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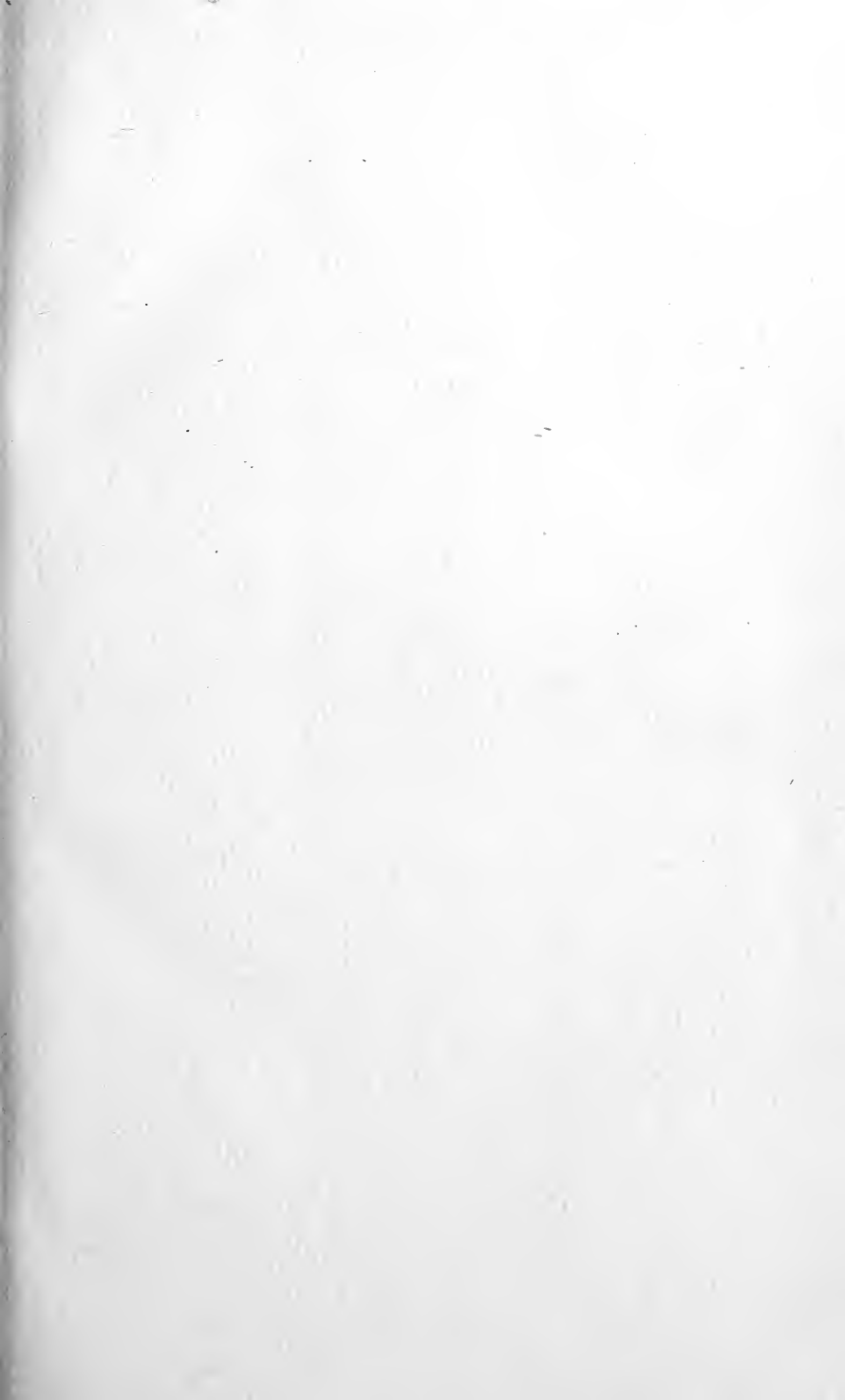


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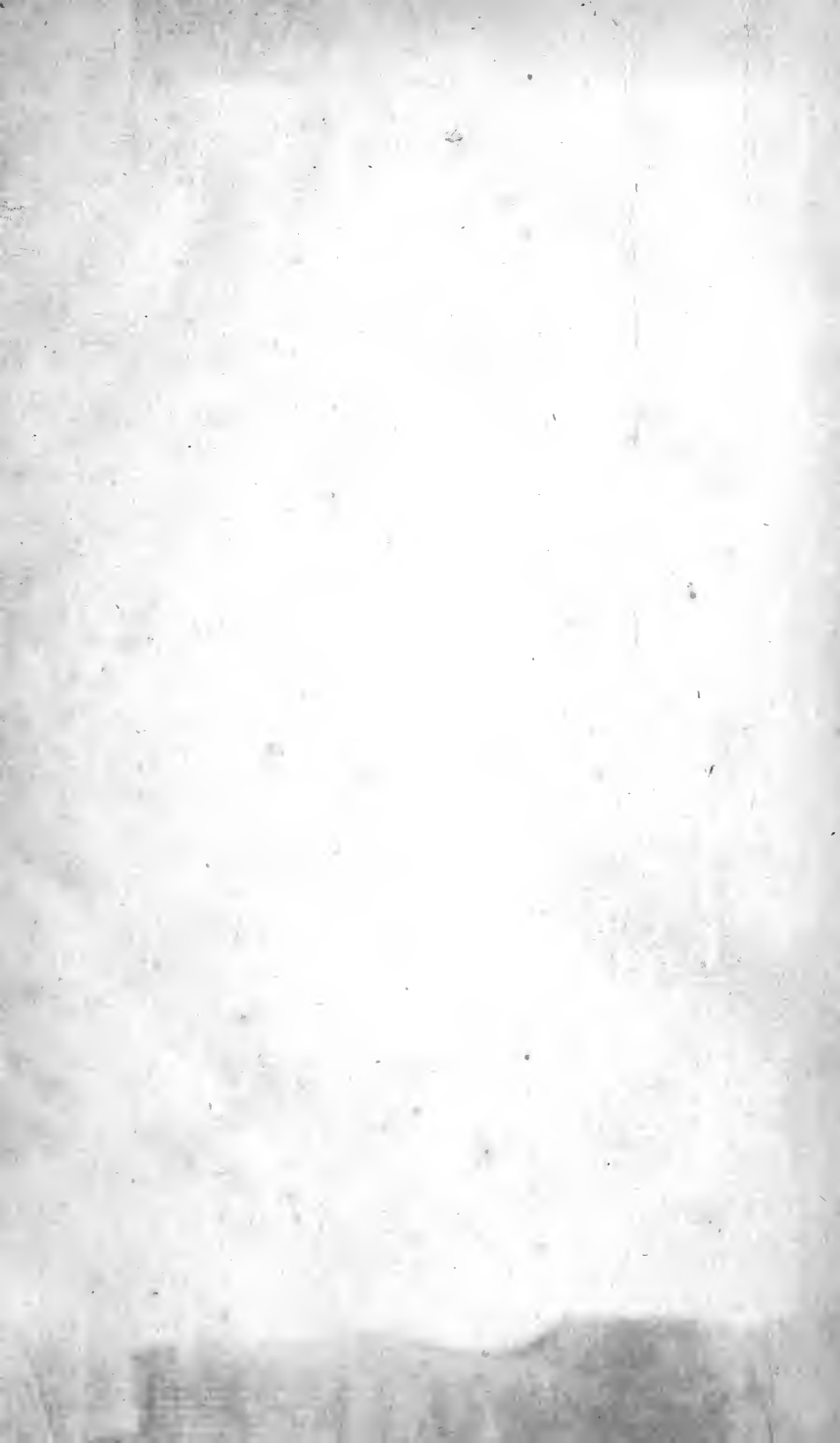
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A
TREATISE
OF
ALGEBRA,
IN
THREE PARTS.

CONTAINING

- I. The fundamental RULES and OPERATIONS.
- II. The COMPOSITION and RESOLUTION of EQUATIONS of all DEGREES; and the different AFFECTIONS of their ROOTS.
- III. The APPLICATION of ALGEBRA and GEOMETRY to each other.

To which is added an

APPENDIX,
Concerning the general PROPERTIES of
GEOMETRICAL LINES.

By COLIN MACLAURIN, M. A.

Late Professor of MATHEMATICKS in the University of
EDINBURGH, and FELLOW of the ROYAL SOCIETY.

The SECOND EDITION.

LONDON,

Printed for A. MILLAR and J. NOURSE, opposite
Katherine-Street in the Strand.

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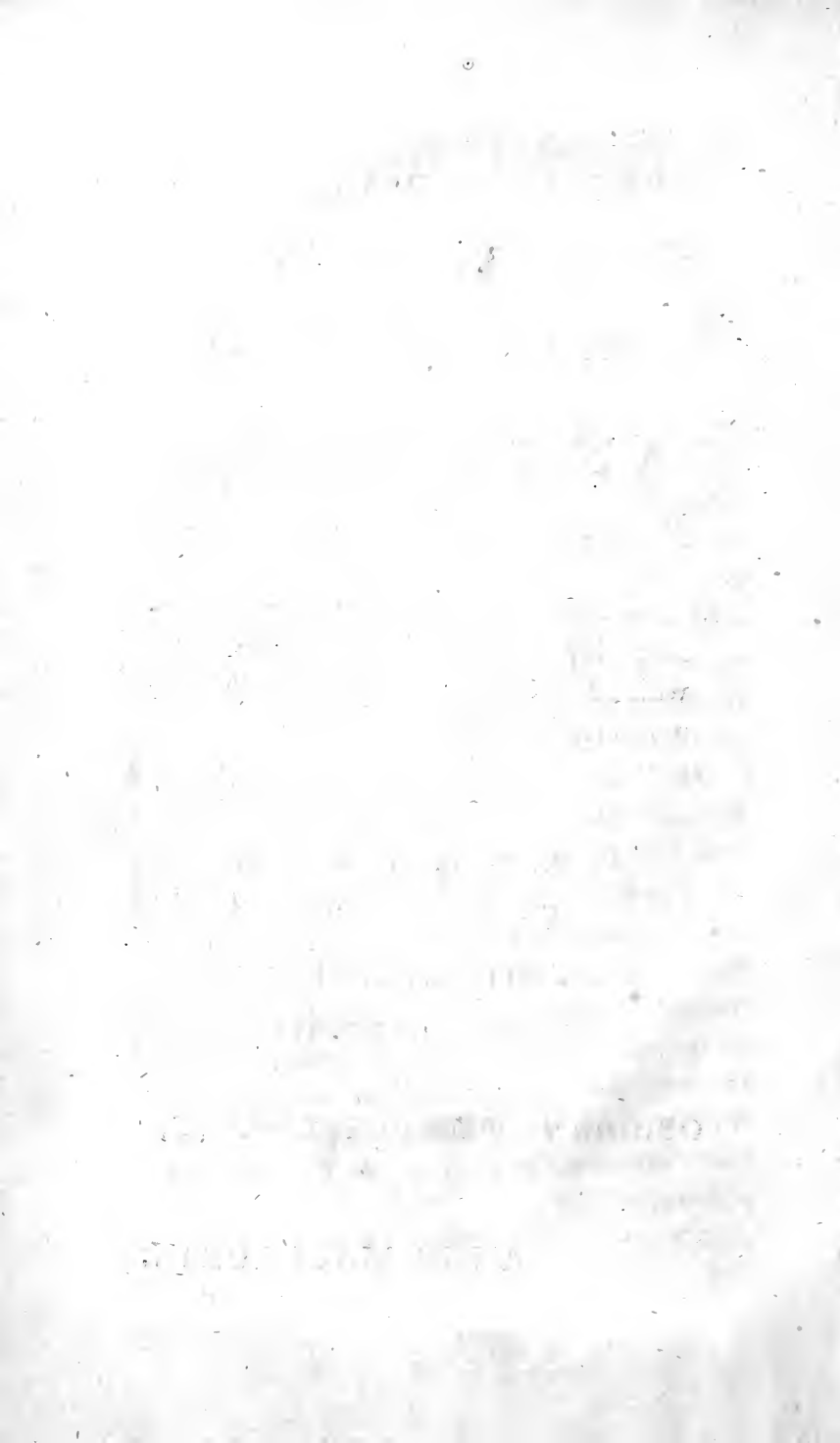
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TO
HIS GRACE
T H O M A S,
L O R D A R C H B I S H O P
O F
C A N T E R B U R Y,
Primate of all ENGLAND:

As a testimony of GRATITUDE for the
friendship and generous protection, with
which HIS GRACE was pleased to honour
my deceased husband,

T H I S
T R E A T I S E
IS INSCRIBED,
BY HIS GRACE'S
MOST OBLIGED
A N D
OBEDIENT HUMBLE SERVANT,

ANNE MACLAURIN;





TO THE
R E A D E R.

THAT Mr. MACLAURIN had, many years ago, intended to publish a Treatise on this subject, appears from a letter, dated April 19, 1729, to his honoured friend MARTIN FOLKES, Esq; now President of the Royal Society *. And we find, in one of his Manuscripts, the Plan of such a Work, agreeing, almost in every article, with the contents of this Volume.

Had the celebrated Author lived to publish his own Work, his name would, alone, have been sufficient to recommend it to the notice of the Publick: but that Task having, by his lamented premature death, devolved to the gentlemen whom he left entrusted with his Papers, the Reader may reasonably expect some account of the materials of which it consists, and of the care that has been taken in collecting and disposing them, so as best to answer the Author's intention, and fill up the Plan he had designed.

* Phil. Transf. N^o 408.

TO the READER.

He seems, in composing this Treatise, to have had these three Objects in view.

I. *To give the general Principles and Rules of the Science, in the shortest, and, at the same time, the most clear and comprehensive manner that was possible. Agreeable to this, though every Rule is properly exemplified, yet he does not launch out into what we may call, a Tautology of examples. He rejects some applications of Algebra, that are commonly to be met with in other writers; because the number of such applications is endless: and, however useful they may be in Practice, they cannot, by the rules of good method, have place in an elementary Treatise. He has likewise omitted the Algebraical solution of particular Geometrical problems, as requiring the knowledge of the Elements of Geometry; from which those of Algebra ought to be kept, as they really are, entirely distinct; reserving to himself to treat of the mutual relation of the two Sciences in his Third Part, and, more generally still, in the Appendix. He might think too, that such an application was the less necessary, that Sir ISAAC NEWTON's excellent Collection of Examples is in every body's hands, and that there are few Mathematical writers,*
who

TO the READER.

who do not furnish numbers of the same kind.

2. Sir ISAAC NEWTON's Rules, in his *Arithmetica Universalis*, concerning the Resolution of the higher equations, and the Affections of their roots, being, for the most part, delivered without any demonstration, Mr. MACLAURIN had designed, that his *Treatise* should serve as a Commentary on that Work. For we here find all those difficult passages in Sir ISAAC's Book, which have so long perplexed the Students of Algebra, clearly explained and demonstrated. How much such a Commentary was wanted, we may learn from the words of a late eminent Author*.

“ The ablest Mathematicians of the last age
“ (says he) did not disdain to write Notes on
“ the Geometry of *DES CARTES*; and surely
“ by Sir ISAAC NEWTON's Arithmetick no
“ less deserves that honour. To excite some
“ one of the many skilful Hands that our
“ times afford to undertake this Work, and
“ to shew the Necessity of it, I give this
“ Specimen, in an explication of two pas-

* s'Gravesande in Præfat. ad Specimen Comment. in Arith. Univerf.

TO THE READER.

“ pages * of the *Aritbmetica Universalis* ;
“ which, however, are not the most diffi-
“ cult in that Book.”

What this learned Professor so earnestly wished for, we at last see executed ; not separately, nor in the loose disagreeable form which such Commentaries generally take, but in a manner equally natural and convenient ; every Demonstration being aptly inserted into the Body of the Work, as a necessary and inseparable Member ; an Advantage which, with some others, obvious enough to an attentive Reader, will, it is hoped, distinguish this Performance from every other, of the kind, that has hitherto appeared.

3. *After having fully explained the Nature of Equations, and the Methods of finding their Roots, either in finite expressions, when it can be done, or in infinite converging series ; it remained only to consider the Relation of Equations involving two variable quantities, and of Geometrical Lines to each other ; the Doctrine of the Loci ; and the Construction of Equations. These make the Subject of the Third Part.*

* *Viz.* The finding of Divisors, and the evolution of Binomial Surds. See § 59---72. Part II. § 127. Part I.

Upon

TO the READER.

Upon this Plan Mr. MACLAURIN composed a System of Algebra, soon after his being chosen Professor of Mathematicks in the University of Edinburgh; which he, thenceforth, made use of in his ordinary Course of Lectures, and was occasionally improving to the Perfection he intended it should have, before he committed it to the Press. And the best Copies of his Manuscript having been transmitted to the Publisher, it was easy, by comparing them, to establish a correct and genuine Text. There were, besides, several detached Papers, some of which were quite finished, and wanted only to be inserted in their proper places. In a few others, the Demonstrations were so concisely expressed, and couched in Algebraical characters, that it was necessary to write them out at more length, to make them of a piece with the rest. And this is the only liberty the Publisher has allowed himself to take; excepting a few inconsiderable additions, that seemed necessary to render the Book more compleat within itself, and to save the trouble of consulting others who have written on the same Subject.

The Rules concerning the Impossible roots of Equations, our Author had very fully considered, as appears from his Manuscript papers: but as he had no where reduced any thing on
that

TO the READER.

that Subject to a better form, than what was long ago published in the Philosophical Transactions, N^o 394, and 408, we thought it best to take the substance of Chap. 11. Part II. from thence; especially as the latter of these Papers furnishes a demonstration of the original rule, which pre-supposes only what the Reader has been taught in a preceding Chapter.

The Paper that is subjoined, on the Sums of the Powers of the Roots of an Equation, is taken from a Letter of the Author (8 Jul. 1743) to the Right Honourable the Earl STANHOPE; communicated to the Publisher, with some things added by his Lordship, which were wanting to finish the Demonstration.

Of these Materials, carefully collected and put in order, the following Elementary Treatise is composed; which we have chosen rather to give in a Volume that is within the reach of every Student, than in one more pompous, which might be less generally useful. And we hope, from the pains it has cost us, its blemishes are not many, nor such as a candid intelligent Reader may not forgive.

The Latin Appendix is a proper Sequel, and a high Improvement, of what had been demonstrated in Part III. concerning the Relation of Curve lines and Equations; a Subject which

our

TO the READER.

our Author had been early and intimately acquainted with; witness his *Geometria Organica*, printed in 1719, when he was not full twenty-one years of age, and which, though so juvenile a work, gained him, at once, that distinguished Rank among Mathematicians, which he thenceforth held with great lustre. Yet he frankly owns, he was led to many of the Propositions in this Appendix, from a Theorem of Mr. COTES, communicated to him, without any demonstration, by the Reverend and Learned Dr. SMITH, Master of Trinity-College, Cambridge. How he has profited of that Hint, the Learned will judge: Thus much we can venture to say, that he himself set some value upon this Performance; having, we are told, employed some of the latest hours he could give to such Studies, in revising it for the Press; to bequeath it as his last Legacy to the Sciences and to the Publick.

The gentlemen to whom Mr. MACLAURIN left the care of his Papers, are MARTIN FOLKES, Esq; President of the Royal Society; ANDREW MITCHEL, Esq; and the Reverend Mr. HILL, Chaplain to his Grace the Archbishop of Canterbury; with whom he had lived in a most intimate friendship. And by their direction this Treatise is published.

C O N -



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APPENDIX.

De Linearum Geometricarum Proprietatibus generalibus.

A T R E A :



A
T R E A T I S E
O F
A L G E B R A.




P A R T I.



C H A P. I.

Definitions and Illustrations.

§ 1.  ALGEBRA is a general method of computation by certain signs and symbols which have been contrived for this purpose, and found convenient. It is called an UNIVERSAL ARITHMETICK, and proceeds by operations and rules similar to those in common arithmetick, founded upon the same principles. This, however, is no argument against its usefulness or evidence; since arithmetick is not to be the less valued

valued that it is common, and is allowed to be one of the most clear and evident of the sciences. But as a number of symbols are admitted into this science, being necessary for giving it that extent and generality which is its greatest excellence; the import of those symbols is to be clearly stated, that no obscurity or error may arise from the frequent use and complication of them.

§ 2. In GEOMETRY, lines are represented by a line, triangles by a triangle, and other figures by a figure of the same kind; but, in ALGEBRA, quantities are represented by the same letters of the alphabet; and various signs have been imagined for representing their affections, relations, and dependencies. In Geometry the representations are more natural, in Algebra more arbitrary: the former are like the first attempts towards the expression of objects, which was by drawing their resemblances; the latter correspond more to the present use of languages and writing. Thus the evidence of Geometry is sometimes more simple and obvious; but the use of Algebra more extensive, and often more ready: especially since the mathematical sciences have acquired so vast an extent, and have been applied to so many enquiries.

§ 3. In those sciences, it is not barely magnitude that is the object of contemplation: but
there

Fig. 1.

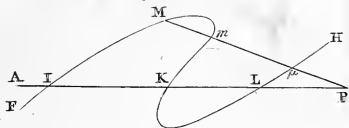


Fig. 2.

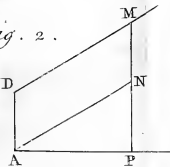


Fig. 3.

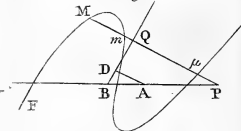


Fig. 4.

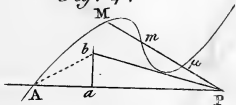


Fig. 5.

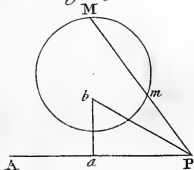


Fig. 6.

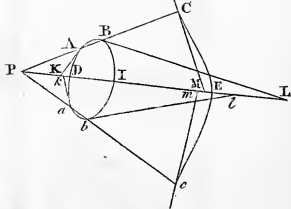


Fig. 7.

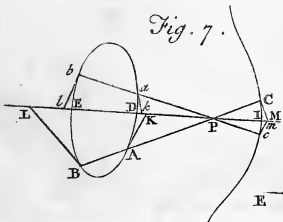


Fig. 8.

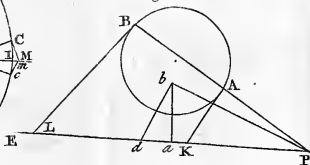
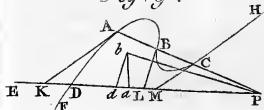
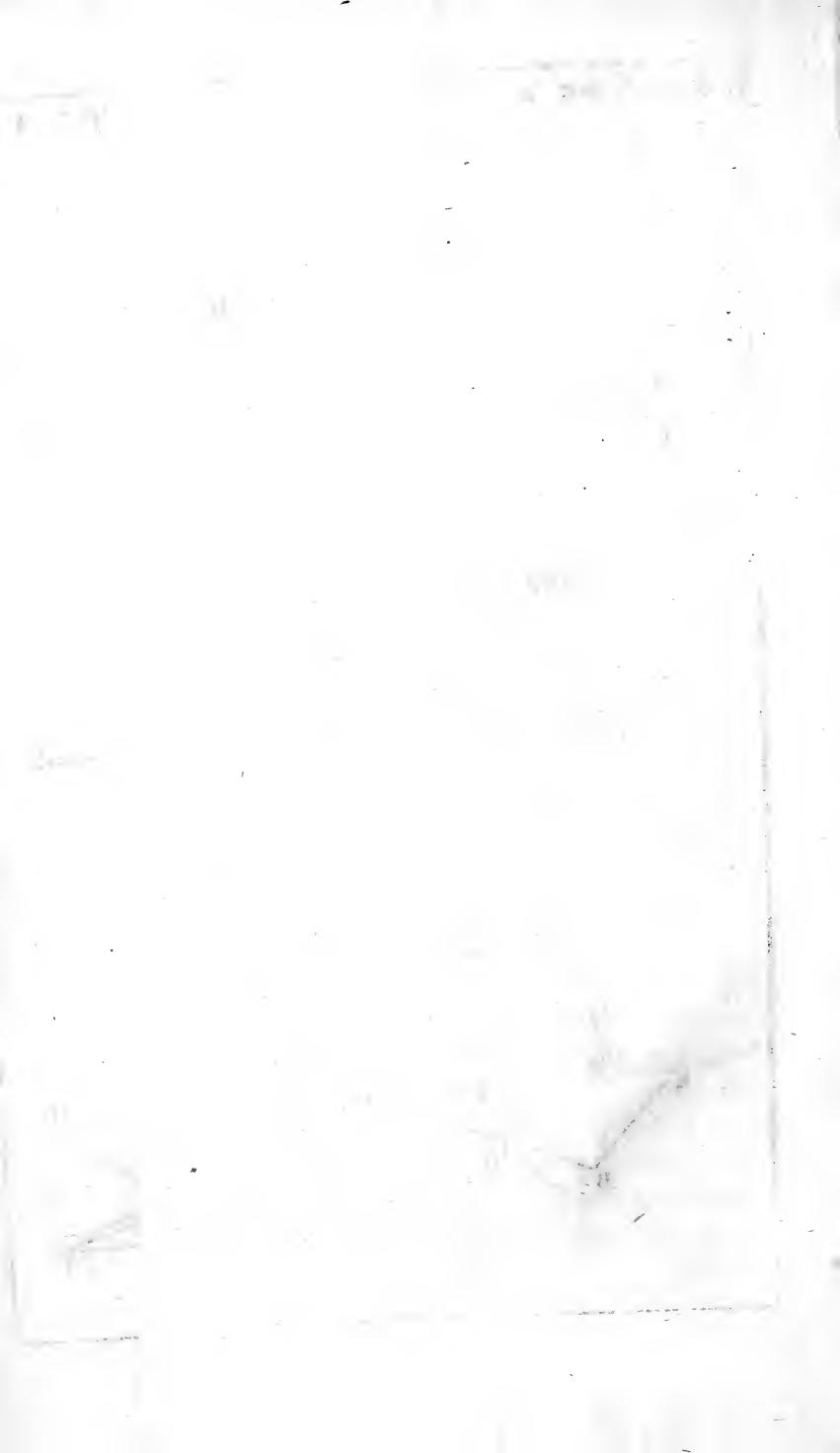


Fig. 9.





there are many affections and properties of quantities, and operations to be performed upon them, that are necessarily to be considered: In estimating the ratio or proportion of quantities, magnitude only is considered (*Elem. 5. Def. 3.*) But the nature and properties of figures depend on the position of the lines that bound them, as well as on their magnitude. In treating of motion, the direction of motion as well as its velocity; and the direction of powers that generate or destroy motion, as well as their forces, must be regarded. In OPTICS, the position, brightness, and distinctness of images, are of no less importance than their bigness; and the like is to be said of other sciences. It is necessary therefore that other symbols be admitted into Algebra beside the letters and numbers which represent the magnitude of quantities.

§ 4. The relation of equality is expressed by the sign $=$; thus to express that the quantity represented by a is equal to that which is represented by b , we write $a = b$. But if we would express that a is greater than b , we write $a > b$; and if we would express algebraically that a is less than b , we write $a < b$.

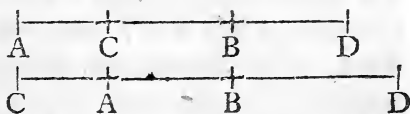
§ 5. QUANTITY is what is made up of parts, or is capable of being greater or less. It is increased by *Addition*, and diminished by *Subtraction*; which are therefore the two primary operations

rations that relate to quantity. Hence it is, that any quantity may be supposed to enter into algebraic computations two different ways which have contrary effects; either as an *increment* or as a *decrement*; that is, as a quantity to be added, or as a quantity to be subtracted. The sign $+$ (*plus*) is the mark of *Addition*, and the sign $-$ (*minus*) of *Subtraction*. Thus the quantity being represented by a , $+ a$ imports that a is to be added, or represents an increment; but $- a$ imports that a is to be subtracted, and represents a decrement. When several such quantities are joined, the signs serve to shew which are to be added, and which are to be subtracted. Thus $+ a + b$ denotes the quantity that arises when a and b are both considered as increments, and therefore expresses the sum of a and b . But $+ a - b$ denotes the quantity that arises when from the quantity a the quantity b is subtracted; and expresses the excess of a above b . When a is greater than b , then $a - b$ is itself an increment; when $a = b$, then $a - b = 0$; and when a is less than b , then $a - b$ is itself a decrement.

§ 6. As addition and subtraction are opposite, or an increment is opposite to a decrement, there is an analogous opposition between the affections of quantities that are considered in the mathematical sciences. As between excess and defect; between the value of effects

or money due to a man, and money due by him; a line drawn towards the right, and a line drawn to the left; gravity and levity; elevation above the horizon, and depression below it. When two quantities equal in respect of magnitude, but of those opposite kinds, are joined together, and conceived to take place in the same subject, they destroy each other's effect, and their amount is *nothing*. Thus 100 *l.* due to a man and 100 *l.* due by him balance each other, and in estimating his stock may be both neglected. Power is sustained by an equal power acting on the same body with a contrary direction, and neither have effect. When two unequal quantities of those opposite qualities are joined in the same subject, the greater prevails by their difference. And when a greater quantity is taken from a lesser of the same kind, the remainder becomes of the opposite kind. Thus if we add the lines AB and BD together, their

sum is AD;
but if we are
to subtract B
D from AB,



then $BC = BD$ is to be taken the contrary way towards A, and the remainder is AC; which, when BD, or BC exceeds AB, becomes a line on the other side of A. When two powers or forces are to be added together, their sum acts

upon the body: but when we are to subtract one of them from the other, we conceive that which is to be subtracted to be a power with an opposite direction; and if it be greater than the other, it will prevail by the difference. This change of quality however only takes place where the quantity is of such a nature as to admit of such a contrariety or opposition. We know nothing analogous to it in quantity abstractly considered; and cannot subtract a greater quantity of matter from a lesser, or a greater quantity of light from a lesser. And the application of this doctrine to any art or science is to be derived from the known principles of the science.

§ 7. A quantity that is to be added is likewise called a *positive* quantity; and a quantity to be subtracted is said to be *negative*: they are equally real, but opposite to each other, so as to take away each other's effect, in any operation, when they are equal as to quantity. Thus $3 - 3 = 0$, and $a - a = 0$. But though $+a$ and $-a$ are equal as to quantity, we do not suppose in Algebra that $+a = -a$; because to infer equality in this science, they must not only be equal as to quantity, but of the same quality, that in every operation the one may have the same effect as the other. A decrement may be equal to an increment, but it has in all operations a contrary effect; a motion downwards

wards may be equal to a motion upwards, and the depression of a star below the horizon may be equal to the elevation of a star above it: but those positions are opposite, and the distance of the stars is greater than if one of them was at the horizon so as to have no elevation above it, or depression below it. It is on account of this contrariety that a negative quantity is said to be less than nothing, because it is opposite to the positive, and diminishes it when joined to it, whereas the addition of 0 has no effect. But a negative is to be considered no less as a real quantity than the positive. Quantities that have no sign prefixed to them are understood to be positive.

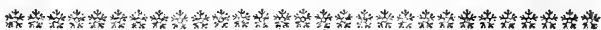
§ 8. The number prefixed to a letter is called the numeral *coefficient*, and shews how often the quantity represented by the letter is to be taken. Thus $2a$ imports that the quantity represented by a is to be taken twice; $3a$ that it is to be taken thrice; and so on. When no number is prefixed, *unit* is understood to be the coefficient. Thus 1 is the coefficient of a or of b .

Quantities are said to be *like* or *similar*, that are represented by the same letter or letters equally repeated. Thus $+3a$ and $-5a$ are like; but a and b , or a and aa are unlike.

A quantity is said to consist of as many *terms* as there are parts joined by the signs $+$

or $-$; thus $a + b$ consists of two terms, and is called a *binomial*; $a + b + c$ consists of three terms, and is called a *trinomial*. These are called *compound* quantities: a *simple* quantity consists of one term only, as $+a$, or $+ab$, or $+abc$.

The other symbols and definitions necessary in *Algebra* shall be explained in their proper places.



CHAP. II.

OF ADDITION.

§ 9. **C**ASE I. To add quantities that are like and have like signs.

Rule. *Add together the coefficients, to their sum prefix the common sign, and subjoin the common letter or letters.*

EXAMPLES.

To	$+5a$	to	$-6b$	to	$a + b$
Add	$+4a$	add	$-2b$	add	$3a + 5b$
Sum	$+9a$	Sum	$-8b$	Sum	$4a + 6b$

To

$$\begin{array}{r}
 \text{To} \quad 3a - 4x \\
 \text{Add} \quad 5a - 8x \\
 \hline
 \text{Sum} \quad 8a - 12x
 \end{array}$$

Case II. To add quantities that are like but have unlike signs.

Rule. Subtract the lesser coefficient from the greater, prefix the sign of the greater to the remainder, and subjoin the common letter or letters.

EXAMPLES.

$$\begin{array}{r|l}
 \text{To} \quad -4a & +5b - 6c \\
 \text{Add} \quad +7a & -3b + 8c \\
 \hline
 \text{Sum} \quad +3a & 2b + 2c
 \end{array}$$

$$\begin{array}{r|l}
 \text{To} \quad a + 6x - 5y + 8 & 2a - 2b \\
 \text{Add} \quad -5a - 4x + 4y - 3 & -2a + 2b \\
 \hline
 \text{Sum} \quad -4a + 2x - y + 5 & 0 \dots 0
 \end{array}$$

This rule is easily deduced from the nature of positive and negative quantities.

If there are more than two quantities to be added together, first add the positive together into one sum, and then the negative (by Case I.) Then add these two sums together (by Case II.)

EXAMPLE.

$$\begin{array}{r}
 + 8a \\
 - 7a \\
 + 10a \\
 - 12a \\
 \hline
 \text{Sum of the positive} \dots + 18a \\
 \text{Sum of the negative} \dots - 19a \\
 \hline
 \text{Sum of all} \dots - a
 \end{array}$$

Case III. To add quantities that are unlike.

Rule. *Set them all down one after another, with their signs and coefficients prefixed.*

EXAMPLES.

$$\begin{array}{r}
 \text{To} \quad + 2a \\
 \text{Add} \quad + 3b \\
 \hline
 \text{Sum} \quad 2a + 3b
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{r}
 + 3a \\
 - 4x \\
 \hline
 3a - 4x
 \end{array}$$

$$\begin{array}{r}
 \text{To} \quad 4a + 4b + 3c \\
 \text{Add} \quad - 4x - 4y + 3z \\
 \hline
 \text{Sum} \quad 4a + 4b + 3c - 4x - 4y + 3z
 \end{array}$$

CHAP. III.

OF SUBTRACTION.

§ 10. **G**eneral rule. *Change the signs of the quantity to be subtracted into their contrary signs, and then add it so changed to the quantity from which it was to be subtracted (by the rules of the last chapter :) the sum arising by this addition is the remainder. For, to subtract any quantity, either positive or negative, is the same as to add the opposite kind.*

EXAMPLES.

From	$+ 5a$	$8a - 7b$
Subtract	$+ 3a$	$3a + 4b$
	$5a - 3a$, or $2a$	$5a - 11b$

From	$2a - 3x + 5y - 6$
Subtract	$6a + 4x + 5y + 4$
	$-4a - 7x \quad 0 - 10$

It is evident, that to subtract or take away a decrement is the same as adding an equal increment. If we take away $-b$ from $a - b$, there remains a ; and if we add $+b$ to $a - b$, the sum is likewise a . In general, the subtraction of a negative quantity is equivalent to adding its positive value.

C H A P. IV.

Of MULTIPLICATION.

§ II. **I**N Multiplication the General rule for the signs is, That *when the signs of the factors are like* (i. e. both $+$, or both $-$,) *the sign of the product is $+$* ; but *when the signs of the factors are unlike, the sign of the product is $-$* .

Case I. When any positive quantity, $+a$, is multiplied by any positive number, $+n$, the meaning is, That $+a$ is to be taken as many times as there are units in n ; and the product is evidently na .

Case II. When $-a$ is multiplied by n , then $-a$ is to be taken as often as there are units in n , and the product must be $-na$.

Case III. Multiplication by a positive number implies a repeated addition: but multiplication by a negative implies a repeated subtraction. And when $+a$ is to be multiplied by $-n$, the meaning is, That $+a$ is to be subtracted as often as there are units in n : therefore

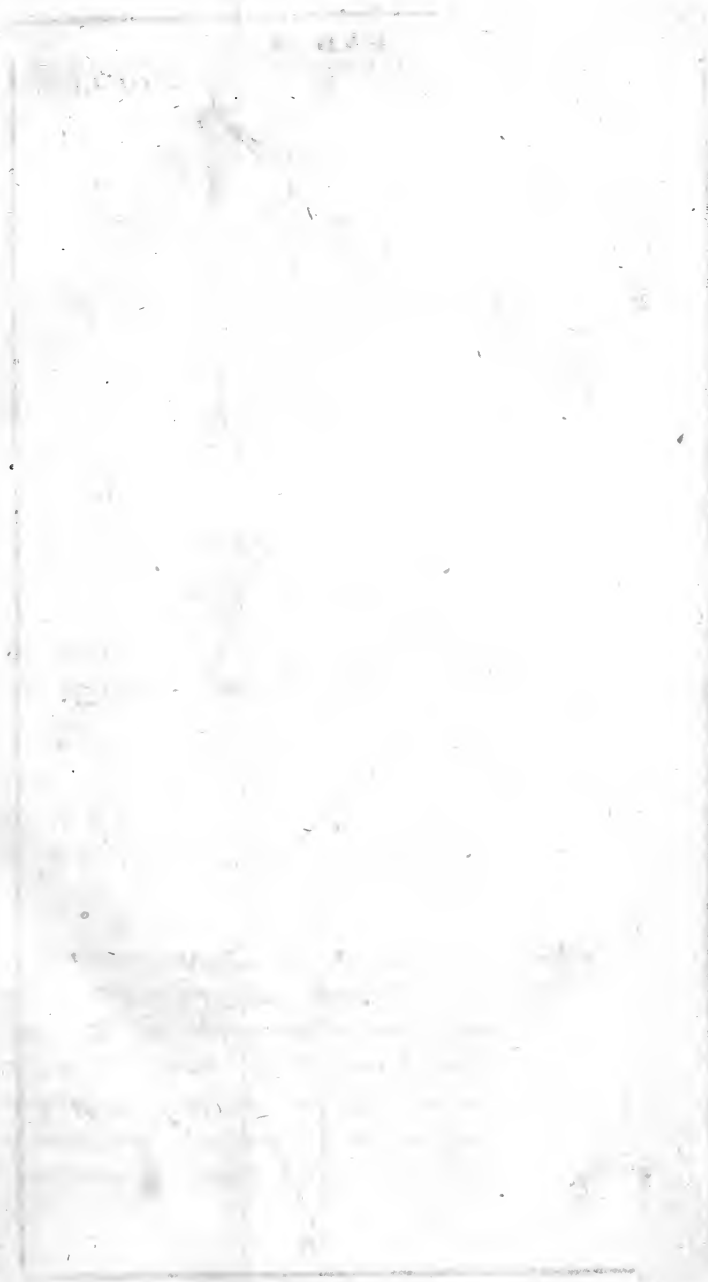


Fig. 10.

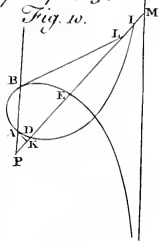


Fig. 11.

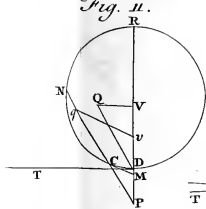


Fig. 12.

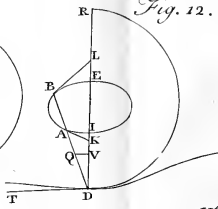


Fig. 13.

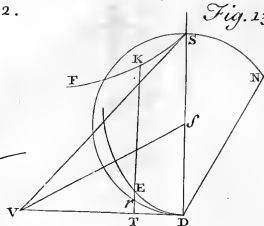


Fig. 14.

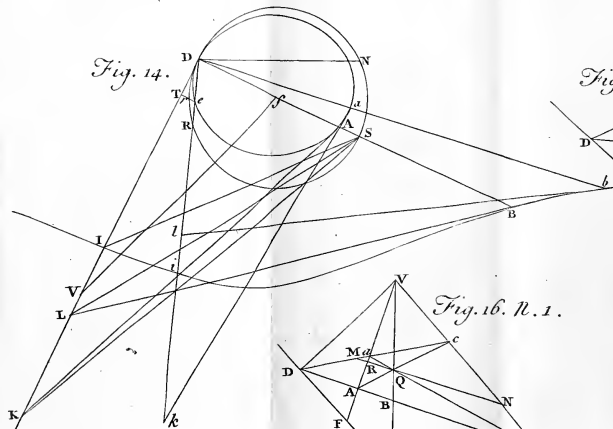


Fig. 15.

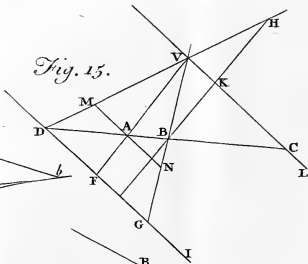


Fig. 16. N. 1.

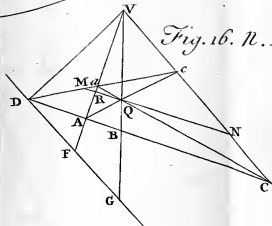
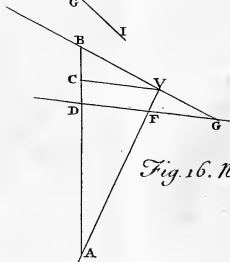


Fig. 16. N. 2.



fore the product is negative, being
 $-na$.

Case IV. When $-a$ is to be multiplied by $-n$, then $-a$ is to be subtracted as often as there are units in n ; but (by § 10.) to subtract $-a$ is equivalent to adding $+a$, consequently the product is $+na$.

The III and IVth cases may be illustrated in the following manner.

By the definitions, $+a - a = 0$; therefore, if we multiply $+a - a$ by n , the product must vanish or be 0, because the factor $a - a$ is 0. The first term of the product is $+na$ (by Case I.) Therefore the second term of the product must be $-na$, which destroys $+na$; so that the whole product must be $+na - na = 0$. Therefore $-a$ multiplied by $+n$ gives $-na$.

In like manner, if we multiply $+a - a$ by $-n$, the first term of the product being $-na$, the latter term of the product must be $+na$, because the two together must destroy each other, or their amount be 0, since one of the factors (*viz.* $a - a$) is 0. Therefore $-a$ multiplied by $-n$ must give $+na$.

In this general doctrine the multiplicator is always considered as a number. A quantity of any kind may be multiplied by a number: but
 a pound

a pound is not to be multiplied by a pound, or a debt by a debt, or a line by a line. We shall afterwards consider the analogy there is betwixt rectangles in Geometry and a product of two factors.

§ 12. If the quantities to be multiplied are *simple* quantities, find the sign of the product by the last rule; after it place the product of the coefficients, and then set down all the letters after one another as in one word.

EXAMPLES.

$$\begin{array}{r|l|l}
 \text{Mult.} & + a & - 2a \\
 \text{By} & + b & + 4b \\
 \hline
 \text{Prod.} & + ab & - 8ab \\
 \end{array}
 \qquad
 \begin{array}{r|l|l}
 & & 6x \\
 & & - 5a \\
 \hline
 & & - 30ax
 \end{array}$$

$$\begin{array}{r|l|l}
 \text{Mult.} & - 8x & + 3ab \\
 \text{By} & - 4a & - 5ac \\
 \hline
 \text{Prod.} & + 32ax & - 15abc
 \end{array}$$

§ 13. To multiply *compound* quantities, you must multiply every part of the multiplicand by all the parts of the multiplier taken one after another, and then collect all the products into one sum: that sum shall be the product required.

EXAMPLES.

$$\begin{array}{r|l}
 \text{Mult.} & a + b \\
 \text{By} & a + b \\
 \hline
 \text{Prod.} & \left\{ \begin{array}{l} aa + ab \\ + ab + bb \end{array} \right. \\
 \hline
 \text{Sum} & aa + 2ab + bb
 \end{array}
 \quad
 \begin{array}{r|l}
 & 2a - 3b \\
 & 4a + 5b \\
 \hline
 & \left\{ \begin{array}{l} 8aa - 12ab \\ + 10ab - 15bb \end{array} \right. \\
 \hline
 & 8aa - 2ab - 15bb
 \end{array}$$

$$\begin{array}{r|l}
 \text{Mult.} & 2a - 4b \\
 \text{By} & 2a + 4b \\
 \hline
 \text{Prod.} & \left\{ \begin{array}{l} 4ac - 8ab \\ + 8ab - 16bb \end{array} \right. \\
 \hline
 \text{Sum} & 4aa \dots 0 - 16bb
 \end{array}
 \quad
 \begin{array}{r|l}
 & xx - ax \\
 & x + a \\
 \hline
 & \left\{ \begin{array}{l} xxx - axx \\ + axx - aax \end{array} \right. \\
 \hline
 & xxx \dots 0 - aax
 \end{array}$$

$$\begin{array}{r|l}
 \text{Mult.} & aa + ab + bb \\
 \text{By} & a - b \\
 \hline
 \text{Prod.} & \left\{ \begin{array}{l} aaa + aab + abb \\ - aab - abb - bbb \end{array} \right. \\
 \hline
 \text{Sum} & aaa \dots 0 \dots 0 - bbb
 \end{array}$$

§ 14. Products that arise from the multiplication of two, three, or more quantities, as abc , are said to be of two, three, or more *dimensions*; and those quantities are called *factors* or *roots*.

If

If all the factors are equal, then these products are called *powers*; as aa or aaa are powers of a . Powers are expressed sometimes by placing above the root to the right-hand a figure expressing the number of factors that produce them. Thus,

$$\begin{array}{l}
 a \\
 aa \\
 aaa \\
 aaaa \\
 aaaaa
 \end{array}
 \left. \vphantom{\begin{array}{l} a \\ aa \\ aaa \\ aaaa \\ aaaaa \end{array}} \right\} \text{is called the}
 \left\{ \begin{array}{l}
 \text{1st} \\
 \text{2d} \\
 \text{3d} \\
 \text{4th} \\
 \text{5th}
 \end{array} \right\}
 \begin{array}{l}
 \text{Power of the} \\
 \text{root } a, \text{ and} \\
 \text{is shortly} \\
 \text{expressed} \\
 \text{thus,}
 \end{array}
 \left\{ \begin{array}{l}
 a \\
 a^2 \\
 a^3 \\
 a^4 \\
 a^5
 \end{array} \right.$$

§ 15. These figures which express the number of factors that produce powers are called their *indices* or *exponents*; thus 2 is the index of a^2 . And powers of the same root are multiplied by adding their exponents. Thus $a^2 \times a^3 = a^5$, $a^4 \times a^3 = a^7$, $a^3 \times a = a^4$.

§ 16. Sometimes it is useful not actually to multiply compound quantities, but to set them down with the sign of multiplication (\times) between them, drawing a line over each of the compound factors. Thus $\overline{a+b} \times \overline{a-b}$ expresses the product of $a + b$ multiplied by $a - b$.

CHAP. V.

Of DIVISION.

§ 17. **T**HE same rule for the signs is to be observed in Division as in Multiplication; that is, *If the signs of the dividend and divisor are like, the sign of the quotient must be +; if they are unlike, the sign of the quotient must be -.* This will be easily deduced from the rule in Multiplication, if you consider that the quotient must be such a quantity as multiplied by the divisor shall give the dividend.

§ 18. The General rule in Division is, *to place the dividend above a small line, and the divisor under it, expunging any letters that may be found in all the quantities of the dividend and divisor, and dividing the coefficients of all the terms by any common measure.* Thus when you divide $10ab + 15ac$ by $2oad$, expunging a out of all the terms, and dividing all the coefficients by 5, the quotient is $\frac{2b + 3c}{4d}$; and

$$2b)ab + bb\left(\frac{a + b}{2}\right).$$

$$12ab)30ax - 54ay\left(\frac{5x - 9y}{2b}\right).$$

4aa)

$$4aa) 8ab + 6ac \left(\frac{4b + 3c}{2a} \right).$$

$$\text{And } 2bc) 5abc \left(\frac{5a}{2} \right).$$

§ 19. Powers of the same root are divided by subtracting their exponents as they are multiplied by adding them. Thus if you divide a^5 by a^2 , the quotient is a^{5-2} or a^3 . And b^6 divided by b^4 gives b^{6-4} or b^2 ; and a^7b^5 divided by a^2b^3 gives a^5b^2 for the quotient.

§ 20. If the quantity to be divided is compound, then you must range its parts according to the dimensions of some one of its letters, as in the following example. In the dividend $a^2 + 2ab + b^2$, they are ranged according to the dimensions of a , the quantity a^2 where a is of two dimensions being placed first, $2ab$ where it is of one dimension next, and b^2 , where a is not at all, being placed last. The divisor must be ranged according to the dimensions of the same letters; then you are to divide the first term of the dividend by the first term of the divisor, and to set down the quotient, which, in this example, is a ; then multiply this quotient by the whole divisor, and subtract the product from the dividend, and the remainder shall give a new dividend, which in this example is $ab + b^2$.

$$a + b)$$

$$\begin{array}{r}
 a + b) a^2 + 2ab + b^2 \quad (a + b \\
 \underline{a^2 + ab} \\
 ab + b^2 \\
 \underline{ab + b^2} \\
 0 \quad 0
 \end{array}$$

Divide the first term of this new dividend by the first term of the divisor and set down the quotient, (which in this example is b) with its proper sign. Then multiply the whole divisor by this part of the quotient, and subtract the product from the new dividend; and if there is no remainder, the division is finished: If there is a remainder, you are to proceed after the same manner till no remainder is left; or till it appear that there will be always some remainder.

Some Examples will illustrate this operation.

EXAMPLE I.

$$\begin{array}{r}
 a + b) a^2 - b^2 \quad (a - b \\
 \underline{a^2 + ab} \\
 - ab - b^2 \\
 \underline{- ab - b^2} \\
 0 \quad 0
 \end{array}$$

C

EX-

E X A M P L E IV.

$$\begin{array}{r}
 3a - 6) 6aaaa - 96(2aaa + 4aa + 8a + 16 \\
 \underline{6aaaa - 12aaa} \\
 12aaa - 96 \\
 \underline{12aaa - 24aa} \\
 24aa - 96 \\
 \underline{24aa - 48a} \\
 48a - 96 \\
 \underline{48a - 96} \\
 0 \quad 0
 \end{array}$$

§ 21. It often happens that the operation may be continued without end, and then you have an *infinite Series* for the quotient; and by comparing the first three or four terms you may find what law the terms observe: by which means, without any more division, you may continue the quotient as far as you please. Thus, in dividing 1 by 1 - a; you find the quotient to be 1 + a + aa + aaa + aaaa + &c. which Series can be continued as far as you please by adding the powers of a.

The Operation is thus:

$$1 - a) 1 (1 + a + aa + aaa, \text{ \&c.}$$

$$\begin{array}{r}
 1 - a \\
 \hline
 + a \\
 + a - aa \\
 \hline
 + aa \\
 + aa - aaa \\
 \hline
 + aaa \\
 + aaa - aaaa \\
 \hline
 + aaaa, \text{ \&c.}
 \end{array}$$

Another Example.

$$a + x) aa + xx (a - x + \frac{2xx}{a} - \frac{2x^3}{a^2} + \frac{2x^4}{a^3} - \text{ \&c.}$$

$$\begin{array}{r}
 aa + ax \\
 \hline
 - ax + xx \\
 - ax - xx \\
 \hline
 + 2xx \\
 + 2xx - \frac{2x^3}{a} \\
 \hline
 - \frac{2x^3}{a} \\
 \frac{2x^3}{a} - \frac{2x^4}{a^2} \\
 \hline
 + \frac{2x^4}{a^2} \\
 \text{ \&c.}
 \end{array}$$

In

Fig. 16. N. 3.

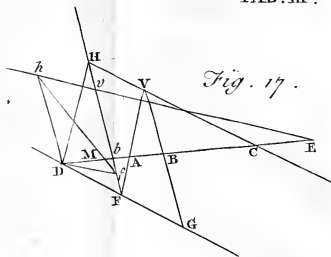
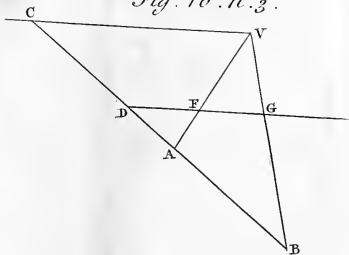


Fig. 17.

Fig. 18.

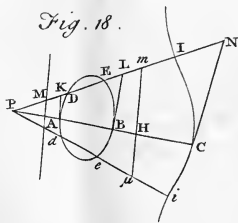


Fig. 19.

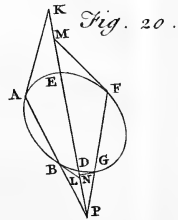
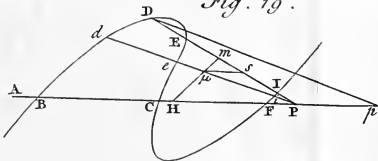


Fig. 20.

Fig. 21.

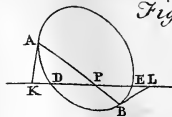


Fig. 22.

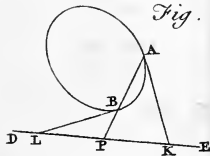
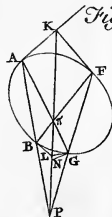
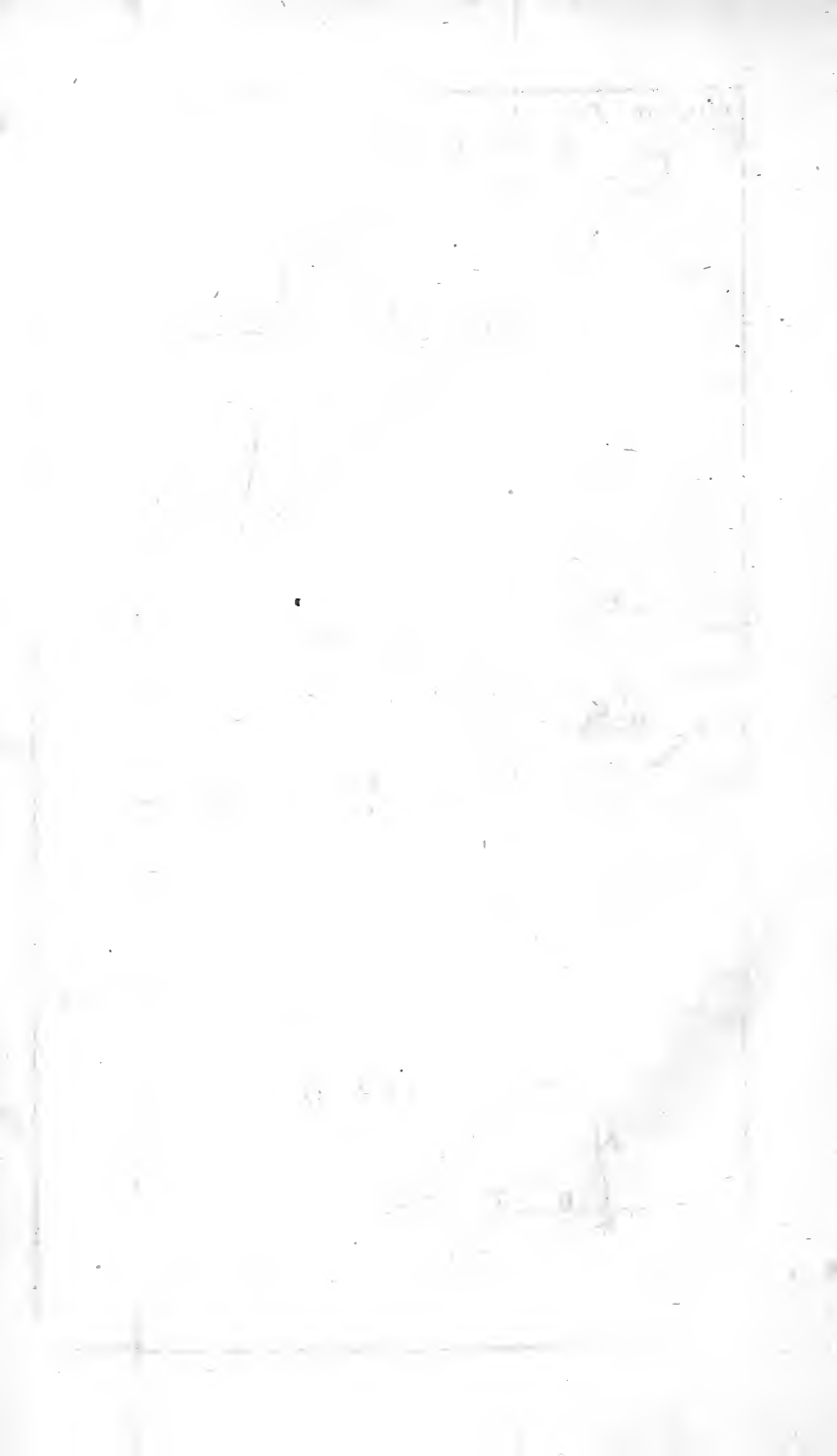


Fig. 23.





In this last example the signs are alternately + and —, the coefficient is constantly 2, after the first two terms, and the letters are the powers of x and a ; so that the quotient may be continued as far as you please without any more division.

But in Division, after you come to a remainder of one term, as $2xx$ in the last example, it is commonly set down with the divisor under it, after the other terms, and these together give the quotient. Thus, the quotient in the

last example is found to be $a - x + \frac{2x^2}{a+x}$. And

$bb + ab$ divided by $b - a$ gives for the quotient

$$b + \frac{2ab}{b-a}.$$

Note, The sign \div placed between any two quantities, expresses the quotient of the former divided by the latter. Thus $\overline{a+b} \div \overline{a-x}$ is the quotient of $a + b$ divided by $a - x$.

C H A P. VI.

O F F R A C T I O N S.

§ 22. **I**N the last Chapter it was said that the quotient of any quantity a divided by b is expressed by placing a above a small line and b under it, thus, $\frac{a}{b}$. These quotients are also called *Fractions*; and the dividend or quantity placed above the line is called the *Numerator* of the fraction, and the divisor or quantity placed under the line is called the *Denominator*. Thus $\frac{2}{3}$ expresses the quotient of 2 divided by 3; and 2 is the numerator and 3 the denominator of the fraction.

§ 23. *If the numerator of a fraction is equal to the denominator, then the fraction is equal to unity. Thus $\frac{a}{a}$ and $\frac{b}{b}$ are equal to unit. If the numerator is greater than the denominator, then the fraction is greater than unit. In both these cases, the fraction is called improper. But if the numerator is less than the denominator, then the fraction is less than unit, and is called proper. Thus $\frac{5}{3}$ is an improper fraction; but $\frac{3}{4}$ and $\frac{2}{3}$ are proper fractions. A mixt quantity is that*
 whereof

whereof one part is an *integer* and the other a *fraction*. As $3\frac{4}{5}$ and $5\frac{2}{3}$, and $a + \frac{a^2}{b}$.

PROBLEM I.

§ 24. To reduce a MIXT quantity to an IMPROPER FRACTION.

Rule. Multiply the part that is an integer by the denominator of the fractional part; and to the product add the numerator; under their sum place the former denominator.

Thus $2\frac{3}{5}$ reduced to an improper fraction, gives $\frac{13}{5}$; $a + \frac{a^2}{b} = \frac{ab + a^2}{b}$; and $a - x + \frac{a^2 - ax}{x} = \frac{a^2 - x^2}{x}$.

PROBLEM II.

§ 25. To reduce an IMPROPER fraction to a MIXT QUANTITY.

Rule. Divide the numerator of the fraction by the denominator, and the quotient shall give the integral part; the remainder set over the denominator shall be the fractional part.

Thus $\frac{12}{5} = 2\frac{2}{5}$; $\frac{ab + a^2}{b} = a + \frac{a^2}{b}$; $\frac{ax + 2xx}{a + x} = x + \frac{x^2}{a + x}$; $\frac{aa + xx}{a - x} = a + x + \frac{2xx}{a - x}$.

PROBLEM III.

§ 26. *To reduce fractions of different denominations to fractions of equal value that shall have the same denominator.*

Rule. *Multiply each numerator, separately taken, into all the denominators but its own, and the products shall give the new numerators. Then multiply all the denominators into one another, and the product shall give the common denominator.*

Thus the fractions $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, are respectively equal to these fractions $\frac{acd}{bcd}$, $\frac{bbd}{bcd}$, $\frac{ccb}{bcd}$, which have the same denominator bcd . And the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, are respectively equal to these $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

PROBLEM IV.

§ 27. *To ADD and SUBTRACT Fractions.*

Rule. *Reduce them to a common denominator, and add or subtract the numerators, the sum or difference set over the common denominator, is the sum or remainder required.*

Thus

$$\begin{aligned} \text{Thus } \frac{a}{b} + \frac{c}{d} + \frac{d}{e} &= \frac{ade + bce + d^2b}{bde}; \\ \frac{a}{b} - \frac{c}{d} &= \frac{ad - bc}{bd}; \quad \frac{2}{3} + \frac{3}{4} = \frac{8 + 9}{12} = \frac{17}{12} \\ &= 1\frac{5}{12}; \quad \frac{3}{4} - \frac{2}{3} = \frac{9 - 8}{12} = \frac{1}{12}; \\ \frac{4}{5} - \frac{3}{4} &= \frac{16 - 15}{20} = \frac{1}{20}; \\ \frac{x}{2} - \frac{x}{3} &= \frac{3x - 2x}{6} = \frac{x}{6}. \end{aligned}$$

PROBLEM V.

§ 28. To MULTIPLY Fractions.

Rule. Multiply their numerators one into another to obtain the numerator of the product; and their denominators multiplied into one another shall give the denominator of the product.

$$\begin{aligned} \text{Thus } \frac{a}{b} \times \frac{c}{d} &= \frac{ac}{bd}; \quad \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}; \quad \text{and} \\ \frac{a+b}{c} \times \frac{a-b}{d} &= \frac{a^2 - b^2}{cd}. \end{aligned}$$

If a mixt quantity is to be multiplied, first reduce it to the form of a fraction (by *Prob. 1.*) And if an integer is to be multiplied by a fraction, you may reduce it to the form of a fraction by placing unit under it.

EXAMPLES.

$$5\frac{2}{3} \times \frac{3}{4} = \frac{17}{3} \times \frac{3}{4} = \frac{51}{12}; \quad 9 \times \frac{2}{3} = \frac{9}{1} \times \frac{2}{3} =$$

$$= \frac{18}{3} = 6 : b + \frac{bx}{a} \times \frac{a}{x} = \frac{ba + bx}{a} \times \frac{a}{x} =$$

$$\frac{a^2b + abx}{ax} = \frac{ab + bx}{x}.$$

PROBLEM VI.

§ 29. To DIVIDE Fractions.

Rule. Multiply the numerator of the dividend by the denominator of the divisor, their product shall give the numerator of the quotient. Then multiply the denominator of the dividend by the numerator of the divisor, and their product shall give the denominator.

Thus $\frac{4}{5} \div \frac{2}{3} \left(\frac{10}{12} ; \frac{3}{7} \right) \frac{5}{8} \left(\frac{35}{24} ; \frac{c}{d} \right) \frac{a}{b} \left(\frac{ad}{cb} ; \right.$

$$\left. \frac{a+b}{a-b} \right) \frac{a-b}{a} \left(\frac{a^2 - 2ab + b^2}{a^2 + ab} \right).$$

§ 30. These last four Rules are easily demonstrated from the definition of a fraction.

1. It is obvious that the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, are respectively equal to $\frac{adf}{bdf}$, $\frac{cbf}{dbf}$, $\frac{ebd}{fbd}$, since if you divide adf by bdf , the quotient will be the same as of a divided by b ; and cbf divided by dbf gives the same quotient as c divided by d ; and ebd divided by fbd the same quotient as e divided by f .

2. Fractions reduced to the same denominator are added by adding their numerators and sub-

subscribing the common denominator. I say $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. For call $\frac{a}{b} = m$, and $\frac{c}{b} = n$, and it will be $a = mb$, $c = nb$, and $mb + nb = a + c$, and $m + n = \frac{a+c}{b}$; that is, $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. After the same manner, $\frac{a}{b} - \frac{c}{b} = m - n = \frac{a-c}{b}$.

3. I say $\frac{a}{b} \times \frac{c}{d} (= m \times n) = \frac{ac}{bd}$; for $bm = a$, $dn = c$; and $bdmn = ac$, and $mn = \frac{ac}{bd}$; that is, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

4. I say $\frac{a}{b}$ divided by $\frac{c}{d}$, or $\frac{m}{n}$, gives $\frac{ad}{cb}$; for $mb = a$, and $mbd = ad$; $nd = c$, and $nbd = cb$; therefore $\frac{mbd}{nbd} = \frac{ad}{cb}$; that is, $\frac{m}{n} = \frac{ad}{cb}$.

PROBLEM VII.

§ 31. To find the greatest common Measure of two numbers; that is, the greatest number that can divide them both without a remainder.

Rule. First divide the greater number by the lesser, and if there is no remainder the lesser number is the greatest common divisor required.

If

If there is a remainder, divide your last divisor by it; and thus proceed continually dividing the last divisor by its remainder, till there is no remainder left, and then the last divisor is the greatest common Measure required.

Thus the greatest common measure of 45 and 63 is 9; and the greatest common measure of 256 and 48 is 16.

$$\begin{array}{r}
 45) 63 (1 \\
 \underline{45} \\
 18) 45 (2 \\
 \underline{36} \\
 9) 18 (2 \\
 \underline{18} \\
 0
 \end{array}
 \qquad
 \begin{array}{r}
 48) 256 (5 \\
 \underline{240} \\
 16) 48 (3 \\
 \underline{48} \\
 0
 \end{array}$$

§ 32. Much after the same manner the greatest common measure of algebraic quantities is discovered; only the remainders that arise in the operation are to be divided by their simple divisors, and the quantities are always to be ranged according to the dimensions of the same letter.

Thus to find the greatest common measure of $a^2 - b^2$ and $a^2 - 2ab + b^2$;

$$\begin{array}{r}
 a^2 - b^2) a^2 - 2ab + b^2 (1 \\
 \underline{a^2 - b^2} \\
 - 2ab + 2b^2 \text{ Remainder,} \\
 \text{which}
 \end{array}$$

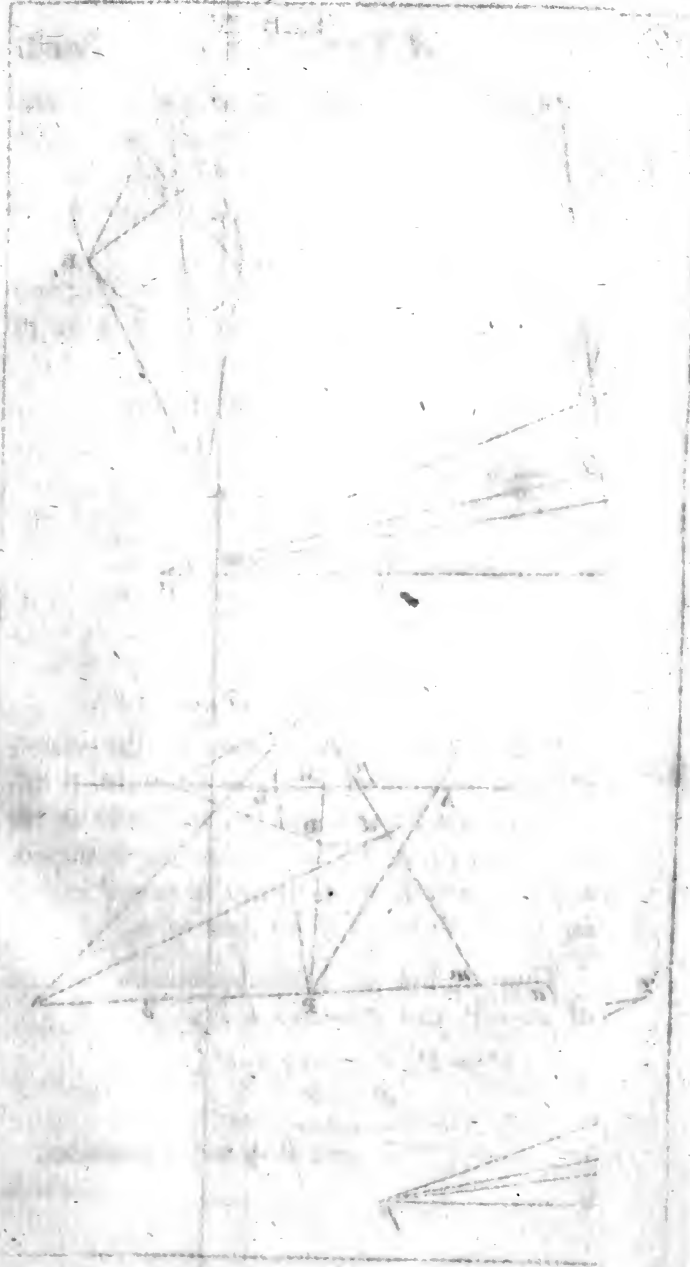


Fig. 25.

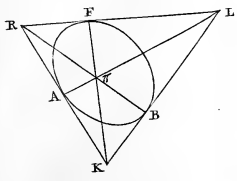


Fig. 24.

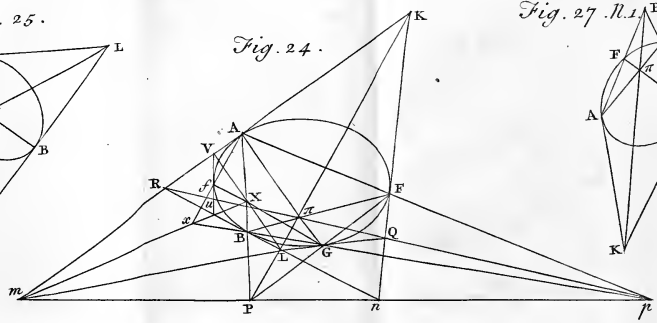


Fig. 27. II. 1.

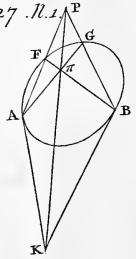


Fig. 27. II. 2.



Fig. 26.

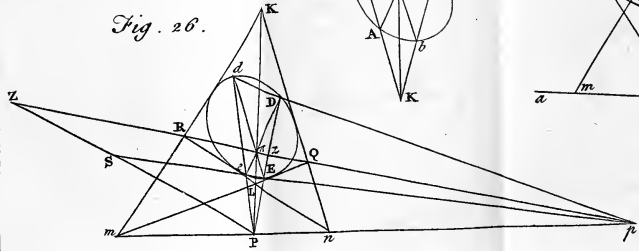
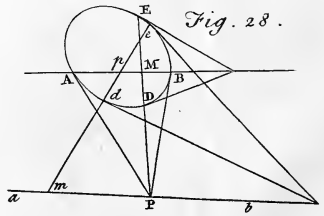


Fig. 28.



which divided by $-2b$ is reduced to

$$\begin{array}{r} a-b \) \ a^2 - b^2 \ (a + b \\ \underline{a^2 - b^2} \\ 0 \quad 0 \end{array}$$

Therefore $a - b$ is the greatest common measure required.

The ground of this operation is, That any quantity that measures the divisor and the remainder (if there is any) must also measure the dividend; because the dividend is equal to the sum of the divisor multiplied into the quotient, and of the remainder added together*. Thus in the last example, $a - b$ measures the divisor $a^2 - b^2$, and the remainder $-2ab + 2b$; it must therefore likewise measure their sum $a^2 - 2ab + b^2$. You must observe in this operation to make that the dividend which has the highest powers of the letter, according to which the quantities are ranged.

PROBLEM VIII.

§ 33. *To reduce any Fraction to its lowest terms.*

Rule. Find the greatest common measure of the numerator and denominator; divide them by that common measure and place the quotients in their room, and you shall have a fraction equivalent

* See Chap. XIV.

to the given fraction expressed in the least terms.

$$\begin{aligned} \text{Thus } \frac{25bc}{25bc} \cdot \frac{75abc}{125bcx} &= \frac{3a}{5x}; \quad \frac{156aa + 156ab}{572aa - 572ab} = \\ & \frac{3a + 3b}{11a - 11b}; \quad \frac{a^2 - b^2}{a^2 - 2ab + b^2} = \frac{a + b}{a - b}; \\ \frac{a^3 - b^2a}{a^2 + 2ab + b^2} &= \frac{a^2 - ba}{a + b}; \quad \frac{a^4 - b^4}{a^5 - a^3b^2} = \frac{a^2 + b^2}{a^3}. \end{aligned}$$

When unit is the greatest common measure of the numbers and quantities, then the fraction is already in its lowest terms. Thus $\frac{3ab}{5dc}$ cannot be reduced lower.

And numbers whose greatest common measure is unit, are said to be *prime* to one another.

§ 34. If it is required to reduce a given fraction to a fraction equal to it that shall have a given denominator, you must *multiply the numerator by the given denominator, and divide the product by the former denominator, the quotient set over the given denominator is the fraction required.* Thus $\frac{a}{b}$ being given, and it being required to reduce it to an equal fraction whose denominator shall be c ; find the quotient of ac divided by b , and it shall be the numerator of the fraction required.

If a Vulgar fraction is to be reduced to a decimal (that is, a fraction whose denominator is 10, or any of its powers) *annex as many cyphers as you please to the numerator, and then divide it by the denominator, the quotient shall give a decimal equal to the Vulgar fraction proposed.*

$$\text{Thus } \frac{2}{3} = .66666, \text{ \&c.}; \quad \frac{3}{5} = .6;$$

$$\frac{2}{7} = .2857142, \text{ \&c.}$$

§ 35. These fractions are added and subtracted like whole numbers; only care must be taken to *set similar places above one another*, as units above units, and tenths above tenths, &c. They are multiplied and divided as integer numbers; only *there must be as many decimal places in the product as in both the multiplicand and multiplier*; and *in the quotient as many as there are in the dividend more than in the divisor*. And in division the quotient may be continued to any degree of exactness you please, by adding cyphers to the dividend. The ground of these operations is easily understood from the general rules for adding, multiplying, and dividing fractions.

C H A P. VII.

Of the INVOLUTION of Quantities.

§ 36. **T**HE products arising from the continual multiplication of the same quantity were called (in *Chap. 4.*) the *powers* of that quantity. Thus $a, a^2, a^3, a^4, \&c.$ are the powers of a ; and $ab, a^2b^2, a^3b^3, a^4b^4, \&c.$ are the powers of ab . In the same Chapter, the rule for the multiplication of powers of the same quantity is to “Add the exponents and make their sum the exponent of the product.” Thus $a^4 \times a^5 = a^9$; and $a^3b^3 \times a^6b^2 = a^9b^5$. In *Chap. 5.* you have the rule for dividing powers of the same quantity, which is, “To subtract the exponents and make the difference the exponent of the quotient.”

Thus $\frac{a^6}{a^4} = a^{6-4} = a^2$; and $\frac{a^5b^3}{a^4b} = a^{5-4}b^{3-1} = ab^2$.

§ 37. If you divide a lesser power by a greater, the exponent of the quotient must, by this Rule, be negative. Thus $\frac{a^4}{a^6} = a^{4-6} = a^{-2}$. But $\frac{a^4}{a^6} = \frac{1}{a^2}$; and hence $\frac{1}{a^2}$ is expressed also by a^{-2} with a negative exponent.

It

Fig. 29.
N. 1.

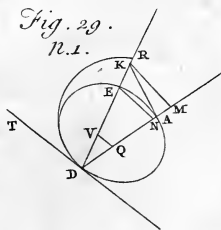


Fig. 29.
N. 2.

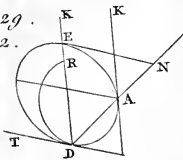


Fig. 31.

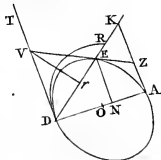


Fig. 30.
N. 1.

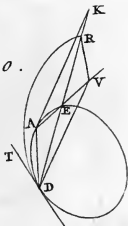


Fig. 30.
N. 2.

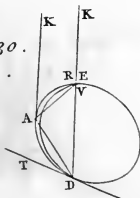


Fig. 32.

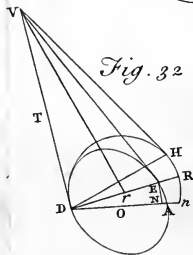


Fig. 33.

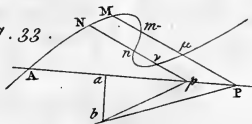


Fig. 34. N. 1.

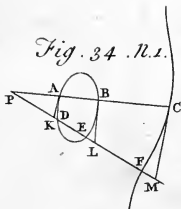
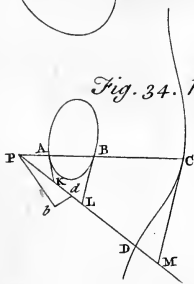


Fig. 34. N. 2.





It is also obvious that $\frac{a}{a} = a^{1-1} = a^0$;
 but $\frac{a}{a} = 1$, and therefore $a^0 = 1$. After
 the same manner $\frac{1}{a} = \frac{a^0}{a} = a^{0-1} = a^{-1}$;
 $\frac{1}{aa} = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$; $\frac{1}{aaa} = a^{0-3} =$
 a^{-3} ; so that the quantities a , 1 , $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$,
 $\frac{1}{a^4}$, &c. may be expressed thus, a^1 , a^0 , a^{-1} ,
 a^{-2} , a^{-3} , a^{-4} , &c. Those are called the
negative powers of a which have negative ex-
 ponents; but they are at the same time *positive*
 powers of $\frac{1}{a}$ or a^{-1} .

§ 38. *Negative powers (as well as positive) are multiplied by adding, and divided by subtracting their exponents.* Thus the product of a^{-2} (or $\frac{1}{a^2}$) multiplied by a^{-3} (or $\frac{1}{a^3}$) is $a^{-2-3} = a^{-5}$ (or $\frac{1}{a^5}$); also $a^{-6} \times a^4 = a^{-6+4} = a^{-2}$ (or $\frac{1}{a^2}$); and $a^{-3} \times a^3 = a^0 = 1$. And, in general, any positive power of a multiplied by a negative power of a of an equal exponent gives UNIT for the product; for the positive and negative destroy each other, and the product gives a^0 , which is equal to unit.

Likewise $\frac{a^{-5}}{a^{-2}} = a^{-5+2} = a^{-3} = \frac{1}{a^3}$; and $\frac{a^{-2}}{a^{-5}} = a^{-2+5} = a^3$. But also, $\frac{a^{-2}}{a^{-5}} = \frac{a^{-2}}{a^{-2} \times a^{-3}} = \frac{1}{a^{-3}}$; therefore $\frac{1}{a^{-3}} = a^3$: And, in general, “any quantity placed in the denominator of a fraction may be transposed to the numerator, if the sign of its exponent be changed.” Thus $\frac{1}{a^3} = a^{-3}$, and $\frac{1}{a^{-3}} = a^3$.

§ 39. The quantity a^m expresses any power of a in general; the exponent (m) being undetermined; and a^{-m} expresses $\frac{1}{a^m}$, or a negative power of a of an equal exponent: and $a^m \times a^{-m} = a^{m-m} = a^0 = 1$ is their product. a^n expresses any other power of a ; $a^m \times a^n = a^{m+n}$ is the product of the powers a^m and a^n , and a^{m-n} is their quotient.

§ 40. To raise any simple quantity to its second, third, or fourth power, is to add its exponent twice, thrice, or four times to itself; therefore the second power of any quantity is had by doubling its exponent, and the third by trebling its exponent; and, in general, *the power expressed by m of any quantity is had by multiplying the exponent by m* , as is obvious from the multiplication of powers. Thus the second power or square of a is $a^{2 \times 1} = a^2$; its third power

power or cube is $a^3 \times 1 = a^3$; and the m th power of a is $a^m \times 1 = a^m$. Also, the square of a^4 is $a^2 \times 4 = a^8$; the cube of a^4 is $a^3 \times 4 = a^{12}$; and the m th power of a^4 is $a^{4 \times m}$. The square of abc is $a^2 b^2 c^2$, the cube is $a^3 b^3 c^3$, the m th power $a^m b^m c^m$.

§ 41. The raising of quantities to any power is called *Involution*; and any simple quantity is involved by multiplying the exponent by that of the power required, as in the preceding Examples.

The coefficient must also be raised to the same power by a continual multiplication of itself by itself, as often as unit is contained in the exponent of the power required. Thus the cube of $3ab$ is $3 \times 3 \times 3 \times a^3 b^3 = 27a^3 b^3$.

As to the Signs, When the quantity to be involved is positive, it is obvious that all its powers must be positive. And when the quantity to be involved is negative, yet all its powers whose exponents are even numbers must be positive, for any number of multiplications of a negative, if the number is even, gives a positive; since $- \times - = +$, therefore $-- \times - \times -- \times - = + \times + = +$; and $- \times - \times - \times - \times - \times - \times - = + \times + \times + = +$.

The power then only can be negative when its exponent is an odd number, though the quantity to be involved be negative. The powers of $-a$ are $-a, +a^2, -a^3, +a^4, -a^5,$

ℰc. Those whose exponents are 2, 4, 6, &c. are positive; but those whose exponents are 1, 3, 5, &c. are negative.

§ 42. The involution of *compound* quantities is a more difficult operation. The powers of any *binomial* $a + b$ are found by a continual multiplication of it by itself as follows.

$$\begin{array}{l}
 a + b = \text{Root.} \\
 \times \begin{array}{l} a + b \\ a^2 + ab \\ ab + b^2 \end{array} \\
 \hline
 a^2 + 2ab + b^2 = \text{the Square or 2d Power.} \\
 \times \begin{array}{l} a + b \\ a^3 + 2a^2b + ab^2 \\ a^2b + 2ab^2 + b^3 \end{array} \\
 \hline
 a^3 + 3a^2b + 3ab^2 + b^3 = \text{Cube or 3d Power.} \\
 \times \begin{array}{l} a + b \\ a^4 + 3a^3b + 3a^2b^2 + ab^3 \\ a^3b + 3a^2b^2 + 3ab^3 + b^4 \end{array} \\
 \hline
 a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = \text{Biquadrate or 4th Power.} \\
 \times \begin{array}{l} a + b \\ a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \\ a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \end{array} \\
 \hline
 a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = \text{5th Power.} \\
 \times \begin{array}{l} a + b \\ a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\ a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\ a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 = \text{6th Power, \&c.} \end{array}
 \end{array}$$

1862

Date	Description	Amount
Jan 1	Balance	100.00
Jan 15	Received from A. B.	50.00
Jan 30	Received from C. D.	25.00
Feb 15	Received from E. F.	75.00
Feb 28	Received from G. H.	30.00
Mar 15	Received from I. J.	40.00
Mar 31	Received from K. L.	60.00
Apr 15	Received from M. N.	20.00
Apr 30	Received from O. P.	80.00
May 15	Received from Q. R.	15.00
May 31	Received from S. T.	90.00
Jun 15	Received from U. V.	35.00
Jun 30	Received from W. X.	55.00
Jul 15	Received from Y. Z.	45.00
Jul 31	Received from AA. BB.	65.00
Aug 15	Received from CC. DD.	25.00
Aug 31	Received from EE. FF.	75.00
Sep 15	Received from GG. HH.	30.00
Sep 30	Received from II. JJ.	85.00
Oct 15	Received from KK. LL.	15.00
Oct 31	Received from MM. NN.	95.00
Nov 15	Received from OO. PP.	35.00
Nov 30	Received from QQ. RR.	55.00
Dec 15	Received from SS. TT.	45.00
Dec 31	Received from UU. VV.	65.00
Total		1000.00

Fig. 35.

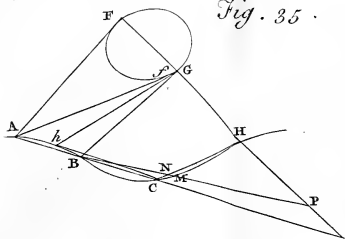


Fig. 36.

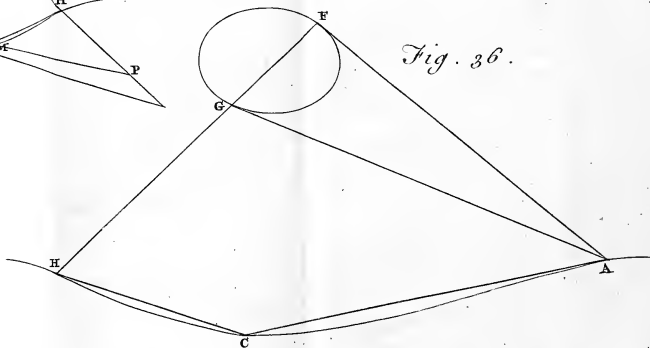
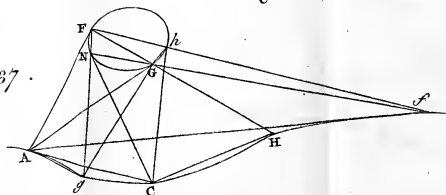


Fig. 37.



§ 43. If the powers of $a - b$ are required, they will be found the same as the preceding, only the terms in which the exponent of b is an odd number will be found negative; “because an odd number of multiplications of a negative produces a negative.” Thus the cube of $a - b$ will be found to be $a^3 - 3a^2b - 3ab^2 - b^3$: where the 2d and 4th terms are negative, the exponent of b being an odd number in these terms. In general, “The terms of any power of $a - b$ are positive and negative by turns.”

§ 44. It is to be observed, That “in the first term of any power of $a + b$, the quantity a has the exponent of the power required, that in the following terms, the exponents of a decrease gradually by the same difference (*viz.* unit) and that in the last terms it is never found. The powers of b are in the contrary order; it is not found in the first term, but its exponent in the second term is unit, in the third term its exponent is 2; and thus its exponent increases, till in the last term it becomes equal to the exponent of the power required.”

As the exponents of a thus decrease, and at the same time those of b increase, “the sum of their exponents is always the same, and is equal to the exponent of the power required.” Thus in the 6th power of $a + b$, *viz.* $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$,

the exponents of a decrease in this order, 6, 5, 4, 3, 2, 1, 0; and those of b increase in the contrary order, 0, 1, 2, 3, 4, 5, 6. And the sum of their exponents in any term is always 6.

§ 45. To find the coefficient of any term, the coefficient of the preceding term being known; you are to "divide the coefficient of the preceding term by the exponent of b in the given term, and to multiply the quotient by the exponent of a in the same term, increased by unit." Thus to find the coefficients of the terms of the 6th power of $a + b$, you find the terms are

$$a^6, a^5b, a^4b^2, a^3b^3, a^2b^4, ab^5, b^6;$$

and you know the coefficient of the first term is unit; therefore, according to the rule, the coefficient of the 2d term will be $\frac{1}{1} \times \overline{5+1} = 6$; that of the 3d term will be $\frac{6}{2} \times \overline{4+1} = 3 \times 5 = 15$; that of the 4th term will be $\frac{15}{3} \times \overline{3+1} = 5 \times 4 = 20$; and those of the following will be 15, 6, 1, agreeable to the preceding Table.

§ 46. In general, if $a + b$ is to be raised to any power m , the terms, without their coefficients, will be, $a^m, a^{m-1}b, a^{m-2}b^2, a^{m-3}b^3, a^{m-4}b^4, a^{m-5}b^5, \&c.$ continued till the exponent of b becomes equal to m .

The

The coefficients of the respective terms, according to the last rule, will be

1, m , $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c. continued until you have one coefficient more than there are units in m .

It follows therefore by these last rules, that $\overline{a+b}^m = a^m + ma^{m-1}b + m \times \frac{m-1}{2} \times a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}b^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4}b^4 +$, &c. which is the general *Theorem* for raising a quantity consisting of two terms to any power m .

§ 47. If a quantity consisting of three, or more terms is to be involved, “you may distinguish it into two parts, considering it as a binomial, and raise it to any power by the preceding rules; and then by the same rules you may substitute instead of the powers of these compound parts their values.”

Thus $\overline{a+b+c}^2 = \overline{a+b+c}^2 = \overline{a+b}^2 + 2c \times \overline{a+b} + c^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

And $\overline{a+b+c}^3 = \overline{a+b}^3 + 3c \times \overline{a+b}^2 + 3c^2 \times \overline{a+b} + c^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

In these Examples, $a + b + c$ is considered as composed of the compound part $a + b$ and the simple part c ; and then the powers of $a + b$ are formed by the preceding rules, and substituted for $\overline{a + b}^3$ and $\overline{a + b}^2$.



CHAP. VIII.

OF EVOLUTION.

§ 48. **T**HE reverse of Involution, or the resolving of powers into their roots, is called *Evolution*. The roots of single quantities are easily extracted by dividing their exponents by the number that denominates the root required. Thus the square root of a^8 is $a^{\frac{8}{2}} = a^4$; and the square root of $a^4b^8c^2$ is a^2b^4c . The cube root of a^6b^3 is $a^{\frac{6}{3}}b^{\frac{3}{3}} = a^2b$; and the cube root of $x^9y^6z^{12}$ is $x^3y^2z^4$. The ground of this rule is obvious from the rule for Involution. The powers of any root are found by multiplying its exponent by the index that denominates the power; and therefore, when any power is given, the root must be found by dividing the exponent of the given power by the number that denominates the kind of root that is required.

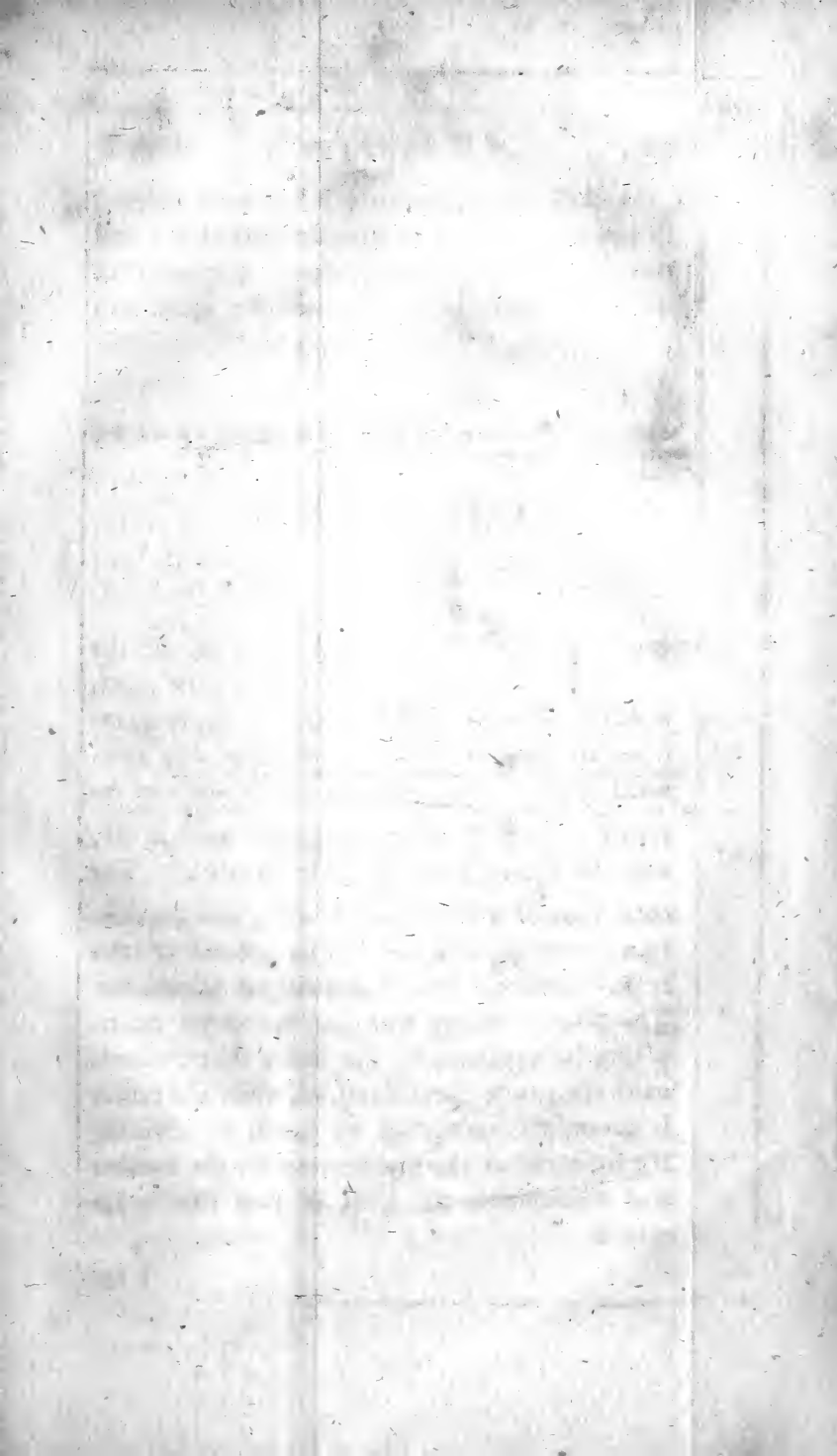


Fig. 38.

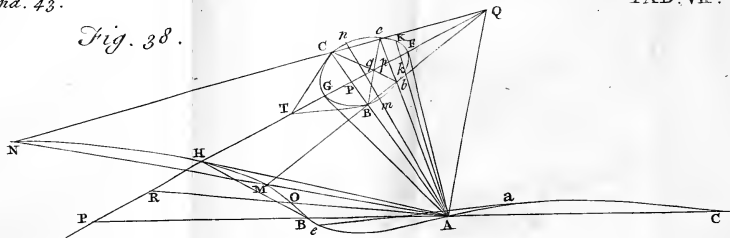
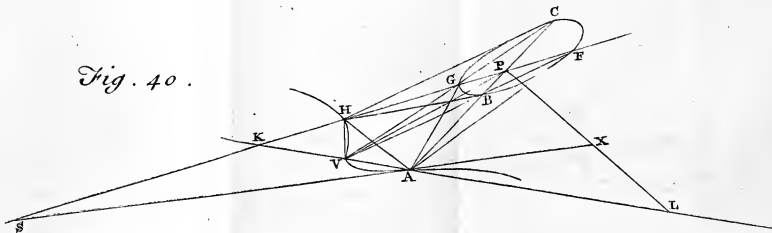


Fig. 39.



Fig. 40.



§ 49. It appears from what was said of Involution, that “*any power that has a positive sign may have either a positive or negative root, if the root is denominated by any even number.*” Thus the square root of $+ a^2$ may be $+ a$ or $- a$, because $+ a \times + a$ or $- a \times - a$ gives $+ a^2$ for the product.

But if a power have a negative sign, “*no root of it denominated by an even number can be assigned,*” since there is no quantity that multiplied into itself an even number of times can give a negative product. Thus the square root of $- a^2$ cannot be assigned, and is what we call an “*impossible or imaginary quantity.*”

But if the root to be extracted is denominated by an *odd* number, *then shall the sign of the root be the same as the sign of the given number whose root is required.* Thus the cube root of $- a^3$ is $- a$, and the cube root of $a^6 b^3$ is $- a^2 b$.

§ 50. If the number that denominates the root required is a divisor of the exponent of the given power, then shall the root be only a “*lower power of the same quantity.*” As the cube root of a^{12} is a^4 , the number 3 that denominates the cube root being a divisor of 12.

But if the number that denominates what sort of root is required is not a divisor of the exponent of the given power, “*then the root required shall have a fraction for its exponent.*”

Thus

Thus the square root of a^3 is $a^{\frac{3}{2}}$; the cube root of a^5 is $a^{\frac{5}{3}}$, and the square of a itself is $a^{\frac{1}{2}}$.

These powers that have fractional exponents are called "*Imperfect powers or surds*;" and are otherwise expressed by placing the given power within the radical sign $\sqrt{\quad}$, and placing above the radical sign the number that denominates what kind of root is required. Thus $a^{\frac{3}{2}} = \sqrt[2]{a^3}$; $a^{\frac{5}{3}} = \sqrt[3]{a^5}$; and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. In numbers the square root of 2 is expressed by $\sqrt[2]{2}$, and the cube root of 4 by $\sqrt[3]{4}$.

§ 51. These imperfect powers or surds are "*multiplied and divided, as other powers, by adding and subtracting their exponents.*" Thus $a^{\frac{3}{2}} \times a^{\frac{5}{2}} = a^{\frac{6}{2}} = a^3$; $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3} + \frac{3}{4}} = a^{\frac{17}{12}} = \sqrt[12]{a^{17}}$; and $\frac{a^{\frac{7}{2}}}{a^{\frac{3}{2}}} = a^{\frac{7-3}{2}} = a^{\frac{4}{2}} = a^2$.

They are *involved* likewise and *evolved* after the same manner as perfect powers. Thus the square of $a^{\frac{3}{2}}$ is $a^{2 \times \frac{3}{2}} = a^3$; the cube of $a^{\frac{2}{5}}$ is $a^{3 \times \frac{2}{5}} = a^{\frac{6}{5}}$. The square root of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3} \times \frac{1}{2}} = a^{\frac{1}{3}}$, the cube root of $a^{\frac{3}{4}}$ is $a^{\frac{1}{4}}$. But we shall have occasion to treat more fully of Surds hereafter.

§ 52. The square root of any compound quantity, as $a^2 + 2ab + b^2$ is discovered after this

this manner. “ *First, take care to dispose the terms according to the dimensions of the alphabet, as in division; then find the square root of the first term aa , which gives a for the first member of the root. Then subtract its square from the proposed quantity, and divide the first term of the remainder ($2ab + b^2$) by the double of that member, viz. $2a$, and the quotient b is the second member of the root. Add this second member to the double of the first, and multiply their sum ($2a + b$) by the second member b , and subtract the product ($2ab + b^2$) from the foresaid remainder ($2ab + b^2$) and if nothing remains, then the square root is obtained; and in this example it is found to be $a + b$.*

The manner of the operation is thus,

$$\begin{array}{r}
 a^2 + 2ab + b^2 \quad (a + b \\
 a^2 \\
 \hline
 2a + b \) \ 2ab + b^2 \\
 \times b \) \ 2ab + b^2 \\
 \hline
 \ 0 \quad 0
 \end{array}$$

But if there had been a remainder, you must have divided it by the double of the sum of the two parts already found, and the quotient would have given the third member of the root.

Thus if the quantity proposed had been $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$, after proceeding as above you would have found the remainder

$2ac$



found if it be under 1000, and its biquadrate if it be under 10000.

The Root	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81
Cube	1	8	27	64	125	216	343	512	729
Biquad.	1	16	81	256	625	1296	2401	4096	6561

But if it is a number above 100, then its square root will consist of two or more figures, which must be found by different operations by the following

R U L E.

§ 53. Place a point above the number that is in the place of units, pass the place of tens, and place again a point over that of hundreds, and go on towards the left hand, placing a point over every 2d figure; and by these points the number will be distinguished into as many parts as there are figures in the root. Then find the square root of the first part, and it will give the first figure of the root; subtract its square from that part, and annex the second part of the given number to the remainder. Then divide this new number (neglecting its last figure) by the double of the first figure of the root, annex the quotient to that double, and multiply the number thence arising by the said quotient, and if the product is less than your dividend, or equal to it, that quotient shall be the second figure of the root. But if the product is greater than the dividend, you must take a less number for the second figure

$\begin{array}{r} \overset{\cdot}{2}\overset{\cdot}{7}\overset{\cdot}{3}\overset{\cdot}{9}475\overset{\cdot}{6} (5234 \\ \underline{25} \\ 102)239 \\ \times 2)204 \\ \underline{204} \\ 1043)3547 \\ \times 3)3129 \\ \underline{3129} \\ 10464\ 41856 \\ \times 4)41856 \\ \underline{41856} \\ 0 \end{array}$	$\begin{array}{r} \overset{\cdot}{5}\overset{\cdot}{2}\overset{\cdot}{9} (23 \\ \underline{4} \\ 43)129 \\ \times 3)129 \\ \underline{129} \\ 0 \end{array}$
--	--

§ 54 In general, to extract any root out of any given quantity, “ *First range that quantity according to the dimensions of its letters, and extract the said root out of the first term, and that shall be the first member of the root required. Then raise this root to a dimension lower by unit than the number that denominates the root required, and multiply the power that arises by that number itself; divide the second term of the given quantity by the product, and the quotient shall give the second member of the root required.*”

Thus to extract the root of the 5th power out of $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, I find that the root of the 5th power out of a^5 gives a , which I raise to the 4th power, and multiplying by 5, the product is $5a^4$; then dividing the second term of the given quantity $5a^4b$ by $5a^4$, I find b to be the second member; and

and raising $a + b$ to the 5th power and subtracting it, there being no remainder, I conclude that $a + b$ is the root required. If the root has three members, the third is found after the same manner from the first two considered as one member, as the second member was found from the first; which may be easily understood from what was said of extracting the square root.

§ 55. In extracting roots it will often happen that the exact root cannot be found in finite terms; thus the square root of $a^2 + x^2$ is found to be

$$a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} +, \text{ \&c.}$$

The operation is thus;

$$a^2 + x^2 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} -, \text{ \&c.} \right.$$

a^2

$$\underline{2a + \frac{x^2}{2a}} * + x^2$$

$$\times \frac{x^2}{2a} = x^2 + \frac{x^4}{4a^2}$$

$$\underline{2a + \frac{x^2}{a} - \frac{x^4}{8a^3}} - \frac{x^4}{4a^2}$$

$$\times - \frac{x^4}{8a^3} = - \frac{x^4}{4a^2} - \frac{x^6}{8a^5} + \frac{x^8}{64a^7}$$

$$\underline{+ \frac{x^6}{8a^5} - \frac{x^8}{64a^7}}$$

\&c.

After

After the same manner, the cube root of $a^3 + x^3$ will be found to be

$$a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}} +, \text{ \&c.}$$

§ 56. “The general Theorem which we gave for the Involution of binomials will serve also for their Evolution;” because to extract any root of a given quantity is the same thing as to raise that quantity to a power whose exponent is a fraction that has its denominator equal to the number that expresses what kind of root is to be extracted. Thus, to extract the square root of $a + b$, is to raise $a + b$ to a power whose exponent is $\frac{1}{2}$: Now since $\overline{a + b}^m = a^m + m \times a^{m-1}b + m \times \frac{m-1}{2} a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}b^3, \text{ \&c.}$

Supposing $m = \frac{1}{2}$, you will find

$$\overline{a + b}^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2} \times a^{-\frac{1}{2}}b + \frac{1}{2} \times -\frac{1}{4} \times a^{-\frac{3}{2}}b^2 + \frac{1}{2} \times -\frac{1}{4} \times -\frac{1}{2} a^{-\frac{5}{2}}b^3, \text{ \&c.} = a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^2}{8a^{\frac{3}{2}}} + \frac{b^3}{16a^{\frac{5}{2}}}, \text{ \&c.}$$

And after this manner you will find that

$$\overline{a^2 + x^2}^{\frac{1}{2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} -, \text{ \&c. as before.}$$

§ 57. The roots of numbers are to be extracted as those of algebraic quantities. “Place a point over the units, and then place points over

E

every

every third, fourth or fifth figure towards the left hand, according as it is the root of the cube, of the 4th or 5th power that is required; and if there be any decimals annexed to the number, point them after the same manner, proceeding from the place of units towards the right hand. By this means the number will be divided into so many periods as there are figures in the root required. Then enquire which is the greatest cube, biquadrate, or 5th power in the first period, and the root of that power will give the first figure of the root required. Subtract the greatest cube, biquadrate, or 5th power from the first period, and to the remainder annex the first figure of your second period, which shall give your dividend.

Raise the first figure already found to a power less by unit than the power whose root is sought, that is, to the 2d, 3d, or 4th power, according as it is the cube root, the root of the 4th, or the root of the 5th power that is required, and multiply that power by the index of the cube, 4th or 5th power, and divide the dividend by this product, so shall the quotient be the second figure of the root required.

Raise the part already found of the root, to the power whose root is required, and if that power be found less than the two first periods of the given number, the second figure of the root is right. But if it be found greater, you must diminish the second figure of the root till that power
be

be found equal to or less than those periods of the given number. Subtract it, and to the remainder annex the next period; and proceed till you have gone through the whole given number, finding the 3d figure by means of the two first, as you found the second by the first; and afterwards finding the 4th figure (if there be a 4th period) after the same manner from the three first."

Thus to find the cube root of 13824; point it 13824; find the greatest cube in 13, viz. 8, whose cube root 2 is the first figure of the root required. Subtract 8 from 13, and to the remainder 5 annex 8 the first figure of the second period; divide 58 by triple the square of 2, viz. 12. and the quotient is 4, which is the second figure of the root required, since the cube of 24 gives 13824, the number proposed. After the same manner the cube root of 13312053 is found to be 237.

OPERATION.

$$\begin{array}{r}
 \overset{\cdot}{1}\overset{\cdot}{3}\overset{\cdot}{8}\overset{\cdot}{2}\overset{\cdot}{4} \quad (24 \\
 \text{Subtr.} \quad 8 = 2 \times 2 \times 2 \\
 \hline
 3 \times 4 = 12) 58 \quad (4 \\
 \text{Subtract } 24 \times 24 \times 24 = 13824 \\
 \hline
 \text{Rem.} \quad 0
 \end{array}$$

$$\begin{array}{r}
 \overset{\cdot}{1}\overset{\cdot}{3}\overset{\cdot}{3}\overset{\cdot}{1}\overset{\cdot}{2}\overset{\cdot}{0}\overset{\cdot}{5}\overset{\cdot}{3} \text{ (237)} \\
 \underline{\quad\quad\quad 8 = 2 \times 2 \times 2} \\
 12) 53 \text{ (4 or) } 3 \\
 \text{Subtract } \underline{12167 = 23 \times 23 \times 23} \\
 3 \times 23 \times 23 = 1587) \underline{11450} \text{ (7)} \\
 \text{Subtract } \underline{13312053 = 237 \times 237 \times 237} \\
 \text{Remain.} \quad \quad \quad 0
 \end{array}$$

In extracting of roots, after you have gone through the number proposed, if there is a remainder, you may continue the operation by adding periods of cyphers to that remainder, and find the true root in decimals to any degree of exactness.



CHAP. IX.

Of PROPORTION.

§ 58. **W**HEN quantities of the same kind are compared, it may be considered either how much the one is greater than the other, and what is their *difference*; or, it may be considered how many times the one is contained in the other; or, more generally, what

what is their *quotient*. The first relation of quantities is expressed by their *Arithmetical ratio*; the second by their *Geometrical ratio*. That term whose ratio is enquired into is called the *antecedent*, and that with which it is compared is called the *consequent*.

§ 59. When of four quantities the difference betwixt the first and second is equal to the difference betwixt the third and fourth, those quantities are called *Arithmetical proportionals*; as the numbers 3, 7, 12, 16. And the quantities, $a, a+b, c, c+b$. But quantities form a *series* in arithmetical proportion, when they “*increase or decrease by the same constant difference.*” As these, $a, a+b, a+2b, a+3b, a+4b, \&c.$ $x, x-b, x-2b, \&c.$ or the numbers, 1, 2, 3, 4, 5, $\&c.$ and 10, 7, 4, 1, $-2, -5, -8, \&c.$

§ 60. In four quantities *arithmetically proportional*, “*the sum of the extremes is equal to the sum of the mean terms.*” Thus $a, a+b, c, c+b$, are arithmetical proportionals, and the sum of the extremes ($a+c$) is equal to the sum of the mean terms ($a+b+c$). Hence, to find the fourth quantity arithmetically proportional to any three given quantities; “*Add the second and third, and from their sum subtract the first term, the remainder shall give the fourth arithmetical proportional required.*”

§ 61. In a *series* of arithmetical proportionals “*the sum of the first and last terms is equal to the sum of any two terms equally distant from the extremes.*” If the first terms are $a, a + b, a + 2b, \&c.$ and the last term x , the last term but one will be $x - b$, the last but two $x - 2b$, the last but three $x - 3b, \&c.$ So that the first half of the terms, having those that are equally distant from the last term set under them, will stand thus ;

$$\begin{array}{cccccc} a, & a + b, & a + 2b, & a + 3b, & a + 4b, & \&c. \\ x, & x - b, & x - 2b, & x - 3b, & x - 4b, & \end{array}$$

$$a + x, a + x, a + x, a + x, a + x, \&c.$$

And it is plain that if each term be added to the term above it, the sum will be $a + x$ equal to the sum of the first term a and the last term x . From which it is plain, that “*the sum of all the terms of an arithmetical progression is equal to the sum of the first and last taken half as often as there are terms,*” that is, the sum of an arithmetical progression is equal to the sum of the first and last terms multiplied by half the number of terms. Thus in the preceding series, if n be the number of terms, the sum of all the terms will be $\overline{a + x} \times \frac{n}{2}$.

§ 62. The common difference of the terms being b , and b not being found in the first term, it is plain that “*its coefficient in any*
term

Fig. 44.

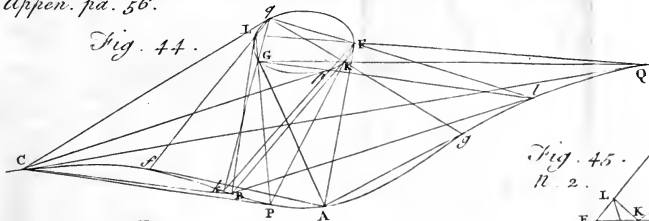


Fig. 45.
N. 2.

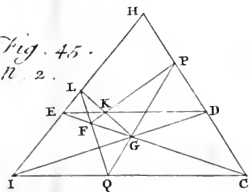


Fig. 45. N. 1.

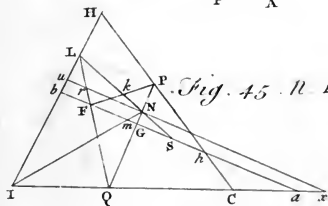
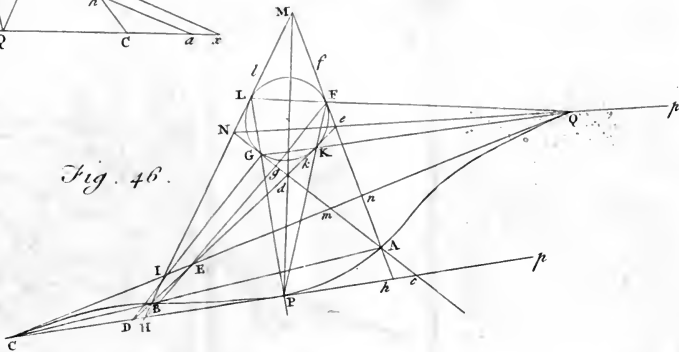


Fig. 46.





term will be equal to the number of terms that precede that term." Therefore in the last term x you must have $\overline{n-1} \times b$, so that x must be equal to $a + \overline{n-1} \times b$. And the sum of all the terms being $\overline{a+x} \times \frac{n}{2}$, it will also be equal to $\frac{2an + n^2b - nb}{2}$, or to $a + \frac{nb-b}{2} \times n$. Thus for example, the series $1 + 2 + 3 + 4 + 5, \&c.$ continued to a hundred, must be equal to $\frac{2 \times 100 + 10000 - 100}{2} = 5050$.

§ 63. If a series have (0) nothing for its first term, then "its sum shall be equal to half the product of the last term multiplied by the number of terms." For then, a being = 0, the sum of the terms, which is in general $\overline{a+x} \times \frac{n}{2}$, will in this case be $\frac{nx}{2}$. From which it is evident, that "the sum of any number of arithmetical proportionals beginning from nothing, is equal to half the sum of as many terms equal to the greatest term."

$$\begin{aligned} \text{Thus } 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 &= \\ 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 &= \frac{10 \times 9}{2} = 45. \end{aligned}$$

§ 64. "If of four quantities the quotient of the first and second be equal to the quotient of the third and fourth, then those quantities are said to be in Geometrical proportion." Such are

the numbers 2, 6, 4, 12; and the quantities a , ar , b , br ; which are expressed after this manner;

$$2 : 6 :: 4 : 12.$$

$$a : ar :: b : br.$$

And you read them by saying, As 2 is to 6, so is 4 to 12; or as a is to ar , so is b to br .

In four quantities geometrically proportional, “*the product of the extremes is equal to the product of the middle terms.*” Thus $a \times br = ar \times b$. And, if it is required to find a fourth proportional to any three given quantities, “*multiply the second by the third, and divide their product by the first, the quotient shall give the fourth proportional required.*” Thus, to find a fourth proportional to a , ar , and b , I multiply ar by b , and divide the product arb by the first term a , the quotient br is the fourth proportional required.

§ 65. In calculations it sometimes requires a little care to place the terms in due order; for which you may observe the following Rule.

“*First set down the quantity that is of the same kind with the quantity sought, then consider, from the nature of the question, whether that which is given is greater or less than that which is sought; if it is greater, then place the greatest of the other two quantities on the left hand; but if it is less, place the least of the other two quantities on the left hand, and the other on the*

the right." Then shall the terms be in due order; and you are to proceed according to the rule, multiplying the second by the third, and dividing their product by the first.

EXAMPLE.

If 30 men do any piece of work in 12 days, how many men shall do it in 18 days?

Because it is a number of men that is sought, I first set down 30, the number of men that is given: I easily see that the number that is given is greater than the number that is sought, therefore I place 18 on the left hand, and 12 on the right; and find a fourth proportional to 18, 30, 12, viz. $\frac{30 \times 12}{18} = 20$.

§ 66. When a series of quantities increase by one common multiplicator, or decrease by one common divisor, they are said to be in "*Geometrical proportion continued.*"

As $a, ar, ar^2, ar^3, ar^4, ar^5, \&c.$ or,

$$a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}, \frac{a}{r^5}, \&c.$$

The common multiplier or divisor is called their "*common ratio.*"

In such a series, "*the product of the first and last is always equal to the product of the second and last but one, or to the product of any two terms equally remote from the extremes.* In the series $a, ar, ar^2, ar^3, \&c.$ if y be the last term, then

then shall the four last terms of the series be $y, \frac{y}{r}, \frac{y}{r^2}, \frac{y}{r^3}$; now it is plain that $a \times y = ar \times \frac{y}{r} = ar^2 \times \frac{y}{r^2} = ar^3 \times \frac{y}{r^3}$, &c.

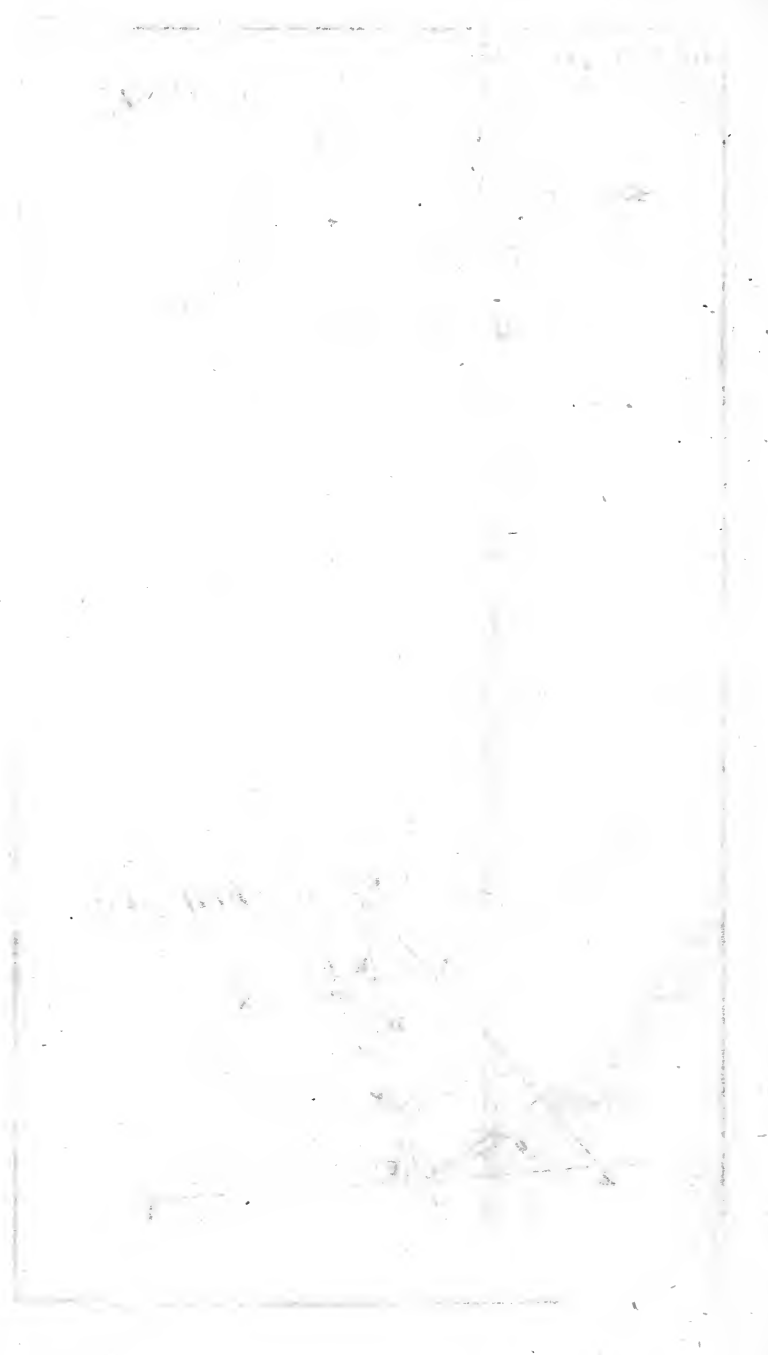
§ 67. "The sum of a series of geometrical proportionals wanting the first term, is equal to the sum of all but the last term multiplied by the common ratio."

For $ar + ar^2 + ar^3, \&c. + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y$
 $= r \times a + ar + ar^2, \&c. + \frac{y}{r^4} + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r}$.
 Therefore if s be the sum of the series, $s - a$ will be equal to $\frac{y}{r} \times r$; that is, $s - a = sr - yr$, or $sr - s = yr - a$, and $s = \frac{yr - a}{r - 1}$ *.

§ 68. Since the exponent of r is always increasing from the second term, if the number of terms be n , in the last term its exponent will be $n - 1$. Therefore $y = ar^{n-1}$; and $yr = ar^{n-1+1} = ar^n$; and $s = \left(\frac{yr - a}{r - 1} \right) = \frac{ar^n - a}{r - 1}$. So that having the first term of the series, the number of the terms, and the common ratio, you may easily find the sum of all the terms.

If it is a decreasing series whose sum is to be found, as of $y + \frac{y}{r} + \frac{y}{r^2} + \frac{y}{r^3}, \&c. + ar^3 + ar^2 + ar + a$, and the number of the terms be

* See the Rules in the following Chapter.



supposed infinite, then shall a , the last term, be equal to nothing. For, because n , and consequently r^{n-1} is infinite, $a = \frac{y}{r^{n-1}} = 0$. The sum of such a series $s = \frac{yr}{r-1}$; which is a finite sum, though the number of terms be infinite.

Thus $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$, &c. $= \frac{1 \times 2}{2-1} = 2$.

and $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$, &c. $= \frac{1 \times 3}{3-1} = \frac{3}{2}$.



CHAP. X.

Of EQUATIONS that involve only one unknown Quantity.

§ 69. **A**N equation is “a proposition asserting the equality of two quantities.” It is expressed most commonly by setting down the quantities, and placing the sign (=) between them.

An equation gives the value of a quantity, when that quantity is alone on one side of the equation: and that value is known, if all those that are on the other side are known. Thus if I find that $x = \frac{4 \times 6}{3} = 8$, I have a known value of x . These are the last conclusions we are to

to seek in questions to be resolved; and if there be only one unknown quantity in a given equation, and only one dimension of it, such a value may always be found by the following Rules.

R U L E I.

§ 70. “ Any quantity may be transposed from one side of the equation to the other, if you change its sign.”

For to take away a quantity from one side, and to place it with a contrary sign on the other side, is to subtract it from both sides; and it is certain, that “ when from equal quantities you subtract the same quantity, the remainders must be equal.”

By this Rule, when the known and unknown quantities are mixed in an equation, you may separate them by bringing all the unknown to one side, and the known to the other side of the equation; as in the following Examples.

$$\text{Suppose } 5x + 50 = 4x + 56,$$

$$\text{By transposit. } 5x - 4x = 56 - 50, \text{ or, } x = 6.$$

$$\text{And if } 2x + a = x + b,$$

$$2x - x = b - a, \text{ or, } x = b - a.$$

R U L E II.

§ 71. “ Any quantity by which the unknown quantity is multiplied may be taken away, if you divide all the other quantities on both sides of the equation by it.”

For

For that is to divide both sides of the equation by the same quantity; and when you divide equal quantities by the same quantity, the quotients must be equal. Thus,

$$\begin{aligned} \text{If } ax &= b, \\ \text{then } x &= \frac{b}{a}. \end{aligned}$$

$$\begin{aligned} \text{And if } 3x + 12 &= 27, \\ \text{by Rule 1. } 3x &= 27 - 12 = 15, \\ \text{and by Rule 2. } x &= \frac{15}{3} = 5. \end{aligned}$$

$$\begin{aligned} \text{Also if } ax + 2ba &= 3cc, \\ \text{by Rule 1. } ax &= 3cc - 2ba, \\ \text{and by Rule 2. } x &= \frac{3cc}{a} - 2b. \end{aligned}$$

R U L E III.

§ 72. *If the unknown quantity is divided by any quantity, that quantity may be taken away if you multiply all the other members of the equation by it.* Thus,

$$\begin{aligned} \text{If } \frac{x}{b} &= b + 5, \\ \text{then shall } x &= bb + 5b. \end{aligned}$$

$$\begin{aligned} \text{If } \frac{x}{5} + 4 &= 10, \\ x + 20 &= 50, \\ \text{and by Rule 1. } x &= 50 - 20 = 30. \end{aligned}$$

If

$$\text{If } \frac{4x}{3} + 24 = 2x + 6,$$

$$4x + 72 = 6x + 18,$$

$$\text{by Rule 1. } 72 - 18 = 6x - 4x, \text{ or } 54 = 2x.$$

$$\text{and by Rule 2. } x = \frac{54}{2} = 27.$$

By this Rule an equation, whereof any part is a fraction, may be reduced to an equation that shall be expressed by integers. If there are more fractions than one in the given equation, you may, by reducing them to a common denominator, and then multiplying all the other terms by that denominator, abridge the calculation thus;

$$\text{If } \frac{x}{5} + \frac{x}{3} = x - 7,$$

$$\text{then } \frac{3x + 5x}{15} = x - 7,$$

$$\text{and by this Rule } 3x + 5x = 15x - 105,$$

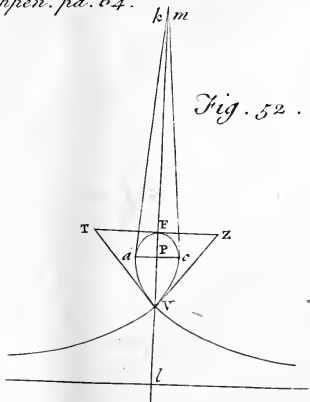
$$\text{and by Rule 1 and 2. } x = \frac{105}{7} = 15.$$

RULE IV.

§ 73. “ If that member of the equation that involves the unknown quantity be a surd root, then the equation is to be reduced to another that shall be free from any surd, by bringing that member first to stand alone upon one side of the equation, and then taking away the radical sign from it, and raising the other side of the equation to the power denominated by the surd.” Thus

k m

Fig. 52.



R

Fig. 53.

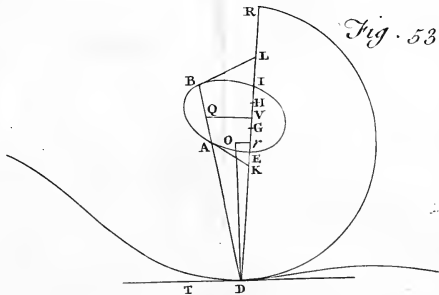


Fig. 54.

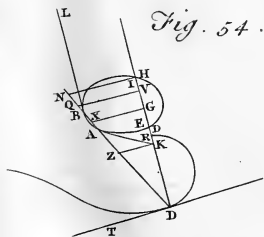
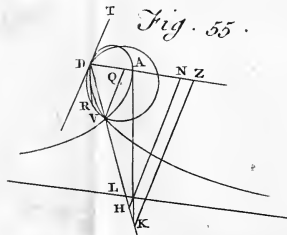


Fig. 55.





$$\begin{aligned} \text{Thus if } \sqrt{4x + 16} &= 12, \\ 4x + 16 &= 144, \\ \text{and } 4x &= 144 - 16 = 128, \\ \text{and } x &= \frac{128}{4} = 32. \end{aligned}$$

$$\begin{aligned} \text{If } \sqrt{ax + b^2} - c &= d, \\ \text{then } \sqrt{ax + b^2} &= d + c, \\ ax + b^2 &= d^2 + 2dc + c^2, \\ \text{and } x &= \frac{d^2 + 2dc + c^2 - b^2}{a}. \end{aligned}$$

$$\begin{aligned} \text{If } \sqrt[3]{a^2x - b^2x} &= a, \\ \text{then } a^2x - b^2x &= a^3, \\ \text{and } x &= \frac{a^3}{a^2 - b^2}. \end{aligned}$$

RULE V.

§ 74. “ If that side of the equation that contains the unknown quantity be a compleat square, cube, or other power; then extract the square root, cube root, or the root of that power, from both sides of the equation, and thus the equation shall be reduced to one of a lower degree.”

$$\begin{aligned} \text{If } x^2 + 6x + 9 &= 20, \\ \text{then } x + 3 &= \pm \sqrt{20}, \\ \text{and } x &= \pm \sqrt{20} - 3. \end{aligned}$$

If

$$\text{If } x^2 + ax + \frac{a^2}{4} = b^2,$$

$$\text{then } x + \frac{a}{2} = \pm b,$$

$$\text{and } x = \pm b - \frac{a}{2}.$$

$$\text{If } x^2 + 14x + 49 = 121,$$

$$\text{then } x + 7 = \pm 11,$$

$$\text{and } x = \pm 11 - 7 = 4, \text{ or } -18.$$

RULE VI.

§ 75. “ A proportion may be converted into an equation, asserting the product of the extreme terms equal to the product of the mean terms; or any one of the extremes equal to the product of the means divided by the other extreme.”

$$\text{If } 12 - x : \frac{x}{2} :: 4 : 1,$$

$$\text{then } 12 - x = 2x \dots 3x = 12 \dots \text{and } x = 4.$$

$$\text{Or if } 20 - x : x :: 7 : 3,$$

$$\text{then } 60 - 3x = 7x \dots 10x = 60 \dots \text{and } x = 6.$$

RULE VII.

§ 76. “ If any quantity be found on both sides of the equation with the same sign prefixt, it may be taken away from both :” “ Also, if all the quantities in the equation are multiplied or divided

Fig. 57.

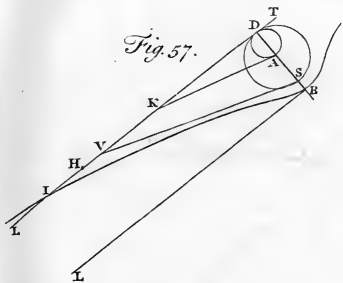


Fig. 56.

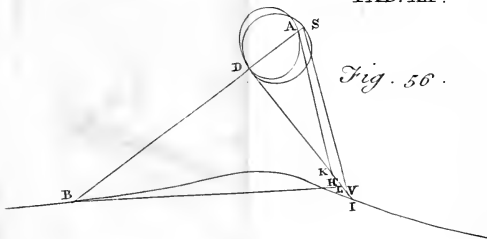


Fig. 58.

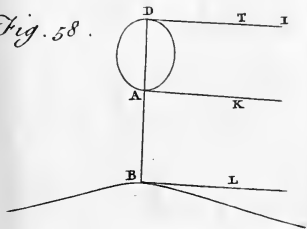
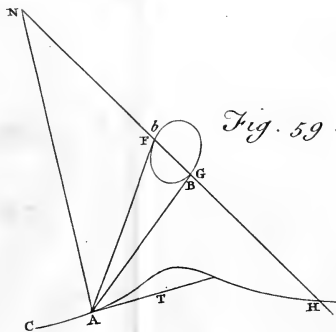


Fig. 59.





divided by the same quantity, it may be struck out of them all." Thus,

$$\text{If } 3x + b = a + b \dots 3x = a \dots \text{and } x = \frac{a}{3}.$$

$$\text{If } 3ax + 5ab = 8ac \dots 3x + 5b = 8c \dots \text{and } x = \frac{8c - 5b}{3}.$$

$$\text{If } \frac{2x}{3} + \frac{8}{3} = \frac{16}{3} \dots 2x + 8 = 16 \dots \text{and } x = 4.$$

RULE VIII.

§ 77. "Instead of any quantity in an equation you may substitute another equal to it."

Thus, if $3x + y = 24$,

and $y = 9$;

then $3x + 9 = 24 \dots x = \frac{24 - 9}{3} = 5.$

If $3y + 5x = 120$,

and $y = 5x$;

then $15x + 5x (= 20x) = 120$,

and $x = \frac{120}{20} = 6.$

The further improvements of this Rule shall be taught in the following chapter.

C H A P. XI.

Of the Solution of questions that produce simple equations.

SIMPLE equations are those “wherein the unknown quantity is only of one dimension:” In the solution of which we are to observe the following directions.

DIRECTION I.

§ 78. “After forming a distinct idea of the question proposed, the unknown quantities are to be expressed by letters, and the particulars to be translated from the common language into the algebraic manner of expressing them, that is, into such equations as shall express the relations or properties that are given of such quantities.”

Thus, if the sum of two quantities must be 60, that condition is expressed thus, $x + y = 60$.

If their difference must be 24, that condition gives $x - y = 24$.

If their product must be 1640, then $xy = 1640$.

If their quotient must be 6, then . . . $\frac{x}{y} = 6$.

If their proportion is as 3 to 2, then $x : y :: 3 : 2$, or $2x = 3y$; because the product of the ex-

tremes

tremes is equal to the product of the mean terms.

DIRECTION II.

§ 79. “After an equation is formed, if you have one unknown quantity only, then, by the Rules of the preceding Chapter, bring it to stand alone on one side, so as to have only known quantities on the other side:” thus you shall discover its value.

EXAMPLE.

A person being asked what was his age, answered that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age gives a product equal to his age. Qu. What was his age?

It appears from the question, that if you call his age x , then shall . . . $\frac{3x}{4} \times \frac{x}{12} = x$,

$$\text{that is . . . } \frac{3x^2}{48} = x;$$

$$\text{and by Rule 3. . . } 3x^2 = 48x,$$

$$\text{and by Rule 7. . . } 3x = 48,$$

$$\text{whence by Rule 2. . . } x = 16.$$

DIRECTION III.

§ 80. “If there are two unknown quantities, then there must be two equations arising from the conditions of the question: Suppose the quantities x and y ; find a value of x or y , from

F 2 each

each of the equations, and then by putting these two values equal to each other, there will arise a new equation involving one unknown quantity; which must be reduced by the Rules of the former Chapter."

EXAMPLE I.

Let the sum of two quantities be s , and their difference d . Let s and d be given, and let it be required to find the quantities themselves. Suppose them to be x and y , then, by the supposition,

$$x + y = s$$

$$x - y = d$$

$$\text{whence } \begin{cases} x = s - y \\ x = d + y \end{cases}$$

$$\text{and } d + y = s - y$$

$$2y = s - d$$

$$y = \frac{s - d}{2}$$

$$\text{and } x = \frac{s + d}{2}.$$

EXAMPLE II.

Let it be required to find two numbers whose sum is s , and their proportion as a to b . Let the numbers be x and y , then shall

Suppos.

$$\text{Suppos. } \begin{cases} x + y = s \\ x : y :: a : b \end{cases}$$

$$\frac{bx = ay}{bx = ay}$$

$$x = \frac{ay}{b}$$

$$x = s - y$$

$$\frac{ay}{b} = s - y$$

$$\frac{ay}{b} + y = s$$

$$ay + by = bs$$

$$\frac{ay + by}{a + b} \times y = bs$$

$$y = \frac{bs}{a + b}$$

$$x = \frac{ay}{b} = \frac{as}{a + b}$$

EXAMPLE III.

A privateer running at the rate of 10 miles an hour, discovers a ship 18 miles off making way at the rate of 8 miles an hour: It is demanded how many miles the ship can run before she be overtaken?

Let the number of miles the ship can run before she be overtaken be called x ; and the number of miles the privateer must run before she come up with the ship, be y ; then shall (by *Supp.*) $y = x + 18$ and $x : y :: 8 : 10$, whence $10x = 8y$ $x = \frac{4y}{5}$ and $x = y - 18$.

F 3

Whence

Whence $y - 18 = \frac{4y}{5}$, and $y = 90 \dots x = y - 18 = 72$.

To find the time, say, if 8 miles give 1 hour; 72 miles will give 9 hours.—Thus, $8 : 1 :: 72 : 9$.

EXAMPLE IV.

Suppose the distance between London and Edinburgh to be 360 miles, and that a courier sets out from Edinburgh running at the rate of 10 miles an hour; another sets out at the same time from London, and runs 8 miles an hour. It is required to know where they will meet? Suppose the courier that sets out from Edinburgh runs x miles, and the other y miles before they meet; then shall

$$\text{by suppos. } \begin{cases} x + y = 360 \\ x : y :: 5 : 4 \end{cases}$$

$$x = \frac{5y}{4}$$

$$x = 360 - y$$

$$\frac{5y}{4} = 360 - y$$

$$\frac{5y}{4} + y = 360$$

$$9y = 1440$$

$$y = \frac{1440}{9} = 160$$

$$x = 360 - y = 200.$$

EXAMPLE V.

Two persons discoursing of their revenues, says A, if B would yield him a post he has of 25l. a year,

year, their revenues would be equal: Says B, if A would give him a place he holds of 22 l. per annum, the revenue of B would be double that of A. Qu. their revenues?

Let the revenue of A be called x , that of B, y ; then,

$$\text{by supp. } \begin{cases} x + 25 = y - 25 \\ y + 22 = 2x - 44 \end{cases}$$

$$\begin{aligned} y &= x + 25 + 25 = x + 50 \\ y &= 2x - 44 - 22 = 2x - 66 \\ 2x - 66 &= x + 50 \\ x &= 66 + 50 = 116 \\ y &= x + 50 = 166. \end{aligned}$$

EXAMPLE VI.

A gentleman distributing money among some poor people, found he wanted 10 s. to be able to give 5 s. to each; therefore he gives each 4 s. only, and finds that he has 5 s. left. Qu. the number of shillings and poor people?

Call the number of the poor x , and the number of shillings y ; then,

$$\text{by supp. } \begin{cases} 5x = y + 10 \\ 4x = y - 5 \end{cases}$$

$$\begin{aligned} y &= 5x - 10 \\ y &= 4x + 5 \\ 5x - 10 &= 4x + 5 \\ 5x - 4x &= 15 \\ x &= 15 \\ y &= 4x + 5 = 65. \end{aligned}$$

EXAMPLE VII.

Two merchants were copartners; the sum of their stock was 300l. One of their stocks continued in company 11 months, but the other drew out his stock in 9 months; when they made up their accounts they divided the gain equally. Qu. What was each man's stock? Suppose the stock of the first to be x , and the stock of the other to be y ; then,

$$\text{by supp. } \begin{cases} x + y = 300 \\ 11x = 9y \end{cases}$$

$$x = \frac{9y}{11} = 300 - y$$

$$11y + 9y = 3300$$

$$20y = 3300$$

$$y = \frac{3300}{20} = 165 \dots x = 300 - y = 135.$$

EXAMPLE VIII.

There are two numbers whose sum is the 6th part of their product, and the greater is to the lesser as 3 to 2. Qu. What are these numbers? Call them x and y ; then,

$$\text{supp. } \begin{cases} x + y = \frac{xy}{6} \\ x : y :: 3 : 2 \end{cases}$$

$$yx = 6x + 6y$$

$$yx - 6x = 6y$$

$$y - 6 \times x = 6y$$

$$x = \frac{6y}{y-6}$$

$$x = \frac{3y}{2}, \text{ whence } *$$

$$* \frac{6y}{y-6} = \frac{3y}{2}$$

$$12y = 3yy - 18y$$

$$30y = 3yy$$

$$30 = 3y$$

$$y = \frac{30}{3} = 10$$

$$x = \frac{3 \times 10}{2} = 15.$$

DIRECTION IV.

§ 81. "When in one of the given equations, the unknown quantity is of one dimension, and in the other of a higher dimension; you must find a value of the unknown quantity from that equation where it is of one dimension, and then raise that value to the power of the unknown quantity in the other equation; and by comparing it, so involved, with the value you deduce from that other equation, you shall obtain an equation that will have only one unknown quantity, and its powers."

That is, when you have two equations of different dimensions, if you cannot reduce the higher to the same dimension with the lower, you must raise the lower to the same dimension with the higher.

EXAMPLE IX.

The sum of two quantities, and the difference of their squares, being given, to find the quantities. Suppose them to be x and y , their sum s , and difference of their squares d . Then,

$$\begin{cases} x + y = s \\ x^2 - y^2 = d \end{cases}$$

$$x = s - y$$

$$x^2 = s^2 - 2sy + y^2$$

$$x^2 = d + y^2$$

$$d + y^2 = s^2 - 2sy + y^2$$

$$d = s^2 - 2sy, \text{ whence } *$$

$$* 2sy = s^2 - d$$

$$y = \frac{s^2 - d}{2s}$$

$$\text{and } x = \frac{s^2 + d}{2s}.$$

EXAMPLE

EXAMPLE X.

Let the proportion of two numbers and the sum of their squares be given, and let it be required to find the numbers themselves. Suppose their proportion to be the same as that of a to b , and let the sum of their squares be c ; that is, let

$$\begin{cases} x : y :: a : b \\ x^2 + y^2 = c \end{cases}$$

$$\text{then } x = \frac{ay}{b},$$

$$\text{and } x^2 = \frac{a^2 y^2}{b^2};$$

$$\text{but } x^2 = c - y^2,$$

$$\text{whence } c - y^2 = \frac{a^2 y^2}{b^2}$$

$$b^2 y^2 + a^2 y^2 = cb^2$$

$$\frac{cb^2}{a^2 + b^2} \times y^2 = cb^2$$

$$y^2 = \frac{cb^2}{a^2 + b^2}$$

$$y = \sqrt{\frac{cb^2}{a^2 + b^2}}, \text{ and } x = \sqrt{\frac{ca^2}{a^2 + b^2}}$$

EXAMPLE XI.

Let the proportion of two numbers be that of a to b , and the difference of their cubes be d .
Qu. What are the numbers? Then,

$$x : y$$

$$\begin{cases} x : y :: a : b \\ x^3 - y^3 = d \end{cases}$$

$$x = \frac{ay}{b}, \text{ and } x^3 = \frac{a^3y^3}{b^3};$$

$$\text{but } x^3 = d + y^3$$

$$\text{whence } d + y^3 = \frac{a^3y^3}{b^3},$$

$$\text{and } a^3y^3 - b^3y^3 = db^3$$

$$y^3 = \frac{db^3}{a^3 - b^3}$$

$$y = \sqrt[3]{\frac{db^3}{a^3 - b^3}}$$

$$\text{and } x = \sqrt[3]{\frac{da^3}{a^3 - b^3}}.$$

DIRECTION V.

§ 82. “If there are three unknown quantities, there must be three equations in order to determine them, by comparing which you may, in all cases, find two equations involving only two unknown quantities; and then, by Direction 3, from these two you may deduce an equation involving only one unknown quantity; which may be resolved by the Rules of the last Chapter.”

From three equations involving any three unknown quantities, x , y , and z , to deduce two equa-

equations involving only two unknown quantities, the following Rule will always serve.

R U L E.

“ Find three values of x from the three given equations; then, by comparing the first and second value, you will find an equation involving only y and z ; again, by comparing the first and third, you will find another equation involving only y and z ;” and lastly, these equations are to be resolved by *Direction 3*.

E X A M P L E XII.

Suppose

$$\left. \begin{array}{l} x + y + z = 12 \\ x + 2y + 3z = 20 \\ \frac{x}{3} + \frac{y}{2} + z = 6 \end{array} \right\} \text{then, } x = \left\{ \begin{array}{l} 12 - y - z \\ 20 - 2y - 3z \\ 18 - \frac{3y}{2} - 3z \end{array} \right. \left. \begin{array}{l} 1 \text{ ft} \\ 2 \text{ d} \\ 3 \text{ d} \end{array} \right\} \text{value.}$$

$$12 - y - z = 20 - 2y - 3z$$

$$12 - y - z = 18 - \frac{3y}{2} - 3z$$

These two last equations involve only y and z , and are to be resolved, by *Direction 3*, as follows.

$$\begin{cases} 2y - y + 3z - z = 20 - 12 = 8 \\ y + 2z = 8 \end{cases}$$

$$36 - 3y - 6z = 24 - 2y - 2z$$

$$12 = y + 4z$$

$$\text{whence } y = \begin{cases} 8 - 2z \dots \text{1st value.} \\ 12 - 4z \dots \text{2d value.} \end{cases}$$

$$8 - 2z = 12 - 4z$$

$$2z = 12 - 8 = 4$$

$$\text{and } z = 2$$

$$y (= 8 - 2z) = 4$$

$$x (= 12 - y - z) = 6.$$

§ 83. This method is general, and will extend to all equations that involve three unknown quantities: but there are often easier and shorter methods to deduce an equation involving one unknown quantity only; which will be best learned by practice.

EXAMPLE XIII.

$$\text{Supposing } \begin{cases} x + y + z = 26 \\ x - y = 4 \\ x - z = 6 \end{cases}$$

$$\text{by addition } 3x = 36$$

$$\begin{cases} x = \frac{36}{3} = 12 \\ y = x - 4 = 8 \\ z = x - 6 = 6. \end{cases}$$

EXAMPLE XIV.

$$\text{Supposing } \begin{cases} x + y = a \\ x + z = b \\ y + z = c \end{cases}$$

$$\underline{x = a - y}$$

$$a - y + z = b$$

$$\underline{y + z = c}$$

$$a + 0 + 2z = b + c$$

$$2z = b + c - a$$

$$\begin{cases} z = \frac{b + c - a}{2} \\ y (= c - z) = \frac{c + a - b}{2} \\ x (= a - y) = \frac{a + b - c}{2} \end{cases}$$

§ 84. It is obvious from the 3d and 5th Directions, in what manner you are to work if there are four, or more, unknown quantities, and four, or more, equations given. By comparing the given equations, you may always at length discover an equation involving only one unknown quantity; which, if it is a simple equation, may always be resolved by the Rules of the last Chapter. We may conclude then, that “When there are as many simple equations given as quantities required, these quantities may be discovered by the application of the preceding Rules.”

§ 85. “ If indeed there are more quantities required than equations given, then the question is not limited to determinate quantities; but is capable of an infinite number of solutions.” And, “ If there are more equations given than there are quantities required, it may be impossible to find the quantities that will answer the conditions of the question;” because some of these conditions may be inconsistent with others.



C H A P. XII.

Containing some General Theorems
for the exterminating unknown
Quantities in given Equations.

IN the following *Theorems*, we call those coefficients of the “ *same order*” that are prefixt to the same unknown quantities in the different equations. Thus, in *Theor. 2.* $a, d, g,$ are of the same order, being the coefficients of x : also $b, e, h,$ are of the same order, being the coefficients of y : and those are of the same order that affect no unknown quantity.

But those are called “ *opposite*” coefficients that are taken each from a different equation,
and

and from a different order of coefficients: As, $a, e,$ and $d, b,$ in the first Theorem; and $a, e, k,$ in the second; also, $a, b, f;$ and $d, b, k, \&c.$

THEOREM I.

§ 86. Suppose that two equations are given, involving two unknown quantities; as

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

$$\text{then shall } y = \frac{af - dc}{ae - db}.$$

Where the numerator is the difference of the products of the opposite coefficients in the orders in which y is not found, and the denominator is the difference of the products of the opposite coefficients taken from the orders that involve the two unknown quantities.

For, from the first equation, it is plain that

$$ax = c - by \dots \text{ and } x = \frac{c - by}{a},$$

$$\text{from the 2d, } dx = f - ey \dots \text{ and } x = \frac{f - ey}{d},$$

$$\text{therefore } \frac{c - by}{a} = \frac{f - ey}{d}, \text{ and } cd - dby = af - aey;$$

$$\text{whence } aey - dby = af - cd,$$

$$\text{and } y = \frac{af - cd}{ae - db};$$

$$\text{after the same manner, } x = \frac{ce - bf}{ae - db}.$$

E X-

EXAMPLE I.

$$\text{Supp. } \begin{cases} 5x + 7y = 100 \\ 3x + 8y = 80 \end{cases}$$

$$\text{then } y = \frac{5 \times 80 - 3 \times 100}{5 \times 8 - 3 \times 7} = \frac{100}{19} = 5\frac{5}{19}$$

$$\text{and } x = \frac{240}{19} = 12\frac{12}{19}.$$

EXAMPLE II.

$$\begin{cases} 4x + 8y = 90 \\ 3x - 2y = 160 \end{cases}$$

$$y = \frac{4 \times 160 - 3 \times 90}{4 \times -2 - 3 \times 8} = \frac{640 - 270}{-8 - 24} = \frac{370}{-32} = -11\frac{9}{16}$$

THEOREM II.

§ 87. Suppose now that there are three unknown quantities and three equations, then call the unknown quantities x , y , and z .

$$\text{Thus } \begin{cases} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{cases}$$

$$\text{Then shall } z = \frac{aep - abn + dbm - dbp + gbn - gem}{aek - ahf + dbc - dbk + gbf - gec}$$

Where the numerator consists of all the different products that can be made of three opposite coefficients taken from the orders in which z is not found; and the denominator consists of all the products that can be made of the three

opposite coefficients taken from the orders that involve the three unknown quantities. For, from the last it appears, that

$$y = \frac{an - afz - dm + dcz}{ae - db}, \text{ and that}$$

$$y = \frac{ap - akz - gm + gcz}{ab - gb}; \text{ therefore}$$

$$\frac{an - afz - dm + dcz}{ae - db} = \frac{ap - akz - gm + gcz}{ab - gb}, \text{ and}$$

$$\frac{an - afz - dm + dcz}{ae - db} \times ab - gb \times \frac{an - afz + gbdm - gbdcz}{ae - db} = \frac{ap - gm - akz + gcz}{ae - db} \times ab - gb \times \frac{ap - akz + gbdm - gbdcz}{ae - db}.$$

Take $gbdm - gbdcz$ from both sides, and divide by a , so shall

$$\frac{an - dm - afz + dcz}{ap - gm - akz + gcz} \times b - gbn + gbfz = \frac{ap - gm - akz + gcz}{e - dbp + dbkz}.$$

Transpose and divide, so shall you find

$z = \frac{aep - abn + dhm - dbp + gen - gem}{ack - ahf + dbc - dbk + gbf - gec}$. The values of x and y are found after the same manner, and have the same denominator. *Ex gr.*

$$y = \frac{afp - akn + dkm - dep + gen - gfm}{ack - ahf + dbc - dbk + gbf - gec}.$$

If any term is wanting in any of the three given equations, the values of z and y will be found more simple. Suppose, for example, that f and k are equal to nothing, then the term fz will vanish in the second equation, and kz in the third, and $z = \frac{aep - anb + dhm - dbp + gbn - gem}{dbc - gec}$;

$$y = \frac{gen - dep}{abc - gec}.$$

If

If four equations are given, involving four unknown quantities, their values may be found much after the same manner, by taking all the products that can be made of four opposite coefficients, and always prefixing contrary signs to those that involve the products of two opposite coefficients.



CHAP. XIII.

Of Quadratic EQUATIONS.

§ 88. **I**N the solution of any question where you have got an equation that involves one unknown quantity, but involves at the same time the square of that quantity, and the product of it multiplied by some known quantity, then you have what is called a *Quadratic equation*; which may be resolved by the following

RULE

1. "Transport all the terms that involve the unknown quantity to one side, and the known terms to the other side of the equation.
2. If the square of the unknown quantity is multiplied by any coefficient, you are to divide all the terms by that coefficient, that the coefficient

of the square of the unknown quantity may be unit.

3. Add to both sides the square of half the coefficient prefixed to the unknown quantity itself, and the side of the equation that involves the unknown quantity will then be a complete square.
4. Extract the square root from both sides of the equation; which you will find, on one side, always to be the unknown quantity with half the foresaid coefficient subjoined to it; so that by transposing this half you may obtain the value of the unknown quantity expressed in known terms." Thus,

$$\text{Suppose } y^2 + ay = b,$$

$$\text{Add the square of } \frac{a}{2} \left\{ \begin{array}{l} y^2 + ay + \frac{a^2}{4} = b + \frac{a^2}{4}, \\ \text{to both sides} \end{array} \right.$$

$$\text{Extract the root, } y + \frac{a}{2} = \pm \sqrt{b + \frac{a^2}{4}},$$

$$\text{Transpose } \frac{a}{2}; y = \pm \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}.$$

§ 89. The square root of any quantity, as $\pm aa$, may be $\pm a$, or $-a$; and hence, "All quadratic equations admit of two solutions." In the last example, after finding that $y^2 + ay + \frac{a^2}{4} = b + \frac{a^2}{4}$, it may be inferred that $y + \frac{a}{2} = \pm \sqrt{b + \frac{a^2}{4}}$, or to $-\sqrt{b + \frac{a^2}{4}}$; since

$-\sqrt{b + \frac{a^2}{4}} \times -\sqrt{b + \frac{a^2}{4}}$ gives $b + \frac{a^2}{4}$, as
 well as $+\sqrt{b + \frac{a^2}{4}} \times +\sqrt{b + \frac{a^2}{4}}$. There
 are therefore two values of y ; the one gives
 $y = +\sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$, the other
 $y = -\sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$.

§ 90. Since the squares of all quantities are
 positive, it is plain that "The square root of a
 negative quantity is imaginary, and cannot be
 assigned." Therefore there are some quadratic
 equations that cannot have any solution. For
 example,

$$\text{Suppose } y^2 - ay + 3a^2 = 0,$$

$$\text{then } y^2 - ay = -3a^2;$$

$$\text{add } \frac{a^2}{4} \text{ to both, } y^2 - ay + \frac{a^2}{4} = -3a^2 + \frac{a^2}{4} = -\frac{11a^2}{4},$$

$$\text{extract the root, } y - \frac{a}{2} = \pm \sqrt{-\frac{11a^2}{4}},$$

$$\text{and } y = \frac{a}{2} \pm \sqrt{-\frac{11a^2}{4}};$$

whence the two values of y must be imaginary or
 impossible, because the root of $-\frac{11a^2}{4}$ cannot
 possibly be assigned.

But of this we shall treat more fully in the
 Second Part.

Suppose that the quadratic equation proposed to be resolved is $y^2 - ay = b$;

$$\text{then } y^2 - ay + \frac{a^2}{4} = b + \frac{a^2}{4},$$

$$y - \frac{a}{2} = \pm \sqrt{b + \frac{a^2}{4}},$$

$$\text{and } y = \frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

If the square root of $b + \frac{a^2}{4}$ cannot be extracted exactly, you must, in order to determine the value of y , nearly approximate to the value of $\sqrt{b + \frac{a^2}{4}}$, by the Rules in *Chap. 8*. The following examples will illustrate the Rule for quadratic equations.

EXAMPLE I.

To find that number, which if you multiply by 8, the product shall be equal to the square of the same number, having 12 added to it.

Call the number y ; then

$$y^2 + 12 = 8y,$$

$$\text{transp. } y^2 - 8y = -12,$$

$$\text{Add the sq. of 4, } y^2 - 8y + 16 = -12 + 16 = 4,$$

$$\text{extract the root } y - 4 = \pm 2,$$

$$\text{transp. } y = 4 \pm 2 = 6, \text{ or } 2.$$

EXAMPLE

EXAMPLE II.

To find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product shall give 21.

Call it y ; then

$$10 - y \times y = 21,$$

that is, $10y - yy = 21;$

transp. $y^2 - 10y = -21,$

add the sq. of 5, $y^2 - 10y + 25 = -21 + 25 = 4,$

extract $y - 5 = \pm \sqrt{4} = \pm 2,$

and $y = 5 \pm 2 = 7, \text{ or } 3.$

EXAMPLE III.

The sum of two quantities is a , their product b . Qu. What are the quantities?

Suppose $\begin{cases} x + y = a \dots \text{ and } x = a - y, \\ xy = b \dots \dots \text{ and } x = \frac{b}{y}, \end{cases}$

therefore $a - y = \frac{b}{y},$

and $ay - y^2 = b;$

transp. $y^2 - ay = -b,$

add $\frac{a^2}{4} \dots y^2 - ay + \frac{a^2}{4} = -b + \frac{a^2}{4},$

extract $\sqrt{}, y - \frac{a}{2} = \pm \sqrt{-b + \frac{a^2}{4}},$

and $y = \frac{a}{2} \pm \sqrt{-b + \frac{a^2}{4}},$

$x (= a - y) = \frac{a}{2} \mp \sqrt{-b + \frac{a^2}{4}}.$

EXAMPLE IV.

The sum of two quantities is a , and the sum of their squares b . Qu. the quantities?

$$\text{Suppose } \begin{cases} x + y = a \dots \text{ and } x = a - y, \\ x^2 + y^2 = b \dots \dots \dots x^2 = b - y^2, \end{cases}$$

$$\text{invol. } x^2 = a^2 - 2ay + y^2,$$

$$\text{whence } a^2 - 2ay + y^2 = b - y^2;$$

$$\text{transp. } \begin{cases} 2y^2 - 2ay = b - a^2, \\ \text{and di-} \left\{ \begin{array}{l} y^2 - ay = \frac{b - a^2}{2}, \\ \text{vide} \end{array} \right.$$

$$\text{add } \frac{a^2}{4}, y^2 - ay + \frac{a^2}{4} = \frac{b - a^2}{2} + \frac{a^2}{4} = \frac{2b - a^2}{4},$$

$$\text{extr. } \sqrt{}, y - \frac{a}{2} = \pm \sqrt{\frac{2b - a^2}{4}}; \text{ and } y = \frac{a}{2}$$

$$\pm \sqrt{\frac{2b - a^2}{4}}; x (= a - y) = \frac{a}{2} \mp \sqrt{\frac{2b - a^2}{4}}.$$

$$\text{Or thus, } y = \frac{a \pm \sqrt{2b - a^2}}{2}, \text{ and } x = \frac{a \mp \sqrt{2b - a^2}}{2}.$$

EXAMPLE V.

A company dining together in an inn, find their bill amounts to 175 shillings; two of them were not allowed to pay, and the rest found that their shares amounted to 10 s. a man more than if all had paid. Qu. How many were in company?

Suppose their number x ; then if all had paid, each man's share would have been $\frac{175}{x}$, seeing $x - 2$ is the number of those that pay. It is therefore, by the question,

$$\frac{175}{x-2} - \frac{175}{x} = 10,$$

and $175x - 175x + 350 = 10x^2 - 20x$;
that is, $10x^2 - 20x = 350$,

and $x^2 - 2x = 35$;

add $1 \dots x - 2x + 1 = 35 + 1 = 36$,

extr. $\sqrt{\dots} x - 1 = \pm 6$,

$x = 1 \pm 6 = 7$, or -5 .

It is obvious that the positive value 7 gives the solution of the question; the negative value -5 being, in the present case, useless.

EXAMPLE VI.

There are three numbers in continual geometrical proportion; the sum of the first and second is 10, and the difference of the second and 3d is 24. Qu. the numbers?

Let the first be x , and the second will be $10 - x$, and the third, $34 - x$; therefore,

$$x : 10 - x :: 10 - x : 34 - x,$$

and $34x - x^2 = 100 - 20x + x^2$;

transp. $\left\{ \begin{array}{l} 54x = 100 + 2x^2, \\ \text{and divid. } \left\{ \begin{array}{l} x^2 - 27x = -50, \end{array} \right. \end{array} \right.$

and divid. $\left\{ \begin{array}{l} x^2 - 27x = -50, \end{array} \right.$

add $\frac{27}{2} \times \frac{27}{2} \dots x^2 - 27x + \frac{729}{4} = \frac{729}{4} - 50 = \frac{529}{4}$,

extract $\sqrt{\dots} x - \frac{27}{2} = \pm \sqrt{\frac{529}{4}} = \pm \frac{23}{2}$,

and $x = \frac{27+23}{2}$, or $= \frac{27-23}{2} = 25$, or 2.

So

So the three continued proportionals are

$$2 : 8 : 32, \text{ or}$$

$$25 : -15 : 9.$$

§ 91. Any equation of this form $yz^m + ay^m = b$, where the greatest index of the unknown quantity y is double to the index of y in the other term, may be reduced to a quadratic $z^2 + az = b$, by putting $y^m = z$, and consequently $y^{2m} = z^2$. And this quadratic resolved as above, gives

$$z = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

$$\text{And seeing } y^m = z = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}},$$

$$y = \sqrt[m]{-\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}}.$$

EXAMPLE. I.

The product of two quantities is a , and the sum of their squares b . Qu. the quantities?

$$\text{Supp. } \begin{cases} xy = a \dots \text{ or, } x = \frac{a}{y}, & x^2 = \frac{a^2}{y^2} \\ x^2 + y^2 = b \dots x^2 = b - y^2, \end{cases}$$

$$\text{whence } b - y^2 = \frac{a^2}{y^2};$$

$$\text{mult. by } y^2 \dots by^2 - y^4 = a^2,$$

$$\text{transp. } y^4 - by^2 = -a^2.$$

Put now $y^2 = z \dots$ and consequently $y^4 = z^2$, and it is

$$z^2 -$$

$$z^2 - bz = -a^2;$$

$$\text{add } \frac{b^2}{4}, z^2 - bz + \frac{b^2}{4} = \frac{b^2}{4} - a^2,$$

$$\text{ext. } \sqrt{\quad}, z - \frac{b}{2} = \pm \sqrt{\frac{b^2}{4} - a^2},$$

$$\text{and } z = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}; \text{ and, seeing } y = \sqrt{z},$$

$$y = \pm \sqrt{\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}}.$$

EXAMPLE II.

To find a number from the cube of which if you subtract 19, and multiply the remainder by that cube, the product shall be 216.

Call the number required x ; and then, by the question,

$$\overline{x^3 - 19} \times x^3 = 216,$$

$$x^6 - 19x^3 = 216.$$

Put $x^3 = z \dots x^6 = z^2$, and it will be

$$z^2 - 19z + \frac{361}{4} = 216 + \frac{361}{4} = \frac{1225}{4},$$

$$\text{and } \sqrt{\quad} \dots z - \frac{19}{2} = \pm \frac{35}{2};$$

$$\text{whence } z = \frac{19 \pm 35}{2} = 27, \text{ or } = -8.$$

$$\text{But } x = \sqrt[3]{z}; \text{ wherefore } x = \pm 3, \text{ or } -2.$$

EXAMPLE III.

To find the value of x , supposing that $x^3 - 7x^{\frac{3}{2}} = 8$.

Put $x^{\frac{3}{2}} = z$, and $x^3 = z^2$;

then $z^2 - 7z = 8$,

$$z^2 - 7z + \frac{49}{4} = \frac{81}{4},$$

$$z - \frac{7}{2} = \pm \frac{9}{2},$$

$$z = 8.$$

But $x^3 = z^2$, and $x = \sqrt[3]{z^2} = \sqrt[3]{64} = 4$.



CHAPTER XIV.

Of SURDS.

§ 92. **I**F a lesser quantity measures a greater so as to leave no remainder, as $2a$ measures $10a$, being found in it five times, it is said to be an *aliquot* part of it, and the greater is said to be a *multiple* of the lesser. The lesser quantity in this case is the *greatest common measure* of the two quantities; for as it measures the greater, so it also measures itself, and no quantity can measure it that is greater than itself.

When a third quantity measures any two proposed quantities, as $2a$ measures $6a$ and $10a$, it is

is said to be a *common measure* of these quantities; and if no greater quantity measure them both, it is called their *greatest common measure*.

Those quantities are said to be *commensurable* which have any common measure; but if there can be no quantity found that measures them both, they are said to be *incommensurable*; and if any one quantity be called *rational*, all others that have any common measure with it, are also called rational: But those that have no common measure with it, are called *irrational* quantities.

§ 93. If any two quantities a and b have any common measure x , this quantity x shall also measure their sum or difference $a \mp b$. Let x be found in a as many times as unit is found in m , so that $a = mx$; and in b , as many times as unit is found in n , so that $b = nx$; then shall $a \mp b = mx \mp nx = \overline{m \mp n} \times x$; so that x shall be found in $a \mp b$, as often as unit is found in $m \mp n$: Now since m and n are integer numbers, $m \mp n$ must be an integer number or unit, and therefore x must measure $a \mp b$.

§ 94. It is also evident, that if x measure any number as a , it must measure any multiple of that number. If it be found in a as many times as unit is found in m , so that $a = mx$, then it will be found in any multiple of a , as na , as many times as unit is found in mn ; for $na = mnx$.

§ 95. If two quantities a and b are proposed, and b measure a by the units that are in m (that is, be found in a as many times as unit is found in m) and there be a remainder c ; and if x be supposed to be a common measure of a and b , it shall be also a measure of c . For by the supposition $a = mb \div c$, since it contains b as many times as there are units in m , and there is c besides of remainder; therefore $a - mb = c$. Now x is supposed to measure a and b , and therefore it measures mb (*Art. 94.*) and consequently $a - mb$ (*Art. 93.*) which is equal to c .

If c measures b by the units in n , and there be a remainder d , so that $b = nc \div d$, and $b - nc = d$, then shall x also measure d ; because it is supposed to measure b , and it has been proved that it measures c , and consequently nc , and $b - nc$ (by *Art. 94.*) which is equal to d . Whence, as after subtracting b as often as possible from a , the remainder c is measured by x ; and after subtracting c as often as possible from b , the remainder d is also measured by x ; so, for the same reason, if you subtract d as often as possible from c , the remainder (if there be any) must still be measured by x : and if you proceed, still subtracting every remainder from the preceding remainder, till you find some remainder which subtracted from the preceding leaves no further remainder, but exactly measures it, this last

last remainder will still be measured by x , any common measure of a and b .

§ 96. The last of these remainders, *viz.* that which exactly measures the preceding remainder, must be a common measure of a and b : suppose that d was this last remainder, and that it measured c by the units in r , then shall $c = rd$, and we shall have these equations,

$$\begin{aligned} a &= mb + c, \\ b &= nc + d, \\ c &= rd. \end{aligned}$$

Now it is plain that since d measures c , it must also measure nc , and therefore must measure $nc + d$, or b . And since it measures b and c , it must measure $mb + c$, or a ; so that it must be a common measure of a and b . But further, it must be their *greatest* common measure; for every common measure of a and b must measure d , by the last article; and the greatest number that measures d , is itself, which therefore is the greatest common measure of a and b .

§ 97. But if, by continually subtracting every remainder from the preceding remainder, you can never find one that measures that which precedes it, exactly, no quantity can be found that will measure both a and b ; and therefore they will be *incommensurable* to each other.

For if there was any common measure of these quantities, as x , it would necessarily measure
all

all the remainders c , d , &c. For it would measure $a - mb$, or c , and consequently $b - nc$, or d ; and so on. Now these remainders decrease in such a manner, that they will necessarily become at length less than x , or any assignable quantity: for c must be less than $\frac{1}{2}a$; because c is less than b , and therefore less than mb , and consequently less than $\frac{1}{2}c + \frac{1}{2}mb$, or $\frac{1}{2}a$. In like manner d must be less than $\frac{1}{2}b$; for d is less than c , and consequently less than $\frac{1}{2}d + \frac{1}{2}nc$, or $\frac{1}{2}b$. The third remainder, in the same manner, must be less than $\frac{1}{2}c$, which is itself less than $\frac{1}{2}a$: thus these remainders decrease so, that every one is less than the half of that which preceded it next but one. Now if from any quantity you take away more than its half, and from the remainder more than its half, and proceed in this manner, you will come at a remainder less than any assignable quantity. It appears therefore that if the remainders c , d , &c. never end, they will become less than any assignable quantity, as x , which therefore cannot possibly measure them, and therefore cannot be a common measure of a and b .

§ 98. In the same way, the greatest common measure of two numbers is discovered. Unit is a common measure of all integer numbers, and two numbers are said to be *prime* to each other, when they have no greater common measure than unit; such as 9 and 25. Such always are

are the least numbers that can be assumed in any given proportion; for if these had any common measure, then the quotients that would arise by dividing them by that common measure would be in the same proportion, and being less than the numbers themselves, these numbers would not be the least in the same proportion; against the supposition.

§ 99. The least numbers in any proportion always measure any other numbers that are in the same proportion. Suppose a and b to be the least of all integer numbers in the same proportion, and that c and d are other numbers in that proportion, then will a measure c , and b measure d .

For if a and b are not aliquot parts of c and d , then they must contain the same number of the same kind of parts of c and d , and therefore dividing a into parts of c , and b into an equal number of like parts of d , and calling one of the first m , and one of the latter n ; then as m is to n , so will the sum of all the m 's be to the sum of all the n 's; that is, $m : n :: a : b$; therefore a and b will not be the least in the same proportion; against the supposition. Therefore a and b must be aliquot parts of c and d . Hence we see that numbers which are prime to each other are the least in the same proportion; for if there were others in the same proportion less than them, these would measure them by the

same number, which therefore would be their common measure against the supposition, for we supposed them to be prime to each other.

§ 100. If two numbers a and b are prime to one another, and a third number c measures one of them a , it will be prime to the other b . For if c and b were not prime to each other, they would have a common measure, which because it would measure c , would also measure a , which is measured by c , therefore a and b would have a common measure, against the supposition.

§ 101. If two numbers a and b are prime to c , then shall their product ab be also prime to c : For if you suppose them to have any common measure as d , and suppose that d measures ab by the units in e , so that $de = ab$, then shall $d : a :: b : e$. But since d measures c , and c is supposed to be prime to a , it follows (by *Art.* 100.) that d and a are prime to each other; and therefore (by *Art.* 99.) d must measure b ; and yet since d is supposed to measure c , which is prime to b , it follows that d is also prime to b : that is, d is prime to a number which it measures, which is absurd.

§ 102. It follows from the last article, that if a and c are prime to each other, then a^2 will be prime to c : For by supposing that a is equal to b , then ab will be equal to a^2 ; and consequently a^2 will be prime to c . In the same manner c^2 will be prime to a .

§ 103.

§ 103. If two numbers a and b , are both prime to other two c, d , then shall the product ab be prime to the product cd ; for (by *Art.* 101.) ab will be prime to c and also to d , and therefore, by the same article, cd will be prime to ab .

§ 104. From this it follows, that if a and c are prime to each other, then shall a^2 be prime to c^2 , by supposing, in the last, that $a = b$, and $c = d$. It is also evident that a^3 will be prime to c^3 , and in general any power of a to any power of c whatsoever.

§ 105. Any two numbers, a and b , being given, to find the least numbers that are in the same proportion with them, *divide them by their greatest common measure x* , and *the quotients c and d shall be the least numbers in the same proportion with a and b* .

For if there could be any other numbers in that proportion less than c and d , suppose them to be e and f , and these being in the same proportion as a and b would measure them: And the number by which they would measure them, would be greater than x , because e and f are supposed less than c and d ; so that x would not be the greatest common measure of a and b ; against the supposition.

§ 106. Let it be required to find the least number that any two given numbers as a and b can measure. First, *if they are prime to each*

other, then their product ab is the least number which they can both measure.

For if they could measure a less number than ab as c , suppose that c is equal to ma , and to nb ; and since c is less than ab , therefore ma will be less than ab , and m less than b ; and nb being less than ab , it follows that n must be less than a ; but since $ma = nb$, and consequently $a : b :: n : m$, and a and b are prime to each other, it would follow that a would measure n , and b measure m ; that is, a greater number would measure a less, which is absurd.

But if the numbers a and b are not prime to each other, and their greatest common measure is x , which measures a by the units in m , and measures b by the units in n , so that $a = mx$, and $b = nx$; then shall an (which is equal to bm , because $a : b :: mx : nx :: m : n$, and therefore $an = bm$) be the least number that a and b can both measure. For if they could measure any number c less than na , so that $c = la = kb$, then $a : b :: m : n :: k : l$; and because x is supposed to be the greatest common measure of a and b , it follows that m and n are the least of all numbers in the same proportion, and therefore m measures k , and n measures l . But as c is supposed to be less than na , that is, la less than na , therefore l is less than n , so that a greater would measure a lesser, which is absurd. Therefore a and b cannot measure any number less than

na ;

na ; which they both measure, because $na = mb$.

It follows from this reasoning, that if a and b measure any quantity c , the least quantity na , which is measured by a and b , will also measure c . For if you suppose as before that $c = la$, you will find that n must measure l , and na must measure la or c .

§ 107. Let a express any integer number, and $\frac{m}{n}$ any fraction reduced to its lowest terms, so that m and n may be prime to each other, and consequently $an + m$ also prime to n , it will follow that $\frac{an + m}{n^2}$ will be prime to n^2 , and consequently $\frac{an + m}{n^2}$ will be a fraction in its least terms, and can never be equal to an integer number. Therefore the square of the mixt number $a + \frac{m}{n}$ is still a mixt number, and never an integer. In the same manner the cube, biquadrate, or any power of a mixt number, is still a mixt number, and never an integer. It follows from this, that *the square root of an integer must be an integer or an incommensurable*. Suppose that the integer proposed is B , and that the square root of it is less than $a + 1$, but greater than a , then it must be an incommensurable; for if it is a commensurable, let it be $a + \frac{m}{n}$, where $\frac{m}{n}$ represents any fraction reduced

to its least terms; it would follow that $a + \frac{m}{n}$ squared would give an integer number B, the contrary of which we have demonstrated.

§ 108 It follows from the last article, that *the square roots of all numbers but of 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, &c.* (which are the squares of the integer numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c.) *are incommensurables*; after the same manner, *the cube roots of all numbers but of the cubes of 1, 2, 3, 4, 5, 6, 7, 8, 9, &c.* are *incommensurables*: and quantities that are to one another in the proportion of such numbers must also have their square roots or cube roots incommensurable.

§ 109. The roots of such numbers being incommensurable are expressed therefore by placing the proper radical sign over them; thus, $\sqrt[2]{2}$, $\sqrt[2]{3}$, $\sqrt[2]{5}$, $\sqrt[2]{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, &c. express numbers incommensurable with unit. These numbers, though they are incommensurable themselves with unit, are *commensurable in power* with it, because their powers are integers, that is, multiples of unit. They may also be commensurable sometimes with one another, as the $\sqrt[2]{8}$, and the $\sqrt{2}$, because they are to one another as 2 to 1: And when they have a common measure, as $\sqrt[2]{2}$ is the common measure of

of both, then their ratio is reduced to an expression in the least terms, as that of commensurable quantities, by dividing them by their greatest common measure. This common measure is found as in commensurable quantities, only the root of the common measure is to be made their common divisor. Thus $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2$, and $\frac{\sqrt{18a}}{\sqrt{2}} = 3\sqrt{a}$.

§ 110. A rational quantity may be reduced to the form of any given surd, by raising the quantity to the power that is denominated by the name of the surd, and then setting the radical sign over it thus, $a = \sqrt{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4} = \sqrt[5]{a^5} = \sqrt[n]{a^n}$, and $4 = \sqrt{16} = \sqrt[3]{64} = \sqrt[4]{256} = \sqrt[5]{1024} = \sqrt[n]{4^n}$.

§ 111. As surds may be considered as powers with fractional exponents, they are reduced to others of the same value that shall have the same radical sign, by reducing these fractional exponents to fractions having the same value and a common denominator. Thus $\sqrt[n]{a} = a^{\frac{1}{n}}$, and $\sqrt[m]{a} = a^{\frac{1}{m}}$, and $\frac{1}{n} = \frac{m}{nm}$, $\frac{1}{m} = \frac{n}{nm}$, and therefore $\sqrt[n]{a}$ and $\sqrt[m]{a}$, reduced to the same radical sign,

figs, become $\sqrt[nm]{a^m}$ and $\sqrt[nm]{a^n}$. If you are to reduce $\sqrt[2]{3}$ and $\sqrt[3]{2}$ to the same denominator, consider $\sqrt[2]{3}$ as equal to $3^{\frac{1}{2}}$, the $\sqrt[3]{2}$ as equal to $2^{\frac{1}{3}}$, whose indices reduced to a common denominator, you have $3^{\frac{1}{2}} = 3^{\frac{3}{6}}$ and $2^{\frac{1}{3}} = 2^{\frac{2}{6}}$, and consequently $\sqrt[2]{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$, and $\sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$; so that the proposed surds $\sqrt[2]{3}$ and $\sqrt[3]{2}$ are reduced to other equal surds $\sqrt[6]{27}$ and $\sqrt[6]{4}$, having a common radical sign.

§ 112. *Surds of the same rational quantity are multiplied by adding their exponents, and divided by subtracting them.*

$$\begin{aligned} \text{Thus } \sqrt[2]{a} \times \sqrt[3]{a} &= a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{3+2}{6}} = a^{\frac{5}{6}} = \sqrt[6]{a^5}; \\ \text{and } \frac{\sqrt[3]{a}}{\sqrt[5]{a}} &= \frac{a^{\frac{1}{3}}}{a^{\frac{1}{5}}} = a^{\frac{1}{3} - \frac{1}{5}} = a^{\frac{5-3}{15}} = a^{\frac{2}{15}} = \sqrt[15]{a^2}; \\ \sqrt[m]{a} \times \sqrt[n]{a} &= a^{\frac{m+n}{mn}}; \quad \frac{\sqrt[m]{a}}{\sqrt[n]{a}} = a^{\frac{n-m}{mn}}; \quad \sqrt[2]{2} \times \sqrt[3]{2} = \\ \sqrt[6]{2^5} &= \sqrt[6]{32}; \quad \frac{\sqrt[2]{2}}{\sqrt[3]{2}} = \sqrt[6]{2}. \end{aligned}$$

§ 113. If the surds are of different rational quantities, as $\sqrt[n]{a^2}$ and $\sqrt[n]{b^3}$, and have the same sign, multiply these rational quantities into one another,

another, or divide them by one another, and set the common radical sign over their product or quotient. Thus $\sqrt[n]{a^2} \times \sqrt[n]{b^3} = \sqrt[n]{a^2 b^3}$;

$$\sqrt{2} \times \sqrt{5} = \sqrt{10}; \quad \frac{\sqrt[m]{a^4}}{\sqrt[m]{b^3 a}} = \sqrt[m]{\frac{a^4}{b^3 a}} = \sqrt[m]{\frac{a^3}{b^3}};$$

$$\frac{\sqrt[3]{9}}{\sqrt[3]{24}} = \sqrt[3]{\frac{9}{24}} = \sqrt[3]{\frac{3}{8}} = \frac{1}{2}\sqrt[3]{3}.$$

If the surds have not the same radical sign, reduce them by the 111th Art. to such as shall have the same radical sign, and proceed as before. Thus

$$\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[nm]{a^n b^m}; \quad \frac{\sqrt[m]{a}}{\sqrt[n]{x}} = \sqrt[mn]{\frac{a^n}{x^m}}; \quad \sqrt{2} \times \sqrt[3]{4} =$$

$$2^{\frac{1}{2}} \times 4^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 4^{\frac{2}{6}} = \sqrt[6]{2^3 \times 4^2} = \sqrt[6]{8 \times 16}$$

$$= \sqrt[6]{128}; \quad \frac{\sqrt[3]{4}}{\sqrt{2}} = \frac{4^{\frac{1}{3}}}{2^{\frac{1}{2}}} = \frac{4^{\frac{2}{6}}}{2^{\frac{3}{6}}} = \sqrt[6]{\frac{4^2}{2^3}} = \sqrt[6]{\frac{16}{8}}$$

$= \sqrt[6]{2}$. If the surds have any rational coefficients, their product or quotient must be prefixed.

Thus $2\sqrt{3} \times 5\sqrt{6} = 10\sqrt{18}$.

§ 114. The powers of surds are found as the powers of other quantities, by multiplying their exponents by the index of the power required. Thus

the square of $\sqrt[3]{2}$ is $2^{\frac{1}{3} \times 2} = 2^{\frac{2}{3}} = \sqrt[3]{4}$; the cube

cube of $\sqrt[2]{5} = 5^{\frac{1}{2} \times 3} = 5^{\frac{3}{2}} = \sqrt[2]{125}$. Or you need only, in involving furds, raise the quantity under the radical sign to the power required, continuing the same radical sign; unless the index of that power is equal to the name of the furd, or a multiple of it, and in that case the power of the furd becomes rational. Evolution is performed by dividing the fraction which is the exponent of the furd by the name of the root required.

Thus the square root of $\sqrt[3]{a^4}$ is $\sqrt[3]{a^2}$, or $\sqrt[6]{a^4}$.

§ 115. The furd $\sqrt[m]{a^m x} = a \sqrt[m]{x}$; and in like manner, if a power of any quantity of the same name with the furd divides the quantity under the radical sign without a remainder, as here a^m divides $a^m x$, and 25 the square of 5 divides 75, the quantity under the sign in $\sqrt[2]{75}$, without a remainder, then place the root of that power rationally before the sign, and the quotient under the sign, and thus the furd will be reduced to a more simple expression. Thus $\sqrt[2]{75} = 5\sqrt{3}$; $\sqrt[2]{48} = \sqrt[2]{3 \times 16} = 4\sqrt{3}$; $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = 3\sqrt{3}$.

§ 116. When furds by the last article are reduced to their least expressions, if they have the same irrational part, they are added or subtracted,

tracted, by adding or subtracting their rational coefficients, and prefixing the sum or difference to the common irrational part.

$$\begin{aligned} \text{Thus } \sqrt[2]{75} + \sqrt[2]{48} &= 5\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}; \\ \sqrt[3]{81} + \sqrt[3]{24} &= 3\sqrt[3]{3} + 2\sqrt[3]{3} = 5\sqrt[3]{3}; \quad \sqrt[2]{150} - \\ \sqrt[2]{54} &= 5\sqrt{6} - 3\sqrt{6} = 2\sqrt{6}; \quad \sqrt{a^2x} + \sqrt{b^2x} \\ &= a\sqrt{x} + b\sqrt{x} = \overline{a+b} \times \sqrt{x}. \end{aligned}$$

§ 117. *Compound surds* are such as consist of two or more joined together. The simple surds are commensurable in power, and by being multiplied into themselves give at length rational quantities; yet compound surds multiplied into themselves commonly give still irrational products. But when any compound surd is proposed, there is another compound surd which multiplied into it gives a rational product. Thus $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$ gives $a - b$, and the investigation of that surd which multiplied into the proposed surd will give a rational product, is made easy by the following Theorems.

THEOREM I.

§ 118. Generally, if you multiply $a^m - b^m$ by $a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m} + a^{n-4m}b^{3m}$, &c. continued till the terms be in number equal to $\frac{n}{m}$, the product shall be $a^n - b^n$: for

$$a^n - b^n$$

$$a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m} + a^{n-4m}b^{3m}, \&c. \dots b^{n-m}$$

$$\times a^m - b^m.$$

$$a^n + a^{n-m}b^m + a^{n-2m}b^{2m} + a^{n-3m}b^{3m}, \&c.$$

$$- a^{n-m}b^m - a^{n-2m}b^{2m} - a^{n-3m}b^{3m}, \&c. - b^n$$

$$a^n \quad * \quad * \quad * \quad - b^n$$

THEOREM II.

$a^{n-m} - a^{n-2m}b^m + a^{n-3m}b^{2m} - a^{n-4m}b^{3m}, \&c.$ multiplied by $a^m + b^m$, gives $a^n + b^n$, which is demonstrated as the other. Here the sign of b^n is positive, when $\frac{n}{m}$ is an odd number.

§ 119. When any Binomial surd is proposed, suppose the index of each number equal to m , and let n be the least integer number that is measured by m , then shall $a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m}, \&c.$ give a compound surd, which multiplied into the proposed surd $a^m + b^m$ will give a rational product.

Thus to find the surd which multiplied by $\sqrt[3]{a} - \sqrt[3]{b}$, will give a rational quantity. Here $m = \frac{1}{3}$, and the least number which is measured by $\frac{1}{3}$ is unit; let $n = 1$, then shall $a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m}, \&c. = a^{1-\frac{1}{3}} + a^{1-\frac{2}{3}}b^{\frac{1}{3}} + a^0b^{\frac{2}{3}} = a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} = \sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$, which multiplied by $\sqrt[3]{a} - \sqrt[3]{b}$, gives $a - b$.

To

To find the surd which multiplied by $\sqrt[4]{a^3} + \sqrt[4]{b^3} = a^{\frac{3}{4}} + b^{\frac{3}{4}}$, gives a rational product. Here $m = \frac{3}{4}$ and $n = 3$, and $a^{n-m} - a^{n-2m}b^m + a^{n-3m}b^{2m}$, &c. $= a^{3-\frac{3}{4}} - a^{3-\frac{6}{4}}b^{\frac{3}{4}} + a^{3-\frac{9}{4}}b^{\frac{3}{2}} - a^{3-3}b^{\frac{9}{4}} = a^{\frac{9}{4}} - a^{\frac{6}{4}}b^{\frac{3}{4}} + a^{\frac{3}{4}}b^{\frac{6}{4}} - b^{\frac{9}{4}} = \sqrt[4]{a^9} - \sqrt[4]{a^6b^3} + \sqrt[4]{a^3b^6} - \sqrt[4]{b^9}$.

THEOREM III.

§ 120. Let $a^m \pm b^l$ be multiplied by $a^{n-m} \mp a^{n-2m}b^l + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l} + \&c.$ and the product shall give $a^n \pm b^{\frac{n}{m}l}$: therefore n must be taken the least integer that shall give $\frac{nl}{m}$ also an integer.

Dem. $a^{n-m} \mp a^{n-2m}b^l + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l} \dots$

$$\times a^m \pm b^l \quad [\dots \&c. a^0 b^{\frac{n}{m}l - 1} \times l]$$

$$a^n \mp a^{n-m}b^l + a^{n-2m}b^{2l}, \&c.$$

$$\pm a^{n-m}b^l - a^{n-2m}b^{2l}, \&c. \pm b^{\frac{n}{m}l}$$

$$a^n \quad * \quad * \quad * \quad \pm b^{\frac{n}{m}l}$$

The sign of $b^{\frac{nl}{m}}$ is positive only when $\frac{nl}{m}$ is an odd number, and the binomial proposed is $a^m \mp b^l$.

§ 121. If any binomial surd is proposed whose two numbers have different indices, let these be m and l , and take n equal to the least integer number that is measured by m and by $\frac{m}{l}$; and $a^{n-m} \mp a^{n-2mb^l} + a^{n-3mb^2l} \mp a^{n-4mb^3l}$, &c. shall give a compound surd, which multiplied by the proposed $a^m \pm b^l$ shall give a rational product. Thus $\sqrt[2]{a} - \sqrt[2]{b}$ being given, suppose $m = \frac{1}{2}$, $l = \frac{1}{3}$, and $\frac{m}{l} = \frac{3}{2}$, therefore you have $n = 3$, and $a^{n-m} + a^{n-2mb^l} + a^{n-3mb^2l} + a^{n-4mb^3l} + \&c. = a^{3-\frac{1}{2}} + a^{3-1}b^{\frac{1}{3}} + a^{3-\frac{3}{2}}b^{\frac{2}{3}} + a^{3-2}b + a^{3-\frac{5}{2}}b^{\frac{4}{3}} + a^0b^{\frac{5}{3}} = a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}} = \sqrt{a^5} + a^2 \times \sqrt[3]{b} + \sqrt{a^3} \times \sqrt[3]{b^2} + ab + \sqrt{a} \times \sqrt[3]{b^4} + \sqrt[3]{b^5} = a^2\sqrt{a} + a^2 \times \sqrt[3]{b} + a\sqrt{a} \times \sqrt[3]{b^2} + ab + b\sqrt{a} \times \sqrt[3]{b} + b \times \sqrt[3]{b^2}$, which multiplied by the $\sqrt[2]{a} - \sqrt[2]{b}$, gives $a^n - b^{\frac{nl}{m}} = a^3 - b^2$.

§ 122. By these Theorems any binomial surd whatsoever being given, you may find a surd which multiplied by it shall give a rational product.

Suppose that a binomial surd was to be divided by another, as $\sqrt[2]{20} + \sqrt[2]{12}$, by $\sqrt[2]{5} - \sqrt[2]{3}$; the

the quotient may be expressed by $\frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}}$.

But it may be expressed in a more simple form by multiplying both numerator and denominator by that surd which multiplied into the denominator gives a rational product. Thus $\frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}} =$

$$\frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} = \frac{\sqrt{100} + 2\sqrt{60} + 6}{5 - 3}$$

$$= \frac{16 + 2\sqrt{60}}{2} = 8 + 2\sqrt{15}.$$

§ 123. In general, when any quantity is divided by a binomial surd, as $a^m \pm b^l$, where m and l represent any fractions whatsoever, take n the least integer number that is measured by m and $\frac{m}{l}$, multiply both numerator and denominator by $a^{n-m} + a^{n-2m}b^l + a^{n-3m}b^{2l}$, &c. and the denominator of the product will become rational, and equal to $a^n - b^{\frac{nl}{l}}$; then divide all the members of the numerator by this rational quantity, and the quote arising will be that of the proposed quantity divided by the binomial surd, expressed in its least terms.

Thus $\frac{3}{\sqrt{5} - \sqrt{2}} = \frac{3\sqrt{5} + 3\sqrt{2}}{3} = \sqrt{5} + \sqrt{2};$

$$\frac{\sqrt{6}}{\sqrt{7} - \sqrt{3}} = \frac{\sqrt{42} + \sqrt{18}}{4}; \quad \frac{\sqrt[3]{20}}{\sqrt{4} - \sqrt{2}} = \frac{\sqrt[3]{20}}{\sqrt{4} - \sqrt{2}}$$

$$\times \frac{\sqrt[3]{16} + 2 + \sqrt[3]{4}}{\sqrt[3]{16} + 2 + \sqrt[3]{4}} = \frac{\sqrt[3]{20}}{\sqrt{4} - \sqrt{2}} \times \frac{2\sqrt[3]{2} + 2 + \sqrt[3]{4}}{2\sqrt[3]{2} + 2 + \sqrt[3]{4}}$$

$$= \frac{2\sqrt[3]{40} + 2\sqrt[3]{20} + \sqrt[3]{80}}{2} = 2\sqrt[3]{5} + \sqrt[3]{20} + \sqrt[3]{10}.$$

$$\text{Also } \frac{\sqrt[2]{10}}{\sqrt[2]{2} - \sqrt[3]{3}} = (\text{because } m = \frac{1}{2}, l = \frac{1}{3}, n = 3,$$

$$\text{and } a^n - b^m = 8 - 9 = -1) =$$

$$\frac{4\sqrt[3]{20} + 4\sqrt[3]{10} \times \sqrt[3]{3} + 2\sqrt[3]{20} \times \sqrt[3]{9} + 6\sqrt[3]{10} + 3\sqrt[3]{20} \times \sqrt[3]{3} + 3\sqrt[3]{10} \times \sqrt[3]{9}}{-1}$$

$$= -8\sqrt[3]{5} - 4\sqrt[3]{10} \times \sqrt[3]{3} - 8\sqrt[3]{5} \times \sqrt[3]{9} - 6\sqrt[3]{10} - 6 \times \sqrt[3]{5} \times \sqrt[3]{3} - 3\sqrt[3]{10} \times \sqrt[3]{9}.$$

§ 124. When the square root of a furd is required, it may be found nearly by extracting the root of a rational quantity that approximates to its value. Thus to find the square root of $3 + 2\sqrt{2}$, we first calculate $\sqrt{2} = 1$, 41421, and therefore $3 + 2\sqrt{2} = 5$, 82842, whose root is found to be nearly 2, 41421: so that $\sqrt[2]{3 + 2\sqrt{2}}$ is nearly 2, 41421. But sometimes we may be able to express the roots of furds exactly by other furds; as in this example the square root of $3 + 2\sqrt{2}$ is $1 + \sqrt{2}$, for $1 + \sqrt{2} \times 1 + \sqrt{2} = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$.

In order to know when and how this may be found, let us suppose that $x + y$ is a binomial furd, whose square will be $x^2 + y^2 + 2xy$: If x and y are quadratic furds, then $x^2 + y^2$ will be rational, and $2xy$ irrational; so that $2xy$ shall

shall always be less than $x^2 + y^2$, because the difference is $x^2 + y^2 - 2xy = \overline{x - y}^2$, which is always positive. Suppose that a proposed surd consisting of a rational part A, and an irrational part B, coincides with this, then $x^2 + y^2 = A$ and $xy = \frac{1}{2}B$: Therefore by what was said of Equations, Chap. 13th,

$$y^2 = A - x^2 = \frac{B^2}{4x^2}, \text{ and therefore}$$

$$Ax^2 - x^4 = \frac{B^2}{4}, \text{ and } x^4 - Ax^2 + \frac{B^2}{4} = 0;$$

from whence we have $x^2 = \frac{A + \sqrt{A^2 - B^2}}{2}$ and

$$y^2 = \frac{A - \sqrt{A^2 - B^2}}{2}. \text{ Therefore when a quantity}$$

partly rational and partly irrational is proposed to have its root extracted, call the rational part A, the irrational B, and the square of the greatest

member of the root shall be $\frac{A + \sqrt{A^2 - B^2}}{2}$, and the

square of the lesser part shall be $\frac{A - \sqrt{A^2 - B^2}}{2}$.

And as often as the square root of $A^2 - B^2$ can be extracted, the square root of the proposed binomial surd may be expressed itself as a binomial surd. For example, if $3 + 2\sqrt{2}$ is proposed, then $A = 3$, $B = 2\sqrt{2}$, and $A^2 - B^2 = 9$

$- 8 = 1$. Therefore $x^2 = \frac{A + \sqrt{A^2 - B^2}}{2} = 2$, and

$$y^2 = \frac{A - \sqrt{A^2 - B^2}}{2} = 1. \text{ Theref. } x + y = 1 + \sqrt{2}.$$

I'

To

To find the square root of $-1 + \sqrt{-8}$, suppose $A = -1$, $B = \sqrt{-8}$, so that $A^2 - B^2 = 9$, and $\frac{A + \sqrt{A^2 - B^2}}{2} = \frac{-1 + 3}{2} = 1$, and $\frac{A - \sqrt{A^2 - B^2}}{2} = \frac{-1 - 3}{2} = -2$, therefore the root required is $1 + \sqrt{-2}$.

§ 125. But though x and y are not quadratic surds or roots of integers, if they are the roots of like surds, as if they are equal to $\sqrt{m\sqrt{z}}$ and $\sqrt{n\sqrt{z}}$, where m and n are integers, then $A = \overline{m+n} \times \sqrt{z}$ and $\frac{1}{2}B = \sqrt{mnz}$; $A^2 - B^2 = \overline{m-n}^2 \times z$, and $x^2 = \frac{A + \sqrt{A^2 - B^2}}{2} = \frac{\overline{m+n} \times z + \overline{m-n}^2 \times z}{2} = m\sqrt{z}$, $y^2 = \frac{A - \sqrt{A^2 - B^2}}{2} = n\sqrt{z}$, and $x + y = \sqrt{m\sqrt{z}} + \sqrt{n\sqrt{z}}$. The part A here easily distinguishes itself from B by its being greater.

§ 126. If x and y are equal to $\sqrt{m\sqrt{z}}$ and $\sqrt{n\sqrt{t}}$, then $x^2 + 2xy + y^2 = m\sqrt{z} + n\sqrt{t} + 2\sqrt{mn\sqrt{zt}}$. So that if z or t be not multiples one of the other, or of some number that measures them both by a square number, then will A itself be a binomial.

§ 127. Let $x + y + z$ express any trinomial surd, its square $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ may be supposed equal to $A + B$ as before. But
rather

rather multiply any two radicals as $2xy$ by $2xz$, and divide by the third $2yz$, which gives the quotient $2x^2$ rational, and double the square of the surd x required. The same rule serves when there are four quantities, $\sqrt{x^2 + y^2 + z^2 + s^2 + 2xy + 2xs + 2xz + 2yz + 2ys + 2zs}$, multiply $2xy$ by $2xs$, and the product $4x^2sy$ divided by $2sy$ gives $2x^2$ a rational quotient, half the square of $2x$. In like manner $2xy \times 2yz = 4y^2xz$, which divided by $2xz$ another member gives $2y^2$, a rational quote, the half of the square of $2y$. In the same manner z and s may be found; and their sum $x + y + z + s$, the square root of the septinomial $\sqrt{x^2 + y^2 + z^2 + s^2 + 2xy + 2xs + 2xz + 2yz + 2ys}$, discovered.

For example, to find the square root of $10 + \sqrt{24} + \sqrt{40} + \sqrt{60}$; I try $\frac{\sqrt{24} \times \sqrt{40}}{\sqrt{60}}$, which I find to be $\sqrt{16} = 4$, the half of the square root of the double of which, viz. $\frac{1}{2} \times \sqrt{8} = \sqrt{2}$, is one member of the square root required; next $\frac{\sqrt{24} \times \sqrt{60}}{\sqrt{40}} = 6$, the half of the square root of the double of which is $\sqrt{3}$, another member of the root required; lastly, $\frac{\sqrt{40} \times \sqrt{60}}{\sqrt{24}} = 10$, which gives $\sqrt{5}$ for the third member of the root required: From which we conclude that the square root of $10 + \sqrt{24} + \sqrt{40} + 60$ is $\sqrt{2} + \sqrt{3} + \sqrt{5}$; and trying you find it succeeds, since

multiplied by itself it gives the proposed quadrinomial.

§ 127. For extracting the higher roots of a binomial, whose two members being squared are commensurable numbers, there is the following

R U L E.

* “ Let the quantity be $A \pm B$, whereof A is the greater part, and c the exponent of the root required. Seek the least number n whose power n is divisible by $AA - BB$, the quotient being Q . Compute $\sqrt[A+B]{\times \sqrt{Q}}$ in the nearest integer number, which suppose to be r . Divide $A\sqrt{Q}$ by its greatest rational divisor, and let the quotient be s , and let $\frac{r + \frac{n}{r}}{2s}$, in the nearest integer number, be t , so shall the root required be $\frac{ts \pm \sqrt{t^2 s^2 - n}}{2c}$, if the c root of $A \pm B$ can be extracted.

E X A M P L E I.

Thus to find the cube root of $\sqrt{968} \pm 25$, we have $A^2 - B^2 = 343$, whose divisors are 7, 7, 7, whence $n = 7$, and $Q = 1$. Further, $\sqrt[A+B]{\times \sqrt{Q}}$, that is, $\sqrt{968} \pm 25$ is a little more

* Arithm. Universal. p. 59.

than

than 56, whose nearest cube root is 4. Wherefore $r=4$. Again, dividing $\sqrt{968}$ by its greatest rational divisor, we have $A\sqrt{Q}=22\sqrt{2}$, and the radical part $\sqrt{2}=s$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{5}{2\sqrt{2}}$, in the nearest integers, is $2=t$. And lastly, $ts=2\sqrt{2}$, $\sqrt{t^2s^2-n}=1$, and $\sqrt[2c]{Q}=\sqrt[6]{1}=1$. Whence $2\sqrt{2}+1$ is the root, whose cube, upon trial, I find to be $\sqrt{968}+25$.

EXAMPLE II.

To find the cube root of $68-\sqrt{4374}$; we have $A^2-B^2=250$, whose divisors are 5, 5, 5, 2. Thence $n=5 \times 2=10$, and $Q=4$, and $\sqrt[A+B]{\sqrt{Q}}$, or $\sqrt[3]{68+\sqrt{4374} \times 2}$ is nearly $7=r$; again, $A\sqrt{Q}$, or $68 \times \sqrt{4}=136 \times \sqrt{1}$, that is, $s=1$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{7+\frac{10}{7}}{2}$, is nearly $=4=t$. Therefore $ts=4$, $\sqrt{t^2s^2-n}=\sqrt{6}$, and $\sqrt[2c]{Q}=\sqrt[6]{4}=\sqrt[3]{2}$, whence the root to be tried is $\frac{4-\sqrt{6}}{\sqrt[3]{2}}$.

EXAMPLE III.

Suppose the fifth root of $29\sqrt{6}+41\sqrt{3}$ is demanded, $A^2-B^2=3$, and $n=3$; $Q=81$,
 $I \ 3$ $r=5$,

$r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{t^2s^2-n}=\sqrt{3}$,
 and $\sqrt[2c]{Q} = \sqrt[10]{81} = \sqrt[5]{9}$. And therefore trial is
 to be made with $\frac{\sqrt{6+\sqrt{3}}}{\sqrt[5]{9}}$.

In these operations, if the quantity is a frac-
 tion, or if its parts have a common divisor, you
 are to extract the root of the *numerator* and *de-*
nominator, or of the *factors* separately. Thus
 to extract the cube root of $\sqrt{242} - 12\frac{1}{2}$,
 this reduced to a common denominator is
 $\frac{\sqrt{968} - 25}{2}$. And the roots of the numerator
 and denominator, separately found, give the
 root $\frac{2\sqrt[2]{2} - 1}{\sqrt[3]{2}}$. And if you seek any root of
 $\sqrt[3]{3993} + \sqrt[6]{17578125}$, divide its parts by the
 common divisor $\sqrt[3]{3}$, and the quotient being
 $11 + \sqrt{125}$, the root of the quantity proposed
 will be found by taking the roots of $\sqrt[3]{3}$ and
 of $11 + \sqrt{125}$, and multiplying them into each
 other.

§ 128. The ground of this Rule may be ex-
 plained from the following

T H E O R E M.

*Let the sum or difference of two quantities x
 and y be raised to a power whose exponent is c ,
 and*

and let the 1st, 3d, 5th, 7th, &c. terms of that power, collected into one sum, be called A, and the rest of the terms, in the even places, call B; the difference of the squares of A and B shall be equal to the difference of the squares of x and y raised to the same power c .

For the terms in the c power of $x \pm y$ (writing for their coefficients, respectively, 1, c , d , e , &c.) are

$$x^c \pm cx^{c-1}y \pm dx^{c-2}y^2 \pm ex^{c-3}y^3 \pm, \&c. = A \pm B,$$

and the same power of $x - y$ (changing the signs in the even places) is

$$x^c - cx^{c-1}y \mp dx^{c-2}y^2 - ex^{c-3}y^3 \pm, \&c. = A - B,$$

and therefore $\overline{x \pm y}^c \times \overline{x - y}^c = \overline{A \pm B} \times \overline{A - B}$

$$= A^2 - B^2 \quad (= \overline{x \pm y \times x - y})^c = \overline{x^2 - y^2}^c.$$

Q. E. D.

Let one, or both, of the quantities x , y , be a quadratic surd, that is, let $x \pm y$, the c root of the proposed binomial $A \pm B$, belong to one of these forms, $p \pm \sqrt{q}$, $k\sqrt{p \pm q}$, or $k\sqrt{p \pm \sqrt{q}}$. And it follows,

1. If $x \pm y = p \pm \sqrt{q}$, that, c being any whole number, A, the sum of the *odd* terms, will be a rational number; and B, the sum of the terms in the *even* places, each of which involves an *odd* power of y will be a rational number multiplied into the quadratic surd \sqrt{q} .

2. Let c , the exponent of the root sought, be an odd number, as we may always suppose

it, because if it is even, it may be halved by the extraction of the square root, till it becomes odd; and let $x + y = k\sqrt{p} + q$. Then A will involve the surd \sqrt{p} , and B will be rational.

3. But if both members of the root are irrational ($x + y = k\sqrt{p} + l\sqrt{q}$) A and B are both irrational, the one involving \sqrt{p} , and the other the surd \sqrt{q} .

And in all these cases, it is easily seen that when x is greater than y , A will be greater than B.

§ 129. From this composition of the binomial $A + B$, we are led to its resolution, as in the foregoing rule, by these steps.

I.

When A is *rational*, and $A^2 - B^2$ is a perfect c power.

1. By the *Theorem*, $A^2 - B^2 = \overline{x^2 - y^2}^c$ accurately; and therefore extracting the c root of $A^2 - B^2$ it will be $x^2 - y^2$. Call this root n .

2. Extract in the nearest integer, the c root of $A + B$, it will be (*nearly*) $x + y$. Which put $= r$.

3. Divide $x^2 - y^2 (= n)$ by $x + y (= r)$ the quotient is (*nearly*) $x - y$; and the sum of the divisor and quotient is (*more nearly*) $2x$; that is, if an integer value of x is to be found, it will

be the nearest to $\frac{r + \frac{n}{r}}{2}$.

$$4. x^2 - \sqrt{x^2 - y^2} = y^2; \text{ OR, } \frac{\sqrt{r + \frac{n}{r}}}{2} - n = y^2:$$

whence $y = \sqrt{\frac{\sqrt{r + \frac{n}{r}}}{2} - n}$, and therefore, putting

$t = \frac{r + \frac{n}{r}}{2}$, the root sought $x + y = t + \sqrt{t^2 - n}$; the same expression as in the rule, when $Q = 1$, $s = 1$, that is, when $A^2 - B^2$ is a perfect c power, and the greater member A is rational.

II.

When A is *irrational*, and $Q = 1$.

By the same process, $x = \frac{r + \frac{n}{r}}{2}$ ($= T$) and $y = \sqrt{T^2 - n}$. But seeing A is supposed irrational, and c an odd number, x will be irrational likewise; and they will both involve the same irreducible surd \sqrt{p} , or s , which is found by dividing A by its greatest rational divisor. Write therefore for x or T , its value $t \times s$, and $x + y = ts + \sqrt{t^2 s^2 - n}$.

III.

If the c root of $A^2 - B^2$ cannot be taken, multiply $A^2 - B^2$ by a number Q , such as that the product may be the (least) perfect c power n^c ($= A^2 Q - B^2 Q$.) And now (instead of $A +$

$A \div B$) extract the c root of $\overline{A + B} \times \sqrt{Q}$, which, found as above, will be $ts + \sqrt{t^2s^2 - n}$; and consequently the c root of $A \div B$ will be $ts \div \sqrt{t^2s^2 - n}$, divided by the c root of \sqrt{Q} ; that is, $\frac{ts + \sqrt{t^2s^2 - n}}{\sqrt[2c]{Q}}$.

It is required in the rule that a perfect c power (n^c) be found which shall be a multiple of $A^2 - B^2$ by the whole number Q . To find this power, let the given number $A^2 - B^2$ be represented by the product $a^m b^p d^f$; whose single divisors let be $a, a, a, \dots, b, b, b, \dots, d, f$; and the product of these divisors raised to the power c , which is $a^c b^c d^c f^c$, divided by $a^m b^p d^f$ will give the quotient $a^{c-m} b^{c-p} d^{c-f} f^{c-1} = Q$ a whole number, provided some index, as m or p , be not greater than c . If it is, take, instead of the single divisor a or b , a^2 or b^2 , a^3 or b^3 , &c. till there be no negative index in the quotient; that is, till Q be a whole number.

§ 130. We may add the following remarks.

1. If the residual $A - B$ is given, it is evident from its genesis by involution, that the same rule gives its root $x - y$.

2. The extracting the c root of $A + B$, or of $A + B \times \sqrt{Q}$, in the nearest integer, neglecting the fractional part, will always give $x \div y$ such, that the value of x which results in the operation shall not differ from its true value by unity;

unity ; that is, it shall be the true integer value sought.

For, f being some proper fraction, let $x + y$

$\pm f$ be the accurate value of $\sqrt[A + B]{c} \times \sqrt{Q}$, and let the quotient of $x^2 - y^2$ divided by it be $x - y \mp g$, then, the sum of the divisor and quotient being $2x \pm f \mp g$, if our reckoning the fractional part could make a difference of unity in the value of x , it would follow that $f - g$ or $g - f = 2$. Which is absurd, g , as well as f , being a proper fraction.

3. If both A and B are irrational ; or, if the lesser of the two members is rational, no root denominated by an even number can be found.

4. When the greater member is rational, and the exponent c is an even number, it is ambiguous whether the greater member of the root is rational or furd. And though a root in the form of $p \mp l\sqrt{q}$ is not found, yet a root in the form of $k\sqrt{p \mp q}$, or, that failing, in the form $k\sqrt{p} \mp l\sqrt{q}$ may be obtained.

If we look for a root $k\sqrt{p \mp q}$, we are now to subtract $x - y$ from $x + y$, and half the remainder will give y (or q) the rational part. And to $x^2 - y^2 (= n)$ adding y^2 , the sum will be x^2 .

So that $y = \frac{r - \frac{n}{r}}{2}$, and $x = \sqrt{\frac{r - \frac{n}{r}}{2}} + n$; the expressions being the same as when c is odd, with

with the sign of n changed. If this does not succeed, and a prime number stands under the radical sign, no farther trial need be made.

But if a composite number stands under the radical sign, the root may possibly belong to the form $k\sqrt{p} + l\sqrt{q}$; and that composite number being $p \times q$, since $k^2p - l^2q = n$, and $k\sqrt{p} = x$, the numbers k, l , may be sought for in the nearest integers, and trial made with $k\sqrt{p} + l\sqrt{q}$; as in this

EXAMPLE.

To find the fourth root of $49849 - 2895\sqrt{224}$.

The 4th root of $A^2 - B^2$ is $157 = x^2 - y^2 = n$, and the 4th root of $A - B$, that is, $x - y = r = 9$ nearly: and $\frac{n}{r} = \frac{157}{9} = 17$ nearly. Whence $x = \frac{9 + 17}{2} = 13$. But now the least radical factor in B being $\sqrt{14} = \sqrt{7 \times 2}$, I put $13 (= x) = k\sqrt{7}$, and k in the nearest integer = 5. Again, $k^2p - l^2q = n = 175 - l^2 \times 2 = 157$; that is, $l^2 \times 2 = 18$, and $l = 3$; which gives the root $5\sqrt{7} - 3\sqrt{2}$.

In this manner the even roots may be sought immediately. But to avoid ambiguity and needless trouble, it is better first to depress them by extracting the square root, as in § 124.

A SUPPLEMENT

TO THIS

CHAPTER.

§ 131. **T**HERE occur sometimes, especially in the resolution of *cubic* equations by *Cardan's* Rule (*Part II.* § 79.) binomials of this form $A \pm B\sqrt{-q}$, whose cube roots must be found. To these the foregoing rule cannot be applied throughout, because of the imaginary factor $\sqrt{-q}$. Yet if the root is expressible in rational numbers, the first step of that rule will often lead us to it in a short way, not merely tentative, the trials being confined to known limits.

For it being, universally, $\sqrt{A^2 - B^2} = x^2 - y^2$

and, in the present case, $\sqrt[3]{A^2 + B^2q} (= x^2 - y^2) = p^2 + l^2 \times q$; if we divide the part under the radical sign by its greatest rational divisor, the quote is the imaginary surd $\sqrt{-q}$, and from $\sqrt[3]{A^2 + B^2q}$, subtracting p^2 , the square of some divisor of A , the remainder is $l^2 \times q$, a known multiple of the square of l a divisor of B .

That

That p and l are divisors of A and B respectively is evident; for cubing $p \pm \sqrt[l]{-q}$, you find $A = p \times p^2 - 3l^2q$, $B = l \times 3p^2 - l^2q$. And the signs of p and l must be such as will give the products of $p \times p^2 - 3l^2q$, $l \times 3p^2 - l^2q$ of the same signs as A and B respectively.

E X A M P L E.

To find the cube root of $81 \pm \sqrt{-2700} = 81 \pm 30\sqrt{-3}$.

Here $A = 81$, $B = 30$, $q = 3$; $\sqrt[3]{81 \times 81 + 2700} = 21 = p^2 + l^2q$. Subtracting therefore from 21 , the square of $(p) \pm 3$, which is a divisor of A , there remains $(l^2 \times q =) 2 \times 2 \times 3$. And $(l =) 2$ is a divisor of 30 . Lastly, $A (= p \times p^2 - 3l^2q)$ being positive, and the factor $p^2 - 3l^2q$ negative, p must have the negative sign; and for the like reason $l = + 2$. So that the root is $-3 \pm 2\sqrt{-3}$.

It will be shewn in the Second Part of this Treatise that "every cube or other power has as many roots, real and imaginary, as there are units in the exponent of the power;" particularly, that *unity* itself has the cube roots 1 , $\frac{-1 + \sqrt{-3}}{2}$, and $\frac{-1 - \sqrt{-3}}{2}$. If therefore we would find the other two cube roots, in this example, seeing $z^3 = z^3 \times 1$, and $\sqrt[3]{z^3} \times \sqrt[3]{1} = z$
(z^3)

(z^3 representing any cube whatever, and z any of its roots) we are to multiply $-3 + 2\sqrt{-3}$, the root already found, by $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and by $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, and the products $-\frac{3}{2} - \frac{5}{2}\sqrt{-3}$, and $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ will be the root required.

Or, because the denominator of the imaginary roots of unity is 2, taking $p = \frac{3}{2}$, one half of a divisor of A , we have $21 - \frac{9}{4} = \frac{75}{4} = \frac{25}{4} \times 3 = l^2q$, that is $l = \frac{5}{2}$; and $p^3 - 3l^2q$ as well as $3p^2 - l^2q$ being negative, both p and l must be negative, and the root is $-\frac{3}{2} - \frac{5}{2}\sqrt{-3}$. Again take $p = \frac{9}{2}$, and you shall find $l = \frac{1}{2}$; so the remaining root is $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$, as before.

We may here observe that the operation ought to be abridged, where it can be done, by dividing the given binomial by the greatest cube that it contains; and finding the root of the quotient; which multiplied by the root of the cube by which you divided, will give the root required. Thus, in the foregoing Example, $81 + \sqrt{-2700} = 27 \times 3 + \sqrt{-\frac{1000}{27}}$, and the roots of $3 + \sqrt{-\frac{1000}{27}}$ being now, more easily, found to be $-1 + 2\sqrt{-\frac{1}{3}}$, $-\frac{1}{2} - \frac{5}{2}\sqrt{-\frac{1}{3}}$, and $\frac{3}{2} + \frac{1}{2}\sqrt{-\frac{1}{3}}$, these multiplied by 3, the cube root of 27, give the roots required the same as above.

“ If the coefficient of the imaginary member of the binomial has a contrary sign, the roots will be the same, with the signs of the imaginary

imaginary parts changed." Thus the cube roots of $81 - \sqrt{-2700}$, or $81 - 30\sqrt{-3}$, will be $-3 - 2\sqrt{-3}$, $-\frac{3}{2} + \frac{5}{2}\sqrt{-3}$, and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$.

And therefore $\sqrt[3]{81 + \sqrt{-2700}} + \sqrt[3]{81 - \sqrt{-2700}} = -3 \times 2 = -6$, or $= -\frac{3}{2} \times 2 = -3$, or $= \frac{9}{2} \times 2 = 9$, the imaginary parts vanishing by the contrariety of their signs.

We may observe likewise, that such roots, whether expressible in rational numbers, or not, may be found by evolving the binomial $A + B\sqrt{-q}$ by the *Theorem* in pag. 41, and summing the alternate terms. As, in the foregoing example,

$\sqrt[3]{81 + 30\sqrt{-3}}$, or rather $\sqrt[3]{81} \times \sqrt[3]{1 + \frac{10}{27}\sqrt{-3}}$, being expanded into a series, the sum of the odd terms will continually approach to $4.5 = \frac{9}{2}$, and the sum of the coefficients of the even terms to $\frac{1}{2}$, which is the coefficient of the imaginary part. But for a general and elegant solution, recourse must be had to Mr. *de Moivre's* Appendix to Dr. *Saunderson's* Algebra, and the continuation of it in *Philos. Trans.* N^o 451. What has been explained above may serve, for the present, to give the Learner some notion of the composition and resolution of those cubes; that he need not hereafter be surpris'd to meet with expressions of real quantities which involve imaginary roots.

End of the FIRST PART.

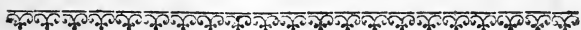


A
T R E A T I S E
O F
A L G E B R A.




P A R T II.

Of the Genesis and Resolution of E-
QUATIONS of all degrees ; and of
the different Affections of the
ROOTS.



C H A P. I.

Of the Genesis and Resolution of Equations
in general ; and the number of roots an
equation of any degree may have.

§ I.  A F T E R the same manner as the
higher powers are produced by the
multiplication of the lower powers
of the same root ; equations of
superior orders are generated by the multiplica-
tion of equations of inferior orders involving the
same unknown quantity. And *an equation of*
K. *any*

any dimensions may be considered as produced by the multiplication of as many simple equations as it has dimensions; or of any other equations whatsoever, if the sum of their dimensions is equal to the dimension of that equation. Thus any cubic equation may be conceived as generated by the multiplication of three simple equations, or of one quadratic and one simple equation. A biquadratic as generated by the multiplication of four simple equations, or of two quadratic equations; or lastly, of one cubic and one simple equation.

§ 2. If the equations which you suppose multiplied by one another are *the same*, then the equation generated will be nothing else but some power of those equations, and the operation is merely *involution*; of which we have treated already: and, when any such equation is given, the simple equation by whose multiplication it is produced is found by *evolution*, or the extraction of a root.

But when the equations that are supposed to be multiplied by each other are *different*, then other equations than powers are generated; which to resolve into the simple equations whence they are generated, is a different operation from involution, and is what is called, *The resolution of equations*.

But as evolution is performed by observing and tracing back the steps of involution; so to discover

discover the rules for the resolution of equations, we must carefully observe their *generation*.

§ 3. Suppose the unknown quantity to be x , and its values in any simple equations to be $a, b, c, d, \&c.$ then those simple equations, by bringing all the terms to one side, become $x - a = 0, x - b = 0, x - c = 0, \&c.$ And, the product of any two of these, as $\overline{x - a} \times \overline{x - b} = 0$ will give a *quadratic* equation, or an equation of two dimensions. The product of any three of them, as $\overline{x - a} \times \overline{x - b} \times \overline{x - c} = 0$ will give a *cubic* equation, or one of three dimensions. The product of any four of them will give a *biquadratic* equation, or one of four dimensions, as $\overline{x - a} \times \overline{x - b} \times \overline{x - c} \times \overline{x - d} = 0$. And, in general, “*In the equation produced, the highest dimension of the unknown quantity will be equal to the number of simple equations that are multiplied by each other.*”

§ 4. When any equation equivalent to this biquadratic $\overline{x - a} \times \overline{x - b} \times \overline{x - c} \times \overline{x - d} = 0$ is proposed to be resolved, the whole difficulty consists in finding the simple equations $\overline{x - a} = 0, \overline{x - b} = 0, \overline{x - c} = 0, \overline{x - d} = 0$, by whose multiplication it is produced; for each of these simple equations gives one of the values of x , and one solution of the proposed equation. For,

if any of the values of x deduced from those simple equations be substituted in the proposed equation, in place of x , then all the terms of that equation will vanish, and the whole be found equal to nothing. Because when it is supposed that $x=a$, or $x=b$, or $x=c$, or $x=d$, then the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$ does vanish, because one of the factors is equal to nothing. There are therefore four suppositions that give $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$ according to the proposed equation; that is, there are four roots of the proposed equation. And after the same manner, "Any other equation admits of as many solutions as there are simple equations multiplied by one another that produce it," or "as many as there are units in the highest dimension of the unknown quantity in the proposed equation."

§ 5. But as there are no other quantities whatsoever besides these four (a, b, c, d) that substituted in the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, in the place of x , will make the product vanish; therefore, the equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$, cannot possibly have more than these four roots, and cannot admit of more solutions than four. If you substitute in that product a quantity neither equal to a , nor b , nor c , nor d , which suppose e , then since neither $e-a$, $e-b$, $e-c$, nor $e-d$ is equal to nothing; their product

product $\overline{e-a} \times \overline{e-b} \times \overline{e-c} \times \overline{e-d}$ cannot be equal to nothing, but must be some real product: and therefore there is no supposition beside one of the foresaid four, that gives a just value of x according to the proposed equation. So that it can have no more than these four roots. And after the same manner it appears, that “*No equation can have more roots than it contains dimensions of the unknown quantity.*”

§ 6. To make all this still plainer by an example, in numbers; suppose the equation to be resolved to be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, and that you discover that this equation is the same with the product of $\overline{x-1} \times \overline{x-2} \times \overline{x-3} \times \overline{x-4}$, then you certainly infer that the four values of x are 1, 2, 3, 4; seeing any of these numbers placed for x makes that product, and consequently $x^4 - 10x^3 + 35x^2 - 50x + 24$, equal to nothing, according to the proposed equation. And it is certain that there can be no other values of x besides these four: since when you substitute any other number for x in those factors $x-1$, $x-2$, $x-3$, $x-4$, none of the factors vanish, and therefore their product cannot be equal to nothing according to the equation.

§ 7. It may be useful sometimes to consider equations as generated from others of an inferior

rior sort beside simple ones. Thus a *cubic* equation may be conceived as generated from the *quadratic* $x^2 - px + q = 0$, and the *simple* equation $x - a = 0$, multiplied by each other; whose product

$$\left. \begin{array}{l} x^3 - px^2 + qx - aq \\ - ax^2 + apx \end{array} \right\} = 0 \text{ may express any cubic}$$

equation whose roots are the quantity (a) the value of x in the simple equation, and the two roots of the quadratic equation, *viz.*

$$\frac{p + \sqrt{p^2 - 4q}}{2} \text{ and } \frac{p - \sqrt{p^2 - 4q}}{2}; \text{ as appears from}$$

Chap. 13. Part I. And, according as these roots are *real* or *impossible*, two of the roots of the cubic equation are *real* or *impossible*.

§ 8. In the doctrine of involution we shewed that “the square of any quantity positive or negative, is always positive,” and therefore “the square root of a negative is impossible or imaginary.” For example, the $\sqrt{a^2}$ is either $+a$ or $-a$; but $\sqrt{-a^2}$ can neither be $+a$ nor $-a$, but must be *imaginary*. Hence is understood that “a quadratic equation may have no impossible expression in its coefficients, and yet, when it is resolved into the simple equations that produce it, they may involve impossible expressions.” Thus the quadratic equation $x^2 + a^2 = 0$ has no impossible coefficient, but the simple equations from which it is produced, *viz.* $x + \sqrt{-a^2} = 0$, and $x - \sqrt{-a^2} = 0$,
both

both involve an imaginary quantity; as the square $-a^2$ is a real quantity, but its square root is imaginary. After the same manner a biquadratic equation, when resolved, may give four simple equations, each of which may give an impossible value for the root: and the same may be said of any equation that can be produced from quadratic equations only; that is, whose dimensions are of the even numbers.

§ 9. But “a cubic equation (which cannot be generated from quadratic equations only, but requires one simple equation besides to produce it) if none of its coefficients are impossible, will have, at least, one real root,” the same with the root of the simple equation whence it is produced. The square of an impossible quantity may be real, as the square of $\sqrt{-a^2}$ is $-a^2$; but “the cube of an impossible quantity is still impossible,” as it still involves the square root of a negative: as, $\sqrt{-a^2} \times \sqrt{-a^2} \times \sqrt{-a^2} = \sqrt{-a^6} = a^3\sqrt{-1}$, is plainly imaginary. From which it appears, that though two simple equations involving impossible expressions, multiplied by one another, may give a product where no impossible expression may appear; yet “if three such simple equations be multiplied by each other, the impossible expression will not disappear in their

product.” And hence it is plain, that though a quadratic equation whose coefficients are all real may have its two roots impossible, yet “ a cubic equation whose coefficients are real cannot have all its three roots impossible.”

§ 10. In general, it appears that the impossible expressions cannot disappear in the equation produced, but when their number is *even*; that there are never in any equations, whose coefficients are real quantities, single impossible roots, or an odd number of impossible roots, but “ that the roots become impossible in pairs;” and that “ an equation of an odd number of dimensions has always one real root.”

§ 11. “ The roots of equations are either *positive* or *negative* according as the roots of the simple equations whence they are produced are positive or negative.” If you suppose $x = -a$, $x = -b$, $x = -c$, $x = -d$, &c. then shall $x + a = 0$, $x + b = 0$, $x + c = 0$, $x + d = 0$; and the equation $\overline{x + a} \times \overline{x + b} \times \overline{x + c} \times \overline{x + d} = 0$ will have its roots, $-a$, $-b$, $-c$, $-d$, &c. negative.

But to know when the roots of equations are positive and when negative, and how many there are of each kind, shall be explained in the next chapter.

C H A P. II.

Of the Signs and Coefficients of
EQUATIONS.

§ 12. **W**HEN any number of simple equations are multiplied by each other, it is obvious that the highest dimension of the unknown quantity in their product is equal to the number of those simple equations; and, the term involving the highest dimension is called the *first* term of the equation generated by this multiplication. The term involving the next dimension of the unknown quantity, less than the greatest by unit, is called the *second* term of the equation; the term involving the next dimension of the unknown quantity, which is less than the greatest by two, the *third* term of the equation, &c. And that term which involves no dimension of the unknown quantity, but is some known quantity, is called the *last* term of the equation.

“*The number of terms is always greater than the highest dimension of the unknown quantity by unit.*” And when any term is wanting, an *asterisk* is marked in its place. The *signs* and *coefficients* of equations will be understood by considering the following TABLE, where the
simple

§ 13. From the inspection of these equations it is plain, that the coefficients of the first term is *unit*.

The coefficient of the second term is *the sum of all the roots* (a, b, c, d, e) *having their signs changed*.

The coefficient of the third term is *the sum of all the products that can be made by multiplying any two of the roots* (a, b, c, d, e) *by one another*.

The coefficient of the fourth term is *the sum of all the products that can be made by multiplying into one another any three of the roots, with their signs changed*. And after the same manner all the other coefficients are formed.

The last term is always *the product of all the roots having their signs changed*, multiplied by one another.

§ 14. Although in the Table such simple equations only are multiplied by one another as have positive roots, it is easy to see, that “the coefficients will be formed according to the same rule when any of the simple equations have negative roots.” And, *in general*, if $x^3 - px^2 + qx - r = 0$ represent any cubic equation, then shall p be the sum of the roots; q the sum of the products made by multiplying any two of them; r the product of all the three: and, if $-p, +q, -r, +s, -t, +u, \&c.$ be the coefficients of the 2d, 3d, 4th, 5th, 6th, 7th, &c.

ℰc. terms of any equation, then shall p be the sum of all the roots, q the sum of the products of any two, r the sum of the products of any three, s the sum of the products of any four, t the sum of the products of any five, u the sum of the products of any six, ℰc.

§ 15. When therefore any equation is proposed to be resolved, it is easy to find the sum of the roots, (for it is equal to the coefficient of the second term having its sign changed :) or, to find the sum of the products that can be made by multiplying any determinate number of them.

But it is also easy “to find the sum of the squares, or of any powers, of the roots.”

The sum of the squares is always $p^2 - 2q$. For calling the sum of the squares B , since the sum of the roots is p ; and “the square of the sum of any quantities is always equal to the sum of their squares added to double the products that can be made by multiplying any two of them,” therefore $p^2 = B + 2q$, and consequently $B = p^2 - 2q$. For example, $\overline{a + b + c}^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$; that is, $p^2 = B + 2q$. And $\overline{a + b + c + d}^2 = a^2 + b^2 + c^2 + d^2 + 2 \times \overline{ab + ac + ad + bc + bd + cd}$, that is again, $p^2 = B + 2q$, or $B = p^2 - 2q$. And so for any other number of quantities. In general therefore, “ B the sum of the squares of the roots

roots may always be found by subtracting $2q$ from p^2 ;” the quantities p and q being always known, since they are the coefficients in the proposed equation.

§ 16. “ The sum of the cubes of the roots of any equation is equal to $p^3 - 3pq + 3r$, or to $Bp - pq + 3r$.” For $B - q \times p$ gives always the excess of the sum of the cubes of any quantities above the triple sum of the products that can be made by multiplying any three of them.

Thus $a^2 + b^2 + c^2 - ab - ac - bc \times a + b + c (= B - q \times p) = a^3 + b^3 + c^3 - 3abc$. Therefore if the sum of the cubes is called C , then shall $B - q \times p = C - 3r$, and $C = Bp - qp + 3r$ (because $B = p^2 - 2q) = p^3 - 3pq + 3r$.

After the same manner, if D be the sum of the 4th powers of the roots, you will find that $D = pC - qB + pr - 4s$: and if E be the sum of the 5th powers, then shall $E = pD - qC + rB - ps + 5t$. And after the same manner the sum of any powers of the roots may be found ; the progression of these expressions of the sum of the powers being obvious.

§ 17. As for the signs of the terms of the equation produced, it appears from inspection that the signs of all the terms in any equation in the Table are alternately $+$ and $-$: these equations are generated by multiplying continually $x - a, x - b, x - c, x - d, \&c.$ by one another.

another. The first term is always some pure power of x , and is positive; the second is a power of x multiplied by the quantities $-a$, $-b$, $-c$, &c. And since these are all negative, that term must therefore be negative. The third term has the products of any two of these quantities ($-a$, $-b$, $-c$, &c.) for its coefficient; which products are all positive, because $- \times -$ gives $+$. For the like reason, the next coefficient, consisting of all the products made by multiplying any three of these quantities, must be negative; and the next positive. So that the coefficients, in this case, will be positive and negative by turns. But "in this case the roots are all positive;" since $x = a$, $x = b$, $x = c$, $x = d$, $x = e$, &c. are the assumed simple equations. It is plain then, that "*when all the roots are positive, the signs are alternately $+$ and $-$.*"

§ 18. But if the roots are all negative, then $x + a \times x + b \times x + c \times x + d$, &c. $= 0$, will express the equation to be produced; all whose terms will plainly be positive; so that "*when all the roots of an equation are negative, it is plain there will be no changes in the signs of the terms of that equation.*"

§ 19. In general, "there are as many positive roots in any equation as there are changes in the signs of the terms from $+$ to $-$, or from $-$ to $+$; and the remaining roots are negative."

negative." The Rule is general, if the impossible roots be allowed to be either positive or negative.

§ 20. In quadratic equations, the two roots are either both positive, as in this

$$(\overline{x-a} \times \overline{x-b} =) x^2 - ax + ab = 0,$$

where there are two changes of the signs: Or they are both negative, as in this

$$(\overline{x+a} \times \overline{x+b} =) x^2 + a } + b \} x + ab = 0,$$

where there is not any change of the signs. Or there is one positive and one negative, as in

$$(\overline{x-a} \times \overline{x+b} =) x^2 - a } + b \} x - ab = 0,$$

where there is necessarily one change of the signs; because the first term is positive, and the last negative, and there can be but one change whether the second term be + or —.

Therefore the rule given in the 19th section extends to all quadratic equations.

§ 21. In cubic equations, the roots may be,

1°. All positive as in this, $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = 0$, in which the signs are alternately + and —, as appears from the Table; and there are three changes of the signs.

2°. The roots may be all negative, as in the equation $\overline{x+a} \times \overline{x+b} \times \overline{x+c} = 0$, where there can be no change of the signs. Or,

3°. There may be two positive roots and one negative, as in the equation $\overline{x-a} \times \overline{x-b} \times \overline{x+c} = 0$; which gives

$$\left. \begin{array}{l} x^3 - a \\ -b \\ +c \end{array} \right\} x^2 - ac \left. \begin{array}{l} +ab \\ -bc \end{array} \right\} x + abc = 0.$$

Here there must be two changes of the signs: because if $a + b$ is greater than c , the second term must be negative, its coefficient being $-a, -b + c$.

And if $a + b$ is less than c , then the third term must be negative, its coefficient $+ab - ac - bc$ ($ab - c \times \overline{a+b}$)* being in that case negative. And there cannot possibly be three changes of the signs, the first and last terms having the same sign.

4°. There may be one positive root and two negative, as in the equation $\overline{x+a} \times \overline{x+b} \times \overline{x-c} = 0$, which gives

$$\left. \begin{array}{l} x^3 + a \\ +b \\ -c \end{array} \right\} x^2 - ac \left. \begin{array}{l} +ab \\ -bc \end{array} \right\} x - abc = 0.$$

Where there must be always one change of the signs, since the first term is positive and the last negative. And there can be but one change of the signs, since if the second term is negative, or $a + b$ less than c , the third must be

* Because the rectangle $a \times b$ is less than the square $\overline{a+b} \times \overline{a+b}$, and therefore much less than $\overline{a+b} \times c$.

negative

negative also, so that there will be but one change of the signs. Or, if the second term is affirmative, whatever the third term is, there will be but one change of the signs. It appears therefore, in general, that in cubic equations, there are as many affirmative roots as there are changes of the signs of the terms of the equation.

The same way of reasoning may be extended to equations of higher dimensions, and the rule delivered in § 19, extended to all kinds of equations.

§ 22. There are several confections of what has been already demonstrated, that are of use in discovering the roots of equations. But before we proceed to that, it will be convenient to explain some transformations of equations, by which they may often be rendered more simple, and the investigation of their roots more easy.



C H A P. III.

Of the Transformation of EQUATIONS; and exterminating their intermediate terms.

§ 23. **W**E now proceed to explain the transformations of equations that are most useful: and first, “*The affirmative roots of an equation are changed into negative roots of the same value, and the negative roots into affirmative, by only changing the signs of the terms alternately, beginning with the second.*” Thus the roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ are $+1, +2, +3, -5$; whereas the roots of the same equation having only the signs of the second and fourth terms changed, *viz.* $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, are $-1, -2, -3, +5$.

To understand the reason of this rule, let us assume an equation, as $\overline{x - a} \times \overline{x - b} \times \overline{x - c} \times \overline{x - d} \times \overline{x - e}$, &c. $= 0$, whose roots are $+a, +b, +c, +d, +e$, &c. and another having its roots of the same value, but affected with contrary signs, as $\overline{x + a} \times \overline{x + b} \times \overline{x + c} \times \overline{x + d} \times \overline{x + e}$, &c. $= 0$. It is plain, that the terms taken alternately, beginning from the first, are the same
in

in both equations, and have the same sign; “being products of an even number of the roots;” the product of any two roots having the same sign as their product when both their signs are changed; as $+a \times -b = -a \times +b$.

But the second terms and all taken alternately from them, because their coefficients involve always the products of an odd number of the roots, will have contrary signs in the two equations. For example, the product of four, *viz.* $abcd$, having the same sign in both, and one equation in the fifth term having $abcd \times +e$, and the other $abcd \times -e$; it follows that their product $abcde$ must have contrary signs in the two equations: these two equations therefore that have the same roots, but with contrary signs, have nothing different but the signs of the alternate terms, beginning with the second. From which it follows, “that if any equation is given; and you change the signs of the alternate terms, beginning with the second, the new equation will have roots of the same value, but with contrary signs.”

§ 24. It is often very useful “to transform an equation into another that shall have its roots greater or less than the roots of the proposed equation by some given difference.”

Let the equation proposed be the cubic $x^3 - px^2 + qx - r = 0$. And let it be required to transform it into another equation whose roots

shall be less than the roots of this equation by some given difference (e), that is, suppose $y = x - e$, and consequently $x = y + e$; then instead of x and its powers, substitute $y + e$ and its powers, and there will arise this new equation,

$$(A) \left. \begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array} \right\} = 0,$$

whose roots are less than the roots of the preceding equation by the difference (e).

If it had been required to find an equation whose roots should be greater than those of the proposed equation by the quantity (e), then we must have supposed $y = x + e$, and consequently $x = y - e$, and then the other equation would have had this form;

$$(B) \left. \begin{array}{l} y^3 - 3ey^2 + 3e^2y - e^3 \\ - py^2 + 2pey - pe^2 \\ + qy - qe \\ - r \end{array} \right\} = 0.$$

If the proposed equation be in this form, $x^3 + px^2 + qx + r = 0$, then by supposing $x + e = y$ there will arise an equation agreeing in all respects with the equation (A), but that the second and fourth terms will have contrary signs.

And

And by supposing $x - e = y$, there will arise an equation agreeing with (B) in all respects, but that the second and fourth terms will have contrary signs to what they have in (B).

The first of these suppositions gives this equation,

$$(C) \left. \begin{array}{l} y^3 - 3ey^2 + 3e^2y - e^3 \\ + py^2 - 2pey + pe^2 \\ + qy - qe \\ + r \end{array} \right\} = 0.$$

The second supposition gives the equation

$$(D) \left. \begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ + py^2 + 2epy + pe^2 \\ + qy + qe \\ + r \end{array} \right\} = 0.$$

§ 25. The first use of this transformation of equations is to shew “how the second (or other intermediate) term may be taken away out of an equation.”

It is plain that in the equation (A) whose second term is $\overline{3e - p} \times y^2$, if you suppose $e = \frac{1}{3}p$, and consequently $3e - p = 0$, then the second term will vanish.

In the equation (C) whose second term is $\overline{-3e + p} \times y^2$, supposing $e = \frac{1}{3}p$, the second term also vanishes.

Now the equation (A) was deduced from $x^3 - px^2 + qx - r = 0$, by supposing $y = x - e$:
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and

and the equation (C) was deduced from $x^3 + px^2 + qx + r = 0$; by supposing $y = x + e$. From which this Rule may easily be deduced for exterminating the second term out of any cubic equation.

R U L E.

§⁶ *Add to the unknown quantity of the given equation the third part of the coefficient of the second term with its proper sign, viz. $\mp \frac{1}{3}p$, and suppose this aggregate equal to a new unknown quantity (y). From this value of y find a value of x by transposition, and substitute this value of x and its powers in the given equation, and there will arise a new equation that shall want the second term.*"

E X A M P L E.

Let it be required to exterminate the second term out of this equation, $x^3 - 9x^2 + 26x - 34 = 0$, suppose $x - 3 = y$, or $y + 3 = x$; and substituting according to the rule, you will find

$$\left. \begin{array}{r} y^3 + 9y^2 + 27y + 27 \\ - 9y^2 - 54y - 81 \\ + 26y + 78 \\ - 34 \end{array} \right\} = 0$$

$$y^3 \quad * \quad - y \quad - 10 = 0.$$

In which there is no term where y is of two dimensions, and an asterisk is placed in the room of the second term, to shew it is wanting.

§ 26. Let the equation proposed be of any number of dimensions represented by (n) ; and let the coefficient of the second term with its sign prefixed be $-p$, then supposing $x - \frac{p}{n} = y$, and consequently $x = y + \frac{p}{n}$, and substituting this value for x in the given equation, there will arise a new equation that shall want the second term.

It is plain from what was demonstrated in *Chap. 2.* that the sum of the roots of the proposed equation is $+p$; and since we suppose $y = x - \frac{p}{n}$, it follows, that in the new equation, each value of y will be less than the respective value of x by $\frac{p}{n}$; and, since the number of the roots is n , it follows that the sum of the values of y will be less than $+p$, the sum of the values of x , by $n \times \frac{p}{n}$ the difference of any two roots, that is, by $+p$: therefore the sum of the values of y will be $+p - p = 0$.

But the coefficient of the second term of the equation of y is the sum of the values of y , viz. $+p - p$, and therefore that coefficient is equal to nothing; and consequently, in the equation

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of y , the second term vanishes. It follows then, that the second term may be exterminated out of any given equation by the following

R U L E.

“Divide the coefficient of the second term of the proposed equation by the number of dimensions of the equation; and assuming a new unknown quantity y , add to it the quotient having its sign changed. Then suppose this aggregate equal to x the unknown quantity in the proposed equation; and for x and its powers, substitute the aggregate and its powers, so shall the new equation that arises want its second term.”

§ 27. If the proposed equation is a quadratic, as $x^2 - px + q = 0$, then, according to the rule, suppose $y + \frac{1}{2}p = x$, and substituting this value for x , you will find,

$$\left. \begin{array}{r} y^2 + py + \frac{1}{4}p^2 \\ - py - \frac{1}{2}p^2 \\ + q \end{array} \right\} = 0$$

$$y^2 \quad * \quad - \frac{1}{4}p^2 + q = 0.$$

And from this example the use of exterminating the second term appears: for commonly the solution of the equation that wants the second term is more easy. And, if you can find the value

value of y from this new equation, it is easy to find the value of x by means of the equation $y + \frac{1}{2}p = x$. For example,

Since $y^2 + q - \frac{1}{4}p^2 = 0$, it follows that

$$y^2 = \frac{1}{4}p^2 - q, \text{ and } y = \pm \sqrt{\frac{1}{4}p^2 - q}, \text{ so}$$

$$\text{that } x = y + \frac{1}{2}p = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q};$$

which agrees with what we demonstrated, *Chapter 13. Part I.*

If the proposed equation is a biquadratic, as $x^4 - px^3 + qx^2 - rx + s = 0$, then by supposing $x - \frac{1}{2}p = y$, or $x = y + \frac{1}{2}p$, an equation shall arise having no second term. And if the proposed is of five dimensions, then you must suppose $x = y \pm \frac{1}{2}p$. And so on.

§ 28. When the second term in any equation is wanting, it follows, that “the equation has both affirmative and negative roots,” and that the sum of the affirmative roots is equal to the sum of the negative roots: by which means the coefficient of the second term, which is the sum of all the roots of both sorts, vanishes, and makes the second term vanish.

In general, “the coefficient of the second term is the difference between the sum of the affirmative roots and the sum of the negative roots:” and the operations we have given serve only to diminish all the roots when the sum of the affirmative is greatest, or increase the roots when the sum of the negative is greatest, so

as

as to balance them, and reduce them to an equality.

It is obvious, that in a quadratic equation that wants the second term, there must be one root affirmative and one negative; and these must be equal to one another.

In a cubic equation that wants the second term, there must be either, two affirmative roots equal, taken together, to a third root that must be negative; or, two negative equal to a third that must be positive.

“ Let an equation $x^3 - px^2 + qx - r = 0$ be proposed, and let it be now required to exterminate the third term.”

By supposing $y = x - e$, the coefficient of the third term in the equation of y is found (see equation *A*) to be $3e^2 - 2pe + q$. Suppose that coefficient equal to nothing, and by resolving the quadratic equation $3e^2 - 2pe + q = 0$, you will find the value of e , which substituted for it in the equation $y = x - e$, will shew how to transform the proposed equation into one that shall want the third term.

The quadratic $3e^2 - 2pe + q = 0$ gives $e = \frac{p \pm \sqrt{p^2 - 3q}}{3}$. So that the proposed cubic

will be transformed into an equation wanting the third term by supposing $y = x - \frac{p - \sqrt{p^2 - 3q}}{3}$,

or $y = x - \frac{p + \sqrt{p^2 - 3q}}{3}$.

If the proposed equation is of n dimensions, the value of e , by which the third term may be taken away, is had by resolving the quadratic equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times n - 1} = 0$, supposing $-p$ and $+q$ to be the coefficients of the second and third terms of the proposed equation.

The fourth term of any equation may be taken away by solving a cubic equation, which is the coefficient of the fourth term in the equation when transformed, as in the second article of this chapter. The fifth term may be taken away by solving a biquadratic; and after the same manner the other terms can be exterminated if there are any.

§ 29. There are other transmutations of equations, that on some occasions are useful.

An equation, as $x^3 - px^2 + qx - r = 0$, may be transformed into another that shall have its roots equal to the roots of this equation multiplied by a given quantity, as f , by supposing $y = fx$, and consequently $x = \frac{y}{f}$, and substituting this value for x in the proposed equation, there will arise $\frac{y^3}{f^3} - \frac{py^2}{f^2} + \frac{qy}{f} - r = 0$, and multiplying all by $f^3 \dots y^3 - fpy^2 + f^2qy - f^3r = 0$, where the coefficient of the second term of the proposed equation multiplied into f makes the coefficient of the

the second term of the transformed equation ; and the following coefficients are produced by the following coefficients of the proposed equation (as q , r , &c.) multiplied into the powers of f (f^2 , f^3 , &c.)

Therefore “ to transform any equation into another whose roots shall be equal to the roots of the proposed equation multiplied by a given quantity” (f), you need only multiply the terms of the proposed equation, beginning at the second term, by f , f^2 , f^3 , f^4 , &c. and putting y instead of x there will arise an equation having its roots equal to the roots of the proposed equation multiplied by (f) as required.

§ 30. The transformation mentioned in the last article is of use when the highest term of the equation has a coefficient different from unity ; for, by it, the equation may be transformed into one that shall have the coefficient of the highest term unit.

If the equation proposed is $ax^3 - px^2 + qx - r = 0$, then transform the equation into one whose roots are equal to the roots of the proposed equation multiplied by (a). That is, suppose $y = ax$, or $x = \frac{y}{a}$, and there will arise

$$\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0 ; \text{ so that } y^3 - py^2 + qay - ra^2 = 0.$$

From

From which we easily draw this

R U L E.

“ Change the unknown quantity x into another y , prefix no coefficient to the highest term, pass the second, multiply the following terms, beginning with the third, by $a, a^2, a^3, a^4, &c.$ the powers of the coefficient of the highest term of the proposed equation, respectively.”

Thus the equation $3x^3 - 13x^2 + 14x + 16 = 0$, is transformed into the equation

$$y^3 - 13y^2 + 14 \times 3 \times x + 16 \times 9 = 0, \text{ or}$$

$$y^3 - 13y^2 + 42x + 144 = 0.$$

Then finding the roots of this equation, it will easily be discovered what are the roots of the proposed equation: since $3x = y$, or $x = \frac{1}{3}y$. And therefore since one of the values of y is -2 , it follows that one of the values of x is $-\frac{2}{3}$.

§ 31. By the last Rule “ an equation is easily cleared of fractions.” Suppose the equation proposed is $x^3 - \frac{p}{m}x^2 + \frac{q}{n}x - \frac{r}{c} = 0$. Multiply all the terms by the product of the denominators, you find

$$mnc \times x^3 - nep \times x^2 + meq \times x - mir = 0.$$

Then (by last section) transforming the equation into one that shall have unit for the coefficient of the highest term, you find

$$y^3 - nep \times y^2 + m^2e^2nq \times y - m^3n^3e^2r = 0.$$

Or,

Or, neglecting the denominator of the last term $\frac{r}{e}$, you need only multiply all the equation by mn , which will give

$$mn \times x^3 - np \times x^2 + mq \times x - \frac{mnr}{e} = 0. \text{ And}$$

$$\text{then } y^3 - np \times y^2 + m^2 nq \times y - \frac{m^3 n^3 r}{e} = 0.$$

Now after the values of y are found, it will be easy to discover the values of x ; since, in the first case, $x = \frac{y}{mne}$; in the second, $x = \frac{y}{mn}$.

For example, the equation

$$x^{3*} - \frac{4}{3}x - \frac{146}{27} = 0, \text{ is first reduced to}$$

this form $3x^{3*} - 4x - \frac{146}{9} = 0$, and then transformed into $y^{3*} - 12y - 146 = 0$.

Sometimes, by these transformations, "*Surds are taken away.*" As for example,

The equation $x^3 - p\sqrt{a} \times x^2 + qx - r\sqrt{a} = 0$,

by putting $y = \sqrt{a} \times x$, or $x = \frac{y}{\sqrt{a}}$, is transformed into this equation,

$$\frac{y^3}{a\sqrt{a}} - p\sqrt{a} \times \frac{y^2}{a} + q \times \frac{y}{\sqrt{a}} - r\sqrt{a} = 0.$$

Which by multiplying all the terms by $a\sqrt{a}$, becomes $y^3 - pay^2 + qay - ra^2 = 0$, an equation free of surds. But in order to make this succeed, the surd (\sqrt{a}) must enter the alternate terms beginning with the second.

§ 32. An equation, as $x^3 - px^2 + qx - r = 0$, may be transformed into one whose roots shall be the quantities reciprocal of x ; by supposing $y = \frac{1}{x}$, and $z = \frac{1}{y}$, or, (by one supposition) $x = \frac{r}{z}$, becomes $z^3 - qz^2 + prz - r^2 = 0$.

In the equation of y , it is manifest that the order of the coefficients is inverted; so that if the second term had been wanting in the proposed equation, the last but one should have been wanting in the equations of y and z . If the third had been wanting in the equation proposed, the last but two had been wanting in the equations of y and z .

Another use of this transformation is, that “the greatest root in the one is transformed into the least root in the other.” For since $x = \frac{1}{y}$, and $y = \frac{1}{x}$, it is plain that when the value of x is greatest, the value of y is least, and conversely.

How an equation is transformed so as to have all its roots affirmative, shall be explained in the following chapter.

◦ CH A P. IV.

Of finding the Roots of Equations when two or more of the roots are equal to each other.

§ 33. **B**EFORE we proceed to explain how to resolve equations of all sorts, we shall first demonstrate “*how an equation that has two or more roots equal, is depressed to a lower dimension ;*” and its resolution made, consequently, more easy. And shall endeavour to explain the grounds of this and many other rules we shall give in the remaining part of this Treatise, in a more simple and concise manner than has hitherto been done.

In order to this, we must look back to § 24. where we find that if any equation, as $x^3 - px^2 + qx - r = 0$, is proposed, and you are to transform it into another that shall have its roots less than the values of x by any given difference, as e , you are to assume $y = x - e$, and substituting for x its value $y + e$, you find the transformed equation,

$$\left. \begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array} \right\} = 0.$$

Where

Where we are to observe,

1°. That the last term ($e^3 - pe^2 + qe - r$) is the very equation that was proposed, having e in place of x .

2°. The coefficient of the last term but one is $3e^2 - 2pe + q$, which is the quantity that arises by multiplying every term of the last coefficient $e^3 - pe^2 + qe - r$ by the index of e in each term, and dividing the product $3e^3 - 2pe^2 + qe$ by the quantity e that is common to all the terms.

3°. The coefficient of the last term but two is $3e - p$, which is the quantity that arises by multiplying every term of the coefficient last found ($3e^2 - 2pe + q$) by the index of e in each term, and dividing the whole by $2e$.

§ 34. These same observations extend to equations of all dimensions. If it is the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ that is proposed, then by supposing $y = x - e$, it will be transformed into this other,

$$\left. \begin{array}{l} y^4 + 4ey^3 + 6e^2y^2 + 4e^3y + e^4 \\ - py^3 - 3pe^2y^2 - 3pe^2y - pe^3 \\ + qy^2 + 2qey + qe^2 \\ - ry - re \\ + s \end{array} \right\} = 0.$$

Where again it is obvious that the last term is the equation that was proposed, having e in place of x . That the last term but one has

for its coefficient the quantity that arises by multiplying the terms of the last quantity by the indices of e in each term, and dividing the product by e . That the coefficient of the last term but two (*viz.* $6e^2 - 3pe + q$) is deduced in the same manner from the term immediately following; that is, by multiplying every term of $4e^3 - 3pe^2 + 2qe - r$ by the index of e in that term, and dividing the whole by e multiplied into the index of y in the term sought, that is, by $e \times 2$. And the next term is $4e - p = \frac{6e^2 \times 2 - 3pe \times 1}{3e}$.

The demonstration of this may easily be made general by the Theorem for finding the powers of a binomial, since the transformed equation consists of the powers of the binomial $y + e$ that are marked by the indices of e in the last term, multiplied each by their coefficients $1, -p, +q, -r, +s, \&c.$ respectively.

§ 35. From the last two articles we can easily find the terms of the transformed equation without any involution. The last term is had by substituting e instead of x in the proposed equation; the next term, by multiplying every part of that last term by the index of e in each part, and dividing the whole by e ; and the following terms in the manner described in the foregoing article; the respective divisors being

being the quantity e multiplied by the index of y in each term.

The demonstration for finding when two or more roots are equal will be easy, if we add to this, that “*when the unknown quantity enters all the terms of any equation, then one of its values is equal to nothing.*” As in the equation $x^3 - px^2 + qx = 0$, where $x - 0 = 0$ being one of the simple equations that produce $x^3 - px^2 + qx = 0$, it follows that one of the values of x is 0. In like manner two of the values of x are equal to nothing in this equation $x^3 - px^2 = 0$; and three of them vanish in the equation $x^4 - px^3 = 0$.

It is also obvious (*conversely*) that “if x does not enter all the terms of the equation, *i. e.* if the last term be not wanting, then none of the values of x can be equal to nothing;” for if every term be not multiplied by x , then $x - 0$ cannot be a divisor of the whole equation, and consequently 0 cannot be one of the values of x . If x^2 does not enter into all the terms of the equation, then two of the values of x cannot be equal to nothing. If x^3 does not enter into all the terms of the equation, then three of the values of x cannot be equal to nothing, &c.

§ 36. Suppose now that two values of x are equal to one another, and to e ; then it is plain that two values of y in the transformed equation

will be equal to nothing: since $y = x - e$. And consequently, by the last article, the two last terms of the transformed equation must vanish.

Suppose it is the cubic equation of § 33. that is proposed, *viz.* $x^3 - px^2 + qx - r = 0$; and because we suppose $x = e$, therefore the last term of the transformed equation, *viz.* $e^3 - pe^2 + qe - r$ will vanish. And since two values of y vanish, the last term but one, *viz.* $3e^2y - 2pey + qy$ will vanish at the same time. So that $3e^2 - 2pe + q = 0$. But, by supposition, $e = x$; therefore, when two values of x , in the equation $x^3 - px^2 + qx - r = 0$, are equal, it follows, that $3x^2 - 2px + q = 0$. And thus “the proposed cubic is depreffed to a quadratic that has one of its roots equal to one of the roots of that cubic.”

If it is the biquadratic that is proposed, *viz.* $x^4 - px^3 + qx^2 - rx + s = 0$, and two of its roots be equal; then supposing $e = x$, two of the values of y must vanish; and the equation of § 34 will be reduced to this form,

$$\left. \begin{array}{l} y^4 + 4ey^3 + 6e^2y^2 \\ - py^3 - 3pey^2 \\ + qy^2 \end{array} \right\}^{**} = 0. \text{ So that}$$

$$4e^3 - 3pe^2 + 2qe - r = 0; \text{ or, because } x = e;$$

$$4x^3 - 3px^2 + 2qx - r = 0.$$

In

In general, when two values of x are equal to each other, and to e , the two last terms of the transformed equation vanish: and consequently, “if you multiply the terms of the proposed equation by the indices of x in each term, the quantity that will arise will be $= 0$, and will give an equation of a lower dimension than the proposed, that shall have one of its roots equal to one of the roots of the proposed equation.”

That the last two terms of the equation vanish when the values of x are supposed equal to each other, and to e , will also appear by considering, that since two values of y then become equal to nothing, the product of the values of y must vanish, which is equal to the last term of the equation; and because two of the four values of y are equal to nothing, it follows also that one of any three that can be taken out of these four must be $= 0$; and therefore, the products made by multiplying any three must vanish; and consequently the coefficient of the last term but one, which is equal to the sum of these products, must vanish.

§ 37. After the same manner, if there are three equal roots in the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$, and if e be equal to one of them; three values of $y (= x - e)$ will vanish, and consequently y^3 will enter all the terms of

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the transformed equation; which will have this form,

$$y^4 + 4ey^3 - py^3 \} *** = 0. \text{ So that here}$$

$6e^2 - 3pe + q = 0$; or, since $e = x$, therefore, $6x^2 - 3px + q = 0$: and one of the roots of this quadratic will be equal to one of the roots of the proposed biquadratic.

In this case, two of the roots of the cubic equation $4x^3 - 3px^2 + 2qx - r = 0$ are roots of the proposed biquadratic, because the quantity $6x^2 - 3px + q$ is deduced from $4x^3 - 3px^2 + 2qx - r$, by multiplying the terms by the indexes of x in each term.

In general, "whatever is the number of equal roots in the proposed equation, they will all remain but one in the equation that is deduced from it, by multiplying all the terms by the indexes of x in them; and they will all remain but two in the equation deduced in the same manner from That;" and so of the rest.

§ 38. What we observed of the coefficients of equations transformed by supposing $y = x - e$, leads to this easy demonstration of this Rule; and will be applied in the next chapter to demonstrate the rules for finding the limits of equations.

It is obvious however, that though we make use of equations whose signs change alternately,
the

the same reasoning extends to all other equations.

It is a consequence also of what has been demonstrated, that “if two roots of any equation, as $x^3 - px^2 + qx - r = 0$, are equal, then multiplying the terms by any arithmetical series, as $a + 3b, a + 2b, a + b, a$, the product will be $= 0$.

For since $ax^3 - apx^2 + aqx - ar = 0$; and $3x^2 - 2px + q \times bx = 0$, it follows that $ax^3 + 3bx^3 - apx^2 - 2bpx^2 + aqx + bqx - ar = 0$. Which is the product that arises by multiplying the terms of the proposed equation by the terms of the series, $a + 3b, a + 2b, a + b, a$; which may represent any arithmetical progression.



C H A P. V.

Of the Limits of EQUATIONS.

§ 39. **W**E now proceed to shew how to discover the limits of the roots of equations, by which their solution is much facilitated.

Let any equation, as $x^3 - px^2 + qx - r = 0$, be proposed; and transform it, as above, into the equation

$$\left. \begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array} \right\} = 0.$$

Where the values of y are less than the respective values of x by the difference e . If you suppose e to be taken such as to make all the coefficients, of the equation of y , positive, *viz.* $e^3 - pe^2 + qe - r$, $3e^2 - 2pe + q$, $3e - p$; then there being no variation of the signs in the equation, all the values of y must be negative; and consequently, the quantity e , by which the values of x are diminished, must be greater than the greatest positive value of x : and consequently must be the limit of the roots of the equation $x^3 - px^2 + qx - r = 0$.

It

It is sufficient therefore, in order to find the limit, to “enquire what quantity substituted for x in each of these expressions $x^3 - px^2 + qx - r$, $3x^2 - 2px + q$, $3x - p$, will give them all positive;” for that quantity will be the limit required.

How these expressions are formed from one another, was explained in the beginning of the last chapter.

EXAMPLE.

§ 40. If the equation $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x + 120 = 0$ is proposed; and it is required to determine the limit that is greater than any of the roots; you are to enquire what integer number substituted for x in the proposed equation, and following equations deduced from it by § 35, will give, in each, a positive quantity.

$$5x^4 - 8x^3 - 30x^2 + 60x + 63$$

$$5x^3 - 6x^2 - 15x + 15$$

$$5x^2 - 4x - 5$$

$$5x - 2.$$

The least integer number which gives each of these positive, is 2; which therefore is the limit of the roots of the proposed equation; or a number that exceeds the greatest positive root.

If

If the limit of the *negative* roots is required, you may, by § 23, change the negative into positive roots, and then proceed as before to find their limits. Thus, in the example, you will find that -3 is the limit of the negative roots. So that the five roots of the proposed equation are betwixt -3 and $\frac{1}{2}$.

§ 41. Having found the limit that surpasses the greatest positive root, call it m . And if you assume $y = m - x$, and for x substitute $m - y$, the equation that will arise will have all its roots positive; because m is supposed to surpass all the values of x , and consequently $m - x (= y)$ must always be affirmative. And by this means, *any equation may be changed into one that shall have all its roots affirmative.*

Or if $-n$ represent the limit of the negative roots, then by assuming $y = x + n$, the proposed equation shall be transformed into one that shall have all its roots affirmative; for $+n$ being greater than any negative value of x , it follows, that $y = x + n$ must be always positive.

§ 42. “*The greatest negative coefficient of any equation increased by unit, always exceeds the greatest root of the equation.*”

To demonstrate this, let the cubic $x^3 - px^2 - qx - r = 0$ be proposed; where all the terms are negative except the first. Assuming $y = x - e$ it will be transformed into the following equation.

(A)

$$(A) \left. \begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ - qy - qe \\ - r \end{array} \right\} = 0.$$

1°. Let us suppose that the coefficients p, q, r , are equal to each other; and if you also suppose $e = p + 1$, then the last equation becomes

$$(B) \left. \begin{array}{l} y^3 + 2py^2 + p^2y + 1 \\ + 3y^2 + 3py \\ + 3y \end{array} \right\} = 0.$$

Where all the terms being positive, it follows that the values of y are all negative, and that consequently e , or $p + 1$, is greater than the greatest value of x in the proposed equation.

2°. If q and r be not $= p$, but less than it, and for e you still substitute $p + 1$ (since the negative part $\left(\begin{array}{l} -qy - qe \\ -r \end{array} \right)$ becomes less, the positive remaining undiminished) *a fortiori*, all the coefficients of the equation (A) become positive. And the same is obvious if q and r have positive signs, and not negative signs, as we supposed. It appears therefore, "that, if, in any cubic equation, p be the greatest negative coefficient, then $p + 1$ must surpass the greatest value of x ."

§ 43. 3°. By the same reasoning it appears, that if q be the greatest negative coefficient of the

the

the equation, and $e = q + 1$, then there will be no variation of the signs in the equation of y : for it appears from the last article, that if all the three (p, q, r) were equal to one another, and e equal to any one of them increased by unit, as to $q + 1$, then all the terms of the equation (A) would be positive. Now if e be supposed still equal to $q + 1$, and p and r to be less than q , then, *a fortiori*, all these terms will be positive, the negative part, which involves p and r being diminished, while the positive part and the negative involving q remain as before.

4°. After the same manner it is demonstrated, that if r is the greatest negative coefficient in the equation, and e is supposed $= r + 1$, then all the terms of the equation (A) of y will be positive; and consequently $r + 1$ will be greater than any of the values of x .

What we have said of the *cubic* equation $x^3 - px^2 + qx - r = 0$, is easily applicable to others.

In general, we conclude that “the greatest negative coefficient in any equation increased by unit, is always a limit that exceeds all the roots of that equation.”

But it is to be observed at the same time, that the greatest negative coefficient increased by unit, is very seldom the *nearest* limit: that is best discovered by the Rule in the 39th article.

§ 44. Having shewn in § 41. how to change any proposed equation into one that shall have all its roots affirmative; we shall only treat of such as have all their roots positive, in what remains relating to the limits of equations.

Any such equation may be represented by $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, &c. = 0, whose roots are *a*, *b*, *c*, *d*, &c.

And of all such equations two limits are easily discovered from what precedes, *viz.* 0, which is less than the least, and *e*, found according to § 39. which surpasses the greatest root of the equation.

But besides these, we shall now shew how “*to find other limits betwixt the roots themselves.*” And, for this purpose, will suppose *a* to be the least root, *b* the second root, *c* the third, and so on; it being arbitrary.

§ 45. If you substitute 0 in place of the unknown quantity, putting $x = 0$, the quantity that will arise from that supposition is the last term of the equation, all the others, that involve *x*, vanishing.

If you substitute for *x* a quantity less than the least root *a*, the quantity resulting will have the same sign as the last term; that is, will be positive or negative according as the equation is of an even or odd number of dimensions. For all the factors $x - a$, $x - b$, $x - c$, &c. will be negative, and their product will be *positive*

or

or *negative* according as their number is *even* or *odd*.

If you substitute for x a quantity greater than the least root a , but less than all the other roots, then the sign of the quantity resulting will be contrary to what it was before; because one factor $(x - a)$ becomes now positive, all the others remaining negative as before.

If you substitute for x a quantity greater than the two least roots, but less than all the rest, both the factors $x - a$, $x - b$, become positive, and the rest remain as they were. So that the whole product will have the same sign as the last term of the equation. Thus successively placing instead of x quantities that are limits betwixt the roots of the equation, the quantities that result will have alternately the signs $+$ and $-$. And, *conversely*, “if you find quantities which substituted in place of x in the proposed equation, do give alternately positive and negative results, those quantities are the limits of that equation.”

It is useful to observe, that, in general, “when, by substituting any two numbers for x in any equation, the results have contrary signs, one or more of the roots of the equation must be betwixt those numbers.” Thus, in the equation $x^3 - 2x^2 - 5 = 0$, if you substitute 2 and 3 for x , the results are -5 , $+4$; whence it follows that the roots are betwixt 2
and

and 3: for when these results have different signs, one or other of the factors which produce the equations must have changed its sign; suppose it is $x = e$, then it is plain that e must be betwixt the numbers supposed equal to x .

§ 46. Let the cubic equation $x^3 - px^2 + qx - r = 0$ be proposed, and let it be transformed, by assuming $y = x - e$, into the equation

$$\left. \begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array} \right\} = 0.$$

Let us suppose e equal successively to the three values of x , beginning with the least value; and because the last term $e^3 - pe^2 + qe - r$ will vanish in all these suppositions, the equation will have this form,

$$\left. \begin{array}{l} y^3 + 3ey + 3e^2 \\ - py - 2pe \\ + q \end{array} \right\} = 0;$$

where the last term $3e^2 - 2pe + q$ is, from the nature of equations, produced of the remaining values of y , or of the excesses of two other values of x above what is supposed equal to e ; since always $y = x - e$. Now,

1°. If e be equal to the least value of x , then those two excesses being both positive, they will give a positive product, and consequently $3e^2 - 2pe + q$ will be, in this case, positive.

2°. If e be equal to the second value of x , then, of those two excesses one being negative and one positive, their product $3e^2 - 2pe + q$, will be negative.

3°. If e be equal to the third and greatest value of x , then the two excesses being both negative, their product $3e^2 - 2pe + q$ is positive. Whence,

If in the equation $3e^2 - 2pe + q = 0$, you substitute successively in the place of e , the three roots of the equation $e^3 - pe^2 + qe - r = 0$, the quantities resulting will successively have the signs $+$, $-$, $+$; and consequently the three roots of the cubic equation are the limits of the roots of the equation $3e^2 - 2pe + q = 0$ (by § 45.) That is, the least of the roots of the cubic is less than the least of the roots of the other; the second root of the cubic is a limit between the two roots of the other; and the greatest root of the cubic is the limit that exceeds both the roots of the other.

§ 47. We have demonstrated that the roots of the cubic equation $e^3 - pe^2 + qe - r = 0$ are limits of the quadratic $3e^2 - 2pe + q$; whence it follows (*conversely*) that the roots of the quadratic $3e^2 - 2pe + q = 0$ are the limits between the first and second, and between the second and third roots of the cubic $e^3 - pe^2 + qe - r = 0$. So that if you find the limit that exceeds the
greatest

greatest root of the cubic, by § 39. you will have (with 0, which is the limit less than any of the roots) four limits for the three roots of the proposed cubic.

It was demonstrated in § 35. how the quadratic $3e^2 - 2pe + q$ is deduced from the proposed cubic $e^3 - pe^2 + qe - r = 0$, viz. by multiplying each term by the index of e in it, and then dividing the whole by e ; and what we have demonstrated of cubic equations is easily extended to all others; so that we conclude, "that the last term but one of the transformed equation is the equation for determining the limits of the proposed equation." Or, that the equation arising by multiplying each term by the index of the unknown quantity in it, is the equation whose roots give the limits of the proposed equation; if you add to them the two mentioned in § 44.

§ 48. For the same reason, it is plain that the root of the simple equation $3e - p = 0$, (*i. e.* $\frac{1}{3}p$) is the limit between the two roots of the quadratic $3e^2 - 2pe + q = 0$. And, as $4e^3 - 3pe^2 + 2qe - r = 0$ gives three limits of the equation $e^4 - pe^3 + qe^2 - re + s = 0$, so the quadratic $6e^2 - 3pe + q = 0$ gives two limits that are betwixt the roots of the cubic $4e^3 - 3pe^2 + 2qe - r = 0$; and $4e - p = 0$ gives one limit that is betwixt the two roots of

the quadratic $6e^2 - 3pe + q = 0$. So that we have a compleat series of these equations arising from a simple equation to the proposed, each of which determines the limits of the following equation.

§ 49. If two roots in the proposed equation are equal, then “the limit that ought to be betwixt them must, in this case, become equal to one of the equal roots themselves.” Which perfectly agrees with what was demonstrated in the last chapter, concerning the Rule for finding the equal roots of equations.

And, the same equation that gives the limits, giving also one of the equal roots, when two or more are equal, it appears, that “if you substitute a limit in place of the unknown quantity in an equation, and, instead of a positive or negative result, it be found $= 0$, then you may conclude, that not only the limit itself is a root of the equation, but that there are two roots in that equation equal to it and to one another.”

§ 50. It having been demonstrated that the roots of the equation $x^3 - px^2 + qx - r = 0$ are the limits of the roots of the equation $3x^2 - 2px + q = 0$, the three roots of the cubic equation, which suppose to be a, b, c , substituted for x in the quadratic $3x^2 - 2px + q$, must give the results positive and negative alternately.

nately. Suppose these three results to be $+N$, $-M$, $+L$; that is, $3a^2 - 2pa + q = N$, $3b^2 - 2pb + q = -M$, $3c^2 - 2pc + q = L$; and since $a^3 - pa^2 + qa - r = 0$, and $3a^3 - 2pa^2 + qa = N \times a$, subtracting the former multiplied into 3 from the latter, the remainder is $pa^2 - 2qa + 3r = N \times a$. In the same manner $pb^2 - 2qb + 3r = -M \times b$, and $pc^2 - 2qc + 3r = +L \times c$. Therefore $px^2 - 2qx + 3r$ is such a quantity that if, for x , you substitute in it successively a, b, c , the results will be $+N \times a, -M \times b, +L \times c$. Whence a, b, c , are limits of the equation $px^2 - 2qx + 3r = 0$ (by § 45.) and, *conversely*, the roots of the equation $px^2 - 2qx + 3r = 0$ are limits between the first and second, and between the second and third roots of the cubic $x^3 - px^2 + qx - r = 0$. Now the equation $px^2 - 2qx + 3r = 0$ arises from the proposed cubic by multiplying the terms of this latter by the arithmetical progression $0, -1, -2, -3$. And in the same manner it may be shewn that the roots of the equation $px^3 - 2qx^2 + 3rx - 4s = 0$ are limits of the equation $x^4 - px^3 + qx^2 - rx + s = 0$.

Or, multiply the terms of the equation

$$x^3 - px^2 + qx - r = 0$$

by $a + 3b, a + 2b, a + b, a,$

$$ax^3 - apx^2 + aqx - ar (= 0)$$

$$+ 3bx^3 - 2bpx^2 + bqx (= 3x^2 - 2px + q \times bx).$$

M 2

Any

Any arithmetical series where a is the least term and b the common difference, and the products (if you substitute for x , successively, a , b , c , the three roots of the proposed cubic) shall be $+N \times bx$, $-M \times bx$, $+L \times bx$. For the first part of the product $a \times x^3 - px^2 + qx - r = 0$; and a , b , c , being limits in the equation $3x^2 - 2px + q = 0$, their substitution must give results N , M , L , alternately positive and negative.

In general, the roots of the equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \dots \pm \mathcal{E}c. = 0$ are limits of the roots of the equation $nx^{n-1} - \overline{n-1} \times px^{n-2} + \overline{n-2} \times qx^{n-3} - \overline{n-3} \times rx^{n-4} + \dots \pm \mathcal{E}c. = 0$; or of any equation that is deduced from it by multiplying its terms by any arithmetical progression $a + b$, $a + 2b$, $a + 3b$, $a + 4b$, &c. And conversely, the roots of this new equation will be limits of the proposed equation

$$x^n - px^{n-1} + qx^{n-2} - \dots \pm \mathcal{E}c. = 0.$$

“If any roots of the equation of the limits are impossible, then must there be some roots of the proposed equation impossible.” For as (in § 46.) the quantity $3e^2 - 2pe + q$ was demonstrated to be equal to the product of the excesses of two values of x above the third supposed equal to e ; if any impossible expression be found in those excesses, then there will of consequence be found impossible expressions in these two values of x .

And

And “from this observation rules may be deduced for discovering when there are impossible roots in equations.” Of which we shall treat afterwards.

§ 51. Besides the method already explained, there are others by which limits may be determined, which the root of an equation cannot exceed.

Since the squares of all real quantities are affirmative, it follows, that “*the sum of the squares of the roots of any equation must be greater than the square of the greatest root.*” And the square root of that sum will therefore be a limit that must exceed the greatest root of the equation.

If the equation proposed is $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \dots = 0$, then the sum of the squares of the roots (by § 15.) will be $p^2 - 2q$. So that $\sqrt{p^2 - 2q}$ will exceed the greatest root of that equation.

Or if you find, by § 16. the sum of the 4th powers of the roots of the equation, and extract the biquadratic root of that sum, it will also exceed the greatest root of the equation.

§ 52. If you find a mean proportional between the sum of the squares of any two roots, a, b , and the sum of their biquadrates ($a^4 + b^4$), this mean proportional will be

$\sqrt{a^6 + a^2b^4 + a^4b^2 + b^6}$. And the sum of the cubes

cubes is $a^3 + b^3$. Now since $a^2 - 2ab + b^2$ is the square of $a - b$, it must be always positive; and if you multiply it by a^2b^2 , the product $a^4b^2 - 2a^3b^3 + a^2b^4$ will also be positive; and consequently $a^4b^2 + a^2b^4$ will be always greater than $2a^3b^3$. Add $a^6 + b^6$, and we have $a^6 + a^4b^2 + a^2b^4 + b^6$ greater than $a^6 + 2a^3b^3 + b^6$; and extracting the root $\sqrt[2]{a^6 + a^4b^2 + a^2b^4 + b^6}$ greater than $a^3 + b^3$. And the same may be demonstrated of any number of roots whatever.

Now if you add the sum of all the cubes taken *affirmatively* to their sum with their *proper* signs, they will give double the sum of the cubes of the affirmative roots. And if you subtract the second sum from the first, there will remain double the sum of the cubes of the negative roots. Whence it follows, that “half the sum of the mean proportional betwixt the sum of the squares and the sum of the biquadrates, and of the sum of the cubes of the roots with their proper signs, exceeds the sum of the cubes of the affirmative roots:” and “half their difference exceeds the sum of the cubes of the negative roots.” And by extracting the cube root of that sum and difference, you will obtain limits that shall exceed the sums of the affirmative and of the negative roots. And since it is easy, from what has
been

been already explained, to diminish the roots of an equation so that they all may become negative but one, it appears how by this means you may approximate very near to that root. But this does not serve when there are impossible roots.

Several other Rules like these might be given for limiting the roots of equations. We shall give one not mentioned by other Authors.

In a cubic $x^3 - px^2 + qx - r = 0$ find $q^2 - 2pr$, and call it e^4 ; then shall the greatest root of the equation always be greater than $\frac{e}{4}$, or $\sqrt[4]{3}$

$\sqrt[4]{\frac{e^4}{3}}$. And,

In any equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \dots = 0$ find $\frac{q^2 - 2pr + 2s}{n}$, and extracting the root of the fourth power out of that quantity, it shall always be less than the greatest root of the equation.



C H A P. VI.

Of the Resolution of Equations, all whose Roots are commensurate.

§ 53. **I**T was demonstrated, in *Chap. 2.* that the last term of any equation is the product of its roots: from which it follows, that the roots of an equation, when commensurable quantities, will be found among the divisors of the last term. And hence we have for the resolution of equations this

R U L E.

Bring all the terms to one side of the equation, find all the divisors of the last term, and substitute them successively for the unknown quantity in the equation. So shall that divisor which, substituted in this manner, gives the result = 0, be the root of the proposed equation.

For example, suppose this equation is to be resolved,

$$\left. \begin{array}{l} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ - bx^2 + 3abx \end{array} \right\} = 0,$$

where the last term is $2a^2b$, whose simple literal

ral divisors are $a, b, 2a, 2b$, each of which may be taken either positively or negatively : but as here we find there are variations of signs in the equation, we need only take them positively. Suppose $x = a$ the first of the divisors, and substituting a for x , the equation becomes

$$\left. \begin{array}{l} a^3 - 3a^2 + 2a^3 - 2a^2b \\ - a^2b + 3a^2b \end{array} \right\} \text{or, } 3a^3 - 3a^2 + 3a^2b - 3a^2b = 0.$$

So that, the whole vanishing, it follows that a is one of the roots of the equation.

After the same manner, if you substitute b in place of x , the equation is

$$\left. \begin{array}{l} b^3 - 3ab^2 + 2a^2b - 2a^2b \\ - b^3 + 3ab^2 \end{array} \right\} = 0,$$

which vanishing shews b to be another root of the equation.

Again, if you substitute $2a$ for x , you will find all the terms destroy one another so as to make the sum $= 0$. For it will then be

$$\left. \begin{array}{l} 8a^3 - 12a^3 + 4a^3 - 2a^2b \\ - 4a^2b + 6a^2b \end{array} \right\} = 0.$$

Whence we find that $2a$ is the third root of the equation. Which, after the first two ($+ a, + b$) had been found, might have been collected from this, that the last term being the product of the three roots, $+ a, + b$ being known, the

the third must necessarily be equal to the last term divided by the product ab , that is, $= \frac{2a^2b}{ab} = 2a$.

Let the roots of the cubic equation

$$x^3 - 2x^2 - 33x + 90 = 0 \text{ be required.}$$

And first the divisors of 90 are found to be 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90. If you substitute 1 for x , you will find $x^3 - 2x^2 - 33x + 90 = 56$; so that 1 is not a root of the equation. If you substitute 2 for x , the result will be 24: but, putting $x = 3$, you have

$$x^3 - 2x^2 - 33x + 90 = 27 - 18 - 99 + 90 = 117 - 117 = 0.$$

So that 3 is one of the roots of the proposed equation. The other affirmative root is $+5$; and after you find it, as it is manifest from the equation, that the other root is negative, you are not to try any more divisors taken positively, but to substitute them, negatively taken, for x : and thus you find that -6 is the third root. For putting $x = -6$, you have

$$x^3 - 2x^2 - 33x + 90 = -216 - 72 + 198 + 90 = 0.$$

This last root might have been found by dividing the last term 90, having its sign changed, by 15, the product of the two roots already found.

§ 55. When one of the roots of an equation is found, in order to find the rest with less trouble, divide the proposed equation by the simple equation which you are to deduce from the root already

already found, and the quotient shall give an equation of a degree lower than the proposed; whose roots will give the remaining roots required.

As for example, the root + 3, first found, gave $x = 3$ or $x - 3 = 0$, whence dividing thus,

$$\begin{array}{r}
 x-3) x^3-2x^2-33x+90 \quad (x^2+x-30 \\
 \underline{x^3-3x^2} \\
 x^2-33x+90 \\
 \underline{x^2-3x} \\
 -30x+90 \\
 \underline{-30x+90} \\
 0 \quad 0
 \end{array}$$

The quotient shall give a quadratic equation $x^2 + x - 30 = 0$, which must be the product of the other two simple equations from which the cubic is generated, and whose roots therefore must be two of the roots of that cubic.

Now the roots of that quadratic equation are easily found by *Chap. 13. Part I.* to be + 5 and - 6. For,

$$x^2 + x = 30;$$

$$\text{add } \frac{1}{4} \dots x^2 + x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4},$$

$$\sqrt{\dots} \dots x + \frac{1}{2} = \pm \sqrt{\frac{121}{4}} = \pm \frac{11}{2},$$

$$\text{and } \dots x = \pm \frac{11}{2} - \frac{1}{2} = + 5, \text{ or } - 6.$$

§ 56. After the same manner, if the biquadratic $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$ is to be resolved; by substituting the divisors of 120 for x , you will find that $+3$, one of those divisors, is one of the roots; the substitution of 3 for x giving $81 - 54 - 225 + 78 + 120 = 279 - 279 = 0$. And therefore dividing the proposed equation by $x - 3$, you must enquire for the roots of the cubic $x^3 + x^2 - 22x - 40 = 0$, and finding that $+5$, one of the divisors of 40, is one of the roots, you divide that cubic by $x - 5$, and the quotient gives the quadratic $x^2 + 6x + 8 = 0$, whose two roots are -2 , -4 . So that the four roots of the biquadratic are $+3$, $+5$, -2 , -4 .

§ 57. This Rule supposes that you can find all the divisors of the last term; which you may always do thus.

“ If it is a simple quantity, divide it by its least divisor that exceeds unit, and the quotient again by its least divisor, proceeding thus till you have a quotient that is not divisible by any number greater than unit. This quotient, with these divisors, are the first or *simple divisors* of the quantity. And the products of the multiplication of any 2, 3, 4, &c. of them are the *compound divisors*.”

As, to find the divisors of 60; first I divide by 2, and the quotient 30 again by 2, then the
next

next quotient 15 by 3, and the quotient of this division 5 is not farther divisible by any integer above units; so that

- The simple divisors are 2, 2, 3, 5.
- The products of two, 4, 6, 10, 15.
- The products of three, 12, 20, 30.
- The product of all four, 60.

The divisors of 90 are found after the same manner.

- Simple divisors, 2, 3, 3, 5.
- The products of two, 6, 9, 10, 15.
- The products of three, 18, 30, 45.
- The product of all four, 90.

The divisors of $21abb$.

- The simple divisors, 3, 7, a , b , b .
- The products of two, 21 , $3a$, $3b$, $7a$, $7b$, ab , bb .
- The products of three, $21a$, $21b$, $3ab$, $3bb$,
 $7ab$, $7bb$, abb .
- The products of four, $21ab$, $21bb$, $3abb$, $7abb$.
- The product of the five, $21ab$.

§ 58. But as the last term may have very many divisors, and the labour may be very great to substitute them all for the unknown quantity, we shall now shew how it may be abridged, by limiting to a small number the divisors you are to try. And first it is plain, from § 42. that “ any divisor that exceeds the
greatest

greatest negative coefficient by unity is to be neglected." Thus in resolving the equation $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$, as 25 is the greatest negative coefficient, we conclude that the divisors of 120 that exceed 26 may be neglected.

But the labour may be still abridged, if we make use of the Rule in § 39; that is, if we find the number, which substituted in these following expressions,

$$\begin{aligned}x^4 - 2x^3 - 25x^2 + 26x + 120, \\2x^3 - 3x^2 - 25x + 13, \\6x^2 - 6x - 25, \\2x - 1,\end{aligned}$$

will give in them all a positive result: for that number will be greater than the greatest root, and all the divisors of 120 that exceed it may be neglected.

That this investigation may be easier, we ought to begin always with that expression, where the negative roots seem to prevail most; as here in the quadratic expression $6x^2 - 6x - 25$; where finding that 6 substituted for x gives that expression positive, and gives all the other expressions at the same time positive, I conclude that 6 is greater than any of the roots, and that all the divisors of 120 that exceed 6 may be neglected.

If

If the equation $x^3 + 11x^2 + 10x - 72 = 0$ is proposed, the Rule of § 42 does not help to abridge the operation; the last term itself being the greatest negative term. But, by § 39. we enquire what number substituted for x will give all these expressions positive :

$$\begin{aligned} x^3 + 11x^2 + 10x - 72, \\ 3x^2 + 22x + 10, \\ 3x + 11. \end{aligned}$$

Where the labour is very short, since we need only attend to the first expression; and we see immediately that 4 substituted for x gives a positive result, whence all the divisors of 72 that exceed 4 are to be rejected; and thus by a few trials we find that + 2 is the positive root of the equation. Then dividing the equation by $x - 2$, and resolving the quadratic equation that is the quotient of the division, you find the other two roots to be - 9, and - 4.

§ 59. But there is another method that reduces the divisors of the last term, that can be useful, still to more narrow limits.

Suppose the cubic equation $x^3 - px^2 + qx - r = 0$ is proposed to be resolved. Transform it to an equation whose roots shall be less than the values of x by unity, assuming $y = x - 1$. And the last term of the transformed equation will be $1 - p + q - r$; which is found by substituting unit, the difference of x and y , for x ,
in

in the proposed equation ; as will easily appear from § 24. where, when $y = x - e$, the last term of the transformed equation was $e^3 - pe^2 + qe - r$.

Transform again the equation $x^3 - px^2 + qx - r = 0$, by assuming $y = x + 1$, into an equation whose roots shall exceed the values of x by unit, and the last term of the transformed equation will be $-1 - p - q - r$, the same that arises by substituting -1 , the difference betwixt x and y , for x , in the proposed equation.

Now the values of x are some of the divisors of r , which is the term left when you suppose $x = 0$; and the values of the y 's are some of the divisors of $+1 - p + q - r$, and of $-1 - p - q - r$, respectively. And these values are in arithmetical progression increasing by the common difference unit ; because $x - 1$, x , $x + 1$, are in that progression. And it is obvious the same reasoning may be extended to any equation of whatever degree. So that this gives a general method for the resolution of equations whose roots are commensurable.

R U L E.

“ *Substitute in place of the unknown quantity successively the terms of the progression 1, 0, -1, &c. and find all the divisors of the sums that result ; then take out all the arithmetical progressions you can find among these divisors, whose*

whose common difference is unit; and the values of x will be among the divisors arising from the substitutions of $x = 0$ that belong to these progressions." The values of x will be affirmative when the arithmetical progression increases, but negative when it decreases.

EXAMPLE.

§ 60. Let it be required to find one of the roots of the equation $x^3 - x^2 - 10x + 6 = 0$. The operation is thus :

Supposit.	Result.	Divisors.	Arith. prog. decr.
$x = 1$	$x^3 - x^2 - 10x + 6 =$	-4	$1, 2, 4$
$x = 0$		$+6$	$1, 2, 3, 6$
$x = -1$		$+14$	$1, 2, 7, 14$
			3 gives $x = -3$

Where the suppositions of $x = 1, x = 0, x = -1$ give the quantity $x^3 - x^2 - 10x + 6$ equal to $-4, 6, 14$; among whose divisors we find only one arithmetical progression, $4, 3, 2$; the term of which opposite to the supposition of $x = 0$, being 3 , and the series decreasing, we try if -3 substituted for x makes the equation vanish; which succeeding one of its roots must be -3 . Then dividing the equation by $x + 3$, we find the roots of the (quadratic) quotient

$$x^2 - 4x + 2 = 0 \text{ are } 2 \pm \sqrt{2}.$$

§ 61. If it is required to find the roots of the equation $x^3 - 3x^2 - 46x - 72 = 0$, the operation will be thus :

0

Suppos.

<i>Suppos. Results</i>	<i>Divisors.</i>	<i>Progressions.</i>
$x = 1$	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120	8 3 4 5
$x = 0$	72, 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.	9 2 3 4
$x = -1$	30, 1, 2, 3, 5, 6, 10, 15, 30.	10 1 2 3

Of these four arithmetical progressions having their common difference equal to unit, the first

first gives $x = 9$, the others give $x = -2$, $x = -3$, $x = -4$; all which succeed except $x = -3$: so that the three values of x are $+9$, -2 , -4 .



CHAP. VII.

Of the Resolution of Equations by finding the equations of a lower degree that are their divisors.

§ 62. **T**O find the roots of an equation is the same thing as to find the *simple* equations, by the multiplication of which into one another it is produced, or, to find the simple equations that divide it without a remainder.

If such simple equations cannot be found, yet if we can find the *quadratic* equations from which the proposed equation is produced, we may discover its roots afterwards by the resolution of these quadratic equations. Or, if neither these simple equations nor these quadratic equations can be found, yet, by finding a *cube* or *biquadratic* that is a divisor of the proposed equation, we may depress it lower, and make the solution more easy.

Now, in order to find the Rules by which these divisors may be discovered, we shall suppose that

$$\left. \begin{array}{l} mx - n \\ mx^2 - nx + r \\ mx^3 - nx^2 + rx - s \end{array} \right\} \text{are the } \left\{ \begin{array}{l} \text{simple} \\ \text{quadratic} \\ \text{cubic} \end{array} \right.$$

divisors of the proposed equation; and if E represent the quotient arising by dividing the proposed equation by that divisor, then

$$E \times \frac{mx - n}{mx^2 - nx + r},$$

or, $E \times \frac{mx^3 - nx^2 + rx - s}{mx^2 - nx + r}$, will represent the proposed equation itself. Where it is plain, that “since m is the coefficient of the highest term of the divisors, it must be a divisor of the coefficient of the highest term of the proposed equation.”

§ 63. Next we are to observe, that, supposing the equation has a simple divisor $mx - n$, if we substitute in the equation $E \times \frac{mx - n}{mx^2 - nx + r}$, in place of x , any quantity, as a , then the quantity that will result from this substitution will necessarily have $ma - n$ for one of its divisors; since, in this substitution, $mx - n$ becomes $ma - n$.

If we substitute successively for x any arithmetical progression, a , $a - e$, $a - 2e$, &c. the quantities that will result from these substitutions, will have among their divisors

$$ma - n,$$

$$ma - me - n,$$

$ma - 2me - n$, which are also in arithmetical progression, having their common difference equal to me .

If, for example, we substitute for x the terms of this progression, 1, 0, -1, the quantities that result have among their divisors the arithmetical progression $m - n$, $-n$, $-m - n$; or, changing the signs, $n - m$, n , $n + m$. Where the difference of the terms is m , and the term belonging to the supposition of $x = 0$ is n .

§ 64. It is manifest therefore, that when an equation has any simple divisor, if you substitute for x the progression 1, 0, -1, there will be found amongst the divisors of the sums that result from these substitutions, one arithmetical progression at least, whose common difference will be unit or a divisor m of the coefficient of the highest term, and which will be the coefficient of x in the simple divisor required: and whose term, arising from the supposition of $x = 0$, will be n the other member of the simple divisor $mx - n$.

From which this Rule is deduced for discovering such a simple divisor, when there is any.

R U L E.

“Substitute for x in the proposed equation successively the numbers $1, 0, -1$. Find all the divisors of the sums that result from this substitution, and take out all the arithmetical progressions you can find amongst them, whose difference is unit, or some divisor of the coefficient of the highest term of the equation. Then suppose n equal to that term of any one progression that arises from the supposition of $x = 0$, and $m =$ the foresaid divisor of the coefficient of the highest term of the equation, which m is also the difference of the terms of this progression; so shall you have $mx - n$ for the divisor required.”

You may find arithmetical progressions giving divisors that will not succeed; but if there is any divisor, it will be found thus by means of these arithmetical progressions.

§ 65. If the equation proposed has the coefficient of its highest term $= 1$, then it will be $m = 1$, and the divisor will be $x - n$, and the rule will coincide with that given in the end of the last chapter, which we demonstrated after a different manner; for the divisor being $x - n$, the value of x will be $+n$, the term of the progression that is a divisor of the sum that arises from supposing $x = 0$. Of this case we gave

gave examples in the last chapter; and though it is easy to reduce an equation whose highest term has a coefficient different from unit, to one where that coefficient shall be unit, by § 30; yet, without that reduction, the equation may be resolved by this rule, as in the following

EXAMPLE.

§ 66. Suppose $8x^3 - 26x^2 + 11x + 10 = 0$, and that it is required to find the values of x ; the operation is thus:

<i>Suppos.</i>	<i>Results.</i>	<i>Divisors.</i>	<i>Progr.</i>
$x = 1$	$8x^3 - 26x^2 + 11x + 10 =$	$+ 3$	1, 3. 3 3
$x = 0$		$+ 10$	1, 2, 5, 10. 2 5
$x = -1$		$- 35$	1, 5, 7, 35. 1 7

The difference of the terms of the last arithmetical progression is 2, a divisor of 8, the coefficient of the highest term x^3 of the equation, therefore supposing $m = 2$, $n = 5$, we try the divisor $2x - 5$; which succeeding, it follows that $2x - 5 = 0$, or $x = 2\frac{1}{2}$.

The quotient is the quadratic $4x^2 - 3x - 2 = 0$, whose roots are $\frac{3 + \sqrt{41}}{8}$, and $\frac{3 - \sqrt{41}}{8}$, so that the three roots of the proposed equation are $2\frac{1}{2}$, $\frac{3 + \sqrt{41}}{8}$, $\frac{3 - \sqrt{41}}{8}$. The other arith-

metical progression gives $x + 2$ for a divisor; but it does not succeed.

§ 67. If the proposed equation has no simple divisor, then we are to enquire if it has not some quadratic divisor (if itself is an equation of more than three dimensions).

An equation having the divisor $mx^2 - nx + r$ may be expressed, as in the first article of this chapter, by $E \times \overline{mx^2 - nx + r}$; and if we substitute for x any known quantity a , the sum that will result will have $ma^2 - na + r$ for one of its divisors; and, if we substitute successively for x the progression $a, a - e, a - 2e, a - 3e, \&c.$ the sums that arise from this substitution will have

$$\begin{aligned} & ma^2 - na + r, \\ & m \times \overline{a - e}^2 - n \times \overline{a - e} + r, \\ & m \times \overline{a - 2e}^2 - n \times \overline{a - 2e} + r, \\ & m \times \overline{a - 3e}^2 - n \times \overline{a - 3e} + r, \\ & \&c. \end{aligned}$$

amongst their divisors, respectively.

These terms are not now, as in the last case, in arithmetical progression; but if you subtract them from the squares of the terms $a, a - e, a - 2e, a - 3e, \&c.$ multiplied by m a divisor of the highest term of the proposed equation, that is from

$$\begin{aligned}
 & ma^2, \\
 & m \times \overline{a - e}^2, \\
 & m \times \overline{a - 2e}^2, \\
 & m \times \overline{a - 3e}^2, \text{ \&c. the remainders,} \\
 \hline
 & na - r,
 \end{aligned}$$

$$\begin{aligned}
 & n \times \overline{a - e} - r, \\
 & n \times \overline{a - 2e} - r, \\
 & n \times \overline{a - 3e} - r, \text{ \&c.}
 \end{aligned}$$

shall be in arithmetical progression, having their common difference equal to $n \times e$.

If, for example, we suppose the assumed progression $a, a - e, a - 2e, a - 3e, \text{ \&c.}$ to be $2, 1, 0, -1$, the divisors will be

$$\begin{aligned}
 & 4m - 2n \div r, \\
 & m - n \div r, \\
 & \quad \quad \quad \div r, \\
 & m \div n \div r, \text{ which subtracted from } 4m,
 \end{aligned}$$

$$\begin{aligned}
 & \hline
 & m, 0, m, \text{ leave } 2n - r, \\
 & \quad \quad \quad n - r, \\
 & \quad \quad \quad - r,
 \end{aligned}$$

— $n - r$, an arithmetical progression, whose difference is $\div n$; and whose term arising from the substitution of 0 for x is $-r$.

From which it follows, that by this operation, if the proposed equation has a quadratic divisor, you will find an arithmetical progression

sion that will determine to you n and r , the coefficient m being supposed known; since it is unit, or a divisor of the coefficient of the highest term of the equation. Only you are to observe, that if the first term mx^2 of the quadratic divisor is negative, then in order to obtain an arithmetical progression, you are not to subtract, but add the divisors $-4m - 2n + r$, $-m - n + r$, $\frac{1}{2}r$, $-m + n + r$, to the terms $4m$, m , 0 , m .

§ 68. The general Rule therefore, deduced from what we have said, is,

“ Substitute in the proposed equation for x the terms $2, 1, 0, -1, \&c.$ successively. Find all the divisors of the sums that result, adding and subtracting them from the squares of these numbers, $2, 1, 0, -1, \&c.$ multiplied by a numerical divisor of the highest term of the proposed equation, and take out all the arithmetical progressions that can be found amongst these sums and differences. Let r be that term in any progression that arises from the substitution of $x = 0$, and let $\bar{+}n$ be the difference arising from subtracting that term from the preceding term in the progression; lastly, let m be the foresaid divisor of the highest term; then shall $\underline{mx^2 + nx - r}$ be the divisor that ought to be tried.” And one or other of the

the divisors found in this manner will succeed, if the proposed equation has a quadratic divisor.

§ 69. Suppose, for example, the biquadratic $x^4 - 5x^3 + 7x^2 - 5x - 6 = 0$ is proposed, which has no simple divisor; then to discover if it has any quadratic divisor, the operation is thus :

Suppos.

<i>Suppos.</i>	<i>Re- sults.</i>	<i>Divisors.</i>	<i>Squares.</i>	<i>Sums and differences of the divisors and squares.</i>	<i>Arithmetical Pro- gressions.</i>
$x = 2$	— 12	1, 2, 3, 4, 6, 12.	4	— 8, — 2, 0, 1, 2, 3, 5, 6, 7, 8, 10, 16.	8 1 — 8 3
$x = 1$	— 8	1, 2, 4, 8.	1	— 7, — 3, — 1, 0, 2, 3, 5, 9.	5 — 1 — 7 2
$x = 0$	— 6	1, 2, 3, 6.	0	— 6, — 3, — 2, — 1, 1, 2, 3, 6.	2 — 3 — 6 1
$x = -1$	+ 12	1, 2, 3, 4, 6, 12.	1	— 11, — 5, — 3, — 2, — 1, 0, 2, 3, 4, 5, 7, 13.	— 1 — 5 — 5 0

The first arithmetical progression gives the divisor $x^2 - 3x - 2$; the second gives $x^2 - 2x + 3$: both which succeed, so that the roots of the two equations $x^2 - 3x - 2 = 0$, and $x^2 - 2x + 3 = 0$, viz. $\frac{3 \pm \sqrt{17}}{2}$ and $1 \pm \sqrt{-2}$, are the four roots of the proposed equation, the two last of which are impossible. The divisors which the other arithmetical progressions give, do not succeed.

§ 70. After the same manner a Rule may be discovered for finding the cubic divisors, or those of higher dimensions, of any proposed equation.

Suppose the cubic divisor to be $mx^3 - nx^2 + rx - s$, and by supposing x equal to the terms of the arithmetical progression, it will be as follows:

<i>Suppos.</i>	<i>Results.</i>	<i>Cubes of terms of progr. xm.</i>	<i>1st Differ.</i>	<i>2d Diff.</i>	<i>3d Diff.</i>
$x = 3$	$27m - 9n + 3r - s$	$27m$	$9n - 3r + s$	$5n - r$	$2n$
$x = 2$	$8m - 4n + 2r - s$	$8m$	$4n - 2r + s$	$3n - r$	$2n$
$x = 1$	$m - n + r - s$	m	$n - r + s$	$n - r$	$2n$
$x = 0$	$-s$	0	$+s$	$-n - r$	
$x = -1$	$-m - n - r - s$	$-m$	$n + r + s$		

Where the first differences are not themselves in arithmetical progression, as in the last case, but the differences of its terms, or the second differences, are in arithmetical progression, the common

common difference being $2n$, whence n is known. The quantity r is found in the column of the second difference, and s is always to be assumed some divisor of the last term of the proposed equation, as m is of the coefficient of the first term. Whence all the coefficients of a divisor $mx^3 - nx^2 + rx - s$, with which trial is to be made, may be determined.

If it is a divisor of four dimensions that is required, by proceeding in like manner, you will obtain a series of differences whose second differences are in arithmetical progression. If it is a divisor of five dimensions that is required, you will obtain, in the same manner, a progression whose third differences will be in arithmetical progression; and by observing these progressions, you may discover rules for determining the coefficients of the divisor required.

The foundation of these Rules being, that, if an arithmetical progression $a, a + e, a + 2e, a + 3e, \&c.$ is assumed, the first differences of their squares will be in arithmetical progression; those differences being $2ae + e^2, 2ae + 3e^2, 2ae + 5e^2, \&c.$ whose common difference is $2e^2$. And the second differences of their cubes, and the third differences of their fourth powers are likewise in arithmetical progression, as is easily demonstrated.

§ 71. Hitherto we have only shewn how to find the divisors of equations that involve but *one letter*. But the same rules serve for discovering the divisors when there are two letters, if all the terms have the same dimensions; for, “*by supposing one of the letters equal to unit, find the divisor by the preceding Rules, and then by compleating the dimensions of the divisor, substituting the letter again for unit, you will have the divisor required.*”

Suppose, for example, you are to find the divisor of $8x^3 - 26ax^2 + 11a^2x + 10a^3 = 0$, by putting $a = 1$, that quantity becomes $8x^3 - 26x^2 + 11x + 10 = 0$; whose divisor was found, § 66. to be $2x - 5$; now multiply the term -5 by $+a$, to bring it to the same dimensions as the other, and the divisor required is $2x - 5a$.

§ 72. Besides the method hitherto explained for finding the divisors of lower dimensions that may divide the proposed equation, there are others that deserve to be considered. The following is applicable to equations of all sorts, though we give it only for those of four dimensions.

Let the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ be the equation proposed; and let us suppose it is the product of these two quadratic equations,

x^2

$$\begin{aligned} x^2 - mx + n &= 0 \\ \times x^2 - kx + l &= 0 \end{aligned}$$

$$\frac{\begin{array}{r} x^4 - k \\ -m \end{array} \left. \begin{array}{l} \} x^3 + l \\ \} + n \\ \} + mk \end{array} \right\} \begin{array}{l} x^2 - ml \\ -nk \end{array} \} x + ln = 0,$$

the terms of which will be equal, respectively, to the terms of the proposed equation.

In this equation, l and n being divisors of the last term s , we may consider one of them (*viz.* l) as known; and in order to find m or k , we need only compare the terms of this equation with the terms of the proposed equation respectively, which gives,

$$1^\circ. \quad k + m = p.$$

$$2^\circ. \quad mk + l + n = q.$$

$$3^\circ. \quad ml + nk = r.$$

$$4^\circ. \quad nl = s.$$

Now in order to find an equation that shall involve only k , and known terms, take the two values of m that arise from the first and third equations, and you will find,

$$m = p - k = \frac{r - nk}{l} \quad (\text{because } n = \frac{s}{l}, \text{ by e-}$$

$$\text{quation the fourth}) = \frac{r - \frac{ks}{l}}{l} = \frac{rl - ks}{l^2}; \text{ whence}$$

$$pl^2 - kl^2 = rl - ks, \text{ and } k = \frac{pl^2 - rl}{l^2 - s}; \text{ and}$$

the quadratic $x^2 - kx + l = 0$ becomes

$$x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l = 0.$$

To apply this to practice, you must substitute successively for l all the divisors of s , the last term of the proposed equation, till you find one of them such, that $x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l$ can divide the proposed equation without a remainder.

EXAMPLE.

§ 73. If the equation $x^4 - 6x^3 + 20x^2 - 34x + 35 = 0$ is proposed. The divisors of 35 are 1, 5, 7, 35; if you put $l = 1$, the quadratic that arises will not succeed. But if you suppose $l = 5$, then the equation $x^2 - kx + l$, that is,

$$x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l = 0 \text{ becomes}$$

$$x^2 - \frac{6 \times 25 - 34 \times 5}{25 - 35} x + 5 = 0 = x^2 - 2x + 5,$$

which divides the proposed equation without a remainder, and gives the quotient $x^2 - 4x + 7 = 0$.

“In this operation it is unnecessary to try any divisor l , that exceeds the square root of s , the last term of the proposed equation.” And, if the proposed equation is literal, “you need only try those divisors of the last term that are of two dimensions.”

If, in any supposition of l , the value of k , viz. $\frac{pl^2 - rl}{l^2 - s}$, becomes a fraction, then that supposition is to be rejected, and another value of l to be tried.

§ 74. By comparing the second and fourth equations of the last article, you may obtain another value of k . For $n = q - l - mk = \frac{s}{l}$; so that

(m being equal to $p - k$) $\frac{s}{l} = q - l - pk + k^2$,

and $k^2 - pk + q - l - \frac{s}{l} = 0$. Which gives

$k = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q + l + \frac{s}{l}}$. So that the quadratic divisor required becomes

$$x^2 - \frac{1}{2}p \mp \sqrt{\frac{1}{4}p^2 - q + l + \frac{s}{l}} \times x + l = 0.$$

This divisor must be tried when $l = \frac{s}{l}$, and

at the same time $l = \frac{r}{p}$, the former expression not serving in that case.

By this formula, divisors may be found whose second terms may be *irrational*.

How the divisors of higher equations may be found, when they have any, may be understood from what has been said of those of four dimensions.

SUPPLEMENT to CHAP. VII.

Of the Reduction of Equations by
Surd divisors.

AN equation of four, six, or more dimensions, although it may admit of no rational divisor, may have one that is irrational. As the biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$, which we suppose to be irreducible by any rational divisor, may yet, by adding a square $k^2x^2 + 2klx + l^2$ multiplied into some quantity n , be completed into a square $\overline{x^2 + \frac{1}{2}px + \mathcal{Q}}^2$. In which case we shall have $x^2 + \frac{1}{2}p + \mathcal{Q} = \sqrt{n} \times \overline{kx + l}$, and x is found by the resolution of an affected quadratic equation.

To reduce a biquadratic equation in this manner, we have the following

R U L E.

* If the biquadratic is $x^4 + px^3 + qx^2 + rx + s = 0$, where p, q, r, s , represent the given coefficients under their proper signs, put $q - \frac{1}{4}p^2 = \alpha$, $r - \frac{1}{2}\alpha p = \beta$, $s - \frac{1}{4}\alpha^2 = \zeta$. And for n take some integer common divisor of β and 2ζ , that is not a square number, and which, if either p or r is an odd number, must be odd, and, di-

* *Arithmet. Univers.* pag. 264.

vided by 4, leave the remainder unity. Write likewise for k some divisor of $\frac{\beta}{n}$ if p is an even number, or the half of an odd divisor if p is odd, or 0 if $\beta = 0$. Subtract $\frac{\beta}{nk}$ from $\frac{1}{2}pk$, and let the remainder be l . For \mathcal{Q} put $\frac{\alpha + nk^2}{2}$, and try if, dividing $\mathcal{Q}^2 - s$ by n , the root of the quotient is rational and equal to l ; if it is, add $nk^2x^2 + 2nkx + nl^2$ to both sides of the equation, and extracting the root you shall have $x^2 + \frac{1}{2}px + \mathcal{Q} = n^{\frac{1}{2}} \times \overline{kx + l}$.

EXAMPLE I.

Let the equation proposed be $x^4 + 12x - 17 = 0$, and because $p=0$, $q=0$, $r=12$, $s=-17$, we shall have $\alpha=0$, $\beta=12$, $\zeta=-17$. And β and 2ζ , that is 12 and -34 , having only 2 for a common divisor, it must be $n=2$. Again, $\frac{\beta}{n} = 6$, whose divisors 1, 2, 3, 6, are to be successively put for k , and -3 , $-\frac{3}{2}$, -1 , $-\frac{1}{2}$ for l respectively.

But $\frac{\alpha + nk^2}{2}$, that is k^2 , is equal to \mathcal{Q} , and $\sqrt{\frac{\mathcal{Q}^2 - s}{n}} = l$. And when the even divisors 2 and 6 are substituted for k , \mathcal{Q} becomes 4 and 36, and

and $\mathcal{Q}^2 - s$ being an odd number, is not divisible by $n (= 2)$. Wherefore 2 and 6 are to be set aside. But when 1 and 3 are written for k , \mathcal{Q} is 1 or 9, and $\mathcal{Q}^2 - s$ is 18 or 98 respectively; which numbers can be divided by 2, and the roots of the quotients extracted, being ± 3 and ± 7 ; but only one of them, *viz.* -3 , coincides with l . I put therefore $k=1$, $l=-3$, $\mathcal{Q}=1$, and adding to both sides of the equation $nk^2x^2 + 2nklx + nl^2$, that is, $2x^2 - 12x + 18$, there results $x^4 + 2x^2 + 1 = 2x^2 - 12x + 18$, and extracting the root of each, $x^2 + 1 = \pm \sqrt{2 \times x - 3}$. And again, extracting the root of this last, the four values of x , according to the varieties in the signs, are

$$-\frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}, \quad -\frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}};$$

$$\frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}, \quad \frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}},$$

being the roots of $x^4 + 12x - 17 = 0$, the equation at first proposed.

EXAMPLE II.

Let the equation be $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$, and writing -6 , -58 , -114 , -11 for p , q , r , s , respectively, we have $-67 = \alpha$, $-315 = \beta$, and $-1133\frac{1}{4} = \zeta$. The numbers β and 2ζ , that is -315 and $-\frac{4533}{2}$, have but one common divisor 3, that is $n = 3$. And the

divisors of $-105 = \frac{\beta}{n}$ are 3, 5, 7, 15, 21, 35, and 105. Wherefore I first make trial with $3 = k$, and dividing $\frac{\beta}{n}$ or -105 , by it get the quotient -35 , and this subtracted from $\frac{1}{2}pk = -3 \times 3$, leaves 26, whose half, 13, ought be equal to l . But $\frac{\alpha + nk^2}{2}$, or $\frac{-67 + 27}{2}$, that is, -20 is equal to \mathcal{Q} ; and $\mathcal{Q}^2 - s = 411$, which is indeed divisible by $n = 3$; but the root of the quotient 137 cannot be extracted. Therefore I reject the divisor 3, and try with $5 = k$; by which dividing $\frac{\beta}{n} = -105$, the quotient is -21 , and this taken from $\frac{1}{2}pk = -3 \times 5$, leaves $6 = 2l$. At the same time, $\mathcal{Q} (= \frac{\alpha + nk^2}{2}) = \frac{-67 + 75}{2} = 4$. And $\mathcal{Q}^2 - s$, or $16 + 11$, is divisible by n , and the root of the quotient 9, that is, 3, coincides with l . Whence I conclude that putting $l = 3$, $k = 5$, $\mathcal{Q} = 4$, $n = 3$, adding to both sides of the equation the quantity $nk^2x^2 + 2nk lx + nl^2$, that is, $75x^2 + 90x + 27$, and extracting the roots, it will be

$$x^2 + \frac{1}{2}px + \mathcal{Q} = \sqrt{n \times \overline{kx + l}}, \text{ or}$$

$$x^2 - 3x + 4 = \pm \sqrt{3 \times 5x + 3}.$$

EXAMPLE III.

In like manner in the equation $x^4 - 9x^3 + 15x^2 - 27x + 9 = 0$ writing $-9, +15, -27, +9$, for

for p, q, r, s , there result $\alpha = -5\frac{1}{4}, -50\frac{5}{8} = \beta,$
 $2\frac{7}{64} = \zeta$. The common divisors of β and 2ζ ,
 that is, of $\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45,
 135; but 9 is a square, and 3, 15, 27, 135 di-
 vided by 4 do not leave unity for a remainder,
 as is required when p is an odd number. Set-
 ting these aside there remain only 5 and 45 to
 be tried for n . First let $n = 5$, and the halves
 of the odd divisors of $\frac{\beta}{n} = -\frac{81}{8}$, that is, $\frac{1}{2},$

$\frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{81}{2}$, are to be tried for k . If $k = \frac{1}{2},$
 the quotient $-\frac{81}{4}$ of $\frac{\beta}{n}$ divided by k , taken from

$\frac{1}{2}pk$ or $-\frac{9}{4}$, leaves $18 = 2l$: and $\mathcal{Q}(= \frac{\alpha + nk^2}{2})$
 $= -2, \mathcal{Q}^2 - s = -5$, which is divisible by 5,
 but the root of the quotient -1 , which should
 be $l = 9$, is imaginary. Put next $k = \frac{3}{2}$, and

the quotient of $\frac{\beta}{n}$ divided by k , or of $-\frac{81}{8}$ by
 $\frac{3}{2}$, is $-\frac{27}{4}$. This subtracted from $\frac{1}{2}pk = -\frac{27}{4}$,
 leaves nothing, that is, $l = 0$. Again,

$\mathcal{Q}(= \frac{\alpha + nk^2}{2}) = 3$, and $\mathcal{Q}^2 - s = 0$, and
 $l(= \sqrt{\frac{\mathcal{Q}^2 - s}{n}}) = 0$. From which coincidence

I infer that $n = 5, k = \frac{3}{2}, l = 0$, and adding
 $nk^2x^2 + 2nlkx + nl^2$, that is, $\frac{45}{4}x^2$ to both sides of
 the equation, I find $x^2 - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

Literal equations may be treated much in the same way. And, if you put $n = 1$, the same Rule will give you the *rational* divisor of a biquadratic equation, if it admits of one. Thus for the equation $x^4 - x^3 + 5x^2 + 12x - 6 = 0$, putting $n = 1$ I find $k = \frac{5}{2}$, $l = -\frac{5}{2}$, and the equation is reduced to the two quadratics $x^2 - 3x + 3 = 0$ and $x^2 + 2x - 2 = 0$.

When the divisors of $\frac{\beta}{n}$ are so many that it would be troublesome to make trial with them all for k , their number may be reduced by finding all the divisors of $\alpha s - \frac{1}{4}r^2$. For to one of these, or to its half when odd, the number \mathcal{Q} must be equal.

The ground of this Rule is as follows.

If a biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, in which p, q, r, s , are the given coefficients with their signs, and the equation is supposed clear of fractions and surds; if this equation can be compleated into a square, in the manner already described, we shall have $x^4 + px^3 + qx^2 + rx + s + nk^2x^2 + 2nklx + nl^2 = nk^2x^2 + 2nklx + nl^2 = \overline{x^2 + \frac{1}{2}px + \mathcal{Q}}^2$, that is, $x^4 + px^3 + \overline{q + nk^2} \times x^2 + \overline{r + 2nkl} \times x + s + nl^2 = x^4 + px^3 + \overline{2\mathcal{Q} + \frac{1}{4}p^2} \times x^2 + p\mathcal{Q} \times x + \mathcal{Q}^2$. And comparing the terms, we get these *three* equations,

$$1. \quad q + nk^2 = 2\mathcal{Q} + \frac{1}{4}p^2,$$

$$2. r + 2nkl = p\mathcal{Q},$$

$$3. s + n^2 = \mathcal{Q}^2;$$

in which there being *four* unknown quantities, they can be found only by trial.

The values of \mathcal{Q} , taken from the first and second equations and made equal to each other,

give $n = \frac{\frac{1}{2}pq - \frac{1}{8}p^3 - r}{2kl - \frac{1}{2}pk^2}$ (writing, as in the Rule,

$$q - \frac{1}{4}p^2 = \alpha, \text{ and } r - \frac{1}{2}\alpha p = \beta) = \frac{\beta}{k \times \frac{1}{2}pk - 2l}$$

Whence, if the quantities n, k, l, \mathcal{Q} , are to be found, it follows, (1°.) That n being a divisor of β , giving the quote $k \times \frac{1}{2}pk - 2l$, k will be a divisor of $\frac{\beta}{n}$, giving the quote $\frac{1}{2}pk - 2l$; and that subtracting this quote from $\frac{1}{2}pk$, l will be equal to half the remainder. (2°.) In the first

equation we had $\mathcal{Q} = \frac{\alpha + nk^2}{2}$, and, from the third,

$$l^2 = \frac{\mathcal{Q}^2 - s}{n}. \quad (3^\circ.) \text{ Because } \mathcal{Q} = \frac{1}{2}\alpha + \frac{1}{2}nk^2 \text{ and}$$

$$nl^2 = \mathcal{Q}^2 - s, \quad n = \frac{\frac{1}{4}\alpha^2 - s}{l^2 - \frac{1}{2}ak^2 - \frac{1}{4}nk^4} = \frac{2s - \frac{1}{2}\alpha^2}{k^2 \times \alpha + \frac{1}{2}nk^2 - 2l^2}$$

$$\text{(if } \zeta = s - \frac{1}{4}\alpha^2) = \frac{2\zeta}{k^2 \times \alpha + \frac{1}{2}nk^2 - 2l^2}, \text{ that is, } n$$

divides 2ζ by $k^2 \times \alpha + \frac{1}{2}nk^2 - 2l^2$. And if the several values of the quantities n, k, l, \mathcal{Q} , answer to those conditions, or coincide, it is a proof that they have been rightly assumed; and that adding to the given equation the quantity $n \times \overline{kx + l}^2$, it will be completed into the square $\overline{x^2 + \frac{1}{2}p + \mathcal{Q}}^2$. It

It was said that \mathcal{Q} will always be some divisor of $\alpha s - \frac{1}{4}r^2$. For $\alpha s = \alpha\mathcal{Q}^2 - \alpha nl^2$, and taking from both $\frac{1}{4}r^2 = \frac{1}{4}p^2\mathcal{Q}^2 - p\mathcal{Q}nkl + n^2k^2l^2$, seeing the remainder $\alpha\mathcal{Q}^2 - \alpha + nk^2 \times nl^2 - \frac{1}{4}p^2\mathcal{Q}^2 + p\mathcal{Q}nkl = \alpha\mathcal{Q}^2 - 2\mathcal{Q} \times nl^2 - \frac{1}{4}p^2\mathcal{Q}^2 + p\mathcal{Q}nkl$, has \mathcal{Q} in every term; the thing is manifest.

It is needless to be particular as to the several limitations in the Rule, seeing they follow easily from the algebraical expressions of the quantities. You are not, for instance, if you seek a surd divisor, to take n a square number, for if n is a square number, $\sqrt{n} \times \overline{kx + l}$ would be rational. Or if n is a multiple of a square, as $v \times m^2$, then, at least, $m \times \overline{kx + l}$ would be rational, and n would be depressed to v .

Let us examine one case, when p is even and r odd; and by the Rule n must be an odd number, a multiple of 4 more unity.

1. Seeing $\beta = r - \frac{1}{2}\alpha p$, or $\beta + \frac{1}{2}\alpha p = r$, of the numbers β and $\frac{1}{2}\alpha p$ one must be even and the other odd, that their sum r may be odd. If β is odd, its divisor n must be odd likewise. Suppose β to be even, then $\frac{1}{2}\alpha p$, and consequently $\frac{1}{2}p$ and α are both odd. But if α is odd, $2\zeta = 2s - \frac{1}{2}\alpha^2$ will be half an odd number, and n its divisor is odd.

In this case, \mathcal{Q} is half an odd number. For let it be an integer, $p\mathcal{Q}$ will be an even number. But if \mathcal{Q} is an integer, so must l , because

$s +$

$s + nl^2 = \mathcal{Q}^2$; and $2nkl$ must be even. And $r + 2nkl$ (an odd number) $= p\mathcal{Q}$ an even number, *which is absurd.*

2. Let N represent any number in general, I an odd number; then I say, "every odd number is a multiple of four, more or less unity," that is, $I = 4N \pm 1$. "The square of an odd number is $4N + 1$," and "if from such a square there be taken any multiple of 4, the remainder, if greater than unity, will be $4N + 1$."

Hence it follows that $n = 4N + 1$. For seeing $nl^2 = \mathcal{Q}^2 - s$; because l and \mathcal{Q} are the halves of odd numbers, we have, according to the present notation, $\frac{n \times I^2}{4} = \frac{I^2 - 4s}{4}$, or without the common denominator $n \times I^2 = I^2 - 4s$, that is, $n \times 4N + 1 = 4N + 1$, and consequently, $n = 4N + 1$. For it is not $4N - 1$ but $4N + 1$ that can give the product $4N + 1$.

In like manner the other limitations may be determined. We shall add only this remark more.

That if $k = 0$, necessarily $\beta = 0$; for in this case $\mathcal{Q} = \frac{1}{2}\alpha$ and $p\mathcal{Q} = r$, whence $\beta (= r - \frac{1}{2}\alpha p) = p\mathcal{Q} - p\mathcal{Q} = 0$.

But the converse of this is not universally true, although it is for most part. For β being $= n \times k \times \frac{1}{2}pk - 2l$, although k be a real quantity, if $\frac{1}{2}pk = 2l$ or $l = \frac{1}{4}pk$, β will vanish.

In this case we may use the following

RULE

R U L E I.

“ Of the compound divisors of 2ζ take one that is a multiple of a square number, but itself not a square, as nk^2 , and try if the quotient $\frac{2\zeta}{nk^2} = \alpha + \frac{1}{2}nk^2 - \frac{1}{3}p^2$; if this happens put $x^2 + \frac{1}{2}px + \frac{1}{2}\alpha + \frac{1}{2}nk^2 = \sqrt{n \times \overline{kx + \frac{1}{4}pk}}$.”

Thus in the equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$, where $\alpha = -38$, $\beta = 0$, $2\zeta = -720$: whose divisors are many, but by inspection only I can reject the least to the purpose, and trying with $45 = 5 \times 3 \times 3$, the quotient $\frac{2\zeta}{n} = -16 = -38 + \frac{45}{2} - \frac{1}{2}$. Which gives $x^2 + x + \frac{7}{2} = \sqrt{5 \times 3x + \frac{3}{2}}$.

R U L E II.

“ When $\beta = 0$, and likewise $k = 0$, n being some divisor of 2ζ , if $\frac{\frac{1}{4}\alpha^2 - s}{n}$ is a square number, the root of this square shall be l .”

As in the equation $x^4 + 2x^3 + 6x^2 + 5x - 5 = 0$, it being $\alpha = 5$, $\beta = 0$, $2\zeta = \frac{45}{2}$, taking $n = 5$, $\frac{\frac{1}{4}\alpha^2 - s}{n} = \frac{9}{4}$, that is, $l = \frac{3}{2}$, $Q = \frac{5}{2}$. And $x^2 + x + \frac{5}{2} = \sqrt{5 \times \frac{3}{2}}$.

What

What has been said may lead to the invention or demonstration of similar Rules for the higher equations of even dimensions, if any one pleases to take the trouble.



C H A P. VIII.

Of the Resolution of Equations by
Cardan's Rule, and others of that
kind.

§ 75. **W**E now proceed to shew how an expression of the root of an equation can be obtained that shall involve only known quantities. In *Chapter 11. Part I.* we shewed how to resolve *simple* equations; and in *Chapter 13.* we shewed how to resolve any *quadratic* equation, by adding to the side of the equation that involves the unknown quantity, what was necessary to make it a compleat square, and then extracting the square root on both sides. In § 27 of this *Part*, we gave another method of resolving *quadratic* equations, by taking away the second term: where it appeared that if $x^2 - px + q = 0$, $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$.

§ 76. The second term can be taken away out of any *cubic* equation, by § 25; so that they all may be reduced to this form, $x^3 + qx + r = 0$.

Let

Let us suppose that $x = a + b$; and $x^3 + qx + r = a^3 + 3a^2b + 3ab^2 + b^3 + qx + r = a^3 + 3ab \times a + b + b^3 + qx + r = a^3 + 3abx + b^3 + qx + r$ (by supposing $3ab = -q$) $= a^3 + b^3 + r = 0$.

But $b = -\frac{q}{3a}$, and $b^3 = -\frac{q^3}{27a^3}$, and consequently, $a^3 - \frac{q^3}{27a^3} + r = 0$; or, $a^6 + ra^3 + \frac{q^3}{27}$.

Suppose $a^3 = z$, and you have $z^2 + rz = \frac{q^3}{27}$; which is a quadratic whose resolution gives

$$z = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}} = a^3,$$

$$\text{and } a = \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}; \text{ and}$$

$$x = a + b = a - \frac{q}{3a} = \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}} - \frac{q}{3 \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}}$$

$$: \text{ in which ex-}$$

$$3 \times \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}$$

pressions there are only known quantities.

§ 77. The values of x may be found a little differently, thus:

Since $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, it follows,

that $a^3 + r = \pm \frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, and

$b^3 (= -a^3 - r) = -\frac{1}{2}r + \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$; so that

$$b = \sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}$$
; and $x (= a + b) =$

$$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}$$
;

which gives but one value of x , because when, in the value of a the surd $\sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ is positive, it is negative in the value of b , and there is only the difference of this sign in their values. So that we may conclude

$$x = \sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}$$

§ 79. * The values of x may be discovered without exterminating the second term.

Any cubic equation may be reduced to this form,

$$\left. \begin{aligned} x^3 - 3px^2 - 3px - 2r \\ + 3p^2x - p^3 \\ + 3pq \end{aligned} \right\} = 0,$$

which, by supposing $x = z + p$, will be reduced to $z^3 - 3qz - 2r = 0$, in which the second term is wanting. But by the last article, since $z^3 - 3qz - 2r = 0$, it follows that

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$$
 (if you

* Vid. Phil. Trans. 309.

suppose

suppose that the cubic root of the binomial $r + \sqrt{r^2 - q^3}$ is $m + \sqrt[3]{n} = m + \sqrt{n} + m - \sqrt{n} = 2m$. And since $x = z + p$, it follows that $x = p + 2m$.

§ 79. But as the square root of any quantity is *twofold*, "the cube root is *threefold*," and can be expressed three different ways.

Suppose the cube root of unit is required, and let $y^3 = 1$, or $y^3 - 1 = 0$, then since unit itself is a cube root of 1, one of the values of y is 1, so that the equation $y - 1 = 0$ shall divide the first equation $y^3 - 1 = 0$, and the quotient

$y^2 + y + 1 = 0$ resolved, gives $y = \frac{-1 + \sqrt{-3}}{2}$;

so that the three expressions of $\sqrt[3]{1}$ are 1, $\frac{-1 + \sqrt{-3}}{2}$,

and $\frac{-1 - \sqrt{-3}}{2}$. And, in general, the cube

root of any quantity A^3 may be A , or $\frac{-1 + \sqrt{-3}}{2} \times A$, or $\frac{-1 - \sqrt{-3}}{2} \times A$; so that the

cube root of the binomial $r + \sqrt{r^2 - q^3}$ may be

$m + \sqrt[3]{n}$, as we supposed above, or $\frac{-1 + \sqrt{-3}}{2}$

$\times m + \sqrt[3]{n}$, or $\frac{-1 - \sqrt{-3}}{2} \times m + \sqrt[3]{n}$. And

hence we have three expressions for x , viz.

$$1^\circ. x = p + 2m,$$

$$2^\circ. x = p - m + \sqrt{-3n},$$

$$3^\circ. x = p - m - \sqrt{-3n};$$

and

and these give the three roots of the proposed cubic equation.

EXAMPLE I.

§ 80. Let it be required to find the roots of the equation $x^3 - 12x^2 + 41x - 42 = 0$.

Comparing the coefficients of this equation with those of the general equation

$$x^3 - 3px^2 + 3p^2x - 3pq \left\{ \begin{array}{l} x - 2r \\ - p^3 \\ + 3pq \end{array} \right\} = 0, \text{ you find,}$$

1°. $3p = 12$, so that $p = 4$,

2°. $3p^2 - 3q (= 48 - 3q) = 41$. . . $q = \frac{7}{3}$;

3°. $3pq - p^3 - 2r (= -36 - 2r) = -42$. . $r = 3$;

and consequently, $r^2 - q^3 = 9 - \frac{343}{27} = -\frac{100}{27}$,

and $r + \sqrt{r^2 - q^3} = 3 + \sqrt{-\frac{100}{27}}$. Now the

cube root of this binomial is found to be

$-1 + \sqrt[3]{-\frac{4}{3}}$ ($= m + \sqrt{n}$)*. Whence,

1°. $x = p + 2m = 4 - 2 = 2$.

2°. $x = p - m - \sqrt{-3n} = 4 + 1 - \sqrt{4} = 5 - 2 = 3$;

3°. $x = p - m + \sqrt{-3n} = 5 + 2 = 7$.

So that the three roots of the proposed equation are 2, 3, 7.

You may find other two expressions of the cube root of $3 + \sqrt{-\frac{100}{27}}$, besides $-1 + \sqrt[3]{-\frac{4}{3}}$,

* Section 131. Part I.

Q

viz.

viz. $\frac{3}{2} + \sqrt{-\frac{1}{12}}$, and $-\frac{1}{2} - \sqrt{-\frac{25}{12}}$; but these substituted for $m + \sqrt{n}$ give the same values for x , as are already found.

EXAMPLE II.

In the equation $x^3 + 15x^2 + 84x - 100 = 0$, you find $p = -5$, $q = -3$, $r = 135$, and $r + \sqrt{r^2 - q^3} = 135 + \sqrt{18252}$, whose cube root is $3 + \sqrt{12}$; so that $x (= p + 2m) = -5 + 6 = 1$. The other two values of x , viz. $-8 + \sqrt{-36}$, $-8 - \sqrt{-36}$, are impossible.

After the same manner, you will find that the roots of the equation $x^3 + x^2 - 166x + 660 = 0$, are -15 , $7 \pm \sqrt{5}$. The Rule by which we may discover if any of the roots of an equation are impossible, shall be demonstrated afterwards.

§ 82. The roots of biquadratic equations may be found by reducing them to cubes, thus.

Let the second term be taken away by the Rule given in *Chap. 3*. And let the equation that results be

$$x^{4*} + qx^2 + rx + s = 0.$$

And let us suppose this biquadratic to be the product of these two quadratic equations,

$$x^2 + ex + f = 0$$

$$x^2 - ex + g = 0$$

$$\left. \begin{array}{l} x^{4*} + f \\ + g \\ - e^2 \end{array} \right\} \times x^2 + \left. \begin{array}{l} eg \\ - ef \end{array} \right\} \times x + fg = 0.$$

Where

Where e is the coefficient of x in both equations but affected with contrary signs; because when the second term is wanting in an equation, the sum of the affirmative roots must be equal to the sum of the negative.

Compare now the proposed equation with the above product, and the respective terms put equal to each other will give $f + g - e^2 = q$, $eg - ef = r$, $fg = s$. Whence it follows, that $f + g = q + e^2$, and $g - f = \frac{r}{e}$; and consequently $f + g + g - f (= 2g) = q + e^2 + \frac{r}{e}$,

and $g = \frac{q + e^2 + \frac{r}{e}}{2}$, the same way, you will

find, by subtraction, &c. $f = \frac{q + e^2 - \frac{r}{e}}{2}$, and

$f \times g (= s) = \frac{1}{4} \times q^2 + 2qe^2 + e^4 - \frac{r^2}{e^2}$; and multiplying by $4e^2$, and ranging the terms, you have this equation,

$$e^6 + 2qe^4 + q^2 - 4s \times e^2 - r^2 = 0.$$

Suppose $e^2 = y$, and it becomes $y^3 + 2qy^2 + q^2 - 4sy - r^2 = 0$, a cubic equation whose roots are to be discovered by the preceding articles. Then the values of y being found, their square roots will give e (since $y = e^2$); and having e ,

you will find f and g from the equations

$$f = \frac{q + e^2 - \frac{r}{e}}{2}, \quad g = \frac{q + e^2 + \frac{r}{e}}{2}.$$

Lastly, extracting the roots of the equations $x^2 + ex + f = 0$, $x^2 - ex + g = 0$, you will find the four roots of the biquadratic $x^4 + qx^2 + rx + s = 0$; for either $x = -\frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - f}$, or, $x = +\frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - g}$.

§ 83. Or if you want to find the roots of the biquadratic without taking away the second term; suppose it to be of this form,

$$x^4 - 4px^3 - 2q \left. \vphantom{x^4} \right\} x^2 - 8r \left. \vphantom{x^4} \right\} x - 4s \left. \vphantom{x^4} \right\} = 0,$$

$$+ 4p^2 \left. \vphantom{x^4} \right\} + 4pq \left. \vphantom{x^4} \right\} + q^2 \left. \vphantom{x^4} \right\}$$

and the values of x will be

$$x = p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}}$$

$$\text{and } x = p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}}, \text{ where}$$

a^2 is equal to the root of the cubic,

$$y^3 - p^2 \left. \vphantom{y^3} \right\} y^2 + 2pr \left. \vphantom{y^3} \right\} y - r^2 = 0.$$

$$- q \left. \vphantom{y^3} \right\} + s \left. \vphantom{y^3} \right\}$$

The demonstration is deduced from the last article, as the 78th is from the preceding.

C H A P. IX.

Of the Methods by which you may approximate to the roots of *numerical* Equations by their limits.

§ 84. **W**HEN any equation is proposed to be resolved, first find the limits of the roots (by *Chap. 5.*) as for example, if the roots of the equation $x^2 - 16x + 55 = 0$ are required, you find the limits are 0, 8, and 17, by § 48; that is, the least root is between 0 and 8, and the greatest between 8 and 17.

In order to find the first of the roots, I consider that if I substitute 0 for x in $x^2 - 16x + 55$, the result is positive, *viz.* $+ 55$, and consequently any number betwixt 0 and 8 that gives a positive result, must be less than the least root, and any number that gives a negative result, must be greater. Since 0 and 8 are the limits, I try 4, that is, the mean betwixt them, and supposing $x = 4$, $x^2 - 16x + 55 = 16 - 64 + 55 = 7$, from which I conclude that the root is greater than 4. So that now we have the root limited between 4 and 8. Therefore I next try 6, and substituting it for x we find $x^2 - 16x + 55 = 36 - 96 + 55 = -5$; which result being negative, I conclude that 6 is greater than the root required, which therefore is limited now between 4

Q 3

and

and 6. And substituting 5, the mean between them in place of x , I find $x^2 - 16x + 55 = 25 - 80 + 55 = 0$; and consequently 5 is the least root of the equation. After the same manner you will discover 11 to be the greatest root of that equation.

§ 85. Thus by diminishing the greater, or increasing the lesser limit, you may discover the true root when it is a commensurable quantity. But by proceeding after this manner, when you have two limits, the one greater than the root, the other lesser, that differ from one another but by unit, then you may conclude the root is *incommensurable*.

We may however, by continuing the operation in fractions, approximate to it. As if the equation proposed is $x^2 - 6x + 7 = 0$, if we suppose $x = 2$, the result is $4 - 12 + 7 = -1$, which being negative, and the supposition of $x = 0$ giving a positive result, it follows that the root is betwixt 0 and 2. Next we suppose $x = 1$; whence $x^2 - 6x + 7 = 1 - 6 + 7 = +2$, which being positive, we infer the root is betwixt 1 and 2, and consequently incommensurable. In order to approximate to it, we suppose $x = 1\frac{1}{2}$, and find $x^2 - 6x + 7 = 2\frac{1}{4} - 9 + 7 = \frac{1}{4}$; and this result being positive, we infer the root must be betwixt 2 and $1\frac{1}{2}$. And therefore we
try

try $1\frac{3}{4}$, and find $x^2 - 6x + 7 = \frac{49}{16} - \frac{42}{4} + 7$
 $= 3\frac{1}{16} - 10\frac{8}{16} + 7 = -\frac{7}{16}$, which is nega-
 tive; so that we conclude the root to be betwixt
 $1\frac{3}{4}$ and $1\frac{1}{2}$. And therefore we try next $1\frac{5}{8}$,
 which giving also a negative result, we conclude
 the root is betwixt $1\frac{1}{2}$ (or $1\frac{4}{8}$) and $1\frac{5}{8}$. We try
 therefore $1\frac{9}{16}$, and the result being positive, we
 conclude that the root must be betwixt $1\frac{9}{16}$ and
 $1\frac{10}{16}$, and therefore is nearly $1\frac{9}{16}$.

§ 86. Or you may approximate more easily
 by transforming the equation proposed into an-
 other whose roots shall be equal to 10, 100, or
 1000 times the roots of the former (by § 29.)
 and taking the limits greater in the same pro-
 portion. This transformation is easy; for you
 are only to multiply the second term by 10, 100,
 or 1000, the third term by their squares, the
 fourth by their cubes, &c. The equation of the
 last example is thus transformed into $x^2 - 600x$
 $+ 70000 = 0$, whose roots are 100 times the
 roots of the proposed equation, and whose li-
 mits are 100 and 200. Proceeding as before,
 we try 150, and find $x^2 - 600x + 70000 =$
 $22500 - 90000 + 70000 = 2500$, so that 150
 is less than the root. You next try 175, which
 giving a negative result must be greater than
 the root: and thus proceeding you find the
 root to be betwixt 158 and 159: from which

Q 4

you

you infer that the least root of the proposed equation $x^2 - 6x + 7 = 0$ is betwixt 1.58 and 1.59, being the hundredth part of the root of $x^2 - 600x + 70000 = 0$.

§ 87. If the cubic equation $x^3 - 15x^2 + 63x - 50 = 0$ is proposed to be resolved, the equation of the limits will be (by § 48) $3x^2 - 30x + 63 = 0$, or $x^2 - 10x + 21 = 0$, whose roots are 3, 7; and by substituting 0 for x the value of $x^3 - 15x^2 + 63x - 50$ is negative, and by substituting 3 for x , that quantity becomes positive, $x = 1$ gives it negative, and $x = 2$ gives it positive, so that the root is between 1 and 2, and therefore incommensurable. You may proceed as in the foregoing examples to approximate to the root. But there are other methods by which you may do that more easily and readily; which we proceed to explain.

§ 88. When you have discovered the value of the root to less than an unit (as in this example, you know it is a little above 1) suppose the difference betwixt its real value and the number that you have found nearly equal to it, to be represented by f ; as in this example. Let $x = 1 + f$. Substitute this value for x in this equation, thus,

$$\begin{aligned} x^3 &= 1 + 3f + 3f^2 + f^3 \\ - 15x^2 &= -15 - 30f - 15f^2 \\ + 63x &= 63 + 63f \\ - 50 &= -50 \end{aligned}$$

$$x^3 - 15x^2 + 63x - 50 = -1 + 36f - 12f^2 + f^3 = 0.$$

Now

Now because f is supposed less than unit, its powers f^2, f^3 , may be neglected in this approximation; so that assuming only the two first terms, we have $-1 + 36f = 0$, or $f = \frac{1}{36} = .027$; so that x will be nearly 1.027.

You may have a nearer value of x by considering, that seeing $-1 + 36f - 12f^2 + f^3 = 0$, it follows that

$$f = \frac{1}{36 - 12f + f^2} \text{ (by substituting } \frac{1}{36} \text{ for } f)$$

$$\text{nearly} = \frac{1}{36 - 12 \times \frac{1}{36} + \frac{1}{36} \times \frac{1}{36}} = \frac{1296}{46225} = .02803.$$

§ 89. But the value of f may be corrected and determined more accurately by supposing g to be the difference betwixt its real value, and that which we last found nearly equal to it. So that $f = .02803 + g$. Then by substituting this value for f in the equation

$f^3 - 12f^2 + 36f - 1 = 0$, it will stand as follows,

$$\left. \begin{array}{l} f^3 = 0.000220226 + 0.002357g + 0.0849g^2 + g^3 \\ -12f^2 = - .00942816 - 0.67272g - 12g^2 \\ +36f = 1.00903 + 36g \\ -1 = -1. \end{array} \right\} = 0$$

$$= -0.0003261374 + 35.329637g - 11.9195g^2 + g^3 = 0.$$

Of which the first two terms, neglecting the rest, give $35.329637 \times g = 0.0003261374$, and

$$g = \frac{0.0003261374}{35.329637} = 0.00000923127. \text{ So that}$$

$f = 0.02803923127$; and $x = 1 + f = 1.02803923127$; which is very near the true root of the equation that was proposed.

If

If still a greater degree of exactness is required, suppose b equal to the difference betwixt the true value of g , and that we have already found, and proceeding as above you may correct the value of g .

§ 90. For another example; let the equation to be resolved be $x^3 - 2x - 5 = 0$, and by some of the preceding methods you discover one of the roots to be between 2 and 3. Therefore you suppose $x = 2 + f$, and substituting this value for it, you find

$$\left. \begin{array}{r} x^3 = 8 + 12f + 6f^2 + f^3 \\ - 2x = -4 - 2f \\ - 5 = -5 \end{array} \right\} = 0$$

$$= -1 + 10f + 6f^2 + f^3;$$

from which we find that $10f = 1$ nearly, or $f = 0.1$. Then to correct this value, we suppose $f = 0.1 + g$, and find

$$\left. \begin{array}{r} f^3 = 0.001 + 0.03g + 0.3g^2 + g^3 \\ 6f^2 = 0.06 + 1.2g + 6g^2 \\ 10f = 1 + 10g \\ - 1 = -1 \end{array} \right\} = 0$$

$$= 0.061 + 11.23g + 6.3g^2 + g^3,$$

$$\text{so that } g = \frac{-0.061}{11.23} = -0.0054.$$

Then by supposing $g = -0.0054 + b$, you may correct its value, and you will find that the root required is nearly 2.09455147.

§ 91. It is not only one root of an equation that can be obtained by this method, but, by making use of the other limits, you may discover the other roots in the same manner. The equation of § 87, $x^3 - 15x^2 + 63x - 50 = 0$, has for its limits 0, 3, 7, 50. We have already found the least root to be nearly 1.028039. If it is required to find the middle root, you proceed in the same manner to determine its nearest limits to be 6 and 7; for 6 substituted for x gives a positive, and 7 a negative result. Therefore you may suppose $x = 6 + f$, and by substituting this value for x in that equation, you find $f^3 + 3f^2 - 9f + 4 = 0$, so that $f = \frac{4}{9}$ nearly. Or since $f = \frac{4}{9 - 3f - f^2}$, it is (by substituting $\frac{4}{9}$ for f) $f = \frac{4}{9 - \frac{4}{3} - \frac{16}{81}} = \frac{324}{605}$, whence $x = 6 + \frac{324}{605}$ nearly. Which value may still be corrected as in the preceding articles. After the same manner you may approximate to the value of the highest root of the equation.

§ 92. "In all these operations, you will approximate sooner to the value of the root, if you take the three last terms of the equation, and extract the root of the quadratic equation consisting of these three terms."

Thus, in § 88, instead of the two last terms of the equation $f^3 - 12f^2 + 36f - 1 = 0$, if
you

you take the three last and extract the root of the quadratic $12f^2 - 36f + 1 = 0$, you will find $f = .028031$, which is much nearer the true value than what you discover by supposing $36f - 1 = 0$.

It is obvious that this method extends to all equations.

§ 93. " By assuming equations affected with general coefficients, you may, by this method, deduce General Rules or *Theorems* for approximating to the roots of proposed equations of whatever degree."

Let $f^3 - pf^2 + qf - r = 0$ represent the equation by which the fraction f is to be determined, which is to be added to the limit, or subtracted from it, in order to have the near value of x . Then $qf - r = 0$ will give $f = \frac{r}{q}$.

But since $f = \frac{r}{f^2 - pf + q}$, by substituting $\frac{r}{q}$ for f , we have this Theorem for finding f nearly, *viz.*

$$f = \frac{r}{\frac{r^2}{q^2} - \frac{pr}{q} + q} = \frac{q^2 \times r}{q^3 - pr + r^2}$$

After the same manner, if it is a biquadratic, by which f is to be determined, as $f^4 - pf^3 + qf^2 - rf + s = 0$, then f being very little, we shall have $f = \frac{s}{r}$; which value is corrected by considering

sidering that $f = \frac{s}{r - qf + pf^2 - f^3}$ (by substituting $\frac{s}{r}$ for f) $= \frac{s}{r - \frac{qs}{r} + \frac{ps^2}{r^2} - \frac{s^3}{r^3}}$, whence we have

this Theorem for all biquadratic equations,

$$f = \frac{r^3 \times s}{-s^3 + ps^2r - qsr^2 + r^4}$$

§ 94. Other Theorems may be deduced by assuming the three terms of the equation, and extracting the root of the quadratic which they form.

Thus, to find the value of f in the equation $f^3 - pf^2 + qf - r = 0$ where f is supposed to be very little, we neglect the first term f^3 , and extract the root of the quadratic $pf^2 - qf + r = 0$, or of $f^2 - \frac{q}{p} \times f + \frac{r}{p} = 0$; and we find

$$f = \frac{q}{2p} \pm \sqrt{-\frac{r}{p} + \frac{q^2}{4p^2}} = \frac{q \pm \sqrt{q^2 - 4pr}}{2p} \text{ nearly.}$$

But this value of f may be corrected by supposing it equal to m , and substituting m^3 for f^3 in the equation $f^3 - pf^2 + qf - r = 0$, which will give $m^3 - pf^2 + qf - r = 0$, and $pf^2 - qf + r - m^3 = 0$; the resolution of which quadratic equation

gives $f = \frac{q \pm \sqrt{q^2 - 4pr + 4pm^3}}{2p}$, very near the true value of f .

After the same manner you may find like Theorems for the roots of biquadratic equations, or of equations of any dimension whatever.

§ 95. In general, let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \dots + A = 0$ represent an equation of any dimensions n , where A is supposed to represent the absolute known term of the equation. Let k represent the limit next less than any of the roots, and supposing $x = k + f$, substitute the powers of $k + f$ instead of the powers of x , and there will arise $\overline{k+f^n} + p \times \overline{k+f^{n-1}} + q \times \overline{k+f^{n-2}} + r \times \overline{k+f^{n-3}} + \dots + A = 0$, or by involution, disposing the terms according to the dimensions of $f \dots \dots \dots$

$$\begin{array}{l}
 k^n + nk^{n-1} \times f + n \times \frac{n-1}{2} k^{n-2} f^2 + \dots \\
 p k^{n-1} + p \times n-1 k^{n-2} \times f + p \times n-1 \times \frac{n-2}{2} k^{n-3} f^2 + \dots \\
 q k^{n-2} + q \times n-2 k^{n-3} \times f + q \times n-2 \times \frac{n-3}{2} k^{n-4} f^2 + \dots \\
 r k^{n-3} + r \times n-3 k^{n-4} \times f + r \times n-3 \times \frac{n-4}{2} k^{n-5} f^2 + \dots \\
 \dots
 \end{array}
 \left. \vphantom{\begin{array}{l} k^n \\ p k^{n-1} \\ q k^{n-2} \\ r k^{n-3} \\ \dots \end{array}} \right\} + A = 0,$$

where

where neglecting all the powers of f after the first two terms, you find

$$f = \frac{A - k^n - pk^{n-1} - qk^{n-2} - rk^{n-3}, \text{ \&c.}}{nk^{n-1} + pxn - 1k^{n-2} + qxn - 2k^{n-3} + rxn - 3k^{n-4}, \text{ \&c.}}, \text{ and}$$

$$x (= k + f) = \frac{-A + n - 1k^n + pxn - 2k^{n-1} + qxn - 3k^{n-2} + rxn - 4k^{n-3}, \text{ \&c.}}{nk^{n-1} + pxn - 1k^{n-2} + qxn - 2k^{n-3} + rxn - 3k^{n-4}, \text{ \&c.}}$$

whence particular Theorems for extracting the roots of equations may be deduced.

§ 96. "By this method you may discover Theorems for approximating to the roots of
pure

pure powers ;” as to find the n root of any number A ; suppose k to be the nearest less root in integers, and that $k + f$ is the true root, then shall $k^n + nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2, \&c. = A$; and, assuming only the two first terms,

$f = \frac{A - k^n}{nk^{n-1}}$: or, more nearly, taking the three first terms,

$$f = \frac{A - k^n}{nk^{n-1} + n \times \frac{n-1}{2}k^{n-2}f}, \text{ and (taking } \frac{A - k^n}{nk^{n-1}} = f \text{)}$$

$$f = \frac{A - k^n}{nk^{n-1} + \frac{n^2-n}{2}k^{n-2} \times \frac{A - k^n}{nk^{n-1}}} = \frac{A - k^n}{nk^{n-1} + \frac{n-n}{2k} \times A - k^n}$$

(putting $m = A - k^n$) = $\frac{km}{nk^n + \frac{n-1}{2} \times m}$; which

is a *rational* Theorem for approximating to f .

You may find an *irrational* Theorem for it, by assuming the three first terms of the power of $k + f$, viz. $k^n + nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2 = A$.

For $nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2 = A - k^n = m$; and resolving this quadratic equation, you find

$$f = -\frac{k}{n-1} \pm \sqrt{\frac{2m}{n \times n-1 \times k^{n-2} + \frac{k^2}{n-1}}} =$$

$$-\frac{k}{n-1} \pm \sqrt{\frac{2mn - 2m + nk^n}{n \times n-1 \times k^{n-2}}}$$

In the application of these Theorems, when a near value of f is obtained, then adding it to k , substitute the aggregate in place of k in the formula, and you will, by a new operation, obtain a more correct value of the root required; and, by thus proceeding, you may arrive at any degree of exactness.

Thus to obtain the cube root of 2, suppose $k = 1$, and $f (= \frac{km}{nk^n + \frac{n-1}{2}m}) = \frac{1}{4} = 0.25$.

In the second place, suppose $k = 1.25$, and f will be found, by a new operation, equal to 0.009921, and consequently, $\sqrt[3]{2} = 1.259921$ nearly. By the irrational Theorem, the same value is discovered for $\sqrt[3]{2}$.



C H A P. X.

Of the Method of *Series* by which you may approximate to the roots of *literal* equations.

§ 97. **I**F there be only two letters, x and a , in the proposed equation, suppose a equal to unit, and find the root of the numeral equation that arises from the substitution, by the rules of the last chapter. Multiply these roots by a , and the products will give the roots of the proposed equation.

Thus the roots of the equation $x^2 - 16x + 55 = 0$ are found, in § 84, to be 5 and 11. And therefore the roots of the equation $x^2 - 16ax + 55a^2 = 0$, will be $5a$ and $11a$. The roots of the equation $x^3 + a^2x - 2a^3 = 0$ are found by enquiring what are the roots of the numeral equation $x^3 + x - 2 = 0$, and since one of these is 1, it follows that one of the roots of the proposed equation is a ; the other two are *imaginary*.

§ 98. If the equation to be resolved involves more than two letters, as $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$, then the value of x may be exhibited in a series having its terms composed of the powers of a and y with their respective coefficients;

ficients; which will “converge the sooner the less y is in respect of a , if the terms are continually multiplied by the powers of y , and divided by those of a .” Or, “will converge the sooner the greater y is in respect of a , if the terms be continually multiplied by the powers of a , and divided by those of y .” Since when y is very little in respect of a , the terms $y, \frac{y^2}{a}, \frac{y^3}{a^2}, \frac{y^4}{a^3}, \frac{y^5}{a^4}, \&c.$ decrease very quickly. If y vanish in respect of a , the second term will vanish in respect of the first, since $\frac{y^2}{a} : y :: y : a$. And after the same manner $\frac{y^3}{a^2}$ vanishes in respect of the term immediately preceding it.

But when y is vastly great in respect of a , then a is vastly great in respect of $\frac{a^2}{y}$, and $\frac{a^2}{y}$ in respect of $\frac{a^3}{y^2}$; so that the terms $a, \frac{a^2}{y}, \frac{a^3}{y^2}, \frac{a^4}{y^3}, \frac{a^5}{y^4}, \&c.$ in this case decrease very swiftly. In either case, the series converge swiftly that consist of such terms; and a few of the first terms will give a near value of the root required.

§ 99. If a series for x is required from the proposed equation that shall converge the sooner, the less y is in respect of a ; to find the first term of this series, we shall suppose y to vanish; and extracting the root of the equation

$$R^2 = x^3$$

$x^3 + a^2x - 2a^3 = 0$, consisting of the remaining parts of the equation that do not vanish with y , we find, by § 97, that $x = a$; which is the true value of x when y vanishes, but is only near its value when y does not vanish, but only is very little. To get a value still nearer the true value of x , suppose the difference of a from the true value to be p , or that $x = a + p$. And substituting $a + p$ in the given equation for x , you will find,

$$\left. \begin{array}{r} x^3 = a^3 + 3a^2p + 3ap^2 + p^3 \\ + a^2x = a^3 + a^2p \\ - 2a^3 = -2a^3 \\ + ayx = a^2y + apy \\ - y^3 = -y^3 \end{array} \right\} = 0$$

$$\left. \begin{array}{r} = 4a^2p + 3ap^2 + p^3 \\ \quad a^2y + apy - y^3 \end{array} \right\} = 0.$$

But since, by supposition, y and p are very little in respect of a , it follows that the terms $4a^2p$, a^2y , where y and p are separately of the *least* dimensions, are vastly great in respect of the rest; so that, in determining a near value of p , the rest may be neglected: and from $4a^2p + a^2y = 0$, we find $p = -\frac{1}{4}y$. So that $x = a + p = a - \frac{1}{4}y$, nearly.

Then to find a nearer value of p , and consequently of x , suppose $p = -\frac{1}{4}y + q$, and substituting this value for it in the last equation, you will find,

p^2

$$\begin{array}{l}
 p^3 = -\frac{1}{6}y^3 + \frac{3}{16}y^2q - \frac{3}{4}yq^2 + q^3 \\
 3ap^2 = \frac{3}{16}ay^2 - \frac{3}{2}ayq + 3aq^2 \\
 4a^2p = -a^2y + 4a^2q \\
 ay p = -\frac{1}{4}ay^2 + ayq \\
 a^2y = a^2y \\
 -y^3 = -y^3
 \end{array}
 \left. \vphantom{\begin{array}{l} p^3 \\ 3ap^2 \\ 4a^2p \\ ay p \\ a^2y \\ -y^3 \end{array}} \right\} = 0$$

$$\begin{array}{l}
 = -\frac{6}{16}y^3 + \frac{3}{16}y^2q - \frac{3}{4}yq^2 + q^3 \\
 -\frac{1}{16}ay^2 - \frac{1}{2}ayq + 3aq^2 \\
 + 4a^2q
 \end{array}
 \left. \vphantom{\begin{array}{l} -\frac{6}{16}y^3 \\ -\frac{1}{16}ay^2 \\ + 4a^2q \end{array}} \right\} = 0.$$

And since, by the supposition, q is very little in respect of p , which is nearly $= -\frac{1}{4}y$, therefore q will be very little in respect of y ; and consequently all the terms of the last equation will be very little in respect of these two, *viz.* $-\frac{1}{16}ay^2$, $+ 4a^2q$, where y and q are of least dimensions separately: particularly the term $-\frac{1}{2}ayq$ is little in respect of $4a^2q$, because y is very little in respect of a ; and it is little in respect of $-\frac{1}{16}ay^2$, because q is little in respect of y .

Neglect therefore the other terms, and supposing $-\frac{1}{16}ay^2 + 4a^2q = 0$, you will have $q = \frac{1}{64} \times \frac{y^2}{a}$; so that $x = a - \frac{1}{4}y + \frac{1}{64} \times \frac{y^2}{a}$. And by proceeding in the same manner you will find $x = a - \frac{y}{4} + \frac{y^2}{64a} - \frac{131y^3}{512a^2} + \frac{509y^4}{16384a^3} -$, &c.

§ 100. When it is required to find a series for x that shall converge sooner, the *greater* y is

in respect of any quantity a , you need only suppose a to be very little in respect of y , and proceed by the same reasoning as in the last example on the supposition of y being very little.

Thus, to find a value for x in the equation $x^3 - a^2x + ayx - y^3 = 0$ that shall converge the sooner the greater y is in respect of a , suppose a to vanish, and the remaining terms will give $x^3 - y^3 = 0$, or $x = y$. So that when y is vastly great, it appears that $x = y$ nearly.

But to have the value of x more accurately, put $x = y + p$, then

$$\left. \begin{array}{r} x^3 = y^3 + 3y^2p + 3yp^2 + p^3 \\ - a^2x = - a^2y - a^2p \\ + ayx = ay^2 + ayp \\ - y^3 = - y^3 \end{array} \right\} = 0$$

$$= + 3y^2p + 3yp^2 + p^3 - a^2y - a^2p + ay^2 + ayp :$$

where the terms $3y^2p + ay^2$ become vastly greater than the rest, y being vastly greater than a or p ; and consequently $p = -\frac{1}{3}a$ nearly.

Again, by supposing $p = -\frac{1}{3}a + q$, you will transform the last equation into

$$\left. \begin{array}{r} - \frac{8}{27}a^3 + 3y^2q + 3yq^2 + q^3 \\ - a^2y - ayq - aq^2 \\ - \frac{2}{3}a^2q \end{array} \right\} = 0 ;$$

where the two terms $3y^2q - a^2y$ must be vastly greater than any of the rest, a being vastly less than

than y , and q vastly less than a , by the supposition; so that $3qy^2 - a^2y = 0$, and $q = \frac{a^2}{3y}$ nearly. By proceeding in this manner, you may correct the value of y , and find that

$$x = y - \frac{1}{3}a + \frac{a^2}{3y} + \frac{a^3}{81y^2} - \frac{8a^4}{243y^3}, \text{ \&c.}$$

which series converges the sooner the greater y is supposed to be taken in respect of a .

§ 101. In the solution of the first *Example* those terms were always compared in order to determine p , q , r , &c. in which y and those quantities p , q , r , &c. were *separately* of fewest dimensions. But in the second *Example*, those terms were compared in which a and the quantities p , q , r , &c. were of least dimensions separately. And these always are the proper terms to be compared together, because they become vastly greater than the rest, in the respective hypotheses.

In general; to determine the first, or any, term in the series, *such terms of the equation are to be assumed together only, as will be found to become vastly greater than the other terms*; that is, which give a value of x , which substituted for it in all the terms of the equation shall raise the dimensions of the other terms all above, or all below, the dimensions of the assumed terms, according as y is supposed to be vastly little, or vastly great in respect of a .

Thus to determine the first term of a converging series expressing the value of x in the last equation $x^3 - a^2x + ayx - y^3 = 0$, the terms ayx and $-y^3$ are not to be compared together, for they would give $x = \frac{y^2}{a}$, which substituted for x , the equation becomes

$$\frac{y^6}{a^3} - ay^2 + y^3 - y^3 = 0,$$

where the first term is of more dimensions than the assumed terms ayx , $-y^3$; and the second of fewer: so that the two first terms cannot be neglected in respect of the two last, neither when y is very great nor very little, compared with a . Nor are the terms x^3 , ayx , fit to be compared together in order to obtain the first term of a series for x , for the like reason.

But x^3 may be compared with $-a^2x$, as also $-a^2x$ with $-y^3$ for that end. These two give the first term of a series that converges the sooner the less y is; as $x^3 = y^3$ gives the first term of a series that converges the sooner the greater y is. The last series was given in the preceding article. The comparing x^3 with $-a^2x$ gives these two series,

$$x = a - \frac{1}{2}y - \frac{y^2}{8a} + \frac{7y^3}{16a^2} - \frac{59y^4}{128a^3}, \text{ \&c.}$$

$$x = -a + \frac{1}{2}y + \frac{y^2}{8a} + \frac{9y^3}{16a^2} + \frac{69y^4}{128a^3}, \text{ \&c.}$$

The comparing $-a^2x$ with $-y^3$ gives

$$x = -\frac{y^3}{a^2} - \frac{y^4}{a^3} - \frac{y^5}{a^4} - \frac{y^6}{a^5} - \text{ \&c.}$$

And

And these series give three values of x when y is very little; the last of which is itself also very little in that case, as it appears indeed from the equation, that when y vanishes, the three values of x become $+a$, $-a$, and 0 , because when y vanishes, the equation becomes $x^3 - a^2x = 0$, whose roots are a , $-a$, 0 .

§ 102. It appears sufficiently from what we have said, that when an equation is proposed involving x and y , and the value of x is required in a converging series, the difficulty of finding the first term of the series is reduced to this; “to find what terms assumed in order to determine a value of x expressed in some dimensions of y and a will give such a value of it, as substituted for it in the other terms will make them all of more dimensions of y , or all of less dimensions of y , than those assumed terms.”

To determine this, draw BA and AC at right angles to each other, compleat the parallelogram $ABCD$ and divide it into equal squares, as in the figure. In these squares place the powers of x from A towards C , and the powers of y from A towards B , and in any other square place that power of x that is directly below it in the line AC , and that power of y that is in a parallel with it in the line AB ; so that the index of x in any square may express its distance from the line AB , and the index of y in any square

square may express its distance from the line AC. Of this square we are to observe,

	Z								D
Z	y^7	y^7x	y^7x^2	y^7x^3	y^7x^4	y^7x^5	y^7x^6	y^7x^7	
	y^6	y^6x	y^6x^2	y^6x^3	y^6x^4	y^6x^5	y^6x^6	y^6x^7	
	y^5	y^5x	y^5x^2	y^5x^3	y^5x^4	y^5x^5	y^5x^6	y^5x^7	
	y^4	y^4x	y^4x^2	y^4x^3	y^4x^4	y^4x^5	y^4x^6	y^4x^7	
	y^3	y^3x	y^3x^2	y^3x^3	y^3x^4	y^3x^5	y^3x^6	y^3x^7	
	y^2	y^2x	y^2x^2	y^2x^3	y^2x^4	y^2x^5	y^2x^6	y^2x^7	
	y	yx	yx^2	yx^3	yx^4	yx^5	yx^6	yx^7	
A	O	x	x^2	x^3	x^4	x^5	x^6	x^7	C

The diagram includes several diagonal lines: a solid line from the top-left corner to the bottom-right corner, a dotted line from the top-left corner to the bottom-right corner, and another dotted line from the top-right corner to the bottom-left corner. Labels 'a', 'b', 'c', 'd', 'e', and 'p' are placed near the intersections of these lines with the grid.

1. That the terms are not only in geometrical progression in the vertical column AB, or the horizontal AC, and their parallels; but also in the terms taken in any oblique straight line whatever; for in any such terms it is manifest that the *indices* of y and x will be in arithmetical progression. The indices of y , because those terms will remove equally from the line AC, or approach equally to it, and the indices of

of y in any such terms are as their distances from that line AC . The indices of x will also be in arithmetical progression, because these terms equally remove from, or approach to the line AB . Thus for Example, in the terms y^7 , y^5x , y^3x^2 , yx^3 , the indices of y decreasing by the common difference 2, while the indices of x increase in the progression of the natural numbers, the common ratio of the terms is $\frac{x}{y^2}$. It follows,

2. From the last observation, that “if any two terms be supposed equal, then all the terms in the same straight line with these terms, will be equal;” because by supposing these two terms equal, the common ratio is supposed to be a ratio of equality; and from this it follows, that “if you substitute every where for x the value that arises for it by supposing any two terms equal, expressed in the powers of y , the dimensions of y in all the terms that are found in the same straight line will be equal;” but “the dimensions of y in the terms *above* that line will be *greater* than in those in that line;” and “the dimensions of y in the terms *below* the said line will be *less* than its dimensions in that line.” Thus, by supposing $y^7 = yx^3$, we find $x^3 = y^6$, or $x = y^2$; and substituting this value for x in all the squares, the dimensions of y in the terms y^7 , y^5x , y^3x^2 , yx^3 , which are all found in the

the same straight line, will be 7, but the dimensions in all the terms above that line will be more than 7, and in all the terms below that line will be less than 7.

§ 103. From these two observations we may easily find a method for discovering what terms ought to be assumed from an equation in order to give a value for x which shall make the other terms all of *higher*, or all of *lower* dimensions of y than the assumed terms: *viz.* “after all the terms of the equation are ranged in their proper squares (by the last article) such terms are to be assumed as lie in a straight line, so that the other terms either lie all above the straight line, or fall all below it.”

For example, suppose the equation proposed is $y^7 - ay^5x + y^4x^3 + a^2yx^4 - ax^6 = 0$, then marking with an asterisk the squares in the last article which contain the same dimensions of x and y as the terms in the equation, imagine a ruler ZE to revolve about the first square marked at y^7 , and as it moves from A towards C, it will first meet the term ay^5x , and while the ruler joins these two terms, all the other terms lie above it: from which you infer, that by supposing these terms equal, you shall obtain a value of x , which substituted for it, will give all the other terms of higher dimensions of y , than those terms: and hence we conclude that the value of x deduced from supposing these terms equal,

equal, viz. $\frac{y^2}{a}$, is the first term of a series that will converge the sooner the less y is in respect of a .

If the ruler be made to revolve about the same square the contrary way from D towards C, it will first meet the term y^4x^3 , and by supposing $y^7 + y^4x^3 = 0$, we find $y = x$, which gives the first term of a series for x , that converges the sooner the greater that y is. And this is the celebrated Rule invented by Sir *Isaac Newton* for this purpose.

§ 104. This Rule may be extended to equations having terms that involve powers of x and y with *fractional* or *surd* indices; “by taking distances from A in the lines AC and AB proportional to these fractions and surds,” and thence determining the situation of the terms of the proposed equation in the parallelogram ABCD.

It is to be observed also, that when the line joining any two terms has all the other terms on one side of it, by them you may find the first term of a converging series for x , and thus “various such series can be deduced from the same equation.” As, in the last Example, the line joining y^5x and yx^4 has all the terms above it; and therefore supposing $-ay^5x + a^2yx^4 = 0$, we find $x^3 = \frac{y^4}{a}$, and $x = \frac{y^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, which is the first

term

term of another converging series for x . Again, the straight line joining yx^4 and x^6 has all the other terms above it, and therefore, supposing $a^2yx^4 - ax^6 = 0$, we find $ay = x^2$, and $x = a^{\frac{1}{2}}y^{\frac{1}{2}}$, the first term of another series for x , converging also the sooner the less y is. There are two series converging the sooner the greater y is, to be deduced from supposing $y^7 = -y^4x^3$, or $y^4x^3 = ax^6$. And, to find all these series, "describe a *polygon Zabcd*, having a term of the equation in each of its angles, and including all the other terms within it, then a series may be found for x , by supposing any two terms equal that are placed in any two adjacent angles of the polygon."

§ 105. If the ruler ZE be made to move parallel to itself, all the terms which it will touch at once will be of the same dimensions of y : for they will bear the same proportion to one another as the terms in the line ZE themselves. The terms which the ruler will touch first will have fewer dimensions of y , than those it touches afterwards in the progress of its motion, if it moves towards D; but more dimensions than they, if it moves towards A. The terms in the straight line ZE, serve to determine the first term of the converging series required. These with the terms it touches afterwards serve to determine the succeeding terms of the converging

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ing series; all the rest vanishing compared with these, when y is very little and the ruler moves from A towards D, or when y is vastly great and the ruler moves from D towards A.

§ 106. The same Author gives another method for discovering the first term of a series that shall converge the sooner the less y is. "Suppose the term where y is separately of fewest dimensions to be Dy^l ; compare it successively with the other terms, as with $Ey^m x^s$, and observe where $\frac{l-m}{s}$ is found *greatest*; and putting $\frac{l-m}{s} = n$, Ay^n will be the first term of a series that shall converge the sooner the less y is:" for in that case Dy^l and $Ey^m x^s$ will be infinitely greater than any other terms of the proposed equation. Suppose $Fy^e x^k$ is any other term of the equation, and, by the supposition, $\frac{l-m}{s} (=n)$ is greater than $\frac{l-e}{k}$, and consequently, multiplying by k , you find nk greater than $l-e$, and $nk + e$ greater than l ; now if for x you substitute Ay^n , then $Fy^e x^k = FA^k y^{nk+e}$, which therefore will vanish compared with Dy^l (since $nk + e$ is greater than l) when y is infinitely little. Thus therefore all the terms will vanish compared with Dy^l and $Ey^m x^s$ which are supposed equal; and consequently they will give the first term of a series that will converge the sooner the less y is.

§ 107.

§ 107. If you observe “when $\frac{l-m}{s}$ is found *least* of all, and suppose it equal to n , then will Ay^n be the first term of a series that will converge the sooner the *greater* y is.” For in that case Dy^l and $Ey^m x^s$ will be infinitely greater than $Fy^e x^k$, because $\frac{l-m}{s} (= n)$ being less than $\frac{l-e}{k}$, it follows that nk is less than $l-e$, and $nk + e$ less than l , and consequently $Fy^e x^k (= FA^k y^{nk} + e)$ vastly less than Dy^l , when y is very great.

After the same manner, if you compare any term $Dy^l x^b$, where both x and y are found, with all the other terms, and observe where $\frac{l-m}{s-b}$ is found *greatest* or *least*, and suppose $\frac{l-m}{s-b} = n$, then may Ay^n be the first term of a converging series. For supposing that $Fy^e x^k$ is any other term of the equation, if $\frac{l-m}{s-b} (= n)$ is greater than $\frac{l-e}{k-b}$, then shall $nk - nb$ be greater than $l-e$, and $nk + e$ greater than $l + nb$. But $nk + e$ are the dimensions of y in $Fy^e x^k$ when $x = Ay^n$, and $l + nb$ are the dimensions of y in $Ey^m x^s$; therefore $Fy^e x^k$ is of more dimensions of y than $Ey^m x^s$, and therefore vanishes compared to it when y is supposed infinitely little. In the same manner, if $\frac{l-m}{s-b}$ is less than $\frac{l-e}{k-b}$, then will

will $Ey^m x^s$ be infinitely greater than $Fy^e x^k$, when y is infinite.

§ 108. When the first term (Ay^n) of the series is found by the preceding method, then by supposing $x = Ay^n + p$, and substituting this binomial and its powers for x and its powers, there will arise an equation for determining p the second term of the series. This new equation may be treated in the same manner as the equation of x , and by the Rule of § 103, the terms that are to be compared in order to obtain a near value of p , may be discovered; by means of which terms p may be found: which suppose equal to $By^n + r$, then by supposing $p = By^n + r + q$, the equation may be transformed into one for determining q the third term of the series, and by proceeding in the same manner you may determine as many terms of the series as you please; finding $x = Ay^n + By^n + r + Cy^n + 2r + Dy^n + 3r$, &c. where the dimensions of y ascend or descend according as r is positive or negative; and always “in arithmetical progression, that this value of x being substituted for it in the proposed equation, the terms involving y and its powers may fall in with one another, so that more than one may always involve the same dimension of y , which may mutually destroy each other and make the whole equation vanish, as it ought to do.”

It is obvious that as the dimensions of y in $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. are in an arithmetical progression whose difference is r , the square, cube, or any power s of $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} + \text{\&c.}$ will consist of terms wherein the dimensions of y will constitute an arithmetical progression having the same common difference r ; for these dimensions will be $sn, sn + r, sn + 2r, sn + 3r$, &c. Therefore, if in any term $Ey^m x^s$ you substitute for x the series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. the terms of the series expressing $Ey^m x^s$ will consist of these dimensions of y , viz. $m + sn, m + sn + r, m + sn + 2r, m + sn + 3r$, &c. and by a like substitution in any other term as $Fy^e x^k$, the dimensions of y will be $e + nk, e + nk + r, e + nk + 2r, e + nk + 3r$, &c. The former series of indices must coincide with the latter series, that the terms in which they are found may be compared together, and be found equal with opposite signs so as to destroy one another, and make the whole equation vanish.

The first series consists of terms arising by adding some multiple of r to $m + sn$, the latter by adding some multiple of r to $e + nk$; and that these may coincide, some multiple of r added to $m + sn$ must be equal to some other multiple of r added to $e + nk$. From which it appears that the difference of $m + sn$ and $e + nk$ is always
a mul-

a multiple of r ; and consequently that r is a divisor of the difference of dimensions of y in the terms $Ey^m x^s$ and $Fy^e x^k$, supposing $x = Ay^n$. It follows therefore "that r is a common divisor of the differences of the dimensions of y in the terms of the equation, when you have substituted Ay^n for x in all the terms." And if r be assumed equal to the *greatest* common divisor (excepting some cases afterward to be mentioned) you will have the true form of a series for x . And now the dimensions $y^n, y^{n+r}, y^{n+2r}, y^{n+3r},$ &c. being known, there remains only, by calculation, to determine the general coefficients $A, B, C, D,$ &c. in order to find the series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} + \dots = x$.

§ 109. This leads us to Sir *Isaac Newton's* second general method of series; which consists in assuming a series with undetermined coefficients expressing x , as $Ay^n + By^{n+r} + Cy^{n+2r} + \dots$ where $A, B, C,$ &c. are supposed as yet unknown, but n and r are discovered by what we have already demonstrated; and substituting this every where for x , you must suppose, in the new equation that arises, the sum of all the terms that involve the same dimension of y to vanish, by which means you will obtain particular equations, the *first* of which will give A ; the *second* B , the *third* C , &c. and these values

being substituted in the assumed series for A , B , C , &c. the series for x will be obtained as far as you please.

Let us apply, for example, this method to the equation (of § 98) $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$. Suppose it is required to find a series converging the sooner the less y is: its first term (by § 99 or 102) is found to be a , so that $n = 0$. Substitute a for x in the equation, and the terms become $a^3 + a^3 - 2a^3 + a^2y - y^3$, and the differences of the indices are 0, 1, 2, 3; whose greatest common measure is 1, so that $r = 1$. Assume therefore $x = A + By + Cy^2 + Dy^3$, &c. and substitute this series for x in the equation. Then

$$\begin{aligned} x^3 &= A^3 + 3A^2By + 3AB^2y^2 + B^3y^3 + \mathcal{E}^2c. \\ &\quad + 3A^2Cy^2 + 3A^2Dy^3 + \mathcal{E}^2c. \\ &\quad + 6ABCy^3 + \mathcal{E}^2c. \\ + a^2x &= a^2A + a^2By + a^2Cy^2 + a^2Dy^3 + \mathcal{E}^2c. \\ + ayx &= aAy + aBy^2 + aDy^3 + \mathcal{E}^2c. \\ - 2a^3 &= - 2a^3 \\ - y^3 &= \dots \dots \dots - 1 \times y^3. \end{aligned}$$

Now since $x^3 + a^2x + ayx - 2a^3 - y^3 = 0$, it follows that the sum of these series involving y must vanish. But that cannot be if the coefficient of every particular term does not vanish. For every term where y is infinitely little, is infinitely greater than the following terms, so that
if

if every term does not vanish of itself, the addition or subtraction of the following terms which are infinitely less than it, or of the preceding terms which are infinitely greater, cannot destroy it; and therefore the whole cannot vanish. It appears therefore that $A^3 + a^2A - 2a^3 = 0$, is an equation for determining A , and gives $A = a$.

In order to determine B , you must suppose the sum of the coefficients affecting y to vanish, viz. $\overline{3A^2B + a^2B + aA} \times y = 0$, or, since $A = a$, $4a^2By + a^2y = 0$, and $B = -\frac{1}{4}$.

To determine C , in the same manner suppose $3AB^2y^2 + 3A^2Cy^2 + a^2Cy^2 + aBy^2 = 0$, or, substituting for A and B their values already found, $\frac{3ay^2}{16} + 4a^2Cy^2 - \frac{ay^2}{4} = 0$, and consequently $C = \frac{1}{64a}$. And, by proceeding in the same manner, $D = \frac{131}{512a^2}$, so that $x = a - \frac{1}{4}y + \frac{1}{64a}y^2 + \frac{131}{512a^2}y^3$, &c. as we found before in § 99.

§ 110. By this method you may transfer series from one undetermined quantity to another, and obtain Theorems for the *reversion* of series.

Suppose that $x = ay + by^2 + cy^3 + dy^4 + \text{Ec}$. and it is required to express y by a series consisting of the powers of x . It is obvious that

when x is very little, y is also very little, and that in order to determine the first term of the series, you need only assume $x = ay$. And therefore $y = \frac{x}{a}$; so that $n = 1$. By substituting $\frac{x}{a}$ for y , you find the dimensions of x in the terms will be 1, 2, 3, 4, &c. so that $r = 1$ also. You may therefore assume $y = Ax + Bx^2 + Cx^3 + Dx^4 + \text{\&c.}$ And by the substitution of this value of y you will find

$$\begin{aligned} ay &= aAx + aBx^2 + aCx^3 + \text{\&c.} \\ by^2 &= bA^2x^2 + 2bABx^3 + \text{\&c.} \\ cy^3 &= cA^3x^3 + \text{\&c.} \\ \text{\&c.} & \qquad \qquad \qquad \text{\&c.} \end{aligned}$$

But the first term being already found to be $\frac{x}{a}$, you have $A = \frac{1}{a}$; and since $aB + bA^2 = 0$, it follows that $B = -\frac{b}{a^3}$. After the same manner you will find $C = \frac{2b^2 - ac}{a^4}$. Whence $y = \frac{x}{a} - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^4}x^3 + \text{\&c.}$

§ 111. Suppose again you have $ax + bx^2 + cx^3 + dx^4 + \text{\&c.} = gy + by^2 + iy^3 + ky^4, \text{\&c.}$ to find x in terms of y . You will easily see, by § 103, that the first term of the series for x is $\frac{yy}{a}$, that $n = 1$, $r = 1$. Therefore assume $x = Ay + By^2 + Cy^3, \text{\&c.}$ and by substituting this

this value for x and bringing all the terms to one side, you will have

$$\begin{aligned} ax &= aAy + aBy^2 + aCy^3 + \mathcal{E}c. \\ bx^2 &= bA^2y^2 + 2bABy^3 + \mathcal{E}c. \\ cx^3 &= cA^3y^3 + \mathcal{E}c. \\ \mathcal{E}c. & \qquad \qquad \qquad \mathcal{E}c. \end{aligned}$$

$$\begin{aligned} -gy &= -gy \\ -by^2 &= \dots\dots\dots -by^2 \\ -iy^3 &= \dots\dots\dots -iy^3 \\ \mathcal{E}c. & \qquad \qquad \qquad \mathcal{E}c. \end{aligned}$$

From whence we see, first, that $aA = g$, and $A = \frac{g}{a}$. 2^o. That $aB + bA^2 - b = 0$, and $B = \frac{b}{a} - \frac{bg^2}{a^3}$. 3^o. That $aC + 2bAB + cA^3 - i = 0$, and therefore $C = \frac{i - 2bAB - cA^3}{a}$. And thus the three first terms of the series $Ay + By^2 + Cy^3, \mathcal{E}c.$ are known*.

§ 112. Before we conclude it remains to clear a difficulty in this method that has embarrassed some late ingenious writers, concerning “the value of r to be assumed when two or more of the values of the first term of a series for expressing x are found equal;” a correction of the preceding Rule being necessary in that case. And the author of that correction having only collected it from experience, and given it us

* See Mr. De Moivre in *Phil. Transf.* 240.

without proof, it is the more necessary to demonstrate it here.

It is to be observed then, that in order that the series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} + \&c.$ may express x , it is not only necessary that when it is substituted for x in the proposed equation $Dy^l + Ey^m x^s + Fy^e x^k = 0$, the indices $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. should fall in with the indices $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. in order that the terms may be compared together to determine the coefficients A , B , C , &c. but it is also necessary, that in the particular equations for determining any of those coefficients, as B for example, those terms that involve B should not destroy each other. Thus the equation $3A^2B - 3A^2B - aA = 0$ can never determine B , because $3A^2B - 3A^2B = 0$, and thus B exterminates itself out of the equation; besides the contradiction arising from $-A = 0$, when A perhaps has been determined already to be equal to some real quantity.

In order to know how to evite this absurdity, let us suppose that the first order of terms in the proposed equation are, as before, Dy^l , $Ey^m x^s$, &c. and if Ay^n is found to be the first term of a series for x , then the dimensions of y in the first order of terms, arising by substituting in them Ay^n for x . will be $m + ns$, and the dimensions of y arising by substituting

Ay^n

$Ay^n + By^{n+r} + Cy^{n+2r}$, &c. for x will be $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. Suppose that $Fy^{e \times k}$ is the next order of terms, and, by the same substitution, the dimensions of y arising from it will be

$$\text{(because } Fy^{e \times k} = Fy^e \times \overbrace{A^n + B^{n+r} + C^{n+2r} + \mathcal{E}c.}^k \text{)} \\ = FA^{ky^e + nk} + kFB A^{k-1}y^{e + nk + r}, \text{ \&c.} \text{) } e + nk,$$

$e + nk + r$, $e + nk + 2r$, &c. Now it is plain that $e + nk$ must coincide with some of the dimensions $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. that the terms involving them may be compared together. And therefore, as we observed in § 108, r must be the difference of $e + nk$ and $m + ns$, or some divisor of that difference. *In general*, r must be assumed such a divisor of that difference as may allow not only $e + nk$ to coincide with some one of the series $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. but as may make all the indices of the other orders besides $e + nk$ likewise to coincide with one of that series: that is, if $Gy^f x^b$ is another term in the equation, r must be so assumed that the series $f + nb$, $f + nb + r$, $f + nb + 2r$, &c. arising by substituting in it $Ay^n + B^{n+r} + C^{n+2r}$, &c. for x , may coincide somewhere with the first series $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. And therefore we said, in § 108, "that r must be assumed so as to be equal to some common divisor of the differences of the indices $m + ns$, $e + nk$, $f + nb$, which arise in the proposed equation

by

by substituting in it for x the first term already known Ay^n ." For by assuming r equal to a common divisor of these differences, the three series

$$m + ns, m + ns + r, m + ns + 2r, m + ns + 3r, \&c.$$

$$e + nk, e + nk + r, e + nk + 2r, e + nk + 3r, \&c.$$

$$f + nb, f + nb + r, f + nb + 2r, f + nb + 3r, \&c.$$

will coincide with one another, since some multiples of r added to $m + ns$ will give $e + nk$ and all that follow it in the *second* series, and some multiples of r added to $m + ns$ will also give $f + nb$ and all that follow it in the *third* series. It is also obvious, that, if no particular reason hinder it, r ought to be assumed equal to the *greatest* common measure of these differences. For example, if the indices $m + ns$, $e + nk$, $f + nb$, happen to be in arithmetical progression, then r ought to be assumed equal to the common difference of the terms, and the first of the second series will coincide with the second of the first, and the first of the third series will coincide with the second of the second series, and with the third of the first, and so on.

§ 113. These things being well understood, we are next to observe that after you have substituted $Ay^n + By^n + r + Cy^n + 2r$, &c. for x in the first order of terms in the equation, the terms that involve $m + ns$ dimensions of y will destroy one another; for $x - Ay^n$ must be a divisor of
the

the aggregate of these terms, since they give Ay^n as one value of x : let $\overline{x - Ay^n} \times P$ represent that aggregate, and, substituting for x its value $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. that aggregate becomes $\overline{Ay^n + By^{n+r} + Cy^{n+2r}}$, &c. $-\overline{Ay^n} \times P = \overline{By^{n+r} + Cy^{n+2r}}$, &c. $\times P$. Now the lowest dimension in $\overline{x - Ay^n} \times P$ was supposed to be $m + ns$, whence the dimension of P , in the same terms, will be $m + ns - n$, and the lowest dimension in $\overline{By^{n+r} + Cy^{n+2r} + \mathcal{E}c.} \times P$ will be $n + r + m + ns - n = m + ns + r$. Suppose again that two values of x , determined from the first order of terms, are equal, and then $\overline{x - Ay^n}^2$ will be a divisor of that aggregate of the first order of terms. Suppose that aggregate now $\overline{x - Ay^n}^2 \times P$, which by substitution of $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. for x will become $\overline{By^{n+r} + Cy^{n+2r} + \mathcal{E}c.}^2 \times P$, in which the lowest term will now be of $m + ns$ dimensions, since in $\overline{x - Ay^n}^2 \times P$ the lowest term is supposed of $m + ns$ dimensions; and consequently, in these terms, the dimension of P itself is $m + ns - 2n$.

In general, if the number of values of x supposed equal to Ay^n be p , then must $\overline{x - Ay^n}^p$ be a divisor of the aggregate of the terms of the first order. And that aggregate being expressed by $\overline{x - Ay^n}^p \times P$, in the lowest terms, the dimensions

fions of y in P will be $m + ns - pn$, that in $\overline{x - Ay^n}^p$ they may be $m + ns$, as we always suppose. Substitute in $\overline{x - Ay^n}^p \times P$ for $x - Ay^n$ its value $By^n + r + Cy^n + 2r + \mathcal{E}c.$ and in the result $\overline{By^n + r + Cy^n + 2r + \mathcal{E}c.}^p \times P$ the lowest dimensions of y will be $pn + pr + m + ns - pn = m + ns + pr.$

§ 114. From what has been said we conclude, that when you have substituted for x in the first order of terms of the equation proposed the series $Ay^n + By^n + r + Cy^n + 2r + \mathcal{E}c.$ the first term of which Ay^n is known, and the values of x whose number is p are found equal, then the terms arising that involve $m + ns, m + ns + r, m + ns + 2r, \&c.$ till you come to $m + ns + pr,$ will destroy each other and vanish; so that the first term with which the terms of the second order $e + nk$ can be compared must be that which involves $m + ns + pr$; and therefore supposing $e + nk = m + ns + pr,$ or $r = \frac{e + nk - m - ns}{p}$ "the highest value you can give r must be the difference of $e + nk$ and $m + ns$ divided by p the number of equal values of the first term of the series." If this value of r is a common measure of all the differences of the indices, then is it a just value of r ; but if it is not, such a value of r must be assumed, as may measure this and all the differences: that is, "such a value as may be

be the greatest common measure of the least difference divided by p (*viz.* $\frac{e + nk - m - ns}{p}$) and of the common measure of all the differences." For thus the indices $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. will coincide with $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. and with $f + nb$, $f + nb + r$, $f + nb + 2r$, &c. and you shall always have terms to be compared together sufficient to determine B , C , D , &c. the general coefficients of the series assumed for x .

§ 115. To all this it may be added, that if $x - Ay^n$ be a divisor of the aggregate of the terms of the *second* order Fy^ex^k , &c. then, by substituting for x the series $Ay^n + By^{n+r} + Cy^{n+2r} + \mathcal{E}c.$ there vanish not only as many terms of the series involving $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. as there are equal values of the first term Ay^n ; but the terms involving $e + nk$ dimensions of y vanish also; and therefore it is then only necessary that $e + nk + r$ coincide with $m + ns + pr$, so that, in that case, you need only take $r = \frac{e + nk - m - ns}{p - 1}$. And if $\overline{x - Ay^n}^{p-1}$ be a divisor of the aggregate of the second order of terms, then the terms (after substituting for x the series $Ay^n + By^{n+r} + Cy^{n+2r}$ &c.) which involve $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. will vanish to the term $e + nk + \overline{p-1} \times r$; so that,

sup-

supposing $e + nk + \overline{p-1} \times r = m + ns + pr$, you have $r = e + nk - m - ns$, that is, to the *least* difference of the indices $m + ns$, $e + nk$, $f + nb$, &c. provided that difference be a measure of the other differences; although there may be as many values of the first term of the series equal as there are units in p . Or, if that does not happen, r must be taken, as formerly, equal to the greatest common measure of the differences.

§ 116. Suppose that the orders of terms of the equation can be expressed the *first* by $\overline{x - Ay^n}^p \times P$, the *second* by $\overline{x - Ay^n}^q \times Q$, the *third* by $\overline{x - Ay^n}^l \times L$, &c. and suppose that $Ey^m x^s$ is one of the first, $Fy^e x^k$ one of the second, $Gy^f x^b$ one of the third, and so on: then it is plain that, substituting for x the series $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. the lowest term that will remain in the first will be $m + ns + pr$ dimensions of y , the lowest term that will remain in the second will be of $e + nk + qr$, and the lowest term remaining in the third of $f + nb + lr$ dimensions of y . For by the same reasoning as we used, in § 113, to demonstrate that, in the first order of terms $\overline{x - Ay^n}^p \times P$, the lowest dimensions of y are $m + ns + pr$, we shall find that, in the subsequent orders, the lowest dimensions of y in the terms $\overline{x - Ay^n}^q \times Q = \overline{By^{n+r} + Cy^{n+2r} \&c.}^q \times Q$ must

must be $e + nk - qn + qn + qr = e + nk + qr$, and so of the other terms $x - Ay^n \times L$ the lowest dimensions must be $f + nb + lr$. The indices therefore of the terms that do not vanish being

$$\begin{aligned} & * * m + ns + pr, \\ & * * * * e + nk + qr, \\ & * * * * * f + nb + lr, \end{aligned}$$

if r be taken equal to $\frac{e + nk - m - ns}{p - q}$, then will

$m + ns + pr$ and $e + nk + qr$ coincide: and if at the same time r be a divisor of $f + nb - m - ns$, and be found in it a number of times greater

than $p - l$, or if r be less than $\frac{f + nb - m - ns}{p - l}$,

then r will be rightly assumed. In general, "take

all the quotients $\frac{e + nk - m - ns}{p - q}$, $\frac{f + nb - m - ns}{p - l}$,

and either the *least* of these, or a number whose denominator, exceeding $p - q$ by an integer,

measures it and all the differences $f + nb - m - ns$, gives r ;" supposing p , q , and l integers. But if p , q , and l are fractions, you are to

"take r so that it be equal to $\frac{e + nk - m - ns}{p - q + K} =$

$\frac{f + nb - m - ns}{p - l + M}$, and so that K and M may be

integers." Suppose, for example, $m + ns = \frac{7}{3}$,

$$p =$$

$p = \frac{5}{2}$; $e + nk = \frac{10}{3}$, $q = \frac{3}{2}$; $f + nb = \frac{9}{2}$, and

$l = \frac{1}{2}$: then putting — — — — — ($r =$)

$$\frac{e + nk - m - ns}{p - q + K} = \frac{1}{1 + K} = \frac{f + nb - m - ns}{p - l + M} = \frac{\frac{1}{6}}{2 + M}$$

$M = \frac{1}{6} + \frac{1}{6}K$; whence it is easily seen that 5

and 11 are the least integers that can be assumed

for K and M . And that $r = \frac{1}{1 + K} = \frac{1}{6}$; and

therefore $m + ns + pr = \frac{33}{12}$, $e + nk + qr = \frac{43}{12}$, and

$f + nb + lr = \frac{55}{12}$. That is, the terms of the first

series whose dimensions are $m + ns + \overline{p + K} \times r$,

$m + ns + \overline{p + M} \times r$ fall in with the first terms

of the second and third series respectively*.

* See on this subject, *Colson*. Epist. in Animadv. D. *Moi-vrei*. *Taylor* Meth. Incr. *Stirling* Lin. iij Ord. *Gravesande* Append. Elem. Algebrae. *Stewart* on the Quadrature of Curves.



C H A P. XI.

Of the Rules for finding the number of impossible Roots in an equation.

§ 117. **T**HE number of *impossible* roots in an equation may, for most part, be found by this

R U L E.

“ Write down a series of fractions whose denominators are the numbers in this progression 1, 2, 3, 4, 5, &c. continued to the number which expresses the dimension of the equation. Divide every fraction in the series by that which precedes it, and place the quotients in order over the middle terms of the equation. And if the square of any term multiplied into the fraction that stands over it gives a product greater than the rectangle of the two adjacent terms, write under the term the sign +, but if that product is not greater than the rectangle, write —; and the signs under the extreme terms being +, there will be as many imaginary roots as there are changes of the signs from + to —, and from — to +.

T

Thus,

Thus, the given equation being $x^3 + px^2 + 3p^2x - q = 0$, I divide the second fraction of the series $\frac{3}{1}$, $\frac{2}{2}$, $\frac{1}{3}$, by the first, and the third by the second, and place the quotients $\frac{1}{3}$ and $\frac{1}{3}$ over the middle terms in this manner;

$$\begin{array}{ccccccc} & & \frac{1}{3} & & \frac{1}{3} & & \\ & & + & & + & & \\ x^3 & + & px^2 & + & 3p^2x & - & q = 0. \\ & + & - & + & + & & \end{array}$$

Then because the square of the second term multiplied into the fraction that stands over it, that is, $\frac{1}{3} \times p^2x^4$ is less than $3p^2x^4$ the rectangle under the first and third terms, I place under the second term the sign $-$: but as $\frac{1}{3} \times 9p^4x^2 (= 3p^4x^2)$ the square of the third term multiplied into its fraction is greater than *nothing*, and consequently much greater than $-pqx^2$ the negative product of the adjoining terms, I write under the third term the sign $+$. I write $+$ likewise under x^3 and $-q$ the first and last terms; and finding in the signs thus marked two changes, one from $+$ to $-$, and another from $-$ to $+$, I conclude the equation has two impossible roots.

In like manner the equation $x^3 - 4x^2 + 4x - 6 = 0$ has two impossible roots;

$$\begin{array}{ccccccc} & & \frac{1}{3} & & \frac{1}{3} & & \\ & & + & & + & & \\ x^3 & - & 4x^2 & + & 4x & - & 6 = 0; \\ & + & + & - & + & & \end{array}$$

and

and the equation $x^{4*} - 6x^2 - 3x - 2 = 0$ the same number ;

$$\begin{array}{ccccccc}
 & \frac{3}{8} & & \frac{4}{9} & & \frac{2}{3} & \\
 x^{4*} & - & 6x^2 & - & 3x & - & 2 = 0. \\
 + & + & + & - & + & &
 \end{array}$$

For the series of fractions $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$ yields, by dividing them as the Rule directs, the fractions $\frac{3}{8}$, $\frac{4}{9}$, $\frac{2}{3}$ to be placed over the terms. Then the square of the second term, which is *nothing*, multiplied by the fraction over it being still *nothing*, and yet greater than $-6x^6$ the negative product of the adjacent terms, I write under (*) the term that is wanting, the sign $\frac{1}{+}$, and proceeding as in the former examples, I conclude, from the two changes that happen in the series $+ + + - +$, that the equation has two of its roots impossible.

The same way we discover two impossible roots in the equation

$$\begin{array}{ccccccc}
 & \frac{2}{5} & & \frac{1}{2} & & \frac{1}{2} & & \frac{2}{5} \\
 x^5 & - & 4x^4 & \frac{1}{+} & 4x^3 & - & 2x^2 & - & 5x & - & 4 = 0. \\
 + & + & - & + & + & + & + & + &
 \end{array}$$

When two or more terms are wanting in the equation, under the first of such terms place the sign $-$, under the second $+$, under the third $-$, and so on alternately; only when the two terms to the right and left of the deficient terms

have contrary signs, you are always to write the sign $+$ under the last deficient term.

As in the equations

$$\begin{array}{cccccc} x^5 + ax^4 & * & * & * & + a^5 = 0 \\ + & + & - & + & - & + \\ \text{and } x^5 + ax^4 & * & * & * & - a^5 = 0 \\ + & + & - & + & + & + \end{array}$$

the first of which has *four* impossible roots, and the other *two*. Thus likewise the equation

$$\begin{array}{cccccccc} x^7 - 2x^6 + 3x^5 - 2x^4 + x^3 & * & * & - 3 = 0 \\ + & - & + & - & + & - & + & + \end{array}$$

has *six* impossible roots.

Hence too we may discover if the imaginary roots lie hid among the affirmative, or among the negative roots. For the signs of the terms which stand over the signs below that change from $+$ to $-$ and $-$ to $+$, shew, by the number of their variations, how many of the impossible roots are to be reckoned affirmative; and that there are as many negative imaginary roots as there are repetitions of the same sign. As in the equation

$$\begin{array}{cccccc} x^5 - 4x^4 + 4x^3 - 2x^2 - 5x - 4 = 0 \\ + & + & - & + & + & + \end{array}$$

the signs ($- + -$) of the terms $- 4x^4 + 4x^3 - 2x^2$ which stand over the signs $+ - +$ pointing

ing out two affirmative roots *, we infer that two impossible roots lie among the affirmative : and the three changes of the signs in the equation (+ — + — —) giving three affirmative roots and two negative, the five roots will be one real affirmative, two negative, and two imaginary affirmatives. If the equation had been

$$\begin{array}{cccccc}
 x^5 & - & 4x^4 & - & 4x^3 & - & 2x^2 & - & 5x & - & 4 & = & 0, \\
 + & & + & & - & & - & & + & & + & &
 \end{array}$$

the terms $-4x^4 - 4x^3$ that stand over the first variation + —, shew, by the repetition of the sign —, that one imaginary root is to be reckoned negative, and the terms $-2x^2 - 5x$ that stand over the last variation — +, give, for the same reason, another negative impossible root ; so that the signs of the equation (+ — — — —) giving one affirmative root, we conclude that of the four negative roots, two are imaginary.

This always holds good unless, which sometimes may happen, there are more impossible roots in the equation than are discoverable by the Rule.”

This Rule hath been investigated by several eminent Mathematicians in various ways ; and others, similar to it, invented and published †. But the

* See § 19.

† See Stirling's *Lineae* iij. Ord. *Newton.* p. 59. *Phil. Transf.* N^o 394, 404, 408.

original Rule being, on account of its simplicity and easy application, if not preferable to all others, at least the fittest for this place, it is sufficient to direct the Reader where he may find the subject more fully treated; and to add the demonstration our Author has given of it towards the end of his Letter to Mr. Folkes, Phil. Trans. N^o 408, as it depends only on what has been demonstrated in Chap. 5. concerning the limits of the roots of equations.

§ 118. Let $ax^2 \pm px \pm q = 0$ be any adfectèd quadratic equation; and, by § 88, Part I. its roots will be $\frac{1}{2a} \times \overline{+p \pm \sqrt{p^2 + 4aq}}$: whence it is plain that, the sign of q in the given equation being $+$, the roots will be impossible as oft as $4aq$ is greater than p^2 , or $\frac{1}{4}p^2$ less than $a \times q$.

§ 119. It was shewn, in general (§ 45 — 50) that the roots of the equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$, are the limits of the roots of the equation $\overline{nx^{n-1} - n-1 \times Ax^{n-2} + n-2 \times Bx^{n-3} \&c. = 0}$, or of any equation that is deduced from it by multiplying its terms by any arithmetical progression $l + d, l + 2d, l + 3d, \&c.$ and conversely the roots of this new equation will be the limits of the roots of the proposed equation $x^n - Ax^{n-1} + Bx^{n-2} \&c. = 0$.

And that if any roots of the equation of the limits are impossible, there must be some roots of the proposed equation impossible.

§ 120. Let $x^3 - Ax^2 + Bx - C = 0$ be a cubic equation, and the *equation of limits* $3x^2 - 2Ax + B = 0$. If the two roots of this last are imaginary, there are two imaginary roots of the given equation $x^3 - Ax^2 + Bx - C = 0$, by the last *Art.* But, by the preceding *Art.* this happens as oft as $\frac{1}{3}A^2$ is less than B ; and, in that case, the given equation has two imaginary roots.

Again, multiplying the terms of the equation by the terms of the progression, 0, -1, -2, -3, we get another *equation of the limits* $Ax^2 - 2Bx + 3C = 0$; whose two roots, and consequently two roots of the given equation, are imaginary when $\frac{1}{3}B^2$ is less than $A \times C$.

Hence likewise the biquadratic $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, will have two imaginary roots, if two roots of the equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ be imaginary; or if two roots of the equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ be imaginary. But two roots of the equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ must be imaginary, when two roots of the quadratic $6x^2 - 3Ax + B = 0$, or of the quadratic $3Ax^2 - 4Bx + 3C = 0$, are imaginary, because the roots of these quadratic equations are the *limits* of the roots of that cubic; and for the same reason two roots of the cubic equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ must be imaginary, when the roots of the quadratic $3Ax^2 - 4Bx + 3C = 0$,

or of the quadratic $Bx^2 - 3Cx + 6D = 0$ are impossible. Therefore two roots of the biquadratic $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ must be imaginary when the roots of any one of these three quadratic equations $6x^2 - 3Ax + B = 0$, $3Ax^2 - 4Bx + 3C = 0$, $Bx^2 - 3Cx + 6D = 0$ become imaginary; that is, when $\frac{3}{8}A^2$ is less than B , $\frac{4}{9}B^2$ less than AC , or $\frac{3}{8}C^2$ less than BD .

§ 121. By proceeding in the same manner, you may deduce from any equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$, as many quadratic equations as there are terms excepting the first and last, whose roots must be all real quantities, if the proposed equation has no imaginary roots. The quadratic deduced from the three first terms $x^n - Ax^{n-1} + Bx^{n-2}$ will manifestly have this form, $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \&c. \times x^2 - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \&c. \times Ax + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \&c. \times B = 0$, continuing the factors in each till you have as many as there are units in $n-2$. Then dividing the equation by all the factors $n-2, n-3, n-4, \&c.$ which are found in each coefficient, the equation will become $n \times \overline{n-1} \times x^2 - \overline{n-1} \times 2Ax + 2 \times 1 \times B = 0$, whose roots will be imaginary, by § 118, when $n \times \overline{n-1} \times 2 \times 4B$ exceeds $\overline{n-1}^2 \times 4A^2$, or when B exceeds $\frac{n-1}{2n}A^2$: so that the proposed equation must have

have some imaginary roots when B exceeds $\frac{n-1}{2n}A^2$. The quadratic equation deduced in the same manner from the three first terms of the equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} \&c. = 0$, will have this form, $\frac{n-1}{n} \times \frac{n-2}{n} \times \frac{n-3}{n} \&c. \times Ax^2 - \frac{n-2}{n} \times \frac{n-3}{n} \times \frac{n-4}{n} \&c. \times 2Bx + \frac{n-3}{n} \times \frac{n-4}{n} \times \frac{n-5}{n} \&c. \times 3C = 0$; which dividing by the factors common to all the terms, is reduced to $\frac{n-1}{n} \times \frac{n-2}{n} \times Ax^2 - \frac{n-2}{n} \times \frac{n-3}{n} \times 2Bx + \frac{n-3}{n} \times \frac{n-4}{n} \times 3C = 0$, whose roots must be imaginary when $\frac{2}{3} \times \frac{n-2}{n-1} \times B^2$ is less than AC ; and therefore in that case some roots of the proposed equation must be imaginary.

§ 122. In general, let $Dx^{n-r+1} - Ex^{n-r} + Fx^{n-r-1}$ be any three terms of the equation, $x^n - Ax^{n-1} + Bx^{n-2} \&c. = 0$, that immediately follow one another; multiply the terms of this equation first by the progression $n, n-1, n-2, \&c.$ then by the progression $n-1, n-2, n-3, \&c.$ then by $n-2, n-3, n-4, \&c.$ till you have multiplied by as many progressions as there are units in $n-r-1$: then multiply the terms of the equation that arises, as often by the progression $0, 1, 2, 3, \&c.$ as there are units in $r-1$, and you will at length arrive at a quadratic of this form;

$$\frac{n-r+1}{n} \times \frac{n-r}{n} \times \frac{n-r-1}{n} \times \frac{n-r-2}{n} \&c. \times \frac{r-1}{r} \times \frac{r-2}{r} \times \frac{r-3}{r} \times \frac{r-4}{r} \&c. \times Dx^2$$

$$- \frac{n-r}{1} \times \frac{n-r-1}{2} \times \frac{n-r-2}{3} \times \frac{n-r-3}{4} \&c.$$

$$\times r \times \frac{r-1}{1} \times \frac{r-2}{2} \times \frac{r-3}{3} \&c. \times Ex$$

$$+ \frac{n-r-1}{1} \times \frac{n-r-2}{2} \times \frac{n-r-3}{3} \times \frac{n-r-4}{4}$$

$$\&c. \times r+1 \times r \times r-1 \times r-2 \&c. \times F = 0:$$
 and dividing by the factors $n-r-1, n-r-2,$
 $\&c.$ and $r-1, r-2, \&c.$ which are found in
 each coefficient, this equation will be reduced to

$$\frac{n-r+1}{1} \times \frac{n-r}{2} \times 1 \times Dx^2 - \frac{n-r}{1} \times 2 \times r$$

$$\times 2Ex + 2 \times 1 \times r + 1 \times rF = 0,$$
 whose roots must
 be imaginary, by § 118, when $\frac{n-r}{n-r+1} \times \frac{r}{r+1} \times E^2$
 is less than DF . From which it is manifest, that
 if you divide each term of this series of fractions
 $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}, \&c. \frac{n-r+1}{r}, \frac{n-r}{r+1},$
 by that which precedes it, and place the quo-
 tients above the terms of the equation $x^n -$
 $Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0,$ beginning with
 the second: then if the square of any term mul-
 tiplied by the fraction over it be found less than
 the product of the adjacent terms, some of the
 roots of that equation must be imaginary quan-
 tities.

§ 123. An equation may have impossible roots
 although none are discovered by the Rule: be-
 cause, “ though *real* roots in the *given equation*
 always give *real* roots in the *equation of limits*;
 yet it does not follow, *conversely*, that when the
 roots of the *equation of limits* are real, those
 of

of the equation from which it is produced must be such likewise. Thus the *cubic*

$$\left. \begin{array}{l} x^3 - 2m \\ - q \end{array} \right\} \times x^2 \left. \begin{array}{l} + m^2 \\ + 2qm \\ + n \end{array} \right\} \times x - q \times \overline{m^2 + n} = 0,$$

has two of its roots imaginary, $m \pm \sqrt{-n}$, $m - \sqrt{-n}$, the third being $+q$: and yet in the *equation of limits* $3x^2 - \overline{4m + 2q} \times x^2 \pm m^2 + 2qm + m = 0$, if $\overline{m - q}^2$ exceeds $3n$, the roots of the equation of limits will be real. Or if the other equation of limits $\overline{2m + q} \times x^2 - 2 \times \overline{m^2 + 2qm + n} \times x \pm 3q \times \overline{m^2 + n} = 0$ is found by multiplying by the progression 0, -1, -2, -3; it will have its roots real as oft as $\overline{m^2 + 2qm + n}^2$ exceeds $\overline{2m + q} \times 3q \times \overline{m^2 + n}$. And the like may be shewn of higher equations.

§ 124. The reason why this Rule, and perhaps every other that depends on the comparison of the square of a term with the rectangles of the terms on either side of it, must sometimes fail to discover the impossible roots, may appear likewise from this consideration; that the number of such comparisons being always less by *unit* than the number of the quantities q , m , n , &c. in the general equation; they cannot include and fix the relations of these quantities,
on

on which the ratio of *greater* or *lesser* inequality of the squares and rectangles depends ; no more than equations fewer in number than the quantities sought can furnish a determinate solution of a problem.

C H A P. XII.

Containing a general demonstration
of Sir *Isaac Newton's* Rule for
finding the sums of the powers of
the roots of an equation *.

L E T the equation be $\overline{x-a} \times \overline{x-b} \times$
 $\overline{x-c} \times \overline{x-d} \times \&c. = 0$, or,

$$\left. \begin{aligned} x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \dots \\ \dots - Ix^3 + Kx^2 - Lx + M \end{aligned} \right\} = 0.$$

It is known that $A = a + b + c + d + \&c.$
 $B = ab + ac + ad + bc + bd + cd + \&c.$
 $C = abc + abd + bcd + \&c.$ $D = abcd + \&c.$
 the *parts* or *terms* of the coefficients $A, B, C,$
 $D, \&c.$ being of 1, 2, 3, 4, &c. *dimensions* ;
 that is, containing as many *roots* or *factors* as
 there are terms of the equation preceding them,
 respectively.

* See *Arith. Univers.* pag. 157. And *Chap. II.*
 § 15-17, of this *Part.*

C A S E

CASE I.

Let r be an *index equal to n , or greater than n* , then, multiplying the equation by x^{r-n} , and substituting successively $a, b, c, d, \&c.$ for x , you obtain

$$\begin{aligned} a^r - Aa^{r-1} + Ba^{r-2} - Ca^{r-3} \dots \dots \dots \} &= 0, \\ \dots \dots - La^{r-n+1} + Ma^{r-n} & \\ b^r - Ab^{r-1} + Bb^{r-2} - Cb^{r-3} \dots \dots \dots \} &= 0, \\ \dots \dots - Lb^{r-n+1} + Mb^{r-n} & \\ c^r - Ac^{r-1} + Bc^{r-2} - Cc^{r-3} \dots \dots \dots \} &= 0, \\ \dots \dots - Lc^{r-n+1} + Mc^{r-n} & \\ \&c. & \end{aligned}$$

Whence, by transposition and addition, this *Theorem* results, that, in this case, “the sum of the powers of the roots, of the exponent r , is equal to the sum of their powers of the exponent $r - 1$ multiplied by A , minus the sum of their powers of the exponent $r - 2$ multiplied by B , + the sum of those of the exponent $r - 3$ multiplied by C , and so on.”

It remains to find the sums of the powers of the roots, when the exponents are *less than n* the exponent of the equation.

CASE II.

If r is less than n , and H be the coefficient, in the equation, of the *dimensions r* ; that is, if H be taken so that the number of terms preceding it in the equation be equal to r , or the number

ber of *factors* in its parts *abcdefghk*, *abcdefghi*, &c. equal to *r*, then the Theorem may be expressed in the following manner.

$$\begin{aligned}
 & a^r + b^r + c^r + d^r + \&c. \\
 = & \left\{ \begin{array}{l} + a^{r-1} \\ + b^{r-1} \\ + c^{r-1} \\ + d^{r-1} \\ + \&c. \end{array} \right\} \times A \left\{ \begin{array}{l} - a^{r-2} \\ - b^{r-2} \\ - c^{r-2} \\ - d^{r-2} \\ - \&c. \end{array} \right\} \times B \left\{ \begin{array}{l} + a^{r-3} \\ + b^{r-3} \\ + c^{r-3} \\ + d^{r-3} \\ + \&c. \end{array} \right\} \times C \dots \\
 & \dots \dots \dots - r \times H.
 \end{aligned}$$

The case when $r = n - 1$ is easily demonstrated; for, dividing the equation by x , we have

$$x^{n-1} - Ax^{n-2} + Bx^{n-3} \dots - L + \frac{M}{x} = 0.$$

Whence

$$a^{n-1} - Aa^{n-2} + Ba^{n-3} \dots - L + \frac{M}{a} = 0,$$

$$b^{n-1} - Ab^{n-2} + Bb^{n-3} \dots - L + \frac{M}{b} = 0,$$

$$c^{n-1} - Ac^{n-2} + Bc^{n-3} \dots - L + \frac{M}{c} = 0,$$

$$\&c. \qquad \qquad \qquad \&c.$$

and (because $L = \frac{M}{a} + \frac{M}{b} + \frac{M}{c} + \frac{M}{d} + \&c.$)

we shall have $a^{n-1} + b^{n-1} + c^{n-1} + \&c.$

$$= \left\{ \begin{array}{l} + a^{n-2} \\ + b^{n-2} \\ + c^{n-2} \\ + \&c. \end{array} \right\} \times A \left\{ \begin{array}{l} - a^{n-3} \\ - b^{n-3} \\ - c^{n-3} \\ - \&c. \end{array} \right\} \times B \left\{ \begin{array}{l} + a^{n-4} \\ + b^{n-4} \\ + c^{n-4} \\ + \&c. \end{array} \right\} \times C \dots$$

$$\dots \dots \dots + \overline{n-1} \times L.$$

When

When $r = n - 2$, the demonstration is derived from hence, that $\overline{a^2 + b^2 + c^2 + d^2 + \mathcal{E}c.} = A^2 - 2B$ (pag. 142.) as follows.

By § 32. transform the given equation, viz.

$$\left. \begin{aligned} x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \dots \\ \dots - Lx + M \end{aligned} \right\} = 0$$
 into the equation

$$\left. \begin{aligned} x^n - \frac{L}{M}x^{n-1} + \frac{K}{M}x^{n-2} - \frac{I}{M}x^{n-3} \dots \\ \dots - \frac{C}{M}x^3 + \frac{B}{M}x^2 - \frac{A}{M}x + \frac{1}{M} \end{aligned} \right\} = 0;$$

the roots $\alpha, \beta, \gamma, \delta, \&c.$ of which new equation shall be respectively equal to the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \&c.$ of the roots of the original equation.

Divide now the original equation by x^2 , and in the quotient substitute for x the roots $a, b, c, d, \&c.$ successively, so shall you have

$$\left. \begin{aligned} a^{n-2} - Aa^{n-3} + Ba^{n-4} - Ca^{n-5} \dots \\ \dots - La + K - \frac{L}{a} + \frac{M}{a^2} \end{aligned} \right\} = 0,$$

$$\left. \begin{aligned} b^{n-2} - Ab^{n-3} + Bb^{n-4} - Cb^{n-5} \dots \\ \dots - Lb + K - \frac{L}{b} + \frac{M}{b^2} \end{aligned} \right\} = 0,$$

$$\left. \begin{aligned} c^{n-2} - Ac^{n-3} + Bc^{n-4} - Cc^{n-5} \dots \\ \dots - Lc + K - \frac{L}{c} + \frac{M}{c^2} \end{aligned} \right\} = 0,$$

&c.

Add all these equations together, and for $n - 2$, substitute its value r , and it will be

a^r

$$\left. \begin{array}{l} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{array} \right\} \times A \left. \begin{array}{l} + a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{array} \right\} \times B \dots$$

$$\dots \left. \begin{array}{l} -a \\ -b \\ -c \\ -\&c. \end{array} \right\} \times I + nK - \frac{L}{M} \times L + M \times \left. \begin{array}{l} + \alpha^2 \\ + \beta^2 \\ + \gamma^2 \\ + \&c. \end{array} \right\} = 0.$$

But by the principle adduced from *pag.* 142, $\alpha^2 + \beta^2 + \gamma^2 + \&c. = \frac{L^2}{M^2} - \frac{2K}{M}$: wherefore, by multiplication and transposition, it will follow that

$$2K - \frac{L}{M} \times L + M \times \left. \begin{array}{l} + \alpha^2 \\ + \beta^2 \\ + \gamma^2 \\ + \&c. \end{array} \right\} = 0.$$

Which equation being subtracted from the preceding, there remains

$$\left. \begin{array}{l} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{array} \right\} \times A \left. \begin{array}{l} + a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{array} \right\} \times B \dots$$

$$\dots \left. \begin{array}{l} -a \\ -b \\ -c \\ -\&c. \end{array} \right\} \times I + r \times K = 0. \text{ Which was}$$

to be proved.

But to shew it *universally*, we may use the following LEMMA:

“ That

“ That if A is the coefficient of one dimension, or the coefficient of the second term, in an equation, G any other coefficient, H the coefficient next after it; the difference of the dimensions of G and A being $r - 2$: if likewise $A' \times G'$ represent the sum of all those terms of the product $A \times G$ in which the square of any root, as a^2 , or b^2 , or c^2 , &c. is found; then will $A' \times G' = AG - rH$.”

This is a particular case of *Prop.* VI. concerning the *impossible roots* in *Phil. Transf.* N^o 408; which, by continuing the Table of Equations in *pag.* 140, and observing how the coefficients are formed, may be thus demonstrated.

Let the coefficient of a term of the equation, as $D (= abcd + abce + abcf, \&c. + bcde + bcdf \&c.)$ be multiplied by $A (= a + b + c + d + \&c.)$ and, in the product $A \times D$, setting aside all the terms, $A' \times D'$, in which $a^2, b^2, c^2, \&c.$ are found, any one of the remaining terms will arise as often as there are factors in the terms of the following coefficient E . Thus the term $abcde$ will arise *five* times: because it is made up of any one of the five roots (or terms of A) a, b, c, d, e , multiplied into the other four that make a term of D : the like is true of every other term, as $abcdf, bcdef, \&c.$ each of which will arise *five* times in the product $A \times D$. And the sum of these terms $abcde + abcdf + \&c.$ making up the coefficient E , it follows that

U

$A \times$

$A \times D - A' \times D' = 5E$, or $A \times D' = AD - 5E$.
And the same holds of any two coefficients G, H , whose dimensions are $r - 1$ and r respectively.

To apply this to the present purpose, it is to be observed that, in each of the coefficients A, B, C, D , &c. except the last M , which is the product of all the roots a, b, c, d , &c. we may distinguish two several portions or members, in one of which any particular root, as a , is contained, but in the whole remaining portion of the same coefficient, that particular root (a) is wholly absent. Now if, for brevity's sake, we denote that portion of any coefficient wherein any root, as a , is contained, by annexing the symbol of the said root with the sign $+$ in an *uncus* to the symbol, as G , of that coefficient (thus $G^{(+a)}$;) and if we denote the remaining portion of the same coefficient, from which the same root a is totally absent by annexing the symbol of the said root with the sign $-$ in an *uncus* to the symbol G of the same coefficient (thus $G^{(-a)}$) it will appear that (if G be any coefficient and H the following coefficient)

$$G = G^{(+a)} + G^{(-a)} \text{ and } H^{(+a)} = aG^{(-a)},$$

$$G = G^{(+b)} + G^{(-b)} \text{ and } H^{(+b)} = bG^{(-b)},$$

&c. &c.

Divide

Divide now the equation proposed by x^{n-r} , and it will become

$$\left. \begin{aligned} & x^r - Ax^{r-1} + Bx^{r-2} - Cx^{r-3} \dots \dots \dots \\ & \dots + Gx - H + \frac{I}{x} - \frac{K}{x^2} + \frac{L}{x^3} - \frac{M}{x^{n-r}} \end{aligned} \right\} = 0;$$

in which substituting $a, b, c, \&c.$ successively for x , we obtain

$$\left. \begin{aligned} & a^r - Aa^{r-1} + Ba^{r-2} - Ca^{r-3} \dots \dots \dots \\ & \dots + Ga - H + \frac{I}{a} - \frac{K}{a^2} + \frac{L}{a^3} - \frac{M}{a^{n-r}} \end{aligned} \right\} = 0,$$

$$\left. \begin{aligned} & b^r - Ab^{r-1} + Bb^{r-2} - Cb^{r-3} \dots \dots \dots \\ & \dots + Gb - H + \frac{I}{b} - \frac{K}{b^2} + \frac{L}{b^3} - \frac{M}{b^{n-r}} \end{aligned} \right\} = 0,$$

$$\left. \begin{aligned} & c^r - Ac^{r-1} + Bc^{r-2} - Cc^{r-3} \dots \dots \dots \\ & \dots + Gc - H + \frac{I}{c} - \frac{K}{c^2} + \frac{L}{c^3} - \frac{M}{c^{n-r}} \end{aligned} \right\} = 0.$$

But, by the notation here used, and explained as above,

$$\begin{aligned} Ga &= aG^{(+a)} + aG^{(-a)} \\ - H &= - aG^{(-a)} - H^{(-a)} \\ + \frac{I}{a} &= + H^{(-a)} + \frac{I^{(-a)}}{a} \\ - \frac{K}{a^2} &= - \frac{I^{(-a)}}{a} - \frac{K^{(-a)}}{a^2} \\ + \frac{L}{a^3} &= + \frac{K^{(-a)}}{a^2} + \frac{L^{(-a)}}{a^3} \\ - \frac{M}{a^{n-3}} &= - \frac{L^{(-a)}}{a^2} \quad \&c. \end{aligned}$$

U 2

Whence

Whence

$$Ga - H + \frac{I}{a} - \frac{K}{a^2} + \frac{L}{a^3} - \frac{M}{a^{n-r}} = aG^{(+a}$$

$$Gb - H + \frac{I}{b} - \frac{K}{b^2} + \frac{L}{b^3} - \frac{M}{b^{n-r}} = bG^{(+b}$$

$$Gc - H + \frac{I}{c} - \frac{K}{c^2} + \frac{L}{c^3} - \frac{M}{c^{n-r}} = cG^{(+c}$$

&c.

And the sum of these = $aG^{(+a} + bG^{(+b} + cG^{(+c} + \mathcal{E}c.$ = (by this notation) $A' \times G' =$ (by the lemma)

$$\left. \begin{array}{l} + a \\ + b \\ + c \\ + \&c. \end{array} \right\} \times G - rH.$$

Compare this last conclusion with that which followed from dividing the proposed equation by x^{n-r} , and substituting for x the roots $a, b, c, \&c.$ and you will have

$$\left. \begin{array}{l} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{array} \right\} \times A \left. \begin{array}{l} + a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{array} \right\} \times C \dots \dots \left. \begin{array}{l} + a \\ + b \\ + c \\ + \&c. \end{array} \right\} \times G - rH \left. \right\} = 0,$$

which was to be demonstrated.

From

From these two Theorems Sir *Isaac Newton's* Rule manifestly follows.

But, to illustrate the reasoning here used by some examples: suppose $r = 3$, then we are to take C for H , because three terms only precede C in the equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \mathcal{E}c. = 0$; and we are to prove that

$$\left. \begin{array}{l} a^3 \\ + b^3 \\ + c^3 \\ + d^3 \\ + \&c. \end{array} \right\} = \left. \begin{array}{l} a^2 \\ + b^2 \\ + c^2 \\ + d^2 \\ + \&c. \end{array} \right\} \times \left. \begin{array}{l} -a \\ -b \\ -c \\ -d \\ -\&c. \end{array} \right\} \times B + 3C.$$

That this may appear, observe that

$$\begin{aligned} a^3 + b^3 + c^3 + d^3 + \mathcal{E}c. &= \overline{a^2 + b^2 + c^2 + d^2 + \mathcal{E}c.} \\ \times \overline{a + b + c + d + \mathcal{E}c.} &- a^2 \times \overline{b + c + d + \mathcal{E}c.} \\ - b^2 \times \overline{a + c + d + \mathcal{E}c.} &- c^2 \times \overline{a + b + d + \mathcal{E}c.} \\ - d^2 \times \overline{a + b + c + \mathcal{E}c.} &- \mathcal{E}c. = (\text{because } AB' = \\ a \times \overline{ab + ac + ad + \mathcal{E}c.} &+ b \times \overline{ab + bc + bd + \mathcal{E}c.} \\ + c \times \overline{ac + bc + dc + \mathcal{E}c.} &+ d \times \overline{ad + bd + cd + \mathcal{E}c.} \\ + \mathcal{E}c.) &= \overline{a^2 + b^2 + c^2 + d^2 + \mathcal{E}c.} \times A - AB' \\ (\text{by the Lemma}) &= \overline{a^2 + b^2 + c^2 + d^2 + \mathcal{E}c.} \\ \times A - AB + 3C. & \end{aligned}$$

$$\begin{aligned} \text{In like manner, } \overline{a^4 + b^4 + c^4 + d^4 + \mathcal{E}c.} &= \\ \overline{a^3 + b^3 + c^3 + d^3 + \mathcal{E}c.} \times \overline{a + b + c + d + \mathcal{E}c.} &- \\ \overline{a^2 + b^2 + c^2 + d^2 + \mathcal{E}c.} \times \overline{ab + ac + ad + bc + bd + cd} & \\ + \mathcal{E}c. + a^2 \times \overline{bc + bd + cd + \mathcal{E}c.} &+ b^2 \times \overline{ac + ad + cd} \\ &+ \mathcal{E}c. \end{aligned}$$

U 3

$$\begin{aligned}
& + \mathcal{E}c. + c^2 \times \overline{ab+ad+bd} + \mathcal{E}c. + d^2 \times \overline{ab+ac+bc} \\
& + \mathcal{E}c. + \mathcal{E}c. = \overline{a^3+b^3+c^3+d^3} + \mathcal{E}c. \times A \\
& - \overline{a^2+b^2+c^2+d^2} + \mathcal{E}c. \times B + AC' = \\
& \overline{a^3+b^3+c^3+d^3} + \mathcal{E}c. \times A - \overline{a^2+b^2+c^2+d^2} \\
& + \mathcal{E}c. \times B + \overline{a+b+c+d} + \mathcal{E}c. \times C - 4D.
\end{aligned}$$

End of the SECOND PART.





A
T R E A T I S E
O F
A L G E B R A.




P A R T III.

Of the Application of *Algebra* and
Geometry to each other.



C H A P. I.

Of the Relation between the equations
of Curve Lines and the figure of those
Curves, in general.

§ 1.  I N the two first parts we considered
Algebra as independent of Geo-
metry; and demonstrated its oper-
ations from its own principles.

It remains that we now explain the use of Al-
gebra in the resolution of geometrical problems;

or reasoning about geometrical figures; and the use of geometrical lines and figures in the resolution of equations. The mutual intercourse of these sciences has produced many extensive and beautiful Theories, the chief of which we shall endeavour to explain, beginning with the relation betwixt curve lines and their equations.

§ 2. We are now to consider quantities as represented by *lines*; a *known* quantity by a *given* line, and an *unknown* by an *undetermined* line.

But as it is sufficient that it be indetermined on one side, we may suppose one extremity to be known.

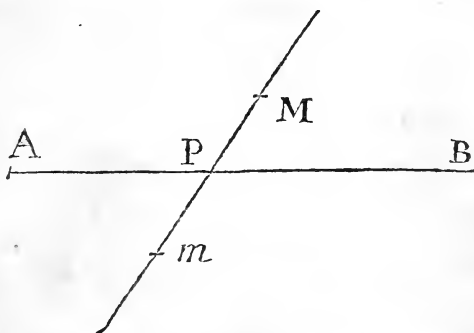


Thus the line AB, whose extremities A and B are both determined, may represent a given quantity: while AP, whose extremity P is undetermined, may represent an undetermined quantity. A lesser undetermined quantity may be represented by AP, taking P nearer to A; and, if you suppose P to move towards A, then will AP, successively, represent all quantities less than the first AP; and after P has coincided with A, if it proceed in the same direction to the place *p*, then will *Ap* represent a negative quantity, if AP was supposed positive.

If

If AP represent x , and $Ap = AP$, then will Ap represent $-x$; and for the same reason, if AB represent $(+a)$, then will $Ab (= AB)$ represent $(-a)$.

§ 3. After the same manner, if PM represent $+y$, and you take Pm , the continuation



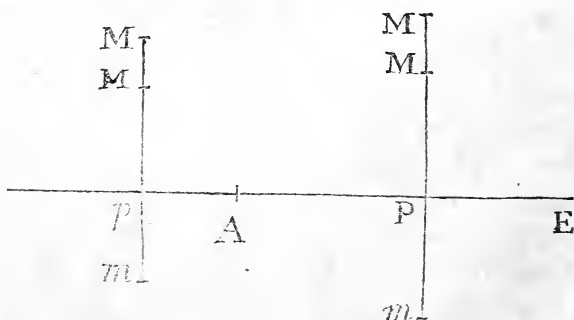
of PM on the other side, equal to PM , then will Pm represent $-y$: for, by supposing M to move towards P , the line PM decreases; when M comes to P , then PM vanishes; and after M has passed P , towards m , it becomes negative.

§ 4. In Algebra, the root of an equation, when it is an impossible quantity, has its expression; but in Geometry, it has none. In Algebra you obtain a general solution, and there is an expression, in all cases, of the thing required; only, within certain bounds, that expression represents an *imaginary* quantity, or rather, “*is the symbol of an operation which, in that case, cannot be performed;*” and serves
only

only to shew the *genesis* of the quantity, and the *limits* within which it is possible.

In the geometrical resolution of a question, the thing required is exhibited only in those cases when the question admits of a *real* solution; and, beyond those limits, no solution appears. So in finding the interfections of a given circle and a straight line, if you determine them by an equation, you will find two general expressions for the distances of the points of interfection from the perpendicular drawn from the center on the given line. But, geometrically, those interfections will be exhibited only when the distance of the straight line from the center is less than the radius of the given circle.

§ 5. “When in an equation there are two undetermined quantities, x and y , then for each particular value of x , there may be as many values of y as it has dimensions in that equation.”



So

So that, if AP (a part of the indefinite line AE) represent x , and the perpendiculars PM represent the corresponding values of y , then there will be as many points (M,) the extremities of these perpendiculars or *ordinates*, as there are dimensions of y in the equation. And the values of PM will be the roots of the equation arising by substituting for x its particular value AP in any case.

From which it appears, how, when an equation is given, you may determine as many of the points M as you please, and draw the line that shall pass through all these points; “ which is called the *locus* of the equation.”

§ 6. When any equation involving two unknown quantities (x and y) is proposed, then substituting for x any particular value AP, if the equation that arises has all its roots positive, the points M will lie on one side of AE; but if any of them are found negative, then these are to be set off on the other side of AE towards m .

If, for x , which is supposed undetermined, you substitute a negative quantity, as Ap , then you will find the points M, m , as before: and the *locus* is not compleat till all the points M, m , are taken in, that it may shew all the values of y corresponding to all the possible values of x .

“ If, in any case, one of the values of y vanish,

vanish, then the point M coincides with P , and the *locus* meets with AE in that point."

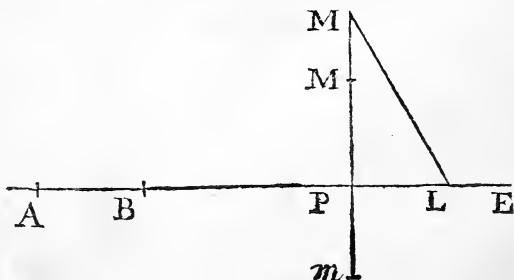
"If one of the values of y becomes infinite, then it shews that the curve has an *infinite arc*: and, in that case, the line PM becomes an *asymptote* to the curve, or touches it at an infinite distance," if AP is itself finite.

"If, when x is supposed infinitely great, a value of y vanish, then the curve approaches to AE produced as an asymptote."

"If any values of y become *impossible*, then so many points M vanish."

§ 7. From what has been said it appears, that when an equation is proposed involving two undetermined quantities (x and y), "there may be as many interfections of the curve that is the *locus* of the equation, and of the line PM as there are dimensions of y in the equation; and as many interfections of the curve and the line AE as there are dimensions of x in the equation."

If you draw any other line LM meeting the



same curve in M , and the line AE in the given

given angle ALM. Suppose LM = u , and AL = z ; "then the equation involving u and z , shall not rise to more dimensions than y and x had in the proposed equation, or, than the sum of their dimensions in any of its terms."

For, since the angles PLM, MPL, PML, are given, it follows that, the sines of these angles being supposed to one another as l, m, n , PM : ML ($y : u$) :: $l : m$; and consequently

$y = \frac{lu}{m}$: and that PL : ML :: $n : m$; so that

PL = $\frac{nu}{m}$, and $x = AP (= AL - PL) = z - \frac{nu}{m}$.

Substitute, for y and x , in the proposed equation

these values $\frac{lu}{m}$ and $z - \frac{nu}{m}$, and it is obvious

(since u and z are of one dimension only in the values of y and x) that in the equation which will arise, z and u will not have more dimensions than the highest dimension of x and y in the proposed equation, or the highest sum of their dimensions taken together in the terms where they are both found: and consequently, "LM drawn any where in the plane of the curve will not meet it in more points than there are units in the highest dimension of x or y , or in the highest sum of their dimensions, in the terms where both are found." Now

the dimension of the equation or curve being denominated from the highest dimension of x or y in it, or from the sum of their dimensions

where

where they are most; we conclude, that “the number of points in which the curve can meet with any straight line, is equal to the number that expresses the dimension of the curve.”

It appears also from this article, how, when an equation of a curve is given expressing the relation of the ordinate PM and abscisse AP, you may transform it, so as to express the relation between any other ordinate ML and the abscisse AL, by substituting for y its value $\frac{lu}{m}$, and for x its value $z - \frac{nu}{m}$.

Or, if you would have the abscisse begin at any other point B, supposing $AB = e$, substitute for x not $z - \frac{nu}{m}$, but $z - \frac{nu}{m} + e$.

§ 8. Those curve lines that can be described by the resolution of equations, the relation of whose ordinates PM and abscisses AP can be expressed by an equation involving nothing but determined quantities besides these ordinates and abscisses, are called “*geometrical or algebraic curves.*”

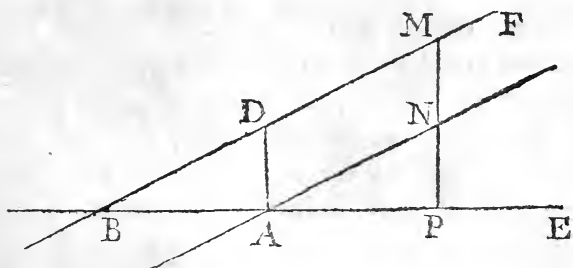
They are divided into *orders* according to the dimensions of their equations, or number of points in which they can intersect a straight line.

The *straight lines* themselves constitute the *first* order of lines; and when the equation expressing the relation of x and y is of one
dimension

dimension only, the points M must be all found in a straight line constituting a given angle with AE.

Suppose, for example, that the equation given is $ay - bx - cd = 0$, and that the *locus* is required.

Since $y = \frac{bx + cd}{a}$, it follows, that, APM being a right angle, if you draw AN making the



angle NAP such that its cosine be to its sine as a to b ; and drawing AD parallel to the ordinates PM, and equal to $\frac{cd}{a}$, through D you draw DF parallel to AN, DF will be the locus required. Where you are to take AD on the same side of the line AE, with PN, if bx and cd have the same sign, but on the contrary side of AE if they have contrary signs.

§ 9. Those curves whose equations are of two dimensions constitute the *second order* of lines, and the *first kind* of curves. Their intersections

terfections with a straight line can never exceed two, by § 7.

The curves whose equations are of *three* dimensions form the *third order* of lines, or *second kind* of curves: and their interfections with a straight line can never exceed Three. And, after the same manner, the curves are determined that belong to the *higher* orders, to infinity.

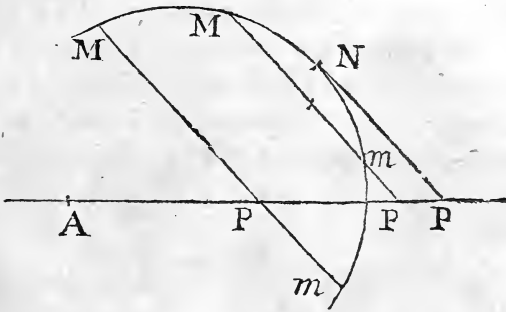
Some curves, if they were completely described, could cut a straight line in an infinite number of points; but these belong to none of the orders we have mentioned; they are not geometrical or algebraic curves, for the relation betwixt their ordinates and abscisses cannot be expressed by a finite equation involving only ordinates and abscisses with determined quantities.

§ 10. As “the roots of an equation become impossible always in pairs, so the interfections of the curve and its ordinate PM must vanish in pairs,” if any of them vanish.

Let PM cut the curve in the points M and *m*, and by moving parallel to itself come to touch it in the point N; then the two points of interfection, M and *m*, go into one point of contact N. If PM still move on parallel to itself, the points of interfection will, beyond

N

N, become imaginary; as the two roots of



an equation first become equal and then imaginary.

§ II. The curves of the 3d, 5th, 7th orders, and all whose dimensions are odd numbers, must have, at least, two infinite arcs; since equations whose dimensions are odd numbers have always one real root *at least*; and consequently, for every value of x , the equation by which y is determined must, at least, have one real root: so that as x (or AP) may be increased *in infinitum* on both sides, it follows that M must go off *in infinitum* on both sides, without limit.

Whereas, in the curves whose dimensions are even numbers, as the roots of their equations may become all impossible, it follows that the figure of the curve may be like a *circle* or *oval*

X

that

that is limited within certain bounds, beyond which it cannot extend.

§ 12. When two roots of the equation by which y is determined become equal, either “the ordinate PM touches the curve,” two points of intersection, in that case, going into a point of contact; or, “the point M is a *punctum duplex* in the curve;” two of its arcs intersecting each other there: or, “some oval that belongs to that kind of curve becoming infinitely little in M , it vanishes into what is called a *punctum conjugatum*.”

If, in the equation, y be supposed $= 0$, then “the roots of the equation by which x is determined, will give the distances of the points where the curve meets AE from A .” And, if two of those roots be found equal, then either “the curve *touches* the line AE ,” or, “ AE passes through a *punctum duplex* in the curve.” When y is supposed $= 0$, if one of the values of x vanish, “the curve, in that case, passes through A .” If two vanish, then either “ AE touches the curve in A ,” or, “ A is a *punctum duplex*.”

“As a *punctum duplex* is determined from the equality of two roots, so is a *punctum triplex* determined from the equality of three roots.

§ 13 A few examples will make these observations very plain. Suppose it is required to describe the line that is the *locus* of this equation,

tion, $y^2 = ax + ab$, or $y^2 - ax - ab = 0$. Since $y = \pm \sqrt{ax + ab}$, and since a and b are given in variable quantities, if you assume AP ($= x$) of a known value, it will be easy to find $\sqrt{ax + ab}$; and setting off PM on one side equal to $\sqrt{ax + ab}$, and Pm on the other equal to PM, the points M and m will belong to the locus required. And for every positive value of AP you will thus obtain a point of the locus on each side. The greater AP ($= x$) is taken, the greater does the $\sqrt{ax + ab}$ become, and consequently PM and Pm become the greater.

If AP be supposed infinitely great, PM and Pm will also become infinitely great; and consequently the locus has two infinite arcs that go off to an infinite distance from AE and from AD. If you suppose x to vanish, $y = \pm \sqrt{ab}$; so that y does not vanish in that case but passes through D and d , taking AD and Ad $= \sqrt{ab}$ a mean proportional betwixt a and b .

If you now suppose that the point P moves to the other side of A, then you must, in the equation, suppose x to become negative, and $y = \pm \sqrt{ab - ax}$; so that y will have two values as before, while x is less than b . But if AB $= b$, and you suppose the point P to come to B, then $ab = ax$, and $y = \pm \sqrt{ab - ax} = 0$. That is, PM and Pm vanish; and the curve there meets the line AE. If you suppose P to move from

have determined this locus to be from the consideration of its equation.

§ 14. Let it be required to describe the line that is the *locus* of this equation, $xy + ay + cy = bc + bx$, or $y = \frac{bc + bx}{a + c + x}$.

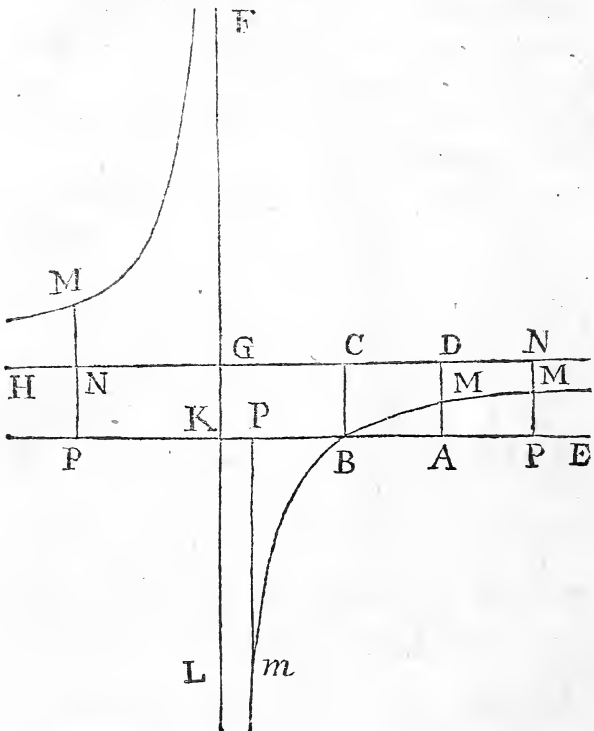
Here, it is plain, the ordinate PM can meet the curve in one point only, there being but one value of y corresponding to each value of x .

When $x = 0$, then $y = \frac{bc}{a + c}$, so that the curve does not pass through A. If x be supposed to increase, then y will increase, but will never become equal to b , since $y = b \times \frac{c + x}{a + c + x}$, and $a + c + x$ is always greater than $c + x$. If x be supposed infinite, then the terms a and c vanish compared with x , and consequently $y = b \times \frac{x}{x} = b$; from which it appears, that taking $AD = b$, and drawing GD parallel to AE , it will be an *asymptote*, and touch the curve at an infinite distance.

If x be now supposed negative, and AP be taken on the other side of A , then shall $y = b \times \frac{c - x}{a + c - x}$; and if x be taken, on that side, $= c$, then shall $y = b \times \frac{c - c}{a} = 0$; so that the curve must pass through B , if $AB = c$.

If x be supposed greater than c , then will $c - x$ become negative, and the ordinate will become

negative and lie on the other side of AE , till x becomes equal to $a + c$; and then $y = b \times \frac{-a}{0}$,



or *infinite*; so that if AK be taken $= a + c$, the ordinate KL will be an *asymptote* to the curve.

If x be taken greater than $a + c$, or AP greater than AK , then both $c - x$ and $a + c - x$ become negative; and consequently $y (= b \times \frac{x - c}{x - a - c})$ becomes positive; and since $x - c$ is always greater

greater than $x - a - c$, it follows that y will be always greater than b or KG , and consequently the rest of the curve lies in the angle FGH . And, as x increases, since the ratio of $x - c$ to $x - a - c$ approaches still nearer to a ratio of equality, it follows that PM approaches to an equality with PN , and the curve to its asymptote GH on that side also.

This curve is the common *hyperbola*; for since $b \times c + x = y \times a + c + x$, by adding ab to both sides $b \times a + c + x = y \times a + c + x + ab$; and $b - y \times a + c + x = ab$; that is, $NM \times GN = GC \times BC$, which is the property of the common hyperbola. And it is easy to see how the figure of the *locus* we have been considering agrees with the figure of the hyperbola.

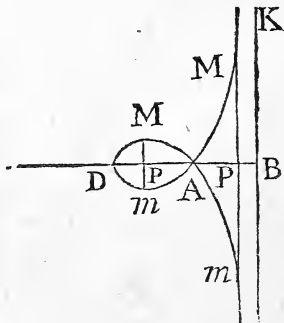
§ 15. Let it be required to describe the *locus* of the equation $cy^2 - xy^2 = x^3 + bx^2$. Where, since $y^2 = \frac{x^3 + bx^2}{c - x}$ and $y = \pm \sqrt{\frac{x^3 + bx^2}{c - x}}$, it follows that PM and Pm must be taken equal, on both sides, to $\sqrt{\frac{x^3 + bx^2}{c - x}}$. But that when x is taken equal to c , if $AB = c$, and BK be perpendicular to AB , then BK must be an *asymptote* to the curve. If x be supposed greater than c , or AP greater than AB , then $c - x$ being negative, the fraction $\frac{x^3 + bx^2}{c - x}$ will become negative, and its square root impossible. So that no part

of the *locus* can be found beyond B. If x be supposed negative, or P taken on the other side of A, then $y = \pm \sqrt{\frac{-x^3 + bx^2}{c + x}}$, the sign of x^3 and x being changed, but not the sign of bx^2 ; because the square of a negative is the same as the square of a positive, but its cube is negative: while x is less than b , the values of y will be real and equal; but if $x = b$, then the values of y vanish, because, in that case,

$$y = \pm \sqrt{\frac{-x^3 + bx^2}{c + x}} = \sqrt{\frac{-b^3 + b^3}{c + x}} = 0;$$
 and consequently, if AD be taken $= b$, the curve will pass through D, and there touch the ordinate.

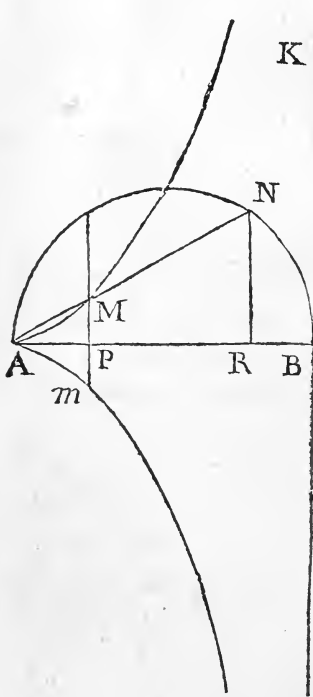
If x be taken greater than b , then $\pm \sqrt{\frac{-x^3 + bx^2}{c + x}}$ will become *imaginary*, so that no part of the curve is found beyond D.

If you suppose $y = 0$, then will $x^3 + bx^2 = 0$



be an equation whose roots are $-b, 0, 0$, from which it appears that the curve passes twice through the point A , and has, in A , a *punctum duplex*. This locus is a line of the third order, BK is its asymptote, and it has a *nodus* betwixt A and D .

If you suppose b to vanish in the equation, so that $cy^2 - xy^2 = x^3$, then will A and D coin-

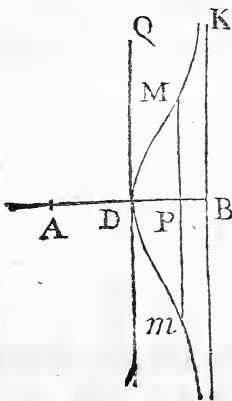


cide, and the *nodus* vanish, and the curve will have in the point A a *cusps*, the two arcs AM and

and Am touching one another in that point. And this is the same curve which by the ancients was called the *cissoïd* of *Diocles*, the line AB being the diameter of the generating circle, and BK the asymptote.

For, if BR be equal to AP , and the ordinate RN be raised meeting the circle in N , and AN be drawn, it will cut the perpendicular PM in M a point of the *cissoïd*. So that if M be a point in the *cissoïd*, $AP : PM :: AR : RN :: \sqrt{AR} : \sqrt{BR} :: \sqrt{BP} : \sqrt{AP}$, and consequently, $BP \times PM^2 = AP^3$. that is, $c - x \times y^2 = x^3$: which is the equation the *locus* of which was required.

If, instead of supposing b positive, or equal to nothing, we now suppose it negative, the equation will be $cy^2 - xy^2 = x^3 - bx^2$, the curve will pass through D , as before, and taking

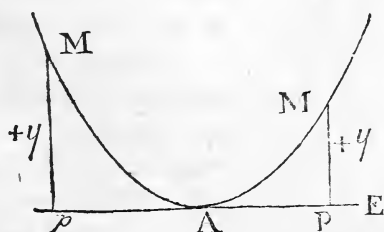


$AB = c$, BK will be its asymptote: it will have a *punctum conjugatum* in A , because when y vanishes, two values of x vanish, and the third becomes equal to b or AD . The whole curve, besides this point A , lies between DQ and BK . These are demonstrated after the same manner as in the first case.

§ 16. If an equation is proposed, as $y = ax^n + bx^{n-1} + cx^{n-2}$, &c. and n is an even number, then will the *locus* of the equation have two infinite arcs lying on the same side of AE. For, if x become infinite, whether positive or negative, x^n will be positive, and ax^n have the same sign in either case; and as ax^n becomes infinitely greater than the other terms bx^{n-1} , cx^{n-2} , &c. it follows that the infinite values of y will have the same sign in these cases; and consequently, the two infinite arcs of the curve will lie on the same side of AE.

But if n be an odd number, then when x is negative, x^n will be negative, and ax^n will have the contrary sign to what it has when x is positive; and therefore the two infinite arcs, in this case, will lie on different sides of AE, and tend towards parts directly opposite.

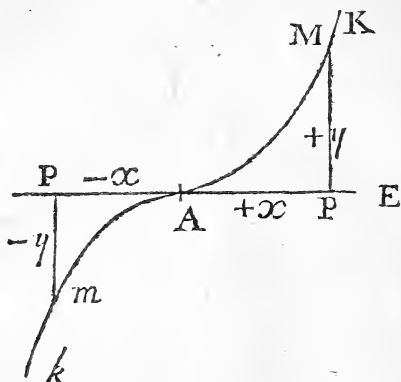
Thus the locus of the equation $ay = x^2$ is the parabola. A is the vertex, AE is the tan-



gent at the vertex; and the two infinite arcs lie manifestly on the same side of AE.

But

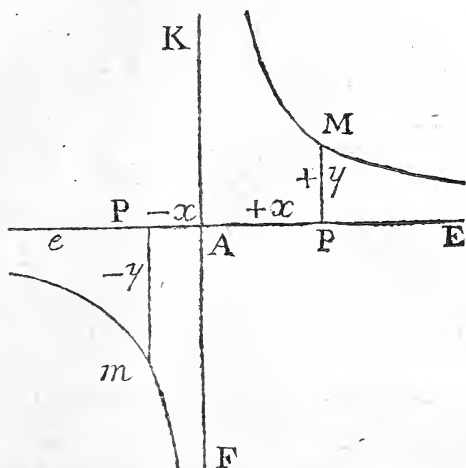
But the *locus* of the equation $a^2y = x^3$, where the index of x is an odd number, has its two



arcs on different sides of AE, tending towards opposite parts, as AMK, and Amk. This curve is called the *cubical parabola*, and is a line of the third order.

The *locus* of the equation $a^2y = x^4$ is of a figure like the common parabola; and “all those loci, in whose equations y is of one dimension, x of an *even* number of dimensions: But those *loci* are like the cubical parabola, in whose equations y is of one dimension only, and x of an *odd* number of dimensions.” And this Rule is even true of the locus of the equation $y = x$, which is a straight line, cutting AE in an angle of 45° ; which manifestly goes off as the cubical parabola does to infinity, towards opposite parts, and on different sides of AE.

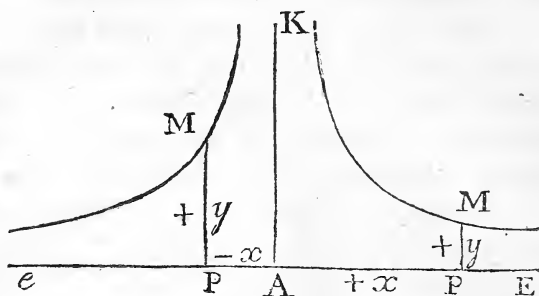
§ 17. If the *locus* of the equation $yx^n = a^{n+1}$ is required;



If n is an *odd* number, then when x is positive, $y = \frac{a^{n+1}}{x^n}$; but when x is negative, then $y = -\frac{a^{n+1}}{x^n}$; so that this curve must all lie in the vertically *opposite* angles KAE, FAE, (as the common *hyperbola* :) FK, Ee, being asymptotes.

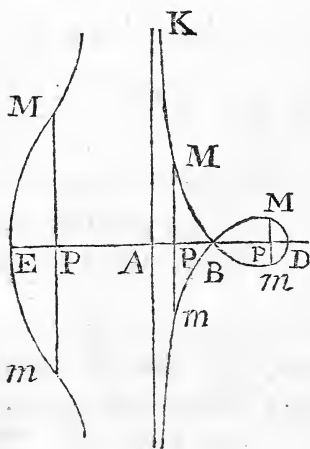
But if n is an *even* number, then y is always positive, whether x be positive or negative, because x^n , in this case, is always positive; and there-

therefore the curve must all lie in the two ad-



acent angles KAE and KAe, and have AK and AE for its two asymptotes.

§ 18. Let the equation given be $\sqrt{a^2 - x^2} \times \sqrt{x - b}^2 = x^2 y^2$; so that $y = \pm \sqrt{a^2 - x^2} \times \frac{x - b}{x}$.



If $x = 0$, then y becomes infinite, and therefore the ordinate at A is an asymptote to the curve. If $AB = b$, and P be taken betwixt A and B, then shall PM and Pm be equal, and lie on different sides of the abscisse AP. If $x = b$, then the two values of y vanish, because

because $x - b = 0$; and consequently, the curve passes through B, and has there a *punctum duplex*. If AP be taken greater than AB, then shall there be two values of y , as before, having contrary signs, that value which was positive before being now become negative, and the negative value being become positive. But if AD be taken $= a$, and P comes to D, then the two values of y vanish, because $\sqrt{a^2 - x^2} = 0$. And if AP is taken greater than AD, then $a^2 - x^2$ becomes negative, and the value of y *impossible*: and therefore, the curve does not go beyond D.

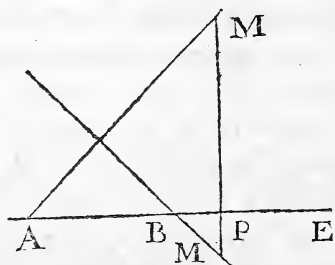
If x now be supposed negative, we shall find $y = \pm \sqrt{a^2 - x^2} \times \overline{b + x} \div x$. If x vanish, both these values of y become infinite, and consequently, the curve has two infinite arcs, on each side of the *asymptote* AK. If x increase, it is plain y diminishes, and if x becomes $= a$, y vanishes, and consequently the curve passes through E, if AE be taken $= AD$, on the opposite side. If x be supposed greater than a , then y becomes *impossible*; and no part of the curve can be found beyond E. This curve is the *conchoid* of the ancients.

If $a = b$, it will have a *cusps* in B, the nodus betwixt B and D vanishing. And if a is less than b , the point B will become a *punctum conjugatum*.

From

From what has been said an error may be corrected of an Author in the *Memoires de l'Acad. Royale des Sciences*, who gives this curve no infinite arcs, but only a *double nodus*. Some other errors of the same kind may be corrected in that Treatise, from what we have said.

§ 19. If the proposed equation can be resolved into two equations of lower dimensions, without affecting either y or x with any radical sign, then the *locus* shall consist of the two *loci* of those inferior equations. Thus the locus of the equation $y^2 - 2xy + by + x^2 - bx = 0$ is found to be two straight lines cutting the ab-



scisse AE in angles of 45° , in the points A and B, whose distance $AB = b$, because that equation is resolved into these two $y - x = 0$, and $y - x + b = 0$.

After the same manner, some *cubic* equations can be resolved into three simple equations, and then the *locus* is *three* straight lines; or may be resolved into a *quadratic* and *simple* equation,

equation, and then the *locus* is a *conic section* and a *straight line*.

In general, “the curves of the superior orders include all the curves of the inferior orders; and whatever is demonstrated generally of any one order, is also true of the inferior orders.” So, for example, any general property of the conic sections holds true of two straight lines as well as of a conic section. Particularly that “the rectangles of the segments of parallels bounded by them, will be always to one another in a given ratio.” The general properties of the lines of the third *order* are true of three straight lines, or of any one straight line and a conic section. And, as the general properties of the higher orders of lines descend also to those of the inferior orders, so there is scarce any property of the inferior orders, but has an analogy to some property of the higher orders; of which it is but a particular case or instance. And hence, the properties of the inferior orders lead to the discovery of those of the superior orders*.

§ 20. We have shewed how to judge of the figure of a *locus* from the consideration of its equation. And when a *locus* is to be described exactly, for every value of x you must, by the resolution of equations, according to the Rules

* See the APPENDIX.

in *Part II.* find the corresponding values of y , and determine from these values the points of the *locus*.

But there are geometrical *constructions* by which the roots of equations can be determined more commodiously for this purpose. And, as by these constructions we describe the *loci* of the equations, so reciprocally when *loci* are described, they are useful in determining the roots of equations; both which shall be explained in the following *Chapter*. Then we shall give an account of the most general and simple methods of describing these *loci* by the mechanical motion of angles and lines, whose interfections trace the curve; or of constructing them by finding geometrically any number of their points.



C H A P. II.

Of the Construction of Quadratic Equations; and of the Properties of the Lines of the second order.

§ 21. **T**HE general equation expressing the nature of the lines of the second order, having all its terms and coefficients, will be of this form;

$$\left. \begin{array}{l} y^2 + axy + cx^2 \\ + by + dx \\ + e \end{array} \right\} = 0.$$

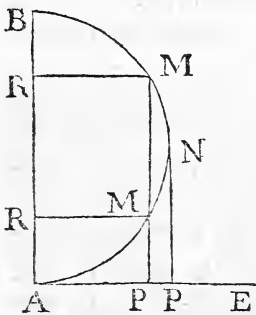
Where a, b, c, d, e , represent any given quantities with their proper signs prefixed to them.

If a quadratic equation is given, as $y^2 + py + q = 0$, and, by comparing it with the preceding, if you take the quantities a, b, c, d, e , and x such that $ax + b = p$, and $cx^2 + dx + e = q$, then will the values of y in the first equation be equal to the values of it in the second; and if the *locus* be described belonging to the first equation, the two values of the ordinate when $ax + b = p$ and $cx^2 + dx + e = q$, will be the two roots of the equation $y^2 + py + q = 0$.

And as *four* of the given quantities a, b, c, d, e , may be taken at pleasure, and the *fifth*,

with the absciffe x , determined, so that $ax + b$ may be still equal to p , and $cx^2 + dx + e = q$; hence there are innumerable ways of constructing the same equation. But those *loci* are to be preferred which are described most easily; and therefore, the *circle*, of all conic sections, is to be preferred for the resolution of quadratic equations.

§ 22. Let AB be perpendicular to AE, and upon AB describe the semicircle BMMA. If AP be supposed equal to x , $AB = a$, and $PM = y$, then making MR, MR, perpendiculars to the diameter AB, since $AR \times RB = RM^2$, and $AR = y$, $RB = a - y$, $RM = x$, it follows that



$a - y \times y = x^2$, and $y^2 - ay + x^2 = 0$. And, if an equation $y^2 - py + q = 0$, be proposed to be resolved, its roots will be the ordinate to the circle, PM and PM, to its tangent AE, if $a = p$, and $x^2 = q$: because then the equation of the circle $y^2 - ay + x^2$

$= 0$, will be changed into the proposed equation $y^2 - py + q = 0$.

We have therefore this construction for finding the roots of the quadratic equation $y^2 - py + q = 0$; take $AB = p$, and on AB describe a semi-

femicircle; then raise AE perpendicular to AB, and on it take $AP = \sqrt{q}$, that is, a mean proportional between 1 and q (by 13 El. 6.) then draw PM parallel to AB, meeting the femicircle in M, M, and the lines PM, PM shall be the roots of the proposed equation.

It appears from the construction that if $q = \frac{p^2}{4}$, or $\sqrt{q} = \frac{1}{2}p$, then $AP = \frac{1}{2}AB$, and the ordinate PN touches the curve in N, the two roots PM, PM, in that case, becoming equal to one another and to PN.

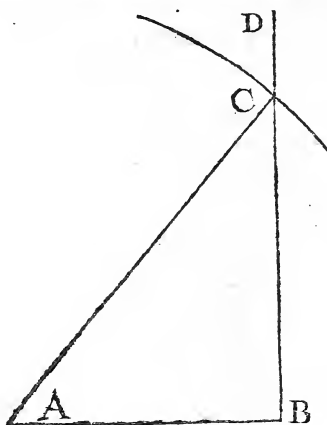
If AP be taken greater than $\frac{1}{2}AB$, that is, when \sqrt{q} is greater than $\frac{1}{2}p$, or q greater than $\frac{1}{4}p^2$, the ordinates do not meet the circle, and the roots of the equation become *imaginary*: as we demonstrated, in another manner, in Part II.

§ 23. The roots of the same equation may be otherwise thus determined.

Take $AB = \sqrt{q}$, and raise BD perpendicular to AB; from A as a center with a radius equal to $\frac{1}{2}p$, describe a circle meeting BD in C, then the two roots of the equation $y^2 - py + q = 0$, shall be $AC + CB$, and $AC - CB$.

For these roots are $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, and $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$; and $AC = \frac{1}{2}p$, $CB = \sqrt{AC^2 - CB^2} = \sqrt{\frac{1}{4}p^2 - q}$, and consequently these roots are $AC \pm CB$.

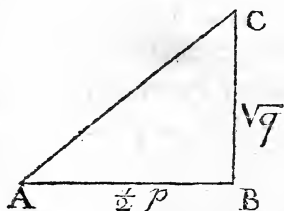
The roots of the equation $y^2 + py + q = 0$ are



— $AC \pm CB$; as is demonstrated in the same manner.

§ 24. The roots of the equation $y^2 - py - q = 0$ are determined by this construction.

Take $AB = \frac{1}{2}p$, $BC = \sqrt{q}$, draw AC ; and the two roots shall be $AB \pm AC$. If the se-



cond term is positive, then the roots shall be — $AB \pm AC$.

And

And all quadratic equations being reducible to these four forms,

$$y^2 - py + q = 0,$$

$$y^2 + py - q = 0,$$

$$y^2 - py - q = 0,$$

$$y^2 + py + q = 0,$$

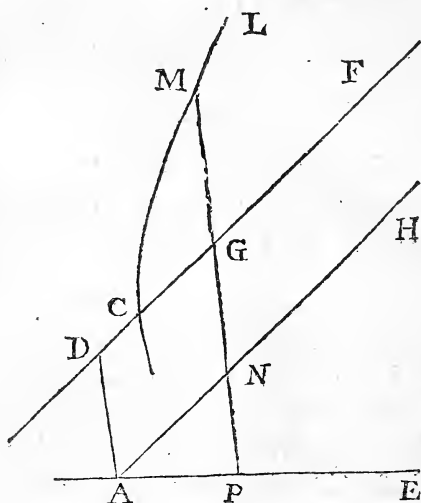
it follows, that they may be all constructed by this and the last two articles.

§ 25. By these geometrical constructions, the locus of any equation of two dimensions may be described; since, by their means, the values of y that correspond to any given value of x may be determined. But if we demonstrate that these *loci* are always *conic sections*, then they may more easily be described by the methods that are already known for describing these curves.

In order to prove this, we shall enquire what equations belong to the different *conic sections*; and, as it will appear that there is no equation of two dimensions but must belong to one or other of them, it will follow that they are *loci* of all equations of two dimensions.

§ 26. Let CML be a *parabola*; AE any line drawn in the same plane; and let it be required to find the equation expressing the relation betwixt the ordinate PM forming any given angle with AE, and the abscisse AP

beginning at A any given point in the line AE.



Let CF be the diameter of the parabola whose ordinates are parallel to PM. Draw AH parallel to CF meeting PM in N; and AD parallel to PM meeting CF in D. Because the angles HAE, APN, ANP, are given, the lines AP, PN, AN, will be in a given ratio to each other: suppose them to be always as a , b , c ; let $AD = d$, $DC = e$; and seeing $AP (= x)$: $PN :: a : b$, $PN = \frac{b}{a}x$; likewise $AP : AN :: a : c$, or $AN = \frac{c}{a}x$. And $GM = PM - PN = NG = y - \frac{b}{a}x - d$. But $CG = DG - DC$

$DC = AN - DC = \frac{c}{a}x - e$. If now the *parameter* of the diameter CF be called p , then, from the nature of the parabola, $p \times CG = GMq$; and consequently, $p \times \frac{c}{a}x - e = y - \frac{b}{a}x - d$,² from which this equation follows,

$$\left. \begin{aligned}
 y^2 - \frac{2b}{a}xy + \frac{b^2}{a^2}x^2 - 2dy + \frac{2bd}{a} \\
 - \frac{pc}{a}
 \end{aligned} \right\} x + \left. \begin{aligned}
 + d^2 \\
 + pe
 \end{aligned} \right\} = 0.$$

Whence, if any equation is proposed, and such values of a , b , c , d , e , p can be assumed as to make *that* equation and *this* coincide, then the *locus* of that equation will be a *parabola*. The construction of which may be deduced from this article.

§ 27. In this general equation for the parabola, the coefficient of x^2 is the square of half the coefficient of xy ; and, “when any equation is proposed that has this property, the *locus* of it is a *parabola*.” For, whatever coefficients affect the three last terms, they may be made to agree with the coefficients of the last terms of the general equation, by assuming proper values of p , c , and e .

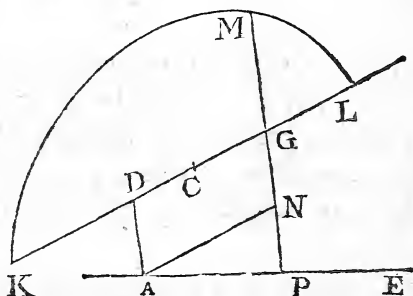
It appears also, that “if the locus be a parabola, and the term xy be wanting, the term x^2 must also be wanting.” And, “if any equation of two dimensions be proposed that

wants

wants both the terms, xy and x^2 , it may be always accommodated to a *parabola*.

§ 28. The general equation for the *ellipse* is deduced from the property of the ordinates of any diameter, in the same manner; the construction of the figure being the same as in § 26. Only, in place of the *parabola*,

Let KML be an *ellipse* whose diameter is KL, having its ordinates parallel to PM, and



let C be the center of the *ellipse*. Suppose $CL = t$, and the *parameter* of that diameter $= p$, then $GMq : CLq - CGq :: p : 2t$. But, as in § 26, $GM = y - \frac{b}{a}x - d$, and $CG = \frac{c}{a}x - e$;

therefore, $\left[y - \frac{b}{a}x - d \right]^2 \times \frac{2t}{p} = t^2 - \frac{c^2}{a^2}x^2 + \frac{2ce}{a}x - e$: whence this equation;

$$y^2 - \frac{2b}{a}xy + \frac{b^2}{a^2}x^2 - 2dy + \frac{2bd}{a}x - \frac{pce}{at} \left. \vphantom{\begin{matrix} y^2 \\ - \frac{2b}{a}xy \\ + \frac{b^2}{a^2}x^2 \\ - 2dy \\ + \frac{2bd}{a}x \\ - \frac{pce}{at} \end{matrix}} \right\} x^2 + \left. \begin{matrix} + \frac{d^2}{2} \\ - \frac{pt}{2} \\ + \frac{pe^2}{2t} \end{matrix} \right\} = 0. \quad \text{And}$$

And if any equation is proposed that can be made to agree with this general equation, by assuming proper values of a, b, c, d, p and e ; then the *locus* of that equation will be an *ellipse*.

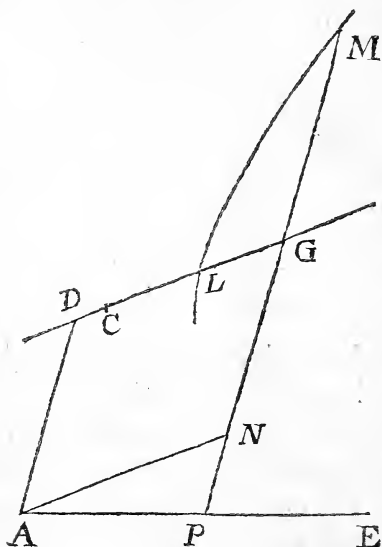
§ 29. “In the general equation for the *ellipse*, the terms x^2 and y^2 have the same sign: and the coefficient of x^2 is always greater than the square of half the coefficient of xy , because $\frac{b^2}{a^2} + \frac{pc^2}{2ta^2}$ is greater than $\frac{b^2}{a^2}$. And although the term xy be wanting, yet the term x^2 must remain, its coefficient, in that case, being $\frac{p}{2t}$, which must be always real and positive. On the other hand, if an equation is proposed in which the coefficient of x^2 exceeds the square of half the coefficient of xy ; or, an equation that wants xy , but has x^2 and y^2 , of the same sign, its *locus* must be an *ellipse*.”

§ 30. In the hyperbola, as $GMq : CGq - CLq :: p : 2t$; when t is a *first* diameter, the equation that arises will differ from the equation of the *ellipse* only in the signs of the values of CGq and CLq , and consequently will have this form,

$$\left. \begin{aligned} y^2 - \frac{2b}{a}xy + \frac{b^2}{a^2}x^2 - 2dy + \frac{2bd}{a}x + d^2 \\ - \frac{pc^2}{2ta^2}x^2 + \frac{pce}{at}x + \frac{pt}{2} \\ - \frac{pe^2}{2t} \end{aligned} \right\} = 0.$$

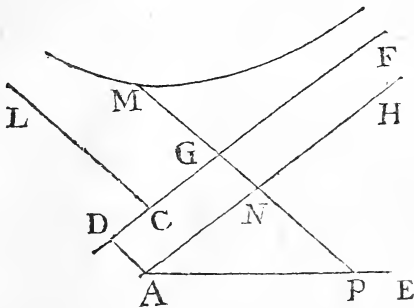
If

If t be a *second* diameter, then $\frac{pt}{2}$ will be negative.



In this equation, it is manifest that the coefficient of the term x^2 is less than the square of half the coefficient of xy ; and, that when the term xy is wanting, the term x^2 must be negative. And, reciprocally, “if an equation is proposed where the coefficient of x^2 is less than the square of half the coefficient of xy ; or where xy is wanting and y^2 and x^2 have contrary signs, the locus of that equation must be an *hyperbola*.”

§ 31. The equation of the *hyperbola* when its ordinates PM are parallel to an asymptote does not come under the general equation of the last article. Let CF and CL be the *asymptotes* of the *hyperbola*, and let PM be parallel to CL.



Then $CG \times GM$ will be equal to a given rectangle (which suppose ga). Then, $CG = DG - DC = \frac{c}{a}x - e$, $GM = y - \frac{b}{a}x - d$; and consequently $y - \frac{b}{a}x - d \times \frac{c}{a}x - e = g \times a$: whence this equation,

$$\left. \begin{array}{l} xy - \frac{b}{a}x^2 + \frac{eb}{c} \\ - d \end{array} \right\} x \quad \left. \begin{array}{l} - \frac{ea}{c}y + \frac{eda}{c} \\ - \frac{ga^2}{c} \end{array} \right\} = 0.$$

Where only one of the terms y^2 , x^2 , can be found with xy ; and where xy will be found without either of these terms, if AE and AH coincide, that is, if AE is parallel to the asymptote DF.

It

It appears from this, that “if an equation is proposed that either has xy the only term of two dimensions; or, has xy and either x^2 or y^2 besides, but not both of them, the locus of the equation shall be an *hyperbola*, one of whose *asymptotes* shall be parallel to y or x according as it is y^2 or x^2 that is wanting in the equation.”

§ 32. From all these compared together, it follows, that “the locus of any equation of two dimensions is a conic section.”

For if the term xy is wanting in the equation, and but one of the terms y^2 , x^2 is found in it, the locus shall be a *parabola*; by § 27.

If xy is wanting, and x^2 , y^2 , have the same sign, then the locus is an *ellipse*. § 29.—But when they have different signs, it is an *hyperbola*. § 30.

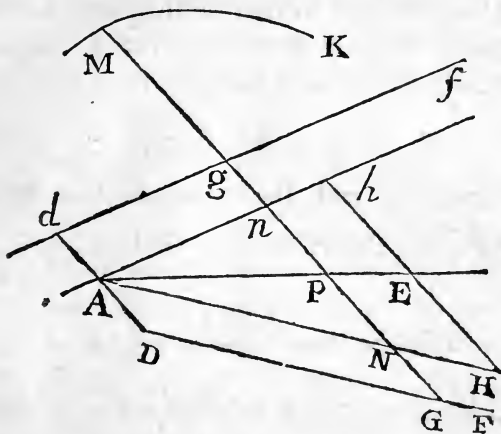
If xy is found in the equation, and x^2 , y^2 , are both wanting, or either of them, the locus is an *hyperbola*. § 31.

If both x^2 and y^2 are found in it, having contrary signs, the locus is still an *hyperbola*.

If y^2 and x^2 have the same signs, then, according as the coefficient of x^2 is *greater*, *equal*, or *less* than the square of half the coefficient of xy , the locus shall be an *ellipse*, *parabola*, or *hyperbola*. § 27, 29, 30.

In any case therefore the locus of the equation is some *conic section*.

§ 33. These may all be demonstrated more directly from the consideration of the general equation of the lines of the *second order* in § 21. For it is obvious that, by § 25. *Part II.*



the second term of that general equation may be exterminated by assuming $z = y + \frac{ax + b}{2}$, and it will be transformed into

$$\left. \begin{aligned} z^{2*} + \frac{4c - a^2}{4} \times x^2 \\ + d - \frac{ab}{2} \times x \\ + e - \frac{b^2}{4} \end{aligned} \right\} = 0,$$

which, by transposing the last term, is

$$z^2 = \frac{a^2}{4} - c \times x^2 + \frac{ab}{2} - d + x \left. \begin{aligned} + \frac{b^2}{4} \\ - e \end{aligned} \right\}.$$

Let

Let MK be the locus of the equation: and if AH be drawn so that HE be to AE as $\frac{1}{2}a$ to unit, and AD, parallel to PM, be $= \frac{1}{2}b$, and through D the line DF be drawn parallel to AH, meeting PM in G, then shall GM ($= PM + PN + NG = y + \frac{1}{2}ax + \frac{1}{2}b$) $= z$. And if AH $= f$, then DG $= AN = fx$.

Suppose DG $= u$, and $x = \frac{u}{f}$. Instead of x substitute $\frac{u}{f}$, and the equation that results will express the relation of GM and DG, of this form,

$$z^2 = \frac{a^2 - 4c}{4f^2} \times u^2 + \frac{ab - 2d}{2f} \times u + \frac{1}{4}b^2 - e = 0.$$

Which will be an *hyperbola*, *parabola*, or *ellipse*, according as the term $\frac{a^2 - 4c}{4f^2}$ is *positive*, *nothing*, or *negative*. That is, according as $\frac{a^2}{4}$ is *greater*, *equal* to, or *less* than c . But a was the coefficient of xy ; from which it appears, that "the locus is an *ellipse*, *parabola*, or *hyperbola*, according as the coefficient of x^2 is *greater*, *equal* to, or *less* than the square of half the coefficient of xy ."

It appears also, that "if the term xy be wanting, or $a = 0$, then the locus will be an *ellipse*, *parabola*, or *hyperbola*, according as the term cx^2 is *positive*, *nothing*, or *negative*."

Hence

Hence likewise, if the term x^2 be wanting, and the term xy not wanting, then the term $\frac{a^2 - 4c}{4f^2}u^2$ being positive (because $\frac{a^2}{4f^2}$ is always positive, whatever a or f be) “the locus must be an hyperbola.”

Note, That part of the figure, on the other side of AE, which is marked with small letters, answers to the case when the coefficient of y , in the general equation, viz. $ax + b$, is negative.

§ 34. The lines of the *second order* have some general properties which may be demonstrated from the consideration of the general equation representing them.

The general equation of § 21. by exterminating the second term can be transformed into the equation,

$$z^2 = \frac{a^2 - 4c}{4f^2} \times u^2 + \frac{ab - 2d}{2f} \times u + \frac{b^2}{4} - e.$$

From which we have

$$z = \pm \sqrt{\frac{a^2 - 4c}{4f^2} \times u^2 + \frac{ab - 2d}{2f} + \frac{b^2}{4} - e.}$$

Where the two values of z are always equal, and have contrary signs, so that the line DF, on which the abscissés are taken, must bisect the ordinates, and consequently, is a *diameter* of the conic section. And, as this has been demonstrated generally, in any situation of the lines PM, it follows that if any parallels, as

Z Mm,

$Mm, Mm,$ be drawn meeting a conic section *, there is a line DF which can bisect all these parallels. And consequently if any two parallels, $Mm, Mm,$ are bisected in G and g , the line Gg that bisects these two, will bisect all the other lines parallel to them, terminated by the curve. "Which is a general property of all the conic sections."

There is one case which must be excepted, when PM is parallel to an asymptote, because in that case it meets with the conic section only in one point.

§ 35. In the general equation of § 21, if you suppose $y=0$, there will remain $cx^2 + dx + e = 0$, by which the points are determined where the curve meets the abscissa AE .

Suppose it meets it in B and D , and that $AB = A$, and $AD = B$. Then shall $-A$ and $-B$ be the two roots of the equation $x^2 + \frac{d}{c}x + \frac{e}{c} = 0$; and therefore $\overline{x + A} \times \overline{x + B} = x^2$



$+ \frac{d}{c}x + \frac{e}{c}$: but $x + A = BP$, and $x + B = DP$;

* Supply the figure.

therefore

therefore $BP \times DP = x^2 + \frac{d}{c}x + \frac{e}{c}$. Now, it is manifest from the nature of equations, that if PM meet the curve in M and m , the rectangle of the roots PM and Pm shall be equal to $cx^2 + dx + e$ the last term of the equation

$$\left. \begin{array}{l} y^2 + axy + cx^2 \\ + by + dx \\ + e \end{array} \right\} = 0.$$

We have therefore $PM \times Pm = cx^2 + dx + e$; and $BP \times DP = x^2 + \frac{d}{c}x + \frac{e}{c}$; so that $PM \times$

$Pm : BP \times DP :: cx^2 + dx + e : x^2 + \frac{d}{c}x + \frac{e}{c} :: c : 1$. That is, "the rectangle of the ordinates PM, Pm is to the rectangle of the segments of the absciffes, as, in a given ratio, c is to 1." Which is another general property of the lines of the *second order*.

In a similar manner the analogous properties of the lines of the higher orders are demonstrated*.

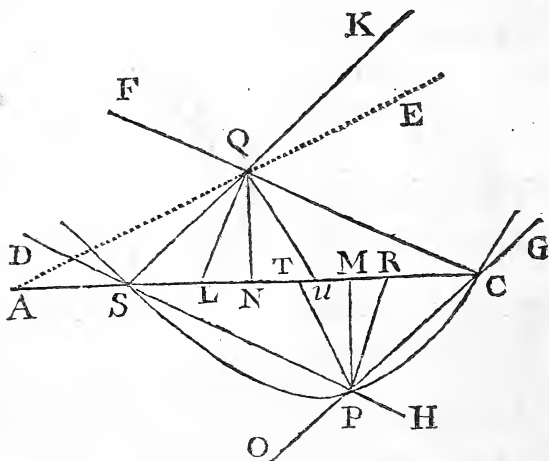
§ 36. There are many different ways of describing the lines of the *second order*, by motion. The following is Sir *Isaac Newton's*.

† Let the two points C and S be given, and the straight line AE in the same plane. Let the

* See the APPENDIX.

† See *Geometria Organica*, Prop. I.

given angles FCO, KSH, revolve about the points C and S as poles, and let the intersection of the sides CF, SK, be carried along the



straight line AE, and the intersection of the sides CO, SH, will describe a line of the *second order*.

Let the sides CF, SK intersect each other in Q, and the sides CO, SH, in P: let PM and QN be perpendicular on CS. Then draw PR, QU; PT, QL; so that $CUQ = CRP = FCG$; and $SLQ = STP = KSD$.

The angle $RCP = CQU$, since RCP makes two right ones with RCQ and QUC. So that the triangles CUQ and CRP will be similar. And after the same manner you may demonstrate

strate that the triangles SLQ, STF are similar; whence,

$$\begin{aligned} \text{CR} : \text{PR} &:: \text{QU} : \text{CU}, \\ \text{and ST} : \text{PT} &:: \text{QL} : \text{SL}. \end{aligned}$$

Suppose $\text{CS} = a$, $\text{CA} = b$, the sine of the angle FCO to its cosine as d to a ; sin. angle CAE to cofin. as c to a , and sin. KSH to cofin. as e to a . Put also $\text{PM} = y$, $\text{CM} = x$, $\text{QN} = z$.

Then $\text{RM} : \text{PM} :: a : d$. $\text{PR} : \text{PM} :: \sqrt{a^2 + d^2} : d$.
 $\text{AN} : \text{QN} :: a : c$. So that $\text{RM} = \frac{ay}{d}$, $\text{CR} (= \text{CM} - \text{RM}) = x - \frac{ay}{d}$, $\text{PR} = y \frac{\sqrt{a^2 + d^2}}{d}$.

Likewise $\text{QU} = \frac{z\sqrt{a^2 + d^2}}{d}$, and $\text{CU} (= \text{CA} - \text{AN} - \text{NU}) = b - \frac{a}{c}z - \frac{a}{d}z$. And it being $\text{CR} : \text{PR} :: \text{QU} : \text{CU}$, it follows that $dx - ay : y\sqrt{a^2 + d^2} :: z\sqrt{a^2 + d^2} : b - \frac{a}{c}z + \frac{a}{d}z$.

So that $z = \frac{bc \times \sqrt{dx - ay}}{dc - a^2 \times y + d + c \times ax}$.

In like manner you will find $\text{ST} = a - x - \frac{a}{e}y$,
 $\text{PT} = \frac{y\sqrt{a^2 + e^2}}{e}$, $\text{QL} = \frac{z\sqrt{a^2 + e^2}}{e}$, and $\text{SL} (= \text{AN} - \text{AS} - \text{NL}) = a - b + \frac{e - c \times az}{ec}$. But

it was $ST : PT :: QL : SL$, that is, $a - x - \frac{a}{e}y : \frac{y\sqrt{a^2 + e^2}}{e} :: \frac{z\sqrt{a^2 + e^2}}{e} : a - b + \frac{e - c \times az}{ec}$.

Whence $QN = z = \frac{a - b \times c \times \overline{ae - ex - ay}}{ec + a^2 \times y + ax \times e - c + a^2 \times c - e}$.

And from the equation of these two values of z this equation results;

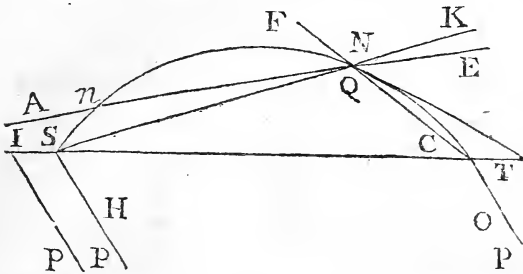
$$\begin{aligned} & \overline{a - b \times ce} \left\{ x^2 + a^2 \times \overline{d + c - e} \right\} + ax \times \overline{-a^2 + ed} \left\{ y^2 \right. \\ & \left. + \overline{ae - bc \times d} \right\} + dce \left\{ xy - bc \times \overline{e + d} \right\} \\ & + abc \times \overline{d + e} \left\{ x - bc \times \overline{a^2 - ed} \right\} \\ & - a^2e \times \overline{d + c} \left\{ x - ae \times \overline{dc - a^2} \right\} y = 0; \end{aligned}$$

where since x and y are only of two dimensions, it appears that the curve described must be a line of the second order, or a conic section, according to what has been already demonstrated.

§ 37. As the angles FCO, KSH revolve about the poles C and S, if the angle CQS becomes equal to the supplement of these given angles to four right ones, then the angle CPS must vanish, that is, the lines CO and SH must become parallel: and the intersection P must go off to an infinite distance. And the lines CO and SH become, in that case, parallel to one of the *asymptotes*.

In order to determine if this may be, describe on CS an arc of a circle that can have inscribed in it an angle equal to the supplement of the angles FCO, KSH, to four right angles. If
this

this arc meet the line AE in two points N, n, then when Q the intersection of the sides CF, SK comes to either of these points, as it is car-



ried along the line AE, the point P will go off to infinity, and the lines SH, CO, become parallel to each other and to an *asymptote* of the curve.

If that arc only touch the line AE, the point P will go off to infinity but once. If the arc neither cut the line AE nor touch it, the point P cannot go off to infinity. In the first case the conic section is an *hyperbola*, in the second a *parabola*, in the third an *ellipse*.

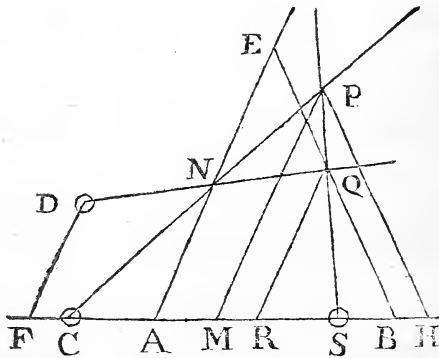
The *asymptotes*, when the curve has any, are determined by the following construction.

Draw NT constituting the angle $CNT = SNA$, meeting SC in T; then take $SI = CT$, and always towards opposite parts, and through I draw IP parallel to SH or CO, and IP will be one asymptote of the curve. The other is determined in like manner, by bringing Q to n.

And the two asymptotes meet in the center, constituting there an angle $= NSn$.

From this construction it is obvious, that when the circular arc $CNnS$ touches the line AE , the angle SNA being then $= SCN$, the line NT will become parallel to CS ; and therefore CT and SI become infinite, that is, the *asymptote* IP going off to infinity, the curve becomes a *parabola*.

§ 38. There is another general method of describing the lines of the *second order*, that deserves our consideration,



Instead of angles we now use three rulers DQ , CN , SP , which we suppose to revolve about the poles D , C , S , and cut one another always in three points N , Q and P ; and carrying any two of these intersections, as N and Q , along the given straight lines AE , BE , the third intersection P will describe a conic section.

Through

Through the points D, P, Q, draw DF, PM, QR, parallel to AE, meeting CS in F, M, R; also through P draw PH parallel to BE meeting CS in H.

Then putting $PM=y$, $CM=x$, $CS=a$, $CA=b$, $SB=c$, $DF=k$, $AF=l$, $AE=d$, $BE=e$, $AB (= a - b + c) = f$; since the triangles PMH, AEB are similar, therefore $PH = \frac{ey}{d}$,

$$MH = \frac{fy}{d}, \quad SH = \frac{dx + fy - ad}{d}. \quad \text{And since}$$

$$CA : AN :: CM : PM, \dots AN = \frac{by}{x}; \quad \text{and since}$$

$$SB : BQ :: SH : PH, \dots BQ = \frac{cey}{dx + fy - ad}$$

But,

$$BQ : QR :: BE : AE \dots QR = \frac{cdy}{dx + fy - ad},$$

$$\text{and } \dots BR = \frac{cfy}{dx + fy - ad}$$

Now $AN - DF : RQ - AN :: AF : AR$; that is,

$$\frac{by}{x} - k : \frac{cdy}{dx + fy - ad} - \frac{by}{x} :: l : f - \frac{cfy}{dx + fy - ad}.$$

And multiplying the extremes and means, and ordering the terms, it is,

$$\left. \begin{aligned} &bfxc - l - f \times y^2 + c \times ld - kf - bd \times l + f + kff \times xy \\ &+ bad \times l + f \times y - adfk \times x + dfk \times x^2 \end{aligned} \right\} = 0.$$

In which equation, the signs of some terms may vary by varying the situation of the poles and

and lines; but x and y not rising to more than two dimensions, it appears that the point P always describes a *conic section*. Only in some particular cases the conic section becomes a straight line. As for example, when D is found in the straight line CS ; for then DF vanishing the terms $dfkx^2 - adfkx$ vanish, and the remaining terms being divisible by y , the equation becomes,

$$bf \times \overline{c-l-f} \times y + \overline{cld-bd \times l + f \times x} + \overline{bad \times l + f} = 0.$$

Which is a *locus* of the *first order*, and shews, that, in this case, P must describe a *straight line*.

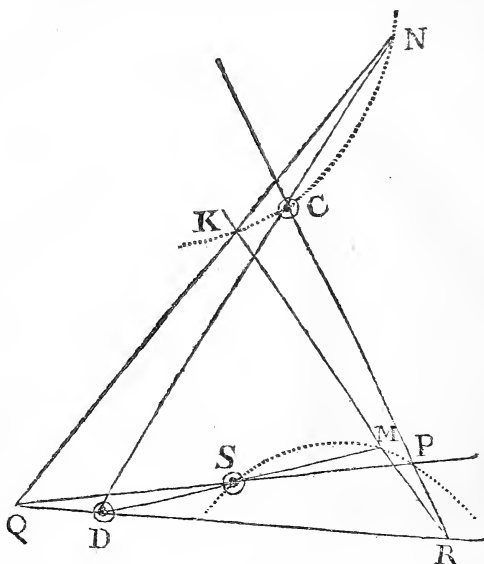
After the same manner it appears that if the point E the intersection of the lines AE , BE , falls in CS , then will P describe a straight line. For in that case d vanishes, and the equation becomes,

$$b \times \overline{c-l-f} \times y - f \times \overline{c-k} \times x = 0.$$

§ 39. These two descriptions furnish, each, a general method of “describing a line of the second order through any five given points whereof three are not in the same straight line.”

Suppose the five given points are C , S , M , K , N ; join any three of them, as C , S , K , and let angles revolve about C and S equal to the angles KCS , KSC . Apply the intersection of the legs CK , SK first to the point N , and let the inter-

And when it comes to Q and L, it passes through N, M.



§ 40. From the second description we have this solution of the same problem.

Let C, S, M, K, N be the five given points : draw lines joining them ; produce two of the lines NC, MS, till they meet in D. Let three rulers revolve about the three poles C, S, D, *viz.* CP, SQ, DR. Let the intersection of the rulers CP, DR, be carried over the given line MK, and the intersection of the rulers SQ, DR be carried through the line NK ; and the point P, the intersection of the rulers that

that revolve about C and S, will describe a conic section that passes through the five points C, S, M, K, N.

§ 41. It is a remarkable property of the conic sections, that “if you assume any number of poles whatsoever, and make rulers revolve about each of them, and all the intersections but one, be carried along given right lines, that one shall never describe a line above a conic section;” if, instead of rulers you substitute given angles which you move on the same poles, the curve described will still be no more than a conic section.

By carrying one of the intersections necessary in the description over a *conic* section, lines of higher orders may be described.



C H A P. III.

Of the Construction of *cubic* and *biquadratic* Equations.

§ 42. “**T**HE roots of any equation may be determined by the interfections of a straight line with a curve of the same dimensions as the equation:” or, “by the interfections of any two curves whose indices multiplied by each other give a product equal to the index of the proposed equation.”

Thus the roots of a *biquadratic* equation may be determined by the interfections of two conic sections; for the equation by which the ordinates from the four points in which these conic sections may cut one another can be determined will arise to four dimensions: and the conic sections may be assumed in such a manner, as to make this equation coincide with any proposed *biquadratic*: so that the ordinates from these four interfections will be equal to the roots of the proposed *biquadratic*.

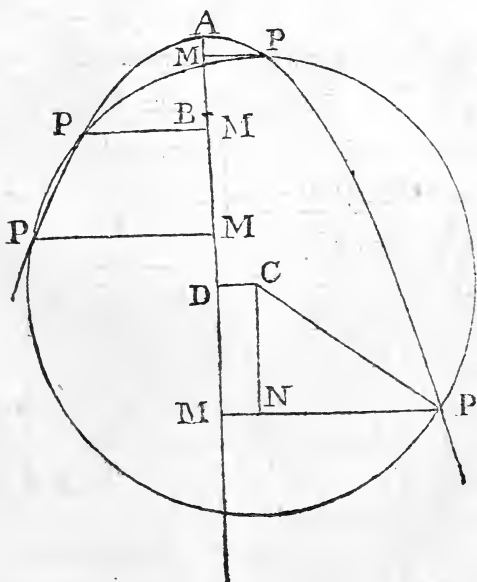
If one of the interfections of the conic section falls upon the *axis*, then “one of the ordinates vanishes, and the equation by which these ordinates are determined will then be of three dimensions only, or a *cubic*,” to which any proposed *cubic* equation may be accommodated.

So

So that the three remaining ordinates will be the three roots of that proposed cubic.

§ 43. Those conic sections ought to be preferred for this purpose that are most easily described. They must not however be both *circles*; for their intersections are only two, and can serve only for the resolution of *quadratic* equations.

Yet the circle ought to be one, as being most easily described; and the *parabola* is commonly assumed for the other. Their intersections are determined in the following manner.



Let

Let APE be the common Apollonian parabola. Take on its axis the line AB = half of its parameter. Let C be any point in the plane of the parabola, and from it as a center describe, with any radius CP, a circle meeting the parabola in P. Let PM, CD, be perpendiculars on the axis in M and D, and let CN, parallel to the axis, meet PM in N.

Then will always $CPq = CNq + NPq$. (47 é r.)

Put $CP = a$, the parameter of the parabola = b , $AD = c$, $DC = d$, $AM = x$, $PM = y$.

Then $CNq = \overline{x + c}^2$, $NPq = \overline{y + d}^2$; and $\overline{x + c}^2 + \overline{y + d}^2 = a^2$ *. That is,

$$x^2 + 2cx + c^2 + y^2 + 2dy + d^2 = a^2.$$

But, from the nature of the parabola, $y^2 = bx$, and $x^2 = \frac{y^4}{b^2}$; substituting therefore these values for x^2 and x , it will be,

$$\frac{y^4}{b^2} + \frac{2cy^2}{b} + y^2 + 2dy + c^2 + d^2 - a^2 = 0.$$

Or, multiplying by b^2 ,

$$y^4 + 2bc + b^2 \times y^2 + 2db^2 \times y + c^2 + d^2 - a^2 \times b^2 = 0.$$

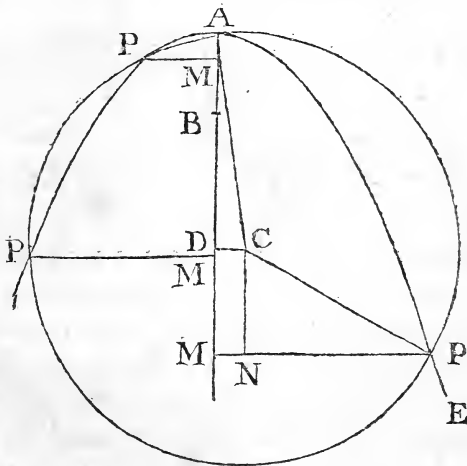
Which may represent any biquadratic equation that wants the second term; since such values may be found for a , b , c , and d , by comparing this with any proposed biquadratic, as to make them coincide. And then the ordinates from the points P, P, P, P, on the axis will be equal to the roots of that proposed biquadratic. And

* $x + c$ is the differences of x and c indefinitely, whichever of the two is greatest.

this

this may be done, though the parameter of the parabola (*viz.* b) be given : that is, if you have a parabola already made or given, by it alone you may resolve all biquadratic equations, and you will only need to vary the center of your circle and its radius.

§ 44. If the circle described from the center C pass through the vertex A , then $CPq = CAq = CDq + ADq$, that is, $a^2 = d^2 + c^2$; and the



last term of the biquadratic ($c^2 + d^2 - a^2$) will vanish; therefore, dividing the rest by y , there arises the cubic,

$$y^{3*} \pm \overline{2bc} + \overline{b^2} \times y \pm 2db^2 = 0.$$

Let the cubic equation proposed to be resolved be $y^{3*} \pm py \pm r = 0$. Compare the terms of
 A a these

these two equations, and you will have $\pm 2bc \mp b^2 = \pm p$, and $\pm 2db^2 = \pm r$, or, $\mp c = \frac{b}{2} \mp \frac{p}{2b}$, and $d = \pm \frac{r}{2b^2}$. From which you have this construction of the cubic $y^{3*} \pm py \pm r = 0$, by means of any given parabola APE.

“From the point B take in the axis (forward if the equation has $-p$, but backwards if p is positive) the line $BD = \frac{p}{2b}$; then raise the perpendicular $DC = \frac{r}{2b^2}$, and from C, describe a circle passing through the vertex A, meeting the parabola in P, so shall the ordinate PM be one of the roots of the cubic $y^{3*} \pm py \pm r = 0$.”

The ordinates that stand on the same side of the axis with the center C are negative or affirmative, according as the last term r is *negative* or *affirmative*; and those ordinates have always *contrary* signs that stand on *different* sides of the axis. The roots are found of the same value, only they have contrary signs, when r is positive as when it is negative; the second term of the equation being wanting; which agrees with what has been demonstrated elsewhere.

§ 45. In resolving *numerical* equations, you may suppose the parameter b to be unit; then
AD

$AD = \frac{1}{2} \mp \frac{1}{2}p$, and $DC = \frac{1}{2}r$; and the ordinate PM must then be measured on a scale where the *parameter*, or $2AB$ is unit. Or, if it be more convenient, the *parameter* may be supposed to express 10, 100, &c. or any other number, and PM will be found by measuring it on a scale where the parameter is 10, 100, &c. or that other number.

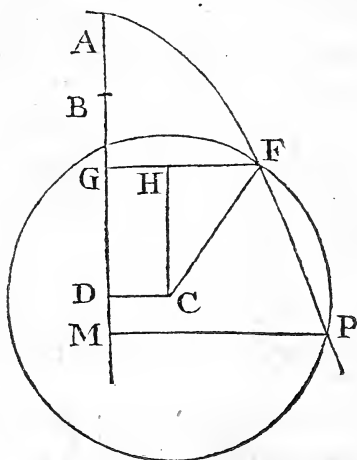
§ 46. "When the circle meets the *parabola* in one point only besides the vertex, the equation has only one real root, and the other two imaginary."

Thus, if the equation has $+p$, or if D falls on the same side of B as A does, the circle can meet the parabola in two points only, whereof A is one; and therefore the equation must have two *imaginary* roots; as we demonstrated elsewhere. If the circle *touch* the *parabola*, then two roots of the equation are equal.

It is also obvious, that the equation must necessarily have *one real* root; because, since the circle meets the *parabola* in the vertex A, it must meet it in one other point, at least, besides A.

§ 47. Instead of making the circle pass thro' the *vertex* A, you may suppose it to pass through some other given point in the parabola, and that intersection being given, the *biquadratic* found for determining the intersections, in § 43, may be reduced to a *cubic*.

Let the ordinate belonging to that given



interfection be g , then one of the values of y being g , it follows that the biquadratic

$$y^{4*} \pm \frac{2bc}{b^2} \left. \vphantom{y^{4*}} \right\} y^2 \pm 2db^2y + \overline{d^2 + c^2 - a^2} \times b^2 = 0$$

will be dividible by $y - g$, which will reduce it to a *cubic* that shall have the second term. And thus we have a construction for *cubic* equations that have all their terms.

For example, let us suppose that the *parameter* is AG , and the ordinate at G is GF meeting the curve in F . Suppose now that the circle is always to pass through F ; then shall $CFq (= a^2) = CHq + HFq = \overline{c \pm b}^2 + \overline{b \pm d}^2 = c^2 + d^2 \pm 2cb \pm 2db + 2b^2$, and substituting
in

in the equation of § 43 this value of a^2 , it becomes

$$y^{4*} \left. \begin{array}{l} \pm 2cb \\ + b^2 \end{array} \right\} y^2 \left. \begin{array}{l} \pm 2db^2 \\ \pm 2cb^3 \\ \pm 2db^3 \end{array} \right\} \times y \left. \begin{array}{l} - 2b^4 \\ \pm 2cb^3 \\ \pm 2db^3 \end{array} \right\} = 0.$$

Where c in the last term has a contrary sign to what it has in the third, and d a contrary sign to what it has in the fourth.

This biquadratic has FG , or b , for one of its roots; and being divided by $y - b$, there arises this cubic,

$$y^3 + by^2 \left. \begin{array}{l} \pm 2cb \\ + 2b^2 \end{array} \right\} y \left. \begin{array}{l} \pm 2db^2 \\ \pm 2cb^2 \\ + 2b^3 \end{array} \right\} = 0,$$

having all its terms complete. If C had been taken on the other side of the axis, the second term by^2 had been negative.

Let now any cubic equation be proposed to be resolved, as $y^3 + py^2 + qy - r = 0$. And, by comparing it with the preceding, you will find

$$\left. \begin{array}{l} p = b \\ q = 2b^2 \pm 2bc \\ -r = 2b^3 \pm 2cb^2 \pm 2db^2 \end{array} \right\} \text{whence} \left\{ \begin{array}{l} b = p, \\ \pm c = p - \frac{q}{2p}, \\ d = \frac{q}{2p} + \frac{r}{2p^2}. \end{array} \right.$$

Therefore, to construct the proposed cubic equation $y^3 + py^2 + qy - r = 0$, let the parameter of your parabola be equal to p , take, on the axis from

the vertex A , the line $AD = p - \frac{q}{2p}$, and raise the perpendicular $DC = \frac{q}{2p} + \frac{r}{2p^2}$, and from C describe a circle through F , meeting the parabola in P , so shall the ordinate PM be a root of the equation.

If the equation proposed is a *literal* equation of this form $y^3 + py^2 + pqy - p^2r = 0$, having all the terms of three dimensions, then this construction will only require $AD = 1 - \frac{1}{2}q$, and $DC = \frac{1}{2}q + \frac{1}{2}r$.

§ 48. If you suppose the parabola to pass through any point F taken any where in the parabola (*vid. Fig. preced.*) and call the ordinate $FG = e$, then $c - \frac{e^2}{b} + e - d = a^2$, and the general biquadratic may have this form,

$$\left. \begin{array}{l} y^{4*} + 2cb \\ + b^2 \end{array} \right\} y^2 - 2db^2y + \left. \begin{array}{l} 2ce^2b \\ + 2deb^2 \\ - e^4 \\ - e^2b^2 \end{array} \right\} = 0.$$

But since $FG = e$ is one of the values of y , the equation will be divisible by $y - e$, and the quotient is found to be this *cubic*,

$$\left. \begin{array}{l} y^3 + cy^2 - 2bc \\ + b^2 \\ + e^2 \end{array} \right\} y - \left. \begin{array}{l} 2db^2 \\ 2ceb \\ + eb^2 \\ + e^3 \end{array} \right\} = 0.$$

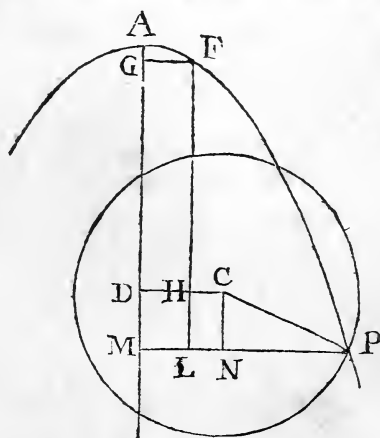
Which

Which compared with $y^3 + py^2 + pqy - p^2r = 0$, gives FG (or e) = p , AD ($= c$) = $\frac{p^2 + b^2 - pq}{2b}$,

and $DC = d = \frac{p^2 \times q + r}{2b^2}$. And by this construction the roots of a compleat cubic equation may be found by any parabola whatsoever.

§ 49. It is easy to see from § 43. how to construct the roots of a biquadratic by any parabola, after the second term is taken away. But “the roots of a biquadratic may be determined by any parabola:” only they cannot be the ordinates on the axis, but “may be equal to the perpendiculars on a line parallel to the axis, meeting the parabola in F , CD in H , and PM in L .”

Let FG be an ordinate to the axis in G ; and



the rest remaining as before, let $FL = x$, $PL = y$,

the parameter = b , $CP = a$, $FH = c$, $CH = d$,
 $FG = e$.

And since $PMq (= \overline{PL + HD})^2 = AM \times b$,
 therefore $y^2 + 2ey + e^2 = \overline{AG + FL} \times b =$
 $\frac{e^2}{b} + x \times b$; and consequently $y^2 + 2ey = bx$.

But $CNq + NPq = CPq$; that is, $\overline{x - c}^2 +$
 $\overline{y - d}^2 = a^2$. And substituting for x^2 and x their
 values $\frac{y^2 + 2ey}{b}$ and $\frac{y^2 + 2ey}{b}$; you will find

$$y^4 + 4ey^3 + 4e^2 \left. \begin{array}{l} - 2cb \\ + b^2 \end{array} \right\} y^2 - 4ecb \left. \begin{array}{l} + c^2b^2 \\ - 2db^2 \end{array} \right\} y + a^2b^2 \left. \begin{array}{l} + c^2b^2 \\ + a^2b^2 \end{array} \right\} = 0,$$

which is a complete biquadratic equation. And
 by comparing with it the equation

$y^4 + py^3 + bqy^2 - b^2ry - b^3s = 0$, you will find

$$FG (= e) = \frac{1}{4}p, \quad FH (= c) = \frac{\frac{1}{4}p^2 + b^2 - bq}{2b},$$

$$HC (= d) = \frac{b^2r - pbc}{2b^2}, \quad \text{and } CP (= a) =$$

$\sqrt{b^3s + c^2 + d^2}$: which gives a general construc-
 tion for any such biquadratic equation by any
 parabola whatsoever. If the signs of p , q , r ,
 or s , are different, it is easy to make the ne-
 cessary alterations in the construction. *Ex. gr.*
 If p is negative, then FG must be taken on the
 other side of the axis.

If you suppose the circle to pass through F ,
 the equation will become a *cubic* having all its
 terms :

terms: the last term $\overline{c^2 + d^2 - a^2} \times b^2$ vanishing, because then $c^2 + d^2 = a^2$. It will have this form,

$$y^3 + 4ey^2 + 4e^2 \left. \begin{array}{l} - 2cb \\ + b^2 \end{array} \right\} y - \frac{4ecb}{2db^2} \left. \right\} = 0;$$

and then "the construction will give the roots of a compleat cubic equation."

§ 50. We have sufficiently shewed, how the roots of cubic and biquadratic equations may be constructed by the parabola and circle; we shall now shew how *other* conic sections may be determined by whose interfections the same roots may be discovered.

Let the equation proposed be $y^{4*} + bpy^2 + b^2qy - b^3r = 0$; and let us suppose, that,

1°. $bx = y^2$; then shall we have by substitution of b^2x^2 for y^4 , and dividing by bp ,

2°. $y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, which has its *locus* an *ellipse*. Then by substituting (in this last) bx for y^2 , and multiplying all the terms by $\frac{p}{b}$, you find,

3°. $x^2 + px + qy - br = 0$, an equation to a *parabola*. Then, adding to this equation $y^2 - bx = 0$, you will have,

4°. $x^2 + y^2 + \frac{p}{b} \left. \begin{array}{l} + px \\ - br \end{array} \right\} + qy - br = 0$, an equation to a *circle*.

The

Let AB be the axis of an ellipsis, equal to $\sqrt{br + \frac{bq^2}{4p}}$, let G be the center of the ellipse, and the axis to the parameter as p to b . At G , raise a perpendicular to the axis, and on it take $GD = \frac{bq}{2p}$, and on the other side in the perpendicular continued take $GK = \frac{1}{2}q \times \frac{p-b}{p}$. Let DE and KC be parallel to the axis: take $KC = \frac{1}{2}b - \frac{1}{2}p$, and from C as a center, with the radius $\sqrt{DCq + br}$ describe a circle meeting the ellipse in P , and the ordinate PM , on the line DE , shall be one of the roots of the proposed equation.

Let $PM (= y)$ produced meet AB in R , and KC in N ; and calling $DM = x$, then $CPq = NPq - NCq$, that is, $\frac{1}{4}q^2 + \frac{1}{4}b^2 - \frac{1}{2}pb + \frac{1}{4}p^2 + br = \frac{1}{2}b - \frac{1}{2}p - x \Big|^2 + y + \frac{1}{2}q \Big|^2$; and therefore,

1°. $y^2 + x^2 + qy - \frac{b}{p} \Big\} x - br = 0$, the equation to the circle, which was to be constructed.

And since $PRq : GBq - GRq :: b : p$, therefore $y + \frac{bq}{2p} \Big|^2 : br + \frac{bq^2}{4p} - x^2 :: b : p$; and consequently,

2°. $y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$; which is the equation that was to be constructed.

Now that their intersections will give the roots required, appears thus.

For

For x^2 in the first equation substitute the value you deduce for it from the second, *viz.*

$br - \frac{p}{b}y^2 - qy$, and there will arise

$$y^2 - \frac{p}{b}y^2 - \frac{b}{p} \left. \vphantom{\frac{p}{b}y^2} \right\} x = 0, \text{ or } \frac{b-p}{b} \times y^2 = \overline{b-p} \times x,$$

that is, $\frac{y^2}{b} = x$, and $x^2 = \frac{y^4}{b^2}$; which substituted for x^2 and x in the first equation, gives

$$\frac{y^4}{b^2} + y^2 + \overline{p-b} \times \frac{y^2}{b} + qy - br = 0; \text{ that is, } y^{4*} + bpy^2 + b^2qy - b^3r = 0.$$

And if you substitute them in the second equation, there will arise

$$\frac{b}{pb^3}y^4 + y^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0, \text{ that is, } y^{4*} + bpy^2 + b^2qy - b^3r = 0, \text{ the very same as before;}$$

and thus it appears that the roots of the equation $y^{4*} - bpy^2 + b^2qy - b^3r = 0$ are the ordinates that are common to the *circle* and *ellipse*, or that are drawn from their intersection.

End of the THIRD PART.





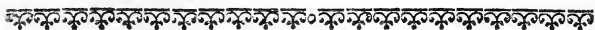
APPENDIX:

DE

Linearum Geometricarum

Proprietatibus generalibus

TRACTATUS.






D E

LINEARUM GEOMETRICARUM

Proprietatibus generalibus.

 E lineis secundi ordinis, five sectionibus conicis, scripserunt uberrime geometræ veteres & recentiores ; de figuris quæ ad superiores linearum ordines referuntur pauca & exilia tantum ante NEWTONUM tradiderunt. Vir illustrissimus, in Tractatu de *Enumerationem Linearum tertii Ordinis*, doctrinam hanc, cum diu jacuisset, excitavit, dignamque esse in qua elaborarent geometris ostendit. Expositis enim harum linearum proprietatibus generalibus, quæ vulgatis sectionum conicarum affectionibus sunt adeo affines ut velut ad eandem normam compositæ videantur, alios suo exemplo impulit ut analogiam hanc five similitudinem quæ tam diversis intercedit figurarum generibus bene cognitam & satis firme animo conceptam atque comprehensam habere studerent. In qua illustranda & ulterius indaganda curam operamque merito posuerunt ; cum nihil sit omnium quæ in disciplinis purè mathematicis tractantur quod pulchrius dicatur, aut ad animum veri investigandi cupidum oblectandum aptius, quam rerum tam diversarum consensus five harmonia, ipsiusque doctrinæ compositio & nexus admirabilis, quo posterius priori convenit, quod sequitur superiori respondet, quæque

quæque simpliciora sunt ad magis ardua viam constanter aperiunt.

Linearum tertij ordinis proprietates generales a *Newtono* traditæ parallelarum segmenta & asymptotos præque spectant. Alias harum affectiones quasdam diversi generis breviter indicavimus in tractatu de fluxionibus nuper edito, Art. 324, & 401. Celeberrimus *Cotesius* pulcherrimam olim detexit linearum geometricarum proprietatem, hucusque ineditam, quam absque demonstratione nobis communicavit vir Reverendus D. *Robertus Smith*, Collegii S. S. Trinitatis apud Cantabrigienses Præfectus, doctrina operibusque suis pariter ac fide & studio in amicos clarus. De his meditantibus nobis alia quoque se obtulerunt theoremata generalia; quæ cum ad arduam hanc geometriæ partem augendam & illustrandam conducere viderentur, ipsa quasi in fasciculum congerenda & una serie breviter exponenda & demonstranda putavimus.



SECTIO I.

De Lineis Geometricis in genere.

§ 1. **L**ineæ secundi ordinis sectione solidi geometrici, conici scilicet, definiuntur, unde earum proprietates per vulgarem geometriam optime derivantur. Verum diversa est ratio figurarum quæ ad superiores linearum ordines referuntur. Ad has definiendas, earumque proprietates eruendas, adhibendæ sunt æquationes generales co-ordinatarum relationem exprimentes. Repræsentet x abscissam AP, y ordinatam PM figuræ FMH, denotentque a, b, c, d, e , &c. coefficients quasunque
 inva-

Fig. 1.

invariabiles; & dato angulo APM si relatio co-ordinatarum x & y definiatur æquatione quæ, præter ipsas co-ordinatas, solas involvat coefficientes invariabiles, linea FMH geometrica appellatur; quæ quidem auctoribus quibusdam linea algebraica, aliis linea rationalis dicitur. Ordo autem lineæ pendet ab indice altissimo ipsius x vel y in terminis æquationis a fractionibus & surdis liberatæ, vel a summa indicis utriusque in termino ubi hæc summa prodit maxima. Termini enim x^2 , xy , y^2 ad secundum ordinem pariter referuntur; termini x^3 , x^2y , xy^2 , y^3 ad tertium. Itaque æquatio $y = ax + b$, sive $y - ax - b = 0$, est primi ordinis & designat lineam sive locum primi ordinis, quæ quidem semper recta est. Sumatur enim in ordinata PM recta PN ita ut PN sit ad AP ut $+a$ ad unitatem; constituatur AD parallela ordinatæ PM æqualis ipsi $+b$, & ducta DM parallela rectæ AN erit locus cui æquatio proposita respondebit. Nam $PM = PN + NM = (a \times AP + AD) = ax + b$. Quod si æquatio sit formæ $y = ax - b$ vel $y = -ax + b$, recta AD, vel PN, sumenda est ad alteram partem abscissæ AP; contrarius enim rectarum situs contrariis coefficientium signis respondet. Si valores affirmativi ipsius x designent rectas ad dextram ductas a principio abscissæ A, valores negativi denotabunt rectas ab eodem principio ad sinistram ductas; & similiter si valores affirmativi ipsius y ordinatas repræsentent supra abscissam constitutas, negativi designabunt ordinatas infra abscissam ad oppositas partes ductas.

Fig. 2.

Æquatio generalis ad lineam secundi ordinis est hujus formæ

$$\begin{aligned}
 yy - axy + cx^2 &= 0 \\
 - by - dx & \\
 + e &
 \end{aligned}$$

& æquatio generalis ad lineas tertii ordinis est $y^3 - ax + b \times y^2 + cxx - dx + e \times y - fx^3 + gx^2 - hx + k = 0$. Et similibus æquationibus definiuntur lineæ geometricæ superiorum ordinum.

§ 2. Linea geometrica occurrere potest rectæ in tot punctis quot sunt unitates in numero qui æquationis vel lineæ ordinem designat, & nunquam in pluribus. Occursus curvæ & abscissæ AP definiuntur ponendo $y = 0$, quo in casu restat tantum ultimus æquationis terminus quem y non ingreditur. Linea tertii ordinis *ex. gr.* occurrit abscissæ AP cum $fx^3 - gx^2 + hx - k = 0$, cujus æquationis si tres radices sint reales abscissa secabit curvam in tribus punctis. Similiter in æquatione generali cujuscunque ordinis index altissimus abscissæ x æqualis est numero qui lineæ ordinem designat, sed nunquam major, adeoque is est numerus maximus occursum curvæ cum abscissa vel alia quavis rectâ. Cum autem æquationis cubicæ unica saltem radix sit semper realis, idemque constet de æquatione quavis quinti aut imparis cujuscunque ordinis (quoniam radix quævis imaginaria aliam necessario semper habet comitem), sequitur lineam tertii aut imparis cujuscunque ordinis rectam quamvis asymptoto non parallelam in eodem plano ductam in uno saltem puncto necessario secare. Si vero recta sit asymptoto parallela, in hoc casu vulgo dicitur curvæ occurrere ad distantiam infinitam. Linea igitur imparis cujuscunque ordinis duo saltem habet crura in infinitum progredientia. Æquationis autem quadraticæ vel paris cujuscunque ordinis radices omnes nonnunquam fiunt imaginariæ, adeoque fieri potest ut recta in plano lineæ paris ordinis ducta eidem nullibi occurrat.

§ 3. Æquatio secundi aut superioris cujuscunque ordinis quandoque componitur ex tot simplicibus, a surdis & fractis liberatis, in se mutuo ductis quot sunt ipsius æquationis propositæ dimensiones; quo in casu figura FMH non est curvilinea sed conflatur ex totidem rectis, quæ per simplices has æquationes definiuntur ut in *Art. I.* Similiter si æquatio cubica componatur ex æquationibus duabus in se mutuo ductis, quarum altera sit quadratica altera simplex, locus non erit linea tertii ordinis proprie sic dicta, sed sectio conica cum rectâ adjunctâ. Proprietates autem quæ de lineis geometricis superiorum ordinum generaliter demonstrantur, affirmandæ sunt quoque de lineis inferiorum ordinum, modo numeri harum ordines designantes simul sumpti numerum compleant qui ordinem dictæ superioris lineæ denotat. Quæ de lineis tertii ordinis (*ex. gr.*) generaliter demonstrantur affirmanda quoque sunt de tribus rectis in eodem plano ductis, vel de sectione conica cum unica quavis recta simul in eodem plano descriptis. Ex altera parte, vix ulla assignari potest proprietas lineæ ordinis inferioris satis generalis cui non respondeat affectio aliqua linearum ordinum superiorum. Has autem ex illis derivare non est cujuscvis diligentix. Pendet hæc doctrina magna ex parte a proprietatibus æquationum generalium, quas hic memorare tantum convenit.

§ 4. In æquatione quacunque coefficientis secundi termini æqualis est excessui quo summa radicum affirmatarum superat summam negativarum; & si desit hic terminus, indicio est summas radicum affirmatarum & negativarum, vel summas ordinatarum ad diversas partes abscissæ constitutarum, æquales esse. Sit æquatio generalis ad lineam ordinis n ,

$y^n - \overline{ax + b} \times y^{n-1} + \overline{cxx - dx + e} \times y^{n-2} - \text{etc.} = 0$,
 supponatur $u = y - \frac{ax + b}{n}$, pro y substituatur ipsius va-

lor $u + \frac{ax + b}{n}$; & in æquatione transformata deerit se-

cundus terminus u^{n-1} ; ut ex calculo, vel ex doctrinâ
 æquationum passim traditâ facile patet: & hinc quo-
 que constat, quod per hypothefin valor quisque ipsius
 u minor sit valore correspondente ipsius y differentia
 $\frac{ax + b}{n}$; unde sequitur summam valorum ipsius u (quo-

rum numerus est n) deficere a summa valorum ipsius
 y (quæ summa est $ax + b$) differentia $\frac{ax + b}{n} \times n = ax$

$+ b$, adeoque priorem summam evanescere & secun-
 dum terminum deesse in æquatione qua u definitur, vel
 affirmativos & negativos valores ipsius u æquales sum-
 mas conficere. Si itaque sumatur $PQ = \frac{ax + b}{n}$, ut fit

$QM = u$, rectæ ex utraque parte puncti Q ad curvam
 terminatæ eandem conficient summam. Locus autem

Fig. 3. puncti Q est recta BD quæ abscissam ultra principium A
 productam secat in B ita ut $AB = \frac{b}{a}$, & ordinatam AD

ipfi PM parallelam in D ita ut fit $AD = \frac{1}{n} \times b$; si enim

hæc recta ordinatæ PM occurrat in puncto Q , erit
 PQ ad PB (seu $\frac{b}{a} + x$) ut AD ad AB vel a ad n , adeo-

que $PQ = \frac{ax + b}{n}$, ut oportebat. Atque hinc constat

rectam semper duci posse quæ parallelas quasvis lineæ
 geometricæ occurrentes in tot punctis quot sunt figuræ
 dimensiones ita secabit ut summa segmentorum cujusvis
 paral-

parallelæ ex una secantis parte ad curvam terminatorum semper æqualis fit summæ segmentorum ejusdem ex altera secantis parte. Manifestum autem est rectam quæ duas quasvis parallelas hac ratione secat ipsam necessario esse quæ similiter alias omnes parallelas secabit. Atque hinc patet veritas theorematis *Newtoniani*, quo continetur proprietas linearum geometricarum generalis, notissimæ sectionum conicarum proprietati analoga. In his enim recta quæ duas quasvis parallelas ad sectionem terminatas bisecat diameter est, & bisecat alias omnes hisce parallelas ad sectionem terminatas. Et similiter recta quæ duas quasvis parallelas lineæ geometricæ occurrentes in tot punctis quot ipsa est dimensionum ita secat ut summa partium ex uno secantis latere consistentium & ad curvam terminatorum æqualis sit summæ partium ejusdem parallelæ ex altero secantis latere consistentium ad curvam terminatorum, eodem modo secabit alias quasvis rectas his parallelas.

§ 5. In æquatione quavis terminus ultimus, five is quem radix y non ingreditur, æqualis est facto ex radicibus omnibus in se mutuo ductis; unde ad aliam ducimur non minus generalem linearum geometricarum proprietatem. Occurrat recta PM lineæ tertii ordinis in M , m & μ , eritque $PM \times Pm \times P\mu = fx^3 - gx^2 + hx - k$. Secet abscissa AP curvam in tribus punctis I, K, L ; & AI, AK, AL erunt valores abscissæ x , posita ordinata $y = 0$, quo in casu æquatio generalis dat $fx^3 - gx^2 + hx - k = 0$ pro his valoribus determinandis, ut in Art. 2. exposuimus. Æquationis igitur $x^3 - \frac{gx^2}{f} + \frac{hx}{f} - \frac{k}{f} = 0$ tres radices sunt AI, AK, AL ; adeoque hæc æquatio componitur ex tribus $x - AI, x - AK, x - AL$ in

Fig. 1.

se mutuo ductis; estque $x^3 - \frac{gx^2}{f} + \frac{bx}{f} - \frac{k}{f} =$
 $\frac{x - AI}{x - AK} \times \frac{x - AL}{x - AL} = \frac{AP - AI}{AP - AK}$
 $\times \frac{AP - AL}{AP - AL} = IP \times KP \times LP = \frac{1}{f} \times PM \times P_m$
 $\times P_\mu$. Factum igitur ex ordinatis PM, P_m, P_μ ad
 punctum P & curvam terminatis est ad factum ex seg-
 mentis IP, KP, LP, rectæ AP, eodem puncto & cur-
 va terminatis in ratione invariabili coefficientis *f* ad
 unitatem. Simili ratione demonstratur, dato angulo
 APM, si rectæ AP, PM, lineam geometricam cujus-
 vis ordinis fecent in tot punctis quot ipsa est dimensio-
 num, fore semper factum ex segmentis prioris ad pun-
 ctum P & curvam terminatis ad factum ex segmentis
 posterioris eodem puncto & curva terminatis in ratione
 invariabili.

§ 6. In articulo præcedente supposuimus, cum *New-*
tono, rectam AP lineam tertii ordinis secare in tribus
 punctis I, K, L; verum ut theorema egregium red-
 datur generalius, supponamus abscissam AP in unico
 tantum puncto curvam secare; fitque id punctum A.

Fig. 4. Quoniam igitur evanescente *y* evanescat quoque *x*, ul-
 timus æquationis terminus, in hoc casu, erit $fx^3 - gx^2 +$

$$bx = fx \times xx - \frac{gx}{f} + \frac{b}{f} = fx \times x - \frac{g^2}{2f} + \frac{b}{f} - \frac{gg}{4ff}$$

(si sumatur Aa versus P æqualis $\frac{g}{2f}$, & ad punctum *a*

$$\text{erigatur perpendicularis } ab = \frac{\sqrt{4fb - gg}}{2f} = f \times$$

$\frac{AP \times aP^2 + ab^2}{2f} = f \times AP \times bP^2$; unde cum $PM \times$
 $P_m \times P_\mu$, fit æqualis ultimo termino $fx^3 - gx^2 + bx$,

ut

ut in articulo præcedente; erit $PM \times Pm \times P\mu$ ad $AP \times bP^2$ in ratione constante coefficientis f ad unitatem. Valor autem rectæ perpendicularis ab est semper realis quoties recta AP curvam in unico puncto fecat; in hoc enim casu radices æquationis quadraticæ $fx^2 - gx + b$ sunt necessario imaginariæ, adeoque $4fb$ major quam gg , & quantitas $\sqrt{4fb - gg}$ realis. Cum igitur recta quævis in unico puncto A fecat lineam tertii ordinis, est solidum sub ordinatis $PM, Pm, P\mu$ ad solidum sub abscissâ AP & quadrato distantie puncti P a puncto dato b in ratione constanti. Junctâ Ab est ad Aa , sive radius ad cosinum anguli bAP , ut $\sqrt{4fb}$ ad g , & $Ab = \sqrt{\frac{b}{f}}$. Idem vero punctum b semper convenit eidem rectæ AP , qualiscunque sit angulus qui abscissâ & ordinatâ continetur.

§ 7. Sit figura sectio conica, cujus æquatio generalis fit $yy - ax - b \times y + cxx - dx + e = 0$ ut supra; & Fig. 5. si æquationis $cxx - dx + e = 0$ radices sint imaginariæ, recta AP sectioni non occurret. In hoc autem casu quantitas $4ec$ semper superat ipsam dd ; unde cum fit $cxx - dx + e = c \times x - \frac{d^2}{2c} + e - \frac{dd}{4c}$ (si sumatur $Aa = \frac{d}{2c}$ & erigatur ab perpendicularis abscissæ in a ita ut $ab = \frac{\sqrt{4ec - dd}}{2c}$) = $c \times aP^2 + ab^2 = c \times bP^2$, fitque $PM \times Pm = cxx - dx + e$, erit $PM \times Pm$ ad bP^2 ut c ad unitatem. Itaque in sectione quavis conica si recta AP sectioni non occurrat, erit, dato angulo APM , rectangulum contentum sub rectis ad punctum B consistentibus & ad curvam terminatis ad quadratum

B b 4

tum distantiae puncti P a puncto dato b in ratione constanti, quæ in circulo est ratio æqualitatis. Manifestum autem est eandem methodum adhiberi posse lineæ quarti ordinis quam abscissa secant in duobus tantum punctis, vel lineæ ordinis cujuscunque quam abscissa secant in punctis binario paucioribus numero qui figuræ ordinem designat.

§ 8. Hisce præmissis, progredimur ad linearum geometricarum proprietates minus obvias exponendas eodem fere ordine quo se nobis obtulerunt. Utebamur autem lemmate sequenti ex fluxionum doctrina petito, quodque in tractatus de hisce nuper editi Art. 717. demonstravimus; harum tamen aliquas per algebraem vulgarem demonstrari posse postea observavimus.

Lemma. Si quantitibus $x, y, z, u, \&c.$ simul fluentibus, ut & quantitibus $X, Y, Z, V, \&c.$ fit factum ex prioribus ad factum ex posterioribus in ratione constanti quacunque, erit $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \frac{\dot{u}}{u} + \&c.$
 $= \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} + \frac{\dot{V}}{V} + \&c.$ Porro, brevitatis gratia, quantitates appellamus sibi mutuo *reciprocas*, quarum in se mutuo ductarum factum est unitas, sic $\frac{1}{x}$ dicimus *reciprocam* esse ipsius x , & $\frac{1}{y}$ ipsius y .

§ 9. Theor. I. Occurrat recta quævis per punctum datum ducta lineæ geometricæ cujuscunque ordinis in tot punctis quot ipsa est dimensionum; rectæ figuram in his punctis contingentes abscindant ab alia rectâ positione data per idem punctum datum ductâ segmenta totidem hoc puncto terminata; & horum segmentorum reciproca eandem

dem semper conficiant summam, modo segmenta ad contrarias partes puncti dati sita contrariis signis afficiantur.

Sit P punctum datum, PA & Pa rectæ quævis duæ Fig. 6. ex P ductæ quarum utraque curvam secant in tot punctis A, B, C, &c. et a, b, c, &c. quot ipsa est dimensionum. Abscindant tangentes AK, BL, CM, &c. et ak, bl, cm, &c. a recta EP per punctum datum P ducta segmenta PK, PL, PM, &c. et Pk, Pl, Pm, &c. dico fore $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \&c. = \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} + \&c.$ atque hanc summam manere semper eandem manente puncto P & recta PE positione data.

Supponamus enim rectas ABC, abc motibus sibi parallelis deferri, ita ut earum occurfus P progrediatur in recta PE positione data; cumque sit semper AP x BP x CP x &c. ad aP x bP x cP in ratione constanti per Art. 5. repræsentet AḐ fluxionem ipsius AP, BḐ fluxionem rectæ BP, & CḐ, EḐ, &c. fluxiones rectarum CP, EP, &c. respectivas, ut vitetur inutilis symbolorum multiplicatio, eritque (per Art. 8.) $\frac{AḐ}{AP} + \frac{BḐ}{BP} + \frac{CḐ}{CP} + \&c. = \frac{aḐ}{aP} + \frac{bḐ}{bP} + \frac{cḐ}{cP} + \&c.$ Verum cum recta AP motu sibi semper parallelo deferatur, notissimum est AḐ fluxionem rectæ AP esse ad EḐ fluxionem rectæ EP ut AP ad subtangentem PK, adeoque $\frac{AḐ}{AP} = \frac{EḐ}{PK}$. Similiter $\frac{BḐ}{BP} = \frac{EḐ}{PL}$, $\frac{CḐ}{CP} = \frac{EḐ}{PM}$, $\frac{aḐ}{aP} = \frac{EḐ}{Pk}$, $\frac{bḐ}{bP} = \frac{EḐ}{Pl}$ & $\frac{cḐ}{cP} = \frac{EḐ}{Pm}$, unde $\frac{EḐ}{PK} + \frac{EḐ}{PL} + \dots$

$$\begin{aligned}
 + \frac{EP}{PM} + \&c. &= \frac{EP}{Pk} + \frac{EP}{Pl} + \frac{EP}{Pm} + \&c. \text{ et } \frac{1}{PK} \\
 + \frac{1}{PL} + \frac{1}{PM} + \&c. &= \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} + \&c.
 \end{aligned}$$

Fig. 7. Hæc ita se habent quoties puncta K, L, M, &c. et k, l, m, &c. sunt omnia ad easdem partes puncti P, adeoque fluxiones rectarum AP, BP, CP, &c. aP, bP, cP, &c. omnes ejusdem signi. Si vero, cæteris mantentibus, puncta quævis M et m cadant ad contrarias partes puncti P, tum crescentibus reliquis ordinatis AP, BP, &c. necessario minuuntur ordinatæ CP & cP, earumque fluxiones pro subdititiis seu negativis habendæ sunt; adeoque in hoc casu $\frac{1}{PK} + \frac{1}{PL} - \frac{1}{PM}$, &c. = $\frac{1}{Pk} + \frac{1}{Pl} - \frac{1}{Pm}$, &c. & generaliter in summis hisce colligendis, termini iisdem vel contrariis signis afficiendi sunt, prout segmenta cadunt ad easdem vel ad contrarias partes puncti dati P.

§ 10. Si recta PE occurrat curvæ in tot punctis D, E, I, &c. quot ipsa est dimensionum, summa $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \&c.$ quam constantem seu invariata manere ostendimus, æqualis erit summæ seu aggregato $\frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} + \&c.$ i. e. summæ reciprocarum segmentis rectæ PE positione datæ puncto dato P & curva terminatis; in qua, si segmentum quodvis sit ad alteras partes puncti P, hujus reciproca subducenda est.

§ 11. Si figura fit sectio conica, cui recta PE nulli occurrat, inveniatur punctum b ut in Art. 7. jungatur Pb , huic ducatur ad rectos angulos bd rectam PE secans in d , eritque $\frac{1}{PK} + \frac{1}{PL} = \frac{2}{Pd}$. Est enim $PA \times PB$ ad bP^2 in ratione constanti, adeoque (per Art. 8.) $\frac{AP}{AP} + \frac{BP}{BP} = \frac{2bP}{bP}$, unde (quoniam AP est ad EP ut AP ad PK , BP ad EP ut BP ad PL , & bP ad EP ut bP ad dP) $\frac{1}{PK} + \frac{1}{PL} = \frac{2}{Pd}$.

Fig. 8.

§ 12. Similiter si recta EP occurrat lineæ tertii ordinis in unico puncto D , inveniatur punctum b ut in Art. 6. recta bd perpendicularis in junctam bP occurrat rectæ EP in d , & quoniam $AP \times BP \times CP$ est ad $DP \times bP^2$ in ratione constanti (*ibid.*) erit $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} = \frac{1}{PD} + \frac{2}{Pd}$. Si autem Pb perpendicularis sit in rectam EP, evanescet $\frac{2}{Pd}$.

Fig. 9.

§ 13. Asymptoti linearum geometricarum ex data plaga crurum infinitorum per hanc propositionem determinantur; eæ enim considerari possunt tanquam tangentes cruris in infinitum producti. Recta PA asymptoto parallela curvæ occurrat in punctis A, B, &c. recta autem PE curvam secet in D, E, I, &c. sumatur in hac recta PM ita ut $\frac{1}{PM}$ sit æqualis excessui quo summa $\frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} + \&c.$ superat summam $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \&c.$ & asymptotos transibit per M, & vero æquales

Fig. 10.

æquales sint hæ summæ, crus curvæ parabolicum erit, asymptoto abeunte in infinitum.

§ 14. Ad *curvaturam* linearum geometricarum unico
 Fig. 11. theoremate generali definiendam, fit CDR circulus cui
 occurrant recta PR in D & R, & recta PC in C & N;
 fecet tangens CM rectam PD in M, atque manente
 recta DR, supponamus rectam PCN deferri motu sibi
 semper parallelo donec coincident puncta P, D, C, &
 quæratulr ultimus valor differentiæ $\frac{1}{PM} - \frac{1}{PD}$. In recta
 PN fumatur punctum quodvis q , occurrat qv parallela
 tangenti CM rectæ DR in v ; ducatur DQ parallela
 ipsi PN, & QV (parallela rectæ circulum contingenti
 in D) secet DR in V. Erit itaque $\frac{1}{PM} - \frac{1}{PD} =$
 $\frac{DM}{PM \times PD}$ (quoniam $DM \times MR = CM^2$) $= \frac{CM^2 \times PM}{PM^2 \times PD \times MR}$
 $= \frac{qv^2 \times PM}{Pv^2 \times MR \times PM + Pv^2 \times MR \times MD}$ (cum MR
 $\times MD$, seu CM^2 , fit ad PM^2 ut qv^2 ad Pv^2) $=$
 $\frac{qv^2 \times PM}{Pv^2 \times MR \times PM + qv^2 \times PM^2} = \frac{qv^2}{Pv^2 \times MR + qv^2 \times PM}$
 cujus ultimus valor, evanescente PM & coincidentibus
 qu & Pv cum QV & DV, est $\frac{QV^2}{DV \times DR}$. Atque idem
 est valor ultimus differentiæ $\frac{1}{PM} - \frac{1}{PD}$ si D & C sint
 in arcu lineæ cujusvis ejusdem curvaturæ cum circulo
 CDR.

Fig. 12. § 15. Theor. II. Ex puncto quovis D lineæ geome-
 tricæ ducantur duæ quævis rectæ DE, DA, quarum
 utraque eam secet in tot punctis D, I, E, &c. & D, A,
 B, &c. quot ipsa est dimensionum; abscindant tangentes
 AK,

AK, BL, &c. a recta DE segmenta DK, DL, &c. Occurrat recta quævis, QV tangenti DT parallela ipsi DA & DE in Q & V, sitque QV^2 ad DV^2 ut m ad 1 ; sumatur in DE recta DR ita ut $\frac{m}{DR}$ æqualis sit excessui summæ $\frac{1}{DE} + \frac{1}{DI} + \text{\&c.}$ supra summam $\frac{1}{DK} + \frac{1}{DL} + \text{\&c.}$ & circulus supra chordam DR descriptus rectam DT contingens erit circulus osculatorius, sive ejusdem curvaturæ cum linea geometrica proposita, ad punctum D.

Ostendimus enim in Art. 10. (Fig. 6.) generaliter summam $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \text{\&c.} = \frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} + \text{\&c.}$ & in Art. præcedente invenimus valorem ultimum differentiæ $\frac{1}{PM} - \frac{1}{PD}$, coincidentibus punctis P, D & C, esse $\frac{QV^2}{DV^2 \times DR} = \frac{m}{DR}$ si circulus ejusdem curvaturæ cum linea geometrica ad punctum D rectæ DE occurrat in R. Unde sequitur fore $\frac{m}{DR} = \frac{1}{DE} + \frac{1}{DI} + \text{\&c.} - \frac{1}{DK} - \frac{1}{DL} - \text{\&c.}$ sive reciprocam ipsi $\frac{1}{m} \times DR$ esse æqualem excessui quo summa reciprocarum segmentis puncto D & curva terminatis superat summam reciprocarum segmentis eodem puncto & tangentibus AK, BL, &c. terminatis. Quoties autem excessus hic evadit negativus, chorda DR sumenda est ad alteras partes puncti D, semperque adhibenda est regula superius descripta pro signis terminorum dignoscendis. Si recta DA bifecet angulum EDT
recta

recta DE & tangente DT^r contentum, theorema fit paulo simplicius. Hoc enim in casu QV = DV, $m=1$, & $\frac{1}{DR}$ æqualis excessui quo $\frac{1}{DE} + \frac{1}{DI} + \&c.$ superat $\frac{1}{DK} + \frac{1}{DL} + \&c.$

§ 16. Ex eodem principio consequitur theorema generale quo determinatur variatio curvaturæ vel mensura anguli contactus curva & circulo osculatorio contenti, in linea quavis geometrica; præmittenda tamen est explicatio brevis variationis curvaturæ, cum hæc non satis dilucide apud auctores descripta sit. Linea quævis curva a tangente flectitur per curvaturam suam, cujus eadem est mensura ac anguli contactus curva & tangente contenti; & similiter curva a circulo osculatorio inflectitur per variationem curvaturæ suæ, cujus variationis eadem est mensura ac anguli contactus curva & circulo osculatorio comprehensi. Occurrat recta TE tangenti DT perpendicularis curvæ in E & circulo osculatorio in r , & variatio curvaturæ erit ultimo ut E_r subtensa anguli contactus ED r si detur DT; cumque dato angulo contactus ED r fit E_r ultimo ut DT^3 , ut ex Art. 369. tractatus de fluxionibus colligitur, generaliter curvaturæ variatio erit ultimo ut $\frac{E_r}{DT^3}$. Utimur circulo ad curvaturam aliarum figurarum definiendam; verum ad variationem curvaturæ mensurandam, quæ in circulo nulla est, adhibenda est parabola vel sectio aliqua conica. Quemadmodum autem ex circulis numero indefinitis qui curvam datam in puncto dato contingere possunt, unicus dicitur osculatorius qui curvam adeo intime tangit ut nullus alius circulus inter hunc & curvam duci possit; similiter omnium parabolæ quæ eandem habent curvaturam cum linea proposita ad punctum datum (sunt

Fig. 13.

(sunt autem hæ quoque numero infinitæ) ea eandem simul habet curvaturæ variationem, quæ, non solum arcum curvæ tangit & osculat, sed adeo premit ut nullus alius arcus parabolicus duci possit inter eas, reliquis omnibus arcibus parabolicis transeuntibus vel extra vel intra utrasque. Qua vero ratione hæc parabola determinari possit, ex iis quæ alibi fusius explicavimus facile intelligitur.

Sit DE arcus curvæ, DT tangens, TEK recta tangenti perpendicularis, sitque rectangulum ET x TK semper æquale quadrato tangenti DT, & curva SKF locus puncti K, qui rectæ DS curvæ normali occurrat in S, quemque tangat in S recta SV tangentem TD secans in V. Recta DS erit diameter circuli osculatorii, & bisecta DS in f , erit f centrum curvaturæ; junctâ autem Vf, si angulus SDN constituatur æqualis angulo fVD ex altera parte rectæ DS, & recta DN circulo osculatorio occurrat in N; tum parabola diametro & parametro DN descripta, quæque rectam DT contingit in D, ipsa erit cujus contactus cum linea proposita in D intimus erit atque maxime perfectus seu proximus. Omnes autem parabolæ aliâ quavis chorda circuli osculatorii tanquam diametro & parametro descriptæ, & rectam DT contingentes in D, eandem habent curvaturam cum linea proposita in puncto D. Qualitas curvaturæ a *Newtono* in opere posthumo nuper edito explicata est potius variatio radii curvaturæ; est enim ut fluxio radii curvaturæ applicata ad fluxionem curvæ, vel (si R denotet radium circuli osculatorii & S arcum curvæ

ut $\frac{\dot{R}}{\dot{S}}$. Ipsa autem curvatura est inverse ut radius R,

& variatio curvaturæ ut $\frac{-\dot{R}}{RR\dot{S}}$, quæ est mensura an-

guli

guli contactus curvâ & circulo osculatorio contenti. Harum autem una ex alterâ datâ facile derivatur. Variatio radii curvaturæ in curvâ quavis DE est ut tangens anguli DVS vel DVf, & in parabolâ quavis est semper ut tangens anguli contenti diametro per punctum contactus transeunte & rectâ ad curvam perpendiculari. Hæc ex theoremate sequenti generali deduci possunt.

Fig. 14. § 17. Theor. III. Sit D punctum in linea quavis geometrica datum, occurrat DS diameter circuli osculatorii per D ducta curvæ in tot punctis D, A, B, &c. quot ipsa est dimensionum; ducatur DT curvam contingens in D, quæ curvam secet in punctis I, &c. binario paucioribus, & occurrat tangentibus AK, BL, &c. in K, L, &c. eritque variatio curvaturæ, sive mensura anguli contactus curva & circulo osculatorio comprehensi, directe ut excessus quo summa reciprocarum segmentis tangentis DT puncto contactus D & tangentibus AK, BL, &c. terminatis superat summam reciprocarum segmentis eodem puncto & curva terminatis, & inverse ut radius curvaturæ, i. e. ut $\frac{1}{DS} \times \frac{1}{DK} + \frac{1}{DL} + \text{&c.} - \frac{1}{DI} - \text{&c.}$

Ducatur enim recta Dk curvam secans in e, i, &c. circulum osculatorium in R; fitque angulus kDT quam minimus; hujus supplementum ad duos rectos bifecetur recta Dab, quæ lineæ geometricæ propositæ occurrat in punctis D, a, b, &c. & ductæ tangentibus ak, bl, &c. fecent rectam Dk in punctis k, l, &c. eritque per propositionem præcedentem $\frac{1}{DK} = \frac{1}{De} + \frac{1}{Di} - \frac{1}{Dk} - \frac{1}{Dl}$, &c. Unde $\frac{1}{DR} - \frac{1}{De}$ (sive $\frac{Re}{DR \times De}$) = $\frac{1}{Di}$

$-\frac{1}{Dk} - \frac{1}{Dl} - \&c.$ Proinde coincidentibus rectis Dk & DK , seu evanescente angulo kDK , erit ultimo $\frac{Re}{DR \times De}$ æqualis $\frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL} - \&c.$ Sit eT perpendicularis tangenti in T , atque occurrat circulo osculatorio in r ; cumque fit re ultimo ad Re ut eT ad De , erit ultimo $\frac{Re}{DR \times De} = \frac{re}{DR \times eT} = \frac{-re \times DS}{DR \times DT^2}$ five $\frac{re \times DS}{DT^2}$. Mensura autem anguli contactus rDe curva & circulo osculatorio contenti, five variatio curvaturæ, est ut $\frac{re}{DT^2}$ adeoque ut $\frac{1}{DS} \times \frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL}$, &c.

§ 18. Variatio autem radii curvaturæ, five hujus qualitas a *Newtono* descripta, ex priori facillime colligitur. Junctis enim SI , SK , SL , &c. erit hæc variatio radii osculatorii ut excessus quo summa tangentium angulorum DKS , DLS , &c. superat summam tangentium angulorum DIS , &c. Crescit autem curvatura a puncto D versus e , & minuitur radius osculatorius, quoties arcus De tangit circulum osculatorium DR interne, vel cum $\frac{1}{DK} + \frac{1}{DL} + \&c.$ superat $\frac{1}{DI} + \&c.$ at contrà minuitur curvatura a D versus e , & augetur radius circuli osculatorii, quoties arcus curvæ De tangit arcum circulare externe vel transit intra circulum & tangentem adeoque cum DR fit ultimo minor quam De vel cum $\frac{1}{DI} + \&c.$ superat $\frac{1}{DK} + \frac{1}{DL} + \&c.$

§ 19. Sumatur igitur in tangente DT recta DV ita ut $\frac{1}{DV} = \frac{1}{DK} + \frac{1}{DL} + \&c. - \frac{1}{DI} - \&c.$ jungatur

$C c$

\sqrt{V} ,

$\int V$, constituatur angulus SDN æqualis DV \int , atque occurrat recta DN circulo osculatorio in N; & parabola diametro DN descripta, cujus parameter est DN, quæque rectam DT contingit in D, eandem habebit variationem curvaturæ cum linea geometrica proposita in puncto D. Ex iisdem principiis alia quoque theoremata deducuntur, quibus variatio curvaturæ in lineis geometricis generaliter definitur.

§ 20. Ut hæc theoremata ad formam magis geometricam reducantur, lemmata quædam sunt præmittenda, quibus doctrina de divisione rectarum harmonicâ amplior & generalior reddatur. In recta quavis DI, sumptis æqualibus segmentis DF & FG, ducantur a puncto quovis V quod non est in rectâ DI tres rectæ VD, VF, VG, & quarta VL ipsi DI parallela, atque hæc quatuor rectæ, a Cl. D. *De la Hire*, Harmonicales dicuntur. Recta vero quævis, quæ quatuor harmonicibus occurrit ab iisdem harmonice secatur. Occurrat recta DC harmonicibus VD, VF, VG & VL in punctis D, A, B, C; eritque DA ad DC ut AB ad BC. Ducatur enim per punctum A recta MAN ipsi DI parallela, quæ occurrat rectis VD & VG in M & N; & ob æquales DF & FG, æquales erunt MA & AN. Est autem DA ad DC ut AM (five AN) ad VC, adeoque ut AB ad BC. Manifestum est rectam, quæ uni harmonicorum parallela est, dividi in æqualia segmenta a tribus reliquis. Occurrat recta BH parallela ipsi VF reliquis VG, VC, VD in B, K & H; eritque VK ad KB ut FG (vel DF) ad VF adeoque ut VK ad KH, & proinde $BK = KH$.

§ 21. Hinc sequitur, si recta quævis a quatuor rectis ab eodem puncto ductis secetur harmonice, aliam quamvis

vis rectam quæ his quatuor rectis occurrit harmonice secari ab iisdem ; eam vero quæ parallela est uni quatuor rectarum in segmenta æqualia dividi a tribus reliquis. Sit DA ad DC ut AB ad BC, jungantur VA, VB, VC & VD ; occurrant rectæ MAN, DFG ipsi VC parallelæ rectis VD, VA & VB in M, A, N & D, F, G ; eritque MA ad VC ut DA ad DC, vel AB ad BC, adeoque ut AN ad VC ; MA = AN, & DF = FG ; &, per præcedentem, recta quævis quæ ipsis VD, VA, VB, VC occurrit harmonice secabitur ab iisdem.

§ 22. Ex puncto D ducantur duæ rectæ DAC, *Dac* Fig. 16.
 rectas VA & VC secantes in punctis A, C atque *a, c* ; n. 1.
 junctæ Ac & *aC* sibi mutuo occurrant in Q, & ducta VQ harmonice secabit rectam DAC vel aliam quamvis rectam ex puncto D ad easdem rectas ductam. Secet enim VQ rectam AC in B, & per punctum Q ducatur recta MQN parallela ipsi DC, quæ occurrat rectis *Da*, VA & VC in punctis M, R & N ; cumque fit MR ad MQ ut DA ad DC, & MQ ad MN in eadem ratione, erit quoque RQ ad QN ut DA ad DC. Sed RQ est ad QN ut AB ad BC. Quare DA est ad DC ut AB ad BC. Hæc est Prop. 20ma, Lib. I. sectionum conicarum Cl. *De la Hire*.

§ 23. Sit DA ad DC ut AB ad BC, eritque $\frac{2}{DB}$
 æqualis summæ vel differentiæ ipsarum $\frac{1}{DA}$ & $\frac{1}{DC}$
 prout puncta A & C sunt ad easdem vel contrarias partes puncti D. Sint imprimis puncta A & C ad easdem partes puncti D, cumque fit $DA \times BC = DC \times AB$, i. e. $DA \times DC - DB = DC \times DB - DA$, vel $DA \times DB - DC = DC \times DA - DB$ erit $2DA \times DC = DA \times DB + DC$

C c 2

$DC \times DB$, adeoque $\frac{2}{DB} = \frac{1}{DA} + \frac{1}{DC}$. Sint nunc
 n. 2 & 3. puncta A & C ad contrarias partes puncti D, eritque
 vel $DA \times \overline{DB - DC} = DC \times \overline{DB + DA}$, vel $DA \times$
 $\overline{DB + DC} = DC \times \overline{DB - DA}$, adeoque $\frac{2}{DB} = \frac{1}{DC}$
 $-\frac{1}{DA}$ cum puncta B & C sunt ad easdem partes puncti
 D, vel $\frac{2}{DB} = \frac{1}{DA} - \frac{1}{DC}$ quoties puncta A & B sunt
 ad easdem partes puncti D. Si igitur, datis puncto D
 & rectis VF & VC positione, ducatur ex puncto D
 recta quævis illis occurrens in punctis A & C, & in
 eadem recta fumatur semper DB ita ut $\frac{2}{DB} = \mp \frac{1}{DA}$
 $\mp \frac{1}{DC}$, ubi supponitur terminos $\frac{1}{DA}$ & $\frac{1}{DC}$ iisdem
 vel contrariis signis afficiendos esse prout puncta A & C
 sunt ad easdem vel contrarias partes puncti D, erit lo-
 cus puncti B ipsa harmonicalis VG quæ rectam DFG
 rectæ VC parallelam secat in G ita $FG = DF$; quæ-
 que transit per punctum Q ubi (ductâ Dac quæ iif-
 dem rectis VF & VC occurrat in a et c) junctæ Ac
 et cC se mutuo decussant.

Fig. 17. § 24. Si in recta DA fumatur semper Db ita ut
 $\frac{1}{Db} = \frac{1}{DA} \mp \frac{1}{DC}$; ducatur DF parallela rectæ VC
 quæ rectæ VF occurrat in F, & DH parallela rectæ VF
 quæ rectæ VC occurrat in H, & ducta diagonalis HF
 erit locus puncti b; nam ex hypothefi $\frac{1}{Db} = \frac{2}{DB}$, &
 $DB = 2Db$; adeoque cum VG fit locus puncti B erit
 punctum b ad rectam HF, si puncta A & C sint ad
 easdem

eadem partes puncti D. Si autem supponatur $\frac{1}{Db} = \frac{1}{DA} - \frac{1}{DC}$, eadem constructio inserviet pro determinando puncto *b*, si substituatnr loco rectæ VC alia *vc* rectæ VC parallela ad æqualem distantiam a puncto D sed ad contrarias partes.

§ 25. Ex puncto dato D ducatur recta quævis DM quæ tribus rectis positione datis occurrat in punctis A, C, E; & sumatur semper DM ita ut $\frac{1}{DM} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$ (ubi termini sunt contrariis signis afficiendi quoties rectæ DA, DC vel DE sunt ad contrarias partes puncti D); supponatur $\frac{1}{DA} + \frac{1}{DC} = \frac{1}{DL}$, eritque L ad rectam positione datam per præcedentem; adeoque, cum sit $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE}$, erit punctum M ad positione datam, per eandem. Compositio autem problematis facile ex dictis perficitur. Sint VA, VC & vE tres rectæ positione datæ, & compleatur parallelogrammum DfVH, ducendo DF & DH rectis VC & VF respectivè parallelas, & occurrat recta vE diagonali in *v*; deinde compleatur parallelogrammum Df**v**h ducendo rectas D*f* & D*b* rectis vE & HF parallelas quæ rectis HF & vE occurrant in punctis *f* & *b*; & diagonalis *bf* erit locus puncti M. Occurrat enim recta DA rectis HF & *bf* in L & M; eritque, ex præcedentibus, $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$. Alia constructio ex Art. 22. deducitur.

§ 26. Recta quævis ex puncto dato D ducta occurrat rectis positione datis in punctis A, B, C, E, &c. et in hac recta sumatur semper $\frac{1}{DM} = \frac{1}{DA} \mp \frac{1}{DB} \mp \frac{1}{DC} \mp \frac{1}{DE}$, &c. eritque locus puncti M semper ad rectam positione datam. Demonstratur ad modum præcedentis.

Fig 18. § 27. Theor. IV. Circa datum punctum P revolvatur recta PD quæ occurrat lineæ geometricæ cujuscunque ordinis in tot punctis D, E, I, &c. quot ipsa est dimensionum, & si in eadem recta sumatur semper PM ita ut $\frac{1}{PM} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp \dots$ (ubi signa terminorum regulam sæpius descriptam observare supponimus) erit locus puncti M linea recta.

Ducatur enim ex polo P recta quævis positione data PA, quæ curvæ occurrat in tot punctis A, B, C, &c. quot ipsa est dimensionum. Ducantur rectæ AK, BL, CN curvam in his punctis contingentes, quæ occurrant rectæ PD in totidem punctis K, L, N, &c. et per Art 10. $\frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp \dots = \frac{1}{PK} \mp \frac{1}{PL} \mp \frac{1}{PN} \mp \dots$ Unde $\frac{1}{PM}$ æqualis est huic summæ, cumque positione detur recta PA, & maneat rectæ AK, BL, CN, &c. dum recta PD circa polum P revolvitur, erit punctum M ad lineam rectam, per articulum præcedentem; quæ per superius ostensa ex datis tangentibus AK, BL, &c. determinari potest.

§ 28. Sicut recta Pm medium est harmonicum inter duas rectas PD & PE, cum $\frac{2}{Pm} = \frac{1}{PD} + \frac{1}{PE}$; si-

militer

militer P_m dicatur *medium* harmonicum inter rectas quaslibet PD, PE, PI, &c. quarum numerus est n , cum $\frac{n}{P_m} = \frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} + \&c.$ Et si ex puncto dato P recta quævis ducta lineam geometricam fecerit in tot punctis quot ipsa est dimensionum, in qua sumatur semper P_m medium harmonicum inter segmenta omnia ductæ ad punctum datum P & curvam terminata, erit punctum m ad rectam lineam. Erit enim $\frac{1}{PM} = \frac{n}{P_m}$ adeoque P_m ad PM ut n ad unitatem; cumque punctum M sit ad rectam lineam, per præcedentem, erit m quoque ad rectam lineam. Atque hoc est theorema *Cotesii*, vel eidem affine.

§ 29. Sint $a, b, c, d, \&c.$ radices æquationis ordinis n , V ultimus ejus terminus quem ordinata seu radix y non ingreditur, P coefficientis termini penultimi, M medium harmonicum inter omnes radices, seu $\frac{n}{M} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \&c.$ Cum igitur sit V factum ex radicibus omnibus $a, b, c, \&c.$ in se mutuo ductis, fitque P summa factorum cum radices omnes unâ dempta in se mutuo ducuntur, erit $P = \frac{V}{a} + \frac{V}{b} + \frac{V}{c} + \frac{V}{d} + \&c. = \frac{nV}{M}$, adeoque $M = \frac{nV}{P}$. Sic, si æquatio sit quadratica, cujus radices duæ sint a et b , erit $M = \frac{2ab}{a+b}$ (assumptâ æquatione generali sectionum conicarum Art. I. propositâ) $= \frac{2cxx - 2dx + 2e}{ax - b}$. In æquatione cubica cujus tres radices sunt a, b, c , erit $M = \frac{3abc}{ab + ac + bc}$ (si assumatur æquatio generalis

$$\text{neralis linearum tertii ordinis ibidem propositi)} = \frac{3fx^3 - 3gx^2 + 3bx - 3k}{cxx - dx + e}.$$

Fig. 19. § 30. Occurrant rectæ quævis duæ Pm & $P\mu$, ex puncto P ductæ, lineæ geometricæ in punctis $D, E, I,$ &c. et $d, e, i,$ &c. fitque Pm medium harmonicum inter segmenta prioris ad punctum P & curvam terminata, & $P\mu$ medium harmonicum inter segmenta similia posterioris rectæ; juncta μm occurrat abscissæ AP in H , eritque $PH = \frac{nV\dot{x}}{V}$ vel PH ad Pm ut P ad $\frac{V}{x}$. Secet enim abscissa curvam in tot punctis $B, C, F,$ &c. quot ipsa est dimensionum; cumque ultimus terminus æquationis (i. e. V) fit ad $BP \times CP \times FP \times \&c.$ in ratione constanti, ut supra (Art. 5.) ostendimus, erit (per Art. 8. $\frac{V}{V} = \frac{\dot{x}}{BP} + \frac{\dot{x}}{CP} + \frac{\dot{x}}{FP} + \&c.$ adeoque $\frac{n}{PH} = \frac{1}{BP} + \frac{1}{CP} + \frac{1}{FP} + \&c. = \frac{V}{Vx}$, & $PH = \frac{nV\dot{x}}{V}$ (quoniam recta $PM = \frac{nV}{P}) = Pm \times \frac{P\dot{x}}{V}$. In sectionibus conicis est PH ad Pm ut $ax - b$ ad $2cx - d$; & in lineis tertii ordinis ut $cxx - dx + e$ ad $3fxx - 2gx + b$.

§ 31. Si desideretur propositionis præcedentis demonstratio ex principiis pure algebraicis petita, ea ope sequentis *Lemmat*is perfici poterit. Sit abscissa $AP = x$, ordinata $PD = y$, ultimus terminus æquationis lineam geometricam definientis $V = Ax^n + Bx^{n-1} + Cx^{n-2} + \&c.$ penultimi coefficientis $P = ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c.$ et fit Q quantitas quæ formatur ducendo terminum quemque quantitatis V in indicem ipsius

fius x in hoc termino & dividendo per x , i. e. fit
 $Q = nAx^{n-1} + \overline{n-1} \times Bx^{n-2} + \overline{n-2} \times Cx^{n-3}$
 $+ \&c.$ (quæ ipsa est quantitas quam $\frac{V}{x}$ dicimus). Du-
 catur ordinata Dp quæ angulum quemvis datum ApD
 cum abscissa constituat, sintque rectæ PD, pD et Pp ut
 datæ l, r et k ; dicatur $pD = u, Ap = z$, & transmu-
 tetur æquatio proposita ad ordinatam u & abscissam z ; &
 æquationis novæ, cum fit $z = AP$, terminus ultimus
 v erit æqualis ipsi V , penultimi autem coefficientis p erit
 æqualis $\pm \frac{Qk + Pl}{r}$.

Cum enim fit $PD (= y)$ ad $pD (= u)$ ut l ad r , erit
 $y = \frac{lu}{r}$; fit autem Pp ad $pD (= u)$ ut k ad r , erit
 $Pp = \frac{ku}{r}$, & $AP = x = Ap \pm Pp = z \pm \frac{ku}{r}$. His
 autem valoribus pro y et x substitutis in æquatione pro-
 posita lineæ geometricæ, prodibit æquatio relationem
 co-ordinatarum z et u definiens. Ad hujus ultimum
 terminum v & penultimum pu determinandum, sufficit
 hos valores substituere in ultimo V , & penultimo Py ,
 æquationis propositæ, atque terminos resultantes colli-
 gere in quibus ordinata u vel non reperitur, vel unius
 tantum dimensionis; horum enim summa dat pu , il-
 lorum v . Substituatur pro x ipsius valor $z \pm \frac{ku}{r}$ in
 quantitate V vel $Ax^n + Bx^{n-1} + Cx^{n-2} + \&c.$ et
 termini resultantes $Az^n \pm Bz \frac{nAz^{n-1}ku}{r} + Bz^{n-1} \pm$
 $\overline{n-1} \times \frac{Bz^{n-2}ku}{r} + Cz^{n-2} \pm \overline{n-2} \times \frac{Cz^{n-3}ku}{r} +$
 $\&c.$ foli ad rem faciunt de qua nunc agitur. Sub-
 stituatur deinde pro x idem valor, & pro y ipsius
 valor

valor $\frac{lu}{r}$, in quantitate $Py = \frac{ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c. \times y}{az^{n-1} + bz^{n-2} + cz^{n-3} + \&c. \times \frac{lu}{r}}$ sunt nobis retinendi. Supponatur nunc $z = x$, & summa priorum fit æqualis $V \pm \frac{Qku}{r}$, & posteriorum summa = $\frac{Plu}{r}$. Unde manifestum est ultimum terminum æquationis novæ $v = V$, & penultimum $pu = \frac{Pl \pm Qk}{r} \times u$.

§ 32. Sit nunc Pm medium harmonicum inter segmenta $PD, PE, PI, \&c.$ et $P\mu$ medium harmonicum inter segmenta $Pd, Pe, Pi, \&c.$ ut in Art. 30. junctâ μm fecet abscissam in H ; & supponamus $P\mu$ ordinatæ pD parallelam esse. Ducatur μs abscissæ parallela, quæ rectæ Pm occurrat in s ; eritque Ps ad $P\mu$ ut PD ad pD vel ut l ad r , et μs ad $P\mu$ ut k ad r . Cumque sit $P\mu = \frac{nV}{p}$ (per Articulum præcedentem) $\frac{nVr}{Pl \pm Qk}$, erit $ms = Pm \pm Ps = \frac{nV}{p} \pm \frac{nvl}{pr} = \frac{nV}{p} \pm \frac{nVl}{Pl \pm Qk} = \frac{nVQk}{P \times Pl \pm Qk}$. Est autem ms ad $s\mu$ ut Pm ad PH , i. e. $\frac{nVQk}{P \times Pl \pm Qk}$ ad $\frac{nVk}{Pl \pm Qk}$ ut Pm ad PH ; adeoque Q ad P ut Pm ad PH , vel $PH = Pm \times \frac{P}{Q}$ vel $\frac{nV}{Q}$. Cum igitur valor rectæ PH non pendeat a quantitatibus l, k et r ; sed, his mutatis, sit semper idem, erit punctum μ ad rectam positione datam, ut in Theor. 4. aliter ostendimus. Quin & valor rectæ PH is est quem in Art. 29. alia methodo definivimus; & recta Hm omnes

nes rectas ex P ductas fecat harmonice, secundum definitionem sectionis harmonicæ in Art. 28. generaliter propositam.



S E C T I O II.

De Lineis secundi ordinis, sive sectionibus conicis.

§ 33. **E**X iis quæ generaliter de lineis geometricis in sectione primâ demonstrata sunt, sponte fluunt proprietates linearum secundi, tertii, & superiorum ordinum. Quæ ad sectiones conicas spectant optime derivantur ex proprietatibus circuli, quæ figura basis est conii. Verum ut usus theorematum præcedentium clarius pateat, & figurarum analogia illustretur, operæ pretium erit harum quoque affectiones ex præmissis deducere. Doctrina autem conica de diametris, earumque ordinatis (quibus parallelæ sunt rectæ sectionem contingentes ad vertices diametri) & de parallelarum segmentis quæ rectis quibuscumque occurrunt, & asymptotis, tota facillime fluit ex iis quæ Art. 4. & 5. ostensa sunt.

§ 34. Rectæ AB & FG sectioni conicæ inscriptæ Fig. 20. occurrant sibi mutuo in puncto P; ductæ AK, BL, FM, GN sectionem contingentes occurrant rectæ PE per P ductæ in punctis K, L, M, N; eritque semper

$$\frac{1}{PK} \pm \frac{1}{PL} = \frac{1}{PM} \mp \frac{1}{PN} \text{ (si recta PE curvæ occurrat in punctis D \& E) } = \frac{1}{PD} \mp \frac{1}{PE}.$$

tem

- tem quæ sunt ad easdem partes puncti P eadem præponuntur signa; iisque quæ sunt ad oppositas partes puncti P signa præponuntur contraria. Hinc si bifecetur DE in P, & ex puncto P ducatur recta quævis sectionem secans in punctis A et B, unde ducantur rectæ AK et BL curvam contingentes quæ rectam DE secant in K et L; erit semper $PK = PL$. Quod si DE sectioni non occurrat, sitque P punctum ubi diameter quæ bifecat rectas ipsi DE parallelas eidem occurrit; erit in hoc quoque casu $PK = PL$.

- Fig. 23. § 35. Concurrant rectæ AB et FG sectioni conicæ inscriptæ in puncto P; ducantur rectæ sectionem contingentes in punctis A et F quæ sibi mutuo occurrant in K, & iuncta PK transibit per occursum rectarum quæ sectionem contingunt in punctis B et G. Si enim recta PK non transeat per occursum rectarum sectionem tangentium in B et G, huic occurrat in N illi in L; cumque $\frac{1}{PK} + \frac{1}{PL} = \frac{1}{PK} + \frac{1}{PN}$ per præcedentem, erit $PL = PN$: & coincidunt puncta L et N contra hypothefin.

§ 36. Eadem ratione patet rectas AG et BF sibi mutuo occurrere in π puncto rectæ LK; adeoque puncta P, K, π , L esse in eadem recta linea. Hinc datis tribus punctis contactus A, B et F, cum duabus tangentibus AK et FK, sectio conica facile describitur. Revolvatur enim recta $K\pi P$ circa tangentium occursum K ut polum, quæ occurrat rectis AB et FB in punctis P et π ; & iunctæ $A\pi$, FP occurso suo G describent sectionem conicam quæ transibit per tria puncta data A, B, F & continget rectas AK et FK in A et F.

§ 37. Cæteris manentibus, occurrant rectæ AF et BG Fig. 24. sibi mutuo in puncto p , tangentes AK et BL in R, atque tangentes FK et GL in Q; & puncta R, π , Q et p erunt in eadem recta linea; similiter occurrant tangentes AK et GQ in m ; tangentes BR et FK in n ; & puncta P; m , n , p erunt in eadem recta linea. Demonstratur ad modum Art. 35.

§ 38. Hinc datis quatuor punctis contactus A, B, F, G cum unica tangente AK, occurfus rectarum AB et FG, AF et BG, atque AG et BF, dabunt puncta P, p , et π ; junctæ autem P p , P π , et $p\pi$ secabunt tangentem datam AK in tribus punctis m , K et R unde ductæ mG, KF, RB sectionem conicam contingent in punctis datis G, F et B.

§ 39. Datis quatuor tangentibus RK, KQ, QL, LR et unico puncto contactus A, occurfus tangentium RK et LQ, LR et QK dabunt puncta m et n . Jungantur LK et nm ; & occurfus rectarum LK et RQ, LK et nm , RQ et nm , dabunt puncta π , P et p ; junctæ vero PA, π A et p A secabunt tangentes RL, QK et QL in punctis contactus B, G et F.

§ 40. Datis quinque punctis contactus A, B, F, G et f , junctæ GF et G f rectæ AB occurrant in punctis P et X; junctæ AF et A f occurrant rectæ BG in p et x ; & junctæ P p , X x occurfu suo dabunt punctum m ; unde ductæ mA et mG sectionem conicam tangent in A et G; & similiter determinantur rectæ quæ curvam contingent in punctis reliquis B, F et f .

§ 41. Dentur quinque rectæ sectionem conicam contingentes, VK, KQ, QL, Lu et uV; occurfus tangentium VK et LQ dabit punctum m ; occurfus tangentium
KQ

KQ et Lu dabit punctum n ; jungantur mn , LK, VL et mu ; recta LK secabit rectam mn in P; & recta LV secabit ipsam mu in X; juncta autem PX secabit tangentes VK et uL in punctis contactus A et B. Similiter reliqua puncta contactus determinantur.

Fig. 25. § 42. Datis tribus tangentibus AK, BK, et RL, cum duobus punctis contactus A et B, facillime determinatur tertium; per Art. 35. Occurrat enim tangens RL reliquis tangentibus in R et L, atque junctæ AL et BR, se mutuo decussent in π , juncta $K\pi$ secabit tangentem RL in tertio puncto contactus F; & sectio conica describi potest ut in Art. 36.

Fig. 26. § 43. Dentur quatuor tangentes KQ, QL, LR et RK cum unico puncto D sectionis conicæ quod non fit in aliquâ quatuor tangentium. Inveniantur puncta P, p et π ut in Art. 39. Jungantur PD, pD , et πD ; & ducta PZ rectæ pD parallela occurrat rectæ RQ in Z; & bifariam secetur PZ in S; & ducta pS secabit rectam PD in E puncto curvæ; vel occurrat PD rectæ RQ in z , et (per Art. 23.) secetur PD harmonice in z et E. Ducta autem $D\pi$ secabit junctam pE in e , et $E\pi$ secabit ipsam pD in d , ita ut hæc quoque puncta d , e sint ad curvam.

Fig. 27. § 44. Ex puncto K ducantur duæ tangentes ad sectionem conicam in A et B; ex puncto A ducantur rectæ duæ AF et AG sectioni occurrentes in F et G; juncta BG secet AF in P, et juncta BF secet rectam AG in π ; eruntque puncta P, K, π in eadem rectæ linea, per Art. 36.

n. 2. Verum propositio hæc generalior est. Si enim a puncto quovis K ducantur duæ rectæ KAa, KBb sectionem

tionem secantes in punctis *A, a* et *B, b*; et ex punctis *A* et *a* ducantur rectæ ad sectionem *AF* et *aG*; juncta autem *BF* secet *aG* in *P*, & ducta *bG* secet *AF* in π ; erunt puncta *P, K, \pi* in eadem recta linea; quod variis modis aliàs demonstravimus, unde expeditam methodum olim deduximus sectionem conicam describendi per data quævis quinque puncta. Sint *A, a, B, b* et *F* puncta quinque data; concurrant rectæ *Aa* et *Bb* in *K*; jungantur *AF* et *BF*; revolvatur recta *PK\pi* circa polum *K*, quæ occurrat his rectis in π et *P*; et ductæ *aP, b\pi* concurfu suo *G* sectionem describent.

§ 45. Sit *P* punctum datum extra sectionem conicam, unde ducta quævis sectioni occurrat in *D* et *E*; Fig. 28.

et si $\frac{2}{PM} = \frac{1}{PD} \mp \frac{1}{PE}$ erit *M* ad lineam rectam *AB* quæ sectioni occurrit in punctis *A* et *B*, ita ut ductæ *PA* et *PB*, erunt contingentes sectionis. Si vero punctum *p* fit in medio puncto rectæ *AB* intra sectionem, sitque $\frac{2}{pm} = \frac{1}{pd} \mp \frac{1}{pe}$, locus puncti *m* erit recta *ab* per *P* ducta ipsi *AB* parallela. Tangentes ad puncta *D* et *E* semper concurrunt in recta *AB*, et tangentes ad puncta *d* et *e* in recta *ab*.

§ 46. Contingat recta *DT* sectionem in *D*, unde ducantur duæ quævis rectæ *DE* et *DA*, quæ sectioni occurrant in *E* et *A*. Occurrat *DE* rectæ *AK* sectionem contingenti, in *K*; et ductæ *EN, KM* tangenti *DT* parallelæ secent *DA* in *N* et *M*, sumatur in recta *DE*, *DR* ad *EN* ut *KM* ad *KE*, & circulus ejusdem curvaturæ cum sectione in *D* transibit per *R*. Nam per Fig. 29. n. 1.

Art. 15. est $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} - \frac{1}{DK} = \frac{KE}{DE \times DK}$ et
DR

$$DR = \frac{DE \times DK}{KE} \times \frac{QV^2}{DV^2} \text{ (quoniam } QV : DV ::$$

n. 2. $KM : DK :: EN : DE) = \frac{KM \times EN}{KE}$. Quod si fuerit tangens AK parallela rectæ DE, (i. e. si DE fit ordinata diametri per A transeuntis) erit $DR = \frac{EN^2}{DE}$; vel DR ad DE ut EN^2 ad DE^2 ; ut alibi demonstravimus Art. 373. tractatus de fluxionibus. Si in hoc casu DE fit diameter, erit $\frac{EN^2}{DE}$, adeoque DR, æqualis parametro diametri DE; ut satis notum est.

Fig. 30. § 47. Ducantur rectæ DT, DE, quarum prior sectionem conicam contingat in D, posterior eidem occurrat in E. Ducatur DA quæ bifecet angulum EDT et sectioni occurrat in A; jungatur AE, cui occurrat in V recta DV parallela rectæ quæ curvam contingit in A; et ducta VR parallelâ rectæ DA, hæc fecabit DE in R ubi circulus osculatorius occurrit rectæ DE; eritque DR diameter curvaturæ si angulus EDT fit rectus. Erit enim VR ad AD ut ER ad DE, et ut DR ad DK; unde DR ad DK ut DE ad EK, adeoque $\frac{1}{DR} = \frac{1}{DE}$ —

n. 2. $\frac{1}{DK}$, ut oportebat, per Art. 15. Si autem fit tangens AK parallela rectæ DE (quo in casu tangentes AK et DT æquales constituunt angulos cum recta DA quæ proinde perpendicularis est axi figuræ) coincident puncta R et E, & circulus osculatorius transibit per punctum E. Sequitur quoque ex dictis rectas EK, DE et ER esse in progressionem geometrica.

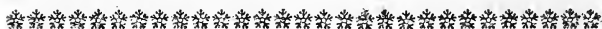
Fig. 31. § 48. Occurrat recta quævis DE sectioni conicæ in D et E, concurrant rectæ curvam contingentes ad D et E
in

in puncto V. Sit DOA diameter per D curvæ, & si constitutur angulus DVr = EDO, erit DR (= 2Dr) chorda circuli osculatorii. Ducatur enim AK sectio- nem contingens quæ rectæ DE occurrat in K, et tan- genti EV in Z; ducatur EN parallela tangenti DT rectam DA secans in N; cumque sit DR ad KA ut EN ad EK; sitque KZ (= $\frac{1}{2}$ AK) ad EK ut VD ad DE, erit VD ad DE ut $\frac{1}{2}$ DR ad EN; adeoque triangu- la DVr et EDN similia et angulus DVr æqualis angulo EDO. Hanc methodum determinandi circulum oscu- latorium demonstravimus in tractatu de fluxionibus, Art. 375. sed non adeo breviter.

§ 49. Variatio curvaturæ, five tangens anguli con- tactus sectione conica & circulo osculatio comp- henfi, est directe ut tangens anguli contenti diametro quæ per contactum ducitur & normali ad curvam, & inverse ut quadratum radii curvaturæ. Sit enim DR Fig. 32. diameter curvaturæ, & hæc variatio ad punctum D erit ut $\frac{1}{DR \times DV}$, per Art. 17. adeoque, cum sit DV ad Dr ut DE ad EN, ut $\frac{EN}{DE \times DR^2}$. Variatio autem ra- dii curvaturæ est ut tangens anguli EDO. Quod si recta DO circulo osculatio occurrat in n, parabola diametro & parametro Dn descripta, quæque contingit rectam DT in D, ea erit cujus contactus cum sectione est intimus, per Art. 19.

§ 50. Cæteris manentibus, ex puncto V ducatur recta Fig. 32. VH circulum osculatio contingens in H; jungatur HD, cumque sit angulus RDH complementum an- guli DrV ad rectum, erit RDH = DVr = EDO; adeoque variatio radii curvaturæ erit ut tangens anguli
D d
RDH;

RDH; & coincidentibus rectis DR et DH variatio evanescit.



SECTIO III.

De Lineis tertii Ordinis.

§ 51. **D**E lineis tertii ordinis five curvis secundi generis, uberius nobis agendum est. Doctrinam conicam, variis modis usque ad fastidium fere, tractarunt permulti. Hanc autem geometriæ universalis partem, pauci adtigerunt; eam tamen nec sterilem esse nec injucundam ex sequentibus, ut spero, patebit, cum præter proprietates harum figurarum a *Newtono* olim traditas, aliæ sint plures geometrarum attentione non indignæ. Ostendimus supra, rectam secare posse lineam tertii ordinis in tribus punctis, quoniam æquationis cubicæ tres sunt radices, quæ omnes reales esse possunt. Recta autem quæ lineam tertii ordinis in duobus punctis secat, eidem in tertio aliquo puncto necessario occurrit, vel parallela est asymptoto curvæ, quo in casu dicitur ei occurrere ad distantiam infinitam: æquationis enim cubicæ si duæ radices sint reales, tertia necessario realis erit. Hinc recta quæ lineam tertii ordinis contingit, eam in aliquo puncto semper secat; cum contactus pro duabus intersectionibus coincidentibus habendus sit. Recta autem quæ curvam in puncto flexus contrarii contingit, simul pro secante habenda est. Ubi duo arcus curvæ sibi mutuo occurrunt, punctum *duplex* formatur, & recta quæ alterum arcum ibi contingit in eodem puncto alterum secat.

fecat. Recta autem alia quævis ex puncto duplice ducta in uno alio puncto curvam fecat, sed non in pluribus.

§ 52. PROP. I. Sint duæ parallelæ, quarum utraque fecet lineam tertii ordinis in tribus punctis; recta quæ utramque parallelam ita fecat ut summa duarum partium parallelæ ex uno secantis latere ad curvam terminatarum æqualis sit tertiæ parti ejusdem ex altero secantis latere ad curvam terminatæ similiter fecabit omnes rectas his parallelas quæ curvæ in tribus punctis occurrunt; per Art. 4.

§ 53. PROP. II. Occurrat recta positione data lineæ tertii ordinis in tribus punctis; ducantur duæ quævis parallelæ quarum utraque curvam fecet in totidem punctis; & solida contenta sub segmentis parallelarum ad curvam & rectam positione datam terminatis erunt in eadem ratione ac solida sub segmentis hujus rectæ ad curvam & parallelas terminatis, per Art. 5.

Hæ duæ proprietates a *Newtono* olim expositæ fuerunt.

§ 54. PROP. III. Cæteris manentibus ut in Fig. 33. propositione præcedente, occurrat recta positione data lineæ tertii ordinis in unico puncto A, & solidum sub segmentis PM, P_m, P_μ unius parallelæ contentum erit semper ad solidum sub segmentis pN, p_n, p_v, alterius parallelæ ut solidum AP × bP² contentum sub segmento AP & qua-

drato distantiae bP puncti P a puncto quodam b ad solidum $Ap \times bp^2$ contentum sub segmento Ap et quadrato distantiae puncti p ab eodem puncto b , per Art. 6.

- Fig. 34.
n. 1. § 55. PROP. IV. Ex dato quovis puncto P ducatur recta PD quæ lineæ tertii ordinis occurrat in tribus punctis D, E, F , & alia quævis recta PA quæ eandem secet in tribus punctis A, B, C . Ducantur tangentes AK, BL, CM , quæ rectæ PD occurrant in K, L , et M ; et medium harmonicum inter tres rectas PK, PL, PM , coincidet cum medio harmonico inter tres rectas PD, PE , et PF , per Art. 10. & 28.
- n. 2. Si autem recta PD curvæ occurrat in unico puncto D , inveniatur punctum d ut in Art. 6. & medium harmonicum inter tres rectas PK, PL, PM , erit ad medium harmonicum inter duas rectas PD et $\frac{1}{2}Pd$ in ratione 3 ad 2, per Art. 12.

§ 56. PROP. V. Revolvatur recta PD circa polum P , sumatur semper PM in recta PD æqualis medio harmonico inter tres rectas PD, PE , et PF , eritque locus puncti M linea recta, per Art. 28.

Atque hæc est proprietas harum linearum a *Cotesio* inventa.

- Fig. 35. § 57. PROP. VI. Sint tria puncta lineæ tertii ordinis in eadem recta linea; ducantur rectæ curvam in his punctis contingentes, quæ eandem fecent

secent in aliis tribus punctis; atque hæc tria puncta erunt etiam in recta linea.

Occurrat recta FGH lineæ tertii ordinis in tribus punctis F, G, et H. Rectæ FA, GB, HC, curvam in his punctis contingentes eandem secent in punctis A, B, C; & hæc puncta erunt in recta linea. Jungatur enim AB, & hæc transibit per C; si enim fieri potest, occurrat curvæ in alio puncto M, tangenti HC in N et rectæ FGH in P; cumque sit $\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PM} = \frac{1}{PA} + \frac{1}{PB} + \frac{1}{PN}$, per Prop. IV. erit PN = PM; quod fieri nequit nisi coincidant puncta N, M, et C. Recta igitur AB transfit per C.

§ 58. *Corol.* Hinc si A, B, C, sint tria puncta lineæ tertii ordinis in eadem recta linea, ductæ autem AF et BG curvam contingant in F et G, & juncta FG curvam denuo secet in H, juncta CH curvam continget in H. Si enim recta curvam contingeret in H quæ eandem secaret non in C sed in alio quovis puncto, foret hoc punctum cum tribus aliis A, B, C, in eadem recta quæ igitur secaret lineam tertii ordinis in quatuor punctis. Hoc autem fieri non potest. Incidi autem primo in hanc propositionem via diversa sed minus expedita, eandem deducendo ex Prop. II. Similiter si recta Af curvam quoque contingat in f, & ducta Gf curvæ occurrat in b, juncta Ch erit tangens ad punctum b. Et si a punctis A, B, C, lineæ tertii ordinis in eadem recta fitis, ducantur tot rectæ curvam contingentes quot duci possunt, erunt semper tres contactus in eadem recta.

§ 59. PROP. VII. Ex puncto quovis lineæ Fig. 36.
tertii ordinis ducantur duæ rectæ curvam con-

tingentes, & recta contactus conjungens denuo secet curvam in alio puncto, rectæ curvam in hoc puncto & in primo puncto contingentes se mutuo secabunt in puncto aliquo curvæ.

Ex puncto A ducantur rectæ curvam contingentes in F et G, juncta FG curvam secet in H, eandemque contingat in H recta HC quæ curvæ occurrat in C, & ducta AC erit curvæ tangens ad punctum A. Sequitur ex *Corollario* præcedente, coincidentibus enim A et B recta CA fit tangens ad punctum A.

§ 60. *Corol. 1.* Si ex puncto curvæ C ducantur duæ rectæ eandem contingentes in A et H, & ex puncto alterutro A contingentes AF et AG ad curvam, recta per contactus F et G ducta transibit per alterum punctum H.

Fig. 37. § 61. *Corol. 2.* Contingat recta AC curvam in A, eamque secet in C, ductæ autem AF et CH curvam contingant in F et H, recta per contactus ducta eam denuo secet in G, & juncta AG curvam continget in G. Quod si alia recta Ch ex puncto C ducatur ad curvam eam contingens in b; & junctæ bF, bG, curvæ occurrant in f et g, ductæ Af et Ag erunt tangentes ad puncta f et g.

Fig. 38. § 62. *Corol. 3.* Sit A punctum flexus contrarii unde ductæ AF et AG curvam contingant in F et G juncta FG secet curvam in H, & ducta AH curvam continget in H. Si enim tangens ad punctum H curvæ in alio quovis puncto ab A diverso occurreret, recta ex hoc occurso ad punctum flexus contrarii A ducta curvam in A contingeret, quod fieri nequit. Manifestum autem est tres tantum duci posse rectas ex puncto flexus contrarii curvam

curvam contingentes præter eam quæ in hoc ipso puncto simul tangit & fecat, atque tres contactus cadere in eandem rectam lineam. Ex solo puncto flexus contrarii tres rectæ ductæ curvam ita contingunt ut tres contactus sint in eadem rectâ. Sint enim F, G, H , in eadem recta, unde tangentes ductæ conveniant in eodem puncto curvæ a , quod non sit punctum flexus contrarii; ducatur ae curvam contingens in a , quæque ei occurrat in e , & juncta eH curvam tanget in H , per hanc propositionem; adeoque rectæ eH et aH curvam contingerent in eodem puncto H . *q. e. a.*

§ 63. PROP. VIII. Ex puncto quovis lineæ tertii ordinis ducantur tres rectæ curvam contingentes in tribus punctis; recta duos quosvis contactus conjungens occurrat denuo curvæ, & ex occurſu ad tertium contactum ducta curvam denuo fecabit in puncto ubi recta ad primum punctum ducta curvam continget.

Ex puncto A , lineæ tertii ordinis ducantur tres rectæ $AF, AG, et Af$, curvam contingentes in tribus punctis $F, G, et f$; recta Gf quæ horum duo quævis conjungit fecet curvam denuo in N , et recta ex hoc puncto ad tertium contactum F ducta curvam fecet in g , tum juncta Ag curvam continget in g . Ducatur enim recta AC curvam contingens in A quæ eandem fecet in C ; cumque puncta $G, N, et f$, sint in eadem recta, & tangentes ad puncta G et f tranſeant per A , ſequitur (per Prop. VII.) tangentem ad punctum N tranſire per C . Cumque puncta F, N, g , sint in eadem rectâ, tangentes autem FA et NC curvæ occurrant in A et C , ſitque AC tangens ad punctum A , tangens ad punctum g tranſibit per A .

§ 64. *Corol.* Hinc si curva describatur, ex datis tribus punctis contactus ubi tres rectæ ex eodem puncto curvæ ductæ eam contingunt, invenitur quartum punctum contactus ubi recta ex eodem puncto curvæ ducta eam contingit. Atque hinc colligitur ex eodem curvæ puncto quatuor tantum rectas duci posse lineam tertii ordinis contingentes præter rectam quæ in hoc ipso puncto curvam contingit. Si enim rectæ ex eodem curvæ puncto duci possent eam in quinque punctis contingentes, plures rectæ numero indefinitæ curvam contingentes ex eodem puncto duci possent; ut ex præmissis facile colligitur. Hoc autem Corollarium postea facilius demonstrabitur. Vide infra, Art. 77.

Fig. 38. § 65. PROP. IX. Ex puncto flexus contrarii ducantur tres tangentes ad curvam, & recta contactus conjungens harmonice secabit rectam quamvis ex puncto flexus contrarii ductam & ad curvam terminatam.

Sit A punctum flexus contrarii, AF, AG, et AH, rectæ curvam contingentes in punctis F, G, et H. Ex puncto A ducatur recta quævis curvam secans in B et C, & rectam FH in P; eritque PB ad PC ut BA ad AC. Cum enim tres tangentes ad puncta F, G, et H, in eodem puncto A conveniant, erit per Prop. IV. $\frac{1}{BF} + \frac{1}{FA} - \frac{1}{PC} = \frac{3}{PA}$, adeoque $\frac{1}{PB} - \frac{1}{PC} = \frac{2}{PA}$, i. e. PA est medium harmonicum inter duas rectas PB et PC ad curvam terminatas. Quæ linearum tertii ordinis proprietas est simplicitatis insignis.

§ 66. *Corol. 1.* Recta quæ duas quasvis rectas ex puncto flexus contrarii ductas ad curvam secat harmonice, secabit quoque alias quasvis rectas ex eodem puncto ductas & ad curvam terminatas.

§ 67. *Corol. 2.* Si recta asymptoto parallela per punctum flexus contrarii ducta occurrat rectæ FH in R & curvæ in O, erit $\frac{1}{RO} = \frac{2}{RA}$, adeoque $RA = 2RO$.

§ 68. PROP. X. Recta duo puncta flexus contrarii conjungens vel transit per 3^{um} punctum flexus contrarii vel dirigitur in eandem plagam cum crure infinito curvæ Fig. 39:

Sint A et a puncta flexus contrarii, juncta Aa curvæ occurrat in a, eritque a quoque punctum flexus contrarii. Si enim tangens figuræ in puncto a curvæ occurreret in alio quovis puncto e, forent A, a, e, in eadem rectâ. Verum ex hypothefi sunt A, a, et a in eadem recta, quæ igitur lineæ tertii ordinis occurreret in punctis quatuor. Sit A punctum flexus contrarii, & recta AO asymptoto parallela curvæ occurrat in O ducatur OQ curvam contingens in O, & secans in Q, juncta AQ, transibit per D ubi curva asymptoton secat.

§ 69. PROP. XI. Ductis ex puncto flexus contrarii A tangentibus ad curvam AF, AG, AH; & duabus secantibus quibuscunque ABC, Abc, junctæ Bb et Cc vel Bc et bC se mutuo secabunt in recta FH quæ contactus conjungit. Fig. 38.

Occurrat enim recta Bb ipsi FH in Q, et BC eidem in P; jungantur QA et QC; cumque sit AB ad AC ut PB ad PC, per Prop. IX. erunt QA, QB, QP et QC, harmo.

harmonicales, adeoque Ab secabit rectam QC in c et ipsam FH in p , ita ut Ab sit ad Ac ut pb ad pc ; & proinde erit c punctum curvæ, per Prop. IX. unde sequitur converse rectas Bb et Cc convenire in puncto Q rectæ FH ; & similiter ostenditur rectas Bc et bC sibi mutuo occurrere in puncto q ejusdem rectæ.

§ 70. *Corol. 1.* Ex puncto quovis Q rectæ FH ducantur ad curvam rectæ QB , QC , eam secantes in punctis B , b , M et C , c , N ; tum junctæ CB , cb , MN , convenient in puncto flexus contrarii A ; junctæ Bc et bC , Mc et Nb , Bb et Cc , NB et MC , convenient in recta FH .

– § 71. *Corol. 2.* Tangentes ad puncta B et C conveniunt in puncto aliquo T rectæ FH ; & si a puncto quovis T in recta FH sito ducantur tangentes ad curvam, rectæ contactus conjungentes vel transibunt per punctum flexus contrarii, vel convenient in recta FH .

§ 72. *Corol. 3.* Dato puncto flexus contrarii A , & punctis B , C , b , c , ubi duæ rectæ ex eo ductæ curvam secant, datur recta FH positione; junctæ enim Bb et Cc , occurfu suo dabunt punctum Q , & junctarum Bc et bC occurfus dabit q , ductaque Qq ea est quæ contactus F , G , et H , conjungit. His autem quinque punctis datis cum aliis duobus M et m , determinatur linea tertii ordinis quæ per hæc septem puncta A , B , C , b , c , M , m , transit & in puncto A habet flexum contrarium. Ex punctis enim M et m dantur N et n , ubi ductæ AM et Am curvam secant, & his novem conditionibus linea determinatur. Si autem dentur tria puncta M , m , et S ; hæc dabunt tria alia N , n , et s ; unde darentur undecim conditiones ad figuram determinandam, quæ nimis sunt.

funt. Similiter dato puncto flexus contrarii A cum punctis F, G, (adeoque tangentibus AF et AG) et punctis M et m quibuscunque, datur recta FG, adeoque puncta N et n, et determinatur curva.

§ 73. *Corol. 4.* Contingant rectæ HB, HC, curvam Fig. 40. in B et C, et junctæ CB transibit per A, junctæ CG et FB concurrent in puncto curvæ V, et ducta VH curvam continget in V. Tangens autem ad punctum flexus contrarii A determinatur ducendo AV cui occurrat in L, recta PL ipsi AH parallela & bifecanda PL in X; junctæ enim AX erit tangens ad punctum A. Occurrat enim tangens ad A rectæ FH in S; eritque $\frac{1}{PS} + \frac{2}{PH} = \frac{1}{PH} + \frac{1}{PG} - \frac{1}{PF}$, adeoque $\frac{1}{PS} + \frac{1}{PH} = \frac{1}{PG} - \frac{1}{PF}$ (quoniam AC secatur harmonice in P et B, adeoque VA, VF, VP, et VG, harmonicales) = $\frac{2}{PK}$. Est igitur PK medium harmonicum inter PS et PH; unde si PL parallela rectæ AH occurrat rectis AV et AS in X et L, erit PX = XL.

§ 74. *PROP. XII.* Ex puncto lineæ tertii Fig. 41. ordinis A ducantur duæ rectæ curvam contingentes in F et G, junctæ FG curvæ occurrat in H, & tangens ad punctum A fecet curvam in M; jungatur HM, cui occurrat FLK ipsi AH parallela in L, & fumatur FK = 2FL; tum junctâ HK, recta quævis AB ex A ducta harmonice secabitur a rectis HK et HF in N, P, et a curva in B, C; ita ut NB erit ad NC ut BP ad PC,

Occurrat

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Occurrat enim recta AB tangenti HM in T, eritque

$$\frac{1}{PB} + \frac{1}{PA} - \frac{1}{PC} = \frac{2}{PA} + \frac{1}{PT},$$
 adeoque $\frac{1}{PB} - \frac{1}{PC}$

$$= \frac{1}{PA} + \frac{1}{PT}$$
 (per constructionem, & *harmonice*)

$$= \frac{2}{PN}.$$
 Unde sequitur rectam NC secari harmonice
 in punctis B et P, vel NB esse ad NC ut BP ad PC.

§ 75. *Corol. 1.* Hinc si duæ quævis rectæ ex A ductæ
 secentur in N harmonice ita ut PC sit ad PB ut CN ad
 BN, omnes rectæ ex A eductæ a rectis HF et HK si-
 militer harmonice secabuntur.

§ 76. *Corol. 2.* Si curva punctum duplex non habeat,
 eamque secet recta HK in duobus punctis *f* et *g*, ductæ
 Af et Ag erunt rectæ curvam contingentes in his pun-
 ctis. Coincidat enim punctum B cum puncto N,
 quando N pervenit ad *f* occursum rectæ HK cum
 curva; adeoque cum $\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PN}$, erit $\frac{1}{PC} =$
 $\frac{1}{PN}$, et coincidit C cum B, et recta ex A ducta cur-
 vam tunc contingit. Ex altera parte, si recta Af cur-
 vam contingat transibit recta HK per *f*; ob æquales
 enim PB, PC, in hoc casu, coincidunt puncta B et
 C cum N.

§ 77. *Corol. 3.* Si recta HK in solo puncto H curvæ
 occurrat, duæ tantum tangentes duci poterunt a puncto
 A ad curvam, viz. AF et AG. Quatuor tantum ad
 summum tangentes duci possunt a puncto quovis lineæ
 tertii ordinis ad curvam ut AF, AG, Af, Ag. Si enim
 alia quævis tangens duci posset a puncto A ad curvam
 ut Aφ, recta HK transiret per punctum φ, et quatuor
 puncta

puncta lineæ tertii ordinis forent in eadem rectâ, viz. $H, f, g, \varphi.$ *Q. E. A.*

§ 78. PROP. XIII. Si ex puncto lineæ tertii ordinis duci possunt quatuor rectæ curvam contingentes, rectæ contactus conjungentes convenient semper in puncto aliquo curvæ, & recta quævis a primo puncto ducta harmonice secabitur a curva & rectis binos contactus conjungentibus:

Sit A punctum curvæ, AF, AG, Af, et Ag, rectæ curvam contingentes in punctis F, G, f, et g. Jungantur FG et fg, quibus occurrat recta quævis ABC (ex A ducta curvamque secans in B et C) in P et N; & recta NC harmonice secabitur in B et P, ita ut semper sit NC ad NB ut CP ad PB: sequitur ex Corol. 2. præcedentis. Rectæ autem FG et fg concurrunt in puncto curvæ H; & similiter rectæ Ff et Gg conveniunt in E, atque Fg et Gf in R; et E, R erunt puncta curvæ, per idem corollarium. Atque hæc est posterior duarum proprietatum linearum tertii ordinis quas descripsimus in tractatu de fluxionibus, Art. 402. Quod si recta AM curvam contingat in A, et secet in M, junctæ ME, MR, MH, curvam tangent in punctis E, R, H; & rectarum AE et HR, AR et HE, AH et RE occurfus erunt quoque in curva*.

§ 79. Corol. Cum igitur sint rectæ HK, HB, HP, et HC, harmonicales; si rectæ HB et HC curvæ occurrant in *b* et *c*; erunt puncta A, *b*, et *c*, in eadem recta linea. Occurrat enim junctæ Ab curvæ in *b* et *c* at-

* Supple quæ desunt in Schemate.

que ipsi HF in p , et HK in n ; cumque sit nc ad nb ut pc ad pb patet c esse in recta HC; & reciproce si c sit in recta HC et b in recta HB, erunt A, b , c , in eadem rectâ.

Fig. 42. § 80. PROP. XIV. Habeat linea tertii ordinis punctum duplex O. Ex puncto quovis curvæ A ducantur duæ rectæ AF et AG curvam contingentes in F et G; ducta FG curvam secet in H; jungatur OH. Recta quævis AB ex A ducta curvæ occurrat in punctis B et C, rectæ FG in P, & rectæ OH in N; & recta NP harmonice secabitur in punctis B et C, ita ut PB sit ad PC ut BN ad NC.

Jungatur enim AO quæ rectæ FG occurrat in p et tangenti HL in t ; cumque sit O punctum duplex, erit $\frac{2}{pO} + \frac{1}{pA} = \frac{1}{pA} + \frac{1}{pt}$, adeoque $\frac{1}{pA} + \frac{1}{pt} = \frac{2}{pO}$. Secatur igitur pA harmonice in t et O, ita ut pt sit ad pA ut tO ad OA, et harmonicales sunt H p , H t , HO, et HA. Occurrat recta PA tangenti LH in T, cumque sit $\frac{1}{PC} + \frac{1}{PB} + \frac{1}{PA} = \frac{2}{PA} + \frac{1}{PT}$, erit $\frac{1}{PC} + \frac{1}{PB} = \frac{1}{PA} + \frac{1}{PT} = \frac{2}{PN}$; consequenter PC est ad NC ut PB ad BN.

§ 81. *Corol.* Si tangens HL occurrat rectæ GZ ipsi AH parallelæ in Z, & sumatur $GV = 2GZ$, ducta HV transibit per punctum duplex O, si modo curva tale punctum habeat. Vel si recta Gra occurrat rectis AH et HR in a et r , junctæ rA et Ra , se decussent in m , juncta H m transibit per punctum duplex O.

§ 82.

§ 82. PROP. XV. Ex puncto lineæ tertii ordinis ducantur duæ tangentes, & ex alio quovis ejusdem puncto ducantur rectæ ad contactus curvam in duobus aliis punctis secantes; tangentes ad hæc duo nova puncta in eodem puncto curvæ convenient.

Ex puncto A ducantur rectæ AF et AG curvam contingentes in F et G. Sumatur punctum quodvis curvæ P, jungantur PF et PG curvam secantes in punctis K et L; atque tangentes ad puncta K et L concurrent in puncto aliquo curvæ B. Determinatur autem punctum B, ducendo rectam PC quæ curvam contingit in P, et fecat in C; si enim jungatur AC occurret denuo curvæ in puncto B. Fig. 43.

Cum enim puncta F, K, P, sint in eadem rectâ, & tangentes ad puncta F et P curvam fecent in A et C; sequitur tangentem ad punctum K transituram per B. Et ob rectam LGP, tangens ad punctum L transibit quoque per B.

§ 83. *Corol.* Sint igitur A et B duo quævis puncta in Fig. 44. linea tertii ordinis; ex utroque ducantur quatuor rectæ curvam in aliis quatuor punctis contingentes, viz. AF, AG, Af, Ag; et BK, BL, Bk, Bl. Junctæ FK et GL, FL et GK, Fl et Gk, Gl et Fk; sibi mutuo occurrent in quatuor punctis curvæ, P, Q, q, p; & si ducantur tangentes ad hæc quatuor puncta, hæc occurrent curvæ & sibi mutuo in puncto C ubi recta AB curvam fecat. Unde si sint tria puncta lineæ tertii ordinis in eadem rectâ, & ex singulis ducantur quatuor rectæ curvam contingentes in quatuor aliis punctis, recta per duo quævis puncta contactus ducta curvam semper secabit in alio aliquo

aliquo puncto contactus; & quatuor hujusmodi rectæ per idem punctum contactus semper transibunt.

Fig. 43. § 84. PROP. XVI. Sint F et G puncta duo lineæ tertii ordinis, ita sumpta ut rectæ FA et GA curvam in his punctis contingentes conveniant in puncto aliquo curvæ A . Sumatur in curva aliud quodvis punctum P , unde ducantur ad puncta F et G rectæ PF et PG quæ curvæ occurrunt in K et L ; jungantur FL et GK , atque harum occurfus Q erit in curva. Tangentes autem ad puncta K et L sibi mutuo & curvæ occurrent in puncto aliquo curvæ B , atque tangentes ad puncta P et Q convenient in puncto curvæ C , ita ut tria puncta A, B, C , sint in eadem recta.

Ducatur enim tangens ad punctum P quæ curvæ occurrat in C , & ducta AC fecet eandem in B ; & ductæ BK, BL , erunt tangentes ad puncta K et L , per præcedentem. Occurrat recta LF curvæ in Q ; & si recta GK non transeat per Q , occurrat curvæ in q . Quoniam igitur tria puncta L, F, Q , sunt in eadem rectâ, tangentes vero ad L et F curvam fecent in B et A , sequitur (per Prop. VII.) tangentium ad punctum Q transire per punctum C . Similiter, cum sint puncta G, K , et q , in eadem recta, tangentes autem ad puncta G & K transeant per A et B , tangens ad punctum q transibit quoque per punctum C . Utraque igitur recta CQ, Cq , curvam contingit prior in Q , posterior in q . Coincidunt igitur puncta Q et q , si enim diversa esse ponamus, sequitur per Prop. VIII. plures quam quatuor tangentes duci posse ad curvam ex eodem puncto C . Sint enim

Af & *Ag* rectæ quæ curvam contingant in *f* & *g*, & ductæ *Lf*, *Lg*, curvam secent in *m* & *n*; & rectæ *Cm*, *Cn*, erunt tangentes ad puncta *m* et *n*. Quare haberemus quinque tangentes ex *C* ad curvam ductas *CP*, *CQ*, *Cm*, *Cn*, & *Cq*; quod repugnat *Corol.* 3. *Prop.* XII.

§ 85. *Corol.* 1. Dato puncto *P*, ubicunque sumantur puncta *F* & *G*, modo tangentes ad hæc puncta in curva convenient, datur punctum *Q*, ubi junctæ *FL* & *GK* occurrunt sibi mutuo & curvæ. Et si a puncto *P* ducatur recta quævis *PNM* quæ curvæ occurrat in *N* et *M*, & junctæ *QM*, *QN*, eam secent in *m* & *n*; erunt puncta *P*, *n*, & *m*, in eadem recta linea. Ostendimus enim tangentes ad puncta *P* & *Q*, se mutuo decussare in puncto curvæ*.

§ 86. *Corol.* 2. Si sumantur quatuor puncta *F*, *G*, *K*, *L*, in linea tertii ordinis, ita ut tangentes ad puncta *F* & *G*, convenient in aliquo puncto curvæ, & tangentes ad puncta *K* et *L*, convenient quoque in aliquo puncto curvæ, ductæ *FK* & *GL* concurrent in puncto curvæ, & ductæ *FL* & *GK* sibi mutuo occurrant in puncto curvæ. Fig. 43.

§ 87. *Prop.* XVII. Sint *F* et *G* duo quævis puncta lineæ tertii ordinis, ubi si rectæ ducantur curvam contingentes, hæc se mutuo secabunt in puncto aliquo curvæ. Sumantur alia quatuor puncta curvæ *L*, *K*, *f*, *g*, ita ut ductæ *LF* et *GK* convenient in curva, atque rectæ *Ff* et *Gg*, in eâ quoque convenient; tunc ductæ *Lf* et *gK*, se mutuo secabunt in curva, ut & ductæ *Lg* et *Kf*.

* Supple quod deest in Schemate.

Tangentes enim ad puncta f et g se mutuo decussant in curva, per Prop. XIV. ut & tangentes ad puncta K et L , per eandem. Adeoque per Corol. 2. præcedentis, junctæ fL et Kg conveniunt in curva, ut et fK et gL .

Fig. 45.
n. 1.

§ 88. *Lemma.* Dentur tres rectæ IC , IH , et CH , positione; & tria puncta F , G , S , quæ sint in eadem recta linea. Sumatur punctum quodvis Q in recta IC , juncta QF occurrat rectæ IH in L , & juncta QG rectæ HC in P ; jungatur FP , ducta SL occurrat rectis FP et QP in k et N ; atque puncta k et N erunt ad rectas positione datas. Jungatur enim IN , quæ occurrat rectæ GS in m , & ducatur per N parallela rectæ FS quæ occurrat rectis IC , IH , et LQ , in punctis x , u , et r ; occurrat recta FG rectis IC , IH , et HC , in a , b , et b . Quoniam Nx est ad Nr ut Ga ad GF , et Nr ad Nu ut SF ad Sb , erit Nx ad Nu (adeoque ma ad mb) ut $Ga \times SF$ ad $GF \times Sb$, i. e. in data ratione. Datur igitur punctum m , adeoque recta INm positione; & similiter est punctum k ad positione datam.

n. 2.

§ 89. *Corol.* Coincidentibus punctis S et G , coincidit quoque punctum m cum puncto G . Jungatur igitur IG quæ rectæ HC occurrat in D , & ducta CF occurrat rectæ HI in E , tum juncta DE erit locus puncti K ubi ductæ GL et FP se mutuo decussant.

Fig. 46.

§ 90. PROP. XVIII. Sit $PGLFQK$ quadrilaterum inscriptum figuræ, cujus sex anguli tangent lineam tertii ordinis ut in Prop. XVI. Ducantur rectæ curvam contingentes IC , CH , HI , in tribus punctis Q , P , L , quæ non sint in eadem rectâ; jungatur IG quæ tangenti CH occurrat

occurrat in D, et HF quæ tangenti CI occurrat in E; erunt puncta D, K, E, in eadem recta linea, quæ quidem curvam in puncto K contingit.

Supponamus enim rectas QFL et FKP moveri circa polum F, & rectas LGP et QKG circa polum G, puncta autem Q, L, et P, deferri in tangentibus QI, LI et PC; tum punctum K movebitur in recta DE, per Corol. præcedens. Unde si puncta Q, L, P, ferantur in curva quæ has rectas QI, LI, et PC, in his punctis contingit, movebitur quoque in curva quam recta DE contingit. Sed per Prop. XV. si puncta Q, L, P, ferantur in lineâ tertii ordinis proposita, punctum K movebitur in eadem, quam igitur recta DE contingit in K.

§ 91. *Corol. 1.* Similiter si rectæ AF et AG (quæ curvam contingunt in F et G) occurrant rectæ IH (quæ curvam contingit in L) in punctis M et N; juncta MP fecet tangentem AG in *d*, & juncta QN tangentem AF in *e*; recta *de* tranſibit per K, & curvam in hoc puncto continget; atque quatuor puncta D, *d*, *e*, E, erunt in eadem rectâ linea.

§ 92. *Corol. 2.* Ex duobus punctis curvæ quibuscunque C et B ducantur ad curvam quatuor contingentes binæ ex ſingulis, CQ et CP ex puncto C, BL et BK ex puncto B, ſintque harum tangentium occurſus I, H, E, et D; tum ductæ LQ et EH ſe mutuo ſecabunt in puncto curvæ F; atque junctarum LP et ID occurſus erit in puncto curvæ G; tangentes autem ad puncta F et G ſe mutuo ſecabunt in puncto curvæ A quod eſt in eadem recta cum punctis C et B.

§ 93. *Corol. 3.* Datis tribus punctis lineæ tertii ordinis quæ ſint in eadem rectâ, & duabus tangentibus ex

horum singulis ductis ad curvam positione datis, sex puncta contactus determinantur per hanc propositionem. Sint A, B, C, tria curvæ puncta data in eadem rectâ, AM et AN tangentibus ex A, BMI, et BDE, tangentibus ex B quæ prioribus occurrant in M, N, *e*, et *d*; sintque CD et CE tangentibus ex tertio puncto C ductæ; atque occurrat CD ipsis BM, BD, AM, et AN, in H, D, *b*, et *c*, & CE iisdem in I, E, *n*, et *m*. His positis, juncta Ne secabit tangentem CI in puncto contactus Q, Md secabit tangentem CD in puncto contactus P, ID secabit tangentem AN in puncto contactus G, EH tangentem AM in contactu F, mb secabit tangentem BH in L, & denique nc tangentem BE in K. Quamvis autem problema in hoc casu determinatum sit, solutiones tamen plures admittit. Diversæ enim lineæ tertii ordinis, sed numero definitæ, per tria puncta A, B, et C, duci possunt contingentes sex rectas positione datas AM, AN, BM, BD, CD, et CE. Occurrat enim Ne tangenti CD in *p*, recta Md tangenti CE in *q*, ID tangenti AM in *f*, EH tangenti AN in *g*, nc tangenti BM in *l*, et mb tangenti BD in *k*; atque linea tertii ordinis quæ conditionibus propositis satisfacit continget rectas CD et CE vel in P et Q, vel in *p* et *q*. Ea continget rectas AM et AN vel in punctis F et G vel in *f* et *g*; rectas autem BM et BD vel in L et K, vel in *l* et *k*. Constat igitur plures lineas tertii ordinis problematis conditionibus satisfacere posse, sed numero determinatas, adeoque problema esse determinatum*.

§ 94. *Corol.* 4. Datis duobus punctis lineæ tertii ordinis A et B, tangentibus quoque AM, AN, BM, BD positione datis cum tribus punctis contactus F, G, et L, datur punctum K ubi recta BD curvam contingit.

* Supple quæ defunt in Schemate.

Si enim ducantur rectæ Ne et LF , harum occurſu dabitur punctum Q , & ducta QG ſecabit contingentem BD in puncto contactus K . Datur quoque punctum P occurſus rectarum LG et Md , vel rectarum Md et FK ; tres enim rectæ LG , Md , et FK , neceſſario conveniunt in puncto P . Sit $MedN$ quadrilaterum quodvis, ſumatur punctum quodvis Q in diagonali Ne et P in diagonali Md , recta quævis QFL ex Q ducta ſecet latera Me et MN in F et L , ducta PL ſecet latus Nd in G , jungatur QG quæ latus de ſecet in K ; atque puncta F , K , P , erunt ſemper in eadem recta linea, per ſuperius oſtenſa. Unde conſtat problema non ideo fieri impoſſibile, quod oporteat tres rectas LG , Md , et FK , in eodem puncto convenire.

§ 95. PROP. XIX. Sint D , E , F , puncta Fig. 47.
lineæ tertii ordinis in eadem recta, ſintque tres
rectæ curvam in his punctis contingentes ſibi
mutuo parallelæ. In recta DF ſumatur punctum
 P ita ut $2PF$ ſit medium harmonicum inter PD
et PE ; & ſi alia quævis recta per P ducta curvæ
occurrat in f , d , et e , erit ſemper $2Pf$ medium
harmonicum inter Pd et Pe . Supponimus autem
puncta d et e eſſe ad eaſdem partes puncti P ,
punctum autem f eſſe ad contrarias.

Occurrant enim tangentes DK , EL , FM , rectæ df
in punctis K , L , et M ; eritque per Art. 9. $\frac{1}{Pf} - \frac{1}{Pd}$
 $-\frac{1}{Pe} = \frac{1}{PM} - \frac{1}{PK} - \frac{1}{PL}$ (ſi recta Qq tangenti-
bus parallela harmonice ſecet rectam PD ita ut PE
ſit ad EQ ut PD ad DQ , et Qq occurrat rectæ fd
E e 3 in

in q) = $\frac{1}{PM} - \frac{2}{Pq}$ (quoniam Pq est ad PM ut PQ ad PF , & ex hypothesi $2PF = PQ$, adeoque $2PM = Pq$)
 = $\frac{1}{PM} - \frac{1}{PM} = 0$; unde $\frac{1}{Pf} = \frac{1}{Pd} + \frac{1}{Pe}$, adeoque
 $2Pf$ est medium harmonicum inter Pd et Pe .

§ 96. *Corol. 1.* Jungantur Dd et Ee quæ convenient in puncto V , junctæ VQ et Ff erunt parallelæ; & productâ VQ donec occurrat rectæ fd in r , erit $Pf = \frac{1}{2}Pr$. Recta enim PD secatur harmonice in E et Q , ex hypothesi, adeoque etiam recta Pd secatur harmonice in e et r , per Art. 21. unde $Pf = \frac{1}{2}Pr$; cumque sit $PF = \frac{1}{2}PQ$; sequitur rectam Ff parallelam esse harmonicali VQr .

§ 97. *Corol. 2.* Similiter si sumatur in recta DF punctum p ita ut $2pD$ sit æqualis medio harmonico inter pE et pF , & recta quævis ex p ducta curvæ occurrat in tribus punctis, erit segmentum hujus rectæ ex una parte puncti p ad curvam terminatum æquale dimidio medii harmonici inter duo segmenta eodem puncto p et curvâ ad alteras partes terminata.

Fig. 48. § 98. *Lemma.* Ex cen ro gravitatis trianguli ducatur recta quævis quæ tribus lateribus trianguli occurrat, & segmentum hujus rectæ centro gravitatis & uno trianguli latere terminatum erit dimidium medii harmonici inter segmenta ejusdem rectæ centro gravitatis & duobus aliis trianguli lateribus terminata. Sit P centrum gravitatis trianguli VTZ ; occurrat recta FDE per P ducta lateribus in F , D , E ; sintque puncta D et E ad easdem partes puncti P ; eritque $\frac{1}{PF} = \frac{1}{PD} + \frac{1}{PE}$. Ducatur enim per punctum P , recta MPL lateri VZ parallela, quæ

quæ lateribus VT, ZT, occurrat in L et M et rectæ VN parallelæ lateri ZT in N; cumque sit MP = PL, et TL = 2VL, ob similia triangula TLM, VLN, erit LM = 2LN, unde LN = LP, et PN = 2PM, proinde si PD occurrat rectæ VN in K, erit (per Art. 21.

$$\& 23.) \frac{1}{PD} + \frac{1}{PE} = \frac{2}{PK} = \frac{1}{PF}.$$

§ 99. PROP. XX. Contingant tres rectæ VT, VZ, TZ, lineam tertii ordinis transeatque eadem recta linea per tres contactus & per P centrum gravitatis trianguli VTZ; recta quævis per hoc centrum ducta curvæ occurrat in puncto *c* ex una parte & in punctis *a* et *b* ex altera ejusdem centri gravitatis parte, eritque 2Pc medium harmonicum inter segmenta Pa et Pb.

Occurrat enim recta Pc lateribus trianguli VTZ in *f*, *d*, et *e*; & rectæ VN lateri TZ parallelæ in *k*; eritque 2Pf = Pk, adeoque $\frac{1}{Pf} = \frac{2}{Pk} = \frac{1}{Pd} + \frac{1}{Pe} = \frac{1}{Pa} + \frac{1}{Pb} - \frac{1}{Pc} + \frac{1}{Pf}$, adeoque $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$, unde Pc est dimidium medii harmonici inter rectas Pa et Pb.

§ 100. PROP. XXI. Sit V punctum duplex in linea tertii ordinis, VT et VZ rectæ curvam in hoc puncto contingentes, quibus in T et Z occurrat recta TZ curvam contingens in F ita ut FT = FZ; jungatur FV, in qua sumatur FP = $\frac{1}{3}$ FV; & si recta quævis per P ductæ curvæ occurrat in tribus punctis *a*, *b*, *c*, quorum *a* et *b* sint ad easdem partes puncti P, *c* ad partes contrarias,

trarias, erit semper $2Pc$ medium harmonicum inter segmenta Pa et Pb , seu $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$.

Cum enim bifecetur TZ in F , sitque $FP = \frac{1}{3}FV$, manifestum est punctum P esse centrum gravitatis trianguli VTZ ; cumque sit punctum P in rectâ FV quæ per contactus transit, sequitur propositio ex præcedente.

§ 101. *Corol. 1.* Si jungatur rectæ Va , Vb , et Fc , erit P quoque centrum gravitatis trianguli hisce rectis contenti, ut & trianguli tribus rectis curvam in a , b , c , contingentibus comprehensi; & si ductæ Va et Vb occurrant rectæ Fc in m et n , erit semper Fm æqualis Fn .

§ 102. *Corol. 2.* Recta per punctum duplex ducta parallela rectæ Fc harmonice secabit ipsam Pa in k ita ut Pa erit ad ak ut Pb ad Pk ; quæ vero ducitur a puncto k ad x occursum rectarum curvam in a et b contingentium parallela est recta cy figuram contingentem in c .

§ 103. *Corol. 3.* Datis duobus punctis a et c ubi recta quævis ex P ducta curvæ occurrit, datur tertium b ; jungantur enim Va et Fc quæ sibi mutuo occurrant in m ; sumatur Fn ex altera parte puncti F æqualis ipsi Fm , et juncta Vn secabit rectam Pa in b .

Fig. 51. § 104. PROP. XXII. Ducatur per punctum quodvis P recta quæ dirigatur in plagam crurum infinitorum & occurrat curvæ in punctis a et c ; ducatur per idem punctum recta quævis curvam secans in punctis D , E , F , quæque rectis curvam in a et c contingentibus occurrat in k et m , atque asymptoto

asymptoto cruris infiniti in l ; & si puncta D, E, k, m, l , sint ad eandem partes puncti P , punctum vero F ad contrarias, erit $\frac{1}{Pl} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF} - \frac{1}{Pk} - \frac{1}{Pm}$, ubi termini cujuscvis signum est mutandum quoties segmentum ad oppositas partes puncti P protenditur.

Sequitur ex Theor. I. Art. 9. est enim per hoc theoremata $\frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF}$.

§ 105. *Corol. 1.* Si recta PD ducatur per concursum tangentium ak et cm ; & sumatur PM æqualis medio harmonico inter rectas PD, PE, PF , secundum Art. 28.

erit $\frac{1}{Pl} = \frac{3}{PM} - \frac{2}{Pk}$, adeoque $\frac{2}{3}PM$ erit medium harmonicum inter Pl et $\frac{1}{2}Pk$. Quod si tangentes ak et cm concurrant in ipso puncto M , asymptotos quoque per M transibit.

§ 106. *Corol. 2.* In casu Prop. XIX. ubi tres contactus Fig. 47. sunt in eadem recta linea & tres tangentes parallelæ, sumatur punctum P ut in Propositione XIX. fitque aPc asymptoto parallela, occurrant ak et cm tangentes rectæ

PD in k et m , eritque $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$, sive Pl æqualis

dimidio medii harmonici inter Pk et Pm . Quod si tangentes ak et cm concurrant in eodem puncto rectæ PD ,

erit $Pl = \frac{1}{2}Pk$; quoniam vero in Prop. XIX. $\frac{1}{Pf} = \frac{1}{Pa}$

$+ \frac{1}{Pe}$, erit $Pa = Pc$.

Fig. 49. § 107. *Corol.* 3. Idem dicendum est de casu Prop. XX. ubi tres contactus D, E, F, sunt in eadem rectâ quæ transit per P centrum gravitatis trianguli VTZ tangentibus contenti. Si autem altera rectarum curvam in *a* vel *c* contingentium (posita *aPc* asymptoto parallelâ) fit rectæ DP parallela, abibit asymptotos in infinitum, eritque crus parabolicum.

§ 108. *Corol.* 4. Iisdem positis ac in Prop. XXI. Sit *cPa* asymptoto parallela, occurrant tangentes *ak*, *cm*, rectæ VF in *k* et *m*, eritque $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$. Unde si

Fig. 52. curva diametrum habet, cum hæc necessario transeat per punctum duplex V, & per punctum curvæ F ubi bifecatur tangens TFZ, sumatur ab F versus V, $FP = \frac{1}{3}FV$, ducatur *cPa* asymptoto parallela, & tangens *ak* quæ diametro occurrat in *k*, & ex altera parte puncti P sumatur, super rectam PV, $Pl = \frac{1}{2}Pk$, & recta per *l* ducta ordinatim applicatis parallela erit asymptotos curvæ. Si vero tangens *ak* fit diametro parallela, erit crus curvæ generis parabolici. Propositio *Newtoni* de segmentis rectæ cujusvis tribus asymptotis & curva terminatis facile sequitur ex Art. 4. ut ab aliis olim ostensum est.

Fig. 53. § 109. PROP. XXIII. Ex puncto quovis D lineæ tertii ordinis ducantur duæ quævis rectæ DEI, DAB, quæ curvæ occurrant in punctis E, I et A, B; ducantur tangentes AK, BL, quæ rectæ DE occurrant in K et L. Sit DG medium harmonicum inter segmenta DE, DI, ad curvam terminata, atque DH medium harmonicum inter segmenta DK, DL, ejusdem rectæ tangentibus abscissa. Sit DV medium geometricum inter DG

DG et DH, ducatur VQ parallela tangenti DT, quæ occurrat rectæ DA in Q; & si circulus ejusdem curvaturæ cum linea tertii ordinis propositâ in puncto D occurrat rectæ DE in R, erunt HG, QV et 2DR continue proportionales.

Nam per Theor. II. (Art. 15.) est $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL} = \frac{2}{DG} - \frac{2}{DH} = \frac{2DH - 2DG}{DG \times DH} = \frac{2HG}{DV^2}$ (quoniam $DV^2 = DG \times DH$;) unde $QV^2 = 2HG \times DR$, adeoque HG ad QV ut QV ad 2DR.

§ 110. *Corol. 1.* Sumatur igitur Dr in recta DE tertia proportionalis rectis HG et $\frac{1}{2}QV$, & perpendicularis rectæ DE ad punctum r secabit normalem tangenti DT ad punctum D in centro circuli osculatorii five circuli ejusdem curvaturæ cum linea proposita, in puncto O. Si puncta E, I, K, L, sint ad easdem partes ejusdem puncti prout DH major est vel minor quam DG, i. e. prout medium harmonicum inter segmenta DK, DL tangentibus abscissa majus est vel minus medio harmonico inter segmenta DE, DI, ad curvam terminata.

§ 111. *Corol. 2.* Si angulus EDT bifecetur recta DA, erit $QV = DV$, et $2HG \times DR = DV^2 = DG \times DH$, adeoque HG ad DG ut DH ad 2DR.

§ 112. *Corol. 3.* Revolvatur recta DA circa polum D, manente rectâ DE, et HG, differentia mediorum harmonicorum DH et DG, augebitur vel minuetur in duplicata ratione rectæ VQ. Quippe ob datam chordam circuli osculatorii DR, manet quantitas $\frac{QV^2}{HG}$ quæ æqualis est 2DR,

Fig. 54. § 113. *Corol.* 4. Si tangentium AK et BL altera, ut BL, fit rectæ DE parallela, ducantur GX et KZ parallelæ rectæ DT curvam in D contingenti, quæ ipsi AB occurrant in X, Z; eritque $\frac{GX \times KZ}{DG \times DK \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} = \frac{2}{DG} - \frac{1}{DK} = \frac{2DK - DG}{DG \times DK}$, adeoque $\frac{GX \times KZ}{DR} = 2DK - DG$, & proinde erit ut $2DK - DG$ ad KZ ita GX ad DR. Si tangens AK evadat quoque parallela rectæ DE (quod in his figuris contingere potest) erit DG ad GX ut GX ad 2DR; nam in hoc casu $\frac{GX^2}{DG^2 \times DR} = \frac{2}{DG}$, adeoque $GX^2 = DG \times 2DR$.

§ 114. *Corol.* 5. Si recta DE fit asymptoto parallela, adeoque curvæ occurrat in uno puncto E præter ipsum D, fitque simul tangens BL asymptoto parallela, ducatur EY parallela tangenti DT quæ occurrat rectæ DA in Y, eritque KE ad KZ ut EY ad DR*.

§ 115. *Corol.* 6. Si fit D punctum flexus contrarii, coincidat punctum H cum G, evanescente linea HG, adeoque evadit DR infinite magna, i. e. curvatura minor est ad punctum flexus contrarii quam in circulo quantumvis magno; ut alibi quoque ostendimus, tractatus de fluxionibus, Art. 378.

Fig. 55. § 116. *Corol.* 7. Sit V punctum duplex, DA asymptoto parallela, & occurrant rectæ VQ, KZ, tangenti DT parallelæ rectæ DA in Q et Z, atque occurrat DV asymptoto in L, fitque DH medium harmonicum inter DK et DL, eritque $2DH - DG$ ad KZ ut DL ad DK,

* Supple figuram.

atque

atque $VH : HN :: VQ : DR$. Si recta DA bifecet angulum TDV , erit $DR : DV :: DH : 2VH$.

§ 117. PROP. XXIV. Sit D punctum quodvis lineæ tertii ordinis, occurrat tangens ad D curvæ in I , fitque DS diameter circuli osculatorii, quæ curvæ occurrat in A et B ; unde rectæ ductæ curvam contingentes fecent DI in K et L ; fit DH medium harmonicum inter DK et DL , & sumatur DV ad DI ut DH ad differentiam rectarum $2DI$ et DH ; eritque variatio curvaturæ inverse ut rectangulum $SD \times DV$; & junctâ VS , variatio radii curvaturæ ut tangens anguli DVS . Fig. 56.

Nam per Theor. III. (Art. 17.) variatio curvaturæ est ut $\frac{1}{DS} \times \frac{1}{DK} + \frac{1}{DL} - \frac{1}{DI} = \frac{1}{DS} \times \frac{2}{DH} - \frac{1}{DI} = \frac{1}{DS} \times \frac{2DI - DH}{DH \times DI} = \frac{1}{DS \times DV}$. Variatio autem radii osculatorii est ut $\frac{DS}{DV}$, adeoque ut tangens anguli DVS , per Art. 18. parabola autem quæ eandem habeat curvaturam & eandem variationem curvaturæ cum linea proposita, determinatur ut in Art. 19.

§ 118. *Corol.* Si tangens BL fit tangenti ad D parallela, erit DV ad DI ut DK ad IK ; & si utraque tangentium AK , BL , fiat parallela ipsi DT , erit $DV = DI$, adeoque variatio curvaturæ inverse ut $DS \times DI$. Quod si in hoc casu fit DT parallela asymptoto curvæ, Fig. 57.
Fig. 58.
 evanescet variatio curvaturæ. Quemadmodum igitur evanescit variatio curvaturæ in verticibus axium sectionum conicarum; ea similiter evanescit in verticibus diame-

diametrorum linearum tertii ordinis quæ ad rectos angulos ordinatim applicatas bifecant.

Fig. 59. *Schol.* Sunt autem alia plurima theoremata de tangentibus & curvatura linearum tertii ordinis. Sint, *ex. gr.* F et G duo puncta lineæ tertii ordinis unde tangentes ductæ concurrunt in curva in A. Producaturs FG donec curvæ occurrat in H. Sit TAC tangens ad punctum A, & constituatur angulus $FAN = GAT$ ad contrarias partes rectarum FA, GA, secetque AN rectam FG in N. Et si circuli osculatorii occurrunt rectæ FG in B et *b*, erit GB ad F*b* ut rectangulum NFH ad NGH. Sit enim puncta *a* ipsi A quamproximum, & puncta *f, g, b*, ipsis F, G, H, quamproxima, eritque $AFa : FGf :: GF : FB$. $FGf (= HGb) : HFb :: FH : GH$. $HFb (= GFg) : AGa :: bG : GF$; unde $AFa : AGa :: FH \times bG : FB \times GH :: GN : FN$; unde $FB : Gb :: NFH : NGH$. *Sed de his satis.*

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