearance, an arrangement which may be owing to the ſuperabundance of calcareous matter.

No. 53. is highly indurated, of a deep red colour, with a conchoidal fracture, and a facetted texture.

No. 54. has the ſame facetted appearance.

No. 55., extremely cloſe-grained, is from one of the contortions north of the dike.

Nos. 56, & 57. Theſe are the varieties of the ſandſtone which have been called *jasper*. This is an improper name, as it confounds two ſubſtances totally different. The moſt compact contains a large proportion of lime, and in aſpect is very ſimilar to ſome of the limeſtones of Gibraltar.

Nos. 58, to 61. are varieties of the ſandſtone, found near the greenſtone.

No. 62. Although this ſpecimen was taken very near the greenſtone, ſtill it does not exhibit the uſual induration. This exception occurs in different places on Saliſbury Craig; and it even ſometimes happens, that the ſtone next the whin is leſs indurated than the one below it.

No. 63. Containing a large proportion of ferruginous matter.

No. 64, to 66. Different ſhades and varieties of the ſandſtone, in an indurated ſtate.

No.

No. 67. In this ſpecimen there is ſomething very like the appearance of an agate; it, however, is not contained in the ſubſtance of the greenſtone, but in the ſtratified matter below it.

No. 68. Another ſpecimen of the ſandſtone, in its unaltered ſtate, taken about thirty feet from the greenſtone.

Dr Hutton conceives, that the induration, ſo very remarkable in the above ſpecimens, was occaſioned by the heat of the whin, when it was injected between the ſtrata of ſandſtone, cauſing it to undergo a certain degree of fuſion; and, to this idea, the facetted texture of ſome of the ſpecimens adds conſiderable weight, ſuch arrangements being very familiar in ſtones which have undergone fuſion.

The Wernerian ſchool, to account for the ſame phenomenon, aſſerts, that as ſandſtone is generally porous, the fluid ſolution of the trap being introduced into the fiſſure, naturally percolated to a greater or leſs extent *. Again, that it is owing to the intermixture of the matter of the vein, with the rock that forms its walls †; and, as a proof of this, it is added, that no induration appears, where the traverſed rock is poſſeſſed of a quartzy baſe.

These arguments occur in different works, but they appear to me very little calculated to ſupport the point in diſpute, if not in ſome reſpects contradictory. On Saliſbury Craig, and generally throughout the neighbourhood of Edinburgh, whereever we find ſandſtone coming in contact with greenſtone, either in beds or veins, we are almoſt certain, that an induration will be exhibited along the edge of the ſtrata.

* Comparative View of the Huttonian and Neptunian Theory, p. 130.

† System of Mineralogy, vol. iii. p. 365.

It has already been obſerved, that there are ſpots on Saliſbury Craig, where this is not ſo apparent as in others, and it very often happens, that ſmall ſeams of clay occur, in a perfectly ſoft ſtate. In Ireland, at Scrabo, in the county of Down, and at Fairhead in that of Antrim, I found ſandſtone in the former, cut by veins, and in the latter, overlayed by a bed 300 feet thick, where no induration was to be ſeen. Now, it appears concluſive, that there could not have been a deficiency of induration in any ſpeck of Saliſbury Craig, far leſs a total abſence, as in the caſes quoted in Ireland, had it in any inſtance been effected either by percolation, or by the intermixture of the matter of the vein. The ſuperincumbent or included matter, if in a fluid ſtate, whatever its chemical powers were, would, to a certain extent, act mechanically, and be, in all circumſtances, poſſeſſed of the ſame power of communicating its moiſture to the ſurrounding maſſes. It is therefore impoſſible to conceive, how it ſhould have withheld it in one inſtance, and parted with it ſo amply in another, how it ſhould have indurated the ſandſtone, and left the thin ſeams of clay in a ſoft and friable ſtate. It is quite unimportant, of what baſe the ſandſtone may be formed; it is a ſubſtance, allowed as above to be generally porous, (and, in the caſes alluded to, it certainly was ſo); into that poroſity, therefore, the fluid muſt have percolated, whatever the baſe may have been.

On the contrary, according to the Huttonian hypotheſis, induration diſtinctly depends, on the compoſition of the ſtrata expoſed to the influence of heat. Some ſtrata may either wholly, or in part, be capable of reſiſting much higher temperatures than others. It is conſequently to the ingredients of which they are formed, that we muſt look either for the cauſe of induration, or the abſence of it. This remark originated in obſerving,

ſerving, that all the indurated ſandſtones of this country, contained more or leſs calcareous matter, while the unindurated ſpecimens from Ireland, did not afford the ſlighteſt indication of that ſubſtance, when ſubjected to the ſame teſt.

Before I take leave of Saliſbury Craig, I muſt notice one more circumſtance, which, ſo far as I have hitherto ſeen, is quite peculiar to the ſpot. I mean the occurrence, in veins, of a ſubſtance in all reſpects ſimilar to the indurated ſandſtones, I have juſt been deſcribing. The firſt of theſe I obſerved, is about thirty paces north of the vein. The ground being cut away, in order to ſee its connection with the ſtrata, it branched out like the prongs of a fork, and had the interſtice filled with a red decompoſed ſubſtance (No. 69.), ſimilar to that which occurred at the extremity of the included ſtripe of greenſtone in the vein. Where the prongs join, it is about three or four inches wide, and is there, partly compoſed of indurated ſandſtone, and partly of hematitic iron-ore and calcareous ſpar. (No. 70.) Higher up, where the vein is narrower, it is wholly compoſed of ſandſtone, the ſpecimen, No. 71., being the entire thickneſs of it. Here the grain is finer than at firſt, and, higher up, it becomes ſtill more ſo, (No. 72.) It ſtill continues to taper upwards, and even when reduced to leſs than half an inch, the ſubſtance retains the uſual aſpect of indurated ſandſtone, (No. 73.) This vein riſes about twenty to thirty feet into the rock, always diminiſhing, and about that height diſappears. I have remarked other veins, alſo containing ſubſtances ſimilar to indurated ſandſtone (No. 74.), one was of a much larger ſize than that above deſcribed (No. 75.), but the grain not near ſo compact, (No. 76.)

These veins all ſet off from the lower ſurface, and ſo long as they are of any conſiderable thickneſs, the including rock is

ſtained

ſtained with ferruginous matter. This fact ſeems connected with the ſingular appearances, which occur in the vein of greenſtone, at the level of the junction of the ſandſtone ſtrata with the incumbent bed.

WITHOUT offering any remarks on a fact as yet ſo inſulated, I content myſelf with merely mentioning it, in hopes that ſimilar appearances may preſent themſelves to geologiſts in other quarters, and perhaps throw ſome light on a phenomenon, which by farther elucidation may prove intereſting.

BEFORE I cloſe this paper, I ſhall take the opportunity of preſenting to the Society, two ſpecimens which were given to me by Sir GEORGE MACKENZIE, and which I eſteem of conſiderable value; one of them, a fragment of the rock of Saliſbury Craig; the other, of the Calton Hill, marked in the handwriting of the late Dr KENNEDY, as the ſubſtances he analyſed, and of which an account was given in the 5th volume of theſe *Tranſactions*. The great variety in the rock, both of Saliſbury Craig and Calton Hill, makes it of importance to aſcertain with preciſion the kind employed in the reſearch of that celebrated chemiſt; and as the moſt proper place for their reception, I depoſite them in the cabinet of this Society, along with my own collection, under the Nos. 77, and 78.

19th

19th March.

Since I had the honour of reading the foregoing paper to the Society, a ſtrong caſe in illuſtration of what is mentioned in the commencement of it, has preſented itſelf; I mean, with reſpect to the conſtant occurrence of new and intereſting matter, even in the moſt frequented ground.

A few days ago, Profeſſor Playfair mentioned to me, that by the cutting down of a ſection of the Craig, within thirty paces of the ſouthern extremity, ſeveral maſſes of ſandſtone had made their appearance, imbedded in the greenſtone. I loſt no time in viſiting the ſpot, and was greatly pleaſed to find, a conſiderable addition to the intereſting facts, already exhibited on Saliſbury Craig.

At this part of the rock, the greenſtone becomes very thin, being no more than twenty to twenty-five feet thick; it has the appearance, however, of having once been of greater extent, the upper part being apparently cut away by ſome operation of nature, of which we have now only to obſerve the effects. It ſlopes rapidly towards the ſouth, and is covered to a conſiderable depth with ſoil and travelled ſtones. It is at the upper ſurface of this, that the imbedded maſſes occur; they appear to be portions of ſtrata, which obſerve the general inclination of the ſandſtone of Saliſbury Craig, that is, dipping towards the north-eaſt, while the expoſed ſections are parallel to each other, and nearly horizontal; conſequently, being near the ſurface, they are cut off, or crop out, on the ſouth ſide. Their appearance,

appearance, however, bespeaks their having been, at some former period, totally included in the greenstone. One mass, indeed, a little towards the north, is unequivocally so; at least we know with certainty, that a short time ago it was inclosed in the greenstone, and not to be seen; and there is at present, great apparent probability, that the next section taken from the same part of the rock, will carry it away altogether.

Till now, we only knew of one included mass in the greenstone of Salisbury Craig; and with this, these now discovered have considerable analogy; they are of the same colour, and although they appear to be only four or five distinct masses, these masses are all intersected vertically and diagonally, and are split through the whole length of the horizontal line; so that in examining a section of about ten feet perpendicular, no less than nine different alternations of sandstone may be reckoned. Some of them are no doubt very minute; but still they were all observable when I examined the rock.

From the most northern mass of included sandstone, I was enabled to procure a few specimens, which I have added to the above collection. The rock rises so rapidly from the south, that although this mass is nearly in the same horizontal line with the others, all of which crop out to the surface, and although it is not distant more than four or five yards, yet it appears to be situated nearly about the middle, between the sandstone and the upper surface, from which it may naturally be inferred, that the masses which now crop out, were like this, once entirely included in the substance of the greenstone. It is highly indurated, and at the extremities, is drawn out into minute veins. The thickness of the principal mass may be from ten to twelve inches, and in length from six to eight feet. This body, as above noticed, is cut in all directions by the greenstone.

greenſtone. The ſpecimen No. 79. ſhews a portion of the ſandſtone, with that ſubſtance traverſing its ſtratified lines diagonally. No. 80. is a maſs of the ſandſtone, containing a ſmall portion of greenſtone, much of the ſame ſhape as the double wedge of St Leonard's Hill, and formed, as I conceive, exactly in the ſame manner. This wedge, on one ſide of the ſpecimen, is two inches long, but, on the oppoſite, it is not one; and in the counter part of the ſame ſpecimen, (No. 81.) it is only to be ſeen on one ſurface; it does not penetrate to the other ſide, though ſcarcely an inch thick.

I AM glad to find, that intereſt has been made to prevent this valuable ſet of facts from being ſoon deſtroyed, as, in a few weeks, the rock in which theſe are contained would have been broken down, and carried off for the repair of the neighbouring roads.

IT is on this account, that much activity is requiſite to keep theſe periſhable phenomena from being loſt, in the neighbourhood of ſuch a town as Edinburgh. Similar things are preſenting themſelves conſtantly, but they are opened only for a day, and if not ſeized and recorded on the inſtant, will be ſhut up, and loſt for ever.

END OF THE SIXTH VOLUME.

DIRECTIONS TO THE BINDER.

The sheets marked a and b, containing the History of the Society and the Laws, to be placed in front of the volume, immediately after the general title-page and general table of Contents.

The temporary title-pages and tables of Contents for Parts I. and II. to be cancelled, as now unnecessary.

The last leaf of signature 3 D, pp. 390, 400, and the first leaf of 3 E, pp. 401, 402, are to be cancelled, and two new leaves substituted.

Plate I. fronting p. 18.

—— II. and III. fronting p. 70.

Sir James Hall's Plates, marked with small numerals, Pl. i, ii, iii, iv, v. to front p. 186.

Plates VI, and VII. fronting p. 244.

——	VIII.	-	-	248.
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——	XII.	-	-	420.

*** The Binder will take notice that the Plates must fold out.