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CONTENTS.

APPENDIX.

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XX.—*On the Thermal Conductivity and Specific Heat of Manganese-Steel.*

By A. Crichton Mitchell, B.Sc.

(Read 1st April and 1st July 1889.)

Introduction.

Until a few years ago it was the general opinion among metallurgists that the presence of manganese in steel exceeding the proportion of 1 per cent. is prejudicial to the value of the steel, inasmuch as a higher percentage of manganese has the effect of lowering markedly its tensile strength and toughness. But in 1884, Messrs Hadfield & Company, of the Hecla Steel Works, Sheffield, exhibited, at a meeting of the Institute of Mechanical Engineers, a number of samples of steel containing upwards of 10 to 15 per cent. of manganese, and submitted the results of experiments, which showed that the samples were, in point of tensile strength and hardness, in no way inferior to steel. Again, in 1888, Mr R. A. Hadfield read to the Institute a paper on the subject, giving the details of a large number of tests, which brought to light some interesting mechanical properties of alloys of manganese and iron. Since its introduction, these alloys (and particularly that containing 10 to 15 per cent. of manganese, known as "manganese-steel") have been studied by several physicists, and further peculiarities have been found. It appeared desirable that the thermal conductivity of so peculiar a substance should be investigated. The present paper is an account of experiments made in the Physical Laboratory, Edinburgh University, with a view to the determination of its thermal conductivity. In the reduction of such experiments a knowledge of the specific heat is necessary, hence there is also given an account of experiments whereby the specific heat was determined.

General Properties of Manganese-Steel.

It will be well to give here a brief summary of the properties of this substance, so far as they have as yet been investigated. In the first place, the peculiar effects of the addition of varying percentages of manganese to steel must be noted. Mr Hadfield's experiments on this point may be shortly summarised as follows:—Ordinary steel contains from 0·6 per cent. to 0·8 per cent. of manganese, besides the usual proportions of carbon, silicon, phosphorus, &c. If the proportion of manganese be increased to 2·5 per cent. a marked falling off in tensile strength takes place, the material becoming at the same time somewhat brittle and "unsound"; if from 2·5 to 7·5 per cent. be present, the steel becomes exceedingly hard, the tensile strength is still lowered, and it becomes so brittle that small samples may be reduced to powder in a mortar; as the proportion of manganese increases to 10 per cent. the brittleness disappears, and the

material gains enormously in toughness and tensile strength, while the hardness which is so marked in the lower percentage is lessened. The same holds with higher proportions up to 25 per cent. Thus, taking into consideration as well the other properties to be immediately mentioned, we find that while iron and manganese alloys containing upwards of 7·5 per cent. of the latter present features more akin to those of steel, others with higher proportions, exceeding 7·5 per cent., have the entirely different characteristics of what is practically a new substance. As already mentioned, it is to that particular alloy, containing from 10 to 15 per cent. of manganese, that the name "manganese-steel" has been given.

The chemical composition of manganese-steel varies, but the following may be taken as an average :—

Carbon,	0·85 per cent.
Silicon,	0·23 „
Sulphur,	0·08 „
Phosphorus,	0·09 „
Manganese,	13·75 „

Its density is 7·83 ; practically the same as wrought iron. Its hardness is of a somewhat peculiar nature. Thus, while to drill a hole in manganese-steel takes 15 or 20 times longer than in ordinary steel, and while it is hard enough to scratch any steel but the hardest-tempered, yet it may easily be indented by a blow from a hand-hammer. It is also strange that, being so difficult to drill, or to cut with a planing-tool, when subjected to a compression-load of 100 tons, cylinders of manganese-steel, 1 inch long, 0·75 inch diameter, were shortened 0·25 inch; while chilled iron cylinders of exactly the same dimensions, and under the same conditions, were scarcely altered. The tensile strength varies from 50 to 65 tons per square inch, according to the mode of treatment. Hard-drawn manganese-steel wire will, however, stand upwards of 110 tons per square inch. The tensile strength is greatly increased by the process known as "water-toughening"; *i.e.*, raising the material to yellow heat, and immediately plunging it into cold water. The elongation under stress is much greater in this alloy than in steel of the ordinary kinds, being in some cases as much as four times. Steel with a tensile strength of 60 tons per square inch seldom gives more than 10 per cent. elongation; while a similar bar of manganese-steel will give 50 per cent. elongation.

When manganese-steel is subjected to the usual process employed in tempering steel, it behaves in an unusual manner. "Water-toughening" makes it softer; heating it, and allowing it to cool in air, hardens it; sudden cooling also increases its ductility. These effects are the reverse of what takes place in ordinary steel.

One of the most peculiar features about this alloy is that it is almost non-magnetic; a fact first pointed out by Mr Bottomley * and Dr Hopkinson.† Since then, Professor Ewing‡ has fully investigated its magnetic properties. He finds that its magnetic suscep-

* B.A. Report, 1885, p. 903. † *Phil. Trans.*, 1885, Part II. ‡ B.A. Report, 1887, p. 587.

tibility is about $\frac{1}{8000}$ of that of iron, *i.e.*, only fractionally greater than that of copper, brass, or air; further, that it is constant; not undergoing any change corresponding to the "breakdown" in resistance to magnetisation which is found in the case of iron.

From the experiments of various observers, it would seem that the electrical resistance of manganese-steel is about eight times that of iron, the temperature coefficient being one-third of that in iron.

Thermal Conductivity of Manganese-Steel.

Experimental Methods and Details.—The bars upon which were made the experiments for the determination of the conductivity were kindly furnished, in the rough cast state, by Messrs Hadfield. The work of bringing them into the required shape and dimensions was one of considerable difficulty. It was at first attempted to reduce them to the desired rectangular section by planing, but this was found to be simply impossible. To begin with, only thick shavings could be taken off at each stroke of the planing-tool, which, when it did get a good grip, seemed rather to tear than to cut the material. Again, the planing-tools were soon all ruined, for, after a few minutes' work, their edges were all but completely turned off. Thus, even with the best tempering of the tools possible, the process of planing was found to be useless. In this extremity, Messrs Hadfield were asked to supply a piece of manganese-steel of harder quality than that of the bar, with which to make a planing-tool, but in reply they stated that although this had been frequently suggested, it had not proved successful. In these circumstances recourse was made to grinding the bar down by means of an emery-wheel revolving at high speed. This method does not, of course, ensure the same uniformity of section that planing would do; but on testing very carefully the finished bar, it was found that the section was quite uniform enough for the purposes of the experiment. The dimensions of the long bar were $50\frac{1}{2}$ inches by $1\frac{1}{3}$ inch by $1\frac{1}{3}$ inch. Eight holes were drilled in this bar for the thermometers, the first being 9 inches from one end, the others 12, 15, 18, 24, 30, and 42 inches respectively from the same end. The holes were $1\frac{1}{8}$ inch in depth and $\frac{17}{64}$ inch in diameter. The length of the short bar was 20 inches, its cross section being, of course, the same as that of the long bar. One hole of the same size as the above was drilled in the short bar, which was fitted with screw-eyes at the ends with which to support it on bearings while being heated. Both bars were finally nickel-plated.

A sample of the turnings from the bar was analysed in Professor Crum Brown's laboratory; the following is the mean of two determinations :—

Iron, ...	87·56 per cent.
Manganese, ...	9·89 „
Carbon, ...	1·30 „
Silicon, ...	0·48 „

The method of finding the conductivity was substantially that originally devised by Forbes, the only difference being that a shorter bar was used, with a cooling bath placed

either at its cooler end, or near its middle, on the cooler side of the fifth thermometer. This was suggested by Professor TAIT in his paper on "Thermal and Electric Conductivity,"* and first carried out in my experiments " On the Thermal Conductivity of Iron, Copper, and German Silver."† The advantages of this improvement were noticed in the latter paper.

The thermometers employed in the experiments were, with one exception, the same, and were used in the same way as in Professor TAIT's and my own previous work. The single exception was that of the thermometer in hole A (*i.e.*, that nearest the source of heat). This instrument was broken at the beginning of the experiments, but was replaced by another of exactly similar make and dimensions, and whose error was carefully ascertained.

Of the eight holes in the bar, only five were used, those being first five reckoned from that end of the bar towards which they are closer together. These holes were, for convenience, named A, B, C, D, E. The first four were separated by intervals of three inches; the fourth and fifth by one of six inches. The cooling bath, through which a stream of water was kept steadily passing, was placed close to hole E.

In all other respects the experiments were conducted on exactly the same lines as formerly, so that nothing remains to be said so far as details of experiment are concerned.

Deduction of Conductivity.—The manner in which the readings of the thermometers are reduced, and from them the curve of stationary temperature excess constructed, has already been fully described in my paper quoted above. It has also been pointed out that this curve furnishes, so far, a test of the extent to which the experiments made on different days agree among themselves. Judged in this way, the experiments were very successful; the only discrepance being in connection with the readings of the thermometer next to the cooling bath. But that this should be so is not at all surprising.

The curve of stationary temperature excess being obtained, the next step is to find the value of the tangents to it at different points corresponding to different sections of the bar. This is best done by finding an equation between v (temperature excess) and x (position along the bar, reckoned from some arbitrary origin), which will represent the curve, and by differentiation finding the value of the tangent of inclination which is simply that of $\frac{dv}{dx}$. In my previous paper, two formulæ were given which have been used for this purpose. These were—

$$\log v = \log a - \frac{bx}{1 + cx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

$$\log v = \log a + \frac{b}{c + x} - ex \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)},$$

where v and x have the above meaning, and where a, b, c, and e are constants. But for the present case, while either would with tolerable accuracy represent any small portion

* *Trans. Roy. Soc. Edin.*, vol. xxviii. † *Trans. Roy. Soc. Edin.*, vol. xxxiii.

of the curve, neither was applicable throughout its whole length with that exactness which is necessary. Accordingly, the curve was treated in sections, each beginning with a point corresponding to one of the holes in the bar, and terminating in another corresponding to the next hole. Each part of the curve was, by addition to or subtraction from its ordinates, converted into a logarithmic. From a comparison of the observed and calculated values of v, it was found that the formula for these separate sections represented curves having a slightly greater curvature than the observational curve, and that the calculated numbers agreed better with the observed in the middle of each section. Hence the values of $\frac{dv}{dx}$ were nearer the truth when found for the middle of each section, were too high for the higher values of x, and too low for the lower values. But by making the sections separately treated overlap one another, three values of $\frac{dv}{dx}$ were obtained; one too high, another too low; and a third, which always lay between these two, and was obviously near the true value.

It must be noted that the data from which the constants in the empirical formulæ were obtained were the ordinates of points merely on the curve—not those particular points given directly by experiment. But the fact that the observations agreed remarkably with each other, and that by using different temperatures at the source of heat, a considerable number of points were given directly by experiment, and all lying well on the curve, justifies the use of such points for data.

The experiments on the cooling of the short bar were carried out, and also reduced in the usual manner. Hence little remark on this point is necessary, save the observation that the curve of rates of cooling at different excesses of temperature exhibited no inclination to fall away, or show any point of contrary flexure. This, of course, is due to the precaution of raising the short bar to a temperature considerably higher than what is actually required to observe the cooling at any particular temperature excess reached by any of the thermometers in the long bar.

Final Results.—

Rates of Cooling of Short Manganese-Steel Bar.

Temperature Excess.	Rate of Cooling.	Temperature Excess.	Rate of Cooling.
5	0·05	110	1·55
10	0·10	120	1·73
20	0·21	130	1·92
30	0·33	140	2·08
40	0·45	150	2·27
50	0·58	160	2·46
60	0·73	170	2·67
70	0·92	180	2·91
80	1·04	190	3·14
90	1·23	200	3·37
100	1·39		

The following table contains nearly all the substance of the further calculations, and sufficiently explains itself:—

Thermometer.	Distance in Feet along Bar.	Temp. Excess ° C.	$-\frac{dv}{dx}$.	Area of Curve of Cooling to next Value of x.	Area of Curve of Cooling to end of Bar.	Area corrected for Change of Specific Heat.
A	0·0	189·2	35·32	6·625	12·603	14·653
B	0·25	108·2	20·29	3·321	5·977	6·619
C	0·5	61·7	11·86	1·636	2·656	2·841
D	0·75	34·1	7·11	1·02	1·02	1·065
E	1·25	1·15	...	...	...	...

From these data, it follows that the thermometric conductivity of manganese-steel is represented by the following numbers:—

(1) From uncorrected areas,

0°	100°	200°
·00221	·00254	·00287

(2) From corrected areas,

·00233	·00285	·00337

It now remains to correct the values of the tangents and the areas of the curve of cooling for the error involved due to variable heating of the thermometer stems. The method by which this error is estimated and applied is detailed by Professor TAIT in his introduction to my former paper. Applying this correction, the above results become—

(1) From areas not corrected for change in specific heat,

0°	100°	200°
·00211	·00246	·00281

(2) From areas corrected for change in specific heat,

·00219	·00272	·00325

For the purposes of comparison with the corresponding results for iron* (FORBES' wrought iron bar, cooled midway), the conductivity of manganese-steel may be taken as above, the results being corrected for change in specific heat, and also for error due to unequal heating of thermometer stems. The figures are as follows:—

	0°	100°	200°
Manganese-steel, . . .	·00219	·00272	·00325
Iron, ...	·0119	·01274	·01358

Hence it appears that the presence of 10 per cent. of manganese in iron or steel lowers its conductivity at 100° to one-fifth, and that the rate of increase of conductivity with temperature is, in manganese-steel, little more than half the corresponding coefficient in iron. That such a proportion, comparatively small, of manganese should have such a distinct effect upon the conductivity of steel, is remarkable, and, while it is on a parallel with the other rather anomalous properties of this substance, it points

* *Trans. Roy. Soc. Edin.*, vol. xxxiii. p. 555.

to the extreme desirability of a thorough examination of the thermal conductivity of alloys, especially of those whose properties differ in a marked way from those of their components.

Specific Heat of Manganese-Steel.

The method employed to determine the specific heat of manganese-steel was practically that known as the "method of cooling"; one which is based on the fact that two bodies, whose surfaces are exactly similar in nature and extent, lose *by radiation* equal quantities of heat, when their excesses of temperature over that of surrounding bodies are equal. That their losses of heat, under such circumstances, due to the combined effects of radiation *and convection*, should be equal, it is necessary that the mode in which the two experiments are performed should be as nearly similar as possible; the same precautions as to air-currents, and other influences tending to alter convection effect, should be adopted to the same degree in both. This was scrupulously attended to in the experiments, of which the following details may be given.

Two cubical masses, one of wrought iron, the other of manganese-steel, were obtained, the latter being a part of the same material as that out of which the bars for the conduction experiments were made. They were made as exactly as possible of the same dimensions, the length of each edge being $1\frac{3}{8}$ inch. A circular hole, 1 inch in depth, $\frac{19}{64}$ inch in diameter, was drilled in each; the axis of the hole being perpendicular to, and in the centre of, one of the faces of the cube. The surfaces of both cubes were made the same by the deposition on them of soot from the smoky flame of a paraffin lamp. One of the cubes was raised to a convenient temperature, say 300° C., by the flame of a Bunsen burner, and then allowed to cool, its temperature being noted at intervals by a thermometer whose bulb was inserted into the hole, which was filled up with a few drops of mercury to ensure good thermal contact of the bulb with the sides of the hole. The other cube was meanwhile placed at a distance, and used, by its thermometric indications, as a means of ascertaining the temperature of surrounding bodies. The thermometer employed for these cooling experiments was that used in the cooling experiments on the short bar in the conduction investigations; its error was well known by comparison with carefully-constructed Kew standards.

By such means the rates of cooling of both cubes were deduced from the observations, throughout a considerable range of temperature excess. Then, if m, m' be the masses of the iron, and manganese-steel cubes, respectively; c, c', their specific heats; r, r', their rates of cooling at the same given temperature excess,

$$m'r'c' = mrc,$$

since each of these quantities represents the amount of heat lost in unit time at the given temperature excess. Hence

$$c' = \frac{mr}{m'r'}c.$$

Thus, if the specific heat of iron be known throughout the range of temperature used,

the other data along with it give the specific heat of manganese-steel. The specific heat of iron may be taken as being 0·114 (1 + ·0014 t), most experimenters agreeing with this result.

The following table embodies the mean results of ten experiments on each of the cubes. Temperature excess is in degrees Centigrade; rates of cooling in degrees per minute :—

Temperature Excess.	Rate of Cooling.		Ratio of these Rates.
	Iron.	Manganese-Steel.	
20	1·02	0·91	1·121
40	2·10	1·88	1·117
60	3·19	2·87	1·111
80	4·41	3·97	1·111
100	5·78	5·17	1·118
120	7·22	6·45	1·119
140	8·76	7·83	1·118
160	10·38	9·27	1·119
180	12·10	10·86	1·114
200	13·98	12·50	1·118
220	15·96	14·28	1·117
240	18·17	16·20	1·121

The fourth column shows that the ratio of the rates of cooling of the two cubes is very nearly constant. The average value of these numbers may be taken for the purposes of calculation as being very near the truth. This average value is 1·117. Then the specific heat of manganese steel is—

$$c' = \frac{m}{m'} \times \frac{r}{r} \times c$$

$$= \frac{252{\cdot}25}{259{\cdot}25} \times 1{\cdot}117 \times 114\,(1 + {\cdot}0014\,t)$$

$$= {\cdot}124\,(1 + {\cdot}0014\,t).$$

Thus the specific heat of manganese-steel is about 1·087 times that of iron, and its rate of rise with temperature is the same as that in iron.

XXI.—*Strophanthus hispidus: its Natural History, Chemistry, and Pharmacology.*

By Thomas R. Fraser, M.D., F.R.S., F.R.S.E., F.R.C.P.E., Professor of Materia Medica in the University of Edinburgh.

Part I.—Natural History and Chemistry. (Plates I.–VII.)

(Read 4th February 1889.)

CONTENTS.

Historical Introduction.

The preliminary notices published by me in 1870 and 1872, on the action and chemistry of Strophanthus, indicated that it was likely to prove of value as a therapeutic agent; and so early as the year 1874, I had applied the substance in a few cases to the treatment of disease. Before sufficient data, however, had been obtained to justify any conclusions regarding its value as a therapeutic agent, the observations were interrupted by my leaving Edinburgh to occupy a public office in England, in connection with which it was impossible to conduct observations on the treatment of disease. In 1879, opportunities were again afforded to resume the interrupted observations, and results confirmatory of the anticipations which had been raised by the earlier physiological observations were gradually collected. The publication of a few of these results at the Cardiff meeting of the British Medical Association in 1885* has led to Strophanthus gaining a wide recognition as an important therapeutic agent, and to the production of numerous papers dealing with its botany, chemistry, pharmacology and therapeutics, not only in this country, but also in the continent of Europe and in America.

In this paper I propose to give, with greater detail than has been attempted in the

* *British Medical Journal*, vol. ii., 1885, p. 904.

previous communications, an account of the observations I have made on the general natural history, the chemistry, and the pharmacology (or physiological action) of Strophanthus. Before doing so, it may be desirable to state what knowledge existed with regard to these departments of its study and consideration previously to the publication of my papers of 1870 and 1872; to reproduce some of the leading statements contained in these papers; and to indicate the extent to which the knowledge regarding Strophanthus was increased during the period of fifteen years which elapsed between the publication of my paper of 1870 and of my subsequent paper read before the British Medical Association in 1885.

Previously to the publication of my preliminary paper of 1870, the knowledge regarding Strophanthus consisted of several botanical descriptions of the plant; of notices by travellers of its use by African tribes, who had discovered its poisonous action, and had employed it as an arrow-poison in the chase, and apparently also in warfare; and of a few brief references to some points relating to its physiological action.

Interest was first attracted to the physiological action of this substance by the introduction into Europe of a few specimens of fruits and seeds reputed to be the source of a remarkable arrow-poison used in several parts of Africa, and termed in some districts the Kombé and in others the Inée poison. The physiologists who first examined the properties of this poison seem to have been SHARPEY, and HILTON FAGGE and STEVENSON of London, and PÉLIKAN of St Petersburg.

SHARPEY's experiments were made in 1862–63, but they were not published, as before his investigation had been completed my preliminary notice of 1870, briefly descriptive of the general results I had then obtained, was communicated to this Society, and, much to my regret, led SHARPEY to refrain from publishing his observations, as they entirely agreed with those contained in my paper. From the notes of his experiments, which he afterwards very kindly sent to me, it is apparent that SHARPEY had determined that the action of Strophanthus was characteristically that of a cardiac poison.

On the 18th of May 1865, HILTON FAGGE and STEVENSON stated, in a note appended to a paper communicated to the Royal Society of London, on the "Application of Physiological Tests for Certain Organic Poisons,"* that the Manganja arrow-poison, obtained during the Zambesi Expedition by Sir JOHN (then Dr) KIRK, acts as a "cardiac poison." By this expression they imply an action on the frog's heart of the same kind as that produced by digitalin, *Antiaris toxicaria*, *Helleborus viridis* and *niger*, *Scilla*, and certain other poisons; and, especially, that the heart is stopped with the ventricle "rigidly contracted and perfectly pale."

In the same year, on the 5th of June, PÉLIKAN,† in a note communicated to the Academy of Sciences of Paris, pointed out that an extract obtained from the seeds which yield the Inée or Onage arrow-poison acts on the frog's heart in the same way as digitalis

* *Proceedings of the Royal Society*, vol. xiv., 1865, p. 274.

† "Sur un nouveau poison du cœur provenant de l'Inée ou Onage, et employé au Gabon (Afrique Occidentale) comme poison des flèches" (*Comptes Rendus de l'Académie des Sciences*, tome lx., 1865, p. 1209).

and other similar cardiac poisons, but with greater activity. The heart's beats were quickly arrested, with the ventricle in systole and with the auricles distended. This effect is attributed by him to an action on the nerve structures of the heart. He also states that his experiments were confirmed by VULPIAN. PÉLIKAN obtained the seeds from the Colonial Exhibition held in Paris in 1865, to which they had been sent by M. GRIFFON DU BELLAY, a surgeon in the French Naval Service, who had obtained them in the Gaboon district of West Africa, where they are used by an elephant-hunting tribe (Pahouins) to poison their small bamboo arrows.

In 1869, a few specimens of ripe follicles were presented to the Materia Medica Museum of the University of Edinburgh by the Rev. HORACE WALLER, who had been a member of the Oxford and Cambridge Universities Mission of 1861–64, superintended by the late Bishop MACKENZIE, with whom had been associated, during the operations of the mission between the River Shiré and Lake Shirwa, the famous traveller LIVINGSTONE and the enterprising botanist KIRK. The follicles were sent with the information that the seeds contained in them constituted the Kombé arrow-poison of South-Eastern Africa. Mr WALLER informs me that, at his suggestion, they had been brought to this country by Mr E. D. YOUNG, R.N., when he went to Africa in 1867, to clear up the story of LIVINGSTONE'S murder. Sir ROBERT CHRISTISON placed these follicles at my disposal for examination, and as in the course of time the insufficient material which they afforded was supplemented by some additional follicles sent to me by Professor SHARPEY and, afterwards, by Mr JOHN BUCHANAN, I was enabled to determine the most important facts in the pharmacological action, as well as in the chemistry of the substance, some of which were communicated to this Society in February 1870, in the form of a preliminary notice, and published in the *Proceedings* of that year,* and also, with a few amplifications, in the *Journal of Anatomy and Physiology* of 1872.† While the investigation was in progress, Sir DOUGLAS MACLAGAN received from Sir JOHN KIRK a poisoned arrow, obtained from the same district of Africa as the follicles; and with this arrow I was enabled to determine that the poison possesses the same action as the seeds contained in the follicles, and thus to confirm the discovery already made by KIRK of the source of the arrow-poison.‡

My experiments were made on cold-blooded animals and on birds and mammals; and the administration was effected by subcutaneous injection, and by introduction into the

* *Proceedings of the Royal Society of Edinburgh*, vol. vii., 1869–70, pp. 99–103.

† *Journal of Anatomy and Physiology*, vol. vii., 1872, pp. 140–155.

‡ In a recently written letter (31st Oct. 1888) Sir JOHN KIRK thus graphically describes the discovery he had made in 1861 of the plant from which the Kombé poison is obtained:—"The source of the poison, namely, *Strophanthus Kombé*, was first identified by me. I had long sought for it, but the natives invariably gave me some false plant, until one day at Chibisa's village, on the river Shiré, I saw the 'Kombé,' then new to me as an East African plant (I had known an allied, or perhaps identical, species at Sierra Leone (1858), where it is used as a poison). There climbing on a tall tree it was in pod, and I could get no one to go up and pick specimens. On mounting the tree myself to reach the Kombé pods, the natives, afraid that I might poison myself if I handled the plant roughly or got the juice in a cut or in my mouth, warned me to be careful, and admitted that this was the 'Kombé' or poison plant. In this way the poison was identified, and I brought specimens home to Kew, where they were described."

stomach and rectum. In the pharmacological portion of the preliminary papers above referred to it was shown that—1. "Strophanthus acts primarily upon the heart, and produces, as a final result of this action, paralysis of that organ with permanence of the ventricular systole." Experiments were quoted to support the view that it "acts in a powerful and direct manner upon the cardiac muscular fibre, greatly prolonging, in the first place, the contraction of these fibres, and ultimately rendering it continuous, and only to be overcome when relaxation occurs as a natural consequence of *post-mortem* decomposition" (p. 148); and that in frogs this action on the heart is independent of any influence exerted through the cerebro-spinal nervous system, as it occurs after destruction of the brain and spinal cord, and after division or paralysis by atropine, of the vagi nerves. It is added that sufficient data had not been obtained to warrant the assertion that no action is exerted upon the intra-cardiac ganglia (p. 149). 2. "Pulmonary respiration continues in cold-blooded animals for several minutes after the heart is paralysed. 3. The striped muscles of the body are acted upon, twitches occur in them, their tonicity is exaggerated, and, finally, their functional activity is destroyed, the muscles being then hard, and, soon afterwards, acid in reaction. These changes are accomplished subsequently to the final effect on the heart. They are the result of direct contact of the substance with the muscles themselves, and are independent of the action on the heart, as well as of any changes that occur in the physiological condition of the cerebro-spinal nervous system. 4. The reflex function of the spinal cord is suspended soon after the heart is paralysed, but the motor conductivity of the spinal cord and of the nerve trunks continues after the striped muscles are paralysed. 5. The lymph hearts of the frog continue to contract for many minutes after the blood heart has been paralysed."

The papers also contain a description of the botanical source and distribution of the arrow-poison and of some of the characters of the plant from which it is obtained, especially of its follicles and seeds, and of the more important of the chemical constituents of the seeds. It is also stated that the seeds contain a large quantity of an inert fixed oil and an active principle of crystalline form, for which, in accordance with the terminology at the time adopted in this country to distinguish neutral or glucosidal active principles from alkaloids, the name Strophanthin, characterising a glucoside, was proposed. This strophanthin was further stated to act in the same way as the extract from the seeds (p. 142).

In the interval of fifteen years that elapsed between the publication of my preliminary papers and of the communication to the British Medical Association, in which the therapeutic uses and value of Strophanthus were pointed out (1870–1885), only two papers were published on Strophanthus.*

* In 1870, however, M. Legros, at meetings of the Société de Biologie, on the 14th and 21st of May, exhibited frogs under the influence of the Inée poison derived from arrows used at the Gaboon, in order to show that the heart is arrested by it, with the ventricle in systole; and at the latter meeting, M. Bert stated that he had observed similar effects in cats under the influence of the same poison (*Comptes Rendus de la Société de Biologie*, 1870, pp. 81 and 84).

The first of these is an admirable essay by MM. POLAILLON and CARVILLE, published in the *Archives de Physiologie* of 1872.* It contains much interesting information regarding the Strophanthus used in the Gaboon as an arrow-poison, and known there, as well as in other districts of West Africa, as the Inée, or Onaye, or Onage poison; but the greater part of it is occupied with a full description of an experimental investigation on the pharmacology of the seeds of the Strophanthus plant. These authors especially examined the action on the heart, on striped and non-striped muscle, and on the cerebro-spinal nervous system. Their results altogether harmonise with those I had already published in the preliminary papers. The most important of them are summarised by MM. POLAILLON and CARVILLE in the following statements:—It acts on the heart, and produces death by paralysing this organ (p. 550). The ventricles are never arrested in diastole; they are always contracted in systole (p. 705). The action on the heart is not produced through the brain, medulla, nor spinal cord (p. 697), but by an effect on the muscular fibre of the heart (p. 704). Inée acts on the muscular fibre, striped and smooth, of which it rapidly destroys the contractility; but it does not appear to act on the nervous system nor on the peripheral blood-vessels. It is essentially a muscle poison. It has no action, or only a secondary action, on the other organs (p. 695). MM. POLAILLON and CARVILLE also state that the Inée poison produces no effect on the sea medusa, a creature unprovided with a central contractile organ for the circulation (p. 707).

The second paper which appeared between the years 1870 and 1885 was that of MM. HARDY and GALLOIS on the active principle of *Strophanthus hispidus*, published in 1877.† The two chief statements contained in this paper are, that the seeds of Strophanthus contain an active principle which is not a glucoside ("ne rentre point dans le groupe des glucosides"),‡ and that the comose appendages of the seeds contain a crystalline substance for which the name "Inéine" is proposed. This "inéine" is stated to give the reactions of an alkaloid, but to be destitute of any action on the heart, and, apparently, of any physiological action whatever.

In the process adopted by them for separating the active principle of the seeds, HARDY and GALLOIS unfortunately used alcohol acidulated with hydrochloric acid. By so doing, they necessarily failed to separate the true active principle, which, as I have shown, is a glucoside easily decomposed by acids, even at an ordinary temperature; and they, therefore, obtained only a decomposition product of the glucosidal active principle —the body, in fact, since described by me as strophanthidin.

In reference to the alkaloid, believed by them to exist in the comose appendages of the seeds, subsequent observers, working with much larger quantities of material than they were able to obtain, have not been successful in discovering its existence. In the chemical portion of this paper I shall have occasion to point out that, even when one pound

* Tome iv. pp. 523 and 681.

† *Journal de Pharmacie et de Chemie*, t. xxv., 1877, p. 177.

‡ *Loc. cit.*, p. 179.

of the comose appendages is manipulated, no alkaloid could be detected in the products that were obtained.

To summarise this historical sketch, in so far as it relates to the pharmacology and chemistry of Strophanthus, previously to the publication of my papers of 1870 and 1872, only two brief notices appeared on its pharmacological action, both of which dealt merely with the nature of the action on the heart; while during the interval of fifteen years that elapsed between the publication of my first paper and the subsequent communication of 1885, one paper of much interest, though adding but little to the existing knowledge, was published on the pharmacological action, and also only one paper on the chemistry of Strophanthus, which, however, did not advance the knowledge of the chemistry of the subject.

Following upon the publication of the paper of 1885, on the therapeutical applications of *Strophanthus hispidus*, the literature of the subject has, however, very rapidly increased, and it now embraces upwards of a hundred separate papers.

Until 1885, also, Strophanthus, elsewhere than in Africa, was a mere curiosity, represented in a few museums by specimens of its flowers, follicles, or seeds. Since that time it has become a not inconsiderable article of commerce, several tons of seeds having been exported from Africa by London merchants alone, in order to supply the requirements of medical practice.

A. NATURAL HISTORY.

1. Use in Africa as an Arrow-Poison, and Description of Arrows.

In nearly every narrative of exploration in uncivilised tropical regions accounts are given of poisonous substances, which in many instances are stated to possess remarkable properties. Usually these poisons are of vegetable origin, and nearly all of them may be included in the two great divisions of *Ordeal* and of *Arrow* poisons. Among the most interesting of the *Ordeal* poisons are the *Physostigma venenosum*, and the Akazga or Akaja, or M'boundou of West Tropical Africa; the Sassy, or Muave, or Casca (*Erythrophlæum*) of wide distribution over Africa; and the *Tanghinia venenifera* of Madagascar: and of the *Arrow* poisons, the *Antiaris toxicaria* and *Strychnos Tieuté* of Java; the *Aconitum ferox* of China and India; and the famous Wourali or Curare poison of South America.

As I have previously stated, it is also to the enterprise and discriminating observation of explorers and missionaries that we are indebted for the interest in the Kombé arrow-poison, which has led to the examination of its properties and to the appreciation of its value as a therapeutic agent; and several of them have collected valuable as well as curious information regarding it.

Dr Livingstone, describing the employment of poisoned arrows for killing buffaloes

by the tribes inhabiting the banks of the Mukuru-Madsé, a tributary of the Shiré River, states that "the animals are wary, from the dread they have of poisoned arrows. . . . . The arrow making no noise, the herd is followed up until the poison takes effect, and the animal falls out. It is then patiently watched till it drops—a portion of meat round the wound is cut away, and all the rest eaten. Poisoned arrows are made in two pieces. An iron barb is fastened to one end of a small wand of wood, ten inches or a foot long, the other end of which, fined down to a long point, is nicely fitted, though not otherwise secured, in the hollow of the reed which forms the arrow shaft. The wood immediately below the iron head is smeared with the poison. When the arrow is shot into an animal, the reed either falls to the ground at once, or is very soon brushed off by the bushes; but the iron barb and poisoned part of the wood remain in the wound. If made in one piece, the arrow would often be torn out, head and all, by the long shaft catching in the underwood, or striking against trees. The poison used here, and called *Kombi*, is obtained from a species of Strophanthus. . . . . It is possible that the Kombi may turn out a valuable remedy. . . . . There is no doubt that all kinds of wild animals die from the effects of poisoned arrows, except the elephant and hippopotamus. The amount of poison that this little weapon can convey into their systems being too small to kill those huge beasts, the hunters resort to the beam-trap instead."* One of the arrows referred to by Dr LIVINGSTONE is represented in Plate I. fig. B.

According to Sir JOHN KIRK, "one poisoned arrow is said to be sufficient to kill a buffalo, but half a day is required for the poison to act. Probably the mechanical state of the poison causes this; for the poison composition is hard, and will require time to be absorbed into the system from the wound. The hippopotamus is killed by it, but the quantity needed seems to be about thrice that on an ordinary arrow. It is driven through the thick skin of the animal by being placed on the barbed head in the lower end of a beam of wood, which falls from a height as the beast passes underneath a trap. The poisoned head is driven well in by the big end of the beam, and is left to act, which it is said to do in about half a day."†

The Rev. HORACE WALLER, who was a member of Bishop MACKENZIE'S Expedition, informs me that in May 1863 he was presented with some pods of the Kombé poison at Chibisa's village, on the Shiré river, by a chief named Dakanamoio.‡ This chief, at the same time, stated that the manner of preparing the poison was "to gather the pods when green, cut off the outside rind, then expose them to the sun till dry, when the seeds were taken out, pounded, mixed with red clay, and the mixture, which is a red paste, packed round the arrow." Mr WALLER also states "that in time of war it is common for the people of a village to place a quantity of the thistle-down appendage about the entrances, to warn the enemy that the villagers have been busy smearing their arrows."

I am indebted for much valuable information to Mr JOHN BUCHANAN, at present

* *Narrative of an Expedition to the Zambesi and its Tributaries*, 1858–1864, by DAVID and CHARLES LIVINGSTONE, 1865, pp. 465–467.

† Unpublished letter to Dr SHARPEY, dated 1st January 1864.

‡ Mr WALLER subsequently gave these pods to Sir JOHN KIRK, who brought them to England in 1863.

Acting-Consul in the Nyassa district, and formerly associated with the Blantyre Mission of the Church of Scotland. In a letter, dated 8th May 1885, he informs me that "the Strophanthus plant is widely known amongst the natives at Blantyre and the surrounding districts as the most powerful poison they have. It is called 'Kombe' by the Manganja and 'Likombe' by the Wayao tribes. I hardly think it is to be found in large quantities. At the chief's village a small quantity may generally be got, for a parcel is always kept in the chief's verandah in case of emergency, along with a number of poisoned arrows, ready to be used against an enemy. Formerly, game was often killed by arrows poisoned with Strophanthus. The flesh round the wound was cut out and thrown away, and the remainder eaten, but the precaution was always taken to boil the meat. . . . . In preserving the Strophanthus, the follicles are taken from the plant before they are quite ripe, and the outer covering is scraped off. A number of follicles are tied together with palm leaves, so that they may not open when put out to dry in the sun. So far as I am aware, only the seeds are used."

In a letter, dated 28th June 1881, Mr BUCHANAN thus describes the method followed in preparing the poison for arrows:—"A man breaks a follicle, and puts the seeds with wool attached into a pot. He then takes a small piece of bamboo, which has two thin splints inserted crosswise in the end, and he revolves this speedily by rubbing it between his hands. The seeds are thus put into motion and fall to the bottom of the pot, and the wool rises and comes out at the top, and is carried away by the least breath of wind. The seeds are then put into a small mortar and pounded into a paste, which is then ready for use. It is common to mix the milky juice of a Euphorbia with it to make it stick on the arrow.* . . . . Poisoned arrows are used in their wars with deadly effect."

During his residence in the Gaboon district of West Africa, Dr VINCENT found that the Pahouins or Fans, a warlike tribe inhabiting the banks of the rivers falling into the estuary of the Gaboon, employ a kind of cross-bow with which they shoot small bamboo arrows that are smeared at one end with a poison called "Inée" or "Onaye." This poison was subsequently discovered to be derived from the fruit of a Strophanthus.†

M. EHRMANN, a merchant of Tchimbié, in the Gaboon country, states that while the Pahouins or Fans, inhabitants of the interior, term the arrow poison "Inée," the Gabonais, inhabitants of the coast, term it "Onaïe." The poison is prepared by drying the pod, removing and pounding the seeds, and forming a paste with water. This paste is used to smear arrows, and also small pieces of iron which are discharged from firearms. M. EHRMANN further states that the inhabitants of the West Coast have largely replaced their bows and arrows by firearms, and that therefore the arrow-poison is now chiefly used by the inhabitants of the interior.‡

* In a letter to Messrs BURROUGHS and WELLCOME of London, of later date than the above, Mr BUCHANAN states that the paste for the arrows is made by mixing the pounded seeds with water, and, to confer adhesiveness, with the juice from the bark of a species of Liliaceæ. He also states that before the flesh of an animal killed with poisoned arrows is eaten, the sap from the bark of the Baobab tree is put into the wound made by the arrow, as it is believed to neutralise any poison that may remain in the wound.

† *Archives de Physiologie normale et pathologique*, tome iv., 1871–72, p. 524.

‡ *Bulletin Général de Thérapeutique*, tome cxiii., 1887, p. 529.

References to the use of poisoned arrows in Africa occur in the writings of many other travellers and explorers, but in most instances the effects of the poison, and the source from which it is derived, are not described with sufficient definiteness to render it possible to identify the poison.*

Only a few poisoned arrows have as yet reached this country from Africa, owing, probably, to some extent to the difficulties of carriage, but certainly much more to the reluctance of the natives to place poisoned arrows in the possession of Europeans. I have, however, been able to examine arrows of eight different forms obtained from various parts of Africa. Two of them were given to me as specimens of arrows the poison of which was known to be the Kombé poison, or Strophanthus. Of the others, either no knowledge of the poison existed, or it was believed to be derived from plants other than Strophanthus. A few details regarding these arrows may prove of interest.

Arrow A (see Plate I.).—Arrow in the Materia Medica Museum of the University of Edinburgh. One of four tied together, and labelled "Poisoned arrows from the interior of Africa, poison unknown." The label has unmistakably been written by Sir ROBERT CHRISTISON, but there is no further information to be found in the Catalogue of the Museum. This arrow has a total length of 38 inches. It has a shaft made of bamboo cane 34 inches in length, with a deep notch for the bowstring, and with eight narrow feathers commencing $1\frac{7}{8}$ inch above the notch, and extending $1\frac{5}{8}$ inch along the shaft. The arrow-head is a formidable-looking weapon made of iron, which is inserted into a hollow in the cane and secured by a cord, apparently consisting of animal tendon, tied round $2\frac{3}{4}$ inches of the cane. The portion of the head not inserted in the cane is almost 4 inches in

* BURTON (*The Lake Regions of Central Africa*, 1860, vol. ii. p. 305), for example, states that the Wanyika of Mombasah, the Wazaramo, the Wak'hutu, the Western Wasagara and the people of Uruwwa use poisoned arrows in warfare, and that the poison is extracted by the Wazaramo and the Wak'hutu from a plant called Mkande-Kande. They sold the poison at an exorbitant price, "but avoided pointing out to the Expedition the plant, which from their description appears to be a variety of Euphorbia." SCHWEINFURTH (*The Heart of Africa*, translated by ELLEN E. TREWER, 1878, vol. i. p. 140) asserts that the Bongo tribe of Central Africa poison their arrows with the milky juice of one of the Euphorbiæ (*venifica*). THOMSON (*To the Central African Lakes and Back*, 1881, vol. ii. pp. 40, 139) describes encounters in which he was threatened with poisoned arrows at Kwakissa, and by a Maranga chief. CAMERON (*Across Africa*, 1885, pp. 59, 242, 291) refers to the employment at Ugambo and Mombassa of poisoned arrows, neatly covered with banana leaves, for killing elephants; to the natives at Nékéto, on the Kaça, possessing arrows deeply barbed and poisoned; and to the inhabitants of Ulegga using poisoned arrows for which they had an antidote. MONTAGU KERR (*The Far Interior*, 1886, vol. i. p. 29) states that the Masarwa bushmen carry small bows and bark pouches containing poisoned arrows, the points of which are made of bone or iron, and the poison is the concentrated milky juice of *Euphorbia arborescens*. And FARINI (*Through the Kalahari Desert*, 1886, pp. 332, 253) gives a description of the preparation of the poison for arrows from the milky juice of a large bulb mixed with serpents' venom, and states that poisoned arrows are used by the M'kabba, a pigmy tribe, and by the Orange River bushmen.

Further, I am indebted to Dr FELKIN for several small arrows, designated "Tikki-Tikki or Akka arrows," obtained by him at Rohl Bahr-el-Ghazal, a province of Central Africa, north of the Equator. They are from $18\frac{1}{2}$ to 20 inches in length, and are furnished with iron heads, of which the straight portion is hollow, and fits on to the end of the wooden shaft, and the true head is oval or obovate, and in some of the arrows provided with wire-like spikes at the base. The poison is applied by dipping the whole head, including the straight part, into a dark brown gummy fluid, stated to be derived from a Euphorbia, which seems to be afterwards removed from the outside of the iron head, as it is found only on the inside of its hollow straight portion, and on the wood of the shaft covered by it. The thin wooden shaft has no feathering, but its extremity is cut into a circular disc of greater diameter than the rest of the shaft, showing apparently that the arrows are projected from a blow-tube. They are reputed to be very active, and are said to be used in warfare as well as for killing game.

length; its straight portion is furnished with two curved and strong spikes pointing downwards and having sharp points; and the true arrow-head is $2\frac{1}{4}$ inches long, elegantly shaped, with a fine tapering extremity and sharp barbs, and with one lateral half of each surface concave and the other convex. The poison covers, in a layer of from $\frac{1}{8}$th to $\frac{1}{16}$th of an inch in thickness, the whole of the exposed straight portion of the barb and the two spikes projecting from it, and also the true arrow-head with the exception of its point and edges. This poison is of a dark greyish-brown colour, and is earthy, though rather tough, in consistence. Only a small portion of it is soluble in water, the solution being faintly acid, bitter, and of a light sherry colour; the remaining undissolved portion appearing under the microscope as a granular débris in which fragments of vegetable tissue, numerous pieces of vegetable hairs, and oil globules could be distinguished.

When examined chemically, the poison produced with solution of potash a faintly yellow fluid, in which pinkish or brownish particles were suspended; and when heated between 116° and 120° F. along with 10 per cent. sulphuric acid, it gradually acquired a greenish colour, which passed into light brown.

When 0·1 grain was thoroughly mixed with four or five drops of distilled water, a clear almost colourless solution was obtained, having a slightly bitter taste. This solution was injected under the skin of a frog weighing 420 grains, and it produced the disorders of motility, fibrillary twitches of muscles, and paralysis of respiration, which are observed under the action of Strophanthus. The heart was exposed one hour and forty-five minutes after the poison had been injected, and it was found to be in complete standstill, with the ventricle small and mottled, and the auricles dark and somewhat distended; and mechanical irritation applied to the ventricle and auricles failed to excite movement of any part of the heart. For some time after complete paralysis of the heart, active general reflex movements could be excited by slight irritation.

Arrow B (see Plate I.) is one of the four tied along with arrow A, and its form is the same as that of other two of these four arrows, and altogether different, as the illustration shows, from the fourth arrow, or arrow A. Arrow B also closely resembles the arrow already referred to (pp. 957 and 961) as having been obtained in Bishop Mackenzie's Expedition, and described by Livingstone.* This circumstance probably indicates that the arrows A and B had also been obtained from the neighbourhood of the River Shiré or of Lake Shirwa.

Arrow B is 37 inches in length. The cane portion of the shaft has no feathers, but they seem to have been removed along with several inches of the extremity of the shaft. The head is of iron, and consists of a long nearly straight portion inserted into a hollow

* An illustration has not been given of this arrow, as it is the same in every important detail as arrow B. The physical characters of the poison are also the same, and it also consists structurally of fragments of vegetable tissue, amorphous yellowish-brown granular matter, oil globules, and incomplete vegetable hairs. The solution obtained by macerating and triturating one-tenth of a grain with water, somewhat quickly produced, in a frog weighing 329 grains, muscular weakness, gaping movements of the mouth, fibrillary twitches, and stoppage of pulmonary respiration. Thirty-six minutes after the solution had been injected, the exposed heart was found to be motionless, with the ventricle contracted, and the auricles large and dark, and no movement of the heart could be excited by mechanical irritation, although general, but feeble, reflex movements still followed irritation of the skin.

in the cane, where it is secured by a cord made of tendon wound round the end of the cane, and of a relatively small barbed head. The poison has been abundantly applied to the straight portion of the iron head, as it surrounds it for a length of nearly $5\frac{1}{4}$ inches in a layer of about $\frac{1}{16}$th of an inch thick; and it has the same appearance as in arrow A. The barbed head is rather more than $1\frac{1}{4}$ inch in length; it has not the elegant tapering form of the barbed head of arrow A, but like it, one of the two wings on each surface is concave and the other convex.

The poison is dark brownish-red on the outside, and paler, with a faint pinkish hue, in the interior; and it is rather tough in consistence, and earthy in structure. It also is only partly dissolved by water, forming a nearly colourless acid solution; and the undissolved portion was found, under the microscope, to consist of fragments of vegetable tissue, numerous pieces of broken hairs, granular particles, and oil globules. When mixed with solution of potash, the fluid part became faintly yellow, having reddish particles suspended in it, and when heated gently with 10 per cent. sulphuric acid, it slowly became green, and afterwards dark brown. One-tenth of a grain rubbed with a few drops of water yielded a nearly colourless clear solution, which produced, in a frog weighing 310 grains, the same symptoms as the poison from arrow A. One hour after administration, the exposed heart was found to be motionless, even when irritated; and the ventricle was small and mottled in colour, and the auricles were dark and distended. Active general reflex movements were obtained fifteen minutes after the heart had been exposed.

Arrow C (see Plate I.).—This arrow is one of two of exactly the same form, kindly given to me by Dr Felkin, along with other two arrows having the form represented in Plate I. fig. D.

Arrow C was brought from a district 75 miles N.N.W. of Zanzibar, by Dr Felkin, and it is reputed to be poisoned with the same substance as arrow D, namely, the poison contained in the packet J, afterwards to be described.

The total length of the arrow is 31 inches, and of this length about $29\frac{1}{2}$ consists of the shaft. The latter is in two unequal pieces spliced together; one piece, carrying the feathering, being about $20\frac{1}{3}$ inches long, and the other, having the head attached to it, being about $9\frac{1}{8}$ inches long. Both pieces of the shaft seem to be made of the same wood, which is about $\frac{3}{8}$ths of an inch in diameter, light, nearly white in colour, and smooth on the surface. The shaft has three narrow parallel feathers $1\frac{3}{4}$ inch long, lashed on to the shaft $1\frac{1}{4}$ inch from the bowstring notch. The head of the arrow is made of iron, and its straight portion is inserted for $\frac{1}{8}$th of an inch into a split made in the wooden shaft, where it is secured by the shaft being lashed for $\frac{1}{2}$ an inch with cord. About $\frac{1}{4}$ inch only of the straight portion of the head is exposed. The barbed portion is about $1\frac{1}{8}$ inch in length, unsymmetrical, and somewhat rudely finished, and both wings of it are flat. The whole of the barbed head and of the short exposed portion of the straight piece of iron is irregularly covered with a thin dark brown incrustation, stated to be the poison, which adheres tenaciously to the head.

On scraping the head with a knife, it was only with difficulty that a small quantity of

a hard gritty powder, of dark colour, could be removed, which appeared under the microscope to consist merely of irregular structureless particles. When the powder was macerated and then triturated with water, a yellowish-brown nearly tasteless solution was obtained, but the greater part of the powder remained undissolved. The solution thus prepared from one-tenth of a grain, along with as much as possible of the undissolved substance, was injected under the skin of a small frog, but it failed to produce any obvious effect. The experiment was repeated with one-fifth of a grain of the scraped substance, and the result was also entirely negative.

Dr Felkin was good enough to place at my disposal other three of the same arrows. On steeping the three heads in distilled water for twenty-four hours, a nearly clear pale yellowish-brown solution was obtained, which, on being evaporated to dryness at 100° F., left a pale reddish-brown residue, weighing only 0·15 grain. This residue was dissolved in 4 minims of distilled water, and injected under the skin of a frog weighing 326 grains; but, as in the previous experiments, no symptoms were produced.

If these arrows, therefore, had originally been poisoned with the same substance as arrow D, which is undoubtedly active, the poison had by some means been removed from them.

Arrow D (see Plate I.).—This arrow is one of two exactly alike, also very kindly given to me by Dr Felkin, and brought by him from the Wanyika country near Mombasa, on the east coast of Africa, north of Zanzibar. The arrow is 29 inches in length. The shaft is made of a nearly white, fined-grained, light wood: it is smooth and round, $23\frac{1}{2}$ inches in length and $\frac{6}{16}$ths of an inch in diameter; and is provided with three rather broad feathers, each nearly 2 inches long and $\frac{1}{2}$ an inch wide, which are neatly lashed to the wooden shaft, immediately above the bowstring notch. The head is made of iron, and consists of a straight portion, merely inserted, without any lashing, for 1 inch into a hollow in the wooden shaft, and of a small barbed head, unsymmetrical, and unprovided with any grooving on the wings. The poison is smeared round the straight portion of the head, which is 4 inches long and $\frac{5}{16}$ths of an inch thick, and it is protected by a covering of skin (like kid) carefully coiled round the whole of the head. The poison is of a greyish-black colour on the surface, and black and resin-like in the interior.* When a little water is added to it, a reddish-brown clear solution is soon produced, which in a few hours becomes very dark in colour and opalescent. The solution has a faintly acid reaction, but no distinct bitterness. On microscopic examination, the poison was found to consist of an abundance of vegetable cells and fibres, numerous oil globules, some amorphous yellowish granular matter, and a few fragments of vegetable hairs. With solution of potash it almost entirely dissolved, and became of a dull orange colour, and with 10 per cent. sulphuric acid it became light brown, and then, on being heated between 116° and 118° F., reddish-brown—the latter colour continuing for twenty-four hours.

* An arrow almost identical with this one was shown to me by the Rev. Ed. H. Baxter, of Mpwapwa, who had obtained it from the Wakamba, a tribe of elephant hunters inhabiting a district adjoining that of the Wanyika tribe The arrow is stated to be used in warfare also.

When the watery solution obtained by triturating one-tenth of a grain with 4 minims of distilled water was injected under the skin of a frog weighing about 400 grains, symptoms appeared similar to those observed with toxic but non-lethal doses of Strophanthus, and the frog afterwards recovered. When, however, the watery solution from one-fifth of a grain was injected under the skin of a frog, weighing 320 grains, the peculiar attitude, the gaping movements of the mouth, the fibrillary twitches of muscles, the slowing of respiration, and the general feebleness of voluntary muscles observed in Strophanthus poisoning manifested themselves; and on exposing the heart, forty-two minutes after the administration, it was found to be motionless and inexcitable by mechanical irritation, and to present the usual appearance of a heart poisoned by Strophanthus. General reflex movements could be produced for many minutes after the heart had ceased to contract, and they were, indeed, particularly sudden and shock-like in character, even when the animal was flaccid and incapable of performing any voluntary movements; but no reflex contractions could be excited by succussion.

Packets or Bags of Wanyika Arrow-Poison, J (see Plate II.).—The poison for arrow D, and it is stated also for arrow C, is stored ready for use in cylindrical packets or bags, constructed of three layers of palm leaf. I am further indebted to Dr Felkin for two specimens of unbroken packets, and also for a separate irregular-shaped, dark, resin-like piece of a substance reputed to be the same poison. One of the packets is represented in fig. J of Plate II. It weighs 834 grains, and the other packet 732 grains. Each packet is neatly tied round and secured at the ends with a cord, which, at one of the ends, is continued into a loop for suspending the packet.

The poison is of a dark brown, nearly black colour, and is hard but yet slightly plastic. A small portion put into a few drops of water sank to the bottom, and at once began to dissolve, the solution being at first clear and pale brown in colour, but afterwards dark brown and opalescent, from suspended minute brown particles. The solution was slightly acid in reaction, but in small quantity it was not distinctly bitter. When the opalescent fluid was examined microscopically, it was found to consist chiefly of minute yellowish-brown granules, and of small masses composed of these granules, mingled with which were a very few fragments of vegetable tissue and apparently of vegetable hairs. It almost entirely dissolved in solution of potash, forming a deep gamboge-yellow solution, which very soon became brownish-yellow; and when heated with 10 per cent. sulphuric acid at a temperature of 110° to 120° F., it became at first light brown, then darker brown, and afterwards brown with a faint violet hue.

One-tenth of a grain mixed with 3 minims of distilled water was injected under the skin at the left flank of a frog weighing 440 grains. The frog soon moved about uneasily, some froth was produced in the glass chamber in which it was confined, the respirations became infrequent and then ceased, the pupils contracted, fibrillary twitches occurred at the flanks and back and subsequently in the posterior extremities, and the movements became greatly impaired. When the frog was lying flaccid and resting on

the chest, it was placed, one hour and nine minutes after the administration, on the back, and it remained in this position after a few feeble struggles; and while in this position, careful inspection failed to reveal any cardiac movement. One hour and thirty-two minutes after the administration, the heart was exposed and found to be motionless and inexcitable, with the ventricle small and pale on the anterior surface and dark on the posterior surface, and with the auricles large and dark. At this time the pupils were small, and the skin much paler than before the experiment. Irritation of the skin over the nates caused reflex movements for ten minutes subsequently, when the observations were discontinued.

Half-a-grain of the same poison, dissolved and suspended in four minims of water, was injected under the skin of a frog weighing 438 grains. Similar symptoms to those last described made their appearance, with the addition of prolonged gaping movements of the mouth. In fifteen minutes, the frog remained on the back, and no cardiac movement could be detected. The exposure of the heart was purposely delayed in order to see if any symptoms of a spasmodic description, or any evidence of reflex exaggeration would appear, but they were not detected. The heart was exposed one hour and twenty-five minutes after the injection of the poison, and it was found to be motionless and inexcitable, with the ventricle small and mottled, and the auricles large and dark; but spinal reflex movements could still be obtained on irritation. A few minutes afterwards, it was found that a section of the ventricle and also a section of the *vastus externus* muscle was acid in reaction.

Rather less than two minims of the dark venous blood that had escaped when the heart was incised in the preceding experiment were injected under the skin at the left flank of a small frog weighing 320 grains. Decided symptoms were manifested in an hour and a half, and they were of the same kind as those described in the preceding experiment. In an hour and forty-five minutes, the respirations had ceased, the frog remained on the back, and careful examination failed to reveal any cardiac impact. The heart, however, was not exposed until the following morning, when the ventricle was found to be pale and contracted, and the auricles dark and distended. Strong general muscular rigidity was also then present.

As the physical and chemical characters of this poison, and also in some respects of the poison of arrow D, reputed to be the same substance, were somewhat different from those of the poison of most of the other arrows, it was considered advisable to perform another experiment, in order to determine if the Sassy or Muave (*Erythrophlæum*) poison might not be present. The latter poison is of wide distribution; and as it is extensively used as an ordeal, its toxic properties are well known to many tribes in Equatorial Africa. It also is a cardiac poison, but in addition it produces spasms by acting on the medulla-centre. As the latter action might be masked by cardiac and muscle actions simultaneously developed, an experiment was made in which, in a frog weighing 435 grains, the blood-vessels of one posterior extremity were carefully ligatured before the watery solution obtained by triturating one-tenth of a grain of the poison J with 4 minims of dis-

tilled water, was injected under the skin at the left flank. Symptoms of the same kind as those manifested in the previous experiment with one-tenth of a grain of the same poison, gradually made their appearance; but at no time was any spasm or any exaggeration of reflex excitability shown, even in the tied leg. The heart ceased to contract in less than one hour after the poison had been administered, and the observations were continued until all reflex excitability had disappeared, as a consequence of stoppage of the circulation.

The irregular-shaped piece of dark resin-like substance reputed to be the same poison as that contained in the packets was found to be inert when given in doses of one-tenth, one-fifth, and one-half of a grain, respectively, to small frogs.

Arrow E (see Plate I.), like arrow A, is in the Materia Medica Museum of the University of Edinburgh. It is one of five similar arrows, tied together, and labelled "Arrows from Negroes of River Gambir, poison unknown," and also on a separate label in Sir Robert Christison's writing, "Poisoned Arrows used by the W. Africans near Macquania Island on the Gambir River. From Dr Neligan, 1856."

The arrow is $32\frac{1}{2}$ inches in length, but was originally of greater length, as all the five arrows have been shortened by being cut across near the bowstring end, there being neither notch nor feathering. The shaft is made of a rather slender bamboo cane, and in its present state it is $28\frac{1}{4}$ inches long. The head is inserted into a hollow in this cane, and the part of the cane receiving the head is strengthened by a lashing of tendon for about $1\frac{1}{4}$ inch. The straight portion of the head projecting from the cane is $2\frac{1}{2}$ inches in length; and the barbed head is nearly $2\frac{1}{2}$ inches long, and $\frac{5}{8}$ of an inch wide at its broadest part. The latter is of an elegant saggitate form, tapering gradually to a long fine point at the distal extremity, and terminating at the base in two narrow and long barbs; and on each surface one of its lateral wings is convex, while the other is concave. The poison surrounds the straight portion of the head, and also extends up the centre of the barbed head almost to its point on each side. It is of a dirty greyish-brown colour externally and nearly black internally, brittle, without odour, and very bitter. When microscopically examined it was found to consist of fragments of vegetable tissue, yellow granular particles, numerous oil globules, and numerous broken pieces of vegetable hairs.

With solution of potash, the fluid part became faintly yellow, with brownish particles diffused through it; and when heated between 116° and 118° with 10 per cent. sulphuric acid, the original brown colour was slowly converted to green, and then became reddish-brown. When the poison was rubbed up with a little water, a sherry-coloured, clear, and faintly acid solution was obtained, but the greater part of it remained undissolved as a reddish-brown débris.

The watery solution from $\frac{1}{10}$th of a grain was injected under the skin of a frog weighing 435 grains, and produced symptoms exactly resembling those following the administration of Strophanthus. The heart was exposed one hour and forty minutes after the poison had been injected, and it was then perfectly motionless and inexcitable to mechanical irritation, the ventricle being small and mottled, and the auricles large

and dark. Fifteen minutes afterwards, fibrillary twitches were still occurring in the muscles, and active reflex contractions could be obtained by irritating any portion of the skin.

Arrow F (see Plate II.).—In 1882, Mr BUCHANAN forwarded from the Shiré district of East Central Africa six poisoned arrow-heads, the poison of which was stated to be derived from a Strophanthus. Unfortunately, the arrows had been packed in a box along with botanical specimens preserved in brine and spirit, and as the jars containing some of the latter had been broken, the arrow-heads were much damaged on their arrival in this country. All the arrow-heads excepting one had the same form as the arrow B. The one of exceptional form is figured in Plate II. fig. F. The portion of the shaft that remains is made of bamboo cane, and the portion of it receiving the arrow-head is gradually thinned to the diameter of the straight portion of the head. This head is a very formidable-looking weapon, on account of the six spikes with points curving downwards, arranged in three tiers of opposite pairs, with which it is armed. It is altogether $4\frac{1}{2}$ inches long, the straight spiked portion occupying $2\frac{3}{4}$ inches, and the true head $1\frac{3}{4}$ inch of this length. The latter is only $\frac{1}{2}$ an inch wide at its broadest part; it is lance-shaped, and has two surfaces, each half of which is slightly concave; and it is not provided with barbs, no doubt because they would be an unnecessary addition to the formidable spikes on the straight portion of the head. The composition or paste which originally contained the poison entirely covers the straight portion of the head and the spikes, and it is also smeared over the lance-shaped head, the encrusting layer having a length of 3 inches. It is now of a dull dark-brown colour, somewhat earthy in structure, easily breaking down to powder, and destitute of bitterness. With solution of potash a doubtful faint yellow was observed in the fluid part, and on heating between 116° and 118° F. with 10 per cent. sulphuric acid no marked change was observed, the colour remaining pale brown. On microscopic examination, it was found to consist of a large quantity of brownish-red particles and granules; of a few oil globules, pieces of vegetable tissue, and small colourless fragments of crystals; and of a large number of broken pieces of vegetable hairs, having a close resemblance to those of *Strophanthus hispidus*.

Although containing structures apparently derived from Strophanthus seeds, it produced no effect when watery solutions from 0·1 grain, 0·2 grain, and 0·5 grain were administered by subcutaneous injection to frogs. This negative result is no doubt to be explained by the long soaking in spirit and brine to which the arrow-heads had accidentally been subjected. At the same time, the microscopic examination seems to confirm Mr BUCHANAN'S statement that the arrows had been poisoned with the Kombé poison.

Arrow G (see Plate II.).—Mr BUCHANAN has more recently (1885) sent me four entire and uninjured arrows, also obtained from the Shiré district. They all have the form represented in Plate II. fig. G. The total length is 37 inches, and the shaft consists of a stout bamboo cane $29\frac{1}{2}$ inches long and from $\frac{7}{16}$th to $\frac{8}{16}$th of an inch in diameter. This shaft is provided with nine feathers, each about 3 inches long, fixed by being inserted into parallel slits in the cane, and also by lashing with tendon at the upper part. The lower

ends of the feathers are $1\frac{1}{8}$ inch above the bowstring notch of the arrow. The iron head is inserted into a hollow in the cane, which is strengthened at the hollowed part by a cord, consisting of a tendon lashed round it for 3 inches. The exposed portion of the head is $7\frac{1}{2}$ inches long, and the poison surrounds its straight portion, which is nearly 6 inches in length, in a layer of $\frac{1}{8}$th of an inch in thickness. The barbed head is altogether $2\frac{1}{4}$ inches long, and it is $1\frac{3}{8}$ inch wide at the broadest part, which is at the ends of the barbs. The barbed head terminates in a rounded extremity, the barbs being sharply pointed, and one lateral half of each surface of the head is concave, while the other is convex. The poison is of a brownish colour, with grey spots; it is smooth on the surface, has an earthy fracture, no odour, but a strongly bitter taste. On microscopic examination it was seen to consist of fragments of vegetable tissue, oil globules, numerous pieces of broken hairs, and yellow granular particles.

With solution of potash, the fluid portion acquired a faint yellowish tinge, and with 10 per cent. sulphuric acid, at a temperature of 110° to 118° F., it became green in colour, but soon the green colour was replaced by brown, and afterwards by a faint dirty violet colour.

The portion dissolved by water from $\frac{1}{10}$th of a grain was administered by subcutaneous injection to a frog weighing 335 grains. It produced in a short time the ordinary phenomena of Strophanthus poisoning. The heart was exposed one hour and thirty minutes after the administration, and found to be motionless and inexcitable by stimulation, with the ventricle pale and small, and the auricles dark and distended. Twenty minutes afterwards, reflex movements could yet be excited by feeble irritations of the skin.

Arrow H (see Plate II.) is one of a pair for which I am indebted to Mr J. K. Tomory, M.B., who, for a short time in 1887, resided at the London Missionary Society's Station in Central Africa. Dr Tomory informs me that the arrows were obtained from one of the Manyuema tribes on the west side of Lake Tanganyika. They were said to be used only for killing game, and the poison was believed to have an action like that of strychnine, and to be derived from a large tree.

The arrow is altogether $30\frac{1}{2}$ inches in length. The shaft is made of a single piece of fine-grained, reddish-brown light wood, $27\frac{1}{2}$ inches in length, and $\frac{6}{16}$ths of an inch in diameter near the head, but $\frac{5}{16}$ths of an inch in diameter near the bowstring end. The feathering commences at $1\frac{3}{8}$ inch from the notch, and extends up the shaft for $1\frac{3}{8}$ inch. It is very elaborate, consisting of fifteen separate feathers placed parallel to each other, and securely lashed to the shaft at each end. The arrow-head is inserted into a hollow in the shaft, so that almost no portion of its straight stem projects beyond the wooden shaft, the base of the expanded barbed head being, therefore, almost in contact with the end of the wooden shaft. The end of the shaft into which the head is inserted is strengthened by a vegetable thong (apparently consisting of a rush) lashed round it for a distance of $4\frac{1}{2}$ inches. The barbed head is 3 inches long and $1\frac{3}{8}$ inch wide at its broadest part; it is of a general oval acuminate shape, and is provided with a sharp spike-like

barb on each side, originating near the junction of the lowest third with the middle third of the head.

The poison is plastered over each surface of the head in a thick layer, for the most part $\frac{1}{8}$th of an inch thick, which covers the whole head, excepting its margins and the spike-like barbs. It is tough and hard, dark brown on the surface, and ochry-brown in the interior. It is partly soluble in water, and gives a slightly gritty sensation when triturated with it; and the watery solution is nearly colourless, acid in reaction, and distinctly bitter.

On microscopic examination, the poison was found to consist of vegetable tissues, pieces of vegetable hairs, oil globules, and brown granular masses. Solution of potash caused it to become yellowish-brown in colour; and, after it had been added, numerous microscopic, oval or kidney-shaped, colourless bodies made their appearance in the mixture. When heated between 110° and 120° F. with 10 per cent. sulphuric acid, it became green, and some time afterwards a faint violet tint could be detected.

The solution obtained by triturating $\frac{1}{10}$th of a grain with 4 minims of distilled water was injected under the skin of a frog weighing 325 grains. In a short time the peculiar attitude of Strophanthus poisoning was assumed, the mouth was frequently opened, the respirations became slow, the pupils contracted, fibrillary twitches occurred, voluntary movements were enfeebled, and the skin became paler in colour. The heart was exposed thirty-six minutes after the poison had been injected; it was found to be motionless, with the ventricle small and mottled, and the auricles large and dark, and irritation of any part of the heart failed to excite contraction. Twenty minutes afterwards, general and feeble reflex movements followed irritation applied to the skin. At no time during the experiment were there any spasms, nor were the reflex movements in the slightest degree exaggerated.

It appeared that some assistance might possibly be obtained in the identification of the poison of the arrows and in the packet, by determining whether a glucoside were present in any of them, especially as the active principle of Strophanthus is a glucoside. It was, however, found that this assistance could not easily be obtained, for each of the poisons reduced Fehling's solution before the poison had been digested with an acid.

In Table I. (p. 973) the results of the examination of the arrow-poisons have been summarised.

From the above experiments I am led to conclude that the poison of arrows A, B, E, F, G, and H consists principally, if not entirely, of a substance made with the seeds of Strophanthus. In reference to arrow C, no results were obtained sufficient to identify the substance with which it had been poisoned; nor, in the meantime, can any more definite statement be made with regard to arrow D than that its poison is a substance closely resembling Strophanthus in pharmacological action. This substance also sufficiently resembles the poison contained in the packet J to lend confirmation to the statement of the natives of the Wanyika tribe, that the poison in the packet is the same as that applied to the arrow D. If this poison be prepared from Strophanthus seeds, the seeds must have

TABLE I.—*Summary of Examination of Arrow-Poisons.*

	A	B	C	D	E	F	G	H	J
Watery Solution. *Colour.*	Pale sherry-brown.	Very pale sherry-brown.	Pale yellowish-brown.	At first reddish-brown, soon very dark brown.	Sherry-brown.	Faint yellowish-brown.	Very pale sherry.	Very pale sherry.	At first yellowish-brown, afterwards darker.
Watery Solution. *Reaction and Taste.*	Distinctly acid and bitter.	Distinctly acid and bitter.	Almost neutral, and not bitter.	Slightly acid, and not bitter.	Slightly acid and bitter.	Faintly acid, and not bitter.	Distinctly acid and bitter.	Distinctly acid and bitter.	Distinctly acid, and not bitter.
Microscopic characters of Poison.	Much vegetable structure, many fragments of hairs, oil globules, and granular masses.	Same as A.	Abundance of yellowish-brown and reddish-brown particles, and several colourless irregularly angular bodies.	Same as A, but no definite fragments of hairs.	Same as A.	Same as A.	Same as A.	Same as A, and oval and kidney-shaped colourless bodies.	Chiefly small yellow and brown granules, and masses of granules.
Effect of Solution of Potash.	Faintly more yellow. Much remains undissolved.	Same as A.	Not tested.	Orange, and greater part dissolved.	Same as A.	Same as A.	Same as A.	Yellowish-brown. Much undissolved.	Deep gamboge-yellow, and then brownish-yellow, with almost complete solution.
Effect of heating with Dilute Sulphuric Acid.	Green and then light brown colour.	Green, and then olive-brown.	Not tested.	Reddish-brown, which afterwards became darker.	Green, and then reddish-brown.	No change in colour.	Green and then brown.	Green, and then green with doubtful violet tint.	Became darker brown, with doubtful violet tint.
Lethal Activity in Frogs.	0·1 gr. fatal.	0·2 gr. fatal.	0·2 gr., and solution from three arrow-heads, no effect.	0·2 gr. fatal.	0·1 gr. fatal.	0·5 gr. no effect.	0·1 gr. fatal.	0·1 gr. fatal.	0·1 gr. fatal.

been subjected to some process, by which, probably, a watery extract had been obtained, almost perfectly free from vegetable structures in the case of the poison contained in the packet J, but less perfectly free from those structures in the case of the poison of arrow D. This, in itself, is sufficient to render it doubtful that these poisons have been derived from Strophanthus; for the preparation of an extract of its seeds for application to arrows would seem a superfluous labour, and it is actually proved to be superfluous in the case of the arrows undoubtedly poisoned with Strophanthus, where the only preparation has been to grind the seeds with water and mix the paste with some adhesive substance. This circumstance, along with the differences in microscopic appearance and in chemical reaction that have been described, renders it possible, if not probable, that the poison of arrow D and the poison contained in the packet J have been obtained from a stem or root. Sir JOHN KIRK informs me that at Nyassa an active poison is prepared from a wood; and it is also known that the Somali tribe, inhabiting an extensive district on the East Coast north from the Wanyika country, employ for their arrows a poison derived from the wood and root of an unknown Apocynaceous plant, apparently belonging to the genus *Carissa*. Further, both Sir JOHN KIRK and Dr FELKIN state that the Strophanthus plant has not been seen in the Wanyika country. These considerations render it advisable to restrict the definition of the poison of arrow D and of packet J to that of a substance acting like, but not demonstrated to be, Strophanthus.

It is, however, a remarkable circumstance that, out of eight arrows of different forms, six arrows, derived from districts so widely separated from each other as the River Gambia, the Tanganyika Lake, and the Zambesi River, should be poisoned with Strophanthus.

Nor do these represent all the known variations in the form of arrows poisoned with this substance, and all the localities in which such arrows are used. Three other forms, derived from the Gaboon district of West Africa, have been described; two made entirely of wood,* and the third, provided with an iron head;† but all three having the shaft feathering replaced by a leaf, and, judging from the absence of a bowstring notch, being adapted for use in crossbows or blow-tubes only.

The wide distribution of the Strophanthus plant over Africa, the great activity of its seeds, and the readiness with which they can be converted into a form suitable for application to arrows, are probable reasons for this extensive use of Strophanthus as an arrow-poison.

While thus widely used, both in the chase and in warfare, as an arrow-poison, it is worthy of remark that no evidence can be found of Strophanthus being used by the natives of Africa as a medicinal substance. On the contrary, Mr BUCHANAN informs me that they have too great a dread of it to use it in the treatment of disease, and that when they were told that the seeds were being used as a medicine in this country they expressed the opinion that the English must be mad to employ so poisonous a substance for medicinal purposes.

* POLAILLON et CARVILLE, *Archives de Physiologie*, tome iv., 1871–72, p. 708.

† R. BLONDEL, *Bulletin Général de Thérapeutique*, tome cxiv., 1888, p. 78.

2. Botanical Description.

Decandolle, in 1802, first described the genus Strophanthus, and gave it this name because of the twisted thong-like prolongations of the lobes of the corolla (στρόφος, a twisted band or cord or thong, ἄνθος, a flower).* It is a genus in the family of the Apocynaceæ, nearly related to Nerium and Echites, and even more so to Roupellia, which differs from it almost alone in being devoid of the prolongations of the corolla-lobes.

The genus is thus described by Bentham and Hooker :†—"Calyx 5-partitus, basi intus 5-∞-glandulosus (rarius eglandulosus ?). Corolla infundibularis, tubo brevi, fauce ample campanulata, squamis ligulisve 10 liberis v. basi per paria connatis instructa; lobi 5, contorti, dextrorsum obtegentes in acumen (seu *caudam*) nunc longissime lineare nunc rarius abbreviatum producti. Stamina summo tubo affixa, fauce inclusa, filamentis brevibus; antheræ sagittatæ, plus minus acuminatæ, circa stigma conniventes et ei medio leviter adhærentes, loculis basi in appendicem vacuam productis. Discus 0. Ovarii carpella 2, distincta; stylus filiformis; stigma crassum basi in membranam reflexam sæpius 5-lobam dilatatum, apicem versus sæpius lobis 5-glanduliferis cinctum, apiculo conico integro v. 5-fido; ovula in quoque carpello numerosa, ∞-seriata. Folliculi oblongi v. elongati, duri, divergentes v. divaricati. Semina compresso-fusiformia, apice in aristam longe plumosam producta, inferne coma decidua appendiculata (rarius ecomosa ?).—Arbusculæ v. frutices sæpe scandentes, glabri pubescentes v. villosi. Folia opposita, pennivenia. Cymæ terminales, nunc dense pauciforæ, nunc corymbosæ multifloræque. Flores speciosi rarius parvi, albi flavicantes aurantiaci rubri v. purpurei."

About twenty species are at present known, of which eight are found in Africa. Decandolle has himself described four species—*S. sarmentosus*, of Sierra Leone; *S. laurifolius*, of Africa; *S. dichotomus*, of India, China, and Java; and *S. hispidus*, of Sierra Leone. Of the others, the best known are *S. Bullenianus*, *S. capensis*, *S. Ledienii*, *S. Petersiana*, and *S. pendulus*, all of Africa; *S. brevicaudatus*, of Burmah; *S. divergens*, of China; *S. Griffithii* and *S. longicaudatus*, of Malacca; and *S. Wightianus*, of India.

Strophanthus Kombé, described by Oliver,‡ and formerly regarded as a distinct species, has not been placed in the list, as I understand that Oliver, after an examination of further and more complete specimens of the flowers and leaves, now regards it as "a variety, a geographical race of *S. hispidus*." His opinion is "that *S. Kombé*, of East Tropical Africa, is but a mere variety of *S. hispidus*, and that the differences between them are not more considerable than it is reasonable to allow to a species of wide distribution." Having submitted to this botanist some of the flowers sent to me by Mr

* *Annales du Muséum National d'Histoire Naturelle*, Paris, 1802, p. 408; *Bulletin des Sciences, par la Société Philomathique*, Paris, 1802, p. 122.

† *Genera Plantarum*, vol. ii. part 2, 1876, p. 714.

‡ Hooker's *Icones Plantarum*, 3rd series, vol. i. part 4, 1871, p. 79, and plate 1098.

Buchanan as the flowers of the plant from which the seeds used in the greatest number of my experiments had been derived, Professor Oliver expresses the opinion that they are the flowers of *S. hispidus*.

It is, therefore, to the species *hispidus* that the greatest amount of interest is at present attached, for, in all probability, it chiefly has furnished the materials for the chemical and pharmacological investigations that have hitherto been made in this country, as well as for the therapeutic experience that has been collected within the last fifteen years.

This species is not known to occur elsewhere than in Africa. It appears to be widely distributed over that continent, in its tropical and subtropical regions; having been found at various places between the east coast and centre of Africa, above the Victoria Falls of the Zambesi (Kirk), on the banks of the Shiré River, in the Manganja country, and extending northwards to the Murchison cataracts (Buchanan); as well as along a large portion of the west coast, in Senegambia, Sierra Leone (Kirk), Guinea, and the Niger and Gaboon districts.

The plant is described by Buchanan* as a strong climber lying in folds on the ground, and climbing to the tops of neighbouring trees by forming coils round them. The stem is about 3 inches in diameter, and young shoots grow from it, as nearly straight rods, of great length. When the stem is cut there exudes from it a milky juice, which is sticky and very bitter. The fruit is arranged in pairs, which have the appearance of gigantic thorns. He believes that, even under favourable circumstances, a plant will not produce flowers and fruit until it is three years of age.

Sir John Kirk—in a letter dated 1st January 1864, which was sent to me by the late Professor Sharpey—gives a similar description. He states that the Kombé plant (*Strophanthus hispidus*) "is a woody climber, growing in the forests both of the valleys and hills. The stem is several inches in diameter, and rough on the outside. It climbs up the highest trees, and hangs from one to the other like a bush vine."

There is considerable diversity of statement with regard to the periods of the year at which flowers and fruit are borne. At Eastern and Central Tropical Africa it is stated by Kirk† and Oliver‡ to flower in October and November, and by Buchanan in January also; while at Western Africa, Baillon§ and Diniau‖ state that the flowering season is in April and May, and Soubeiran,¶ on the authority of G. Fontaine, a pharmacist employed in the French Naval service, in December. The plant is said to bear fruit at East and Central Africa, in June, by Kirk; and in July, August, and September, by Buchanan and Consul Hawes:** and at West Africa, in June, by Blondel,†† Cazaux,‡‡ and Diniau.

* Unpublished letter.

† Unpublished letter to Dr Sharpey, 1st January 1864.

‡ *Loc. cit.*, p. 79.

§ *Archives de Physiologie*, tome iv., 1871–72, p. 526.

‖ *Bulletin Général de Thérapeutique*, tome cxiii., 1887, p. 172.

¶ *Journal de Pharmacie et de Chimie*, 15 Juin 1887, p. 593.

** *Pharmaceutical Journal and Transactions*, March 3, 1888, p. 748.

†† *Bulletin Général de Thérapeutique*, tome cxiv., 1888, p. 81.

‡‡ *Contributions à l'histoire médicale des Strophanthus.* Thèse. Paris, 1887, p. 15.

Mr Buchanan has at various times sent me specimens of the root, stem, branches, leaves, flowers, and fruit,* and has thus provided me with materials for a description of the different parts of the plant. I am also indebted for specimens and for valuable information to Sir John Kirk, Mr John Moir of the African Lakes Company, and Messrs Burroughs and Wellcome and Messrs Christy & Co., drug merchants, London.

Root.

The root consists of a main portion, which is swollen and constricted at irregular intervals, and of secondary roots, some of which are also swollen and marked by narrow constrictions like the main root. The specimens received from Mr Buchanan were preserved in spirit, and when they were compared with fresh roots taken from plants growing in the Botanic Garden of Edinburgh, it was seen that they had retained their original shape. They are in pieces of from $3\frac{1}{2}$ to 15 inches in length; but, as the extremities are broken, the length of the entire root cannot be ascertained. The pieces are straight or slightly curved, of a dark brown colour, and wrinkled by furrows extending in the direction of the long axis of the root. The extremities of the pieces are from $\frac{10}{16}$ths to $\frac{3}{16}$ths of an inch in diameter. The swollen portions are from $\frac{6}{16}$ths to $2\frac{12}{16}$ths of an inch in length, and from $\frac{6}{16}$ths to $1\frac{14}{16}$ths of an inch in diameter, and they have an irregularly oval, ovoid, or spindle shape. The portions of the root occupied by the constrictions have a diameter varying from $\frac{3}{16}$ths to $\frac{11}{16}$ths of an inch (Plate III. figs. 1 and 2). When sections are made through the root, it is seen that the swellings or enlargements are caused by a development of the cellular rind of food-storing cells, which at the constricted portions is present only as a relatively thin layer surrounding the central wood cylinder (Plate III. figs. 3 and 4). In addition to the constrictions or deep furrows involving the entire circumference of the root, there are other transverse furrows which are less deep, and which extend along a portion only of the circumference of the swollen parts of the roots (Plate III. figs. 1 and 2).

In specimens of the dried root, of which I have received several from Sir John Kirk, the masses of hypertrophied cellular-rind occur as soft, friable, and very irregularly-shaped and wrinkled swellings, separated from each other in many places by intervals of a quarter of an inch, where the hard cylindrical core of woody tissue is exposed.

The microscopic structure of the root is illustrated in Plate V. fig. 1.

* The obstacles which Mr Buchanan has had to contend with in procuring and sending specimens, owing to tribal feuds and the difficulties of carriage, are illustrated in the following extracts from one of his letters:—"Your letter about Strophanthus came duly to hand when I was at Zomba, standing by my property in case of an attack from the Mangoni, who were ravaging the country on the high lands on this side (Blantyre) of the Shiré River. One of my boys who belongs to the river was actually on the road to the river to get a supply when news came of the death of Mr Fenwick and Chipitula, and for long no person dared to go to the river. When these difficulties were got over, the Mangoni came, and I had to go to Zomba, and remain there until they took their departure. Lately, however, I have got a supply, and as I have arranged to come home in December, I shall take the Strophanthus with me. Unless special care be taken, it is sure to be damaged owing to leaky boats and canoes. The African Lakes Company's steamer is at present undergoing repairs, otherwise I should have sent the Strophanthus by her." (21st October 1884.)

Stem and Branches.

Specimens of the stem were received, both dry and preserved in spirit. They vary in diameter from $1\frac{2}{16}$ths to $1\frac{6}{16}$ths of an inch. The dry specimens, equally with those preserved in spirit, have a cork-like surface, which is profusely furrowed by deep branching grooves (Plate IV. figs. 1 and 2). In the dry specimens, the cork layer has shrunk so as to lay bare at the extremities the underlying hard structures of the stem. In a portion of dry stem $1\frac{6}{16}$ths of an inch in diameter, the cork layer was $\frac{2}{16}$ths of an inch in thickness. The structure of the stem has been further illustrated in Plate V. figs. 2 and 3, representing transverse sections, and fig. 4, a longitudinal section.

The branches are opposite. Their surface is nearly smooth, and the cork layer is thin, thus presenting a marked contrast to the stem; and they have numerous small, irregularly-shaped, pale (greyish-brown) markings.

Juice of Root, Stem, and Branches.—On examining fresh young plants, raised in the Edinburgh Botanic Garden from seeds sent from the Shiré district, I found that when incisions were made into the roots, stem, or branches, there exuded a considerable quantity of juice. From each of these parts it is acid and very bitter, and at first quite clear and almost colourless. The juice of the stem is, however, very sticky, and in a short time it becomes milky; but that from the root and branches remains unadhesive and non-opalescent.

Through the kindness of Mr John Moir, I have obtained from Africa a small quantity of the exuded juice from the stem of a growing plant. It consists of a slightly opalescent, bitter, and acid fluid, in which there is a mass of a plastic caoutchouc-like substance.

Leaves.

The leaves are entire, and generally oval acuminate, though occasionally they are ovate or obovate and shortly acuminate. The largest of those sent from East Central Africa is $5\frac{2}{16}$ths by $2\frac{10}{16}$ths of an inch, and the smallest $1\frac{9}{16}$ths by $\frac{8}{16}$ths of an inch. They are opposite, and have usually short petioles, but those attached to the extremities of branches are sessile or almost so. Both surfaces, the lower rather more so than the upper, are well covered with short fine hairs, which are most abundant along the veins and margins. The petioles, flower bracts, and terminal branches are also profusely hirsute (see Plate III. figs. 5 and 6).

Flowers.

The flowers are grouped in terminal cymes, which sometimes contain only four or five flowers, but often as many as eight or nine (see Plate III. fig. 6). In the specimens in my possession, unexpanded flowers are present with expanded ones in the same cyme. The calyx is gamosepalous and five-lobed, each lobe being oval acuminate in the expanded flower, and almost linear in the unexpanded flower-bud; and the calyx and its lobes are covered on the outside with numerous fine hairs. The corolla is gamopetalous, funnel-shaped, and five-lobed, each lobe being pro-

longed into a singular-looking narrow tail (see Plate III. fig. 7). In many of my specimens the corolla tails are so long as nine inches, but even these have obviously been broken, owing to their brittle condition when in the dry state. In the unexpanded flower-bud, each of the prolongations of the corolla appears to be doubled on itself, and the five doubled prolongations are twisted together to form a cord-like structure, which projects upwards from the flower-bud for a distance of from half an inch to two inches, according to the age of the flower (see *a* in figs. 6 and 7, Plate III.). As has been stated, the genus received from DECANDOLLE the name Strophanthus on account of this very singular character of the flower; but the drawings accompanying his original descriptions represent the prolongations in expanded flowers as projecting vertically upwards from the extremities of each corolla lobe, whereas they do so only in the unexpanded flowers. In the expanded flowers, the prolongations are no longer bent upon themselves and twisted together, but they are unfolded and hang downwards as thread-like tails, probably more than 10 inches in length, and about the $\frac{1}{20}$th of an inch in diameter, which give not only a singular, but, also, a very graceful appearance to the flowers.

The corolla is about $\frac{3}{4}$ths of an inch in length in its undivided portion, and each lobe is about $\frac{5}{8}$ths of an inch in length from its base to the point where it narrows into the thread-like prolongation. Within the corolla, and immediately below the points of junction of contiguous corolla lobes, are five deeply bifid scales or appendages, each division of which projects upwards and inwards, and is terminated by a rounded blunt extremity (Plate III. fig. 8). Below these appendages, at the base of the corolla, are seen the five stamens closely surrounding the pistil (Plate III. fig. 8, *a*). The forms of the stamens and pistil are represented in Plate III. figs. 9 and 10.

In dried specimens, the corolla varies in colour from a brownish to a reddish yellow, and the inside is of the same colour as the outside. In the fresh, natural condition, judging from the descriptions of BUCHANAN and HUEDELOT,* it appears to be of a general creamy white colour, with yellow at the base and a few purple spots above. DECANDOLLE,† however, describes the colour of the corolla as orange, and KIRK as yellowish-white or pale yellow,‡ or pale yellowish-green.§

The external surface of the corolla and of its lobes is hirsute, the hairs being extremely fine, short, and pointed, and in the lobes most abundant along their margins. The internal surface has only a few very minute hairs, but the ovary is distinctly hirsute. The tail-like prolongations of the corolla lobes are likewise hirsute, but very sparsely in their unfolded state, although markedly so when they are doubled up and twisted together.

Fruit.

The fruit is arranged in pairs of follicles. The follicles in each pair are united together at the ventral surfaces in the young state, but they gradually become separated as maturity advances by a hinge-like movement at their bases, until when ripe each separated follicle

* *Archives de Physiologie*, tome iv., 1871–72, p. 526.
† *Loc. cit.*, pp. 412 and 123.
‡ Letter to Dr SHARPEY, 1st January 1864.
§ Letter, dated 4th November 1888.

projects from the fruit-stalk almost at a right angle with it, the two follicles forming together a nearly straight line, whose extremities are the apices of the follicles (see Plate IV. figs. 3, 4). The mature follicles have a general fusiform shape, but frequently they taper fairly regularly from the base nearly to the apex, and so present a lanceolate contour. The middle portion, however, is generally thicker than either extremity, and the base is always much thicker than the apex. The latter is terminated by an irregular bifid disc or expansion, measuring transversely about $\frac{7}{16}$ths by $\frac{5}{16}$ths of an inch, the sulcus of which is at right angles to the ventral surface of the follicle. This bifid expansion may be produced by an indentation remaining after the style has fallen off, or, in the event of the style being persistent, it may represent the cleft apex of the stigma. Each follicle has two surfaces; one rounded and occupying the greater part of the circumference, and the other flattened, concave, or even wedge-shaped, and representing the surface originally in apposition to the follicle developed along with it.

The rounded surface is of a dark greyish-brown colour, smooth and fleshy-looking in the undried follicles, but rough and marked by numerous small and nearly white spots* in the dried follicles (see Plate IV. fig. 3). The flattened or concave surface is in the dried follicles of a pale brown nearly white colour. It consists of a thin parchment-like and brittle membrane, whose margins are depressed below the contiguous margins of the rounded surface, and it presents a central longitudinal slit, and occasionally, when the follicle is very ripe, several slits at different parts of its surface, through which the silky hairs of the seed-appendages project here and there.

The dimensions and weights of the entire and dry mature follicles vary considerably. Of sixteen sent by BUCHANAN from the Shiré district, it was found that the average length was 11·2 inches, the average diameter at the middle 1·2 inch, and the average weight 377 grains: but the extremes were represented by a follicle $12\frac{1}{2}$ inches in length, 1 inch in diameter, and 512 grains in weight; and by one $9\frac{1}{2}$ inches in length, 0·85 inch in diameter, and 160 grains in weight.

Only a few entire follicles have, however, been brought to this country. The great bulk of those imported have had the outer part of the pericarp, comprising the epi- and mesocarp scraped off before importation, and while they are in a fresh and soft condition (see Plate II. fig. I.). They are thus customarily treated in Africa to enable them more easily to be dried and stored for use; and, when dried, they are tied together with ribbon-like strips of palm leaves, so disposed as to encircle the follicles in pairs. Long bands of follicles arranged parallel to each other are in this way produced, which are then rolled into cylindrical bundles each containing from two to three hundred follicles.

The scraped follicles retain the general form of the entire ones, but their distal extremities are not terminated by the bifid expansions found in the follicles that have not been scraped. Many of them are fusiform, others lanceolate, and a few almost cylindrical in shape. They also possess a rounded or convex dorsal, and a flattened or concave ventral

* Dr MACFARLANE has suggested to me that these spots are scars marking the positions of the roots of the fallen off hairs, formerly attached to the carpels.

surface. The former represents the exterior of the endocarp which has been left as a covering for the scraped follicle. This covering is of brittle consistence, only about the $\frac{1}{100}$th of an inch in thickness, externally of a pale brown colour, and marked by irregular shallow furrows produced in the scraping of the follicles, and internally smooth and of a uniform dull lemon colour. The flattened or concave ventral surface possesses the same characters as in the entire follicles.

Before describing in detail the structures contained within the follicles, some particulars will be given of the dimensions and weights of the follicles and of their constituent parts. The chief supplies of follicles were obtained from BUCHANAN in 1879, 1881, and 1885, and they were all collected in the Shiré district. No important differences could be detected between the follicles, or their constituent parts, received at these several dates, in respect of their general characters, chemical or pharmacological properties, or microscopic structure.

Entire Follicles.—The dimensions and weights were ascertained of the constituent parts of two only of the dry entire follicles.

TABLE II.

Follicle.			Seeds.		Weight of Pericarp.	Weight of Placenta.	Weight of Comose Appendages.
Length.	Maximum Diameter.	Weight.	Number.	Weight.			
11·65 inches	1·5 inch.	509 grains	222	100 grains	352 grains	17 grains	40 grains
10·75 „	1·25 „	401 „	212	84 „	266 „	15 „	35 „

Scraped Follicles.—The dimensions and weights of a large number of scraped follicles and of their constituent parts have, however, been ascertained. I have below tabulated, separately, the results of the detailed examination of considerable numbers of follicles from the supplies received in each of the above three years, giving only the averages for each supply:—

TABLE III.

Supply of	Number Examined.	Follicles.			Seeds.		Weight of Endocarp and Placenta (grains).	Weight of Comose Appendages (grains).
		Length (inches).	Maximum Diameter (inch).	Weight (grains).	Number.	Weight (grains).		
1879	116	10·44	0·75	143·2	169·3	63·88	46·16	31·08
1881	149	11·8	0·88	200·8	185·8	70·4	83	45·9
1885	72	10·3	0·82	191·7	189·1	76·3	70·7	40·9

The figures in the above table, it will be understood, are merely the average

dimensions and weights obtained by dividing the totals for each group of follicles by the number of follicles. They show considerable differences in the averages of the follicles obtained in different years from the same district. The differences in the follicles of each year are still more considerable, and this may be illustrated by stating the dimensions and weights of the largest and smallest follicles, and of their constituent parts, among those examined from the supplies obtained in each of the three years :—

Table IV.

		Follicle.			Seeds.		Weight of Endocarp and Placenta.	Weight of Comose Appendages.
		Length.	Maximum Diameter.	Weight.	Number.	Weight.		
1879.	Largest,	11·0 ins.	0·75 in.	200 grs.	187	106·5 grs.	60 grs.	32·5 grs.
	Smallest,	8·25 „	0·62 „	53·5 „	51	14 „	29 „	11 „
1881.	Largest,	12·5 „	1·12 „	317 „	187	134 „	126 „	53 „
	Smallest,	10·25 „	0·62 „	110 „	54	44 „	49 „	16·5 „
1885.	Largest,	13·12 „	1·0 „	340·5 „	230	164 „	112 „	62 „
	Smallest,	8·87 „	0·63 „	114 „	144	32 „	54 „	28 „

From the data given in Table III., it appears that the average length of the 337 scraped follicles there represented is 10·8 inches, and the average weight 178·6 grains. In order to obtain some indication of the dimensions and weights of the constituent parts of average follicles, twenty follicles, whose size and weight were about the average, were selected from the supply of 1885, and an analysis, similar to that shown in Table III., was made of their constituent parts. The twenty selected follicles varied in length from 9·5 to 11·5 inches, and in weight from 172·25 to 198 grains. The results are stated below in averages.

Table V.—*Average Weight and Dimensions of Twenty Follicles of Average Size, and Averages of their Constituent Parts.*

Follicles.			Seeds.		Weight of Endocarp.	Weight of Placenta.	Weight of Comose Appendages.
Length.	Maximum Diameter.	Weight.	Number.	Weight.			
10·75 ins.	0·76 in.	181·5 grs.	197·2	90·5 grs.	43·7 grs.	16·4 grs.	30·3 grs.

On comparing the figures in this Table with those in Table III., the most noteworthy difference between them is seen to be that in the twenty follicles selected as average-sized

follicles, the average number and weight of the seeds in each follicle is greater than in the 337 unselected follicles.

A few of the scraped follicles in my possession, obtained from Mr Buchanan in 1885, are, however, of greater length than any of those represented in Tables III. and IV. Their form, also, is nearly cylindrical; but, while the seeds contained in them have the same characters as those in the fusiform and lanceolate follicles, these seeds are of relatively light weight, and, apparently, insufficiently ripened. Several of the follicles are from 15 to 17 inches in length, but one of 17 inches weighed only 165 grains, and one of 15 inches only 146 grains. In a follicle 15 inches long, the diameter at the base is 0·75 inch, at the middle 0·65, and at the apex 0·59; and in another, also 15 inches long, the respective diameters are 0·59, 0·62, and 0·53 inch.

The average number of seeds in each follicle of the 337 represented in Table III. is 180·8, and the average weight of the seeds in each follicle is 69·4 grains. In the twenty follicles of average size and weight represented in Table V., the average number of seeds in each follicle is 197·2, and the average weight of the seeds in each follicle is 90·5 grains.

These averages are, however, much above and below the numbers and weights actually found in many follicles. This is illustrated by the following examples:—

Table VI.

Follicles containing the	Largest Number of Seeds.		Smallest Number of Seeds.		Greatest Weight of Seeds.		Smallest Weight of Seeds.	
	Number of Seeds.	Weight of Seeds.	Number of Seeds.	Weight of Seeds.	Weight of Seeds.	Number of Seeds.	Weight of Seeds.	Number of Seeds.
In the 337 Follicles represented in Table III. — Follicles of 1879	228	40 grs.	51	14 grs.	106 grs.	187	14 grs.	51
	223	53 „	76	19 „	100 „	204	19 „	76
	215	79·5 „	91	36 „	99 „	194	24 „	137
	205	74 „	98	58·5 „	99 „	192	29 „	109
„ „ 1881	281	60 „	54	44 „	179 „	219	27·5 „	215
	265	45·5 „	61	49 „	153 „	220	32·5 „	207
	251	105 „	74	17 „	145·5 „	244	33 „	187
	250	53 „	75	43 „	141·5 „	222	36·5 „	145
	246	75 „	85	57 „	134 „	187	36·5 „	87
	245	81 „	101	53·5 „	128 „	154	38 „	124
„ „ 1885	244	67 „	92	61 „	155 „	174	32 „	144
	234	48·5 „	109	57 „	139 „	178	34·5 „	146
	234	90 „	114	101 „	137 „	173	36 „	170
	232	109 „	129	57 „	122 „	193	37 „	154
In the 20 Follicles of average size and weight represented in Table V.	235	96·5 „	112	73 „	107·5 „	204	72 „	142
	229	98·5 „	141	79 „	98·5 „	229	73 „	112
	226	90 „	142	72 „	97 „	202	79 „	141

A transverse section of a green immature follicle is shown, unmagnified, in Plate IV. fig. 5, and of one slightly magnified in Plate V. fig. 5. In the latter figure, the several

layers of the pericarp become apparent, and it is seen how great is the thickness of the mesocarp relatively to the other layers. When the follicle is mature and dry, the pericarp becomes much thinner, the reduction in thickness being mainly caused by shrinking of the mesocarp. This is rendered apparent when the above illustrations are compared with figs. 3, 4, 5, and 6 in Plate VII., and also by the measurements given below of the transverse section of the pericarp and its layers in a mature and in an immature follicle :—

Immature Undried Follicle.

Epicarp, ...	0·0097 inch.	0·00025 metre.
Mesocarp, ...	0·2559 „	0·0065 „
Endocarp, ...	0·0078 „	0·0002 „
	0·2734 „	0·00695 „

Mature Dry Follicle.

Epicarp, ...	0·01 inch.	0·00025 metre.
Mesocarp, ...	0·05 „	0·0013 „
Endocarp, ...	0·01 „	0·00025 „
	0·07 „	0·0018 „

The mesocarp contains numerous vascular fibres arranged in isolated bundles (see Plate V. figs. 5 and 5*b*). The structure of the endocarp is illustrated in Plate V. figs. 5, 5*c*, and 5*d*, and the explanation of its hardness is seen in the elongated indurated cells of which it is composed. The circumferential direction of the cells in the inner of its two layers (next the cavity of the follicle) and the longitudinal direction of those in its outer layer (next the mesocarp) is also shown in these illustrations.

Contents of the Follicles.—When the interior of a mature follicle is examined, it is found to contain three different structures, namely, (*a*) the placenta, (*b*) the seeds with their attached comose appendages, and (*c*) a quantity of fine downy hairs, for the most part loose in the interior.

(*a*) *The placenta* is attached to the inverted carpellary margins, which project into the interior of the follicle in its immature condition, and which, as maturity advances, become split into the two plates that together form the placental or ventral surface of the follicle. The placenta also subdivides into two portions, each of which curves round into one of the lateral halves of the interior of the follicle. In a transverse section of the follicle, each lateral half of the interior is, therefore, seen to be occupied by a curved, almost completely spiral placental membrane (Plates IV. fig. 5 ; V. fig. 5 ; and VII. figs. 3, 4, 5, and 6). In the dry follicle, the concave surface of the curved portion of the placenta is smooth, shining, and, like the interior surface of the endocarp, of a pale lemon colour ; and the convex surface is of much the same grey colour as the nearly flat or concave exterior ventral surface of the follicle, with which surface, indeed, it is continuous. This curved portion of the placenta is marked by a number of depressions or pittings, caused by the pressure of the seeds in contact with it.

When a transverse section of the placenta in a mature follicle is examined microscopically, it is seen to consist of a broad cellular centre in which are numerous islands of vascular bundles, bounded on each side by several layers of elongated cells. On the concave surface of the placenta, the cells constituting the surface layer are of larger size than the other elongated cells next them, and they appear to be continuous with the circumferentially elongated and indurated cells of the inner surface of the endocarp, from which, however, they differ in being less elongated and nonindurated. In a longitudinal section, the cells on both surfaces of the placenta are seen to be of considerably greater length than in a transverse section, and the actual long axis of each cell is, therefore, parallel with the long axis of the follicle.

(*b*) *The seeds* which in the green condition of the follicles are attached to the convex surface of the curved portion of the placenta are, in the mature, dry follicles, unattached, and, therefore, merely in contact with the placenta and the interior of the endocarp, from the latter of which, however, they are separated by the loose downy hairs referred to above (described under *c*, p. 989). They are of a brownish-fawn colour, but in certain lights they are nearly white, owing to the numerous shining hairs which closely cover their surface, and give to the seeds a soft velvety feeling. When placed in water they float on the surface, and, if left in the water, they remain floating for many days. They have an intensely bitter taste, but no odour until they are bruised, when at first the odour is not unpleasant, having some resemblance to oatmeal, but after a considerable time, especially if the bruised seeds be exposed to the air, it becomes oily and somewhat rancid. They are flattened and have two surfaces, but their shape varies considerably, especially in immature seeds, owing to the distortions that occur during the change from the moist to the dry state. Most frequently, in well-matured seeds, the shape is oval acuminate, though occasionally it is elliptical. The dorsal surface is usually convex or nearly flat, and has a depression near the apex, and frequently also several slight longitudinal ridges, no doubt caused by puckering of the testa and shrinking of the albumen during drying (Plate IV. fig. 6). The ventral or placental surface is always irregular, the chief and most constant irregularity being caused by a ridge near its middle line occupying two-thirds or three-fourths of the upper part of the seed (Plate IV. fig. 7*a*). On this ridge, generally at the junction of the upper fourth with the lower three-fourths of the seed, there is a minute whitish spot or projection (Plate IV. figs. 7*a* and 8*b*, *x*). The ridge is produced by the raphe of the seed, and the minute spot on it represents the funiculus, broken off at its attachment to the seed. The base of the seed is usually pointed, sometimes acutely, at other times bluntly; but, occasionally, it is quite rounded and even flattened. Towards its apex, the seed most frequently tapers gradually to a fine extremity, which is continued as a slender shaft or stalk, whose summit is crowned by a tuft of long silky hairs, this shaft and tuft or coma forming the peculiar plumose appendage of the seed. The entire seed, with its plumose appendage, has a striking and beautiful appearance, and in its general form it closely resembles an arrow: the seed representing the head; the slender prolongation of the testa of the seed,

the shaft; and the long silky hairs that crown the summit of the stalk, the feather of the arrow (Plate IV. fig. 6).

The tuft of hairs forming the coma of the seeds has generally a conical outline, with the apex of the cone pointing downwards; and the summit of the tuft is usually dome-shaped, but at times it has the form of a hollow cone. The hairs next to the seed proceed from the stalk at an angle of about 70°, but higher up the stalk the angle gradually becomes more acute, so that at the upper third it is about 40°, while at the further extremity the hairs are nearly parallel with the axis of the stalk.

In a few follicles the coma has been found to be nearly cylindrical in form, with all the hairs placed nearly at right angles with the stalk.

The dimensions and weights of the seeds vary greatly.

Of twenty selected on account of their large size, the average length was 0·8 inch, the average maximum width 0·156 inch, the average thickness from one flat surface to the other 0·065 inch, and the average weight 0·78 grain.

Of twenty selected on account of their small size, the average length was 0·5 grain, the average maximum width 0·118 inch, the average thickness 0·058 inch, and the average weight 0·15 grain.

In order, if possible, to obtain a nearer approach to the average dimensions and weight of the seeds, twenty were taken without selection from the seeds contained in the twenty average-sized follicles from Buchanan, referred to at page 982, and it was found that the average length was 0·686 inch, the average maximum width 0·143 inch, the average thickness 0·083 inch, and the average weight 0·586 grain. These great variations are, indeed, such as might have been anticipated in seeds obtained from plants in which, of necessity, the conditions of maturity, season, and locality of growth could not be the same. The averages which the figures represent cannot, therefore, be regarded as true for all collections of seeds. This was further exemplified in the case of a large quantity of seeds very liberally given to me in 1886 by Mr Moir, of the African Lakes Company, of which twenty, likewise taken without any selection, had an average length of 0·7 inch, an average maximum width of 0·16 inch, an average thickness of 0·088 inch, and an average weight of 0·66 grain.

It may, however, be stated that good mature seeds have a length of from 0·6 to 0·7 inch, and a maximum width of from 0·14 to 0·16 inch; although they may be so large as 1 inch in length and 0·18 inch in maximum width.

It is also difficult to define the average weight of the seeds. Deduced from the figures in Table III., the average weight of each seed in the examined follicles of 1879 is 0·376 grain, in those of 1881 it is 0·379 grain, and in those of 1885, 0·403 grain; and for the collective follicles of these three years, 0·386 grain. The average weight, however, of each seed in the largest follicle of 1879 is 0·57 grain, and in the smallest 0·27 grain; in the largest follicle of 1881 it is 0·71 grain, and in the smallest 0·81 grain; and in the largest follicle of 1885 it is 0·71 grain, and in the smallest 0·22 grain (Table IV.). When these figures are considered along with the circumstance that the average weight of each seed

in the twenty follicles of average size (Table V.) is 0·458 grain, the inference may be drawn that the average weight of a mature seed of *Strophanthus hispidus* is from about 0·4 to 0·6 grain.

Magnified longitudinal and transverse sections of the seed, and illustrations of its microscopical structure, are given in Plate VI., and a somewhat diagrammatic representation of a longitudinal antero-posterior section in Plate VII. fig. 1. In the latter, the raphe and funiculus are shown.

While great variations are met with in the length of the plumose appendage of the seed, it may in general terms be stated to be shortest in the smallest and lightest follicles, and longest in the largest and heaviest. The variations may be illustrated by the following measurements of the appendages of seeds from the upper, middle, and lower parts of three follicles, the follicles themselves being representative of those of small, medium, and large size (Table VII.)

TABLE VII

Follicle.		Position of Seed in Follicle.	Comose Appendage.			Size of Seed.	Weight of Seed.
Length.	Weight.		Length of Naked Portion.	Length of Tuft.	Total Length.		
8 inches,	135 grains,	Top,	0·96 inch. 1·03 „	1·18 inch. 1·53 „	2·14 inches 2·56 „	0·53 × 0·12 inch. 0·5 × 0·09 „	0·32 gr. 0·3 „
		Middle,	1·28 „ 1·09 „	1·56 „ 1·81 „	2·84 „ 2·9 „	0·53 × 0·14 „ 0·56 × 0·14 „	0·295 „ 0·32 „
		Base,	0·96 „ 0·81 „	1·4 „ 1·18 „	2·36 „ 1·99 „	0·43 × 0·14 „ 0·4 × 0·13 „	0·26 „ 0·15 „
10 „	194 „	Top,	1·18 „ 1·25 „	1·81 „ 1·5 „	2·99 „ 2·75 „	0·68 × 0·15 „ 0·68 × 0·14 „	0·46 „ 0·51 „
		Middle,	1·21 „ 1·31 „	1·96 „ 1·84 „	3·17 „ 3·15 „	0·65 × 0·156 „ 0·68 × 0·13 „	0·39 „ 0·44 „
		Base,	1·56 „ 1·09 „	1·96 „ 1·81 „	3·116 „ 2·9 „	0·68 × 0·12 „ 0·65 × 0·09 „	0·39 „ 0·31 „
16 „	312 „	Top,	2·87 „ 2·81 „	2·68 „ 2·75 „	5·55 „ 5·56 „	0·81 × 0·2 „ 0·93 × 0·18 „	0·45 „ 0·88 „
		Middle,	3·12 „ 2·53 „	2·81 „ 2·78 „	5·93 „ 5·31 „	0·87 × 0·2 „ 0·92 × 0·19 „	0·8 „ 0·88 „
		Base,	2·09 „ 2·25 „	2·12 „ 2·218 „	4·21 „ 4·468 „	0·53 × 0·156 „ 0·59 × 0·18 „	0·65 „ 0·25 „

The measurements in the above Table show that the longest comose appendages occur at the middle of each follicle, and that the shortest are more frequently at the base than at the top. They also show that the tufted portion of the appendage is generally of

greater length than the naked portion, although the difference between them is never great. On consideration of details, it is seen that the tuft is slightly longer than the naked shaft in the smallest and medium sized follicles, and that in the largest follicle the tuft is in some seeds longer and in others shorter than the naked shaft.

Comose appendages have, however, been met with of greater length than any represented in the above Table (VII.). Not uncommonly they are found slightly over 6 inches in length in large follicles. I give below the measurements of the comose appendages of seeds taken from a follicle 17 inches in length.

TABLE VIII.

Position of Seed in Follicle.	Comose Appendage.			Size of Seed.
	Length of Naked Portion.	Length of Tuft.	Total Length.	
Top,	2·62 inches.	3·25 inches.	5·87 inches.	0·75 × 0·156 inch.
	2·68 ,,	3·12 ,,	5·8 ,,	0·84 × 0·156 ,,
	2·92 ,,	2·75 ,,	5·67 ,,	0·81 × 0·175 ,,
Middle,	2·78 ,,	3·25 ,,	6·03 ,,	0·85 × 0·156 ,,
	2·43 ,,	3·12 ,,	5·55 ,,	0·75 × 0·175 ,,
	2·31 ,,	3·25 ,,	5·56 ,,	0·62 × 0·175 ,,
Base,	2·34 ,,	2·4 ,,	4·74 ,,	0·65 × 0·14 ,,
	2·31 ,,	2·31 ,,	4·62 ,,	0·65 × 0·156 ,,
	2·5 ,,	3·06 ,,	4·56 ,,	

The measurements in the Table (VIII.) also give further evidence in favour of the statement that the tufted portion of the comose appendage is generally longer than the naked portion.

Two or three follicles have been observed in which all the seeds have comose appendages of unusually small and uniform size. The dimensions of two entire seeds taken from the top, middle, and base, respectively, of one of these follicles were found to be as follows :—

TABLE IX.

Position of Seed in Follicle.	Comose Appendage.			Size of Seed.	Weight of Seed.
	Length of Naked Portion.	Length of Tuft.	Total Length.		
Top,	0·96 inch.	1·03 inch.	1·99 inch.	0·46 × 0·12 inch.	0·37 grain.
	1·06 ,,	1·06 ,,	2·12 ,,	0·53 × 0·15 ,,	0·41 ,,
Middle,	0·87 ,,	1·09 ,,	1·96 ,,	0·56 × 0·12 ,,	0·39 ,,
	0·93 ,,	1·06 ,,	1·99 ,,	0·5 × 0·14 ,,	0·41 ,,
Base,	0·78 ,,	0·81 ,,	1·59 ,,	0·5 × 0·15 ,,	0·4 ,,
	0·87 ,,	0·87 ,,	1·74 ,,	0·5 × 0·12 ,,	0·38 ,,

The seeds with the small appendages attached to them are very beautiful and graceful miniature representations of seeds with larger appendages. The follicle containing the above seeds was 10 inches in length, and the seeds themselves were fairly mature, as their sizes and weights show.

The shaft of the comose appendage consists of a prolongation of the testa of the seed. It is extremely brittle, and has a diameter of about the one-hundredth of an inch, which remains pretty uniform throughout the whole length of the naked portion of the shaft.*

The hairs in the tufted portion are unicellular, white, silky in lustre, and flexible, but yet delicate and easily broken. They vary in length from a little less than 1 inch (0·9) to 3 inches. Their diameter at the base is about 0·0022 inch (0·055 mm.), and they gradually taper to a fine point of about 0·00012 inch (0·0032 mm.). At the base, there is often a slight swelling or bulb. The structure of the hairs is represented in Plate IV. figs. 9, 10*a*, and 10*b*.

c. Fine Hairs covering, and longer than the Seeds, and interposed between the Seeds and the inner surface of the Endocarp.—When a mature follicle is opened without any special precaution, a large quantity of a soft down is seen to be mingled with the seeds, but not attached to them. If the dorsal surface of the endocarp be carefully removed so as to expose the interior of the follicle, especially after the follicle has been soaked in water for a few days, the seeds are found to be concealed by a padding of hairs, interposed between the seeds and the inner surface of the endocarp (see Plate VII. fig. 2). The hairs constituting this padding appear to originate at the base of each seed, where the collected hair-roots are seen as opaque transverse lines. For the most part, the hairs proceed directly upwards, so as entirely to cover the dorsal surface, and a portion of the comose appendage of the seed from whose base they appear to originate, and, also, a part of the next seeds immediately higher up in the follicle and the interspaces between them. A few hairs extend downwards. Nearly all the hairs are therefore interposed between the dorsal surface of the seeds and the inner surface of the endocarp, and between contiguous seeds; only a small number lying upon the placental (or ventral) surface of the seeds.

The roots of the hairs are generally curved, and as the hairs become swollen above the roots, the roots have very commonly a beak-like appearance (see Plate IV. fig. 12*a*). Above the swollen portion, the hairs gradually taper to a somewhat blunt extremity. Their length is from 0·657 to 1·12 inch, and their diameter at the end of the root is about 0·00052 inch (0·013 mm.), at the bulb-like swelling 0·0026 inch (0·065 mm.), and at the tip or apex 0·00052 inch (0·013 mm.). Their naked eye and microscopic appearances are illustrated in Plate IV. figs. 11, 12*a*, and 12*b*.

* The following are the diameters of the naked portion of the shaft of two comose appendages removed from two average-sized follicles :—

	Near Base.	Near Middle.	Near Tuft.
No. 1, . .	0·0112 inch. 0·28 mm.	0·015 inch. 0·25 mm.	0·0093 inch. 0·23 mm.
No. 2, . .	0·0108 inch. 0·27 mm.	0·0092 inch. 0·23 mm.	0·008 inch. 0·22 mm.

The roots of the hairs are firmly pressed against each other, forming the bases of tuft-like groups, the individual hairs of which, in undried and immature follicles, are in contact and parallel with each other. In dry follicles, however, the hairs, while still remaining in contact at the roots, diverge from each other above the roots to an extent directly proportional to the dryness and advancement in dehiscence of the follicle (Plate IV. fig. 11).

The structural relationship of these hairs to the seeds is indicated, even in dry mature follicles, by the circumstances that the tips of their roots are pointed towards the base of the seed from which they appear to originate, that those hairs whose roots point towards the centre of the base of seeds curve round the base, and thus acquire their usual vertical direction, and that the hairs are found only in the portion of the interior of the follicle where the seeds are placed, and, therefore, not at the upper part.* Their relationship to the seeds is, however, unambiguously shown when an immature undried follicle is examined; for it is then seen that a tuft of hairs of considerable length originates at the base of each seed, and is firmly adherent to it. A group of these hairs attached to the base of a seed removed from an immature green follicle is illustrated in Plate IV. fig. 13, the hairs having been drawn down from the surface of the seed, in order to display them more distinctly. Their probable function is referred to in the succeeding paragraphs.

Dehiscence of the Follicles and Dissemination of the Seeds.—As the follicle matures, its ventral or placental surface enlarges by the inverted fused edges of the carpels, which project into the interior of the follicle in its immature condition, splitting up more and more, and so expanding this surface. The tearing asunder of the inverted carpellary edges appears to be mainly produced by the gradual separation from each other of the ventral margins of the pericarp, which becomes less and less rounded as maturity and drying proceed. The changes produced in this process are illustrated in Plate VII. figs. 3, 4, 5, and 6, which represent transverse sections of dry follicles in four stages of maturition. On comparing fig. 5, Plate IV., or fig. 5, Plate V., with figs. 3, 4, 5, and 6, Plate VII., it will also be seen how greatly the pericarp shrinks, especially in its mesocarp portion, as maturition and drying advance. Several entire follicles in my possession exhibit a still greater degree of flattening of the pericarp and opening up of the ventral surface than is shown in Plate VII. fig. 6, but in them some of the contents of the follicle, including even a portion of the placenta, have generally escaped from the interior. No doubt, the assumption by the pericarp of a flat or nearly flat shape occurs in natural conditions when perfect maturity has been attained, and thereby the complete extrusion of the seeds is rendered possible.

That the condition of roundness or flatness of the pericarp is greatly dependent on the moisture or dryness of its structures may be shown by immersing a dry mature follicle in

* For example, in a follicle 11 inches in length, these hairs were present only in the lower 6½ inches, and this exactly corresponded to the part of the follicle where seeds occurred. Above this part the follicle contained only placenta and comose appendages.

water. The effect of immersion is a gradual transverse rounding of the pericarp, with a corresponding approximation of the placental or ventral edges and a consequent narrowing of the placental or ventral surface of the follicle. An originally flat entire pericarp, one inch and a quarter wide, may in a few hours become so greatly rounded that its edges approach to within three-quarters of an inch from each other. If the pericarp be then allowed to dry, by merely exposing it to the air at an ordinary temperature, it gradually resumes its original flat shape, its edges gradually separating from each other to their original distance. If the endocarp alone be immersed in water and then removed and allowed to dry, it assumes the same changes of form as the entire pericarp, but they are more rapidly accomplished. If the endocarp be removed from the pericarp, the conjoined meso- and epicarp also react in the same way as the entire pericarp, or as the endocarp alone, under the influence of moisture and dryness.

The above experiments seem to show that the changes in form are not dependent on the anatomical structure of any one part of the pericarp. As the changes occur, however, most rapidly and completely in the detached endocarp, the presence in it of elongated indurated cells, arranged in circumferential and longitudinal directions (see Plate V. figs. 5*c* and 5*d*), may confer, as BLONDEL * has supposed, on this portion of the pericarp a special facility of movement during the change from the condition of moistness to that of dryness. It cannot, however, be overlooked that the existence of indurated cells in the endocarp would strengthen the entire pericarp, and, by enabling it more effectually to resist any bursting force operating in the interior of the follicle, would prevent splitting of the follicle elsewhere than at the feebly resisting placental or ventral surface. It will afterwards be pointed out that, for the satisfactory extrusion of the seeds, it is of importance that dehiscence should occur at the ventral surface.

When the separation of the inverted carpellary edges and the resulting expansion of the ventral or placental surface of the follicle has advanced to a certain stage, the latter opens either at the middle line alone, by complete separation of the edges of the two previously united carpellary margins, or both at the middle line and at other parts of the dried and brittle ventral surface, by several longitudinal splittings.

The dragging of the inverted carpellary edges from the interior to the surface of the follicle induces a change in the position and form of the placenta, which is attached to these edges. By this movement it is brought nearer to the ventral surface of the follicle, and, as it is being drawn from its original position, its spiral is unfolded (Plate VII. figs. 3, 4, 5, and 6). The seeds, imbedded at maturity in the elastic hairs which surround them, and fixed in position, also, by their comose appendages, are unable to accompany the placenta in its changes of position, and they thus become detached from it by rupture of the now dry and brittle funiculi.

The actual extrusion of the seeds appears to be produced by the pressure exerted upon them by the hairs contained in the follicle, and especially by the long basal seed-hairs, which separate the seeds from the endocarp and from each other. These hairs, in

* *Bulletin Général de Thérapeutique*, 1888, pp. 100–103.

the immature moist follicle, are arranged parallel to and in close contact with each other, but in the mature follicle, as the process of drying advances, they acquire elasticity and a tendency to diverge and become separated from each other. The basal seed-hairs, being interposed between the seeds and the endocarp, thus press the seeds towards the ventral or placental surface, and through the openings in this surface; and the movement outwards of the entire seeds, as well as the extrusion of the placenta, is aided by a similar elastic force acquired, during drying, by the hairs of the comose appendages.

In order to convince oneself of the adequacy of the extruding force of these hairs to produce dissemination, it is sufficient to observe a follicle from which the ventral surface has been removed. In a short time the contents of the follicle protrude through the vacant space, and the protruded seeds, with their appendages, expand into a large loose heap, consisting of the seeds mingled with the widely separated basal hairs, and of the comose appendages with their hairs now widely diverging from the stalk of the coma. On moistening the large heap of seeds and hairs, it soon again shrinks into a small bulk, owing to the hairs losing their elasticity, and again becoming closely approximated to each other along their whole length.

The basal seed-hairs, which separate the seeds from the endocarp and from each other, and which in mature follicles are no longer attached to the seeds, seem to possess the additional function of preventing fracture of the long and brittle shaft of the comose appendages, by forming a soft and yielding bed for the seeds, during their changes in position before they escape from the follicle. They thus insure that the seeds shall be disseminated with their comose appendages attached to them.

I have not considered it necessary to give a detailed and systematic description of the histological characters of each part of the *Strophanthus hispidus* plant. These characters are fully illustrated by the figures relating to histological structure in Plates III. to VII., in whose preparation I owe much to the kind assistance of Dr Macfarlane, of the Botanical Department of the University. The description of the figures (pp. 1025–1027) will sufficiently explain their more important details.

B. CHEMISTRY.

1. Seeds.

Composition.—In order to ascertain the general composition of the seeds, a weighed quantity, after having been carefully powdered, was dried at 100° F. and extracted by percolation, first with petroleum ether, boiling below 50° C. (100° F.), and then with anhydrous ethyl ether. After the ether had been completely removed by exposure to the air and to a moderate heat, the residue was divided into two equal portions, one of which was extracted with rectified spirit, and the other with distilled water, and in the latter solution the mucilage and albumen were estimated. The water was estimated by heating a separate quantity of ground seeds to 212° F.; and this, also, was used for the determination of the inorganic matter by combustion. Stated in percentages, the results were—

Analysis No. 1.

Water,	6·7	per cent.
Petroleum ether extract (chiefly fat),	31·81	,,
Ethyl ether extract (resin, chlorophyle, &c.),	0·845	,,
Rectified spirit extract (20 of rectified spirit to 1 of seeds),	8·94	,,
Water extract, { Mucilage,	7·35	,,
Water extract, { Albumen,	1·95	,,
Ash,	3·514	,,
	61·109	
Undetermined constituents,	38·891	
	100·000	

Analysis No. 2.

In a second analysis, in which the same processes were adopted, with seeds from the same parcel as those and in the first analysis, the chief results were—

Water,	6·35	per cent.
Petroleum ether extract,	31·725	,,
Ethyl ether extract,	0·905	,,
Rectified spirit extract (20 of rectified spirit to 1 of seeds),	9·1	,,
Water extract, { Mucilage,	7·142	,,
Water extract, { Albumen,	2·03	,,

In many other analyses, no attempt was made to estimate the water, mucilage, albumen, and inorganic matter, but the seeds were merely extracted with ethyl ether followed by rectified spirit, or with rectified spirit alone. It was early found that the fat and mucilage present in the seeds rendered water an inappropriate menstruum for removing the active principle; and for the same reason even dilute alcohol, in the form of proof

spirit, could not conveniently be used, especially when extraction by percolation was attempted. The results of a few of these analyses are given below, and in all of them the extraction was accomplished by the process of percolation.

Extraction by Anhydrous Ethyl Ether (sp. gr. 0·730) followed by Rectified Spirit.

Analysis No. 3.

581 grains of seeds from Buchanan, 1885, lost on drying 37·99 grains = water, 6·538 per cent.

500 grains of the above dried seeds yielded—

Ethyl ether extract (18 of ether to 1 of seeds), 160·82 grains,	= 33·96	per cent.
Rectified spirit extract (18 of spirit to 1 of seeds), 54·93 grains,	= 10·986	,,
Undetermined constituents, 271·32 grains,	= 54·26	,,
	99·206	,,

Analysis No. 4.

500 grains of dried seeds from Buchanan, 1885—

Ethyl ether extract (13 of ether to 1 of seeds), 165 grains,	= 33·0	per cent.
Rectified spirit extract (18 of spirit to 1 of seeds), 53 grains,	= 10·6	,,
Undetermined constituents, 280 grains,	= 56·0	,,
	99·6	,,

Analysis No. 5.

6000 grains of seeds from Buchanan, 1885—

Ethyl ether extract (12 of ether to 1 of seeds), 2164 grains,	= 36·066	per cent.
Rectified spirit extract (6 of rectified spirit to 1 of seeds), 487·48 grains,	= 8·124	,,

Extraction by Rectified Spirit and subsequent Removal of Fat, &c., from the Alcoholic Extract mixed with Water by frequent agitation with Ethyl Ether.

In the next analysis, the ground seeds, after having been dried at 100° F., were first extracted with rectified spirit, and then the fat, &c., was removed from the alcoholic extract by mixing it with a little water and agitating the mixture with successive quantities of ether. The residue obtained on the evaporation of this ether is the "ethyl ether extract" mentioned in the following analyses, and the "rectified spirit extract" is the dry residue obtained by evaporation of the watery solution of the alcoholic extract, after this extract mixed with water had been agitated with successive quantities of ether.

Analysis No. 6.

4000 grains of seeds from Buchanan, 1885—

Rectified spirit extract (12 of rectified spirit to 1 of seeds), 362·19 grains,	= 9·06	per cent.
Ethyl ether extract, 536·865 grains,	= 13·421	,,

Analysis No. 7.

8000 grains of seeds from Buchanan, 1885—

Rectified spirit extract (12 of rectified spirit to 1 of seeds), 701·655 grains, = 8·77 per cent.
Ethyl ether extract, 1013·416 grains, ... = 12·667 „

In this analysis the extraction of the seeds by rectified spirit had been effected by two successive percolations. The first percolate of 1·3 of rectified spirit to 1 of seeds yielded—

Rectified spirit extract, 388·755 grains, ... = 4·859 per cent. of seeds.
Ethyl ether extract, 260·65 grains, ... = 3·258 „

The second percolate of 10·7 of rectified spirit to 1 of seeds yielded—

Rectified spirit extract, 312·9 grains, ... = 3·911 per cent. of seeds.
Ethyl ether extract, 752·765 grains, ... = 9·409 „

Analysis No. 8.

1500 grains of seeds from Buchanan, 1885—

Rectified spirit extract (8 of rectified spirit to 1 of seeds), 114·66 grains, = 7·664 per cent.
Ethyl ether extract, 499 grains, ... = 33·266 „

In analyses 3, 4, and 5 the extraction with ether was continued until a colourless percolate had been obtained, and the subsequent extraction with spirit was continued until the percolate was free, or almost free, from bitterness. In analyses 6, 7, and 8, the percolation with spirit was continued until the tinctures were colourless and free, or almost free, from bitterness.

In the analyses in which extraction with ether preceded extraction with rectified spirit, the results were fairly concordant. They show that the ether extract, consisting mainly of fat, with a small quantity of chlorophyll and of resin, amounts to about 34 per cent., and that the alcohol extract, containing the active principle, amounts to about 9·5 per cent. of the seeds.

When the seeds were first extracted with rectified spirit, and the substances in the extract soluble in ether then removed from it, the results varied considerably. This was specially apparent in the case of the ether product, but it was also observed in the alcohol extract from which the substances soluble in ether had been removed. As will, however, be pointed out, in so far as the latter product is concerned, these differences are of comparatively little importance as indications of corresponding variations in the actual quantities of active principle present in the seeds.

Ether Extract.

The ether extract, whether obtained with ethyl or petroleum ether, consists mainly of a liquid fat or oil containing chlorophyll and other colouring matters; and when obtained with ethyl ether, of a small quantity also of resin. It gives a permanent translucent stain to paper. Its colour varies considerably, the lightest coloured specimens

being very pale greenish-yellow, and the darkest, brown with a faint tint of green; the chief intermediate shades being grass-green and pale and deep olive-green. The lighter coloured ether extracts were usually derived from the later percolates, and the dark coloured from the earlier percolates of the same seeds. The extract is translucent and clear, but after standing for some time a nearly colourless sediment usually separates, which disappears when the extract is heated to 120° F. Ethyl alcohol, amyl alcohol, acetone, chloroform, ethyl and petroleum ether, and bisulphide of carbon dissolve the extract freely. It has an oleaginous odour, and when dissolved in ether and washed by shaking several times with water, it has an oleaginous taste without bitterness. The well-washed ether extract does not possess any toxic action, nor indeed any other action than that of a bland oil. Its viscosity, and in the paler specimens, its appearance and other characters, are very similar to those of olive oil. The specific gravity, determined in a pale yellow or greenish-yellow oil, was found to be 0·975, in a pale green oil 0·954, and in a dark greenish-brown oil 0·9267. The two former or light-coloured oils, when heated to 120° F., and then allowed to cool to the temperature of the laboratory (about 60° F.), became semi-solid and uniformly opaque, although previously to this heating they had remained for more than twelve months liquid, and, with the exception of a slight deposit, clear and translucent, in the same laboratory. When microscopically examined in the opaque condition, the oil was found to contain numerous small aggregations of slender, needle-shaped crystals.

In several of the analyses, when the ethereal solution of the oil and other substances was shaken with water a thick and persistent emulsion or magma was produced, from which, however, the greater portion of the ether, holding oil and chlorophyll in solution, gradually separated itself. After this emulsion had been decanted and washed by shaking with ether, it was found to contain a small quantity of active principle and of resin, and a considerable quantity of mucilage and of a substance possessing the characters of caoutchouc. While, as will afterwards be stated, neither common or anhydrous ethyl ether dissolve appreciable quantities of the previously separated active principle, when the seeds are percolated with ether, or when the alcoholic extract mixed with water is shaken with ether, a very small quantity of the active principle appears in the ethereal solution. It may, however, be entirely removed from the ethereal solution by shaking it several times with water.

Alcohol Extract.

On evaporating, with the aid of a gentle heat, the concentrated tincture of the seeds previously freed from substances soluble in ether, or a watery solution of this tincture, a sweetish mucilaginous and somewhat nutty odour is developed. The extract then assumes the appearance of a translucent brownish-yellow or yellowish-brown hard substance, having some tenacity. If it be further dried by being placed *in vacuo* over sulphuric acid, it gradually loses its translucency, and becomes opaque, lighter in colour, and brittle. The extract is intensely bitter. It is freely soluble in water and in rectified spirit,

sparingly soluble in absolute ethyl alcohol and in amyl alcohol, and insoluble in chloroform and in ethyl and petroleum ether. The watery solution has usually a distinctly acid reaction. When ether or chloroform is added to the solution of the extract in ethyl or amyl alcohol, the solution immediately becomes opalescent, and an amorphous deposit is by and by formed. Occasionally, when ether is added to a very dilute solution in rectified spirit, the opalescence is succeeded by the formation at the sides and bottom of the vessel of groups of colourless glassy crystals, which, when magnified, have the appearance represented in Plate VII. fig. 7. These crystals possess the chemical and pharmacological properties, afterwards described, of the active principle, strophanthin.

In several of the processes in which the seeds were extracted with rectified spirit alone, after the alcoholic extract had been concentrated to a syrupy consistence, rounded tufts or nodules of crystals appeared in it. Under the microscope, these tufts or nodules were found to consist of long and very slender radiating crystals. Their appearance, when magnified, is shown in Plate VII. fig. 8. The crystals are intensely bitter, very soluble in water and in rectified spirit, but much less so in absolute alcohol, and they are insoluble in petroleum and ethyl ether and in chloroform.

In every process in which the seeds were extracted with rectified spirit, and the concentrated extract mixed with water and shaken with successive quantities of ether, the dried watery solution was found to consist largely of crystals having the above appearance and characters.

In a few of the analyses where this plan of extraction was adopted, when the extract was mixed with a very small quantity of water, and then shaken with ethyl ether, the ether assumed a fluorescent satiny appearance, which was found to be caused by the diffusion through it of an enormous number of minute particles, which, when the mixture was allowed to remain at rest for a short time, formed a deposit at the bottom of the ether and therefore at the surface of the underlying strong watery solution. On microscopic examination, these particles were also found to consist of minute slender acicular crystals, usually united together in small bundles, and having the same general characters as those represented in Plate VII. fig. 8.

When, however, the seeds were extracted with ether previously to being extracted with rectified spirit, the alcohol extract, on being concentrated, in no instance exhibited to the unaided eye the formation of groups of crystals in it; and when dried, although having the same general appearance and characters as extracts obtained without previous percolation of the seeds with ether, this extract, when broken down and examined under the microscope, was seen to consist, not of slender acicular crystals, but of irregular crystalline plates, whose appearance was similar to that represented in Plate VII. fig. 9.

However careful may have been the extraction with ether of the seeds or of the alcoholic extract, this extract does not consist of a crystalline substance alone. The crystals are mixed with, or imbedded in, other substances of non-crystalline structure, whose existence is rendered clear when the extract is further analysed. The further

analysis also shows that the quantity of non-crystalline substances varies in different extracts, and that one important cause of this variation is the amount of spirit percolated through the seeds in the preparation of the extract.

Composition of the Alcoholic Extract.—On adding to the dried alcohol extract a small quantity of rectified spirit, the extract does not entirely dissolve, but a residue remains, which is insoluble in a moderate quantity of rectified spirit. When the clear alcohol solution thus obtained is mixed with ether, it becomes densely opalescent, but in a short time the opalescence clears away, and a translucent amorphous and intensely bitter deposit occurs. The decanted and usually clear alcohol-ether solution also yields a residue when distilled and evaporated.

The first of these products is freely soluble in water, forming a mucilaginous solution, which reduces Fehling's solution after it has been digested for some hours with a little dilute sulphuric acid. The second product agrees with the active principle, for which I have proposed the name, strophanthin, in the chemical and pharmacological characters afterwards described. The third product is insoluble in water and in acids, soluble in rectified spirit and in dilute alkalis, and precipitated from the latter solution by acids, and it is, therefore, a resin.

The quantity of each of these products, in two out of several extracts that have been analysed, is stated in Analysis 9 and 11 below—

Analysis No. 9.

Alcohol Extract of Analysis No. 1 (p. 993).—Total alcohol extract, 8·94 per cent. of seeds.

Impure strophanthin, ...	63·367	per cent. of alcohol extract.	
Mucilage, ...	16·275	„	„
Resin, ...	14·542	„	„
	94·184	„	„

The total extract (8·94 per cent. of seeds) was, however, the sum total of the extracts of three successive percolations of the same seeds; the first having been obtained by a percolation of 10 parts of rectified spirit to 1 part of seeds, the second by a subsequent percolation of 5 parts of rectified spirit to 1 of the same seeds, and the third by a subsequent percolation of 5 of spirit to 1 of seeds. It is interesting to note the total quantity of alcoholic extract obtained from each of these percolates, and the composition of each extract.

Analysis No. 10.

	1st Percolate, 10 : 1. Total Extract, 7·9 per cent. of Seeds.	2nd Percolate, 5 : 1. Total Extract, 0·674 per cent. of Seeds.	3rd Percolate, 5 : 1. Total Extract, 0·37 per cent. of Seeds.		
Impure strophanthin, .	68·16	27·44	25·67	per cent. of extract.	
Mucilage, . . .	12·27	42·28	52·7	„	„
Resin, ...	13·79	23·36	14·18	„	„
	94·22	93·08	92·12	„	„

The first percolate of 10 of spirit to 1 of seeds, therefore, yielded 7·9 per cent. of the 8·94 per cent. of total alcohol extract; and this 7·9 per cent. contained a much larger percentage of strophanthin than either of the subsequent percolates.

Analysis No. 11.

Alcohol Extract of Analysis No. 3 (p. 994).—500 grains of seeds yielded 54·93 grains, or 10·98 per cent., of alcohol extract—

Impure strophanthin, 36·516 grains, ...	= 66·479	per cent.	of alcohol extract.
Mucilage, 7·46 grains, ...	= 13·6	,,	,,
Resin, 7·14 grains, ...	= 13·01	,,	,,
	93·089	,,	,,

The seeds had been extracted by three successive percolations with rectified spirit. In the first and second percolations, 4 of spirit to 1 of seeds was used, and in the third 10 of spirit to 1 of seeds. The total alcohol extract of each percolate, and the quantity of strophanthin, mucilage, and resin in it are stated below.

Analysis No. 12.

	1st Percolate, 4 : 1. Total Alcohol Extract, 42·43 grains = 8·486 per cent. of Seeds.	2nd Percolate, 4 : 1. Total Alcohol Extract, 5·1 grains = 1·02 per cent. of Seeds.	3rd Percolate, 10 : 1. Total Alcohol Extract, 7·4 grains = 1·48 per cent. of Seeds.		
Impure strophanthin, .	78·34 (33·236 grains)	31·568 (1·61 grain)	22·567 (1·67 grain)	per cent.	of extract.
Mucilage, . . .	3·37 (1·42 grain)	48·647 (2·48 grains)	48·4 (3·56 grains)	,,	,,
Resin, ...	12·39 (4·9 grains)	12·549 (0·64 grain)	21·621 (1·6 grain)	,,	,,

Of the 10·98 per cent. of alcoholic extract, 8·48 per cent. was, therefore, obtained by the first four ounces of percolate, and only 2·5 per cent. by the subsequent fourteen ounces. The extract from the first percolate was also much richer in active principle than the extract from subsequent percolates.

It appears from the above analyses that, by the process of percolation, nearly all the active principle is extracted by the first small quantity of spirit, and that this percolate yields an extract consisting chiefly of active principle. Further percolates contain only small quantities of the active principle, even although they may be of decidedly bitter taste; but they contain much mucilage, resin, and other undetermined substances. It is also to be noted that the extract obtained from the first percolate with a moderate quantity of rectified spirit differs from the extracts obtained from further percolates, not only in chemical composition, but also in physical characters. After having been dried by spontaneous evaporation and by exposure *in vacuo* over sulphuric acid, both extracts may be opaque, brittle, and only slightly coloured, although the extract from the first percolate is less coloured than those from subsequent percolates; but while the former retains for an indefinite time the appearance and physical characters it had acquired on becoming dry,

the latter become much darker in colour, they lose their opacity and brittleness, and acquire a plastic amorphous character and a dark reddish-brown colour. These changes occur independently of exposure, as they have been observed with extracts protected from the effects of exposure by being placed as soon as dried in well-stoppered bottles.

The impure strophanthin, precipitated by ether from an alcoholic solution of extract, is also a much purer substance when it is derived from the first percolate than when it is derived from subsequent percolates. In the former case, it is pale, brittle, crystalline, and opaque, and it retains these characters for an indefinite period; while in the latter case, it is, from the first, translucent and of a brownish-yellow colour, and if dried so as to admit of being reduced to a powder, it soon afterwards becomes an adherent homogeneous mass of dark colour.

Analyses of the Testa and of the Cotyledons and Embryos of the Seeds.

The next analyses were made in order to ascertain the quantity of each of the above ingredients present in the testa and in the combined cotyledons and embryos, respectively, and especially to ascertain whether the former or the latter contains the largest quantity of active principle. When the testa was carefully separated from the rest of the seeds it was found, in 119·48 grains of seeds, that the testa weighed 52·6 grains, or 44 per cent., and the combined cotyledons and embryos, 66·88 grains, or 55·97 per cent., of the seeds.

Analysis No. 13.

52·6 grains of *testa* yielded—

Anhydrous ether extract (28 : 1) 9·58 grains = 18·212 per cent. of testa, or 8·016 per cent. of seeds.
Rectified spirit extract (20 : 1) 4·58 grains = 8·707 „ „ or 4·873 „ „

66·75 grains of *cotyledons* and *embryos* yielded—

Anhydrous ether extract (26 : 1) 31·15 grains = 46·666 per cent. of cotyledons and embryos, or 26·118 per cent. of seeds.
Rectified spirit extract (20 : 1) 4·865 grains = 7·288 per cent. of cotyledons and embryos, or 4·07 per cent. of seeds.

The testa therefore yielded a much smaller quantity of ether extract, but a somewhat larger quantity of spirit extract, than the combined cotyledons and embryos. The ether extract derived from the testa was, however, a very different substance from that derived from the cotyledons and embryos. The former was of a dark greenish-brown colour, and not quite clear; the latter was of a very pale yellow colour, with a tinge of green, and at a temperature of 60° F., the greater part of it was perfectly clear and translucent, there being only a small whitish sediment.

The alcohol extracts also possessed marked differences in character and composition. That from the testa, when perfectly dry, was yellowish-brown in colour, semi-translucent, only partly brittle, and faintly aromatic; but in a short time, even in a stoppered bottle, it became dark reddish-brown, adhesive, and soft. The alcohol extract from the coty-

ledons and embryos, on the other hand, was opaque, pale yellowish-white, brittle, and inodorous; and it retained these characters without change for several months. The differences in composition are stated below.

Analysis No. 14.

Alcoholic extract of *testa*, 4·58 grains, yielded—

Impure strophanthin,	2·7	grains = 58·95	per cent. of extract, or	5·13	per cent. of testa.
Mucilage,	0·875	grain = 19·104	,,	,,	1·663	,,	,,
Resin,	0·94	grain = 20·545	,,	,,	1·663	,,	,,

Alcoholic extract of *cotyledons* and *embryos*, 4·865 grains, yielded—

Impure strophanthin,	3·765	grains = 77·4	per cent. of extract, or	5·65	of cotyledons and embryos.
Mucilage,	0·48	grain = 9·866	,,	,,	0·719	,,	,,
Resin,	0·44	grain = 9·044	,,	,,	0·67	,,	,,

On comparing the above analyses, it is seen that the alcoholic extract of the testa contains less active principle and much more mucilage and resin than the alcoholic extract of the cotyledons and embryos. When derived from the testa, each of these products is also much more coloured than when derived from the cotyledons and embryos. The alcoholic extract of the seeds, therefore, obtains most of its colouring matter, mucilage and resin from the testa, and most of its strophanthin and oil from the cotyledons and embryos.

Reactions of the Alcoholic Extract.

The action of a considerable number of reagents has been tested upon both the dry extract and a watery solution of it.

Dry Extract.

1. Moistened with **strong sulphuric acid**, it first became pale yellow, then brown, and in a few seconds emerald-green. In about a minute the green was almost completely displaced by brownish-black, and in about an hour dark green became the predominating colour, but it passed in another hour into greyish-green.

When, after the addition of strong sulphuric acid, the extract was placed in a chamber heated to 105° F., and the temperature was gradually raised to 120° F., the green colour soon became much intensified, and in about an hour and a half it passed into a dirty green, and ultimately into a nondescript grey, through which numerous black particles were diffused.

2. With **dilute sulphuric acid** (10 or 2 per cent.) no material colour change was produced within several hours, provided the extract were originally only a slightly coloured one. When the solution was then heated to 120° F., it gradually became light green, dark green, bluish-green, deep blue, violet-blue, deep violet, and ultimately violet-black and brownish-grey.

The final coloured products of 1 and 2 are insoluble in water.

3. Strong **nitric acid** (Ph. Brit.) produced a pale brown solution.

4. Dilute **nitric acid** (10 per cent.) slowly dissolved the extract, forming a pale yellow solution.

When this solution was heated between 115° and 130° F., it gradually became faintly red, then blue appeared at the margins, and the centre became canary-yellow, then pinkish streaks extended across the yellow centre, and, finally, the whole became permanently of a gamboge-yellow colour.

5. Strong **hydrochloric acid** (Ph. Brit.) produced a yellowish solution.

6. Dilute **hydrochloric acid** (10 per cent.) also produced a yellowish solution.

When this solution was heated between 115° and 130° F., it became greenish-yellow, brownish-green with faint blue patches, deep violet, and, finally, very dark green.

7. **Acetic acid** (Ph. Brit.) produced a pale brownish solution.

8. **Iodic acid** produced a pale brownish solution, in which, however, no blue colour was developed by starch.

9. Strong **sulphuric acid** and **bichromate of potassium** produced a greenish-brown colour.

10. No material change was caused by strong **sulphuric acid, rectified spirit,** and neutral solution of **ferric chloride**; nor by **sulphuric acid** and **bromine water.**

11. Solution of **potash, soda,** or **ammonia** produced a bright yellow solution, but the yellow colour immediately disappeared on the addition of dilute sulphuric, hydrochloric, or acetic acid. When the alkaline yellow solution was boiled, it evolved a methylamine odour and alkaline fumes, and in a short time it became reddish-brown in colour, and lost much of its bitterness.

12. **Phospho-molybdic acid** produced a green colour, which immediately changed to blue on the addition of an alkali.

Solution of Extract in Water (2 *per cent.*).

1. **Acetate of lead** produced a faintly yellow flocculent precipitate.
2. **Subacetate of lead** produced an abundant yellowish flocculent precipitate.

After the lead precipitates in 1 and 2 had subsided, the supernatent fluid was nearly colourless and intensely bitter.

3. Solution of **ferric chloride** (Ph. Br.) caused a greenish-yellow colour, and afterwards a slight precipitate.

4. **Nitrate of silver** produced a faint opalescence, which afterwards became a dark precipitate.

5. **Mercurous nitrate** produced a white cloudiness, which afterwards subsided as a slight grey sediment.

6. **Cupric sulphate** produced a slight haziness, which, on subsidence, left a pale green fluid.

7. **Platinic chloride** failed to produce any change within several hours, but on the following day a slight brownish opalescence had occurred.

8. **Phospho-molybdic acid** produced a very pale greenish-yellow precipitate, permanent only with a considerable quantity of reagent. When the precipitate had subsided, the supernatant fluid was seen to be emerald-green; and the precipitate dissolved on boiling, and reappeared on cooling.

9. **Molybdate of ammonium** produced a faint yellow tint, and, after several hours, a considerable yellowish-white precipitate, the supernatant fluid continuing to be yellow. The precipitate dissolved on boiling, and appeared again on cooling.

10. **Tannic acid** produced a copious white precipitate.

11. Solution of **potash, soda, ammonia, lime,** and **baryta,** and of **carbonate of potash** and **carbonate of soda,** each produced a bright orange-yellow colour. **Carbonate of ammonium, carbonate of baryta,** and **bicarbonate of potash** produced a less marked yellow. In each case, the yellow colour was immediately discharged by dilute acetic acid. The alkaline yellow fluids did not reduce Fehling's solution when boiled with it.

12. **Sulphuric acid** (10 per cent.), dilute **hydrochloric acid** (Ph. Br.), dilute **nitric acid** (Ph. Br.), and dilute **phosphoric acid** (Ph. Br.) each rendered the solution paler, and slowly produced a slight flocculence, which disappeared in great part on boiling. When afterwards neutralised and tested with Fehling's solution, a well-marked reduction occurred.

13. Dilute **acetic acid** (Ph. Br.) produced no obvious change; and after boiling for a few seconds, and neutralising with sodium carbonate, only a slight reduction of Fehling's solution was obtained.

No obvious change was produced by **picric acid, carbonateof b aryta, phosphate of sodium, chloride of gold, mercuric chloride, potassio-mercuric iodide, metatungstate of sodium, tri-iodide of potassium, potassio-bismuthic iodide,** nor **potassio-cadmic iodide.**

Absence of any Alkaloid from the Extract.

The failure, already described, of many reagents for alkaloids to produce change in the watery solution of the extract, although it is naturally acid in reaction, affords sufficient evidence of the absence from the seeds of any alkaloidal principle.

In addition to this negative evidence, ten grains of the extract were treated according to Stas' method for separating alkaloids, ether and chloroform being used as the separating solvents; but the result was also entirely negative, only 0·035 grain of total product (=0·35 per cent.) having been obtained, which with sulphuric acid and heat gave merely colour changes characteristic of strophanthin.

At the same time, the extract contains nitrogen in small quantity, but this is by no means remarkable when its composition is borne in recollection.

Further, when the extract is made alkaline by solution of potash and then heated, alkaline vapours, having a distinctly ammoniacal or, rather, methylamine odour, are evolved.

Some interest is attached to the circumstance that when the extract has been in contact with a dilute acid for a short time, slight and inconclusive alkaloidal reactions may be obtained with it. Thus, in a 2 per cent. watery solution, acidulated with weak sulphuric acid, potassio-mercuric iodide, platinic chloride, auric chloride, and tri-iodide of potassium, each produced a slight haze, becoming in a few hours a faint precipitate, and metatungstate of sodium produced in a few minutes a scanty, but well-marked precipitate. The solution was originally free from glucose, but was found to contain it soon after the addition of the acid, and before the above reactions were obtained.

Presence of a Glucoside in the Extract.

The reduction of Fehling's reagent by solutions of the extract in dilute acids having indicated the presence of glucose in these solutions, it became of importance to determine if this glucose is usually and normally present in the extract, or is produced in it by the decomposition of one or more of its constituents.

Some alcoholic extract, prepared by percolating the seeds with ethyl ether and then with rectified spirit, was dissolved in distilled water so as to constitute a 2 per cent. solution. When heated with Fehling's reagent it failed to give any evidence of reduction. A portion of the same solution of extract was then acidified with sulphuric acid, and left at the ordinary temperature. After three days, the now slightly turbid solution was filtered, and after having been neutralised with carbonate of sodium it also was tested with Fehling's solution, when it immediately produced a copious reduction.

Evidence was thus obtained in an extract originally free from glucose, of a decomposition having been caused by dilute acid, of which one of the products is glucose, and the presence of a glucoside in the extract was accordingly indicated.

Similar evidence was also obtained in one of the dark extracts derived from late percolates of the seeds. When dissolved in water, it failed to reduce Fehling's solution, but it did so after it had been acidulated with weak sulphuric acid.

The production of this decomposition in the cold by the action of dilute acids was further examined.

It was found that when a 3 or 4 per cent. solution of alcoholic extract in water is acidified with sulphuric acid, so that the acid is present as a 0·3 to 2 per cent. solution, the mixture in a short time becomes turbid, an apparently amorphous deposit forms in it, and in from two to four days the solution becomes clear and less coloured, and small crystalline tufts appear at the bottom and sides of the vessel, which increase in size until a considerable crystallisation has been produced. To this crystalline substance I have given the name Strophanthidin. On examining the solution in which the crystals have appeared, it is now found to contain much glucose.

When a minute quantity of the extract dissolved in a drop of water is placed on a microscope slide provided with a shallow cup, and a drop of 2 per cent. sulphuric acid is added to it before the cover-glass has been applied, in one or two days a large number of small and translucent globular bodies make their appearance, and in three or four days a

beautiful crystallisation of strophanthidin may be observed in the solution. The crystals have for the most part the appearance represented in Plate VII. fig. 10, but in portions of the field, where only a thin layer of fluid is interposed between the cover-glass and the slide, their form is modified to that of groups of long and slender radiating needles, and of fan-shaped crystalline plates.

A well-defined crystallisation produced in such circumstances in a solution of a pharmacologically active substance, is of so rare occurrence that it may usefully be applied as a test for strophanthus extract. It would probably, also, be an easy matter to devise a quantitative test for all strophanthus preparations, depending on the amount of crystalline strophanthidin produced in this reaction.

If a watery solution of the extract be acidified, and, after a short time, shaken with chloroform or amyl alcohol, the separated chloroform or amyl alcohol yields on evaporation a crystalline substance. It is, accordingly, an easy matter to obtain a well-defined crystalline product from the extract; but it will, at the same time, be found that the original solution now contains glucose, and the crystalline substance thus obtained is not therefore strophanthin, but strophanthidin, one of the products of its decomposition.

Notwithstanding the circumstance that dilute acids so readily decompose the glucoside present in the extract, this decomposition does not appear to occur spontaneously in a watery solution of the extract, although its reaction is decidedly acid. In such a solution, a fungous growth makes its appearance in a short time; but even in a solution made more than two months previously and containing an abundant fungous growth, no glucose could be detected by Fehling's reagent. At this time, also, the solution appears to be as intensely bitter as when it was first prepared. When, however, the naturally acid solution of the extract in water is boiled for a short time, a small quantity of glucose is produced; but the decomposition is so slight that no formation of crystalline strophanthidin can be observed. (See Table X. Analysis No. 15.)

On the other hand, the glucoside in the extract is quickly and completely decomposed by the addition of many acids; and weak solutions of several acids may, even at the ordinary temperature, produce this decomposition sufficiently to cause crystalline strophanthidin to appear in the solution.

The following experiment illustrates the action in the cold of a weak solution of sulphuric acid:—

A 5 per cent. solution of extract in water was acidulated with sulphuric acid so as to produce a 0·4 per cent. solution of acid. The solution very soon became slightly turbid; in two days, the turbidity had greatly increased; in four days, a few small crystalline rosettes had appeared at the bottom of the solution, which was now less turbid; and in five days, there was a considerable pale brownish-yellow incrustation at the bottom and sides of the vessel, consisting chiefly of round groups of lancet-shaped crystals, while the fluid was now only faintly turbid, and much less coloured. The crystals, which could not be removed from the vessel without some loss, when washed and dried, weighed 18·3 per cent. of the extract used. The filtered solution, after having been neutralised, was

found, on estimation with Fehling's reagent, to contain 23·4 per cent. of glucose. (Table X. Analysis No. 16.)

In another experiment with the same strength of solution of extract and of sulphuric acid, but in which the acidulated solution was at once boiled for half an hour, an abundant formation of large and only slightly coloured crystals occurred while the solution was being raised to the boiling point, and before the temperature of 180° F. had been attained; and, at the same time, a peculiar odour, like that of cooked raisins, was developed. The crystals became broken and more deeply coloured by the boiling; but still, when dried, they weighed 27·4 per cent. of the extract. The glucose produced amounted to 27·9 per cent. (Table X. Analysis No. 17.)

In an experiment in which 0·32 per cent. of sulphuric acid was used, on raising the temperature to 170° F., the solution became opalescent; but after it had cooled, only an amorphous sediment, amounting to 6·3 per cent., was deposited. When, however, the filtered solution was made slightly alkaline with sodium carbonate, a precipitate, consisting of minute and perfectly formed crystals, was thrown down, which amounted to 13·7 per cent. of extract. 21 per cent. of glucose was produced. (Table X. Analysis No. 20.)

In another experiment, where all the conditions were the same as in the preceding one, except that the percentage of sulphuric acid was 1·6 instead of 0·32, no crystalline strophanthidin was produced, but merely an amorphous brown substance, which weighed 9 per cent. The quantity of glucose formed was exactly the same as in the preceding experiment, namely, 21 per cent. (Table X. Analysis No. 21.)

The production in the cold as well as at an elevated temperature of crystalline strophanthidin and glucose was observed with other acids, and with different degrees of acidity. In many of the experiments, the acidulated solution was left at the ordinary temperature for several days, and then decanted from any crystals that had formed; and the decanted solution, after having been filtered, was divided into two equal parts, in one of which the glucose was at once estimated, while in the other this estimation was not made until the solution had been boiled for half an hour. By this plan, the production or non-production of crystalline strophanthidin and of glucose in the cold, and of glucose at a temperature of 212° F., and the quantity of each substance produced in these conditions could be ascertained. Even when made with the same acid, the various experiments are not, however, always comparable, as the percentage of acid and of extract in the solutions, the duration of contact, the temperature to which the solutions was subjected, and even the extracts used, were not the same, but, indeed, were intentionally varied.

In estimating the quantity of glucose by Fehling's reagent, a difficulty was encountered, due to the absence of a sharp indication of the point when the whole of the reagent had been reduced. The disappearance of the blue colour, on which reliance generally is placed, is masked by the production of a greenish-blue, which, on further additions of the glucose solution, gradually passes into distinct green, yellowish-green, yellow, and reddish-yellow. Control experiments appeared to show that complete reduction of Fehling's reagent is not produced until a reddish-yellow colour has appeared, but the shades of colour pass

TABLE X.—*Summary of Analyses in which Extract of Strophanthus was decomposed by Acids.*

No. of Analysis.	Quantity of Extract in Grammes.	Percentage of Extract in the Solution.	Acid and its Percentage.	Strophanthidin produced in the Cold.	Glucose produced in the Cold.	Strophanthidin produced after Heating.	Glucose produced after Heating.
15	0·25	10 %	No added acid.	None.	None.	No appreciable production.	After boiling for half an hour, about 1 %.
16	0·25	5 %	Sulphuric acid, 0·4 %.	In five days, 18·3 % of crystalline strophanthidin.	23·4 %	...	...
17	0·25	5 %	Do., 0·4 %.	...	...	After boiling for half an hour, 27·4 % of crystalline, brownish-yellow strophanthidin.	After boiling for half an hour, 27·9 %.
18	1·0	2·5 %	Do., 1 %.	...	...	After boiling for four hours, 31·6 % of yellowish-brown amorphous deposit.	After boiling for four hours, 21·3 % by Fehling's reagent, 22·4 % by fermentation.
19	0·5	3·4 %	Do., 1·6 %.	In five days, 30·45 % of yellowish-brown crystalline strophanthidin.	26 %	After boiling one half of the filtered solution for half an hour, only a little brown amorphous substance.	After boiling for half an hour, 25 %.
20	0·25	3·2 %	Do., 0·32 %.	...	...	After heating to 170°, 6·3 % of brown amorphous substance ; on rendering the cold filtered solution alkaline, 13·73 % of crystalline strophanthidin.	After heating to 170°, 21 %.
21	0·25	3·2 %	Do., 1·6 %.	...	...	After heating to 170°, 9 % of amorphous brown substance.	After heating to 170°, 21 %.
22	0·25	4 %	Do., 2 %.	In three days, 27·21 % of brownish-yellow crystalline strophanthidin.	20 %	After boiling one half of the filtered solution for half an hour, a slight amorphous deposit.	After boiling for half an hour, 20·8 %.
23	0·25	4 %	Do., 2 %.	In five days, 20·7 % of crystalline strophanthidin.	20·5 %	...	...
24	0·25	1 %	Hydrochloric acid, 0·1 %.	...	...	After heating for half an hour between 98° and 100° F., no crystallisation.	After heating for half an hour between 98° and 100° F., 5·5 %.
25	0·5	4 %	Do., 0·1 %.	In six days, a slight non-crystalline deposit.	24·5 % (?)	After heating one half of the filtered solution between 120° and 140° F. for half an hour, 16·32 % of crystalline strophanthidin. The filtrate boiled for half an hour, deposited a further small quantity of crystalline strophanthidin.	After boiling for half an hour, 25·7 %.
26	0·5	5 %	Nitric acid, 2 %.	In five days, 29·9 % of crystalline strophanthidin.	24 %	After boiling one half of the filtered solution for half an hour, 1·5 % of crystalline strophanthidin.	After boiling for half an hour, 27·2 %.
27	0·5	5 %	Phosphoric acid, 2 %.	In five days, 6 % of crystalline strophanthidin.	12·8 %	After boiling one half of the filtered solution for half an hour, 26·42 % of crystalline strophanthidin.	After boiling for half an hour, 22 %.
28	0·5	5 %	Acetic acid, 2 %.	In six days, only a slight amorphous deposit.	2 %	After boiling one half of the filtered solution for half an hour, 12·9 % of crystalline strophanthidin.	After boiling for half an hour, 15·34 %.
29	0·25	2·5 %	Oxalic acid, 2 %.	In five days, a slight amorphous deposit.	16 %	After boiling one half of the fitered solution for half an hour, 10·4 % of crystalline strophanthidin.	After boiling for half an hour, 19·4 %.
30	0·25	2·5 %	Hydrocyanic acid, 2 %.	In five days, a very slight amorphous deposit.	Glucose not sufficient to be estimated.	After boiling one half of the filtered solution for half an hour, only a minute, apparently amorphous deposit.	After boiling for half an hour, 3 %.

so gradually into each other that accurate determinations by this method are extremely difficult.

Exact estimations of the quantity of strophanthidin produced in the experiments were not attempted. If unattached to the vessel in which the decomposition had been effected, the strophanthidin crystals were collected as carefully as possible by filtration, and if they adhered to the vessel they were removed by scraping; but as the crystals are somewhat soluble in water, while the quantity of water used in the experiments varied considerably, some loss, which was not the same in each experiment, undoubtedly occurred.

In fact, the object of the experiments was to determine the qualitative rather than the quantitative changes produced in the extract by the influence of acids, and above all to demonstrate clearly that glucose and strophanthidin are produced, and that the latter substance can with great facility be obtained in a crystalline form.

A summary of the experiments is given in Table X., p. 1007.

It has thus been shown (*a*) that the extract contains a glucoside, which is readily decomposed by weak solutions of acids so as to produce glucose, and the crystalline body, strophanthidin; (*b*) that glucose is produced in the extract by sulphuric, hydrochloric, nitric, phosphoric, acetic, and oxalic acids, both in the cold and at an elevated temperature, but by two per cent. hydrocyanic acid only at an elevated temperature; (*c*) that crystalline strophanthidin is produced in the cold by sulphuric, nitric, and phosphoric acids, but as only very weak solutions of hydrochloric acid were used, it cannot be stated that this acid is unable in the cold also to produce crystalline strophanthidin; and (*d*) that crystalline strophanthidin is produced at an elevated temperature by sulphuric, hydrochloric, nitric, phosphoric, acetic, and oxalic acids.

It may be added that while carbonic acid fails to decompose the glucoside, it is decomposed with the formation of crystalline strophanthidin and of glucose, by sulphuretted hydrogen.

The crystalline form assumed by strophanthidin was found to vary considerably. When produced in the cold, the strophanthidin was usually in the form of rosettes or nodules, consisting of lancet-shaped crystals (see Plate VII. fig. 10); but in the experiments where an elevated temperature was employed, beautiful, long and slender acicular needles, perfect minute prisms, and prisms grouped in stellar arrangements were also produced. In the experiments at an elevated temperature with sulphuric acid, a brown amorphous substance, and no crystalline strophanthidin, appeared when the acid was of greater strength than 0·4 per cent.; but in the cold, even 2 per cent. sulphuric acid caused the formation of crystalline strophanthidin.

Strophanthin.

Preparation.—The well-defined crystals produced during the evaporation of non-acidulated watery solutions of the extract (pp. 996 and 997) consist, no doubt, of the active principle, strophanthin.

It is, however, extremely difficult to separate the crystals from adhering impurities by the use of any solvents; and even when that is accomplished, the great solubility of strophanthin in water and in rectified spirit, entails much loss, if separation by repeated crystallisation be attempted. The crystalline products obtained when ether is added to a very dilute alcoholic solution of the extract, and when ether is added to a strong solution of the alcoholic extract in water, also both represent nearly pure forms of strophanthin; but, on several times repeating each process, it was found that sometimes only did it succeed in yielding a crystalline product, while, frequently, it failed to do so. Nice adjustments, extremely difficult to determine, are obviously required of the proportional quantities of active principle, water, alcohol and ether, and of active principle, water and ether, respectively, in order to ensure the separation of the active principle in the form of colourless crystals.

It was therefore found necessary to devise some other process. In the first place, at an early stage in the research, the removal of impurities and the isolation of the active principle by subacetate of lead was attempted; as it had been found that after the removal of the copious precipitate formed in solutions of the extract by subacetate of lead an intensely bitter, clear, and only slightly coloured filtrate, of great pharmacological activity, was obtained. When, however, sulphuretted hydrogen was passed through this filtrate, in order to precipitate lead, the active principle was necessarily subjected to the action both of sulphuretted hydrogen and of free acetic acid; and accordingly it was decomposed, glucose appeared in the solution, and strophanthidin crystallised out in great abundance.

As the extract obtained by small quantities of rectified spirit from the seeds previously percolated with ether, appeared to consist chiefly of active principle, the removal of the inconsiderable quantity of impurity present in it was attempted by treatment with pure animal charcoal; but this process also proved unsatisfactory both in the quantity and quality of the product obtained.

After several other attempts, the following was adopted as a tolerably satisfactory, though, no doubt at the same time laborious, process for separating the active principle in a pure form.

The active principle was precipitated by a solution of tannin from a strong solution of the extract in water; the well-washed tannate was thoroughly mixed with recently precipitated, carefully washed, and moist oxide of lead, which was added in the quantity calculated to be necessary for the conversion of the tannin into tannate of lead; the mixture was digested for several days at a low temperature; and, after it had been dried, it was thoroughly exhausted with rectified spirit, and occasionally with proof spirit. If the alcoholic solution still contained any tannin, as it usually did, it was evaporated to a syrupy consistence, and again treated as above with a smaller quantity of oxide of lead. It was frequently necessary to adopt a third such treatment before every trace of tannin had been removed. The product was now dissolved in weak alcohol, and, if necessary, decanted and filtered from sediment; and through the clear and usually almost colourless solution, a gentle stream of well-washed carbonic acid was passed for two or three days,

in order to remove traces of lead. The solution was then evaporated to dryness, and the residue dissolved in rectified spirit, and, after filtration, ether was added to the solution to precipitate the active principle. The precipitate was dissolved in absolute alcohol, which usually left a further slight sediment, and the clear alcoholic solution was finally dried by spontaneous evaporation, and by being placed in a partial vacuum over sulphuric acid.

By this process, about 65 per cent. of the active principle, strophanthin, was usually obtained from the extract. This quantity, undoubtedly, does not represent the whole of the active principle present in the extract; but the result otherwise is satisfactory, in so far as the quality of the product is concerned.

Characters.—Strophanthin thus obtained is a colourless, opaque, and brittle substance, having an appearance suggestive of a crystalline body, but exhibiting no crystals to the naked eye. Under the microscope, however, it is found to consist of minute irregular crystalline plates, whose appearance is illustrated in Plate VII. fig. 9.

When ether is added to very dilute alcoholic solutions of it, and the faintly turbid mixture is put aside in a stoppered bottle for a few days, beautiful stellar groups of colourless and transparent crystals frequently form on the sides and at the bottom of the bottle. Some of these groups, as seen with a lens magnifying about six times, have been represented in Plate VII. fig. 7.

Strophanthin is very freely soluble in water and in rectified spirit, losing its opacity when a very small quantity of either solvent is added to it, and becoming a viscous, clear, and faintly yellow solution on further minute additions. It is soluble in 55 parts of absolute alcohol (sp. gr. ·796), in 300 parts of acetone, and in 1000 parts of amyl alcohol (sp. gr. ·820). It is almost insoluble in chloroform (sp. gr. 1·497), in absolute (sp. gr. ·723) and common (sp. gr. ·730) ethyl ether, in petroleum ether boiling below 120° F., and in bisulphide of carbon.* Glycerine (sp. gr. 1·26) dissolves it freely; but when small quantities are placed in strophanthus oil and in olive oil, respectively, they remain unchanged for several months, although, afterwards, the particles appear to dissolve very slowly.

Solutions in rectified spirit and in amyl alcohol are precipitated by the addition of chloroform, absolute or common ethyl ether, petroleum ether, and bisulphide of carbon. A solution in absolute alcohol is precipitated by ethyl and by petroleum ether, and is rendered slightly turbid by bisulphide of carbon; but neither chloroform nor acetone produce any change in the appearance of the solution. A solution in acetone is precipitated by ethyl ether, petroleum ether, chloroform, and bisulphide of carbon, but not by absolute alcohol nor by amyl alcohol.

Strophanthin is intensely bitter. When dissolved in distilled water, the bitterness is slightly appreciable in a solution of 1 part in 300,000. Its solution in water or rectified spirit is acid in reaction. When a dilute solution in water is shaken, a persistent froth is produced. Solutions in ordinary or in distilled water soon lose their perfect trans-

* In the experiments that were made, chloroform dissolved 1 part in 10,000, and absolute and common ethyl ether, petroleum ether, and bisulphide of carbon about 1 part in 20,000.

lucency by the growth of a fungus in them; but, notwithstanding this circumstance, as has already been stated, even after several months, no glucose appears in the solutions, and they apparently retain their original bitterness and pharmacological activity.

Strophanthin melts at a temperature of 343° F. Below this temperature, at about 295° F., it acquires a faintly yellow colour, which becomes yellowish-brown at the melting point. When the temperature is further raised, it evolves fumes having at first a caramel and then a disagreeable empyreumatic odour, becomes charred, and finally disappears without almost any residue.

When heated in a test-tube with soda lime, and when tested by Lassaigne's cyanogen process, it was found to contain no nitrogen.

Ultimate Analysis—In order to determine its percentage composition, several combustions were made, of which the three following agree closely in their results:—

Analysis No. 31.—0·3 gramme, yielded CO_2, 0·610 = 55·45 per cent. C.
„ „ H_2O, 0·204 = 7·55 per cent. H.
Analysis No. 32.—0·1789 gramme, yielded CO_2, 0·3635 = 55·41 per cent. C.
„ „ H_2O, 0·1222 = 7·58 per cent. H.
Analysis No. 33.—0·1893 gramme, yielded CO_2, 0·3849 = 55·45 per cent. C.
„ „ H_2O, 0·1288 = 7·56 per cent. H.

These percentages correspond with the formula $C_{16}H_{26}O_8$.

	Found (average of above three analyses).	Calculated for $C_{16}H_{26}O_8$.
Carbon, ...	55·43	55·49 per cent.
Hydrogen, ...	7·56	7·51 „
Oxygen (by subtraction), .	37·01	37·0 „

$C_{16}H_{26}O_8$ may, therefore, provisionally be adopted as the formula of strophanthin, until at any rate more complete knowledge has been obtained of its constitution.

Reactions.

When various reagents were applied to strophanthin in the dry state, and also in 2 per cent. solution, the results were as follows:—

Dry Strophanthin.

1. When a minute quantity, in the form of powder, was moistened with a drop of **strong sulphuric acid**, a bright green colour was immediately produced, which in a few seconds became greenish-yellow, and then brown at the centre with green at the edges; in twenty minutes, the whole was brownish-green; in a few minutes afterwards, it was grey, with a greenish tint; and in the course of an hour or two, dirty brown, without any green.

When strophanthin moistened with strong sulphuric acid was heated to between 110°

and 120° F., the green colour first produced soon became dark olive, changing to very dark brown, with green at the parts which had dried, then to violet and dark violet-blue, and, finally, to black with a violet tint.

2. With **10 per cent. sulphuric acid** a nearly colourless solution was produced, which remained unchanged in appearance for several hours.

When heated between 110° and 120° F. with 10 per cent. sulphuric acid, it soon became light green, grass green, dark green, deep bluish-green, deep greenish-violet, very dark violet, and in about two hours, black with a violet tint. When allowed to cool, the last colour remained for more than twelve hours.

These colour changes were quite distinctly obtained with even the $\frac{1}{15,000}$ of a grain of strophanthin.

3. Strong **nitric acid**, in the cold, produced a pale brown solution.

4. Dilute **nitric acid** (10 per cent.) merely dissolved strophanthin, without obvious change of colour.

When heated between 115° and 130° F. with 10 per cent. nitric acid, a violet colour was first developed, in which blue streaks appeared; the whole then became violet for a few minutes, then yellow appeared at the margins, the violet gave place to yellowish-brown, and, finally, in about forty minutes, the whole became gamboge-yellow, and remained this colour for several hours.

5. **Strong hydrochloric acid** dissolved strophanthin, forming a pale yellow solution, which afterwards became brownish-yellow.

6. **Dilute hydrochloric acid** (10 per cent.) dissolved it, and produced a colourless solution.

When heated between 115° and 130° F., changes were very slowly produced; in about twenty minutes, a yellow colour appeared, which, however, soon passed into green, and then into blue (Turnbull's), and the last colour remained for several hours.

7. **Strong sulphuric acid** and **bichromate of potassium**, in the cold, produced successively green, orange-brown, dark brown with green at the edges, and emerald-green. When now heated to between 115° and 120°, the green slowly became bluish-violet.

8. When to a minute particle of strophanthin there was added a small drop of distilled water and also of dilute solution of **ferric chloride**, and then a drop of **strong sulphuric acid**, a deep yellow colour appeared, which changed to pink. On mixing the whole with a glass rod the pink disappeared.

9. Solution of **phospho-molybdic acid** developed rather slowly a green tint, which on prolonged exposure became a pure blue of considerable intensity. If an alkali was added along with or after the phospho-molybdic acid, the blue colour was immediately developed.

10. Solution of **potash, soda**, and **ammonia**, and of other alkalies and their carbonates, produced a faint yellow colour, which disappeared on the addition of acids.

11. Negative results were obtained on the addition of **iodic acid** and **starch, nitrate of silver, sulphate of zinc, sulphate of copper**, and **Nessler's reagent**.

Solution of Strophanthin in Water (1 or 2 per cent.).

1. Concentrated, or 10 per cent. solutions of, **sulphuric, nitric, hydrochloric, phosphoric,** and **chromic acid,** and concentrated **acetic acid,** each produced a slight haze even in a one per cent. solution of strophanthin. When the solution was afterwards neutralised and tested with Fehling's reagent, the reagent was in each case reduced.

2. **Sulphuric acid** and **bichromate of potassium** also produced a slight opalescence, and the solution, on being neutralised, reduced Fehling's reagent.

3. Solutions of **potash, soda, ammonia, lime,** and **baryta,** of **carbonate of ammonium,** and of **phosphate of sodium,** each caused the solution of strophanthin to become of a light yellow colour; but even after prolonged contact, the yellow solutions did not reduce Fehling's reagent. The alkaline yellow fluids became deep reddish-brown when heated to 212° F., and, at the same time, they lost much of their original bitterness, and apparently also of their pharmacological activity.

4. Solution of **ferric chloride** produced no change until sulphuric acid had been added, when a faint opalescence occurred. When a drop of 0·5 per cent. solution of strophanthin in water was placed on a white porcelain slab, and a minute drop of solution of ferric chloride, followed by a small drop of strong sulphuric acid, was added to it, a yellow colour was first produced, and then streaks or patches of pink and blue were quickly developed. In a short time, the whole assumed a dirty pale greenish-blue colour.

5. Solution of **nitrate of silver** very slowly produced a reddish-brown colour and a slight dark deposit.

6. **Phospho-molybdic acid** slowly produced a bright green colour, which gradually passed into greenish-blue.

7. **Tannic acid** solution threw down a copious yellowish-white precipitate, which redissolved until an excess of the acid had been added.

8. **Molybdate of ammonium** in sulphuric acid produced a slight opalescence, and Fehling's reagent afterwards revealed the presence of glucose in the neutralised solution.

9. Negative results were obtained on the addition of **chloride of gold, platinic chloride, cobaltous chloride, acetate** and **subacetate of lead, mercuric chloride, mercurous nitrate, cupric sulphate, ferro-** and **ferricyanide of potassium, chloride of barium, acid carbonate of potassium, iodide of potassium, tri-iodide of potassium, tri-bromide of potassium, potassio-mercuric iodide, metatungstate of sodium, potassio-bismuthic iodide,** and **potassio-cadmic iodide.**

Decomposition of Strophanthin by Acids, &c.

Glucose having been produced by the application to strophanthin of such of the above reagents as were acid in reaction, it was indicated that this substance is a glucoside. This indication has been rendered clear and unambiguous by the results of other experiments, of which the following are given by way of illustration.

To a colourless and clear 3·3 per cent. of strophanthin in distilled water, sulphuric

acid was added so as to make the solution a 0·3 per cent. one of acid, and the solution was then left to itself at the ordinary temperature. On the following day, it had become slightly turbid, and two days afterwards several colourless rosettes of lancet-shaped crystals had formed at the bottom of the flask. On the fourth day, the rosettes had increased in size, and now also a general crystalline incrustation had occurred over the bottom and sides of the flask, while the solution had lost its turbidity, and had again become quite clear. The crystals increased in quantity during the next twenty-four hours; and, on the sixth day, when they were collected as carefully as possible, they weighed 33·7 per cent. of the strophanthin used. The filtered solution, after having been neutralised, was found to contain 22 per cent. of glucose. (Table XI. Analysis No. 34.)

In an experiment with the same quantity of strophanthin and of sulphuric acid in solution as in the last experiment, as soon as the solution had been made it was placed in the water-bath and gradually heated. While the temperature rose from 150° to 165° F., a beautiful crystallisation appeared in the solution, consisting of slender colourless crystals, many of them being three-fourths of an inch in length, and the solution at the same time became slightly turbid. The temperature was raised to 212°, and maintained at this point for half an hour; during the boiling, an odour like that of cooked raisins was given off, and the solution became slightly yellow in colour. When the solution had cooled, it was found that, in addition to now partially broken, long and slender crystals, a further crystallisation had formed, consisting of colourless nodules or tufts attached to the bottom of the flask. The washed and dried crystals weighed 36·2 per cent. of the strophanthin used, and there was found in the filtered solution 27·5 per cent. of glucose. (Table XI. Analysis No. 35.)

On gradually heating a solution of strophanthin which contained 0·5 per cent. sulphuric acid, the solution became slightly turbid at 130°, and then, at 152° F., long and slender colourless crystals began to form in it. The crystals increased in size until the temperature had risen to 160° F., when also the turbidity of the solution greatly diminished. After allowing the solution to become cold, 34·9 per cent. of crystals were obtained, and the colourless filtrate was found to contain from 15 to 16 per cent. of glucose. (Table XI. Analysis No. 36.)

Strophanthin dissolved in 1·5 per cent. sulphuric acid was allowed to stand in the cold. On the following day, a small crystalline nodule and tuft, both consisting of colourless transparent crystals, had formed at the bottom of the flask. On the third day, several large nodules or rosettes had formed, the crystals of which had to the naked eye an acicular lancet-shape (Plate VII. fig. 10).* The crystals were collected and dried on the fourth day, when they weighed 37·58 per cent. of the strophanthin used. The filtered solution was divided into two portions. In one of them the glucose was found to be 21·3 per cent. The second portion was boiled for four hours; during

* The crystals formed at the bottom of the flask have been represented, unmagnified, in Pl. VII. fig. 10, as they appeared on looking down upon them through the fluid in the flask.

boiling it evolved a cooked raisin odour, became turbid, and, when cooled, deposited a yellowish-brown amorphous substance, all of which was not collected. The filtrate of the second portion was slightly coloured and only faintly bitter; and 26·65 per cent. of glucose was found in it by Fehling's reagent, and 23·64 by fermentation. (Table XI. Analysis No. 40.)

A summary of the preceding and of several other experiments is given in Table XI.

TABLE XI.—*Summary of Analyses in which Strophanthin was decomposed by Sulphuric Acid.*

No. of Analysis.	Quantity of Strophanthin in Grammes.	Percentage of Strophanthin in Solution.	Percentage of Sulphuric Acid.	Strophanthidin produced in the Cold.	Glucose produced in the Cold.	Strophanthidin produced after Heating.	Glucose produced after Heating.
34	0·25	3·3 %	0·33 %	In six days, 33·7 % of crystalline strophanthidin.	22 %	...	...
35	0·25	3·3 %	0·33 %	...	...	After boiling for half an hour, 36·2 % of crystalline strophanthidin.	After boiling for half an hour, 27·5 %.
36	0·25	2·5 %	0·5 %	...	...	After heating to 160° F., 34·9 % of crystalline strophanthidin.	After heating to 160° F., 15 %.
37	0·25	2·5 %	0·5 %	...	...	After heating to 170° F., 32·14 % of crystalline strophanthidin.	After heating to 170° F., 16·5 %.
38	0·25	2·5 %	0·5 %	...	...	After boiling for half an hour, 22·3 % of strophanthidin, chiefly in round particles.	After boiling for half an hour, 21·2 %.
39	0·5	3·3 %	1·3 %	In five days, 37·5 % of crystalline strophanthidin.	In five days, 19 %.	After boiling one half of the filtered solution for four hours, 4·6 per cent. of amorphous yellowish-brown substance.	After boiling one half of the filtered solution for four hours, 24·6 %.
40	0·5	3·3 %	1·5 %	In four days, 37·58 % of crystalline strophanthidin.	In four days, 21·3 %.	After boiling one half of the filtered solution for four hours, 4·3 % of yellowish-brown amorphous substance.	After boiling one half of the filtered solution for four hours, 26·65 % by Fehling's reagent, and 23·64 % by fermentation.
41	0·25	3·5 %	2 %	...	...	After boiling for half an hour, 37·7 % of yellowish-brown amorphous substance.	After boiling for half an hour, 22·02 %.

Sulphuric acid was the only acid employed to decompose strophanthin; as it seemed unnecessary, for the present purpose, to multiply the experiments, in view of the evidence already described regarding the action of other acids on the extract of strophanthus, there being no reason to doubt that the decomposition of strophanthin into glucose and strophanthidin will occur under the influence of those other acids which have been shown to decompose strophanthin in the extract.

It is seen from the analyses summarised in the above Table (XI.) that large quantities of crystalline strophanthidin, and considerable quantities of glucose, were produced by the prolonged contact in the cold of strophanthin with from 0·3 to 1·5 per cent. sulphuric acid. Crystalline strophanthidin was also abundantly produced when strophanthin was boiled for a short time with 0·3 per cent. sulphuric acid; and much crystalline strophanthidin, but proportionally less glucose, when strophanthin was heated between 160° and

170° F. with 0·5 per cent. sulphuric acid. When, however, strophanthin was boiled with stronger than 0·5 per cent. sulphuric acid, although much glucose was generally produced, no crystalline strophanthidin, but only a brown amorphous substance, appeared as a result of the decomposition.

The crystals of strophanthidin produced in the cold by sulphuric acid were usually in the form of colourless nodules or rosettes, consisting of moderately thick, lancet-shaped crystals (Plate VII. fig. 10). The finest crystals, of long and slender form, were obtained when strophanthin was heated to 160° or 170° F. with 0·3 and 0·5 per cent. sulphuric acid.

When the decomposition is produced so as to allow the physical changes to be observed under the microscope, the changes are found to be much the same as those already described in the decomposition, in similar circumstances, of the extract (p. 1004), except that the crystals that are formed are absolutely colourless. Thus, when a drop of 2 per cent. sulphuric acid was added to a small drop of solution of strophanthin, slender rods appeared on the second day, and, on the third day, small circular crystalline masses having a radiating structure, which increased in size and number during the two following days. In another experiment, the addition of 10 per cent. sulphuric acid immediately produced an abundant precipitation of minute particles; and round, clear bodies, and colourless circular crystalline masses, having radial markings, appeared during several subsequent days.

In addition to the influence of acids on strophanthin, that of ptyaline was also examined. 0·1 gramme of strophanthin was dissolved in 5 c.c. of distilled water, and to the clear solution 2 c.c. of filtered saliva* was added. The now decidedly alkaline mixture was digested for an hour at a temperature ranging between 99° and 100° F. The digestion did not produce any obvious change, nor could any strophanthidin be detected in the fluid after it had cooled. When, however, it was tested with Fehling's reagent, reduction immediately occurred, and an estimation showed that rather less than one per cent. of glucose had been produced. Prolonged contact with saliva at the body temperature is therefore able to cause only a slight decomposition. As a large quantity of saliva of great diastatic activity had been used in this experiment, it is reasonable to infer that in the ordinary administration of strophanthus, decomposition will not be produced to any appreciable extent by admixture with the mouth secretions.

The relatively slight decomposition which has been shown to occur (Table X. Analysis No. 24) when extract of strophanthus is digested for half an hour with 0·1 per cent. hydrochloric acid at a temperature ranging between 98° and 100° F., appears also to justify the inference that when strophanthus is introduced into the stomach it will be absorbed into the blood before any important part of the dose has undergone decomposition.

* The saliva was obtained, with the usual precautions to exclude impurity, from an adult to whom pilocarpine had been administered. It was alkaline in reaction, and neither before nor after prolonged heating, did it affect Fehling's solution. A small quantity rapidly and abundantly produced glucose in starch solution.

Still, these experiments, and indeed all the experiments in which strophanthin was shown to be decomposed by acids, render it not only of interest but probably of practical importance to determine, as I propose on some early occasion to do, the pharmacological action of strophanthidin itself.

Strophanthidin.

In the meantime, in addition to those physical and chemical characters of crystalline strophanthidin that have incidentally been mentioned, it may be added that it has an intensely bitter taste and a neutral reaction; that it is slightly soluble in cold water, moderately soluble in cold rectified spirit, chloroform, and amyl alcohol, and freely soluble in warm rectified spirit; that it becomes of a green colour when heated with 2 per cent. sulphuric acid; that it does not give a glucose reaction with the phenylhydrazin test, nor with Fehling's solution, either before or after prolonged digestion with 2 per cent. sulphuric acid between 200° and 212° F.; and that it is extremely active as a pharmacological agent, 0·0025 and 0·00125 grain producing death in frogs weighing 350 grains and 345 grains, respectively, with symptoms closely resembling those produced by strophanthin. Further, it can readily be obtained in colourless crystals by the spontaneous evaporation of a solution in rectified spirit.

As a solution of recrystallised strophanthidin, produced by the decomposition of strophanthin by sulphuric acid, remained unchanged when solution of chloride of barium was added to it, strophanthidin cannot be regarded as a combination of some substance present in strophanthin with the acid employed in decomposing it.

The amorphous brown substance obtained by boiling strophanthin with moderately strong acids has not been examined further than to determine that it is much less bitter than either strophanthin or strophanthidin, and that it is insoluble or nearly so in water and acids, and soluble in alkalies and rectified spirit.

Kombic Acid.

Basic and neutral acetate of lead have been enumerated among the reagents which produce precipitates in solutions of the extract in water. The precipitate obtained by the former reagent has not been examined. That produced by neutral acetate of lead, after having been carefully washed with distilled water, was decomposed by sulphuretted hydrogen, and the filtrate from lead sulphide was concentrated by evaporation at a low temperature, and then dried *in vacuo* over sulphuric acid. There was thus obtained a scaly brownish-yellow substance, representing 1·6 per cent. of the extract, of strongly acid reaction, and freely soluble in water. For this acid, the name Kombic acid is suggested.

2. Chemical Composition of other Parts of the Plant.

An examination was made of the comose appendages, pericarp, and other parts of the plant, mainly for the purpose of determining if strophanthin is present in them, and, if present, in what quantity.

Taking advantage of the circumstance that ethyl ether precipitates the glucoside from its solutions in alcohol, the process described at page 998 was adopted in the analyses, as it appeared to be one that was likely to yield sufficiently accurate results for the purpose immediately in view.

Comose Appendages.

7000 grains of comose appendages, carefully separated from all other parts of the seed, were reduced to a powder by being passed through a Burroughs & Wellcome's drug-mill, and the powder was macerated for six weeks with rectified spirit. The spirit was removed by pressure in a tincture-press, and the marc was twice afterwards saturated with fresh rectified spirit, which also was removed by strong pressure. The extract obtained by distilling and evaporating this tincture weighed 136 grains, and was of a dark reddish-brown colour, acid reaction, and bitter taste. Distilled water imperfectly dissolved it, a dark resinous substance remaining undissolved. The watery solution was filtered and several times carefully shaken with ether, and thereby an ether extract, weighing 47·1 grains, was obtained, which was reddish-brown in colour, and had a pleasant aromatic odour. The watery solution was then evaporated, and during its concentration a considerable quantity of a dark pitch-like substance separated from it, which, when dried, became hard and brittle. The extract obtained on the further evaporation of the watery solution had a sweet mucilaginous odour and an acid reaction. It was, for the most part, soluble in a small quantity of rectified spirit, the insoluble residue weighing 6·5 grains, and having the characters of mucilage. The addition of ether to the alcoholic solution produced an abundant precipitate, the alcohol-ether becoming at the same time densely milky. After standing for several hours the milkiness disappeared, and the decanted alcohol-ether yielded on distillation a further small quantity of resin. The precipitate thrown down by ether from the alcohol solution weighed when dry 50·35 grains; and it was bitter, markedly acid, amorphous, and of a dark brown colour.

The chief results of this analysis are stated below.

Analysis No. 42.

7000 grains of comose appendages yielded—

Alcohol extract (8 of rectified spirit to 1 of comose appendages), 136 grains = 1·94 per cent.

136 grains of alcohol extract yielded—

Ethyl ether extract,	47·1	grains = 34·63	per cent. of	extract, or	0·67	per cent. of	comose appendages.
Mucilage,	6·5	grains = 4·78	,,	,,	0·092	,,	,, ,,
Resin,	30·36	grains = 22·32	,,	,,	0·43	,,	,, ,,
Impure strophanthin,	50·35	grains = 37·02	,,	,,	0·71	,,	,, ,,

As the impure strophanthin was found to have a sweet as well as a bitter taste, it was examined for glucose; and this substance was found to be abundantly present in it, the amount indicated by Fehling's solution being so much as 40·3 per cent. In the 50·35 grains of *very* impure strophanthin there were, therefore—

Impure strophanthin, 30·04 grains = 59·66 per cent.
Glucose, 20·31 grains = 40·33 „

Or, otherwise stated, 136 grains of alcohol extract actually yielded—

Impure strophanthin, 30·04 grains = 22·08 per cent. of alcohol extract, or 0·42 per cent. of comose appendages.
Glucose, 20·31 grains = 14·93 „ „ „ 0·29 „ „ „

The presence of strophanthin, or of a body acting like strophanthin, in the above ether precipitate (impure strophanthin) was demonstrated by administering 0·005 grain of it by subcutaneous injection to a frog weighing 450 grains, when the usual general and cardiac actions of a small dose of this active principle were manifested, and this dose proved to be a lethal one.

Chemical tests were less conclusive, owing no doubt to the large quantity of glucose present. Dilute sulphuric acid and dilute hydrochloric acid (10 per cent.), with gentle heat, each produced a green colour, but in both cases this passed into a dark brown, almost black, without intermediate colour changes having been observed; and dilute nitric acid, with heat, produced a brownish-yellow colour, which soon passed into gamboge-yellow.

Search for an Alkaloid.—A small portion of the above impure strophanthin (or ether precipitate) was heated with soda lime, when it evolved alkaline fumes, which formed a white cloud with strong hydrochloric acid. The precipitate, therefore, contained nitrogen. A 5 per cent. solution was accordingly tested with a number of reagents for alkaloids. The results were altogether negative with mercuric chloride, potassio-mercuric iodide, tri-iodide of potassium, potassio-bismuthic iodide, potassio-cadmic iodide, meta-tungstate of sodium and phosphoric acid, picric acid, sulphate of zinc, and cobaltous chloride. On the other hand, tri-bromide of potassium very slowly produced a slight amorphous, yellowish-white precipitate; nitrate of silver, a fairly abundant yellowish-white precipitate; platinic chloride, after several hours, a slight diffused haziness; and chloride of gold and cupric sulphate, each a very faint precipitate.

The greater part, 40·5 grains, of the ether precipitate (impure strophanthin) was then treated by STAS' method for separating alkaloids. The acid solution in water was made distinctly alkaline by carbonate of sodium; it became much darker in colour, and at the same time *a strong odour similar to that of ethylamine or methylamine* was given off. The alkaline solution was carefully shaken with three successive quantities of ethyl ether; the decanted ethers were washed with distilled water and distilled; and the residue was dried. The alkaline solution was then similarly treated with three successive quantities of chloroform. The products thus obtained were—

Analysis No. 43.

Ether extract, 0·04 grain = 0·098 per cent. of 40·5 grains of impure strophanthin.
Chloroform extract, 0·18 grain = 0·44 „ 40·5 „ „

Assuming that the whole of the 50·35 grains of impure strophanthin had been subjected to STAS' process, the results, on the above basis, would have been—

Ether extract, 0·49 grain = 0·036 per cent. of alcohol extract, or 0·0007 per cent. of comose appendages.
Chloroform extract, 0·223 grains = 0·163 per cent. of alcohol extract, or 0·003 per cent. of comose appendages.

Only an insignificant product, therefore, was obtained when the extract derived from a very large quantity of the comose appendages was subjected to STAS' process for the separation of alkaloids. So small, indeed, was this product that its alkaloidal or other characters could not satisfactorily be determined. The following, however, were ascertained :—

The *ether product* was amorphous, of a brownish-yellow colour, slightly bitter, and with a strong aromatic odour. It became opaque when distilled water or dilute acetic or hydrochloric acid was brought into contact with it, and it was insoluble in each of these liquids, but it was readily dissolved by weak alkalies and again precipitated by acids.

The *chloroform product* was also amorphous and brownish-yellow, and it had a faint aromatic odour and a decidedly bitter taste. It also became opaque when moistened with water, or with dilute acetic or hydrochloric acid; but while it was insoluble in water, it was partly soluble in a large bulk of either acid. Prolonged contact with dilute acetic acid resulted in a very bitter, yellowish solution being obtained; and after several washings with this acid, the evaporated solutions gave an amorphous slightly coloured residue, which weighed only 0·04 grains. When administered by subcutaneous injection, 0·01 grain of this residue was found to be a lethal dose for a frog weighing 470 grains; and the symptoms, including the changes in the heart's action, were the same as those that are produced by strophanthin. The remainder of the 0·04 grains was dissolved in distilled water, and tested with reagents for alkaloids with the following results :—Auric chloride caused a haziness, and tri-bromide of potassium and metatungstate of sodium with phosphoric acid each a very slight haze; but no change was caused by potassio-mercuric iodide, platinic chloride, potassio-bismuthic iodide, potassio-cadmic iodide, nor by picric acid.

In another process, in which also 7000 grains of comose appendages were examined in the same manner as has above been described, the results were equally inconclusive of the existence of an alkaloid. It is a significant fact that when the alcoholic extract of

the appendages is made alkaline, a volatile body of alkaline reaction is disengaged, which has the peculiar odour of ethylamine or methylamine.

That an alkaloid exists in the appendages has been frequently asserted on the authority of HARDY and GALLOIS, who have gone so far as to designate it "Inéine." They, however, give very imperfect evidence of its existence, although they appear to have had a sufficient quantity of the substance to enable them to make several pharmacological experiments with it. They state that it does not act on the heart as Strophanthus does, but at the same time they do not state how it does act, and even if it has any pharmacological action whatever.

The very minute product which I have obtained by STAS' process is, however, an active substance, with so close pharmacological resemblances to strophanthin itself, that it might well be merely impure strophanthin or an impure decomposition product of that glucoside. At any rate, the endeavours I have made, working with very large quantities of materials, to separate an alkaloid from the comose appendages, have failed to give evidence of the existence of an alkaloid.

Placenta.

By percolating 274 grains of placenta with 7 ounces of rectified spirit (=11 of spirit to 1 of placenta), an extract weighing 8·4 grains (=3·06 per cent. of placenta) was obtained. When this extract was mixed with a little water and several times shaken with ethyl ether, 1·86 grains of ethereal extract (=0·67 per cent. of placenta) was separated from it, leaving 6·39 grains of dry alcohol extract (=2·33 per cent. of placenta).

Analysis No. 44.

From this 6·39 grains of alcohol extract there was obtained—

Impure strophanthin,	1·7	grain	= 26·5	per cent. of alcohol extract,	or	0·62	per cent. of placenta.
Mucilage,	3·6	grains	= 56·3	„	„	1·31	„	„
Resin, &c.,	0·895	grain	= 14	„	„	0·32	„	„

This impure strophanthin became of a dark violet colour when it was heated with 10 per cent. sulphuric acid, and one-tenth of a grain was rapidly fatal to a frog, and produced the ordinary pharmacological effects of strophanthin.

Endocarp.

The tincture obtained by percolating 548 grains of powdered endocarp of the follicle with 5 ounces of rectified spirit (=4 of spirit to 1 of endocarp) yielded 13·44 grains, or 2·45 per cent. of extract, from which there was obtained—

Analysis No. 45.

Ethyl ether extract,	3·7	grains	= 27·529	per cent. of extract,	or	0·675	per cent. of endocarp.
Alcohol extract,	10·54	grains	= 78·422	„	„	1·923	„	„

On further examination, the 10·54 grains of alcohol extract was found to contain—

Impure strophanthin,	5·7	grains = 54·07	per cent.	of	alcohol extract,	or	1·03	per cent. of endocarp.
Mucilage,	3·67	grains = 34·81	„	„	„		0·67	„ „
Resin,	1·	grain = 9·48	„	„	„		0·18	„ „

This impure strophanthin was freely soluble in water, acid in reaction, and strongly bitter, though at the same time sweetish in taste. It was examined for glucose, and 12 per cent. of this substance was found in it; so that in 5·7 grains there was 5·02 grains, or 88·07 per cent. of impure strophanthin and 0·68 grains of glucose.

Stated otherwise, 5·7 grains of very impure strophanthin contained—

Impure strophanthin,	5·02	grains = 47·62	per cent.	of	alcohol extract,	or	0·91	per cent. of endocarp.
Glucose,	0·68	grains = 6·45	„	„	„		0·124	„ „

This impure strophanthin was amorphous, and of a pale brown colour and acid reaction. It gave indistinctly the chemical reactions of strophanthin, and 0·002 grain of it, by subcutaneous injection, was found to be a lethal dose for a frog, the pharmacological effects being those of strophanthin.

Pericarp.

548 grains of the entire pericarp (including endocarp) of the follicle (see p. 981), reduced to a coarse powder, gave, by percolation with four parts of rectified spirit, an extract which weighed 9·92 grains, or 1·8 per cent. of the pericarp. This extract yielded—

Analysis No. 46.

Ether extract,	3·8	grains = 38·306	per cent.	of	extract,	or	0·693	per cent. of pericarp.
Alcohol extract,	5·9	grains = 59·475	„	„			1·076	„ „

From this 5·9 grains of alcohol extract there was obtained—

Impure strophanthin,	3·25	grains = 55·06	per cent.	of	alcohol extract,	or	0·59	per cent. of pericarp.
Mucilage,	2·32	grains = 39·32	„	„	„		0·42	„ „
Resin,	0·23	grain = 3·89	„	„	„		0·04	„ „

The impure strophanthin was pale brown, acid, and hygroscopic; and it had a saline and only faintly bitter taste, and an aromatic odour. It contained an undetermined quantity of glucose. In its dry state a large number of minute acicular crystals were present in it, but these crystals disappeared when it became soft and liquid on exposure to the air. Chemical tests gave indistinct evidence of the presence of strophanthin. It possessed very feeble pharmacological activity, as 0·2 grain, administered by subcutaneous injection to a frog, weighing 427 grains, did not cause death, although this dose was sufficient to cause slight strophanthin symptoms.

Leaves.

100 grains of dried and well-preserved leaves, obtained from Mr Buchanan, were ground to a coarse powder, and extracted with rectified spirit. The alcoholic extract

obtained from this tincture was of a dark green colour and acid reaction, and it weighed 5·51 grains. It was further examined so as to determine the quantity of its chief constituents.

Analysis No. 47.

Impure strophanthin,	1·27 grain	= 23·04	per cent. of alcoholic extract.
Crystalline substance,	0·36 grain	= 6·53	„ „
Mucilage,	0·27 grain	= 4·9	„ „
Resin, chlorophyll, fat (?), &c.	3·405 grains	= 61·79	„ „

The leaves, therefore, contain only a small quantity of strophanthin, and a large quantity of resin mixed with chlorophyll, &c. The impure strophanthin, which was precipitated from an alcoholic solution by ethyl ether, was freely soluble in water, translucent, brownish-yellow, and bitter, and it gave a characteristic but dirty violet colour when heated with dilute sulphuric acid, and also produced the ordinary pharmacological effects of strophanthin when the one-hundredth of a grain was injected under the skin of a frog.

The crystalline substance was obtained during the evaporation of a watery solution of the impure strophanthin. It crystallised in minute tufts of a pale brown colour, which, on microscopic examination, were found to consist of slender radiating needles. The crystals were nearly insoluble in rectified spirit, but were slightly soluble in water, and they were destitute of bitterness. When heated with dilute sulphuric acid, they for the most part dissolved and formed a bright yellow solution, which remained unchanged for many hours. One-twentieth of a grain, administered by subcutaneous injection, produced very slight effects in a frog, which recovered after exhibiting for two days symptoms of motor weakness with spastic phenomena, and slowing of the respirations.

Bark of the Branches.

When extracted with rectified spirit, the bark of the slender branches, sent by Mr Buchanan (p. 978), yielded 3·42 per cent. of extract, which was destitute of bitterness, having only a taste like that of wood. From a concentrated alcoholic solution of this extract, ether threw down a small precipitate, which amounted only to ·75 per cent. of the bark, or 22 per cent. of the extract. The precipitate was soluble in water, and while the watery solution was being evaporated several long needle-shaped crystals appeared in it. Chemical and pharmacological examination of the precipitate failed, however, to give any evidence of the presence of strophanthin in it; and even 0·2 grain produced no effect when injected under the skin of a frog.

Bark of Stem.

Even a smaller quantity of alcoholic extract (1·5 per cent.) was obtained from the bark of the stem (p. 978). It also was devoid of bitterness, and no strophanthin could be detected in it by chemical or pharmacological tests.

Root.

Some fresh roots from plants grown in the Edinburgh Botanic Garden were grated and partially dried at from 100° to 103° F. The water lost was found to represent 33·34 per cent. of the weight. 1191·2 grains of this incompletely dried root yielded 65 grains of dark reddish-brown alcohol extract, representing 5·457 per cent.

Further analysis of this extract gave the following results—

Analysis No. 48.

Sixty-five grains alcohol extract yielded—

Ethyl ether extract,	8·6	grains =	13·23	per cent. of alcohol extract, or	0·722	per cent. of dried root.
Alcohol extract,	56·12	grains =	86·34	„ „	4·712	„ „

The ether extract was of a reddish-brown colour, and it had a peculiar aromatic odour, and an acrid and slightly bitter taste. The alcohol extract was reddish-brown, acid and amorphous, and it had a distinctly bitter taste. A small quantity dissolved in water was tested with the following results :—Tri-bromide of potassium produced a slight haze, which afterwards subsided as a yellowish-brown precipitate, leaving a nearly colourless supernatant fluid ; tannic acid produced a slight precipitate ; ferric chloride, a slight haze, becoming a brown precipitate ; and phospho-molybdic acid and potash, a blue colour : but no important change occurred with potassio-mercuric chloride, tri-iodide of potassium, platinic chloride, mercuric chloride, potassio-bismuthic iodide, chloride of gold, or picric acid.

Analysis No. 49.

The 56·12 grains of alcohol extract yielded on further examination—

Impure strophanthin,	13·26	grains =	23·627	per cent. of alcohol extract, or	1·113	per cent. of dried root.
Mucilage,	40·	grains =	71·22	„ „	3·35	„ „
Resin,	2·7	grains =	4·81	„ „	0·226	„ „

The impure strophanthin was bitter, but it obviously contained much impurity, being brown in colour, and very difficult to dry, even *in vacuo*, over sulphuric acid. It, however, gave the pharmacological, and less distinctly the chemical, reactions of strophanthin.

When examined for glucose, so large a quantity as 41·8 per cent. was found to be present ; so that in the 13·26 grains of this very impure strophanthin there were 5·55 grains of glucose, and only 8·71 grains (or 65·68 per cent.) of impure strophanthin.

Stated otherwise, from 65 grains of alcohol extract (Analysis No. 48) there were obtained—

Impure strophanthin,	8·71	grains =	13·4	per cent. of alcohol extract, or	0·731	per cent. of dried root.
Glucose,	5·5	grains =	8·54	„ „	0·46	„ „

The foregoing account of an examination of various parts of the *Strophanthus hispidus* plant has shown that strophanthin is present in many other parts besides the seeds, as,

of the parts examined, the bark of the stem and branches alone failed to give evidence of its presence. The endocarp and placenta of the follicles and the comose appendages of the seeds were found to contain larger quantities than the roots, leaves, or epi- and mesocarp. In none of these parts of the plant, however, is it so largely present as in the seeds, and none of them can so conveniently be used as the seeds to produce strophanthin or the pharmaceutical preparations of Strophanthus.

(The pharmacology of *Strophanthus hispidus* will be described in a future part of the *Transactions*.)

EXPLANATION OF PLATES.

Plate I.

A. One of four arrows in the Materia Medica Museum, University of Edinburgh, tied together, and labelled by Sir Robert Christison, "Poisoned arrows from the interior of Africa, poison unknown." Poison found to be Strophanthus.

(The brown colour in this and the other arrows indicates the poison composition smeared on the arrow.)

B. One of above four arrows. Poison also found to be Strophanthus. As this arrow closely resembles Kirk's arrow (p. 957), it is probable that it, as well as arrow *A*, has been obtained from the Zambesi country.

C. Arrow from a district 75 miles N.N.W. of Zanzibar. From Dr Felkin. Stated to be poisoned with the substance contained in packet J, Plate II. Found to be inert.

D. Arrow from Wanyika country, north of Zanzibar. From Dr Felkin. Also stated to be poisoned with the substance contained in packet J, Plate II. Found to be active.

E. One of five similar arrows in the Materia Medica Museum, University of Edinburgh, labelled "Arrows from Negroes of River Gambir, poison unknown." Poison found to be Strophanthus.

Plate II.

F. Arrow from the Shiré District of East Africa. From Mr Buchanan. Poisoned with Strophanthus.

G. Arrow also from the Shiré District. From Mr Buchanan. Poisoned with Strophanthus.

H. Arrow from the West Side of Lake Tanganyika. From Dr Tomory. Poison unknown. Found to have a Strophanthus action.

I. Scraped mature follicle of *Strophanthus hispidus*. Hairs of the comose appendages of the seeds are seen protruding through the partially split placental surface of the follicle.

J. Packet of Wanyika poison. From Dr Felkin. Poison very active, and similar in action to Strophanthus.

Plate III.

Fig. 1. Root from small *Strophanthus hispidus* plant, one year old, grown in Royal Botanic Garden, Edinburgh. Natural size.

Fig. 2. Portion of root from mature plant, sent preserved in spirit from the Shiré District, East Africa. Natural size.

Fig. 3. Transverse section through a constricted portion of above root from mature plant, from Africa. × 6.

Fig. 4. Transverse section through a swollen portion of same root from Africa, showing great development of food-storing cellular tissue. × 6.

Fig. 5. Leaf from a plant, one year old, grown in Royal Botanic Garden, Edinburgh. Natural size.

Fig. 6. Inflorescence, sent as a dried specimen from the Shiré District of East Africa. In the expanded flowers, the prolongations of the corolla-lobes are seen as long and drooping tails; whereas in the flower-buds they are seen (*a*) to be twisted together, and to project upwards. Natural size.

Fig. 7. Flower-bud and fully developed flower. The contrast in appearance between the prolongations of the corolla-lobes in the fully developed flower and in the flower-bud (*a*) is clearly exhibited. × 1½.

Fig. 8. Longitudinal section of fully developed flower, showing (*a*) the five stamens closely surrounding the pistil. × 1½.

Fig. 9. Stamen. × 4.

Fig. 10. Pistil, showing cleft stigma and hirsute carpels. × 2.

Plate IV.

Fig. 1. Surface-view of stem, from Africa. Natural size.

Fig. 2. Transverse section of stem, showing thick cork layer. Natural size.

Fig. 3. Follicles dehiscing. Natural size.

Fig. 4. Mature follicles, reduced one-third, showing position of the two follicles when ripe.

Fig. 5. Transverse section of unripe follicle, from the Shiré District, preserved in spirit. The line of future dehiscence is indicated at *x*. Natural size.

Fig. 6. Seed with comose appendage. Dorsal view of seed. Natural size.

Fig. 7. *a*, ventral or parietal, and *b*, dorsal views of seeds. Natural size. In 7 *a*, the spot indicating the entrance of the funiculus is shown at *x*.

Fig. 8. *a*, dorsal, and *b*, lateral view of specially large seeds. Natural size. In 8 *b*, the position of the funiculus is shown at *x*.

Fig. 9. Magnified view of the base or root of one of the hairs of the tuft of the comose appendage of the seed. × 230.

Fig. 10. *a*, base, and *b*, apex of the same hair as fig. 9, less magnified. × 52.

Fig. 11. Small tufts or groups of the fine basal seed-hairs, which are interposed between the seeds and the inner surface of the endocarp. Natural size.

Fig. 12. *a* and *b*, magnified tufts of above hairs: 12 *a*, showing the roots, and 12 *b*, the apices. × 52.

Fig. 13. Immature seed showing the fine and relatively long hairs attached to the base of the seed. The hairs have been teased out a little, in order to display them more distinctly. × 6.

Plate V.

Fig. 1. Transverse section of swollen portion of root of mature plant, from Africa. Same section as fig. 4, Plate III., but more highly magnified. *a*, layer of cork tissue; *b*, cork cambium; *d*, very broad cellular rind of food-storing cells, many containing conglomerate crystals; *e*, wedge-shaped masses of bast tissue, with conglomerate crystals in a few of the cells; *f*, cambium layer; and *g*, central wood cylinder, exhibiting annual (?) growth rings. × 80.

Fig. 2. Transverse section of one-year old stem, from Edinburgh Botanic Garden. *a*, cork; *b*, cork cambium; *c*, indurated cells of cellular layer; *d*, ordinary cells of cellular layer, many containing conglomerate crystals, with an outer zone of shaded cells containing numerous chlorophyll corpuscles; *e*, bast tissue; *f*, cambium; *g*, wood; *h*, internal bast; *i*, pith. × 75.

Fig. 3. Transverse section of outer part of old stem, from Africa. Same lettering as in fig. 2. × 75.

Fig. 4. Longitudinal section of same stem as in fig. 3. A large portion of the inner wood region is not figured. Lettering the same as in figs. 2 and 3. In the bast tissue, *e*, besides bast cells, a sieve vessel and laticiferous cell are shown; and in the wood, *g*, dotted and spiral vessels are seen at the interior portion. × 300.

Fig. 5. Transverse section of an unripe follicle, sent in spirit from the Shiré District, showing the placenta and immature seeds, as well as the layers of the pericarp. *a*. epicarp; *b*, mesocarp; *c*, endocarp; *z*, tufts of basal seed-hairs. The line of future dehiscence is faintly indicated at *x*. Many of the seeds have fallen out. × 6.

Fig. 5*a*. Transverse section of epicarp and of outer mesocarp cells, from a hard mature follicle. *a*, epicarp; *b*, mesocarp. × 150.

Fig. 5*b*. Longitudinal section of mesocarp, composed of latex cells embedded in cells of matrix, from an immature follicle. × 150.

Fig. 5*c*. Longitudinal section of endocarp, from a hard mature follicle, consisting of indurated cells, which, *a*, in the external layer (next mesocarp) are arranged longitudinally, and *b*, in the internal layer are arranged transversely (circumferentially). × 230.

Fig. 5*d*. Transverse section of endocarp from a hard mature follicle, showing the longitudinal direction in this section of the internal cells (*b*), and the transverse (circumferential) direction of the external cells (*a*). × 52.

Plate VI.

Fig. 1. Longitudinal section of seed cut parallel with its flat surfaces. *a*, testa, bearing hairs, and prolonged above to form the base of the stalk of the comose appendage; *b*, tegmen, slightly developed; *c*, albumen; *d*, root-cap of embryo; *e*, radicle of embryo; *f*, one of the two cotyledons. × 30.

Fig. 2. Transverse section of seed, near position of the letter *f* in fig. 1. Lettering as in fig. 1; and *ff*, cotyledons, *g*, raphe. × 50.

Fig. 3. Longitudinal section of testa and tegmen. *a*, testa; *b*, tegmen. × 150.

Fig. 4. Internal view of testa, showing hoops of thickening on the primary membrane of each cell. × 350.

Fig. 5. Longitudinal section of albumen. The starch granules have been stained with iodine. × 150.

Fig. 6. Transverse section of testa, tegmen, and albumen. *a*, testa; *b*, tegmen; *c*, albumen. The starch granules have been stained with iodine. × 150.

Fig. 7. Transverse section of cotyledon. The lower part shows the natural appearance, and the upper part (*a*) the appearance after treatment with $\frac{1}{4}$ per cent. osmic acid solution, indicating a large quantity of oil. × 350.

Fig. 8. Transverse section of cotyledon, after treatment with ether. × 350.

Plate VII.

Fig. 1. Semi-diagrammatic representation of longitudinal section of mature seed, cut parallel with the narrow diameter. Same lettering as in Plate VI. figs. 1 and 2. *g*,*g*, raphe, with vascular bundle of funiculus passing into the seed at the upper *g*. × 20.

Fig. 2. Interior of the dorsal aspect of a follicle, displayed by removing the pericarp. The seeds are nearly concealed by the fine and relatively long basal seed-hairs covering them, which, in the entire follicle, are interposed between the seeds and the endocarp. The upper portion exhibits the compressed stalks and tufts of the comose appendages. Natural size.

Figs. 3, 4, 5, and 6. Transverse sections of mature follicles in various stages of dehiscence, showing stages of opening of the follicle at its ventral surface, and of uncoiling of the placenta, *p*. Natural size.

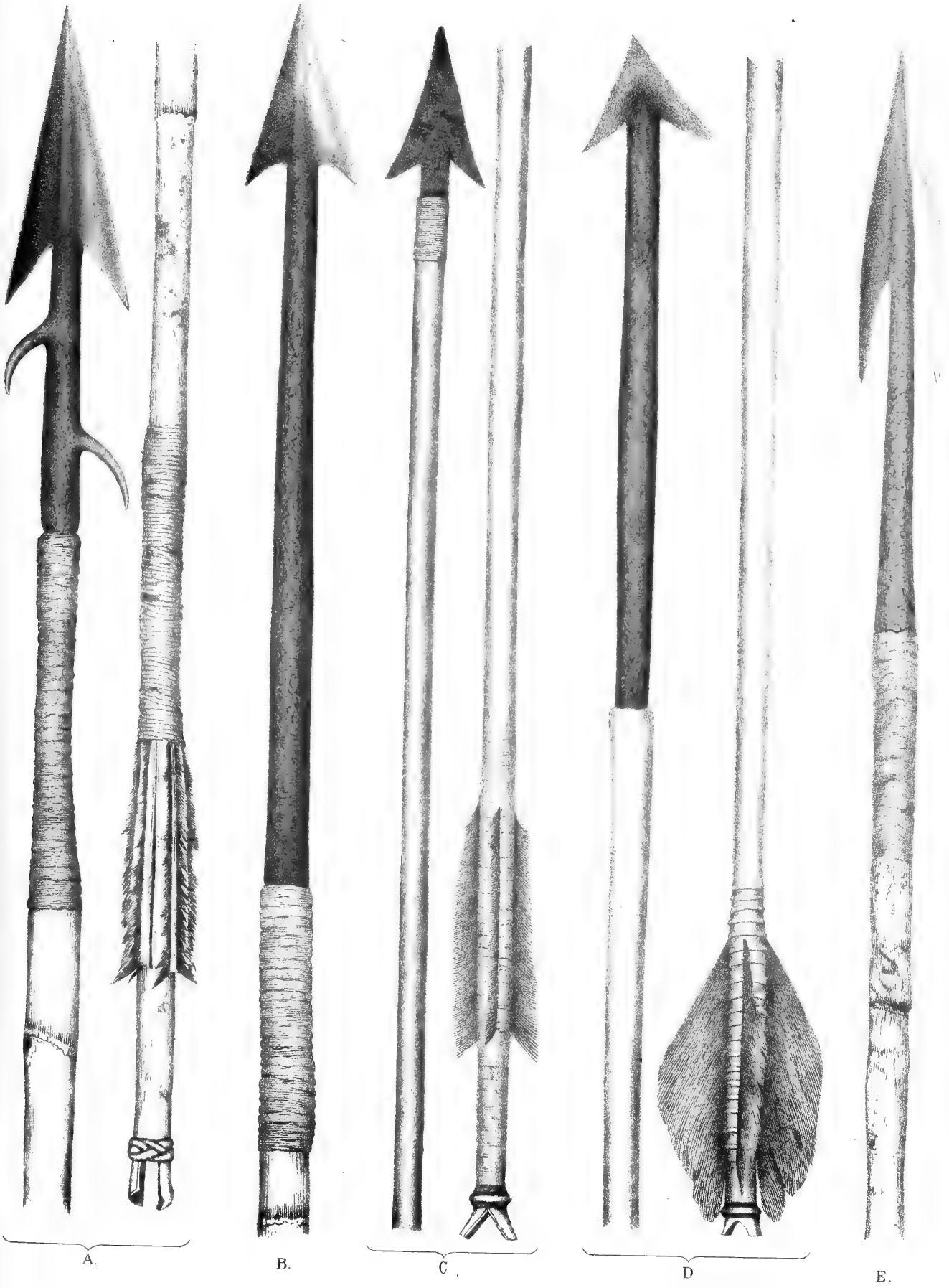
Fig. 7. Crystalline groups of Strophanthin, slowly formed in a dilute alcoholic solution after the addition of ether. × 6.

Fig. 8. Crystalline Strophanthin (impure?) obtained by the evaporation of a watery solution of alcoholic extract of Strophanthus. × 195.

Fig. 9. Usual microscopic appearance of Strophanthin obtained by the process described at page 1008. × 52.

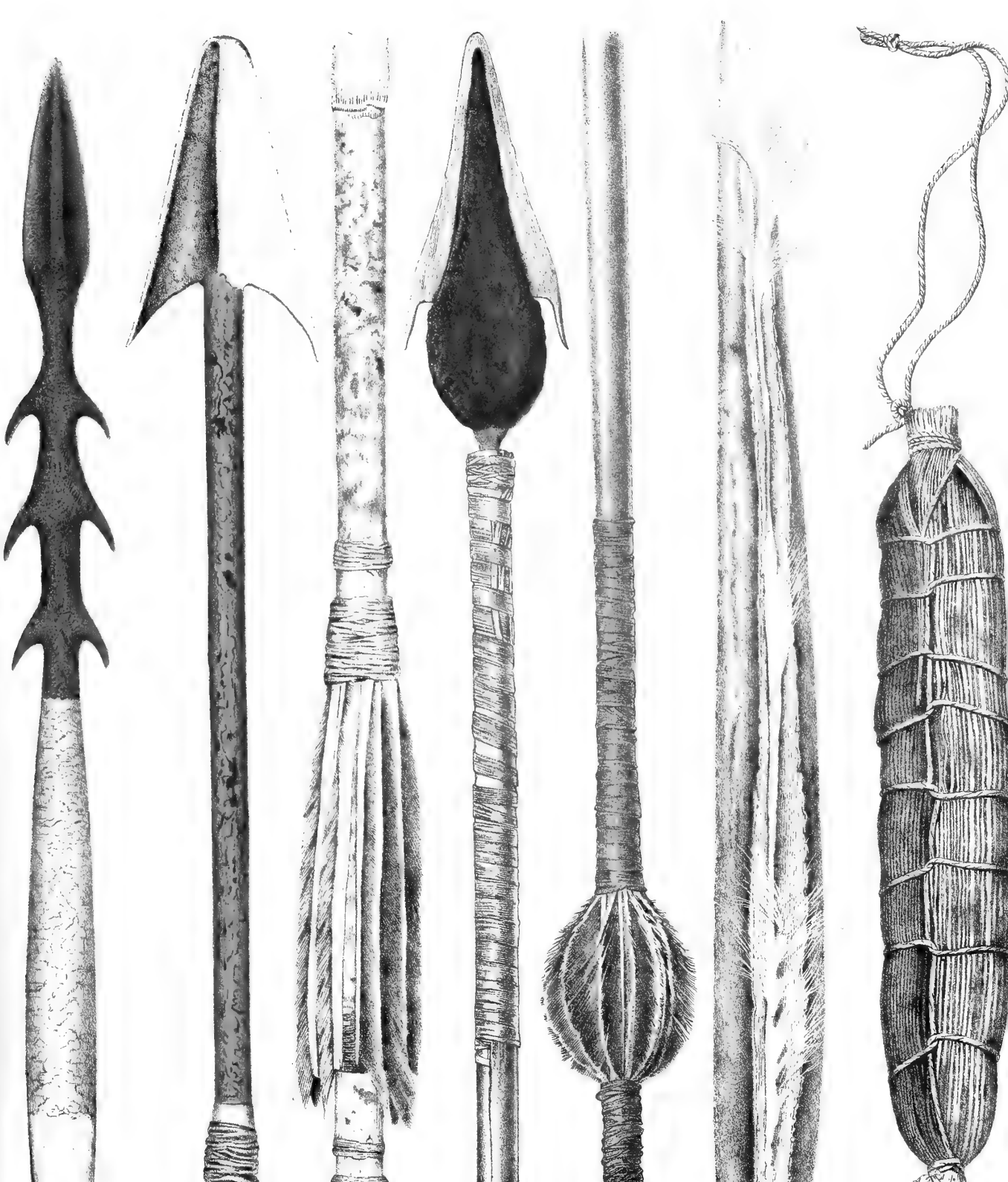
Fig. 10. Stropanthidin formed spontaneously in a solution of Strophanthin acidulated with sulphuric acid. Natural size.

FRASER ON STROPHANTHUS HISPIDUS. — PLATE I.



M^cFarlane & Erskine, Lith^rs, Edin^r

FRASER ON STROPHANTHUS HISPIDUS. — PLATE II.



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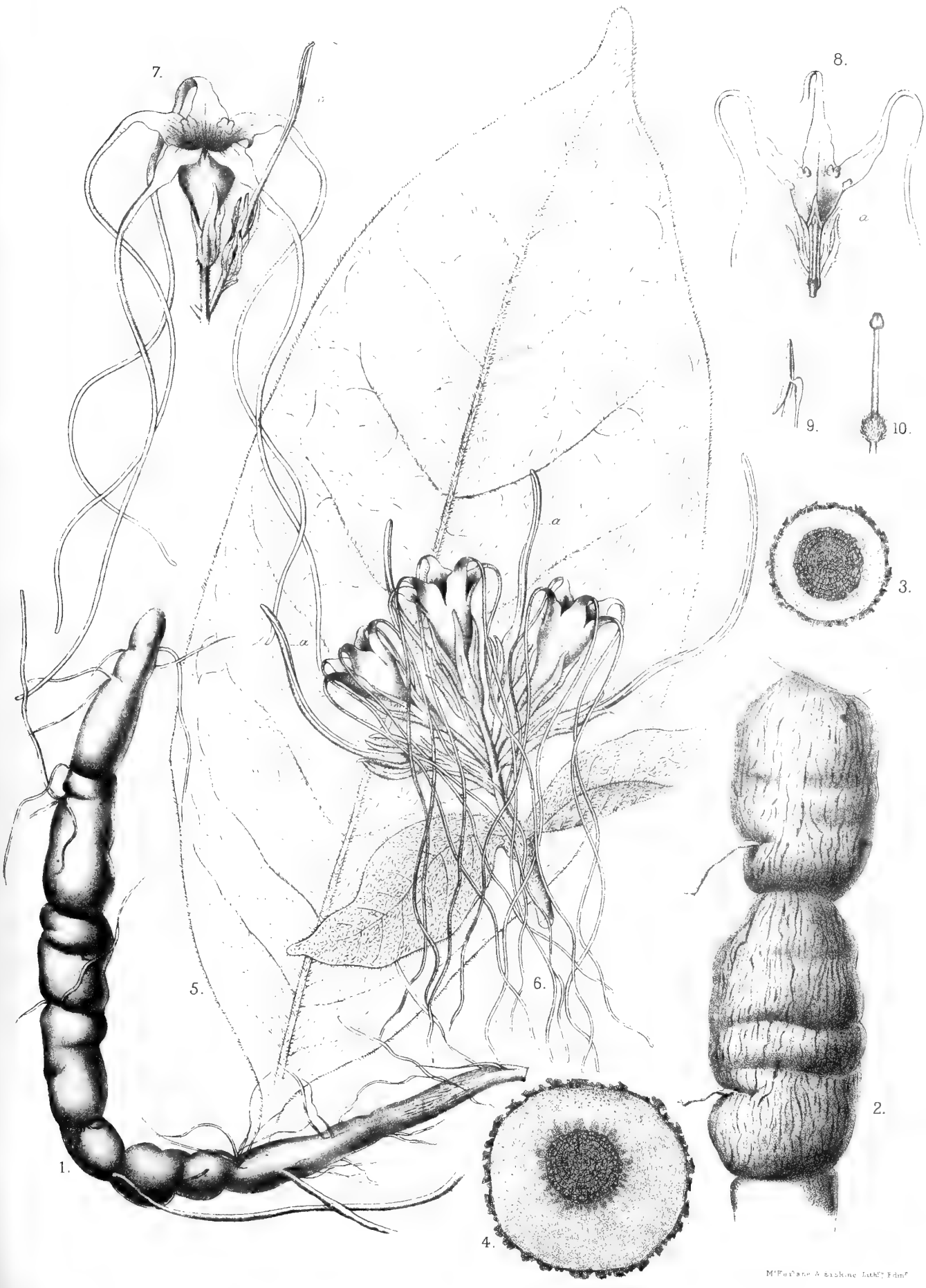
FRASER ON STROPHANTHUS HISPIDUS. — PLATE III.



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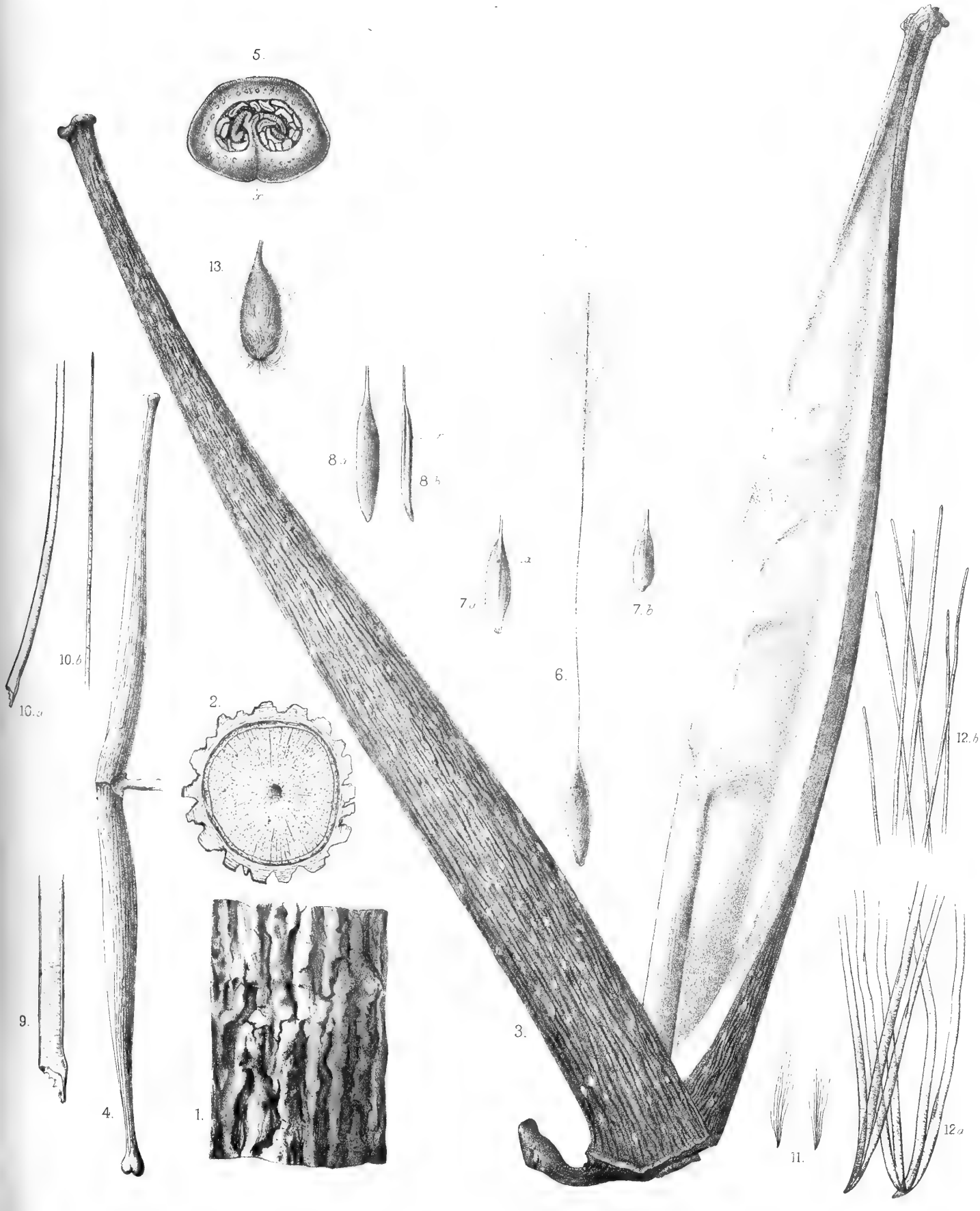
FRASER ON STROPHANTHUS HISPIDUS. — PLATE IV.



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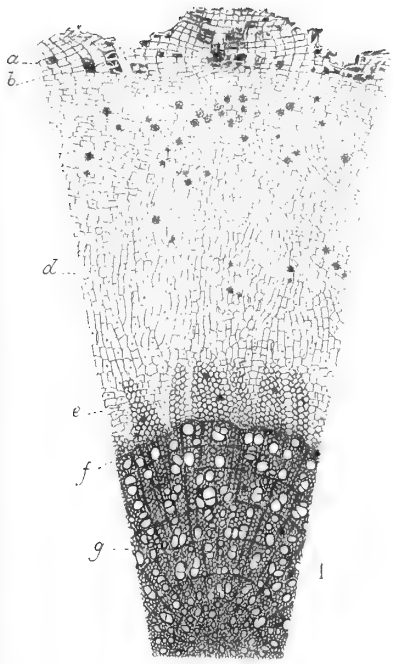
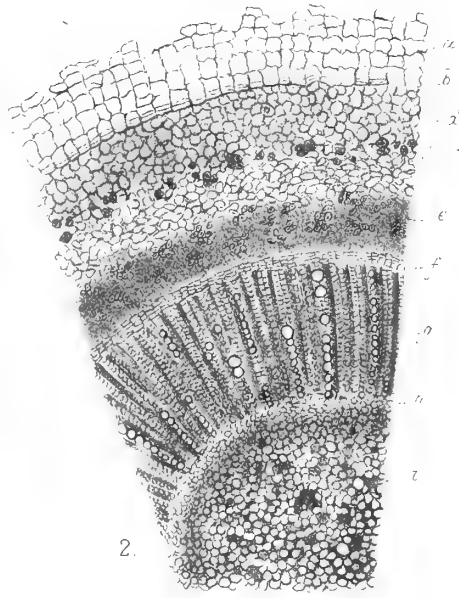
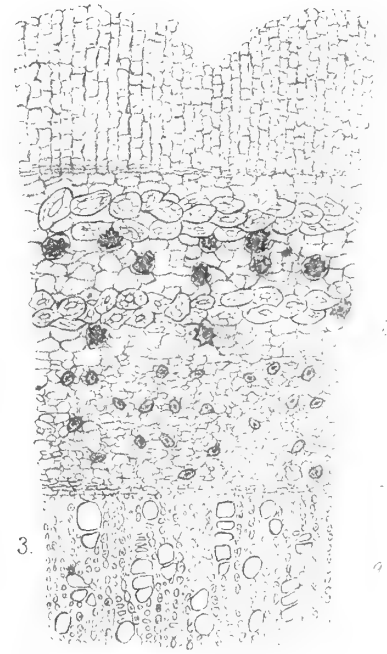
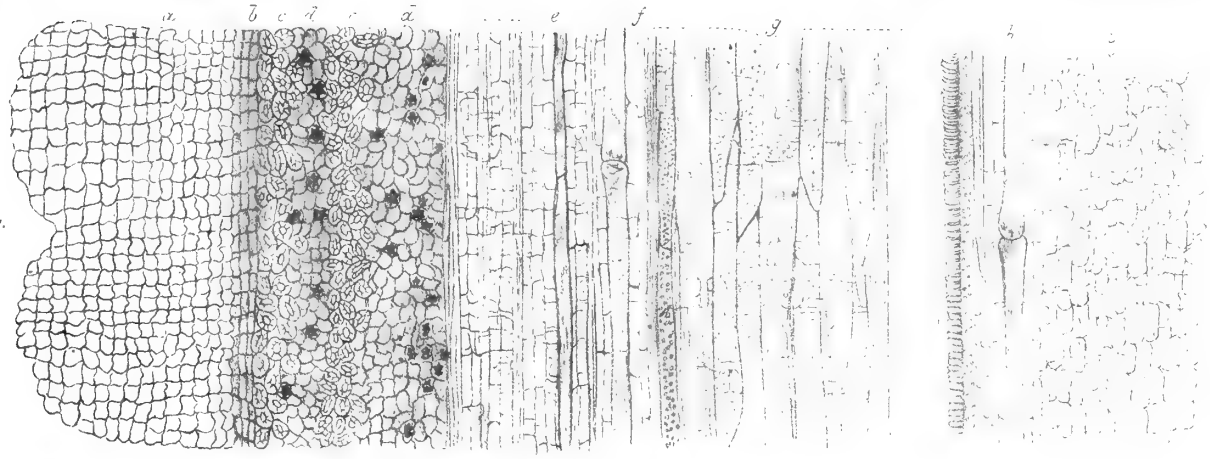
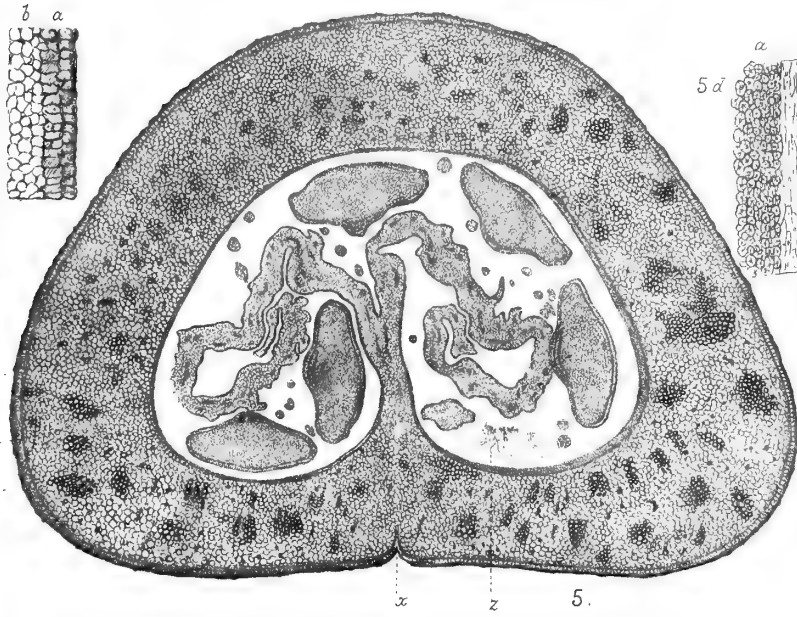
Fraser on Strophanthus hispidus. — Plate V.

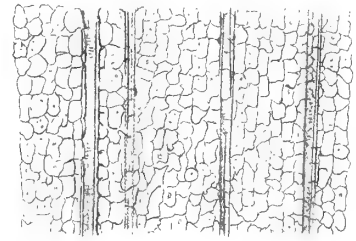


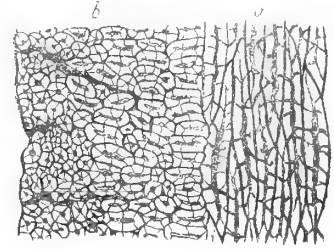






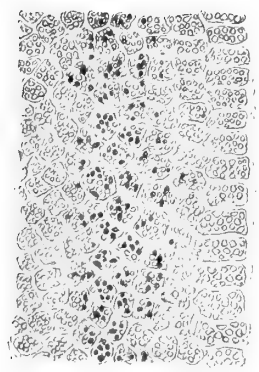






M Farlane & Erskine, Lith^rs Edin^r

Fraser on Strophanthus hispidus. — Plate VI.



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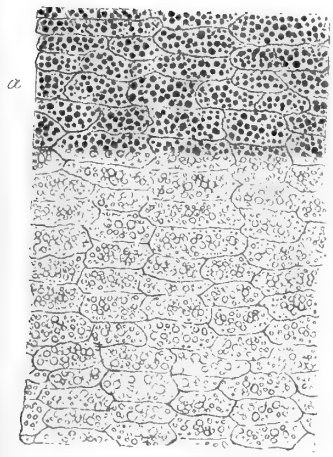

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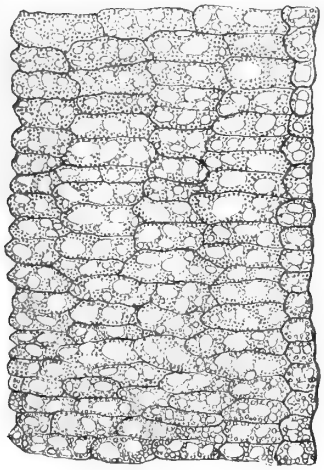

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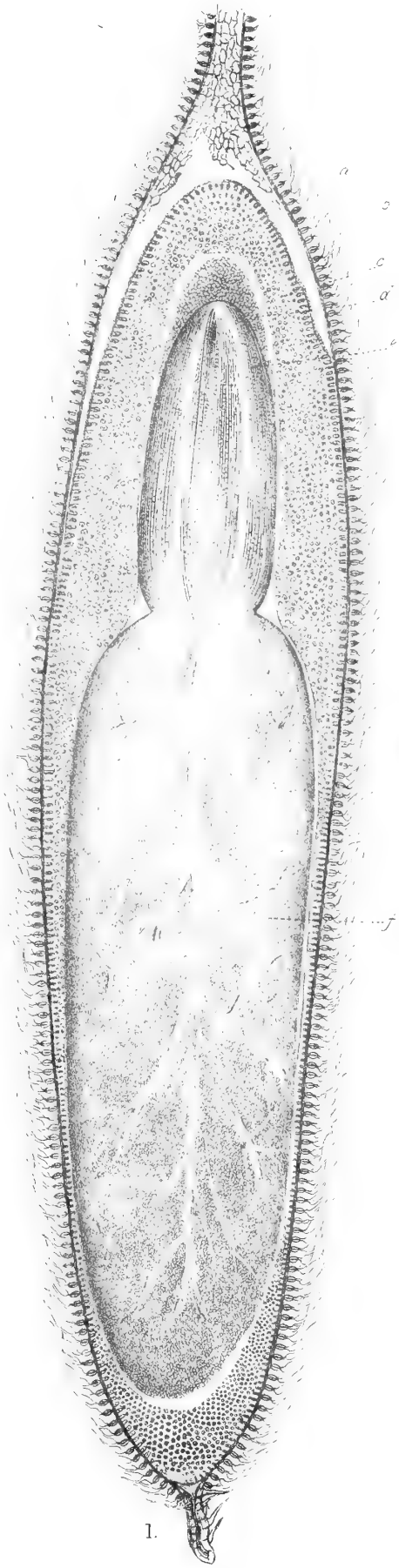

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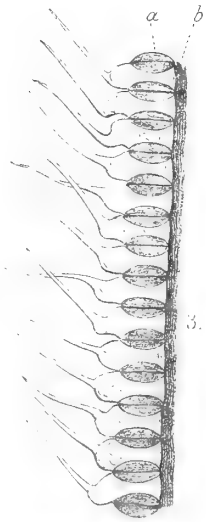

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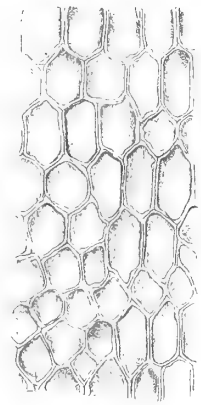

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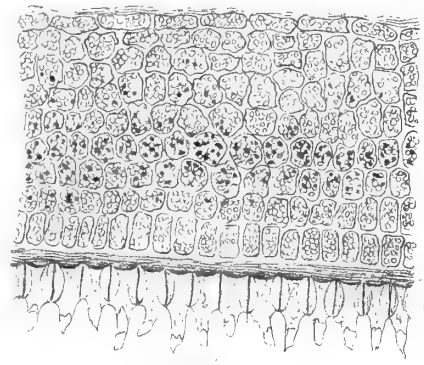

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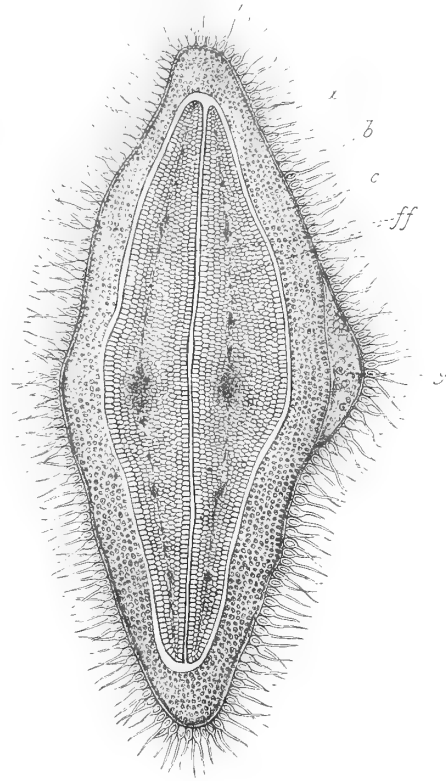

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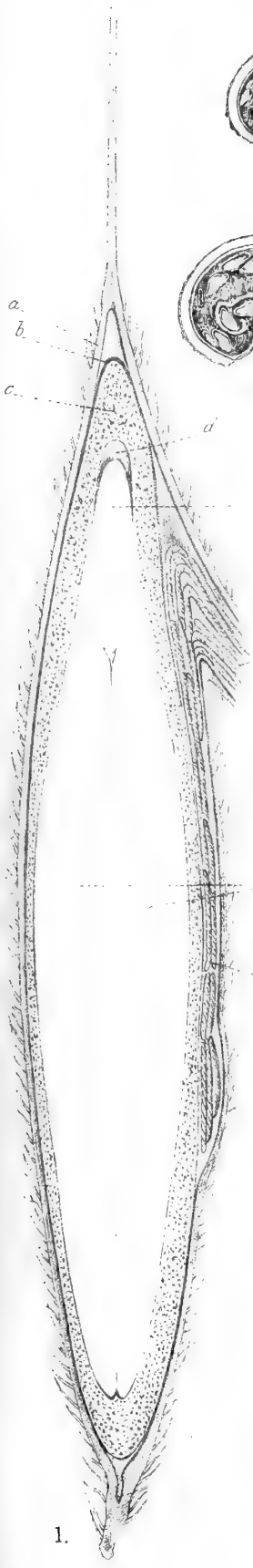
M^{c}Farlane & Erskine Lith^{rs} Edin^{r}

FRASER ON STROPHANTHUS HISPIDUS. — PLATE VII.



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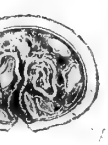

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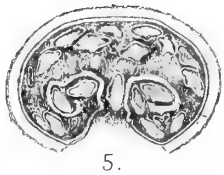

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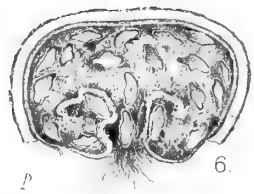

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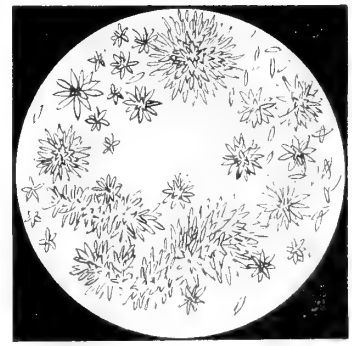

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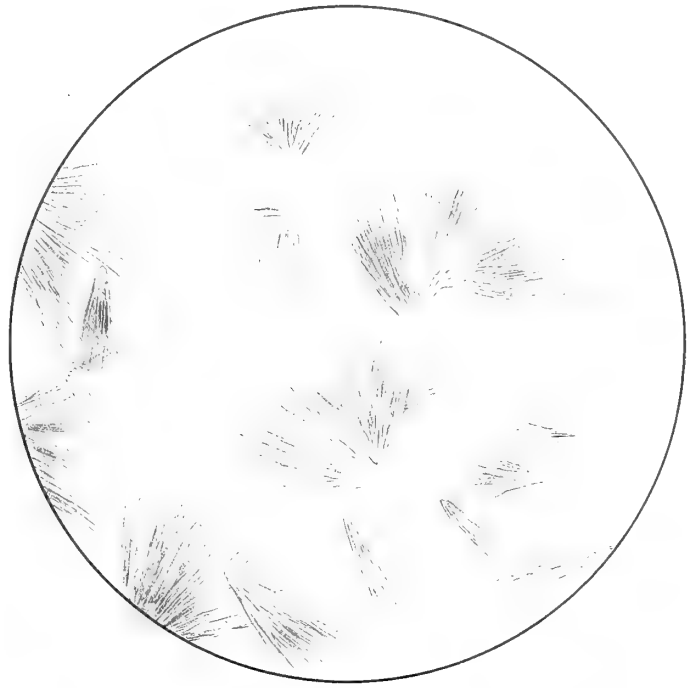

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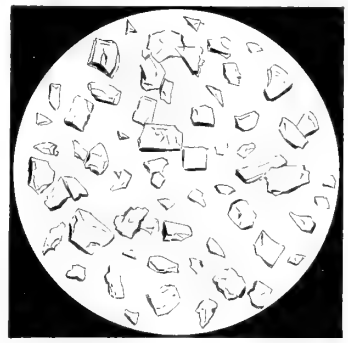

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XXII.—*On the Foundations of the Kinetic Theory of Gases.* III.

By Professor Tait.

INDEX TO CONTENTS.

I have explained at some length, in my "Reply to Prof. Boltzmann," * the circumstances under which the present inquiry originated and has been pursued. Of these I need now only mention two:—*first*, the very limited time which I can spare for such work; *second*, the very meagre acquaintance I possessed of what had been already done with regard to the subject. My object has been to give an easily intelligible investigation of the *Foundations* of the Kinetic Theory; and I have, in consequence, abstained from reading the details of any investigation (be its author who he may) which seemed to me to be unnecessarily complex. Such a course has, inevitably, certain disadvantages, but its manifest advantages far outweigh them.

In August 1888, however, I was led in the course of another inquiry † to peruse rapidly the work of Van der Waals, *Die Continuität des gasförmigen und flüssigen Zustandes.* This shows me that Lorenz had anticipated me in making nearly the same correction of the Virial equation as that given in the earlier part of § 30 of my first paper. His employment of the result is a totally different one from mine; he uses it to find a correction for the number of impacts. The desire to make, at some time, this investigation arose with me when I was writing my book on *Heat*, as will be seen in the last paragraphs of § 427 of that book. It was caused by my unwillingness to contemplate the existence of molecular *repulsion* in any form, and my conviction that the effects

* *Proc. R. S. E.*, January 1888; *Phil. Mag.*, March 1888.

† "Report on some of the Physical Properties of Water," *Phys. Chem. Chall. Exp.*, Part IV.

ascribed to it could be explained by the mere resilience involved in the conception of impacts.

The present paper consists of instalments read to the Society at intervals during the years 1887, 8. The first of these, which is also the earliest in point of date, deals with a special case of molecular attraction, on which, of course, depends the critical temperature, and the distinction between gases and vapours. Here the particles which, at any time, are under molecular force have a greater average kinetic energy than the rest. Mathematical, or rather *numerical*, difficulties of a somewhat formidable nature interfered with the exact development of these inquiries. I found, for instance, that in spite of the extreme simplicity of the special assumption made as to the molecular force, the investigation of the average time between the encounter of two particles and their final disengagement from one another involves a quadrature of a very laborious kind. Thus the correction of the number of impacts could not easily be made except by some graphic process.

One reason for the postponement of publication of the present part was the hope that I might be enabled to append tables of the numerical values of the chief integrals which it involves, especially the peculiarly interesting one

$$y = \varepsilon^{-x^2} \int_0^x \varepsilon^{x^2} dx .$$

Want of time, however, forced me to substitute for complete tables mere graphical representations of the corresponding curves, drawn from a few carefully calculated values. These are not fitted for publication, though they were quite sufficient to give a general notion of the numerical values of the various results of the investigation; and enabled me to take the next step:—viz. the approximate determination of the form of the Virial equation when molecular attraction is taken account of. Part IV. of this investigation, containing this application, was read to the Society on Jan. 21, 1889, and an Abstract has appeared in the *Proceedings*. It appears that the difference of average kinetic energy between a free, and an entangled, particle is of special importance in the physical interpretation of the Virial Equation.

An Appendix is devoted to the consideration of the modification which the previous results undergo when the coefficient of restitution is supposed to be less than 1. This extension of the investigation was intended as an approximation to the case of radiation from the particles of a gas, and the consequent loss of energy. But, so far as I have developed it, no results of any consequence were obtained. I met with difficulties of a very formidable order, arising mainly from the fact that the particles after impact do not always separate from one another. The full treatment of the impact of a single particle with a double one is very tedious; and the conditions of impact of two double particles are so complex as to be totally unfit for an elementary investigation like the present.

The remainder of the Appendix is devoted to two points, raised by Professors Newcomb and Boltzmann, respectively:—the first being the problem of distribution of

speed in the "special" state;—the other involving a second approximation to the estimates of Viscosity and Thermal Conductivity already given in Part II.

XV. *Special Assumption as to Molecular Force.*

§ 57. To simplify the treatment of the molecular attraction between two particles, let us make the assumption that the kinetic energy of their relative motion changes by a constant (finite) amount at the instant when their centres are at a distance a apart. This will be called an *Encounter*. There will be a refraction of the direction of their relative path, exactly analogous to that of the path of a refracted particle on the corpuscular theory of light. To calculate the term of the virial (§ 30) which corresponds to this, we must find

(a) The probability that the relative speed before encounter lies between u and $u+du$.

(b) The probability that its direction is inclined from θ to $\theta+d\theta$ to the line of centres at encounter.

(c) The magnitude of the encounter under these conditions, and its average value.

Next, to find the (altered) circumstances of impact, we must calculate

(d) The probability that an encounter, defined as above, shall be followed by an impact.

(e) The circumstances of the impact.

(f) The magnitude of the impact, and its average value per encounter.

In addition to these, we should also calculate the number of encounters per second, and the average duration of the period from encounter to final disentanglement, in order to obtain (from the actual speeds before encounter) the correction for the length of the free path of each. This, however, is not easy. But it is to be observed that, in all probability, this correction is not so serious as in the case when no molecular force is assumed. For, in that case the free path is *always* shortened; whereas, in the present case it depends upon circumstances whether it be shortened or lengthened. Thus, if the diameters of the particles be *nearly* equal to the encounter distance, there will in general be shortening of the paths, and consequent diminution of the time between successive impacts:—if the diameters be *small* in comparison with the encounter distance, the whole of the paths will be lengthened and the interval between two encounters may be lengthened or shortened. Thus if we assume an intermediate relation of magnitude, there will be (on the average) but little change in the intervals between successive impacts. Hence also the time during which a particle is wholly free will be nearly that calculated as in § 14, with the substitution, of course, of a for s.

XVI. *Average Values of Encounter and of Impact.*

§ 58. The number of encounters of a v, with a v_1, in directions making an angle β with one another, is by § 21 proportional to

$$\nu\nu_1 v_0 \sin\beta d\beta,$$

where

$$v_0^2 = v^2 + v_1^2 - 2vv_1\cos\beta$$

Hence the number of encounters for which the relative speed is from u to $u+du$ proportional to

$$u^2du\int\frac{\nu\nu_1}{vv_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The limits of v_1 are $v\pm u$, or $u\pm v$, according as $v\gtrless u$, and those of v are 0 to ∞, so that the integral is

$$\int_0^\infty\frac{\nu}{v}\int_{v-u}^{v+u}\frac{\nu_1}{v_1}=\int_0^\infty\frac{\nu}{2hv}\left(\varepsilon^{-h(v-u)^2}-\varepsilon^{-h(v+u)^2}\right)$$

$$=\frac{\varepsilon^{-hu^2/2}}{2h}\int_0^\infty vdv\left(\varepsilon^{-2h\left(v-\frac{u}{2}\right)^2}-\varepsilon^{-2h\left(v+\frac{u}{2}\right)^2}\right).$$

The first term of this integral may be written as

$$\int_{-\frac{u}{2}}^\infty\left(x+\frac{u}{2}\right)dx\varepsilon^{-2hx^2}.$$

and the second as

$$-\int_{\frac{u}{2}}^\infty\left(x-\frac{u}{2}\right)dx\varepsilon^{-2hx^2}.$$

Together, these amount to

$$\int_{-\frac{u}{2}}^{\frac{u}{2}}xdx\varepsilon^{-2hx^2}+u\int_0^\infty dx\varepsilon^{-2hx^2}.$$

The first term vanishes, and the second is

$$\frac{u}{2}\sqrt{\frac{\pi}{2h}}.$$

Thus the value of (1) is

$$\frac{u^3du}{4}\varepsilon^{-hu^2/2}\sqrt{\frac{\pi}{2h^3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But, on the same scale, the whole number of encounters in the same time is

$$\int\nu\nu_1v_0\sin\beta d\beta=\int\frac{\nu\nu_1}{vv_1}v_0^2dv_0=\frac{\mathrm{I}_3}{3}=\frac{1}{4}\sqrt{\frac{2\pi}{h^7}}.$$

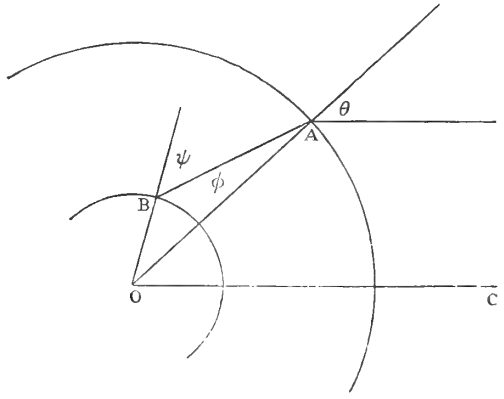
Thus the fraction of the whole encounters, which takes place with relative speed u to $u+du$, is

$$\frac{h^2}{2}u^3du\varepsilon^{-hu^2/2};$$

whose integral, from 0 to ∞, is 1 as it ought to be.

§ 59. Now these relative motions are before encounter distributed equally in all directions. Let us deal therefore only with those which are parallel to a given line. The final result will be of the same character relative to all such lines; and therefore the encounters will not disturb the even distribution of directions of motion.

Refer the motion to the centre, O, of one of the encountering particles. Let A be the point midway between the particles at encounter, B that of impact, the encountering particle coming parallel to CO. Let OA$=a/2$, OB (as before)$=s/2$. Let θ, ϕ be the



angles of incidence and refraction at encounter, ψ that of incidence at impact, u and w the relative speeds before and after the encounter. Then

$$u \sin\theta = w \sin\phi ;$$

and, if Pc^2 represent double the work done in the encounter by the molecular forces,

$$u^2 \cos^2\theta + c^2 = w^2 \cos^2\phi$$

so that

$$u^2 + c^2 = w^2 .$$

Also it is obvious from the diagram that

$$s \sin\psi = a \sin\phi = \frac{au}{w} \sin\theta .$$

Hence the encounter will not be followed by an impact if

$$\sin\theta > \frac{sw}{au} .$$

§ 60. We must next find the average value of an encounter, and also of an impact; in the latter case taking account of all the encounters whether or not they involve an impact.

The numerical value of the encounter-impulse in the above figure is evidently

$$P(w \cos\phi - u \cos\theta)/2 ,$$

which must be doubled to include the repetition on separation; and the average value, when the relative speed is u, is

$$2P\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta (w \cos\phi - u \cos\theta) d\theta$$

$$= \frac{2P}{3u^2}\left((c^2+u^2)^{\frac{3}{2}} - c^3 - u^3\right) \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The value of the subsequent impact is

$$-\mathrm{P}w\cos\psi,$$

and the average value

$$-2\mathrm{P}w\int\cos\theta\sin\theta\sqrt{1-\frac{a^2u^2}{s^2w^2}\sin^2\theta}d\theta.$$

When $sw>au$, the limits are 0 and $\frac{\pi}{2}$, and the value is

$$-\frac{2}{3}\mathrm{P}w\frac{s^2w^2}{a^2u^2}\left(1-\left(1-\frac{a^2u^2}{s^2w^2}\right)^{\frac{3}{2}}\right) \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But when $sw<au$, the limits are 0 and $\sin^{-1}\frac{sw}{au}$, and the value is

$$-\frac{2}{3}\mathrm{P}w\frac{s^2w^2}{a^2u^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

By (2) and (3) we find as the average value of the encounter, taking account of all possible relative speeds,

$$+\frac{\mathrm{P}}{3}h^2\int_0^\infty udu\,\varepsilon^{-hu^2/2}\left((c^2+u^2)^{\frac{3}{2}}-c^3-u^3\right),$$

$$=\frac{\mathrm{P}}{3}h^2\left\{\frac{\varepsilon^{hc^2/2}}{\sqrt{h^5}}\left(3\sqrt{\frac{\pi}{2}}-4\sqrt{2}\int_0^{c\sqrt{\frac{h}{2}}}\varepsilon^{-y^2}y^4dy\right)-\frac{c^3}{h}-3\sqrt{\frac{\pi}{2h^5}}\right\}$$

or, if we write for simplicity,

$$e^2=hc^2/2,$$

$$=\frac{\mathrm{P}}{3\sqrt{h}}\left\{\varepsilon^{e^2}\left(3\sqrt{\frac{\pi}{2}}-4\sqrt{2}\int_0^e\varepsilon^{-y^2}y^4dy\right)-2\sqrt{2}e^3-3\sqrt{\frac{\pi}{2}}\right\}$$

$$=\frac{\mathrm{P}}{\sqrt{h}}\left\{\sqrt{\frac{\pi}{2}}(\varepsilon^{e^2}-1)+e\sqrt{2}-\sqrt{2}\,\varepsilon^{e^2}\int_0^e\varepsilon^{-y^2}dy\right\}$$

$$=\frac{\mathrm{P}}{\sqrt{h}}\left\{\sqrt{2}\varepsilon^{e^2}\int_e^\infty\varepsilon^{-y^2}dy+e\sqrt{2}-\sqrt{\frac{\pi}{2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6).$$

The expression obviously vanishes, as it ought to do, when $e=0$. And it is always positive, for its differential coefficient with respect to e is

$$\frac{\mathrm{P}}{\sqrt{h}}2^{\frac{3}{2}}e\varepsilon^{e^2}\int_e^\infty\varepsilon^{-y^2}dy.$$

In a similar way (4) and (5) give, with (2), as the average impact *per encounter*,

$$\mathrm{R}=-\frac{\mathrm{P}}{3}\frac{s^2h^2}{a^2}\left\{\int_0^{\frac{sc}{\sqrt{a^2-s^2}}}w^3udu\varepsilon^{-hu^2/2}\left(1-\left(1-\frac{a^2u^2}{s^2w^2}\right)^{\frac{3}{2}}\right)+\int_{\frac{sc}{\sqrt{a^2-s^2}}}^\infty w^3udu\varepsilon^{-hu^2/2}\right\};$$

$$=-\frac{\mathrm{P}}{3}\frac{s^2h^2}{a^2}\left\{\int_0^\infty w^3udu\varepsilon^{-hu^2/2}-\int_0^{\frac{sc}{\sqrt{a^2-s^2}}}w^3udu\varepsilon^{-hu^2/2}\left(1-\frac{a^2u^2}{s^2w^2}\right)^{\frac{3}{2}}\right\}.$$

The first integral we have already had as part of the encounter. To simplify the second, let $s/a = \cos\alpha$, and it becomes

$$\int_0^{c\cot\alpha} u du \varepsilon^{-hu^2/2}(u^2+c^2-u^2\sec^2\alpha)^{\frac{3}{2}},$$

which, with $c^2-u^2\tan^2\alpha = z^2$, gives

$$\cot^2\alpha \int_0^c z^4 dz \varepsilon^{+\frac{h}{2}(z^2-c^2)\cot^2\alpha}$$

or

$$\left(\frac{2}{h}\right)^{\frac{5}{2}} \tan^3\alpha \quad \varepsilon^{-hc^2\cot^2\alpha/2} \int_0^{c\sqrt{\frac{h}{2}}\cot\alpha} x^4 dx \varepsilon^{x^2}.$$

The whole is now

$$\mathrm{R} = -\frac{\mathrm{P}}{3}h^2\cos^2\alpha \left\{ \frac{\varepsilon^{hc^2/2}}{\sqrt{h^5}}\left(3\sqrt{\frac{\pi}{2}} - 4\sqrt{2}\int_0^{c\sqrt{\frac{h}{2}}} y^4 dy \varepsilon^{-y^2}\right) - \left(\frac{2}{h}\right)^{\frac{5}{2}} \tan^3\alpha \varepsilon^{-\frac{hc^2}{2}\cot^2\alpha} \int_0^{c\sqrt{\frac{h}{2}}\cot\alpha} x^4 dx \varepsilon^{x^2} \right\}$$

$$= -\frac{\mathrm{P}\cos^2\alpha}{\sqrt{h}} \left\{ \varepsilon^{e^2}\sqrt{\frac{\pi}{2}} + \sqrt{2}e - \sqrt{2}\varepsilon^{e^2}\int_0^e \varepsilon^{-y^2} dy + \sqrt{2}e\tan^2\alpha - \sqrt{2}\varepsilon^{-e^2\cot^2\alpha}\tan^3\alpha \int_0^{e\cot\alpha} \varepsilon^{x^2} dx \right\}$$

$$= -\frac{\mathrm{P}}{\sqrt{h}}\cos^2\alpha \left\{ \varepsilon^{e^2}\sqrt{\frac{\pi}{2}} + \sqrt{2}e\sec^2\alpha - \sqrt{2}\varepsilon^{e^2}\int_0^e \varepsilon^{-y^2} dy - \sqrt{2}\varepsilon^{-e^2\cot^2\alpha}\tan^3\alpha \int_0^{e\cot\alpha} \varepsilon^{x^2} dx \right\},$$

which, when $e=0$ and $\cos\alpha = 1$, becomes

$$-\mathrm{P}\sqrt{\frac{\pi}{2h}}$$

as in § 30.

It would at first sight appear that the value of the impact is finite $\left(=-\mathrm{P}e\sqrt{\frac{2}{h}}\right)$ when there is *no nucleus* $\left(i.e.\ \alpha=\frac{\pi}{2}\right)$. But, in such a case, we must remember that the second part of the first expression for R above has no existence. In fact the value of the second of the two integrals is $\sqrt{2}\tan^3\alpha . e\cot\alpha$, when $e\cot\alpha$ is small; and this destroys the apparently non-vanishing term.

XVII. *Effect of Encounters on the Free Path.*

§ 61. If two particles of equal diameters impinge on one another, the *relative* path must obviously be shortened on the average by

$$s\frac{\int_0^{\frac{\pi}{2}} 2\pi\sin\theta\cos^2\theta d\theta}{\int_0^{\frac{\pi}{2}} 2\pi\sin\theta\cos\theta d\theta} = \frac{2s}{3},$$

But if v, v_1 be their speeds, and v_0 their relative speed, the paths are shortened respectively by the fractions v/v_0 and v_1/v_0 of this. The average values must be equal, so that we need calculate one only.

Now the average value of v/v_0 is obviously

$$\frac{\int vv_1 v \sin\beta d\beta}{\int vv_1 v_0 \sin\beta d\beta};$$

where β is the angle between the directions of motion, so that

$$vv_1 \sin\beta d\beta = v_0 dv_0.$$

Hence the average above is

$$\frac{\int \frac{vv_1 v_0 dv_0}{v_1}}{\int \frac{vv_1 v_0^2 dv_0}{vv_1}} = \frac{2\int vv_1 v}{I_3/3} = \frac{\frac{\sqrt{\pi}}{4}h^{-\frac{7}{2}}}{\frac{\sqrt{2\pi}}{4}h^{-\frac{7}{2}}} = \frac{1}{\sqrt{2}}$$

Hence the mean of the free paths during a given period becomes

$$\frac{1}{\sqrt{2}n\pi s^2} - \frac{\sqrt{2}s}{3};$$

that is, it is shortened in the ratio

$$1 - \frac{2}{3}n\pi s^3 : 1$$

or

$$1 - 4\,(\text{sum of vols. of spheres in unit vol.}) : 1 = 1 - \frac{a}{V} : 1 \text{ say.}$$

Hence the number of collisions per second, already calculated, is too small in the same ratio.

Thus the value of $\Sigma(R)$ in § 30 must be increased in the ratio $1 : 1 - \frac{a}{V}$, and the virial equation there given becomes

$$n\overline{Pv^2}/2 = \frac{3}{2}p\left(V - \frac{a}{1 - \frac{a}{V}}\right).$$

If this were true in the limit, the ultimate volume would be double of that before calculated, *i.e.* 8 times the whole volume of the particles.

§ 62. Another mode of obtaining the result of § 61 is to consider the particles as mere points, and to find the average interval which elapses between their being at a distance s from one another and their reaching the positions where their mutual distance is least. The space passed over by each during that time will have to be subtracted from the length of the *mean free* path calculated as in § 11 when the particles were regarded as mere circular discs.

The average interval just mentioned is obviously

$$\frac{1}{u}\frac{\int_0^{\frac{\pi}{2}} s\cos\theta \,.\, \sin\theta\cos\theta d\theta}{\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta} = \frac{2s}{3u}.$$

Hence the average space passed over in that interval is

$$\frac{2s}{3u}\int vv\,.\,\frac{v_1v_0^2}{vv_1}\,dv\,.\,\Big/\frac{\mathrm{I}_3}{3}=\frac{\sqrt{2}\,s}{3}\,.$$

If we put a for s in this expression we have the amount to be subtracted from the average path between two encounters in consequence of the finite size of the region of encounter.

XVIII. *Average Duration of Entanglement, and consequent Average Kinetic Energy.*

§ 63. We have next to find the average duration of entanglement of two particles :— *i.e.*, the interval during which their centres are at a distance less than a.

The whole relative path between the entering and leaving encounters is

$$2(a\cos\phi-s\cos\psi)\,,$$

or

$$2a\cos\phi\,,$$

according as there is, or not, an impact.

Hence the whole time of entanglement is the quotient, when one or other is divided by w. And the average value, for relative speed u, is

$$\tau=\frac{4}{w^2}\int_0^{\frac{\pi}{2}}\Big(a\sqrt{w^2-u^2\sin^2\theta}-\sqrt{w^2s^2-a^2u^2\sin^2\theta}\Big)\cos\theta\sin\theta d\theta,$$

$$=\frac{4}{3w^2}\left\{\frac{a}{u^2}(w^3-c^3)-\frac{1}{a^2u^2}\Big(w^3s^3-(w^2s^2-a^2u^2)^{\frac{3}{2}}\Big)\right\}$$

when $$ws>au;$$

and

$$=\frac{4}{w^2}\left\{\int_0^{\frac{\pi}{2}}a\sqrt{w^2-u^2\sin^2\theta}\cos\theta\sin\theta d\theta-\int_0^{\sin^{-1}\frac{ws}{au}}\sqrt{w^2s^2-a^2u^2\sin^2\theta}\cos\theta\sin\theta d\theta\right\},$$

$$=\frac{4}{3w^2}\Big(\frac{a}{u^2}(w^3-c^3)-\frac{1}{a^2u^2}w^3s^3\Big),$$

when $$ws<au\,.$$

These must be multiplied by the chance of relative speed u, as in § 58, and the result is

$$\frac{2h^2}{3}\left\{\int_0^\infty\frac{udu}{w^2}\Big(a(w^3-c^3)-\frac{w^3s^3}{a^2}\Big)\varepsilon^{-hu^2/2}+\frac{1}{a^2}\int_0^{\frac{cs}{\sqrt{a^2-s^2}}}\frac{udu}{w^2}(w^2s^2-a^2u^2)^{\frac{3}{2}}\varepsilon^{-hu^2/2}\right\};$$

or, with the notation of § 60,

$$=\frac{2ah^2}{3}\varepsilon^{hc^2/2}\left\{\int_c^\infty\frac{dw}{w}\Big(w^3(1-\cos^3\alpha)-c^3\Big)\varepsilon^{-hw^2/2}+\int_c^{c\,\mathrm{cosec}\,\alpha}\frac{dw}{w}\Big(c^2-w^2\sin^2\alpha\Big)^{\frac{3}{2}}\varepsilon^{-hw^2/2}\right\}$$

$$=\frac{2ah^2}{3}\varepsilon^{hc^2/2}\int_c^\infty\frac{dw}{w}\Big(w^3(1-\cos^3\alpha)-c^3\Big)\varepsilon^{-hw^2/2}+\frac{2ah^2}{3}\varepsilon^{-\frac{hc^2}{2}\cot^2\alpha}\int_0^{c\cos\alpha}\frac{z^4dz}{c^2-z^2}\varepsilon^{+hz^2\,\mathrm{cosec}^2\alpha/2}$$

As the value of this expression depends in no way on the length of the free path, it is clear that the average energy of all the particles is greater than that of the free particles, by an amount which increases rapidly as the length of the free path is diminished.

APPENDIX.

A. *Coefficient of Restitution less than Unity.*

Let us form again the equations of § 19, assuming e to be the coefficient of restitution. We have

$$P(u'-u)=-\frac{PQ(1+e)}{P+Q}(u-v)=-Q(v'-v),$$

so that

$$P(u'^2-u^2)=-\frac{PQ(1+e)}{(P+Q)^2}(u-v)\big((2P+Q\overline{1-e})u+Q(1+e)v\big)$$

$$Q(v'^2-v^2)=\frac{PQ(1+e)}{(P+Q)^2}(u-v)\Big(P(1+e)u+\big(2Q+P(1-e)\big)v\Big).$$

The whole energy lost in the collision is half the sum of these quantities, viz.,

$$-\tfrac{1}{2}\frac{PQ(1-e)^2}{P+Q}(u-v)^2.$$

With the help of the expressions in § 22, we find for the average changes of energy of a P and of a Q, respectively,

$$\tfrac{1}{2}P(\overline{u'^2}-\overline{u^2})=-\frac{PQ(1+e)}{2hk(P+Q)^2}\big(2(Pk-Qh)+Q(1-e)(h+k)\big)$$

$$\tfrac{1}{2}Q(\overline{v'^2}-\overline{v^2})=\frac{PQ(1+e)}{2hk(P+Q)^2}\big(2(Pk-Qh)+P(1-e)(h+k)\big).$$

The first term on the right is energy exchanged between the systems; and, as in the case of $e=1$, it vanishes when the average energy per particle is the same in the two systems. The second term (intrinsically negative for each system) is the energy lost, and is always greater for the particles of smaller mass. The average energy lost per collision is

$$\frac{PQ(1-e^2)}{2(P+Q)}\left(\frac{1}{h}+\frac{1}{k}\right).$$

It is easy to make for this case an investigation like that of § 23. But we must remember that there is loss of energy by the internal impacts of each system, which must be taken into account in the formation of the differential equations. This is easily found from the equations just written, by putting $Q=P$:—but the differential equations become more complex than before, and do not seem to give any result of value. [Shortly

after Part I. was printed off, Prof. BURNSIDE called my attention to the fact that the equations of interchange of energy in § 23 are easily integrable without approximation. But the approximate solution in the text suffices for the application made.]

B. *The Law of Distribution of Speed.*

In addition to what is said on this subject in the Introduction to Part II., it may be well to take the enclosed (from *Proc. R. S. E.*, Jan. 30, 1888).

"The behaviour parallel to y and z (though not the number) of particles whose velocity-components are from x to $x+dx$, must *obviously* be independent of x, so that the density of 'ends' in the velocity space diagram is of the form

$$f(x)\ \mathrm{F}(y, z).$$

The word I have underlined may be very easily justified. No collisions count, except those in which the line of centres is practically perpendicular to x (for the others each dismiss a particle from the minority; and its place is instantly supplied by another, which behaves exactly as the first did), and therefore the component of the relative speed *involved in the collisions which we require to consider* depends wholly on y and z motions. Also, for the same reason, the frequency of collisions of various kinds (so far as x is concerned) does not come into question. Thus the y and z speeds, not only in one x layer but in all, are entirely independent of x; though the *number* of particles in the layer depends on x alone."

C. *Viscosity.*

In my "Reply to Prof. BOLTZMANN" I promised to give a further approximation to the value of the coefficient of Viscosity, by taking account of the alteration of permeability of a gas which is caused by (slow) shearing disturbance. I then stated that a rough calculation had shown me that the effect would be to change my first, *avowedly approximate,* result by 11 or 12 per cent. only. I now write again the equations of § 36, modifying them in conformity with the altered point of view.

The exponential expression in that section for the number of particles crossing the plane of yz, must obviously now be written

$$\epsilon^{-\sec\theta\int_0^x ev_0 d\xi/v}\sin\theta d\theta/2,$$

where v_0 is the velocity *relative* to the absorbing layer at ξ, and e also is no longer constant. But we have at once

$$v_0 = v + \mathrm{B}\xi\sin\theta\cos\phi,$$

so that the exponent above is

$$-\frac{\sec\theta}{v}\int_0^x (ev + (ev)'\mathrm{B}\xi\sin\theta\cos\phi)d\xi.$$

Thus the differential of the whole y-momentum which comes to unit surface on $x=0$ from the layer x, $x+dx$, is

$$\frac{d\phi}{4\pi}\mathrm{P}nvev\varepsilon^{-ex\sec\theta}\left(1-\frac{(ev)'}{2v}\mathrm{B}x^2\frac{\sin\theta\cos\phi}{\cos\theta}\right)\left(v\sin\theta\cos\phi+\mathrm{B}x\right)\sin\theta d\theta\,.$$

Integrating with respect to ϕ from 0 to 2π, to x from 0 to ∞, and to θ from 0 to $\frac{\pi}{2}$, and doubling the result, we have

$$\mathrm{BP}n\int_0^\infty vv\left(\frac{1}{3e}-\frac{(ev)'}{15e^2}\right).$$

The first term expresses my former result, viz.

$$\frac{\mathrm{BPC}_1}{3\pi s^2\sqrt{h}}.$$

But the whole is

$$\frac{\mathrm{BP}n}{15}\int_0^\infty vv\left(\frac{4}{e}-\frac{ve'}{e^2}\right)=\frac{2\mathrm{BP}n}{15}\int_0^\infty\frac{\nu hv^3}{e}=\frac{2\mathrm{BPC}_3}{15\pi s^2\sqrt{h}}.$$

The ratio is $2\mathrm{C}_3/5\mathrm{C}_1=3{\cdot}704/4{\cdot}19=0{\cdot}882$.

It is worthy of remark that the term

$$\int_0^\infty vv\frac{(ev)'}{15e^2}$$

has the value

$$\frac{5\mathrm{C}_1-2\mathrm{C}_3}{15n\pi s^2\sqrt{h}},$$

and that 4/5ths of the C_1 term are due to e'.

D. *Thermal Conductivity.*

Applying a process, such as that just given, to the expressions in § 39, we find that the exponential in the integral for the *number* of particles must be written

$$\varepsilon^{-e_0x\sec\theta+\frac{\alpha'_0x^2e_0}{2v}-\frac{e'_0x^2}{2}\sec\theta}=\varepsilon^{-e_0x\sec\theta}\left(1-e'_0x^2\sec\theta/2v+\frac{e_0\alpha'x^2}{2}\right)$$

to the required degree of approximation. [Properly, the superior limit of the θ integration should be $\cos^{-1}\frac{a}{v}$; but this introduces quantities of the order a^2 only.] Thus equation (1) becomes

$$a=\int_0^\infty\nu\left(\frac{n'}{n}+\frac{\nu'}{\nu}\right)v/3e-\frac{a'}{4}\int\frac{\nu}{e}.$$

In the same way equation (3) of § 41 becomes

$$\mathrm{E}=-\frac{\mathrm{P}}{6}\int_0^\infty nvv^3\left(\left(\frac{n'}{n}+\frac{\nu'}{\nu}\right)/e-5a/v-9a'/4ev\right).$$

Thus equations (1′) and (3′) of § 42 become, respectively,

$$a=\frac{h'}{\sqrt{h^5}}\frac{\mathrm{P}}{6\pi\rho s^2}\left(\frac{5}{2}\mathrm{C}_1-\mathrm{C}_3\right)-\frac{\mathrm{PC}_0a'}{8\pi\rho hs^2},$$

and

$$\mathrm{E}=\frac{h'}{\sqrt{h^5}}\frac{\mathrm{P}}{6\pi s^2}\left(\frac{25}{4}\mathrm{C}_1-5\mathrm{C}_3+\mathrm{C}_5\right)+\frac{3\mathrm{P}}{8}\frac{\mathrm{C}_2a'}{\pi hs^2}-\frac{5\mathrm{P}}{16}\frac{\mathrm{C}_0a'}{\pi hs^2}.$$

Thus we have finally to deal with the new forms of (1″) and (3″) of § 43, viz. :—

$$a=\frac{h'}{\sqrt{h^5}}\frac{\rho\lambda}{p}0{\cdot}06-\frac{\rho\lambda a'}{ph}0{\cdot}12,$$

$$\mathrm{E}=\frac{h'}{\sqrt{h^5}}\rho\lambda 0{\cdot}45+\frac{\rho\lambda a'}{h}0{\cdot}44.$$

When similar methods are applied to the diffusion equations, they become hopelessly complicated.

XXIII.—*On Systems of Solutions of Homogeneous and Central Equations of the* nth *Degree and of two or more Variables; with a Discussion of the Loci of such Equations.* By the Hon. Lord M'Laren. (Plates I.–VI.)

(Read 6th July 1888.)

The purpose of the present paper is to ascertain how far it is possible to find exact solutions or values of x, y, &c., in equations between variables, so that the forms of plane curves and contour-lines of surfaces may be exactly determined. No approximate methods have been admitted, and only those methods have been used which are applicable to algebraic equations of every degree and any number of variables. In the examples given I have generally selected equations of even degree and even powers of the variables. But every such solution evidently includes the solution of the non-central equation of half the degree having corresponding terms and equal coefficients. The methods of solution employed are founded on the following introductory theorem or principle, which may be described as that of Homogeneous or Linear Variation of the quantities.

The paper, as laid before the Royal Society in July 1888, embraced only the solution of homogeneous equations in which one of the quantities was given explicitly in terms of the others. The preparation of the paper for the press having been interrupted by my absence abroad for a considerable time, I have resumed the investigation from a more general point of view.

1. *Principle of Homogeneous or Linear Variation of a Function.*

If $F(\alpha, \beta, \gamma \ldots)^n = 0$ be a homogeneous function of the n^{th} degree of any number of quantities, $\alpha, \beta, \gamma, \ldots$; and if $\alpha_1, \beta_1, \gamma_1$, &c., be known values of these quantities satisfying the equation, then may another set of values, $\alpha_2, \beta_2, \gamma_2$, satisfying the equation, be found by multiplying or dividing each term by any desired factor, m^n.

Let the function consist of a series of homogeneous terms of powers of the quantities $\alpha, \beta, \gamma, \ldots$ multiplied by coefficients $p_1\ q_1 \ldots$ and equated to zero. Let $\alpha_1, \beta_1, \gamma_1$ be values satisfying the equation; which accordingly will be of the form

$$\alpha^n + p_1\alpha_1^{n-1}\beta_1 + p_2\alpha_1^{n-1}\beta^2 + \ldots q_0\beta_1^n + q_1\beta_1^{n-1}\gamma_1 + q_2\beta_1^{n-2}\gamma_1^2 + \ldots + r_r\alpha_1^{n-p-q}\beta_1^p\gamma_1^q + \ldots = 0;$$

where the last term is the generalised term for three quantities.

To find a new series of values satisfying the equation, we have only to multiply every term by the same numerical quantity, m^n. The equation is, of course, unaltered in value, and is now of the form

$$(m\alpha_1)^n + p_1(m\alpha_1)^{n-1}(m\beta_1) + p_2(m\alpha_1)^{n-2}(m\beta)^2 + \ldots q_0(m\beta_1)^n + q_1(m\beta_1)^{n-1}(m\gamma_1) + r_r(m\alpha_1)^{n-p-q}(m\beta_1)^p(m\gamma_1)^q + \ldots = 0,$$

where the term in the second line is the generalised form of a term resulting from the multiplication of the function by m^n.

By writing α_2 for $m\alpha_1$, β_2 for $m\beta_1$, γ_2 for $m\gamma_1$, &c., the equation is restored to its original form, with a new set of values, $\alpha_2, \beta_2, \gamma_2$, of these indeterminate quantities satisfying the equation. Comparing the two sets of values, we find the relation

$$\frac{\alpha^2}{\alpha_1} = \frac{\beta^2}{\beta_1} = \frac{\gamma_2}{\gamma_1} = m,$$

which was to be proved.

The preceding proof evidently includes the cases of negative, reciprocal, and fractional indices.

In the preceding theorem it is not assumed that all the quantities $\alpha, \beta, \gamma, \ldots$ are variables; and the proof is evidently the same, whether all the quantities are conceived as being subject to indefinite variation, or whether some of them are conceived as having only certain definite values from which values of the other quantities are to be obtained. For example, if $\alpha_1\beta_1$ are variable coordinates, and γ_1 is a parameter, the set of values $\alpha_1, \beta_1, \gamma_1$ represents a point on a plane curve of the nth degree having the parameter γ_1, and the set of values $\alpha_2, \beta_2, \gamma_2$ represents a corresponding point on a similar curve whose parameter is γ_2. But if the three quantities α, β, γ are all conceived as being subject to indefinite variation, γ being then a third coordinate, the function represents a conical surface of the nth degree, and the two sets of values then represent corresponding points on parallel, and therefore similar, plane sections of this surface.

Again, certain of the quantities may represent the coordinates of a point on a central

plane curve or central surface, while others of these quantities may represent the coordinates of the centre. If now the function represents a central surface whose centre is variable in position, we may have seven quantities, whereof one is a parameter, and six are variables. When the number of variables exceeds three there must, of course, be other relations between the coordinates, otherwise the problem becomes indeterminate. In the case supposed, a second equation between the coordinates of the centre and one of the coordinates of the surface determines the curve or surface which the centre is supposed to describe, and supplies the necessary elements for the solution of the first equation. I have introduced this illustration because every homogeneous equation of even degree of three or four quantities represents a central curve or central surface respectively referred to the centre; and it is easily seen that, if the origin be changed to any point, whether exterior or interior to the curve, the left-hand side of the resulting equation is a homogeneous function of the original coordinates, and the coordinates of the centre.

2. *Application of the Principle to finding Solutions of Homogeneous Equations of one part.* (Case I.)

The most obvious application of the method of homogeneous variation is to the exact determination of a series of points on a curve or surface whose equation is given in the form of a homogeneous function equated to an arbitrary term. The method, however, is purely analytical, and it is not necessary that the quantities should have a geometrical interpretation. The arbitrary term is to be expressed as the n^{th} power of a number w, and the equation is then of the form—

$$x^n + A_1 x^{n-1} y + A_2 x^{n-2} y^2 + \ldots + A_n y^n = w^n .$$

The quantity w is evidently a parameter, being the value of x when $y = 0$. It is required to find a series of exact values of x and y to the given parameter w. The values to be found may be denoted by $x_1, y_1, x_2, y_2 \ldots$ Let ξ_1, η_1 be any values arbitrarily assumed; these values are to be inserted in the given function, and the value of the parameter computed by summing the terms and extracting the nth root of the sum. The equation formed may be called auxiliary equation (1); and may be written—

$$\xi_1^n + A_1 \xi_1^{n-1} \eta_1 + A_2 \xi_1^{n-2} \eta_1^2 + \ldots + A_n \eta_1^n = w_1^n .$$

Then by the preceding theorem we have the relation $x_1/\xi_1 = y_1/\eta_1 = w/w_1$, which gives for the coordinates of the first point (or first set of values of the original equation) $x_1 = \xi_1 w/w_1$; $y_1 = \eta_1 w/w_1$.

A second auxiliary equation being formed from new assumed values $\xi_2 \eta_2$, and the parameter w_2 computed, we find from these data the coordinates of a second point (or second set of values of the original equation), viz., $x_2 = \xi_2 w/w_2$; $y_2 = \eta_2 w/w_2$, and so on. These are true *algebraic* solutions of the given equation.

This method of finding solutions of indeterminate quantities is hereafter referred to as the method of Homogeneous Variation, because all the quantities are varied proportionately in order to obtain a new series of values.

Although only two variables, x and y, are here expressed, the explanation of the method of solution is intended to cover the case of an equation of three or more variables. In order to simplify the illustrations as much as possible, I shall generally suppose two variable quantities x and y; or $r\cos\theta$ and $r\sin\theta$; w is then the parameter or intercept on the axis of X. It is convenient to take this quantity = unity, which can always be done by dividing out.

In order that the series of points to be found may correspond to equal angular intervals, it is best to assume $\xi_1..$ and $\eta_1..$ equal respectively to the cosine and sine of an angle. Then $x_1...$ $y_1...$ are proportional to the same cosine and sine, and are the rectangular coordinates of the curve to the argument θ.

Example 1.

$$x^4+2x^3y+3x^2y^2+4xy^3+2y^4=w^4=1.$$

For the sake of clearness, I shall, in this example only, dispense with the use of logarithmic tables, and find two values of x and y from auxiliary equations in which the assumed values are whole numbers.

(1) Let $\xi_1=1$; $\eta_1=1$. The sum of the terms of the auxiliary equation is 12; $\therefore w_1=12^{1/4}$; $x_1=\xi/w_1=\frac{1}{12^{1/4}}=y_1$.

This may be verified as follows:—Let x_1 and y_1 have the values here found. Then taking the terms of the equation in their order,

$$x^4=\left(\frac{1}{12^{1/4}}\right)^4=\frac{1}{12};\quad 2x^3y=2\left(\frac{1}{12^{1/4}}\right)^3\left(\frac{1}{12^{1/4}}\right)=\frac{2}{12},$$

and so on; and the terms are as under—

x^4	$2x^3y$	$3x^2y^2$	$4xy^3$	$2y^4$
$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{2}{12}$

Sum of the terms $=\frac{1+2+3+4+2}{12}=1$, as it should be.

(2) Let $\xi_2=1$; $\eta_2=2$; the sum of the terms of the auxiliary equation is

$$1+4+12+32+32=81;\ \therefore w_2=81^{1/4}=3;\ x_2=\xi_2/w_2=\frac{1}{3};\ y_2=\eta/w_2=\frac{2}{3}.$$

This solution may be verified in the same way as the preceding without the use of logarithms.

In the next example I shall take the assumed quantities $\xi_{1,2}\ldots\ \eta_{1,2}\ldots$ from the trigonometrical tables; so that $\log\xi_1=\log\cos\theta_1$; $\log\eta_1=\log\sin\theta_1$; and so on, and thence determine $x_{1,2}\ldots\ y_{1,2}\ldots$ for parameter $=1$.

Example 2.

$$x^{10}+x^8y^2+x^6y^4+y^{10}=w^{10}=1\,.$$

Values of x and y will be found to the arguments, $\theta=20^\circ$; $\theta=30^\circ$.

1. Let $\theta_1=20^\circ$; $\xi_1=\cos 20^\circ$; $\eta_1=\sin 20^\circ$; $x_1=\xi_1/w_1$; $y_1=\eta_1/w_1$.

Log cos 20° =	$\bar{1}\cdot9730$	Nat. numbers.	$\log\cos^8=1\cdot7840$
$\log\cos^6$	$\bar{1}\cdot8380$		$\log\sin^2+\bar{1}\cdot0682$
$\log\cos^8$	$\bar{1}\cdot7840$	$\xi^8\eta^2=0\cdot0712$	$2\cdot8522$
$\log\cos^{10}$	$\bar{1}\cdot7300$	$\xi^{10}\quad 0\cdot5370$	$=\log\xi^8\eta^2$
Log sin 20° =	$\bar{1}\cdot5341$		$\log\cos^6=1\cdot8380$
$\log\sin^2$	$\bar{1}\cdot0682$		$\log\sin^4+\bar{1}\cdot1364$
$\log\sin^4$	$\bar{2}\cdot1364$	$\xi^6\eta^4\quad 0\cdot0094$	$\bar{3}\cdot9744$
$\log\sin^{10}$	$\bar{5}\cdot3410$	η^{10} (insensible)	
$\log w^{10}$	$=1\cdot7907$	$w^{10}=0\cdot6176$	$=\log\xi^6\eta^4$
$\log w$	$1\cdot9791$		

$\log\xi_1=$	$\bar{1}\cdot9730$	$\log\eta_1=$	$1\cdot5341$
$-\log w_1$	$1\cdot9791$	$-\log w_1$	$1\cdot9791$
$\log x_1$	$1\cdot9939$	$\log y_1$	$1\cdot5550$

2. Let $\theta=30^\circ$; $\xi_2=\cos 30^\circ$; $\eta_2=\sin 30^\circ$; $x_2=\xi_2/w_2$; $y_2=\eta_2/w_2$.

Log cos 30° =	$\bar{1}\cdot9375$	Nat. numbers.	$\log\cos^8=1\cdot5000$
$\log\cos^6$	$\bar{1}\cdot6250$		$\log\sin^2+1\cdot3980$
$\log\cos^8$	$\bar{1}\cdot5000$	$\xi^8\eta^2=0\cdot0791$	$\bar{2}\cdot8980$
$\log\cos^{10}$	$\bar{1}\cdot3750$	$\xi^1\quad 0\cdot2371$	$=\log\xi^8\eta^2$
Log sin 30°	$\bar{1}\cdot6990$		$\log\cos^6\ \bar{1}\cdot6250$
$\log\sin^2$	$\bar{1}\cdot3980$		$\log\sin^4+\bar{2}\cdot7960$
$\log\sin^4$	$\bar{2}\cdot7960$	$\xi^6\eta^4\quad 0\cdot0264$	$\bar{2}\cdot4210$
$\log\sin^{10}$	$\bar{4}\cdot9900$	η^{10} (insensible)	
$\log w^{10}$	$\bar{1}\cdot5348$	$w^{10}=0\cdot3426$	$=\log\xi^6\eta^4$
$\log w$	$1\cdot9534$		

$\log\xi_2=$	$1\cdot9375$	$\log\eta_2=$	$1\cdot6990$
$-\log w_2$	$1\cdot9534$	$\log w_2$	$1\cdot9534$
$\log x_2$	$1\cdot9841$	$\log y_2$	$1\cdot7456$

Results $\begin{cases} x_1=\cdot9860\,;\ y_1=\cdot3589\,; \\ x_2=\cdot9640\,;\ y_2=\cdot5567\,. \end{cases}$

The following independent analytical proof of the general theorem, including its

extension to any number of variable quantities, was communicated to me by Dr Thomas Muir after reading the first sketch of this paper:—

Since $x=a$, $y=b$ is manifestly a solution of the equation

$$A_0x^n+A_1x^{n-1}y+\ldots+A_ny^n=A_0a^n+A_1a^{n-1}b+\ldots+A_nb_n,$$
$$=p \text{ say},$$

then

$$x=\frac{a}{p^{\frac{1}{n}}},\quad y=\frac{b}{p^{\frac{1}{n}}}$$

is a solution of the equation

$$A_0x^n+A_1x^{n-1}y+\ldots+A_ny^n=1.$$

For, substituting $a/p^{\frac{1}{n}}$, $b/p^{\frac{1}{n}}$ for x, y, the left-hand side becomes

$$A_0\cdot\frac{a^n}{p}+A_1\frac{a^{n-1}b}{p}+\ldots+A_n\frac{b^n}{p};\ i.e.,\ \frac{p}{p}=1.$$

This proof, as well as that formerly given, is applicable to functions of any number of variables. For example, the equation

$$x^4+y^4+6y^3z+7xyz^2=1,$$

has the algebraic solution

$$x=\frac{a}{\sqrt[4]{a^4+b^4+6b^3c+7abc^2}},$$

$$y=\frac{b}{\sqrt[4]{a^4+b^4+6b^3c+7abc^2}},$$

$$z=\frac{c}{\sqrt[4]{a^4+b^4+6b^3c+7abc^2}}.$$

And quite generally we can formulate as follows:—

If ϕ be a homogeneous function of the n*th degree in* r *variables, the equation*

$$\phi(x_1,x_2,x_3,\ldots,x_r)=1$$

has the algebraic solution

$$x_1=\frac{a_1}{\sqrt[n]{\phi(a_1,a_2,\ldots,a_r)}},$$

$$x_2=\frac{a_2}{\sqrt[n]{\phi(a_1,a_2,\ldots,a_r)}},$$

$$x_3=\frac{a_3}{\sqrt[n]{\phi(a_1,a_2,\ldots,a_r)}},$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$x_r=\frac{a_r}{\sqrt[n]{\phi(a_1,a_2,\ldots,a_r)}}.$$

3. *Solution of Equations consisting of two Homogeneous parts of Different Degrees.* (Case II.)

These equations represent a class of central curves or surfaces essentially different from the preceding. In the case of equations of even degree the curve $F_n(x, y) = A^n$ cannot pass through the centre; while curves which are of the form $F_n(x, y) = F_{n-p}(x, y)$, when reduced to the lowest terms, generally pass through the centre, because their equations are satisfied by the values $x=0$; $y=0$.

Equations of the form $F_n(x, y)=F_{n-p}(x, y)$ are really *homogeneous*, or, at least, are reducible to homogeneous form by division.

Take, for example, the equation

$$Ax^5+Exy^4=Q(Hx^3+Ky^3).$$

By division we have

$$\frac{Ax^5+Exy^4}{Hx^3+Ky^3}=Q,$$

where the left-hand side is homogeneous, and of the 2nd degree. Consequently, by the results of the preceding analysis (sections 1 and 2), a solution is

$$\left.\begin{aligned} x&=a.\sqrt{\frac{Q(Ha^3+Kb^3)}{Aa^5+Eab^4}}\\ y&=b.\sqrt{\frac{Q(Ha^3+Kb^3)}{Aa^5+Eab^4}}\end{aligned}\right\},$$

where a and b are any quantities whatever. This may be directly verified as follows:—

Calling the expression under the root-sign ω and substituting, we have

$$\begin{aligned} Ax^5+Exy^4&=Aa^5(\sqrt{\omega})^5+Eab^4(\sqrt{\omega})^5\\ &=(Aa^5+Eab^4)(\sqrt{\omega})^5\end{aligned}$$

and

$$\begin{aligned} Q(Hx^3+Ky^3)&=Q\left(Ha^3(\sqrt{\omega})^3+Kb^3(\sqrt{\omega})^3\right)\\ &=Q(Ha^3+Kb^3)(\sqrt{\omega})^3.\end{aligned}$$

Hence

$$\begin{aligned} \frac{Ax^5+Exy^4}{Q(Hx^3+Ky^3)}&=\frac{(Aa^5+Eab^4)(\sqrt{\omega})^5}{Q(Ha^3+Kb^3)(\sqrt{\omega})^3}\\ &=\frac{Aa^5+Eab^4}{Q(Ha^3+Kb^3)}\cdot\omega\\ &=\frac{Aa^5+Eab^4}{Q(Ha^3+Kb^3)}\cdot\frac{Q(Ha^3+Kb^3)}{Aa^5+Eab^4}\\ &=1, \text{ as it should be.}\end{aligned}$$

This quasi-extension of the original theorem may be formulated as follows:—

If ϕ be a homogeneous function of the n^{th} degree in r variables, and ψ a homogeneous function of the $(n-p)^{th}$ degree in r variables, the equation

$$\phi(x_1,x_2,\ldots,x_r)=\psi(x_1,x_2,\ldots,x_r)$$

has the solution

$$x_1=a_1\sqrt[p]{\frac{\psi(a_1,\ldots,a_r)}{\phi(a_1,\ldots,a_r)}}$$

$$x_2=a_2\sqrt[p]{\frac{\psi(a_1,\ldots,a_r)}{\phi(a\ ,\ldots,a_r)}}$$

$$x_3=\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

The cases here examined evidently include the following forms:—

$$F_n(x,\,y)=Ax^{n-p};\ F_n(x,\,y)=Ax^{n+p}$$

and

$$F_n(x,\,y)=z^p,$$

where z^p is a soluble function of other quantities, whose numerical value can be found and stated as a power of z.

4. *Another mode of Solution; viz., by expressing each of the* r *Variables in terms of* r—1 *New Variables.*

(1) Where there are only two variables X and Y, we have the relations $Y=X\tan\theta$; $X=Y\operatorname{cotan}\theta$, from which by substitution and division we may at once write the transformed equations of the homogeneous function $f(x,\,y)^n=1$,

$$1\ +a_1\tan\theta\qquad +a_2\tan^2\theta\qquad +\ldots=\frac{1}{X^n}\quad .\quad .\quad .\quad (1)$$

$$\operatorname{cotan}^n\theta+\ +a_1\operatorname{cotan}^{n-1}\theta\ +a_2\operatorname{cotan}^{n-2}\theta\quad +\ldots=\frac{1}{Y^n}\quad .\quad .\quad .\quad (2)$$

Supposing a series of values of X to be formed from (1), and tabulated for the argument θ, then the column of values of Y is found by adding to each value of log X the corresponding log tan θ.

(2) When there are three or more variables α, β, γ, &c., they may, in like manner, be all expressed as functions of one of them, α, and new quantities. For this purpose, assume $\beta=\alpha\tan\phi$; $\gamma=\alpha\tan\psi$, &c., or more generally, $\beta=l\alpha$; $\gamma=m\alpha$; $\delta=n\alpha$, &c. (3)

Substituting these values, and dividing by α^n, the transformed equation will then consist of a series of powers of l, m, n, &c., equated to $1/\alpha^n$. Values of α may then be directly computed for any arguments or assumed values of l, m, n; and the other quantities, β, γ, δ, &c., are formed from (3).

The manner of doing this is shown by the following examples. (1) Let the equation be

$$x^8+x^4y^2z^2+3x^2y^2z^4+y^8=w^8=1\quad .\quad .\quad .\quad .\quad .\quad (4)$$

x is the quantity of which values are to be directly found; θ is the angular coordinate in

the plane xy; ϕ is the angular coordinate in the plane xz; $y=x\tan\theta$; $z=x\tan\phi$; and the transformed equation is

$$x^8\{1+\tan^2\theta.\tan^2\phi+3\tan^2\theta.\tan^4\phi+\tan^8\theta\}=1$$

$$\therefore\ \frac{1}{x}=\{1+\tan^2\theta\tan^2\phi+3\tan^2\theta\tan^4\phi+\tan^8\theta\}^{1/8}.$$

Suppose we want a series of values of x and y to the argument of $\tan\phi=1$ or $z=x$, the reduced equation is $1/x=\{2+4\tan^2\theta\}^{1/8}$, from which the following values are directly found:—

$\theta=$°.	15°	30°	45°	60°
Log $\tan\theta$	= $\bar{1}$·4281	$\bar{1}$·7614	0·0000	0·2386
Log $\tan^2\theta$	= $\bar{2}$·8562	$\bar{1}$·5228	0·0000	0·4772
$\mathrm{Tan}^2\theta$	= 0·0718	0·3333	1·0000	3·0000
$4\tan^2\theta+2$	= 2·2872	3·3332	6·0000	14·0000
Log $(4\tan^2\theta+2)$	= 0·3593	0·5227	0·7782	1·1461
$1/8=\log 1/x$	= 0·0449	0·0653	0·0973	0·1433
Log x	= $\bar{1}$·9551	$\bar{1}$·9347	$\bar{1}$·9027	$\bar{1}$·8567
Log y	= $\bar{1}$·3832	$\bar{1}$·6961	$\bar{1}$·9027	0·0953
x	= 0·9018	0·8604	0·7993	0·7190
y	= 0·2416	0·7251	0·7993	5·2360

(2) Let the equation be

$$\alpha^{10}+\beta^4\gamma^3\delta^3+\alpha^4\beta^3\gamma^3=1\ .\quad .\quad .\quad .\quad .\quad .\quad (5)$$

$\beta=l\alpha$; $\gamma=m\alpha$; $\delta=n\alpha$; and the transformed equation is

$$1+l^4m^3n^3+l^3m^3=\frac{1}{\alpha^{10}}.$$

If it is desired to find values of α and δ to the arguments $l=1$, $m=1$, the reduced equation is $2+n^3=1/\alpha_{10}$ $[n=\tan v.]$

v	20°	40°	60°	80°
Log $\tan^3 v$	= $\bar{1}$·5611	$\bar{1}$·9238	0·2386	0·7537
Log $\tan^3 v$	= $\bar{2}$·6833	$\bar{1}$·7714	0·7158	2·2611
$2+\tan^3 v$	= 2·0482	2·5907	7·1980	184·4500
Log $(2+\tan^3 v)$	= 0·3113	0·4135	0·8572	2·2658
$1/10=\log 1/\alpha$	= 0·03113	0·04135	0·08572	0·22658
Log α	= $\bar{1}$·96887	$\bar{1}$·95865	$\bar{1}$·91428	$\bar{1}$·77342
Log $\tan v$	= $\bar{1}$·5611	$\bar{1}$·9238	0·2386	0·7537
Log δ	= $\bar{1}$·5300	$\bar{1}$·8825	0·1529	0·5271
α	= 0·9309	0·9093	0·8210	0·5934
δ	= 0·3388	0·7630	1·4220	3·3660

An equation consisting of a single homogeneous part may also be reduced to polar coordinates and solved for r. If we write $r\cos\theta$ for x, and $r\sin\theta$ for y, and divide by r^n, the resulting equation is

$$\mathrm{Cos}^n\theta+\mathrm{A}_1.\cos^{n-1}\theta.\sin\theta+\mathrm{A}_2\cos^{n-2}\theta.\sin^2\theta\pm\ldots\pm\sin^n\theta=\frac{w}{r^n}=\frac{1}{r^n},$$

whence $1/r^n$ is found by summing the terms. But for purposes of computation the formulæ of the preceding paragraph are preferable, because they contain only half the number of trigonometrical quantities that are contained in the polar expression.

Where there are three variables, and it is desired to obtain values of a radius vector in terms of θ and the spherical angle ϕ, the computation may also be simplified by making use of cylindro-polar coordinates. In this system r is the radius vector *in the plane of* XY; thence $x = r\cos\theta$; $y = r\sin\theta$; $z = r\tan\phi$. Each term of x, y, z then contains at the most only *three* trigonometrical quantities to be computed, instead of *five*, as in the ordinary spherical system, and the angles θ and ϕ are the same.

The spherical radius vector, if required, can be afterwards found by the relation, spherical radius-vector $= r\sec\phi$. The equation of three variables transformed to cylindro-polar coordinates is of the form

$$\cos^n\theta + \{A_0\cos^{n-2}\theta.\sin\theta + A_1\cos^{n-3}\theta.\sin^2\theta + \ldots\}\tan\phi + \{\cos^{n-3}\theta.\sin\theta + \ldots\}\tan^2\phi$$
$$+ \ldots + \sin^n\theta + \tan^n\phi = 1/r^n.$$

Examples of solutions effected by transformation to polar and cylindro-polar coordinates will be given in the sequel.

5. *Solution of certain Homogeneous Equations by the introduction of a New Variable.* (Case III.)

It is only in the case of homogeneous equations that the n^{th} root of the arbitrary term is a parameter or value of x when $y = 0$. In all other cases the parameter is determined by an equation in x or y (as the case may be), which in the case of the higher degrees can only be solved by approximation. Hence the method of homogeneous variation is not directly applicable. In applying the principle of homogeneous variation to functions which are not homogeneous, we must consider the function of two variables, as a particular value of a function of three variables in which z has become unity. Thus, if we suppose a surface to be represented by an implicit homogeneous function $f(x, y, z)$, a plane section, parallel to the plane xy and at a distance from the origin $z = 1$, will be represented by the heterogeneous equation formed by the disappearance of the quantity z.

In order to solve an equation of the form $u_n + u_{n-1}$ &c. $= 0$, we must first restore it to the homogeneous form by introducing such powers of z as will make the equation homogeneous, and then endeavour to reduce z to unity by homogeneous variation.

Consider the two following equations, in which the brackets include terms of the same degree in x and y:—

$$\{x + A_1x^{n-1}y \pm \ldots \pm A_ny^n\} + \{x^{n-1} + B_1x^{n-2}y^2 \pm \ldots \pm B_{n-1}y^{n-1}\} + \{x^{n-2} + \&c.\} = 0. \quad . \quad . \quad (1)$$

$$\{\xi^n + A_1\xi^{n-1}\eta \pm \ldots \pm A_n\eta^n\} + \{\xi^{n-1} + B_1\xi^{n-2}\eta \pm \ldots \pm B_{n-1}\eta^{n-1}\}\zeta + \{\xi^{n-2} + \&c.\}\zeta^2 + \&c. = 0 \quad (2)$$

The first form is a thoroughly heterogeneous equation, containing terms of degrees

n, $n-1$, $n-2$, &c. The second form is a homogeneous surface equation, from which (1) may be derived by giving to z the value unity. Suppose ξ and η to be arbitrarily assumed quantities, and that we can by any known method find a value of the third coordinate ζ, which will make the equation zero, then, dividing by ζ^n, we eliminate ζ. Thus, by division, the second of the above equations becomes

$$\left\{\left(\frac{\xi}{\zeta}\right)^n + A_1\left(\frac{\xi}{\zeta}\right)^{n-1}\left(\frac{\eta}{\zeta}\right) \mp \ldots \mp \left(\frac{\eta}{\zeta}\right)^n\right\} + \left\{\left(\frac{\xi}{\zeta}\right)^{n-1} + B_1\left(\frac{\xi}{\zeta}\right)^{n-2}\left(\frac{\eta}{\zeta}\right) \mp \ldots \mp\right\}\frac{\xi}{\zeta} + \left\{\left(\frac{\xi}{\zeta}\right)^{n-2} + \left(\frac{\xi}{\zeta}\right)^{n-3}\left(\frac{\eta}{\zeta}\right) \mp \ldots \mp\right\}\frac{\zeta^2}{\zeta^2} = 0.$$

The quantities outside the brackets are unity, and the quantities inside the brackets constitute a solution or value of equation (1), where ξ/ζ is a value of x, and η/ζ is a value of y, while z as a separate quantity has disappeared. ζ may be considered either as a third dimension or as a variable parameter.

Accordingly if ζ can be found and the arbitrary equation formed, the solution is at once obtained by dividing ξ and η respectively by ζ. Let S_n represent the numerical value of the homogeneous part within the first bracket formed by assuming arbitrary quantities ξ, η; S_{n-1} is the numerical value of the terms within the second bracket (which are all of the degree $n-1$), and so on, and the equation is

$$S_n + S_{n-1}\zeta + S_{n-2}\zeta^2 + S_{n-3}\zeta^3 + S_{n-4}\zeta^4 + \&c. = 0 \quad . \quad . \quad . \quad . \quad (3).$$

It is easily seen that the possibility of solution does not at all depend on the degree of the given equation, but upon the *relative degrees* of the terms of ζ, which it is necessary to introduce. If the equation consists of only two homogeneous parts, suppose of the 9th and 3rd degrees, we have a simple equation to determinate ζ, as in this example $x^9 + A_1x^8y + \ldots = x^3 + B_1x^2y + \ldots$ &c., which may be written $u_9 = u_3$. By introducing the quantity $z=1$ this becomes

$$\{x^9 + A_1x^8y + \ldots\} = \{x^3 + B_1x^2y + \ldots\}z^5$$

$$\zeta = \frac{\{\xi^9 + A_1\xi^8\eta + \&c.\}^{1/5}}{\{\xi^3 + B_1\xi^2\eta + \&c.\}^{1/5}},$$

Then by the introductory theorem we find

$$x_1 = \xi/\zeta; \; y_1 = \eta/\zeta.$$

This is the case already considered in section 3. Similarly, if the auxiliary equation in ξ, η, and ζ contains only the first and second powers of ζ, we have a quadratic equation between ζ and the sums of the numerical terms of the assumed quantities, whence ζ may be found, and thence exact solutions of x and y. If the auxiliary equation contains ζ^2 and ζ^4 or ζ^3 and ζ^6, we have a quadratic equation to determine ζ^2 or ζ^3, whose root may then be extracted. Or, finally, there may be a soluble cubic or biquadratic equation in ζ or some power of ζ.

If the equation contains an arbitrary term, this is equivalent to an additional

homogeneous part. The arbitrary term may either be treated as a coefficient or reduced to unity by division, and in the auxiliary equation it is replaced by ζ^n in order to form a homogeneous function equated to zero.

The method of this section is essentially the same as is implied in the following :—

A solution of the equation

$$x^5+6x^4y+3x^4+5xy^3+2x^3+4y^3=0$$

is

$$x=\frac{a}{\mathrm{R}}$$

$$y=\frac{b}{\mathrm{R}},$$

where R is a root of the equation

$$(a^5+6a^4b)+(3a^4+5ab^3)\mathrm{R}+(2a^3+4b^3)\mathrm{R}^2=0.$$

For on substituting we have the left-hand side

$$=\frac{a^5}{\mathrm{R}^5}+\frac{6a^4b}{\mathrm{R}^5}+\frac{3a^4}{\mathrm{R}^4}+\frac{5ab^3}{\mathrm{R}^4}+\frac{2a^3}{\mathrm{R}^3}+\frac{4b^3}{\mathrm{R}^3}$$

$$=\frac{a^5+6a^4b+(3a^4+5ab^3)\mathrm{R}+(2a^3+4b^3)\mathrm{R}^2}{\mathrm{R}^5}$$

$$=\frac{0}{\mathrm{R}^5}$$

$=0$, as it should be.

And generally—

If $\phi_n(x, y)$ denote a homogeneous equation of the n^{th} degree, the equation

$$\phi_n(x, y)+\phi_{n-1}(x, y)+\ldots+\phi_1(x, y)+\phi_0(x, y)=0$$

has for solution

$$x=\frac{a}{\mathrm{R}}$$

$$y=\frac{b}{\mathrm{R}},$$

where R is a root of the equation

$$\phi_n(a, b)+\mathrm{R}\phi_{n-1}(a, b)+\mathrm{R}^2\phi_{n-2}(a, b)+\ldots+\mathrm{R}^{n-1}\phi_1(a, b)+\mathrm{R}^n\phi_0(a, b)=0.$$

In the case of a heterogeneous function of three variables, there is a choice of four solutions. (1) A quantity P may be introduced which will make the equation homogeneous as a whole; and the equation may then be solved for P by treating the arbitrarily assumed quantities, ξ, η, ζ, as known quantities. Then, dividing by P^n, values of x, y, z are found for $\mathrm{P}=1$. (2, 3, and 4) Any one of the quantities (say z) may be assumed as constant during the operation, or $z=a$ and a quantity P is then to be intro-

duced which will make the equation a homogeneous function of P and the other two quantities, x and y. Then dividing by P^n, x and y are found for $z=a$; $P=1$. In this way a series of values of x and y may be found to any argument z. The choice of the method will of course be determined by the possibility of solving the equation in P. Unless the degree of the equation is very high, or the terms very numerous, it will generally be found that an equation can be formed from which P may be determined, and the corresponding values of x, y, and z deduced by homogeneous variation.

As in the case of plane curves, we see that in the case of surfaces also, any surface may be expressed as a homogeneous equation of the three variables and the intercept P on one of the axes. Also, any function of identical form, with a different value of P, is a similar surface.

The method is evidently capable of extension to equations of any number of unknown quantities.

There are two distinct geometrical interpretations of the processes here given, according as we consider the new quantity z as being in a different plane from x, y, or in the same plane.

(1) In the former case z is a third coordinate, and the 3-dimensional homogeneous equation $f(x,y,z)=0$ always and necessarily represents a conical surface. This may be proved (without drawing on the methods of the differential calculus) by transforming the equation to cylindrical coordinates. XY is the reference plane in which r and θ are measured; z is then perpendicular to that plane. Then writing $r\cos\theta$ for x, $r\sin\theta$ for y, we have a homogeneous equation in r and z with trigonometrical coefficients. Accordingly if r and z be varied, while θ remains unchanged, we have by the introductory proposition $r_1/z_1=r_2/z_2=r_3/z_3$, &c. This can only be true if r and z are coordinates of the same generating line, which of course lies in a plane passing through the axis of z and making the angle θ with the plane XZ. More simply, as the result of the transformation to cylindrical coordinates is to form a homogeneous function, $f(r.z)=0$, this is known to be the equation of two right lines, and the surface is then shown to be made up of generating lines passing through the origin, which is the definition of a conical surface.

In order that the equation $f(x,y,z)=0$ may have real solutions, the highest power of one of the quantities must be negative; and it is easy to see that the homogeneous function of the n^{th} degree, $f(x,y,z)=0$ is the asymptotic cone of all the concentric and similar surfaces which can be found by equating the same function to an arbitrary term P^n. It is in fact the limiting form of this series of concentric and similar surfaces when the parameter P vanishes.

(2) I began by observing that we might conceive the quantity z (which was introduced for the purpose of rendering the equation to be solved homogeneous) as being either in a different plane from x and y, or in the same plane. If it is considered as being in the same plane, it is the parameter of the non-homogeneous curve, and may be denoted by P. The proof is as follows:—Compare the two subjoined equations, in which

the original heterogeneous equations xy, $\xi\eta$, have been made homogeneous by introducing supplementary powers of P and Π,

$$\{x^n + A_1x^{n-1}y \mp\mp y^n\} + \{B_1x^{n-1}P + B_1x^{n-2}yP \mp\mp B_1y^{n-1}P\} + \{C_1x^{n-2}P^2 + C_1x^{n-3}yP^2 \mp\mp y^{n-2}\} = 0$$

$$\{\xi^{n-1} + A_1\xi^{n-1}\eta \mp\mp \eta^n\} + \{B\xi^{n-1}\Pi + B_1\xi^{n-2}\eta\Pi \mp\mp B_1\eta^{n-1}\Pi\} + \{C\xi^{n-2}\Pi^2 + C\xi^{n-3}\eta\Pi^2 \mp\mp \eta^{n-2}\} = 0.$$

As the equations are homogeneous and identical in form, they represent similar curves; and according to the fundamental theorem of this paper the one form may be derived from the other by multiplying every term by a constant, that is by $(P/\Pi)^6$. Hence x, y, and P are obtained from ξ, η, and Π by multiplying each by the factor P/Π. This can only be true if P and Π are the parameters or the same multiple of parameters of the respective curves.

It may occur as a difficulty that in the case of heterogeneous curves, the quantity P does not correspond to the value x_0 of x when y is equated to zero. But P can easily be shown to be proportional to x_0. For suppose y and η in the two curves of the example equated to zero, the equations are then of the form

$$x_0^n + Bx_0^{n-1}P + Cx_0^{n-2}P^2 \mp\mp = P^n$$

$$\xi_0^n + B\xi_0^{n-1}\Pi + C\xi_0^{n-2}\Pi^2 \mp\mp = \Pi^n.$$

Dividing by the highest powers of P and Π, we have

$$\left(\frac{x_0}{P}\right)^n + B\left(\frac{x_0}{P}\right)^{n-1} + C\left(\frac{x_0}{P}\right)^{n-2} \mp\mp = 1$$

$$\left(\frac{\xi_0}{\Pi}\right)^n + B\left(\frac{\xi_0}{\Pi}\right)^{n-1} + C\left(\frac{\xi_0}{\Pi}\right)^{n-2} \mp\mp = 1.$$

Hence by the known law of expansions, $x_0/P = \xi_0/\Pi$, or the quantities P and Π of similar curves have a constant ratio to the intercepts x_0 ξ_0. They are therefore virtual parameters.

(3) The case of a homogeneous equation of the n^{th} degree equated to a term Z^n or P^n, with which the paper commences, is now seen to be merely an explicit form of the general conical or parametral equation, $f(x, y, z) = 0$. If the explicit term be considered as a third coordinate (z), the conical surface is referred to a plane of symmetry, xy, and an axis of symmetry z. In the implicit function the *projections* of the similar parallel sections in the plane xy are neither similar nor symmetrical; and the similar sections are only found by taking z into account.

So with the implicit function considered as of three quantities in one plane. The parameter, P, is evidently not the principal parameter of the curve, but is the value of the intercept on the axis of x in the system of axes proposed.

From this investigation we see that any plane curve whatever may be expressed as a homogeneous function of rectangular coordinates *and the intercept on one of the axes.* When so expressed, it is a similar and similarly situated curve with respect to any other curve expressed as the same function with a different value of P.

6. *Soluble Cases of the Homogeneous Function* $f(x, y, z)^n = 0$.

This function, as has been observed, represents a conical surface, being the asymptotic cone of all the concentric and similar surfaces that can be formed by equating the same function of x, y, z, to different arbitrary terms.

Unless the equation contains a large number of terms, it can in general be solved by taking arbitrary values of those two quantities which are most involved, and solving for the one which is least involved.

These solutions represent points on the conical surface, and if it is desired to obtain such solutions in series, so as to represent a plane section or curve, they may be reduced by division to the argument $x=1$, $y=1$, or $z=1$ as desired. It is only necessary to tabulate one such series; because the surface is conical, and values of y and z may be obtained to any other argument $x=a$, by merely multiplying the tabular values of y and z by a. Consider, for example, the equation of a homogeneous surface of this form,

$$x^8 + Ax^6y^2 + Bx^5z^3 + Cx^2z^6 + Cy^2z^6 + By^5z^3 + Ax^2y^6 + y^8 = 0.$$

Here the equation is symmetrical in x and y, but contains no powers of z except the 6th and the 3rd. Accordingly, we may form an equation by assuming values x_1 and y_1, and then solving the quadratic equation in z^3, the root of which, being extracted, is a solution. That is to say, the values x_1, y_1, and z_1 thus found satisfy the equation.

7. *Solution of Homogeneous Equations of Functions of the Variables.* (Case IV.)

Assume

$$v_1 = ax^p + by^q + c\,;\ v_2 = dx^p + ey^q + f\,;\ v_3 = \ldots\ .$$

Any homogeneous expression in $v_1v_2\ldots$ equated to an arbitrary term, or to another homogeneous expression in $v_1v_2\ldots$ of a different degree, can be solved by the methods previously given. Values of $v_1v_2\ldots$ being first found, we have then two simple equations for determining x^p and y^q in terms of these values, whence x and y are found.

The original equation is of course heterogeneous when the quantities $ax^p + by^q + c$, &c., are substituted in place of v, &c.; and by means of this new application of the fundamental theorem, an endless variety of heterogeneous equations may be formed and solutions in series obtained. It is evidently a condition of the possibility of solving such equations that the number of factors v_1v_2, &c., shall not exceed the number of constituent quantities, x, y, &c., of which values are to be found.

If the indices p, q, &c., are even, the curve or surface is central; but the converse does not necessarily hold. Thus $v_1 = ax + b$; $v_2 = cy + d$, gives a central curve from an excentric origin, of which Pl. VI. figs. 5 and 6 are illustrations. If one of the quantities, v_1, be taken $= a + \sqrt{x}$ the curve will only have single symmetry.

If we take $v_1^2 = x^2 + z^2$; $v_2^2 = y^2 + z^2$, thus a series of values of x and y may be found to an invariable value of z, and the series of points so determined will trace out one or

more contour lines of the homogeneous surface in x, y, z, which is represented by the equation. Examples of these are given in the sequel.

Other applications of the combination of soluble functions of x and y, or r and θ, will readily suggest themselves. The following may suffice as illustrations :—

(1) Let the equation consist of powers of quantities (x^2+y^2) and (x^2-y^2) as

$$(x^2+y^2)^n+A_1(x^2+y^2)^{n-1}+A_2(x^2+y^2)^{n-2}++\ldots=A_n(x^2-y^2)^p\,.$$

This is equivalent to

$$r^2\ +A_1r^{2n-2}+A_2r^{2n-4}++\ldots=r^{2p}\cos^p 2\theta\,.\,A_n\,,$$

whereby θ is determined in terms of r, and thence x and y.

(2) $$(x^l+y^m)^n+A_1(x^l+y^m)^{n-1}+A_2(x^l+y^m)^{n-2}++=A_n(x^l-y^m)^p\,.$$

Take $u^2=x^l$; $v^2=y^m$; and solve the resulting equation in u and v, which is of the above form. Then u and v are found from r and θ, whence x and y are determined.

(3) This solution is evidently capable of extension to any function in the form on the left side of the sign of equality, where the quantity on the right side can be expressed as the power of a cosine or sine of a multiple of θ, or a soluble function of such a sine or cosine. In these equations each term is a homogeneous function, and the solution depends partly on this circumstance.

(4) If the right-hand term consists of a power of x alone or of y alone, the equation is solved by writing $r^p\cos^p\theta$ for x^p, or $r^p\sin^p\theta$ for y^p.

(5) If the left side of the equation consists of powers of the quantity (x^l-y^m), and the right side of a single term x^p or y^q, or $(x^l+y^m)^p$ solutions of r can be found to the argument θ, and thence u_1 v_2 and xy are similarly found.

8. *To find the Condition under which parallel Sections of a given Surface may be similar Curves.*

The condition is evidently fulfilled if one of the quantities x, y, z be given explicitly in the equation. The surface may be conceived as traced by the motion of a generating curve controlled by a guiding curve.

Suppose the generating curve to be a homogeneous function of x, y, equated to a function of z only. Then, as the sections parallel to XY are to be similar curves, the generating curve must move parallel to itself with varying parameter, and so as always to touch a guiding curve in the place XZ. Let the equation of the guiding curve be of the form

$$x^n-z^n\pm B_1z^{n\pm1}\pm B_2z^{n\pm2}\pm\ldots+B_nz^{n\pm p}=0\,.$$

Then the equations of the generating and guiding curves are

$$f(xy)\ =x^n+A_1x^{n-1}y+A_2x^{n-2}y^2\mp\mp A_ny^n=p \quad . \quad . \quad . \quad . \quad (1).$$

$$f(x'y')=x'^n-z'^n\pm B_1z'^{n\pm1}\pm\ldots+B_nz'^{n\pm p}=0 \quad . \quad . \quad . \quad . \quad (2).$$

In equations (1) and (2) p is the variable parameter of the section parallel to xy, and is evidently the ordinate in the principal plane xz. Hence, $x' = p$; $x'^n = p^n = f(xy)$.

Also the value of z in the two sections is the same.

Substituting in (2) the value of x' above found, we have for the equation of the surface

$$x^n + A_1 x^{n-1} y + A_2 x^{n-2} y^2 \mp \mp A_n y^n = z^n \mp B_1 z^{n-1} \pm \ldots + B_n z^{n \pm p};$$

whence values of x and y can be found by homogeneous variation to any argument z. For, we have only to suppose z constant, and to state the sum of the terms of z as an arbitrary quantity in the form P^n, and the problem is reduced to that of the introductory proposition.

In the example given, the plane XY is a plane of symmetry, and the axis z is an axis of symmetry; but these considerations are insufficient to determine similar parallel sections unless the quantities x and y are combined homogeneously. It is easy to see that in the example given the sections parallel to XZ and YZ are not similar, because they are neither homogeneous functions of the respective pairs of variables, nor of these quantities and the parameter.

On trial it will be found that no other generating curve, except a homogeneous curve, gives similar parallel sections. If possible, let the sections of the unknown surface taken parallel to the following symmetrical equation be similar curves

$$x^n + A_1 x^{n-2} y^2 + A_2 x^{n-4} y^4 \mp \mp A_2 x^4 y^{n-4} + A_1 x^2 y^{n-2} + y^n = P^n.$$

In order that the sections parallel to the plane of xy may be similar, their equations must be homogeneous functions of x, y and P, as has been proved. Hence the condition is satisfied by substituting $z = P$ or $f(z) = P$.

It thus appears that a central surface, other than a conical surface, will not furnish sections parallel to xy which are similar curves, unless the third quantity, z, be given explicitly, so that the terms of x and y alone constitute a homogeneous equation. In treating of equations in this perfectly general form, one is apprehensive of some possible exception or flaw in the demonstration; I have accordingly taken pains to verify this conclusion, by endeavouring to find values of x and y from the surfaces generated by various non-homogeneous curves, choosing the most symmetrical forms referred to conjugate axes as being those which were most likely to give results.

In every instance the values of x and y, found *on the assumption that the parallel sections were similar*, failed to satisfy the original equation, although they must necessarily have done so had the hypothesis been correct.

I consider it then demonstrable that the sections of homogeneous surfaces are only similar curves when the sections are homogeneous functions of x and y equated to powers of z uncombined with x or y. In other words, the homogeneous function must be of the form $f(x, y) = f(z, P)$; otherwise parallel sections will be dissimilar.

9. *Classification and Forms of Curves considered as Sections of Surfaces whose Equations are Homogeneous.*

In this chapter I do not enter on the question of the singularities of curves, a theme which has already been the subject of much learned investigation. My purpose is (1) to discover the different elementary and symmetrical forms of the curves of a given degree, which may be considered as the sections of a homogeneous surface parallel to its principal planes; and (2) from these elementary forms to show how by variation of the unknown quantities corresponding types of unsymmetrical curves of the n^{th} or given degree may be obtained, and the surfaces traced in series of contour-lines.

I ought here to point out that the motive of this investigation is somewhat different from that of Mr Frost's valuable work on Curve Tracing.

In a treatise on Curve-Tracing in general, the exact determination of the locus of the curve is of course unattainable, and only approximate methods are used.

In the present paper, only those curves are considered whose loci can be exactly traced, by solving these equations rigorously for successive positions of x and y. In the diagrams, which are photographic reductions of the original tracings on diagram-paper, the error at any point ought not to exceed $\frac{1}{500}$ of an inch.

It is perhaps unnecessary to prove that every homogeneous equation of three variables represents a surface symmetrical about three principal axes of symmetry, which it is convenient to consider as placed perpendicularly to one another. This follows from the consideration that when the homogeneous equation is transformed to polar coordinates, it contains only the highest power of r, which in the case of a curve of even degree has always equal positive and negative roots. In the case of curves of uneven degree, the same results are obtained by considering *the sign of the arbitrary term indeterminate*,—as it evidently ought to be, because by so treating it, we obtain from the equations of uneven degrees forms which are strictly analogous to those of the nearest even degrees.

This being premised, if in the surface represented by the given homogeneous equation we take for the axis of Z, the direction of the radius vector of maximum length; and if the surface be referred to this axis Z and to a central plane perpendicular to Z in which angles are denoted by θ, then for every value of θ and ϕ there are four equal values of r corresponding to the four permutations of the positive and negative values of θ and ϕ and also other four formed from $\pi - \theta$ and ϕ. Hence, for any plane through the axis of Z, there are four equal values of r, and the curve is symmetrical about Z and the diameter in the plane XY. By transforming to an axis X coinciding with the maximum radius vector of the central plane and a plane perpendicular to it, similar conclusions are obtained for all diametral sections through X, and also for all diametral sections through Y, the line of intersection of the first and second reference planes. Thus the symmetry of the surface with reference to three principal planes and their intersections is established.

A homogeneous surface may have more than one set of axes of symmetry. Some of these may be conjugate diameters meeting the surface at finite points, and some of them may be asymptotic lines.

In a central equation the axes of reference are asymptotic lines, if the equation wants the highest powers of the three variables; because then, dividing the equation by the lowest power of any of the variables, suppose x^p, we find for the values, $y=0$, $x=0$, the corresponding value $z=A/x^p=\infty$.

When a homogeneous central surface is referred to axes of symmetry, its equation must consist either entirely of terms of even powers of each variable z, x or y, or entirely of terms of uneven powers of the same variable; because then only will the value of the term be unaltered when x is changed to $-x$, or y is changed to $-y$. It is of course only in curves of even degree that x and y are both even or both uneven, and therefore curves of uneven degree have in general only single symmetry unless the sign of the arbitrary term is treated as indeterminate. Accordingly,

1. If the equation of a homogeneous surface is of uneven degree, and consists of terms of even powers of x and uneven powers of y, the axes of reference are asymptotic lines.

2. If the homogeneous surface equation, being of uneven degree, consists of terms of uneven powers of the variables, the axes of reference coincide with finite conjugate diameters; but this condition can only be fulfilled if the equation is of the form—

$$x^n \mp y^n \mp z^n = \mp A^n,$$

where the sign A^n is indeterminate.

3. Again, if the equation of a homogeneous surface be of even degree, and consists entirely of terms of uneven powers of the variables, the axes of reference are asymptotic lines.

4. If the equation being of even degree consists of terms of even powers of the variables, the axes of reference coincide with finite conjugate diameters, unless the highest powers of the variables are awanting.

5. If, in any of these cases, the surface is expressed by a symmetrical equation,—that is, if the equation consists of pairs of homologous terms, the signs in all the pairs being either like or unlike,—the three axes are equal; and the surface is also symmetrical about six secondary conjugate axes, which bisect the angles between each pair of the first set. Moreover, there are two planes through the axis of Z and a secondary axis lying between the axes X and Y, which are planes of symmetry; and two for each similar combination; that is, six planes of symmetry in addition to those originally given.

6. If the equation be of the form $f^n(x/a, y/b, z/c)=1$, and be a symmetrical function of these ratios, the surface will of course be a "3-dimensional projection," or homogeneous transformation of the corresponding function of x, y, z. It is evident from known principles that all lines and planes of rectangular symmetry will be projected into lines and planes of oblique symmetry; and the secondary planes and lines will bisect

those parallel to conjugate planes and lines, but will not bisect the angles between the principal diameters and principal planes.

7. Paragraphs 1, 2, 3, and 4 may be applied to plane curves by suppressing the element Z; 6 also applies to plane curves, and it will be shown that the inclination of the secondary axis, $\underline{x}$, to the principal axis, X, is given by the relation, $\tan\theta = b/a$. In the further discussion of the subject I shall use the term "Diametral Equation" to express the equation of a curve when referred to axes of symmetry. If an equation containing only even powers of the variables x and y be also a symmetrical expression, the curve has fourfold symmetry, because the symmetrical form of the equation shows that the axes of reference are equally inclined to a second pair of conjugate axes. There are then eight points at which the value of R is either a maximum or a minimum. This is a property which is not lost by projection. A Symmetrical Diametral Equation is an equation which is itself symmetrical; where therefore the curve is equiaxial and has fourfold symmetry.

10. *Classification of Central Curves of the Form* $\mathrm{F}(x, y)^n = \mathrm{A}^n$.

A central function of two variables equated to an arbitary term may be either homogeneous or heterogeneous. In the first case, the equation may represent either a central section of the general homogeneous surface, or a section taken parallel to a principal plane of any homogeneous surface whose equation contains only the highest power of Z. In the second case, the equation represents a section taken parallel to a principal plane of the general homogeneous surface. Reference is made on this point to the preceding part of the paper.

If we begin by considering homogeneous symmetrical forms, or forms which are the projections of these, it is evident that the equations must be composed by the multiplication of factors of the forms,

$$\left(\frac{x}{a} \pm \frac{y}{b}\right)^p ; \left(\frac{x^2}{a^2} \pm \frac{y^2}{b^2}\right)^q ; \cdots \left(\frac{x^n}{a^n} \pm \frac{y^n}{b^n}\right)^r .$$

The number of possible symmetrical equations is, however, very much less than the possible permutations of such factors; and it is not difficult to see that the required number is that of the permutations of the positive and negative signs in a symmetrical equation containing only even, or only uneven powers of x or y. From the preceding remarks it is seen that a diametral symmetrical equation represents a curve which has two pairs of conjugate axes, each axis bisecting the angles between the axes of the other pair; and that such axes are either asymptotes or finite conjugate diameters.

(1) *Oval Forms.*—If the given equation is homogeneous, and if the *four axes* of symmetry coincide with *finite diameters*, the equiaxial curve is generally a symmetrical oval entirely concave to the centre. In this species, if the original equation consists of terms of even powers, the equation of the curve when transformed to secondary axes as

axes of reference, consists also of even powers of the variables. In this type of curve, when referred to either pair of conjugate axes, all the terms of the variables are positive. It will presently be seen that where some of the homologous pairs of terms are positive and some are negative, the curve may be an inflexional oval of double symmetry, passing into an inflexional hyperbolic for certain values of the coefficients.

(2) *Hyperbolic Forms.*—If the axes consist of a pair of finite diameters and a pair of asymptotes, the curve consists of two or more infinite branches symmetrically placed, which may be either all equal or of two sets. These may be termed continuous or discontinuous hyperbolics, according as the branches are all real, or consist of real and imaginary (or conjugate) branches in alternate order. If in each pair of homologous terms the signs are *unlike*, the branches are entirely convex to the centre, the inflexional hyperbolic being represented by an equation of pairs of homologous terms, some of which pairs are positive and some negative, or of unlike signs.

All equiaxial curves, whether of the first or the second class, complete their phases within a quadrant. In curves of the second class, the secondary axes, although asymptotic, are true diameters, because the form of the equation shows that each asymptote bisects all ordinates drawn parallel to the other; that is, it bisects the intercepts made by two adjacent branches, which may be either both real, or one of them real and the other conjugate.

(3) *Projections of Equiaxial Forms.*—By writing x/a for x and y/b for y in any homogeneous equiaxial equation, the equation of the projection of the equiaxial curve is formed. The curves of the series which may be formed by projection have the same general resemblance to the primitive equiaxial forms that ellipses and common hyperbolas have to the primitive forms of the circle and the equilateral hyperbola.

(4) *Heterogeneous Central Equations.*—Every equation of even degree, and containing only even powers of the variables (although not homogeneous), represents a central curve; and if the equation be a symmetrical expression, the curve is equiaxial. I shall here only consider those heterogeneous central forms which represent sections of the symmetrical homogeneous surface equation.

It has been pointed out that every heterogeneous central expression represents a section of a homogeneous and central surface taken parallel to a principal plane. There is then no specific difference between homogeneous and heterogeneous central curves pertaining to the same surface. The highest homogeneous part of the equation is the limiting equatorial section, where the terms compounded with Z disappear, and the general form of the curve depends solely on the highest homogeneous part of its equation.

(5) With regard to those curves whose equations are not symmetrical functions of x and y, or x/a and y/b, it is in general not possible to find secondary axes to these. But the curves of unsymmetrical expression are assimilated to those whose equations are symmetrical by the Rule of Signs, which will presently be deduced, and their traces, computed by the homogeneous method, prove that they follow the same classification.

(6) The inclination of the asymptotes of a *heterogeneous central curve* is always the same as the inclination of the asymptotes of the curve represented by its highest homogeneous part. Because, if we transform to polar coordinates, and divide the polar equation by r^n, the resulting equation is of the form—

$$F_0(\cos\theta, \sin\theta)^n + F_1(\cos\theta, \sin\theta)^{n-2} \cdot \frac{1}{r^2} + F_2(\cos\theta, \sin\theta)^{n-4} \cdot \frac{1}{r^4} \ldots = \frac{1}{r^n}.$$

Now, r can only become infinite when $F_0(\cos\theta, \sin\theta)^n = 0$.

But this is also the condition for r becoming infinite when the equation is reduced to its highest homogeneous part.

It follows that for all parallel sections of a central surface the inclination of the asymptotes to the axis of symmetry of the section is the same, and it is evident that all such asymptotes lie in two intersecting planes.

(7) In the case of the homogeneous central equations with an arbitrary term, it is evident that the curve cannot pass through the centre.

11. *Transformation to Secondary Axes—Rule of Signs.*

A symmetrical equation is evidently equiaxial; that is, the intercepts on the axes of reference are equal.

If a diametral symmetrical equation be transformed to secondary axes (bisecting the angles between the primary), the secondary axes are also diametral and symmetrical, and the curve consists of eight equal and similar segments. This might be inferred from general considerations as to symmetry, but it is desirable to prove it analytically. It may be here convenient to transcribe certain known formulæ of transformation of axes (with unchanged origin) of which I am to make use. If θ be the inclination of X to x—

$$X = x\cos\theta - y\sin\theta \quad Y = x\sin\theta + y\cos\theta \quad . \quad . \quad . \quad . \quad (1)$$

$$X = (x-y)\cos\theta \quad Y = (x+y)\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$X = (x-y)\sqrt{\tfrac{1}{2}} \quad Y = (x+y)\sqrt{\tfrac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(1) Is the formula for transformation in the same plane from any system of rectangular coordinates to any other rectangular system.

(2) Is the formula for transformation to symmetrical axes; *i.e.*, axes equally inclined to the original rectangular axes.

(3) Is the formula for transformation to axes which are at once symmetrical and rectangular, and which accordingly bisect the angles between the original rectangular axes; whence, $\cos\theta = \sin\theta = \sqrt{\tfrac{1}{2}}$.

In order to prove that a symmetrical diametral equation is of the same form when transformed to secondary axes, it is only necessary to write the generalised form of the expression in lines and columns. As the original axes are always supposed rectangular,

the transformation to secondary axes is effected by substituting in every term the values $\sqrt{\frac{1}{2}}(x-y)$ for X and $\sqrt{\frac{1}{2}}(x+y)$ for Y, and expanding. Let $A_p\{X^{n-p}Y^p+X^pY^{n-p}\}$ be any pair of homologous terms; their equivalent in the transformed equation is

$$\left(\frac{1}{2}\right)^{\frac{n}{2}}A_p\{(x-y)^{n-p}(x+y)^p+(x-y)^p(x+y)^{n-p}\} \quad . \quad . \quad . \quad . \quad (4).$$

Expanding the first term within the bracket in columns, the coefficients are—

$$\begin{aligned} &1+ \quad p \quad +\frac{p.(p-1}{1.2}+\frac{p.(p-1)(p-2)}{1.2.3}+ \&c. \\ &-(n-p)-(n-p)\times p-(n-p)\times\frac{p.(p-1)}{1.2}- \&c. \\ &\qquad +\frac{(n-p)(n-p-1)}{1.2}+\frac{(n-p)(n-p-1)}{1.2}\times p+ \&c. \quad . \quad . \quad . \quad . \quad (5). \end{aligned}$$

The coefficients of the expansion of the second term within the brackets are—

$$\begin{aligned} &1- \quad p \quad +\frac{p(p-1)}{1.2}-\frac{p.(p-1)(p-2)}{1.2.3} \qquad + \quad . \quad , \quad . \quad . \quad . \quad (6) \\ &+(n-p)-(n-p)\times p+(n-p)\times\frac{p.(p-1)}{1.2} \qquad - \\ &\qquad +\frac{(n-p)(n-p-1)}{1.2}-\frac{(n-p)(n-p-1)}{1.2}\times p+ \quad , \end{aligned}$$

where in the first set of terms the sign $\times$ is used to separate the factors derived from the expansion of $(x-y)^{n-p}$ from those derived from the expansion $(x+y)^p$, and similarly in the second set of terms.

The quantity in the first column (always unity) is the coefficient of x^n in the transformed equation. The sum of the quantities in the second column is the coefficient of $x^{n-1}y$ in the transformed equation, and so on. We see that the expansion of the second term of the pair is the same as that of the first term, except that in the second, fourth, and every other alternate column, the signs + and − are interchanged, and therefore the sums of these columns in the two expansions is zero. If the pair of homologous terms have contrary signs, then in the expansion the sum of the first, third, &c., columns is zero. From this analysis is derived the following abstract of results, hereafter referred to as the Rule of Signs.

1. From the mode of formation of the transformed equation it is always symmetrical if the original equation is symmetrical.

2. If in the original equation the terms constituting a symmetrical pair are of even degree and have *like signs, i.e.*, both positive or both negative, then in the expansion of these terms in the transformed equation the sum of the partial coefficients is *zero for all terms of uneven powers.*

3. If in the original equation the terms constituting a symmetrical pair are of even

degree with *unlike signs*, then in the expansion of these terms in the transformed equation the sum of the partial coefficients is *zero for all terms of even powers.*

4. If the sum of the indices in each term be uneven, then the expansion consists of a homogeneous expression containing only the even powers of one of the variables, and the uneven powers of the other variable. In my notation, if the terms are both positive, the transformed expression will consist of even powers of y and uneven powers of x.

5. If we consider these equations only to be symmetrical where the terms of all the homologous pairs have like signs, or where for all homologous pairs the terms have unlike signs, then in the complete expansion of the transformed symmetrical equation of even degree, the sum of the partial coefficients is zero for uneven powers in the first case, and is zero for even powers in the second case; in other words, if in the original equation, being of even degree, the homologous terms have *like signs*, the equation of the curve, when referred to secondary axes, consists entirely of *terms of even powers.* If in the original equation the homologous terms have *contrary signs*, the equation of the same curve, when referred to secondary axes, consists entirely of terms of *uneven powers.* These results are independent of the degree of the curve, and it will hereafter be shown that they are applicable to the projection of any symmetrical equation obtained by writing x/a for x and y/b for y (page 1069). Thus from the order of the signs of any symmetrical equation it is immediately known to which of the previously named classes the equation belongs, *i.e.*, whether the curve represented is elliptic, hyperbolic, or inflexional.

6. These results are evidently true for any diametral, symmetrical equation, although not homogeneous, because it is only necessary to the proof that the equation should consist entirely of even or entirely of uneven terms.

12. *Diameters in Central Curves of the Fourth Degree.*

I shall now give a proof that every central curve of the fourth degree has two pairs of axes of symmetry, and in general only two such pairs.

(1) Let the equation be homogeneous, or of the form

$$Ax^4+Bx^2y^2+Cy^4=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

To prove that in general the curve has not a pair of conjugate axes equally inclined to a *given line.* Let the equation be referred to the given line and an axis perpendicular to that line. It may then be written

$$Dx^4+Ex^3y+Fx^2y^2+Gxy^3+Hy^4=1 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If we now transform to axes equally inclined to the line by formula (2) p. 1064, it will be seen whether the unknown angle θ can be determined so as to make the uneven terms of the resulting equation disappear, so that the resulting equation should be one referred to conjugate axes. To this effect we are to make the x of equation (2) $=(x-y)\cos\theta$, and

the y of equation $(2) = (x+y).\sin\theta$, and expand in lines and columns. The expansion of the first term of (2) forms the first line, that of the second term is the second line, and so on.

	x^4	x^3y	x^2y^2	xy^3	y^4	
Term 1	{ 1	−4	+6	−4	+1 }	D cos θ
Term 1	{ 1	−2		+2	−1 }	E.cos$^3\theta$ sin θ
Term 3	{ 1		−2		+1 }	F.cos$^2\theta$ sin$^2\theta$
Term 4	{ 1	+2		−2	−1 }	G.cos θ sin$^3\theta$
Term 5	{ 1	+4	+6	+4	+1 }	H.sin θ.

The coefficients of the new equation are of course the numerical quantities multiplied by the quantities outside the brackets. The two uneven terms in the new equations are accordingly

$$\{-4\mathrm{D}\cos\theta - 2\mathrm{E}\cos^3\theta\sin\theta + 2\mathrm{G}\cos\theta\sin^3\theta + 4\mathrm{H}\sin\theta\}x^3y,$$

and

$$\{-4\mathrm{D}\cos\theta + 2\mathrm{E}\cos^3\theta\sin\theta - 2\mathrm{G}\cos\theta\sin^3\theta + 4\mathrm{H}\sin\theta\}xy^3.$$

The coefficient of x^3y cannot be changed into that of xy^3 by interchanging the signs + and —. Hence a value of θ which makes the term x^3y disappear will not in general make the term xy^3 disappear. In order that both terms may disappear, we must have $\mathrm{D} = \mathrm{H}$; $\mathrm{E} = \mathrm{G}$; $\cos\theta = \sin\theta$. Hence the condition of the existence of a pair of conjugate axes equally inclined to the axes of (2) is that the equation (2) be symmetrical. The conjugate diameters thus found are evidently the principal axes of symmetry of the curve.

If in equation (2) the second and fourth terms are supposed to be wanting, so that the curve is already referred to its principal axes of symmetry, then in the new equation the two uneven terms will be

$$\{-4\mathrm{D}\cos\theta + 4\mathrm{H}\sin\theta\}x^3y \quad \text{and} \quad \{-4\mathrm{D}\cos\theta + 4\mathrm{H}\sin\theta\}xy^3.$$

Their coefficients are *identical*, and the value, $\tan\theta = \mathrm{D}/\mathrm{H}$, will reduce both terms to zero, leaving an equation consisting of even powers, and therefore referred to conjugate diameters. The diameters thus found are the secondary axes of the curve; and the relation, $\tan\theta = \mathrm{D}/\mathrm{H}$ shows that they are the diagonals of the circumscribing parallelogram whose sides are parallel to the principal axes. When the highest power of y is negative, a real solution is impossible, and the diagonals in question are the asymptotes of the hyperbolic curve of the fourth degree. These results might have been found directly by considering that every quartic homogeneous equation of even powers is necessarily a projection of an equiaxial form. Because the given equation, when referred to conjugate axes, is of the form

$$\frac{x^4}{a^4} + px^2y^2 + \frac{y^4}{b^4} = 1.$$

We have then only to take $\mathrm{P} = pa^2b^2$, or $\mathrm{P}/a^2b^2 = p$, in order to obtain the equation in projection form.

(2) Where the central equation (being of the 4th degree) also contains terms of the 2nd degree $P(x^2+y^2+xy)$ the term in xy disappears by transformation, and the transformed expression consists entirely of even powers. The new axes are accordingly conjugate axes.

More generally an equation of any even degree, consisting of the highest terms of x and y, and of one other term of even powers, may, in like manner, be immediately reduced to the projection of an equiaxial curve. Such a projection always has a pair of secondary axes, which are the diagonals of the circumscribing parallelogram.

13. *Diameters in Central Curves of Higher Degrees.*

For an equation of a degree higher than the fourth, secondary conjugate axes cannot in general be found. Because, if we transform and expand as before, there are for every even degree above the fourth more than two terms of uneven powers; and it is impossible, unless some relation amongst the coefficients be given, to determine θ so as to make the coefficients of more than two of the uneven terms vanish. The required relation is easily found. The equation must be a symmetrical function of x/a and y/b. This being premised, if θ be the angle between X and x, the inclination of the secondary axis, x, to the primary is given by the relation, $\tan\theta = b/a$. This will he made clear by two examples.

(13*a*). *To find the Secondary Axes of the Curve* $X^n/a^n \pm Y^n/b^n = 1$.

(*a*) Taking first the upper sign, and transforming to axes equally inclined to the primary, by the formula $X=(x-y)\cos\theta$; $Y=(x+y)\sin\theta$, we find for the transformed equation the expression

$$\frac{\cos {}^n\theta}{a^n}\left\{x^n - nx^{n-1}y + \frac{n(n-1)}{1.2}x^{n-2}y^2 - \&c.\right\} + \frac{\sin\theta}{b^n}\left\{x^n + nx^{n-1}y + \&c.\right\} = 1.$$

Let $\cos {}^n\theta/a^n = \sin {}^n\theta/b^n$, or $\tan\theta = b/a$; then all the uneven terms disappear, and the equation is accordingly referred to conjugate diameters, which are equal in length, and symmetrically placed with reference the principal axes. These diameters coincide with the diagonals of the circumscribing parallelogram, whose sides are parallel to the principal axes of the oval of the n^{th} degree.

(*b*) If the equation be taken with the negative sign, it is the terms of even powers which disappear in the transformation, and the secondary axes found are asymptotes of the hyperbolic curve.

(*c*) It is to be observed that the values above found for the inclination of the asymptotes of homogeneous symmetrical functions equated to an arbitrary term, are *solutions* of the relative functions equated to zero. Because the equation of the asymptote

of a homogeneous curve is always that of the curve deprived of its arbitrary term. Hence $y/x = b/a$ is a solution of an equation in either of the forms

$$\frac{x^n}{a^n} \mp \frac{x^{n-2}}{a^{n-2}}\frac{y^2}{b^2} \mp \frac{x^{n-4}}{a^{n-4}}\frac{y^4}{b^4} - \frac{y^n}{b^n} = 0$$

$$\frac{x^n}{a^n} \mp \frac{x^{n-2}}{a^{n-2}} \mp \frac{x^{n-4}}{a^{n-4}} \mp + \frac{y^{n-4}}{b^{n-4}} \mp \frac{y^{n-2}}{b^{n-2}} - \frac{y^n}{b^n} = 0 .$$

If we divide the first of these by y^n we obtain the form

$$\frac{1}{a^n}\left(\frac{x}{y}\right)^n \mp \frac{1}{a^{n-2}}\frac{1}{b^2}\left(\frac{x}{y}\right)^{n-2} \mp \frac{1}{a^{n-4}}\frac{1}{b^4}\left(\frac{x}{y}\right)^{n-4} \mp \mp = 1 .$$

Treating x/y as a single quantity u, we see that an equation of descending powers of u equated to unity is always soluble, if its coefficients constitute a homogeneous function of a and b, in which case a/b is a real root of the equation.

(13*b*). *To find the Secondary Axes of any Curve of Symmetrical Expression referred to principal Axes.*

The equation if homogeneous is of the form,

$$x^n \pm A_2 x^{n-2}y^2 \pm A_4 x^{n-4}y^4 \pm \ldots \mp A_4 x^4 y^{n-4} \mp A_2 x^2 y^{n-2} + y^n = 1 .$$

In this symmetrical expression (after transformation to secondary axes equally inclined to the primary) every pair of homologous terms produces an expansion of the form (5) or (6) of p. 1065 above. The coefficients of the corresponding terms in the two expansions are equal, and the sum of the alternate columns is zero.

In the same way for curves of the form

$$A_0\frac{x^n}{a^n} \pm A_2\frac{x^{n-2}}{a^{n-2}}\frac{y^2}{b^2} \pm \ldots \mp A_2\frac{x^2}{a^2}\frac{y^{n-2}}{b^{n-2}} \pm \frac{y^n}{b^n} = 1 ,$$

it may be shown, by taking $u = x/a$, and $v = y/b$, that, when the curve is transformed by the formula for axes equally inclined to the primary axes, the coefficients of the alternate columns disappear, or are neutralised, when $\cos^n\theta/a^n = \sin^n\theta/b^n$; or $\tan\theta = b/a$.

The solutions here given are applicable to symmetrical heterogeneous curves in any of the above forms, as may be verified by expanding separately the several homogeneous parts u_1 u_2, &c., which are of the above form; because in the proof of the Rule of Signs it is not assumed that the equation is homogeneous, but only that it consists of pairs of homologous terms equated to a constant.

14. *Sextic Curves of the Homogeneous Form* $F(x, y)^6 = A^6$.

The equation is understood to be referred to principal axes when it consists of terms of even powers only; but in the case of the symmetrical oval it will be seen that this description applies to each of the two pairs of conjugate axes; and there is, geometrically speaking, no reason why either pair should be considered principal axes preferentially to the other pair. The curves of this class do not pass through the centre.

In order to obtain fundamental forms, symmetrical equations are first to be considered. Of these there are in strictness only *four* species, corresponding to the four symmetrical combinations of the positive and negative signs of the terms. There are, for the sixth degree, two other forms, (γ) and (δ), in which the coefficients are symmetrical, but the signs are not symmetrical. In the form (γ) the extreme terms are positive, and the intermediate terms have unlike signs, the order of the signs being + + − +. In the form (δ) the extreme terms have unlike signs and the intermediate terms have like signs, the order being + + + −. The forms obtained by changing all the signs of the variables are, of course, the same curves. Also (γ) is a variety of (β), and (δ) is a variety of (ζ). As the equations are symmetrical, the equal coefficients of the highest powers disappear by division; the equations of the equiaxial curves of the sixth degree may then be written—

$$x^6 + Px^4y^2 + Px^2y^4 + y^6 = A^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$
$$x^6 - Px^4y^2 - Px^2y^4 + y^6 = A^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$
$$x^6 + Px^4y^2 - Px^2y^4 + y^6 = A^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$
$$x^6 + Px^4y^2 + Px^2y^4 - y^6 = A^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$
$$x^6 - Px^4y^2 + Px^2y^4 - y^6 = A^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$
$$x^6 + Px^4y^2 - Px^2y^4 - y^6 = A^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\zeta)$$

Referring to p. 1062, where the principle of classification is indicated, (α) is the Sextic Oval; (β) is the Inflexional Oval, passing into the Continuous Hyperbolic; (ϵ) is the Discontinuous Hyperbolic, consisting of alternate real and imaginary branches; (γ), (δ), and (ζ) are Inflexional Hyperbolics.

(14a). *To find the Equations of the Equiaxial Curves referred to Secondary Axes.*

The coordinates of the original equations being denoted by capitals, if, in the formula of transformation, we were to make $Y = (x+y)\sqrt{\frac{1}{2}}$, we should obtain negative values of A^6 in the transformed functions (ϵ) and (ζ). Therefore, let $X = (x+y)\sqrt{\frac{1}{2}}$; $Y = (x-y)\sqrt{\frac{1}{2}}$, and transform to bisecting axes.

In the transformation of equations (α) and (β), we have for the sum of the 1st and 4th terms,

$$1/4\{x^6 + 15x^4y^2 + 15x^2y^4 + y^6\} = X^6 + Y^6;$$

and for the sum of the 2nd and 3rd terms,

$$1/4\mathrm{P}\{x^6-x^4y^2-x^2y^4+y^6\}=\mathrm{PX}^4\mathrm{Y}^2+\mathrm{PX}^2\mathrm{Y}^4;$$

whence, by addition and subtraction we find for (α') and (β') (being the curves, α and β, referred to their secondary axes).

$$(1+\mathrm{P})x^6+(15-\mathrm{P})x^4y^2+(15-\mathrm{P})x^2y^4+(1+\mathrm{P})y^6=4\mathrm{A}^6 \quad . \quad . \quad . \quad (\alpha');$$
$$(1-\mathrm{P})x^6+(15+\mathrm{P})x^4y^2+(15+\mathrm{P})x^2y^4+(1-\mathrm{P})y^6=4\mathrm{A}^6 \quad . \quad . \quad . \quad (\beta').$$

Again, in the transformation of (ϵ) and (ζ) we have for the sum of the two extreme terms,

$$1/4\{6x^5y+20x^3y^3+6xy^5\}=\mathrm{X}^6-\mathrm{Y}^6,$$

and for the sum of the two intermediate terms,

$$1/4\mathrm{P}\{\mp 2x^5y\pm 4x^3y^3\mp 2xy^5\}\mp\mathrm{PX}^4\mathrm{Y}^2\pm\mathrm{PX}^2\mathrm{Y}^4,$$

where the upper and the lower signs belong respectively to the equations (ϵ) and (ζ).

Observing that the numerical coefficients are divisible by 2, we have for (ϵ') and (ζ'), by addition,

$$(3-\mathrm{P})x^5y+(10+2\mathrm{P})x^3y^3+(3-\mathrm{P})xy^5=2\mathrm{A}^6. \quad . \quad . \quad . \quad . \quad (\epsilon')$$
$$(3+\mathrm{P})x^5y+(10-2\mathrm{P})x^3y^3+(3+\mathrm{P})xy^5=2\mathrm{A}^6 \quad . \quad . \quad . \quad . \quad (\zeta')$$

The above are all the forms of symmetrical diametral equations that can be formed with *four* or *three* terms. If we seek for those that may be formed with only two terms, it is evident that

(1) The form $x^6+y^6=\mathrm{A}^6$, is a limiting form of (α) and (β) when $\mathrm{P}=0$. (α_1);

(2) $x^6-y^6=\mathrm{A}^6$, is a limiting form of (ϵ) and (ζ) when $\mathrm{P}=0$ . . . (ϵ_1);

(3) $4x^4y^2+4x^2y^4=\mathrm{A}^6$, is a limiting form of (β'), and therefore of (β) when $\mathrm{P}=1$, and is also a limiting form of (α'), (α), when $\mathrm{P}=-1$ . . . (β_2);

(4) $4x^5y+4xy^5=\mathrm{A}^6$, is a limiting form of (ζ'), (ζ), when $\mathrm{P}=5$, and of (ϵ'), (ϵ) when $\mathrm{P}=-5$ . . . . . (ζ_2).

Now we cannot directly obtain the last two forms *with the negative sign* from any of preceding equations. Hence, there are apparently two independent limiting forms,

$$4x^4y^2-4x^2y^4=\mathrm{A}^6;\ (\epsilon'') \text{ and } 4x^5y-4xy^5=\mathrm{A}^6 \quad . \quad . \quad . \quad (\beta'').$$

On further consideration, it is seen that (ϵ'') is derivable from (ϵ) or (ϵ'), if the coefficients of the intermediate terms in the fundamental equations are supposed invariable, while the extreme terms, x^6, y^6 are supposed to be multiplied by coefficients which are *indefinitely diminished.* Transforming to secondary axes, we find

(1) The equation of (ϵ'') is unaltered in form and value by transformation (ϵ'')

(2) The equation of β'' is $2x^5y-4x^2y^3+2x'y^5=1$ . . . . . (β''),

which is different from any of the previously given forms. But it has been found, as the result of the computation of values of r and θ, that this curve is identical with No. 10 of the table given below, which is of the form (β); the explanation being that the curve consists of four real and four conjugate branches, and has accordingly four pairs of conjugate diameters.

As the result of a study of the fundamental forms here given, I have found that there are certain other critical values of P which produce characteristic forms of the equiaxial curves. These I proceed to enumerate. From the drawings and relative tabular places (computed by the method of homogeneous variation) a very complete conception may be obtained of the possible variations of this family of sextic curves and their projections.

(14*b*). *Limiting Forms of the Equiaxial Curves.*

If n, the index number of a curve, be divisible into factors, p and q, a symmetrical function of the p^{th} or q^{th} degree may be a limiting form of the symmetrical curve of the n^{th} degree; for we have only to raise the equation of the p^{th} degree to the power q, or to raise the equation of the q^{th} degree to the power p to obtain such a limiting form.

Thus, (1) by raising the equation $x^2 \pm y^2 = A^2$ to the 3rd power, we obtain the circle and the equilateral hyperbola in the sextic form,

$$x^6 \pm 3x^4y^2 + 3x^2y^4 \pm y^6 = A^6;$$

where, the upper sign being taken, we see that *the circle* is a limiting form of the equiaxial curve (α), when $P = 3$. The lower sign being taken, the curve is *the equilateral hyperbola*, which is thus shown to be a limiting form of the equiaxial curve (ϵ) when $P = 3$. Similarly by writing x/a for x, and y/b for y, it may be shown that any ellipse or hyperbola is a limiting form of the sextic curve which is the projection of (α), or (ϵ) to the principal axes a and b.

(2) It might be expected that a symmetrical cubic would also be a limiting form of an equiaxial sextic. This, however, is not universally true. I shall, however, write down the limiting forms obtained by squaring the symmetrical cubics $x^3 \pm y^3 = A^3$, and $x^2y \pm xy^2 = A^3$.

These forms, with the equivalent forms obtained by transforming to secondary axes, are as follows:—

$$\left.\begin{matrix}(x^3 \pm y^3)^2 \\ x^6 \pm 2x^3y^3 + y^6\end{matrix}\right\} = A^6; \quad \left\{\begin{matrix}x^6 + 6x^4y^2 + 9x^2y^4 \\ y^6 + 6y^4x^2 + 9y^2x^4\end{matrix}\right\} = 2A^6$$

$$\left.\begin{matrix}(x^2y \pm y^2x)^2 \\ x^4y^2 \pm 2x^3y^3 + x^2y^4\end{matrix}\right\} = A^6 \quad \left\{\begin{matrix}x^6 - 2x^4y^2 + x^2y^4 \\ y^6 = 2y^4x^2 + y^2x^4\end{matrix}\right\} = 2A^6.$$

The forms in the second column are limiting cubic forms for a sextic curve referred to a transverse diameter X, and an asymptote Y, or the converse, as is made evident by dividing the equations by x^2 or y^2 respectively.

(3) Another series of critical forms is the series where the coefficient $P=1$. By making $P=1$ in the four equations (α), (β), (ϵ), and (ζ) we obtain curves which may also be obtained by the multiplication of the factors (x^2+y^2), (x^2-y^2) with (x^4+y^4) and (x^4-y^4). The forms (β) and (ζ) are also obtained by the multiplication of $(x^2 \pm y^2)^2$ with $(x^2 \mp y^2)$.

(4) A fourth series of critical forms are those which correspond with the polar equations of sines and cosines of multiple arcs. These will be noticed in their order.

(14c). *Form and Variations of the Equiaxial Curves* (α) *and* (β).

It is desirable to give a name to the variation of the curve consequent on the variation of the single coefficient P.

In equations of the 2nd degree the term "eccentricity" has relation to the variation due to projection, which is the only kind of variation of which these curves admit.

But in curves of the higher degrees, where we consider only those characteristics that are unaltered by projection, this kind of eccentricity is not considered at all. Hence, without ambiguity, I may make use of the term Quadrantal Eccentricity to denote the variation within each quadrant in the magnitude or direction of the secondary axes due to the variation of P, while the principal parameters remain unaltered.

The quadrants referred to are of course those which are marked out by rectangular reference lines, coinciding with the equal principal diameters or parameters of the curve.

Length of a Secondary Diameter in terms of the principal Diameter and P.—The variation of P in the oval of fourfold symmetry, has no effect on the *direction* of the secondary diameters, but only alters the ratio of their length to that of the primary. It is convenient always to put the length of a principal diameter $=1$, which is then also the value of the arbitrary term. The ratio of the secondary diameter to the primary may be denoted by Γ, which is also numerically the length of the former. In the oval forms of (α) and (β) we then find for the length of the secondary diameters, by putting $x=y$; $\Gamma^2=2x^2$, the relation,

$$2P+2=\frac{1}{x^6}=\frac{2^3}{\Gamma^6}; \quad \therefore \frac{1}{\Gamma^6}=\frac{P+1}{4}; \quad P=\frac{4}{\Gamma^6}-1.$$

In curves of the hyperbolic type where the variation of P affects the direction of the asymptotes, the quadrantal eccentricity might be measured by the tangent of their inclination to the principal axis; but this relation has not been fully investigated.

Examples of the Curves, (α) *and* (β).—The following equiaxial curves of the forms (α) and (β) have been computed and traced. The number in the first column is a reference number corresponding with that in the second table; P and Γ are as above;

P^1 is the coefficient of the intermediate terms when the equation is transformed to secondary axes, the arbitrary term then being Γ^6.

No.	P.	Γ.	Equation of Curve.
1.	4	1	$4x\,y^2+4x^2y\,=1$
2.	31	$\sqrt{\frac{1}{2}}$	$x^6+31x^4y^2+31x^2y^4+y^6=1$
3.	15	$(\frac{1}{4})^{1/6}$	$x^6+15x^4y^2+15x^2y^4+y^6=1$
4.	7	$(\frac{1}{2})^{1/6}$	$x^6+\ 7x^4y^2+\ 7x^2y^4+y^6=1$
5.	3	1	$x^6+\ 3x^4y^2+\ 3x^2y^4+y^6=1$
6.	1	$(2)^{1/6}$	$x^6+\ x^4y^2+\ x^2y^4+y^6=1$
7.	0	$(4)^{1/6}$	$x^6+\ +y^6=1$
8.	$-\frac{1}{2}$	$\sqrt{2}$	$x^6-\ \frac{1}{2}x^4y^2-\ \frac{1}{2}x^2y^4+y^6=1$
9.	-1	∞	$x^6-\ x^4y^2-\ x^2y^4+y^6=1$
10.	-5	-1	$x^6-\ 5x^4y^2-\ 5x^2y^4+y^6=1$

No. 5 is the limiting circle. Nos. 1, 2, 3, 4 are the transformed equations 9, 8, 7, and 6, with the curves turned round through an angle of 45°. On referring to Plate I., where the numerals attached to the curves are those of the first column of the table, it is seen that starting from the circle, as P falls from 3 to 0, the curve approaches more nearly to the circumscribing square (Pl. I. figs. 6, 7). For all negative values of P between 0 and -1 the curve is inflexional (fig. 8), the secondary axis becoming more and more elongated, until at the value $P=-1$ it passes into the continuous equilateral hyperbolic (fig. 9).

For values of P from -1 to -5 the curve is the *discontinuous* hyperbolic, where the angle between asymptote and axis ranges from 45° to 22° 30′, or 90° to 45° between the asymptotes.

In the diagram the curves 1, 2, 3, 4 are seen to be 9, 8, 7, 6, diminished and turned round through 45°.

For the value $P=-5$ (No. 10 of the Table) we have *four* equal *discontinuous* hyperbolics, having angle between asymptotes $=45°$ (Pl. II. fig. 2). The intervening angular spaces may be made to contain four equal and similar conjugate curves by changing the signs of all the variable terms in the equation. The equation of No. 10 referred to secondary axes contains only uneven powers (see No. 15). This curve also has the polar equation, $r^6\cos 4\theta = A^6$.

For negative values of P exceeding 5 we find hyperbolics of greater eccentricity, which are the conjugate curves of the series found for values of P between -1 and -5. The equation of these may also be obtained in another form, from the last-mentioned series, by changing the signs of all the variable terms. It is easily seen that a similar series of curves are obtained from the form (γ), because one negative term suffices to make the oval inflexional.

The annexed table contains the places of x and y computed by the homogeneous method for curves 6 to 10 of the preceding table, so far as necessary, viz., from 0° to 45°.

From 45° to 90°, the places are obtained by changing x for y, and the places in the other three quadrants by changing the signs of x and y.

Table of Computed Places.

θ		10°	15°	20°	25°	30°	35°	40°	45°
Curve 6.	x	0·9947	0·9877	0·9766	0·9603	0·9365	0·9026	0·8557	0·7936
	y	0·1754	0·2646	0·3555	0·4478	0·5406	0·6320	0·7180	0·7936
Curve 7.	x	0·9999	0·9998	0·9996	0·9984	0·994	0·982	0·951	0·891
	y	0·176	0·268	0·364	0·466	0·574	0·687	0·798	0·891
Curve 8.	x	1·002	1·007	1·013	1·022	1·037	1·048	1·049	1·0 *
	y	0·177	0·270	0·369	0·477	0·599	0·734	0·880	1·0
Curve 9.	x	1·006	1·014	1·028	1·051	1·091	1·171	1·373	∞ †
	y	0·178	0·272	0·374	0·490	0·630	0·820	1·152	∞

	θ	5°	7° 30′	10°	12° 30′	15°	17° 30′	20°	22° 30′
Curve 10.	x	1·007	1·016	1·030	1·051	1·084	1·140	1·258	∞
	y	0·088	0·134	0·182	0·233	0·291	0·360	0·458	∞
	$\log r$	0·005	0·010	0·019	0·032	0·050	0·078	0·127	∞

(14*d*). *Examples of the other Equiaxial Curves.*

The following equiaxial curves of the forms ϵ and ζ have been computed and traced: the second column is a reference number, corresponding with that in the subjoined table of computed places: I, is the inclination of asymptote to transverse axis. The equation of each curve being given in different forms, those containing even powers are referred to a transverse axis X; those containing odd powers are referred

* For No. 8, the following additional values are necessary to trace the second inflexion—

	41°	42°	43°	44°
x	1·045	1·039	1·030	1·016
y	0·908	0·935	0·961	0·981

† For No. 9 we find also, $\theta = 44° 59'$; $x = 4·165$; $y = 4·163$. And for other curves additional places have been computed where necessary for tracing the inflexions.

to an asymptote X; the polar equations in ϵ and ζ are referred to the transverse axis.*

Class.	No.	I.	Equations of the Curves.
ϵ	11.	45°	$\begin{cases} x^6-x^4y^2+x^2y^4-y^6 & =1 \\ x^5y+6x^3y^3+xy^5 & =1 \end{cases}$
ϵ	12.	15°	$\begin{cases} x^6-15x^4y^2+15x^2y^4-y^6 & =1 \\ \mp 6x^5y \pm 20x^3y^3 \mp 6xy^5 & =1 \\ r^6.\cos(6\theta)=1 : r^6.\sin(6\theta) & =1 \end{cases}$
ϵ or ζ	13.	45°	$\begin{cases} x^6-y^6=1 & =1 \\ 3x^5y+10x^3y^3+3xy^5 & =2 \end{cases}$
ζ	14.	45°	$\begin{cases} x^6+x^4y^2-x^2y^4-y^6 & =1 \\ 2x^5y+4x^3y^3+2xy^5 & =1 \\ r^6.\cos(2\theta)=1 : r^6.\sin(2\theta) & =1 \end{cases}$
β''	15.	22° 30′	$\begin{cases} 4x^5y-4xy^5 & =1 \\ 4x'^5y'-4x'y'^5 & =1 \\ r^6.\sin(4\theta)=1 : r^6\cos(4\theta) & =1 \end{cases}$
ϵ''	16.	...	$\begin{cases} 4x^4y^2-4x^2y^4 & =1 \\ 2x^5y-4x^3y^3+2xy^5 & =1 \end{cases}$

The mode of variation of these curves is very remarkable; and it is the more deserving of attention, because it results from the rule of signs (p. 1065) that *for any even degree*

* The identity of the polar equations in the annexed tables with the Cartesian equations is proved as follows:—

$\text{Cos}(2\theta)=\cos^2\theta-\sin^2\theta;$
$\text{Cos}(4\theta)=\cos^4\theta-6\cos^2\theta.\sin^2\theta+\sin^4\theta;$
$\text{Cos}(6\theta)=\cos^6\theta-15\cos^4\theta.\sin^2\theta+15\cos^2\theta.\sin^4\theta-\sin^6\theta.$

$\therefore$ (1) $A^6=r^6.\cos(2\theta)=r^4(r^2.\cos\overline{2\theta})=(x^2+y^2)^2.r^2(\cos^2\theta-\sin^2\theta)$
$=x^6+x^4y^2-x^2y^4-y^6.$ [No. 14 of Table.]

(2) $A^6=r^6.\cos(4\theta)=r^2.(r^4.\cos\overline{4\theta})=(x^2+y^2).r^4.(\cos^4\theta-6\cos^2\theta.\sin^2\theta+\sin^4\theta)$
$=x^6-5x^4y^2-5x^2y^4+y^6.$ [No 10 of Table.]

(3) $A^6=r^6.\cos(6\theta)=r^6(\cos^6\theta-15\cos^4\theta.\sin^2\theta+15\cos^2\theta.\sin 4\theta-\sin^6\theta)$
$=x^6-15x^4y^2+15x^2y^4-y^6.$ [No. 12 of Table.]

Again, observing that—

$\text{Sin}(2\theta)=2\sin\theta.\cos\theta;$
$\text{Sin}(4\theta)=4\cos^3\theta.\sin\theta-4\cos\theta.\sin^3\theta;$ and
$\text{Sin}(6\theta)=6\cos^5\theta.\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta.\sin^5\theta.$

$\therefore$ (1) $A^6=r^6.\sin(\overline{2\theta})=r^4(r^2.\sin\overline{2\theta})=(x^2+y^2)2xy$
$=2x^5y+4x^3y^3+2xy^5.$ [No. 14 of Table.]

(2) $A^6=r^6.\sin(4\theta)=r^2(r^4.\sin 4\theta)=(x^2+y^2)(4x^3y-4xy^3)$
$=4x^5y-4xy^5.$ [No. 15 of Table.]

(3) $A^6=r^6.\sin(6\theta)=6x^5y-20x^3y^3+6xy^5.$ [No. 12 of Table.]

there is a series of equivalent curves which go through corresponding phases. The different forms are shown in Pl. II. figs. 1 to 4.

In their complete forms the equiaxial curves (ϵ) of the 6th degree have three pairs of asymptotic axes; one pair being the secondary axes, whose inclination to the primary axes is always 45°; the other two pairs having an inclination depending on the value of P (fig. 3).

Table of Computed Places.

Curve 11.	θ	10°	15°	20°	25°	30°	35°	40°	44° 59′	45°
	x	1·005	1·012	1·021	1·034	1·054	1·079	1·146	2·470	∞
	y	0·177	0·271	0·372	0·482	0·608	0·756	0·961	2·468	∞
Curve 12.	θ	2°	4°	6°	8°	10°	12°	14°	15°	
	x	1·002	1·013	1·030	1·059	1·106	1·190	1·414	∞	
	y	0·035	0·071	0·108	0·149	0·195	0·253	0·353	∞	
Curve 13.	θ	10°	15°	20°	25°	30°	35°	40°	44° 59′	45°
	x	1·000	1·000	1·001	1·002	1·007	1·021	1·074	2·461	∞
	y	0·176	0·268	0·364	0·467	0·581	0·715	0·901	2·460	∞
Curve 14.	θ	5°	10°	15°	20°	25°	30°	35°	40°	45°
	x	0·9986	0·9951	0·9895	0·9824	0·9754	0·9721	0·9797	1·024	∞
	y	0·0874	0·1755	0·2652	0·3576	0·4548	0·5611	0·6860	0·861	∞
Curve 15.*	θ	0°	2° 30′	5°	7° 30′	10°	12° 30′	15°	17° 30′	22° 30′
	x	∞	1·338	1·191	1·113	1·060	1·021	0·989	0·964	0·924
	y	0°	0·058	0·104	0·147	0·187	0·226	0·265	0·304	0·383
	$\log r$	∞	0·127	0·078	0·050	0·032	0·019	0·010	0·005	0·0011
Curve 16.	θ	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°
	x	∞	1·786	1·421	1·247	1·139	1·066	1·020	1·000	1·030	∞
	y	0	0·156	0·251	0·334	0·414	0·497	0·589	0·700	0·865	∞

In the form (ϵ), when P = 3, the curve has the limiting form of the equilateral hyperbola (of 2nd degree), the three pairs of asymptotes being there *coincident*.

When P>3, there are six equal real branches, and six conjugate branches. If P exceeds 3 by a very small quantity, the first real branch (bisected by X) is nearly rectangular, and the first conjugate branch is extremely acute. The secondary axis divides this from a similar acute real branch; and then there is a nearly rectangular conjugate branch bisected by Y. When P = 7, the inclination of the first pair of asymptotes is

* These values of $\log r$ are identical with those of No. 10 of the preceding table. In No. 15 the curve is referred to asymptotes.

$\pm 22^\circ\ 30'$, and the first real branch is contained within an angle of 45°. The first conjugate branch and the second real branch are contained within angles of $22^\circ\ 30'$; and the second conjugate branch (bisected by Y) is contained within the angle 45°, and so on.

When $P = 15$ there are six real and six alternate conjugate branches all equal, each contained within an angle of 30° (fig. 3). When P exceeds 15, we have a series of pairs of unequal curves (which have not been fully investigated), but are probably the conjugates of the preceding set.

Returning to the neutral form of the equilateral hyperbola, and varying P by diminishing it indefinitely:—If $P>0$ and <3, x and y can only become infinite for $\theta = \pm 45^\circ$; and we have a series of equilateral forms consisting of two real and two conjugate branches. The variation of P between these limits only affects the quadrantal eccentricity. The form $x^6 - y^6 = 1$ is the limit between the forms (ϵ) and (ζ). In Pl. II. fig. (1) the curve which is nearest the centre is No. 13 of the Table; the curve furthest from the centre is the limiting equilateral hyperbola; and the intermediate curve is No. 11.

In the series (ζ), where the signs of the two intermediate terms of the equation do not follow in alternate order, the curves are inflexional and equilateral, the only asymptotes being the secondary axes.

In Plate II. fig. 4, the curve which is nearest the centre is No. 14 of the Table. The curve next it, having the same pair of asymptotic axes, is traced from an equation of the same form with a different coefficient, ($P = 15$). The curve which has the axis of X for one of its asymptotes is evidently a limiting form of the same series, and is No. 16 of the Table. Its minimal radius-vector corresponds to

$$\theta = \tan^{-1}\cdot 5 = 26^\circ\ 34', \quad \text{nearly.}$$

The variety (δ) resembles (ζ) in its forms and inflexions, but is not equilateral, as (ζ) is. One of these forms is figured, Pl. IV. fig. 4. Its equation is

$$x^6 + x^4y^2 + x^2y^4 - y^6 = 1;$$

and for the curve figured ($P = 1$) the inclination of asymptotes to axis X (which depends on the value of P) is $53^\circ\ 37'$, nearly.

All the curves here traced have been computed by the tangent formula, which is the best for studying the transitions from one of the enumerated forms to the other or others.

General Results.—It is evident that the results which have been obtained are in the main independent of the degree of the symmetrical homogeneous equation. For equations of curves of even degree, referred to axes of symmetry, these results may be generalised as follows:—

(1) If all the terms are positive, the curve is an oval of fourfold symmetry, entirely concave to the centre, and having the circle as a limiting form.

(2) If all the pairs of homologous terms of the symmetrical expression have like signs, but some of the pairs are positive and some are negative, the curve is the Inflexional oval, for all *positive* values of the coefficients *which are less than those of the binomial expansion*, and for all fractional negative values. Outside these limits the curve is a hyperbolic, with alternate real and conjugate branches, the limit between the closed and open forms being the continuous hyperbolic in which all the branches are *equal and real.*

(3) If all the pairs of homologous terms have unlike signs, and if the equation when arranged in binomial form has the terms (being all even) alternately positive and negative, the curve consists generally of n hyperbolic real branches, with alternating conjugate branches; but for certain values of the coefficients, the number may be reduced to two equilateral real branches, having the equilateral hyperbola as a limiting form.

(4) If all the pairs of homologous terms have unlike signs, but if the positive and negative terms do not follow in alternate order, the curve consists of two equilateral inflected branches, the curve being *concave* to the centre and to the asymptotes where it crosses the axis of X, but after inflexion on either side of that axis becoming convex to the asymptotes.

The same form, where some of the pairs of terms have like signs, and some have unlike signs, except that the assymptotic axes are not rectangular.

(5) In all cases where the equation is reducible to the two-term Polar form,

$$r^n \cos(p\theta) = 1 ,$$

the curve consists of a number of alternate real and conjugate branches, which are all equal. The number of such forms evidently is $n/2$, because p may have the series of values, n, $n-2$, $n-4$, &c.

(6) If the equation is not a symmetrical expression, but is homogeneous, the curves fall into the above categories, but have not in general secondary axes.

15. *Determination of Contour-lines of Homogeneous Surfaces.*

If v_1 v_2 be coordinate quantities of any symmetrical diametral equation (suppose of the form α), and if x^2+z^2 be substituted for v_1^2, and y^2+z^2 for v_2^2, and the equation be expanded in terms of powers of x^2, y^2, z^2, we obtain the equation of a symmetrical homogeneous surface referred to conjugate diameters. The equation then takes successively the three forms which follow—

$$v_1^6 + \mathrm{P} v_1^4 v_2^2 + \mathrm{P} v_1^2 v_2^4 + v_2^6 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(x^2+z^2)^3 + \mathrm{P}(x^2+z^2)^2(y^2+z^2) + \mathrm{P}(x^2+z^2)(y^2+z^2)^2 + (y^2+z^2)^3 = 1 \quad . \quad . \quad (2)$$

$$x^6 + 3x^4y^2 + 3x^2y^4 + z^6 + \mathrm{P}(x^4y^2 + x^4z^2 + 2x^2y^2z^2 + 2x^2z^4 + z^4y^2 + z^6)$$
$$+ \mathrm{P}(y^4x^2 + y^4z^2 + 2x^2y^2z^2 + 2y^2z^4 + x^2z^4 + z^6) + y^6 + 3y^4z^2 + 3y^2z^4 + z^6 = 1 \quad . \quad . \quad (3).$$

If we suppose the equation to be given in the form (3), we can only find values of

x, y, z for a central plane, and it is evidently impossible to determine a contour-line of the surface parallel to a central plane. Because if, for example, we make $z=a$, the equation in x and y is thoroughly heterogeneous, containing in fact all the even terms of the general equation of the 6th degree.

But if the equation be presented for solution in the form (2), we can find values of x and y in contour series. For we have then only to find a series of values of v_1^2 v_2^2 (or x^2+z^2 and y^2+z^2); then, making $z=a$, we find from the series of values $v_{1,1}$ $v_{2,1}$; $v_{1,2}$ $v_{2,2}$; $v_{1,3}$ $v_{2,3}$, &c., the coordinates

$$x_1=\sqrt{v_{1,1}^2-a^2};\quad x_2=\sqrt{v_{1,2}^2-a^2};\quad x_3=\sqrt{v_{1,3}^2-a^2};\quad \&c.$$
$$y_1=\sqrt{v_{2,1}^2-a^2};\quad y_2=\sqrt{v_{2,2}^2-a^2};\quad y_3=\sqrt{v_{2,3}^2-a^2};\quad \&c.$$

Through such a series of points a contour-line of the surface, in the plane $z=a$, may be traced.

In the same manner contour-lines may be traced for other planes parallel to XY, viz. $z=b$; $z=c$, &c.

Such contour-lines have been computed and traced for surfaces derived from the curves (α) and (ϵ).

Plate VI. fig. 1, represents four contour-lines of the above surface (Eq. 2), with the coefficient P = 1. The values of v_1 v_2 were taken from the preceding Tables (curve α, 6). The maximum value of z was found to be, $z_0=\cdot7938$; and the three inner contour-lines were found by taking z successively equal to $\frac{1}{2}z_0$, $\frac{2}{3}z_0$, and $\frac{3}{4}z_0$. The outermost contour-line of this figure is the equatorial section of the surface, in the plane, $z=0$, and is identical with the curve of the Table, which is also figured in Pl. I.

It will be observed that as the circumference of the contour-lines decreases, the variation of curvature within the curve becomes less, the limiting form being evidently circular.

Plate VI. fig. 2, represents a series of contour-lines for the hyperbolic surface of two sheets, derived from (ϵ, 12) of the Tables by writing x^2+z^2 for v_1^2 and y^2+z^2 for v_2^2. In this instance I have been less fortunate in the choice of contour-lines, because the lines are not far enough apart to give a clear notion of the figure of the surface. The values of z^2, from which the computations were made, are ·003, ·0125, and ·0275, and the results are shown in the figure.

I may here observe that, while the preceding illustrations are confined to symmetrical forms, it is apparent that if the analytical expressions were varied by merely altering the coefficients of the terms, such a variation would only affect the symmetry of the curves, and would not in general produce a curve of a different type. There is no difficulty in forming any number of systems of unsymmetrical curves or contour-lines of surfaces, as we have only to fix on any unsymmetrical homogeneous expression in v_1v_2, and to replace these quantities by $\sqrt{ax^2+z^2}$ and $\sqrt{by^2+z^2}$, giving such values to z as may be desired; x and y are then found from v_1 v_2.

16. *Central Curves whose Equations are of the Form* $F_1(x, y)^n = F_2(x, y)^{n-p}$.

The first function may be divisible by the second, without remainder, the equation being then reducible to one of lower degree. This will generally be the case where the equation consists of pairs of homologous terms, all of which have like signs, or all unlike signs. I here suppose that the equation is not divisible.

Confining our attention, as before, to symmetrical diametral equations, it is evident that such equations always contain at least one uncombined power of the variables, because, if the equation be given in composite terms, we can always divide out the lowest powers of x and y. When the equation after reduction consists of only two homogeneous parts its form is easily determined. Transforming to polar coordinates and dividing by the lowest power of r, we obtain an equation of the form $r^{n-p} = \frac{f^p(\cos\theta, \sin\theta)}{f^n(\cos\theta, \sin\theta)}$. The denominator of this fraction is formed from the terms of the highest homogeneous part, and if its terms be all positive, r cannot become infinite; but if the numerator be wholly positive and the denominator contains positive and negative terms, there will be certain values of θ for which r is infinite, these being the same as were found for the curves β, ϵ, ζ (p. 1076). Again, if the numerator consists only of positive terms, the curve cannot pass through the origin, as it necessarily does where some of the terms in the numerator are positive and some are negative. If the terms of the highest homogeneous part be all positive, and the terms of the lower degree be partly positive and partly negative, the curve will be of the "foliated" type, consisting of a series of loops symmetrically arranged about the centre-origin, and having no inflexions except at the centre where the trace passes from one loop into another.

More generally, for symmetrical expressions of any even degree, and any number of pairs of homologous terms of even powers of the variables equated to zero; which may be written $u_n + u_{n-2} + \ldots u_2 = 0$: and are supposed to be reduced to their lowest terms,—

(1) If any pair or pairs of homologous terms of the part u_n have unlike signs, while the terms of lower degree are all positive, or are all negative, then, by transforming to polar coordinates and dividing by r^n, we find that $r^n = \infty$ is a solution of the equation where $u_n = 0$. The curve, therefore, consists of branches of infinite extent resembling those already described under the character of contour-lines of surfaces formed from the equations (ϵ) and (ζ).

(2) If u_n consists entirely of positive terms, and if any pair or pairs of terms in the parts of lower degree have unlike signs, and the other pairs are all positive, then the curve consists of finite branches or loops passing through the centre. Because (1) the radius-vector cannot become infinite, since u_n consists of positive terms, and (2) when $\cos\theta = \sin\theta$, all the negative terms are neutralised by the homologous positive terms, and there remains a series of positive terms equated to zero; whence $r = 0$. In this case, since $\cos\theta = \sin\theta$ when $r = 0$, the tangents at the centre bisect the angles between the axes of reference, and are secondary axes, and the centre is a point of inflexion.

(3) A curve whose equation is strictly symmetrical, and consists of terms of even powers, whereof only one homologous pair have unlike signs, can have only two loops; but if any of the homogeneous expressions u_{n-2}, u_{n-4}, &c., has a middle term, the curve may have a number of loops depending on the degree of the equation, because then the angle for which $r = 0$ depends on a relation between three terms.

(4) A curve, consisting of loops passing through the centre, is also the result where u_n is positive, and $u_{n-2} + u_4$, &c., consists of pairs of positive terms and pairs of terms which are both negative; because evidently there must be definite values of θ which render $r = 0$.

(5) If the terms $u_{n-2} + u_{n-4}$, &c., can be resolved into factors, while u_n consists of pairs of unlike terms, the hyperbolic branches may break up into detached ovals sometimes with an infinite branch extending beyond these and within the same angular space.

These seem to exhaust the possible combinations for symmetrical equations without an arbitrary term.

(6) If we transform to axes equally inclined to the original symmetrical axes, the curve will be symmetrical about the new axes also, and the new equation will consist of even or of uneven powers of the variables, according to the rule of signs given above (p. 1065). In applying the rule, each homogeneous part of the equation is to be considered separately; so that, if one homogeneous part consist of positive terms, and the other of alternate positive and negative, their equivalents in the transformed equation will consist respectively of even and uneven powers.

(7) There are limiting parabolic forms where the highest homogeneous part contains only one of the variables, *i.e.*, consists of a single term.

(8) In the case of axes which do not meet the curve except at the centre, these are, notwithstanding, true diameters, as the form of the equation proves. Accordingly, every such *Exterior Diameter*, if I may so term it, bisects the intercepts made by the adjacent branches or chords drawn parallel to the conjugate Exterior Diameter, and therefore bisects the Bitangents.

(9) These results are manifestly true, with the necessary restriction as to angles, for all projections of the curves in question.

(10) By an easy extension of (8) we have for all symmetrical equations of this type, and their projections, this relation: *Each pair of Bitangents is parallel to one axis of symmetry, and is bisected by the axis conjugate to it.*

(16*a*). *Examples of such Curves* (*Sixth Degree*).

Any of the functions on the left side of the sign of equality may be combined with any on the right; but of course the terms, when equated to zero, cannot all be positive.

The limits suited to this paper have been already so far exceeded that I shall not

attempt to illustrate all the varieties. The following illustrations of curves of the 6th degree of two homogeneous parts include the most characteristic forms :—

$$\left.\begin{aligned}(1)\quad & x^6+3x^4y^2+3x^2y^4+y^6=x^2-y^2\\ & r^4=\cos 2\theta\end{aligned}\right\}$$

$$\left.\begin{aligned}(1a)\quad & x^6+3x^4y^2+3x^2y^4+y^6=2xy\\ & r^4=\sin 2\theta\end{aligned}\right\}$$

$$\left.\begin{aligned}(2)\quad & x^6+3x^4y^2+3x^2y^4+y^6=x^4-6x^2y^2+y^4\\ & r^2=\cos 4\theta\end{aligned}\right\}$$

$$\left.\begin{aligned}(2a)\quad & x^6+3x^4y^2+3x^2y^4+y^6=4x^3y-4xy^3\\ & r^2=\sin 4\theta\end{aligned}\right\}$$

$$(3)\quad x^6-6x^4y^2+6x^2y^4-y^6=A^6(x^2-y^2)$$

The following Tables contain the computed places for the symmetrical half of a foliation or loop of each curve :—

Equation (1).

$\theta=$	0°	5°	10°	15°	20°	25°	30°	35°	40°	44°·47	45°
$r=$	1·0	·996	·985	·965	·936	·895	·841	·765	·646	·363	0·

From 45° to 135° values of r are impossible.
From 135° to 180° we obtain the above series reversed.
Similar results from 180° to 360°.
The curve consists of two loops, and there are two inflexions at the centre.
Pl. III. fig. (1) is this curve, and fig. (2) is a projection of it.

Equation (1a).

$\theta=$	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°
$r=$	0·	·646	·765	·841	·895	·936	·965	·985	·996	1·0

(1a) is therefore (1) transformed to secondary axes, which are the tangents at the central point of inflexion.

Equation (2).

$\theta=$	0°	3°	6°	9°	12°	15°	18°	21°	$22°\frac{3}{8}$	$22°\frac{1}{2}$
$r=$	1·	·989	·956	·900	·818	·707	·556	·323	·132	0·

Equation (2a).

$\theta=$	0°	1°·5	4°·5	7°·5	10°·5	13°·5	16°·5	19°·5	22°·5
$r=$	0	·323	·556.	·707	·818	·900	·956	·989	1·0

Equation (2a) represents the same curve as (2) transformed to axes inclined to the former at the angle 22°·5, and having the axes tangents at the centre origin. Values of r are impossible for each alternate arc of 45°. The curve consists of four equal, similar, and symmetrical loops or foliations, and the centre is a point of inflexion for the four intersecting lines (fig. 3). Fig. (4) is a linear projection of the same curve.

In this and the preceding figure the bitangents are seen to be parallel to the conjugate axes.

The same construction is evident in fig. 4, which is a projection of the last-mentioned curve.

Equation (3).

$\theta=$	0°	10°	15°	20°	22°·5	23°	23°·10
$r=$	1	1·03	1·15	1·37	1·68	3·26	∞

Plate III. fig. 5, represents this curve, which consists of four hyperbolic branches without inflexions. As these branches do not pass through the centre, although u_2 contains a negative term, it is evident that the equation is reducible to one of lower degree with an arbitrary term. Accordingly, by division we find for the reduced form of equation (3), $-x^4+y^4-5x^2y^2=A^6$, or say, $=1$.

17. *Contour-lines of Surfaces derived from Central Curves passing through their Centres.*

Having already explained the mode of derivation of such lines, it is here only necessary to describe the illustrative figures (Pl. VI. figs. 3 and 4).

Figure 3 represents a surface with a central core or axis, being the axis of Z. It is formed from equation (1), above, by taking $r^2=r'^2+z^2$. $z_0=1$ is a maximum, and the four contour-lines are sections of the derived surface in the planes, $z=0$, $z=·65$, $z=·75$, and $z=·9$.

Figure 4 is formed from Equation 2 (above) by transforming to x-and-y coordinates, and then taking $x^2=x^2+z'^2$; $y^2=y'^2+z^2$. $z_0=1$ is a maximum, and the contour-lines are for sections of the derived surface in the planes, $z=0$, $z=·1$, and $z=·14$.

In this surface the pear-shaped figures are only united at the cusps, which are also points of inflexion, and the sections consist of detached loops. The diagram makes clear what is the kind of variation of an equation by which a continuous looped curve may break up into detached loops or ovoids. We see that these only become continuous through the disappearance of the quantity z, and by the equation becoming a homogeneous function of x and y of the form,

$$F_n(x, y)/F_{n-p}(x, y)=1\,.$$

It will be understood that these contour-lines are all traced from a sufficient number

of computed values of x and y, although I have not printed the tables of computed places.

18. *The Wave-Surface.*

This surface, as usually given, is of the form

$$a^2x^2/(r^2-a^2)+b^2y^2/(r^2-b^2)+c^2z^2/(r^2-c^2)=0 \quad . \quad . \quad . \quad . \quad (1).$$

This, when cleared of fractions and expanded, is an equation of the 6th degree, containing all the terms of even powers of the general sextic equation of three variables.

If, however, the equation be merely cleared of its fractional form, and the terms be arranged in powers of r, it has the form

$$\{a^2x^2+b^2y^2+c^2z^2\}r^4-a^2b^2c^2r^4+a^2b^2c^2r^2=0;$$

whence, dividing by $a^2b^2c^2r^2$, and writing α, β, and γ, for b^2c^2, c^2a^2, and a^2b^2, we have

$$\{x^2/\alpha+y^2/\beta+z^2/\gamma\}r^2-r^2+1=0; \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where

$$r^2=x^2+y^2+z^2.$$

This may be written

$$u_4-u_2+1=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

where u_4 and u_2 are homogeneous functions of x^2, y^2, and z^2 (or of x^2 and y^2 in the plane curve), *consisting entirely of positive terms.*

The generalised form of the wave-surface, or wave-curve of any even degree, evidently is

$$u_m+u_n+\ldots-u_p-u_q+1=0 \quad . \quad . \quad . \quad . \quad . \quad (4);$$

where u is defined as above. The equation has an arbitrary term. The definition of u implies that each homogeneous part of the equation consists of *terms of like signs*, and under this condition this equation of different homogeneous parts represents an oval (though it is usual only to consider the semi-oval) entirely concave to the centre. If any of the homogeneous parts u_p should consist of a homologous pair of negative terms and a homologous pair of positive terms, the curve would be the inflexional oval (Pl. IV. fig. 5); but, as already seen, so long as each homologous pair of terms have like signs, r can neither become 0 or ∞; and the curve or surface is always and necessarily a continuous closed curve of double symmetry.

Plate III. (fig. 7) is a representation of the limiting form of the 4th degree, obtained from equation (1) by suppressing the 3rd term. The reduced equation is

$$a^2x^4+(a^2+b^2)x^2y^2+b^2y^4=a^2b^2r^2,$$

whence

$$a^2\cos^4\theta+(a^2+b^2)\cos^2\theta.\sin^2\theta+b^2\sin^4\theta=a^2b^2/r^2. \quad [a=\tfrac{1}{2}\,;\ b=\tfrac{1}{3}].$$ Dividing by a^2b^2,

$$\therefore\ 9\cos^4\theta+13\cos^2\theta\sin^2\theta+4\sin^4\theta=1/r^2.$$

$\theta=$	0°	5°	10°	15°	20°	25°	30°	35°	40°	
$r=$	·333	·33	4·336	·340	·345	·351	·359	·369	·380	
$\theta=$	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
$r=$	·392	·406	·421	·437	·452	·467	·480	·491	·498	·500

Returning to the equation of the wave-surface in its usual form (1), the curve of any section through an axis, Z, is most easily computed by transforming to polar coordinates. If, as usual, we make $x=r\cos\theta.\cos\phi$; $y=r\sin\theta.\cos\phi$; $z=r\sin\phi$; and then divide by $r^2\cos^2\phi$, we obtain

$$\left(\frac{c^2-r^2}{c^2}\right)\left\{\left(\frac{a^2.\cos^2\theta}{r^2-a^2}\right)+\left(\frac{b^2\sin^2\theta}{r^2-b^2}\right)\right\}=\tan^2\phi\,,$$

whence ϕ may be found for any given values of r and θ.

In the following illustration, Pl. III. fig. (6), I suppose a section through Z making the angle

$$\theta=45\,;\ \sin\theta=\cos\theta=\sqrt{\tfrac{1}{2}}\,.\ c=3\,;\ b=2\,;\ a=1.$$

The form of the equation shows that r must be >2 ; <3 .*

$r=$	2·0	2·05	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0
$\theta=$	90°	73°·2	58°	47°·1	39°·7	33°·8	28°·7	24°·2	19°·9	15°·5	10°·5	0°·

The two curves are shown in figs. (7) and (6), and although the first is of the 4th degree and the 2nd is of the 6th degree, the resemblance is very apparent. These may be compared with the curve of Pl. II. fig. (5), which represents the symmetrical equation

$$x^4+x^2y^2+y^4=\tfrac{1}{4}(y^2+y^2).$$

19. *Curves Symmetrical about One Axis.*

It has been observed that an ordinary section of a central surface is only central when it is taken parallel to a principal plane. But now, if a central section be taken in any direction through an axis of symmetry (Y) of the central surface, then all sections parallel to this will be symmetrical about y, but will not necessarily or usually be symmetrical

* The numerical equation is $\frac{9-r^2}{18r^2-18}+\frac{18-2r^2}{9r^2-36}=\tan^2\varphi$.

about the axis perpendicular to Y. To obtain a curve of single symmetry from any central surface referred to conjugate axes XYZ, we have only to transform to new axes x and z, leaving Y unaltered. If we then make $z=$ unity, or any arbitrary value, within proper limits, we obtain an equation in x and y, which is the required equation. If the given central surface have all its terms positive, then the form of the curves of the oblique sections (parallel to an axis) resembles that of the Cartesian oval; that is, it is a symmetrical closed curve without inflexions, but more pointed towards the positive direction of x than towards the negative.

I shall give an illustration of such a curve of the 6th degree. Let the surface equation be

$$X^6+Y^6+Z^6=1\,.$$

This, when transformed to secondary axes in the plane XZ has the equation

$$x^6+15x^4z^2+15x^2z^4+z^6y^6=1\,.$$

In the equation, as first given, let Y remain unchanged, and let the equation be tranformed to axes x and z, each having the inclination 60° to the original plane XY. The formula of transformation is

$$X=(x-z).\cos 60^\circ;\ Z=(x+z).\sin 60^\circ.$$

If in the transformed equation z be taken equal to unity, and the equation cleared of fractions, the resulting expression for the plane curve is

$$16y^6=9-\{7x^6+39x^5+105x^4+130x^3+105x^2+39x\},$$

where the new arbitrary term, 9, is the difference between z^6 or unity and the arbitrary term of the transformed surface equation. The new plane xy is then inclined at 60° to XY.

The following values of x and y have been computed :—

$-x=$	0	$-\cdot1$	$-\cdot2$	$-\cdot3$	$-\cdot4$	$-\cdot5$	$-\cdot6$	$-\cdot7$	$-\cdot8$	$-\cdot9$
$\pm y=$	$(9/16)^{1/6}$ $=\cdot9086$	·953	·972	·978	·976	·967	·950	·919	·881	·801

$-x=$	$-\cdot99$	-1	$+x$	·1	·2
$\pm y=$	·561	0·	$\pm y$	·761	impossible

The approximate value of $+x$ when $y=0$ is ·154.

The value $-x=-1$, when $y=0$, is exact.

If the equation of the derivative surface contains the terms $Px^4y^2+Px^2y^4$, the equation of the section may contain additional terms of the form $y^2(x^4+x^3+x^2+x)$ and $y^4(x^2+x)$, where the coefficients are omitted.

If in the original equation y^6 be taken negative, the surface will be a sextic hyperboloid of one sheet, y being the axis. Taking a section whose inclination to the original plane XY is 60°, we obtain the same expression for the plane curve as that last given, except that the terms on the right side of the sign of equality have their signs changed.

The following values of x and y have been computed for this hyperbolic curve of single symmetry: the first value only being approximate, those from ·2 to 2·0 being exact.

$+x =$	·154	·2	·4	·6	·8	1·0	2·0 ∞
$\mp y =$	0	·80	1·14	1·35	1·54	1·72	2·60 ∞ .

The values of x of the immediately preceding table give an opposite and dissimilar branch.

The two curves are shown in figs. (1) and (2) of Pl. IV.

(19*a*). *To find a Symmetrical Expression for the Oval of Single Symmetry.*

Referring to the figures of the curves, since for every pair of equal positive and negative values of y there are two values of x, it is always possible to inscribe a square in such a figure. Let the oval be referred to the diagonals of the inscribed square as coordinate axes. Then the equation must satisfy three conditions:—(1) The uncombined terms of x and y are all terms of even powers, otherwise the values of $+x$ and $-x$ would not be equal when $y = 0$; (2) the equation is a symmetrical expression, because the axes of reference are equally inclined to the axis of symmetry; (3) the terms are all positive, because, according to the rule of signs (above), it is only positive pairs of even terms which, when transformed to bisecting axes, produce *exclusively even powers of y*, as must be the case here. There is a fourth condition. I may here anticipate what is proved in the section on radial coordinates, that the algebraic equations of these curves, when referred to their axes of symmetry, contain no *uneven powers* in the *even terms; i.e.*, these even terms are of the form $x^{2p}y^{2q}$, and we have already shown that the transformation to axes having the inclination 45°, does not introduce uneven powers. Hence (4) our symmetrical equation may consist of pairs of the terms x^6, y^6; x^4y^2, x^2y^4; x^2y^2 and x^2, y^2 together with composite uneven terms. If an equation satisfying these conditions be made homogeneous by supplying powers of z, it is seen that the axes of the plane curve lie in principal planes of the homogeneous closed symmetrical surface, and that the origin of the plane curve is in a diameter of this surface.

Unless all the uneven terms are present, the oval will be inflexional. See figure (3) of Plate IV.

Similar results are obtained for the hyperbolic curve of single symmetry.

These curves are best investigated by means of radial equations from two foci, as given in the sequel. The origin or point to which the last equation is referred has no

direct relation that I can discover to the foci. It is certainly not the mean point between the foci in the curve of single symmetry, because then the radial equation would be a symmetrical expression with equal coefficients, which is of course not the case in curves of single symmetry.

It is evident that the sextic hyperboloid of one sheet, and indeed a similarly constructed surface of any degree, will furnish either oval or asymptotic curves of the quasi-Cartesian type, according to the angle at which the section is taken.

If the equation of such a surface contains only the highest power of one of the *positive* terms, x, then when $x^n =$ the arbitrary term, the equation reduces to a pair of right lines or *rules*. But the number of such rules apparently cannot exceed that of the conjugate diameters for a hyperboloid of any even degree above the second.

(19*b*). *Examples of Curves of Single Symmetry.*

To form the equation of the oval of single symmetry of any degree, it is not necessary to go through the process of forming a surface equation and then transforming to new axes. I have only done this to illustrate the theorem that every plane curve is a section of a homogeneous surface of the same degree.

From the mode of formation of the preceding expressions, it is easily seen that a symmetrical binomial function of x and z, with the highest power of y added, becomes an oval of single symmetry when a definite value is given to z, as in the following illustration:—

$$\mathrm{P}(x+z)^8 = \mathrm{P}\{x^8+8x^7z+28x^6z^2+56x^5z^3+70x^4z^4+56x^3z^5+28x^2z^6+8xz^7+z^8\} \pm y^8 = 1,$$

By making $z=1$, we obtain

$$\mathrm{P}(x+z)^8 = \mathrm{P}\{x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x\} + \mathrm{P} - 1 = \mp y^8.$$

If all the signs of the second equation be changed, then the positive sign of y gives the closed oval, and the negative sign of y the asymptotic form.

In this equation for any possible value of x, the positive and negative roots of y are equal; but for a given value of y the roots of x are unequal.

On these considerations the following methods have been devised for obtaining the curves of single symmetry of any degree (1). In any diametral homogeneous equation in v_1v_2 we may take $v_1 = z + \sqrt{x}$; $v_2 = y$; $\therefore x = (v_1 - z)^2$; whence values of x and y are found from $v_1 v_2$ for any required value of z. Or we may take $v_2^2 = y^2$; $v_1^2 = z^2 + x$, whence $x = v_1^2 - z^2$; and the equation consists of even powers of y and uneven powers of x.

The curve of Pl. VI. fig. 7, which is of the form of a rifle-ball, was obtained from (α, 6) by substituting $\cdot 5 + \sqrt{x}$ for v_1 after transforming the origin to the extremity of the axis of X. It is of the 12th degree. I might have taken $z + x^{3/2} = 0$, or $x^3 = v_1 - z$.

Each of the homogeneous curves, α, β, ϵ, and ζ, may be made to furnish by deriva-

tion curves of single symmetry of different degrees. Again, by giving different values to z in any of the derived equations, a series of contour-lines may be traced representing a surface which is symmetrical about one axis.

(2) An equation also represents a curve of single symmetry when it is of the homogeneous form $u_n/u_{n-p}=1$, and (1) the function u_n is of even, and u_{n-p} of uneven degree, and also (2) the terms contain only even powers of one of the quantities, y. Thus the equation

$$x^6+Px^4y^2+Px^2y^4=x^5-Qx^3y^2+Rxy^4,$$

represents a non-central symmetrical equation, from which contour-lines of a derived surface may be obtained by substituting $x'+z$ for x and y'_2+z^2 for y in the equation, and finding values of x' and y' from x and y for any required value of z.

I ought, perhaps, to refer here to the case of curves composed of factors; but the subject has been already fully investigated; and I could scarcely hope to add anything material to what has been found by writers of higher authority in these matters. I only make this observation, that when equations expressed in terms of factors are expanded in terms connected by additive or subtractive signs, it is generally necessary to give *alternative signs* to some of the terms, consistently with the original equation, otherwise the complete ovals will not be obtained. I have given an illustration of such a curve of the 6th degree, in which the expansion of the terms of factors does not lead to alternative signs, and which forms an elegant symmetrical closed curve having thirty-two inflexions in its orbit. Its equation is

$$(x^2-1)(x^2-2)(x^2-3)+(y^2-1)(y^2-2)(y^2-3)=1. \qquad \text{(Pl. IV. fig. 5.)}$$

20. *Parabolic Limiting Forms.*

Considered as a section of the homogeneous central surface, a parabola of the n^{th} degree is evidently a section of such a surface parallel to any tangent plane of the asymptotic cone. For such a section the inclination of the asymptotes (which is the same as that of a parallel section through the centre) vanishes, which proves that the curve is parabolic. It is not quite correct to describe a parabola (as is sometimes done) as being a curve whose equation wants the highest power of one of the variables. Homogeneous equations are always central curves, although they may not contain the highest powers of both variables. An equation in x, y, represents a parabolic curve when one of the variables does not occur in the highest homogeneous part; in other words, *when the highest part consists of a single term*, y^n, or is reducible to a single term by transformation of axes. Because, by transforming to polar coordinates, and dividing by $\sin^n\theta$, we see that when $\sin\theta$ is equated to zero r becomes infinite, and that there are no asymptotes, because u_n consists of a single term. When in the equation of a parabola of any degree, u_n consists of more than one term, then since u_n must be derived by transformation of axes from a single term, y'^n (where y' becomes $px+qy$) the homogeneous part, u_n ought to be a

complete square, or a complete binomial expansion of $px+qy$. This would seem to be the proper criterion by which an equation not referred to principal axes may be known to be the expression of a parabolic curve.

If the equation consists entirely of terms of even powers, there are two parabolas, one on each side of the axis of X, which may in a sense be considered to be two branches of a central curve, as in the following easy example:—

$$y^6+4y^4=4x^4$$

$y=$ 0	1	2	3	4	5	6
$x=$ 0	1·057	2·379	4·027	5·981	8·206	10·670.

21. *Biradial Coordinates.*

The homogeneous equations hitherto treated have been solved for loci determined by Cartesian coordinates. The same equations and the same series of values as are above found may be represented graphically under different coordinate systems, and so as to produce curves differing widely in form and geometric properties from the curves of the x-and-y system.

If to avoid ambiguity we call the coordinates for which values were found v_1,v_2, these may represent radii, angles, or trigonometrical quantities instead of lines drawn to coordinate axes; and the equations may be equations in r_1 and r_2; θ_1 and θ_2; r and x; or r and z (where r is the radius vector from a pole, and z is a perpendicular on a directrix). This last system again may be immediately transformed into trilinear coordinates by substituting x^2+y^2 for r^2. The homogeneous equation in $\sin\theta_1$ and $\sin\theta_2$ is evidently identical with the homogeneous equation of corresponding terms in r_2 and r_1; because in the variable triangle composed of the two radii and the line joining the foci the sides are proportional to the sines of the opposite angles.

As an illustration of what may be done in a new direction with the homogeneous equations already examined, the following chapter on a class of Biradials has been written:—

The radial coordinates, from foci F_1, F_2, are denoted by r_1r_2; and the distance F_1F_2 by $2c$.

The equations here considered are of the form

$$r_1^n \pm r_2^n = A_1^n \qquad (1); \qquad r_1^n/\lambda^n \pm r_2^n/\mu^n = c^n/\nu^n = A^n \quad . \quad . \quad (2),$$

where the index n is an even number.

I have not been able to come to a clear conclusion regarding biradial equations of uneven degrees. On the one hand, if we seek to transform these to rectangular or ordinary polar coordinates, it is necessary to square the equation twice to remove the

radical, so that a biradial equation of the 3rd degree corresponds to an equation of the 12th degree in rectangular Cartesian coordinates; while a biradial of the 4th degree can be transformed into a quadric in Cartesian coordinates. On the other hand, when the biradial curves of the 3rd, 4th, 5th, and 6th degrees are traced, they are found to be in series; and in this case, apparently, the degree of the Cartesian equation (contrary to the general understanding of mathematicians on this subject) is not a criterion of the true order of the curve.

Plate V. fig. 8, represents this series of curves, that of the 3rd degree being the nearest to the centre. The dotted curve is the biradial curve of the fractional degree 7/2. Each curve is laid down from nine computed points for the quadrant.

Equation (1) is the *Oval of double symmetry.* Its equation in ordinary polar coordinates reduces to a simple and easily remembered expression. Taking O, the centre of the oval for the origin of Cartesian and polar coordinates (x, y, R, θ); $OF_1 = OF_2 = c$. For a point P in the curve, we have from the triangles OF_1P, OF_2P,

$$r_1^2 = OP^2 + OF_1^2 - 2OF_1.OP.\cos F_1OP = R^2 + c^2 - 2cR\cos\theta\,;$$
$$r_2^2 = OP^2 + OF_2^2 + 2OF_2.OP.\cos F_2OP = R^2 + c^2 + 2cR\cos\theta\,.$$

In the expansions of r_1^n, r_2^n the uneven terms of $R\cos\theta$ disappear. Thus by transformation $r_1^2 + r_2^2 = A^2$ becomes

$$(R^2 + c^2 - 2cR.\cos\theta) + (R^2 + c^2 + 2cR.\cos\theta) = A^2,$$
$$\therefore \qquad R^2 = (A^2 - 2c^2)/2\,, \text{ the circle.}$$

For the radial equation of the 4th degree we have

$$r_1^4 + r_2^4 = A\,; \quad (R^2 + c^2 - 2cR.\cos\theta)^2 + (R^2 + c^2 + 2cR.\cos\theta)^2 = A^4$$
$$(R^2 + c^2)^2 + 4c^2R^2\cos^2\theta = A^4/2\,;$$
$$\therefore \qquad (x^2 + y^2 + c^2) + (2cx)^2 = A^4/2 \quad . \quad . \quad . \quad . \quad (3).$$

For the radial equation of the 6th degree we have

$$r_1^6 + r_2^6 = A^6\,; \qquad (R^2 + c^2 - 2cR\cos\theta)^3 + (R^2 + c^2 + 2cR.\cos\theta)^3 = A^6,$$
$$\therefore \qquad (R^2 + c^2)^3 + (R^2 + c^2)(2cR\cos\theta)^2 = A^6/2\,;$$
$$\therefore \qquad (x^2 + y^2 + c^2)^3 + (x^2 + y^2 + c^2)(2cx)^2 = A^6/2 \quad . \quad . \quad . \quad . \quad (4).$$

Cognate polar equations may be formed in the same way for radial equations of any even degree, whence the equations in x and y may be written out. The equations are homogeneous functions of the composite quantities $(x^2 + y^2 + c^2)$ and $2cx$.

If we write v_1v_2 for these expressions, and solve the homogeneous equation in v_1v_2 for any point, we may then find $x = v_2/2c$, and $y = \sqrt{v_1 - x^2 - c^2}$.

These equations do not presuppose any relation between A and c. Accordingly, by varying the distance of F and O, the equation may be made to represent an oval from any two points on the major or minor axis taken as poles, and these poles may be either interior to, or on the oval, or (within certain limits) exterior to it. If a and b represent the lengths of the principal semi-axes of the oval, the limiting position of an exterior pole or focus is $c = 2a$, where the curve is reduced to a line. The limiting positions of interior poles or foci is of course $c = 0$, where the foci coincide, and the curve becomes a circle.

The principal foci of the oval are determined under the same conditions as the foci of an ellipse; by taking $c^2 = a^2 - b^2$. Then, for the pair of equal radii drawn from the foci to the extremity of the minor axis, we have the relation $r_1 = r_2 = a$. If the major axis be taken as of unit value, then $c^2 = (a^2 - b^2)/a^2 = e^2$, and the polar equations for the 4th and 6th degree curves may be written

$$R^4 + 2e^2R^2(1 + 2\cos^2\theta) = (A^4 - 2e^4)/2 \quad . \quad . \quad . \quad . \quad . \quad (5);$$

$$R^6 + (3e^2R^4 + 3e^4R^2)(1 + 4\cos^2\theta) = (A^6 - 2e^6)/2 \quad . \quad . \quad . \quad (6),$$

where e is the eccentricity estimated in the same manner as in the case of the ellipse, the cognate curve of the 2nd degree. And similarly for any curve of even degree in which the foci are properly taken. It may here be noticed that the equation $R^2(1 + c^2\cos^2\theta) = A^2$ represents an ellipse, because it may be immediately changed to $y^2 + c^2x^2 = A^2$. The ellipse then belongs to this family of curves, of which it is of course the lowest form.

Plate V. fig. 4, represents the sextic curve having the equation

$$r_1^6 + 15r_1^4r_2^2 + 15r_1^2r_2^4 + r_2^6 = 1,$$

and referred to its principal foci. For its construction the values of r_1r_2 are used, which are transcribed in the ensuing table, p. 1096. But as the curve was to be referred to its principal foci, it was necessary to adopt as the maximum and minimum radii the pair of values whose sum is equal to twice the mean radius, r_{21}. Hence the only available values were

$r_2 =$	·7652	·7115	·6603	·6103	·5611
$r_1 =$	·3569	·4108	·4623	·5122	·5611

*

The curves here considered have a general resemblance to ellipses; and if the equation in x-and-y coordinates be referred to oblique axes, the curve resembles an ellipse referred to conjugate inclined axes. The greatest and least diameters are thus apparently conjugate, but are not really so; because it has been found impossible by analysis to reduce the locus of mid-points of parallel chords to a simple equation. It will be seen from Plate V. that the difference between the biradial curve of the 6th degree and the ellipse described on the same axes is very small, and it is probable that the class of homo-

* These coordinates are very nearly the same as those of the ellipse described on the same axes.

geneous biradial curves may be resolved into functions of curves of the second degree with variable elements.

Hitherto I have considered the radial curve as the geometrical expression of a homogeneous radial equation of the simple form $r_1^n + r_2^n = A$. It will now be shown that all symmetrical homogeneous equations in $r_1 r_2$ of the same degree are identical curves, the eccentricity being dependent only on the choice of foci, or, which is the same thing, on the ratio of c to A. This identity is proved by transformation to polar coordinates. For this purpose, let $r_1^p . r_2^{n-p}$ and $r_1^{n-p} . r_2^p$ be any pair of homologous terms of the symmetrical radial equation. If definite values be given to the indices n and p, and the transformation to polar coordinates be effected by the formula $r_{1,2} = R^2 c^2 \pm 2Rc . \cos\theta$, it will be found that, in the addition of the transformed terms, all the terms of uneven powers disappear, and that the resulting polar expression is identical with that obtained from the sum of the terms r_1^n and r_2^n. Thus, from each pair of homologous terms we have the same polar expression multiplied by a coefficient, and the transformed homogeneous symmetrical radial equation has the form already found for the equation of three terms, with a new value of A.

Thus, if we transform the equation $r_1^6 + P r_1^4 r_2^2 + P r_1^2 r_2^4 + r_2^6 = A^6$ to polar coordinates, we obtain from the two extreme terms the terms of the left side of equation (4), which may be denoted by $F(r_1 r_2)$; and from the mean terms we obtain $P \times F(r_1 r_2)$. The polar equation then is of the form (4) with the right-hand term divided by $(P+1.)$, or

$$(R^2+c^2)^3 + (R^2+c^2)(2cR\cos\theta)^2 = \frac{A^6}{2P+2}.$$

The proposition that the same curve or trace may be obtained from different homogeneous equations of the same degree is illustrated by fig. (1). This figure, as is the case with all the illustrations, is drawn by tracing the curve through a series of computed points laid down on diagram paper, never less than nine points for a complete *phase*. This figure, when referred to the two marked exterior foci, satisfies the equation $r_1^6 + r_1^4 r_2^2 + r_1^2 r_2^4 + r_2^6 = 1$; and when referred to the two marked interior foci it satisfies the equation $r_1^6 + r_2^6 = 1$. The computed values of r_1 and r_2 are those of the table, p. 1074, for the equations of these forms in x and y.

If the homogeneous radial equation contains a middle term of even powers of $r_1 r_2$, its equivalent in polar coordinates differs only from the expression found for a pair of homologous terms in having the *negative sign* prefixed to all the terms containing $\cos^2\theta$. I must, therefore, qualify the statement of the preceding paragraph by adding that, in the case of homogeneous equations of the 4th and 8th degrees (and generally where the index is divisible by 4), there are apparently two forms, one without and the other with a middle term. But it does not appear that this variation can have any other effect than that of varying the coefficients of the terms multiplied by cosine$^2\theta$. The examples which I have worked out are confined to equations of the 6th degree, in which, of course, there is no middle term.

If the equation consists of only a middle term equated to unity, or $r_1^{\frac{n}{2}} . r_2^{\frac{n}{2}} = 1$, this is immediately reducible to $r_1^2 . r_2^2 = 1$, which gives by transformation,

$$(x^2+y^2+c^2)^2 - 4(x^2+y^2)c^2\cos^2\theta\,,$$

the equation of the oval of Cassini.

There is a curious relation between radial equations and equations of the same form in rectangular coordinates, which is connected with the value of the coefficients.

It has been seen (p. 1072), in the case of the sextic equation of positive even powers of x and y, (1) that if P, the coefficient of the pair of intermediate terms be = 3, the curve is a limiting circle; (2) that by varying P from 0 to 3 we obtain every form of the non-inflexional oval; and (3) that for values of P exceeding 3 and less than 15, or $n(n-1)/2$, we obtain the same series of curves turned round through an angle of 45°. In the case of the sextic radial equation of positive even powers of $r_1\, r_2$, if P, the coefficient of the pair of intermediate terms = 3, the curve also reduces to the circle $(r_1^2+r_2^2=1)$. If P be less than 3, the radial equation represents a curve referred to foci, or poles, in the line of the minor axis, which may be either interior or exterior or on the curve (see figs. 1 and 2); and this includes the case of the equation of three terms, where P = 0. But, if P exceeds 3, the radial equation represents a curve referred to foci, or poles, in the line of the major axis, which may be either interior or exterior or on the curve; and this includes the case of the curve referred to its principal foci.

It appeared to me that the radial curves, as traced, were a little more rounded at the apses than ellipses, and this impression has been confirmed by the numerical computation and comparison of the forms of the ellipse and of the sextic oval described on identical major and minor axes, which will be immediately given. It would be interesting to make a cognate comparison for elliptic ovals of different degrees. There are two ways in which such a comparison may be instituted.

(1) If the vertices of the curves be taken for foci, or poles of radial coordinates, the arbitrary term is then a parameter, and a series of curves of different degrees may be described upon the same principal axes, a and b. (2) If the foci of the normal position $(c^2=a^2-b^2)$ be taken for poles, it is difficult to prearrange the equations so that the curves to be formed shall have the same amplitude. We may, however, compute each curve independently, and compare it with the ellipse described on the same axes, and thus find out for the curve of any degree how far its coordinates differ from those of the ellipse of equal amplitude. In either case, it is necessary to reduce the radial coordinates to rectangular. This is easily done. Referring to p. 1092, we see that

$$r_2^2 - r_1^2 = 4c\mathrm{R}\cos\theta = 4cx;\ \therefore\ x = \frac{r_2^2-r_1^2}{4c}\cdot$$

Also,

$$r_2^2+r_1^2 = 2(x^2+y^2+c^2);\ \therefore\ y^2 = \frac{r_2^2+r_1^2}{2} - (x^2+c^2)\cdot$$

To determine a and b, and c (the distance between the foci), we may denote by $r_{0,2}$ $r_{0,1}$ the greatest and least radial coordinates, being those which are drawn to the extremities of the axis of X; and by $r_{2,1}$ the radii of equal length, being those which are drawn to the extremities of the axis of Y. Then,

For any interior foci, ... $c=\frac{r_{02}-r_{01}}{2}$; $a=\frac{r_{02}+r_{01}}{2}$;

For exterior foci, ... $c=\frac{r_{02}+r_{01}}{2}$; $a=\frac{r_{02}-r_{01}}{2}$;

For foci on the curve, ... $r_{01}=0$; $c=a=\frac{r_{02}}{2}$;

For foci in the normal position $r_{02}+r_{01}=2r_{21}=2a$; and $r_{21}-r_{01}=c$;

In all cases, ... $b=\sqrt{r_{21}^2-c^2}$.

Values of a, b, c, x and y being thus found for the elliptic oval of any degree from its radial coordinates r_1r_2, the comparison with the ellipses described on the same axes is made by taking identical values of x (or $x=x'$), and thence computing the relative values of

$$y'=\sqrt{b^2-\frac{b^2}{a^2}\cdot x^2}.$$

Such a comparison has been made for the curve $r^6+15r_1^4r_2^2+15r_1^2r_2^4+r_2=1$, and the ellipse having the same axes, a and b; and the values of y and y' to the argument $x=x'$, together with the difference ($\Delta=y-y'$) are given in the subjoined table. Exact values of r_1r_2 were found by the homogeneous method, whence exact values of x, y, and y' were found by the preceding formulæ. Taking the highest and lowest computed values of r_2r_1 for the axial radii, and with interior foci, we have $r_{02}=\cdot9367$; $r_{01}=\cdot1651$; $c=\cdot3858$; $a=\cdot5509$; $b=\cdot4075$. From these elements and the tabular values of r_2 and r_1, the corresponding values of x, y, and y' are as in the annexed table.

Arguments, r_2	·9367	·8798	·8215	·7652	·7115	·6603	·6103	·5611
Arguments, r_1	·1651	·2358	·2990	·3569	·4108	·4623	·5122	·5611
$x=x'$	·5509	·4656	·3794	·2970	·2187	·1440	·0715	0·0
Sextic Oval, y	0·0	·2218	·2986	·3455	·3754	·3941	·4043	·4075
Ellipse, y'	0·0	·2178	·2954	·3432	·3741	·3932	·4040	·4075
$\Delta=(y-y')$	0·0	·0040	·0032	·0023	·0013	·0009	·0002	0·0

When the eccentricity is increased by giving a different value to c, the difference between the y coordinates of the oval and the ellipse also increases, as shown in the following table, where $r_{02} = \cdot 9819$; $r_{01} = \cdot 0859$; $c = \cdot 4480$; $a = \cdot 5339$; $b = \cdot 3380$; other values of r_1 r_2 being as above.

$x = x'$	·5339	·4743	·4009	·3267	·2577	·1882	·1240	·0615	0·0
y	0·0	·1634	·2313	·2733	·3007	·3184	·3298	·3362	·3380
y'	0·0	·1550	·2232	·2670	·2967	·3162	·3287	·3357	·3380
$\Delta = (y - y')$	0·0	·0084	·0081	·0063	·0030	·0022	·0011	·0005	·0000

In Pl. IV. fig. 6, the exterior curve of each pair represents the sextic oval, and the interior curve the ellipse, as traced through the points here given.

Further researches as to the properties of elliptic ovals may be expected to yield interesting results; and it appears to me that these curves, and the curves obtained from them by linear transformation, are capable of expressing the facts of a certain class of physical problems with greater accuracy than is obtainable by the tables at present in use for the purpose.

The three curves shown in Plate V. fig. 6, are linear transformations of the symmetrical oval. Their equation is $r_1^6/\lambda + r_2^6/\mu = 1$; λ was taken $= 1$; and μ was taken successively $= 2$, $\frac{1}{2}$ and $\frac{2}{3}$ to obtain the three curves. In the diagram each curve is connected with its foci by lines drawn for the purpose. The same equation with different values of either c or μ gives a different curve, as the figure shows. The complete equation is

$$\frac{r_1^n}{\lambda^n} + \frac{P r_1^{n-2}}{\lambda^{n-2}} \frac{r_2^2}{\mu^2} + + \ \ldots \ \frac{r_2^n}{\mu^n} = A^n ,$$

and the curve may be described as the Oval of single symmetry. The Cartesian oval is a limiting form of the oval of single symmetry of the 4th degree, as may be verified by twice squaring its equation $r_1/\lambda + r_2/\mu = A$.

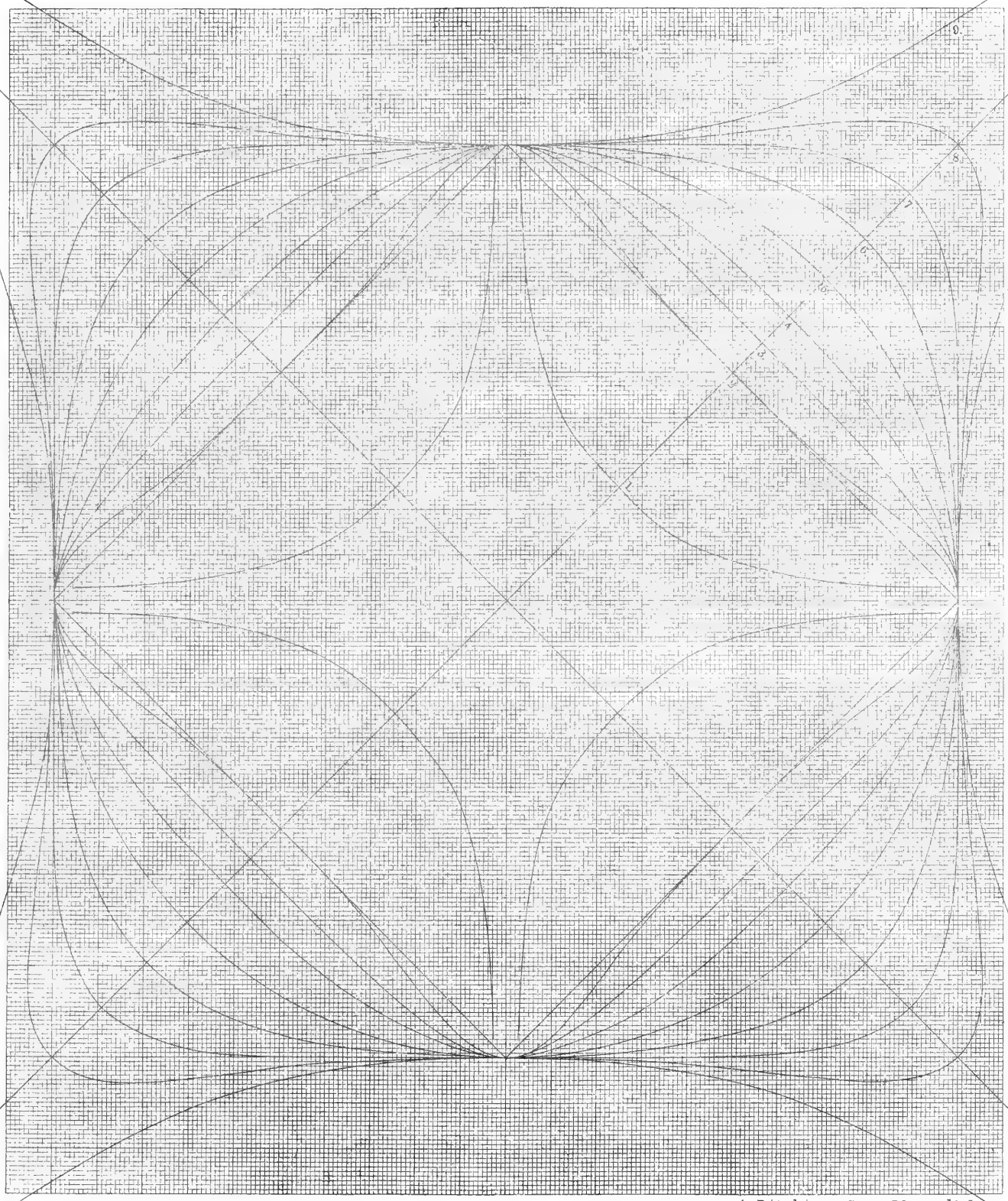
The curves formed by giving negative values to one of the quantities, or to one of each pair of homologous terms, are remarkable for their varied and fantastic forms; but I have not been able to discover any properties which are common to the class.

The curve of the symmetrical equation $r_2^6 - r_1^6 = A$ (shown in Pl. V. fig. 5) is an inflexional curve of two branches. Each branch crosses the axis of X, and is symmetrical on either side of it; after being inflected on either side, the branches continue to approach to the asymptote Y, which accordingly has double contact with the curve at the point infinity and also at negative infinity. Generally, for the symmetric radial equation of any number of terms of even powers, I have found (1) in the form (β) or ($+ - - +$), if the coefficients of the intermediate terms are fractional, the curve is an oval entirely concave to the centre; (2) if the coefficient (in the form β) exceed unity

the curve is of the form last described (Pl. V. fig. 5), having the axis of Y for its asymptote; (3) in the forms (ϵ) and (ζ), or $(+-+-)$ and $(++--)$, the curve is of the same form, having the axis of Y for its asymptote. The linear transformation of the radial equation with negative terms has been found to be a closed curve in all the examples I have tried; it apparently only becomes asymptotic when the coefficients λ and μ are equal. The oval and the mushroom-like forms of Plate V. fig. 7, are traced from the equation $r_1^6/\lambda - r_2^6/\mu = 1$ by giving different values to μ and c.

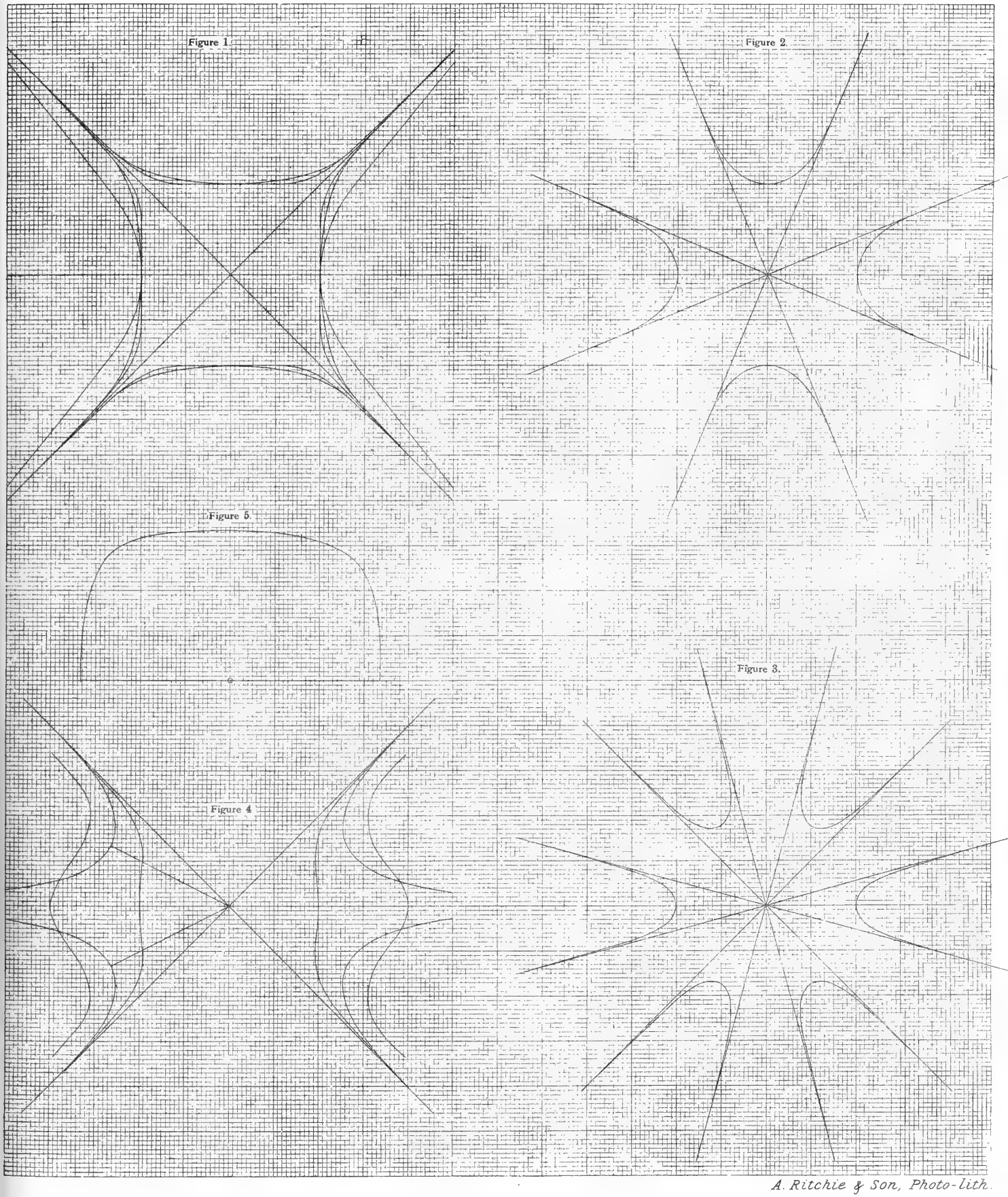
It is easily seen that in the case of bi-radial curves with diverse coefficients, λ and μ, the transformation to polar coordinates will not give rise to a simplified expression, because the uneven terms of the expansions do not disappear. The equations in x and y contain in general all the even powers of y and all the powers, even and uneven, of x, and are similar in form to those which have been considered as resulting from an oblique section of a central surface parallel to a plane through one of its axes of symmetry.

LORD McLAREN ON HOMOGENEOUS EQUATIONS—POSITIVE SYMMETRY. —PLATE I.



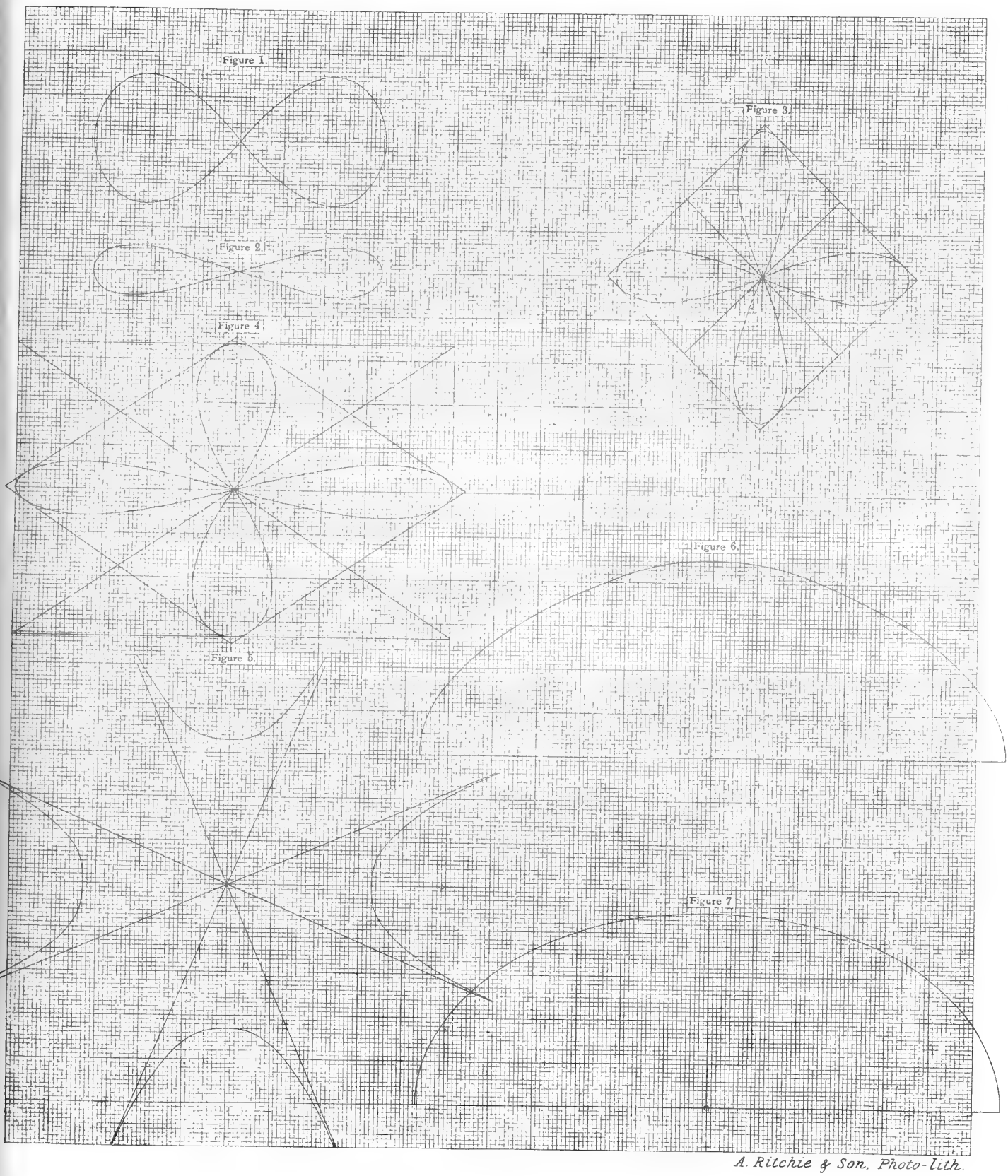
A. Ritchie & Son, Photo-lith.

LORD McLAREN ON HOMOGENEOUS EQUATIONS—NEGATIVE SYMMETRY.—PLATE II.



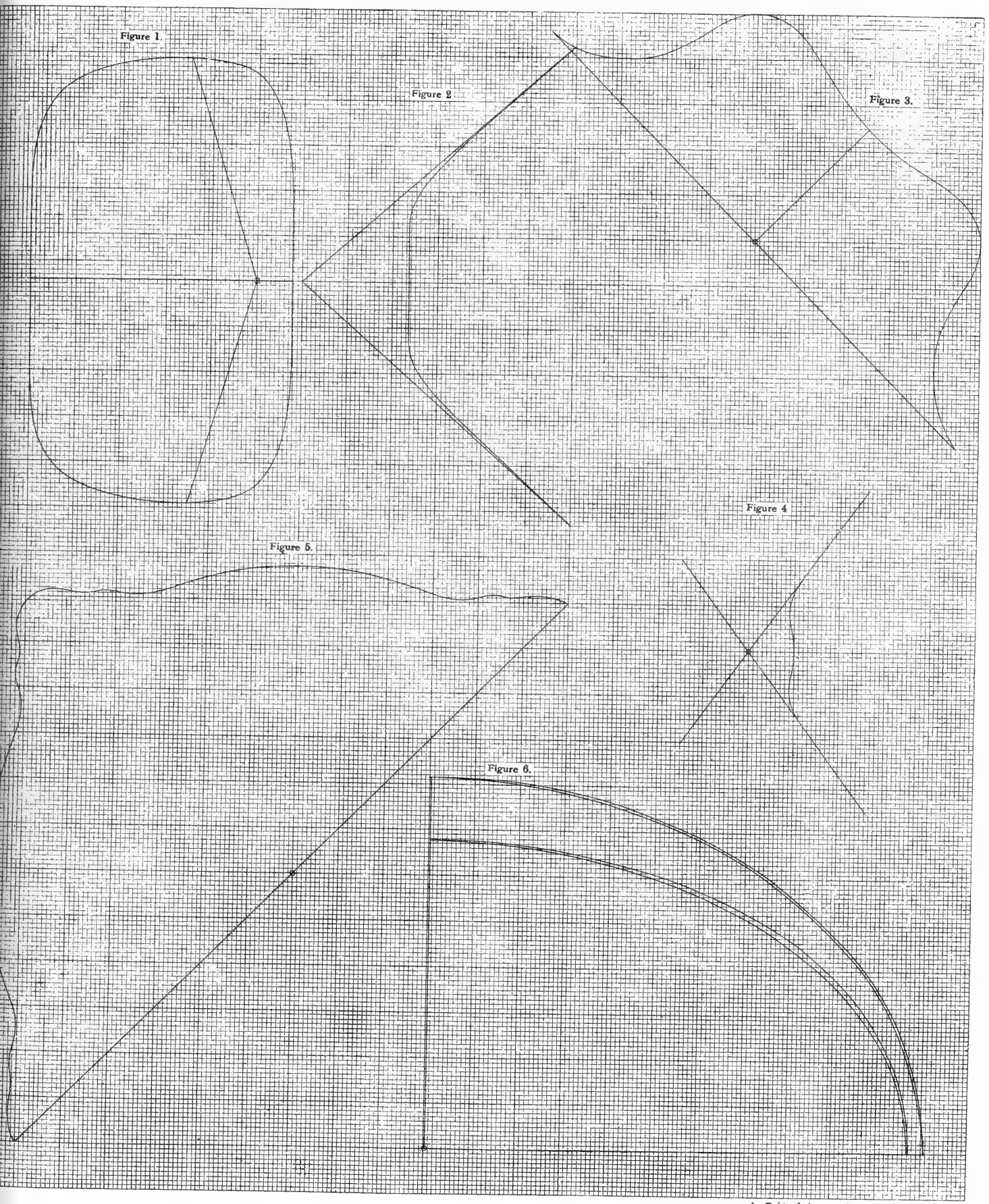
A. Ritchie & Son, Photo-lith.

LORD McLAREN ON EQUATIONS OF TWO HOMOGENEOUS PARTS.—PLATE III.



A. Ritchie & Son, Photo-lith.

LORD McLAREN ON EQUATIONS OF SINGLE, OR IMPERFECT SYMMETRY.—PLATE IV.



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LORD McLAREN ON BIRADIAL EQUATIONS.—PLATE V.

Figure 3.

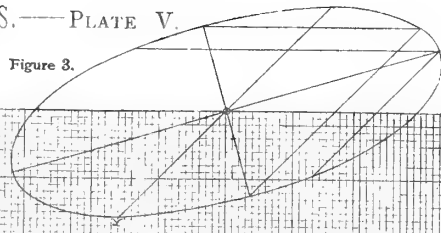
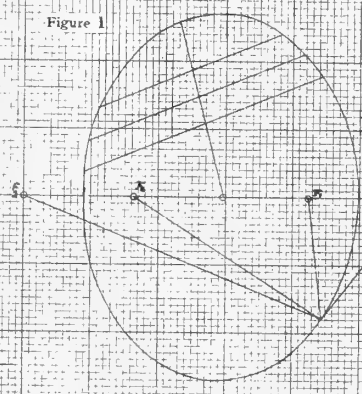


Figure 1



Figure

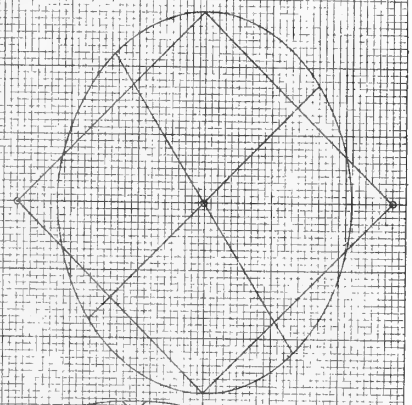
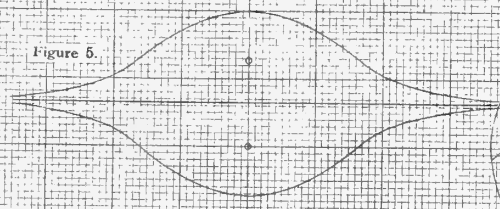


Figure 5.



Figure

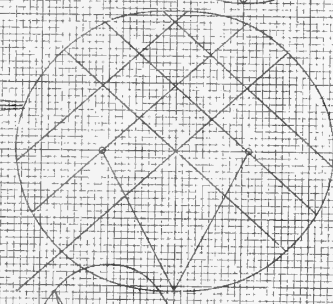


Figure 6.

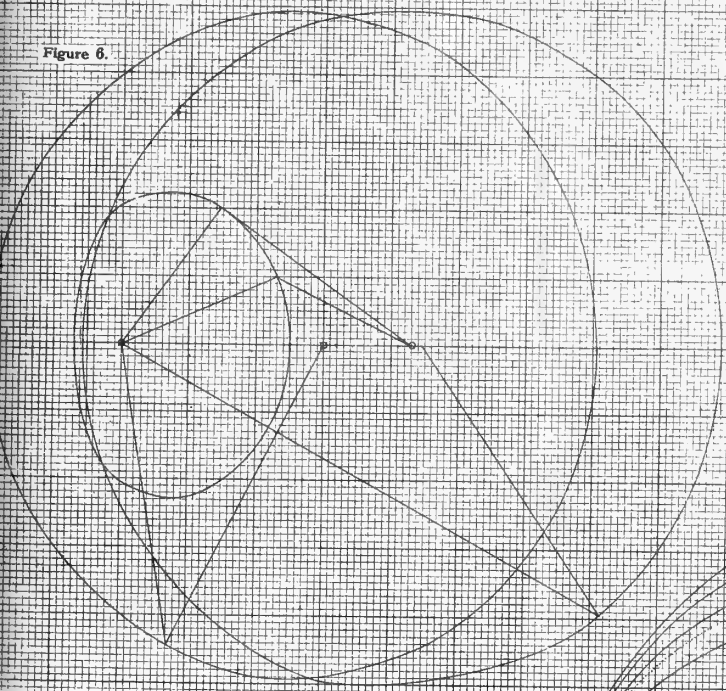


Figure 7

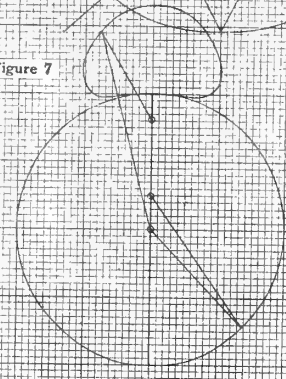
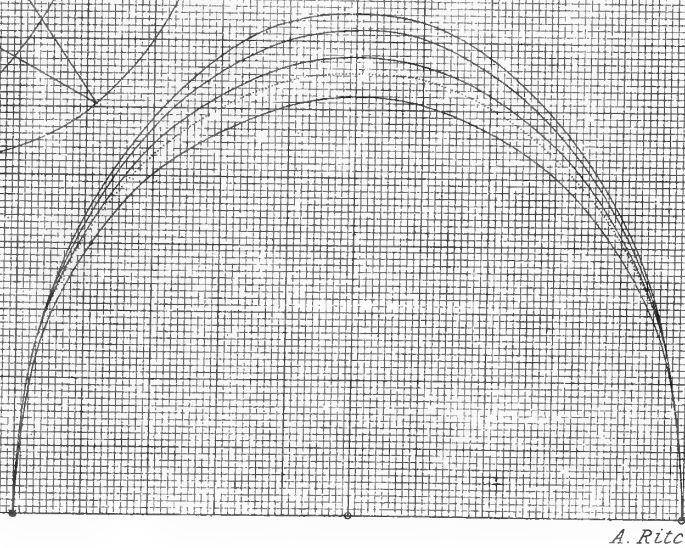


Figure 8.



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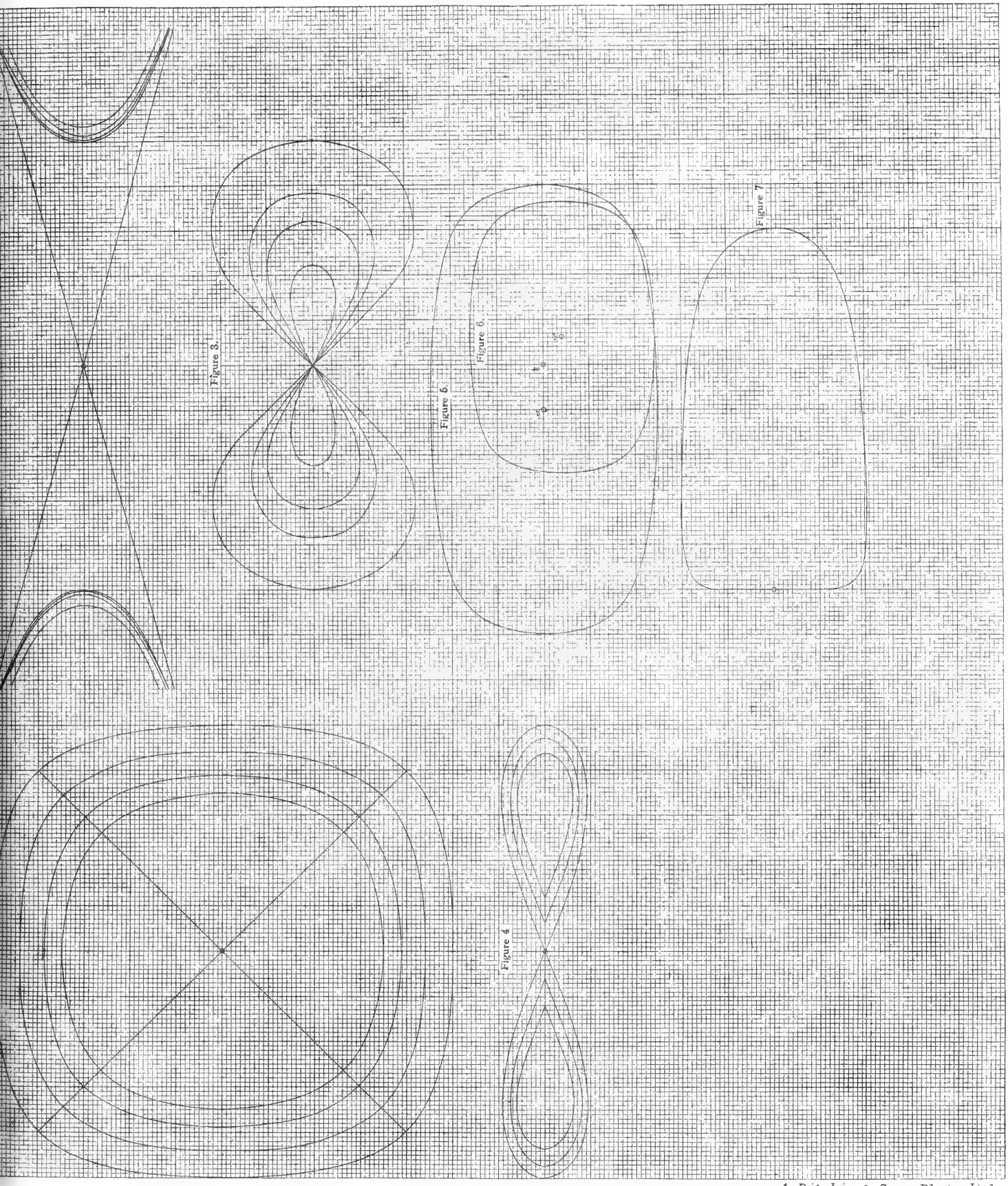
LORD McLAREN ON CONTOUR-LINES OF SURFACES—HOMOGENEOUS EQUATIONS.—PLATE VI.



A. Ritchie & Son, Photo-lith.

APPENDIX.

TRANSACTIONS

OF THE

ROYAL SOCIETY OF EDINBURGH.

CONTENTS.

LIST OF MEMBERS.

COUNCIL,

ALPHABETICAL LIST OF ORDINARY FELLOWS,

AND LIST OF HONORARY FELLOWS,

At November 1889.

THE COUNCIL

OF

THE ROYAL SOCIETY OF EDINBURGH,

NOVEMBER 1889.

PRESIDENT.

SIR WILLIAM THOMSON, LL.D., D.C.L., Grand Officer of the Legion of Honour of France, Member of the Prussian Order *Pour le Mérite*, F R.S., Foreign Associate of the Institute of France, Regius Professor of Natural Philosophy in the University of Glasgow.

HONORARY VICE-PRESIDENTS, HAVING FILLED THE OFFICE OF PRESIDENT.

HIS GRACE THE DUKE OF ARGYLL, K.G., K.T., D.C.L. Oxon., LL.D., F.R.S., F.G.S.
THE RIGHT HON. LORD MONCREIFF, of Tullibole, LL.D.

VICE-PRESIDENTS.

SIR DOUGLAS MACLAGAN, M.D., F.R.C.P.E., Professor of Medical Jurisprudence in the University of Edinburgh.
THE HON. LORD MACLAREN, LL.D. Edin. and Glas., F.R.A.S., one of the Senators of the College of Justice.
THE REV. PROFESSOR FLINT, D.D., Corresponding Member of the Institute of France.
GEORGE CHRYSTAL, M.A., LL.D., Professor of Mathematics in the University of Edinburgh.
THOMAS MUIR, M.A., LL.D., Mathematical Master in the High School of Glasgow.
SIR ARTHUR MITCHELL, K.C.B., M.A., M.D., F.R.C.P.E., LL.D., Commissioner in Lunacy.

GENERAL SECRETARY.

P. GUTHRIE TAIT, M.A., Professor of Natural Philosophy in the University of Edinburgh.

SECRETARIES TO ORDINARY MEETINGS.

SIR WILLIAM TURNER, M.B., LL.D., F.R.C.S.E., F.R.S., Professor of Anatomy in the University of Edinburgh.
ALEXANDER CRUM BROWN, M.D., D.Sc., F.R.C.P.E., F.R.S., Professor of Chemistry in the University of Edinburgh.

TREASURER.

ADAM GILLIES SMITH, Esq., C.A.

CURATOR OF LIBRARY AND MUSEUM.

ALEXANDER BUCHAN, Esq., M.A., LL.D., Secretary to the Scottish Meteorological Society.

COUNCILLORS.

J. BATTY TUKE, M.D., F.R.C.P.E.
FREDERICK O. BOWER, M.A., F.L.S., Regius Professor of Botany in the University of Glasgow.
GERMAN SIMS WOODHEAD, M.D., F.R.C.P.E.
ROBERT COX, Esq. of Gorgie, M.A.
ISAAC BAYLEY BALFOUR, D.Sc., M.D., C.M., F.R.S., Professor of Botany in the University of Edinburgh.
JAMES ALFRED EWING, B.Sc., F.R.S., Professor of Engineering and Drawing in University College, Dundee.
WILLIAM JACK, M.A., LL.D., Professor of Mathematics in the University of Glasgow.
JAMES GEIKIE, LL.D., F.R.S., F.G.S., Professor of Geology in the University of Edinburgh.
W. H. PERKIN, Ph.D., Professor of Chemistry in the Heriot-Watt College.
A. BEATSON BELL, Chairman of the Prison Commission, Scotland.
The Right Hon. LORD KINGSBURGH, C.B., LL.D., F.R.S., M.S.T.E. and E., Lord Justice-Clerk, and Lord President of the Second Division of the Court of Session.
JOHN MURRAY, LL.D., Ph.D., Director of the Challenger Expedition Commission.

ALPHABETICAL LIST

OF

THE ORDINARY FELLOWS OF THE SOCIETY,

CORRECTED TO NOVEMBER 1889.

N.B.—*Those marked * are Annual Contributors.*

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. ,, ,, ,, Keith Medal.
N. ,, ,, ,, Neill Medal.
V. J. ,, ,, ,, the Victoria Jubilee Prize.
P. ,, ,, ,, contributed one or more Papers to the TRANSACTIONS.

Date of Election.		
1879		Abernethy, Jas., Memb. Inst. C.E., Prince of Wales Terrace, Kensington
1871		* Agnew, Stair, C.B., M.A., Registrar-General for Scotland, 22 Buckingham Terrace
1888		* Aikman, C. M., M.A., B.Sc., F.I.C., F.C.S., Lecturer on Agricultural Chemistry in Glasgow and West of Scotland Technical College, 183 St Vincent Street, Glasgow
1881		Aitchison, James Edward Tierney, C.I.E., M.D., LL.D., F.R.S., F.L.S., Brigade-Surgeon, retired, H.M. Bengal Army; F.R.C.S. Edin., M.R.C.P. Edin., Corresp. Fell. Obstet. Soc. Edin., 20 Chester Street
1878		* Aitken, Andrew Peebles, M.A., Sc.D., F.I.C., 57 Great King Street 5
1875	K. P.	* Aitken, John, F.R.S., Darroch, Falkirk
1889		* Alison, John, M.A., Secretary to the Edinburgh Mathematical Society, 33 Woodburn Terrace
1888		* Allardice, R. E., M.A., 16 Nile Grove, Morningside
1878		Allchin, W. H., M.B., F.R.C.P.L., Physician to the Westminster Hospital, 5 Chandos Street, Cavendish Square, London
1856	B. P.	Allman, George J., M.D., F.R.S., M.R.I.A., F.L.S., Emeritus Professor of Natural History, University of Edinburgh, Ardmore, Parkstone, Dorset 10
1886		* Anderson, Arthur, M.D., C.B., Ex-Inspector-General of Hospitals, Pitlochry
1874		Anderson, John, M.D., LL.D., F.R.S., late Superintendent of the Indian Museum and Professor of Comparative Anatomy in the Medical College, Calcutta, 71 Harrington Gardens, London
1883		* Anderson, Robert Rowand, LL.D., 19 St Andrew Square
1883	P.	Andrews, Thomas, Memb. Inst. C.E., F.R.S., F.C.S., Ravencrag, Wortley, near Sheffield
1881		Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork 15

Date of Election.		
1867		* Annandale, Thomas, M.D., F.R.C.S.E., Professor of Clinical Surgery in the University of Edinburgh, 34 Charlotte Square
1883		Archibald, John, M.D., C.M., F.R.C.S.E., Woodhouse-Eaves, Loughborough
1886		* Armstrong, George Frederick, Professor of Engineering in the University of Edinburgh
1849		Argyll, His Grace the Duke of, K.G., K.T., D.C.L., LL.D., F.R.S. (Hon. Vice-Pres.), Inveraray Castle
1887		* Ashdown, Herbert H., M.B., 49 Upper Bedford Place, Russell Square, London 20
1885		* Baildon, H. Bellyse, B.A., Duncliffe, Murrayfield, Edinburgh
1879		* Bailey, James Lambert, Royal Bank of Scotland, Ardrossan
1875		* Bain, Sir James, 3 Park Terrace, Glasgow
1879		* Balfour, George W., M.D., LL.D., F.R.C.P.E., 7 Walker Street
1877	P.	* Balfour, I. Bayley, Sc.D., M.D., C.M., F.R.S., Professor of Botany in the University of Edinburgh 25
1870		* Balfour, Thomas A. G., M.D., F.R.C.P.E., 51 George Square
1889		* Barbour, A. H. F., M.A., M.D., F.R.C.P.E., 24 Melville Street
1886		* Barclay, A. J. G., M.A., 5 Ethel Terrace
1872		* Barclay, George, M.A., 17 Coates Crescent
1883		* Barclay, G. W. W., M.A., 40 Princes Street 30
1887		Barlow, W. H., Memb. Inst. C.E., High Combe, Old Charlton, Kent
1882		Barnes, Henry, M.D., 6 Portland Square, Carlisle
1874		Barrett, William F., M.R.I.A., Professor of Physics, Royal College of Science, Dublin
1889		Barry, T. D. Collis, Staff Surgeon, M.R.C.S., F.L.S., Prof. of Chemistry and Medical Jurisprudence to the Grant Medical College, Bombay, and Acting Chemical Adviser to the Indian Government
1887		* Bartholomew, J. G., F.R.G.S., 12 Blacket Place 35
1857		Batten, Edmund Chisholm, of Aigas, M.A., Thornfaulcon, near Taunton, Somerset
1880		* Bayly, General John, C.B., R.E., 58 Palmerston Place
1888		* Beare, Thomas Hudson, B.Sc., Assoc. Memb. Inst. C.E., Professor of Engineering and Mechanical Technology in University College, Gower Street, London
1882	P.	Beddard, Frank E., M.A. Oxon., Prosector to the Zoological Society of London, Zoological Society's Gardens, Regent's Park, London
1887		* Begg, Ferdinand Faithful, 13 Earl's Court Square, London, S.W. 40
1886		* Bell, A. Beatson, Chairman of Prison Commission, 143 Princes Street
1874		* Bell, Joseph, M.D., F.R.C.S.E., 2 Melville Crescent
1888		* Bell, William James, of Scatwell, B.A., LL.M., F.C.S., Barrister-at-Law, Scatwell, Muir of Ord, and 1 Plowden Buildings, Temple, London
1887		* Bernard, J. Mackay, 25 Chester Street
1875		Bernstein, Ludwik, M.D., Lismore, New South Wales 45
1881		* Berry, Walter, K.D., Danish Consul-General, 11 Atholl Crescent
1880		* Birch, De Burgh, M.D., Professor of Physiology, Yorkshire College, Victoria University, 16 De Grey Terrace, Leeds
1884		* Black, Rev. John S., 6 Oxford Terrace
1850		Blackburn, Hugh, M.A., LL.D., Emeritus Professor of Mathematics in the University of Glasgow, Roshven, Ardgour
1863	P.	Blackie, John S., Emeritus Professor of Greek in the University of Edinburgh, 9 Douglas Crescent 50

Date of Election.		
1862		Blaikie, The Rev. W. Garden, M.A., D.D., LL.D., Professor of Apologetics and Pastoral Theology, New College, Edinburgh, 9 Palmerston Road
1878	P.	* Blyth, James, M.A., Professor of Natural Philosophy in Anderson's College, Glasgow
1884		Bond, Francis T., M.D., B.A., M.R.C.S., 1 Beaufort Buildings, Spa, Gloucester
1872		* Bottomley, J. Thomson, M.A., F.R.S., F.C.S., Lecturer on Natural Philosophy in the University of Glasgow, 13 University Gardens, Glasgow
1869		* Bow, Robert Henry, C.E., 7 South Gray Street 55
1886		* Bower, Frederick O., M.A., F.L.S., Regius Professor of Botany in the University of Glasgow, 45 Kerrsland Terrace, Hillhead, Glasgow
1884		Bowman, Frederick Hungerford, D.Sc., F.R.A.S., F.C.S., F.L.S., F.G.S., West Mount, Halifax, Yorkshire
1871		* Boyd, Sir Thomas J., Chairman of the Scottish Fishery Board, 41 Moray Place
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1886		Brittle, John Richard, Memb. Inst. C.E., Vanbrugh Hill, Blackheath, Kent
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1881		* Brown, J. A. Harvie, of Quarter, Dunipace House, Larbert, Stirlingshire
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1885		* Brown, J. Macdonald, M.B., F.R.C.S.E., Apsley Lodge, 12 South Mansionhouse Road
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1870		Browne, Sir James Crichton, M.D., LL.D., F.R.S., 7 Cumberland Terrace, Regent's Park, London 70
1883		* Bruce, Alexander, M.A., M.B., M.R.C.P.E., 13 Alva Street
1878		Brunlees, Sir James, Memb. Inst. C.E., 5 Victoria Street, Westminster
1867		* Bryce, A. H., LL.D., D.C.L., 42 Moray Place
1888		* Bryson, William A., Electrical Engineer, 196 St Vincent Street, Glasgow
1869	B. P.	* Buchan, Alexander, M.A., LL.D., Secretary to the Scottish Meteorological Society (CURATOR OF LIBRARY), 72 Northumberland Street 75
1870	K. P.	* Buchanan, John Young, M.A., F.R.S., 10 Moray Pl., Edinburgh, and Christ's Coll., Cambridge
1882		* Buchanan, T. R., M.A., M.P. for the West Division of the City of Edinburgh, 10 Moray Place, Edinburgh, and 36 Upper Brook Street, London
1887		* Buist, J.B., M.D., F.R.C.P.E., 1 Clifton Terrace
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1888		* Burns, Rev. T., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, 13 Cumin Pl. 80
1887		* Burton, Cosmo Innes, B.Sc., F.C.S., 7 Montpelier, Viewforth, Edinburgh
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1878		Campbell, John Archibald, M.D., Garland's Asylum, Carlisle

Date of Election.		
1874		Carrington, Benjamin, M.D., Eccles, Lancashire
1882		* Cay, W. Dyce, Memb. Inst. C.E., 107A Princes Street
1876		* Cazenove, The Rev. John Gibson, M.A., D.D., 22 Alva Street, Chancellor of St Mary's Cathedral
1866		* Chalmers, David, Redhall, Slateford 90
1874		* Chiene, John, M.D., F.R.C.S.E., Professor of Surgery in the University of Edinburgh, 26 Charlotte Square
1875		* Christie, John, 19 Buckingham Terrace
1872		Christie, Thomas B., M.D., F.R.C.P.E., Royal India Asylum, Ealing, London
1880	K. P.	* Chrystal, George, M.A., LL.D., Professor of Mathematics in the University of Edinburgh (Vice-President), 5 Belgrave Crescent
1875		* Clark, Robert, 7 Learmonth Terrace 95
1886		* Clark, Sir Thomas, Bart., 11 Melville Crescent
1863	P.	Cleghorn, Hugh F. C., of Stravithie, M.D., LL.D., F.L.S., St Andrews, United Service Club, 14 Queen Street
1875		* Clouston, T. S., M.D., F.R.C.P.E., Tipperlin House, Morningside
1882		* Coats, Sir Peter, of Auchendrane, President of the Glasgow and West of Scotland Horticultural Society, Auchendrane, Ayr
1887		* Cockburn, John, F.R.A.S., 6 Atholl Crescent 100
1888		Collie, John Norman, Ph.D., F.C.S., University College, London
1886		Connan, Daniel M., M.A., Education Department, Cape of Good Hope
1872		* Constable, Archibald, 11 Thistle Street
1879		* Cox, Robert, of Gorgie, M.A., 34 Drumsheugh Gardens
1875		* Craig, William, M.D., F.R.C.P.E., F.R.C.S.E., Lecturer on Materia Medica to the College of Surgeons, 7 Bruntsfield Place 105
1886		* Croom, John Halliday, M.D., F.R.C.P.E., 25 Charlotte Square
1887		* Crawford, William Caldwell, Lockharton Gardens, Slateford, Edinburgh
1887		* Cumming, A. S., M.D., F.R.C.P.E., 18 Ainslie Place
1878		* Cunningham, Daniel John, M.D., Professor of Anatomy in Trinity College, 69 Harcourt Street, Dublin
1886		* Cunningham, David, Memb. Inst. C.E., Harbour Chambers, Dock Street, Dundee 110
1877		* Cunningham, George Miller, Memb. Inst. C.E., 2 Ainslie Place
1884	P.	* Cunningham, J. T., B.A., Marine Biological Laboratory, Plymouth
1871		* Cunynghame, R. J. Blair, M.D., 18 Rothesay Place
1885		* Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, 3 Great King Street
1867		* Davidson, David, Somerset Lodge, Wimbledon Common, Wimbledon 115
1884		Davy, Richard, F.R.C.S., Surgeon to the Westminster Hospital, 33 Welbeck St., Cavendish Square, London
1870		* Day, St John Vincent, C.E.
1876		* Denny, Peter, Memb. Inst. C.E., Dumbarton
1869	P.	* Dewar, James, M.A., F.R.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fullerian Professor of Chemistry at the Royal Institution of Great Britain, London
1884		* Dickson, Charles Scott, Advocate, 4 Heriot Row 120
1888		* Dickson, H. N., 38 York Place
1876	P.	* Dickson, J. D. Hamilton, M.A., Fellow and Tutor, St Peter's College, Cambridge

Date of Election.		
1863	P.	Dittmar, W., LL.D., F.R.S., Professor of Chemistry, Anderson's College, 11 Hillhead Street, Glasgow
1885		Dixon, J. M., M.A., Prof. of English Literature in the University of Tokio, Japan
1881		* Dobbin, Leonard, Ph.D., 16 Kilmaurs Road
1867	P.	* Donaldson, J., M.A., LL.D., Principal of the United College of St Salvador and St Leonard, St Andrews
1882		* Dott, D. B., Memb. Pharm. Soc., 7 Victoria Terrace, Musselburgh
1866		* Douglas, David, 22 Drummond Place
1880		* Drummond, Henry, F.G.S., Professor of Natural History in the Free Church College, 3 Park Circus, Glasgow
1860		Dudgeon, Patrick, of Cargen, Dumfries
1863	P.	Duncan, J. Matthews, M.A., M.D., F.R.C.P.E. (Lond. and Edin.), LL.D., F.R.S., 71 Brook Street, London
1876		* Duncan, James, of Benmore, Kilmun, 9 Mincing Lane, London
1889		* Duncan, James Dalrymple, F.S.A. Scot., 211 Hope Street, Glasgow
1870		* Duncan, John, M.D., F.R.C.P.E., F.R.C.S.E., 8 Ainslie Place
1878		* Duncanson, J. J. Kirk, M.D., F.R.C.P.E., 22 Drumsheugh Gardens
1859		Duns, Rev. Professor, D.D., New College, Edinburgh, 14 Greenhill Place
1888		* Durham, James, F.G.S., Wingate Place, Newport, Fife
1874		* Durham, William, Seaforth House, Portobello
1869		* Elder, George, Knock Castle, Wemyss Bay, Greenock
1885		* Elgar, Francis, Memb. Inst. C.E., LL.D., The Admiralty, London
1875		Elliot, Daniel G., New York
1880		* Elliot, T. Armstrong, M.A., 6 Sanderson Road, Newcastle-on-Tyne
1855		Etheridge, Robert, F.R.S., Assistant-Keeper of the Geological Department at the British Museum of Natural History, 14 Carlyle Square, Chelsea, London
1884		* Evans, William, F.F.A., Secretary Royal Physical Soc., 18A Morningside Park, Edinburgh
1863	P.	Everett, J. D., M.A., D.C.L., F.R.S., Prof. of Nat. Philosophy, Queen's Coll., Belfast
1879		* Ewart, James Cossar, M.D., F.R.C.S.E., Professor of Natural History, University of Edinburgh, 2 Belford Park
1878	P.	* Ewing, James Alfred, B.Sc., F.R.S., Professor of Engineering and Drawing in University College, Dundee
1875		Fairley, Thomas, Lecturer on Chemistry, 8 Newton Grove, Leeds
1888	P.	* Fawsitt, Charles A., 4 Maule Terrace, Partick, Glasgow
1859		Fayrer, Sir Joseph, K.C.S.I., M.D., F.R.C.P.L., F.R.C.S.L. and E., LL.D., F.R.S., Honorary Physician to the Queen, 53 Wimpole Street, London
1883		* Felkin, Robert W., M.D., F.R.G.S., Fellow of the Anthropological Society of Berlin, 20 Alva Street, Edinburgh
1888		* Ferguson, John, M.A., LL.D., Professor of Chemistry in the University of Glasgow
1868		* Ferguson, Robert M., Ph.D., 12 Moray Place
1874		* Ferguson, William, of Kinmundy, F.L.S., F.G.S., Kinmundy House, Mintlaw
1886		Field, C. Leopold, F.C.S., Upper Marsh, Lambeth, London
1852		Fleming, Andrew, M.D., Deputy Surgeon-General, 3 Napier Road
1876		* Fleming, J. S., 16 Grosvenor Crescent

Date of Election.		
1880		* Flint, Robert, D.D., Corresponding Member of the Institute of France, Corresponding Member of the Royal Academy of Sciences of Palermo, Professor of Divinity in the University of Edinburgh (Vice-President), Johnstone Lodge, 54 Craigmillar Park
1872		Forbes, Professor George, M.A., Memb. Inst. C.E., M.S.T.E. and E., F.R.S., F.R.A.S., 34 Great George Street, Westminster
1859		Forlong, Major-Gen. J. G., F.R.G.S., R.A.S., Assoc. C.E., &c., 11 Douglas Crescent
1828		Foster, John, Liverpool
1887		Fowler, Sir John, Bart., K.C.M.G., Memb. Inst. C.E., LL.D., Thornwood Lodge, Kensington, London
1858		Fraser, A. Campbell, M.A., LL.D., D.C.L., Professor of Logic and Metaphysics in the University of Edinburgh, Gorton House, Hawthornden
1867	B. P.	* Fraser, Thomas R., M.D., F.R.C.P.E., F.R.S., Professor of Materia Medica in the University of Edinburgh, 13 Drumsheugh Gardens
1885		* Fraser, A. Y., M.A., care of Dr Kennedy, 25 Newington Road, Edinburgh
1888		* Galt, Alexander, B.Sc., F.C.S., Gowanbrae, Dunoon
1867		Gayner, Charles, M.D., Oxford
1889		* Geddes, George H., Mining Engineer, 8 Douglas Crescent
1880	P.	* Geddes, Patrick, Professor of Botany in University College, Dundee, and Lecturer on Zoology, 6 James' Court, Lawnmarket
1861	B. P.	Geikie, Archibald, LL.D., F.R.S., F.G.S., Corresponding Member of the Royal Academy of Berlin, Director of the Geological Surveys of Great Britain, and Head of the Geological Museum, 28 Jermyn Street, London
1871	B. P.	* Geikie, James, LL.D., D.C.L., F.R.S., F.G.S., Professor of Geology in the University of Edinburgh, 31 Merchiston Avenue
1881		* Gibson, G. A., D.Sc., M.D., F.R.C.P.E., 17 Alva Street
1877		* Gibson, John, Ph.D., 15 Dick Place
1885	P.	* Gibson, R. J. Harvey, M.A., Lecturer on Botany, Victoria University, 44 Sydenham Avenue, Sefton Park, Liverpool
1887		* Gilmour, William, 10 Elm Row
1879		* Gilray, Thomas, M.A., Professor of English Language and Literature in the University of Otago, New Zealand
1880		* Gilruth, George Ritchie, Surgeon, 48 Northumberland Street
1850		Gosset, Major-General W. D., R.E., 70 Edith Road, West Kensington, London
1867		* Graham, Andrew, M.D., R.N., Army and Navy Club, 36 Pall Mall, London
1880		* Graham, James, 198 West George Street, Glasgow
1851		Grant, The Rev. James, D.D., D.C.L., 15 Palmerston Place
1883		* Gray, Andrew, M.A., Professor of Physics in University College, Bangor, North Wales
1880	P.	Gray, Thomas, B.Sc., Professor of Physics, Rose Polytechnic Institute, Indiana, U.S.
1886		* Greenfield, W. S., M.D., Professor of General Pathology in the University of Edinburgh, 7 Heriot Row
1884		* Grieve, John, M.A., M.D., F.L.S., 212 St Vincent Street, Glasgow
1886		* Griffiths, Arthur Bower, Ph.D., Lecturer on Chemistry in the School of Science of the City and County of Lincoln, Richmond House, Charlotte Road, Edgbaston, Birmingham
1883		Gunning, R. H., Grand Dignitary of the Order of the Rose of Brazil, M.D., LL.D., 12 Addison Crescent, Kensington
1888	P.	Guppy, Henry Brougham, M.B., 17 Woodlane, Falmouth

Date of Election.		
1886		* Haddington, The Right Hon. the Earl of, Tyninghame House, Haddington
1867		* Hallen, James H. B., F.R.C.S.E., F.R.P.S.E., Inspecting Veterinary Surgeon in H.M. Indian Army, Pebworth, near Stratford-on-Avon 190
1881	P.	* Hamilton, D. J., M.B., F.R.C.S.E., Professor of Pathological Anatomy in the University of Aberdeen, 1A Albyn Place, Aberdeen
1876	P.	* Hannay, J. Ballantyne, Cove Castle, Loch Long
1886		* Hare, Arthur W., M.B., F.R.C.S.E., Professor of Surgery, Owens College, 3 Adelphi Terrace, Salford, Manchester
1888		* Hart, D. Berry, M.D., F.R.C.P.E., 29 Charlotte Square
1869		Hartley, Sir Charles A., K.C.M.G., Memb. Inst. C.E., 26 Pall Mall, London 195
1877		Hartley, Walter Noel, F.R.S., Professor of Chemistry, Royal College of Science for Ireland, Dublin
1875		Hawkshaw, Sir John, Memb. Inst. C.E., F.R.S., F.G.S., 33 Great George Street, Westminster
1880	P.	* Haycraft, J. Berry, M.D., D.Sc., Lecturer on Physiology in the University of Edinburgh, 20 Ann Street
1870		Heathfield, W. E., F.C.S., 1 Powis Grove, Brighton
1862		Hector, Sir James, K.C.M.G., M.D., F.R.S., Director of the Geological Survey, Wellington, New Zealand 200
1876	K. P.	* Heddle, M. Forster, M.D., Emeritus Professor of Chemistry in the University of St Andrews
1884		* Henderson, John, jun., Meadowside Works, Partick, Glasgow
1881	N. P.	* Herdman, W. A., D.Sc., Professor of Natural History in University College, Liverpool
1889		Hewitt, William Morse Graily, M.D., Emeritus Professor of Obstetric Medicine in University College, London, 36 Berkeley Square, London
1871		Higgins, Charles Hayes, M.D., M.R.C.P., F.R.C.S., Alfred House, Birkenhead 205
1859		Hills, John, Lieut.-Colonel, C.B., Bombay Engineers, United Service Club, London
1879		Hislop, John, Secretary to the Department of Education, Wellington, New Zealand
1885		Hodgkinson, W. R., Ph.D., F.I.C., F.C.S., Professor of Chemistry and Physics at the Royal Military Academy and Royal Artillery College, Woolwich, 75 Vanbrugh Park, Blackheath, London
1828	P.	Home, David Milne, of Milne-Graden, LL.D., F.G.S., 10 York Place
1881	P.	* Horne, John, F.G.S., Geological Survey of Scotland, 41 Southside Road, Inverness 210
1883	P.	* Hoyle, William Evans, M.A., M.R.C.S., 25 Brunswick Road, Withington, Manchester
1886		Hunt, Rev. H. G. Bonavia, Mus. D. Dublin, Mus. B. Oxon., F.L.S., La Belle Sauvage, London
1872		* Hunter, Lieut.-Col. Chas., Pläs Cöch, Llanfairpwll, Anglesea, and 17 St George's Sq., London
1887		* Hunter, James, F.R.C.S.E., F.R.A.S., 20 Craigmillar Park
1887		* Hunter, William, M.D. 215
1864		Hutchison, Robert (Carlowrie Castle), and University Club
1855		Inglis, Right Hon. John, LL.D., D.C.L., Lord Justice-General of Scotland, and Chancellor of the University of Edinburgh, 30 Abercromby Place
1882		* Inglis, J. W., Memb. Inst. C.E., 19 Montpelier, Edinburgh
1874		* Irvine, Alex. Forbes, of Drum, LL.D., Advocate, Sheriff of Argyll (Vice-President), 25 Castle Terrace
1886		* Irvine, Robert, Royston, Granton, Edinburgh 220
1875		Jack, William, M.A., LL.D., Professor of Mathematics in the University of Glasgow

Date of Election.		
1889		* James, Alexander, M.D., F.R.C.P.E., 44 Melville Street
1882		* Jamieson, A., Memb. Inst. C.E., Professor of Engineering in The Glasgow and West of Scotland Technical College, Glasgow
1860		Jamieson, George Auldjo, Actuary, 24 St Andrew Square
1880		Japp, A. H., LL.D., The Limes, Elmstead, near Colchester 225
1865		Jenner, Charles, Easter Duddingston Lodge
1869		Johnston, John Wilson, M.D., Surgeon-Major, Dacre House, Shrewsbury Road, Oxton, Birkenhead
1867		* Johnston, T. B., F.R.G.S., Geographer to the Queen, 9 Claremont Crescent
1874		Jones, Francis, Lecturer on Chemistry, Monton Place, Manchester
1888		Jones, John Alfred, Memb. Inst. C.E., Vice-President, and Engineer, City of Madras, Peter's Road, Madras 230
1877		* Jolly, William, H.M. Inspector of Schools, F.G.S., Ardgowan, Pollokshields
1866		* Keiller, Alexander, M.D., F.R.C.P.E., LL.D., 21 Queen Street
1886	P.	* Kidston, Robert, F.G.S., 24 Victoria Place, Stirling
1877		* King, Sir James, of Campsie, Bart., LL.D., 12 Claremont Terrace, Glasgow
1880		* King, W. F., Lonend, Russell Place, Trinity 235
1886		* Kingsburgh, The Right Hon. Lord, C.B., LL.D., F.R.S., M.S.T.E. and E., Lord Justice-Clerk, and Lord President of the Second Division of the Court of Session, 15 Abercromby Place
1883		* Kinnear, The Hon. Lord, one of the Senators of the College of Justice, 2 Moray Place
1878		* Kintore, The Right Hon. the Earl of, M.A. Cantab., Keith Hall, Inglismaldie Castle, Laurencekirk
1880	P.	* Knott, C. G., D.Sc., Prof. of Natural Philosophy in the Imperial University of Tokio, Japan
1875		* L'Amy, John Ramsay, of Dunkenny, Forfarshire, 107 Cromwell Road, London 240
1886		* Laing, Rev. George, 17 Buckingham Terrace
1878		* Lang, P. R. Scott, M.A., B.Sc., Professor of Mathematics in the University of St Andrews
1885		* Laurie, A. P., B.A., B.Sc., Lecturer on Chemistry at the People's Palace Technical School, London
1870		* Laurie, Simon S., M.A., Professor of Education in the University of Edinburgh, Nairne Lodge, Duddingstone
1881		* Lawson, Robert, M.D., Deputy-Commissioner in Lunacy, 24 Mayfield Terrace 245
1872		* Lee, Alexander H., C.E., Blairhoyle, Stirling
1872		* Lee, The Hon. Lord, one of the Senators of the College of Justice, 12 Rothesay Place
1882		* Leslie, Alexander, Memb. Inst. C.E., 12 Greenhill Terrace
1883		* Leslie, George, M.B., C.M., Old Manse, Falkirk
1863		Leslie, Hon. G. Waldegrave, Leslie House, Leslie 250
1858		Leslie, James, Memb. Inst. C.E., 2 Charlotte Square
1874	P.	* Letts, E. A., Ph.D., F.I.C., F.C.S., Professor of Chemistry, Queen's College, Belfast
1889		* Lindsay, Rev. James, B.D., B.Sc., F.G.S., Minister of St Andrews Parish, Springhill Terrace, Kilmarnock
1870	B. P.	* Lister, Sir Joseph, Bart., M.D., F.R.C.S.L., F.R.C.S.E., LL.D., D.C.L., F.R.S., Professor of Clinical Surgery, King's College, Surgeon Extraordinary to the Queen, 12 Park Crescent, Portland Place, London
1882		* Livingston, Josiah, 4 Minto Street 255

Date of Election.		
1861	P.	Lorimer, James, M.A., Advocate, Professor of Public Law in the University of Edinburgh, 1 Bruntsfield Crescent
1884		* Low, George M., Actuary, 15 Chester Street
1888		* Lowe, D. F., M.A., Headmaster of Heriot's Hospital School, Lauriston
1849		Lowe, W. H., M.D., F.R.C.P.E., Woodcote, Inner Park, Wimbledon
1886		Lyster, George Fosbery, Memb. Inst. C.E., Gisburn House, Liverpool 260
1855		Macadam, Stevenson, Ph.D., Lecturer on Chemistry, Surgeons' Hall, Edinburgh, 11 East Brighton Crescent, Portobello
1888		* Macadam, W. Ivison, Lecturer on Chemistry, 6 East Brighton Crescent, Portobello
1887		M'Aldowie, Alexander M., M.D., Brook Street, Stoke-on-Trent
1888	P.	M'Arthur, John, Battersea, London
1885		* M'Bride, Charles, M.D., Wigtown 265
1883		* M'Bride, P., M.D., F.R.C.P.E., 16 Chester Street
1867		* M'Candlish, John M., W.S., 27 Drumsheugh Gardens
1886		* Macdonald, William J., M.A., 6 Lockharton Terrace
1847		Macdonald, W. Macdonald, of St Martin's, Perth
1888		* M'Fadyean, John, M.B., B.Sc., Lecturer on Anatomy, 9 East Hermitage Place, Leith 270
1878	P.	Macfarlane, Alex., M.A., D.Sc., LL.D., Professor of Physics in the University of the State of Texas, Austin, Texas
1885	P.	* Macfarlane, J. M., D.Sc., 15 Scotland Street
1877		* Macfie, Robert A., Dreghorn Castle, Colinton
1878		* M'Gowan, George, F.I.C., Ph.D., University College of North Wales, Bangor
1886		* MacGregor, Rev. James, D.D., 11 Cumin Place, Grange 275
1880	P.	MacGregor, J. Gordon, M.A., D.Sc., Professor of Physics in Dalhousie College, Halifax, Nova Scotia
1879		* M'Grigor, Alexander Bennett, LL.D., 19 Woodside Terrace, Glasgow
1869	N. P.	* M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Professor of Natural History in the University of St Andrews, 2 Abbotsford Crescent, St Andrews
1882		* Mackay, John Sturgeon, M.A., LL.D., Mathematical Master in the Edinburgh Academy, 69 Northumberland Street
1873	P.	* M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Professor of the Institutes of Medicine in the University of Glasgow 280
1840		Mackenzie, John, New Club, Princes Street
1843	P.	Maclagan, Sir Douglas, M.D., F.R.C.S.E., Professor of Medical Jurisprudence in the University of Edinburgh (VICE-PRESIDENT), 28 Heriot Row
1853		Maclagan, General R., Royal Engineers, LL.D., 4 West Cromwell Road, S. Kensington, London, S.W.
1869		* Maclagan, R. Craig, M.D., 5 Coates Crescent
1864		M'Lagan, Peter, of Pumpherston, M.P., Clifton Hall, Ratho 285
1869	P.	* M'Laren, The Hon. Lord, LL.D. Edin. and Glasg., F.R.A.S., one of the Senators of the College of Justice (VICE-PRESIDENT), 46 Moray Place
1888		* Maclean, Magnus, M.A., Assistant to the Professor of Natural Philosophy in the University of Glasgow, 21 Hayburn Crescent, Partick
1870		* Macleod, Sir George H.B., M.D., F.R.C.S.E., Regius Prof. of Surgery in the University of Glasgow, and Surgeon in Ordinary to the Queen in Scotland, 10 Woodside Crescent, Glasgow

Date of Election.		
1876		* Macleod, Rev. Norman, D.D., 7 Royal Circus
1883		* Macleod, W. Bowman, L.D.S., 16 George Square 290
1872		* Macmillan, Rev. Hugh, D.D., LL.D., Seafield, Greenock
1876		* Macmillan, John, M.A., B.Sc., 6 St Vincent Street
1884		* Macpherson, Rev. J. Gordon, M.A., D.Sc., Ruthven Manse, Meigle
1883		* M'Roberts, George, F.C.S., Bath House, Ardrossan, Ayrshire
1888		Mactear, James, F.C.S., 2 Victoria Mansions, Hyde Park, London 295
1858		Malcolm, R. B., M.D., F.R.C.P.E., 126 George Street
1880	P.	Marsden, R. Sydney, M.B., C.M., D.Sc., F.I.C., F.C.S., Pembroke House, King Street, Stockton-on-Tees
1882	P.	Marshall, D. H., M.A., Professor of Physics in Queen's University and College, Kingston, Ontario, Canada
1869		Marshall, Henry, M.D., Clifton, Bristol
1888		* Marshall, Hugh, D.Sc., Assistant to the Professor of Chemistry in the University of Edinburgh, 1 Lorne Terrace 300
1864		Marwick, Sir James David, LL.D., Town-Clerk, Glasgow
1866		* Masson, David, LL.D., Professor of Rhetoric and English Literature in the University of Edinburgh, 58 Great King Street
1885	P.	* Masson, Orme, D.Sc., Professor of Chemistry in the University of Melbourne
1883		* Matthews, James Duncan, Springhill, Aberdeen
1888		* Methven, C. W., Memb. Inst. C.E., Engineer's Office, Harbour Works, Port Natal 305
1885		* Mill, Hugh Robt., D.Sc., F.C.S., Scot. Marine Station, Granton, Braid Road, Morningside, Edinburgh
1886		* Miller, Hugh, H.M. Geological Survey Office, George IV. Bridge
1852		Miller, Thomas, M.A., LL.D., Emeritus Rector of Perth Academy, Inchbank House, Perth
1833		Milne, Admiral Sir Alexander, Bart., G.C.B., Inveresk
1886		* Milne, William, M.A., B.Sc., Mathematical and Science Teacher, High School, Glasgow 310
1866		* Mitchell, Sir Arthur, K.C.B., M.A., M.D., LL.D., Commissioner in Lunacy (Vice-President), 34 Drummond Place
1889	P.	* Mitchell, A. Crichton, B.Sc., 2 Baxter's Place
1865		Moir, John J. A., M.D., F.R.C.P.E., 52 Castle Street
1870		* Moncreiff, The Right Hon. Lord, of Tullibole, LL.D. (Honorary Vice-President), 15 Great Stuart Street
1871		* Moncrieff, Rev. Canon William Scott, of Fossaway, Christ's Church Vicarage, Bishop-Wearmouth, Sunderland 315
1868		* Montgomery, Very Rev. Dean, M.A., D.D., 17 Atholl Crescent
1887		Moos, Nanabhay A. F., L.C.E., B.Sc., Assistant Professor of Engineering, College of Science, Bombay
1887		More, Alexander Goodman, M.R.I.A., F.L.S., 74 Leinster Road, Dublin
1873		* Muir, M. M. Pattison, Prælector on Chemistry, Caius College, Cambridge
1874	K.P.	* Muir, Thomas, M.A., LL.D. (Vice-President), Mathematical Master, High School, Glasgow, Beechcroft, Bothwell, Glasgow 320
1888		* Muirhead, George, Mains of Haddo, Aberdeen
1877		Mukhopâdhyay, Âsûtosh, M.A., F.R.A.S., Examiner in Mathematics in the University of Calcutta, Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta
1870		* Munn, David, M.A., 2 Ramsay Gardens

Date of Election.		
1889		* Munro, Rev. Robert, M.A., B.D., F.S.A. Scot., Free Church Manse, Old Kilpatrick
1857		Murray, John Ivor, M.D., F.R.C.S.E., M.R.C.P.E., 24 Huntriss Row, Scarborough 325
1877	N. P.	* Murray, John, LL.D., Ph.D., Director of the Challenger Expedition Commission, 28 Douglas Crescent, and United Service Club. Office, 45 Frederick Street
1888		* Murray, R. Milne, M.A., M.B., F.R.C.P.E., 10 Hope Street
1887		Muter, John, M.A., F.C.S., Winchester House, 397 Kennington Road, London
1884		Mylne, R. W., C.E., F.R.S., 7 Whitehall Place, London
1888		Napier, A. D. Leith, M.D., C.M., M.R.C.P.L., 67 Grosvenor Street, Grosvenor Square, London 330
1877		* Napier, John C., Audley Mansions, Grosvenor Square, London
1887		* Nasmyth, T. Goodall, M.B., C.M., D.Sc., Foulford, Cowdenbeath, Fife
1866		* Nelson, Thomas, St Leonard's, Dalkeith Road
1883		* Newcombe, Henry, F.R.C.S.E., 5 Dalrymple Crescent, Edinburgh
1884		* Nicholson, J. Shield, Professor of Political Economy in the University of Edinburgh, Eden Lodge, Eden Lane, Newbattle Terrace 335
1880	P.	* Nicol, W. W. J., M.A., D.Sc., Lecturer on Chemistry, Mason College, Birmingham
1878		Norris, Richard, M.D.
1888		* Ogilvie, F. Grant, M.A., B.Sc., Principal of the Heriot-Watt College, 27 Blacket Place
1888		* Oliphant, James, M.A., 50 Palmerston Place
1886		Oliver, James, M.D., C.M., M.R.C.P., Assistant Physician, Hospital for Women, 18 Gordon Square, London 340
1884		* Omond, Robert Traill, Superintendent of Ben Nevis Observatory, Fort-William, Inverness
1877		Panton, George A., 73 Westfield Road, Edgbaston, Birmingham
1886		* Paton, D. Noel, M.D., B.Sc., F.R.C.P.E., 4 Walker Street
1889		* Patrick, David, M.A., 25 Gillespie Crescent
1881	N.P.	* Peach, B. N., F.G.S., Acting Palæontologist of the Geological Survey of Scotland, 13 Dalrymple Crescent 345
1889		* Peck, William, F.R.A.S., Town's Astronomer, Murrayfield, Edinburgh
1863		Peddie, Alexander, M.D., F.R.C.P.E., 15 Rutland Street
1887		* Peddie, Wm., D.Sc., Assistant to the Professor of Natural Philosophy, Edinburgh University
1886		* Peebles, D. Bruce, Tay House, Bonnington, Edinburgh
1869		Pender, Sir John, 18 Arlington Street, Piccadilly, London 350
1888		* Perkin, W. H., junior, Ph.D., Prof. of Chemistry in the Heriot-Watt College
1889		* Philip, R. W., M.A., M.D., F.R.C.P.E., 4 Melville Crescent
1883		Phillips, Charles D. F., M.D., 10 Henrietta Street, Cavendish Square, London, W.
1859	P.	Playfair, The Right Hon. Sir Lyon, K.C.B., M.P., LL.D., F.R.S., 68 Onslow Gardens, London
1877		Pole, William, Memb. Inst. C.E., Mus. Doc., F.R.S., 31 Parliament St., Westminster 355
1886		* Pollock, Charles Frederick, M.D., F.R.C.S.E., 1 Buckingham Terr., Hillhead, Glasgow
1874		Powell, Baden Henry Baden-, Forest Department, India
1852		Powell, Eyre B., C.S.I., M.A., 28 Park Road, Haverstock Hill, Hampstead, London
1888		Prain, David, Surgeon, Indian Medical Service, and Curator of the Herbarium, Royal Botanic Gardens, Shibpur, Calcutta
1880		* Prentice, Charles, Actuary, C.A., Edinburgh, Athenæum, Glasgow 360
1875		Prevost, E. W., Ph.D., The Poplars, Shuttington, Tamworth

Date of Election.		
1849		Primrose, Hon. B. F., C.B., 22 Moray Place
1882		* Pryde, David, M.A., LL.D., Head Master of the Ladies' College, 10 Fettes Row, Edinburgh
1885		* Pullar, J. F., Rosebank, Perth
1880		* Pullar, Robert, Tayside, Perth 365
1884		Ramsay, E. Peirson, M.R.I.A., F.L.S., C.M.Z.S., F.R.G.S., F.G.S., Fellow of the Imperial and Royal Zool. and Bot. Soc. of Vienna, Curator of Australian Museum, Sydney, N.S.W.
1882		* Rattray, James Clerk, M.D., 61 Grange Loan
1885	P.	* Rattray, John, M.A., B.Sc., 31 Belsize Avenue, South Hampstead, London
1869		Raven, Rev. Thomas Milville, M.A., The Vicarage, Crakehall, Bedale
1883		* Readman, J. B., D.Sc., F.C.S., 9 Moray Place 370
1889		Redwood, Boverton, Glenwathen, Ballard's Lane, Finchley, Middlesex
1875		* Richardson, Ralph, W.S., 10 Magdala Place
1872		Ricarde-Seaver, Major F. Ignacio, Conservative Club, St James' Street, London, and 2 Rue Laffitte, Boulevard des Italiens, Paris
1883		* Ritchie, R. Peel, M.D., Pres. R.C.P.E., 1 Melville Crescent
1880		Roberts, D. Lloyd, M.D., F.R.C.P.L., 23 St John Street, Manchester 375
1872		* Robertson, D. M. C. L. Argyll, M.D., F.R.C.S.E., Surgeon Oculist to the Queen for Scotland, and President of the Royal College of Surgeons, 18 Charlotte Square
1859		Robertson, George, Memb. Inst. C.E., Athenæum Club, Pall Mall, London
1886		* Robertson, Right Hon. J. P. B., Q.C., LL.D., M.P., Lord Advocate of Scotland, 19 Drumsheugh Gardens
1877	P.	* Robinson, George Carr, F.I.C., Lecturer on Chemistry in the College of Chemistry, Royal Institution, Hull
1881		* Rogerson, John Johnston, B.A., LL.B., Merchiston Castle Academy 380
1881		Rosebery, The Right Hon. the Earl of, LL.D., Dalmeny Park, Edinburgh
1880		Rowland, L. L., M.A., M.D., President of the Oregon State Medical Society, and Professor of Physiology and Microscopy in Williamette University, Salem, Oregon
1880		* Russell, J. A., M.A., B.Sc., M.B., F.R.C.P.E., Woodville, Canaan Lane
1869	P.	* Rutherford, Wm., M.D., F.R.C.P.E., F.R.S., Professor of the Institutes of Medicine in the University of Edinburgh, 14 Douglas Crescent
1863		Sanderson, James, Deputy Inspector-General of Hospitals, F.R.C.S.E., 8 Manor Place 385
1864		Sandford, The Right Rev. Bishop D. F., LL.D., Boldon Rectory, Newcastle-on-Tyne
1849	B. P.	Sang, Edward, C.E., LL.D., 31 Mayfield Road
1846		Schmitz, Leonard, LL.D., 53 Gloucester Road, Regent's Park, London
1887		* Schulze, Adolf P., 2 Doune Gardens, Kelvinside, Glasgow
1885		Scott, Alexander, M.A., D.Sc., 4 North Bailey, Durham 390
1880		Scott, J. H., M.B., C.M., M.R.C.S., Professor of Anatomy in the University of Otago, New Zealand
1888		* Scott, John, C.B., Shipbuilder, Hawkhill, Greenock
1875		Scott, Michael, Memb. Inst. C.E., care of Alexander Grahame, Esq., 30 Great George Street, Westminster
1889		* Scougal, Andrew E., M.A., H.M. Inspector of Schools, 12 Blantyre Terrace
1864		Sellar, W. Y., M.A., LL.D., Professor of Humanity in the University of Edinburgh, 15 Buckingham Terrace 395
1872		* Seton, George, M.A., Advocate, 42 Greenhill Gardens

Date of Election.		
1887		* Sexton, A. H., F.C.S., Professor of Chemistry, College of Science and Arts, 38 Bath Street, Glasgow
1872		Sibbald, John, M.D., Commissioner in Lunacy, 3 St Margaret's Road, Whitehouse Loan
1870		* Sime, James, M.A., South Park, Fountainhall Road
1871		* Simpson, A. R., M.D., F.R.C.P.E., Professor of Midwifery in the University of Edinburgh, 52 Queen Street 400
1888		* Sinclair, D. S., 328 Renfrew Street, Glasgow
1859	P.	Skene, W. F., W.S., LL.D., D.C.L., Historiographer-Royal for Scotland, 27 Inverleith Row
1876		* Skinner, William, W.S., Town-Clerk of Edinburgh, 35 George Square
1868		* Smith, Adam Gillies, C.A. (Treasurer), 64 Princes Street
1882	P.	Smith, C. Michie, B.Sc., Professor of Physical Science, Christian College, Madras, India 405
1885		* Smith, George, F.C.S., Polmont Station
1883		Smith, James Greig, M.A., M.B., 16 Victoria Square, Clifton
1871		* Smith, John, M.D., F.R.C.S.E., LL.D., President of the Medico-Chirurgical Society, 11 Wemyss Place
1886		* Smith, Major-General Sir R. Murdoch, K.C.M.G., R.E., Director of Museum of Science and Art, Edinburgh
1871	P.	* Smith, Rev. W. Robertson, M.A., LL.D., Professor of Arabic in the Univ. of Cambridge 410
1880		Smith, William Robert, M.D., D.Sc., Barrister-at-Law, Professor of Forensic Medicine in King's College, 74 Great Russell Street, Bloomsbury Square, London
1846	K.B. P.	Smyth, Piazzi, LL.D., Ex-Astronomer-Royal for Scotland, and Emeritus Professor of Astronomy in the University of Edinburgh, Clova, Ripon
1880		Sollas, W. J., M.A., D.Sc., F.R.S., late Fellow of St John's College, Cambridge, and Professor of Geology and Mineralogy in the University of Dublin, Talbot House, Merrion Avenue, Blackrock, County Dublin
1889		* Somerville, William, Dr Oec., B.Sc., of Comiston, Lecturer on Forestry in the University of Edinburgh, 1 Braid Crescent
1882		* Sorley, James, F.F.A., C.A., 18 Magdala Crescent 415
1874	P.	* Sprague, T. B., M.A., Actuary, 29 Buckingham Terrace
1850	P.	Stark, James, M.D., F.R.C.P.E., of Huntfield, Underwood, Bridge of Allan
1885		* Steggall, J. E. A., Prof. of Mathematics and Natural Phil. in University College, Dundee
1886		* Stevenson, C. A., B.Sc., Assoc. Memb. Inst. C.E., 45 Melville Street
1884		* Stevenson, David Alan, B.Sc., Memb. Inst. C.E., 45 Melville Street 420
1877		* Stevenson, James, F.R.G.S., 4 Woodside Crescent, Glasgow
1888		* Stevenson, Rev. John, LL.D., Minister of Glamis, Forfarshire
1868		Stevenson, John J., 4 Porchester Gardens, London
1888		* Stewart, Charles Hunter, M.B., B.Sc., 2 Bellevue Terrace
1868		Stewart, Major-General J. H. M. Shaw, R.E., F.R.G.S., 61 Lancaster Gate, London, W. 425
1878		* Stewart, James R., M.A., 10 Salisbury Road
1866		* Stewart, T. Grainger, M.D., F.R.C.P.E., Professor of the Practice of Physic in the University of Edinburgh, 19 Charlotte Square
1873		* Stewart, Walter, 22 Torphichen Street
1848		Stirling, Patrick J., LL.D., Kippendavie House, Dunblane
1877		* Stirling, William, D.Sc., M.D., Brackenbury Professor of Physiology and Histology in Owens College and Victoria University, Manchester 430
1889		* Stockman, Ralph, M.D., F.R.C.P.E., 5 Bellevue Crescent
1823		Stuart, Captain T. D., H.M.I.S.

Date of Election.		
1870		* Swan, Patrick Don, Ex-Provost of Kirkcaldy
1848	P.	Swan, Wm., LL.D., Emeritus Professor of Natural Philosophy in the University of St Andrews, Ardchapel, Helensburgh
1844		Swinton, A. Campbell, of Kimmerghame, LL.D., Duns 435
1875		* Syme, James, 9 Drumsheugh Gardens
1885		* Symington, Johnson, M.D., F.R.C.S.E., 2 Greenhill Park
1872		Tait, the Very Rev. A., D.D., LL.D., Provost of Tuam, Moylough Rectory, County Galway, Ireland
1861	K.P.	Tait, P. Guthrie, M.A., Professor of Natural Philosophy in the University of Edinburgh (General Secretary), 38 George Square
1870		* Tatlock, Robert R., City Analyst's Office, 156 Bath Street, Glasgow 440
1872		* Teape, Rev. Charles R., M.A., Ph.D., 15 Findhorn Place
1885		* Thompson, D'Arcy W., Professor of Natural History in University College, Dundee
1884		* Thoms, George Hunter, of Aberlemno, Advocate, Sheriff of the Counties of Orkney and Zetland, 13 Charlotte Square
1870		* Thomson, Rev. Andrew, D.D., 63 Northumberland Street
1887		* Thomson, Andrew, M.A., D.Sc., Assistant to the Professor of Chemistry in the University College, Dundee, 10 Comly Bank, Bridge End, Perth 445
1875	P.	* Thomson, James, LL.D., F.R.S., 2 Florentine Gardens, Hillhead, Glasgow
1887	P.	* Thomson, J. Arthur, M.A., Lect. on Zoology, School of Medicine, Edin., 30 Royal Circus
1880		Thomson, John Millar, King's College, London
1863		Thomson, Murray, M.D., Professor of Chemistry, Thomason College, Roorkee, India, 22 Victoria Road, Gipsy Hill, London, S.E.
1870		* Thomson, Spencer C., Actuary, 10 Eglinton Crescent 450
1847	V.J. K.P.	Thomson, Sir William, LL.D., D.C.L., F.R.S. (President), Foreign Associate of the Institute of France, Regius Professor of Natural Philosophy in the University of Glasgow, Grand Officer of the Legion of Honour of France, and Member of the Prussian Order *Pour le Mérite*
1882		Thomson, Wm., M.A., B.Sc., Professor of Mathematics, Victoria College, Stellenbosch, Cape Colony
1870		* Thomson, Wm. Burns, F.R.C.P.E., F.R.C.S.E., 112 Newington Green Road, London
1876		Thomson, William, Royal Institution, Manchester
1878		Thorburn, Robert Macfie, Uddevalla, Sweden 455
1874	N.P.	* Traquair, R. H., M.D., F.R.S., F.G.S., Pres. Royal Physical Soc., Keeper of the Natural History Collections in the Museum of Science and Art, Edinburgh, 8 Dean Park Crescent
1874		* Tuke, J. Batty, M.D., F.R.C.P.E., 20 Charlotte Square
1888		* Turnbull, Andrew H., Actuary, The Elms, Whitehouse Loan
1879		* Turnbull, John, of Abbey St Bathans, W.S., 49 George Square
1861	N.P.	Turner, Sir William, M.B., LL.D., F.R.C.S.E., F.R.S., Professor of Anatomy in the University of Edinburgh, and President of the Royal Physical Society (Secretary), 6 Eton Terrace 460
1877		* Underhill, Charles E., B.A., M.B., F.R.C.P.E., F.R.C.S.E., 8 Coates Crescent
1889		Underhill, T. Edgar, M.D., F.R.C.S.E., Broomsgrove, Worcestershire
1875		Vincent, Charles Wilson, F.I.C., F.C.S., M.R.I., Librarian of the Reform Club, Pall Mall, and Royal Institution, Albemarle Street, London

Date of Election.		
1888		Walker, James, Memb. Inst. C.E., Engineer's Office, Harbour Works, Douglas, Isle of Man
1873		* Walker, Robert, M.A., University, Aberdeen 465
1886		* Wallace, Robert, Prof. of Agriculture and Rural Economy in the University of Edinburgh
1883		* Watson, Charles, Redhall, Slateford
1870		* Watson, James, C.A., 45 Charlotte Square
1866		* Watson, John K., 14 Blackford Road
1866		* Watson, Patrick Heron, M.D., F.R.C.S.E., LL.D., 16 Charlotte Square 470
1862	P.	Watson, Rev. Robert Boog, B.A., Free Church Manse, Cardross, Dumbartonshire
1887		* Webster, H. A., Librarian to University of Edinburgh, 7 Duddingstone Park, Portobello
1873		Welsh, David, Major-General, R.A., 1 Barton Terrace, Dawlish
1882		* Wenley, James A., Treasurer of the Bank of Scotland, 5 Drumsheugh Gardens
1887		* White, Arthur Silva, Secretary to the Royal Scottish Geographical Society, 22 Duke St. 475
1881		Whitehead, Walter, F.R.C.S.E., 202 Oxford Road, Manchester
1883		Wickham, R. H. B., M.D., F.R.C.S.E., Medical Superintendent, City and County Lunatic Asylum, Newcastle-on-Tyne, Dawlish, South Devon
1887		* Wieland, G. B., Whitehill, Rosewell, Mid-Lothian
1879		* Will, John Charles Ogilvie, M.D., 305 Union Street, Aberdeen
1868		* Williams, W., Principal and Professor of Veterinary Medicine and Surgery, New Veterinary College, Leith Walk 480
1888		* Williamson, George, F.A.S. Scot., 37 Newton Street, Finnart, Greenock
1879		* Wilson, Andrew, Ph.D., Lecturer on Zoology and Comparative Anatomy in the Edinburgh Medical School, 118 Gilmore Place
1878		* Wilson, Rev. John, M.A., 27 Buccleuch Place
1875		Wilson, Sir Daniel, LL.D., President of the University of Toronto, and Professor of English Literature in that University
1882		Wilson, George, M.A., M.D., 23 Claremont Road, Leamington 485
1834		Wilson, Isaac, M.D.
1889		Wilson, Robert, Memb. Inst. C.E., St Stephen's Club, and 7 Westminster Chambers, Victoria Street, London
1870		Winzer, John, Chief Surveyor, Civil Service, Ceylon, 7 Dryden Place, Newington
1881		* Wise, Thos. Alex., M.D., F.R.C.P.E., F.R.A.S., Thornton, the Beulah, Upper Norwood
1886		* Woodhead, German Sims, M.D., F.R.C.P.E., 6 Marchhall Crescent 490
1884		Woods, G. A., M.R.C.S., Lansdowne, 36 Hoghton Street, Southport
1864		Wyld, Robert S., LL.D., 19 Inverleith Row
1887		* Yeo, John S., Carrington House, Fettes College
1882		* Young, Andrew, F.G.S., 22 Elm Row
1882		* Young, Frank W., F.C.S., Lecturer on Natural Science, High School, Dundee, Woodmuir Park, West Newport, Fife 495
1882		* Young, Thomas Graham, Westfield, West Calder

LIST OF HONORARY FELLOWS

AT NOVEMBER 1889.

His Royal Highness The Prince of Wales.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW X.).

Elected		
1884	Pierre J. van Beneden,	*Louvain.*
1889	Marcellin Pierre Eugène Berthelot,	*Paris.*
1864	Robert Wilhelm Bunsen,	*Heidelberg.*
1877	Alphonse de Candolle,	*Geneva.*
1883	Luigi Cremona,	*Rome.*
1889	Ernst Curtius,	*Berlin.*
1858	James D. Dana,	*New Haven, Conn.*
1877	Carl Gegenbaur,	*Heidelberg.*
1888	Ernst Haeckel,	*Jena.*
1883	Julius Hann,	*Vienna.*
1884	Charles Hermite,	*Paris.*
1864	Hermann Ludwig Ferdinand von Helmholtz,	*Berlin.*
1879	Jules Janssen,	*Paris.*
1875	August Kekulé,	*Bonn.*
1864	Albert Kölliker,	*Würzburg.*
1875	Ernst Eduard Kummer,	*Berlin.*
1876	Ferdinand de Lesseps,	*Paris.*
1864	Rudolph Leuckart,	*Leipzig.*
1881	Sven Lovén,	*Stockholm.*
1889	James Russell Lowell,	*Cambridge, U.S.*
1888	Demetrius Ivanovich Mendeléef,	*St Petersburg.*
1886	Alphonse Milne-Edwards,	*Paris.*
1864	Theodore Mommsen,	*Berlin.*
1881	Simon Newcomb,	*Washington.*
1886	H. A. Newton,	*Yale College.*
1874	Louis Pasteur,	*Paris.*
1886	Alphonse Renard,	*Gand.*
1889	Georg Hermann Quincke,	*Heidelberg.*
1881	Johannes Iapetus Smith Steenstrup,	*Copenhagen.*
1878	Otto Wilhelm Struve,	*Pulkowa.*
1886	Tobias Robert Thalén,	*Upsala.*
1874	Otto Torell,	*Lund.*
1868	Rudolph Virchow,	*Berlin.*
1874	Wilhelm Eduard Weber,	*Göttingen.*

Total, 34.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW X.).

Elected.		
1849	John Couch Adams, LL.D., F.R.S., Corresp. Mem. Inst. of France,	*Cambridge.*
1835	Sir George Biddell Airy, K.C.B., LL.D., D.C.L., F.R.S., Foreign Associate Inst. of France,	*Greenwich.*
1889	Ball, Sir Robert Stawell, Kt., LL.D., M.R.I.A., Professor of Astronomy in the University of Dublin, and Royal Astronomer for Ireland,	*Dublin.*
1865	Arthur Cayley, LL.D., D.C.L., F.R.S., Corresp. Memb. Inst. of France,	*Cambridge.*
1884	Edward Frankland, D.C.L., LL.D., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1874	John Anthony Froude, LL.D.,	*London.*
1881	The Hon. Justice Sir William Robert Grove, LL.D., D.C.L., F.R.S.,	*London.*
1883	Sir Joseph Dalton Hooker, K.C.S.I., M.D., LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1884	William Huggins, LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1876	Thomas Henry Huxley, LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1845	Sir Richard Owen, K.C.B., M.D., LL.D., D.C.L., F.R.S., Foreign Associate Inst. of France,	*London.*
1886	The Lord Rayleigh, D.C.L., LL.D., Sec. R.S., Corresp. Mem. Inst. of France,	*London.*
1881	The Rev. George Salmon, D.D., LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	*Dublin.*
1884	J. S. Burdon Sanderson, M.D., LL.D., F.R.S.,	*Oxford.*
1864	Sir George Gabriel Stokes, Bart., M.P., LL.D., D.C.L., Pres. R.S., Corresp. Mem. Inst. of France,	*Cambridge.*
1874	James Joseph Sylvester, LL.D., F.R.S., Corresp. Mem. Inst. of France,	*Oxford.*
1864	The Right Hon. Lord Tennyson, D.C.L., LL.D., F.R.S., Poet Laureate,	*Isle of Wight.*
1883	Alexander William Williamson, LL.D., F.R.S., Corresp. Mem. Inst. of France,	*London.*
1883	Colonel Henry Yule, C.B., LL.D.,	*London.*

Total, 19.

ORDINARY FELLOWS ELECTED

During Session 1887–88,

Arranged according to the Date of their Election.

5th December 1887.

D. S. Sinclair.

A. D. Leith Napier, M.D., C.M.

Alexander Galt, B.Sc., F.C.S.

16th January 1888.

John Norman Collie, Ph.D., F.C.S.

Principal Grant Ogilvie, M.A., B.Sc.

R. E. Allardice, M.A.

D. F. Lowe, M.A.

Charles Hunter Stewart, M.B., B.Sc.

6th February 1888.

James MacTear, F.C.S.

W. H. Perkin, Ph.D.

John M'Arthur.

H. N. Dickson.

Charles A. Fawsitt.

David Prain.

George Brook, F.L.S.

George Muirhead.

Cathcart W. Methven, M. Inst. C.E.

5th March 1888.

James Durham, F.G.S.

Professor Thomas Hudson Beare, B.Sc., Assoc. M. Inst. C.E.

William James Bell, B.A., LL.M., F.C.S.

Rev. John Stevenson.

Andrew H. Turnbull, Actuary.

Henry Brougham Guppy, M.B.

John Alfred Jones, M. Inst. C.E.

Professor John Ferguson, M.A., LL.D.

2nd April 1888.

Rev. Thomas Burns, F.S.A. Scot.

R. Milne Murray, M.A., M.B., F.R.C.P.E.

William A. Bryson.

John M'Fadyean, M.B., B.Sc.

7th May 1888.

John Scott, C.B.

Magnus Maclean, M.A.

Hugh Marshall, B.Sc.

D. Berry Hart, M.D., F.R.C.P.E.

James Walker, M. Inst. C.E.

4th June 1888.

George Williamson, F.A.S. Scot.

C. M. Aikman, M.A., B.Sc., F.I.C., F.C.S.

2nd July 1888.

W. Ivison Macadam.

James Oliphant, M.A.

FELLOWS DECEASED OR RESIGNED

During Session 1887–88.

ORDINARY FELLOWS DECEASED.

Colonel Balfour of Balfour and Trenabie.
Robert Chambers.
Professor Alexander Dickson.
Samuel Drew, M.D.
R. M. Smith.
William Wallace, Ph.D.
Allan A. M. Welwood, LL.D.
Charles Edward Wilson, LL.D
Professor John Wilson.

RESIGNED.

John W. Capstick.
Thomas Harvey, LL.D.
Robert Tennent.
Peter Waddell.

HONORARY FELLOWS DECEASED.

Session 1887–88.

FOREIGN.

Rudolf Clausius. Asa Gray. J. N. Madvig.

BRITISH.

Professor Balfour Stewart.

ORDINARY FELLOWS ELECTED

During Session 1888–89,

Arranged according to the Date of their Election.

3rd December 1888.

William Somerville, B.Sc.
David Patrick, M.A.
Alexander James, M.D.
A. H. F. Barbour, M.D.
Ralph Stockman, M.D.
A. Crichton Mitchell, B.Sc.

7th January 1889.

James Dalrymple Duncan, F.S.A. Scot.

4th March 1889.

Boverton Redwood.
Rev. James Lindsay, M.A., B.D., B.Sc., F.G.S.

1st April 1889.

John Alison, M.A.
R. W. Philip, M.D., M.A., F.R.C.P.E.
T. Edgar Underhill, M.D., F.R.C.S.E.

6th May 1889.

William Morse Graily Hewitt, M.D., F.R.C.P.

3rd June 1889.

George H. Geddes, C.E.
Robert Wilson, Memb. Inst. C.E.
William Peck, F.R.A.S.
Rev. Robert Munro, M.A., B.D., F.S.A. Scot.

1st July 1889.

Professor T. D. Collis Barry, F.Z.S., F.C.S.
Andrew E. Scougal, M.A.

FELLOWS DECEASED OR RESIGNED

During Session 1888–89.

ORDINARY FELLOWS DECEASED.

John Frederic Latrobe Bateman, Memb. Inst. C.E., F.R.S.
Joseph James Coleman.
Charles Cowan of Westerlea.
James Dalmahoy.
Henry Davidson of Muirhouse.
William Dickson.
Sir James Falshaw, Bart., Assoc. Inst. C.E.
T. H. Cockburn Hood, F.G.S.
William Miller, S.S.C.
Prof. Sir James Roberton, LL.D.
Edmund Ronalds, LL.D.

RESIGNED.

Rev. F. E. Belcombe.
Alan MacDougall, Memb. Inst. C.E.

HONORARY FELLOWS DECEASED.

Session 1888–89.

FOREIGN.

Michel Eugène Chevreul.
Franz Cornelius Donders.

BRITISH.

James Prescott Joule.

LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH,

AS REVISED 20TH FEBRUARY 1882.

LAWS.

[By the Charter of the Society (printed in the *Transactions*, Vol. VI. p. 5), the Laws cannot be altered, except at a Meeting held one month after that at which the Motion for alteration shall have been proposed.]

I.

Title.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows.

II.

The fees of Ordinary Fellows residing in Scotland.

Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter.*

III.

Payment to cease after 25 years.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from further payment.

IV.

Fees of Non-Resident Ordinary Fellows.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt

* A modification of this rule, in certain cases, was agreed to at a Meeting of the Society held on the 3rd January 1831.

At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s. to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future annual Contributions.

Case of Fellows becoming Non-Resident.

from any further payment. In the case of any Resident Fellow ceasing to reside in Scotland, and wishing to continue a Fellow of the Society, it shall be in the power of the Council to determine on what terms, in the circumstances of each case, the privilege of remaining a Fellow of the Society shall be continued to such Fellow while out of Scotland.

V.

Defaulters.

Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

Privileges of Ordinary Fellows.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

VII.

Numbers Unlimited.

The number of Ordinary Fellows shall be unlimited.

VIII.

Fellows entitled to Transactions.

The Ordinary Fellows, upon producing an order from the TREASURER, shall be entitled to receive from the Publisher, gratis, the Parts of the Society's Transactions which shall be published subsequent to their admission.

IX.

Mode of Recommending Ordinary Fellows.

Candidates for admission as Ordinary Fellows shall make an application in writing, and shall produce along with it a certificate of recommendation to the purport below,* signed by at least *four* Ordinary Fellows, two of whom shall certify their recommendation from personal knowledge. This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall afterwards be printed in the circulars for three Ordinary Meetings of the Society, previous to the day of election, and shall lie upon the table during that time.

* "A. B., a gentleman well versed in Science (*or Polite Literature, as the case may be*), being "to our knowledge desirous of becoming a Fellow of the Royal Society of Edinburgh, we hereby "recommend him as deserving of that honour, and as likely to prove a useful and valuable Member."

X.

Honorary Fellows shall not be subject to any contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states. Honorary Fellows, British and Foreign.

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X. Royal Personages.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one meeting, and printed in the circulars for two ordinary meetings of the Society, previous to the day of election. Recommendation of Honorary Fellows. Mode of Election.

XIII.

The election of Ordinary Fellows shall only take place at the first Ordinary Meeting of each month during the Session. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote. Election of Ordinary Fellows.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of every month from December to July inclusively; excepting when there are five Mondays in January, in which case the Meetings for that month shall be held on its third and fifth Mondays. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make. Ordinary Meetings.

* We hereby recommend_____
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

_____

_____

_____

To the President and Council of the Royal Society of Edinburgh.

XV.

The Transactions.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

The Council shall have power to regulate the private business of the Society. At any Meeting of the Council the Chairman shall have a casting as well as a deliberative vote.

XVI.

How Published.

The Transactions shall be published in parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society.

XVII.

The Council.

That there shall be formed a Council, consisting—First, of such gentlemen as may have filled the office of President; and Secondly, of the following to be annually elected, viz.:—a President, Six Vice-Presidents (two at least of whom shall be resident), Twelve Ordinary Fellows as Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

XVIII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XIX.

Election of Office-Bearers.

An Extraordinary Meeting for the Election of Office-Bearers shall be held on the fourth Monday of November annually.

XX.

Special Meetings: how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XXI.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in November, he shall present the accounts for the preceding year, duly audited.

At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXII.

Auditor.

At the Extraordinary Meeting in November, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXIII.

General Secretary's Duties.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes he shall, when necessary, employ a clerk, to be paid by the Society.

XXIV.

Secretaries to Ordinary Meetings.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the Donors' names, to the Curator of the Library and Museum; they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty.

XXV.

Curator of Museum and Library.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows.

XXVI.

Use of Museum and Library.

All Articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations, as the Council from time to time shall appoint.

XXVII.

Register Book.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND VICTORIA JUBILEE PRIZES.

The above Prizes will be awarded by the Council in the following manner :—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1889–90 for the "best communication on a scientific subject, communicated, in the first instance, to the Royal Society during the Sessions 1887–88 and 1888–89." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1890–91, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 1st June 1890.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented to the Society during the Sessions 1888–89, 1889–90, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate,

1. The NEILL PRIZE, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1889–90.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented to the Society during the three years preceding the 1st May 1889,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded triennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize is open to men of science resident in or connected with Scotland.

The first award shall be in the year 1887, and shall consist of a sum of money. In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND VICTORIA JUBILEE PRIZES, FROM 1827 TO 1888.

I. KEITH PRIZE.

1st Biennial Period, 1827–29.—Dr Brewster, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2nd Biennial Period, 1829–31.—Dr Brewster, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3rd Biennial Period, 1831–33.—Thomas Graham, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4th Biennial Period, 1833–35.—Professor J. D. Forbes, for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5th Biennial Period, 1835–37.—John Scott Russell, Esq., for his Researches "on Hydrodynamics," published in the Transactions of the Society.

6th Biennial Period, 1837–39.—Mr John Shaw, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7th Biennial Period, 1839–41.—Not awarded.

8th Biennial Period, 1841–43.—Professor James David Forbes, for his papers "on Glaciers," published in the Proceedings of the Society.

9th Biennial Period, 1843–45.—Not awarded.

10th Biennial Period, 1845–47.—General Sir Thomas Brisbane, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11th Biennial Period, 1847–49.—Not awarded.

12th Biennial Period, 1849–51.—Professor Kelland, for his papers "on General Differentiation, including his more recent communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13th Biennial Period, 1851–53.—W. J. Macquorn Rankine, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14th Biennial Period, 1853–55.—Dr Thomas Anderson, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15th Biennial Period, 1855–57.—Professor Boole, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16th Biennial Period, 1857–59.—Not awarded.

17th Biennial Period, 1859–61.—John Allan Broun, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18th Biennial Period, 1861–63.—Professor William Thomson, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19th Biennial Period, 1863–65.—Principal Forbes, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20th Biennial Period, 1865–67.—Professor C. Piazzi Smyth, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21st Biennial Period, 1867–69.—Professor P. G. Tait, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22nd Biennial Period, 1869–71.—Professor Clerk Maxwell, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23rd Biennial Period, 1871–73.—Professor P. G. Tait, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24th Biennial Period, 1873–75.—Professor Crum Brown, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25th Biennial Period, 1875–77.—Professor M. Forster Heddle, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26th Biennial Period, 1877–79.—Professor H. C. Fleeming Jenkin, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II. having appeared in the volume for 1877–78.

27th Biennial Period, 1879–81.—Professor George Chrystal, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28th Biennial Period, 1881–83.—Thomas Muir, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29th Biennial Period, 1883–85.—John Aitken, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30th Biennial Period, 1885–87.—John Young Buchanan, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, &c.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

II. MAKDOUGALL-BRISBANE PRIZE.

1st Biennial Period, 1859.—Sir Roderick Impey Murchison, on account of his Contributions to the Geology of Scotland.

2nd Biennial Period, 1860–62.—William Seller, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3RD BIENNIAL PERIOD, 1862–64.—JOHN DENIS MACDONALD, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4TH BIENNIAL PERIOD, 1864–66.—Not awarded.

5TH BIENNIAL PERIOD, 1866–68.—Dr ALEXANDER CRUM BROWN and Dr THOMAS RICHARD FRASER, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6TH BIENNIAL PERIOD, 1868–70.—Not awarded.

7TH BIENNIAL PERIOD, 1870–72.—GEORGE JAMES ALLMAN, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8TH BIENNIAL PERIOD, 1872–74.—Professor LISTER, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9TH BIENNIAL PERIOD, 1874–76.—ALEXANDER BUCHAN, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10TH BIENNIAL PERIOD, 1876–78.—Professor ARCHIBALD GEIKIE, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11TH BIENNIAL PERIOD, 1878–80.—Professor PIAZZI SMYTH, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1880–82.—Professor JAMES GEIKIE, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13TH BIENNIAL PERIOD, 1882–84.—EDWARD SANG, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculation of Logarithms both of Numbers and Trigonometrical Ratios, —the former communication being published in the Proceedings of the Society.

14TH BIENNIAL PERIOD, 1884–86.—JOHN MURRAY, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15TH BIENNIAL PERIOD, 1886–88.—ARCHIBALD GEIKIE, Esq., LL.D., for numerous communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

III. THE NEILL PRIZE.

1st Triennial Period, 1856–59.—Dr W. Lauder Lindsay, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2nd Triennial Period, 1859–62.—Robert Kaye Greville, LL.D., for his Contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3rd Triennial Period, 1862–65.—Andrew Crombie Ramsay, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and Memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geographical Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4th Triennial Period, 1865–68.—Dr William Carmichael M'Intosh, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5th Triennial Period, 1868–71.—Professor William Turner, for his papers "on the great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6th Triennial Period, 1871–74.—Charles William Peach, for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7th Triennial Period, 1874–77.—Dr Ramsay H. Traquair, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton), published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8th Triennial Period, 1877–80.—John Murray, for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9th Triennial Period, 1880–83.—Professor Herdman, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10th Triennial Period, 1883–86.—B. N. Peach, Esq., for his Contributions to the Geology and Palaeontology of Scotland, published in the Transactions of the Society.

IV. VICTORIA JUBILEE PRIZE.

1st Triennial Period, 1884–87.—Sir William Thomson, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

PROCEEDINGS

OF THE

STATUTORY GENERAL MEETINGS,

28TH NOVEMBER 1887

AND

26TH NOVEMBER 1888.

STATUTORY MEETING.

HUNDRED AND FIFTH SESSION.

Monday, 28th November 1887.

At a General Statutory Meeting,

Sir WILLIAM THOMSON in the Chair.

The Minutes of last General Statutory Meeting of 22nd November 1886 were read approved, and signed.

The Secretary read a letter of apology for absence from Professor M'INTOSH.

On the motion of Dr Buchan, the LORD PROVOST and Mr ANDREW YOUNG were named Scrutineers of the Balloting Lists. They reported that the following Council had been unanimously elected:—

Sir WILLIAM THOMSON, LL.D., F.R.S., President.
DAVID MILNE HOME, LL.D.,
JOHN MURRAY, Ph.D.,
Professor Sir DOUGLAS MACLAGAN,
The Hon. Lord MACLAREN,
The Rev. Professor FLINT, D.D.,
Professor CHRYSTAL,
} Vice-Presidents.
Professor TAIT, M.A., General Secretary.
Professor Sir WM. TURNER, F.R.S.,
Professor CRUM BROWN, F.R.S.,
} Secretaries to Ordinary Meetings.
ADAM GILLIES SMITH, Esq., C.A., Treasurer.
ALEXANDER BUCHAN, Esq., M.A., LL.D., Curator of Library and Museum.

COUNCILLORS.

Professor BUTCHER, M.A.
Professor M'KENDRICK, F.R.S.
THOMAS MUIR, Esq., M.A., LL.D.
Professor M'INTOSH, F.R.S.
Sir ARTHUR MITCHELL, C.B.
STAIR A. AGNEW, Esq., C.B., M.A.
ROBERT M. FERGUSON, Esq., Ph.D.
A. FORBES IRVINE, Esq. of Drum, LL.D.
Dr J. BATTY TUKE, F.R.C.P.E.
Professor BOWER, M.A., F.L.S.
Dr G. SIMS WOODHEAD.
ROBERT COX, Esq. of Gorgie, M.A.

The TREASURER'S Accounts for the past Session, with the Auditor's Report thereon, were read. On the motion of Sir DOUGLAS MACLAGAN, seconded by Sheriff FORBES IRVINE, these were unanimously approved.

Sheriff IRVINE, seconded by the LORD PROVOST, moved that the Auditor be reappointed. Agreed to.

On the motion of the GENERAL SECRETARY, a vote of thanks was passed to the Chairman for presiding.

DOUGLAS MACLAGAN, *V.-P.*

STATUTORY MEETING.

HUNDRED AND SIXTH SESSION.

Monday, 26th November 1888.

At a General Statutory Meeting,

Sir DOUGLAS MACLAGAN in the Chair.

The Minutes of last General Statutory Meeting of 28th November 1887 were read, approved, and signed.

On the motion of Dr Buchan, Messrs FORBES IRVINE and YOUNG were invited to act as Scrutineers.

A Ballot having been taken, the Scrutineers reported that the following new Council had been unanimously elected :—

Sir WILLIAM THOMSON, LL.D., F.R.S., President.
JOHN MURRAY, Esq., LL.D.,
Professor Sir DOUGLAS MACLAGAN,
Hon. Lord M'LAREN, LL.D.,
Rev. Professor FLINT, D.D.,
Professor CHRYSTAL, LL.D.,
THOMAS MUIR, Esq., LL.D., } Vice-Presidents.
Professor TAIT, M.A., General Secretary.
Professor Sir WM. TURNER, F.R.S.,
Professor CRUM BROWN, F.R.S., } Secretaries to Ordinary Meetings.
ADAM GILLIES SMITH, Esq., C.A., Treasurer.
ALEXANDER BUCHAN, Esq., M.A., LL.D., Curator of Library and Museum.

COUNCILLORS.

Sir Arthur Mitchell, K.C.B.
Stair Agnew, Esq., C.B.
Robert M. Ferguson, Esq., Ph.D.
A. Forbes Irvine, Esq. of Drum, LL.D.
Dr J. Batty Tuke, F.R.C.P.E.
Professor Bower, F.L.S.
Dr G. Sims Woodhead, F.R.C.P.E.
Robert Cox, Esq. of Gorgie, M.A.
Professor Isaac B. Balfour, F.R.S.
Professor Ewing, F.R.S.
Professor Jack, LL.D.
Professor James Geikie, LL.D., F.R.S.

The Treasurer's Accounts with the Auditor's Report were presented. On the motion of Mr Forbes Irvine, seconded by Dr Buchan, the Auditor was reappointed.

On the motion of the General Secretary, a vote of thanks was given to the Chairman.

John Murray, *V.-P.*

The following Public Institutions and Individuals are entitled to receive Copies of the Transactions and Proceedings of the Royal Society of Edinburgh:—

London, British Museum.
... Royal Society, Burlington House, London.
... Anthropological Institute of Great Britain and Ireland, 3 Hanover Square, London.
... British Association for the Advancement of Science, 22 Albemarle Street, London.
... Society of Antiquaries, Burlington House.
... Royal Astronomical Society, Burlington House.
... Royal Asiatic Society, 22 Albemarle Street.
... Society of Arts, John Street, Adelphi.
... Athenæum Club.
... Chemical Society, Burlington House.
... Institution of Civil Engineers, 25 Great George Street.
... Royal Geographical Society, Burlington Gardens.
... Geological Society, Burlington House.
... Royal Horticultural Society, South Kensington.
... Hydrographic Office, Admiralty.
... Royal Institution, Albemarle Street, W.
... Linnean Society, Burlington House.
... Royal Society of Literature, 4 St Martin's Place.
... Medical and Chirurgical Society, 53 Berners Street, Oxford Street.
... Royal Microscopical Society, King's College.
... Museum of Economic Geology, Jermyn Street.
... Royal Observatory, Greenwich.
... Pathological Society, 53 Berners Street.
... Statistical Society, 9 Adelphi Terrace, Strand, London.
... Royal College of Surgeons of England, 40 Lincoln's Inn Fields.
London, United Service Institution, Whitehall Yard.
... University College, Gower Street.
... Zoological Society, 11 Hanover Square.
... The Editor of *Nature*, 29 Bedford Street, Covent Garden.
... The Editor of the *Electrician*, 396 Strand.
Cambridge Philosophical Society.
... University Library.
Leeds Philosophical and Literary Society.
Manchester Literary and Philosophical Society.
Oxford, Bodleian Library.
Yorkshire Philosophical Society.

SCOTLAND.

Edinburgh, Advocates Library.
... University Library.
... College of Physicians.
... Highland and Agricultural Society, 3 George IV. Bridge.
... Royal Medical Society, 7 Melbourne Place, Edinburgh.
... Royal Observatory.
... Royal Physical Society, 40 Castle Street.
... Royal Scottish Society of Arts, 117 George Street.
... Royal Botanic Garden, Inverleith Row.
Aberdeen, University Library.
Dundee, University College Library.
Glasgow, University Library.
... Philosophical Society, 207 Bath Street.
St Andrews, University Library.

IRELAND.

Royal Dublin Society.
Royal Irish Academy, 19 Dawson Street, Dublin.
Library of Trinity College, Dublin.

COLONIES, DEPENDENCIES, &c.

Bombay, Royal Asiatic Society.
... Elphinstone College.
Calcutta, Asiatic Society of Bengal.
... Geological Survey of India.
Madras, Literary Society.
Canada, Library of Geological Survey.
... Queen's University, Kingston.
... Royal Society of Canada, Parliament Buildings, Ottawa, Canada.
... Quebec, Literary and Philosophical Society.
... Toronto, Literary and Historical Society.
... ... The Canadian Institute.
Cape of Good Hope, The Observatory.
Melbourne, University Library.
Sydney, University Library.
... Linnæan Society of New South Wales.
... Royal Society of New South Wales.
Wellington, New Zealand Institute.

CONTINENT OF EUROPE.

Amsterdam, Koninklijke Akademie van Wetenschappen.
... Koninklijk Zoologisch Genootschap.
Athens, University Library.
Basle, Die Schweizerische Naturforschende Gesellschaft.
Bergen, Museum.
Berlin, Königliche Akademie der Wissenschaften.
... Physicalische Gesellschaft.
Bern, Allgemeine Schweizerische Gesellschaft für die gesammten Naturwissenschaften.
Bologna, Accademia delle Scienze dell' Istituto.
Bordeaux, Société des Sciences Physiques et Naturelles.
Brussels, Académie Royale des Sciences, des Lettres et des Beaux-arts.
... Musée Royal d'Histoire Naturelle de Belgique.
.. L'Observatoire Royal.
... La Société Scientifique.
Bucharest, Academia Romana.
Buda-Pesth, Magyar Tudományos Akadémia—Die Ungarische Akademie der Wissenschaften.
... Königliche Ungarische Naturwissenschaftliche Gesellschaft.
Catania, Accademia Gioenia di Scienze Naturali.
Christiania, University Library.
Christiania, Meteorological Institute.
Coimbra, University Library.
Copenhagen, Royal Academy of Sciences.
Danzig, Naturforschende Gesellschaft.
Dorpat, University Library.
Ekatherinebourg, La Société Ouralienne d'Amateurs des Sciences Naturelles.
Erlangen, University Library.
Frankfurt-am-Main, Senckenbergische Naturforschende Gesellschaft.
Gand (Ghent), University Library.
Geneva, Société de Physique et d'Histoire Naturelle.
Genoa, Museo Civico di Storia Naturale.
Giessen, University Library.
Göttingen, Königliche Gesellschaft der Wissenschaften.
Graz, Naturwissenschaftlicher Verein für Steiermark.
Haarlem, Société Hollandaise des Sciences Exactes et Naturelles.
... Musée Teyler.
Halle, Kaiserliche Leopoldino - Carolinische Deutsche Akademie der Naturforscher.
... Naturforschende Gesellschaft.
Hamburg, Naturwissenschaftlicher Verein, 6 Domstrasse.
Helsingfors, Sällskapet pro Fauna et Flora Fennica.
... Societas Scientiarum Fennica (Société des Sciences de Finlande).
Jena, Medicinisch-Naturwissenschaftliche Gesellschaft.
Kasan, University Library.
Kiel, University Library.
... Ministerial-Kommission zur Untersuchung der Deutschen Meere.
Kiev, University of St Vladimir.
Königsberg, University Library.
Leyden, Neerlandsche Dierkundige Vereeniging.
... The University Library.
Leipzig, Prof. Wiedemann, Königliche Sächsische Akademie.
Lille, Société des Sciences.
Lisbon, Academia Real das Sciencias de Lisboa.
... Sociedade de Geographia, 5 Rua Capello.
Louvain, University Library.
Lucca, M. Michelotti.
Lund, University Library.

Lyons, Académie des Sciences, Belles Lettres et Arts.
... Société d'Agriculture.
Madrid, Real Academia de Ciencias.
... Comision del Mapa Geologico de España.
Milan, Reale Istituto Lombardo di Scienze, Lettere, ed Arti.
Modena, Regia Accademia di Scienze, Lettere, ed Arti.
Montpellier, Académie des Sciences et Lettres.
Moscow, Société Impériale des Naturalistes de Moscou.
... Société Impériale des Amis d'Histoire Naturelle, d'Anthropologie et d'Ethnographie.
... Musée Polytechnique.
... L'Observatoire Impérial.
Munich, Königlich-Bayerische Akademie der Wissenschaften (2 copies).
Naples, Zoological Station, Dr Anton Dohrn.
... Società Reale di Napoli—Accademia delle Scienze Fisiche e Matematiche.
Neufchatel, Société des Sciences Naturelles.
Nice, L'Observatoire.
Palermo, Signor Agostino Todaro, Giardino Botanico.
... Società di Scienze Naturali ed Economiche.
Paris, Académie des Sciences de l'Institut.
... Académie des Inscriptions et Belles Lettres de l'Institut.
... Association Scientifique de France.
... Société d'Agriculture, 18 Rue de Bellechasse.
... Société Nationale des Antiquaires de France.
... Société de Biologie.
... Société de Géographie, 184 Boulevard St Germain.
... Société Géologique de France, 7 Rue des Grands Augustins.
... Société d'Encouragement pour l'Industrie Nationale.
... Bureau des Longitudes.
... Dépôt de la Marine.
... Société Mathématique, 7 Rue des Grands Augustins.
... École des Mines.
... Ministère de l'Instruction Publique.
Paris, Musée Guimet, 30 Avenue du Trocadéro.
... Muséum d'Histoire Naturelle, Jardin des Plantes.
... L'Observatoire.
... École Normale Supérieure, Rue d'Ulm.
... Société Française de Physique, 44 Rue de Rennes.
... École Polytechnique.
... Société Zoologique de France, 7 Rue des Grands Augustins.
Prague, Königliche Sternwarte.
... Königlich-Böhmische Gessellschaft der Wissenschaften.
Rome, R. Accademia dei Lincei.
... Accademia Ponteficia dei Lincei.
... Società Italiana delle Scienze detta dei XL.
... Società degli Spettroscopisti Italiani.
... Comitato Geologico.
Rotterdam, Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.
St Petersburg, Académie Impériale des Sciences.
... Commission Impériale Archéologique.
... Comité Géologique.
... L'Observatoire Impérial de Pulkowa.
... Physikalisches Central-Observatorium.
... Physico-Chemical Society of the University of St Petersburg.
Stockholm, Kongliga Svenska Vetenskaps-Academien.
Strasbourg, University Library.
Stuttgart, Verein für Vaterländische Naturkunde zu Würtemberg.
Throndhjem, Videnskabernes Selskab.
Toulouse, Faculté des Sciences.
... L'Observatoire.
Tübingen, University Library.
Turin, Reale Accademia delle Scienze.
Upsala, Kongliga Vetenskaps-Societeten.
Venice, Reale Istituto Veneto di Scienze, Lettere ed Arti.
Vienna, Kaiserliche Akademie der Wissenschaften.
... Novara Commission.
... Oesterreichische Gesellschaft für Meteorologie, Hohe Warte, Wien.

Vienna, Geologische Reichsanstalt.
... Zoologisch-Botanische Gesellschaft.
Zurich, University Library.
... Commission Géologique Suisse.

ASIA.

Java, Bataviaasch Genootschap van Kunsten en Wetenschappen.
... The Observatory.
Japan, The Imperial University of Tokio (Teikoku-Daigaku).

UNITED STATES OF AMERICA.

Albany, New York State Library.
American Association for the Advancement of Science.
Baltimore, Johns Hopkins University.
Boston, The Bowditch Library.
... American Academy of Arts and Sciences, Beacon Street, Boston.
... Society of Natural History.
California, Academy of Sciences, San Francisco.
Cambridge, Mass., Harvard University.
... Harvard College Observatory.
Chicago Observatory.
Clinton, Litchfield Observatory, Hamilton College.
Concord, Editor of Journal of Speculative Philosophy.
Denison, University and Scientific Association.
Philadelphia, American Philosophical Society.
Philadelphia, Academy of Natural Sciences, Logan Square.
... Geological Survey of Pennsylvania.
Salem, The Peabody Academy of Science.
St Louis, Academy of Sciences.
Washington, United States National Academy of Sciences.
... Bureau of Ethnology.
... United States Coast Survey.
... United States Fishery Commission.
... United States Naval Observatory.
... United States Geological Survey of the Territories.
... United States Signal Office.
... The Smithsonian Institution.
... Surgeon-General's Office, United States Army.
Wisconsin, University (Washburn Observatory), Madison.
Yale College, Newhaven, Connecticut.

MEXICO.

Mexico, Observatorio Meteorologico Central.

SOUTH AMERICA.

Buenos Ayres, Public Museum, per Dr Burmeister.
Cordoba, Argentine Republic, Academia Nacional de Ciencias.
... The Observatory.
Rio de Janeiro, The Astronomical Observatory.

All the Honorary and Ordinary Fellows of the Society are entitled to the Transactions and Proceedings.
See Notice at foot of page 1150.

The following Institutions and Individuals receive the Proceedings only :—

SCOTLAND.

Edinburgh, Botanical Society, 5 St Andrew Sq.
... Geological Society, 5 St Andrew Sq.
... Scottish Fishery Board, 101 George Street.
... Scottish Geographical Society.
... Mathematical Society, 8 Queen Street.
... Scottish Meteorological Society, 122 George Street.
... Pharmaceutical Society, 36 York Pl.
Geological Society of Glasgow, 207 Bath Street.
The Glasgow University Observatory.
Berwickshire Naturalists' Club, Old Cambus, Cockburnspath.

ENGLAND.

London, Geologists' Association, University College.
... Mathematical Society, 22 Albemarle Street, London, W.
... Institution of Mechanical Engineers, 10 Victoria Chambers, Victoria Street, Westminster.

London, Meteorological Office, 116 Victoria Street.
... The Meteorological Society, 25 Great George Street, Westminster.
... *Nautical Almanac* Office, 3 Verulam Buildings, Gray's Inn.
... Pharmaceutical Society, 17 Bloomsbury Square, London.
Birmingham Philosophical Society, King Edward's Grammar School.
Cornwall, Geological Society.
... Royal Institution of Cornwall, Truro.
Epping Forest and County of Essex Naturalists' Field Club.
Halifax, Geological and Polytechnic Society of Yorkshire.
Liverpool, Literary and Philosophical Society.
... Biological Society, University College.
Manchester, Geological Society, 36 George Street.
... Microscopical Society, care of the Secretary, 131 Embden Street.
Newcastle, Philosophical Society.
... North of England Institute of Mining and Mechanical Engineers.
Norfolk and Norwich Naturalists' Society, The Museum, Norwich.
Oxford, Ashmolean Society.
... Radcliffe Observatory.
Scarborough, Philosophical Society.
Whitby, Philosophical Society.

IRELAND.

Dublin, Royal Geological Society.
... Dunsink Observatory.
Belfast, Natural History and Philosophical Society.

COLONIES, DEPENDENCIES, ETC.

Adelaide, South Australia, University Library.
... Royal Society.
Bombay, Natural History Society.
Brisbane, Royal Society of Queensland.
Canada, Natural History Society of Montreal.
... Canadian Society of Civil Engineers, Toronto.
Halifax, Nova Scotian Institute of Natural Science.
Melbourne, Royal Society of Victoria.
Sydney, The Australian Museum.
Hong Kong, China Branch of the Asiatic Society.
... The Observatory.
Madras, Superintendent of Government Farms of Madras Presidency.
Queensland, Royal Society.
... Queensland Branch of Geographical Society.
... Government Meteorological Office.
Tasmania, Royal Society.

CONTINENT OF EUROPE.

Amsterdam, Genootschap der Mathematische Wetenschappen.
Berlin, Deutsche Meteorologische Gesellschaft.
... K. Technische Hochschule.
Bonn, Naturhistorischer Verein der Preussischen Rheinlande und Westfalens.
Bern, Naturforschende Gesellschaft.
Bordeaux, Société de la Géographie Commerciale.
Brunswick, Verein für Naturwissenschaft.
Bucharest, Institut Météorologique de Roumanie.
Cassel, Verein für Naturkunde.
Chemnitz, Naturwissenschaftliche Gesellschaft.
Cherbourg, Société Nationale des Sciences Naturelles.
Copenhagen, Naturhistoriske Forening.
Cracow, Académie des Sciences.
Delft, École Polytechnique.
Dijon, Académie des Sciences.
Erlangen, Physico-Medical Society.
Gratz, Chemisches Institut der K. K. Universität.
Halle, Verein für Erdkunde.
... Naturwissenschaftlicher Verein für Sachsen und Thüringen.
Hamburg, Verein für Naturwissenschaftliche Unterhaltung, 29 Steindamm, St Georg.
Helsingfors, Société de Géographie Finlandaise.
Iceland, Islenzka Fornleifafelag, Reikjavik, Iceland.
Lausanne, Société Vaudoise des Sciences Naturelles.
Leipzig, Naturforschende Gesellschaft.
Lille, Société Géologique du Nord.
Luxembourg, Société des Sciences Naturelles.
Lyons, Société Botanique.
... Société Linnéenne, Place Sathonay.
Marseilles, Société Scientifique Industrielle, 61 Rue Paradis.
Milan, Società Crittogamologica Italiana.
Modena, Società dei Naturalisti.
Nijmegen, Nederlandsche Botanische Vereeniging.
Oberpfalz und Regensburg, Historischer Verein.

Odessa, New Natural History Society.
Offenbach, Verein für Naturkunde.
Paris, Société d'Anthropologie (4 Rue Antoine Dubois).
... Société Philomathique.
... École Libre des Sciences Politiques.
... Bureau des Ponts et Chaussées.
... Sociétés des Jeunes Naturalistes et d'Études Scientifiques, 35 Rue Pierre-Charron.
Pisa, Nuovo Cimento.
St Petersburg, Imperatorskoe Russkoe Geographicheskoe Obtshéstvo.
Stockholm, Svenska Sällskapet för Anthropologi och Geografi.
Tiflis, Physical Observatory.
Trieste, Società Adriatica di Scienze Naturali.
... Museo Civico di Storia Naturale.
Tromsö, The Museum.
Utrecht, Provinciaal Genootschap van Kunsten en Wetenschappen.
Vienna, K. K. Naturhistorisches Hofmuseum., I., Burgring, 7.
Zurich, Naturforschende Gesellschaft.

ASIA.

China, Shanghai, North China Branch of the Royal Asiatic Society.
Japan, Tokio, The Seismological Society.
... Yokohama, Deutsche Gesellschaft für Natur- und Völkerkunde Ostasiens.
Java, Koninklijke Natuurkundige Vereeniging, Batavia.

UNITED STATES.

Annapolis, Maryland, St John's College.
California, State Mining Bureau, Sacramento.
... The Lick Observatory, Mount Hamilton, *viâ* San José, San Francisco.
... University of California (Berkeley).
Chapel Hill, North Carolina, Elisha Mitchell Scientific Society.
Chicago Observatory.
Cincinnati, Observatory.
... Society of Natural History.
... Ohio Mechanics' Institute.
Colorado, Scientific Society.
Connecticut, Academy of Arts and Sciences.
Davenport, Academy of Natural Sciences.
Iowa, The State University of Iowa.
Minnesota, The Geological and Natural History Survey of Minnesota, Minneapolis, Minnesota.
Nebraska, The University of Nebraska, Lincoln.
New Orleans, Academy of Sciences.
New York, The American Museum of Natural History.
... The American Geographical and Statistical Society, No. 11 West 29th Street, New York.
Philadelphia, Wagner Free Institute of Science.
Salem, The Essex Institute.
... Peabody Academy of Science.
Trenton, Natural History Society.
Washington, Philosophical Society.
... American Museum of Natural History, Central Park.
... United States National Museum.
Wisconsin, Academy of Sciences, Arts, and Letters

SOUTH AMERICA.

Rio de Janeiro, Museo Nacional.
Santiago, Deutscher Wissenschaftlicher Verein.

MEXICO.

Mexico, Observatorio Metorologico-Magnetico Central.
Sociedad Cientifica, "Antonio Alzate."
Tacubaya, Observatorio Astronomico.

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INDEX TO VOL. XXXV.

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Part 4.	0 10 0	0 7 6	XXXIV.	.....	.....
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Part 2.	0 10 0	0 7 6	„ Part 2.	.....	.....
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