A TREATISE

ON

HYDRAULICS.

BY

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PREFACE.

THE present treatise is the outcome of lectures delivered in McGill University during the last ten or twelve years, and although intended primarily for the use and convenience of the student of hydraulics, it is hoped that it may also prove acceptable to the engineer in general practice.

In order to render the treatment of the subject more complete, free reference has been made to standard authors on the subject. The examples introduced to illustrate the text have also been selected in part from the works of such well-known writers as Weisbach, Osborne Reynolds, and Cotterill, but the greater number are such as have occurred in the course of the author's own experience. The tables of coefficients of discharge have been prepared from the results of experiments carried out in the Hydraulic Laboratory of the University. These experiments are still being continued and may probably form the subject of a special paper.

The author desires to acknowledge many suggestions offered by Professor Bamford, and to express his deep obligation to Professor Chandler for much labor and time given to the revision of proof sheets.

HENRY T. BOVEY.

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HYDRAULICS.

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CHAPTER I.

FLOW THROUGH ORIFICES, OVER WEIRS, ETC.

1. **Fluid Motion.**—The term "hydraulics," as its derivation (ὕδωρ, water; αὐλός, a tube or pipe) indicates, was primarily applied to the conveyance of water in a tube or pipe, but its meaning now embraces the experimental theory of the motion of fluids.

The motion of a fluid is said to be *steady* or *permanent* when the molecules successively arriving at any given point are animated with the same velocity, are subjected to the same pressure, and are the same in density. As soon as the motion of a stream becomes steady a *permanent régime* is said to be established, and hydraulic investigations are usually made on the hypothesis of a permanent régime. With such an hypothesis any portion of the fluid mass which leaves a given region is replaced by a like portion under conditions which are identically the same.

The terms "steady motion" and "permanent régime" are often considered to be synonymous.

The general problem of flow is the determination of the relation which exists at any point between the density, pressure, and velocity of the molecules which successively pass that point.

The actual motion of a fluid is exceedingly complex, and in order to simplify the investigations various assumptions are made as to the nature of the flow.

2. (*a*) **Stream-line Motion.**—The molecules may be regarded as flowing along definite paths, and a succession of such molecules will form a continuous fluid rope which is termed an *elementary stream* or a fluid filament, or, if the motion is steady and the paths therefore fixed in space, a *stream-line.*

Experiment shows that the velocity of flow in any cross-section varies from point to point, and hence it is often assumed that the section is made up of an infinite number of indefinitely small areas, each area being the section of a fluid filament.

(*b*) **Motion in Plane Layers.**—In this motion it is assumed that the molecules which at any given moment are found in a plane layer will remain in a plane layer after they have moved into any new position.

(*c*) **Laminar Motion.**—On this hypothesis the stream is supposed to consist of an infinite number of indefinitely thin layers. The variation in velocity from point to point of a cross-section may then be allowed for by giving the several layers different velocities based upon the law of fluid resistance between consecutive layers.

3. **Density; Compressibility; Head; Continuity.**

The weight of ice	per cubic foot at 23° F.	is 57.2 lbs.;
" " " fresh water	" " " " 39.2° F.	is 62.425 lbs.;
" " " salt	" " " " " 53° F.	is 64 lbs.;
" " " fresh	" " " " " 53° F.	is 62.4 lbs.,
		or 1000 kilog. per cubic metre.

The following table from the article on "Hydromechanics" in the Encyc. Brit. gives the density of water at different temperatures:

Temperature. Cent.	Temperature. Fahr.	Density.	Weight in Lbs. per Cu. Ft.	Temperature. Cent.	Temperature. Fahr.	Density.	Weight in Lbs. per Cu. Ft.
0	32	.999884	62.417	20	68	.998272	62.316
1	33.8	.999941	62.420	22	71.6	.997839	62.289
2	35.6	.999982	62.423	24	75.2	.997380	62.261
3	37.4	1.000004	62.424	26	78.8	.996879	62.229
4	39.2	1.000013	62.425	28	82.4	.996344	62.196
5	41	1.000003	62.424	30	86	.995778	62.161
6	42.8	.999983	62.423	35	95	.994690	62.093
7	44.6	.999946	62.421	40	104	.992360	61.947
8	46.4	.999899	62.418	45	113	.990380	61.823
9	48.2	.999837	62.414	50	122	.988210	61.688
10	50	.999760	62.409	55	131	.985830	61.540
11	51.8	.999668	62.403	60	140	.983390	61.387
12	53.6	.999562	62.397	65	149	.980750	61.222
13	55.4	.999443	62.389	70	158	.977950	61.048
14	57.2	.999312	62.381	75	167	.974990	60.863
15	59	.999173	62.373	80	176	.971950	60.674
16	60.8	.999015	62.363	85	185	.968800	60.477
17	62.6	.998854	62.353	90	194	.965570	60.275
18	64.4	.998667	62.341	100	212	.958660	59.844
19	66.2	.998473	62.329				

Fluids are sensibly compressed under heavy pressures, and the compression is proportional to the pressure up to about 1000 lbs. (65 atmospheres) per square inch. Grassi's experiments indicate that the compressibility of water diminishes as the temperature increases.

TABLE OF ELASTICITY OF VOLUME OF LIQUIDS.

(Reduced from Grassi's results.)

Liquid.	Elasticity of Volume.	Temperature.
Mercury ...	717,000,000	0° C.
Water. ...	42,000,000	0° C.
	45,900,000	18° C.
Sea-water..	52,900,000	
Ether	16,280,000	0° C.
	15,000,000	14° C.
Alcohol....	25,470,000	7.3° C.
	23,380,000	13.1° C.
Oil	44,090,000	

N. B.—The value for mercury is probably erroneous.

If a volume V of a fluid is compressed by an amount ΔV under an increase Δp of the pressure, then

mgh_v $\rho Ahg = A\rho g r$

mg $m = \rho Ah$

$\dfrac{\Delta V}{V}$ is called the cubical compression, and

$V\dfrac{\Delta p}{\Delta V}$ is termed the elasticity of volume. This is sensibly constant.

The vertical distance between the free surface of a mass of water and any datum plane is called the *head* with respect to that plane. If the water extends down to the level of the plane, a pressure p is produced at that level, and the value of p, so long as the water is at rest, is given by the equation

$$\frac{p}{w} = h + \frac{p_0}{w},$$

$p = \rho h + p_0$

$\frac{p}{\rho g} = h + \frac{p_0}{\rho g}$

w being the specific weight of the water and p_0 the pressure at the free surface. Thus the pressure may be measured in terms of the head, and hence the expression "head due to pressure or pressure head."

The mean value of the atmospheric pressure is 14.7 lbs. per square inch.

A head of	is equivalent to a pressure of
2.3 ft. of fresh water	1 lb. per sq. in.
2.25 ft. of salt water	1 lb. per sq. in.
About 34 ft. of fresh water	14.7 lbs. per sq. in.
" 33 ft. of salt "	14.7 lbs. per sq. in.

A head of water is a source of energy. A volume of water descending from an upper to a lower level may be employed to drive a machine which receives energy from the water and utilizes it again in overcoming the resistances of other machines doing useful work.

Let Q cu. ft. of water per second fall through a vertical distance of h ft. Then the total power of the fall $= wQh$ ft.-lbs. $= \dfrac{wQh}{550}$ h. p., w being the weight of the water in pounds per cubic foot.

Let K be the proportion of the total power which is absorbed in overcoming frictional and other resistances. Then

Qwh Energy
K Qwh) Energy loss
= Qwh(1-K)

the *effective* power of the fall $= wQh\,(1 - K)$, and the efficiency is $1 - K$.

Imagine a bounding surface enclosing a space of invariable volume in the midst of a moving mass of fluid. The principle of continuity affirms that in any interval of time the flow into the space must be equal to the outflow during the same interval. Giving the inflow a positive and the outflow a negative sign, the principle may be expressed symbolically by

$$\Sigma Q = 0.$$

The continuity of a mass of water will be preserved so long as the pressure exceeds the tension of the air held in solution. It is on account of the pressure of this air that pumps cannot draw water to the full height of the water barometer, or about 34 ft.

Generally speaking, the pressure at every point of a continuous fluid must be positive. A negative pressure is equivalent to a tension which will tend to break up the continuity presupposed by the formulæ; and should negative pressures result from the calculations, the inference would be that the latter are based upon insufficient hypotheses.

The pressure in water flowing through the air cannot at any point fall below the atmospheric pressure. There are cases, however, as in water flowing through a closed pipe (Art. 3, Chap. III), in which the pressure may fall below this limit and become almost nil. But there is then a danger of the air held in solution being set free, thus tending to interrupt the continuity of the flow, which may be wholly stopped if the air is present in sufficient volume.

Consider a length of a canal or stream bounded by two normal sections of areas A_1, A_2, and let v_1, v_2 be the mean normal velocities of flow across these sections. Then by the principle of continuity

$$A_1v_1 = Q = A_2v_2,$$

and the velocities are inversely as the sectional areas.

Again, assume that a moving mass of fluid consists of an infinite number of stream-lines, and consider a portion of the mass bounded by stream-lines and by two planes of areas A_1, A_2 at right angles to the direction of flow. If v_1, v_2 are the mean velocities of flow across the planes,

$$v_1A_1 = Q = v_2A_2 \text{ if the fluid is incompressible.}$$

Assuming that the fluid is compressible, and that the mean specific weights at the two planes are w_1 and w_2, then the *weight* of fluid flowing across A_1 is equal to the *weight* which flows across A_2, since the weight of fluid *between* the two planes remains constant. Hence

$$w_1A_1v_1 = w_2A_2v_2.$$

4. **Bernouilli's Theorem.**—This theorem is based on the following assumptions:

(1) That the fluid mass under consideration is a *steadily* moving stream made up of an infinite number of stream-lines whose paths in space are necessarily fixed.

(2) That the velocities of consecutive stream-lines are not widely different, so that viscosity, or the frictional resistance between the stream-lines, is sufficiently small to be disregarded.

(3) That the fluid is incompressible, so that there can be no *internal work* due to a change of volume.

In any given stream-line let a portion AB, Fig. 1, of the fluid move into the position $A'B'$ in t seconds.

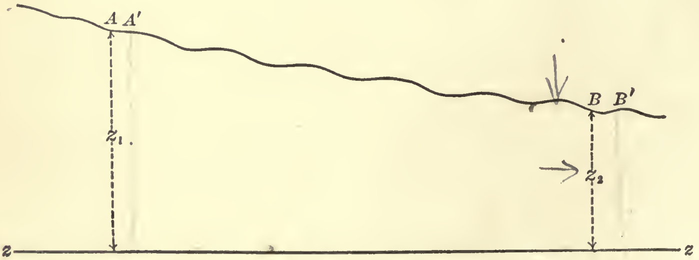


FIG. 1.

Let a_1, p_1, v_1, z_1 be the normal sectional area, the intensity of the pressure, the velocity of flow, and the elevation above a datum plane ZZ of the fluid at A. Let a_2, p_2, v_2, z_2 denote similar quantities at B.

Since the internal work is nil, the work done by *external* forces must be equivalent to the change of kinetic energy.

Now the external work

$=$ the work done by gravity $+$ the work done by pressure.

But when the fluid AB passes into the position $A'B'$, the work done by gravity is equivalent to the work done in the transference of the portion BB', and therefore, t beng the time.

$$\textit{the work done by gravity} = wa_1 \cdot AA' \cdot z_1 - wa_2 \cdot BB' \cdot z_2$$
$$= wa_1 \cdot v_1 t \cdot z_1 - wa_2 \cdot v_2 t \cdot z_2$$
$$= wQt\,(z_1 - z_2),$$

since $AA' = v_1 t$, $BB' = v_2 t$, and $a_1 v_1 = Q = a_2 v_2$.

Again, *the work done by the pressures* on the ends A and B

$$= p_1 a_1 v_1 t - p_2 a_2 v_2 t$$
$$= Qt\,(p_1 - p_2).$$

The work done by the pressure on the surface of the stream-line between A and B is nil, since the pressure is at every point normal to the direction of motion.

The change of kinetic energy

$=$ kinetic energy of $A'B'$ — kinetic energy of AB

$=$ kinetic energy of BB' — kinetic energy of AA',

since the motion is steady, and there is therefore no change in the kinetic energy of the intermediate portion $A'B$. Thus,

$$\textit{the change of kinetic energy} = \frac{w}{g} a_2 BB' \frac{v_2^2}{2} - \frac{w}{g} a_1 AA' \frac{v_1^2}{2}$$
$$= \frac{w}{g} a_2 v_2 t \frac{v_2^2}{2} - \frac{w}{g} a_1 v_1 t \frac{v_1^2}{2}$$
$$= \frac{w}{g} Qt \left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right).$$

Hence, equating the external work and the change of kinetic energy,

$$wQt\,(z_1 - z_2) + Qt\,(p_1 - p_2) = \frac{w}{g} Qt \left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right),$$

which may be written in the form

$$wz_1 + p_1 + \frac{w}{g}\frac{v_1^2}{2} = wz_2 + p_2 + \frac{w}{g}\frac{v_2^2}{2}, \quad . \quad . \quad . \quad (1)$$

or

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But A and B are arbitrarily chosen points, and therefore, at any point of a stream-line, the motion being steady and the viscosity nil, the gradual interchange of the energies due to head, pressure, and velocity is expressed by the equation

$$wz + p + \frac{w}{g}\frac{v^2}{2} = wH, \text{ a constant}; \quad \frac{\text{ft lbs}}{\text{cu ft}} \quad . \quad (3)$$

or

$$z + \frac{p}{w} + \frac{v^2}{2g} = H, \text{ a constant}; \quad \frac{\text{ft lbs}}{\text{lb fluid}} \quad . \quad (4)$$

z being the elevation of the point above the datum line, p the pressure at the point, w the specific weight, and v the velocity of flow. This is Bernouilli's theorem.

Thus the total constant energy of wH ft.-lbs. per cubic foot of fluid, or H ft.-lbs. per pound of fluid, is distributed uniformly along a stream-line, wH being made up of wz ft.-lbs. due to head, p ft.-lbs. due to pressure, $\frac{w}{g}\frac{v^2}{2}$ ft.-lbs. due to velocity, and H being made up of z ft.-lbs. due to head, $\frac{p}{w}$ ft.-lbs. due to pressure, and $\frac{v^2}{2g}$ ft.-lbs. due to velocity.

Assuming that

(*a*) the motion is steady,
(*b*) the frictional resistance may be disregarded,
(*c*) the fluid is incompressible,

Bernouilli's theorem may be applied to currents of finite size at any normal section, if the stream-lines across that section are sensibly rectilinear and parallel. There is then no interior work due to a change of volume, and the distribution of the pressure in the section under consideration will be the same as if the fluid were at rest, that is, in accordance with the hydrostatic law. This is also true whether the flow takes place under atmospheric pressure only, or whether the fluid is wholly or partially confined by solid boundaries, as in pipes and canals, or whether the flow is through another medium already occupied by a volume of the fluid at rest or moving steadily in a parallel direction. In the last case there must necessarily be a lateral connection between the two fluids, but the pressure over the section must follow the hydrostatic law throughout the separate fluids, and there can be no sudden change of pressure at the surface of separation, as this would lead to an interruption of the continuity.

The hypotheses, however, upon which these results are based are never exactly realized in actual experience, and the results can only be regarded as tentative. Further, they can only apply to an indefinitely short length of the current, as the viscosity, which is proportional to the surface of contact, would otherwise become too great to be disregarded.

5. **Applications.**—If a glass tube, open at both ends, and called a piezometer (*πιέζειν*, to press; *μέτρον*, a measure) is inserted vertically in the current, Fig. 2, at a point N, z ft. above the point O in the datum line, the water will rise in the tube to a height MN dependent upon the pressure at N. The effect of the eddy motion produced at N by obstructing the stream-lines may be diminished by making this end of the tube parallel to the direction of flow. Neglecting altogether the effect of the eddies, and taking p to be the intensity of the pressure at N, and p_0 the intensity of the atmospheric pressure, then,

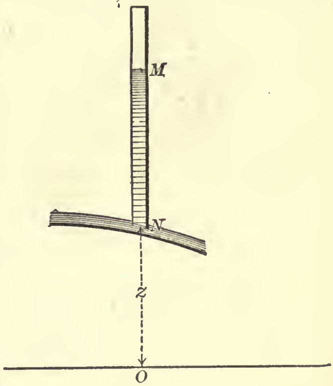


FIG. 2.

$$\frac{p}{w} = MN + \frac{p_0}{w},$$

and therefore

$$\begin{aligned} z + \frac{p}{w} &= z + MN + \frac{p_0}{w} \\ &= ON + MN + \frac{p_0}{w} \\ &= OM + \frac{p_0}{w}. \qquad \cdots\cdots\cdots \quad (5) \end{aligned}$$

The locus of all such points as M is often designated "the line of hydraulic gradient," or the "virtual slope," terms also used when friction is taken into account.

Let the two piezometers AB, CD, Fig. 3, be inserted in the current at any two points B and D, z_1 ft., and z_2 ft. respectively above the points E and F in the datum line.

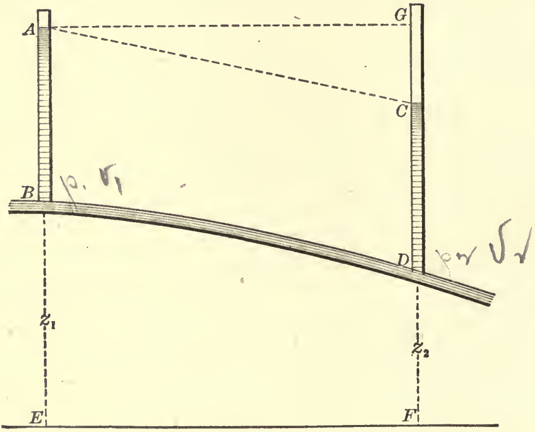


FIG. 3.

Let p_1 be the intensity of the pressure at B in pounds per square foot, p_2 that at D, and let the water rise in these tubes to the heights BA, DC. Then

$$\frac{p_0}{w} + AE = z_1 + \frac{p_1}{w}, \quad \text{and} \quad \frac{p_0}{w} + CF = z_2 + \frac{p_2}{w},$$

and therefore

$$\left(z_1 + \frac{p_1}{w}\right) - \left(z_2 + \frac{p_2}{w}\right) = AE - CF = CG, \quad . \quad . \quad (6)$$

the line AG being parallel to the datum line.

Thus, $\left(z_1 + \frac{p_1}{w}\right) - \left(z_2 + \frac{p_2}{w}\right)$ is equal to the fall of the free surface level between the points B and D.

Let v_1, v_2 be the velocities of flow at B and D. Then by Bernouilli's theorem

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g}, \quad . \quad . \quad . \quad . \quad (7)$$

and therefore the fall of free surface level between B and D

$$= \left(z_1 + \frac{p_1}{w}\right) - \left(z_2 + \frac{p_2}{w}\right) = \frac{v_2^2 - v_1^2}{2g}.$$

Equation (7) may also be written in the form

$$\frac{v_2^2}{2g} = \frac{v_1^2}{2g} + \left(z_1 + \frac{p_1}{w}\right) - \left(z_2 + \frac{p_2}{w}\right) = \frac{v_1^2}{2g} + CG, \quad . \quad (8)$$

so that the velocity at D is equal to that acquired by a body with an initial velocity v_1 falling freely through the vertical distance CG.

Froude illustrated Bernouilli's theorem experimentally by means of a tube of varying section, Fig. 4, conveying a current

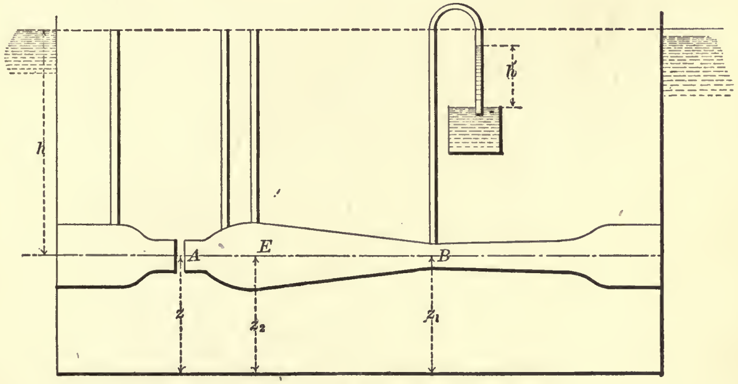


FIG. 4.

between two cisterns. The pressure at different points along the tube is measured by piezometers, and it is found that the water stands higher and the pressure is therefore greater, where the cross-section is larger and the velocity consequently less. If the section of the throat at A is such that the velocity is that acquired by a body falling freely through the vertical distance h between A and the surface level of the water in the cistern, and if p be the pressure at A, and z the elevation of A above datum, then, neglecting friction,

$$z + \frac{p}{w} + \frac{v^2}{2g} = H = z + h + \frac{p_0}{w}.$$

But $v^2 = 2gh$, and therefore $p = p_0$, so that the pressure at A is that due to atmospheric pressure only. Thus, a portion of the pipe in the neighborhood of A may be removed, as in the throat of the injector.

Again, let the cross-section in the throat at B be less than that at A. The pressure at B will be less than the atmospheric pressure, and a column of water will be lifted up in the curved piezometer to a height h'.

Let a_1, z_1, p_1, v_1 be the sectional area, elevation above datum, pressure, and velocity at B.

Let a_2, z_2, p_2, v_2 be similar symbols at E.

Then

$$z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g} = z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_1 + \frac{p_0}{w} - h' + \frac{v_1^2}{2g}. \quad (9)$$

Put $H_2 = z_2 + \frac{p_2}{w}$, the height above datum to which the water is observed to rise in the piezometer inserted at E, and also let $H_1 = z_1 + \frac{p_0}{w} - h'$. Then

$$H_2 - H_1 = \frac{1}{2g}(v_1^2 - v_2^2) = \frac{v_1^2}{2g}\frac{a_2^2 - a_1^2}{a_2^2},$$

since $a_2 v_2 = a_1 v_1$, a_2 being the sectional area at E. Therefore

$$v_1^2 = \frac{2ga_2^2}{a_2^2 - a_1^2}(H_2 - H_1),$$

an equation giving the *theoretical* velocity of flow at the throat B. Hence the *theoretical* quantity of flow across the section at B is

$$a_1 v_1 = \frac{a_1 a_2}{\sqrt{a_2^2 - a_1^2}} \sqrt{2g(H_2 - H_1)}. \quad . \quad . \quad . \quad (10)$$

This is the principle of the Venturi water-meter and also of the aspirator.

The actual quantity of flow is found by multiplying equation (10) by a coefficient C whose value is to be determined by experiment.

If the pressure at E is positive, then H_2 is merely the height to which the water is observed to rise in an ordinary piezometer inserted at E.

Again, Froude also points out that when any number of combinations of enlargements and contractions occur in a pipe, the pressures on the converging and diverging portions of the pipe will balance each other if the sectional areas and directions of the ends are the same.

6. Orifice in a Thin Plate.—If an opening is made in the wall or bottom of a tank containing water, the fluid particles

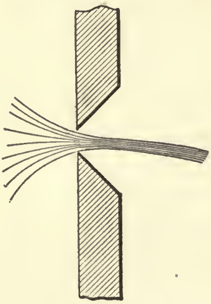


FIG. 5.

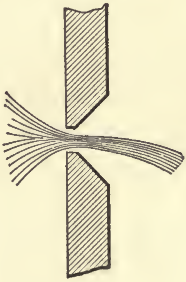


FIG. 6.

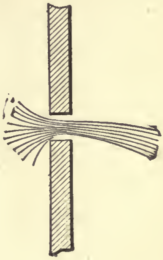


FIG. 7.

immediately move towards the opening, and arrive there with a velocity depending upon its depth below the free surface. The opening is termed an "orifice in a thin plate" when the water springs clear from the inner edge, and escapes without again touching the sides of the orifice. This occurs when the bounding surface is changed to a *sharp edge*, as in Fig. 5, and also when the ratio of the thickness of the bounding surface to the least transverse dimension of the orifice does not exceed a certain amount which is usually fixed at unity, as in Figs. 6 and 7.

Owing to the inertia acquired by the fluid filaments there will be no sudden change in their direction at the edge of the orifice, and they will continue to converge to a point a little in front of the orifice, where the jet is observed to contract to the smallest section. This portion of the jet is called the *vena contracta* or contracted vein, and the fluid filaments flow across the minimum section in sensibly parallel lines, so that here, if the motion is steady, Bernouilli's theorem is applicable.

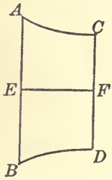


FIG. 8.

The dimensions of the contracted section and its distance from the orifice depend upon the form and dimensions of the orifice and upon the head of water over the orifice.

Let Fig. 8 represent the portion of the jet between a circular orifice of diameter AB and the contracted section of diameter CD, EF being the distance between AB and CD. Then, taking the *average* results of a number of observations, it is found that AB, CD and EF are in the ratios of 100 to 80 to 50.

Thus the areas of the contracted section and of the orifice are in the ratio of 16 to 25, and, generally speaking, this is assumed to be the ratio whatever may be the form of the orifice.

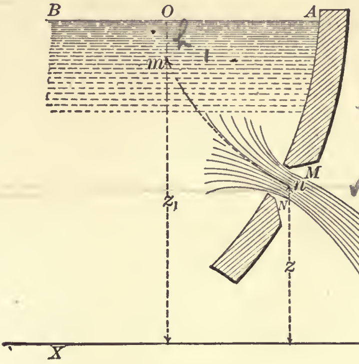
7. **Torricelli's Theorem.**—Let Fig. 9 represent a jet issuing from a thin-plate orifice in the side of a vessel containing water kept at a constant level AB.

Let XX be the datum line, MN the contracted section, and consider any stream-line mn, m being in a region where the velocity is sensibly zero, and n in the contracted section. Then by Bernouilli's theorem, the motion being steady,

$$z_1 + \frac{p_1}{w} + \frac{0}{2g} = z + \frac{p}{w} + \frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad (1)$$

p, p_1 being the pressures at n and m, and z, z_1 their elevations above datum. Hence

$$\frac{v^2}{2g} = z_1 - z + \frac{p_1 - p}{w}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$



h

$h_1 + 0 = h + v$

$v^2/2g = h - h$

$h + \frac{v^2}{2g} = h_1$

FIG. 9.

If the flow is into the atmosphere,

$$p = \text{the atmospheric pressure} = p_0, \text{ and}$$
$$p_1 = w.Om + p_0,$$

O being the point in which the vertical through m intersects the free surface. Thus,

$$\frac{v^2}{2g} = z_1 - z + Om = h, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

h being the depth of n below the free surface.

The result given by equation (3) was first deduced by Torricelli.

The depth below the free surface is very nearly the same for all points of the contracted vein, and the value of v as given by (3) is taken to be the theoretical mean velocity of flow across the contracted section.

Equation (3) is equivalent to the statement that when the orifice is opened the hydrostatic energy of the water, viz., h ft.-lbs. per pound, is converted into the kinetic energy of $\frac{v^2}{2g}$

ft.-lbs. per pound. Thus, if the jet is directed vertically upwards, it will very nearly rise to the level of the free surface, and would reach that level were it not for air resistance, or for viscosity, or for friction against the sides of the orifice, or for a combination of these retarding causes.

If the jet issues in any other direction, it describes a parabolic arc of which the directrix lies in the free surface.

Let OTV, Fig. 10, be such a jet, its direction at the orifice

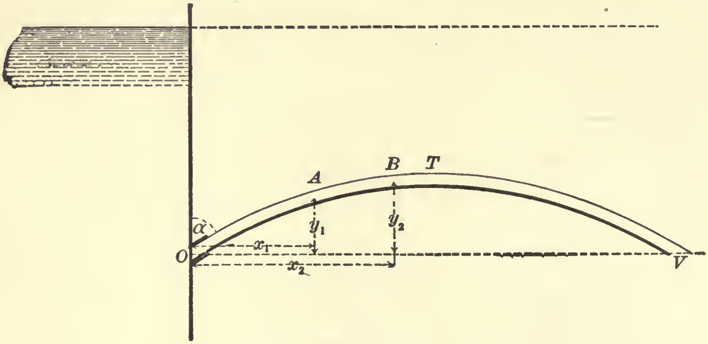


FIG. 10.

at O making an angle α with the vertical. With a properly formed orifice a greater or less length of the jet will have the appearance of a glass rod, and if this portion were suddenly solidified and supported at the ends, it would stand as an arch without any shearing stress in normal sections.

Again, the horizontal component of the velocity of flow at any point of the jet is constant ($= v \sin \alpha$), so that, for the unbroken portion of the jet, equidistant vertical planes will intercept equal amounts of water, and the height of the C. G. of the jet above the horizontal line OV will be two thirds of the height of the jet.

8. Efflux through an Orifice in the Bottom or in the Side of a Vessel in Motion.—If a vessel containing water z ft. deep ascend or descend vertically with an acceleration f, the pressure p at the bottom is given by the equation

$$\pm \frac{w}{g} zf = p - p_0 - wz,$$

p_0 being the atmospheric pressure. Therefore

$$\frac{p - p_0}{w} = z\left(1 \pm \frac{f}{g}\right).$$

If now an orifice is opened at the bottom, the velocity of efflux v is still taken as due to the head of the pressure p, and therefore by Torricelli's Theorem

$$\frac{v^2}{2g} = z\left(1 \pm \frac{f}{g}\right).$$

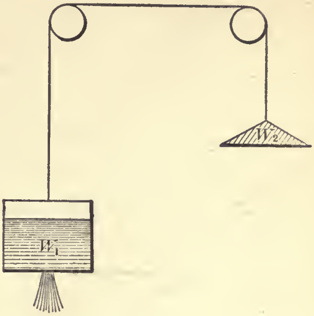


FIG. 11.

Let W_1 be the weight of the vessel and water, and let the vessel be connected with a counterpoise of weight W_2 by meansof a rope passing over a pulley. Then by Newton's second law of motion, and neglecting pulley friction,

$$\frac{f}{g} = \frac{\pm T \mp W_1}{W_1} = \frac{\pm W_2 \mp T}{W_2} = \frac{\pm W_2 \mp W_1}{W_2 + W_1},$$

T being the tension of the rope.

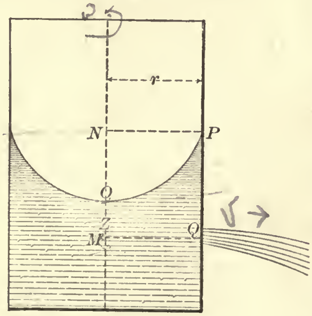


FIG. 12.

Next let a cylindrical vessel, Fig. 12, of radius r and containing water, rotate with an angular velocity ω about its axis. The surface of the water assumes the form of a paraboloid of which the latus rectum is $\frac{2g}{\omega^2}$. If an orifice is made at Q in the side of the vessel, the water will flow out with a velocity v due to the head of pressure at the orifice. This head is PQ, and

$$PQ = ON \pm z = \frac{\omega^2 r^2}{2g} \pm z,$$

z being the vertical distance OM between the orifice and the vertex of the paraboloid. Hence by Torricelli's theorem

$$\frac{v^2}{2g} = \frac{\omega^2 r^2}{2g} \pm z,$$

or

$$v^2 = \omega^2 r^2 \pm 2gz.$$

9. Application to the Flow through a Frictionless Pipe of Gradually Changing Section (Fig. 13).—Let the pipe be supplied from a mass of water of which the free surface is H ft. above datum.

Let a_1, p_1, v_1 be the sectional area, pressure, and velocity of flow at any point A, z_1 ft. above datum and h_1 ft. below the free surface.

Let a_2, p_2, v_2 be similar symbols for a second point B, z_2 ft. above datum and h_2 ft. below the free surface.

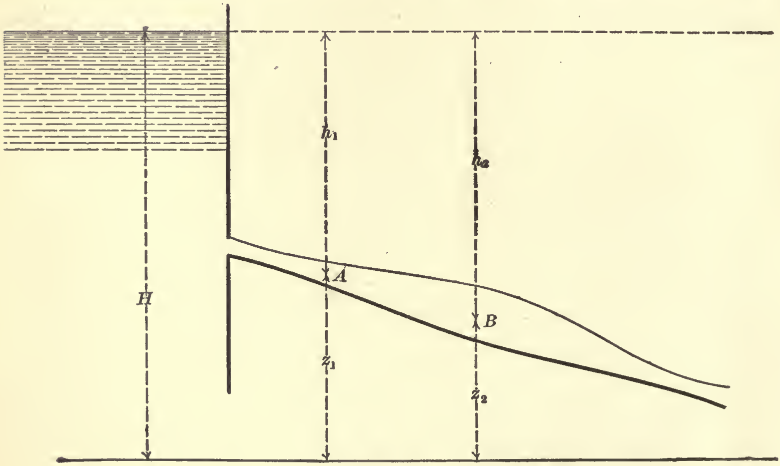


FIG. 13.

Then by the condition of continuity

$$a_1 v_1 = a_2 v_2,$$

and by Torricelli's theorem

$$\frac{v_1^2}{2g} = h_1 + \frac{p_0 - p_1}{w},$$

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho} + z_2$$

$=$

and

$$\frac{v_2^2}{2g} = h_2 + \frac{p_0 - p_2}{w}.$$

Hence

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \cancel{z_1 + h_1 + \frac{p_0}{w}} = H + \frac{p_0}{w}$$

$$= \cancel{z_2 + h_2 + \frac{p_0}{w}} = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2,$$

so that Bernouilli's theorem, viz.,

$$\frac{v^2}{2g} + \frac{p}{w} + z = H + \frac{p_0}{w} = \text{a constant},$$

holds true for the assumed conditions.

10. Hydraulic Resistances—(*a*) *Coefficient of Velocity.*—In reality, the velocity v at the vena contractâ is a little less than $\sqrt{2gh}$ (Art. 7, eq. 3) and the ratio of v to $\sqrt{2gh}$ is called the coefficient of velocity, and may be denoted by c_v, so that $v = c_v \sqrt{2gh}$. C_v

Again, the equations for the velocity of discharge in the case of moving vessels now become

$$\frac{v^2}{2g} = c_v^2 \left\{ z \left(1 \pm \frac{f}{g} \right) \right\},$$

and

$$\frac{v^2}{2g} = c_v^2 \left(\frac{\omega^2 r^2}{2g} \pm z \right).$$

A mean value of c_v for well-formed simple orifices is .974.

An easy method of determining the value of c_v, experimentally, may be indicated by reference to the jet represented in Fig. 10, p. 16.

Measure the vertical and horizontal distances from the orifice of any two points A, B in the jet.

Let y_1, x_1 denote the co-ordinates of A.

Let y_2, x_2 denote the co-ordinates of B.

Then if t_1 is the time occupied by a fluid particle in moving from the orifice to A, and t_2 the time from the orifice to B,

$$x_1 = v \sin \alpha \, . \, t_1 ; \quad y_1 = v \cos \alpha \, . \, t_1 - \frac{1}{2} g t_1^2 ;$$

$$x_2 = v \sin \alpha \, . \, t_2 ; \quad y_2 = v \cos \alpha \, . \, t_2 - \frac{1}{2} g t_2^2 .$$

Hence

$$y_1 = x_1 \cot \alpha - \frac{g}{2} \frac{x_1^2}{v^2 \sin^2 \alpha}, \quad = \frac{v \cos \alpha t_1 x}{v \sin \alpha t_1} - \frac{1}{2} g t_1^2 \times \frac{x_1}{v^2 \ldots}$$

and

$$y_2 = x_2 \cot \alpha - \frac{g}{2} \frac{x_2^2}{v^2 \sin^2 \alpha}.$$

By means of the two last equations

$$\cot \alpha = \frac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{g}{2} \frac{x_1^2}{\sin^2 \alpha \, (x_1 \cot \alpha - y_1)} = v^2 = c_v^2 \;\; 2gh;$$

so that

$$c_v^2 = \frac{x_1^2}{4h \sin^2 \alpha \, (x_1 \cot \alpha - y_1)}; \quad . \quad . \quad . \quad . \quad (2)$$

and since the values of x_1, y_1, x_2, y_2 are known, equation 1 will give the value of α, and equation 2 the value of c_v.

Note.—If the jet issues from the orifice horizontally, $\alpha = 90°$, and the last equation becomes

$$c_v^2 = \frac{x_1^2}{4hy_1},$$

so that the position of *one* point only relatively to the orifice need be observed.

(*b*) *Coefficient of Resistance.*—Let h_v be the head required to produce the velocity v. Let h_r be the head required to overcome the frictional resistance. Then

$$h, \text{ the total head, } = h_v + h_r = h_v (1 + c_r),$$

where $h_r = c_r h_v$.

c_r is termed the coefficient of resistance, and is approximately constant for varying heads with simple sharp-edged orifices. Again,

$$c_v^2 h = \frac{v^2}{2g} = h_v.$$

Hence

$$h = c_v^2 h(1 + c_r),$$

and therefore

$$\frac{1}{c_v^2} = 1 + c_r;$$

so that c_r can be found when c_v is known, and *vice versa*.

(*c*) *Coefficient of Contraction.*—The ratio of the area a of the vena contracta to the area A of the orifice is called the coefficient of contraction, and may be denoted by c_c. C_c

The value of c_c must be determined in each case, but in sharp-edged orifices an average value of c_c, as already pointed out, is $\frac{16}{25} = .64$. *Cæteris paribus*, c_c increases as the orifice area and the head diminish.

The following are some of the conditions which tend to modify the value of c_c:

(1) The contraction is *imperfect* and will be suppressed over the lower edge of a square orifice at the bottom of a vessel, and over a side as well if the orifice is in a corner. In fact, the contraction is more or less imperfect for any orifice within three diameters from the side or bottom of the vessel. Thus, the cross-section of the vena contracta is increased, and experiment shows that the discharge is also increased.

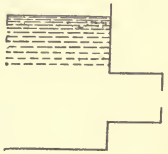


FIG. 14.

(2) c_c is increased or diminished according as the surface surrounding the orifice is convex or concave to the interior of the vessel.

(3) The contraction is imperfect and c_c is increased if the orifice is placed in a confined part of the vessel or if it approaches the orifice through a channel, as in Fig. 14, the velocity of the fluid filaments being thereby considerably increased.

(4) If the inner edge of an orifice is rounded, as shown by Figs. 15 and 16, the contraction is more or less imperfect.

FIG. 15.

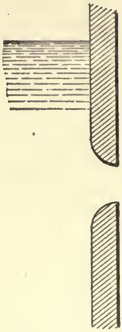


FIG. 16.

The value of c_c varies from .64 for a sharp-edged orifice to very nearly *unity* for a perfectly rounded orifice.

(5) The contraction is *incomplete* when a border or rim is placed round a part of the edge of the orifice, projecting inwards or outwards. According to Bidone,

$$c_c = .62\left(1 + .152\frac{n}{p}\right) \text{ for rectangular orifices,}$$

and

$$c_c = .62\left(1 + .128\frac{n}{p}\right) \text{ for circular orifices,}$$

n being the length of the edge of the orifice over which the border extends, and p the perimeter of the orifice.

(6) If the sides of the orifice are curved so as to form a bell-mouth of the proportions shown by Fig. 17, and corre-

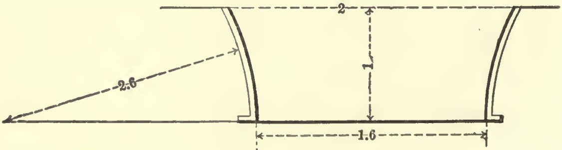


FIG. 17.

sponding approximately to the shape of the vena contracta, the whole of the contraction will take place within the bell-mouth, and c_c is unity if the area of the orifice is taken to be the area of the smaller end.

For such an orifice Weisbach gives the following table of values of c_v:

Head over Orifice in Feet.	c_v.
.66	.959
1.64	.967
11.48	.975
55.77	.994
337.93	.994

The dimensions of the jet at the contracted section or at any other point may be directly measured by means of set-screws of fine pitch, arranged as in Fig. 18. The screws are adjusted so as to touch the surface of the jet, and the distance between the screw-points is then measured.

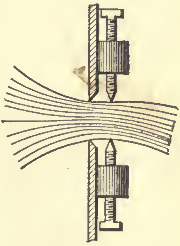


FIG. 18.

(*d*) *Coefficient of Discharge.*—If Q is the quantity of flow per second across the contracted section, then

$$Q = av = c_c A c_v \sqrt{2gh} = c_c c_v A \sqrt{2gh} = cA \sqrt{2gh},$$

where $c = c_c c_v$ is the coefficient of discharge, and is to be determined by experiment.

The values of c in thousandths for orifices of different forms, given in tables A and B, have been deduced by the author from an extended series of experiments carried out in the hydraulic laboratory, McGill University.

The experimental tank is about 30 ft. in height and its horizontal section is square, with an interior area of 25 sq. ft. The inside faces of the tank are plumb, and there are no projections to interfere with the stream-lines.

The letters T and S at the head of the columns respectively indicate that the orifice is in a plate of thickness .16 in., or is sharp-edged.

TABLE A.

OF VALUES OF c FOR ORIFICES OF .197 SQ. IN. IN AREA.

Form of Orifice.	Circular.		Equilateral Triangle with Horizontal Side uppermost.		Square with Vertical Sides.		Square with Diagonal Vertical.	Rectangle with Vertical Sides equal to Twice the Width.		Rectangle with Vertical Sides equal to One Half of the Width.	Rectangle with Vertical Sides equal to Four Times the Width.	Rectangle with Vertical Sides equal to One Fourth of the Width.
Head in Feet.	T	S	T	S	T	S	S	T	S	S	S	S
1	624	618	627	627	623	628	623	635	640	641	658	659
2	616	611	620	621	613	621	619	626	633	632	646	646
4	610	607	615	615	606	617	614	619	629	629	637	637
6	607	605	613	613	604	614	612	616	625	627	634	633
8	606	604	612	612	603	612	612	614	625	625	631	631
10	606	604	611	611	602	610	611	612	624	623	630	629
12	605	603	611	611	601	610	611	611	622	622	627	626
14	604	603	610	610	600	610	609	611	622	621	624	625
16	606	602	610	610	600	610	609	610	620	621	624	624
18	605	602	610	610	600	610	609	609	620	620	623	623
20	604	601	609	609	600	610	609	602	620	620	622	622

TABLE B.

OF VALUES OF c FOR FOUR ORIFICES OF .0625 SQ. IN. IN AREA, AND FOR ONE TRIANGULAR ORIFICE OF .05 SQ. IN. IN AREA.

Form of Orifice.	Circular.		Equilateral Triangle with Horizontal Base uppermost.		Square with Vertical Sides		Rectangle with Vertical Sides equal to Twice the Width.		Rectangle with Vertical Sides equal to Four Times the Width.	
Head in Feet.	T	S	T	S	T	S	T	S	T	S
1	678	620	657	631	643	627	662	640	688	671
2	618	613	646	623	631	621	643	629	655	657
4	610	605	628	616	620	615	631	620	642	643
6	607	601	628	613	615	612	627	616	634	636
8	606	601	621	610	612	609	624	613	631	632
10	604	600	618	608	613	608	621	613	629	629
12	603	598	617	607	611	606	621	611	626	627
14	602	598	617	607	610	606	620	610	623	625
16	602	598	616	606	609	606	619	609	622	625
18	601	597	615	605	607	605	618	608	622	623
20	601	597	615	605	607	604	618	608	621	622

T = thickness

S = sharp edged

The jet springs clear from the orifice in all cases represented in Tables A and B.

The following inferences may be drawn from an inspection of Tables A and B:

(1) The coefficient of discharge diminishes as the head increases, but at a diminishing rate.

(2) The coefficients for the thick-plate orifices are in all cases greater than the corresponding coefficients for sharp-edged orifices, excepting in the case of the longest rectangular orifice in Table B. Under a head of 1 ft. the coefficient of discharge for this orifice still exceeds that of the same orifice with sharp edge, but for heads exceeding 1 ft. the coefficient seems to be a little less, but is practically the same. It may be noted that the thickness of the plate is 2.56 times the width of the orifice, and the contraction for the thick-plate orifice is consequently increased.

(3) The coefficient for rectangular orifices seems to be practically the same whether the longest side is vertical or horizontal.

(4) The coefficient increases with the area of the orifice, excepting when the head is very small. The coefficient for orifices of small area then rapidly increases, as shown in Table B.

(5) With rectangular orifices the coefficient increases as the width of the orifice diminishes, i.e., as the orifice becomes more elongated.

The two last results are in accordance with similar results deduced by Weisbach, Buff, and others.

The coefficient of discharge is modified when the edges of the orifices are not sharp, but have a sensible thickness, and the formula giving the discharge may be written

$$Q = cA\sqrt{2gH},$$

H being the depth of the axis of the orifice below the free surface.

11. Miner's Inch.—The miner's inch is a term applied to the flow of water through a standard vertical aperture, one square inch in section, under an average head of $6\frac{1}{2}$ inches.

Taking $c = .62$,

$$\text{the flow} = Q = .62A\sqrt{2gh}$$

$$= .62 \times \frac{1}{144}\sqrt{2 \times 32.2 \times \frac{13}{24}}$$

$$= 1\tfrac{1}{2} \text{ cu. ft. per minute, approximately.}$$

The term is more or less indefinite, as the different companies in disposing of water to their customers do not always use the

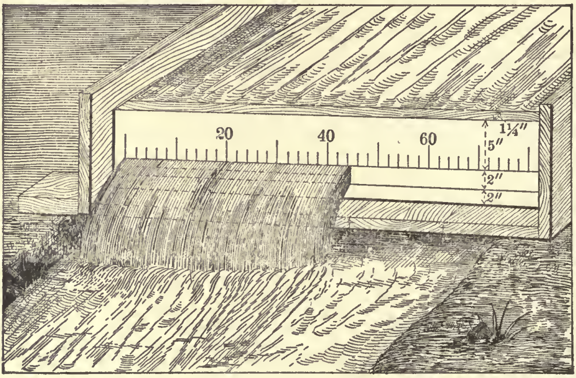


FIG. 19.

same head, and the flow is thus found to vary from 1.36 to 1.73 cu. ft. for each square inch of aperture.

The aperture is usually 2 in. deep and may be of any required width, Fig. 19. The upper and lower edges of the aperture are formed by $1\frac{1}{4}$-in. planks, the lower edge being 2 in. above the bottom of the channel, and the plank forming the upper edge being 5 to $5\frac{1}{2}$ in. deep, so that the head over the centre of the aperture is from 6 to $6\frac{1}{2}$ inches.

12. Energy and Momentum of the Jet.

$$\text{The energy of the jet} = wav\,\frac{v^2}{2g} \text{ ft.-lbs. per second}$$

$$= \frac{wav^3}{2g} \text{ ft.-lbs. per second}$$

$v = c\sqrt{2gh}$ $\frac{v^2}{2g} = c_v h$

$$= wavhc_v^2 \text{ ft.-lbs. per second}$$

$$= \frac{wavhc_v^2}{550} \text{ h. p. (horse-power)}$$

$$= \frac{pavc_v^2}{550} \text{ h. p.,}$$

$p\,(= wh)$ being the hydrostatic pressure due to the head h.

The momentum of the jet $= \dfrac{w}{g}av\,.\,v = wa\dfrac{v^2}{g} = 2wahc_v^2$

$$= 2pac_v^2,$$

and this is equal to the pressure in pounds produced by the jet against a fixed plane perpendicular to its direction. Neglecting c_v^2, the thrust is double the hydrostatic pressure due to the head h.

13. Inversion of the Jet.—The phenomenon of the inversion of the jet was first noticed by Bidone, and has been subsequently investigated by Poncelet, Lesbros, Magnus, Lord Rayleigh, the author, and others. When a jet issues from an orifice in a vertical surface, the sections of the jet at points along its path assume singular forms dependent upon the nature of the orifice.

Figs. 20 to 27 are from photographs (taken from the same point) of jets issuing under the same head, viz., 12 ft., from orifices of different forms and sizes. The dimensions of these jets are comparable with the jets shown by Figs. 20 and 21, which are issuing from circular orifices of 1 in. and $\frac{1}{2}$ in. diameter, respectively.

With a square orifice, Fig. 22 (side = 1 in.), Fig. 23 (side = .443 in.), and Fig. 24 (side = .25 in.), the section is a star of four sheets at right angles to the sides.

With a triangular orifice, Fig. 25 (side = .676 in.), the section is a star of three sheets at right angles to the sides.

In general, with a polygonal orifice of n sides the section will be a star of n sheets at right angles to the sides.

Fig. 26 is a jet from a rectangular orifice ($\frac{1}{2}$ in. $\times$ $\frac{1}{8}$ in.), its section near the orifice being a star of four sheets.

Fig. 27 is a jet from a semi-circular orifice (diar. = .388 in.),

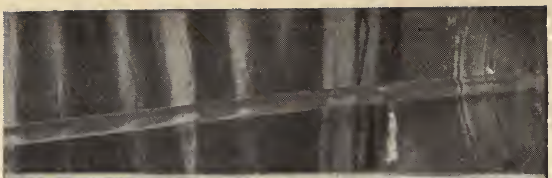


Fig. 20.

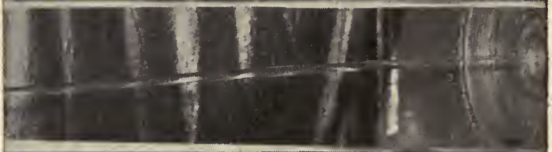


Fig. 21.

Fig. 22.

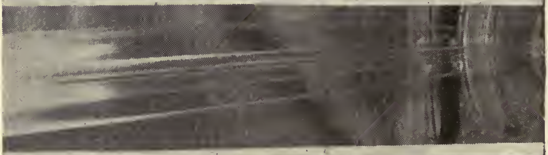


Fig. 23.

Fig. 24.

Fig. 25.

Fig. 26.

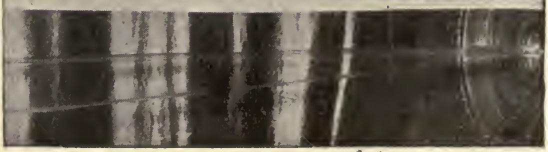


Fig. 27.

the section near the orifice being a rounded boundary and a single sheet at right angles to the diameter.

The changes in the form of the jet are doubtless due to the mutual action between the fluid particles. A filament issuing horizontally and freely at B, Fig. 28, has a velocity $2g . AB$, and

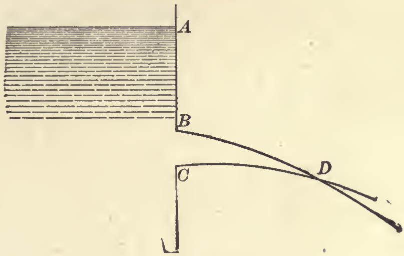


FIG. 28.

describes a certain parabola BD. A filament issuing horizontally and freely at a lower level C has a velocity $2g . AC$, and describes a parabola CD of less curvature than BD. Now the two filaments cannot pass simultaneously through the point of intersection D, and must necessarily press upon each other. They are thus deviated out of their natural paths, and the jet spreads out into sheets, as described above.

If the orifice is small and the head not large, the jet, on leaving the contracted section at the orifice, spreads out into sheets, and then diminishes to a contracted section similar to the first, after which it again spreads out into sheets, bisecting the angles between the first set of sheets, and again diminishes to a contracted section. This action is repeated so long as the jet remains unbroken.

14. Emptying and Filling a Canal Lock.—When the head varies, as in filling or emptying a reservoir or a lock, in filling a vessel by means of an orifice under water, or in emptying water out of a vessel through a spout, Torricelli's theorem is still employed.

If the lock or vessel is to be filled, Fig, 29, let X sq. ft. be the area of the water-surface when it is x ft. below the surface of the outside water.

If the lock or vessel is to be emptied, Fig. 30, then X sq. ft. is the area of the water-surface when it is x ft. above the orifice.

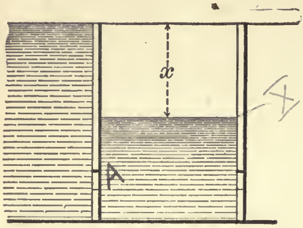


FIG. 29.

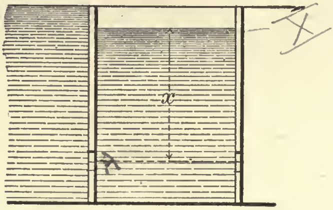


FIG 30.

In each case x ft. is the effective head over the orifice, and is the head under which the flow takes place.

In the time dt the water-surface in the lock or vessel will rise or fall by an amount dx. Then

$$-A \,.\, dx = \text{quantity which has entered the lock}$$

$$= cA\sqrt{2gx} \,.\, dt,$$

A being the area of the orifice.

Hence

$$t = \int_x^h \frac{Xdx}{cA\sqrt{2gx}},$$

an equation giving the time of filling or emptying the lock between the level x and h. The value of c for submerged orifices seems to be somewhat less than when the flow occurs freely, but it is usual to take .6 or .625 as a mean value.

15. General Equations.—Bernouilli's theorem may be easily deduced from the general equations of fluid motion, as follows:

Let p be the pressure and ρ the density at any point whose co-ordinates parallel to the axes are x, y, z.

Let u, v, w be the velocities of flow at the same point parallel to the axes, and let X, Y, Z be the accelerating forces. Then three equations result from the principle of the equality of pressure in all directions, viz.:

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{d(u)}{dt} = X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}; \quad (1)$$

$$\frac{1}{\rho}\frac{dp}{dy} = Y - \frac{d(v)}{dt} = Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz}; \quad (2)$$

$$\frac{1}{\rho}\frac{dp}{dz} = Z - \frac{d(w)}{dt} = Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}; \quad (3)$$

If the motion is steady, so that the velocity at any point is a function of the position only, then $\frac{du}{dt} = 0 = \frac{dv}{dt} = \frac{dw}{dt}$, and u, v, w may be expressed as the differential coefficients of a function F. Thus,

$$u = \frac{dF}{dx}; \quad v = \frac{dF}{dy}; \quad w = \frac{dF}{dz};$$

and therefore

$$\frac{du}{dy} = \frac{d^2F}{dydx} = \frac{dv}{dx};$$

$$\frac{du}{dz} = \frac{d^2F}{dzdx} = \frac{dw}{dx};$$

$$\frac{dv}{dz} = \frac{d^2F}{dzdy} = \frac{dw}{dy}.$$

Hence equations 1, 2, and 3 may be written

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{dv}{dx} - w\frac{dw}{dx}; \quad . \quad . \quad . \quad (4)$$

$$\frac{1}{\rho}\frac{dp}{dy} = Y - u\frac{du}{dy} - v\frac{dv}{dy} - w\frac{dw}{dy}; \quad . \quad . \quad . \quad (5)$$

$$\frac{1}{\rho}\frac{dp}{dz} = Z - u\frac{du}{dz} - v\frac{dv}{dz} - w\frac{dw}{dz}. \quad . \quad . \quad . \quad (6)$$

Multiplying eq. 4 by dx, eq. 5 by dy, and eq. 6 by dz, and adding, then

$$\frac{dp}{\rho} = Xdx + Ydy + Zdz - u\left(\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz\right)$$
$$- v\left(\frac{dv}{dx}dx + \frac{dv}{dy}dy + \frac{dv}{dz}dz\right)$$
$$- w\left(\frac{dw}{dx}dx + \frac{dw}{dy}dy + \frac{dw}{dz}dz\right),$$

which may be written

$$\frac{dp}{\rho} = Xdx + Ydy + Zdz - (u\,du + v\,dv + w\,dw).$$

Integrating, and assuming the fluid to be homogeneous,

$$\frac{p}{\rho} = \int (Xdx + Ydy + Zdz) - \frac{u^2 + v^2 + w^2}{2} + \text{a constant.}$$

Hence, if gravity is the only force, and if V is the resultant velocity at the point, then

$$X = 0 = Y; \quad Z = -g; \quad u^2 + v^2 + w^2 = V^2;$$

and the last equation becomes

$$\frac{p}{\rho} = -\int gdz - \frac{V^2}{2} + \text{a constant}$$
$$= -gz - \frac{V^2}{2} + \text{a constant};$$

and therefore

$$z + \frac{p}{\rho g} + \frac{V^2}{2g} = \text{a constant.}$$

16. Loss of Energy in Shock.—An abrupt change of section at any point in a length of piping destroys the parallelism of the fluid filaments, breaks up the fluid, and energy is dissipated in the production of eddy and other motions. The energy thus wasted is termed "*energy lost in shock.*"

In a short length of piping, where the section suddenly changes from $A'B'$ to EF, consider the fluid mass between the two transverse sections AB, where the motion of the fluid fila-

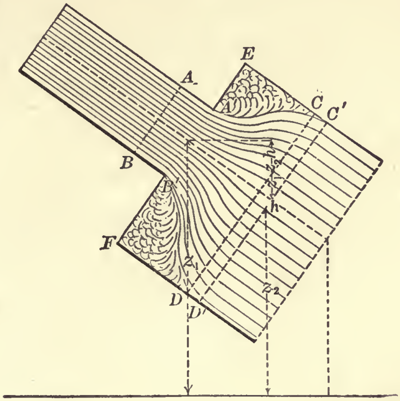


FIG. 31.

ments has been undisturbed and is in parallel lines, and CD, where the parallelism has been again re-established.

In an indefinitely short interval of time t let the mass move forward into the position bounded by the plane sections $A'B'$ and $C'D'$.

Let a_1, v_1, p_1 be the sectional area, velocity of flow, and mean intensity of pressure at $A'B'$.

Let a_2, v_2, p_2 be similar symbols for $C'D'$.

Let z_1, z_2 be the elevation above datum of the C. G.s of the sectional areas at $A'B'$ and $C'D'$.

Let h be the vertical distance between the C. G.s of the areas EF and $A'B'$.

Let P be the mean intensity of pressure over the annular surface between EF and $A'B'$.

The resultant force acting in the direction of motion upon the mass of fluid under consideration

= component of weight of mass in this direction
+ pressure on $A'B'$
+ pressure on annular surface between EF and $A'B'$
− pressure on $C'D'$

$$= wa_2 . EC' \frac{z_1 - z_2 - h}{EC'} + p_1 a_1 + P(a_2 - a_1) - p_2 a_2$$
$$= wa_2(z_1 - z_2 - h) + a_2(p_1 - p_2),$$

assuming that $P = p_1$, or that the mean intensity of pressure is unchanged throughout the whole of the section EF.

The normal reaction of the pipe-surface between EF and $C'D'$ has no component in the direction of motion, and frictional resistances are disregarded.

Hence the impulse of the resultant force

$= wa_2(z_1 - z_2 - h)t + a_2(p_1 - p_2)t$

= change of momentum in the same direction of the fluid masses $CDD'C'$ and $ABB'A'$, since the momentum of the mass between $A'B'$ and CD remains unchanged

$$= \frac{w}{g} a_2 v_2 . v_2 t - \frac{w}{g} a_1 v_1 . v_1 t$$
$$= \frac{w}{g} a_2(v_2^2 - v_1 v_2)t,$$

since by the condition of continuity

$$a_1 v_1 = a_2 v_2.$$

Dividing throughout by the factor $wa_2 t$, the equation becomes

$$z_1 - z_2 - h + \frac{p_1}{w} - \frac{p_2}{w} = \frac{v_2^2}{g} - \frac{v_1 v_2}{g},$$

which may be written in the form

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + h + \frac{p_2}{w} + \frac{v_2^2}{2g} + \frac{(v_1 - v_2)^2}{2g}.$$

Now the pipes are nearly always axial, and in such case $h = 0$, so that the last equation becomes

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g} + \frac{(v_1 - v_2)^2}{2g}.$$

If there had been no abrupt change of section, or if the change between $A'B'$ and CD had been gradual, then no internal work would have been done in destroying the parallelism of the fluid filaments, and no energy wasted. Therefore, by Bernouilli's theorem, the relation

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g}$$

would have held good.

Thus, $\frac{(v_1 - v_2)^2}{2g}$ ft.-lbs. of energy per pound of fluid is the *loss in shock* between $A'B'$ and CD.

Experiment justifies the assumption $P = p_1$.

17. Mouthpieces.—(*a*) *Borda's Mouthpiece.*—This is merely a short pipe projecting inwards, as in Fig. 32, representing

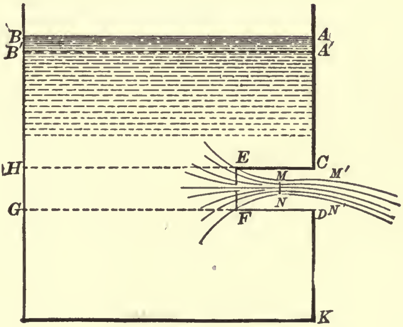


FIG. 32.

a jet flowing through a re-entrant mouthpiece of sectional area A, fixed in the vertical side of a vessel of constant horizontal section and containing water kept at a constant level. The mouthpiece is as long as will allow of the jet springing clear from the end EF without adhering to the inside surface.

The velocity of the fluid molecules along AC and DK is sufficiently small to be disregarded, so that the pressure over this portion of the vessel is distributed in accordance with the hydrostatic law. The same may also be said of the pressure on the remainder of the vessel's surface.

Again, the only unbalanced pressure is that on the surface HG immediately opposite the mouthpiece, and the resultant horizontal force in the direction of the axis of the mouthpiece

$$= (p_0 + wh)A - p_0A = whA,$$

h being the depth of the axis below the water-surface and p_0 the intensity of the atmospheric pressure.

The jet converges to a minimum or contracted section MN of area a.

In a unit of time let the fluid mass between AB and MN take up the position bounded by $A'B$ and $M'N'$. Then

whA = impulse of force in direction of motion
= change of momentum in same direction in a unit of time.
= difference between the momenta of $MNN'M'$ and $ABB'A'$, since the momentum of the mass between $A'B'$ and MN remains unchanged
= momentum of $MNN'M'$, since the momentum of $ABB'A'$ is vertical

$$= \frac{w}{g}av.v = \frac{w}{g}av^2,$$

v being the mean velocity of flow across the contracted section.

Hence

$$whA = \frac{w}{g}av^2 = \frac{w}{g}a.2gh,$$

and therefore

$$A = 2a,$$

or

$$\frac{1}{2} = \frac{a}{A} = \text{coefficient of contraction.}$$

This result has been very closely verified by experiment, the coefficient having been found to be .5149 by Borda, .5547 by Bidone, and .5324 by Weisbach.

Borda's mouthpiece gives a smaller discharge than a sharp-edged orifice, but a discharge which is much more uniform, and hence it is generally used in vessels from which water is to be distributed by measure.

Note.—Let Fig. 33 represent a jet flowing through a re-entrant mouthpiece of sectional area A fixed in the sloping side of a reservoir containing water kept at a constant level, and suppose that the reservoir is of such size that $GHKL$ may represent a cylindrical fluid mass coaxial with the mouthpiece and so large that the velocity at its surface is sensibly nil. Let h', h be the depths below the water-surface of the C. G.'s of the areas GH and KL, respectively.

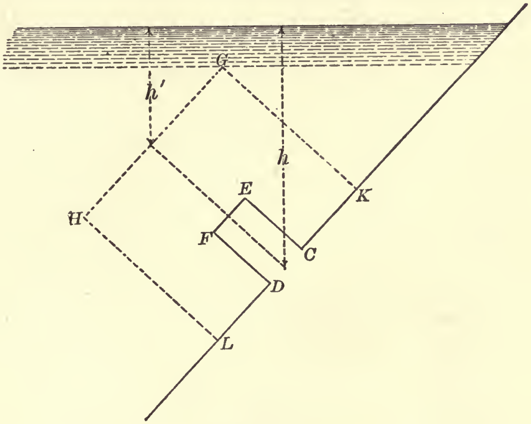


Fig. 33.

Then the resultant force along the axis of the mouthpiece

$=$ pressure on GH — pressure on CK and on DL

— pressure on EF

$+$ component of the weight of the fluid mass $GHKL$

$$= (p_0 + wh') \text{ area } GH - (p_0 + wh) (\text{area } CK + \text{area } DL)$$

$$- p_0 \,.\, \text{area } EF + w \,.\, \text{area } GH \,.\, GK \,.\, \frac{h - h'}{GK}, \text{ very nearly}$$

$$= whA.$$

Hence, in a unit of time,

$$wh A = \text{impulse of this force}$$
$$= \text{change of momentum in direction of axis}$$
$$= \frac{w}{g}av \,.\, v = \frac{w}{g}av^2 = \frac{w}{g}a \,.\, 2gh,$$

a being the area of the contracted section, while h is also very approximately the depth of its C.G. below the water-surface.

Thus, as before,

$$\text{the coefficient of contraction} = \frac{a}{A} = \frac{1}{2}.$$

(*b*) *Ring-Nozzle.*—The ring-nozzle (see Fig. 34) is often used with a fire-engine jet, and consists of a re-entrant pipe of sectional area a_1 fixed in a pipe of sectional area a_2. The length of the re-entrant portion is such that the water springs clear from the inner end and, without again touching the surface of the mouthpiece, converges to a minimum or contracted section of area a at MN.

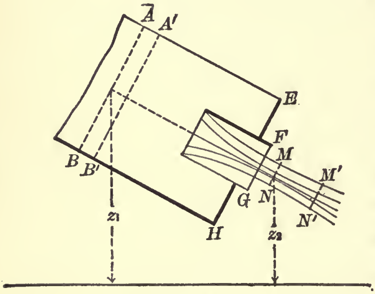


FIG. 34

Consider the fluid mass between MN and a transverse section AB, and in a unit of time let it move into the position bounded by the planes $M'N'$ and $A'B'$.

It is assumed that the motion is steady and that there is no internal work due to the production of eddies or other motions.

Let p_0, v be the intensity of the atmospheric pressure and the velocity at MN.

Let p_1, v_1 be the mean intensity of pressure and the velocity at AB.

Let P be the mean intensity of the pressure over the annular surface EF, GH.

Let z_0, z_1 be the elevations above datum of the C. G.s of the sections MN and AB.

Then

$$wa_2(z_1 - z_0) + p_1a_2 - P(a_2 - a_1) - p_0a_1$$

= impulse in direction of motion
= change of momentum in same direction in a unit of time
= difference of the momenta of the fluid masses $MNN'M'$ and $ABB'A'$

$$= \frac{w}{g}(av^2 - a_2v_1^2).$$

Assuming that $P = p_1$, the last equation becomes

$$wa_1(z_1 - z_0) + a_1(p_1 - p_0) = \frac{w}{g}(av^2 - a_2v_1^2). \quad . \quad . \quad (1)$$

By Bernouilli's theorem,

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_0 + \frac{p_0}{w} + \frac{v^2}{2g},$$

and therefore

$$z_1 - z_0 + \frac{p_1 - p_0}{w} = \frac{v^2 - v_1^2}{2g}. \quad . \quad . \quad . \quad . \quad (2)$$

Now $z_1 - z_0$ is very small and may be disregarded without sensible error, and then by eqs. (1) and (2)

$$\frac{v^2 - v_1^2}{2g} = \frac{p_1 - p_0}{w} = \frac{1}{g}\frac{av^2 - a_2v_1^2}{a_1}.$$

Hence

$$\frac{2}{a_1} = \frac{v^2 - v_1^2}{av^2 - a_2v_1^2} = \frac{(a_2^2 - a^2)v_1^2}{(aa_2^2 - a^2a_2)v_1^2} = \frac{1}{a} + \frac{1}{a_2},$$

since $a_2v_1 = av$.

If the sectional area a_2 of the pipe is very large as compared with a, so that $\frac{1}{a_2}$ may be disregarded without sensible error, then $\frac{2}{a_1} = \frac{1}{a}$, and therefore the coefficient of contraction $= \frac{a}{a_1} = \frac{1}{2}$, as before.

(*c*) *Cylindrical Mouthpiece.*—When water issues from a cylindrical mouthpiece (see Fig. 35) at least two to two and one half diameters in length, the jet issues full bore or without contraction at the point of discharge.

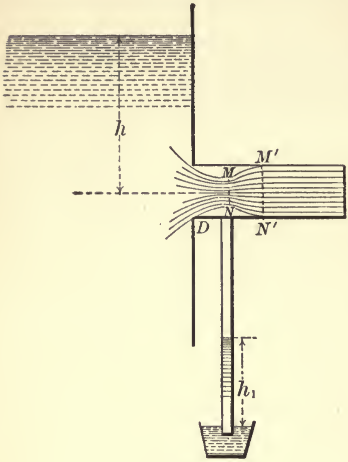


FIG 35.

If A be the sectional area of the mouthpiece, h the depth of its axis below the water-surface, and Q the amount of the discharge. Then experiment shows that

$$Q = .82A\sqrt{2gh}. \quad . \quad (1)$$

The coefficient .82 is the product of the coefficients of velocity and contraction, but the coefficient of contraction is unity, and therefore the coefficient of velocity is .82. Now the mean coefficient of velocity in the case of a simple sharp-edged orifice is .947, and the difference between .947 and .82 cannot be wholly accounted for by frictional resistances, but is in part due to a loss of head. In fact, the water as it clears the inner edge of the mouthpiece converges to a minimum section MN of area a and then swells out until at $M'N'$ it again fills the mouthpiece.

Energy is wasted in eddy motions between MN and $M'N'$, where the action is similar to that which occurs at an abrupt change of section.

Let p, v be the intensity of the pressure and the mean velocity of flow at the point of discharge.

Let p_1, v_1 be similar symbols for the contracted section MN.

Let p_0 be the intensity of the atmospheric pressure.

Remembering that $\dfrac{(v_1 - v)^2}{2g}$ is the loss of head "due to shock" between MN and $M'N'$, then by Bernouilli's theorem

$$h + \frac{p_0}{w} = \frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p}{w} + \frac{v^2}{2g} + \frac{(v_1 - v)^2}{2g}. \quad . \quad . \quad (2)$$

Hence

$$\frac{p_0 - p_1}{w} = \frac{v_1^2}{2g} - h, \quad \ldots \ldots \quad (3)$$

and

$$h + \frac{p_0 - p_1}{w} = \frac{v^2}{2g}\left\{1 + \left(\frac{v_1}{v} - 1\right)^2\right\}$$

$$= \frac{v^2}{2g}\left\{1 + \left(\frac{A}{a} - 1\right)^2\right\}$$

$$= \frac{v^2}{2g}\left\{1 + \left(\frac{1}{c_c} - 1\right)^2\right\},$$

where c_c = coefficient of contraction $= \frac{a}{A} = \frac{v}{v_1}$. Therefore

$$v^2 = \frac{2g\left(h + \frac{p_0 - p}{w}\right)}{1 + \left(\frac{1}{c_c} - 1\right)^2}, \quad \ldots \ldots \quad (4)$$

an equation giving the velocity of flow at the point of discharge.

If the discharge is into the atmosphere, $p_0 = p$ and equation 4 becomes

$$v^2 = \frac{2gh}{1 + \left(\frac{1}{c_c} - 1\right)^2} = c_v^2 \, . \, 2gh, \quad \ldots \ldots \quad (5)$$

where

$$\frac{1}{c_v^2} = 1 + \left(\frac{1}{c_c} - 1\right)^2. \quad \ldots \ldots \quad (6)$$

If $c_c = .62$, then $c_v = .85$, while experiment gives .82 as the value of c_v. The small difference between .85 and .82 is probably due to frictional resistance. The value .82 for c_v makes c_c approximately .617.

Again, the discharge from a simple sharp-edged orifice of same sectional area as the mouthpiece is $.62A\sqrt{2gh}$, or more than 24 per cent less than the discharge from the cylindrical mouthpiece.

The loss of head between MN and $M'N'$

$$= \frac{(v_1 - v)^2}{2g} = \frac{v^2}{2g}\left(\frac{1}{c_c} - 1\right)^2$$

$$= hc_v^2\left(\frac{1}{c_v^2} - 1\right) \qquad \text{(by eqs. 5 and 6)}$$

$$= h(1 - c_v^2) = h \times .3276$$

$$= \frac{h}{3}, \text{ approximately.}$$

Thus the effective head is only $\frac{2}{3}h$, instead of h.

By eq. 3 the difference between the pressure-heads at MN and at the point of discharge

$$= \frac{p_0 - p}{w} = \frac{v_1^2}{2g} - h$$

$$= \frac{1}{c_c^2}\frac{v^2}{2g} - h = h\left(\frac{c_v^2}{c_c^2} - 1\right)$$

$$= \tfrac{3}{4}h, \text{ very nearly.}$$

Now if one end of a tube is inserted in the mouthpiece at the contracted section (Fig. 35) and the other end immersed in a vessel of water, the water will at once rise to a height h_1 in the tube, showing that the pressure at the contracted section is less than that due to the atmosphere. By careful measurement it is found that h_1 is very nearly equal to $\frac{3}{4}h$, which verifies the theory.

(*d*) *Divergent Mouthpiece.*—Suppose that for the cylindrical mouthpiece in (*c*) there is substituted a divergent mouthpiece of the exact form of the issuing jet (see Fig. 36). Then—

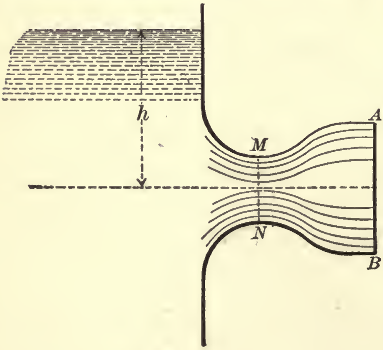


FIG. 36.

(1) The mouthpiece will run full bore.

(2) There will be no loss of head between the minimum section MN and the plane of discharge AB, as there is now no abrupt change of section.

Hence by Bernouilli's theorem, and retaining the same symbols as in (c),

$$\frac{p_0}{w} + h = \frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p}{w} + \frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad (1)$$

If the discharge is into the atmosphere, $p = p_0$, and therefore

$$v^2 = 2gh; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or introducing a coefficient c_v ($= .98$, nearly, for a smooth well-formed mouthpiece),

$$v^2 = c_v^2 2gh, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the discharge is

$$Q = c_v A \sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From the last equation it would appear as if the discharge would increase indefinitely with A, but this is manifestly impossible.

In fact, by eq. 1, the flow being into the air, and taking $c_v = 1$,

$$\frac{p_1}{w} = \frac{p_0}{w} - \frac{v^2}{2g}\left(\frac{v_1^2}{v^2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$= \frac{p_0}{w} - h\left(\frac{A^2}{a^2} - 1\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

since $av_1 = Av$. But p_1 cannot be negative, and therefore

$$\frac{p_0}{w} \geqq h\left(\frac{A^2}{a^2} - 1\right),$$

so that

$$\frac{A}{a} = \sqrt{\frac{p_0}{wh} + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

gives a maximum limit for the ratio of A to a.

Now $\frac{p}{w} = 34$ feet very nearly, and the last equation may be written

$$\frac{A}{a} = \sqrt{\frac{34 + h}{h}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By eqs. 4 and 7,

$$Q = c_v a \sqrt{2g\left(h + \frac{p_0}{w}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which is also the expression for the discharge through the minimum section a into a vacuum.

If, however, the sectional areas of the mouthpiece at the point of discharge and at the throat are in the ratio of A to a, as given by eq. 7, it is found that the full-bore flow will be interrupted either by the disengagement of air, or by any slight disturbance, as, for example, a slight blow on the mouthpiece, and hence, in practice, it is usual to make the ratio of A to a sensibly less than that given by eq. 7.

(*e*) *Convergent Mouthpiece.*—With a convergent mouthpiece (Fig. 37) two points are to be noted:

(1) There is a contraction within the mouthpiece, followed by a swelling out of the jet until it again fills the mouthpiece.

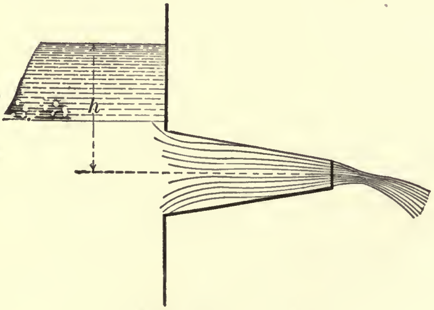


FIG. 37.

Thus, as in the case of cylindrical mouthpieces, there is a "loss of head" between the contracted section and the point of discharge, and also a consequent diminution in the velocity of discharge.

(2) There is a second contraction outside the mouthpiece due to the convergence of the fluid filaments. The mean velocity of flow (v') across the section is

$$v' = C_v' \sqrt{2gh},$$

C_v' being the coefficient of velocity and h the effective head above the centre of the section.

Also, the area of this section

$$= C_c' \times \text{ area of mouthpiece at point of discharge}$$
$$= C_c' \,.\, A,$$

C_c' being the coefficient of contraction. Hence the discharge Q is given by

$$Q = C_v'C_c'A \sqrt{2gh} = C'A \sqrt{2gh},$$

$C'(= C_v'C_c')$ being the coefficient of discharge.

The coefficients C_v' and C_c' depend upon the angle of convergence, and Castel found that a convergence of 13° 24′ gave a maximum discharge through a mouthpiece 2–6 diameters in length, the smallest diameter being .05085 foot.

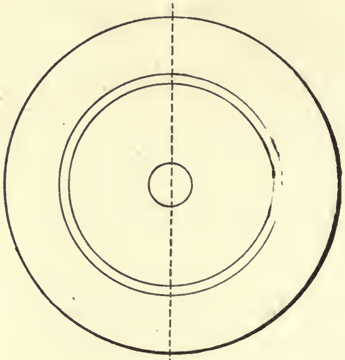
TABLE GIVING CASTEL'S RESULTS.

Angles of Convergence.	C_c'	C_v'	C'	Angles of Convergence.	C_c'	C_v'	C'
0° 0′	.999	.830	.829	13° 24′	.983	.962	.946
1 36	1.000	.866	.866	14 28	.979	.966	.941
3 10	1.001	.894	.895	16 36	.969	.971	.938
4 10	1.002	.910	.912	19 28	.953	.970	.924
5 26	1.004	.920	.924	21 0	.945	.971	.918
7 52	.998	.931	.929	23 0	.937	.974	.913
8 58	.992	.942	.934	29 58	.919	.975	.896
10 20	.987	.950	.938	40 20	.887	.980	.869
12 4	.986	.955	.942	48 50	.861	.984	.847

18. Radiating Current.—As an application of Bernouilli's theorem, consider the steady plane motion of a body of water flowing radially between two horizontal planes a ft. apart and symmetrical with respect to a central axis (Fig. 38).

Let v ft. per second be the velocity at the surface of a cyl-

FIG. 38.



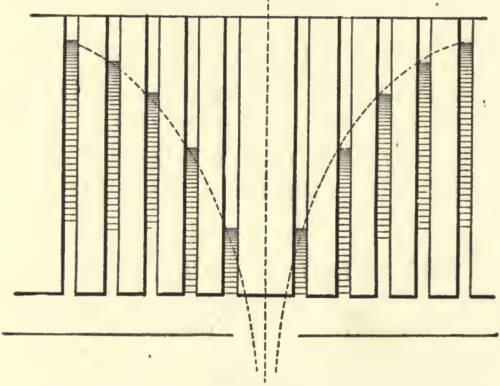


FIG. 39.

inder of radius r ft. described about the same axis. Then the volume Q crossing the second per surface is

$$Q = 2\pi r \,.\, av,$$

and therefore

$$rv = \frac{Q}{2\pi a} = \text{a constant},$$

since Q is constant.

Thus v increases as r diminishes, and becomes infinitely great at the axis; but it is evident that the current must take a new course at some finite distance from the axis.

If p is the pressure at any point of the cylindrical surface z ft. above datum, then, by Bernouilli's theorem,

$$z + \frac{p}{w} + \frac{v^2}{2g} = \text{a constant} = h = y + \frac{v^2}{2g},$$

denoting the *dynamic head* $z + \frac{p}{w}$ by y. Hence

$$h - y = \frac{v^2}{2g} = \frac{Q^2}{8\pi^2 a^2 r^2 g} = \frac{\text{a constant}}{r^2},$$

and therefore

$$r^2(h - y) = \text{a constant}$$

is an equation giving the free surfaces of the pressure columns (Fig. 39). These surfaces are thus generated by the revolution of Barlow's curve.

The surfaces of equal pressure are also given by an equation of the same form.

19. Vortex Motion.—A vortex is a mass of rotating fluid, and the vortex is termed *free* when the motion is produced naturally and under the action of the forces of weight and pressure only.

In the radiating current already discussed, assume that the direction of motion at each point is turned through a right angle, so that the mass of water will now revolve in circular layers about the central axis. Also, if there is a slow radial movement, so that fluid particles travel from one circular stream-line to another, it is assumed that these particles freely take the velocities proper to the stream-lines which they join. Such a motion is termed a free *circular* vortex.

The motion being steady and horizontal, the equation

$$z + \frac{p}{w} + \frac{v^2}{2g} = \text{a constant} = H, \quad . \quad . \quad . \quad . \quad (1)$$

holds good at every point of a circular stream of radius r.

Again,

$$w \,.\, d\left(z+\frac{p}{w}\right) = \text{increment of dynamic pressure between two consecutive elementary stream-lines}$$
$$= \text{deviating force}$$
$$= \text{centrifugal force of an element between the two stream-lines}$$
$$= \frac{w}{g}\frac{v^2}{r} \,.\, dr.$$

But, by eq. 1,

$$d\left(z+\frac{p}{w}\right) = -\frac{v \,.\, dv}{g}.$$

Hence

$$w \,.\, d\left(z+\frac{p}{w}\right) = -\frac{w}{g}vdv = \frac{w}{g}\frac{v^2}{r} \,.\, dr,$$

and therefore

$$\frac{dv}{v}+\frac{dr}{r}=0,$$

so that vr = a constant, and v varies inversely as r, as in the case of the radiating current. Therefore the curves of equal pressure will also be the same as in a radiating current.

Free Spiral Vortex.—Suppose that the motion of a mass of water with respect to an axis O is of such a character that at any point M the components of the velocity in the direction of OM, and perpendicular to OM, are each inversely proportional to the distance OM from O. The motion is thus equivalent to the superposition of the motions in a radiating current and in a free circular vortex; and if θ is the angle between OM and the direction of the stream-line at M, $v \cos \theta$ and $v \sin \theta$ are each inversely proportional to OM, and therefore θ must be constant. Hence the stream-lines must be equiangular spirals and the motion is termed a free spiral vortex.

This result is of value in the discussion of certain turbines and centrifugal pumps. A *steady* free surface in the case of a free spiral vortex is impossible, as the stream-lines cross the surfaces of equal pressure, which are the same as before.

Also if p_0, v_0, r_0 are the pressure, radius, and velocity at any other point at the same elevation z above datum, then

$$z + \frac{p}{w} + \frac{v^2}{2g} = z + \frac{p_0}{w} + \frac{v_0^2}{2g},$$

and the increase of pressure-head

$$= \frac{p - p_0}{w} = \frac{v_0^2 - v^2}{2g} = \frac{v^2}{2g}\left(\frac{r^2}{r_0^2} - 1\right) = \frac{v_0^2}{2g}\left(1 - \frac{r_0^2}{r^2}\right).$$

Forced Vortex.—A forced vortex is one in which the law of motion is different from that in a free vortex. The simplest and most useful case is that in which all the particles have an equal angular velocity, so that the water will revolve bodily, the velocity at any point being directly proportional to the distance from the axis.

As before,

$$d\left(z + \frac{p}{w}\right) = \frac{v^2}{g}\frac{dr}{r}.$$

But

$$v \propto r = \omega r,$$

ω being the constant angular velocity of the rotating mass. Therefore

$$d\left(z + \frac{p}{w}\right) = \frac{\omega^2}{g}r \,.\, dr.$$

Integrating,

$$z + \frac{p}{w} = \frac{\omega^2 r^2}{2g} + \text{a constant} = \frac{v^2}{2g} + \text{a constant}.$$

Hence, if p_0, r_0, v_0 are the pressure, radius, and velocity for any second point at the same elevation z above datum, then

$$\frac{p - p_0}{w} = \frac{\omega^2}{2g}(r^2 - r_0^2) = \frac{1}{2g}(v^2 - v_0^2).$$

$$z + \frac{p}{w} + \frac{\omega^2 r_0^2}{2g} = z + \frac{p_0}{w} + \frac{\omega^2 r^2}{2g}$$

If the second point is on the axis of revolution, then $r_0 = 0$, and the last equation becomes

$$\frac{p - p_0}{w} = \frac{\omega^2}{2g} r^2.$$

Thus the free surface of the pressure columns is evidently a paraboloid of revolution with its vertex at O, as in Fig. 40.

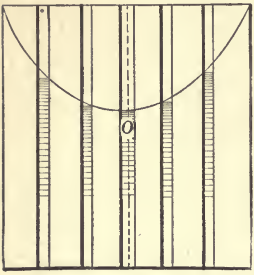


FIG. 40.

A *compound* vortex is produced by the combination of a central forced vortex with a free circular vortex, the free surface being formed by the revolution of a Barlow curve and a parabola.

For example, the fan of a centrifugal pump draws the water into a forced vortex and delivers it as a free spiral vortex into a whirlpool-chamber (Chap. VII.).

In this chamber there is thus a gain of pressure-head, and the water is therefore enabled to rise to a corresponding additional height. James Thomson adopted the theory of the com-compound vortex as the principle of the action of his vortex turbine.

20. Large Orifices in Vertical Plane Surfaces.—The issuing jet is approximately of the same sectional form as the orifice, and the fluid filaments converge to a minimum section as in the case of simple sharp-edged orifices.

(*a*) *Rectangular Orifice* (Fig. 41).—Let E, F be the upper and lower edges of a large rectangular orifice of breadth B, and let H_1, H_2 be the depths of E and F, respectively, below the free surface at A. If u be the velocity with which the water reaches the orifice, then $H = \frac{u^2}{2g}$ is the fall of free surface which must have been expended in producing the velocity u.

Hence, $H_1 + H$ and $H_2 + H$ are the true depths of the edges E and F below the surface of still water.

Let MN be the minimum or contracted section, and assume that it is a rectangle of breadth b.

Let h_1, h_2 be the depths of M and N, respectively, below the free surface at A.

Then $h_1 + H$, $h_2 + H$ are the true depths of M and N below the surface of still water.

First, let the flow be into the air, the orifice being clear above the tail-water level.

Consider a lamina of the fluid at the section MN of the

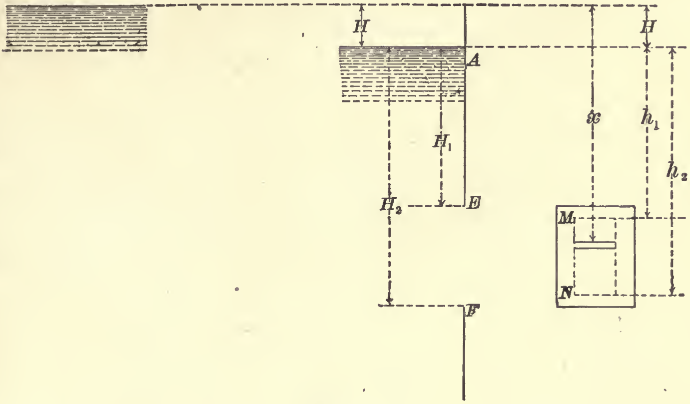


FIG. 41.

width of the section and between the depths x and $x + dx$ below the surface of still water.

The elementary discharge dq, in this lamina, is

$$dq = bdx\sqrt{2gx},$$

and therefore the total discharge Q across the section MN is

$$Q = \int dq = \int_{h_1 + H}^{h_2 + H} b\,.\,dx\sqrt{2gx}$$

$$= \tfrac{2}{3}b\sqrt{2g}\{(h_2 + H)^{\frac{3}{2}} - (h_1 + H)^{\frac{3}{2}}\}$$

Put $c = \dfrac{b\{(h_2 + H)^{\frac{3}{2}} - (h_1 + H)^{\frac{3}{2}}\}}{B\{(H_2 + H)^{\frac{3}{2}} - (H_1 + H)^{\frac{3}{2}}\}}$.

Then

$$Q = \tfrac{2}{3}cB\sqrt{2g}\{(H_2 + H)^{\frac{3}{2}} - (H_1 + H)^{\frac{3}{2}}\}. \quad . \quad . \quad (1)$$

The coefficient c is by no means constant, but is found to vary both with the head of water and also with the dimensions of the orifice, and can only be determined by experiment.

Second, let the orifice be partially (Fig. 42) submerged, and and let H_3 be the depth between the surface of the tail-race water and the free surface at A.

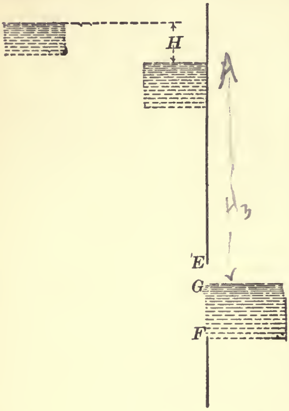


FIG. 42.

By what precedes, the discharge Q_1 through EG, the portion of the orifice clear above the tail-race, is

$$Q_1 = c_1 B\sqrt{2g}\{(H_3+H)^{\frac{3}{2}} - (H_1+H)^{\frac{3}{2}}\}. \quad (2)$$

Every fluid filament flows through the portion GF of the orifice under an effective head $H_3 + H$, and therefore with a velocity equal to

$$\sqrt{2g(H_3+H)}.$$

Hence the discharge Q_2 through GF is

$$Q_2 = c_2 B(H_2 - H_3)\sqrt{2g(H_3+H)}, \quad . \quad . \quad . \quad (3)$$

and the total discharge Q is equal to $Q_1 + Q_2$.

The coefficients c_1, c_2 are to be determined by experiment, and if $c_1 = c_2 = c$,

$$Q = Q_1 + Q_2 = cB\sqrt{2g}\Big[\frac{2}{3}\{(H_3+H)^{\frac{3}{2}} - (H_1+H)^{\frac{3}{2}}\}$$
$$+ (H_2 - H_3)\sqrt{H_3+H}\Big]. \quad (4)$$

Third, let the orifice be wholly submerged (Fig. 43). Then the total discharge Q is evidently

$$Q = cB\sqrt{2g}(H_2 - H_1)\sqrt{H_3+H}, \quad . \quad . \quad . \quad (5)$$

c being a coefficient to be determined by experiment.

If the velocity of approach, u, is sufficiently small to be

$\Rightarrow$ $Q = cAv$ equiv-to.
$= c\{B(H_2-H_1)\sqrt{2g(H_3+H)}$

disregarded without sensible error, then $H = 0$, and equations 1, 4, and 5, respectively, become

$$Q = \frac{2}{3}cB\sqrt{2g}(H_2^{\frac{3}{2}} - H_1^{\frac{3}{2}}); \quad . \quad . \quad (6)$$

$$Q = cB\sqrt{2g}\left\{H_3^{\frac{1}{2}}\left(H_2 - \frac{H_3}{3}\right) - \frac{2}{3}H_1^{\frac{3}{2}}\right\}; (7)$$

$$Q = cB\sqrt{2g}(H_2 - H_1)H_3^{\frac{1}{2}}. \quad . \quad . \quad (8)$$

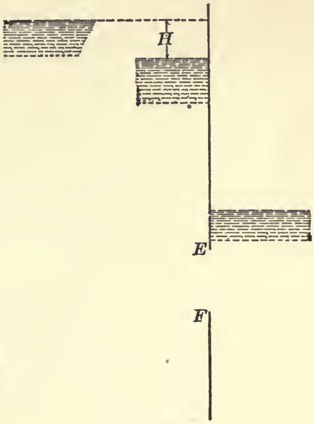


FIG. 43.

(*b*) *Circular Orifices.*—Let Fig. 44 represent the minimum section of the circular jet issuing from a circular orifice.

Let 2θ be the angle subtended at the centre by the fluid

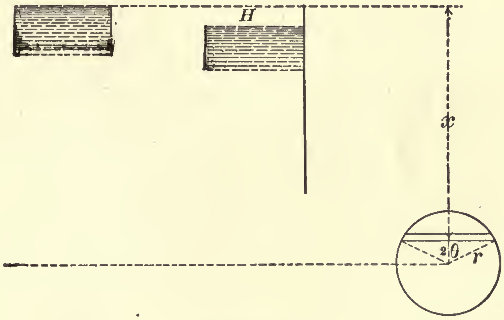


FIG. 44.

lamina between the depths x and $x + dx$ below the surface of still water.

Let r be the radius of the section so that $2r = h_2 - h_1$, h_1 and h_2 being, as in (*a*), the depths of the highest and lowest points of the orifice below the free surface at A.

H, as before, is the head corresponding to the velocity of approach.

Then the area of the lamina under consideration

$$= 2r \sin \theta . dx,$$

and the elementary discharge, dq, in this lamina, is

$$dq = 2r \sin \theta . dx \sqrt{2gx}.$$

But $x = \dfrac{h_1 + H + h_2 + H}{2} - r \cos \theta = \dfrac{h_1 + h_2 + 2H}{2} - r \cos \theta,$

and therefore

$$dx = r \sin \theta . d\theta.$$

Hence

$$dq = 2r^2 \sin^2\theta \sqrt{2g\left(\frac{h_1 + h_2 + 2H}{2} - r \cos \theta\right)} d\theta,$$

and the total discharge Q is

$$Q = 2r^2 \sqrt{2g} \int_0^\pi \sin^2\theta \left(\frac{h_1 + h^2 + 2H}{2} - r \cos \theta\right)^{\frac{1}{2}} d\theta. \qquad (9)$$

21. **Notches and Weirs.**—When an orifice extends up to the free-surface level it becomes what is called a *notch.*

A *weir* is a structure over which the water flows, the discharge being in the same conditions as for a notch.

Rectangular Notch or Weir.—The discharge may be found by putting $H_1 = 0$.

Thus equation 1 becomes

$$Q = \frac{2}{3} cB\sqrt{2g}\{(H_2 + H)^{\frac{3}{2}} - H^{\frac{3}{2}}\}. \quad . \quad . \quad . \quad (10)$$

If the velocity of approach be disregarded, then $H = 0$, and the last equation becomes

$$Q = \frac{2}{3} cB\sqrt{2g} H_2^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and H_2 is the depth to the bottom of the notch or to the crest of the weir.

The effective sectional area of the water flowing through a rectangular notch, or over a weir, is less than BH_2, because of (*a*) crest contraction, (*b*) end contraction, (*c*) the fall of the free surface towards the point of discharge.

It is reasonable to assume that the diminution of the actual sectional area, BH_2, due to crest contraction and to the fall of the free-surface level is proportional to the width B of the opening, and that the effect of end contractions is very nearly the same both for wide and narrow openings.

Francis, in his Lowell weir experiments, found that for depths $H_2 + H$ over the crest, varying from 3 in. to 24 in., and for widths B not less than *three* times the depth, a perfect end contraction had the effect of diminishing the width of the fluid section by an amount approximately equal to one tenth of the depth, or $\frac{H_2 + H}{10}$, so that the effective width $= B - \frac{H_2 + H}{10}$.

Thus, if there are n end contractions, the effective width $= B - \frac{n}{10}(H_2 + H)$, and the equation giving the discharge becomes

$$Q = \frac{2}{3}c\left\{B - \frac{n}{10}(H_2 + H)\right\}\sqrt{2g}\{(H_2 + H)^{\frac{3}{2}} - H^{\frac{3}{2}}\}. \quad (12)$$

According to Francis, the average value of c in this equation is .622.

Circular Notch.—In equation 9, Art. 20, put $h_1 = 0$ and $h_2 = 2r$. Then

$$Q = 2r^2\sqrt{2g}\int_0^\pi \sin^2\theta\left(H + 2r\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}} d\theta,$$

and if the velocity of approach be disregarded, so that $H = 0$,

$$Q = 2r^{\frac{5}{2}}\sqrt{4g}\int_0^{\pi} \sin^2\theta \sin\frac{\theta}{2} \,.\, d\theta$$

$$= \sqrt{r^5 g}\int_0^{\pi}\left(2\sin\frac{\theta}{2} - \sin\frac{5\theta}{2} + \sin\frac{3\theta}{2}\right)d\theta$$

$$= \frac{64}{15}\sqrt{r^5 g}. \qquad (13)$$

22. Triangular Notch.—Disregard the velocity of approach and let B be the width of the free surface.

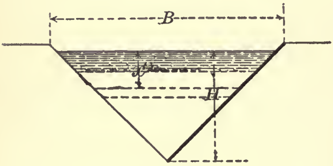


FIG. 45.

As before, consider a lamina of fluid between the depths x and $x + dx$.

The area of the lamina $= \frac{B}{H_2}(H_2 - x)dx$, and the discharge in this lamina is

$$dq = c\frac{B}{H_2}(H_2 - x)dx\sqrt{2gx}.$$

Hence the total discharge Q is

$$Q = c\int_0^{H_2}\frac{B}{H_2}(H_2 - x)dx\sqrt{2gx}$$

$$= c\frac{B}{H_2}\sqrt{2g}\int_0^{H_2}(H_2 x^{\frac{1}{2}} - x^{\frac{3}{2}})dx$$

$$= \frac{4}{15}cB\sqrt{2g}H_2^{\frac{5}{2}}. \qquad (14)$$

c is a coefficient introduced to allow for contraction, etc., and Professor James Thomson gives .617 as its mean value for a sharp-edged triangular notch.

Now the ratio $\frac{B}{H_2}$ is constant in a triangular notch and varies in a rectangular notch. Hence Thomson inferred and proved by experiment that the value of c is more constant for triangular than for rectangular notches, so that a triangular notch is more suitable for accurate measurements.

Example.—A sharp-edged triangular notch is opened in the side of a reservoir, and the water flows out until the free-surface level sinks to the bottom of the notch.

The discharge in the short interval dt, when the depth of water in the notch is x ft.,

$$= \frac{4}{15} cmx \sqrt{2g} x^{\frac{3}{2}} dt$$

$$= \frac{4}{15} \sqrt{2g} cmx^{\frac{5}{2}} dt,$$

mx being the width of the free surface corresponding to the depth x, and m a coefficient depending upon the angle of the notch.

Again, $S \,.\, dx$ is the quantity of the water which leaves the reservoir in the same time dt, S being the horizontal sectional area of the reservoir corresponding to the depth x. Hence

$$\frac{4}{15} \sqrt{2g} cmx^{\frac{5}{2}} dt = - Sdx,$$

and therefore

$$\frac{4}{15} \sqrt{2g} cmdt = - Sx^{-\frac{5}{2}} dx,$$

so that the time in which the free surface sinks to the required level

$$= - \frac{15}{4\sqrt{2g}\, cm} \int_0^X Sx^{-\frac{5}{2}} dx,$$

X being the initial depth.

If S is constant, then

$$\text{the time} = \frac{5SX^{-\frac{3}{2}}}{2\sqrt{2g}\,cm}.$$

23. Broad-crested Weir.—Let Fig. 46 represent a stream flowing over a broad-crested weir. On the up-stream side the

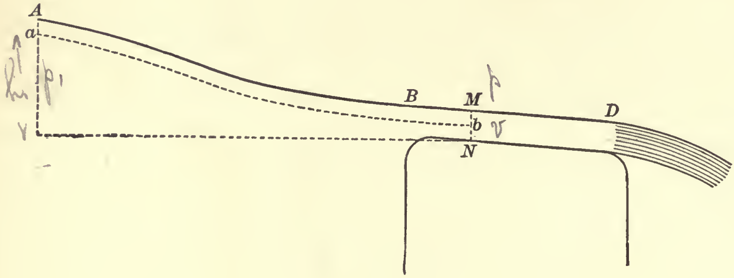


FIG. 46.

free surface falls from A to B. For a distance BD on the crest the fluid filaments are sensibly rectilinear and parallel; the inner edge of the crest is rounded so as to prevent crest contraction.

Consider a filament ab, the point a being taken in a part of the stream where the velocity of flow is so small that it may be disregarded without sensible error.

Let λ be the thickness MN of the stream at b.

Let the horizontal plane through N be the datum plane.

Let z_1, z be the depths below the free surface of a and b.

Let h_1 be the elevation of a above datum.

Let p_0, p_1, p be the atmospheric pressure and the pressures at a and b.

Let v be the velocity of flow at b.

Then, by Bernouilli's theorem,

$$h_1 + \frac{p_1}{w} = \lambda - z + \frac{p}{w} + \frac{v^2}{2g}.$$

But

$$\frac{p_1}{w} = z_1 + \frac{p_0}{w} \quad \text{and} \quad \frac{p}{w} = z + \frac{p_0}{w};$$

therefore

$$h_1 + z_1 + \frac{p_0}{w} = \lambda - z + z + \frac{p_0}{w} + \frac{v^2}{2g},$$

and hence

$$\frac{v^2}{2g} = h_1 + z_1 - \lambda = H_2 - \lambda,$$

H_2 being the depth of the crest of the weir below the surface of still water.

Thus, if B be the width of the weir, the discharge Q is

$$Q = B\lambda\sqrt{2g(H_2 - \lambda)}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From this equation it appears that Q is nil both when $\lambda = 0$ and when $\lambda = H_2$. Hence there must be some value of λ between 0 and H_2 for which Q is a maximum. This value may be found by putting

$$dQ = 0 = B\sqrt{2g}\left(\sqrt{H_2 - \lambda} - \frac{1}{2}\frac{\lambda}{\sqrt{H_2 - \lambda}}\right)d\lambda,$$

and therefore

$$\lambda = \frac{2}{3}H_2,$$

and the expression for the discharge becomes

$$Q = \frac{2}{3\sqrt{3}}BH_2\sqrt{2gH_2} = .385B\sqrt{2g_2}H^{\frac{3}{2}}, \quad . \quad . \quad (17)$$

which is the maximum discharge for the given conditions.

Experiment shows that the more correct value for the discharge is

$$Q = .35B\sqrt{2g_2}H^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This formula agrees with the ordinary expression for the discharge over a weir as given by equation 11, if $c = .525$.

It might be inferred that for broad-crested weirs and large masonry sluice-openings the discharge should be determined by means of equation 18 rather than by the ordinary weir formula, viz., equation 11.

It must be remembered, however, that in deducing equation 17 frictional resistances have been disregarded and the gratuitous assumption has been made that the stream adjusts itself to a thickness t which will give a maximum discharge. The theory is therefore incomplete.

EXAMPLES.

1. A frictionless pipe gradually contracts from a 6-in. diameter at *A* to a 3-in. diameter at *B*, the rise from *A* to *B* being 2 ft. If the delivery is 1 cubic foot per second, find the difference of pressure between the two points *A* and *B*. *Ans.* 500 lbs. per sq. ft.

2. In a frictionless horizontal pipe discharging 10 cubic feet of water per second, the diameter gradually changes from 4 in. at a point *A* to 6 in. at a point *B*. The pressure at the point *B* is 100 lbs. per square inch; find the pressure at the point *A*. *Ans.* 4118 lbs. per sq. ft.

3. A ½-in. horizontal pipe is gradually reduced in diameter to ⅛ in. and then gradually expanded again to its mouth, where it is open to the atmosphere. Determine the maximum quantity of water which can be forced through the pipe (*a*) when the diameter of the mouth is ½ in., (*b*) when the diameter is ¾ in. Also determine the corresponding velocities at the throat and the total heads (neglect friction, which, however, is very considerable). *Ans.* (*a*) .24 cub. ft. per min.; 46.7 ft. per. sec.
(*b*) .239 cub. ft. per. min.; 46.66 ft. per sec.

4. A short horizontal pipe *ABC* connecting two reservoirs gradually contracts in diameter from 1 inch at *A* to ½ inch at *B* and then enlarges to 1 inch again at *C*. If the height of the water in the reservoir over *C* be 12 inches, determine the maximum flow through the pipe and sketch the curve of pressures. Also obtain an equation for this curve, assuming the rates of contraction and expansion of the pipe to be equal and uniform. *Ans.* 3.75 cub. ft. per min.

5. The pipe *DE* in the figure is gradually contracted in diameter from *D* to *E*, where it is enclosed in another pipe *ABC*, expanding from *B* towards *A* and *C*; at *C* it is open to the atmosphere and at *A* it is connected with a reservoir *R*; the water surface in *R* being h' below the horizontal axis of *DE*. If the velocity in *DE* at *E* be v and the velocity in *AB* at *B* be *V*, what will be the common velocity after uniting? Explain what becomes of the energy lost in impact. If the diameters at *E*, *B*, and *C* are ½ in., ¾ in., and 1 in., the distance between the outside of *E* and inside

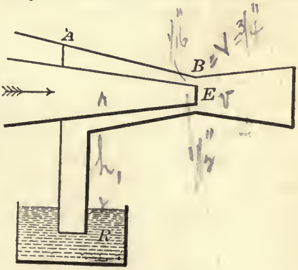


FIG. 47.

of B being $\frac{1}{16}$ inch, find the ratio of the quantity pumped from R to the flow through DE.

6. A 3-in. pipe gradually expands to a bell-mouth; if the total head, H, be 40 ft., find the greatest diameter of the mouth at which it will run full when open to the atmosphere. Compare the discharge from this pipe with the discharge when the pipe is not expanded at the mouth.

Ans. 4.8 in.; discharge is 18.63 cub. ft. per minute with bell-mouth and 7.337 cub. ft. per minute without bell-mouth.

7. The pressure in a 12-in. pipe at A is 50 lbs.; the pipe then enlarges to a 15-in. pipe at B, the rise from A to B being 3 ft.; the discharge is Q cubic feet per minute. Find the pressure at B; also find the pressure at a point C, the rise from B to C being 6 ft.

Ans. $\left(6637.5 + \frac{49Q^2}{1393920}\right)$ lbs. per sq. ft.

8. Two equal pipes lead, one from the steam-space, the other from the water-space of a boiler at pressure p; S_s is the density of the steam and S_w that of the water. Assuming Torricelli's theory to hold for rate of efflux of steam and water, show that

$$\frac{\text{vel. of steam-jet}}{\text{vel. of water-jet}} = \sqrt{\frac{S_w}{S_s}} = \frac{\text{quantity of water-jet}}{\text{quantity of steam-jet}} = \frac{\text{energy of steam-jet}}{\text{energy of water-jet}},$$

and that the momentum of each jet is the same.

9. Find the head required to give 1 cub. ft. of water per second through an orifice of 2 square inches area, the coefficient of discharge being .625. ($g = 32$.) *Ans.* 206 ft.

10. The area of an orifice in a thin plate was 36.3 square centimetres, the discharge under a head of 3.396 metres was found to be .01825 cubic metre per second, and the velocity of flow at the contracted section, as determined by measurements of the axis of the jet, was 7.98 metres per second. Find the coefficients of velocity, contraction, discharge, and resistance. ($g = 9.81$.) *Ans.* .978; .631; .617; .045.

11. The piston of a 12-in. cylinder containing salt water is pressed down under a force of 3000 lbs. Find the velocity of efflux and the volume of discharge at the end of the cylinder through a well-rounded 1-in. orifice. Also find the power exerted.

Ans. 60.373 ft. per sec.; .1691 cub. ft. per sec.; 1.166 H. P.

12. In the condenser of a marine engine there is a vacuum of $26\frac{1}{2}$ in. of mercury; the injection orifices are 6 ft. below the sea-level. With what velocity will the injection-water enter the condenser? (Neglect resistance.) *Ans.* 25.3 ft. per sec.

13. Water in the feed-pipe of a steam-engine stands 12 ft. above the

surface of the water in the boiler; the pressure per sq. in. of the steam is 20 lbs., of the atmosphere 15 lbs. Find the velocity with which the water enters the boiler. *Ans.* 5.376 ft. per sec.

14. The injection orifice of a jet condenser is 5 ft. below sea-level and vacuum = 27 in. of mercury. Find velocity of water entering condenser, supposing three fourths of the head lost by frictional resistance. *Ans.* 23.86 ft. per sec.

15. A vessel containing water is placed on scales and weighed. How will the weight be affected by opening a small orifice in the bottom of the vessel?

16. Water is supplied by a scoop to a locomotive tender at 7 ft. above trough. Find lowest speed of train at which the operation is possible. *Ans.* 14.44 miles per hour.

Also find the velocity of delivery when train travels at 40 miles per hour, assuming half the head lost by frictional resistance. *Ans.* 35.68 ft. per. sec.

17. The head in a prismatic vessel at the instant of opening an orifice was 6 ft. and at closing it had decreased to 5 ft. Determine the mean constant head h at which, in the same time, the orifice would discharge the same volume of water. *Ans.* 5.434 ft.

18. A prismatic vessel 5.747 in. in diameter has an orifice of .2 in. diam. at the bottom; the surface sinks from 16 in. to 12 in. in 53 seconds. Find the coefficient of discharge. *Ans.* .6.

19. A prismatic basin with a horizontal sectional area of 9 sq. ft. has an orifice of .09 sq. ft. at the bottom; it is filled to a depth of 6 ft. above the centre of the orifice. Find the time required for the surface to sink 2 ft., $3\frac{1}{2}$ ft., 5 ft. *Ans.* 260 sec.; 502 sec.; 838 sec.

20. The water in a cylindrical cistern of 144 sq. in. sectional area is 16 ft. deep. Upon opening an orifice of 1 sq. in. in the bottom the water fell 7 ft. in 1 minute. Find the coefficient of discharge. The coefficient of contraction being .625, find the coefficients of velocity and resistance. *Ans.* .6; .96; 0.85.

21. How long will it take to fill a paraboloidal vessel up to the level of the outside surface through a hole in the bottom 2 ft. under water? ($g = 32$ and $c = .625$.)

Ans. $\frac{176\sqrt{2}}{105}\frac{B}{A}$, B being the parameter of the parabola and A the sectional area of the orifice.

22. How long will it take to fill a spherical vessel of radius r up to the level of the outside surface through a hole of area A at bottom 2 ft. under water? ($g = 32$ and $c = .625$.)

Ans. $\frac{22}{35A}(7.54r - 6.53)$.

23. A vessel full of water weighs 350 lbs. and is raised vertically by means of a weight of 450 lbs. Find the velocity of efflux through an orifice in the bottom, the head being 4 ft. *Ans.* 17.02 ft. per sec.

24. A vessel full of water makes 100-revols. per min. Find the velocity of efflux through an orifice 2 ft. below the surface of the water at the centre. *Ans.* 33.4 ft. per sec.

What will be the velocity if the vessel is at rest?

Ans. 11.35 ft. per sec.

25. The jet from a circular sharp-edged orifice, $\frac{1}{2}$ in. in diameter, under a head of 18 ft., strikes a point distant 5 ft. horizontally and 4.665 in. vertically from the orifice. The discharge is 98.987 gallons in 569.218 seconds. Find the coefficients of discharge, velocity, contraction, and resistance. *Ans.* .6014; .945; .636; .118.

26. A square box 2 ft. in length and 1 ft. across a diagonal is placed with a diagonal vertical and filled with water. How long will it take for the whole of the water to flow out through a hole at the bottom of .02 sq. ft. area? ($c = .625$.) *Ans.* 97.48 secs.

27. A pyramid 2 ft. high, on a square base, is inverted and filled with water. Find the time in which the water will all run out through a hole of .02 sq. ft. at the apex. A side of the base is 1 ft. in length. ($c. = .625$.) *Ans.* 15.08 sec.

28. Find the discharge under a head of 25 ft. through a thin-lipped square orifice of 1 sq. in. sectional area, (*a*) when it has a border on one side, (*b*) when it has a border on two sides.

Ans. (*a*) .3575 cu. ft. per sec.; (*b*) .3706 cu. ft. per sec.

29. A vessel in the form of a paraboloid of revolution has a depth of 16 in. and a diam. of 12 in. at the top. At the bottom is an orifice of 1 sq. in. sectional area. If water flows into the vessel at the rate of $2\frac{1}{12}$ cubic feet per minute, to what level will the water ultimately rise? How long will it take to rise (*a*) 11 in., (*b*) 11.9 in., (*c*) 11.99 in., (*d*) 12 in. above the orifice? If the supply is now stopped, how long (*e*) will it take to empty the vessel?

Ans. 12 inches; (*a*) 83.095 sec.; (*b*) 124.2 sec.; (*c*) 263.9 sec.; (*d*) an infinite length of time; (*e*) 11.3 sec.

30. If the vessel in Question 29 is a semi-sphere 1 ft. in diameter, to what height will the water rise? How long will it take for the water to rise (*a*) 11 in., (*b*) 12 in. above the orifice? How long (*c*) will it take to empty the vessel?

Ans. 12 inches; (*a*) 67.16 sec.; (*b*) 81.46 sec.; (*c*) 24.13 sec.

31. In a vortical motion two circular filaments of radii r_1, r_2, of velocities v_1, v_2, and of equal weight W are made to change place. Show that a stable vortex is produced if $\frac{v^2}{r}$=const.; and if $r_2 > r_1$, show that the surfaces of equal pressure are cones.

32. Prove that for a Borda's mouthpiece running full the coefficient of discharge is $\frac{1}{\sqrt{2}}$.

33. The surface of the water in a tank is kept at the same level; obtain the discharge at 60 in. below the surface (*a*) through a circular orifice 1 sq. in. in area, (*b*) through a cylindrical ajutage of the same sectional area fitted to the outside, (*c*) through the same ajutage fitted to the inside, and determine the mechanical effect of the efflux in each case. *Ans.* (*a*) 4 36 lbs. per sec.; 20.514 ft.-lbs. per sec.
(*b*) 6.356 " " "; 21.369 " " "
(*c*) 3.488 " " "; 17.44 " " "

34. Water is discharged under a head of 64 ft. through a short cylindrical mouthpiece 12 in. in diameter. Find (*a*) the loss of head due to shock, (*b*) the volume of discharge in cubic feet per second, (*c*) the energy of the issuing jet. ($g = 32$.)
Ans. (*a*) 20.96 ft.; (*b*) 51.54 cub. ft.; (*c*) 393.8 H. P.

35. If a bell-mouth is substituted for the mouthpiece in the preceding question, find the discharge and the mechanical effect of the jet.
Ans. 61.6 cub. ft. per sec.; 470.6 H. P.

36. Compare the energies of a jet issuing under an effective head of 100 ft. through (1) a 12-in. cylindrical ajutage, (2) a 12-in. divergent ajutage, (3) a 12-in. convergent ajutage, the angle of convergence being 21°. Draw the plane of charge in each case.
Ans. (1) 393.8 H. P.; (2) 672.28 H. P.; (3) 618.23 H. P.

37. Find the discharge through a rectangular opening 36 in. wide and 10 in. deep in the vertical face of a dam, the upper edge of the opening being 10 ft. below the water surface.
Ans. 40.2 cub. ft. per sec.

38. Find the discharge in pounds per minute through a Borda's mouthpiece 1 in. in diameter, the lip being 12 in. below the water-surface. *Ans.* 87.714 lbs.

39. Sometimes the crest of a dam is raised by floating a stick L into the position L_1, where it is supported against the verticals. The stick then falls of itself into position L_2 and rests on the crest. Explain the reason of this.

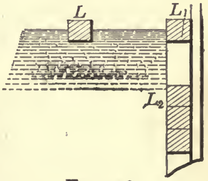


FIG. 48.

40. A sluice 3 ft. square and with a head of 12 ft. over the centre has, from the thickness of the frame, the contraction suppressed on all sides when fully open; when partially open, the contraction exists on the upper edge, i.e., against the bottom of the gate, which is formed of a thin sheet of metal. Find the discharge in cubic feet when opened 1 ft., 2 ft. and also when fully open. *Ans.* 57.77; 114.45; 175.9.

41. What quantity of water flows through the vertical aperture of a dam, its width being 36 in. and its depth 10 in.; the upper edge of the aperture is 16 ft. below the surface. *Ans.* 50.65 cub. ft. per sec.

42. 264 cubic feet of water are discharged through an orifice of 5 sq. ins. in 3 min. 10 sec. Find the mean velocity of efflux.

Ans. 64 ft. per sec.

43. One of the locks on the Lachine Canal has a superficial area of about 12,150 sq. ft., and the difference of level between the surfaces of the water in the lock and in the upper reach is 9 feet. Each leaf of the gates is supplied with one sluice, and the water is levelled up in 2 min. 48 secs. Determine the proper area of the sluice-opening. (Centre of sluice 20 ft. below surface of upper reach.)

Ans. Area of one sluice = 43.73 sq. ft.

44. The horizontal section of a lock-chamber may be assumed a rectangle, the length being 360 ft. When the chamber is full, the surface width between the side walls, which have each a batter of 1 in 12, is 45 ft. How long will it take to empty the lock through two sluices in the gates, each 8 ft. by 2 ft., the height of the water above the centre of the sluices being 13 feet in the lock and 4 feet in the canal on the down-stream side. *Ans.* 594 sec., c being .625.

45. Water approaches a rectangular opening 2 ft. wide with a velocity of 4 ft. per second. At the opening the head of water over the lower edge = 13 ft., and over the surface of the tail-race = 12 ft.; the discharge through the opening is 70 cub. ft. per second. Find the height of the opening. *Ans.* 1.022 ft.

46. The water in a regulating-chamber is 8 ft. below the level of the water in the canal and 8 ft. above the centre of the discharging-sluice. Determine the rise in the canal which will increase the discharge by 10 per cent. *Ans.* 1.68 ft.

The horizontal sectional area of the chamber is constant and equal to 400 sq. ft.; in what time will the water in the chamber rise to the level of that in the canal, if the discharging-sluice is closed; the sluice between the canal and chamber being 3 sq. ft. in area? *Ans.* 150.83 sec.

47. A lock on the Lachine Canal is 270 ft. long by 45 ft. wide and has a lift of $8\frac{3}{8}$ ft.; there are two sluices in each leaf, each $8\frac{3}{4}$ ft. wide by $2\frac{1}{2}$ ft. deep; the head over the horizontal centre line of the sluices is 19 ft. Find the time required to fill the lock. *Ans.* 164.6 sec.

48. Show that the energy of a jet issuing through a large rectangular orifice of breadth B is $125B(H_2^{\frac{5}{2}} - H_1^{\frac{5}{2}})$, H_1, H_2 being the depths below the water-surface of the upper and lower edges of the orifice, and the coefficient of discharge being .625.

49. A reservoir at full water has a depth of 40 ft. over the centre of the discharging-sluice, which is rectangular and 24 in. wide by 18 in. deep. Find the discharge in cubic feet per second at that depth, and also

when the water has fallen to 30, 20, and 10 ft., respectively; find the mechanical effect of the efflux in each case.

Ans. 94.8 cub. ft.; 82.1 cub. ft.; 67 cub. ft.; 47.4 cub. ft.; 431.2 H.P.; 280 H.P.; 152.5 H.P.; 53.95 H.P.

50. Require the head necessary to give 7.8 cubic feet per second through an orifice 36 sq. in. in sectional area. *Ans.* 38.9 ft.

51. The upper and lower edges of a vertical rectangular orifice are 6 and 10 feet below the surface of the water in a cistern, respectively; the width of the orifice is 1 ft. Find the discharge through it.

Ans. 5642 cub. ft. per sec.

52. To find the quantity of water conveyed away by a canal 3 ft. wide, a board with an orifice 2 ft. wide and 1 ft. deep is placed across the canal and dams it back until it attains a height of $2\frac{1}{4}$ ft. above the bottom and $1\frac{3}{4}$ ft. above the lower edge of the orifice. Find the discharge. ($c = .625$.) *Ans.* 17.59 cub. ft. per sec.

53. Six thousand gallons of water per minute are forced through a line of piping *ABC* and are discharged into the atmosphere at *C*, which is 6 ft. vertically above *A*. The pipe *AB* is 12 in. in diameter and 12 ft. in length; the pipe *BC* is 6 in. in diameter and 12 ft. in length. Disregarding friction, find the "loss in shock" and draw the plane of charge.

Ans. Loss of head in shock = 57.9 ft.

54. What should be the height of a drowned weir 400 ft. long, to deepen the water on the up-stream side by 50 per cent, the section of the stream being 400 ft. × 8 ft., and the velocity of approach 3 ft. per second? *Ans.* 8.396 ft.

55. The two sluices each 4 ft. wide by 2 ft. deep in a lock-gate are submerged one half their depth. The constant head of water above the axis of the sluice is 12 ft. Find the discharge through the sluice, the velocity of approach being 4 ft. per second.

Ans. 16626.2 cub. ft. per minute.

56. Find the flow through a square opening, one diagonal being vertical and 12 in. in length, and the upper extremity of the diagonal being in the surface of the water. *Ans.* 1.727 cub. ft. per sec.

57. The locks on the Montgomeryshire Canal are 81 ft. long and $7\frac{3}{4}$ ft. wide; at one of the locks the lift is 7 ft.; a 24-in. pipe leads the water from the upper level and discharges below the surface of the lower level into the lock-chamber; the mouth of the pipe is square, 2 ft. in the side, and gradually changes into a circular pipe 2 ft. in diameter. Find time of filling the lock. ($c = 1$.) *Ans.* 130 secs.

58. A canal lock is 115.1 ft. long and 30.44 ft. wide; the vertical depth from centre of sluice to lower reach is 1.0763 ft., the charge being 6.3945 ft.; the area of the two sluices is 2 × 6.766 sq. ft. Find the time of filling up to centre of sluices. ($c = .625$ for the sluice, but is reduced

to .548 when both are opened.) Also, find time of filling up to level of upper reach from centre of sluice-doors. *Ans.* 25 sec.; 298 secs.

59. A reservoir half an acre in area with sides nearly vertical, so that it may be considered prismatic, receives a stream yielding 9 cub. ft. per second, and discharges through a sluice 4 ft. wide, which is raised 2 ft. Calculate the time required to lower the surface 5 ft., the head over the centre of the sluice when opened being 10 feet. *Ans.* 1079 secs.

60. Show that in a channel of V section an increment of 10 per cent in the depth will produce a corresponding increment of 5 per cent in the velocity of flow and of 25 per cent in the discharge.

61. The angle of a triangular notch is 90°. How high must the water rise in the notch so that the discharge may be 1000 gallons per minute? *Ans.* 12 inches very nearly.

62. Show that upon a weir 10 feet long with 12 inches depth of water flowing over, an error of 1/1000 of a foot in measuring the head will cause an error of 3 cubic feet per minute in the discharge, and an error of 1/100 of a foot in measuring the length of the weir will cause an error of 2 cubic feet in the discharge.

63. In the weir at Killaloe the total length is 1100 ft., of which 779 ft. from the east abutment is level, while the remainder slopes 1 in 214, giving a total rise at the west abutment of 1.5 ft. Calculate the total discharge over the weir when the depth of water on the level part is 1.8 ft., which gives .3 ft. on highest part of weir. (Divide slope into 8 lengths of 40 ft. each, and assume them severally level, with a head equal to the arithmetic mean of the head at the beginning and end of each length.)

Ans. 7483 cub. ft. per sec.

64. A watercourse is to be augmented by the streams and springs above its level. The latter are severally dammed up at suitable places and a narrow board is provided in which an opening 12 in. long by 6 in. deep is cut for an overfall; it was surmised that this would be sufficient for the largest streams; another piece attached to the former would reduce the length to 6 in. for smaller streams. Calculate the delivery by the following streams:

In No. 1 stream with the 12-in. notch, depth over crest = .37 ft.
" No. 2 " " " 6-in. " " " " = .41 ft.
" No. 3 " " " 12-in. " " " " = .29 ft.
" No. 4 " " " 6 in. " " " " = .19 ft.

(Take into account the side contractions.)

Ans. No. 1, .695 cub. ft.; No. 2, .3658 cub. ft.; No. 3, .4904 cub. ft.; No. 4, .1275 cub. ft.

65. The horizontal sectional area of a reservoir is constant and = 10,000 square feet. When the reservoir is full, a right-angled notch 2 ft. deep is opened. Find the time in which the level of the water falls to the bottom of the notch. *Ans.* 15.3 min.

66. A weir passes 6 cubic feet per second, and the head over the crest is 8 inches. Find the length of the weir. *Ans.* 3.3068 ft.

67. A weir 400 ft. long, with a 9-in. depth of water on it, discharges through a lower weir 500 ft. long. Find the depth of water on the latter. *Ans.* .6457 ft.

68. A stream 30 ft. wide, 3 ft. deep, discharges 310 cubic feet per second; a weir 2 feet deep is built across the stream. Find increased depth of latter, (*a*) neglecting velocity of approach, (*b*) taking velocity of approach into account. *Ans.* (*a*) 1.26 ft. to 1.265 ft.; (*b*) 1.19 ft.

69. A weir is 545 ft. long; how high will the water rise over it when it rises .68 ft. upon an upper weir 750 ft. long? *Ans.* .8413 ft.

70. In a stream 50 ft. wide and 4 ft. deep water flows at the rate of 100 ft. per minute; find the height of a weir which will increase the depth to 6 ft., (1) neglecting velocity of approach, (2) taking velocity of approach into account. *Ans.* (1) 4.4126 ft; (2) 4.4509 ft.

71. A stream 50 ft. wide and 4 ft. deep has a velocity of 3 ft. per second; find the height of the weir which will double the depth, (1) neglecting velocity of approach, (2) taking velocity of approach into account. *Ans.* (1) 5.615 ft.; (2) 5.7688 ft.

72. A stream 80 ft. wide by 4 ft. deep discharges across a vertical section at the rate of 640 cubic feet per second; a weir is built in the stream, increasing its depth to 6 ft. Find the height of the weir. *Ans.* 4.233 ft.

73. Salmon-gaps are constructed in a weir; they are each 10 ft. wide and their crests are 18 in. below the weir crest. Calculate the discharge down three of these gaps, the water on the level part of the weir being 8 in. deep. *Ans.* 238.15 cub. ft. per sec.

74. A pond whose area is 12,000 sq. ft. has an overfall outlet 36 in. wide, which at the commencement of the discharge has a head of 2.8 ft. Find the time required to lower the surface 12 in. *Ans.* 354.72 sec.

75. How much water will flow in an hour through a rectangular notch 24 in. wide, the surface of still water being 8 in. above the crest of the notch? (Take into account side contraction.) *Ans.* 3.386 ft.

76. Show that when the water flowing over has a depth greater than .3874 ft. it is carried completely over the longitudinal opening, .83 ft. in width. At what depth does *all* the water flow in? *Ans.* .221 ft.

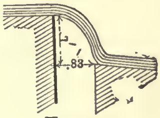


FIG. 49.

CHAPTER II.

FLUID FRICTION.

1. Fluid Friction.—The term fluid friction is applied to the resistance to motion which is developed when a fluid flows over a solid surface, and is due to the viscosity of the fluid. This resistance is necessarily accompanied by a loss of energy caused by the production of eddies along the surface, and similar to the loss which occurs at an abrupt change of section, or at an angle in a pipe or channel.

Froude's experiments on the resistance to the edgewise motion of planks in a fluid mass, the planks being $\frac{3}{16}$ in. thick, 19 in deep, and 1 to 50 ft. long, each plank having a fine cutwater and run, are summarized in the following table:

Nature of Surface Covering.	Length of Surface in Feet.											
	2 Feet.			8 Feet.			20 Feet.			50 Feet.		
	A	B	C	A	B	C	A	B	C	A	B	C
Varnish	2.00	.41	.390	1.85	.325	.264	1.85	.278	.240	1.83	.250	.226
Paraffine		.38	.370	1.94	.314	.260	1.93	.271	.237			
Tinfoil	2.16	.30	.295	1.99	.278	.263	1.90	.262	.244	1.83	.246	.232
Calico	1.93	.87	.725	1.92	.626	.504	1.89	.531	.447	1.87	.474	.423
Fine sand	2.00	.81	.690	2.00	.583	.450	2.00	.480	.384	2.06	.405	.337
Medium sand	2.00	.90	.730	2.00	.625	.488	2.00	.534	.465	2.00	.488	.456
Coarse sand	2.00	1.10	.880	2.00	.714	.520	2.00	.588	.490			

Columns A give the power of the speed (v) to which the resistance is approximately proportional.

Columns B give the mean resistance in lbs. per square foot of the whole surface of a board of the lengths stated in the table.

Columns C give the resistance, in pounds, of a square foot of surface at the distance sternward from the cutwater stated in the heading, each plank having a standard speed of 10 ft. per second. The resistance at other speeds can be easily calculated.

An examination of the table shows that the mean resistance per square foot diminishes as the length of the plank increases. This may be explained by the supposition that the friction in the forward portion of the plank develops a force which drags the water along with the surface, so that the relative velocity of flow over the rear portion is diminished. Again, the decrease of the mean resistance per square foot is .132 lb. when the length of a varnished plank is increased from 2 to 20 ft., while it is only .028 lb. when the length increases from 20 to 50 ft. Hence, for greater lengths than 50 ft. the decrease of resistance may be disregarded without much, if any, practical effect.

Thus, generally speaking, these experiments indicate that the mean resistance is proportional to the nth power of the relative velocity, n varying from 1.83 to 2.16, and its average value being very nearly 2.

Colonel Beaufoy, as a result of experiments at Deptford, also assumed the mean resistance to be proportional to the nth power of the relative velocity, the value of n in three series of observations being 1.66, 1.71 and 1.9.

The frictional resistance is evidently proportional to some function of the velocity, $F(v)$, which should vanish when v is nil, as when the surface is level, and should increase with v.

Coulomb assumed the function $F(v)$ to be of the form $av + bv^2$, a and b being coefficients to be determined by experiment. Experiment shows that when v does not exceed 5 ft. per minute the resistance is directly proportional to the velocity, but that it is more nearly proportional to the square of the velocity when the velocity exceeds 30 ft. per minute; or,

$$F(v) = av \text{ when } v \leqq 5 \text{ ft. per minute,}$$

and

$$F(v) = bv^2 \text{ when } v \geqq 30 \text{ ft. per minute.}$$

Again, observations on the flow of water in town mains indicate that no difference of resistance is developed under widely varying pressures, and this independence of pressure is also verified by Coulomb's experiment showing that if a disk is oscillated in water there is no apparent change in the rate of decrease of the oscillations, whether the water is under atmospheric pressure or not.

From the preceding and other similar experiments the following general laws of fluid friction have been formulated:

(1) The frictional resistance is independent of the pressure between the fluid and the surface over which it flows.

(2) The frictional resistance is proportional to the area of the surface.

(3) The frictional resistance is proportional to some function, usually the square, of the velocity.

To these three laws may probably be added a fourth, viz.:

(4) The frictional resistance is proportional to the density of the fluid.

A fifth law, viz., that "the frictional resistance is independent of the nature of the surface against which the fluid flows," has been sometimes enunciated, and at very low velocities the law is approximately true. At high velocities, however, such as are common in engineering practice, the resistance has been shown by experiment, and especially by the experiments carried out by Darcy, to be very largely influenced by the nature of the surface.

Let p be the frictional resistance in pounds per square foot of surface at a velocity of 1 ft. per second.

Let A be the area of the surface in square feet.

Let v be the relative velocity of the surface and the water in which it is immersed.

Let R be the total frictional resistance.

Then from the laws of fluid friction

$$R = p \, . \, Av^2.$$

Take $f = \frac{2g}{w}p$, w being the specific weight of the fluid. Then

$$R = fwA\frac{v^2}{2g}.$$

The coefficient f is approximately constant for any given surface, and is termed the coefficient of fluid friction. The power absorbed by the frictional resistance

$$= pAv^2 \times v = pAv^3 = fwA\frac{v^3}{2g}.$$

TABLE GIVING THE AVERAGE VALUES OF f IN THE CASE OF LARGE SURFACES MOVING IN AN INDEFINITELY LARGE MASS OF WATER.

Surface.	Coefficient of Friction (f).
New well-painted iron plate..............	.00489
Painted and planed plank	.0035
Surface of iron ships	.00362
Varnished surface........................	.00258
Fine sand surface........................	.00418
Coarse sand surface......................	.00503

2. **Surface Friction of Pipes.**—Assuming that the laws of fluid friction already enunciated hold good when water flows through a pipe, it has been shown by numerous experiments that the coefficient of friction f lies between the limits .005 and .01, its average value under ordinary conditions being about .0075. No single value of f is applicable to very different cases. Indeed, f depends not only upon the condition of the surface, but also upon the diameter of the pipe and the velocity of the water. Some authorities have expressed its value by a relation of the form

$$\frac{f}{g} = a + \frac{b}{v},$$

a and b being constants whose values are to be determined by experiment.

The following table gives some of the best numerical results obtained for a and b:

Authority.	a	b
Prony	.00021230	.00003466
D'Aubuisson.	.0002090	.000037608
Eytelwein.	.00017059	.00004441

In pipes of small diameter in which the velocity of flow is less than 4 in. per second the term a may be disregarded so that

$$\frac{f}{g}=\frac{b}{v}.$$

In ordinary practice and when the pipes have been in use for some time, the velocity usually exceeds 4 in. per second, and the term $\frac{b}{v}$ may then be disregarded, so that

$$\frac{f}{g}=a.$$

Now Darcy's experiments have shown that it is more correct to assume that a and b, instead of being constant, are variable, and Darcy expressed them as functions of the diameter of the pipe.

Thus, for pipes in which the velocity exceeds 4 in. per second, Darcy took

$$\frac{f}{g}=a=\alpha+\frac{\beta}{d},$$

d being the diameter of the pipe, and α and β coefficients.

Darcy also gave the following values for α and β:

	α	β
For drawn wrought-iron or smooth cast-iron pipes	.0001545	.000012973
For pipes with surfaces covered by light incrustations	.0003093	.00002598

These coefficients can be put into the following very simple form without sensibly altering their values :

For clean pipes........... $f = .005\left(1 + \frac{1}{12d}\right)$

For slightly incrusted pipes $f = .01\left(1 + \frac{1}{12d}\right)$

d being the diameter in feet.

Darcy proposed to include all cases by expressing f more generally in the form

$$\frac{f}{g} = \alpha + \frac{\beta}{d} + \left(\alpha' + \frac{\beta'}{d^2}\right)\frac{1}{v},$$

in which, for new and smooth iron pipes,

$$\alpha = .00003959, \qquad \beta = .00002603125\,;$$
$$\alpha' = .000064375, \qquad \beta' = .000000335625.$$

These values are rarely of any practical use.

TABLE GIVING DARCY'S VALUES OF f FOR VELOCITIES EXCEEDING 4 IN. PER SECOND.

Diam. of Pipe in Inches.	Value of f. New Pipes.	Value of f. Incrusted Pipes.	Diam. of Pipe in Inches.	Value of f. New Pipes.	Value of f. Incrusted Pipes.	Diam. of Pipe in Inches.	Value of f. New Pipes.	Value of f. Incrusted Pipes.
2	.0075	.0150	9	.00556	.01111	27	.00519	.01037
3	.00667	.01333	12	.00542	.01083	30	.00517	.01033
4	.00625	.0125	15	.00533	.01067	36	.00514	.01028
5	.0060	.012	18	.00528	.01056	42	.00512	.01024
6	.00583	.01167	21	.00524	.01048	48	.00510	.01021
7	.00571	.01143	24	.00521	.01042	54	.00509	.01019
8	.00563	.01125						

Again, Weisbach has proposed the formula

$$f = a + \frac{b}{\sqrt{v}},$$

where $a = .003598$ and $b = .004289$.

3. **Resistance of Ships.**—The motion of a ship through water causes the production of waves and eddies, and the total resistance to the movement of a ship is made up of a frictional resistance, a wave-making resistance, and an eddy-making resistance. Although there is no theory by which the resistance at a given speed of a ship of definite design can be absolutely determined, Froude's experiments render it possible to make certain inferences and furnish some useful data.

According to Froude, the frictional resistance is sensibly the same as that of a rectangular surface moving with the same speed, of the same length as the ship in the direction of motion, and of an area equal to the immersed surface of the ship. Experiments seem to indicate that as the speed increases, the frictional resistance of well-designed ships with clean bottoms is from 90 to 60 per cent of the total resistance, and that the percentage is greater when the bottoms become foul.

The wave-making resistance is especially affected by the form and proportions of the ship, depending, for a given length, upon the proportions of the entrance, middle body, and run. For every ship there is a limit of speed below which the resistance is approximately proportional to the square of the speed, being chiefly due to friction, and beyond which it increases more rapidly than as the square.

The eddy-resistance in the case of well-formed ships should not exceed about 10 per cent of the total resistance, and is often much less.

Froude's law of resistance may be enunciated as follows:

Let l_1, l_2 be the lengths of a ship and its model.

Let Δ_1, Δ_2 be the displacements of a ship and its model.

Let R_1, R_2 be the resistances of a ship and its model at the speeds v_1 and v_2.

Then, if

$$\frac{v_1}{v_2} = \frac{l_1^{\frac{1}{2}}}{l_2^{\frac{1}{2}}} = \frac{\Delta_1^{\frac{1}{6}}}{\Delta_2^{\frac{1}{6}}},$$

the resistances are in the ratio of

$$\frac{R_1}{R_2} = \frac{\Delta_1}{\Delta_2} = \frac{l_1^3}{l_2^3}.$$

Hence, too, the H. P., and therefore also the coal consumption per hour, is proportional to Rv, that is, to

$$\Delta^{\frac{7}{6}} \quad \text{or} \quad l^{\frac{7}{2}},$$

and the coal consumption per mile is proportional to

$$\Delta \quad \text{or to} \quad l^3.$$

Again, R is proportional to l^3;
that is, to $l \times l^2$;
that is, to $v^2 \times \Delta^{\frac{2}{3}}$;

and it is sometimes convenient to express the resistance in pounds in the form

$$R = k \,.\, v^2 \Delta^{\frac{2}{3}},$$

v being the speed in knots, Δ the displacement in tons, and k a coefficient depending upon the type of ship and varying from .55 to .85 when the bottom is clean.

CHAPTER III.

FLOW OF WATER IN PIPES.

1. Assumptions.—In the ordinary theory of the flow of water in a pipe it is assumed that the water consists of thin plane layers perpendicular to the axis of the pipe, that each layer is driven through the pipe by the action of gravity and by the difference of pressure on its plane faces, and that the liquid molecules in any layer at any given moment will also be found in a plane layer after any interval of time. In such motion the internal work done in deforming a layer may be generally disregarded.

It is further assumed that there is no variation of velocity over the surface of a layer, and this is equivalent to saying that each liquid molecule in a cross-section has the same mean velocity.

The disagreement of these assumptions with the results of recent experimental researches will be referred to in a subsequent article.

2. Steady Motion in a Pipe of Uniform Section.—Since the motion is to be steady, the same volume Q cub. ft. of water will always arrive at any given cross-section of A square feet with the same mean velocity v ft. per second. Then

$$Q = Av.$$

But since the pipe is of constant diameter, A is constant, and hence also v is constant, so that the mean velocity is the same throughout the whole length of the pipe.

Consider an elementary mass of the fluid $AABB$, bounded by the pipe and by the two cross-sections AA, BB. Let dl

be the length AB of the element, the length l ft. of the pipe being measured along the axis from any origin O.

Let z, $z + dz$ be the elevations in feet above a datum line of the centres of pressure in the cross-sections AA, BB, respectively.

Let p, $p + dp$ be the intensities of the pressures on these cross-sections in pounds per square foot.

Let P be the perimeter of the pipe.

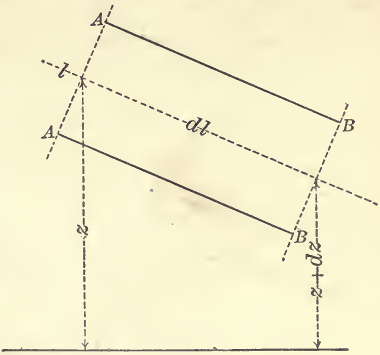


FIG. 50.

Let w be the specific weight of the water in pounds per cubic foot.

Work Done by Gravity.—In one second wQ lbs. of water are transferred from AA to BB, falling through a vertical distance of dz ft. Thus the work done by gravity per second

$$= - wQ \,.\, dz,$$

a positive quantity if dz is negative, and *vice versa.*

Work Done by Pressure.—The total pressure on AA parallel to the axis $= pA$; the total pressure on BB parallel to the axis $= (p + dp)A$.

Therefore the total resultant pressure parallel to the axis in the direction of motion $= - A \,.\, dp$, and the work done per second on the volume Q by this pressure $= - Q \,.\, dp$.

Note.—The work done by the pressure at the pipe surface is nil, as its direction is at right angles to the line of motion.

Work Absorbed by Frictional Resistance.—From the laws of fluid friction this work per second is evidently

$$= - P \,.\, dl \,.\, F(v) \times v = - \frac{P}{A} \,.\, Q \,.\, F(v) \,.\, dl,$$

the sign being negative as the work is done against a resistance.

Since the motion is steady, the work done by the external forces must be equivalent to the work absorbed by the frictional resistance, and hence

$$-wQ \, . \, dz - Q \, . \, dp - \frac{P}{A} Q \, . \, F(v) \, . \, dl = 0,$$

or

$$dz + \frac{dp}{w} + \frac{P}{A} \, . \, \frac{F(v)}{w} \, . \, dl = 0.$$

Integrating,

$$z + \frac{p}{w} + \frac{P}{A} \, . \, \frac{F(v)}{w} \, . \, l = \text{a constant} = H,$$

so that H ft.-lbs. per pound of fluid is the uniformly distributed total constant energy.

$\frac{A}{P}$ is called the hydraulic mean radius of a pipe and will be denoted by m.

Take

$$\frac{F(v)}{w} = f \frac{v^2}{2g},$$

the value adopted in ordinary practice, f being the coefficient of friction. Then

$$z + \frac{p}{w} + \frac{fl}{m} \frac{v^2}{2g} = H.$$

Let z_1, A_1, p_1 be the elevation above datum, the area of the cross-section, and the intensity of the pressure at any point X on the axis of the pipe distant l_1 from the origin (Fig. 51).

Let z_2, A_2, p_2 be the elevation above datum, the area of the cross-section, and the intensity of the pressure at any other point Y on the axis distant l_2 from the origin (Fig. 51).

Then, from the equation just deduced,

$$z_1 + \frac{p_1}{w} + \frac{fl_1}{m}\frac{v^2}{2g} = H = z_2 + \frac{p_2}{w} + \frac{fl_2}{m}\frac{v^2}{2g}.$$

Hence

$$\left(z_1 + \frac{p_1}{w}\right) - \left(z_2 + \frac{p_2}{w}\right) = \frac{f}{m}\frac{v^2}{2g}(l_2 - l_1) = \frac{fL}{m}\frac{v^2}{2g},$$

L being the length $l_2 - l_1$ of the pipe between the two points X and Y.

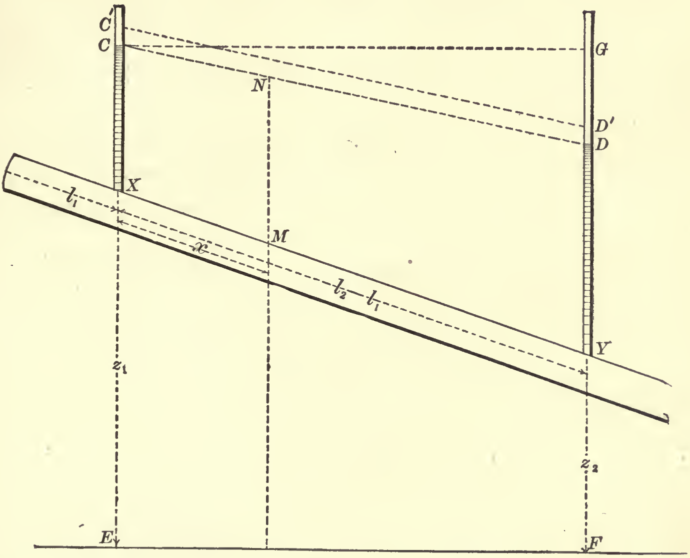


FIG. 51.

Let vertical tubes (pressure-columns) be inserted in the pipe at X and at Y. The water will rise in these tubes to the levels C and D, and evidently

$$p_1 = w \, . \, CX + p_0,$$
$$p_2 = w \, . \, DY + p_0,$$

p^0 being the intensity of the atmospheric pressure.

Hence, if CX and DY are produced meet the datum line in E and F,

$$z_1 + \frac{p_1}{w} = z_1 + CX + \frac{p_0}{w} = CE + \frac{p_0}{w}$$

and

$$z_2 + \frac{p_2}{w} = z_2 + DY + \frac{p_0}{w} = DF + \frac{p_0}{w}.$$

Therefore

$$\left(z_0 + \frac{p_1}{w}\right) - \left(z_2 + \frac{p_2}{w}\right) = CE - DF = DG = \frac{fL}{m}\frac{v^2}{2g},$$

G being the point in which the horizontal through C meets FD produced.

DG is called the "virtual fall" of the pipe, being the fall of level in the pressure-columns; and since there would be no fall of level if the friction were nil, DG is said to be the head lost in friction in the distance XY.

Denote this head by h; then

$$h = \frac{fL}{m}\frac{v^2}{2g},$$

and therefore

$$\frac{h}{L} = \frac{f}{m}\frac{v^2}{2g}.$$

This ratio $\frac{h}{L}$ is designated the virtual slope of the pipe, and is the head lost in friction per unit of length It will be denoted by i, so that

$$\frac{h}{L} = i = \frac{f}{m}\frac{v^2}{2g}.$$

If the section of the pipe is a circle of diameter d, or a square with a side of length d, then

$$m = \frac{A}{P} = \frac{d}{4},$$

and

$$\frac{h}{L} = i = \frac{4f}{d}\frac{v^2}{2g}.$$

3. Influence upon the Flow of the Pipe's Position and Inclination.—In Fig. 51 join CD. Now since the fall of level (h) is proportional to L, the free surface in any other column between X and Y must also be on the line CD. Thus the pressure p' at any intermediate point M distant $x\,(=XM)$ from X is given by

$$\frac{p'}{w} = MN + \frac{p_0}{w} = CX + \frac{x}{L}(DY - CX) + \frac{p_0}{w}.$$

Hence, at every point of a pipe laid below CD, the fluid pressure (p') exceeds the atmospheric pressure (p_0) by an amount $w\,.\,MN$, so that if holes are made in such a pipe the water will flow out and there will be no tendency on the part of the air to flow in. In pipes so placed vertical bends may be introduced, care being taken to provide for the removal of the air which may collect in the upper parts of the bends.

If the line of the pipe coincides with CD, i.e., with the virtual slope or line of free surface level, $MN = 0$, and the fluid pressure is equal to that of the atmosphere. If holes are now made in the pipe it can easily be shown by experiment that there will be neither any tendency on the part of the water to flow out nor on the part of the air to flow in.

Next take $CC' = DD' = \frac{p_0}{w}$, and join $C'D'$.

If the pipe is placed in any position between CD and $C'D'$ MN becomes negative, and the fluid pressure in the pipe is less than that of the atmosphere. If holes are made in this pipe, there will be no tendency on the part of the water to flow out, but the air will flow in. Thus, if a pipe rises above the line of virtual slope, there is a danger of air accumulating in the pipe and impeding, or perhaps wholly stopping, the flow. No vertical bends should be introduced, as the air is easily set free and would collect in the upper parts of the bends, with the effect of impeding the flow and of acting detrimentally upon the water itself, which the liberation of the air renders less wholesome. If the line of pipe coincide with $C'D'$, then the fluid pressure is nil.

Finally, if the pipe at any point rises above $C'D'$, the pressure becomes negative, which is impossible. In fact, the continuity of flow is destroyed, and the pipe will no longer run full bore. Air will be disengaged and will rise and collect at the point in question, so that in order to prevent the flow being wholly impeded, it will be necessary to introduce an air-chamber at this point from which the air can be removed when required.

Note.—In the preceding it has been assumed that the pipe is straight. If the pipe is curved, so also is the line of virtual slope. In ordinary practice, however, the vertical changes of level in a pipe at different points are small as compared with the length of the pipe, and distances measured along the pipe are sensibly proportional to distances measured along the horizontal projection of the pipe. Hence the line of virtual slope may be assumed to be a straight line without error of practical importance.

4. Transmission of Energy by Hydraulic Pressure.—Let Q cub. ft. of water per second be driven through a pipe of diameter d ft. and length L ft. under a total head of H ft. Also let n per cent. of the total head be absorbed in overcoming the frictional resistance in the pipe. Then

the head expended in useful work $= H - h$

$h = nH/100$

$$= H\left(1 - \frac{n}{100}\right),$$

$$\text{and the efficiency} = \frac{H-h}{H} = 1 - \frac{n}{100}.$$

Again,

$$\frac{nH}{100} = h = \frac{4fL}{d}\frac{v^2}{2g} = \frac{fLQ^2}{\pi^2 d^5}.$$

Since $Q = \frac{\pi d^2}{4}v$, and g is assumed to be 32, thus,

$$Q = \frac{\pi}{10}\sqrt{\frac{nHd^5}{fL}},$$

and the work transmitted in foot-pounds per second

$$= wQH = \frac{275}{14}\sqrt{\frac{nH^3d^5}{fL}}.$$

If $N =$ the number of horse-power transmitted, then

$$N = \frac{1}{550}\frac{275}{14}\sqrt{\frac{nH^3d^5}{fL}} = \frac{1}{28}\sqrt{\frac{nH^3d^5}{fL}},$$

and this equation also gives the distance L to which N horse-power can be transmitted with a loss of n per cent of the total head.

Again,

$$\text{the efficiency} = 1 - \frac{h}{H} = 1 - \frac{2fL}{gH}\frac{v^2}{d} = 1 - \frac{2fLw}{g}\frac{v^2}{pd},$$

$p(= wH)$ being the pressure corresponding to the head H. Thus, the efficiency is constant if $\frac{v^2}{pd}$ is constant.

Assuming this to be the case, take $v^2 = c^2 . pd$. Then the total energy transmitted $= wQH = w\frac{\pi d^2}{4}vH$

$$= \frac{\pi c}{4}p^{\frac{3}{2}} d^{\frac{5}{2}}.$$

If it be also assumed that the thickness t of the pipe-metal is so small that the formula

$$pd = 2f't$$

holds true, f' being the circumferential stress induced in the metal, then

$$\text{the energy transmitted} = \frac{\pi c}{4} p^{\frac{3}{2}} d^{\frac{5}{2}}$$

$$= \frac{\pi c f' t d}{2} \sqrt{pd}$$

$$= \frac{c f' V}{2} \sqrt{pd},$$

V being the volume of the pipe per unit of length.

Hence, for a given volume (V) of metal and a constant efficiency, the energy transmitted is a maximum when pd is a maximum.

If p is increased beyond a certain limit, the ratio $\frac{t}{d}$ is no longer small and the thickness t will have a greater value than that given by the equation $pd = 2f't$. Then the cost of the pipe will also increase. On the other hand, if d is increased the ratio $\frac{t}{d}$, and therefore also the pressure p, will remain small, and thus the cost of the pipe will not increase. Hence it is more economical to employ large pipes and low pressures than small pipes and high pressures.

Note.—The efficiency diminishes as v increases, so that, as far as the efficiencv is concerned, it is advantageous to transmit the energy at a low speed.

5. Flow in a Pipe of Uniform Section and of Length L, connecting two Reservoirs at Different Levels.—Let z ft. be the difference of level between the water-surface in the two reservoirs.

FIG. 52.

The work done per second is evidently equal to the work done by the fall of wQ pounds of water through the vertical distance z, and is expended—

(1) In producing the velocity of flow v feet per second which requires a head of z_1 feet and an expenditure of wQz_1 foot-pounds of work per second;
(2) In overcoming the resistance at the entrance from the upper reservoir into the pipe, which requires a head of z_2 feet and an expenditure of wQz_2 foot-pounds of work per second.
(3) In overcoming the frictional resistance which requires a head of z_3 feet and an expenditure of wQz_3 foot-pounds of work per second. Thus

$$wQz = wQz_1 + wQz_2 + wQz_3,$$

or

$$z = z_1 + z_2 + z_3.$$

Now $z_1 = \frac{v^2}{2g}$ feet, and the corresponding energy wQz_1 is ultimately wasted in producing eddy motions, etc., in the lower reservoir.

z_2 may be expressed in the form $n\frac{v^2}{2g}$ feet, n being a coefficient whose value varies with the nature of the construction of the entrance into the pipe. If the pipe-entrance is bell-mouth in form, $n = .08$, but if it is cylindrical, $n = .5$. Finally,

$$z_3 = \frac{L}{m}\frac{F(v)}{w} \text{ ft.} = \frac{4fL}{d}\frac{v^2}{2g} \text{ ft.},$$

taking $\frac{F(v)}{w} = f\frac{v^2}{2g}$, as is usual in practice. Hence

$$z = \frac{v^2}{2g}\left(1 + n + \frac{4fL}{d}\right)$$

$$= \frac{Q^2}{4\pi^2 d^4}\left(1 + n + \frac{4fL}{d}\right),$$

since $Q = \frac{\pi d^2}{4}v$, and g is assumed to be 32.

For given values of Q and z a first approximate value of d may be obtained from the last equation by neglecting the term $\frac{Q^2}{4\pi^2 d^4}(1 + n)$. Call this value d_1, and substitute it for the d in the term $\frac{4fL}{d}$ within the brackets. A second approximation may now be made by deducing d from the formula

$$z = \frac{Q^2}{4\pi^2 d^4}\left(1 + n + \frac{4fL}{d_1}\right),$$

and the operation may be again repeated if desired.

Generally speaking, $1 + n$ is usually very small as compared with $\frac{4fL}{d}$, and may be disregarded without error of practical importance.

The formula then becomes

$$z = \frac{4fL}{d}\frac{v^2}{2g},$$

which is known as Chezy's formula for long pipes.

In fact, the term $1 + n$ need only be taken into account in the case of short pipes and high velocities.

6. Losses of Head due to Abrupt Changes of Section, Elbows, Valves, etc.—When the velocity or the direction of motion of a mass of water flowing through a pipe is abruptly changed, the water is broken up into eddies or irregular motions which are soon destroyed by viscosity, the corresponding energy being wasted.

CASE I. *Loss due to a sudden contraction.* (Art. 16, Chap. I.)

(*a*) Let water flow from a pipe (Fig. 53), or from a reservoir (Fig. 54) into a pipe of sectional area A.

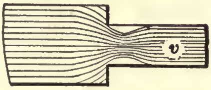


FIG. 53.

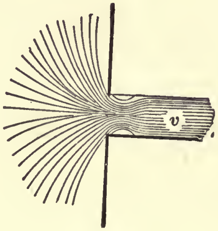


FIG. 54.

Let c_c be the coefficient of contraction.

Then the area of the contracted section $= c_c A$, and

$$\text{the loss of head} = \frac{1}{2g}\left(\frac{v}{c_c} - v\right)^2$$

$$= \frac{v^2}{2g}\left(\frac{1}{c_c} - 1\right)^2$$

$$= m\frac{v2}{2g},$$

where $m = \left(\frac{1}{c_c} - 1\right)^2$.

The value of m has not been determined with any great degree of accuracy; but if $c_c = .64$, then $m = .316$.

When the water enters a cylindrical (not bell-mouthed) pipe from a large reservoir, the value of m is about .505.

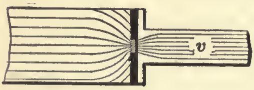


FIG. 55.

(*b*) Let the water flow across the abrupt change of section through a central orifice in a diaphragm placed as in Fig. 55.

Let a be the area of the orifice.

Then $c_c a$ is the area of the contracted section, and

$$\text{the loss of head} = \left(\frac{A}{c_c a} - 1\right)^2 \frac{v^2}{2g} = m\frac{v^2}{2g},$$

where $m = \left(\dfrac{A}{c_c a} - 1\right)^2$.

According to Weisbach,

if $\frac{a}{A} =$	.1	.2	.3	.4	.5
$c_c =$	.616	.614	.612	.610	.607
$m =$	231.7	50.99	19.78	9.612	5.256
if $\frac{a}{A} =$	.6	.7	.8	.9	1.00
$c_c =$	.605	.603	.601	.598	.596
$m =$	3.077	1.876	1.169	.734	.48

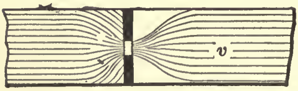


FIG. 56.

(*c*) A diaphragm with a central orifice of area a, placed in a cylindrical pipe of sectional area A as in Fig. 56.

The "contracted area" of the water $= c_c a$ and

$$\text{the loss of head} = \frac{1}{2g}\left(\frac{vA}{c_c a} - v\right)^2 = \frac{v^2}{2g}\left(\frac{A}{c_c a} - 1\right)^2$$

$$= m\frac{v^2}{2g},$$

where $m = \left(\dfrac{A}{c_c a} - 1\right)^2$.

Generally m must be determined by experiment, but Weisbach gives the following results :

if $\frac{a}{A} =$	.1	.2	.3	.4	.5
$c_c =$	.624	.632	.643	.659	.681
$m =$	225.9	47.77	30.83	7.801	3.753
if $\frac{a}{A} =$	.6	.7	.8	.9	1.00
$c_c =$	.712	.755	.813	.892	1.00
$m =$	1.796	.797	.29	.06	00

CASE II. *Loss due to a Sudden Enlargement.* (Fig. 57.)

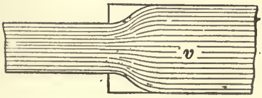


FIG. 57.

Let A_1 = external area of small pipe.
" A_2 = " " " large "

Then, loss of head $= \frac{1}{2g}\left(\frac{vA_2}{A_1} - v\right)^2 = \frac{v^2}{2g}\left(\frac{A_2}{A_1} - 1\right)^2$

$$= m\frac{v^2}{2g},$$

where $m = \left(\frac{A_2}{A_1} - 1\right)^2$.

Note.—The losses of head in Case I (a) and in Case II may be avoided by substituting a gradual and regular change of section for the abrupt changes.

CASE III. *Loss of Head due to Elbows.* (Fig. 58.)—The loss of head due to the disturbance caused by an elbow is expressed by Weisbach in the form $m\frac{v^2}{2g}$,

where $m = .9457 \sin^2 \frac{\phi}{2} + 2.047 \sin^4 \frac{\phi}{2}$,

ϕ being the elbow angle.

Weisbach deduced this formula from the results of experiments with pipes 1.2 in. in diameter.

The velocity v_1 with which the water flows along the length AB may be resolved into a component v with which the water flows along BC and a component u at right angles to the

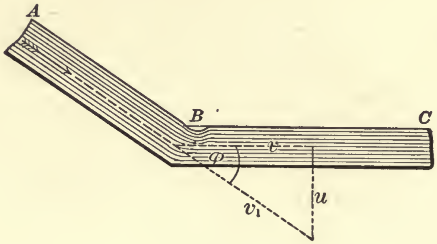


FIG. 58.

direction of v. The component u and therefore the corresponding head, viz., $\frac{u^2}{2g}$, is wasted. The component u evidently diminishes with the angle ϕ and becomes nil when a gradually and continuously curved bend is substituted for the elbow.

CASE IV. Weisbach gives the following empirical formula for the loss of head at a bend in a pipe:

$$h_b = m_b \frac{v^2}{2g},$$

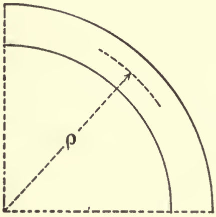


FIG. 59.

where $m = .131 + 1.847\left(\frac{d}{2\rho}\right)^{\frac{1}{2}}$

for a circular pipe of diameter d, ρ being the radius of curvature of the bend, and

$$m = .124 + 3.104\left(\frac{s}{2\rho}\right)^{\frac{1}{2}}$$

for a pipe of rectangular section, s being the length of a side of the section parallel to the radius of curvature (ρ) of the bend.

CASE V. *Valves, Cocks, Sluices, etc.*—The loss of head in each of the cases represented by the several figures may be traced to a contraction of the stream similar to the contraction which occurs in the case of an abrupt change of section. The loss may be expressed in the form $m\frac{v^2}{2g}$, and the following tables give the results obtained by Weisbach.

(*a*) *Sluice in Pipe of Rectangular Section.* (Fig. 60.)—Area of pipe $= a$; area of sluice $= s$.

FIG. 60.

$\frac{s}{a} =$	1	.9	.8	.7	.6	.5	.4	.3	.2	.1
$m =$	.00	.09	.39	.95	2.08	4.02	8.12	17.8	44.5	193

(*b*) *Sluice in Cylindrical Pipe.* (Fig. 61).—$s =$ ratio of height of opening to diameter of pipe.

$s =$	1	.875	.75	.625	.5	.375	.25	.125
$m =$	.00	.07	.26	.81	2.06	5.52	17.00	97.8

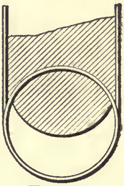


FIG. 61.

(*c*) *Cock in Cylindrical Pipe* (Fig. 62).

$s =$ ratio of cross-sections;
$\theta =$ angle through which cock is turned.

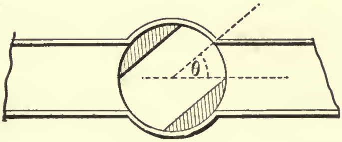


FIG. 62.

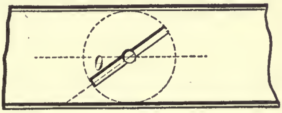


FIG 63.

If $\theta =$	5°	10°	15°	20°	25°	30°	35°
$s =$	.926	.85	.772	.692	.613	.535	.458
$m =$	.05	.29	.75	1.56	3.1	5.47	9.68

If $f =$	40°	45°	50°	55°	60°	65°	82°
$s =$	.385	.315	.25	.19	.137	.091	00
$m =$	17.3	31.2	52.6	106	206	486	∞

(*d*) *Throttle-valve in Cylindrical Pipe* (Fig. 63)
$\theta =$ angle through which valve is turned.

If $\theta =$	5°	10°	15°	20°	25°	30°	35°	40°
$m =$	.24	.52	.90	1.54	2.51	3.91	6.22	10.8

If $\theta =$	45°	50°	55°	60°	65°	70°	90°
$m =$	18.7	32.6	58.8	118	256	751	∞

CASE VI. The fall of free surface-level, or loss of head, due to sudden changes of section, frictional resistance, etc., may be graphically represented as in Fig. 64.

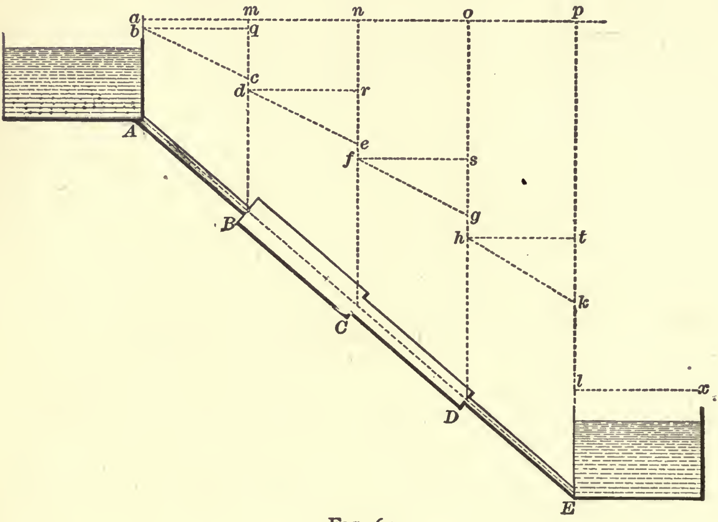


FIG. 64.

Let a length of piping AE connect two reservoirs, and let h be the difference of surface-level of the water in the reservoirs.

Let L_1, r_1 be length and radius of portion AB of pipe.
" L_2, r_2 " " " " " " BC " "
" L_3, r_3 " " " " " " CD " "
" L_4, r_4 " " " " " " DE " "
" u_1, u_2, u_3, u_4 be the velocities of flow in AB, BC, CD, DE, respectively.

The reservoir opens abruptly into the pipe at A.

There is an abrupt change at B from a pipe of radius r_1 to one of radius r_2.

There is an abrupt change at C from a pipe of radius r_2 to one of radius r_3.

At D the water flows through an orifice of area A in a diaphragm. At E the velocity of the water as it enters the lower reservoir is immediately dissipated in eddies or vortices.

Draw the horizontal plane *amnop* at a distance from the water-surface in the upper reservoir equal to the head due to atmospheric pressure.

Draw vertical lines at A, B, C, D, E. Take

ab = loss of head at the entrance A $= .49\dfrac{u_1^2}{2g}$;

qc = " " " due to friction from A to B $= \dfrac{2f}{r_1}\dfrac{u_1^2}{2g}L_1$;

cd = " " " due to change of section at $B = \left(\dfrac{r_2^2}{r_1^2} - 1\right)^2 \dfrac{u_2^2}{2g}$;

re = " " " due to friction from B to C $= \dfrac{2f}{r^2}\dfrac{u_2^2}{2g}L_2$;

ef = " " " due to change of section at $C = .316\dfrac{u_3^2}{2g}$;

sg = " " " due to friction from C to D $= \dfrac{2f}{r_3} \cdot \dfrac{u_3^2}{2g}L_3$;

gh = " " " due to change of section at $D = \left(\dfrac{\pi r_3^2}{c_c A} - 1\right)^2 \dfrac{u_4^2}{2g}$;

tk = " " " due to friction from D to E $= \dfrac{2f}{r_4}\dfrac{u_4^2}{2g}L_4$;

kl = " " " corresponding to u $= \dfrac{u_4^2}{2g}$.

Through l draw a horizontal plane lx. This plane must evidently be at a distance from the water-surface in the lower reservoir equal to the pressure-head due to the atmosphere.

Then the *total* loss of head $= lp$

$$= ab + cd + ef + gh + kl + qc + re + sg + tk,$$

$$= .49\frac{u_1^2}{2g} + \left(\frac{r_2^2}{r_1^2} - 1\right)^2\frac{u_2^2}{2g} + .316\frac{u_3^2}{2g} + \left(\frac{\pi r_3^2}{c_cA} - 1\right)^2\frac{u_4^2}{2g} + \frac{u_4^2}{2g}$$

$$+ \frac{2f}{r_1}\frac{u_1^2}{2g}L_1 + \frac{2f}{r_2}\frac{u_2^2}{2g}L_2 + \frac{2f}{r_3}\frac{u_3^2}{2g}L_3 + \frac{2f}{r_4}\frac{u_4^2}{2g}L_4$$

$$= \frac{u_1^2}{2g}\left\{.49 + \left(\frac{r_2^2}{r_1^2} - 1\right)^2\frac{r_1^4}{r_2^4} + .316\frac{r_1^4}{r_3^4} + \left(\frac{\pi r_3^2}{c_cA} - 1\right)^2\frac{r_1^4}{r_4^4} + \frac{r_1^4}{r_4^4}\right\}$$

$$+ \frac{f}{g}u_1^2\left\{\frac{L_1}{r_1} + \frac{L_2}{r_2}\frac{r_1^4}{r_2^4} + \frac{L_3}{r_3}\frac{r_1^4}{r_3^4} + \frac{L_4}{r_4}\frac{r_1^4}{r_4^4}\right\}$$

$$= \frac{Q^2}{2\pi^2 g}\left\{\frac{.49}{r_1^4} + \left(\frac{r_2^2}{r_1^2} - 1\right)^2\frac{1}{r_2^4} + \frac{.316}{r_3^4} + \left(\frac{\pi r_3^2}{c_cA} - 1\right)^2\frac{1}{r_4^4} + \frac{1}{r_4^4}\right\}$$

$$+ \frac{fQ^2}{\pi^2 g}\left\{\frac{L_1}{r_1} + \frac{L^2}{r_2} + \frac{L_3}{r_3} + \frac{L_4}{r_4}\right\}.$$

The broken line *abcdefghkl* is the hydraulic gradient.

7. **Remarks on the Law of Resistance.**—Poiseuille's experiments on the flow of water through capillary tubes showed that the loss of head was directly proportional to the velocity.

In the case of pipes used in ordinary practice the loss is undoubtedly more nearly proportional to the square of the velocity, and must be mainly due to the formation of eddies. These eddies, again, are formed more or less readily according as the water possesses less or greater viscosity.

The experiments of Unwin and others have shown that the surface friction is diminished by about 1% for every rise of 5° F. in the temperature, and it is also known that the viscosity diminishes as the temperature rises and *vice versa*. Reynolds has propounded a single law of resistance to the flow through pipes, which embraces the results of Poiseuille and of Darcy, and takes into account the effects of viscosity, temperature, etc. This law may be expressed in the form

$$\text{slope} = i = \frac{B^n}{AP^{n-1}} \frac{v^n}{d^{3-n}},$$

where d is the diameter of the pipe, $A = 67,700,000$, $B = 396$, and $P = (1 + .0336t + .000221t^2)$, the units being metres and degrees centigrade (t).

Unwin considers that the index of the diameter d is not exactly $3 - n$, and should be determined independently. For a rough surface $n = 2$, for a smooth cast-iron pipe $n = 1.9$, and for a lead pipe $n = 1.723$; a limitation which is analogous to that found by Froude in his experiments upon surface friction.

Experimenting with glass tubes, Reynolds found for velocities below a certain *critical* velocity given by the formula

$$v_c = \mu \frac{P}{d},$$

that the motion of the water is undisturbed, i.e., that it was in parallel stream-lines. At and above this critical velocity eddies are formed, and the parallel stream-line motion is completely broken up within a very short distance from the mouth of the tube.

$$\text{In capillary tubes } \frac{1}{\mu} = 43.79.$$

$$\text{In ordinary pipes } \frac{1}{\mu} = 278.$$

8. Flow of Water in a Pipe of Varying Diameter.— The variation in the diameter is supposed to be so gradual that the fluid filaments may still be assumed to flow in sensible parallel lines.

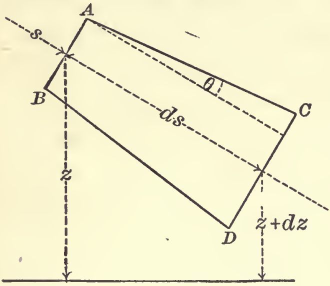


Fig. 65.

Consider a thin slice of the moving fluid, bounded by the transverse sections AB, CD, distant s and $s + ds$, respectively, from an origin on the axis of the pipe.

Let p be the mean intensity of pressure, A the water area, P the wetted perimeter for the section AB.

Let these symbols become $p + dp$, $A + dA$, $P + dP$, respectively, for the section CD.

Let z be the height of the C. of G. of the section AB above datum.

Let $z + dz$ be the height of the C. of G. of the section CD above datum.

Let u, $u + du$ be the velocities of flow across the sections AB, CD, respectively.

Then

$$\left.\begin{array}{l}\text{The rate of increase of}\\ \quad\text{momentum of the slice}\\ \quad ABCD \text{ in the direction of}\\ \quad\text{the axis}\end{array}\right\} = \left\{\begin{array}{l}\text{momentum generated by}\\ \quad\text{the effective forces acting}\\ \quad\text{upon the slice in the same}\\ \quad\text{direction.}\end{array}\right.$$

The acceleration in time $dt = \frac{w}{g}Au \,.\, dt\frac{du}{dt} = \frac{w}{g}Au \,.\, du.$

The total pressure on $AB = p \,.\, A$, and acts along the axis.

The total pressure on $CD = (p + dp)(A + dA)$, and acts along the axis.

The total normal pressure on the surface $ACBD$ of the pipe

$$= 2\pi\left(r + \frac{dr}{2}\right)\left(p + \frac{dp}{2}\right)AC = 2\pi rp \,.\, AC, \text{ very nearly.}$$

The component of this pressure along the axis

$$= 2\pi r p AC \,.\, \sin\theta$$
$$= 2\pi p r \,.\, dr, \text{ nearly,}$$

θ being the angle between AC and the axis.

Thus the *total resultant pressure along the axis*

$$= pA - (p + dp)(A + dA) + 2\pi p r \,.\, dr$$
$$= -p \,.\, dA - A \,.\, dp + 2\pi p r \,.\, dr$$
$$= -A \,.\, dp,$$

since $A = \pi r^2$, and therefore $dA = 2\pi r \,.\, dr$.

The component of *the weight of* the slice along the axis

$$= \left(A + \frac{dA}{2}\right) ds \,.\, w \sin i = -\left(A + \frac{dA}{2}\right) w \,.\, dz = -wA \,.\, dz.$$

The *frictional resistance* $= P \,.\, AC \,.\, F(u) = P \,.\, ds \,.\, F(u)$, very nearly. Hence

$$\frac{wAu \,.\, du}{g} = -A \,.\, dp - wA \,.\, dz - P \,.\, ds \,.\, F(u),$$

and therefore

$$dz + \frac{dp}{w} + \frac{u \,.\, du}{g} + \frac{P}{A}\frac{F(u)}{w} ds = 0.$$

Integrating,

$$z + \frac{p}{w} + \frac{u^2}{2g} + \int \frac{P}{A}\frac{F(u)}{w} ds = \text{a constant.}$$

Take $\dfrac{F(u)}{w} = f\dfrac{u^2}{2g} = \dfrac{f}{2g}\dfrac{Q^2}{\pi^2 r^4}$.

Then

$$z + \frac{p}{w} + \frac{u^2}{2g} + \int \frac{f}{g}\frac{Q^2}{\pi^2 r^5} ds = \text{a constant.}$$

The integration can be effected as soon as the relation between r and s is fixed.

Example.—Take $r = a + bs$, and assume f and Q to be constant. Then

$$z + \frac{p}{w} + \frac{u^2}{2g} + \frac{1}{b}\frac{fQ^2}{g\pi^2}\int\frac{dr}{r^5} = \text{a constant},$$

and therefore

$$z + \frac{p}{w} + \frac{u^2}{2g} + \frac{1}{4b}\frac{fQ^2}{g\pi^2}\frac{1}{r^4} = \text{a constant}.$$

9. Equivalent Uniform Main.—A water-main usually consists of a series of lengths of different diameters.

As a first approximation the smaller losses of head due to changes of section, etc., may be disregarded, and the calculations may be further simplified by substituting for the several lengths a single pipe of uniform diameter giving the same frictional loss of head. Such a pipe is called an equivalent main.

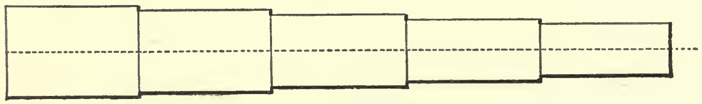


FIG. 66.

Let l_1, l_2, l_3 be the successive lengths of the main.

Let d_1, d_2, d_3 be the diameters of these lengths.

Let v_1, v_2, v_3 be the velocities of flow in these lengths.

Let h_1, h_2, h_3 be the frictional losses of head in these lengths.

Let L, d, v, h be the corresponding quantities for the equivalent uniform main.

Then

$$h = h_1 + h_2 + h_3 + \ldots,$$

and therefore

$$\frac{4f}{d}\frac{v^2}{2g}L = \frac{4f}{d_1}\frac{v_1^2}{2g}l_1 + \frac{4f}{d_2}\frac{v_2^2}{2g}l_2 + \frac{4f}{d_3}\frac{v_3^2}{2g}l_3 + \ldots$$

Hence

$$L\frac{v^2}{d} = l_1\frac{v_1^2}{d_1} + l_2\frac{v_2^2}{d_2} + l_3\frac{v_3^2}{d_3} + \ldots,$$

where it is assumed that f is the same for the several lengths of the main and also for the equivalent pipe.

But

$$\frac{\pi d^2}{4}v = Q = \frac{\pi d_1^2}{4}v_1 = \frac{\pi d_2^2}{4}v_2 = \text{etc.}$$

Hence

$$\frac{L}{d^5} = \frac{l_1}{d_1^5} + \frac{l_2}{d_2^5} + \frac{l_3}{d_3^5} + \text{etc.},$$

an equation giving the length L of an equivalent pipe having the same total frictional loss of head.

10. Branch Main of Uniform Diameter.—In a branch main AB of length L and diameter d, receiving its supply at A.—

Let Q_w be the way-service, i.e., the amount of water given up to the service-pipes on each side.

Let Q be the end-service, i.e., the amount of water discharged at the end B.

Then it may be assumed, and it is approximately true, that the way-service per lineal foot, viz., $\frac{Q_w}{L}$, is constant.

Thus the amount of water consumed in way-service in a length AC of the main, where $BC = s$, is

$$\frac{Q_w}{L}(L - s),$$

while the total amount of water flowing across the section of the pipe at C

$$= Q_e + \frac{Q_w}{L}s = \frac{\pi d^2}{4}v,$$

v being the velocity of flow at C.

Now dh, the frictional loss of head at C for an elementary length ds of the pipe, is given by the equation

$$dh = \frac{4f}{d}\frac{v^2}{2g}.ds$$

$$= \frac{f}{\pi^2 d^5}\left(Q_e + \frac{Q_w}{L}s\right)^2 ds,$$

if $g = 32$.

Integrating, the total loss of head is

$$h = \frac{fL}{\pi^2 d^5}\left(Q_e^2 + Q_e Q_w + \frac{Q^2_w}{3}\right).$$

SPECIAL CASES.

CASE I. Let Q_e' be the total discharge for the same frictional loss of head, h, when the whole of the way-service is stopped. Then

$$\frac{fL}{\pi^2 d^5} Q_e'^2 = h = \frac{fL}{\pi^2 d^5}\left(Q_e^2 + Q_e Q_w + \frac{Q^2_w}{3}\right),$$

and therefore

$$Q_e'^2 = Q_e^2 + Q_e Q_w + \frac{Q_w^2}{3}.$$

Hence

$$Q_e'^2 > \left(Q_e + \frac{Q_w}{2}\right)^2 \quad\text{and}\quad < \left(Q_e + \frac{Q_w}{\sqrt{3}}\right)^2,$$

and Q_e' lies between $Q_e + \dfrac{Q_w}{2}$ and $Q_e + \dfrac{1}{\sqrt{3}}Q_w$, its mean value being $Q_e + .55Q_w$.

CASE II. If there is no end-service, all the water having been absorbed in way-service, $Q_e = 0$, and therefore $Q'_e = \dfrac{Q_w}{\sqrt{3}}$ and

$$h = \frac{1}{3}\frac{fLQ_w^2}{\pi^2 d^5}.$$

CASE III. If $Q_e = 0$,

$$dh = \frac{fQ_w^2}{\pi^2 d^5 L^2} s^2 ds = \text{elementary frictional loss of head.}$$

Integrating between 0 and s,

$$h = \frac{1}{3} \frac{fQ_w^2 s^3}{\pi^2 d^5 L^2},$$

and the vertical slope, or line of free pressure, becomes a cubical parabola.

CASE IV. Let the main receive its supply at A from a reservoir X in which the surface of the water is h^1 above datum, and let it discharge at the end B into a reservoir Y with its surface h^2 above datum.

Since $(Q_e')^2 = Q_e^2 + Q_e Q_w + \frac{Q_w^2}{3}$, therefore

$$Q_e = -\frac{Q_w}{2} + \sqrt{(Q_e)'^2 - \frac{Q_w^2}{12}}.$$

If $Q_w = \sqrt{3Q_e'}$, $Q_e = 0$; and if $Q_w > \sqrt{3Q_e'}$, then the reservoir Y will furnish a portion of the way-service.

Suppose that X gives the supply for the distance AO $(= l_1)$ and that Y supplies BO $(= l_2)$.

Let z be the height above datum of the surface in a pressure column inserted at O.

Then, neglecting the loss of head at entrance,

$$h_1 - z = \left(h_1 + \frac{p_0}{w}\right) - \left(z + \frac{p_0}{w}\right)$$

$$= \text{loss of head between } A \text{ and } O = \frac{1}{3} \frac{fQ_w^2 l_1^3}{\pi^2 d^5 L^2},$$

and

$$h_2 - z = \left(h_2 + \frac{p_0}{w}\right) - \left(z + \frac{p_0}{w}\right)$$

$$= \text{loss of head between } B \text{ and } O = \frac{1}{3} \frac{fQ_w^2 l_2^3}{\pi^2 d^5 L^2}.$$

Also $l_1 + l_2 = L$.

11. Nozzles.—Let a pipe AB, of length l and diameter d, lead from a reservoir h ft. above the end B.

First, let the pipe be open to the atmosphere at B.

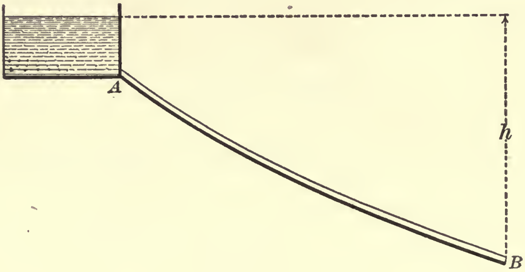


FIG. 68.

Then

$h =$ head to overcome resistance to entrance at $A\left(= n\frac{v^2}{2g}\right)$

$+$ head to overcome resistance due to bends, etc. $\left(= m\frac{v^2}{2g}\right)$

$+$ head to overcome frictional resistance $\left(= \frac{4fl}{d}\frac{v^2}{2g}\right)$

$+$ head corresponding to the velocity v in the pipe and at the outlet $\left(= \frac{v}{2g}\right)$

$$= \frac{v^2}{2g}\left(n + m + \frac{4fl}{d}\right) + \frac{v^2}{2g}.$$

Hence the height to which the water is capable of rising at B

$$= \frac{v^2}{2g} = h - \frac{v^2}{2g}\left(n + m + \frac{4fl}{d}\right),$$

or, again, is

$$= \frac{h}{1 + n + m + 4f\frac{l}{d}}.$$

Second, let a nozzle be fitted on the pipe at B.

Let V be the velocity with which the water leaves the nozzle.

Let D be the diameter of the nozzle-outlet.

This diameter is very small as compared with the diameter d of the pipe. But

$$\frac{\pi D^2}{4}V = \frac{\pi d^2}{4}v,$$

and therefore

$$V = \frac{d^2}{D^2}v,$$

so that V is very large as compared with v.

Also,

$h =$ head to overcome the resistance to entrance at A
+ head to overcome the resistance due to bends, etc.
+ head to overcome the frictional resistance in pipe
+ head to overcome the frictional resistance in nozzle $\left(= m'\frac{V^2}{2g}\right)$
+ head corresponding to the velocity V with which the water leaves the nozzle $\left(= \frac{V^2}{2g}\right)$

$$= \frac{v^2}{2g}\left(n + m + \frac{4fl}{d}\right) + m'\frac{V^2}{2g} + \frac{V^2}{2g},$$

and the height to which the water is now capable of rising at B is

$$\frac{V^2}{2g} = h - \frac{v^2}{2g}\left(n + m + \frac{4fl}{d}\right) - m'\frac{V^2}{2g}$$

$$= \frac{h}{1 + m' + \frac{D^4}{d^4}\left(n + m + \frac{4fl}{d}\right)}.$$

Let $\frac{p_n}{w}, = h_n,$ be the pressure-head at the entrance to the nozzle. Then the effective head at the same point

$$= h_n + \frac{v^2}{2g} = (1 + m')\frac{V^2}{2g}.$$

Hence

$$\frac{V^2}{2g} = \frac{h_n}{1 + m' - \frac{D^4}{d^4}}.$$

It will be observed that the delivery from the nozzle is less than that from the pipe before the nozzle was attached, but that the velocity-head at the nozzle-outlet is enormously increased. The actual height to which the water rises on leaving a nozzle is less than the calculated height, owing to air-resistance and to the impact of particles of water as they fall back.

The force required to hold the nozzle is evidently

$$\frac{wQ}{g}V = \frac{w}{g}\frac{\pi D^2}{4}V^2.$$

If the water flowing through a pipe, or hose, of length l ft., with a velocity of v ft. per second, is quickly and uniformly shut off by a stop-valve t sec., the pressure in the pipe near the valve is increased by an amount $\dfrac{wlV}{gt}$ lbs. per square foot.

Of two forms of nozzle in general use, the one (Fig. 70) is a

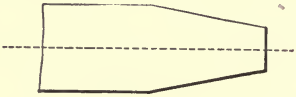


FIG. 69.

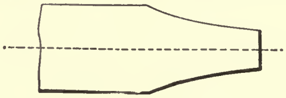


FIG. 70.

surface of revolution with a section which gradually diminishes to the outlet, while the other (Fig. 69) is a frustum of a cone, having a diaphragm with a small circular orifice at the outlet. Denoting the former by A and the latter by B, the following table gives the results of Ellis's experiments:

Pressure in lbs. per sq. in.	Head in feet.	Height of jet from 1-inch Nozzle.		Height of jet from 1⅛-inch Nozzle.		Height of jet from 1¼-inch Nozzle.	
		A	*B*	*A*	*B*	*A*	*B*
10	23	22	22	22	22	23	22
20	46	43	42	43	43	43	43
30	69	62	61	63	62	63	63
40	92	79	78	81	79	82	80
50	115	94	92	97	94	99	95
60	138	108	104	112	108	115	110
70	161	121	115	125	121	129	123
80	184	131	124	137	131	142	135
90	207	140	132	148	141	154	146
100	230	148	136	157	149	164	155

Third, if an engine, working against a pressure of p_c lbs. per square foot, pumps Q cubic feet of water per second through a nozzle at the end of a hose l feet in length, then

$$\text{the pumping H.P. of the engine} = \frac{Qp_c}{550}.$$

The total head at the engine end of the hose = the head corresponding to the pressure p in the hose + the head required to produce the velocity of flow v

$$= \frac{p}{w} + \frac{v^2}{2g},$$

and this head is expended in overcoming the frictional resistance of the hose (all other resistances are disregarded) and in producing the velocity of flow V at the outlet. Hence

$$\frac{p_c}{w} = \frac{p}{w} + \frac{v^2}{2g} = \frac{4fl}{d}\frac{v^2}{2g} + \frac{V^2}{2g},$$

and therefore

$$\frac{p}{w} = \frac{4fl}{d}\frac{v^2}{2g} + \frac{V^2}{2g} - \frac{v^2}{2g},$$

$$= \frac{8Q^2}{g\pi^2}\left(\frac{1}{D^4} - \frac{1}{d^4} + \frac{4fl}{d^5}\right),$$

$$\text{since } Q = \frac{\pi d^2}{4}v = \frac{\pi D^2}{4}V.$$

The pumping H.P.

$$= \frac{8wQ^3}{550g\pi^2}\left(\frac{1}{D^4} + \frac{4fl}{d^5}\right).$$

12. Motor Driven by Water from a Pipe. — Let the nozzle in the preceding article be replaced by a cylinder having its piston driven by the water from the pipe.

Let u = the velocity of the piston per second.

Let p_m = unit pressure at the end of the pipe, i.e., in the cylinder.

Let d_m = diameter of cylinder.

Then, velocity of flow in pipe $= \frac{d_m^2}{d^2}u$. Hence

$$h = \frac{d_m^4}{d^4}\frac{u^2}{2g} + \frac{4fl}{d}\frac{d_m^4}{d^4}\frac{u^2}{2g} + \frac{p_m}{w}$$

(other losses of head being disregarded).

13. **Siphons.**—A siphon is a bent tube, $ABCD$, Fig. 71, and is often employed to convey water from one reservoir to another at a lower level.

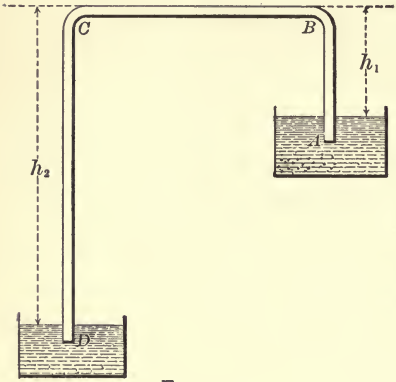


FIG. 71.

Let h_1, h_2, respectively, be the differences of level between the top of the siphon and the entrance A and outlet D to the siphon. Then, so long as the height h_1 does not exceed the head of water (= 32.8 ft.) which measures the atmospheric pressure, the water will flow along the tube in the direction of the arrow, with a velocity v given by the equation

$$h_2 - h_1 = \frac{4fl}{d}\frac{v^2}{2g},$$

l being the length of the tube $ABCD$, and all resistances, except that due to frictional resistance, being disregarded.

If $h_1 > 32.8$ feet, each of the branches AB and DC becomes a water-barometer, and the siphon will no longer work.

Even when the siphon does work, an arrangement must be made for withdrawing the air which will always collect at the upper part of the siphon.

14. **Inverted Siphons.**—The existence of a cutting or a valley sometimes renders it necessary to convey the water from a course AB to a course DE by means of an inverted siphon BCD of length.

Let u be the velocity of flow in AB, and h the height of B above a datum line.

Let v be the velocity of flow in the siphon, and h_2 the height of D above datum.

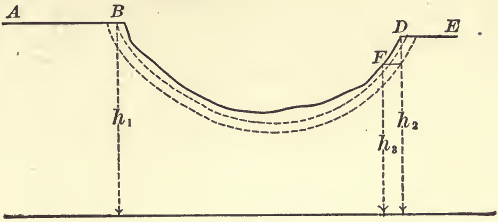


FIG. 72.

Then

$$\begin{aligned} h_1 - h_2 &= \text{loss of head at } B \\ &\quad + \text{frictional loss of head in siphon} \\ &\quad + \text{loss of head at } D \\ &= \frac{u^2}{2g} + \frac{4fl}{d}\frac{v^2}{2g} + \frac{v^2}{2g} \\ &= \frac{4fl}{d}\frac{v^2}{2g}, \text{ approximately,} \end{aligned}$$

assuming the entrance and outlet to the siphon formed in such a manner as to considerably reduce the losses $\dfrac{u^2}{2g}$ and $\dfrac{v^2}{2g}$, and to allow of these losses being disregarded without practical error. Find, by chaining along the ground, the length of the siphon from B up to a point F not far from D. Call this length l_1, and let h_3 be the height above datum of F, obtained with a level. Generally speaking, DF is nearly always of uniform slope. Call the slope α. Then,

$$DF = (h_2 - h_3) \operatorname{cosec} \alpha.$$

But

$$\begin{aligned} \frac{4fl}{d}\frac{v^2}{2g} = \frac{4f}{d}\frac{v^2}{2g}(l_1 + DF) = h_1 - h_2 &= h_1 - h_3 - (h_2 - h_3) \\ &= h_1 - h_3 - DF \,.\, \sin \alpha, \end{aligned}$$

an equation from which DF can be found, as $h_1 - h_3$ can be determined by means of a level.

15. Air in a Pipe.—The effect of an air-bubble in a pipe $ABCD$ may be discussed as follows:

Let the air occupy the portion BC of a pipe.

Let the surface of the water in the reservoir supplying the pipe be h_1 ft. vertically above E, and h_2 ft. above D.

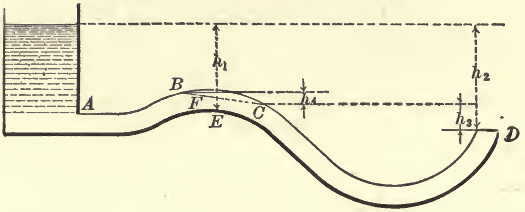


FIG. 73.

Also, let h_3 be the difference of level between C and D, h_4 the difference of level between B and C, and t the thickness of the water-layer EF.

Let H designate the head equivalent to the elastic resistance of the air in BC. Then, approximately,

$$h_1 + \frac{p_0}{w} - H - t = \frac{4fl_1}{d}\frac{v^2}{2g}$$

and

$$H + h_3 - \frac{p_0}{w} = \frac{4fl_2}{d}\frac{v^2}{2g},$$

l_1 being length of portion of pipe from A to E, and l_2 the length from E to D.

Adding the two equations,

$$h_1 + h_3 - t = \frac{4f}{d}\frac{v^2}{2g}(l_1 + l_2) = \frac{4fl}{d}\frac{v^2}{2g},$$

l being total length of pipe.

But $h_1 - t + h_4 = h_2 - h_3$, very nearly. Hence

$$h_2 - h_4 = \frac{4fl}{d}\frac{v^2}{2g},$$

an equation showing the variation of v with a variation in the height h_4 of the space occupied by the air.

Note.—H of course varies with the temperature.

16. Three Reservoirs at Different Levels connected by a Branched Pipe.—Let a pipe *DO* of length l_1 ft. and radius r_1 ft., leading from a reservoir *A* in which the water stands h_1 ft. above datum, divide at *O* into two branches, the one, *OE*, of length l_2 ft. and radius r_2 ft., leading to a reservoir *B* in which the water stands h_2 ft. above datum, the other, *OF*, of length l_3 ft. and radius r_3 ft., leading to a reservoir *C* in which the water stands h_3 ft. above datum.

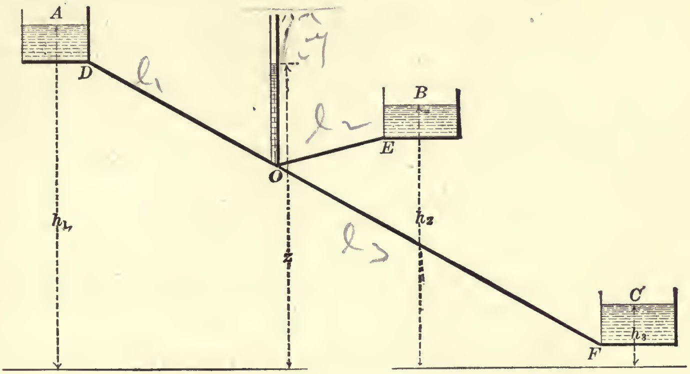


FIG. 74.

Let v_1, v_2, v_3 be the velocities of flow in *DO*, *OE*, *OF*, respectively.

Let Q_1, Q_2, Q_3 be the quantities of flow in *DO*, *OE*, *OF*, respectively.

Let z be the height above datum to which the water will rise in a tube inserted at the junction.

Two problems will be considered, and all losses of head excepting those due to frictional resistance will be disregarded.

PROBLEM I. Given h_1, h_2, h_3; r_1, r_2, r_3; to find Q_1, Q_2, Q_3; v_1, v_2, v_3, and z.

For the pipe *DO*, $\frac{h_1 - z}{l_1} = \alpha \frac{v_1^2}{r_1}$ . . (1) and $Q_1 = \pi r_1^2 v_1$. . . (2)

" " " *OE*, $\frac{\pm z \mp h_2}{l_2} = \alpha \frac{v_2^2}{r_2}$ . . (3) " $Q_2 = \pi r_2^2 v_2$. . . (4)

$\lambda = 4f \quad \alpha = \frac{f}{g} \quad f = \alpha g$

$\therefore \lambda = 4\alpha g.$

$h - z \quad l_1 v_1^2 \quad 4\alpha g \frac{l_1}{} \frac{v_1^2}{} = l_1 \alpha \frac{v_1^2}{r}$

For the pipe OF, $\frac{z-h_3}{l_3}=\alpha\frac{v_3^2}{r_3}$ . . (5) and $Q_3=\pi r_3^2 v_3$. . . (6)

Also, $$Q_1 = \pm Q_2 + Q_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From these seven equations the seven required quantities can be found.

In equations (3) and (7), the upper or lower signs are to be taken according as the flow is from O towards E or from E towards O.

This may be easily determined as follows:

Assume $z = h_2$, and then find v_1 and v_3 by means of equations (1) and (5), and hence Q_1 and Q_3 by means of equations (2) and (6). If it is found that $Q_1 > Q_3$, then the flow is from O to E, and equations (3) and (7) become

$$\frac{z-h_2}{l_2} = \alpha\frac{v_2^2}{r_2} \quad \text{and} \quad Q_1 = Q_2 + Q_3;$$

while if $Q_1 < Q_3$, the flow is from E to O, and the equations are

$$\frac{h_2 - z}{l_2} = \alpha\frac{v_2^2}{r_2} \quad \text{and} \quad Q_1 + Q_2 = Q_3.$$

Note.—It is assumed that $\alpha\left(=\frac{f}{g}\right)$ is the same for each pipe.

Special Case. Fig. 75.—Suppose the pipe OE closed at E.

Also let $r_1 = r_2 = r_3 = r$, and let V be the velocity of flow from A to C.

The "plane of charge" for the reservoir A is a horizontal plane MQ distant $\frac{p_0}{w}$ from the water surface, p_0 being the atmospheric pressure.

The "plane of charge" for the reservoir C is a horizontal plane TS distant $\frac{p_0}{w}$ from the water-surface.

In the vertical line VTQ, take $TN = \frac{V^2}{2g}$, and join MN. Then, neglecting the loss of head at entrance, MN is the "line of charge," or hydraulic gradient, for the pipe DF, and is approximately a straight line.

Let the "plane of charge" KK for the reservoir B, distant $\frac{p_0}{w}$ from the water-surface, meet MN in G.

If the junction O is vertically below G, there is no head

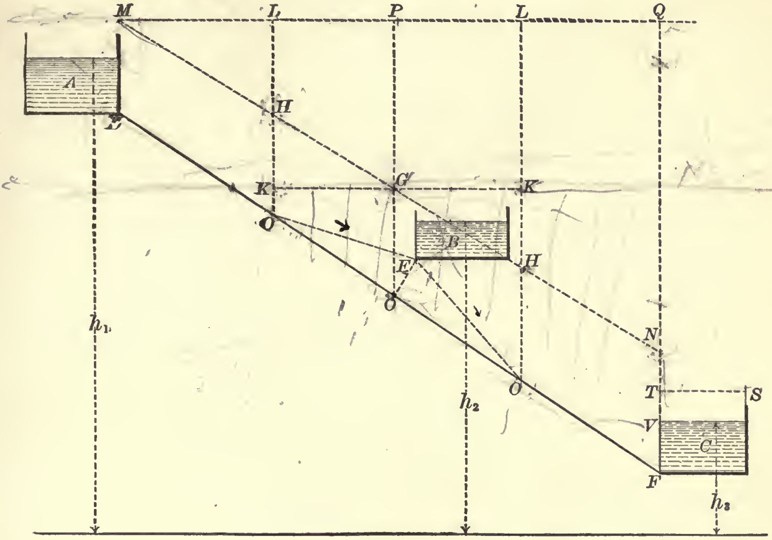


FIG. 75.

available for producing flow either from E towards O or from O towards E, and hydrostatic equilibrium is established.

If the junction O is on the left of G, and a vertical line $OKHL$ is drawn intersecting KK, MN, and MQ in the points K, H, and L, there is the head HK available for producing flow from O towards E.

If the junction O is on the right of G, and the vertical line $OHKL$ is drawn, the head HK is now available for producing flow from E towards O.

Let the vertical through G meet MQ in P, and take $PG = Y$. Then, approximately,

$$\frac{l_1}{l_1 + l_3} = \frac{MG}{MN} = \frac{PG}{QN} = \frac{Y}{h_1 - h_3},$$

and therefore

$$Y = \frac{h_1 - h_3}{l_1 + l_3} \cdot l_1.$$

If $HL < Y$, the flow is from O towards E.
If $HL > Y$, " " " " E " O.
Again,

$$\left(h_1 + \frac{p_0}{w}\right) - \left(h_3 + \frac{p_0}{w} + \frac{V^2}{2g}\right) = \alpha\frac{V^2}{r}(l_1 + l_3),$$

and therefore, approximately,

$$h_1 - h_3 = \alpha\frac{V^2}{r}(l_1 + l_3). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next assume the junction O to be on the left of G, and open the valve at E. Then

$$\frac{h_1 - z}{l_1} = \alpha\frac{v_1^2}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{z - h_2}{l_2} = \alpha\frac{v_2^2}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{z - h_3}{l_3} = \alpha\frac{v_3^2}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and $\qquad Q_1 = Q_2 + Q_3,$

or $\qquad v_1 = v_2 + v_3.$

Thus

$$\alpha\frac{V^2}{r}(l_1 + l_3) = h_1 - h_3 = \frac{\alpha}{r}(l_1 v_1^2 + l_3 v_3^2) = \frac{\alpha}{r}\left\{ l_1(v_2 + v_3)^2 + l_3 v_3^2 \right\};$$

and therefore

$$v_3^2(l_1 + l_3) + 2l_1 v_2 v_3 + l_1 v_2^2 - (i_1 + l_3)V^2 = 0.$$

Hence, assuming v_2 very small as compared with V,

$$v_3 = V - \frac{l_1 v_2}{l_1 + l_3},$$

or

$$Q_3 = Q - \frac{l_1 Q_2}{l_1 + l_3},$$

where $Q = \pi r^2 V$.

Thus it appears that if a quantity Q_2 of water is drawn off by means of a branch from a main capable of giving a total end service Q, this end service will be diminished by $\frac{1}{2}Q_2$, $\frac{1}{3}Q_2$, $\frac{1}{4}Q_2$, etc. according as the junction O divides the pipe DF into two portions in the ratio of 1 to 1, 1 to 2, 1 to 3, etc.

Note.—The more correct value of v_3 is

$$v_3 = -\frac{l_1 v_2}{l_1 + l_3} + \sqrt{V^2 - \frac{l_1 l_3 v_2^2}{(l_1 + l_3)^2}},$$

and the maximum value of $\frac{l_1 l_3}{(l_1 + l_3)^2}$ does not exceed $\frac{1}{4}$.

Orifice Fed by Two Reservoirs.—Neglect all losses of head except the losses due to frictional resistance.

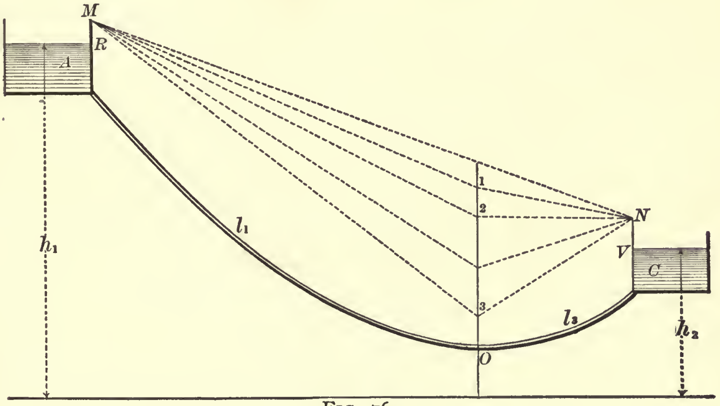


FIG. 76.

When the valve at O is closed the flow is wholly from A to C, and the delivery is

$$Q = \sqrt{\frac{\pi^2 r^5}{\alpha}\frac{h_1 - h_2}{l_1 + l_3}}.$$

The line of charge (hydraulic gradient) is MN, where

$$MR = \frac{p_0}{w} = NV.$$

Open the valve a little: a volume Q_2 will now flow through O, and a volume Q_3 into C, where

$$Q_3 = Q - \frac{l_1 Q_2}{l_1 + l_3}.$$

The "line of charge" becomes the broken line $M1N$.

As the opening of the valve continues, the pressure-head at O diminishes, and when it is equal to $h_3 + \frac{p_0}{w}$ the line of charge is $M2N$, $2N$ being horizontal. Hydrostatic equilibrium is now established between O and C, and the whole of the water from A passes through O, the delivery being given by

$$Q_2 = \sqrt{\frac{\pi_2 r^5}{\alpha}\frac{h_1 - h_2}{l_1}} = Q\sqrt{\frac{l_1 + l_3}{l_1}}.$$

Opening O still further, both reservoirs will serve the orifice, and the line of charge will continue to fall.

When the valve is full open the "line of charge" is $M3N$, where $3O = \frac{p_0}{w}$, and the discharge is

$$= \sqrt{\frac{\pi^2 r^5}{\alpha}}\left\{\left(\frac{h_1}{l_1}\right)^{\frac{1}{2}} + \left(\frac{h_2}{l_3}\right)^{\frac{1}{2}}\right\}.$$

The supply from A is equal to that from C when $\frac{h_1}{l_1} = \frac{h_2}{l_3}$.

The above investigation shows the advantage of a second reservoir in emergent cases when an excessive supply is suddenly demanded, as, e.g., on the occasion of a fire.

PROBLEM II. Given h_1, h_2, h_3; Q_2, Q_3, and therefore $Q_1 (= \pm Q_2 + Q_3)$; to find r_1, r_2, r_3, v_1, v_2, v_3, z.

As before, let z be the pressure-head at O. Then

$$\frac{h_1 - z_1}{l_1} = \alpha\frac{v_1^2}{r_1} \quad . \quad . \quad . \quad (1) \quad \text{and} \quad Q_1 = \pi r_1^2 v_1; \quad . \quad . \quad . \quad (2)$$

$$\frac{\pm z \mp h_2}{l_2} = \alpha\frac{v_2^2}{r_2} \quad . \quad . \quad . \quad (3) \quad \text{"} \quad Q_2 = \pi r_2^2 v_2; \quad . \quad . \quad . \quad (4)$$

$$\frac{z - h_3}{l_3} = \alpha\frac{v_3^2}{r_3} \quad . \quad . \quad . \quad (5) \quad \text{"} \quad Q_3 = \pi r_3^2 v_3. \quad . \quad . \quad . \quad (6)$$

These six equations contain the seven required quantities, viz., r_1, r_2, r_3, v_1, v_2, v_3, and z. Thus a seventh equation must be obtained before their values can be found. This equation is given by the condition "that the cost of the piping laid in place should be a minimum," it being assumed that the cost of a pipe laid in place is proportional to its diameter.

Hence

$$l_1r_1 + l_2r_2 + l_3r_3 = \text{a minimum.} \quad . \quad . \quad . \quad . \quad (7)$$

From equations (1) and (2), $\dfrac{h_1 - z}{l_1} = \dfrac{\alpha Q_1^2}{\pi^2 r_1^5}$;

" " (3) " (4), $\dfrac{\pm z \mp h_2}{l_2} = \dfrac{\alpha Q_2^2}{\pi_2 r_2^5}$;

" " (5) " (6), $\dfrac{z - h_3}{l_3} = \dfrac{\alpha Q_3^2}{\pi^2 r_3^5}$.

Differentiating these three equations,

$$\frac{dz}{l_1} = \frac{5\alpha Q_1^2}{\pi^2 r_1^6} \,.\, dr_1;$$

$$\frac{dz}{l_2} = \mp \frac{5\alpha Q_2^2}{\pi^2 r_2^6} \,.\, dr_2;$$

$$\frac{dz}{l_3} = - \frac{5\alpha Q_3^2}{\pi^2 r_3^6} \,.\, dr_3.$$

But by equation (7)

$$l_1dr_1 + l_2dr_2 + l_3dr_3 = 0.$$

Hence

$$\frac{r_1^6}{Q_1^2} = \pm \frac{r_2^6}{Q_2^2} + \frac{r_3^6}{Q_3^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is the seventh equation required.

This last equation may be written in the forms

$$\frac{r_1^2}{v_1^2} = \pm \frac{r_2^2}{v_2^2} + \frac{r_3^2}{v_3^2}$$

and

$$\frac{Q_1}{v_1^3} = \pm \frac{Q_2}{v_2^3} + \frac{Q_3}{v_3^3}.$$

17. Mains with any Required Number of Branches.

Let there be n junctions and m pipes.

Let $h_1, h_2, \ldots h_m$ be the m pressure-heads at the end of each successive length of pipe.

Let $z_1, z_2, \ldots z_n$ be the n pressure-heads at the 1st, 2d, 3d, . . . nth junctions.

Let $l_1, l_2, \ldots l_m$ be the lengths of the m pipes.

PROBLEM I. Given $h_1, h_2, \ldots h_m$, $r_1, r_2, \ldots r_m$, to find $v_1, v_2, \ldots v_m$, $z_1, z_2, \ldots z_n$.

There are m equations of the type $\frac{\pm h \mp z}{l} = \alpha \frac{v^2}{r}$.

Also, the quantity flowing through the first portion of the main is equal to the sum of the quantities flowing through all the branches at the first junction, and an analogous equation will hold for each of the remaining $n - 1$ junctions. Thus n additional equations are obtained.

From these $m + n$ equations, $v_1, v_2, \ldots v_m$, $z_1, z_2, \ldots z_n$ may be found analytically or by the method of repeated approximation.

PROBLEM II. Given $h_1, h_2, \ldots h_m$, $Q_1, Q_2, \ldots Q_m$, to find $r_1, r_2, \ldots r_m$, $z_1, z_2, \ldots z_n$.

There are now only m equations of the type

$$\frac{\pm h \mp z}{l} = \alpha \frac{v^2}{r},$$

involving $m + n$ unknown quantities, and the problem admits of an infinite number of solutions.

It is therefore assumed that the cost of the piping laid in place is to be a *minimum*. Thus n new equations are obtained, and the $m+n$ equations may be solved analytically or by repeated trial.

18. Variation of Velocity in a Transverse Section.—*Assumption.*—That the water in any portion of a pipe is made up of an infinite number of hollow concentric cylinders of fluid, each moving parallel to the axis with a certain definite velocity.

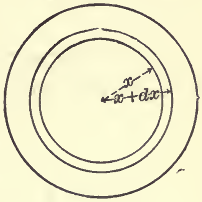


FIG. 77.

Let u be the velocity of one of these cylinders of radius x and thickness dx. Then the flow across a transverse section is given by the equation

$$dq = 2\pi x\,dx\,.\,u,$$

and the total flow

$$Q = 2\pi \int_0^r ux\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

r being the radius of the pipe.

If v_m be the mean velocity for the whole transverse section of the pipe,

$$v_m = \frac{Q}{\pi r^2} = \frac{2\int_0^r ux\,dx}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, assuming with Navier that the surface resistance between two concentric cylinders is of the nature of a viscous resistance and may be represented by $k\dfrac{du}{dx}$ per unit of area at the radius x, k being a coefficient called the coefficient of viscosity, then the total resistance at the radius x for a length ds of the cylinder

$$= -\,2\pi x\,.\,ds\,.\,k\frac{du}{dx} = -\,2\pi k\,.\,ds\,.\,x\frac{du}{dx}.$$

The total resistance at the radius $x+dx$

$$= +\,2\pi k\,.\,ds\left[x\frac{du}{dx} + \frac{d}{dx}\left(x\frac{du}{dx}\right)dx\right].$$

Hence the total resultant resistance for the length ds of the cylinder under consideration

$$= 2\pi k ds \frac{d}{dx}\left(x\frac{du}{dx}\right)dx.$$

The component of the weight of the slice of the cylinder in the direction of the axis

$$= w \,.\, 2\pi x \,.\, dx \,.\, ds \,.\, \sin\theta,$$

θ being the inclination of the axis to the horizon.

Let $-dz$ be the fall of level in the distance ds. Then

$$-dz = ds \,.\, \sin\theta.$$

Therefore, component of weight in direction of axis

$$= -w \,.\, 2\pi x\, dx \,.\, dz.$$

The resultant pressure on the slice in the direction of motion

$$= p - (p + dp) \,.\, 2\pi x \,.\, dx = 2\pi x \,.\, dx \,.\, dp.$$

Then, since the motion is uniform,

$$w \,.\, 2\pi k \,.\, ds \,.\, \frac{d}{dx}\left(x\frac{du}{dx}\right)dx - w \,.\, 2\pi x \,.\, dx \,.\, dz - 2\pi x \,.\, dx \,.\, dp = 0,$$

and therefore

$$\frac{k \,.\, ds}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) - dz - \frac{dp}{w} = 0.$$

Integrating only for the cylinder under consideration,

$$\frac{ks}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) - \left(z + \frac{p}{w}\right) = \text{a constant}.$$

But $z + \frac{p}{w}$ is evidently independent of x, and is a linear function of s (Art. 2, Chap. III.). Hence

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) = \text{a constant} = A, \text{ suppose}.$$

Therefore

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = Ax. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating,

$$x\frac{du}{dx} = A\frac{x^2}{2} + B.$$

Assuming that the central fluid filament is the filament of maximum velocity, then when $x = 0$, $\frac{du}{dx}$ is also nil. Therefore

$$B = 0 \quad \text{and} \quad x\frac{du}{dx} = Ax^2,$$

and therefore

$$\frac{du}{dx} = A\frac{x}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Integrating,

$$u = A\frac{x^2}{4} + C.$$

Let $u_{\max}$ be the velocity of the central filament, i.e., the value of u when $x = 0$.

Then

$$u_{\max} - u = -\frac{A}{4}x^2 = Dx^2, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $D = -\frac{A}{4}$.

Again, by equation 1,

$$Q = 2\pi\int_0^r (u_{\max} - Dx^2)x.dx = \pi r^2\left(u_{\max} - \frac{Dr^2}{2}\right):$$

and by equation 2,

$$v_m = u_{\max} - \frac{Dr^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If u_s = surface velocity, then, by equation 5,

$$u_s = u_{\max} - Dr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence, by equations 6 and 7,

$$u_s + u_{\max} = 2v_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$h = \frac{v^2}{2g} + \lambda \frac{l}{d} \frac{v^2}{2g}$

$f = .06 + 3$

$f = 3.06$

$Q = av$

$v = \sqrt{\frac{2gh}{\lambda\, l/d}}$

EXAMPLES.

1. A water-main is to be laid with a virtual slope of 1 in 850, and is to give a maximum discharge of 35 cubic feet per second. Determine the requisite diameter of pipe and the maximum velocity, taking $f = .0064$.
Ans. 3.679 ft.; 3.2888 ft. per sec.

2. Find the loss of head due to friction in a pipe: diameter of pipe = 12 in., length of pipe = 5280 ft., velocity of flow = 3 ft. per second; $f = .0064$.; Also find the discharge.
Ans. 19.008 ft.; 2.3562 cub. ft. per sec.

3. A pipe has a fall of 10 ft. per mile; it is 10 miles long and 4 ft. in diameter. Find the discharge, assuming $f = .0064$.
Ans. 54.7 cub. ft. per sec.

4. A pipe discharges 250 gallons per minute and the head lost in friction is 3 ft.. Find approximately the head lost when the discharge is 300 gallons per minute; also find the work consumed by friction in both cases.
Ans. 4.32 ft.; 7500 ft.-lbs.; 12,960 ft.-lbs.

5. What is the mean hydraulic depth in a circular pipe when the water rises to the height $\frac{\text{diameter}}{2\sqrt{2}}$ above the centre?
Ans. $\frac{10}{33} \times$ diameter.

6. A 12-inch pipe has a slope of 12 feet per mile; find the discharge. ($f = .005$.)
Ans. 2.118 cub. ft. per sec.

7. The mean velocity of flow in a 24-in. pipe is 5 ft. per second; find its virtual slope, f being .0064.
Ans. 1 in 200.

8. Calculate the discharge per minute from a 24-in. pipe of 4000 ft. length under a head of 80 ft., using a coefficient suitable for a clean iron pipe.
Ans. 34.909 cub. ft. per sec.

9. How long does it take to empty a dock, whose depth is 31 ft. 6 ins. and which has a horizontal sectional area of 550,000 sq. ft., through *two* 7-ft. circular pipes 50 ft. long, taking into account resistance at entrance?
Ans. 214 min. 6 sec.

10. The virtual slope of a pipe is 1 in 700; the delivery is 180 cubic feet per minute. Find the diameter and velocity of flow.
Ans. 1.26 ft.; 2.401 ft. per sec.

11. Determine the diameter of a clean iron pipe, 100 feet in length, which is to deliver .5 cub. ft. of water per second under a head of 5 feet. Assume $f = .006$.
Ans. .326 ft.

12. A reservoir has a superficial area of 12,000 ft. and a depth of 60 ft.; it is emptied in 60 minutes through *four* horizontal circular pipes, equal in diameter and 50 ft. long. Find the diameter. *Ans.* 1.75 ft.

Explain how the *total* head is made up, and draw the plane of charge.

13. A 3-inch pipe is very gradually reduced to $\frac{1}{2}$ inch. If the pressure-head in the pipe is 40 ft., find the greatest velocity with which the water can flow through. *Ans.* 1.4 ft. per sec.

14. Water flows through a 24-inch pipe 5000 yards in length. At 1000 yards it yields up 300 cubic feet per minute to a branch. At 2800 yards it yields up 400 cubic feet per minute to a second branch. At 4000 yards it yields up 600 cubic feet per minute to a third branch. The delivery at the end is 500 cubic feet per minute. Find the head absorbed by friction. ($f = .0075$.) *Ans.* 176.801 ft.

15. Find the H. P. required to raise 550 gallons per minute to a height of 60 feet, through a pipe 100 feet in length and 6 in. in diameter, the coefficient of friction being .0064. *Ans.* 10.74.

16. What head of water is required for a 5-in. pipe, 150 ft. in length, to carry off 25 cub. ft. of water per minute? *Ans.* 1.56223 ft.

What head will be required if the pipe contains two rectangular knees? *Ans.* 1.84918 ft.

17. Determine the delivery of a 2-in. pipe, 48 ft. long, under a 5-ft. head. *Ans.* .1349 cub. ft. per sec.

What will be the delivery if the pipe has five small curves of 90° curvature, the ratio of the radius of the pipe to that of the curves being 1 : 2? *Ans.* .1327 cub. ft. per sec.

18. The curved buckets of a turbine form channels 12 in. long, 2 in. wide, and 2 in. deep; the mean radius of curvature of the axis is 8 in. the water flows along the channel with a velocity of 50 ft. per minute. What is the head lost through curvature? *Ans.* .00138 ft.

19. Find the maximum power transmitted by water in a 36-inch pipe, the metal being $1\frac{1}{2}$ inches thick and the allowable stress 2800 lbs. per square inch. If the pipe is $1\frac{1}{4}$ miles in length, find the loss of power. *Ans.* 576 H. P.; 720.2 ft.-lbs.

20. Find the diameter of a pipe $\frac{1}{2}$ mile long to deliver 1500 gallons of water per minute with a loss of 20 feet of head. ($f = .005$.) *Ans* .1.0135 ft.

21. Water is to be raised 20 ft. through a 30-ft. pipe of 6 in. diameter. Find the velocity of flow, assuming that 10 per cent of additional power is required to overcome friction. *Ans.* 8.44 ft. per sec.

22. In a pipe 3280 ft. in length the loss of head in friction is 83 ft. Taking $f = .0064$, find the diameter. *Ans.* 1.527 ft.

23. A pipe 2000 ft. long and 2 ft. in diameter discharges at the rate of 16 ft. per second. Find the increase in the discharge if for the last

1000 ft. a second pipe of same size be laid by the side of the first and connected with it so that the water may flow equally well along either pipe. *Ans.* 7.24 cub. ft. per sec.

24. A pipe of length l and radius r gives a discharge Q. How will the discharge be affected (1) by doubling the radius for the whole length; (2) by doubling the radius for half the length; (3) by dividing it into three sections of equal length, of which the radii are r, $\frac{r}{2}$, and $\frac{r}{4}$, respectively? (f = coefficient of friction.)

Ans. 1. New discharge $= 4Q\left(\frac{3r + 4fl}{3r + 2fl}\right)^{\frac{1}{2}}$;

2. " " $= Q\left(\frac{48r + 64fl}{48r + 33fl}\right)^{\frac{1}{2}}$;

3. " " $= Q\left(\frac{9r + 4fl}{9r + 4228fl}\right)^{\frac{1}{2}}$.

25. A 24-inch pipe 2000 ft. long gives a discharge of Q cubic feet of water per minute. Determine the change in Q by the substitution for the foregoing of either of the following systems: (1) two lengths, each of 1000 ft., whose diameters are 24 in. and 48 in. respectively; (2) four lengths, each of 500 ft., whose diameters are 24 in., 18 in., 16 in., and 24 in.

Draw the "plane of charge" in each case.

Ans. (1) Discharge is increased 33.2 per cent taking loss at change of section into account;
Discharge is increased 35.7 per cent disregarding loss at change of section.
(2) Discharge is diminished 45 per cent disregarding losses at change of section.

26. Q is the discharge from a pipe of length l and radius r; examine the effect upon Q of increasing r to nr for a length ml of the pipe.

$$\textit{Ans.}\ \text{New discharge} = Q\left\{\frac{\frac{3}{2} + \frac{2fl}{r}}{\frac{3}{2} + \frac{2fl}{r}\left(1 - m + \frac{m}{n^5}\right) + \frac{(n^2 - 1)^2}{n^4}}\right\}^{\frac{1}{2}}.$$

27. A *reducer*, l ft. in length, discharges at the rate of 400 gallons per minute, and its diameter diminishes from 12 in. to 6 in.; find the total loss of head due to friction. *Ans.* $.005529l$.

28. A reservoir of 10,000 square feet superficial area and 100 feet deep discharges through a pipe 24 in. in diameter and 2000 feet long. Find the velocity of flow in the pipe.

What should be the diameter of the pipe in order that the reservoir might be emptied in two hours? *Ans.* 15.36 ft. per sec.; 3.67 ft.

29. Eight cubic feet of ore is to be raised at the rate of 900 ft. per

minute by a water-pressure engine with four single-acting cylinders of 6 in. diameter and 18 in. stroke, making 60 revolutions per minute. Find the diameters of a supply-pipe 230 ft. long for a head of 230 ft., disregarding friction of machinery, etc. *Ans.* 4 in.

30. A 2-inch pipe A suddenly enlarges to a 3-inch pipe B, the quantity of water flowing through being 100 gallons per minute. Find the loss of head and the difference of pressure in the pipes (1) when the flow is from A to B; (2) when the flow is from B to A.

Ans. (1) Loss of head = 8.639 in.
Gain of pressure-head = 13.83 "
(2) Loss of head = 7.428 "
Diminution of pressure-head = 29.88 "

31. A 3-inch horizontal pipe rapidly contracts to a 1-inch mouthpiece, whence the water emerges into the air, the discharge being 660 lbs. per minute. Find the pressure in the 3-inch main.

If the 3-inch pipe is 200 ft. in length and receives water from an open tank, find the height of the tank.

Ans. 1003.5 lbs. per sq. ft.; 19.92 ft.

32. The efficiency of an engine is $\frac{3}{5}$; it burns 8 lbs. of coal per hour per H.P., and works 8 hours a day for 300 days in the year; the cost of the engine is $12.00 per H.P., and the cost of the coal is $3.00 per ton; 4500 gallons of water per minute have to be raised a height of 200 ft. through a pipe of which the diameter is to be a minimum. Cost of piping $= \$D$ per lineal foot, D being the diameter. Find the value of D.

Ans. 2.923 ft.

33. A reservoir is to be supplied with water at the rate of 11,000 gallons per minute, through a vertical pipe 30 ft. high; find the minimum diameter of pipe consistent with economy. Cost of pipe per foot $= \$d$, d being the diameter; cost of pumping = 1 cent per H.P. per hour; original cost of engine per H.P. = $100.00; add 10 per cent for depreciation. Engine works 12 hours per day for 300 days in the year. *Ans.* 4.375 ft.

34. A horizontal pipe 4 in. in diameter suddenly enlarges to a diameter of 6 in.; find the force required to cause a flow of 300 gallons of water per minute through the sudden enlargement.

Ans. .06 H.P.

35. 1000 gallons per minute is to be forced through a system of pipes AB, BC, CD, of which the lengths are 100 ft., 50 ft., 120 ft., and the radii 4 in., 6 in., and 3 in., respectively. Draw the plane of charge.

Ans. Loss in friction from A to B = 111.96 ft.; loss at B = 4.499 ft.;
" " " " B to C = 7.372 " " " C = 14.56 "
" " " " C to D = 566.17 "

36. A pipe 4 in. in diameter suddenly contracts to one 3 in. in diameter; find the power necessary to force 250 gallons per minute through the sudden contraction. *Ans.* 1.23997 H.P.

37. If a pipe whose diameter is 8 in. suddenly enlarges to one whose diameter is 12 ins., find the power required to force 1000 gallons per minute through the enlargement, and draw to scale the plane of charge.

Ans. Energy expended = .1377 H.P.

38. 1000 gallons per minute are forced through a system of pipes *AB*, *BC*, *CD*, of which the lengths are 100 ft., 50 ft., and 120 ft., and the radii 6 in., 3 in., and 4 in., respectively. Draw to scale the plane of charge.

Ans. Loss in friction from A to $B =$ 14.744 ft.; loss at $B =$ 14.56 ft.
" " " " B to $C =$ 235.9 "; " " $C =$ 8.819 "
" " " " C to $D =$ 134.36 "

39. Water flows from a 3-inch pipe through a $1\frac{1}{2}$-inch orifice in a diaphragm into a 2-inch pipe. What head is required if the delivery is to be 8 cubic feet of water per minute? *Ans.* 2.826 ft.

40. 500 gallons of water per minute are forced through a continuous line of pipes *AB*, *BC*, *CD*, of which the radii are 3 in., 4 in., 2 in., and the lengths 100 ft., 150 ft., and 80 ft., respectively. Find the *total* loss of head (*a*) due to the sudden changes of form at B and C, (*b*) due to friction. Find (*c*) the diameter of an equivalent uniform pipe of the same total length.

Ans. (*a*) .1378 ft.; 1.152 ft.
(*b*) 3.688 ft. in *AB*; 1.313 ft. in *BC*; 22.393 ft. in *CD*.
(*c*) .4212 ft.

41. *AB*, *BC*, *CD* is a system of three pipes of which the lengths are 1000 ft., 50 ft., and 800 ft., and the diameters 24 in., 12 in., and 24 in., respectively; the water flows from *CD* through a 1-inch orifice in a thin diaphragm, and the velocity of flow in *AB* is 2 ft. per second. Draw the plane of charge and find the mechanical effect of the efflux.

Ans. Loss at $B = \frac{9}{16}$ ft.; at $C = \frac{81}{256}$ ft.; in friction from A to $B =$ .8 ft.; from B to $C \doteq$ 1.28 ft.; from C to $D =$ .64 ft.; energy of jet = $14{,}811\frac{3}{7}$ H.P.

42. 1000 gallons per minute flows through a sudden contraction from 12 inches to 8 inches at A, then through a sudden enlargement from 8 inches to 12 inches at B, the intermediate pipe *AB* being 100 ft. long. Draw the plane of charge.

Ans. Loss at $A =$ .288 ft.; at $B =$ .281 ft.; in friction from A to $B =$ 3.499 ft.

43. Water flows from one tube into another of *twice* the diameter; the velocity in the latter is 10 ft. Find the head corresponding to the resistance. *Ans.* 14.0625 ft.

44. In a given length l of a circular pipe whose inner radius is r and thickness e, a column of water flowing with a velocity v is suddenly checked by the shutting off of cocks, etc. Show that

$$gh = \frac{Ee\lambda^2}{r}\left\{1 + \frac{1e}{2r}\left(1 + \frac{E}{E_1}\right) + \frac{e^2}{r^2}\right\},$$

in which h = head due to the velocity v, E = coefficient of elasticity, E_1 = coefficient of compressibility of water, λ = extension of pipe circumference due to E.

45. A 100-gallon tank, 100 feet above the ground, is filled by a $1\frac{1}{2}$-in. pipe connected with an accumulator consisting of a 3-ft. cylinder with a piston loaded with 50 tons. How long will it take to fill the tank, assuming that frictional resistances absorb nine tenths of the head and that the mean height of the piston above the ground is 10 feet?

Ans. 13.9 secs.

46. Determine the discharge from a pipe of 12 in. radius and 3280 ft. in length which connects two reservoirs having a difference of level of 128 ft. Take into account resistance at entrance. Draw the plane of charge. *Ans.* 48.571 cub. ft. per sec.

47. Determine the diameter of a clean iron pipe 5000 ft. in length which connects two reservoirs having a total head of 40 ft. and discharges into the lower at the rate of 20 cub. ft. per second. Draw to scale the line of charge. *Ans.* 1.9219 ft.

48. The difference of level between the two reservoirs is 100 ft., and they are connected by a pipe 10,000 ft. long. Find the diameter of the pipe so as to give a discharge of 2000 cubic feet per minute (*a*) by Darcy's formula, (*b*) assuming $f = .0064$. (Allow for loss of head at entrance.) *Ans.* (*a*) 2.266 ft.; (*b*) 2.360 ft.

49. Two reservoirs are connected by a 12-inch pipe $1\frac{1}{4}$ miles long. For the first 500 yards it has a slope of 1 in 30, for the next half mile a slope of 1 in 100, and for the remainder of its length it is level. The head of water over the inlet is 55 ft. and that over the outlet is 15 ft. Determine the discharge in gallons per minute. (Take $f = .0064$.)

Ans. 1950.66.

50. Two reservoirs are connected by a 6-inch pipe in three sections, each section being three quarters of a mile in length. The head over the inlet is 20 ft., that over the outlet 9 ft. The virtual slope of the first section is 1 in 50, of the second 1 in 100, and the third section is level. Find the velocity of flow, and the delivery.

Ans. 4.5 ft. per sec.; 332 gallons per minute.

51. A pipe 5 miles long, of uniform diameter equal to 12 in., conveys water from a reservoir in which the water stands at a height of 300 ft. above Trinity high-water mark, to a reservoir in which the water stands

at a height of 150 ft. above the same datum. To what height will water rise in a supply-pipe taken one mile from the lower end? For what pressure would you design the main at this point, if it lies 20 ft. above the level of the lower reservoir? *Ans.* 179.93 ft.; 19 lbs. per sq. in.

52. The water surface in one reservoir is 500 ft. above datum, and is 100 ft. above the surface of the water in a second reservoir 20,000 ft. away, and connected with the first by an 18-in. main. Find the delivery per second, taking into account the loss of head at the upper entrance.

53. Water surface of a reservoir is 300 ft. above datum, and a 4-in. pipe 600 ft. long leads from reservoir to a point 200 ft. above datum. Find the height to which the water would rise (*a*) if end of pipe is open to atmosphere, (*b*) if it terminates in a 1-inch nozzle. In latter case find longitudinal force on nozzle. *Ans.* (*a*) $2\frac{3}{8}$ ft.; (*b*) 87.52 ft.; 59.693 lbs.

54. The surface of the water in a tank is 388 ft. above datum and is connected by a 4-in. pipe 200 ft. long with a turbine 146 ft. above datum. Determine the velocity of the water in the pipe at which the power obtained from the turbine will be a maximum. Assuming the efficiency of the turbine to be 85 per cent, determine the power.

Ans. 19.928 ft. per sec.; 31.895 H. P.

55. A pipe 12 in. in diameter and 900 ft. long is used as an inverted siphon to cross a valley. Water is led to it and away from it by an aqueduct of rectangular section 3 ft. broad and running full to a depth of 2 ft. with an inclination of 1 in 1000. What should be the difference of level between the end of one aqueduct and the beginning of the other? *Ans.* 575.8 ft.

56. Water flows through a pipe 20 ft. long with a velocity of 10 ft. per second. If the flow is stopped in $\frac{1}{10}$ sec. and if retardation during the stoppage is uniform, find the increase in the pressure produced. ($g = 32$ and the density of the water $= 62.5$ lbs. per cub. ft.)

Ans. $62\frac{1}{2}$ cu. ft. of water.

57. An hydraulic motor is driven by means of an accumulator giving 750 lbs. per square inch. The supply-pipe is 900 ft. long and 4 in. in diameter. Find the maximum power attainable, and velocity in pipe. ($f = .0075$.) *Ans.* 242.4 H. P.; 21.203 ft. per sec.

58. A 2-inch hose conveys 2 gallons of water per second. Find the longitudinal tension in the hose. *Ans.* 9.18 lbs.

59. Find the pumping H. P. to deliver 1 cub. ft. of water per second through a 1-inch nozzle at end of a 3-inch hose 200 ft. long, f being .016.

Ans. 97.335 H. P.

60. A volume of water 50 ft. in length flowing through a pipe with a velocity of 24 ft. per second is quickly and uniformly stopped in one tenth of a second by closing a stop-valve. Find the increase of pressure per square inch in the pipe near the valve. *Ans.* 162.5 lbs.

61. The surface of the water in a tank is 286 ft. above datum. The

tank is connected by a 4-in. pipe 500 ft. long with a 36-in. cylinder 170 ft. above datum. Find (*a*) the velocity of flow in the pipe for which the available power will be a maximum; (*b*) the power. If the piston moves at the rate of 1 ft. per minute, find (*c*) the pressure on the piston. Also find the height to which the water would rise if (*d*) the cylinder end of the pipe were open to the atmosphere and if (*e*) the pipe terminated in a nozzle 1 inch in diameter, neglecting the frictional resistance of the nozzle. Finally, find (*f*) the power required to hold the nozzle. (Coeff. of friction = .005.)

Ans. (*a*) 8.93 ft. per sec.; (*b*) 6.85 H. P.; (*c*) 22.8 tons per sq. ft.; (*d*) 3.74 ft.; (*e*) 103.8 ft.; (*f*) 70.8 lbs.

62. The conduit-pipe for a fountain is 250 ft. long and 2 in. in diameter; the coefficient of resistance for the mouthpiece is .32; the entrance orifice is sufficiently rounded, and the bends have sufficiently long radii of curvature to allow of our neglecting the corresponding coefficient of resistance. How high will a $\frac{1}{2}$-in. jet rise uuder a head of 30 ft.?

Ans. 19.14 ft.

63. The difference in level of two reservoirs is 250 ft. and they are connected by a 24-inch pipe *AB*, 6000 ft. long. If $f = .0064$, draw the plane of charge. A third reservoir is so placed that the difference between its level and that of the first (or highest) is 100 ft., and is connected to the main at a point *O* by a branch *OC*, 3000 ft. long and 12 in. in diameter. Examine the distribution.

Ans. Upper reservoir will supply the two lower reservoirs if $AO < \frac{3}{5}BO$.

The two upper reservoirs will both discharge into the lower reservoir if $AO > \frac{3}{5}BO$.

If $AO = 2000$ ft., the pressure-head at $O = 161$ ft.; $v_1 = 14.9$ ft.; $v_2 = 3.02$ ft.; $v_3 = 14.18$ ft.

If $AO = 4000$ ft., the pressure-head at $O = 96$ ft.; $v_1 = 13.8$ ft.; $v_2 = 6.7$ ft.; $v_3 = 15.4$ ft.

64. A pipe 24 in. in diameter and 2000 ft. long leads from a reservoir in which the level of the water is 400 ft. above datum to a point *B*, at which it divides into two branches, viz., a 12-in. pipe *BC*, 1000 ft. long, leading to a reservoir in which the surface of the water is 250 ft. above datum, and a branch *BD*, 1500 ft. long, leading to a reservoir in which the surface of the water is 50 ft. above datum. Determine the diameter of *BD* when the free surface-level at *B* is (*a*) 300 ft., (*b*) 250 ft., and (*c*) 200 above datum. *Ans.* (*a*) 1.454 ft.; (*b*) 1.783 ft.; (*c*) 2.096 ft.

65. Two reservoirs *A* and *B* are connected by a line of piping *MON*, 2000 ft. in length. From the middle point *O* of this pipe a branch *OP*, 1000 feet in length, leads to a reservoir *C*. The reservoirs *A* and *C* are 200 feet and 100 feet, respectively, above the level of *C*. The deliveries in *MO*, *OP*, *ON*, in cubic feet per second, are $\frac{25}{9}\pi$, $\frac{16}{9}\pi$, and π, respec-

$v = \sqrt{\frac{2gh}{f}}$

tively. Find (*a*) the velocities of flow in *MO*, *OP*, *ON*; (*b*) the radii of these lengths; (*c*) the height of the free surface-level at *O* above *C*.

Ans. (*a*) 11.121 ft. per sec. in *MO*; 10.158 ft. per sec. in *OP*; 14.145 ft. per sec. in *ON*.

(*b*) .49976 ft.; .41831 ft.; .26588 ft.

(*c*) 150.5 ft., very nearly.

66. A main, 1000 ft. long and with a fall of 5 ft. discharges into two branches, the one 750 ft. long with a fall of 3 ft., the other 250 ft. long with a fall of 1 ft. The longer branch passes twice as much water as the other and the total delivery is $47\frac{1}{7}$ cu. ft. per minute. The velocity of flow in the main is $2\frac{1}{2}$ ft. per second. Find the diameters of the main and branches. *Ans.* .63245 ft.; .288 ft.; .488 ft.

67. How far can 100 H.P. be transmitted by a $3\frac{1}{2}$ in. pipe with a loss of head not exceeding 25 per cent under an effective head of 750 lbs. per square inch? *Ans.* 5426.3 ft.

68. A city is supplied with water by means of an aqueduct of rectangular section, 24 ft. wide, running 4 ft. deep, and sloping 1 in 2400. *One-fourth* of the supply is pumped into a reservoir through a pipe 3000 ft. long, rising 25 ft. in the first 1500 ft., and 75 ft. in the second 1500 ft. The pumping is effected by an engine burning $2\frac{1}{2}$ lbs. or coal per H.P. per hour, and working constantly through the year. A percentage is to be allowed for repairs and maintenance; the cost of the coal per ton of 2000 lbs. is $4; the prime cost of the engine is $100 per H.P.; the efficiency of the engine is $\frac{3}{5}$; the coefficient of pipe friction is .0064, the cost of the piping is $30 per ton. Determine the most economical diameter of pipe, and the H.P. of the engine. *Ans.* 4.84 ft.; 456.455 H.P.

CHAPTER IV.

FLOW OF WATER IN OPEN CHANNELS.

1. **Flow of Water in Open Channels.**—A transverse section of the water flowing in an open channel may be supposed to consist of an infinite number of elementary areas representing the sectional areas of fluid filaments or stream-lines. The velocities of these stream-lines are very different at different points of the same transverse section, and the distribution of the pressure is also of a complicated character. Generally speaking, the side and bed of a channel exert the greatest retarding influence on the flow, and therefore along these surfaces are to be found the stream-lines of minimum velocity. The stream-lines of maximum velocity are those farthest removed from retarding influences. If the stream-line velocities for any given section are plotted, a series of equal velocity-curves may be obtained. In a channel of symmetrical

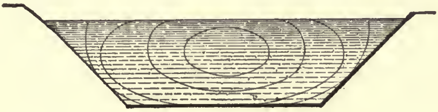


FIG. 78.

section, the depth of the stream-line of maximum velocity below the water-surface is less than one fourth of the depth of the water, while the mean velocity-curve cuts the central vertical line at a point below the surface about three fourths of the depth of the water.

In the ordinary theory of flow in open channels, the variation of velocity from point to point in a transverse section is disregarded, and it is assumed that all the stream-lines are sensibly parallel and move normally to the section with a common velocity equal to the mean velocity of the stream. With this assumption, it also necessarily follows that the distribution of pressure over the section is in accordance with the hydrostatic law.

Again, it is assumed that the laws of fluid friction already enunciated are applicable to the flow of water in open channels. Thus, the resistance to flow is proportional to some function of the velocity ($F(v)$), to the area (S) of the wetted surface, is independent of the pressure, and may be expressed by the term $S.F(v)$. An obvious error in this assumption is that v is the *mean* velocity of the stream and not the velocity of the stream-lines along the bed and sides of the channel. In practice, however, the errors in the formulæ based upon these imperfect hypotheses are largely neutralized by giving suitable values to the coefficient of friction (f).

When a constant volume (Q) of water feeds a channel of given form, the water assumes a definite depth. A permanent régime is said to be established and the flow is *steady*. If the transverse sectional area (A) is also constant, then, since $Q = vA$, the velocity v is constant from section to section and the flow is said to be uniform. Usually the sectional area A is variable and therefore the velocity v also varies, so that the motion is steady with a varying velocity. Any convenient short stretch of a channel, free from obstructions, may be selected, and treated without error of practical importance, as being of a uniform sectional area equal to that of the mean section for the whole length under consideration.

2. Steady Flow in Channels of Constant Section (A).—The flow is evidently uniform; and since A is constant, the depth of the water is also constant, so that the water-surface is parallel to the channel-bed. Consider a portion of the stream, of length l, between the two transverse sections aa, bb.

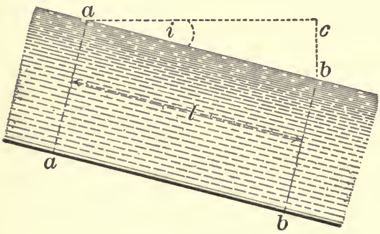


FIG. 79.

Let i be the inclination of the bed (or water-surface) to the horizon.

Let P be the length of the wetted perimeter of a cross-section.

Then, since the motion is uniform, the external forces acting upon the mass between *aa* and *bb* in the direction of motion must be in equilibrium.

These forces are:

(1) The component of the weight of the mass, viz.,

$$wAl \sin i = wAli = wAl\frac{h}{l} = wAh,$$

h being the fall of level in the length l.

Note.—When i is small, as is usually the case in streams,

$$\frac{h}{l} = \tan i = \sin i = i, \text{ approximately.}$$

(2) The pressures upon the areas *aa* and *bb*, which evidently neutralize each other.

(3) The frictional resistance developed by the sides and bed, viz.,

$$P \,.\, l \,.\, F(v).$$

Hence

$$wAh - PlF(v) = 0,$$

or

$$\frac{F(v)}{w} = \frac{Ah}{Pl} = mi,$$

m being the hydraulic mean depth.

It now remains to determine the form of the function $F(v)$. In ordinary English practice it is usual to take

$$\frac{F(v)}{w} = f\frac{v^2}{2g},$$

f being the coefficient of friction. Then

$$f\frac{v^2}{2g} = mi$$

or

$$v = \sqrt{\frac{2g}{f}}\sqrt{mi} = c\sqrt{mi},$$

c being a coefficient whose value depends upon the roughness of the channel surface and upon the form of its transverse section.

Prony and Eytelwein adopted the formula

$$\frac{F(v)}{w} = av + bv^2 = mi,$$

and carried out different experiments to determine the values of a and b.

According to Prony, $\frac{1}{a} = 22472.5$ and $\frac{1}{b} = 10607.02$,

" " Eytelwein, $\frac{1}{a} = 41688.02$ " $\frac{1}{b} = 8975.43$.

For a velocity of about 70 ft. per minute Prony's and Eytelwein's results give the same value for mi. For other velocities, Prony's values of mi are greater or less than those of Eytelwein, according as the velocity v is greater or less than 70 ft. If v, however, does not differ very widely from 70 ft., the change of value is small and of no practical importance.

For values of v exceeding 20 ft. per minute the term av may be disregarded without practical error, and the formula then becomes

$$mi = bv^2,$$

or

$$v = \frac{1}{\sqrt{b}}\sqrt{mi}.$$

Hence

$$v = 105\sqrt{mi}, \text{ according to Prony,}$$

and

$$v = 95\sqrt{mi}, \text{ according to Eytelwein,}$$

giving as a mean

$$v = 100\sqrt{mi}, \text{ which is Beardmore's formula.}$$

The total head H in a stream is made up of two parts, the one required to produce the velocity of flow, and the other absorbed by the frictional resistance. Thus,

$$H = \frac{v^2}{2g} + \frac{l}{m}\frac{F(v)}{w}.$$

In long canals, and in rivers with slopes not exceeding 3 ft. per mile, the term $\frac{v^2}{2g}$ is very small as compared with the term $\frac{l}{m}\frac{F(v)}{w}$, and may be disregarded without sensible error.

Note.—The retarding effect of the air upon the free surface of a stream or river has yet to be determined by careful observation and experiment. It may, however, be assumed that the resistance offered by calm air per unit of free surface is approximately one tenth of the resistance offered by similar units at the bottom and sides of smooth channels. Thus, in smooth channels, if X is the width of the free surface, the wetted perimeter is more correctly $P + \frac{X}{10}$.

In general, the wetted perimeter may be expressed in the form $P + \frac{X}{\beta}$, β being 10 for smooth channels and greater than 10 for rough channels. The value of β is obviously diminished by opposing winds and increased by following winds.

3. **On the Form of a Channel.**—In the formula

$$mi = \frac{F(v)}{w},$$

$m\left(= \frac{A}{P}\right)$ and $i\left(= \frac{h}{l}\right)$ are similarly related in the determination of v, the mean velocity of flow. If v is constant, the product mi must also be constant, so that if m increases i must diminish, and *vice versa*. Thus, in a very flat country the flow may be maintained by making m sufficiently large, while again if the channel-bed is steep m is small.

The erosion caused by a watercourse increases with the rapidity of flow. At the same time the sectional area (A) of the waterway also increases, so that the velocity of flow v diminishes. Thus there is a tendency to approximate to a "permanent régime" when the resistance to erosion balances the tendency to scour.

Hence, throughout any long stretch of a river, passing through a specific soil, the mean velocity of flow will be very nearly constant if the amount of flow (Q) does not vary. Generally speaking, the volume conveyed by a river increases from source to mouth on account of the additions received from tributaries, etc. Since Q increases, A must also increase; and if mi or v is to remain constant, i must diminish. It is also observed that the surface slopes of large rivers diminish gradually from source to mouth.

Again, various problems relating to the proper sectional form of a channel may be discussed by means of the formulæ

$$v = c\sqrt{mi} = c\sqrt{\frac{A}{P}i}$$

and

$$Q = Av = c\sqrt{\frac{A^3}{P}i}.$$

Suppose the slope to be constant. Then

$$v^2 \text{ is proportional to } \frac{A}{P}$$

and

$$Q^2 \text{ is proportional to } \frac{A^3}{P}.$$

PROBLEM I. The section of the waterway being a rectangle of width x and depth y, and of given area ($A = xy$), it is required to find the ratio of x to y for which the velocity of flow (v) will be a maximum. Then $dv = 0$, and therefore

$$d\left(\frac{A}{P}\right) = 0 = \frac{P\,.\,dA - A\,.\,dP}{P^2}.$$

Hence

$$PdA - AdP = 0.$$

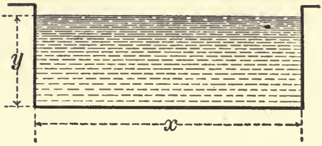


FIG. 80.

But $dA = 0 = xdy + ydx$, and therefore also

$$dP = 0 = dx + 2dy,$$

since

$$P = x + 2y.$$

Hence,

$$\frac{x}{y} = -\frac{dx}{dy} = 2,$$

and the mean hydraulic depth

$$= \frac{A}{P} = \frac{xy}{x + 2y} = \frac{y}{2}$$

$$= \text{one half of the depth of the water.}$$

The same results follow if the discharge Q instead of v is to be a maximum. In such case

$$dQ = 0 = d\left(\frac{A^3}{P}\right) = \frac{3A^2P.dA - A^3.dP}{P^2},$$

and therefore $3PdA - AdP = 0$.

But $dA = 0$, and therefore $dP = 0$. Hence, etc.

Note.—The same results also follow if, instead of A being given, the wetted perimeter P is to be a minimum, since then $dP = 0$, and therefore also $dA = 0$.

PROBLEM II. The waterway being trapezoidal in section,

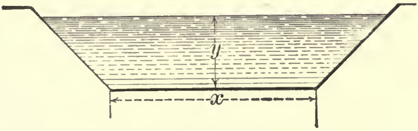


FIG. 81.

of bottom width x, depth y, and sides sloping at a given angle θ to the horizontal, it is required to find the ratio of x to y which, for a given wetted perimeter (P) or area (A), will make the velocity of flow or the discharge a maximum.

As in Problem I,

$$dA = 0 \quad \text{and} \quad dP = 0.$$

But $A = (x + y \cot \theta)y$ and $P = x + 2y \operatorname{cosec} \theta$.
Hence

$$dA = 0 = ydx + dy(x + 2y \cot \theta)$$

and

$$dP = 0 = dx + dy \,.\, 2 \operatorname{cosec} \theta.$$

Therefore

$$\frac{x + 2y \cot \theta}{y} = -\frac{dx}{dy} = 2 \operatorname{cosec} \theta.$$

Hence

$$x = 2y(\operatorname{cosec} \theta - \cot \theta) = 2y\frac{1 - \cos \theta}{\sin \theta} = 2y \tan \frac{\theta}{2},$$

and therefore $\dfrac{x}{y} = 2 \tan \dfrac{\theta}{2}$.

Then mean hydraulic depth

$$= \frac{A}{P} = \frac{(x + y \cot \theta)y}{x + 2y \operatorname{cosec} \theta} = \frac{y}{2}\left(\frac{2 - \cos \theta}{2 - \cos \theta}\right) = \frac{y}{2}$$

= one half of the depth of the water.

The section may be easily sketched as in Figs. 82 and 83.

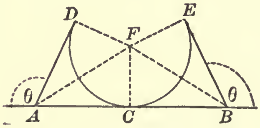


FIG. 82.

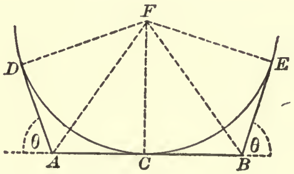


FIG. 83.

From the middle point C of AB, the bottom width, draw CF at right angles to AB and equal in length to the depth of the water. Then

$$\frac{AB}{CF} = 2 \tan \frac{\theta}{2},$$

θ being the given slope of the sides.

With F as centre and FC as radius describe a circle. From the points A and B draw tangents to touch this circle at D and E. FA evidently bisects the angle CAD. Therefore

$$\tan\frac{CAD}{2} = \tan CAF = \frac{CF}{AC} = \frac{CF}{\frac{1}{2}AB} = \cot\frac{\theta}{2}.$$

Hence $\pi - CAD = \theta$, and AD, BE have the slope required.

PROBLEM III. To find the proper sectional form of a channel of bottom width $2a$ so that the mean velocity of flow may be constant for all depths of water.

Let x, y, Fig. 84, be the co-ordinates of any point P in the profile referred to the middle point O of AB, the bottom width, as origin and let s be the length of AP.

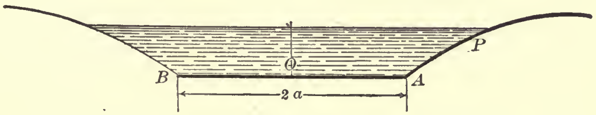


FIG. 84.

Since v is to be constant m must also be constant, and therefore

$$\frac{A}{P} = \frac{\int ydx}{s+a} = \text{a const.} = m,$$

which may be written

$$\int ydx = m(s+a).$$

Differentiating,

$$ydx = mds = m(dx^2 + dy^2)^{\frac{1}{2}},$$

and therefore

$$\frac{dx}{m} = \frac{dy}{(y^2 - m^2)^{\frac{1}{2}}}.$$

Integrating,

$$\frac{x}{m} = \log_e\left(y + \sqrt{y^2 - m^2}\right) + c,$$

c being a constant of integration.

When $x = 0$, $y = a$, and therefore

$$0 = \log_e (a + \sqrt{a^2 - m^2}) + c = \log_e b + c,$$

where $b = a + \sqrt{a^2 - m^2}$. Hence

$$\frac{x}{m} = \log_e \frac{y + \sqrt{y^2 - m^2}}{b},$$

or

$$y + \sqrt{y^2 - m^2} = be^{\frac{x}{m}},$$

is the equation to the required profile, which, as may be easily shown, is a curve which flattens very rapidly.

PROBLEM IV. If water flows through a circular aqueduct, find the angle θ subtended at the centre by the wetted perimeter, for which the velocity of flow is a maximum.

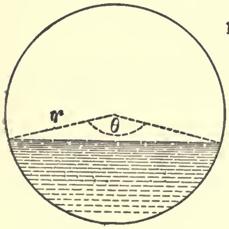


FIG. 85.

Let r = radius of aqueduct.

Area of waterway $= \frac{r^2}{2} (\theta - \sin \theta)$.

Wetted perimeter $= r\theta$.

Then

$$m = \frac{r}{2} \frac{\theta - \sin \theta}{\theta} = \frac{r}{2}\left(1 - \frac{\sin \theta}{\theta}\right).$$

Now v is to be a maximum and therefore $\frac{\sin \theta}{\theta}$ must be a minimum. Hence

$$d\left(\frac{\sin \theta}{\theta}\right) = 0 = \frac{\theta \cos \theta - \sin \theta}{\theta^2} d\theta,$$

and therefore $\theta \cos - \sin \theta = 0$.

Hence $\theta = \tan \theta$, and the angle θ in degrees is about $77° \; 27'$.

Also, the mean hydraulic depth $= \frac{r}{2}\left(1 - \frac{\sin\theta}{\theta}\right)$

$$= \frac{r}{2}(1 - \cos\theta)$$

$$= r\sin^2\frac{\theta}{2} = .39 \times r.$$

PROBLEM V. A channel of given slope has a given surface-width AC, vertical sides AB $(= y_1)$ and CD $(= y_2)$ of given depths, and a curved bed BD $(= L)$ of given length.

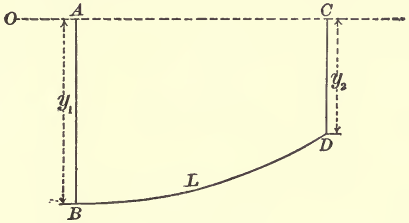


FIG. 86.

The amount and velocity of flow in the channel will be a maximum when the form of the bed BD is a circular arc. This can be easily proved as follows:

Since the slope is constant, $v \propto \sqrt{m} \propto \sqrt{\frac{A}{P}}$.

But $P\,(= L + y_1 + y_2)$ is a constant quantity, and therefore v and also Q will be a maximum when A is a maximum.

Hence, too, the area between the chord BD and the curve must be a maximum, and therefore the curve must be a circular arc. The proof of this by the Calculus of Variations is as follows:

Take O in CA produced as the origin, OC as the axis of x, and the vertical through O as the axis of y. Then

$$A = \int_{x_1}^{x_2} y\,dx \text{ is to be a maximum.}$$

Also,

$$L = \int_{x_1}^{x_2} \frac{ds}{dx} \cdot dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + p^2}\, dx,$$

is a given quantity, OA being $= x_1$, $OC = x_2$, and $\frac{dy}{dx} = p$.

Let $V = y + a\sqrt{1 + p^2}$, a being some constant.

Then

$$\int_{x_1}^{x_2} V \,.\, dx \text{ is to be a maximum,}$$

and therefore

$$V = p\frac{dV}{dp} + c_1;$$

that is,

$$y + a\sqrt{1 + p^2} = \frac{ap^2}{\sqrt{1 + p^2}} + c_1.$$

and thus

$$y + \frac{a}{\sqrt{1 + p^2}} = c_1.$$

Therefore

$$\frac{dx}{dy} = \frac{1}{p} = \frac{c_1 - y}{\sqrt{a^2 - (c_1 - y)^2}}.$$

Integrating,

$$x + c_2 = \sqrt{a^2 - (c_1 - y)^2},$$

the equation to a circle of radius a.

Hence the profile BD is a circular arc.

The maximum depth of the channel is $c_1 - a$.

The constants c_1, c_2, a can be found from the three conditions that the arc is of given length and has to pass through the two fixed points B and D.

4. **Flow in Aqueducts.**—The velocity v depends upon m ($m = \frac{A}{P}$ and therefore upon the depth of the water in the

aqueduct. For some definite depth the velocity will be a maximum. If the water fills the aqueduct, the aqueduct becomes a pipe, and the formula for channel-flow *ought* to change suddenly so as to agree with that for pipe-flow. The theory is thus imperfect.

5. **River-bends.**—The following explanation is due to Professor James Thomson (Inst. Mechl. Engs., 1879; Proc. Royal Soc. 1877). In rivers flowing in alluvial plains, the curvature of the windings which already exist tends to increase owing to the scouring away of material from the outer bank and to the deposition of detritus along the inner bank. The sinuosities often increase until a loop is formed, with only a narrow isthmus of land between two encroaching banks of a river. Finally a cut-off occurs, a short passage for the water is opened through the isthmus, and the loop is separated from the river-course, taking the form of a horseshoe-shaped lagoon or swamp. The ordinary supposition that the water always tends to move forward in a straight line, rushing against the outer bank and wearing it away, and at the same time causing deposits at the inner bank, is correct, but it is very far from being a complete explanation of what takes place.

When water flows round a circular curve under the action of gravity only, it takes a motion like that in a free vortex. Its velocity parallel to the axis of the stream is greater at the inner than at the outer side of the curve.

Thus, too, the water in a river-bank flows more quickly along courses adjacent to the inner bank of the bend than

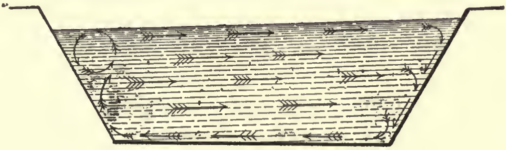


FIG. 87.

along courses adjacent to the outer. The water, in virtue of centrifugal force, presses outwards so that the water-surface of a transverse section (Fig. 87) has a slope rising upwards from the inner to the outer bank. Hence the free level for any particle of the water near the outer bank is higher than the free level for any particle in the same transverse section near the inner bank, but the tendency to flow from the higher to the lower level is counteracted by centrifugal action. Now the water immediately in contact with the bottom and sides of the course is retarded, and its centrifugal force is not sufficient to balance the pressure due to the greater depth at the outside of the bend. This water therefore tends to flow from

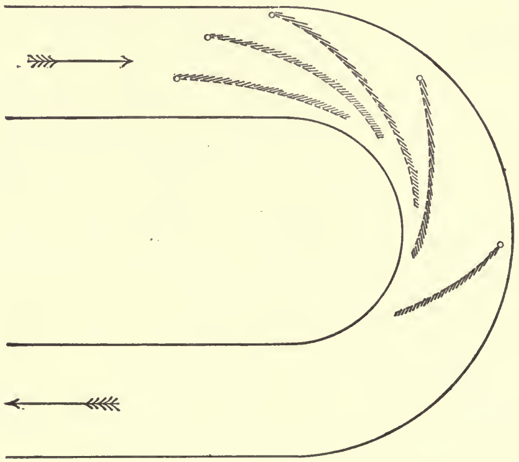


Fig. 88.

the outer bank towards the inner (Fig. 88), carrying with it detritus which is deposited at the inner bank. Simultaneously with the flow of water inwards, the mass of the water must necessarily flow outwards to take its place.

6. **Value of** f.—The value of f depends upon

(*a*) the roughness of the sides and bed;

(*b*) the velocity of flow;

(*c*) the dimensions of the transverse section;

(*d*) the slope of the channel-bed.

An average mean value of f is .00757.

Weisbach has proposed to take

$$f = \alpha\left(1 + \frac{\beta}{v}\right),$$

the values of α and β, obtained as the results of 255 experiments, being $\alpha = .007409$, $\beta = .192$, so that

$$f = .007409 + \frac{.0014225}{v}.$$

Darcy and Bazin assume f to be given by an expression of the form

$$f = \alpha\left(1 + \frac{\beta}{m}\right),$$

giving the following values of α and β as the results of their experiments:

In very smooth channels, with sides of planed timber or rendered in cement,

$$\alpha = .00316, \ \beta = .1; \quad \therefore f = .00316 + \frac{.000316}{m}.$$

In smooth channels with sides of planks, brick-work, or ashlar

$$\alpha = .00401, \ \beta = .23; \quad \therefore f = .00401 + \frac{.0009223}{m}.$$

In rough channels with sides of rubble masonry or pitched with stone

$$\alpha = .00507, \quad \beta = .82; \quad \therefore f = .00507 + \frac{.0041574}{m}.$$

In very rough channels in earth

$$\alpha = .00592, \quad \beta = 4.1; \quad \therefore f = .00592 + \frac{.024272}{m}.$$

In torrential streams encumbered with detritus

$$\alpha = .00846, \quad \beta = 8.2; \quad \therefore f = .00846 + \frac{.069372}{m}.$$

Ganguillet and Kutter, taking the formula

$$v = c\sqrt{mi} = \sqrt{\frac{2g}{f}}\sqrt{mi},$$

have endeavored to obtain a more correct value of c by a careful investigation of:

(*a*) The experimental results of Darcy and Bazin. These results show that the value of c depends upon the roughness of the channel and also upon its dimensions. The values given for α and β are different for different classes of channel even when the dimensions are infinite. But while in small channels the influence of differences of roughness upon the flow must be very great, it is certainly more than probable that this influence diminishes as the section of the channel increases, and that it will be nil in the case of an indefinitely large channel.

(*b*) The measurements of Humphreys and Abbott on the Mississippi, a stream of very large section and of very low slope.

(*c*) Their own gaugings in the regulated channels of certain Swiss torrents with exceptionally steep slopes and running through extremely rough channels.

(*d*) The effect of the slope.

From the Mississippi data it was found that

$c = 256$ for a slope of .0034 per 1000

and

$c = 154$ " " " " .02 " "

Thus c, and therefore also the discharge, will be subject to considerable variations in the case of large streams with low slopes. The value of c does not vary much with the slope in small rivers. Proceeding in a purely empirical manner, Ganguillet and Kutter arrived at the formula

$$c = \frac{A + \frac{l}{n}}{1 + \frac{An}{\sqrt{m}}},$$

where n is a coefficient depending only on the roughness of the channel sides and bed, while A and l are new coefficients whose values remain to be determined.

Now c depends upon the slope i and decreases as i increases. This may be allowed for by taking

$$A = a + \frac{p}{i},$$

so that

$$c = \frac{a + \frac{l}{n} + \frac{p}{i}}{1 + \left(a + \frac{p}{i}\right)\frac{n}{\sqrt{m}}},$$

the form finally adopted by Ganguillet and Kutter.

The values given for the constants, the unit being a foot, are

$a = 41.6$; $l = 1.811$; $p = .00281$; $n = .008$ to $.05$.

The following table gives the values of n which will be found of most use in practice:

In a channel with sides of well-planed timber.......	$n = .009$
" " " " " rendered with cement......	$n = .01$
In a channel with sides rendered with a mixture of 3 of cement to 1 of sand........................	$n = .011$
In a channel with sides of unplaned planks...........	$n = .012$
" " " " " " ashlar or brickwork......	$n = .013$
" " " " " " canvas on frames.........	$n = .015$
" " " " " " rubble masonry..........	$n = .017$

In rivers and canals in very firm gravel.............. $n = .02$
In rivers and canals in perfect order and free from detritus (stones and weeds).................... $n = .025$
In rivers and canals in moderately good order, not quite free from stones and weeds.............. $n = .03$
In rivers and canals in bad order, with weeds and detritus .... $n = .035$
In torrential streams encumbered with detritus..... $n = .05$

To the above Jackson adds the following classification for artificial canals:

In canals in very firm gravel in perfect order....... $n = .02$
" " " earth above the average order.......... $n = .0225$
" " " " in fair order................... $n = .025$
" " " " below the average order........ $n = .0275$
In canals in earth in rather bad order, partially overgrown with weeds and obstructed with detritus, $n = .03$

The difficulty of properly selecting the value of n is due to the fact that there is no absolute measure of the roughness of channel-beds.

In Cunningham's experiments on the Ganges c varied from 48 to 130.

In Humphreys and Abbott's experiments on the Mississippi c varied from 53 to 167, the units in each case being a foot and a second.

7. **Variation of Velocity** in different parts of the transverse section of a stream.

Assumptions.—(*a*) That the stream is of uniform depth h and of indefinite width.

(*b*) That the fluid filaments flow across the section in sensibly parallel lines.

(*c*) That a permanent régime has been established, and that the flow is uniform. The pressure in the section is therefore distributed in accordance with the hydrostatic law.

(*d*) That the resistance to the relative sliding of consecutive filaments is of the nature of viscous resistance.

Let Fig. 89 represent a portion of a vertical longitudinal section of the stream intersected by two transverse sections *AB*, *CD*, l being the distance between them.

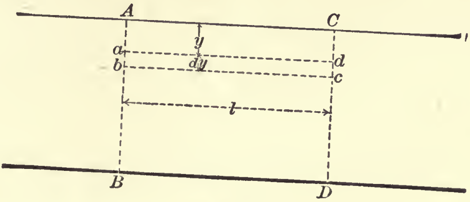


FIG. 89.

Consider a thin layer *abcd* of thickness dy and width b, bounded by the sections *AB*, *CD*, and by the planes *ad*, *bc*, at depths y and $y + dy$, respectively, below the free surface.

The forces acting upon the layer in the direction of motion are:

(1) The pressures on the ends *ab*, *cd*, which evidently neutralize each other.

(2) The component of the weight $= wbl.dy.\sin i = wbli.dy$; i being the slope of the bed.

(3) The viscous resistances on the lateral faces of the layer under consideration. These are nil, since in a stream of indefinite width there will be no relative sliding between *abcd* and the vertical faces on each side.

(4) The viscous resistances along the planes *ad* and *bc*.

The frictional resistance to distortion, i.e., to shearing, along such planes is found to be proportional to the shear per unit of time, and is measured by the shear per unit of area at the actual rate of shearing. The coefficient of viscosity, or simply the viscosity, is the quotient $\dfrac{\text{shear per unit of area}}{\text{shear per unit of time}}$, and defines that quality of the fluid in virtue of which it resists a change of shape.

Adopting Navier's hypothesis,

$$\text{the viscous resistance along } ad = -kbl\frac{dv}{dy},$$

k being the coefficient of viscosity. The sign is negative as, since v increases with y, $\frac{dv}{dy}$ is positive, and, at the same time, the action of the layers above ad is of the character of a retardation.

The viscous resistance along $bc = kbl\frac{dv}{dy} + kbl \, . \, d\left(\frac{dv}{dy}\right)$

$$= kbl\frac{dv}{dy} + kbl\frac{d^2v}{dy^2}dy.$$

Then, as the motion is uniform,

$$wbli \, . \, dy - kbl\frac{dv}{dy} + kbl\frac{dv}{dy} + kbl\frac{d^2v}{dy^2}dy = 0.$$

Hence

$$\frac{d^2v}{dy^2} + \frac{wi}{k} = 0.$$

Integrating twice,

$$v = -\frac{wi}{2k}y^2 + ay + v_s, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a and v_s being constants of integration.

It is evident that v_s is the surface-velocity, i.e., the value of v when $y = 0$.

The equation may be written in the form

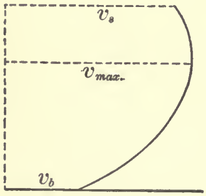


FIG. 90.

$$v - v_s - \frac{ka^2}{2wi} = -\frac{wi}{2k}\left(y - \frac{ka}{wi}\right)^2. \quad (2)$$

Thus the velocity-curve is a parabola having a horizontal axis at a depth $Y = \frac{ka}{wi}$ below the free surface. This is also the depth of the filament of maximum velocity $\left(\frac{dv}{dy} = 0\right)$ and

$$v_{\max} = v_s + \frac{ka^2}{2wi} = v_s + \frac{wi}{2k}Y^2. \quad . \quad . \quad . \quad (3)$$

Hence, by equations 1 and 3,

$$v = v_{\max} - \frac{wi}{2k}(y - Y)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let v_m be the "mean" velocity for the whole depth h. Let $v_{\frac{1}{2}}$ be the mid-depth velocity. Then

$$v_m = \frac{\int_0^h \left\{ v_{\max} - \frac{wi}{2k}(y - Y)^2 \right\} dy}{h}$$

$$= v_{\max} - \frac{wi}{6k}(h^2 - 3hY + 3Y^2), \quad . \quad . \quad . \quad . \quad (5)$$

and

$$v_{\frac{1}{2}} = v_{\max} - \frac{wi}{2k}\left(\frac{h}{2} - Y\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence

$$v_{\frac{1}{2}} - v_m = \frac{wih^2}{24k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

a result upon which Humphreys and Abbott have based a rapid method of gauging rivers.

Let v_b be the bottom velocity, i.e., the value of v when $y = h$. Then by equation 4,

$$v_b = v_{\max} - \frac{wi}{2k}(h - Y)^2,$$

and therefore

$$v_{\max} - v_b = \frac{wi}{2k}(h - Y)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When the filament of maximum velocity was below the free surface Bazin found the value of the difference $v_{\max} - v_b$ to be constant. Take

$$v_{\max} - v_b = N = \frac{wi}{2k}(h - Y)^2.$$

Then the general equation (4) of the velocity-curve becomes

$$v = v_{\max} - N\left(\frac{y - Y}{h - Y}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now if $Y = 0$, i.e., if the filament of maximum velocity is in the free surface,

$$v = v_{\max} - N\frac{y^2}{h^2}.$$

But in such case Bazin's experiments led to the relation

$$v = v_{\max} - 36.3\sqrt{hi}\frac{y^2}{h^2}.$$

Hence

$$N = 36.3\sqrt{hi},$$

and the general equation of the velocity-curve becomes

$$v = v_{\max} - 36.3\sqrt{hi}\left(\frac{y - Y}{h - Y}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is Bazin's formula, and it agrees well with his experiments on artificial channels and also with the results of experiments on the Saone, Seine, Garonne, and Rhine. It was found that

$\dfrac{v_{\max}}{v_m} = 1.17$ in the Rhine at Basle and ranged from 1.1 to 1.13 in the others;

$\dfrac{36.3h^2\sqrt{hi}}{(h - Y)^2}$ lay between 13 and 20;

$\dfrac{Y}{h} = .33$ in some artificial channels and ranged from 0 to 0.2 in the other cases;

$v_{\max} - v_b$ ranged from $\frac{1}{4}v_{\max}$ to $\frac{1}{2}v_{\max}$.

These results are not in agreement with the Mississippi measurements.

Note.—When the filament of maximum velocity is in the free surface, $Y = 0$, and therefore, by equation 5,

$$v_m = v_{\max} - \frac{wih^2}{6k},$$

and by equation 8,

$$v_{\max} - v_b = \frac{wih^2}{2k}.$$

Hence, combining these two equations,

$$v_m = \tfrac{1}{3}(2v_{\max} + v_b). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Boileau assumes that the velocity-curve is given by the equation

$$v = A - By^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

below the filament of maximum velocity, being MM_1 in Fig. 91, and by the equation

$$v = a - By^2 + Cy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

above the filament of maximum velocity, being MM_2 in Fig. 92.

Let v_s be the surface-velocity, i.e., the value of v when $y = 0$. Then, by equation 13,

$$v_s = a.$$

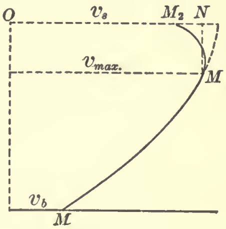


FIG. 91.

Also, the two equations (12) and (13) must each give the same value for the maximum velocity ($v_{\max}$), and therefore

$$A - BY^2 = v_{\max} = a - BY^2 + CY,$$

from which

$$C = \frac{A - a}{Y} = \frac{A - v_s}{Y}.$$

Again, taking $A = v_{\max} + d$, Boileau deduced experimentally that d is sensibly constant for different streams.

But $A = v_{\max} + d = A - BY^2 + d$, and therefore $B = \dfrac{d}{Y^2}$.
Hence Boileau's equation becomes

$$v = v_{\max} + d - d\left(\frac{y}{Y}\right)^2$$

for points *below* the filament of maximum velocity, and

$$v = v_s - d\left(\frac{y}{Y}\right)^2 + (v_{\max} + d - v_s)\frac{y}{Y}$$

for points *above* the filament of maximum velocity.

8. Relations between Surface, Mean, and Bottom Velocities.—Bazin deduced from his experiments on canals the relation

$$v_m = v_s - 25.4\sqrt{mi} = v_s - 25.4\frac{v^m}{c},$$

where $c = \sqrt{\dfrac{2g}{f}}$. Therefore

$$v_m = \frac{cv_s}{c + 25.4}.$$

Darcy and Bazin give the relation

$$v_m = v_b + 10.87\sqrt{mi} = v_b + 10.87\frac{v_m}{c}.$$

Therefore

$$v_m = \frac{cv_b}{c - 10.87}.$$

A mean value of c is 45.7, which makes

$$v_m = 1.312 \cdot v_b.$$ V_b = 76% of V

Dubuat gives the following table of maximum bottom velocities consistent with stability:

Nature of Canal Bed.	v_b.
Soft earth	0.25
Loam	0.50
Sand	1.00
Gravel	2.00
Pebbles	3.40
Broken stone, flint	4.00
Chalk, soft shale	5.00
Rock in beds	6.00
Hard rock	10.00

TABLE OF MAXIMUM VELOCITIES FROM INGENIEURS TASCHENBUCH.

Nature of Canal-bed.	v_s	v_m	v_b
Slimy earth or brown clay	.49	.36	.26
Clay	.98	.75	.52
Firm sand	1.97	1.51	1.02
Pebbly bed	4.00	3.15	2.30
Boulder bed	5.00	4.03	3.08
Conglomerate of slaty fragments	7.28	6.10	4.90
Stratified rocks	8.00	7.45	6.00
Hard rocks	14.00	12.15	10.36

TABLE OF VISCOSITY OF WATER AND MERCURY.

(From Everett's System of Units.)

WATER.				MERCURY.			
Temp. (Cent.)	Viscosity.	Temp. (Cent.)	Viscosity.	Temp. (Cent.)	Viscosity.	Temp. (Cent.)	Viscosity.
0°	.0181	35°	.0073	0°	.0169	315°	.00918
5	.0154	40	.0067	10	.0162	340	.00897
10	.0133	45	.0061	18	.0156		
15	.0116	50	.0056	99	.0123		
20	.0102	60	.0047	154	.0109		
25	.0091	80	.0036	197	.0102		
30	.0081	90	.0032	249	.00964		

The viscosity is given by

$$\frac{.0183}{1 + .0369\,t}, \text{ according to Meyer;}$$

and by

$$\frac{.5212}{26 + t} - .00131, \text{ according to Slotte;}$$

t being the temperature centigrade.

9. Flow of Water in Open Channels of Varying Cross-section and Slope.

Assumptions.—(*a*) That the motion is steady.

Thus the mean velocity is constant for any given cross-section, but varies *gradually* from section to section.

(*b*) That the change of cross-section is also gradual.

(*c*) That, as in cases of *uniform* motion, the work absorbed by the frictional resistance of the channel-bed and sides is the only *internal* work which need be taken into consideration.

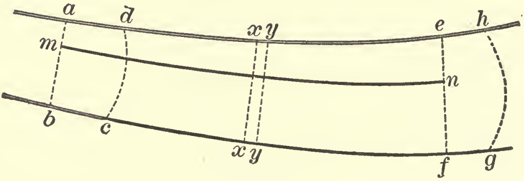


FIG. 92.

Let Fig. 92 represent a longitudinal section of the stream. The fluid molecules which are found in any plane section *ab* at the commencement of an interval will be found in a curved surface *dc* at the end of the interval, on account of the different velocities of the fluid filaments.

Suppose that the mass of water bounded by the two transverse sections *ab*, *ef*, comes into the position *cdhg* in a unit of time. Then the change of kinetic energy in this mass is equal to the algebraic sum of the work done by gravity, of the work done by pressure, and of the work done against the frictional resistance.

Change of Kinetic Energy.—This is evidently the difference between the kinetic energies of the masses *efgh* and *abcd*, since, as the motion is steady, the kinetic energy of the mass between *cd* and *ef* remains constant.

Let A_1 be the area of the cross-section *ab*.
" u_1 " " mean velocity across this section.
" v " " velocity at this section of any given fluid, filament of sectional area a.

Let $v = u_1 \pm V$.
Then

$$A_1 u_1 = \Sigma(av) \quad \text{and} \quad \Sigma(aV) = 0.$$

The kinetic energy of the mass *abcd*

$$= \frac{w}{2g}\Sigma(av^3) = \frac{w}{2g}\Sigma\{a(u_1 \pm V)^3\}$$

$$= \frac{w}{2g}\Sigma\{a(u_1^3 \pm 3u_1^2 V + 3u_1 V^2 \pm V^3)\}$$

$$= \frac{w}{2g}\Sigma a\{u_1^3 + V^2(2u_1 + v)\}.$$

Since $\qquad \Sigma(aV) = 0 \quad \text{and} \quad 3u_1 \pm V = 2u_1 + v.$

Now $2u_1 + v$ is evidently positive. Hence the kinetic energy of the mass *abcd*

$$= \frac{w}{2g}\Sigma av^3 > \frac{w}{2g}\Sigma au_1^3$$

$$> \frac{w}{2g}A_1 u_1^3$$

$$= \alpha\frac{w}{2g}A_1 u_1^3,$$

α being a coefficient of correction whose value depends upon the law of the distribution of the velocity throughout the section *ab*. It is positive and greater than unity. *Assume* that α has the same value for the sections *ab* and *ef*. Then if A_2,

u_2, are the area and mean velocity at the transverse section ef, the kinetic energy of the mass $efgh$

$$= \alpha\frac{w}{2g}A_2u_2^3.$$

Hence the change of kinetic energy in the mass under consideration

$$= \alpha\frac{w}{2g}(A_2u_2^3 - A_1u_1^3)$$

$$= \alpha\frac{wQ}{g}\frac{u_2^2 - u_1^2}{2},$$

since $\qquad A_2u_2 = Q = A_1u_1.$

Work done by Gravity.—Consider any fluid filament mn, the depth of m below the surface being y_1, and of n, y_2.

Let z be the fall in the surface-level from a to e.

Then the fall from m to n

$$= z + y_2 - y_1,$$

and the work done by gravity on the elementary volume dQ in a unit of time

$$= + w \,.\, dQ(z + y_2 - y_1).$$

Work done by Pressure.

The pressure per unit of area at $m = wy_1 + p_0$;
" " " " " " " $n = wy_2 + p_0$;

p_0 being the atmospheric pressure.

Hence the work due to these pressures per unit of time

$$= dQ(wy_1 + p_0) - dQ(wy_2 + p_0),$$
$$= w \,.\, dQ(y_1 - y_2).$$

Thus the *total* work done by gravity and by pressure

$$= \Sigma\{w \,.\, dQ(z + y_2 - y_1) + w \,.\, dQ(y_1 - y_2)\}$$
$$= \Sigma(w \,.\, dQ \,.\, z) = wQz,$$

for the mass under consideration.

Work absorbed by Friction.—Consider a thin lamina of water of thickness ds, bounded by the transverse planes xx, yy, the distance of xx from ab being s.

Since the change of velocity is gradual, the mean velocity from xx to yy may be assumed to be constant.

Let u be this mean velocity.

" P be the wetted perimeter at the section xx.

" A be the area of the waterway at the section xx.

Then the work absorbed by friction per second from xx to yy

$$= P \, . \, ds \, . \, u \, . \, F(u).$$

and the total work absorbed between ab and ef

$$= Q\int_0^l \frac{P}{A}F(u)ds,$$

l being the distance between ab and ef. Hence

$$\alpha\frac{wQ}{g}\frac{u_2^2 - u_1^2}{2} = wQz - Q\int_0^l \frac{P}{A}F(u)ds,$$

and therefore $z = \alpha\dfrac{u_2^2 - u_1^2}{2g} + \displaystyle\int_0^l \frac{P}{A}\frac{F(u)}{w}ds.$

Take $\dfrac{F(u)}{w} = f\dfrac{u^2}{2g}$ and $\dfrac{A}{P} = m$. Then

$$z = \alpha\frac{u_2^2 - u_1^2}{2g} + \int_0^l \frac{f}{m}\frac{u^2}{2g}ds. \quad . \quad . \quad . \quad . \quad (1)$$

If the two planes ab and ef are indefinitely near one another (Fig. 93), the last equation evidently gives,

$$dz = \frac{\alpha}{g}u \, . \, du + \frac{f}{m}\frac{u^2}{2g}ds, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the fundamental differential equation of steady varied motion, dz being the fall of surface level in the distance ds.

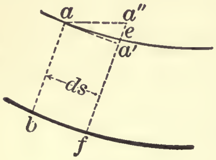


Fig. 93.

In the figure aa' is drawn parallel to the bed and aa'' is horizontal. The distance $a''e$ may, without sensible error, be assumed equal to dz.

Also $a''a' = i \,.\, aa' = i \,.\, ds$, very nearly.

Hence

$$ids = a'a'' = a'e + a''e = dh + dz. \quad . \quad . \quad . \quad (3)$$

Substituting the value of dz from this equation in equation 2,

$$i \,.\, ds - dh = \frac{\alpha}{g} u \,.\, du + \frac{f}{m}\frac{u^2}{2g} \,.\, ds. \quad . \quad . \quad . \quad . \quad (4)$$

Also, since $Au = Q$, a constant,

$$A \,.\, du + u \,.\, dA = 0,$$

and $dA = x \,.\, dh$, very nearly, if x is the width of the stream. Therefore

$$Adu + ux \,.\, dh = 0,$$

and hence, by equation 4,

$$i \,.\, ds - dh = -\,\alpha\frac{u^2}{g}\frac{x}{A} \,.\, dh + \frac{f}{m}\frac{u^2}{2g}ds.$$

Therefore

$$\frac{dh}{ds} = \frac{i - \dfrac{f}{m}\dfrac{u^2}{2g}}{1 - \alpha\dfrac{u^2}{g}\dfrac{x}{A}} = i\frac{1 - \dfrac{f}{m}\dfrac{u^2}{2gi}}{1 - \alpha\dfrac{u^2x}{gA}}. \quad . \quad . \quad . \quad . \quad (5)$$

Let the position of any point a in the surface be defined by its perpendicular distance h from the bed and by the distance s of the transverse section at a from an origin in the bed. Then $\dfrac{dh}{ds}$ is the tangent of the angle which the tangent to the surface at a makes with the bed. It is positive or negative according as the depth increases or diminishes in the direction of flow, thus defining two states of steady varied motion.

Between these there is an intermediate state defined by

$$\frac{dh}{ds} = 0 = i - \frac{f}{m}\frac{u^2}{2g},$$

and $i = \frac{f}{m}\frac{u^2}{2g}$ is the equation for steady flow with *uniform* motion.

Let U, M, H be the corresponding values of u, m, h in the case of uniform motion. Then

$$i = \frac{f}{M}\frac{U^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and equation 5 becomes

$$\frac{dh}{ds} = i\frac{1 - \frac{M}{m}\frac{u^2}{U^2}}{1 - \alpha\frac{u^2}{g}\frac{x}{A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

EXAMPLE.—Consider a stream of *rectangular section* and of a width x which is very great as compared with the depth. Then

$$A = xh; \; P = x \text{ very nearly}; \; m = \frac{xh}{x} = h; \; M = \frac{xH}{x} = H.$$

Hence

$$\frac{dh}{ds} = i\frac{1 - \frac{H}{h}\frac{u^2}{U^2}}{1 - \alpha\frac{u^2}{gh}} = i\frac{1 - \left(\frac{H}{h}\right)^3}{1 - \alpha\frac{u^2}{gh}}, \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

since $xhu = xHU$ and therefore $\frac{u}{U} = \frac{H}{h}$.

Note.—In each of the following cases the line PQ drawn parallel to the bed, represents the surface of uniform motion, H being the distance between PQ and the bed.

CASE I. $\alpha u^2 < gh$ and $H < h$.

$\dfrac{dh}{ds}$ is positive, and therefore h increases in the direction of flow. Thus the actual surface MN of the stream is wholly above the line PQ.

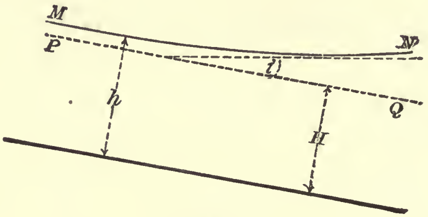


FIG. 94.

Proceeding up stream, h becomes more and more nearly equal to H, so that the numerator of equation 8, and therefore also $\dfrac{dh}{ds}$, approximates more and more closely to *zero*.

Again, proceeding down-stream, h increases and u diminishes, so that the numerator and denominator in equation 8 approximate each more and more closely to the value *unity*, and therefore $\dfrac{dh}{ds}$ becomes more and more nearly equal to i, the slope corresponding to uniform motion.

Hence up-stream, MN is asymptotic to PQ, and down-stream MN is asymptotic to a horizontal line. This form of water-surface is produced when a weir is built across a channel in which the water had previously flowed with a uniform motion.

CASE II. $\alpha u^2 < gh$ and $H > h$.

$\dfrac{dh}{ds}$ is now negative, and the depth diminishes in the direction of flow.

Up-stream, h increases and approaches H in value, so that MN is asymptotic to PQ.

Down-stream, h diminishes, u increases, and therefore the value of $\frac{\alpha u^2}{gh}$ is more and more nearly equal to *unity*.

Thus, in the limit, the denominator in equation 8 becomes zero, and therefore $\frac{dh}{ds} = \infty$. Hence theory indicates that at a certain point down-stream the surface line MN takes a direction which is at right angles to the general direction of flow. This is contrary to the fundamental hypothesis that the fluid filaments flow in sensibly parallel lines. In fact, before the

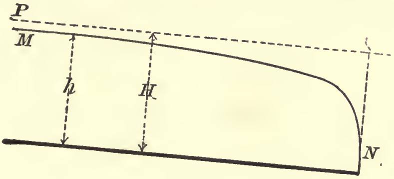


FIG. 95.

limit could be reached this hypothesis would cease to be even approximately true, and the general equation would cease to be applicable. This form of water-surface is produced when there is an abrupt depression in the bed of the stream.

Fig. 96 shows one of the abrupt falls in the Ganges canal as at first constructed. The surface of the water flowing freely

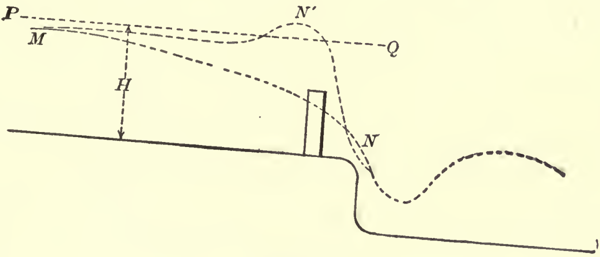


FIG. 96.

over the crest of the fall took a form similar to MN below the line PQ of uniform motion. The diminution of depth in the approach to the fall caused an increase in the velocity of flow, with the result that for several miles above the fall a serious erosion of the bed and sides took place. In order to remedy this, temporary weirs were constructed so as to raise the level of the water until the surface-line assumed a form MN' corresponding approximately to PQ. In some cases the water was raised above its normal height and a backwater produced.

CASE III. $\alpha u^2 > gh$ and $H < h$.

$\frac{dh}{ds}$ is negative and the surface-line of the stream is wholly above PQ.

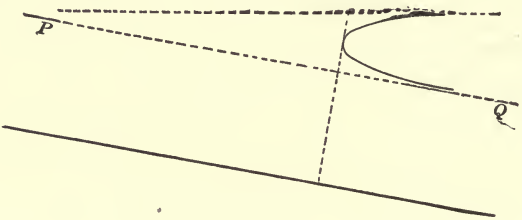


FIG. 97.

If h gradually increases, u diminishes and $\frac{dh}{ds}$ approximates to $-i$ in value.

If h gradually diminishes it approximates to H in value, and in the limit $\frac{dh}{ds} = 0$.

Between these two extremes there is a value of h for which the denominator of equation 8 becomes nil, viz.,

$$h = \alpha\frac{u^2}{g},$$

and the corresponding value of $\frac{dh}{ds}$ is infinity.

Thus one part of the surface line is asymptotic to PQ, the line of uniform motion, another part is asymptotic to a horizontal line, while at a certain point at which the depth is

$$h = \alpha\frac{u^2}{g}$$

the surface of the stream is normal to the bed.

This is contrary to the fundamental hypothesis that the fluid filaments flow in sensibly parallel lines, and the general equation no longer represents the true condition of flow.

In cases such as this, there has been an abrupt rise of the surface of the stream, and what is called a "standing wave" has been produced.

In a stream of depth H flowing with a uniform velocity U which is $> \sqrt{\frac{gH}{\alpha}}$, construct a weir so as to increase the depth to h_1 which is $> \frac{\alpha U^2}{g}$.

Then in one portion of the stream near the weir the depth is $> \frac{\alpha U^2}{g}$, while further up the stream the depth is $< \frac{\alpha U^2}{g}$. Thus at some intermediate point the depth $= \alpha\frac{U^2}{g}$, the cor-

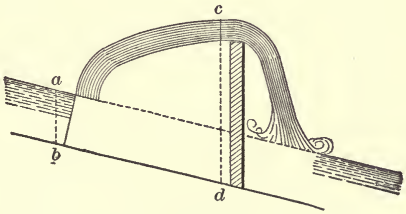


FIG. 98.

responding value of $\frac{dh}{ds}$ being ∞, so that at this point a *standing wave is produced.*

Now

$$\frac{fU^2}{2g} = Mi = Hi;$$

and since

$$H < \alpha\frac{U^2}{g},$$

$$\frac{fU^2}{2g} < \alpha\frac{U^2}{g}i,$$

and therefore

$$i > \frac{f}{2\alpha},$$

which condition must be fulfilled for a standing wave.

Bazin gives the following table of values of f:

Nature of Bed.	Slope $\left(\frac{h}{l} = i\right)$ below which standing wave is impossible. In Metres per Metre.	Standing Wave Produced.	
		Slope in Metres per Metre (or Feet per Foot).	Least Depth in Metres.
Very smooth cemented surface....	.00147	.002	.08
		.003	.03
		.004	.02
Ashlar or brickwork............	.00186	.003	.12
		.004	.06
		.006	.03
Rubble masonry................	.00235	.004	.36
		.006	.16
		.010	.08
Earth.........................	.00275	.006	1.06
		.010	.47
		.015	.28

A standing wave rarely occurs in channels with earthen beds, as their slope is almost always less than the limit, .00275.

The formation of a standing wave was first observed by Bidone in a small masonry canal of rectangular section.

The width of the canal $= 0^m.325 = x$.

" slope $\left(= \frac{h}{l}\right)$ of the canal $=$ .023;

" uniform velocity of flow $= 1^m.69 = U$;

" depth corresponding to $U = 0^m.064 = H$.

A weir built across the canal increased the depth of the water near the weir to $0^m.287 = h_1$.

It was found that the "uniform régime" was maintained up to a point within $4^m.5$ of the weir. At this point the depth suddenly increased from $0^m.064$ to about $0^m.170$, and between the point and the weir the surface of the stream was slightly convex in form (Fig. 98).

With the preceding data and taking $\alpha = 1.1$, $\dfrac{\alpha U^2}{gH} = 5$ and is therefore > 1 at a section *ab*, Fig. 99.

At the section *cd*,

$$u = \frac{H}{h}U = \frac{.064}{.287} \times 1.69 = 0^m.377,$$

and $\dfrac{\alpha U^2}{gh_1} = .055$ and is therefore < 1.

Thus the expression $1 - \dfrac{\alpha u^2}{gh}$ is negative for a section *ao* and positive for a section *cd*, so that i must change sign between these sections, and $\dfrac{dh}{ds}$ will then become infinite.

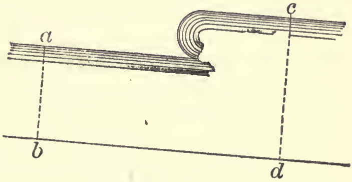


FIG. 99.

Consider a portion of a stream bounded by two transverse sections *ab*, *cd*, in which a standing wave occurs, Fig. 99.

Assume that the fluid filaments flow across the sections in sensibly parallel lines.

Let the velocities and area at section *ab* be distinguished by the suffix 1, and those at *cd* by the suffix 2. Then

$$\left.\begin{array}{l}\text{Change of momentum in di-}\\ \qquad\text{rection of flow}\end{array}\right\} = \text{impulse in same direction.}$$

Hence

$$\frac{w}{g}\left\{\Sigma(av_2)v_2 - \Sigma(av_1)v_1\right\} = w(A_1y_1 - A_2y_2),$$

and therefore

$$\frac{1}{g}\left(\Sigma av_2^2 - \Sigma av_1^2\right) = A_1y_1 - A_2v_2, \quad . \quad . \quad . \quad (9)$$

y_1, y_2, being the depths below the surface of the centres of gravity of the sections *ab*, *cd*, respectively.

Now, $v_1 = u_1 + V_1$. Therefore

$$\Sigma a v_1^2 = \Sigma a(u_1^2 + 2u_1V_1 + V_1^2)$$
$$= u_1^2A_1 + \Sigma a V_1^2.$$

Also, as already shown,

$$\alpha_1 A_1 u_1^3 = \Sigma a v_1^3 = A_1 u_1^3 + \Sigma a V_1^2(3u_1 + V_1),$$

and, neglecting V_1 as compared with $3u_1$,

$$\alpha A_1 u_1^3 = A_1 u_1^3 + 3u_1 \Sigma a V_1^2.$$

Thus

$$\Sigma a V_1^2 = \frac{A_1 u_1^2}{3}(\alpha - 1),$$

and hence

$$\Sigma a v_1^2 = u_1^2 A_1 + \frac{A_1 u_1^2}{3}(\alpha - 1)$$

$$= \frac{u_1^2 A_1}{3}(\alpha + 2) = \alpha' A_1 u_1^2,$$

where $\alpha' = \dfrac{\alpha + 2}{3}$, and is 1.033 if $\alpha = 1.1$.

Similarly it may be shown that

$$\Sigma a v_2^2 = \alpha' A_2 u_2^2.$$

Thus equation 9 becomes

$$\frac{\alpha'}{g}(A_2 u_2^2 - A_1 u_1^2) = A_1 y_1 - A_2 y_2. \quad . \quad . \quad . \quad (10)$$

Let the section of the canal be a rectangle of depth H_1 at *ab* and H_2 at *cd*. Then

$$u_1 H_1 = u_2 H_2; \quad \frac{H_1}{2} = y_1; \quad \frac{H_2}{2} = y_2.$$

Therefore, by equation 10,

$$\frac{\alpha'}{g}A_1u_1^2\left(\frac{H_1}{H_2} - 1\right) = A_1\left(\frac{H_1}{2} - \frac{H_2^2}{2H_1}\right),$$

which reduces to

$$\frac{\alpha' u_1^2}{gH_2}(H_2 - H_1) = \frac{1}{2H_1}(H_2^2 - H_1^2).$$

$H_2 = H_1$ satisfies the equation and corresponds to a condition of uniform motion.

Also

$$\frac{\alpha' u_1^2}{g} = \frac{H_2}{H_1}\frac{H_2 + H_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In Bidone's canal, $u_1 = 1^m.69$, $H_1 = 0^m.064$. Substituting these values in equation 11, the value of H_2 is found to be $0^m.16$, which agrees somewhat closely with the actual measurements.

N.B.—The coefficients α and α' have not been very accurately determined, but their exact values are not of great importance. They are often taken equal to *unity*.

EXAMPLES.

1. What fall must be given to a canal 2600 ft. long, 7 ft. wide at the top, 3 ft. wide at the bottom, $1\frac{1}{2}$ ft. deep, and conveying 40 cubic ft. of water per second? $f = \frac{1}{64}$. *Ans.* 1 in 135.

2. Determine the fall of a canal 1500 ft. long, of 2 ft. lower, 8 ft. upper breadth, and 4 ft. deep, which is to convey 70 cubic feet of water per second. *Ans.* 1 in 1365.4.

3. For a distance of 300 ft. a brook with a mean water perimeter of 40 ft. has a fall of 9.6 in.; the area of the upper transverse profile is 70 sq. ft., that of the lower 60 sq. ft. Find the discharge.

Ans. 662.87 cub. ft. per sec.

4. In a horizontal trench 5 ft. broad and 800 ft. long it is desired to carry off 20 cub. ft. discharge and to let it flow in at a depth of 2 ft.; what must be the depth at the end of the canal? (f = .008.)

Ans. 1.64 ft.

5. Water flows along an open channel 12 ft. wide and 4 ft. deep, at the rate of 2 ft. per second. What is the fall? A dam 12 ft. by 3 ft. high is formed across the channel; how high will the water rise over the crest of the dam? *Ans.* 1 in 480, f being .08; .899 ft.

6. A stream is rectangular in section, 12 ft. wide, 4 ft. deep, and falls 1 in 100. Determine the discharge (1) with an air-perimeter; (2) without air-perimeter. *Ans.* (1) 645.398 cub. ft. per sec.

(2) 665.088 cub. ft. per sec.

7. A canal 20 ft. wide at the bottom and having side slopes of $1\frac{1}{2}$ to 1 has 8 ft. of water in it; find the hydraulic mean depth. *Ans.* 5.24 ft.

8. The water in a semicircular channel of 10 ft. radius, when full flows with a velocity of 2 ft. per second; the fall is 1 in 400. Find the coefficient of friction. *Ans.* .2.

9. Calculate the flow per minute across a given section of a rectangular canal 20 ft. deep, 45 ft. wide, the slope of the bed being 22 in. per mile and the coefficient of friction per square foot = .008.

Ans. 279,229 cub. ft.

10. Why does the water of the St. Lawrence rise on the formation of the ice?

11. Find the depth and width of a rectangular stream of 900 sq. ft. sectional area, so that the flow might be a maximum; also find the flow, f being .008 and the slope 22 in. per mile.

Ans. 21.21 ft.; 42.42 ft.; 4885 cub. ft. per second.

12. Water flows along a symmetrical channel, 20 ft. wide at top and 8 ft. wide at bottom; the friction at the sides varies as the square of the velocity, and is 1 lb. per square foot for a velocity of 16 ft. per second. Find the proper slope, so that the water may flow at the rate of 2 ft. per second when its depth is 6 ft. *Ans.* 1 in 3445.

13. Calculate the flow across the vertical section of a stream 4 ft. deep, 18 ft. wide at top, 6 ft. wide at bottom, the slope of the surface being 18 in. per mile. ($f = .008$.) *Ans.* 110.9376 cub. ft. per second.

14. The sewers in Vancouver are square in section and are laid with one diagonal vertical. To what height should the water rise so that (*a*) the velocity of flow may be a maximum; (*b*) the discharge may be a maximum? (A side of the square = 12 in.)

Ans. (*a*) .292 ft. above horizontal diameter.
(*b*) .5797 ft. " " "

15. The sides of an open channel of given inclination slope at 45° and the bottom width is 20 ft. Find the depth of water which will make the velocity of flow across a vertical section a maximum.

Ans. 6.73 ft.

17. The banks of a channel slope at 45°; the flow across a transverse section is to be at the rate of 100 cubic feet at a maximum velocity of 5 ft. per second. Determine the dimensions of the transverse profile.

Ans. 11.05 ft. wide at bottom; 2.28 ft. deep.

18. What dimensions must be given to the transverse profile of a canal whose banks slope at 40°, and which has to conduct away 75 cubic feet with a mean velocity of 3 ft. per second?

Ans. Depth = 3.6 ft.; width at bottom = 2.62 ft.

19. The section of a canal is a regular trapezoid; its slope is 1 in 500; its width at the bottom is 8 ft.; the sides are inclined at 30° to the vertical. On one occasion when the water was 4 ft. deep a wind was blowing up the canal, causing an air-resistance for each unit of free surface equal to one fifth of that for like units at the bottom and sides, where the coefficient of friction may be taken to be .08.

Determine the discharge. How will the discharge be affected when the canal is frozen over? *Ans.* 75.34 cub. ft. per sec.

20. The section of a channel is a rhombus with diagonal vertical. How high must the water rise in the channel (*a*) to give a maximum of flow, and (*b*) to give a maximum discharge?

Ans. If D is the length of the horizontal diameter, and if θ is the inclination of a side to the vertical, the water must rise above the horizontal diameter to the height $D \cot \theta \times .207$ in (*a*) and to the height $D \cot \theta \times .4099$ in (*b*).

21. In the transverse section $ABCD$ of an open channel with a vertical slope of 1 in 300, the bottom width is 20 ft., the angle $ABC = 90°$

and the angle $BCD = 45°$. Find the height to which the water will rise so that the velocity of flow may be a maximum; also find the discharge across the section, f being .008.

Ans. 11.715 ft.; 1584 cub. ft. per second.

22. A canal is 20 ft. wide at the bottom, its side slopes are $1\frac{1}{2}$ to 1, its longitudinal slope is 1 in 360; calculate H.M.D. and the flow per minute across any given vertical section when there is a depth of 8 ft. of water in the canal. (Coeff. of friction = .008.)

Ans. 5.24 ft.; 2762.7776 cub. ft. per second.

23. If a weir 2 ft. high were built across the canal in the preceding question, what would be the increase in the depth of the water?

Ans. 2.79 ft.

24. For a small tachometer the velocities are .163, .205, .298, .366, .61 metre; the number of revolutions per second are .6, .835, 1.467, 1.805, 3.142. Find the constants corresponding to the wheel.

Ans. .162; .202; .309; .367; .595.

25. If the head of water in a channel increase by one tenth, show that the velocity and discharge, respectively, increase by $\frac{1}{20}$ and $\frac{3}{20}$, approximately.

If the depth diminish by 8%, show that the velocity and discharge, respectively, diminish by 4% and 12%, approximately.

26. Assuming (1) that a river flows over a bed of uniform resistance to source; (2) that to maintain stability the velocity is constant from source to mouth; (3) that the river sections at all points are similar; (4) that the discharge increases uniformly in consequence of the supply from affluents—determine the longitudinal section of such a river.

Ans. A parabola.

CHAPTER V.

METHODS OF GAUGING.

1. Gauging of Streams and Watercourses. — The amount of flow Q in cubic feet per second across a transverse section of A sq. ft. in area is given by the expression

$$Q = Au,$$

u being the mean velocity of flow in the section in feet per second. Various methods are employed for the determination of u.

METHOD I. The most convenient method for gauging small streams, canals, etc., is by means of a temporarily constructed weir, which usually takes the form of a rectangular notch. The greatest care should be exercised to ensure that the crest of the weir is truly level and properly formed and that the sides are truly vertical. The difference of level between the crest of the weir and the surface of the water at a point where it has not begun to slope down towards the weir is best estimated by means of Boyden's hook gauge, Fig. 100.

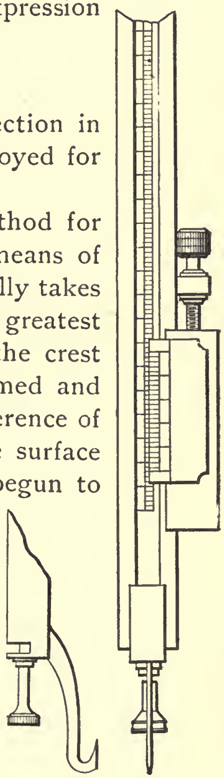


FIG. 100.

This gauge consists of a carefully graduated rod, or of a rod with a scale attached, having at the lower end a hook with a thin flat body and a fine point. The rod slides in vertical supports, and a slow vertical movement is given by means of a screw of fine pitch. In an experiment, the hook point is set truly level with the crest of the weir, and a reading is taken. The gauge is then moved away from the weir, about 2 to 4 ft. for small weirs and about 6 to 8 ft. for large weirs. The hook is then slowly raised, until a capillary elevation of the surface is produced over the point. The hook is now lowered until this elevation is barely perceptible, and a second reading is taken. The difference between the two readings is the difference of level required.

In ordinary light, differences of level as small as the one-thousandth of a foot, can be easily detected by the hook gauge, while with a favourable light it is said that an experienced observer can detect a difference of two ten-thousandths of a foot.

METHOD II. A portion of the stream which is tolerably straight and of approximately uniform section is defined by two transverse lines O_1AB, O_2CD, at any distance S ft. apart.

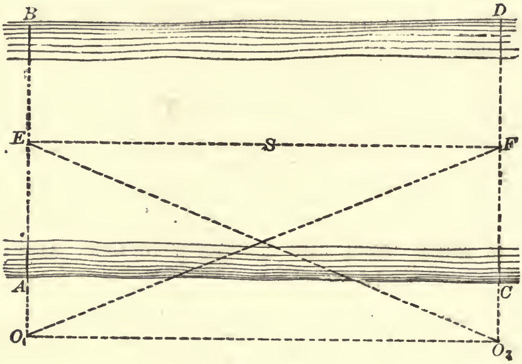


FIG. 101.

The base-line O_1O_2 is parallel to the thread EF of the stream, and observers with chronometers and theodolites (or sextants) are stationed at O_1, O_2. The time T and path EF taken by a float between AB and CD can now be determined. At the moment the float leaves AB the observer at O_1 signals the observer at O_2, who measures the angle O_1O_2E, and each marks the time. On reaching CD the observer at O_2 signals the observer at O_1, who measures the angle O_2O_1F, and each again marks the time.

Experience alone can guide the observer in fixing the distance S between the points of observation. It should be remembered that although the errors of time observations are diminished by increasing S, the errors due to a deviation from lines parallel to the thread of the stream are increased.

A number of floats may be sent along the same path, and their velocities $\left(\frac{S}{T}\right)$ are often found to vary as much as 20 per cent and even more.

Having thus found the velocities along any required number of paths in the width of the stream, the mean velocity for the whole width can be at once determined.

Surface-floats are small pieces of wood, cork, or balls of wax, hollow metal and wood, colored so as to be clearly seen, and ballasted so as to float nearly flush with the water-surface and to be little affected by the wind.

Subsurface-floats.—A subsurface-float consists of a heavy float with a comparatively large intercepting area, maintained at any required depth by means of a very fine and nearly vertical cord attached to a suitable surface-float of minimum immersion and resistance. Fig. 102 shows such a combination, the lower float consisting of two pieces of galvanized iron soldered together at right angles, the upper float being merely a wooden ball.

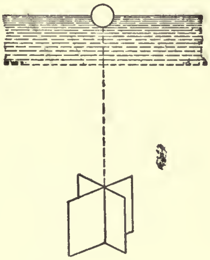


FIG. 102.

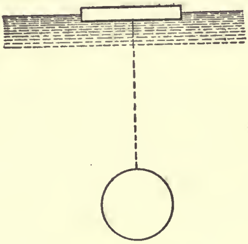


FIG. 103.

Another combination of a hollow metal ball with a piece of cork is shown by Fig. 103.

The motion of the combination is sensibly the same as that of the submerged float, and gives the velocity at the depth to which the heavy float is submerged.

Twin-floats.—Two equal and similar floats (Fig. 104), one denser and the other less dense than water, are connected by a fine cord. The velocity (v_t) of the combination is approximately the mean of the surface-velocity (v_s) and of the velocity (v_d) at the depth to which the heavier float is submerged. Thus

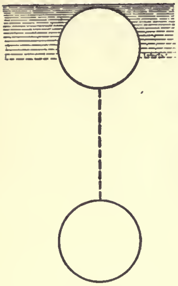


FIG. 104.

$$v_t = \frac{v_s + v_d}{2},$$

and therefore

$$v_d = 2v_t - v_s,$$

so that v_d can be determined as soon as the value of v_t has been observed and the value of v_s found by surface-floats.

Velocity-rod. — This is a hollow cylindrical rod of adjustable length and ballasted so as to float nearly vertical. It sinks almost to the bed of the stream, and its velocity (v_m) is approximately the mean velocity for the whole depth.

Francis gives the following empirical formula connecting the mean velocity (v_m) with the observed velocity (v_r) of the rod:

$$v_m = v_r\left(1.012 - .116\sqrt{\frac{d'}{d}}\right),$$

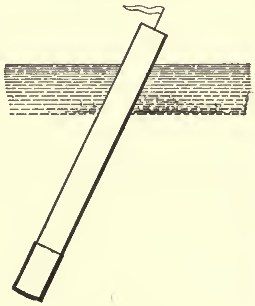


FIG. 105.

d being depth of stream, and d' the depth of water below bottom of rod; but d' should not exceed about one fourth of d.

METHOD III. *Pitot Tube and Darcy Gauge.*—A Pitot tube (Figs. 106 to 108) in its simplest form is a glass tube with a right-angled bend. When the tube is plunged vertically into the stream to any required depth z below the free surface, with its mouth pointing up-stream and normal to the direction of flow, the water rises in the tube to a height h above the outside surface, and the weight of the column of water $z+h$

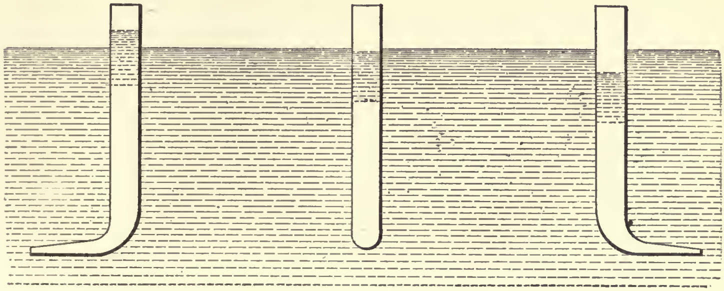


FIG. 106. FIG. 107. FIG. 108.

high, is balanced by the impact of the stream on the mouth. Hence, (Chap. VI.),

$$wA(z+h) = wAz + kwA\,\frac{u^2}{2g},$$

and therefore

$$h = k\frac{u^2}{2g},$$

A being the sectional area of the tube, u the velocity of flow at the given depth, and k a coefficient to be determined by experiment.

A mean value of k is 1.19. With a funnel-mouth or a bell-mouth, Pitot found k to be 1.5. This form of mouth, however, interferes with the stream-lines, and the velocity in front of the mouth is probably a little different from that in the unobstructed stream.

The advantages of tubes of small section are that the disturbance of the stream-lines is diminished and the oscillations of the column of water are checked. Darcy found by careful measurement that the difference of level between the surfaces of the water-column in a tube of small section placed as in Fig. 106, and of the water-column placed as in Fig. 107 with

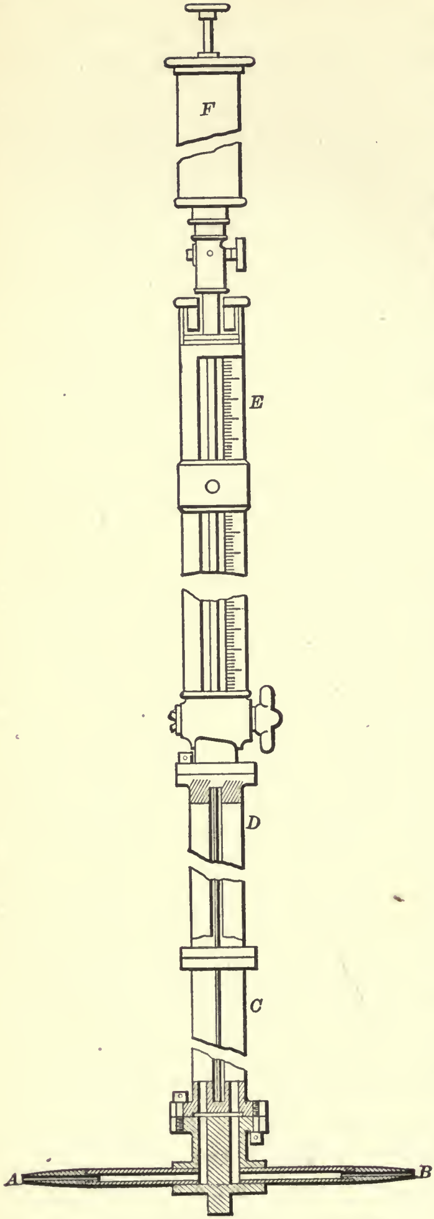


FIG. 109.

its mouth parallel to the direction of flow, is almost exactly equal to $\frac{u^2}{2g}$.

When the tube is placed as in Fig. 108 with its mouth pointing down-stream and normal to the direction of flow, the level of the surface of the water in the tube is at a depth h' below the outside surface, and

$$h' = k'\frac{u^2}{2g},$$

where k' is a coefficient to be determined by experiment and a little less than unity.

In this case the tube again obstructs the stream-lines. Pitot's tube does not give measurable indications of very low velocities. A serious objection to the simple Pitot tube is the difficulty of obtaining accurate readings near the surface of the stream. This objection is removed in the case of Darcy's gauge, shown in the accompanying sketch, Fig. 109.

A and *B* are the water-inlets; *C* and *D* are two double tubes; *E* is a brass tube containing two glass pipes which communicate at the bottom with the water-inlets and at the top with each other, and with a pump F by which the air can be drawn out of the glass, pipes thus allowing the water to rise in them to any convenient height.

Thus Darcy's gauge really consists of two Pitot tubes connected by a bent tube at the top and having their mouths at right angles or pointing in opposite directions. If h is the difference of level between the water-surfaces in the tubes when the mouths are at right angles, then

$$k\frac{u^2}{2g} = h,$$

and Darcy's experiments showed that k does not sensibly differ from unity.

When the mouths point in opposite directions, let h_1, h_2 be the differences of level between the stream-surface and the surfaces of the water in the tube pointing up-stream and the tube pointing downstream, respectively. Then

$$h_1 = k_1\frac{u^2}{2g};$$

$$h_2 = k_2\frac{u^2}{2g};$$

and therefore

$$h_1 + h_2 = (k_1 + k_2)\frac{u^2}{2g}$$

$$= k\frac{u^2}{2g};$$

where $k = k_1 + k_2$.

k having been determined experimentally once for all, the difference of level $(= h_1 + h_2)$ between the columns for any given case can be measured on the gauge and then u can be at once found.

A cock may be inserted in the bend connecting the two tubes, and through this cock air may be exhausted and a partial vacuum created in the upper portion of the gauge. The water-columns will thus rise to higher levels, but the difference between them will remain constant. Thus the surface of the column in the down-stream tube may be brought above the level of the outside surface, and the reading is then easily made.

Sometimes the gauge is furnished with cocks at the lower parts of the tubes, and if these cocks are closed when the measurement is to be made, the gauge may be removed from the stream for the readings to be taken.

METHOD IV. *Current-meters.*—The velocity of flow in large streams and rivers is most conveniently and most accurately ascertained by means of the current-meter. The earliest form of meter, the Woltmann mill, is merely a water-mill with flat vanes, similar in theory and action to the wind-mill. When the Woltmann is plunged into a current, a counter registers the number of revolutions made in a given interval of time, and the corresponding velocity can then be determined. This form of meter has gone out of use and has been replaced by a variety of meters of greater accuracy, of finer construction, and much better suited to the work. In its simplest form the present meter consists of a screw-propeller wheel (Fig. 110), or a wheel with three or more vanes mounted on a spindle and connected by a screw-gearing with a counter which registers the number of revolutions. The meter is put in or out of gear by means of a string or wire. When a current velocity at any given point is to be found, the reading of the counter is noted, the meter is sunk to the required position, and is then set and kept in gear for any specified interval of time. At the end of the interval the meter is put out of gear and is raised to the surface when the reading of the counter is again noted. The difference between the readings gives the number of revolutions made during the interval, and the velocity is given by an empirical formula connecting the velocity and the number of revolutions in a unit of time.

The vane *V* is introduced to compel the meter to take its proper direction.

In order to prevent the mechanism of the meter from being

FIG. 110.

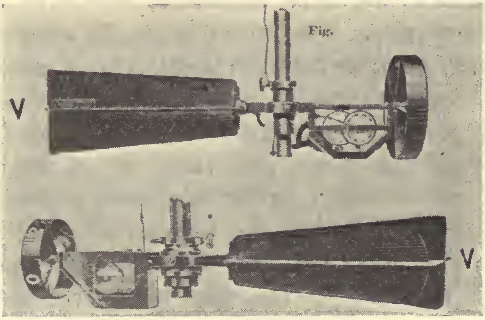


FIG. 111.

injuriously affected by floating particles of detritus, Revy enclosed the counter in a brass box, Fig. 111, with a glass face,

FIG. 112.

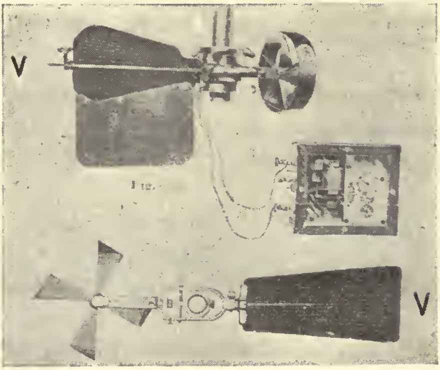


FIG. 113.

and filled the box with pure water so as to ensure a constant coefficient of friction for the parts which rub against each other. In the best meters, however, the record of the number of revolutions is kept by means of an electric circuit, Fig. 112, which is made and broken once, or more frequently, each revolution, and which actuates the recording apparatus. The time at which an experiment begins and ends is noted, and the revolutions made in the interval are read on the counter, which may be kept in a boat or on the shore, as the circumstances of the case may require. The meter is usually attached to a suitably graduated pole, so that the depth of the meter below the water-surface can be directly read. The mean velocity for the whole depth at any point of a stream may be found by moving the meter vertically down and then up, at a uniform rate. The mean of the readings at the two surface positions and at the bottom position will be the number of revolutions corresponding to the mean velocity required. The mean velocity for the whole cross-section may also be determined by moving the meter uniformly over all parts of the section.

Before the meter can be used it must be rated. This is done by driving the meter at different uniform speeds through still water. Experiment shows that the velocity v and the number of revolutions n are approximately connected by the formula

$$v = an + b,$$

where a and b are coefficients to be determined by the method of least squares or otherwise.

Exner gives the formula

$$v^2 = c^2n^2 + v_0^2,$$

v_0 being the velocity at which the meter just ceases to revolve.

OTHER METHODS.—Many other pieces of apparatus for the measurement of current velocities have been designed.

Perrodil's hydrodynamometer, for example, gives the velocity directly in terms of the angle through which a vertical torsion-rod is twisted, and in this respect is superior to the current-meter.

The hydrometric pendulum (Fig. 114), again, connects the velocity with the angular deviation from the vertical of a heavy ball suspended by a string in the current.

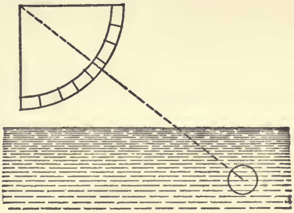


FIG. 114.

Hydrometric and torsion balances have also been devised, but they must be regarded rather as curiosities than as being of any real practical use.

2. Gauging of Pipe Flow.—A variety of meters have been designed to register the quantity of water delivered by a pipe. The principal requisites of such a meter are:

1. That it should register with accuracy the quantity of water delivered under different pressures.

2. That it should not appreciably diminish the effective pressure of the water.

3. That it should be compact and adaptable to every situation.

4. That it should be simple and durable.

The Venturi Meter (Fig. 115) is so called from Venturi, who first pointed out the relation between the pressures and velocities of flow in converging and diverging tubes.

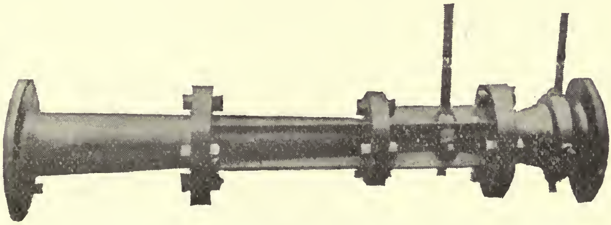


FIG. 115.

As shown by the longitudinal section, Fig. 116, this meter consists of two truncated cones joined at the smallest sections by a short throat-piece. At *A* and *B* there are air-chambers with holes for the insertion of piezometers, by which the fluid pressure may be measured. By Art. 5, Chap. I, the theoretical quantity Q of flow through the throat at A is

$$Q = \frac{a_1 a_2}{\sqrt{a_2^2 - a_1^2}} \sqrt{2g(H_2 - H_1)},$$

a_1, a_2 being the sectional areas at A and B, respectively, and $H_2 - H_1$ the difference of head in the piezometers, or the "head on Venturi," as it is called.

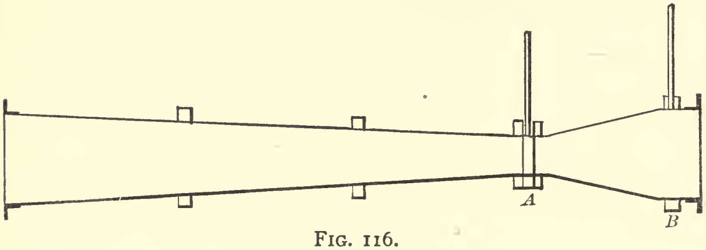


FIG. 116.

Introducing a coefficient of discharge C, the actual delivery through A is

$$Q = C\frac{a_1 a_2}{\sqrt{a_2^2 - a_1^2}} \sqrt{2g(H_2 - H_1)}.$$

An elaborate series of experiments by Herschel gave C values varying between .94 and 1·04, but the great majority of the values lay between .96 and .99.

The piezometers may be connected with a recorder, and thus a continuous register of the quantity of water passing through the meter may be obtained at any convenient position within a radius of 1000 ft. This distance may be extended to several miles by means of an electric device.

Other meters may be generally classified as Piston or Reciprocating Meters and Inferential or Rotary Meters. They are all provided with recorders which register the delivery with a greater or less degree of accuracy.

The piston meter (Fig. 118) is the more accurate and gives a positive measurement of the actual delivery of water as recorded by the strokes of the piston in a cylinder which is filled from each end alternately. Thus an additional advan-

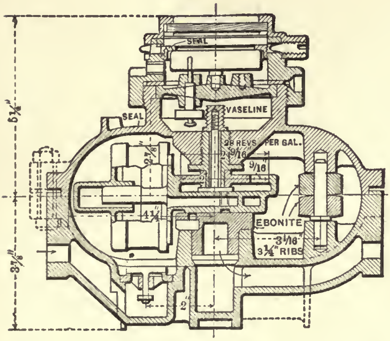


FIG. 118. — SCHONHEYDER'S POSITIVE METER.

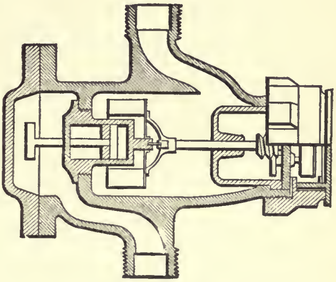


FIG. 119.—THE UNIVERSAL METER.

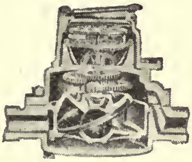


FIG. 120.—THE BUFFALO METER.

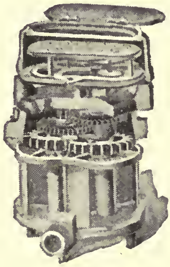


FIG. 121.—THE UNION ROTARY PISTON METER.

tage possessed by a water-engine is that the working cylinder will also serve as a meter.

In inferential meters, a drum or turbine is actuated by the force of the current passing through the pipe, but it often happens that when the flow is small the force is insufficient to cause the turbine to revolve, and there is consequently no register of the corresponding quantity of water passing through the meter.

CHAPTER VI.

IMPACT.

Note.—The following symbols are used:

v_1 = the velocity of the jet before impact;

v_2 = " " " " " after leaving the vane;

u = " " " " vane;

V = " " " " water relatively to the vane;

A = sectional area of the impinging jet;

m = mass of the water reaching the vane per second.

1. Impact of a Jet upon a Flat Vane oblique to the Direction of the Jet.—Let θ be the angle between the normal to the vane and the direction of the impinging jet, ϕ the angle between the normal to vane and the direction of the vane's motion, and α the angle between the vane and the vertical.

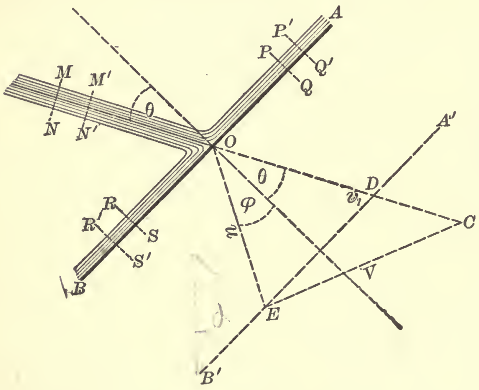


FIG. 122.

The jet moving with its stream-lines parallel, swells out near the vane, over which it spreads and with which it travels along in the direction of the vane's motion, and finally again flows along with its stream-lines sensibly parallel to the vane.

The problem is still further complicated by the production of eddies and vortices for which allowance can only be made in a purely empirical manner.

Let N be the normal pressure on the vane due to the impact.

Let N' be the total normal pressure on the vane.

Let W be the weight of water on the vane.

Then

$N = N' - W \sin \alpha =$ change of momentum in direction of the normal

$$= mv_1 \cos \theta - mu \cos \phi.$$

or

$$N = m(v_1 \cos \theta - u \cos \phi). \quad . \quad . \quad . \quad . \quad (1)$$

(*N. B.*—The sign in front of $u \cos \phi$ will be plus if the jet and vane move in opposite directions.)

The term $W \sin \alpha$ may be designated the *static* pressure and the term $m(v_1 \cos \theta - u \cos \phi)$ the *dynamic* pressure which causes the deviation of the stream-lines.

Note.—The pressure when a jet *first* strikes the plane is greater than when the flow has become steady, or permanent régime is established.

This is made evident by the following consideration:

At any moment let MN, PQ, RS be the bounding planes across which the water is flowing with its stream-lines sensibly parallel.

In a unit of time let the bounding planes of the mass be $M'N'$, $P'Q'$, $R'S'$.

Then, initially, the reaction of the plane must destroy the motion of the mass of the fluid bounded by $M'N'$, $P'Q'$, and $R'S'$.

Take OC to represent v_1 in direction and magnitude.
" OE " " u " " " "

In one second the vane AB moves parallel to itself into the position $A'B'$. Let $A'B'$ intersect OC in D.

Then

$$m = \frac{w}{g} A \, . \, DC = \frac{w}{g} A(v_1 - OD)$$

$$= \frac{w}{g} A\left(v_1 - u \frac{\cos \phi}{\cos \theta}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus equation 1 becomes

$$N = \frac{w}{g} \frac{A}{\cos \theta}(v_1 \cos \theta - u \cos \phi)^2. \quad . \quad . \quad . \quad (3)$$

Let P be the pressure in the direction of the vane's motion, then

$$P = N \cos \phi = \frac{w}{g} A \frac{\cos \phi}{\cos \theta}(v_1 \cos \theta - u \cos \phi)^2, \quad . \quad (4)$$

and the useful work done on the vane per second

$$= Pu = \frac{w}{g} A \frac{\cos \phi}{\cos \theta} u (v_1 \cos \theta - u \cos \phi)^2. \quad . \quad . \quad (5)$$

$$\text{The total available work} = \frac{w}{g} A \frac{v_1^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\text{Hence, the efficiency} = \frac{\frac{w}{g} A \frac{\cos \phi}{\cos \theta} u (v_1 \cos \theta - u \cos \phi)^2}{\frac{w}{g} A \frac{v_1^3}{2}}$$

$$= 2 \frac{\cos \phi}{\cos \theta} \frac{u}{v_1^3} (v_1 \cos \theta - u \cos \phi).^2 \quad (7)$$

This is a maximum when

$$v_1 \cos \theta = 3u \cos \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and therefore

$$\text{the maximum efficiency} = \frac{8}{27} \cos^2 \theta. \quad . \quad . \quad . \quad (9)$$

If the vane is of small sectional area a portion of the water will escape over the boundary and the pressure must necessarily be less than that given by equation 3.

Instead of one vane moving before the jet, let a series of vanes be introduced at short intervals at the same point in the path of the jet.

The quantity of water now reaching the vane per second is evidently

$$m = \frac{w}{g} A v_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and, by equation 1, the normal pressure

$$N = N\frac{w}{g}Av_1(v_1 \cos\theta - u\cos\phi). \quad . \quad . \quad . \qquad (11)$$

Also, the pressure in the direction of the motion of the vane

$$= P = N\cos\phi = \frac{w}{g}A\cos\phi\, v_1(v_1\cos\theta - u\cos\phi). \qquad (12)$$

The useful work done per second

$$= Pu = \frac{w}{g}A\cos\phi\, v_1 u(v_1\cos\theta - u\cos\phi), \quad . \quad . \qquad (13)$$

and the efficiency

$$= \frac{\frac{w}{g}A\cos\phi\, v_1 u(v_1\cos\theta - u\cos\phi)}{\frac{w}{g}A\frac{v_1^3}{2}}$$

$$= \frac{2\cos\phi\, u(v_1\cos\theta - u\cos\phi)}{v_1^2}. \quad . \quad . \quad . \quad . \qquad (14)$$

This is a maximum when $v_1\cos\theta = 2u\cos\phi$, . . (15)
and therefore

$$\text{the maximum efficiency} = \frac{\cos^2\theta}{2}. \quad . \quad . \quad . \quad . \qquad (16)$$

SPECIAL CASE 1. Let a single vane be at right angles to, and move in the line of, the jet's motion, Fig. 123.

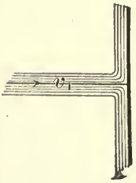


FIG. 123.

Then $\theta = 0 = \phi$.
Hence

$$\text{the pressure} = P = N = \frac{w}{g}A(v_1 - u)^2; \quad . \qquad (17)$$

$$\text{the useful work} = Pu = \frac{w}{g}Au(v_1 - u)^2; \quad . \qquad (18)$$

$P = \frac{Q\gamma}{g}(r)$

$$\text{the efficiency} \qquad = 2\frac{w}{v_1^2}(v_1 - u); \quad . \quad . \quad (19)$$

$$\text{the maximum efficiency} = \frac{8}{27} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Again, if $u = 0$, i.e., if the vane be fixed, and if H be the head corresponding to the velocity v_1, then, by equation 17,

$$P = \frac{w}{g}Av_1^2 = 2wAH$$

= *twice* the weight of a column of water of height H and sectional area A.

SPECIAL CASE 2. Let each of a series of vanes be at right angles to and move in the line of the jet's motion at the instant of impact.

Then $\theta = 0 = \phi$.

$$\text{The pressure} = N = P = \frac{w}{g}Av_1(v_1 - u). \quad . \quad (21)$$

$$\text{The useful work} = Pu \quad = \frac{w}{g}Av_1u(v_1 - u). \quad . \quad (22)$$

$$\text{The efficiency} \qquad = \frac{2u(v_1 - u)}{v_1^2}. \quad . \quad . \quad (23)$$

$$\text{The maximum efficiency} = \frac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

2. **Reaction—Jet Propeller.**—The term reaction is employed to denote the pressure upon a surface due to the direction and velocity with which the water leaves the surface. Water, for example, issues under the head h and with the

velocity v (at contracted section) from an orifice of sectional area A in the vertical side of a vessel, Fig. 124.

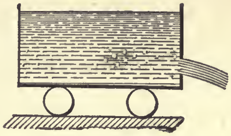


FIG. 124.

Let R be the reaction on the opposite vertical side of the vessel, and let Q be the quantity of water which flows through the orifice per second. Then

$$R = \text{horizontal change of momentum}$$

$$= \frac{wQ}{g}v = \frac{w}{g}c_c A v^2 = 2wc_c c_v\, Ah = 2wAh, \quad . \quad . \quad . \quad (1)$$

disregarding the contraction and putting $c_v = 1$.

Thus the reaction is double the corresponding pressure when the orifice is closed (Special Case I, Art. 1).

Again, let the vessel be propelled in the opposite direction with a velocity u relatively to the earth.

Then $v_1 - u$ is the velocity of the jet at the contracted section relatively to the earth and

$$R = \text{horizontal change of momentum}$$

$$= \frac{w}{g}Q(v_1 - u). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The useful work done by the jet

$$= Ru = \frac{w}{g}Qu(v_1 - u). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The energy carried away by the issuing water

$$= \frac{w}{g}Q\frac{(v_1 - u)^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence

$$\text{the total energy} = \frac{w}{g}Qu(v_1 - u) + \frac{w}{g}Q\frac{(v_1 - u)^2}{2}$$

$$= \frac{w}{g}Q\frac{v_1^2 - u^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\text{the efficiency} = \frac{\frac{w}{g}Qu(v_1 - u)}{\frac{w}{g}Q\frac{v_1^2 - u^2}{2}} = \frac{2u}{v_1 + u}. \quad . \quad . \quad . \quad (6)$$

Thus the more nearly v_1 is equal to u, and therefore the larger the area A of the orifice, the greater is the efficiency.

If the vessel is driven in the same direction as the jet, then $v_1 + u$ is the relative velocity of the jet with respect to the earth, and the reaction is

$$R = \text{horizontal change of momentum}$$

$$= \frac{w}{g}Q(v_1 + u) = \frac{w}{g}c_c c_v A v_1(v_1 + u)$$

$$= \frac{w}{g}Av_1(v_1 + u), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

disregarding the contraction and putting $c_v = 1$.

3. A Jet of Water impinging upon a Surface of Revolution moving in the Direction of its Axis and also in the Line of the Jet's Motion.—Disregarding friction, the water flows over the surface without any change in the magnitude of the relative velocity $v_1 - u$, but the stream-lines are deviated from their original direction through an angle β.

(*N.B.*—The sign before u is plus if the surface and jet are moving in opposite directions.)

Let the water leave the surface at D, and in the direction of the tangent at D take DE to represent $v_1 - u$ in direction and magnitude. Also draw DF parallel to the axis of the surface and take DF to represent u.

Complete the parallelogram EF.

The diagonal DG evidently represents in direction and magnitude the absolute velocity v_2 with which the water leaves the surface. Hence, from the triangle DFG, since the angle $DFG = \pi - \beta$,

$$v_2^2 = (v_1 - u)^2 + u^2 - 2(v_1 - u)u \cos(\pi - \beta),$$

from which

$$v_1^2 - v_2^2 = 2u(v_1 - u)(1 - \cos\beta) = 4u(v - u)\sin^2\frac{\beta}{2}. \quad (1)$$

Also, $\frac{w}{g}A(v_1 - u)$ = the quantity of water reaching the surface per second.

Hence, if P is the pressure in the direction of motion, the useful work done per second

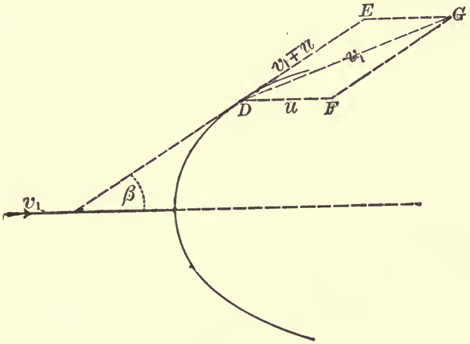


FIG. 125

$$= Pu = m\left(\frac{v_1^2 - v_2^2}{2}\right) = 2\frac{w}{g}Au(v_1 - u)^2\sin^2\frac{\beta}{2}; \quad . \quad (2)$$

and the pressure

$$P = 2\frac{w}{g}A(v_1 - u)^2\sin^2\frac{\beta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The efficiency

$$= \frac{2\frac{w}{g}Au(v_1 - u)^2\sin^2\frac{\beta}{2}}{\frac{w}{g}A\frac{v_1^3}{2}} = 4\frac{u}{v_1^3}(v_1 - u)^2\sin^2\frac{\beta}{2}. \quad . \quad . \quad (4)$$

This is a maximum when

$$v_1 = 3u, \quad \ldots \ldots \ldots \quad (5)$$

and therefore

$$\text{the maximum efficiency} = \frac{16}{27} \sin^2 \frac{\beta}{2}. \quad \ldots \quad (6)$$

If, instead of one surface, a series of surfaces are successively introduced at short intervals at the same point in the path of the jet, the quantity of water reaching each surface per second becomes

$$m = \frac{w}{g} A v_1, \quad \ldots \ldots \ldots \quad (7)$$

and hence the useful work, pressure, and efficiency also respectively become

$$2 \frac{w}{g} A v_1 u (v_1 - u) \sin^2 \frac{\beta}{2}; \quad \ldots \ldots \quad (8)$$

$$\frac{2w}{g} A v_1 (v_1 - u) \sin^2 \frac{\beta}{2}; \quad \ldots \ldots \quad (9)$$

$$4 \frac{u(v_1 - u)}{v_1^2} \sin^2 \frac{\beta}{2}. \quad \ldots \ldots \ldots \quad (10)$$

The efficiency is a maximum when

$$u = \frac{v_1}{2}; \quad \ldots \ldots \ldots \quad (11)$$

its value then being $\sin^2 \dfrac{\beta}{2}$.

It will be observed that the results given by equations 2 to 11 are identical with those given by equations 17 to 20 and 21 to 24, Art. 1, except that in each case there is an additional factor $2 \sin^2 \dfrac{\beta}{2}$ or $1 - \cos \beta$. This factor is greater than unity, and therefore the pressure, useful work, and efficiency are each increased, if $\beta > 90°$, i.e., in the case of a concave vane; while in the case of a convex vane, β being $< 90°$, the factor is also less than unity and they are each diminished.

Special Case. Let $\beta = 180°$, i.e., let the vane be of the cup type and in the form of a hemisphere.

The maximum efficiency is $\sin^2 \dfrac{180°}{2}$ = unity, and is perfect. The water should therefore leave the surface without velocity; and this is the case; for, by equation 1,

$$v_1^2 - v_2^2 = 4u(v_1 - u), \quad \text{and} \quad u = \frac{v_1}{2}.$$

Hence

$$v_1^2 - v_2^2 = v_1^2, \quad \text{and therefore} \quad v_2 = 0.$$

4. Impact of a Jet of Water upon a Vane with Borders.—Let the vane in Art. 1 be provided with borders, Figs. 126, 127, so as to produce a further deviation of the stream-lines, and let the water finally flow off with a velocity v^2 in a direction making an angle θ' with the normal to the vane.

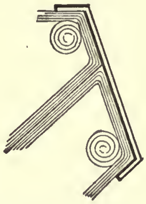


Fig. 126.

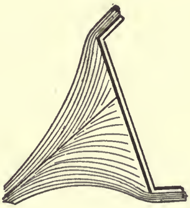


Fig. 127.

Then

the normal pressure $= N$

$$= mv_1 \cos\theta \mp mv_2 \cos\theta' \mp mu \cos\phi$$

$$= m(v_1 \cos\theta \mp v_2 \cos\theta' \mp u \cos\phi),$$

the sign of the second term being plus or minus according to the direction in which the stream-lines are finally deviated.

The effect of the borders is therefore to increase or diminish the normal pressure, and hence also the useful work and the efficiency.

SPECIAL CASE. Let the vane be at rest, i.e., let $u = 0$, and let the final and initial directions of the jet be parallel.

Also, let $v_1 = v_2$. Then

$$N = m(v_1 \cos\theta + v_1 \cos\theta)$$

$$= 2\frac{w}{g}Av_1^2 \cos\theta$$

$$= 4wAH \cos\theta.$$

Hence, if $\theta = 0$, the normal pressure $N = 4wAH =$ *four* times the weight of a column of water of height H and sectional area A.

5. Pressure of a Steady Stream in a Uniform Pipe against a Thin Plate *AB* Normal to the Direction of Motion.—The stream-lines in front of the plate are deviated and a contraction is formed at C_2C_2. They then converge, leaving a mass of eddies behind the plate.

Consider the mass bounded by the transverse planes C_1C_1, C_3C_3, where the stream-lines are again parallel.

At C_1C_1 let p_1, A_1, v_1, z_1 be the mean intensity of the pressure, the sectional area of the waterway, the velocity of flow, and the elevation of the C. of G. of the section above datum.

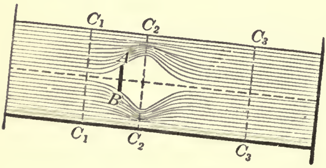


FIG. 128.

Let p_2, A_2, v_2, z_2 be corresponding symbols at C_2C_2.

Let p_3, A_1, v_1, z_3, be corresponding symbols at C_3C_3.

Let a be the area of the plate.

Let c_c be the coefficient of contraction.

Neglect the skin and fluid friction between C_1C_1 and C_3C_3.

Then by Bernouilli's theorem,

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g} = z_3 + \frac{p_3}{w} + \frac{v_1^2}{2g} + \frac{(v_2 - v_1)^2}{2g},$$

the term $\dfrac{(v_2-v_1)^2}{2g}$ representing the loss of head due to the bending of the stream-lines between C_2C_2 and C_3C_3.

Hence

$$z_1 - z_3 + \frac{p_1 - p_3}{w} = \frac{(v_2 - v_1)^2}{2g}.$$

Again, let R be the total pressure on the plane. Then

$$p_1A_1 - p_3A_1 = (p_1 - p_3)A_1 = \left\{ \begin{array}{l} \text{fluid pressure in the direction} \\ \quad \text{of the axis.} \end{array} \right.$$

$$wA_1C_1C_3\frac{z - z_3}{C_1C_3} = wA_1(z_1 - z_3)$$

$$= \text{component of the weight in the direction of the axis.}$$

Thus

$$(p_1 - p_3)A_1 + wA_1(z_1 - z_3) - R = \text{change of motion in direction of axis}$$
$$= 0,$$

since the motion is steady.

Hence

$$\frac{R}{wA_1} = \frac{p_1 - p_3}{w} + z_1 - z_3 = \frac{(v_2 - v_1)^2}{2g}.$$

But $A_1v_1 = A_2v_2 = c_c(A_1 - a)v_2$. Therefore

$$R = wA\frac{v_1^2}{2g}\left\{\frac{A_1}{c_c(A_1 - a)} - 1\right\}$$

$$= wa\frac{v_1^2}{2g}m\left\{\frac{m}{c_c(m - 1)} - 1\right\}^2,$$

where $m = \dfrac{A}{a}$, or

$$R = Kwa\frac{v_1^2}{2g},$$

where $K = m\left\{\dfrac{m}{c_c(m - 1)} - 1\right\}^2$

6. Pressure of a Steady Stream in a Uniform Pipe on a Cylindrical Body about Three Diameters in Length.—The stream-lines in front of the body are deviated and a contraction is formed at C_2C_2. They then converge, flow in parallel lines, and converge a second time at C_3C_3, leaving a mass of eddies behind the body.

Consider the mass bounded by the planes C_1C_1, C_4C_4.

As in the previous article, let

p_1, A_1, v_1, z_1 be the intensity of pressure, sectional area of the waterway, velocity of flow, and elevation of C. of G. above datum at C_1C_1.

FIG. 129.

p_2, A_2, v_2, z_2 be similar symbols for C_2C_2.

p_3, A_3, v_3, z_3 be similar symbols for C_3C_3.

p_4, A_1, v_1, z_4 be similar symbols for C_4C_4.

Neglect the skin and fluid friction between C_1C_1 and C_4C_4. Then, by Bernouilli's theorem,

$$z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g} = z_3 + \frac{p_3}{w} + \frac{v_3^2}{2g} + \frac{(v_2 - v_3)^2}{2g}$$

$$= z_4 + \frac{p_4}{w} + \frac{v_1^2}{2g} + \frac{(v_3 - v_1)^2}{2g} + \frac{(v_2 - v_3)^2}{2g},$$

$\dfrac{(v_2 - v_3)^2}{2g}$ being the loss of head between C_2C_2 and C_3C_3 and

$\dfrac{(v_3 - v_1)^2}{2g}$ being the loss of head between C_3C_3 and C_4C_4.

Hence

$$z_1 - z_4 + \frac{p_1 - p_4}{w} = \frac{(v_3 - v_1)^2}{2g} + \frac{(v_2 - v_3)^2}{2g}.$$

But $\qquad A_1v_1 = A_2v_2 = A_3v_3, \quad c_c(A_1 - a) = A_2,$

and $\qquad A_3 = A_1 - a.$

Therefore

$$z_1 - z_4 + \frac{p_1 - p_4}{w} = \frac{v_1^2}{2g}\left[\left(\frac{A_1}{A_1 - a} - 1\right)^2 + \left\{\frac{A_1}{c_c(A_1 - a)} - \frac{A_1}{A_1 - a}\right\}^2\right]$$

$$= \frac{v_1^2}{2g}\left[\left(\frac{m}{m-1} - 1\right)^2 + \left\{\frac{m}{c_c(m-1)} - \frac{m}{m-1}\right\}^2\right],$$

where $m = \frac{A_1}{a}$.

Also, as in the preceding article,

$$(p_1 - p_4)A_1 + wA_1(z_1 - z_4) - R = 0.$$

Hence

$$R = wa\frac{v_1^2}{2g}m\left\{\frac{1}{(m-1)^2} + \frac{m^2}{(m-1)^2}\left(\frac{1}{c_c} - 1\right)^2\right\}$$

$$= Kwa\frac{v_1^2}{2g},$$

where $m = \frac{A_1}{a}$, and

$$K = m\left\{\frac{1}{(m-1)^2} + \frac{m^2}{(m-1)^2}\left(\frac{1}{c_c} - 1\right)^2\right\}.$$

This value of K is always less than the value of K for the plate in the preceding article for the same values of m, a, and c_c.

Hence the pressure on the cylinder is also less than the corresponding pressure on the plate.

In every case K should be determined by experiment.

7. Jet impinging upon a Curved Vane and deviated wholly in one Direction—Best Form of Vane.—Let the jet, of sectional area A, moving in the direction AB with a velocity v_1, drive the vane AD in the direction AC with a velocity u.

Take AB to represent v_1 in direction and magnitude.
" AC " " u " " " "
Join CB.

Then CB evidently represents V, the velocity of the water relatively to the vane, in direction and magnitude. If CB is parallel to the tangent to the vane at A, there will be no sud-

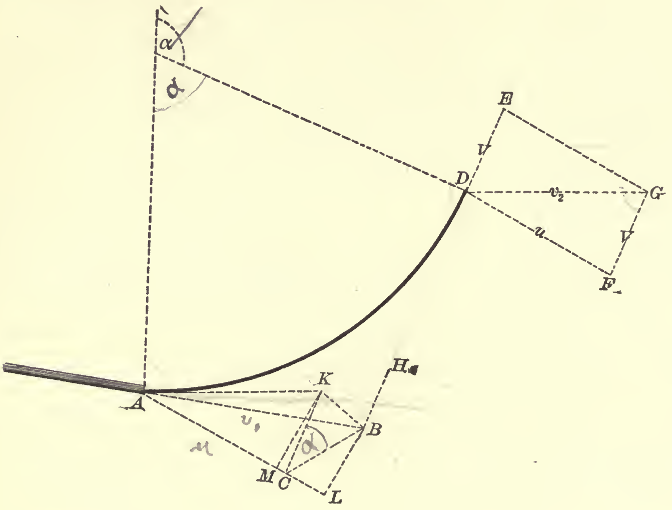


FIG. 130.

den change in the direction of the water as it strikes the vane, and, disregarding friction, the water will flow along the vane from A to D without any change in the magnitude of the relative velocity V (= CB). The vane is then said to "receive the water without shock."

Again, from the triangle ABC, denoting the angles BAC, ABC, ACB, by A, B, C, respectively.

$$\frac{u}{v_1} = \frac{AC}{AB} = \frac{\sin B}{\sin C} = \frac{\sin B}{\sin (A + B)}, \quad . \quad . \quad . \quad . \quad (1)$$

and therefore

$$\cot B = \frac{v_1}{u} \operatorname{cosec} A - \cot A, \quad . \quad . \quad . \quad . \quad (2)$$

a formula giving the angle between the lip and the direction of the impinging jet, which will ensure the water being received "without shock."

In the direction of the tangent to the vane at D, take $DE = CB (= V)$.

Draw DF parallel and equal to $AC (= u)$.

Complete the parallelogram EF.

Then the diagonal DG evidently represents in direction and magnitude the absolute velocity v_2 with which the water leaves the vane.

Draw AK equal and parallel to $DG (= v_2)$.

Join BK. Then BK represents the total change of velocity between A and D in direction and magnitude.

Thus, if R is the resultant pressure on the vane, then $R = m \,.\, BK$.

Let ML be the projection of BK upon AC.

Then ML represents the total change of velocity in the direction of the vane's motion.

Let P be the pressure upon the vane in this direction. Then

$$P = m \,.\, LM. \qquad (3)$$

$$\text{The useful work} = Pu = mu \,.\, LM = m\frac{v_1^2 - v_2^2}{2}. \qquad (4)$$

$$\text{The total available work} = \frac{w}{g} A \frac{v_1^3}{2}. \qquad (5)$$

$$\text{The efficiency } \frac{mu \,.\, LM}{\frac{w}{g} A \frac{v_1^3}{2}} = 2mg\frac{v_1^2 - v_2^2}{wAv_1^3}. \qquad (6)$$

Again, join CK.

Then, since AC is equal and parallel to DF, and AK to DG, the line CK is equal and parallel to DE, and is therefore equal to CB.

Thus in the isosceles triangle CBK, CB is equal and parallel to the relative velocity V at A, CK is equal and parallel to the relative velocity V at D, and the base BK represents the total change of motion.

Let δ be the angle through which the direction of the water is deviated, i.e., the angle between AB and AK. Then

$$V^2 = CK^2 = AK^2 + AC^2 - 2AK \,.\, AC \cos(A+\delta)$$

$$= v_2^2 + u^2 - 2v_2u \cos(A+\delta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and also

$$V^2 = CK^2 = CB^2 = AB^2 + AC^2 - 2AB \,.\, AC \cos A$$

$$= v_1^2 + u^2 - 2v_1u \cos A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence

$$\frac{v_2^2 - v_1^2}{2} = u\{v_2 \cos(A+\delta) - v_1 \cos A\}. \quad . \quad . \quad (9)$$

If BH is drawn parallel to the tangent at D, BK evidently bisects the angle between BC and BH, and this angle is equal to the angle between the tangents to the vane at A and D.

Let α be ~~the supplement of~~ the angle between the normals at A and D. Then the angle $KCB = \alpha$, and

$$\text{the angle } CBK = \frac{1}{2}(180° - \alpha) = 90° - \frac{\alpha}{2}.$$

Therefore

$$BK = 2CB\left(\cos 90° - \frac{\alpha}{2}\right) = 2V \sin\frac{\alpha}{2}.$$

Hence

$$R = m \,.\, BK = 2mV \sin\frac{\alpha}{2}. \quad . \quad . \quad . \quad . \quad (10)$$

Let X, Y be the components of R in the direction of the normal at A and at right angles to this direction. Then

$$Y = R \cos \frac{\alpha}{2} = mV \sin \alpha; \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$X = R \sin \frac{\alpha}{2} = 2mV \sin^2 \frac{\alpha}{2} = mV(1 - \cos \alpha). \quad . \quad (12)$$

The efficiency is a maximum when

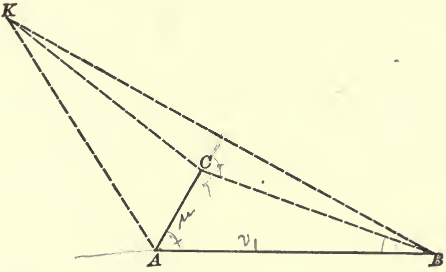
$$\frac{d(Pu)}{du} = 0 = u\frac{dP}{du} + P. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The efficiency is nil when

$$Pu = 0, \quad \text{i.e., when } u = 0 \text{ or } P = 0. \quad . \quad . \quad . \quad (14)$$

In the latter case, since $P = m.LM$, the projection LM must be nil, and therefore BK must be at right angles to AC, as in Fig. 131.

FIG. 131.



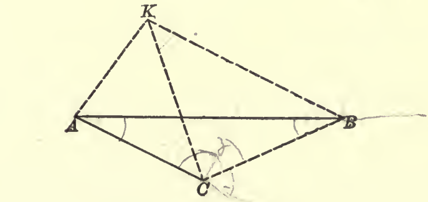


FIG. 132.

The angle ACB is now $= 180° - \frac{\alpha}{2}$, and therefore

$$\frac{u}{v_1} = \frac{\sin ABC}{\sin ACB}$$

$$= \frac{\sin\left(180° - \frac{\alpha}{2} + A\right)}{\sin\left(180° - \frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha}{2} - A\right)}{\sin\frac{\alpha}{2}}. \qquad (15)$$

If BK is parallel to AC (Fig. 132), then

$$\text{the angle } ACB = \frac{1}{2}(180° - \alpha) + \alpha = 90° + \frac{\alpha}{2},$$

and therefore

$$\frac{u}{v_1} = \frac{\sin ABC}{\sin ACB} = \frac{\sin\left(90° + \frac{\alpha}{2} + A\right)}{\sin\left(90° + \frac{\alpha}{2}\right)} = \frac{\cos\left(\frac{\alpha}{2} + A\right)}{\cos\frac{\alpha}{2}}. \qquad (16)$$

Special Case.—Let the direction of the impinging jet be tangential to the vane at A, and let the jet and vane move in the same direction. Then

$$V = v_1 - u, \qquad m = \frac{w}{g}A(v_1 - u);$$

$$P = Y = \frac{w}{g}A(v_1 - u)^2(1 - \cos\alpha) = 2\frac{w}{g}A(v_1 - u)\sin^2\frac{\alpha}{2};$$

$$\text{useful work} = Pu = 2\frac{w}{g}Au(v_1 - u)^2\sin^2\frac{\alpha}{2};$$

$$\text{efficiency} = 4\frac{u(v_1 - u)^2}{\quad}\sin^2\frac{\alpha}{2}.$$

This is a maximum and equal to $\frac{16}{27}\sin^2\frac{\alpha}{2}$ when $v_1 = 3u$.

These results are identical with those for a concave cup when $\alpha = 180°$.

Instead of one vane let a series of vanes be successively introduced at short intervals at the same point in the path of the jet. Then

$$m = \frac{w}{g}Av_1,$$

and hence the pressure P, useful work, and efficiency respectively become

$$\frac{w}{g}Av_1 . LM;$$

$$\frac{w}{g}Av_1 . \frac{v_1^2 - v_2^2}{2};$$

and

$$\frac{v^2 - v_2^2}{v_1^2}.$$

8. **Friction.**—The effect of friction has been disregarded, and nothing definite is known as to its action or law of distribution. It has been suggested to assume that the loss of head due to friction is a fraction of the head due to the velocity of the jet relatively to the surface over which it spreads. Thus in Art. 7

$$\text{the loss of head due to friction} = f\frac{V^2}{2g}$$

$$\text{and the corresponding loss of energy} = wQ . f\frac{V^2}{2g}.$$

9. **Resistance to the Motion of Solids in a Fluid Mass.**—The preceding results indicate that the pressure due to

the impact of a jet upon a surface may be expressed in the form

$$P = KwA\frac{V^2}{2g},$$

A being the sectional area of the jet, V the velocity of the jet relatively to the surface, and K a coefficient depending on the position and form of the surface.

Again, the normal pressure (N) on each side of a thin plate, completely submerged in an indefinitely large mass of still water, is the same. If the plate is made to move horizontally with a velocity V, a forward momentum is developed in the water immediately in front of the plate, while the plate tends to leave behind the water at the back. A portion of the water carried on by the plate escapes laterally at the edges and is absorbed in the neighboring mass, while the region it originally occupied is filled up with other particles of water. Thus the normal pressure N, in front of the plate, is increased by an amount n, while at the back eddies and vortices are produced, and the normal pressure N at the back is diminished by an amount n'. The total resultant normal pressure, or the normal resistance to motion, is $n + n'$, and this increases with the speed. In fact, as the speed increases, n' approximates more and more closely to N, and in the limit the pressure at the back would be nil, so that a vacuum might be maintained.

Confining the attention to a plate moving in a direction normal to its surface, the resistance is of the same character as if the plate is imagined to be at rest and the fluid moving in the opposite direction with a velocity V. So, if both the water and the plate are in motion, imagine that a velocity equal and opposite to that of the water is impressed upon every particle of the plate and of the water. The resistance is then of the same character as that of a plate moving in still water, the velocity of the plate being the velocity relatively to the water. Thus, in general, the resistance to the motion of such a plane moving in the direction of the normal to its surface, with a velocity V relatively to the water, may be expressed in the form

$$R = KwA\frac{V^2}{2g},$$

A being the area of the plate, and K a coefficient depending upon the form of the plate and also upon the relative sectional areas of the plate and of the water in which it is submerged.

According to the experiments of Dubuat, Morin, Piobert, Didion, Mariotte, and Thibault, the value of K may be taken at 1.3 for a plate moving in still water, and at 1.8 for a current moving on a fixed plate. Unwin points out the unlikelihood of such a difference between the two values, and suggests that it might possibly be due to errors of measurement.

Again, reasoning from analogy, the resistance to the motion of a solid body in a mass of water, whether the body is wholly or only partially immersed, has been expressed by the formula

$$R = KwA\frac{V^2}{2g},$$

V being the relative velocity of the body and water, A the greatest sectional area of the immersed portion of the body at right angles to the direction of motion, and K a coefficient depending upon the form of the body, its position, the relative sectional areas of the body and of the mass of water in which it is immersed, and also upon the surface wave-motion.

The following values have been given for K:

$K = 1.1$ for a prism with plane ends and a length from 3 to 6 times the least transverse dimension;

$K = 1.0$ for a prism, plane in front, but tapering towards the stern, the curvature of the surface changing gradually so that the stream-lines can flow past without any production of eddy motion, etc.;

$K = .5$ for a prism with tapering stern and a cut-water or semi-circular prow;

$K = .33$ for a prism with a tapering stern and a prow with a plane front inclined at 30° to the horizon;

$K = .16$ for a well-formed ship.

Froude's experiments, however, show that the resistance to the motion of a ship, or of a body tapering in front and in the rear, so that there is no abrupt change of curvature leading to the production of an eddy motion, is almost entirely due to skin-friction (see Art. 1, Chap. II).

EXAMPLES.

1. A stream with a transverse section of 24 square inches delivers 10 cubic feet of water per second against a flat vane in a normal direction. Find the pressure on the vane. *Ans.* $1171\frac{7}{8}$ lbs.

2. If the vane in question 1 moves in the same direction as the impinging jet with a velocity of 24 ft. per second, find (*a*) the pressure on the vane; (*b*) the useful work done; (*c*) the efficiency.
Ans. (*a*) $421\frac{7}{8}$ lbs.; (*b*) 10,125 ft.-lbs.; (*c*) .288.

3. What must be the speed of the vane in question 2, so that the efficiency of the arrangement may be a maximum? Find the maximum efficiency. *Ans.* 20 ft. per sec.; $\frac{8}{27}$.

4. Find (*a*) the pressure, (*b*) the useful work done, (*c*) the efficiency, when, instead of the single vane in question 2, a series of vanes are introduced at the same point in the path of the jet at short intervals.
Ans. (*a*) $703\frac{1}{8}$ lbs.; (*b*) 16,875 ft.-lbs.; (*c*) .48.

What must be the speed of the vane to give a maximum efficiency? What will be the maximum efficiency? *Ans.* 30 ft. per sec.; .5.

5. A stream of water delivers 7,500 gallons per minute at a velocity of 15 ft. per second and strikes an indefinite plane. Find the normal pressure on the vane when the stream strikes the vane (*a*) normally; (*b*) at an angle of 60° to the normal. *Ans.* (*a*) 585.9 lbs.; 292.9 lbs.

6. A railway truck, full of water, moving at the rate of 10 miles an hour, is retarded by a jet flowing freely from an orifice 2 in. square in the front, 2 ft. below the surface. Find the retarding force.
Ans. 7.97 lbs.

7. A jet of water of 48 sq. in. sectional area delivers 100 gallons per second against an indefinite plane inclined at 30° to the direction of the jet; find the total pressure on the plane, neglecting friction. How will the result be affected by friction? *Ans.* 750 lbs.

8. If the plane in question 7 move at the rate of 24 ft. per second in a direction inclined at 60° to the normal to the plane, find the useful work done and the efficiency. *Ans.* 2250 ft.-lbs.; $\frac{1}{16}$.

At what angle should the jet strike the plane so that the efficiency might be a maximum? Find the maximum efficiency.
Ans. $\sin^1\frac{3}{4}$; $\frac{1}{6}$.

9. A stream of 32 square inches sectional area delivers 32 cub. feet of water per second. At short intervals a series of flat vanes are intro-

duced at the same point in the path of the stream. At the instant of impact the direction of the jet is at right angles to the vane, and the vane itself moves in a direction inclined at 45° to the normal to the vane. Find the speed of the vane which will make the efficiency a maximum. Also find the maximum efficiency and the useful work done. *Ans.* 15.08 ft. per sec.; $\frac{8}{27}$; $2106\frac{242}{243}$ ft.-lbs.

10. In a railway truck, full of water, an opening 2 in. in diameter is made in one of the ends of the truck, 9 ft. below the surface of the water. Find the reaction (*a*) when the truck is standing; (*b*) when the truck is moving at the rate of 10 ft. per second in the same direction as the jet; (*c*) when the truck is moving at the rate of 10 ft. per second in a direction opposite to that of the jet. If this movement of the truck is produced by the reaction of the jet, find the efficiency.

Ans. (*a*) 24.55 lbs. per sq. in.; (*b*) 34.78 lbs. per sq. in.; (*c*) 14.3 lbs. per sq. in.; .588.

11. From a ship moving forward at 6 miles an hour a jet of water is sent astern with a velocity relative to the ship of 30 feet per second from a nozzle having an area of 16 square inches; find the propelling force and the efficiency of the jet as a propeller without reference to the manner in which the supply of water may be obtained.

Ans. $138\frac{1}{48}$lbs.; $\frac{1}{11.6}$.

12. A stream of 64 sq. in. section strikes with a 40-ft. velocity against a fixed cone having an angle of convergence = 100°; find the hydraulic pressure. *Ans.* 492.1 lbs.

13. A jet of 9 sq. in. sectional area, moving at the rate of 48 ft. per second, impinges upon the convex surface of a paraboloid in the direction of the axis and drives it in the same direction at the rate of 16 ft. per second. Find the force in the direction of motion, the useful work done, and the efficiency. The base of the paraboloid is 2 ft. in diameter and its length is 8 inches. *Ans.* 25 lbs.; 400 ft.-lbs.; $\frac{8}{135}$.

14. A stream of water of 16 sq. in. sectional area delivers 12 cubic feet of water per second against a vane in the form of a surface of revolution, and drives in the same direction, which is that of the axis of the vane. The water is turned through an angle of 120° from its original direction before it leaves the vane. Neglecting friction, find the speed of vane which will give a maximum effect. Also find impulse on vane, the work on vane, and the velocity with which the water leaves the vane.

Ans. 36 ft. per sec.; $562\frac{1}{2}$ lbs.; 20,250 ft.-lbs.; 95.24 ft. per sec.

15. At 8 knots an hour the resistance of the Water-witch was 5500 lbs.; the two orifices of her jet propeller were each 18 in. by 24 in. Find (*a*) the velocity of efflux; (*b*) the delivery of the centrifugal pump;

(*c*) the useful work done; (*d*) the efficiency; (*e*) the propelling H.P., assuming the efficiency of the pump and engine to be .4.

Ans. (*a*) 29.4 ft. per sec.; (*b*) 1104.6 gallons per sec.; (*c*) 74,393 ft.-lbs.; (*d*) .63; (*e*) 532.

16. If feathering-paddles are substituted for the jet propeller in question 15, what would be the area of stream driven back for a slip of 25%? Find the efficiency and the water acted on in gallons per minute.

Ans. 34 sq. ft.; .75; 236,000.

17. A vane moves in the direction ABC with a velocity of 10 ft. per second, and a jet of water impinges upon it at B in the direction BD with a velocity of 20 ft. per second; the angle between BC and BD is 30°. Determine the direction of the receiving-lip of the vane, so that there may be no shock.

Ans. The angle between lip and $BC = 23°47'$.

18. A jet moves in a direction ABC with a velocity V and impinges upon a vane which it drives in the direction BD with a velocity $\frac{1}{2}V$. The angle ABD is 165°. Determine the direction of the lip of the vane at B, so that there may be no shock at entrance.

Ans. The angle between lip and direction of stream $= 14°3'$.

19. A jet issues through a thin-lipped orifice 1 sq. in. in sectional area in the vertical side of a vessel under a pressure equivalent to a head of 900 ft. and impinges on a curved vane, driving it in the direction of the axis of the jet. The water enters without shock and turns through an angle of 60° before it leaves the vane. Find (*a*) the speed of the vane which will give a maximum effect; (*b*) the pressure on the vane; (*c*) the work done; (*d*) the absolute velocity with which the water leaves the vane; (*e*) the reaction on the vessel, disregarding contraction.

Ans. (*a*) 80 ft. per sec.; (*b*) 320.9 lbs.; (*c*) 46.68 H. P.; (*d*) 184 ft. per sec.; (*e*) 781.25 lbs.

20. A stream moving with a velocity v impinges without shock upon a curved vane and drives it in a direction inclined at an angle to the direction of the stream. The angle between the lip of the vane and the direction of the stream is x, and V is the relative velocity of the water with respect to the vane. If the speed of the vane is changed by a small amount, say n per cent, show that the corresponding change in the direction of the lip, in order that the water might still strike the vane without shock, is $n\frac{v}{V}\sin x$.

21. A jet of water under a head of 20 feet, issuing from a vertical thin-lipped orifice 1 in. in diameter, impinges upon the centre of a vane 3 ft. from the orifice. Determine the position of the vane and the force of the impact (*a*) when the vane is a plane surface; (*b*) when the vane is 6 in. in diameter and in the form of a portion of a sphere of 6 in. radius.

22. A stream of water of 36 sq. in. section moves in a direction ABC and delivers 4 cub. ft. of water per second upon a vane moving in a direction BD with a velocity of 8 ft. per second, the angle between BC and BD being 30°. Find (*a*) the best form to give to the vane; (*b*) the velocity of the water as it leaves the vane; (*c*) the mechanical effect of the impinging jet; and (*d*) the efficiency, the angle turned through by the jet being 90°.

Ans. (*a*) The angle between lip and $BC = 23°48'$; (*b*) 2.946 ft. per sec.; (*c*) 966.098 ft. per sec.; (*d*) .966.

23. A stream of thickness t and moving with the velocity v impinges without shock upon the concave surface of a cylindrical vane of a length subtending an angle 2α at the centre. Determine the total pressure upon the vane (*a*) if it is fixed; (*b*) if it is moving in the same direction as the stream with the velocity u. In case (*b*) also find (*c*) the work done on the vane.

Ans. (*a*) $2\frac{w}{g}btv^2 \sin \alpha$; (*b*) $2\frac{w}{g}bt(v-u)^2 \sin \alpha$; (*c*) $2\frac{w}{g}btu(v-u)^2 \sin^2 \alpha$.

24. Two cubic feet of water are discharged per second under a pressure of 100 lbs. per sq. in. through a thin-lipped orifice in the vertical side of a vessel, and strike against a vertical plate. Find the pressure on the plate and the reaction on the vessel. *Ans.* 475.82 lbs.

25. A stream moving with a velocity of 16 ft. per second in the direction ABC, strikes obliquely against a flat vane and drives it with a velocity of 8 ft. per second in the direction BD, the angle CBD being 30°. Find (*a*) the angle between ABC and the normal to the plane for which the efficiency is a maximum; (*b*) the maximum efficiency; (*c*) the velocity with which the water leaves the vane; (*d*) the useful work per cubic foot of water.

Ans. (*a*) 21° 44′; (*b*) .25664; (*c*) 12.6 ft. per sec.; (*d*) 256.64 ft.-lbs.

CHAPTER VII.

HYDRAULIC MOTORS AND CENTRIFUGAL PUMPS.

1. **Hydraulic Motors** are machines designed to utilize the energy possessed by a moving mass of water in virtue of its position, pressure, and velocity.

The motors may be classified as follows:

(1) *Bucket Engines.*—In this now antiquated form of motor weights are raised and resistances overcome by allowing water to flow into suspended buckets, thus causing them to descend vertically.

(2) *Rams and Jet-pumps,* in which the impulsive effect of one mass of water is utilized to drive a second mass of water.

(3) *Water-pressure Engines* are especially adapted for high pressures and low speeds, and necessarily have very heavy moving parts. With low pressures the engine becomes unwieldy and costly.

Pressure-engines are either reciprocal or rotative. The latter are very convenient with moderately high pressures and especially when they are to drive machinery which is to be used intermittently. They also give an exact measurement of the water used.

Direct-acting pressure-engines are of great advantage where a slow and steady motion is required, as, for example, in working cranes, lifts, etc.

(4) *Vertical Water-wheels,* in which the water acts almost wholly by weight, or partly by weight and partly by impulse, or wholly by impulse.

(5) *Turbines,* in which the water acts wholly by pressure or wholly by impulse.

2. Hydraulic Rams.—By means of the hydraulic ram a quantity of water falling through a vertical distance h_1 is made to force a smaller weight of water to a higher level.

The water is brought from a reservoir through a supply-pipe *S*. At the end *B* of this pipe there is a check- or clack-valve opening into an air-chamber *A*, which is connected with a discharge-pipe *D*. At *C* there is a weighted check- or clack-valve opening inwards, and the length of its stem (or the stroke) is regulated by means of a nut or cottar at *E*. When the waste-valve at *C* is open the water begins to escape with a velocity due to the head h_1 and suddenly closes the valve. The momentum

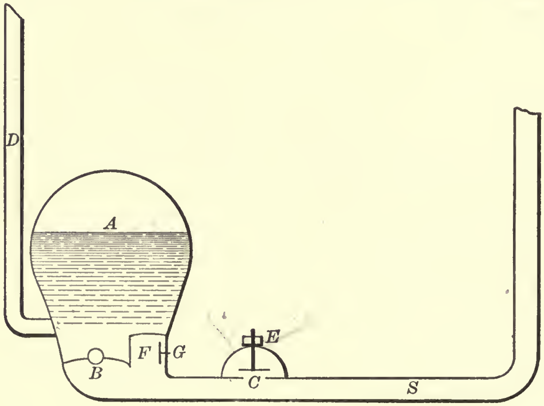


FIG. 133.

of the water in the pipe opens the valve at *B*, and a portion of the water is discharged into the air-vessel. From this vessel it passes into the discharge-pipe in consequence of the reaction of the compressed air. At the end of a very short interval of time the momentum of the water has been destroyed, the valve at *B* closes, the waste-valve again opens, and the action commences as before. It is found that the efficiency of the ram is increased by introducing a small air-vessel at *F*, supplied with a check- or clack-valve opening inwards at *G*. The wave-motion started up in the supply-pipe by the opening and closing of the valve at *B* has been utilized in driving a piston so as to pump up water from some independent source.

Let v be the velocity of flow in the supply-pipe at the moment when the valve at C is closed.

" W_1 be the weight of the mass of water in motion.

Then $\frac{W_1}{g}\frac{v_1^2}{2}$ is the energy of the mass, and this energy is expended in opening the valve at B, forcing the water into the air-chamber, compressing the air, and finally causing the elevation of a weight W_2 of the water through a vertical distance h'.

Let h_f be the head consumed in frictional and other hydraulic resistances.

Then

$$W_2(h' + h_f) = \text{the actual work done} = \frac{W_1}{g}\frac{v^2}{2}.$$

This equation shows that, however great h' may be, W_2 has a definite and positive value, and therefore water may be raised to any required height by the hydraulic ram.

The efficiency of the machine $= \frac{W_2 h'}{W_1 h_1}$, and may be as much as 66 per cent if the machine is well made.

3. **Pressure-engines.**—The energy required to drive a pressure-engine is usually supplied by means of steam-pumps, but an accumulator is often interposed between the pumps and the motor in order to store up the pressure energy of the water. Indeed, it is perhaps to the introduction of the accumulator that the success of hydraulic transmission is especially due. Its cost, however, only allows of its use in cases where the demand for energy is for short intervals of time.

In its simplest form the accumulator is merely a vertical cylinder into which the water is pumped and from which it is then discharged by the descent of a heavily loaded piston. The water-pressure thus developed in ordinary hydraulic machinery is from 700 to 800 lbs. per square inch, but in riveting and other similar machinery pressures of 1500 lbs. per square inch and upwards are often employed.

Fig. 134 represents an accumulator designed by Tweddell for these higher pressures.

The loaded cylinder *A* slides upon a fixed spindle *B*. The water enters near the base, passes up the hollow spindle, and fills the annular space surrounding the spindle. Thus

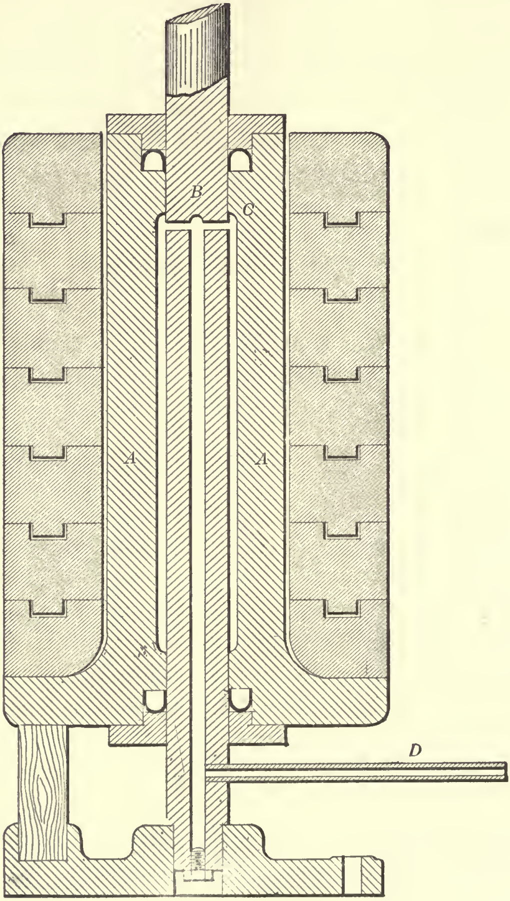


FIG. 134.

the whole of the weight is lifted by the pressure of the water upon a shoulder *C*. The water section being small, any large demand for water will cause the loaded cylinder to fall rapidly, so that when it is brought to rest there will be a considerable increase of pressure which is of advantage in punching, riveting, etc.

Let W be the weight of the loaded cylinder.

Let F be the friction of each of the two cup-leathers.

Let r_1 be the radius of the cylinder, r_2 the radius of the spindle.

Let h be the height of the column of water above the pipe D.

Let w be the specific weight of the water.

Then p_1, the intensity of the pressure in D when the cylinder is rising,

$$= wh + \frac{W + 2F}{\pi(r_1^2 - r_2^2)},$$

and p_2, the intensity of the pressure in D when the cylinder is falling,

$$= wh + \frac{W - 2F}{\pi(r_1^2 - r_2^2)}.$$

Hence an approximate measure of the variation of the pressure is $p_1 - p_2 = \dfrac{4F}{\pi(r_1^2 - r_2^2)}$, which ordinarily varies from about 1% of the pressure for a 16-in. ram to 4% for a 4-in. ram.

In a direct-acting pressure-engine let A be the sectional area of the working cylinder (Fig. 135).

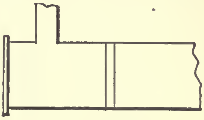


FIG. 135.

Let a be the sectional area of the supply-pipe.

Let $A = na$.

Let W be the weight of the water, piston, and other reciprocating parts in the working cylinder.

Let l be the length of the supply pipe.

Let f be the acceleration of the piston. Then nf is the acceleration of the water in the supply-pipe.

The force required to accelerate the piston

$$= \frac{W}{g}f,$$

and the corresponding pressure in feet of water

$$=\frac{W}{wA}\frac{f}{g}.$$

The force required to accelerate the water in the supply pipe

$$=\frac{wal}{g}nf,$$

and the corresponding pressure in feet of water

$$=nl\frac{f}{g}.$$

Similarly, if l' is the length of the discharge-pipe and $\frac{A}{n'}$ its sectional area, the pressure-head due to the inertia of the discharge-water

$$=n'l'\frac{f}{g}.$$

Hence the total pressure in feet of water required to overcome inertia in the supply-pipe and cylinder

$$=\frac{f}{g}\left(\frac{W}{wA}+nl\right).$$

The quantity $\frac{W}{wA}+nl$ has been designated the length of working cylinder equivalent to the inertia of the moving parts. Let the engine drive a crank of radius r, and assume that the velocity V of the crank-pin is approximately constant. Then the acceleration of the piston when it is at a distance x from its central position

$$=\frac{V^2}{r^2}x,$$

and the pressure due to inertia

$$=\frac{V^2}{gr^2}\left(\frac{W}{wA}+nl\right)x.$$

Let v be the velocity of the piston in the working cylinder.

Let u be the velocity of the water in the supply-pipe.

Let h be the vertical distance between the accumulator-ram and the motor.

Let p_0 be the unit pressure at the accumulator-ram.

Let p be the unit pressure in the working cylinder.

Then

$$\frac{p_0}{w}+\frac{u^2}{2g}+h=\frac{p}{w}+\frac{v^2}{2g}+\left\{\begin{array}{l}\text{losses due to friction, sudden changes}\\ \quad\text{of section, etc.}\end{array}\right.$$

Thus

$$\frac{p_0-p}{w}=\frac{v^2-u^2}{2g}-h+\text{losses.}$$

The term $\dfrac{v^2-u^2}{2g}+$ losses may be approximately expressed in the form $K\dfrac{v^2}{2g}$, K being the coefficient of hydraulic resistance. Hence

$$\frac{p_0-p}{w}=K\frac{v^2}{2g}=\frac{KV^2}{2gr^2}(r^2-x^2), \quad . \quad . \quad . \quad . \quad (1)$$

the term h being disregarded as it is usually very small as compared with $\dfrac{p_0}{w}$.

Thus the total pressure-head in feet required to overcome inertia and the hydraulic resistances

$$=\frac{V^2}{gr^2}\left\{\left(\frac{W}{nA}+nl\right)x+\frac{K}{2}(r^2-x^2)\right\}, \quad . \quad . \quad . \quad (2)$$

and is represented by the ordinate between the parabola *ced* and the line ab in Fig. 136, in which $afgb$ is a rectangle, ab representing the stroke $2r$,

$$ac = bd = \frac{V^2}{gr}\left(\frac{W}{nA} + nl\right)$$

the pressure due to inertia at the end of the stroke, and

$$oe = K\frac{V^2}{2g}$$

the pressure required to overcome the hydraulic resistances at the centre of the stroke.

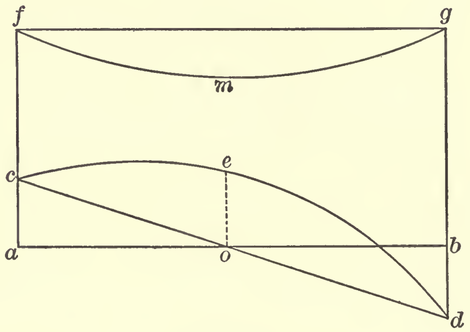


FIG. 136.

The ordinate between the parabola fmg and the line fg represents the back pressure, which is necessarily proportional to the square of the piston-velocity, i.e., to $\frac{V^2}{r^2}(r^2 - x^2)$. Hence the effective pressure-head on the piston, transmitted to the crank-pin, is represented by the ordinate between the curves amg and ced. The diagram shows that the pressure at the end of the stroke is very large and may become excessive. It is therefore usual to introduce relief-valves or air-vessels to prevent violent shocks. In certain cases, however, as, e.g., in a riveting-machine, a heavy pressure at the end of the stroke, just where it is most needed to close the rivet, is of great advantage, and therefore the inertia effect is increased by the use of a supply-pipe of small diameter and an accumulator with a small water section (Fig. 134).

The effective pressure should be as great as possible, and therefore the pressures due to inertia and frictional resistance, and the back pressure, which are each proportional to v^2, should be as small as possible, and hence it is of importance to fix a low value for the speed of the piston, which in practice rarely exceeds 80 ft. per minute. The exhaust port should also be made of large area, as the back pressure diminishes as the area of the port increases.

By equation 1,

$$v^2 = \frac{2g}{wK}(p_0 - p). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This speed v can be regulated at will by the turning of a cock, as in this manner the hydraulic resistances may be indefinitely increased.

Let the engine be working steadily under a pressure P, and let v_0 be the speed of steady motion. Then

$$v_0^2 = \frac{2g}{wK}(p_0 - P),$$

and

$$P = \begin{cases} \text{useful resistance overcome by the piston} \\ + \text{ friction between piston and accumulator-cylinder.} \end{cases}$$

If P is diminished, the speed v_0 will be slightly increased, but in no case can it exceed $\sqrt{\dfrac{2gp_0}{wK}}$.

4. Losses of Energy.—The losses may be enumerated as follows:

(*a*) *The Loss L_1 due to Piston-friction.*—It may be assumed that piston-friction consumes from 10 to 20 per cent of the total available work.

(*b*) *The Loss L_2 due to Pipe-friction.*—The loss of head in the supply-pipe of diameter d_1

$$= \frac{4fl}{d_1}\frac{(nv)^2}{2g}.$$

The loss of head in the discharge-pipe of diameter d_2

$$= \frac{4fl'}{d_2}\frac{(n'v)^2}{2g}.$$

Hence the total loss of head in pipe-friction is

$$L_2 = 4f\left(\frac{n^2l}{d_1} + \frac{(n')^2l'}{d_2}\right)\frac{v^2}{2g} = f_2\frac{v^2}{2g}.$$

The loss in the relatively short working cylinder is very small and may be disregarded.

(*c*) *The Loss L_3 due to Inertia.*—The work expended in moving the water in the supply-pipe

$$= \frac{wA}{gn}l\,\frac{v^2}{2},$$

and in moving the water in the discharge-pipe

$$= \frac{wA}{gn'}l'\,\frac{v^2}{2}.$$

The total work thus expended

$$= wA\left(\frac{l}{n} + \frac{l'}{n'}\right)\frac{v^2}{2g},$$

and it may be assumed that nearly the whole of this is wasted.

Hence the corresponding loss of head is

$$L_3 = \frac{wA}{A2r}\left(\frac{l}{n} + \frac{l'}{n'}\right)\frac{v^2}{2g} = \frac{w}{2r}\left(\frac{l}{n} + \frac{l'}{n'}\right)\frac{v^2}{2g} = f_3\frac{v^2}{2g}.$$

(*d*) *The Loss L_4 due to Curves and Elbows.*—The losses due curves and elbows may be expressed in the form

$$L_4 = f_4 \frac{v^2}{2g} \text{ (Chap. III, Art. 6).}$$

(*e*) *The loss L_5 due to sudden Changes of Section.*—The loss of head in the passage of the water through the ports may be expressed in the form $f' \dfrac{v^2}{2g}$.

The loss occasioned by valves may also be expressed by $f'' \dfrac{v^2}{2g}$.

Thus the total loss is

$$L_5 = (f' + f'') \frac{v^2}{2g} = f_5 \frac{v^2}{2g}.$$

The coefficient f'' may be given any desired value between 0 and ∞ by turning a valve, so that any excess of pressure may be destroyed and the speed regulated at will.

(*f*) *The Loss L_6 due to the Velocity with which the Water leaves the Discharge-pipe.*

$$L_6 = \frac{(n'v)^2 v^2}{2g} = f_6 \frac{v^2}{2g}.$$

Hence

$$\text{the effective head} = \frac{p_0}{w} - (L_1 + L_2 - L_3 + L_4 + L_5 + L_6),$$

$$\text{and the efficiency} = 1 - \frac{w}{p_0}(L_1 + L_2 + L_3 + L_4 + L_5).$$

The volume of water used per stroke is a constant quantity, and the efficiency, which may be as great as *eighty* per cent when the engine is working under a full load, may fall below *forty* per cent when the load is light.

5. **Brakes.**—Hydraulic resistances absorb energy which is proportional to the square of the speed. This property has

been taken advantage of in the design of hydraulic brakes for arresting the motion of a rapidly moving mass, as a gun or a train, of weight W. In Fig. 137 the fluid is allowed to pass from one side of the piston to the other through orifices in the piston.

Let m be the ratio of the area of the piston to the effective area of the orifices.

Let v be the velocity of the piston when moving under a force P.

Let A be the sectional area of the cylinder.

FIG. 137.

Then

$$\begin{aligned}\text{the work done per second} &= Pv \\ &= \text{the kinetic energy produced} \\ &= wAv\frac{(m-1)^2v^2}{2g},\end{aligned}$$

and therefore

$$P = wA(m-1)^2\frac{v^2}{2g},$$

and is the force required to overcome the hydraulic resistance at the speed v.

Let V be the initial value of v, and P_1 the maximum value of P. Then

$$P_1 = wA(m-1)^2\frac{V^2}{2g}.$$

Let F be the friction of the slide. Then

$$P + F = wA(m-1)^2\frac{v^2}{2g} + F,$$

and $P_1 + F$ is the maximum retarding force. It would certainly be an advantage if the retarding force could be constant. In order that this might be the case $(m - 1)v$ must be constant, and therefore as v diminishes m should increase and consequently the orifice area diminish. Various devices have been adopted to produce this result.

Assuming the retarding force to be constant, let x be the piston's distance from the end of the stroke when its velocity is v. Then

$$\frac{wv^2}{2g} = (P + F)x,$$

and therefore v^2 is proportional to x.

But $(m - 1)v$ is constant.

Therefore $(m - 1)$ is inversely proportional to $\sqrt{x}$.

6. **Water-wheels.**—Water-wheels are large vertical wheels which are made to turn on a horizontal axis by water falling from a higher to a lower level. These wheels may be divided into three classes:

(*a*) *Undershot Wheels*, in which the water is received near the bottom and acts *by impulse*.

(*b*) *Breast Wheels*, in which the water is received a little below the axis of rotation and acts *partly by impulse and partly by its weight*.

(*c*) *Overshot Wheels*, in which the water is delivered nearly at the top and acts *chiefly by its weight*.

7. **Undershot Wheels.**—Wheels of this class, with plane floats or buckets, are simple in construction, are easily kept in repair, and were in much greater use formerly than they are now. They are still found in remote districts where there is an abundance of water-power, and are also employed to work floating mills, for which purpose they are suspended in an open current by means of piles or suitably moored barges. They are made from 10 to 25 feet in diameter, and the floats, which are from 24 to 28 in. deep, are fixed either normally to the periphery of the wheel, or with a slight slope towards the supply-sluice, the angle between the float and radius being

from 15° to 30°. Generally from one half to one third of the total depth of float is acted upon by the water.

Let Fig. 138 represent a wheel with plane floats working in an open current.

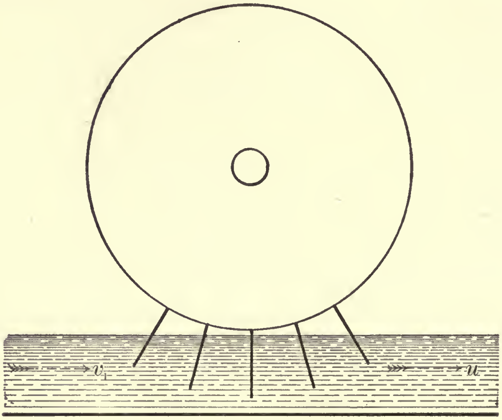


FIG. 138.

Let v_1 be the velocity of the current.

Let u be the velocity of the wheel's periphery.

Let Q be the delivery of water in cubic feet per second.

The water impinges upon a float, is reduced to relative rest, and is carried along with the velocity u. Thus

$$\text{the impulse} = \frac{wQ}{g}(v_1 - u),$$

and

$$\text{the useful work per second} = \frac{wQ}{g}u(v_1 - u).$$

Hence

$$\text{the efficiency} = \frac{\frac{wQ}{g}u(v_1 - u)}{\frac{wQ}{g}\frac{v_1^2}{2}} = \frac{2u(v_1 - u)}{v_1^2},$$

which is a maximum and equal to $\frac{1}{2}$ when $u = \frac{1}{2}v_1$.

Theoretically, therefore, the wheel works to the best advantage when the velocity of its periphery is one half of the current velocity. Even then its maximum theoretic effect is only 50%, and in practice this is greatly reduced by frictional and other losses, so that the useful effect rarely exceeds 30%. Undershot wheels with plane floats are cumbrous, have little efficiency, and should not be used for falls of more than 5 feet.

Again, let A be the water-area of a float, and w be the specific weight of the water.

wQ is somewhat less than wAv_1, as there will be an escape of water on both sides of the float.

Let $wQ = kwAv_1$, k being some coefficient (< 1) to be determined by experiment. Then

$$\text{the useful work per second} = kAw\frac{v_1 u}{g}(v_1 - u),$$

$$\text{and its maximum value} = \frac{kA}{4g}v_1^3 w.$$

According to Bossut's and Poncelet's experiments a mean value of k is $\frac{4}{5}$, and the best effect is obtained when $u = \frac{2}{5}v_1$, the corresponding useful work being $\frac{24}{125}\frac{wAv_1^3}{g}$ and the efficiency $\frac{48}{125}$.

8. Wheels in Straight Race.—Generally the water is let on to the wheel through a channel made for the purpose, and closely fitting the wheel, so as to prevent the water escaping without doing work. For this reason also, the space between the ends of the floats in their lowest positions and the channel is made as small as is practicable and should not exceed 2 in. Hence k, and therefore also the efficiency, will be increased. Assume the channel to be of a uniform rectangular section and to have a bed of so slight a slope that it may be regarded as horizontal without sensible error.

The wheel is usually from 24 to 48 ft. in diameter, with 24 to 48 floats, either radial or inclined. The floats are 12 to 20 inches deep, or about $2\frac{1}{2}$ to 3 times the depth of the approaching stream. The fall should not exceed 4 ft. Let the floats be radial, Fig. 139.

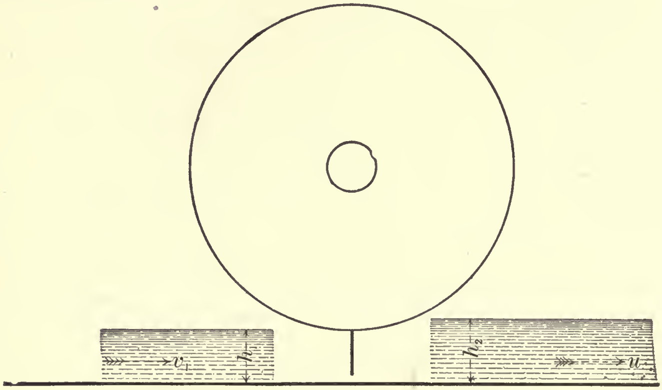


FIG. 139.

Let h_1 be the depth of the water on the *up-stream* side of the wheel.

Let h_2 be the depth of the water on the *down-stream* side of the wheel.

Let b_1 be the width of the race.

The *impulse* = impulse due to change of velocity + impulse due to change of pressure

$$= \frac{wQ}{g}(v_1 - u) + \frac{wQ}{2}\left(\frac{h_1}{v_1} - \frac{h_2}{u}\right),$$

and the *useful work per second*

$$= \text{impulse} \times u = \frac{wQ}{g}u(v_1 - u) + \frac{wQ}{2}\left(\frac{h_1}{v_1} - \frac{h_2}{u}\right)u.$$

The second term is negative, since $h_2 > h_1$, and the maximum theoretic efficiency may be easily shown to be $<.5$.

Three losses have been disregarded, viz.:

(1) The loss of Q_1 cubic feet of the deeper fluid elements which do not impinge upon some of the foremost floats.

According to Gerstner,

$$Q_1 = \frac{cQ}{n_1^2}\left(\frac{v_1}{v_1 - u}\right)^2,$$

n_1 being the number of the floats immersed, and c being $\frac{1}{3}$ or $\frac{2}{3}$ according as the bottom of the race is straight or falls abruptly at the lowest point of the wheel.

(2) The loss of Q_2 cubic feet of water which escape between the wheel and the race-bottom.

Approximately, the play at the bottom may be said to vary from a minimum, $s_1 = BC$, when a float AB is in its lowest position, Fig. 140, to a maximum, $B_1C_1 = CD = B_2C_2$, when

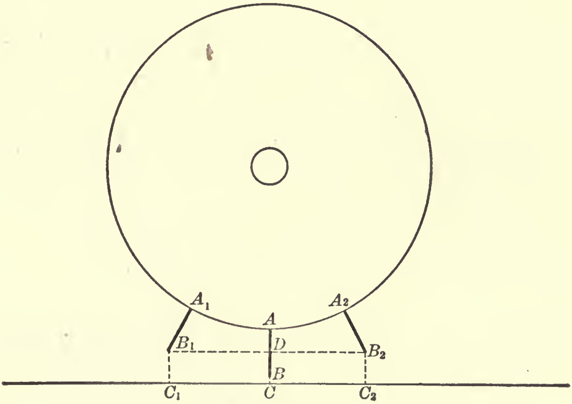


Fig. 140.

two floats A_1B_1, A_2B_2 are equidistant from the lowest position, Fig. 140. Thus the mean clearance

$$= \tfrac{1}{2}(BC + B_1C_1) = \tfrac{1}{2}(S_1 + CD)$$

$$= \tfrac{1}{2}(2S_1 + BD) = S_1 + \frac{1}{4}\frac{(B_1D)^2}{r_1}, \text{ nearly},$$

r_1 being the wheel's radius.

But $\frac{2\pi r_1}{n}$ = distance between two consecutive floats

$$= 2 \,.\, B_1D, \text{ very nearly,}$$

n being the total number of floats. Hence

$$B_1D = \frac{\pi r_1}{n},$$

and therefore the mean clearance $= S_1 + \frac{1}{4}\frac{\pi^2 r_1}{n^2}$.

Again, the difference of head on the up-stream and down-stream sides

$$= h_2 - h_1 = h_1\left(\frac{v_1 - u}{u}\right),$$

and the velocity of discharge, v_d, through the clearance is given by the equation

$$v_d^2 = v_1^2 - 2g(h_2 - h_1) = v_1^2 - 2g\left(\frac{v_1}{u} - 1\right).$$

Hence

$$Q_2 = b_1\left(s_1 + \frac{1}{4}\frac{\pi^2 r_1}{n^2}\right)v_d.$$

Introducing .7 as a coefficient of hydraulic resistance,

$$Q_2 = .7b_1\left(s_1 + \frac{1}{4}\frac{\pi^2 r_1}{n^2}\right)v_d.$$

If the depth of the stream is the same on both sides of the wheel, i.e., if $h_1 = h_2$, then

$$v_d = v_1.$$

(3) The loss of Q_3 cubic feet of water which escape between the wheel and the race-sides.

Let s_2 be the clearance on each side. Then

$$Q = .7 \times 2h_1 s_2 v_d = 1.4 h_1 s_2 v_d,$$

.7 being a coefficient of hydraulic resistance.

Finally, if W lbs. is the weight on the wheel-journals, the loss due to journal friction

$$= \mu W \frac{\rho}{r_1} u,$$

μ being the journal coefficient of friction, and ρ the journal radius.

Thus the actual delivery of the wheel in foot-pounds

$$= \left\{ \frac{u(v_1 - u)}{g} + \frac{u}{2}\left(\frac{h_1}{v} - \frac{h_2}{v_2}\right) \right\} \left\{ w(Q - Q_1 - Q_2 - Q_3 \right\} - \mu W \frac{\rho}{r_1} u.$$

These wheels are most defective in principle, as they utilize only about one third of the total available energy. They may be made to work to somewhat better advantage by introducing the following modifications:

(*a*) The supply may be so regulated by means of a sluice-board, that the mean thickness of the impinging stream is about 6 or 8 inches. If the thickness is too small, the relative loss of water along the channel will be very great. If the thickness is too great, the floats, as they emerge, will have to raise a heavy weight of water. The sluice-board is inclined at an angle of 30° to 40° to the vertical, so that the sluice-opening may be as near the wheel as possible, thus diminishing the loss of head due to channel friction, and is rounded at the bottom to prevent a contraction of the issuing fluid. Neglecting frictional losses, etc.,

$$\text{the useful effect} = wQ\left(H + \frac{v_1^2}{2g} - \frac{u^2}{2g}\right) - \left\{ \begin{array}{l} \text{loss of energy} \\ \text{due to shock} \end{array} \right.$$

$$= wQ\left(H + \frac{v_1^2}{2g} - \frac{u^2}{2g}\right) - \frac{wQ}{g}\frac{(v_1 - u)^2}{2}$$

$$= wQ\left\{ H + \frac{u}{g}(v_1 - u) \right\},$$

H being the difference of level between the point at which the water enters the wheel and the surface of the water in the tail-race, i.e., the fall. H is usually very small and may be negative.

If the vanes are inclined, the resistance to emergence is not so great, and the frictional bed resistance between the sluice and float is practically reduced to *nil*. With a straight bed and small slope (1 in 10) the minimum convenient diameter of wheel is about 14 ft.

(*b*) The bed of the channel for a distance at least equal to the interval between two consecutive vanes may be curved to the form of a circular arc concentric with the wheel, with the view of preventing the escape of the water until it has exerted its full effect upon the wheel. When the bed is curved, the minimum convenient diameter of wheel is about 10 ft.

An undershot wheel with a curb is in reality a low breast-wheel, and its theory is the same as that described in Arts. 13 and 14.

(*c*) The down-stream channel may be deepened so that the velocity of the water as it flows away becomes $> v_1$. The *impulse due to pressure* is then positive, which increases the useful work and therefore also the efficiency.

(*d*) The down-stream channel may be widened and a slight counter-inclination given to the bed. What is known as a *standing-wave* is then produced, in virtue of which there is a sudden rise of surface-level on the down-stream side above that on the up-stream side. This allows of the wheel being lowered by an amount equal to the difference of level between the surfaces of the standing-wave and of the water-layer as it leaves the wheel, thus giving a corresponding gain of head.

(*e*) The introduction of a sudden fall has been advocated in order to free the wheel from back-water, but it must be borne in mind that all such falls diminish the available head.

Thus undershot wheels with plane floats have little effect because of loss of energy by shock at entrance and the loss of energy carried away by the water on leaving the floats. These losses have been considerably modified in Poncelet's wheel, which is often the best motor to adopt when the fall does not exceed 6 ft., and which, in its design, is governed by two principles which should govern every perfect water-motor, viz.:

(1) *That the loss of energy by shock at entrance should be a minimum.*

(2) *That the velocity of the water as it leaves the wheel should be a minimum.*

The vanes are curved and are comprised between two crowns, at a slightly greater distance apart than the vane-width; the inner ends of the vanes are radial, and the water acts in nearly the same manner as in an impulse turbine.

First. Assume that the outer end of a vane is tangential to the wheel's periphery, that the impinging layer is infinitely thin, and that it strikes a float tangentially.

Let af (Fig. 141) be a float, and aq the tangent at a.

The velocity of the water relatively to the float $= v_1 - u$.

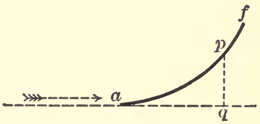


FIG. 141.

The water, in virtue of this velocity, ascends on the bucket to a height $pq = \frac{(v_1 - u)^2}{2g}$, then falls back and leaves the float with the *relative* velocity $v_1 - u$ and with an *absolute* velocity $v_1 - 2u$. This absolute velocity is nil when the speed of the wheel is such that $u = \frac{1}{2}v_1$, and the theoretical height of a float is $pq = \frac{1}{4}\frac{v_1^2}{2g}$. The total available head is thus changed into useful work, and the efficiency is *unity*, or perfect.

Taking R as the mean radius of the crown and u_1 as the corresponding linear velocity, the mean centrifugal force on each unit of fluid mass is $\frac{u_1^2}{R}$ and acts very nearly at the direction of gravity, so that the height pq of a float may be approximately expressed in the form

$$pq = \frac{V^2}{2\left(g + \frac{u_1^2}{R}\right)},$$

V being the velocity with which the water commences to rise on the float.

Practically, however, the float is *not* tangential to the periphery at a, as the water could not then enter the wheel. Also the impinging water is of sensible thickness, strikes the periphery at some appreciable angle, and in rising and falling on the floats loses energy in shocks, eddies, etc.

Let the water impinge in the direction ac, Fig. 142, and take $ac = v_1$.

Take ad in the direction of and equal to u, the velocity of the wheel's periphery.

Complete the parallelogram bd.

Then $cd = ab = V$ is the velocity of the water relatively to the float.

That there may be no shock at entrance, ab must be a tangent to the vane at a.

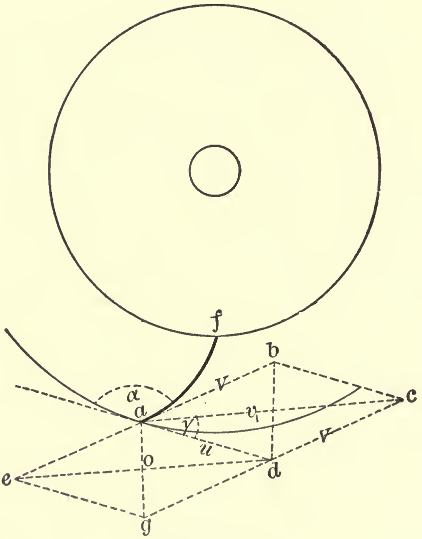


FIG. 142.

Again, the water leaves the vane in the direction of ba produced, and with a relative velocity $ae = ab = V$.

Complete the parallelogram *de*. Then $ag (= v_2)$ is the *absolute* velocity of the water leaving the wheel.

Evidently *cdg* is a straight line.

Let the angle $cad = \gamma$, and the angle $bad = \pi - \alpha$.

From the triangle *adc*,

$$V^2 = v_1^2 + u^2 - 2v_1 u \cos\gamma\,; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v_1^2 = V^2 + u^2 - 2Vu\cos\alpha\,; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{V}{v_1} = \frac{\sin\gamma}{\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From the triangle *adg*,

$$v_2^2 = V^2 + u^2 + 2Vu\cos\alpha. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By equations 1, 2, and 4,

$$\frac{v_1^2 - v_2^2}{2} = -2Vu\cos\alpha = v_1^2 - V^2 - u^2 = 2u(v_1\cos\gamma - u).$$

Therefore the useful work per second

$$= \frac{wQ}{g} 2u(v_1\cos\gamma - u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is a maximum and equal to $\dfrac{wQ}{g}\,\dfrac{v_1^2\cos^2\gamma}{2}$ when $u = \dfrac{v_1\cos\gamma}{2}$, and the maximum efficiency is $\cos^2\gamma$. Hence, too, by equations 1 and 3,

$$\tan(\pi - \alpha) = 2\tan\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also,

$$\frac{V_0}{u} = \frac{\sin\gamma}{\sin(\alpha + \gamma)} = \frac{1}{\cos(\pi - \alpha)}, \text{ by equation 6.}$$

The efficiency is perfect if γ is nil, and therefore $\alpha = 180°$. Practically this is an impossible value, but the preceding calculations indicate that γ should not be too large (usually $< 30°$), and that the speed of the wheel should be a little less than one half of the velocity of the inflowing stream.

Take $\gamma = 15°$ as a mean value. Then

$$u = v_1 \times .484, \text{ and the efficiency} = .993.$$

Actually the efficiency does not exceed 68 per cent. Indeed it must be borne in mind that the theory applies to one elementary layer only, say the mean layer, and that all the other layers enter the wheel at angles differing from 15°, thus giving rise to "losses of energy in shock." The losses of energy in frictional resistance, eddy motion, etc., in the vane passages, have also been disregarded. The layers of water, flowing to the wheel under an adjustable sluice and with a velocity very nearly equal to that due to the total head, may be all made to enter at angles approximately equal to 15°, and the corresponding losses in shock reduced to a minimum by forming the *course* as follows:

The first part of the course FG, Fig. 143, is curved in such a manner that the normal pqr at any point p makes an angle of 15° with the radius oq. The water moves sensibly parallel to the bottom FG, and therefore in a direction at right angles

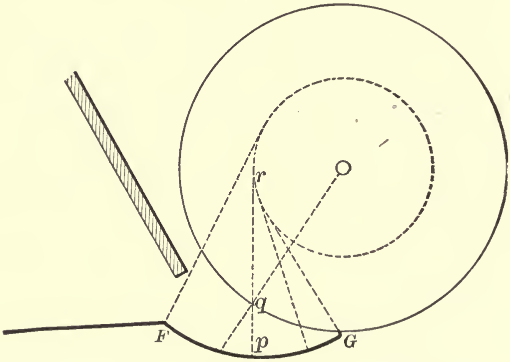


FIG. 143.

to pr. Hence at q the direction of motion makes an angle of 15° with the tangent to the wheel's periphery. If or is drawn perpendicular to pr, then $or = oq \sin 15° =$ a constant.

Thus the normal pqr touches at r a circle concentric with the wheel and of a certain constant diameter.

The initial point F of the curve FG is the point in which the tangent to this circle, passing through the upper edge of the sluice-opening, cuts the bed of the supply-channel.

If t is the thickness (or depth of sluice-opening) and b the breadth of the layer of water as it leaves the sluice, then

$$Q = btv_1,$$

and according to Grashof

$$t = \tfrac{1}{6}H,$$

H being the available fall.

The thickness should not exceed 12 to 15 inches, and is generally from 8 to 10 inches.

Neglecting float thickness, the capacity of the portion of the wheel passing in front of the entering stream per second $= bdu_1$, very nearly.

Only a portion of this space can be occupied by the water, so that

$$Q = mbdu_1,$$

m being a fraction whose value may be taken to be $\frac{1}{2}$ or $\frac{2}{5}$. Hence

$$mbdu_1 = btv_1,$$

and therefore

$$t = md\frac{u_1}{v_1} = \frac{md}{2}\cos\gamma\frac{u_1}{u}$$

$$= \frac{md}{2}\cos\gamma\frac{R}{r_1}.$$

According to Morin,

$$r_1 = 2d \text{ to } 3d.$$

The mean velocity at entrance $= c_v\sqrt{2g(H - \frac{1}{2}t)}$, an average value of c_v being .9.

Thus if $t = \dfrac{H}{16}$,

$$v_1 = .9\sqrt{2g(H - \tfrac{1}{2}t)} = .9\sqrt{2gH\tfrac{31}{32}}.$$

The diameter of the wheel is often taken to be $4H$.

The area of the sluice-opening is usually from $1\frac{1}{4}bt$ to $1.3bt$.

The inside width of the wheel is about $(b + \frac{1}{3})$ ft.

The water should not rise over the top of the buckets, and in order to prevent this the depth of the shrouding is from $\frac{1}{2}H$ to $\frac{2}{3}H$.

If λ is the angle subtended at the centre O of the wheel by the water-arc between the point of entrance A and the lowest

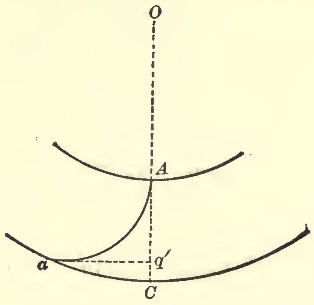


FIG. 144.

point C, Fig. 144, of the wheel, and if Aq' is drawn horizontally, then Aq' is approximately the height of the float, and the theoretic depth d of the crown is given by

$$d = AC = Aq' + Cq' = Aq' + OC - Oq'$$
$$= \frac{V^2}{2\left(g + \frac{u_1^2}{R}\right)} + r_1(1 - \cos\lambda).$$

In practice it is usual to increase this depth by t, the thickness of the impinging water-layer.

Again,

$$d = \frac{2}{3}\frac{V^2}{g + \frac{u_1^2}{R}} + r_1(1 - \cos\lambda) + \text{a few inches, approximately.}$$

The buckets are usually placed about 1 ft. apart, measured along the circumference, but the number of the buckets is not a matter of great importance. There are generally 36 buckets in wheels of 10 to 14 ft. diameter, and 48 buckets in wheels of 20 to 23 ft. diameter.

It may be assumed that the water-arc is equally divided by the lowest point C of the wheel, so that

$$\text{the length of the water-arc} = 2\lambda r = 2uT,$$

T being the time of the ascent or descent of the water in the bucket.

In the middle position, the upper end of the bucket should be vertical, and if the float is in the form of a circular arc, its radius $r' = d \sec(\pi - \alpha)$, α being the angle between the bucket's lip and the wheel's periphery.

The time of ascent or descent is also given by

$$T = \frac{9\psi + \sin\psi}{16}\sqrt{\frac{r'}{g + \frac{u_1^2}{R}}},$$

where $\sin\psi = \sqrt{\cos(\pi - \alpha)}$.

9. Efficiency corresponding to a Minimum Velocity of Discharge (v_2).—From Fig. 142,

$$\frac{ao\ (= \frac{1}{2}ag)}{ad} = \frac{\sin\gamma}{\sin aod} = \frac{\frac{1}{2}(v_2)}{u}.$$

Hence for any given values of u and γ, v_2 is a minimum when $\sin aod$ is greatest, that is, when $aod = 90°$, or ag is at right angles to de. Then also $ad = ae = ab$, or $u = V$, and ac bisects the angle bad. Thus,

$$v_1 = 2u\cos\gamma \quad \text{and} \quad v_2 = 2u\sin\gamma.$$

The useful work

$$= \frac{W}{g}\cdot\frac{v_1^2 - v_2^2}{2} = \frac{W}{g}2u^2\cos 2\gamma = \frac{W}{g}\frac{v_1^2}{2}\frac{\cos 2\gamma}{\cos^2\gamma}.$$

The total available work

$$= \frac{W}{g}\frac{v_1^2}{2}.$$

Therefore the efficiency

$$= \frac{\cos 2\gamma}{\cos^2 \gamma} = \eta.$$

Ex.—If $\gamma = 15°$, the efficiency $= .928$ and $u = .516v_1$.

In practice the best value of u is found to lie between. $50v_1$ and $.60v_1$.

The horse-power of the wheel

$$= \eta \frac{62\frac{1}{2}QH}{550},$$

η being the efficiency with an average value of 60%.

Although, under normal conditions of working, the efficiency of a Poncelet wheel is a little less than that of the best turbines, the advantage is with the former when working with a reduced supply.

10. Form of Bucket.—The form of the bucket is arbitrary, and may be assumed to be a circular arc. In practice there are various methods of tracing its form.

Method I (Fig. 145). The tangent am to the bucket at a

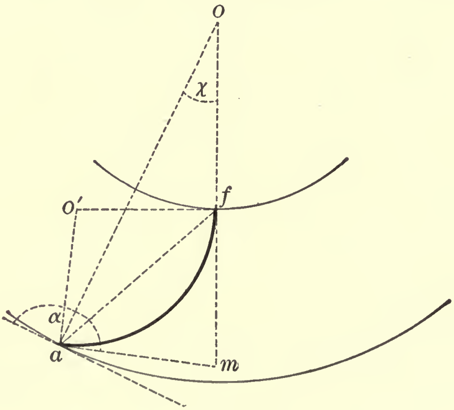


Fig. 145.

makes a given angle α with the tangent at a to the wheel's outer periphery. The radius of is also a tangent to the bucket at f. If the angle aof is known the position of f on the inner periphery is at once fixed, and the form of the bucket can be easily traced.

Let the angle $aof = x$. Join af and let the tangents to the bucket at a and f meet in m. Then

the angle $oam = \alpha - 90°$.

" " $oma = 180° - oam - aom = 270° - \alpha - x$.

" " $mfa =$ the angle $maf = \frac{1}{2}(180° - fma)$

$$= \frac{\alpha + x}{2} - 45°$$

Let r_1, r_2 be the radii of the outer and inner peripheries of the wheel. Then

$$\frac{r_1}{r_2} = \frac{oa}{of} = \frac{\sin ofa}{\sin oaf} = \frac{\sin mfa}{\sin oaf} = \frac{\sin\left(\frac{\alpha + x}{2} - 45°\right)}{\sin\left(\frac{\alpha - x}{2} - 45°\right)},$$

since the angle $oaf = oam - maf = \frac{\alpha - x}{2} - 45°$.

Hence

$$\frac{r_1 - r_2}{r_1 + r_2} = \frac{\sin\left(\frac{\alpha}{2} - 45° + \frac{x}{2}\right) - \sin\left(\frac{\alpha}{2} - 45° - \frac{x}{2}\right)}{\sin\left(\frac{\alpha}{2} - 45° + \frac{x}{2}\right) + \sin\left(\frac{\alpha}{2} - 45° - \frac{x}{2}\right)}$$

$$= \frac{\tan\frac{x}{2}}{\tan\left(\frac{\alpha}{2} - 45°\right)},$$

an equation giving x.

The point o' in which the perpendicular $o'f$ to of meets the perpendicular $o'a$ to am is the centre of the circular arc required and $o'f$ $(= o'a)$ is the radius.

METHOD II (Fig. 146). Take $mad = 150°$, and in ma produced take $ak = of$. With k as centre and a radius equal to *ao* describe the arc of a circle intersecting the inner periphery in the point *f*. Join *kf*, *of*, and *af*. The two triangles *aof* and *akf* are evidently equal in every respect, and therefore the angle *kaf* is equal to the angle *ofa*. Drawing *ao'* at right angles to *ak* and *fo'* tangential to the periphery at *f*, the angle $o'af$ $(= kaf - 90°)$ is equal to the angle $o'fa$ $(= ofa - 90°)$, and therefore $o'a = o'f$. Thus *o'* is the centre of the circular arc required and $o'a$ $(= o'f)$ is the radius.

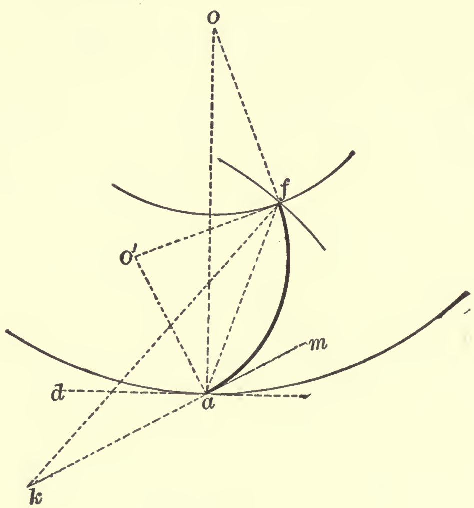


FIG. 146.

METHOD III (Fig. 147). Let the bed with a slope of, say, 1 in 10 extend to the point *C*, and then be made concentric with the wheel for a distance *CC* subtending an angle of 30° at the centre of the wheel. Let the mean layer, half way between the sloping bed and the surface of the advancing water, strike the outer periphery at the point *f*. Draw *fk* making an angle of 23° with *of*, and take *fk* equal to *one half* or seven tenths of the available fall. *k* is the centre of the circular arc required and *kf* is its radius.

11. Breast-wheels.—These wheels are usually adopted for falls of from 5 to 15 feet, and for a delivery of from 5 to 80 cubic feet per second.

The diameter should be at least 11 ft. 6 in., and rarely exceeds 24 ft. The velocity (u) of the wheel's periphery is generally from $3\frac{1}{2}$ ft. to 5 ft. per second, the most useful average velocity being about $4\frac{1}{4}$ ft. per second.

The width of the wheel should not exceed from 8 to 10 ft.

It is of great importance to retain the water in the wheel as long as possible, and this is effected by surrounding the

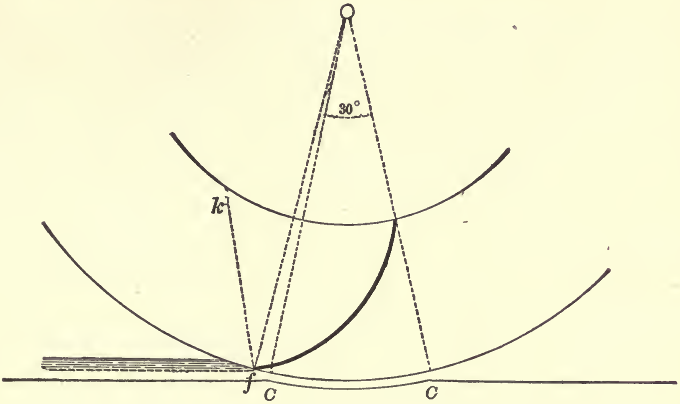


FIG. 147.

water-arc with an apron, or a curb, or a *breast*, which may be constructed of timber, iron, or stone. Hence, too, the buckets may be plane floats, but they should be set at an angle to the periphery of the wheel, so as to rise out of the water with the least resistance (Art. 8).

The depth of a float should not be less than 2.3 ft., and the space between two consecutive floats should be filled to at least one half, and even to two thirds, of its capacity. The head (measured from still water) over the sill or lip should be about 9 in.

The play between the outer edge of the floats and the curb varies from $\frac{1}{2}$ in. in the best constructed wheels to 2 inches.

The distances between the floats is from $1\frac{1}{3}$ to $1\frac{2}{3}$ times the head over the sill.

Breast-wheels are among the best of hydraulic motors, giving a practical efficiency which may be as large as 80 per cent.

12. **Sluices.**—The water is rarely admitted to the wheel without some sluice arrangement, which may take the form of an overfall sluice (Fig. 148), an underflow sluice (Fig. 149), or a bucket or pipe sluice (Fig. 150).

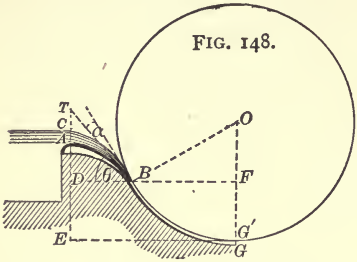


FIG. 148.

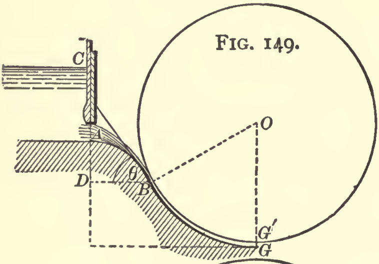


FIG. 149.

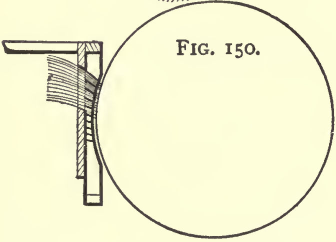


FIG. 150.

The pipe sluice is especially adapted for a varying supply, being provided, for a certain vertical distance, with a series of short tubes, so inclined as to ensure that the water enters the wheel in the right direction. Taking .85 as the mean coefficient of hydraulic resistance for these tubes, the head h_1 required to produce the velocity of entrance v_1 is

$$h_1 = \left(\frac{1}{.85}\right)^2 \frac{v_1^2}{2g};$$

and if H is the total available fall,

$$H - h_1 = H - \frac{1}{(.85)^2}\frac{v_1^2}{2g}$$

= remainder of fall available for pressure-work.

The profile AB in an overfall and an underflow sluice, should coincide with the parabolic path of the lowest stream-lines of the jet. The crest of the overfall should be properly curved, and the inner edges of the underflow opening should be carefully rounded so as to eliminate losses due to contraction.

The underflow sluice-opening should also be normal to the axis of the jet.

Let h_0 be the head above the crest of an overfall sluice. Then

$$Q = \frac{2}{3} c b_1 \sqrt{2g} h_0^{\frac{3}{2}},$$

b_1 being the width of the crest and c the coefficient of discharge. The width b_1 is usually 3 or 4 inches less than the width b of the wheel.

From this equation

$$h_0 = \left(\frac{3Q}{2cb_1\sqrt{2g}}\right)^{\frac{2}{3}},$$

and the depth of water over the crest or lip is usually about 9 inches.

Again, the head $h_1 (= CD)$ required to produce the velocity v_1 at the point of entrance B is

$$CD = h_1 = \frac{11}{10}\frac{v_1^2}{2g},$$

10 per cent being allowed for loss due to friction.

Thus the height of the crest A above B, the point of entrance,

$$= AD = CD - CA = h_1 - h_0$$

$$= \frac{11}{10}\frac{v_1^2}{2g} - \left(\frac{3Q}{2cb_1\sqrt{2g}}\right)^{\frac{2}{3}}.$$

But BA is a parabola with its vertex at A, and therefore, if θ is the angle between the horizontal BD and the tangent BT to the parabola at B,

$$\frac{v_1^2 \sin^2\theta}{2g} = AD = \frac{11}{10}\frac{v_1^2}{2g} - \left(\frac{3Q}{2cb_1\sqrt{2g}}\right)^{\frac{2}{3}}.$$

Also

$$BD = \frac{v_1 \sin 2\theta}{2g}.$$

The head available for pressure work

$$= DE = FG = H - h_1.$$

Let α be the angle between BT and the tangent to the wheel's periphery at B. Then

$$\alpha + \theta = \text{the angle } BOF,$$

BO being the radius to the centre of the wheel and OFG' vertical.

If the lowest point G' of the wheel just clears the tail-race, the head available for pressure work

$$= H - h_1 = FG' = OG' - OF$$

$$= r_1(1 - \cos BOF) = 2r_1 \sin^2 \frac{BOF}{2},$$

r_1 being the radius to the outer periphery of the wheel.

If, again, the water enters the wheel tangentially,

$$\alpha = 0, \text{ and the angle } BOF = \theta,$$

so that

$$H - h_1 = 2r_1 \sin^2 \frac{\theta}{2}.$$

If the sluice-opening is not at the vertex of the parabola, the axis of the opening should be tangential to the parabola.

13. Speed of Wheel.—The water leaves the buckets and flows away in the race with a velocity not sensibly different from the velocity u of the wheel's periphery.

Let b be the breadth of the wheel (Fig. 151).

Let x be the depth of the water in the lowest bucket.

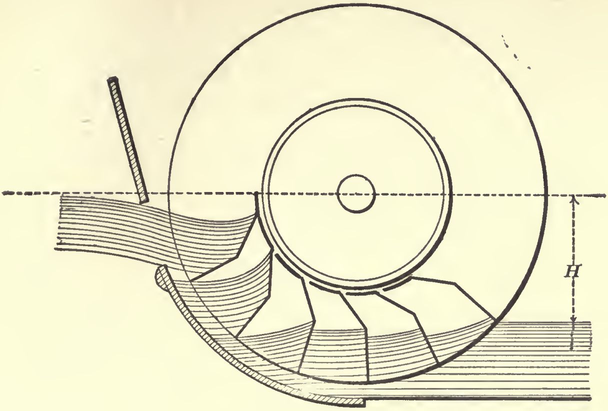


FIG. 151.

Allowing for the thickness of the buckets, the play between the wheel and curb, etc.,

$$Q = cbxu,$$

c being an empirical coefficient whose average value is about .9. Hence

$$u = \frac{10}{9}\frac{Q}{bx}.$$

In practice b is often taken to be $\frac{Q}{15}$ to $\frac{Q}{25}$. It is important that b should be as small as possible and hence x should be as large as possible, its value being usually $1\frac{1}{2}$ ft. to 2 ft.

It must be borne in mind, however, that any increase in the value of x will cause an increase in the weight of water lifted by the buckets as they emerge from the race, and will therefore tend to diminish the efficiency.

14. **Mechanical Effect.**—Theoretically, the total mechanical effect

$$= wQ\left(H - \frac{v_2^2}{2g}\right) = wQ\left(H - \frac{u^2}{2g}\right),$$

H being the fall from the surface of still water in the supply-channel to the surface of the water in the tail-race.

This, however, is reduced by the following losses:

(*a*) Owing to frictional resistance, etc., there is a loss of head in the supply-channel which may be measured by $\nu\frac{v_1^2}{2g}$, ν being approximately $\frac{1}{20}$ to $\frac{1}{10}$.

The head required to produce the velocity at entrance, v_1,

$$= (1 + \nu)\frac{v_1^2}{2g}.$$

(*b*) Let *af*, Fig. 152, represent in direction and magnitude v, the velocity of the water entering the bucket.

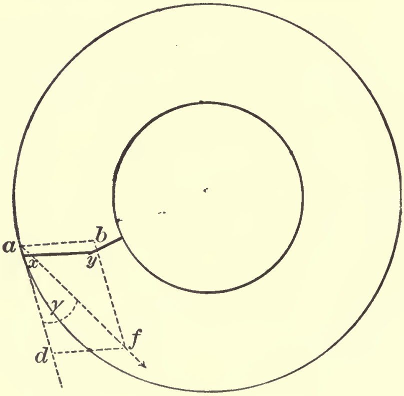


FIG. 152.

Let *ad*, in the direction of the tangent to the wheel's periphery, represent the velocity u of the periphery in direction and magnitude.

Complete the parallelogram *bd*. Then *ab* evidently represents the velocity V of the water relatively to the wheel. This velocity V is rapidly destroyed, the corresponding loss of head being

$$\frac{V^2}{2g} = \frac{u^2 + v_1^2 - 2uv_1 \cos\gamma}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

γ being the angle *daf*.

Assuming that the water enters the race with the velocity u of the wheel, the theoretical useful work per pound per second due to impact

$$= \frac{v_1^2 - V^2 - u^2}{2g}$$

$$= \frac{u}{g}(v_1 \cos \gamma - u).$$

If the loss $\frac{V^2}{2g}$ is to be a minimum for a given speed of wheel,

$$v_1 dv_1 - u \cos \gamma \,.\, dv_1 = 0, \quad \text{or} \quad v_1 = u \cos \gamma. \quad . \quad . \quad (2)$$

Hence, by equation 1, $V = u \sin \gamma$, and therefore

$$\tan \gamma = \frac{V}{v_1} = \frac{df}{af},$$

so that for a velocity of entrance $v_1 = u \cos \gamma$ the angle *afd* should be 90°. But this value is inadmissible, as the water would arrive tangentially and consequently would not enter the buckets. In order that the loss in shock at entrance may be as small as possible, *ab*, the direction of the relative velocity V, should be parallel to the arm *xy* of the bucket, and should therefore be approximately normal to the wheel's periphery. This is equivalent to the assumption that the water arrives in a given direction (γ) with a given velocity (v_1), and that the speed (u) of the wheel is to be such as will make V a minimum. Thus, by equation 1,

$$0 = udu - v_1 \cos \gamma \,.\, du, \quad \text{or} \quad u = v_1 \cos \gamma,$$

and therefore

$$V = v_1 \sin \gamma.$$

Hence $\tan \gamma = \frac{V}{u} = \frac{df}{ad}$, and therefore the angle $adf = 90°$.

In practice γ is generally 30°, and the corresponding loss of head $= \frac{V^2}{2g} = \frac{v_1^2}{2g}\sin^2\gamma = \frac{v_1^2}{2g}.\frac{1}{4} = \frac{u^2}{2g}.\frac{1}{3}$.

At point of entrance x falls below y, the water flows up the inclined plane xy, and V, instead of being wholly destroyed in eddy motion, is partially destroyed by gravity. This velocity, destroyed by gravity, is again restored to the water on its return, and thus adds to the efficiency of the wheel.

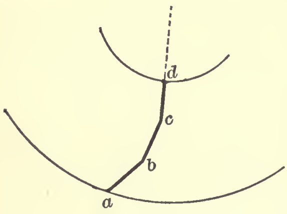


FIG. 153.

It will be found advantageous to use curved or polygonal buckets and not plane floats. A bucket, for example, may consist of three straight portions, *ab*, *bc*, *cd*, Fig. 153. Of these the inner portion *cd* shoud be radial; the outer portion *ab* is nearly normal to the periphery of the wheel, and the central portion *bc* may make angles of about 135° with *ab* and *cd*.

Disregarding all other losses, the theoretical delivery of the wheel in foot-pounds

$$= wQ\left\{\frac{u(v_1\cos\gamma - u)}{g} + h_2\right\},$$

where $h_2 =$ total fall — fall (h_1) required to produce the velocity v_1.

If η be the efficiency, then, according to the results of Morin's experiments,

$$\eta = .40 \text{ to } .45 \text{ if } h_1 = \frac{1}{4}H;$$

$$\eta = .42 \text{ to } .49 \text{ if } h_1 = \frac{2}{5}H;$$

$$\eta = .47 \qquad \text{if } h_1 = \frac{2}{3}H;$$

$$\eta = .55 \qquad \text{if } h_1 = \frac{3}{4}H.$$

(*c*) There is a loss of head due to frictional resistance along the channel in which the wheel works.

Let $l =$ length of the channel (or curb).

Let $t =$ thickness of water-layer leaving the wheel.

Let $b =$ breadth of wheel.

The mean velocity of flow in this curb channel is approximately $\frac{4}{3}u$, and *the loss of head due to channel friction*

$$= f\frac{b+2t}{bt}\frac{\left(\frac{4}{3}u\right)^2}{2g}l = \frac{4}{3}f\frac{b+2t}{bt}\frac{v_1^2}{2g}l,$$

where $f =$ coefficiency of friction, $b + 2t =$ wetted perimeter, $bt =$ water area, and γ being 30°.

(*d*) There is a loss of head due to the escape of water over the ends and sides of the buckets.

Let s_1 be the play between the ends of the buckets and the channel.

Let s_2 be the play at the sides. ($s_1 = s_2$, approximately.)

Let $z_1, z_2, \ldots z_n$ be the depths of water in a bucket corresponding to n successive positions in its descent from the receiving to the lowest points.

Let $l_1, l_2, \ldots l_n$ be the corresponding water-arcs measured along the wheel's periphery.

The orifice of discharge at end of a bucket $= bs_1$.

The mean amount of water escaping from a bucket over its end

$$= cbs_1\sqrt{2g}\frac{\sqrt{z_1} + \sqrt{z_2} + \ldots \sqrt{z_n}}{n},$$

c being the coefficient of discharge.

The water escapes at the sides as over a series of weirs, and the mean amount of water escaping from a bucket over the sides

$$= 2 \times \frac{2}{3}cs_2\sqrt{2g}\frac{l_1\sqrt{z_1} + l_2\sqrt{z_2} + \ldots l_n\sqrt{z_n}}{n}.$$

Hence the total loss of effect from escape of water

$$c\sqrt{2g}\frac{wh}{n}\left\{bs_1\left(\sqrt{z_1}+\sqrt{z_2}+\ldots\sqrt{z_n}\right)\right.$$
$$\left.+\frac{4}{3}s_2\left(l_1\sqrt{z_1}+l_2\sqrt{z_2}+\ldots l_n\sqrt{z_n}\right)\right\}$$

per sec., h being the vertical distance between the point of entrance and the surface of the water in the tail-race

$$= H - (1+\nu)\frac{v_1^2}{2g}.$$

(e) There is a loss of head due to journal friction.

Let $W =$ weight of wheel.

Let $w_1 =$ weight of water on the wheel.

Let $r_1 =$ radius of wheel's outer periphery.

Let $r' =$ radius of axle.

Loss per second of mechanical effect due to journal friction

$$= \mu(W + w_1)\frac{r'}{r_1}u,$$

r being the coefficient of journal friction.

There is a loss of mechanical effect due to the resistance of the air to the motion of the floats (buckets), but this is practically very small, and may be disregarded without sensible error. A deepening of the tail-race produces a further loss of effect, and should only be adopted when back-water is feared.

Hence the total actual mechanical effect,
putting

$$Z = bs_1(\sqrt{z_1}+\sqrt{z_2}+\ldots\sqrt{z_n})+\frac{4}{3}s_2(l_1\sqrt{z_1}+{}_2l\sqrt{z_2}+\ldots+l_n\sqrt{z_n}),$$

is $$= wQ\left(H-\frac{u^2}{2g}\right) - wQ\left(\nu\frac{v_1^2}{2g}+\frac{u^2+v_1^2-2uv_1\cos\gamma}{2g}\right)$$
$$-f\frac{b+2t}{bt}\frac{4}{3}\frac{v_1^2}{2g}l - c\sqrt{2g}\frac{wh}{n}Z - \mu(W+w_1)\frac{r'}{r_1}u,$$

$$= wQ\left\{H - (1+\nu)\frac{v_1^2}{2g}\right\} + \frac{wQ}{g}u(v_1 \cos\gamma - u)$$

$$- f\frac{b+2t}{bt}\frac{4}{3}\frac{v_1^2}{2g}l - c\sqrt{2gh}\frac{w}{n}Z - \mu(W+w_1)\frac{r'}{r_1}u$$

$$= \left(wQ - c\sqrt{2g}\frac{wZ}{n}\right)\left(H - \overline{1+\nu}\frac{v_1^2}{2g}\right) + \frac{wQ}{g}u(v_1 \cos\gamma - u)$$

$$- f\frac{b+2t}{bt}\frac{4}{3}\frac{v_1^2}{2g}l - \mu(W+w_1)\frac{r'}{r_1}u.$$

Hence, for a given value of v_1, the mechanical effect (omitting the last term) is a maximum when

$$u = \frac{v_1 \cos\gamma}{2} \; (= .433 \times v_1, \text{ if } \gamma = 30°).$$

In practice the speed of the wheel is made about *one half* of the velocity with which the water enters the wheel.

For a given speed of wheel, and disregarding the loss of effect due to curb friction, which is always small, the mechanical effect is a maximum for a value of v_1 given by

$$-\left(wQ - c\sqrt{2g}\frac{wZ}{n}\right)\frac{1+\nu}{g}v_1 + \frac{wQ}{g}u\cos\gamma = 0,$$

or

$$v_1 = \frac{u\cos\gamma}{(1+\nu)\left(1 - \frac{c\sqrt{2g}}{nQ}Z\right)}.$$

The loss by escape of water, viz., $c\sqrt{2g}\frac{Z}{n}$, varies, on an average, from 10 to 15 per cent of the whole supply, so that $c\sqrt{2g}\frac{Z}{n}$ varies from $\frac{Q}{10}$ to $\frac{3Q}{20}$.

15. **Sagebien Wheels** have plane floats inclined to the radius at from 40° to 45° in the direction of the wheel's rotation. The floats are near together and sink slowly into the fluid mass. The level of the water in the float-passages grad-

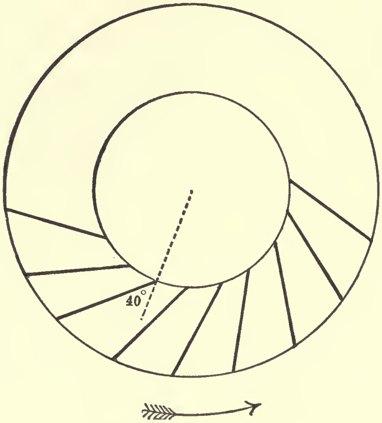


FIG. 154.

ually varies and the volume discharged in a given time may be very greatly changed. The efficiency of these wheels is over 80 per cent, and has reached even 90 per cent. The action is almost the same as if the water were transferred from upper to lower race, without agitation, frictional resistance, etc., flowing away without obstruction, into the tail-race.

16. **Overshot Wheels.**—These wheels are among the best of hydraulic motors for falls of 8 to 70 ft. and for a delivery of 3 to 25 cub. ft. per second. They are especially useful for falls of 12 to 20 ft. The efficiency of overshot wheels of the best construction is from .70 to .85.

If the level of the head-water is liable to a greater variation than 2 ft., it is most advantageous to employ a pitch-back or high breast-wheel, which receives the water on the same side as the channel of approach.

17. **Wheel-velocity.**—This evidently depends upon the work to be done, and upon the velocity with which the water

arrives on the wheel. Overshot wheels should have a low circumferential speed, varying from 10 ft. per sec. for large wheels to 3 ft. per sec. for small wheels, and should not be less than $2\frac{1}{2}$ ft. per sec.

In order that the water may enter the buckets easily, its velocity should be greater than the peripheral velocity of the wheel.

18. Effect of Centrifugal Force.—Consider a molecule of weight W in the "unknown" surface of the water in a

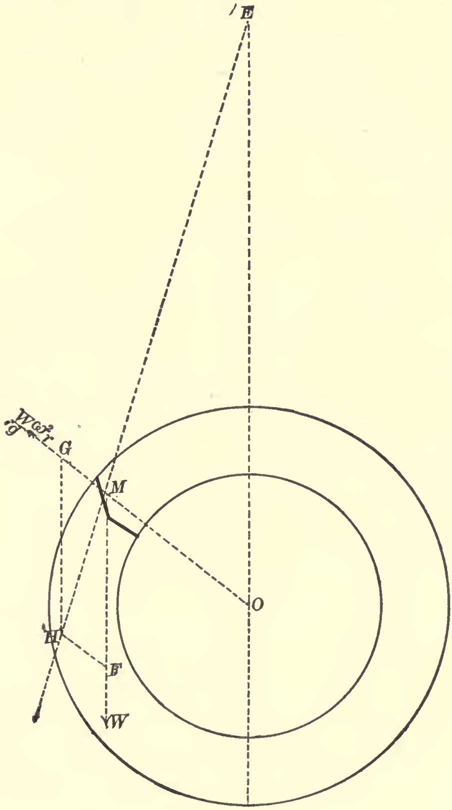


FIG. 155.

bucket (Fig. 155). At each moment there is a dynamical equilibrium between the "forces" acting on m, viz.: (1) its weight w; (2) the centrifugal force $\frac{w}{g}\omega^2 r$; (3) the resultant T of the neighboring reactions.

Take $MF = w$, $MG = \frac{w}{g}\omega^2 r$, and complete parallelogram FG. Then $MH = T$. The direction of T is, of course, normal to the surface of the water in the bucket.

Let HM produced meet the vertical through the axis O of the wheel in E. Then

$$\frac{MG}{MF} = \frac{\frac{w}{g}\omega^2 r}{w} = \frac{FH}{MF} = \frac{OM}{OE} = \frac{r}{OE},$$

and therefore

$$OE = \frac{g}{\omega^2} = \frac{2915}{n^2} \text{ ft.},$$

taking $g = 32$ ft. and n being the number of revolutions per minute.

Thus the position of E is independent of r and of the position of the bucket, so that all the normals to the water-surface in a bucket meet in E, and the surface is the arc of a circle having its centre at E, or, rather, a cylindrical surface with axis through E parallel to the axis of rotation.

19. Weight of Water on Wheel and Arc of Discharge.—Let Q = volume supplied per sec., and N = number of buckets.

Then $\frac{N\omega}{2\pi}$ = number of buckets fed per sec.,

and $\frac{2\pi Q}{N\omega}$ = volume of water received by each bucket per sec.

Hence the area occupied by the water until spilling commences $= \frac{2\pi Q}{bN\omega}$, b being the bucket's width (= width of wheel between the shroudings).

The water flows on to the wheel through a channel (Fig. 156), usually of the same width b as the wheel, and the supply is regulated by means of an adjustable sluice, which may be either vertical, inclined, or horizontal.

When the water springs clear from the sluice, as in Fig. 156, the axis of the sluice should be tangential to the axis of the

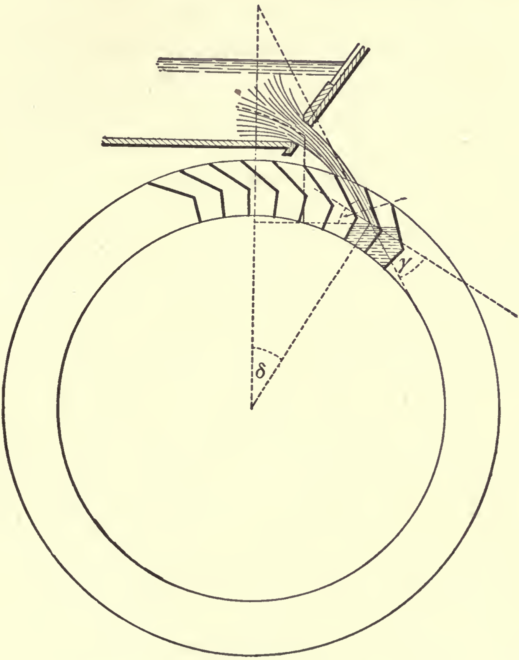


FIG. 156.

jet, and the inner edges of the sluice-opening should be rounded so as to eliminate contraction.

Let y, z be the horizontal and vertical distances between the sluice and the point of entrance.

Let T be the time of flow between the sluice and entrance.

Let v_0, v_1 be the velocities of flow on leaving the sluice and on entering the bucket.

Then

$$v_1 \cos(\gamma + \delta)T = y,$$

$$v_1 \sin(\gamma + \delta)T - \tfrac{1}{2}gT^2 = z,$$

and

$$v_1^2 = v_0^2 + 2gz,$$

δ being angular deviation of point of entrance from summit, and γ the angle between the direction of motion of the water and the wheel at the point of entrance.

Assume the bed of the channel to be horizontal, and the sluice vertical and of the same width b as the wheel. The sluice is also supposed to open upwards from the bed. Then

$$Q = bx\sqrt{2gh_1},$$

x being the depth of sluice-opening and h_1 the effective head over the sluice. This effective head is about $\frac{9}{10}$ths of the actual head.

Thus, taking $g = 32$, $\frac{Q}{b} = 8xh_1^{\frac{1}{2}}$ gives the delivery per foot width of wheel.

Taking .6 ft. and 3.6 ft. as the extreme limits between which h_1 should lie, and .2 ft. and .33 ft. as the extreme limits between which x should lie, then $\frac{Q}{b}$ must lie between the limits 1.24 and 5, and an average value of $\frac{Q}{b}$ is 3. Thus the width of the wheel should be on the average $\geqq \frac{Q}{3}$.

Again, neglecting the thickness of the buckets, the capacity of the portion of the wheel passing in front of the water-supply per second

$$= b\omega\left\{\frac{r_1^2}{2} - \frac{(r_1 - d)^2}{2}\right\} = bd\omega\left(r_1 - \frac{d}{2}\right) = bdr_1\omega, \text{ approximately,}$$

$$= bdu_1 = bd\frac{\pi r_1 n}{30},$$

r_1 being the radius and u_1 the velocity of the outer circumference of the wheel, d the depth of the shrouding, and n the number of revolutions per minute.

Only a portion, however, of the space can be occupied by the water, so that the capacity of a bucket is *mubd*, *m* being a fraction less than unity and usually $\frac{1}{3}$ or $\frac{1}{4}$. For very high wheels *m* may be $\frac{1}{5}$. Hence

$$mbdu_1 = \frac{2\pi Q}{N\omega}.$$

Again, since the thickness of the buckets is disregarded,

$$Nu = 2\pi r_1 = Nr_1\omega.$$

Therefore $$mdu_1 = \frac{Q}{b}.$$

The delivery $\left(\frac{Q}{b}\right)$ per foot of width must not exceed a certain limit, otherwise either *d* or *u* will be too great. In the former case the water would acquire too great a velocity on entering the buckets, which would lead to an excessive loss in eddy motion and a corresponding loss of efficiency; while if the speed *u* of the wheel is too great the efficiency is again diminished and might fall even below 40%.

The depth of a bucket or of the shrouding varies from 10 to 16 in., being usually from 10 to 12 in., and the buckets are spread along the outer circumference at intervals of 12 to 14 inches. The number of the buckets is approximately 5*r* or 6*r*, *r* being the radius of the wheel in feet.

The efficiency of the wheel necessarily increases with the number of the buckets, but the number is limited by certain considerations, viz.: (*a*) the bucket thickness must not take up too much of the wheel's periphery; (*b*) the number of the buckets must not be so great as to obstruct the free entrance of the water; (*c*) the form of the bucket essentially affects the number.

Let the bucket, Fig. 157, consist of two portions, an inner portion *bc*, which is radial, and an outer portion *cd*; *c* being a point on what is called the division circle. The length *bc* is usually one half or two thirds of the depth *d* of the shrouding.

Take $bc = \frac{1}{2}d$.

It may also be assumed without much error that the water-surface ad is approximately perpendicular to the line ed, so that the angle edc is approximately a right angle.

The spilling evidently commences when the cylindrical surface, having its axis at e and cutting off from the bucket a water-area equal to $\dfrac{2\pi Q}{N\omega}$, passes through the outer edge d of the bucket.

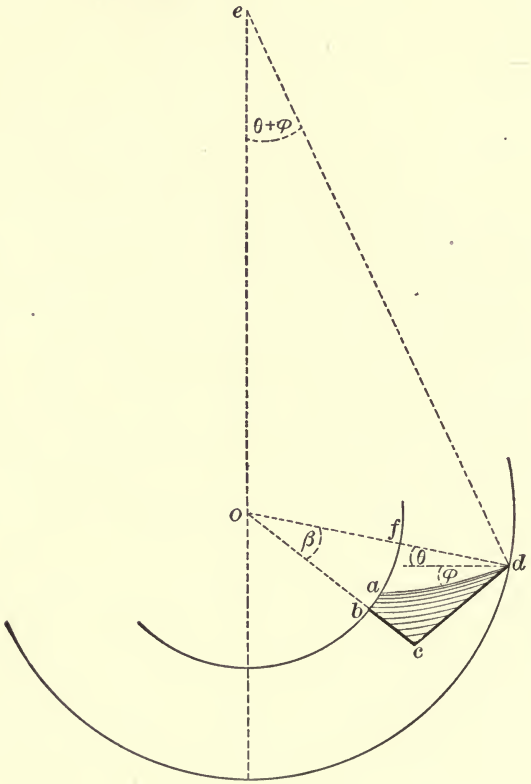


FIG. 157.

Let β be the bucket angle cOd.

Let θ be the inclination of Od to the horizon.

Let ϕ be the inclination of ad to the horizon.

Let r_1 be the radius of the outer periphery.
Let R be the radius of the division circle.
Let r_2 be the radius of the inner periphery.
Then

$$\frac{od}{oe} = \frac{r_1}{\frac{g}{\omega^2}} = \frac{\sin\phi}{\sin\{90^\circ - \overline{\theta+\phi}\}} = \frac{\sin\phi}{\cos(\theta+\phi)};$$

and therefore

$$r_1\frac{\omega^2}{g} = \frac{\sin\phi}{\cos(\theta+\phi)}. \quad \ldots \ldots \quad (1)$$

Again,

$$af = fd\tan(\theta+\phi), \text{ approximately.}$$

Therefore

$$\text{the area } dfa = \frac{fd^2}{2}\tan(\theta+\phi) = \frac{d^2}{2}\tan(\theta+\phi),$$

where $d = r_1 - r_2$. Hence

$$\text{the area } abcd = \text{area } cod - \text{area } bof - \text{area } afa$$

$$= \frac{r_1 R}{2}\sin\beta - \frac{r_2^2}{2}\beta - \frac{d^2}{2}\tan(\theta+\phi) = \frac{2\pi Q}{bN\omega}. \quad . \quad (2)$$

Equations (1) and (2) give θ and ϕ, and therefore the position of the bucket when spilling commences.

The bucket will be completely emptied when it has reached a position in which cd is perpendicular to a line from e to middle point of cd, or, approximately, when edc is a right angle.

Let θ_1, ϕ_1 be the corresponding values of θ and ϕ, and let

γ_1 be the angle between cd and the tangent at d to the wheel's periphery. Then

$$\gamma_1 = 90^\circ - (\theta_1 + \phi_1)$$

and

$$\frac{\sin \gamma_1}{\sin \phi_1} = \frac{g}{r_1\omega^2},$$

two equations giving ϕ_1 and θ_1.

Also, if ce is drawn perpendicular to od,

$$\tan \gamma = \cot cde = \frac{de}{ce} = \frac{r_1 - R \cos \beta}{R \sin \beta}$$

The vertical distance between the points where spilling begins and ends, viz., $r_1 (\sin \theta_1 - \sin \theta)$ can now be determined.

The pitch-angle($= \psi$) is the angle between two consecutive buckets so that $\psi = \frac{360^\circ}{N}$. In order to obtain a small angle ($= \gamma_1$) between the lip of the bucket and the wheel's periphery, it is usual to make the bucket angle β greater than ψ.

For example,

$$\beta = \frac{5}{4}\psi = \frac{5}{4}\frac{360^\circ}{N} = \frac{450^\circ}{N}.$$

The interval between the buckets should be at least sufficient to prevent any bucket dipping into the one below at the moment the latter begins to spill.

Let coc', Fig. 158, be the division angle and t the thickness of the bucket.

Then

$$ff' = \frac{fa}{2} = \frac{d}{2} \tan (\theta + \phi) = \frac{d}{2} \tan \theta,$$

approximately, and therefore

$$N \left(r_2 \beta + t - \frac{d}{2} \tan \theta\right) = 2\pi r_2. \quad . \quad . \quad . \quad . \quad (3)$$

Also, by equation 2,

$$\frac{r_1 R}{2}\sin\beta - \frac{r_2^2}{2}\beta = \frac{d^2}{2}\tan\theta + \frac{2\pi Q}{bN\omega}. \quad . \quad . \quad . \quad (4)$$

These two last equations give N and θ.

The number of buckets may also be approximately found from the formula

$$N = \frac{2\pi r_1}{d}.$$

In practice the bucket may be delineated as follows:

Let dd' = distance between two buckets.

Take $dd'' = \frac{5}{4}dd'$ to $\frac{6}{5}dd'$; also take $bc = \frac{d}{2}$, and join dc.

This gives the form of a suitable wooden bucket.

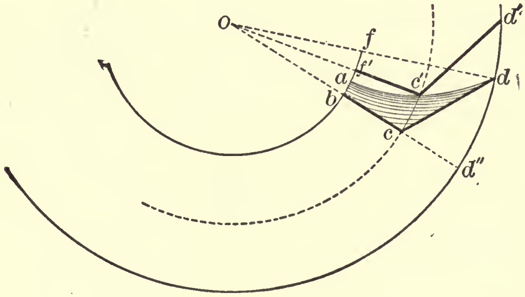


FIG. 158.

If the bucket is of iron, a circular arc is substituted for the portions *bc*, *cd*.

Again, let *pm*, Fig. 159, be the thickness of the stream just before entering the bucket.

Let *dn* be the thickness of the stream just after entering the bucket.

Let λ be the angle between the bucket's lip and the wheel's periphery.

Then

$$mbdu_1 = \text{capacity of bucket} = bv_1 \,.\, pm = bV \,.\, dn$$
$$= bv_1 dp \sin\gamma = bV \,.\, dp \,.\, \sin\lambda,$$

and therefore

$$dp = \frac{mdu_1}{v_1 \sin\gamma} = \frac{mdu_1}{V \sin\lambda}.$$

Now overshot wheels cannot be ventilated, and it is conse-

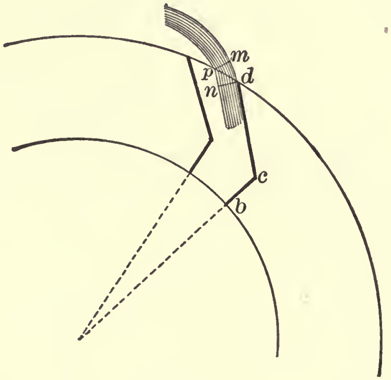


FIG. 159.

quently necessary to leave ample space above the entering stream for the free exit of air. Thus, neglecting float thickness,

$$\frac{2\pi r_1}{N} = \text{distance between consecutive floats}$$

$$= dd' \text{ (Fig. 158)} > dp > \frac{mdu_1}{V \sin\lambda},$$

and N, the number of buckets,

$$< \frac{2\pi r_1 V \sin\lambda}{mdu_1}.$$

For efficient action the number of the buckets is much less than the limit given by this relation, often not exceeding one half of such limit.

If γ is very small, $V = v_1 - u_1$, approximately, and therefore

$$N < \frac{2\pi r_1 \sin \lambda}{md}\left(\frac{v_1}{u_1} - 1\right).$$

The capacity of a bucket depends upon its form; and the bucket must be so designed that the water can enter freely and without shock, is retained to the lowest possible point, and is finally discharged without let or hindrance. Hence flat buckets, Fig. 160, are not so efficient as the curved iron bucket in Fig. 163 and as the compound bucket made of three or two

FIG. 160.

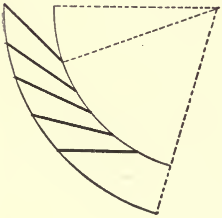


FIG. 161.

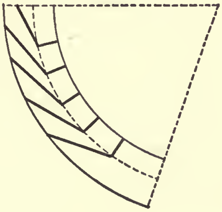
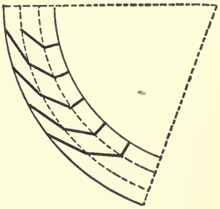


FIG. 162.



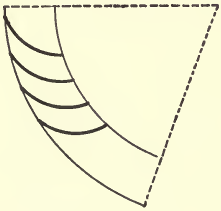


FIG. 163.

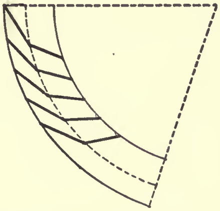


FIG. 164.

pieces in Figs. 161, 162, 164. The resistance to entrance is least in the curved bucket, as there are no abrupt changes of direction due to angles. The capacity of a compound bucket may be increased, without diminishing the ease of entrance, by making the inner portion strike the inner periphery at an acute angle, Fig. 164. The objection to this construction, especially if the relative velocity V is large, is that the water tends to return in the opposite direction and escape from the bucket.

Let *bcd*, *efg*, Fig. 165, represent two consecutive buckets of an overshot wheel turning in the direction shown by the arrow.

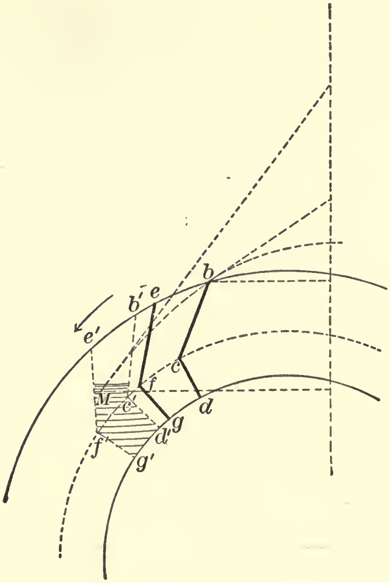


FIG. 165.

Water will cease to enter the bucket-space between *bcd* and *efg*, and impact will therefore cease, when the upper parabolic boundary of the supply-stream intersects the edge b. The last fluid elements will then strike the water already in the bucket at a point M, whose vertical distance below b may be designated by z. The velocity v' with which the entering particles reach M is given by the equation

$$v_1' = \sqrt{v_1^2 + 2gz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, while the fluid particles move from b to M let the buckets move into the positions $b'c'd'$, $e'f'g'$.

Let arc $bb' = s_1 = ee'$.

Let arc $bM = s_2$.

Let T be the time of movement from b to b' (or b to M). Then

$$s_1 = uT$$

and

$$s_2 = \frac{v_1 + v_1'}{2} T,$$

assuming that the mean velocity from b to M is an arithmetic mean between the initial and final velocity of entrance. Thus

$$\frac{2s_2}{v_1 + v_1'} = T = \frac{s_1}{u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, since the angle between bM and the wheel's periphery is small, it may be assumed that

$$\text{the arc } bM = be + ef + ee', \text{ approximately,}$$

$$= \frac{2\pi r_1}{N} + \frac{2\pi r_1}{N} \cdot \frac{v_1 - u}{u} + s_1.$$

$$\left(\textit{Note.}—ef = eb\frac{V}{u} = eb\frac{v_1 - u}{u} = \frac{2\pi r_1}{N} . \frac{v_1 - u}{u}, \text{ nearly.}\right)$$

Thus

$$s_2 = \frac{2\pi r_1}{N}\frac{v_1}{u} + s\ ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and by equations 2 and 3,

$$s_1\left(\frac{v_1 + v_1' - 2u}{2u}\right) = \frac{2\pi r_1}{N}\frac{v_1}{u},$$

an equation giving approximately the distance s_1 passed through by a float during impact. The buckets can now be plotted in the positions they occupy at the end of the impact. The amount of water in each bucket being also known, the water-surface can be delineated, and hence the vertical distance z can be at once found.

20. Useful Effect.—(*a*) *Effect of Weight.*—The wheel should hang freely, or just clear the tail-water surface, and the total fall is measured from the surface of the water in the tail-race to the water-surface just in front of the sluices through which the water is brought on to the wheel.

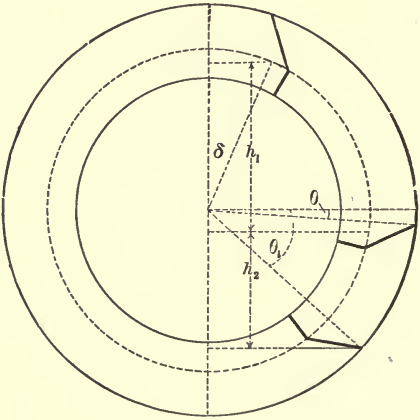


FIG. 166.

Let h_1, Fig. 166, be the vertical distance between the centres of gravity of the water-areas of the first and last buckets before spilling commences. Then

$$h_1 = R \cos \delta + r_1 \sin \theta, \text{ very nearly.}$$

Let h_2 be the vertical distance between the centres of gravity of the water-area of the bucket which first begins to spill, and the point at which the spilling is completed. Then

$$h_2 = r_1(\sin \theta_1 - \sin \theta), \text{ very nearly.}$$

The useful work per sec. $= wQ(h_1 + kh_2)$, k being a fraction < 1 and approximately $= .5$.

Let A_0 be the water-area in the bucket which first begins to spill.

Between this bucket and the one which is first emptied, i.e., in the vertical distance h_2, insert an even number s of buckets, and let their water-areas $A_1, A_2, A_3, \ldots A_s$ be carefully calculated.

Let Q_m be the mean amount of water per bucket in the discharging arc.

Let A_m be the mean water-area per bucket in the discharging arc.

Then

$$A_m = \frac{A_0 + A_1 + A_3 + \ldots + A_{s-1} + A_s}{s}.$$

The value of k can now be easily found, since

$$k = \frac{Q_m}{Q} = \frac{A_m}{A_0}.$$

Let q be the varying amount of water in a bucket from which spilling is taking place, and at any moment let y be the vertical distance between the outer edge of the bucket and the surface of the water in the tail-race.

q is a function of y and depends upon the contour of the water in the bucket.

Let Y be the *mean* value of y between the points where spilling begins and ends, i.e., for values y_1 and y_2 of y. Then

$$Y\left(\frac{2\pi Q}{Nw}\right) = Y\int_0^q dq = \int_0^q y dq = y_1\frac{2\pi Q}{Nw} - \int_{y_1}^{y_2} q \, . dy,$$

since

$$\int y \, . dq = yq - \int q \, . dy.$$

Again, the elementary quantity of water, dq, having an initial velocity equal to that of the wheel, viz., u, falls a distance y and acquires a velocity $= \sqrt{u^2 + 2gy}$.

Thus it flows away in the tail-race causing a loss of energy $= \frac{w \,.\, dq}{g}(u^2 + 2gy) = w \,.\, dq\left(\frac{u^2}{2g} + y\right)$.

Hence the *total* loss of energy between the points where spilling begins and ends

$$= \int w \,.\, dq\left(\frac{u^2}{2g} + y\right) = \frac{wu^2}{2g}\int dq + w\int y \,.\, dq = \left(\frac{wu^2}{2g} + Y\right)kQ.$$

Overshot and pitch-back wheels do not work well in back-water, as they lift a greater or less weight of water in rising above the surface.

If the water-level in the race is liable to variation it is better to diminish the diameter of the wheel and design it so that it may never be immersed to a greater depth than 12 inches.

(*b*) *Effect of Impact.*—The head h' required to produce the velocity v with which the water reaches the wheel is theoretically $\frac{v_1^2}{2g}$; but as there is a loss of at least 5 per cent in the most perfect delivery, it is usual to take $h' = \nu\frac{v_1^2}{2g}$, an average value of ν being 1.1.

Let the water enter the bucket in the direction ac, Fig. 167. Take $ac = v_1$. The water now moves round with a velocity u (assumed the same as that of the division circle), and leaves the wheel with the same velocity. Take ab in the direction of the tangent to the division circle at the point of entrance $= u$. The component bc represents the relative velocity V of the water with respect to the bucket, and this velocity is wholly destroyed. ab must necessarily be parallel to the outer arm of the bucket, so that there may be no loss of shock at entrance. Then the impulsive effect

$$= \frac{wQ}{g}\left(\frac{v_1^2}{2} - \frac{V^2}{2} - \frac{u^2}{2}\right).$$

But

$$V^2 = v_1^2 + u^2 - 2v_1u \cos \gamma,$$

γ being the angle through which the water is deviated from its original direction at the point of entrance.

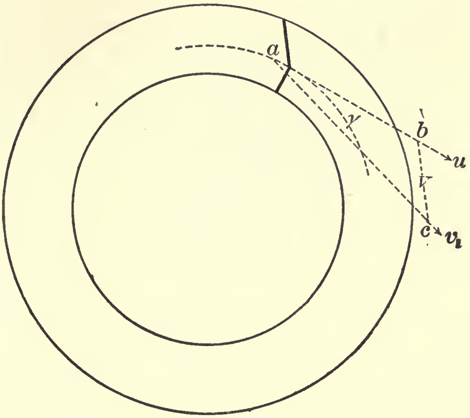


FIG. 167.

Hence the impulsive effect

$$= \frac{wQ}{g} u(v_1 \cos \gamma - u),$$

and the TOTAL USEFUL EFFECT

$$= wQ(h_1 + kh_2) + \frac{wQ}{g} u(v_1 \cos \gamma - u) - \text{loss due to journal friction.}$$

The loss due to journal friction

$$= \mu\{(1 + k)wQ + W\}\frac{\rho}{r_1} u,$$

ρ being the radius of the axle and W the weight of the wheel.

21. A **pitch-back** or **high breast** wheel is to be preferred to an overshot wheel when the surface-levels of the head- and tail-water are liable to very considerable variation.

In the pitch-back wheel the water is admitted by an adjustable sluice into the buckets on the same side as the supply-channel, Fig. 168. Thus the wheel revolves in the direction

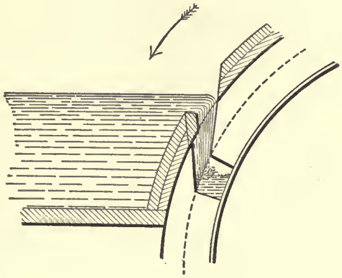


FIG. 168.

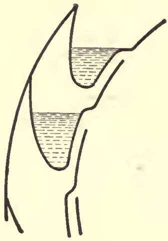


FIG. 169.

in which the water leaves, and the drowning of the wheel is prevented. Further, the buckets may be now ventilated, Fig. 169, and may therefore be placed closer together than in the unventilated overshot wheel.

The efficiency of the pitch-back is at least equal to that of the overshot.

22. **The Jet Reaction Wheel (Scotch Turbine).**—In this form of motor the water enters the centre of the wheel, spreads out radially in tubular passages, and issues from openings at the ends tangentially to the direction of rotation.

Fig. 170 represent the simplest wheel of this class. In England it is known as Barker's mill, and in Germany it is called Segner's water-wheel.

A reaction wheel may have several tubular passages, as in Fig. 172, and the vertical chamber XY may be cylindrical, rectangular, or conical.

Let r be the horizontal distance between the axis of an orifice and the axis of the vertical chamber.

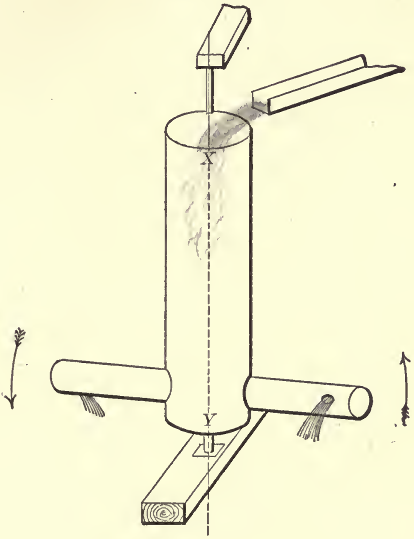
Let h be the head of water over the orifices when closed.

Let v be the velocity of efflux relatively to the tube when the orifices are open, and let V be the corresponding linear velocity of rotation at the centre of an orifice. Then

$$v^2 = c_v^2(V^2 + 2gh), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

c_v being the coefficient of velocity.

FIG. 170.



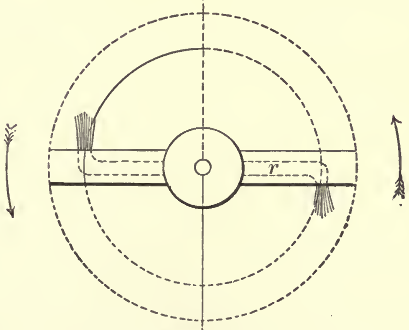


FIG. 171.

The absolute velocity of efflux $= v - V$.

The angular momentum of each pound of water $= \dfrac{v - V}{g} r$.

The useful work of each pound of water

$$= \frac{v - V}{g} r \frac{V}{r} = \frac{V}{g}(v - V).$$

The total work of each pound of water $= h$.

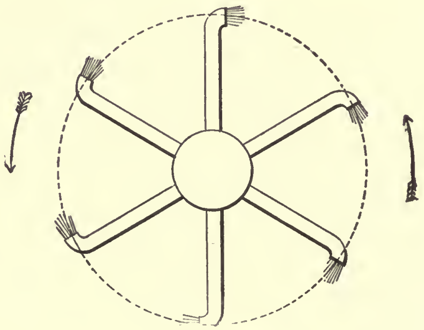


FIG. 172.

The efficiency

$$= \frac{\text{useful work}}{\text{total work}} = \frac{V(v - V)}{gh} = \frac{2V(v - V)}{\dfrac{v^2}{c_v^2} - V^2} = \eta, \text{ suppose.} \qquad (3)$$

The reaction $= \dfrac{\text{useful work}}{\text{linear velocity of rotation}} = \dfrac{v - V}{g}$.

For a maximum efficiency

$$d\eta = 0 = V\left(\frac{v^2}{c_v^2} - V^2\right)dv - 2V\left(v - V\right)\frac{v}{c_v^2}dv.$$

Hence

$$v^2 - 2vV + c_v^2 V^2 = 0,$$

and therefore

$$v = V(1 + \sqrt{1 - c_v^2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Experience indicates that the greatest efficiency corresponds to a speed of rotation equal to the velocity due to a head h, i.e., to a value of V given by

$$V^2 = 2gh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By equations (1) and (5)

$$v^2 = 4c_v^2 gh, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and therefore, by equations (4), (5), and (6),

$$c_v^2 = \frac{8}{9}, \quad \text{or} \quad c_v = .94. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence, by equations (3), (5), (6), and (7),

$$\text{the maximum efficiency} = \frac{2}{3}.$$

Thus one third of the head is lost, and of this amount the portion $\dfrac{(v - V)^2}{2g}\left(= \dfrac{h}{9}\right)$ is carried away by the effluent water. The portion $\dfrac{h}{3} - \dfrac{h}{9}\left(= \dfrac{2}{9}h\right)$ is lost in frictional resistance, etc.

Again,

$$\text{the efficiency} = \frac{V}{gh}\left(v - V\right) = \frac{V^2}{gh}\left\{ c_v\left(1 + \frac{2gh}{V^2}\right)^{\frac{1}{2}} - 1 \right\}$$

$$= \frac{V^2}{gh}\left\{ c_v\left(1 + \frac{gh}{V^2} - \text{terms cont'g higher powers of } \frac{1}{V}\right) - 1 \right\}.$$

The efficiency therefore increases with V, but the value of V is limited by the practical consideration that, even at moderately high speeds, so much of the head is absorbed by friction as to sensibly diminish the efficiency.

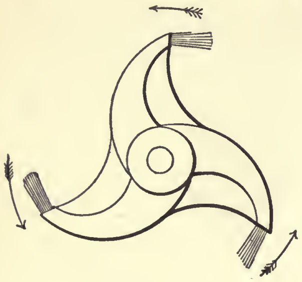


Fig. 173.

The serious practical defects of this wheel are that its speed is most unstable and that it admits of no efficient system of regulation for a varying supply of water.

The Scotch or Whitelaw's turbine, Fig. 173, excepting in the curved arms, does not differ essentially from the reaction wheel just considered.

23. Reaction and Impulse Turbines.—All turbines belong to one of two classes, viz., *Reaction Turbines* and *Impulse Turbines*, and are designed to utilize more or less of the available energy of a moving mass of water.

In a *reaction* turbine a portion of the available energy is converted into kinetic energy at the inlet surface of the wheel. The water enters the wheel-passages formed by suitably curved vanes, and acts upon these vanes by pressure, causing the wheel to rotate. The proportions of the turbine are such that there is a particular pressure (hence the term pressure-turbine) at the inlet surface corresponding to the best normal condition of working. Any variation from this pressure, caused, e.g., by the partial closure of the passages through which the water passes to the wheel, changes the working conditions and diminishes the efficiency. In order to avoid such a variation of pressure, it is essential that there should be a continuity of flow in every part of the turbine; the wheel-passages should be kept completely filled with water, and therefore must receive the water simultaneously. Such turbines are said to have complete admission. The admission is partial when the water is received over a portion of the inlet surface only.

In an *impulse* (Girard) turbine, Figs. 174, 175, the energy of the water is wholly converted into kinetic energy at the inlet surface. Thus the water enters the wheel with a velocity due to the total available head and therefore without pressure, is received upon the curved vanes, and imparts to the wheel the whole of its energy by means of the impulse due to the

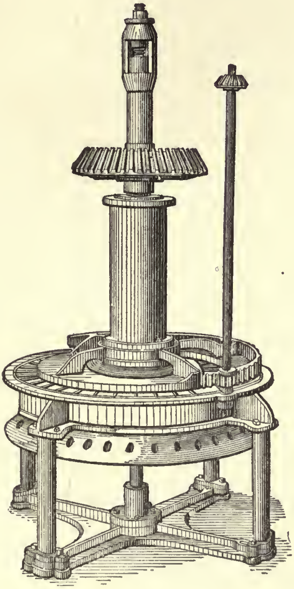


FIG. 174.
Girard Turbine for Low Falls.

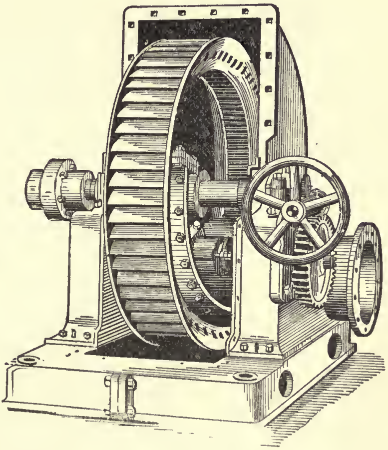


FIG. 175.
Girard Turbine for High Falls.

gradual change of momentum. Care must be taken to ensure that the water may be freely deviated on the curved vanes, and hence such turbines are sometimes called turbines with free deviation. For this reason the water-passages should never be completely filled, and the water should flow through under a pressure which remains constant. In order to ensure an unbroken flow through the wheel-passages and that no eddies are formed at the backs of the vanes, ventilating holes are arranged in the wheel sides, Fig. 177. Figs. 176 and 177 also show the relative path *AB* and the absolute path *CD* traversed by the water in an inward-flow and a downward-flow turbine.

If there is a sufficient head, the wheel may be placed clear above the tail-water, when the stream will be at all times under atmospheric pressure. With low falls the wheel may be placed in a casing supplied with air from an air-pump by which the surface of the water may be kept at an invariable level below the outlet orifices, which is essential for perfectly free deviation. While the wheel-passages of a reaction turbine should be kept completely filled with water, no such restriction is necessary with an impulse turbine. The supply may be partially checked and the water may be received by one or more vanes without affecting the efficiency. Thus the dimensions of an impulse turbine may vary between very wide

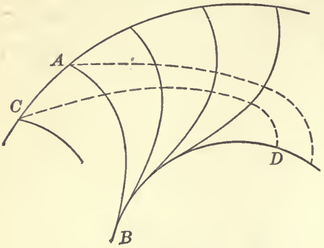


FIG. 176.

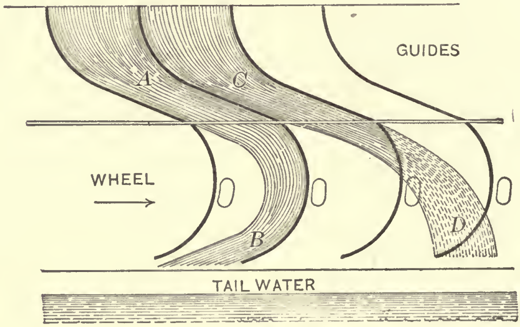


FIG. 177.

limits, so that for high falls with a small supply, a comparatively large wheel with low speed may be employed. The speed of a reaction turbine under similar conditions would be disadvantageously great, and any considerable increase of the diameter would largely increase the fluid friction and would also render the proper proportioning of the vane-angles almost impracticable. Impulse turbines may have complete or partial admission, while in reaction turbines the admission should be always complete, as in Fig. 178, which shows the relative path AB and absolute path CD traversed by the water. When there is an ample supply of water the reaction turbine is usually to be preferred, but on very high falls its speed

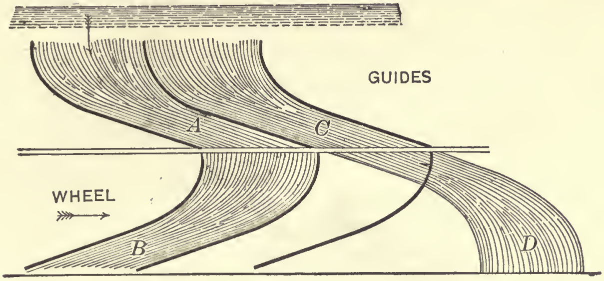


FIG. 178.

becomes inconveniently great and it is then better to adopt a turbine of the impulse type. The diameter of the wheel can then be increased and the speed proportionately diminished.

The *Hurdy-gurdy* is the name popularly given to an impulse wheel which was introduced into the mining districts of California about the year 1865. Around the periphery of the wheel is arranged a series of flat iron buckets, about 4 to 6 in. in width, which are struck normally by a jet of water often not more than three eighths of an inch in diameter. Theoretically, the efficiency of such an arrangement cannot exceed 50 per cent (Art. 7), while in practice it rarely reaches 40 per cent. The best speed of the wheel, in accordance with both theory and practice, is one half of that of the jet. Although the efficiency is so low, the wheel found great favor for many reasons. Any required speed could be obtained by a suitable choice of diameter; the plane of the wheel could be placed in any convenient position; the wheel could be cheaply constructed and was largely free from liability to accident. Hence it was of the utmost importance to increase, if possible, the efficiency of a wheel possessing such advantages. Obviously a first step was to substitute cups for the flat buckets, the immediate result necessarily being a very large increase in the efficiency. This was increased still further by the adoption of double buckets, Fig. 179, that is, curved buckets divided in the middle so that the water is equally deflected on both sides.

Thus developed, the wheel is widely and most favorably known as the Pelton wheel, Fig. 179. Its efficiency is at least 80 per cent, and it is claimed that it often rises above 90 per cent. The power of the wheel does not depend upon its diameter, but upon the available quantity and head of water. The water passes to the wheel through one or more nozzles,

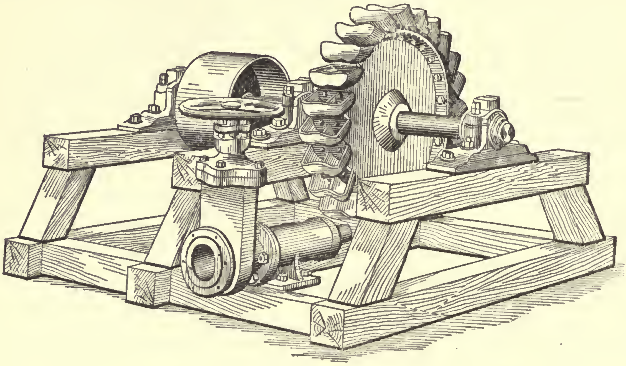


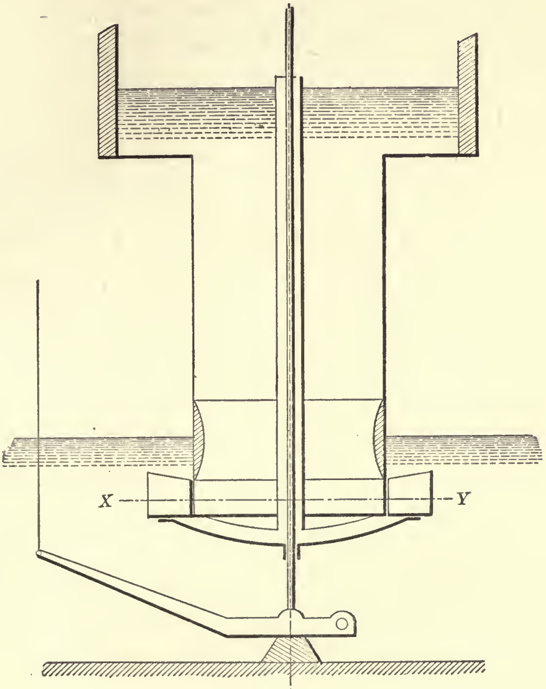
FIG. 179.

having tips bored to suit any required delivery. These tips are screwed into the nozzles and can be easily and rapidly replaced by others of larger or smaller size, so that the Pelton is especially well adapted for a varying supply of water. It is claimed that in this manner the power may be varied from a maximum down to 25 per cent of the same without appreciable loss of efficiency.

The character of the construction of turbines has led to their being classified as (1) Radial-flow turbines; (2) Axial-flow turbines; (3) Mixed-flow turbines.

In *Radial-flow* turbines the water flows through the wheel in a direction at right angles to the axis of rotation and approximately radial. The two special types of this class are the *Outward-flow* turbine, invented by Fourneyron, and the *Inward-flow* or *Vortex* turbine, invented by James Thomson. In the former, Figs. 180 and 181, the water enters a cylindrical chamber and is led by means of fixed guide-blades outwards from the axis. It is distributed over the inlet-surface, passes through the curved passages of an annular wheel closely sur-

FIG. 180.



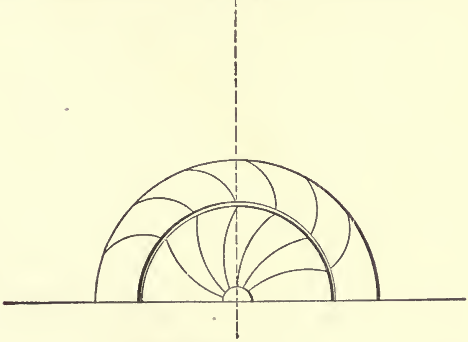


FIG. 181.

rounding the chamber, and is finally discharged at the outer surface. The wheel works best when it is placed clear above the tail-water. A serious practical defect is the difficulty of constructing a suitable sluice for regulating the supply over the inlet-surface. Fourneyron was led to the design of this turbine by observing the excessive loss of energy in the ordinary Scotch turbine, or reaction wheel, and introduced guide-blades in order to give the water an initial forward velocity and thus cause a diminution of the velocity of the water leaving the outlet-surface.

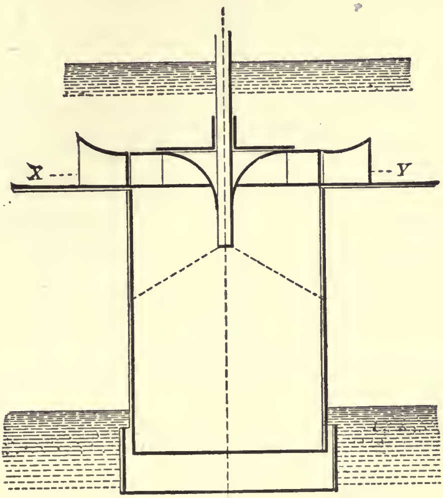
In the *Inward-flow* or *Vortex* turbine, Figs. 182 and 183, the wheel is enclosed in an annular space, into which the water flows through one or more pipes, and is usually distributed over the inlet-surface of the wheel by means of four guide-blades. The water enters the wheel, flows towards the space around the axis, and is there discharged. This turbine possesses the great advantage that there is ample space outside the wheel for a perfect system of regulating-sluices.

Axial-flow turbines, Figs. 184, are also known as *Parallel* and *Downward-flow* turbines and are sometimes called by the names of the inventors, Jonval and Fontaine. In these the water passes downward through an annular casing in a direction parallel to the axis of rotation, and is distributed by means of guide-blades over the inlet-surface of an adjacent wheel. It enters the wheel-passages and is finally discharged vertically, or nearly so, at the outlet-surface. The sluice regulations are worse even than in the case of an outward-flow turbine, but there is this advantage, that the turbine may be placed either below the tail-water, or, if supplied with a suction-pipe, at any point not exceeding 30 ft. above the tail-water.

If a turbine is designed so that the pressure at the clearance between the casing and the wheel is nil, and with curved passages in the form of a freely deviated stream, it becomes what is called a *Limit* turbine. In its normal condition of working it is an Impulse turbine, but when drowned, it is a Reaction turbine, with a small pressure at the clearance. For moderate falls with a varying supply its average efficiency is higher than that of a pressure turbine.

The *Mixed- or Combined-flow* (Schiele) turbine is a combination of the radial and axial types. The water enters in a nearly radial direction and leaves in a direction approximately

FIG. 182.



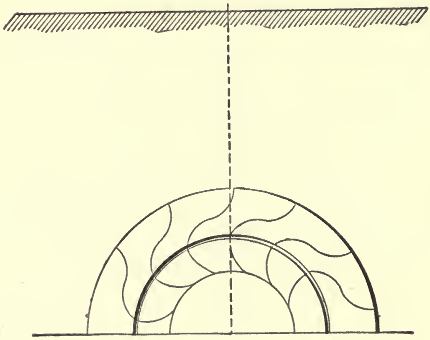


FIG. 183.

parallel to the axis of rotation. This type of turbine admits of a good mode of regulation and is cheap to construct.

24. **Theory of Turbines** (Figs. 185 to 188).—Denote in-

ward-flow, outward-flow, and axial-flow turbines by I. F., O. F., and A. F., respectively.

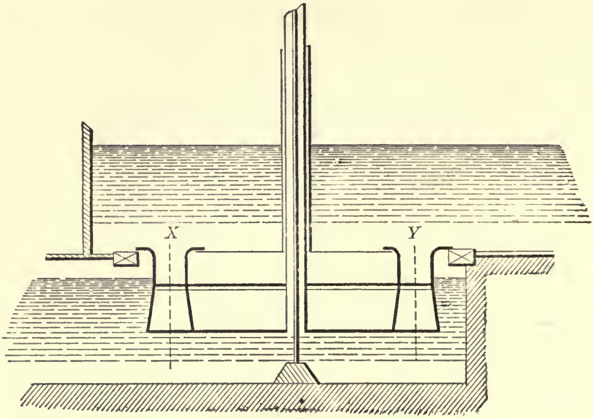


FIG. 184.

Let r_1, r_2 be the radii of the wheel inlet and outlet surfaces of an I. F. or O. F.

Let r_1, r_2 be the outer and inner radii of the wheel inlet-surface of an A. F.

Let R be the mean radius $\left(= \frac{r_1 + r_2}{2}\right)$ of an A. F., assumed constant throughout.

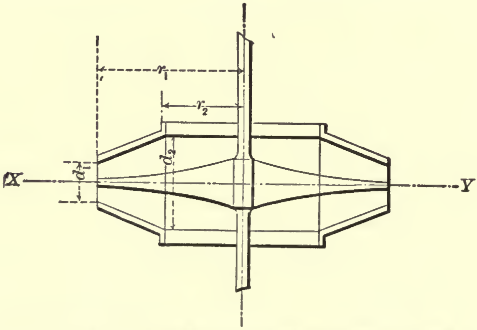


FIG 185.—Section of an inward-flow turbine.

Let A_1, A_2 be the areas of the wheel inlet and outlet orifices.

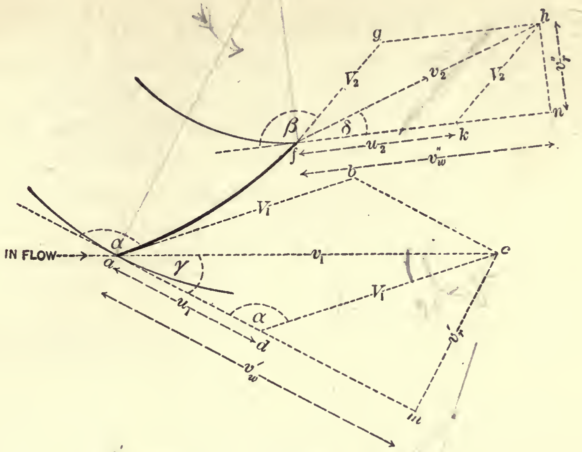


FIG. 186.—Enlarged portion of the section through XY, Fig. 185.

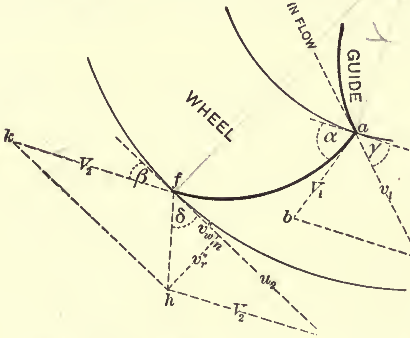


FIG. 187.—Enlarged portion of a section through XY, Fig. 180, of an outward-flow turbine.

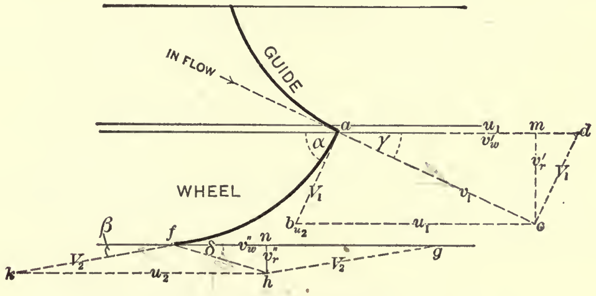


FIG. 188.—Enlarged portion of a cylindrical section XY, Fig. 184, of a downward-flow turbine developed in the plane of the paper.

Let d_1, d_2 be the depths of the same in an I. F. or O. F.

Let d_1, d_2 be the widths of the same in an A. F.

Let h be the thickness of the wheel in an A. F.

Let H_1 be the effective head over the inlet-surface of the wheel. This is the total head over the inlet-surface diminished by the head consumed in frictional resistance in the supply-channel, and by the head lost in bends, sudden changes of section, etc.

Let H_2 be the fall from the outlet-surface to the surface of the water in the tail-race. If the turbine is submerged, then H_2 is negative.

Let v_1, v_2 be the absolute velocities of the water at the inlet- and outlet-surfaces.

Let u_1, u_2 be the absolute velocities of the inlet- and outlet-surfaces.

Let V_1, V_2 be the velocities of the water relatively to the wheel, at the inlet- and outlet-surfaces.

Let ω be the angular velocity of the wheel.

Let the water enter the wheel in the direction ac, making an angle γ with the tangent ad. Take ac to represent v_1 and ad to represent u_1. Complete the parallelogram bd. The side ab represents V_1, and in order that there may be *no shock at entrance*, ab must be tangential to the vane at a. Again, at f draw fg, a tangent to the vane, and fk, a tangent to the wheel's periphery.

Take fg and fk to represent V_2 and u_2 respectively. Complete the parallelogram gk. The diagonal fh must represent in direction and magnitude the absolute velocity v_2 with which the water leaves the wheel. Let the angle $hfk = \delta$.

The *tangential* component of the velocity of the water as it enters or leaves the wheel is termed the *velocity of whirl*, and the *radial* component the *velocity of flow*. Denote these components respectively by

v_ω', v_r' at the inlet-surface;

v_ω'', v_r'' at the outlet-surface.

Let the angle $bad = 180° - \alpha$.
Let the angle $gfk = 180° - \beta$.
Draw cm perpendicular to ad, and hn to fk.

Then at the inlet-surface,

$$v_\omega' = v_1 \cos \gamma = ac \cos \gamma = am = ad \pm dm = u_1 - V_1 \cos \alpha; \quad (1)$$
$$v_r' = v_1 \sin \gamma = cm = V_1 \sin \alpha; \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

and at the outlet-surface

$$v_\omega'' = v_2 \cos \delta = fn = fk \pm kn = u_2 - V_2 \cos \beta; \ . \quad (3)$$
$$v_r'' = v_2 \sin \delta = hn = V_2 \sin \beta. \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4)$$

Let Q be the volume of water passed per second. Then

in an I. F. or O. F.	in an A. F.
$vr'A_1 = v_r'2\pi r_1 d_1 = Q$ $= v_r''2\pi r_2 d_2 = vr''A_2.$ (5)	$vr'A_1 = vr'2\pi Rd_1 = Q$ $= vr''2\pi Rd_2 = vr''A_2.$. (5)

In equations (5) the thickness of the vanes has been disregarded. If θ is the angle between the vane, of thickness BC, and the wheel's periphery AB, then the space occupied by the vane along the wheel's periphery is $AB = BC \operatorname{cosec} \theta$.

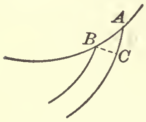


FIG. 189.

Let n be the number of the guide-vanes and t their thickness.

Let n_1 be the number of the wheel-vanes and t_1, t_2 their thickness at the inlet- and outlet-surfaces, respectively.

Then, in a radial-flow turbine,

$$A_1 = \tfrac{9}{10} d_1 \{2\pi r_1 - nt \operatorname{cosec} \gamma - n_1 t_1 \operatorname{cosec} \alpha\} \ . \ . \quad (6)$$

and

$$A_2 = \tfrac{9}{10} d_2 \{2\pi r_2 - n_1 t_2 \operatorname{cosec} \beta\}, \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

$\frac{9}{10}$ being a fraction depending on practical considerations.

In an axial-flow turbine R is to be substituted for r_1 and r_2 in the values of A_1 and A_2.

n_1 may be made equal to $n + 1$ or $n + 2$.

Again, as the water flows through the wheel its angular momentum relatively to the axis of rotation is changed from $\frac{wQ}{g} r_1 v_\omega{}'$ at the inlet- to $\frac{wQ}{g} r_2 v_\omega{}''$ at the outlet-surface.

Hence, if T is the effective work done by the water on the turbine, and ω the angular velocity of the turbine,

in an I. F. or O. F.

$$T = \frac{wQ}{g}(v_w{}' r_1 - v_w{}'' r_2)\omega$$

$$= \frac{wQ}{g}(v_w{}' u_1 - v_w{}'' u_2), \quad . \quad (8)$$

since

$$\frac{u_1}{r_1} = \frac{u_2}{r_2} = \omega, \quad . \quad . \quad (9)$$

and the hydraulic efficiency

$$= \frac{T}{wQH_1} = \frac{v_w{}' u_1 - v_\omega{}'' u_2}{gH_1}. \quad (10)$$

in an A. F.

$$T = \frac{wQ}{g}(v_w{}' R - v_w{}'' R)\omega$$

$$= \frac{wQ}{g}(v_w{}' - v_w{}'')u_1, \quad . \quad . \quad (8)$$

since

$$\frac{u_1}{R} = \frac{u_2}{R} = \omega, \quad . \quad . \quad (9)$$

and the hydraulic efficiency

$$= \frac{T}{wQ(H_1 + h)} = \frac{(v_w{}' - v_w{}'')u_1}{g(H_1 + h)}. \quad (10)$$

Equation 10 is the fundamental equation upon which the whole design of turbines depends.

From the triangle abc,

$$V_1^2 = v_1^2 + u_1^2 - 2v_1u_1 \cos \gamma, \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\frac{V_1}{v_1} = \frac{\sin \gamma}{\sin \alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From the triangle fkh,

$$v_2^2 = u_2^2 + V_2^2 - 2u_2V_2 \cos \beta. \quad . \quad . \quad . \quad . \quad (13)$$

Again, if $\frac{p_1}{w}, \frac{p_2}{w}$ are the pressure-heads at the inlet- and outet-surfaces of the wheel of a REACTION TURBINE,

$$\frac{v_1^2}{2g} = H_1 - \frac{p_1 - p_2}{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In an IMPULSE TURBINE the water is under atmospheric pressure only, and therefore

$$\frac{v_1^2}{2g} = H_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

To make allowance for hydraulic resistances $k_1\frac{v_1^2}{2g}$ may be substituted for $\frac{v_1^2}{2g}$ in equations 14 and 15, a mean value of k_1 being $\frac{10}{9}$.

Applying Bernouilli's theorem to the filament from a to f, and taking account of the head $\frac{u_2^2 - u_1^2}{2g}$ due to centrifugal force—

In a *reaction* I. F. or O. F.

$$\frac{p_1}{w_1} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} - \frac{u_2^2 - u_1^2}{2g}, \quad (16)$$

and therefore

$$\frac{V_2^2 - V_1^2}{2g} = \frac{p_1 - p_2}{w} + \frac{u_2^2 - u_1^2}{2g}. \quad (17)$$

In words, the change of energy from a to f = work due to pressure + work due to centrifugal force.

In an *impulse* I. F. or O. F.

$$\frac{V_2^2 - V_1^2}{2g} = \frac{u_2^2 - u_1^2}{2g}. \quad (18)$$

In a *reaction* A. F.

$$h + \frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g}, \quad (16)$$

and therefore

$$\frac{V_2^2 - V_1^2}{2g} = \frac{p_1 - p_2}{w} + h. \quad (17)$$

In words, the change of energy from a to f = work due to pressure + work due to gravity. The work due to centrifugal force is evidently nil.

In an *impulse* A. F.

$$\frac{V_2^2 - V_1^2}{2g} = h. \quad . \quad . \quad (18)$$

To make allowance for hydraulic resistances $k_2 V_2^2$ may be substituted for V_2^2 in equations 17 and 18, a mean value of k_2 being 1.1.

For a maximum effect the water should leave the wheel without velocity, i.e., v_2 should be nil. But this value of v_2 is impracticable, as no water could then pass through the wheel. It is usual either to make the velocity of whirl (v_w'') at the outlet-surface equal to nil, or to make the relative (V_2) and circumferential (u_2) velocities at the outlet-surface, equal and opposite. In each case v_2 is small. First let

$$v_w'' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

so that the water leaves the wheel with a much-reduced velocity in a direction normal to the outlet-surface. Thus (Fig. 194),

$$\delta = 90^\circ ; \ v_2 = v_r'',$$

and

u_2 $$u_2 = v_2 \cot \beta = V_2 \cos \beta. \qquad (20)$$

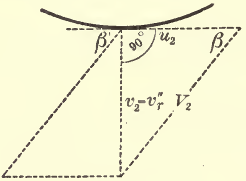


FIG. 189.

Also, by equations 2, 4, 5, and 20—

In an I. F. or O. F.

$$\frac{Q}{2\pi} = v_1 \sin \gamma \, r_1 d_1 = V_2 \sin \beta r_2 d_2$$
$$= u_2 \tan \beta r_2 d_2. \quad (21)$$

In an A. F.

$$\frac{Q}{2\pi R} = v_1 \sin \gamma d_1 = V_2 \sin \beta d_2$$
$$= u_2 \tan \beta d_2. \quad (21)$$

The following results are now easily obtained:

In an I. F. or O. F.:

Relation between the Vane-angles.

By equations 9 and 21, and from the triangle *acd*,

$$\frac{r_1 d_1 \sin \gamma}{r_2 d_2 \tan \beta} = \frac{u_2}{v_1} = \frac{r_2}{r_1}\frac{u_1}{v_1}$$
$$= \frac{r_2}{r_1} \frac{\sin(\alpha + \gamma)}{\sin \alpha}, \quad (22)$$

In an A. F.:

Relation between the Vane-angles.

By equations 9 and 21, and from the triangle *acd*,

$$\frac{d_1 \sin \gamma}{d_2 \tan \beta} = \frac{u_2}{v_1} = \frac{u_1}{v_1}$$
$$= \frac{\sin(\alpha + \gamma)}{\sin \alpha}, \quad (22)$$

and therefore

$$\frac{r_1^2 d_1}{r_2^2 d_2} \cot \beta = \cot \alpha + \cot \gamma. \quad (23)$$

and therefore

$$\frac{d_1}{d_2} \cot \beta = \cot \alpha + \cot \gamma. \quad (23)$$

IN REACTION TURBINES.

In an I. F. or O. F.:

Speed of Turbine.

By equations 1, 10, and 19,

$$wQ\left(H_1 - \frac{v_2^2}{2g}\right) = \text{effective work}$$

$$= \frac{wQ}{g} v_w' u_1 = \frac{wQ}{g} u_1 v_1 \cos \gamma, \quad (24)$$

and therefore

$$H_1 - \frac{v_2^2}{2g} = \frac{u_1 v_1}{g} \cos \gamma. \quad (25)$$

Hence, by equations 20, 22, and 25,

$$u_2^2 = \frac{r_2^2}{r_1^2} u_1^2$$

$$= \frac{2gH_1 \cot \beta}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad (26)$$

Note.—If the water is to have no velocity of whirl (v_w') relatively to the wheel at the inlet-surface, then

$$u_1 - v_w' = 0, \quad (27)$$

and therefore

$$\alpha = 90^\circ$$

and

$$\tan \gamma = \frac{V_1}{u_1} = \frac{v_r'}{v_w'}.$$

Also, the efficiency

$$= \frac{v_w' u_1}{gH_1} = \frac{u_1^2}{gH_1},$$

and thus

$$u_1^2 = gH_1 \quad (28)$$

In an A. F.:

Speed of Turbine.

By equations 1, 10, and 19,

$$wQ\left(H_1 + h - \frac{v_2^2}{2g}\right) = \text{effective work}$$

$$= \frac{wQ}{g} v_w' u_1 = \frac{wQ}{g} u_1 v_1 \cos \gamma, \quad (24)$$

and therefore

$$H_1 + h - \frac{v_2^2}{2g} = \frac{u_1 v_1}{g} \cos \gamma. \quad (25)$$

Hence, by equations 20, 22, and 25,

$$u_2^2 = u_1^2$$

$$= \frac{2g(H_1 + h) \cot \beta}{\tan \beta + 2\frac{d_1}{d_1} \cot \gamma}. \quad (26)$$

Note.—If the water is to have no velocity of whirl (v_w') relatively to the wheel at the inlet-surface, then

$$u_1 - v_w' = 0, \quad (27)$$

and therefore

$$\alpha = 90^\circ$$

and

$$\tan \quad = \frac{V_1}{u_1} = \frac{v_r'}{v_w'}.$$

Also, the efficiency

$$= \frac{v_w' u_1}{g(H_1 + h)} = \frac{u_1^2}{g(H_1 + h)},$$

an thus

$$u_1^2 = g(H_1 + h) \quad (28)$$

if the efficiency is perfect.

Usually the efficiency of good turbines is about .85.

Velocity of Efflux.

$$v_2^2 = u_2^2 \tan^2 \beta$$

$$= \frac{2gH_1 \tan \beta}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad . \quad (29)$$

Useful Work

$$= wQ\left(H_1 - \frac{v_2^2}{2g}\right)$$

$$= wQH_1 \frac{2\frac{d_2}{d_1} \cot \gamma}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad (30)$$

Efficiency

$$= \frac{2\frac{d_2}{d_1} \cot \gamma}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad . \quad (31)$$

Amount Q of water passing through turbine

$$= 2\pi r_2 d_2 V_2 \sin \beta = 2\pi r_2 d_2 u_2 \tan \beta$$

$$= 2\pi r_2 d_2 \sqrt{\frac{2gH_1 \tan \beta}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}}. \quad (33)$$

The pressure-head at the inlet-surface

$$= \frac{p_1}{w} = \frac{p_2}{w} + H_1 - \frac{v_1^2}{2g}$$

$$= \frac{p_2}{w}$$

$$+ H_1 \left\{ 1 - \frac{r_2^2 d_2^2}{r_1^2 d_1^2} \frac{\tan \beta}{\sin^2 \gamma (\tan \beta + 2\frac{d_2}{d_1} \cot \gamma)} \right\} \quad (34)$$

if the efficiency is perfect.

Usually the efficiency of good turbines is about .85.

Velocity of Efflux.

$$v_2^2 = u_2^2 \tan^2 \beta$$

$$= \frac{2g(H_1 + h) \tan \beta}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad . \quad (29)$$

Useful Work

$$= wQ\left(H_1 + h - \frac{v_2^2}{2g}\right)$$

$$= wQ(H_1 + h) \frac{2\frac{d_2}{d_1} \cot \gamma}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad (30)$$

Efficiency

$$= \frac{2\frac{d_2}{d_1} \cot \gamma}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}. \quad . \quad (31)$$

Amount Q of water passing through turbine

$$= 2\pi R d_2 V_2 \sin \beta = 2\pi R d_2 u_2 \tan \beta$$

$$= 2\pi R d_2 \sqrt{\frac{2g(H_1 + h) \tan \beta}{\tan \beta + 2\frac{d_2}{d_1} \cot \gamma}}. \quad (33)$$

The pressure-head at the inlet-surface

$$= \frac{p_1}{w} = \frac{p_2}{w} + H_1 - \frac{v_1^2}{2g}$$

$$= \frac{p_2}{w} - h +$$

$$(H_1 + h) \left\{ 1 - \frac{d_2^2}{d_1^2} \frac{\tan \beta}{\sin^2 \gamma (\tan \beta + 2\frac{d_2}{d_1} \cot \gamma)} \right\} (34)$$

When the turbine is working freely in space above the surface of the tail-water, there will be no inflow of air if $p_1 > p_2$, i.e., if

$$1 > \frac{r_2^2 d_2^2}{r_1^2 d_1^2} \frac{\tan\beta}{\sin^2\gamma(\tan\beta + 2\frac{d_2}{d_1}\cot\gamma)}.$$

If the turbine is drowned with a head h' of water over the outlet-surface, there will be no back-flow of water if

$$\frac{p_1}{w} > \frac{p_2}{w} + h',$$

that is, if

$$\frac{H_1 - h'}{H_1} > \frac{r_2^2 d_2^2}{r_1^2 d_1^2} \frac{\tan\beta}{\sin^2\gamma(\tan\beta + 2\frac{d_2}{d_1}\cot\gamma)}.$$

When the turbine is working freely in space above the surface of the tail-water, there will be no inflow of air if $p_1 > p_2$, i.e., if

$$\frac{H_1}{H_1 + h} > \frac{d_2^2}{d_1^2} \frac{\tan\beta}{\sin^2\gamma(\tan\beta + 2\frac{d_2}{d_1}\cot\gamma)}.$$

If the turbine is drowned with a head h' of water over the outlet-surface there will be no back-flow of water if

$$\frac{p_1}{w} > \frac{p_2}{w} + h',$$

that is, if

$$\frac{H_1 - h'}{H_1 + h} > \frac{d_2^2}{d_1^2} \frac{\tan\beta}{\tan\beta + 2\frac{d_2}{d_1}\cot\gamma}.$$

IN IMPULSE TURBINES.

In an I. F. or O. F.:

Speed of Turbine.

Since

$$v_1^2 = 2gH_1, \quad . \quad . \quad (35)$$

by equation 22,

$$\frac{r_1^2 d_1^2 \sin^2\gamma}{r_2^2 d_2^2 \tan^2\beta} = \frac{u_2^2}{v_1^2} = \frac{r_2^2}{r_1^2}\frac{u_1^2}{v_1^2},$$

and therefore

$$u_2^2 = \frac{r_2^2}{r_1^2}u_1^2 = 2gH_1\frac{r_1^2 d_1^2 \sin^2\gamma}{r_2^2 d_2^2 \tan^2\beta}. \quad (36)$$

Velocity of Efflux.

$$v_2^2 = u_2^2 \tan^2\beta$$

$$= 2gH_1\frac{r_1^2 d_1^2}{r_2^2 d_2^2}\sin^2\gamma. \quad (37)$$

In an A. F.:

Speed of Turbine.

Since

$$v_1^2 = 2gH_1, \quad . \quad . \quad (35)$$

by equation 22,

$$\frac{d_1^2 \sin^2\gamma}{d_2^2 \tan^2\beta} = \frac{u_2^2}{v_1^2} = \frac{u_1^2}{v_1^2},$$

and therefore

$$u_2^2 = u_1^2 = 2gH_1\frac{d_1^2 \sin^2\gamma}{d_2^2 \tan^2\beta}. \quad (36)$$

Velocity of Efflux.

$$v_2^2 = u_2^2 \tan^2\beta$$

$$= 2gH_1\frac{d_1^2}{d_2^2}\sin^2\gamma. \quad . \quad (37)$$

Useful Work	*Useful Work*
$= wQ\left(H_1 - \frac{v_2^2}{2g}\right)$	$= wQ\left(H_1 + h - \frac{v_2^2}{2g}\right)$
$= wQH_1\left(1 - \frac{r_1^2 d_1^2}{r_2^2 d_2^2}\sin^2\gamma\right)$. (38)	$= wQ\left(H_1 + h - H_1\frac{d_1^2}{d_2^2}\sin^2\gamma\right)$. (38)
Efficiency	*Efficiency*
$= 1 - \frac{r_1^2 d_1^2}{r_2^2 d_2^2}\sin^2\gamma = \eta$. (39)	$= 1 - \frac{H_1}{H_1 + h}\frac{d_1^2}{d_2^2}\sin^2\gamma = \eta$. (39)

Second, let

$$u_2 = V_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

so that the water again leaves the wheel with a much-reduced velocity. Evidently also

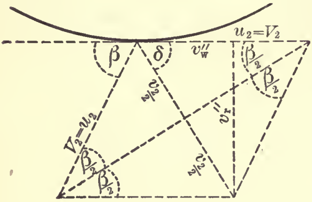


FIG. 190.

$$\left.\begin{aligned} \delta = 90° - \frac{\beta}{2}; \quad v_r'' = u_2 \sin\beta; \\ v_w'' = 2u_2\sin^2\frac{\beta}{2}; \end{aligned}\right\} \quad (41)$$

and

$$v_2 = 2u_2\cos\delta = 2u_2\sin\frac{\beta}{2} = 2V_2\sin\frac{\beta}{2}. \quad . \quad . \quad (42)$$

Also, by eqs. 2, 4, 5, and 42—

In an I. F. or O. F.	In an A. F.
$\frac{Q}{2\pi} = v_1 \sin\gamma r_1 d_1 = V_2 \sin\beta\, r_2 d_2$	$\frac{Q}{2\pi} = v_1 \sin\gamma\, R d_1 = V_2 \sin\beta\, R d_2$
$= u_2 \sin\beta\, r_2 d_2$. (43)	$= u_2 \sin\beta\, R d_2$. (43)

The following results are now easily obtained:

In an I. F. or O. F.:

Relation between the Vane-angles.

By eqs. 9 and 43 and from the triangle *acd*

$$\frac{r_1 d_1 \sin\gamma}{r_2 d_2 \sin\beta} = \frac{u_2}{v_2} = \frac{r_2}{r_1}\frac{u_1}{v_1}$$

$$= \frac{r_2}{r_1}\frac{\sin(\alpha+\gamma)}{\sin\alpha}, \quad . \quad (44)$$

and therefore

$$\frac{r_1^2 d_1}{r_2^2 d_2}\operatorname{cosec}\beta = \cot\alpha + \cot\gamma. \quad (45)$$

Relation between the Vane-angles.

By eqs. 9 and 43 and from the triangle *acd*

$$\frac{d_1 \sin\gamma}{d_2 \sin\beta} = \frac{u_2}{v_1} = \frac{u_1}{v_1}$$

$$= \frac{\sin(\alpha+\gamma)}{\sin\alpha}, \quad (44)$$

and therefore

$$\frac{d_1}{d_2}\operatorname{cosec}\beta = \cot\alpha + \cot\gamma. \quad (45)$$

IN REACTION TURBINES.

In an I. F. or O. F.:

Speed of Turbine.

By eqs. 14, 17, and 40

$$u_1 v_1 \cos\gamma = gH_1 = u_1 v_w'. \quad (46)$$

Also,

$$\frac{u_1}{v_1} = \frac{\sin(\alpha+\gamma)}{\sin\alpha}.$$

Hence,

$$u_1^2 = gH_1\frac{\sin(\alpha+\gamma)}{\sin\alpha\cos\gamma}$$

$$= gH_1(1 + \cot\alpha\tan\gamma)$$

$$= gH_1\frac{r_1^2 d_1}{r_2^2 d_2}\tan\gamma\operatorname{cosec}\beta. \quad (47)$$

Note.—If the velocity of whirl (v_w') relatively to the wheel at the inlet-surface is to be nil,

$$u_1 - v_w' = 0, \quad . \quad . \quad (48)$$

and then

$$u_1^2 = v_w'^2 = gH_1. \quad . \quad (49)$$

In an A. F.:

Speed of Turbine.

By eqs. 14, 17, and 40

$$u_1 v_1 \cos\gamma = g(H_1 + h) = u_1 v_w'. \quad (46)$$

Also,

$$\frac{u_1}{v_1} = \frac{\sin(\alpha+\gamma)}{\sin\alpha}.$$

Hence

$$u_1^2 = g(H_1 + h)\frac{\sin(\alpha+\gamma)}{\sin\alpha\cos\gamma}$$

$$= g(H_1 + h)(1 + \cot\alpha\tan\gamma)$$

$$= g(H_1 + h)\frac{d_1}{d_2}\tan\gamma\operatorname{cosec}\beta. \quad (47)$$

Note.—If the velocity of whirl (v_w') relatively to the wheel at the inlet-surface is to be nil,

$$u_1 - v_w' = 0, \quad . \quad . \quad (48)$$

and then

$$u_1^2 = v_w'^2 = g(H_1 + h). \quad (49)$$

Velocity of Efflux.

By equations 42 and 47

$$v_2^2 = 4u_2^2 \sin^2 \frac{\beta}{2}$$

$$= 2gH_1 \frac{d_1}{d_2} \tan \frac{\beta}{2} \tan \gamma. \quad (50)$$

Useful Work

$$= wQ\left(H_1 - \frac{v_2^2}{2g}\right)$$

$$= wQH_1\left(1 - \frac{d_1}{d_2} \tan \frac{\beta}{2} \tan \gamma\right). \quad (51)$$

Efficiency

$$= 1 - \frac{v_2^2}{2gH_1} = 1 - \frac{d_1}{d_2} \tan \frac{\beta}{2} \tan \gamma. \quad (52)$$

Amount Q of Water passing through Turbine

$$= 2\pi r_2 d_2 v_r'' = 2\pi r_2 d_2 V_2 \sin \beta$$

$$= 2\pi r_2 d_2 u_2 \sin \beta$$

$$= 2\pi r_2 \sqrt{gH_1 d_1 d_2 \tan \gamma \sin \beta}. \quad (53)$$

Pressure-head at Inlet-surface

$$= \frac{p_1}{w} = \frac{p_2}{w} + H_1 - \frac{v_1^2}{2g}$$

$$= \frac{p_2}{w} + H_1\left(1 - \frac{r_2^2 d_2}{r_1^2 d_1} \frac{\sin \beta}{\sin 2\gamma}\right), \quad (54)$$

by equations 44 and 47.

When the turbine is working freely in space there will be no inflow of air if $p_1 > p_2$, i.e., if

$$1 > \frac{r_2^2 d_2}{r_1^2 d_1} \frac{\sin \beta}{\sin 2\gamma}.$$

When the turbine is drowned, with a head h' of water over the outlet-surface,

Velocity of Efflux.

By equations 42 and 47

$$v_2^2 = 4u_2^2 \sin^2 \frac{\beta}{2}$$

$$= 2g(H_1 + h)\frac{d_1}{d_2} \tan \frac{\beta}{2} \tan \gamma. \quad (50)$$

Useful Work

$$= wQ\left(H_1 + h - \frac{v_2^2}{2g}\right)$$

$$= Q(H_1 + h)\left(1 - \frac{d_1}{d_2} \tan \frac{\beta}{2} \tan \gamma\right). \quad (51)$$

Efficiency

$$= 1 - \frac{v_2^2}{2g(H_1+h)} = 1 - \frac{d_1}{d_2} \tan \frac{\beta}{2} \tan \gamma. \quad (52)$$

Amount Q of Water passing through Turbine

$$= 2\pi R d_2 v_r'' = 2\pi R d_2 V_2 \sin \beta$$

$$= 2\pi R d_2 u_2 \sin \beta$$

$$= 2\pi R \sqrt{g(H_1+h) d_1 d_2 \tan \gamma \sin \beta}. \quad (53)$$

Pressure-head at Inlet-surface

$$= \frac{p_1}{w} = \frac{p_2}{w} + H_1 - \frac{v_1^2}{2g}$$

$$= \frac{p_2}{w} - h + (H_1 + h)\left(1 - \frac{d_2}{d_1} \frac{\sin \beta}{\sin 2\gamma}\right), \quad (54)$$

by equations 44 and 47.

When the turbine is working freely in space there will be no inflow of air if $p_1 > p_2$, i.e., if

$$\frac{H_1}{H_1 + h} > \frac{d_2}{d_1} \frac{\sin \beta}{\sin 2\gamma}.$$

When the turbine is drowned, with a head h' of water over the outlet-surface, there will be no back-flow of water if

$$\frac{p_1}{w} > \frac{p_2}{w} + h',$$

that is, if

$$\frac{H_1 - h'}{H_1} > \frac{r_2^2 d_2}{r_1^2 d_1} \frac{\sin \beta}{\sin 2\gamma}.$$

there will be no back-flow of water if

$$\frac{p_1}{w} > \frac{p_2}{w} + h',$$

that is, if

$$\frac{H_1 - h'}{H_1 + h'} > \frac{d_2}{d_1} \frac{\sin \beta}{\sin 2\gamma}.$$

IN IMPULSE TURBINES.

In an I. F. or O. F.:

Speed of Turbine.

Since

$$v_1^2 = 2gH_1, \quad . \quad . \quad (55)$$

$$\frac{r_1^2}{r_2^2} u_2^2 = u_1^2 = v_1^2 \frac{\sin^2(\alpha + \gamma)}{\sin^2 \alpha}$$

$$= 2gH_1 \frac{\sin^2(\alpha + \gamma)}{\sin^2 \alpha}$$

$$= 2gH_1 \frac{r_1^4 d_1^2}{r_2^4 d_2^2} \frac{\sin^2 \gamma}{\sin^2 \beta}. \quad . \quad (56)$$

Velocity of Efflux.

$$v_2^2 = 4u_2^2 \sin^2 \frac{\beta}{2}$$

$$= 2gH_1 \frac{r_1^2 d_1^2}{r_1^2 d_1^2} \frac{\sin^2 \gamma}{\cos^2 \frac{\beta}{2}}. \quad . \quad (57)$$

Useful Work

$$= wQ\left(H_1 - \frac{v_2^2}{2g}\right)$$

$$= wQH_1\left(1 - \frac{r_1^2 d_1^2}{r_2^2 d_2^2} \frac{\sin^2 \gamma}{\cos^2 \frac{\beta}{2}}\right). \quad (58)$$

Efficiency

$$= 1 - \frac{r_1^2 d_1^2}{r_2^2 d_2^2} \frac{\sin^2 \gamma}{\cos^2 \frac{\beta}{2}}. \quad . \quad (59)$$

In an A. F.:

Speed of Turbine.

Since

$$v_1^2 = 2gH_1, \quad . \quad . \quad (55)$$

$$u_2^2 = u_1^2 = v_1^2 \frac{\sin^2(\alpha + \gamma)}{\sin^2 \alpha}$$

$$= 2gH_1 \frac{\sin^2(\alpha + \gamma)}{\sin^2 \alpha}$$

$$= 2gH_1 \frac{d_1^2}{d_2^2} \frac{\sin^2 \gamma}{\sin^2 \beta}. \quad . \quad . \quad (56)$$

Velocity of Efflux.

$$v_2^2 = 4u_2^2 \sin^2 \frac{\beta}{2}$$

$$= 2gH_1 \frac{d_1^2}{d_2^2} \frac{\sin \gamma}{\cos^2 \frac{\beta}{2}}. \quad . \quad (57)$$

Useful Work

$$= wQ\left(H_1 + h - \frac{v_2^2}{2g}\right),$$

$$= wQ\left\{h + H_1(1 - \frac{d_1^2}{d_2^2} \frac{\sin^2 \beta}{\cos^2 \frac{\gamma}{2}}\right\}. (58)$$

Efficiency

$$= 1 - \frac{H_1}{H_1 + h} \frac{d_1^2}{d_2^2} \frac{\sin^2 \gamma}{\cos^2 \frac{\beta}{2}}. \quad (59)$$

The great advantages possessed by turbines over vertical wheels on horizontal axes are shown by a consideration of the expressions for the *useful work* and *efficiency*. The former involves the *available head* only, while the latter is independent even of that. Thus a turbine will work equally well under water or above water, while its efficiency remains the same, whatever the available head may be.

The efficiency, also, increases as the ratio $\frac{d_1}{d_2}$ diminishes. The value of d_1, however, must not be too small, as there might be a loss of energy due to a contracted section at entrance, while if d_2 is made too large, the vane-passages will no longer run full bore.

Finally, the efficiency increases as the angles β and γ diminish.

In practice γ usually ranges from 10° to 30° in an I. F., and from 20° to 50° in an O. F. and A. F., an average value being 20° for an I. F., and 25° for an O. F. and A. F.

In an I. F. β generally ranges from 135° to 150° if $u_2 = V_2$, or from 30° to 45° if $v_\omega'' = 0$, and in an O. F. and P. F. from 20° to 30°, an average value being 145° or 35° for an I. F., according as $u_2 = V_2$, or $v_\omega'' = 0$, and 25° for an O. F. and A. F.

25. Remarks on the Centrifugal Head $\frac{u_2^2 - u_1^2}{2g}$.

From equations 14 and 17

$$\frac{v_1^2}{2g} = \left\{ H_1 - \frac{V_2^2 - V_1^2}{2g} + \frac{u_2^2 - u_1^2}{2g} \right\}.$$

In an I. F. $u_2 < u_1$, and the term $\frac{u_2^2 - u_1^2}{2g}$ is *negative*.

Hence the velocity v_1 diminishes as the speed of the turbine increases and *vice versa*. The centrifugal head $\frac{u_1^2 - u_2^2}{2g}$ therefore tends to secure a steady motion in the case of an I. F., and also to diminish the frictional loss of head. For this reason it should be made as large as possible consistent with practical requirements, and $\frac{r_1}{r_2}$ is usually made equal to 2.

In an O. F., on the other hand, $u_2 > u_1$ and the centrifugal head is *positive*. The velocity v_1 will, therefore increase and diminish with the speed of the turbine (u). Thus the centrifugal head is adverse to a steady motion, and tends both to augment a variation from the normal speed and to increase the frictional loss of head. It follows that $\frac{u_2^2 - u_1^2}{2g}$ should be as small as possible consistent with practical requirements, and a common value of $\frac{r_2}{r_1}$ is 1.25.

Again, eq. 5 shows that the *velocity of flow* v_r (and therefore also v_1) increases as the size of the wheel diminishes, and is accompanied by a corresponding increase in the frictional loss of head. Hence it would seem advisable to employ large wheels; but if the size of a wheel is increased, it must be borne in mind that the skin-friction (if the turbine works under water), the weight, and consequently the journal friction, will all increase. Belanger has suggested that the efficiency of an A. F. may be increased by so forming the vane-passages that the path of a fluid particle gradually approaches the axis of rotation.

26. Practical Values of the Velocities, etc.—Let v be the theoretical velocity due to the head H; i.e., let $v^2 = 2gH$.

Experience indicates that the following values will give good results in reaction turbines:

In I. F., $v_r' = v_r'' = \frac{v}{8}$; $\qquad u_1 = \frac{r_1}{r_2}u_2 = \frac{2}{3}v.$

In O. F., $v_r' = \frac{v}{4}$; $v_r'' = .21v$ to $.17v$; $\quad u_1 = \frac{r_1}{r_2}u_2 = .56v.$

In A. F., $v_r' = v_r'' = .15v$ to $.2v$; $\qquad u_1 = u_2 = \frac{3}{5}v$ to $\frac{2}{3}v.$

Again, in reaction and impulse turbines the thickness of the vanes varies from $\frac{1}{8}$ inch to $\frac{3}{8}$ inch if of wrought iron, and from $\frac{1}{2}$ inch to $\frac{5}{8}$ inch if of cast iron. In the latter case the vanes are usually tapered at the ends.

In axial-flow turbines the mean radius R is often made to vary

from $\frac{3}{2}\sqrt{A_1 \sin\gamma}$ to $2\sqrt{A_1 \sin\gamma}$ if $A_1 \sin\gamma < 2$ square feet;

from $\frac{5}{4}\sqrt{A_1 \sin\gamma}$ to $\frac{3}{2}\sqrt{A_1 \sin\gamma}$ if $A_1 \sin\gamma > 2$ sq. ft. < 16 sq. ft.;

from $\sqrt{A_1 \sin\gamma}$ to $\frac{5}{4}\sqrt{A_1 \sin\gamma}$ if $A_1 \sin\gamma > 16$ square feet.

In axial-impulse turbines the mean radius R is often made to vary from $\frac{5}{4}\sqrt{A_1 \sin\gamma}$ to $2\sqrt{A_1 \sin\gamma}$.

Also, the depth h of the wheel varies from $\frac{2R}{8}$ to $\frac{2R}{11}$ but must be determined by experience.

Again,

$$A_1 \sin\gamma = \frac{9}{8}\frac{Q}{v_1}.$$

For a delivery of 30 to 60 cubic feet and a fall of 25 ft. to 40 ft. γ should be 15° to 18°, and β should be 13° to 16°.

For a delivery of 40 to 200 cubic feet, and a fall of 5 ft. to 30 ft. γ should be 18° to 24°, and β should be 16° to 24°.

For a delivery of more than 200 cubic feet, and lower falls, γ should be 24° to 30°, and β 24° to 28°.

In axial-impulse turbines it may also be assumed as a first approximation that

$$\text{work per pound} = \frac{v_1^2}{2g} = \frac{v_w' u_1}{g},$$

and therefore

$$v_1 = 2u_1 \cos\gamma = 2V_1 \cos\gamma.$$

27. Theory of the Suction (or Draught) Tube.—Vortex and axial-flow turbines sometimes have their outlet orifices opening into a suction (or draft) tube which extends downwards and discharges *below* the surface of the tail-water. By such an arrangement the turbine can be placed at any convenient height above the tail-water and thus becomes easily accessible, while at the same time a shorter length of shafting will suffice. The suction tube is usually cylindrical and of constant diameter, so that there is an abrupt change of section at the outlet surface of the turbine, producing a corresponding loss of energy by eddies, etc. This loss may be prevented by so forming the tube at the upper end that there is no abrupt change of section, and by gradually increasing the diameter downwards. The cost of construction is greater, but the action of the tube is much improved.

Let h' be the head above the inlet orifices of the wheel.

Let h'' be the head between the inlet orifices and the surface of the tail-water.

Let L_1 be the loss of head up to the inlet surface.

Let L_2 be the loss of head between the wheel and the tube outlet.

Let v_4 be the velocity of discharge from the outlet at bottom of tube.

Let P be the atmospheric pressure.

Then, assuming that there is no sudden change of section at the outlet surface,

$$h' + \frac{P}{w} - \frac{p_1}{w} = \frac{v_1^2}{2g} + L_1,$$

$$h'' + \frac{p_2}{w} + \frac{v_2^2}{2g} = \frac{v_4^2}{2g} + L_2 + \frac{P}{w},$$

and therefore

$$\frac{p_1 - p_2}{w} = h' + h'' - \frac{v_1^2 - v_2^2 + v_4^2}{2g} - L_1 - L_2,$$

$$= H - \frac{v_1^2}{2g}(1 - \mu_3 + \mu_4 + \mu_5 + \mu_6),$$

where $H = h' + h'' =$ total head above tail-water surface; and v_2^2, v_4^2, L_1, L_2 are expressed in the forms

$$\mu_2 v_1^2,\ \mu_4 v_1^2,\ \mu_5 \frac{v_1^2}{2g},\ \mu_6 \frac{v_1^2}{2g},$$

$\mu_2, \mu_4, \mu_5, \mu_6$ being empirical coefficients.

Again, the effective head

$$H_1 = \frac{v_1^2}{2g} + \frac{p_1 - p_2}{2} = H + \frac{v_1^2}{2g}\left(\mu_2 - \mu_4 - \mu_5 - \mu_6\right),$$

and is entirely independent of the position of the turbine in the tube.

Also, if A_4 is the area of the outlet from the suction-tube,

$$A_4 v_4 = Q = A_1 v_1 \sin \gamma,$$

so that v_1 can be expressed in terms of v_4, and hence $\frac{p_1 - p_2}{w}$ is also independent of the position of the turbine in the tube.

Suppose the velocity of flow to be so small that v_4, v_2, L_2 may be each taken equal to *nil.* Then

$$h'' + \frac{p_2}{w} = \frac{P}{w};$$

and since the minimum value of p_2 is also *nil*, the maximum theoretical height of the wheel above the tail-water surface is equal to the head due to one atmosphere. Again,

$$g(h' + h'') = gH = v_w' u_1 - v_w'' u_2 + \frac{v_4^2}{2}$$

$$= v_1 \cos \gamma u_1 - u_2(u_2 - V_2 \cos \beta) + \frac{v_4^2}{2}.$$

But

$$A_1 v_1 \sin \gamma = Q = A_2 v_2 \sin \delta = A_2 V_2 \sin \beta = A_4 v_4;$$

and hence, taking

$$v_2 = \sqrt{\mu_2} \, . \, v_1, \quad V_2 = \sqrt{\mu_3} \, . \, v_1, \quad v_4 = \sqrt{\mu_4} \, . \, v_1,$$

$$gH = v_1(u_1 \cos \gamma + \sqrt{\mu_3} \, . \, u_2 \cos \beta) - u_2^2 + \frac{\mu_4 v_1^2}{2},$$

and therefore

$$\frac{2}{\mu_4}\left(gH + u_2^2\right) = v_1^2 + 2v_1 \frac{u_1 \cos \gamma + \sqrt{\mu_3} \, . \, u_2 \cos \beta}{\mu_4}$$

$$= v_1^2 + 2v_1 u_2 \, . \frac{\frac{r_1}{r_2} \cos \gamma + \sqrt{\mu_3} \, . \cos \beta}{\mu_4}$$

$$= v_1^2 + 2v_1 u_2 B,$$

where $B = \frac{1}{\mu_4}\left(\frac{r_1}{r_2} \cos \gamma + \sqrt{\mu_3} \cos \beta\right)$.

Hence it follows that v_1 increases with u_2, i.e., with the speed of the turbine, if

$$\frac{u_2^2}{gH} > \frac{B^2 \mu_4}{2 + B^2 \mu_4}.$$

A suction-tube is not used with an outward-flow turbine, but a similar result is obtained by adding a surrounding stationary casing with bell-mouth outlet. A similar diffusor might be added with effect to a Jonval working without a suction-tube below the tail-water. The theory of the diffusor is similar to that of the suction-tube.

28. Losses and Mechanical Effect.—The losses may be enumerated as follows:

I. The loss (L_1) of head in the channel by which the water is taken to the turbine.

$$L_1 = f_1 \frac{l}{m} \frac{v_0^2}{2g},$$

f_1 being the coefficient of friction with an average value of

.0067, l the length of the channel of approach m its mean hydraulic depth, and v_0 the mean velocity in the channel.

L_1 is generally inappreciable in the case of turbines of the inward- and axial-flow types, as they are usually supplied with water from a large reservoir in which v_0 is sensibly nil.

If A_0 is the sectional area of the supply-channel, then

$$A_0 v_0 = Q = A_1 v_1 \sin \gamma,$$

and

$$L_1 = f_1 \frac{l}{m}\left(\frac{A_1 \sin \gamma}{A_0}\right)^2 \frac{v_1^2}{2g}.$$

II. The loss (L_2) of head in the guide-passages.

This loss is made up of:

(*a*) The loss due to resistance at the entrance into the passages;

(*b*) The loss due to the friction between the fluid and the fixed blades;

(*c*) The loss due to the curvature of the blades;

(*d*) The loss of head on leaving the guide-passages.

These four losses may be included in the expression

$$L_2 = f_2 \frac{v_1^2}{2g},$$

f_2 being a coefficient which has been found to vary from .025 to .2 and upwards. An average value of f_2 is .125, but this is somewhat high for good turbines.

Note.—In *Impulse* turbines f_2 has been found to vary from .11 to .17.

III. The loss (L_3) due to shock at entrance into the wheel. In order that there may be no shock at entrance, the relative velocity (V_1) must be tangential to the lip of the vane. For any other velocity ($v_1' = ac'$) and direction ($dac' = \gamma'$) of the water at entrance, evidently

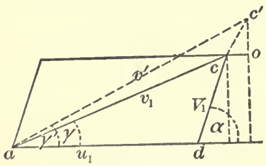


FIG. 191.

L_3 = the loss of head

$$= \frac{(cc')^2}{2g} = \frac{(c'o)^2 + (co)^2}{2g}$$

$$= \frac{(v' \sin \gamma' - v_1 \sin \gamma)^2}{2g} + \frac{(v' \cos \gamma' - v_1 \cos \gamma)^2}{2g}$$

$$= \frac{(v' \sin \gamma' - V_1 \sin \alpha)^2}{2g} + \frac{(v' \cos \gamma' - \overline{v_1 - V_1 \cos \alpha})^2}{2g}.$$

Generally co is small, and L_3 is always nil when the turbine is working at full pressure and at the normal speed.

This loss of head in shock caused by abrupt changes of section, and also at an angle, may be avoided by causing the section to vary gradually, and by substituting a continuous curve for the angle.

IV. The loss (L_4) of head due to friction, etc., in passing through the wheel-passages, including the loss due to leakage in the space between the guides and the inlet-surface. This loss is expressed in the form

$$L_4 = f_4 \frac{V_2^2}{2g} = f_4 \left(\frac{A_1 \cos \gamma}{A_2 \sin \beta}\right)^2 \frac{v_1^2}{2g},$$

where f_4 varies from .10 to .20.

Note.—The loss of head due to skin-friction often governs the dimensions of a turbine, and renders it advisable, in the case of high falls, to employ small high-speed turbines.

V. The loss of head (L_5) due to the abrupt change of section between the outlet-surface and the suction-tube.

As in III, v_2 $(= fh)$ is suddenly changed into v_2' $(= fh')$, and loss of head is

$$L_5 = \frac{(hh')^2}{2g} = \frac{(hx)^2 + (h'x)^2}{2g} = \frac{(hx)^2}{2g},$$

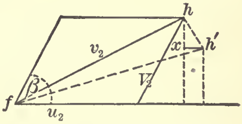


FIG. 192.

since $h'x$ is very small and may be disregarded. Thus,

$$L_5 = \frac{(V_2 \sin \beta - v_3')^2}{2g}$$

v_3' being the component of v_2' (fh') in the direction of the axis of the suction-tube.

If there is no abrupt change of section between the outlet-surface and the tube, L_5 is nil.

VI. The loss of head (L_6) due to friction the in suction-tube. Assume that the velocity v_3 of flow in the tube is equal to v_2', the velocity with which the water leaves the turbine. Also let A_3 be the sectional area of the tube. Then

$$L_6 = f_6 \frac{l'}{m'} \frac{v_3^2}{2g} = f_6 \frac{l'}{m'} \left(\frac{A_1 \sin \gamma}{A_3}\right)^2 \frac{v_1^2}{2g},$$

$f_6(=f_1)$ being the coefficient of friction with an average value of .0067, l' the length of the tube, and m' its mean hydraulic depth.

VII. The loss (L_7) of head due to entrance to sluice at base of tube. This loss may be expressed in the form

$$L_7 = f_7 \frac{v_4^2}{2g} = f_7 \left(\frac{A_1 \sin \gamma}{A_4}\right)^2 \frac{v_1^2}{2g},$$

the average value of f_7 being about .03.

VIII. The loss (L_8) of head due to the energy carried away by the water on leaving the suction-tube.

$$L_8 = \frac{v_4^2}{2g},$$

and v_4 usually varies from $\frac{1}{5}\sqrt{2gH}$ to $\frac{2}{7}\sqrt{2gH}$.

In good turbines the loss should not exceed 6%. It might be reduced to 3%, or even to 1%, but this would largely increase the skin-friction.

IX. The loss of head (L_9) produced by the friction of the bearings.

$$L_9 = \mu W \frac{\rho}{r_1} u_1,$$

μ being the coefficient of journal friction, W the weight of the turbine and of the water it contains, and ρ the radius of the journal.

Hence the total loss of head

$$= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 + L_9 = L,$$

and the total *mechanical effect*

$$= wQ\left(\frac{v_1^2}{2g} - L\right).$$

Note.—If there is no suction-tube, $L_5 = O = L_6 = L_7 = L_8$, and the total loss becomes

$$L_1 + L_2 + L_3 + L_4 + L_9 + \frac{v_2^2}{2g} + \left\{ \begin{array}{l}\text{fall from outlet-surface to} \\ \quad\text{tail-water surface.}\end{array}\right.$$

29. Centrifugal Pumps.—If an hydraulic motor is driven in the reverse direction, and supplied with water at the point from which the water originally proceeded, the motor becomes a pump. All turbines are reversible, and may, therefore, be converted into pumps, but no pump has yet been constructed of an inward-flow type. The ordinary centrifugal pump, Fig. 193, is an outward-flow machine. It is more economical and less costly for low falls than a reciprocating pump, and has been known to give good and economic results for falls as great as 40 feet.

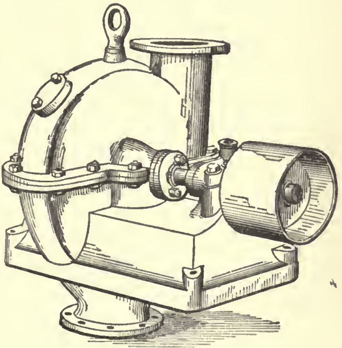


FIG. 193.

With compound centrifugal pumps very much greater lifts are economically possible.

There are three main differences between centrifugal pumps and turbines:

1st. The gross lift with a pump is greater, on account of frictional resistances, than the fall in the case of a turbine.

2d. The water enters the pump-fan without any velocity of whirl ($v_w' = 0$) and leaves the fan with a velocity of whirl (v_w'') which should be reduced to a minimum in the act of lifting, but which is by no means small. In a turbine, on the other hand, the water has a considerable velocity of whirl (v_w') at entrance, while at exit the velocity of whirl (v_w'') is reduced to a minimum, and is generally *nil*.

3d. In a turbine the direction of the water as it flows into the wheel is controlled by guide-blades; whereas in the case of a pump, the direction of the water, as it flows towards the discharge-pipe, is controlled by a single guide-blade, which forms the outer surface of the volute, or chamber, into which the water flows on leaving the fan.

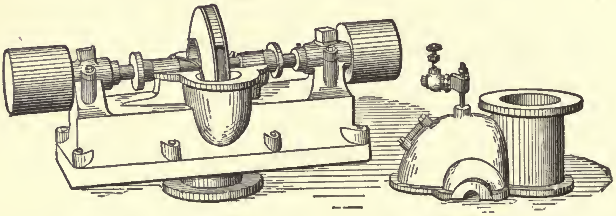


FIG. 194.—Experimental Centrifugal Pump in the Hydraulic Laboratory, McGill University.

Before the pump can be put into action it must be filled, and this can be effected through an opening (closed by a plug) in the casing when the pump is under water, or, if the pump is above water, by creating a vacuum in the pump-case by means of an air pump or a steam-jet pump, when the water must necessarily rise in the suction-tube.

At first the water rotates as a solid mass, and delivery commences when the speed is such that the head due to centrifugal force $\left(\frac{u_2^2 - u_1^2}{2g}\right)$ exceeds the lift. This speed may be afterwards reduced, providing a portion of the energy is utilized at exit.

As soon as the pump, which is keyed on to a shaft driven by a belt or by gearing, commences to work, the water rises in the suction-tube and divides so as to enter the eye of the pump-disc on both sides. As in turbines, the revolving pump-disc is provided with vanes curved so as to receive the water at the inlet-surface, for a given normal condition of working, *without shock*. Experiment has also tended to show that the angle between the tangents to a vane and the disc circumference at the outlet-surface, may be advantageously made as small even as 15°, but manufacturers hold different opinions on this point. The water leaves the disc with a more or less considerable velocity, and impinges upon the fluid mass flowing round the volute, or spiral casing surrounding the disc, towards the discharge-pipe. This volute should have a section gradually increasing to the point of discharge, in order that the delivery across any transverse section of the volute may be uniform. This volute is also so designed as to compel rotation in one direction only, with a velocity corresponding to the velocity of whirl (v_w'') on leaving the fan. There are examples of pumps in which the delivery is effected in all directions, and the water is guided to the outlet by a number of spiral blades.

In these pumps an important advantage is gained by the addition of a vortex or whirlpool chamber surrounding the pump-disc. The water discharged from the disc then continues to rotate in this chamber, and a portion of the kinetic energy is thus converted into pressure energy, which would otherwise be largely wasted in eddies in the volute or discharge-pipe. The water leaves the vortex chamber with a diminished whirling velocity which cannot be very different in direction and magnitude from the velocity of the mass of water in the volute. The vortex chamber is provided with guide-blades following the direction of free vortex stream-lines (equiangular spirals) so as to prevent irregular motion. A conical suction-pipe is advantageous, as it allows of a gradual increase of velocity, and a still greater advantage is to be found in the use of a conical discharge-pipe. The velocity in the discharge-pipe should not be too great, as it leads to a waste of energy. A velocity of 3 to 6 feet is found to give the best results.

Pumps work under different conditions from turbines, and hence there are corresponding differences necessary in their design. They work best for the particular lift for which they are designed, and any variation from this lift causes a rapid reduction in the efficiency.

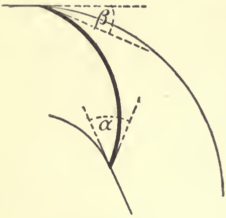


FIG. 195.

30. Theory of Centrifugal Pump.—Denote the velocities at the inlet- and outlet-surfaces of the pump-fan by the same symbols as in turbines.

Let Q be the delivery of the pump.

Let H_g be the gross lift, including the actual lift (H_a), the head due to the velocity of delivery, the heads due to the frictional resistances in the ascending main, in the suction-pipe and in the wheel-passages, and the head corresponding to the losses "in shock" at entrance and exit.

Let H_a be the actual lift.

The *total* work done on the wheel

$$= \frac{wQ}{g}(v_w{}''u_2 - v_w{}'u_1).$$

The *useful* work done by the pump $= wQH_g$.

Hence

$$\text{the efficiency } (\eta) = \frac{gH_g}{v_w{}''u_2 - v_w{}'u_1}.$$

At the inlet-surface the flow is usually radial, so that $\gamma = 90°$, and the velocity of whirl $v_w{}'$ is *nil.*

Thus,

$$\text{the efficiency} = \frac{gH_g}{v_w{}''u_2} = \eta,$$

and the equation

$$\eta v_w{}''u_2 = gH_g,$$

is the fundamental equation governing the design of centrifugal pumps.

Again,

$$\text{the efficiency } \eta = \frac{H_g}{v_w'' u_2} = 1 - \frac{v_2^2}{2g v_w'' u_2}.$$?

For a given speeed (u_2) this is a maximum when

$$\frac{v_2^2}{v_w''} = \text{a minimum} = \frac{(v_w'')^2 + (v_r'')^2}{v_w''} = \frac{(v_w'')^2 + (u_2 - v_w'')^2 \tan^2 \beta}{v_w''}.$$

Hence, differentiating,

$$-\frac{u_2^2 \tan^2 \beta}{(v_w'')^2} + \sec^2 \beta = 0,$$

and therefore

$$v_w'' = u_2 \sin \beta,$$

and is the velocity of whirl at exit which, for a given speed (u_2), will give a maximum efficiency.

Note.—If $u_2 = v_w''$, then

$$(v_w'')^2 = u_2^2 = gH_g,$$

and the water leaves the fan with a velocity equal to that due to at least one half of the gross lift. The efficiency must therefore be necessarily less than .5.

Again, since $v_r'' \cot \beta = u_2 - v_w''$, β must be 90° if $u_2 = v_w''$; but β is generally much less than 90°, and therefore v_w'' is generally less than u_2. Let $v_w'' = ku_2$, k being an empirical coefficient less than unity.

Then $ku_2^2 = gH_g$ and the efficiency $= \dfrac{gH_a}{ku_2^2}$.

Consider two cases.

CASE I. *Pump without a vortex-chamber.*

When the water is discharged into the volute, the velocity of flow (v_r'') is wasted and the velocity of whirl (v_w'') is suddenly changed to the velocity v_s of the mass of water in the volute assumed to be moving in a direction tangential to the pump-disc. Thus,

$$\text{the gain of pressure-head} = \frac{(v_w'')^2 - (v_s)^2}{2g} - \frac{(v_w'' - v_s)^2}{2g}$$

$$= \frac{v_s(v_w'' - v_s)}{g},$$

which is a maximum and equal to $\frac{1}{4}\frac{(v_w'')^2}{g}$ when $v_s = \frac{v_w''}{2}$.

This gain of head is always very small and may be disregarded as being almost inappreciable. Neglecting also the losses due to frictional resistances, etc., then, precisely as in the case of turbines,

$$\frac{v_1^2}{2g} + H_g = \left\{ \begin{array}{l} \text{variation of pressure-head between} \\ \quad \text{outlet and inlet surfaces.} \end{array} \right.$$

$$= \frac{u_2^2 - u_1^2}{2g} - \frac{V_2^2 - V_1^2}{2g}.$$

But $V_1^2 = u_1^2 + v_1^2$, since $\gamma = 90°$, and therefore

$$H_g = \frac{u_2^2 - V_2^2}{2g} = \frac{u_2^2 - (u_2 - v_w'')^2 \sec^2 \beta}{2g},$$

and

$$\text{the efficiency} = \frac{u_2^2 - (u_2 - v_w'')^2 \sec^2 \beta}{2u_2 v_w''},$$

which is a maximum for a given speed u_2 and equal to $\frac{1}{1 + \sin \beta}$ when $v_w'' = u_2 \sin \beta$.

Thus the efficiency increases as β diminishes.

When $\beta = 90°$, or $v_w'' = u_2$, the maximum efficiency is $\frac{1}{2}$, and therefore one half of the work done in driving the pump is wasted.

Note.—Loss of head

= loss due to hydraulic friction

+ loss due to abrupt change from v_w'' to v_s

+ loss due to dissipation of v_r''

+ loss due to v_s carried away

= loss due to friction (hydraulic)

$$+\frac{v_s(v_w''-v_s)}{2g}+\frac{(v_s)^2}{2g}+\frac{(v_r'')^2}{2g}$$

= loss due to friction (hydraulic)

$$+\frac{(v_w'')^2}{4g}+\frac{(v_r'')^2}{2g},$$

when $v_s=\frac{1}{2}v_w''$.

CASE II. *Pump with a vortex-chamber* (Fig. 199).

The diameter ($=2r_3$) of the outer surface of this chamber should be at least twice that of the outlet-surface of the pump-disc.

Assuming that the motion in the chamber is a free vortex, then

$$\left.\begin{array}{r}\text{the gain of}\\ \text{pressure-head}\end{array}\right\}=\frac{v_2^2}{2g}\left(1-\frac{r_2^2}{r_3^2}\right),$$

and hence

$$\text{the efficiency}=\frac{gH_g+\frac{v_2^2}{2}\left(1-\frac{r_2^2}{r_3^2}\right)}{u_2v_w''}.$$

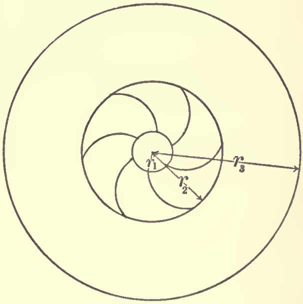


FIG. 196.

This, again, is a maximum for a given speed, when $v_w'=u_2\sin\beta$, its value being

$$\frac{1+\left(1-\frac{r_2^2}{r_3^2}\right)\sin\beta}{1+\sin\beta}.$$

This expression increases as β diminishes, but the value of β is not of so much importance as in Case I, and it is very common to make β equal to 30° or 40°.

When $\beta = 90°$ the maximum efficiency $= \frac{1}{2}\left(2 - \frac{r_2^2}{r_3^2}\right) = \frac{7}{8}$ if $r_3 = 2r_2$.

31. Practical Values.—The following values are often adopted:

$$v_r' = \tfrac{1}{4}\sqrt{2gH_a};$$

$$v_2 = 2r_1;$$

$d_2 = d_1$ when faces of pump-disc are parallel;

$d_2 = \frac{1}{2}d_1$ when pump-disk is coned.

EXAMPLES.

1. An accumulator ram is 9 inches diameter and 21 feet stroke. Find the store of energy in foot-pounds when the ram is at the top of its stroke, and is loaded till the pressure is 750 lbs. per square inch.

Ans. 958,000 ft.-lbs.

2. In a differential accumulator the diameters of the spindle are 7 inches and 5 inches; the stroke is 10 feet. Find the store of energy when full and loaded to 2000 lbs. per square inch. *Ans.* 377,000 ft.-lbs.

3. A direct-acting lift has a ram 9 inches diameter, and works under a *constant* head of 73 feet, of which 13 per cent. is required by ram-friction and friction of mechanism. The supply-pipe is 100 feet long and 4 inches diameter. Find the speed of steady motion when raising a load of 1350 lbs., and also the load it would raise at double that speed.

If a valve in the supply-pipe is partially closed so as to increase the coefficient of resistance by $5\frac{1}{2}$, what would the speed be?

Ans. Speed = 2 ft. per second; load = 150 lbs.

4. Eight cwt. of ore is to be raised from a mine at the rate of 900 feet per minute by a water-pressure engine, which has four single-acting cylinders, 6 inches diameter, 18 inches stroke, making 60 revolutions per minute. Find the diameter of a supply-pipe 230 feet long for a head of 230 feet, not including friction of mechanism.

Ans. Diameter = 4 inches.

5. If λ be the length equivalent to the inertia of a water-pressure engine, F the coefficient of hydraulic resistance, both reduced to the ram, v_0 the speed of steady motion, find the velocity of ram after moving from rest through a space x against a constant useful resistance. Also find the time occupied.

$$Ans.\ v^2 = v_0^2\left(1 - e^{-\frac{F}{\lambda}x}\right);\ \ t = \frac{\lambda}{Fv_0}\log_e \frac{v_0 + v}{v_0 - v}.$$

6. An hydraulic motor is driven from an accumulator, the pressure in which is 750 lbs. per square inch, by means of a supply-pipe 900 feet long, 4 inches diameter; what would be the maximum power theoretically attainable, and what would be the velocity in the pipe corresponding to that power? Find approximately the efficiency of transmission at half power. *Ans.* H.P. = 240; v = 22 ft.; efficiency = .96 nearly.

7. A gun recoils with a maximum velocity of 10 feet per second. The area of the orifices in the compressor, after allowing for contraction, may be taken as one twentieth the area of the piston. Find the initial pressure in the compressor in feet of liquid.

Assuming the weight of the gun to be 12 tons, friction of slide 3 tons, diameter of compressor 6 inches, fluid in compressor, water, find the recoil.

Find the mean resistance to recoil. Compare the maximum and mean resistances, each exclusive of friction of slide.

Ans. 621; 4 ft. $2\frac{1}{2}$ in.; total mean resistance = 4.4 tons; ratio = 2.5.

8. A reaction wheel is inverted and worked as a pump. Find the speed of maximum efficiency and the maximum efficiency, the coefficient of hydraulic resistance referred to the orifices being .125.

Ans. Speed = twice that due to lift; .758.

9. A reaction wheel with orifices 2 in. in diameter makes 80 revolutions per minute under a head of 5 ft. The distance between the centre of an orifice and the axis of rotation is 12 inches. Find the H.P. and the efficiency. *Ans.* .146; .596.

10. In a reaction wheel the speed of maximum efficiency is that due to the head. In what ratio must the resistance be diminished to work at $\frac{4}{3}$ this speed, and what will then be the efficiency? Obtain similar results when the speed is diminished to three fourths its original amount. *Ans.* .949; .8896; 1.071; .753.

11. In a reaction wheel, determine the per cent of available effect lost, (1) if $u^2 = 2gH$; (2) if $u^2 = 4gH$; (3) if $u^2 = 8gH$.

What conclusion may be drawn from the results?

Efficiencies are respectively .828, .9, .945.

12. An undershot water-wheel with straight floats works in a straight rectangular channel of the same width as the wheel, viz., 4 ft.; the stream delivers 28 cub. ft. of water per second, and the efficiency is $\frac{1}{3}$. Find the relation between the up-stream and down-stream velocities. If the velocity of the inflowing water is 2 ft. per second, find the velocity on the down-stream side and determine the mechanical effect of the wheel, its diameter being 20 ft., the diameter of the gudgeons being 4 in., and the coefficient of friction .008.

13. A vane rotates about an axis with an angular velocity A, and and water moves freely along the vane. Show that the work per unit of weight of water, due to centrifugal force, in moving from a point distant r_1 ft. from the axis to a point distant r_2 ft. from the axis is $\dfrac{A^2(r_2^2 - r_1^2)}{2g}$.

14. Determine the effect of a low *breast* or undershot wheel 15 ft. in diameter and making 8 revols. per minute; the fall is 4 ft. and the delivery 20 cub. ft. per second; the velocity of the stream before coming on the wheel is double that of the wheel. *Ans.* 1490 ft.-lbs.

15. The efficiency of an undershot water-wheel working in a straight rectangular channel with horizontal bed is $\frac{1}{3}$. Find the relation between the up- and down-stream velocities of flow.

16. Determine the mechanical effect of an undershot wheel of 12 ft. diam. making 10 revols. per minute, the fall being 3 ft. and the quantity of water passed per second 15 cub. ft.

17. Ascertain the general proportions of a Poncelet wheel, being given: height of fall $= 4\frac{1}{2}$ ft.; delivery of water $= 40$ cub. ft. per second; radius of exterior circumference $= 9$ ft.; thickness of the stream $= 9$ in.

18. Design a Poncelet wheel for a fall of 4.5 ft. and 24 cub. ft. of water per second, using the formulæ on pages 237–239, taking $\gamma = 20°$, and also $\lambda = 20°$ as a first approximation.

Ans. $\alpha = 143° \ 57'$; depth of crown $= 1.73$ ft.; depth of stream $= .386$ ft.; $b = 4.14$ ft.; radius of bucket $= 2.35$ ft.; $\psi = 128° \ 6'$; $\lambda = 18.69°$; number of buckets $= 44$; mechanical effect $= 8.5$ H.P.; efficiency $= .69$.

19. 15 cub. ft. of water per second with a fall of 8_2 ft. are brought on a breast-wheel revolving with a linear velocity of 5 ft.; depth of shrouding $= 12$ in.; the buckets are half filled, and $v_1 = 2u$; also $r_1 = 12$ ft. Find the theoretical mechanical effect. *Ans.* 7240 ft.-lbs.

20. A wheel is to be constructed for a 30-ft. fall having an 8-ft. velocity at circumference and taking on the water at 12° from the summit with a velocity of 16 ft. Determine the radius of the wheel and the number of revols. *Ans.* 12.8 ft.; 5.94.

21. If for the wheel in example 20 the number of revols. is 5, and $v_1 = 2u$, the water being again taken on at 12°, find the radius and u.

Ans. 13.98 ft.; 7.3 ft. per sec.

22. A breast wheel passes 12 cub. ft. of water per second, and for the speed $= \frac{4}{5}v_1 = 4$ ft. per second the loss of mechanical effect due to the relative velocity V being destroyed is a minimum. Find this effect.

23. In a breast-wheel $Q = 10$ cub. ft. per second; $H = 10$ ft.; $v_1 = \frac{8}{5}u$; $u = 4\frac{1}{4}$ ft. per second; $\gamma = 30°$; diam. of gudgeon $= 6$ in.; diam. of wheel $= 30$ ft.; $\mu = .08$; weight of wheel and water $= 20{,}000$ lbs. Find the mechanical effect of the wheel. (Neglect loss of effect due to escape of water from buckets and to frictional resistance along the curb.)

24. The quantity of water laid on a breast-wheel by an overfall sluice $= 6$ cub. ft. per second, the total fall being 4 ft. 6 in., and the velocity of the periphery 5 ft. per second; also $5v_1 = 8u$, and if d be the depth of the shrouding $2bdu = 5Q$ (in the present case $d = 12$ in.). Find the effective fall, the height of the lip of the guide, the angle of inclination at the end of the guide-curve, the breadth of the lip of the guide-curve, and the radius of the wheel that the water may enter tangentially. If the radius is limited to 12 ft. 6 in., find the deviation of the direction of motion of the water from that of the wheel at the point of entrance.

Ans. 6.9 ft.; .325 ft.; $34° \ 46'$; $2\frac{3}{4}$ ft.; 38.6 ft.; $28° \ 36'$.

25. In an overshot wheel $r_1 = 15$ ft., $d = 10$ in., $\beta = \frac{5}{4}\psi$. If the

division circle is at one half of the depth of the crown, find the angle (γ_1) between the bucket-lip and the wheel's periphery.

Ans. $\gamma_1 = 18° 1'$.

26. An overshot wheel in which $r_1 = 18$ ft. makes 4 revolutions per minute, and the velocity of the water on entering the buckets is twice that of the wheel's periphery. If $\gamma_1 = 20°$, find α, and also find the relative velocity (V) of the entering water.

Ans. $\alpha = 10° 9'$; $V = 7.78$ ft. per second.

27. If one fourth of the theoretic capacity of a bucket is filled by the water, find the greatest number of buckets theoretically possible, the depth of the crown being 1 ft., the radius (r_1) to the outer periphery 12 ft., the angle γ_1 20°, and the velocity of the entering water twice that of the wheel's periphery.

Ans. 103.1. Making allowance for exit of air, the number of buckets might be about two thirds of this amount, or, say, 69.

28. A wheel of 30-ft. diam. with 72 buckets makes 7 revolutions per minute, Q being 5 cub. ft. per second. The division circle is halfway between the outer and inner peripheries. If $d = 1$ ft. and $v_1 = 2u$, find the effect due to impact.

29. A 30-ft. wheel weighs 24,000 lbs. and makes 6 revolutions per minute; its gudgeons are 6 in. in diameter and the coefficient of friction is .08. The water enters the wheel with a velocity of 15 ft. per second, and in a direction making an angle of 10° with the direction of motion of the wheel at the point of entrance. The deviation from the summit of the point of entrance is 12°, of the point where spilling begins is 150°, of the point where all is spilt is 160°, and 5 cub. ft. of water enter the wheel per second, of which the partially filled buckets contain one half. Determine the total mechanical effect. *Ans.* 9305.6 ft.-lbs.

30. The velocity of the pitch circle is $9\frac{3}{8}$ ft.; the angle between the directions of motion of stream and wheel is 15°. Find impulsive action of wheel. *Ans.* 91 ft.-lbs. per cub. ft. of water.

31. An overshot wheel 40 ft. in diameter makes 4 revolutions per minute and passes 300 cub. ft. of water per minute. Show how to determine the mechanical effect of the wheel. (Neglect friction of gudgeons.)

If the gudgeons are 6 in. in diameter and the wheel weighs 30,000 lbs., by how much will the mechanical effect be diminished ($f = .008$)?

Ans. 25 ft.-lbs. per sec.

32. The diameter of an overshot wheel = 30 ft.; $v_1 = 15$ ft.; $u = 9\frac{3}{8}$ ft., deviation of impinging water from direction of motion of wheel $(\gamma) = 8\frac{1}{3}°$; deviation of point of entrance from summit = 12°; deviation of point where spilling begins from the centre = $58\frac{1}{2}°$; deviation of point where spilling ends = $70\frac{1}{2}°$; $Q = 5$ cub. ft. Find total effect of impact and weight. *Ans.* 17 H. P.

33. An overshot wheel with a radius of 15 ft. and a 12-in. crown takes 10 cub. ft. of water per second and makes 5 revolutions per

minute. If $m = \frac{1}{4}$, find the width of the wheel and the number of the buckets. *Ans.* $5\frac{1}{11}$ ft.; 75 or 90.

34. An overshot wheel of 32 ft. diameter makes 5 revolutions per minute. Find the angle between the water-surface in a bucket and the horizontal when the lip is 140° from the summit. *Ans.* 4° 33′.

35. An overshot wheel of 10 ft. diameter makes 20 revolutions per minute. Find the angle between the water-surface and the horizontal when the lip is (1) 90° from the summit, (2) 45° 26′ from the summit.

Ans. (1) 34° 16′; (2) 45° 26′.

36. The water enters an overshot wheel at 12° from the summit with a velocity of 16 ft. per second and the linear velocity of the wheel's periphery is 8 ft. per second. The fall is 30 ft. Find the diameter of the wheel and the number of revolutions per minute.

Ans. 25.68 ft.; 5.94.

37. An overshot wheel of 36 ft. diameter and with 96 buckets has a peripheral velocity of $7\frac{1}{2}$ ft. per second. The water enters with a velocity of 15 ft. per second and acquires in the wheel a velocity of 16.49 ft. per second Find the distance through which the float moves during impact.

Ans. 2.15 ft.

38. The sluice for a 10-ft. overshot wheel is vertically above the centre and inclined at 45° to the vertical. The water enters the buckets at a point 2 ft. vertically below the sluice and 10° from the summit of the wheel. Find the angle between the directions of motion of the entering water and of the wheel's circumference. Also find the velocity of the water as it enters the wheel.

39. In an overshot wheel $v_1 = 17$ ft.; $u = 11$ ft. per second; elbow-angle $= 70°$; division-angle $= 5°$; water enters the first bucket at 12° from summit of wheel. Find (*a*) the relative velocity V so that water may enter unimpeded; (*b*) the direction of the entering water; (*c*) the diameter of the wheel, which makes 5 revolutions per minute; (*d*) the position and direction of the sluice, which is 2 ft., measured horizontally from the point of entrance.

40. In an overshot wheel the deviation of the impinging water from the direction of motion of the wheel is 10°; the velocity (v_1) of the impinging stream = 15 ft. per second; of the circumference of the wheel (u) = 15 cos 10°. What proportion of the head is sacrificed?

41. A 30-ft. water-wheel with 72 buckets and a 12-in. shrouding makes 5 revolutions and receives 240 cub. ft. of water per minute. Find the width and sectional area of a bucket. The fall is 30 ft.; at what point does the water enter the wheel, the inflowing velocity being $1\frac{1}{2}$ times that of the wheel's periphery? Also find the deviation of the water-surface from the horizontal at the point at which discharging commences, i.e., 140° from the summit.

42. What number of buckets should be given to an overshot wheel of

40 ft. diameter and 12 in. width in wheel, pitch-angle $= 4°$, thickness of bucket lip $= 1$ in., water area $= 24\frac{1}{2}$ sq. in.?

43. A wheel makes 5 revolutions per minute, the radius is 16 ft., and the discharging angle 50°. Find deviation of water-surface from the horizon. *Ans.* 4° .29.

44. A wheel makes 20 revolutions per minute; radius $= 5$ ft., angle of discharge $= 0°$. Find deviation of water-surface from horizon. Also find deviation at 44° 35′ above centre. *Ans.* 4° 33′; 44° 34′.

45. The water in a head-race stands 4.66 ft. above the sole and leaves the race under a gate which is raised 6 in. above the sole, the coefficient of velocity (v_1) being .95. The water enters a breast wheel in a direction making an angle of 30° with the tangent to the wheel's periphery at the point of entrance. The speed (u) of the periphery is 10 ft. per second, the breadth of the wheel is 5 ft., the depth of the water beneath the axle is 8 in., and the length of the flume is 8.2 ft. Find the loss of head (a) due to the destruction of the relative velocity (V) at entrance; (b) due to the velocity of flow in the tail-race; (c) in the circular flume. *Ans.* (a) 1.11 ft.; (b) 1.57 ft.; (c) .44 ft.

46. In the preceding example, find how the losses of head would be modified if the flume were lowered 1.03 ft., and if the point of entrance were raised so as to make $u = v_1 \cos 30°$.

47. A water-wheel has an internal diameter of 4 ft. and an external diameter of 8 ft.; the direction of the entering water makes an angle of 15° with the tangent to the circumference. Find the angle subtended at the centre of the wheel by the bucket, which is in the form of a circular arc, and also find the radius of the bucket.

48. An overshot wheel 5 ft. wide, 30 ft. in diameter, having a 12-in. crown and 72 buckets, receives 10 cub. ft. of water per second and makes 5 revolutions per minute. Determine the deviation from the horizontal at which the water begins to spill, and also the corresponding depression of the water-surface.

49. An overshot wheel makes $\frac{15}{2\pi}$ revolutions per minute; its mean diameter is 32 ft.; the water enters the buckets with a velocity of 8 ft. per second at a point 12° 30′ from the summit of the wheel. At the point of entrance the path of the inflowing water makes an angle of 30° with the horizontal. Show that the path is horizontal vertically above the centre. The sluice-board is placed at a point whose horizontal distance from the centre is one half that of the point of entrance. Find its position relatively to the centre and its inclination to the horizon. (Sin 12° 30′ $= .2165$).

50. The water enters the buckets of the wheel in the preceding example without shock. Find the elbow-angle. Also, if the buckets

begin to spill at 150° from the summit, find where the bucket is empty and the number of buckets. (Depth of crown = 12 in.; thickness of bucket = $1\frac{1}{8}$ in.)

51. Given $v_1 = 15$ ft. per second, and $\delta = 20\frac{1}{4}°$. Find the position of the centre of the sluice, which is 4 in. above the point of entrance.

Ans. .097 ft. vertically below and 1.114 ft. horizontally from the summit. The axis of the sluice is inclined at 9° 58′ to the horizontal.

52. In an overshot water-wheel $v_1 = 15$ ft.; $u = 10$ ft.; elbow-angle $= 70\frac{1}{2}°$; division-angle $= 4\frac{1}{2}°$; deviation from summit of point of entrance = 12°. Find the deviation of the layer from that of the arm, so that the water might enter unimpeded; also find the inclination of the layer to the horizon, and the value of V. If the centre of the sluice-aperture is to be 4 in. above point of entrance, find its vertical and horizontal distance from the vertex of the stream's parabolic path which is vertically above the centre of the wheel, and also find inclination of sluice-board to horizon.

Ans. $15\frac{3}{4}°$; $20\frac{1}{4}°$; 5.3 ft. per sec.; .42 ft.; 1.04 ft.; 9° 34′.

53. In an overshot wheel $Q = 18$ cub. ft.; $r_1 = 6$ ft.; $d = 1$ ft.; $b = 4$ ft.; $N = 24$. At the moment spilling commences the area $afd = 1.025$ sq. ft.; between this point and the point where the spilling is completed three buckets are interposed, the sectional areas of the water being .501, .409, and .195 sq. ft., respectively. Find (*a*) the sectional area of bucket, (*b*) the point where the spilling commences, (*c*) the point where the spilling is completed, (*d*) the height of the arc of discharge, (*e*) the mechanical effect due to the fall of the water through the arc of discharge.

Ans. (*a*) .662 sq. ft; (*b*) $\theta = 7° 26'$, $\phi = 28° 33'$;
(*c*) $\theta = 73° 15'$, $\phi = 5° 59'$; (*d*) 4.49 ft.; (*e*) 4.93 H.P.

54. In the preceding example, if the water enters with a velocity of 20 ft. per second at 20° below the summit, and if the direction of the inflowing stream makes an angle of 25° with the wheel's periphery at the point of entrance, find the mechanical effect (*a*) due to impulse, (*b*) due to the fall to the point where spilling commences.

Ans. (*a*) 5.34 H.P.; (*b*) 12.15 H.P.

55. 300 cub. ft. of water per minute enter the buckets of a 40-ft. overshot wheel with a 12-in. crown and making four revolutions per minute. The wheel has 136 buckets. At the moment when spilling commences the area *afd* (Fig. 156) = 102 sq. in., and the area *abcd* = 24.5 sq. in. The spilling is completed when the angle between the horizontal and the radius to the lip of the bucket = 62° 30′. Between these two positions three buckets are interposed, the sectional areas of the water in the buckets being 24.5, 14.48, and 6.6 sq. in., respectively. The vertical distance between the water-surface in the first bucket and the centre is 18 ft. Find (*a*) the width of the wheel, (*b*) the cross-section of

a bucket, (c) the angle between the horizontal and the radius to the lip of the bucket when spilling commences, (d) the height of the discharging arc, (e) the mechanical effect due to weight.

Ans. (a) 2.4 ft.; (b) 33.09 sq. ft.; (c) $\theta = 52° 2\frac{1}{2}'$; ($d$) 1.9 ft.; ($e$) 19.48 H.P.

56. As the bucket arm cd moves downwards from the horizontal position, show that while the wheel moves through an angle the last particle of water at c will move through a distance approximately equal to $\frac{r(gr + u^2)}{u^2}(\theta - \sin\theta)$, r being the distance (assumed constant) of the particle of water from the axis, and u being the linear velocity of the wheel at the radius.

57. If the last particle of water leaves the buckets just as the lip d reaches the lowest point of the wheel, and if the arm is 1 ft. in length, find the angle between the lip and the wheel's periphery (1) for a wheel of 20 ft. diameter, the peripheral velocity being 5 ft. per second; (2) for a wheel of 40 ft. diameter, the peripheral velocity being 10 ft. per second; (3) for a wheel of 10 ft. diameter, the peripheral velocity being 8 ft. per second. *Ans.* (1) $20\frac{1}{3}°$; (2) $20°$; (3) $40°$.

58. In an overshot wheel of 30 ft. diameter, 5 cub. ft. of water per second enter the buckets with a velocity of 16 ft. per second and the wheel's velocity at the division circle is 7 ft. per second. The point of entrance is 18° from the summit, and the angle between the directions of the inflowing water and the wheel's periphery at the point of entrance is 12°. The water begins to spill at $148\frac{1}{2}°$ from the summit and the spilling is complete at $160\frac{1}{2}°$ from the summit. Find the total mechanical effect due to impulse and weight. What is the tangential force at the outer periphery? *Ans.* 16.28 H.P.; 1194 lbs.

59. 20 cub. ft. of water per second enter an undershot wheel of 30 ft. diameter, making 8 revolutions per minute through an underflow sluice. The velocity of the entering water is twice that of the wheel's periphery. Find (a) the head of water behind the sluice, (b) the fall, (c) the theoretical mechanical effect, (d) the actual mechanical effect, disregarding axle friction.

Ans. (a) 2.779 ft.; (b) 1.221 ft.; (c) 5.57 H.P.; (d) 2.62 H.P.

60. 20 cub. ft. of water per second enter a breast wheel of 32 ft. diameter and having a peripheral velocity of 8 ft. per second, at an angle of $25\frac{1}{2}$ with the circumference. The depth of the crown is $1\frac{1}{4}$ ft.; the buckets are half filled, and the fall is 9 ft. The velocity of the entering water is 12 ft. per second. The centre of the sluice-opening is .54 ft. above the point of entrance, and the width of the sluice is $3\frac{3}{4}$ ft. The wheel has 48 buckets. The distance between the wheel and breast is $\frac{1}{2}$ inch. The bucket passes through .9 ft. while receiving water, and the depth of the water-surface in the bucket below the point of entrance is

1.25 ft. Find (*a*) the angular distance of the point of entrance from the horizontal, (*b*) the fall in the breast, (*c*) the head of water over the sluice, (*d*) the velocity of the water in the bucket the moment entrance ceases, (*e*) the total mechanical effect, disregarding axle friction.

Ans. (*a*) 53° 53′; (*b*) 6.525 ft.; (*c*) 1.935 ft.; (*d*) 14.9 ft.; (*e*) 15.59 H.P.

61. In the preceding question, if the energy absorbed by axle friction, etc., is 743 ft.-lbs., find the efficiency of the wheel. *Ans.* $\frac{2}{3}$.

62. 20 cub. ft. of water per second enter an undershot wheel of 20 ft. diameter in a straight race, the fall being 3 ft. The depth of the entering stream is $\frac{1}{3}$ ft. The width of the wheel is $4\frac{3}{4}$ ft., and the clearance is $\frac{3}{4}$ inch. The number of the floats, of which four are immersed, is 48, and each is 1 ft. long. The weight of the wheel is 7200 lbs., the radius of the axle is $1\frac{3}{4}$ in., and the coefficient of friction is .1. Find (*a*) the best speed for the wheel, (*b*) the corresponding mechanical effect, (*c*) the efficiency.

Ans. (*a*) 6 ft. per second; (*b*) 2.32 H.P., assuming the speed of wheel reduced to 5.74 ft. per second by axle friction; (*c*) .34.

63. A downward-flow turbine of 24 in. internal diameter passes 10 cub. ft. of water per second under a head of 31 ft; the depth of the wheel is 1 ft. and its width 6 in. Find the efficiency, assuming the whirling velocity at outlet to be nil. *Ans.* .997.

64. A downward-flow turbine of 5 ft. external diameter passes 20 cub. ft. of water per second under a head of 4 ft., the depth of the wheel being 5 ft. The water enters the wheel at an angle of 60° with the vertical, the receiving-lip of the wheel-vanes is vertical, and the velocity of whirl at outlet is nil. Find the internal diameter and the speed in revolutions per minute. *Ans.* 4.68 ft.; 46.53.

65. A downward-flow turbine has an internal diameter of 24 in.; the breadth of the wheel is 6 in.; the turbine passes 33 cub. ft. per second under an effective head of 16 ft. Assuming the whirling velocity at outlet to be nil, find the efficiency and power of the turbine. If the vane-lip at inlet is radial, find the direction of the vane at outlet, and the speed of the turbine in revolutions per minute.

Ans. .931; 55.865 H.P.; $\beta = \gamma = 21° \ 2'$; 166.7.

66. Discuss the preceding question on the assumption that the peripheral speed at outlet (u_2) is equal to the speed of the water at that point relatively to the wheel (V_2).

Ans. .928; 55.715 H.P.; $\beta = 21° \ 47'$ and $\gamma = 20° \ 21'$.

67. An axial-flow impulse turbine of 5 ft. mean diameter passes 170 cub. ft. of water per second under an effective head of 8.6 ft.; the depth of the wheel is .9 ft. At what angle should the water enter the wheel to give an efficiency of 81 per cent, the width of the wheel being constant and disregarding hydraulic resistances? *Ans.* = 27 16′.

68. In example 67, find (*a*) the velocity with which the water enters

the wheel (*b*) the speed of the turbine in revolutions per minute, (*c*) the directions of the vane edges at inlet and outlet, (*d*) the velocity of the water as it leaves the wheel, (*e*) the power of the turbine.

Ans. (*a*) 23.46 ft. per sec.; (*b*) 45.08; (*c*) $\alpha = 130° \ 05'$, $\beta = 42° \ 19'$; (*d*) 10.748 ft. per sec.; (*e*) 148.65 H.P.

69. In example 67, if instead of assuming that the whirling velocity at exit is nil, it is assumed that the peripheral speed (u_2) of the wheel at the mean radius is equal to the relative velocity (V_2) of the water at exit, show how the several results are affected.

Ans. $\gamma = 25° \ 6'$; (*a*) 23.46 ft. per second; (*b*) 54.638; (*c*) $\alpha = 124° \ 54'$, $\beta = 44° \ 7'$; (*d*) 10.748 ft. per second; (*e*) 148.65 H.P.

70. In examples 68 and 69, assuming that the hydraulic resistances necessitate an increase of $12\frac{1}{2}$ per cent in the head equivalent to the velocity with which the water enters the wheel, and an increase of 10 per cent in the head equivalent to the relative velocity (V_2) at outlet, show how the several results are affected.

Ans. Question 68. (*a*) 22.12 ft. per sec.; (*b*) 47.82; (*c*) $\alpha = 121° \ 30'$, $\beta = 40° \ 9'$; (*d*) 10.748 ft. per sec.; (*e*) 148.65 H.P.

Question 69. (*a*) 22.119 ft. per sec.; (*b*) 50.97; (*c*) $\alpha = 124° \ 91'$, $\beta = 47° \ 28'$; (*d*) 10.748 ft. per sec.; (*e*) 148.65 H.P.

71. The efficiency of an axial-flow turbine is 90 per cent, and it passes 12 cub. ft. per second under an effective head of 40 ft. At the mean radius the water enters at an angle of 30° with the wheel's face, and the whirling velocity at outlet is nil. Find (*a*) the velocity with which the water enters and leaves the wheel, (*b*) the directions of the vane at inlet and outlet, (*c*) the sectional areas of the inlet- and outlet-orifices, (*d*) the speed of the wheel in revolutions per minute, (*e*) the power of the turbine.

Ans. (*a*) $42\frac{2}{3}$ ft. per second; 16 ft. per second; (*b*) $\alpha = 49° \ 6'$, $\beta = 21° \ 3'$; (*c*) .75 sq. ft.; (*d*) 198.39; (*e*) $49\frac{1}{11}$ H.P.

72. An axial-flow turbine of 5 ft. mean radius passes 212 cub. ft. of water per second under a total effective head of 12.1 ft. At the mean radius, the direction of the inflowing water makes an angle of 70° with the vertical, and the vane-lip at the outlet makes an angle of 17° with the wheel's periphery. If the whirling velocity at the outlet-surface is nil, find (*a*) the velocity with which the water must enter the wheel to give an efficiency of .953 per cent. Also find (*b*) the direction of the vane-lip at outlet, (*c*) the speed of the wheel in revolutions per minute, (*d*) the widths and areas of the inlet- and outlet-orifices, (*e*) the power of the turbine.

Ans. (*a*) 19.9 ft. per sec.;
(*b*) $\alpha = 81° 22'$; (*c*) 37.67;
(*d*) .991 ft.; 31.148 sq. ft.; 1.181 ft.; 35.14 sq. ft.;
(*e*) 277.799 H.P.

73. An axial-flow impulse turbine passes 170 cub. ft. of water per second under an effective head of 9.5 ft., the depth of the wheel being .9 ft. and its mean radius 4.2 ft. The vane-lip at the outlet makes an angle of 72° with the vertical. Assuming that the whole of the effective head is transformed into useful work, and that the whirling velocity at the outlet-surface is nil, find (*a*) the inclination to the vertical of the outlet-lip of the guide-vane, (*b*) the direction of the inlet-lip of the wheel-vane, (*c*) the efficiency; *first* neglecting hydraulic resistances, and *second* taking these resistances into account.

Ans. First. (*a*) 59° 52′; (*b*) 60° 16′; (*c*) .905;
Second. (*a*) 52° 52′; (*b*) 74° 16′; (*c*) .804.

74. In the preceding example find the inlet- and outlet-orifice areas in the two cases. *Ans.* First. 8.12 sq. ft.; 22.4 sq. ft.;
Second. 9.64 sq. ft.; 28.56 sq. ft.

75. An axial-flow turbine passes 200 cub. ft. of water per second under a head of 14 ft., the depth of the wheel being 1 ft. The mean radius of the wheel is 3 ft.; the areas of the inlet- and outlet-surfaces are in the ratio of 7 to 8; the water enters the wheel at an angle of 21° to the wheel face, and the outlet edge of the vane makes an angle of 16° with the face. Find the speed, efficiency, and power of the turbine, and also the direction of the inlet-lip of the vanes.

Ans. 73.69 revolutions per minute; .954; 325.243 H.P.;
$\alpha = 65° 57'$.

76. In question 11, if there are 62 wheel, and 66 guide-vanes, the thickness of the latter being .2 in. and of the former .4 in., find the width of the inlet-orifices.

77. Water is delivered to an O. F. turbine at a radius of 24 in. with a whirling velocity of 20 ft. per second, and leaves in a reverse direction at a radius of 4 ft. with a whirling velocity of 10 ft. per second. If the linear velocity of the inlet-surface is 20 ft. per second, find the head equivalent to the work done in driving the wheel. *Ans.* 24.8 ft.

78. An outward-flow turbine of 9.5 in. external diameter works under an effective head of 270 ft. Find the speed in revolutions per minute, assuming that the whirling velocity at the inlet-surface relatively to the wheel is nil and that the efficiency is unity. *Ans.* 2242.

79. An outward-flow turbine, whose external and internal diameters are 8 ft. and 5½ ft. respectively, makes 26 revolutions per minute under an effective head of 4 ft. The water enters the wheel in a direction making an angle of 36° (γ) with the direction of motion at the point of entrance. Determine the angles of the *moving vane* at ingress and egress,

the efficiency being .85. Also find the energy per pound of water carried away by the water as it leaves the turbine.

Ans. $\alpha = 129°\ 59'$, $\beta = 29°\ 38'$; .6 ft.-lbs.

80. Construct an outward-flow turbine from the following data: the fall = 5 ft.; internal diameter = 1.8 ft.; external diameter = 2.45 ft.; quantity of water passed per second = 30 cub. ft., $\gamma = 30°$; efficiency = .9.

Ans. $\alpha = 108°\ 15'$; $\beta = 21°\ 3'$; $A_1 = 3.897$ sq. ft.; $A_2 = 5.303$; $d_1 = d_2 = .688$ ft., neglecting thickness of vanes.

81 Assuming that the intensities of the pressure at the receiving and discharging edges of the moving vanes of a Fourneyron turbine are equal, and also that the rim velocity and the velocity of the water relatively to the wheel at the discharging-surface are equal, show that the direction of the impinging stream must bisect the angle between the direction of motion and the tangent to the vane at the receiving edge. Also show that $\dfrac{r_1^2}{r_2^2} = \dfrac{\sin\beta}{\sin 2\gamma}$.

82. If the areas of the inlet- and outlet-orifices of an inward- or outward-flow impulse turbine are equal, show that the efficiency of the turbine is $\cos^2\gamma$, γ being the angle which the direction of the entering water makes with the wheel's periphery.

83. A radial impulse turbine of 4.5 ft. and 4 ft. external and internal radii passes $8\frac{1}{2}$ cub. ft. of water per second under an effective head of 560 ft. The direction of the entering water is inclined at 17° to the wheel's periphery, and the wheel has the same depth at the inlet- and outlet-surfaces. If the peripheral speed at the outlet-surfe (u_2) is equal to the relative velocity of the water (V_2) with respect to the wheel, find (*a*) the efficiency, (*b*) the speed of the turbine in revolutions per minute, (*c*) the sectional areas of the stream at inlet and outlet, (*d*) the direction of the vane-outlet edge, (*e*) the velocity of the water as it leaves the wheel, (*f*) the power of the turbine.

Ans. (*a*) .873; (*b*) 209.94; (*c*) .15357 sq. ft.; .13651 sq. ft.; (*d*) $\beta = 45°\ 2'$; (*e*) 67.39 ft. per sec.; (*f*) 472.33 H.P.

84. In the preceding question examine how the results will be affected when hydraulic resistances are taken into account, allowing .94 as a coefficient of velocity for the water on entering the wheel, and assuming that the head equivalent to the relative velocity (V_2) on leaving the wheel is increased by 10 per cent.

Ans. (*a*) .886; (*b*) 193.76 revols. per minute;
(*c*) .163 sq. ft.; .145 sq. ft.;
(*d*) $\beta = 46°\ 18'$; (*e*) 63.842 ft. per sec.;
(*f*) 479.39 H.P.

85. A radial outward-flow turbine of the impulse type passes $8\frac{1}{2}$ cub. ft. of water per minute under an effective head of 560 ft.; the width of

the wheel is $7\frac{1}{2}$ in.; the radius to the outlet-surface is 1.15 times the radius to the inlet-surface; the linear velocity of the inlet-surface is 87 ft. per second; the direction of the water at entrance makes an angle of 17° with the wheel's periphery. Find (*a*) the efficiency, (*b*) the lip-angles, (*c*) the areas of the inlet- and outlet-orifices, neglecting *first* hydraulic resistances, and *second* taking these resistances into account.

Ans. First. (*a*) .878; (*b*) $\alpha = 149° 31'$ and $\beta = 33° 20'$;
(*c*) .1535 sq. ft. and .1283 sq. ft.;
Second. (*a*) .826; (*b*) $\alpha = 148°$ and $\beta = 17° 19'$;
(*c*) .183 sq. ft. and .306 sq. ft.

86. Construct a Fourneyron turbine for a fall of 5 ft. with 30 cub. ft. of water per second, $\alpha = 80°$, $\gamma = 30°$, $\frac{r_2}{r_1} = 1.35$. Assume $u_2 = V_2$, and neglect hydraulic resistances.

Ans. $\beta = 16° 42'$; $A_1 = 4.29$ sq. ft.; $A_2 = 5.8189$ ft.; $\eta = .915$;
if $r_1 = 1.8$ ft., then $d_1 = d_2 = .38$ ft.

87. In an inward-flow turbine passing 400 gallons of water per minute, the slope of the guide-vane lips is 1 in 5, the radii to the inlet- and outlet-surfaces are 1 ft. and 6 ins., respectively; the breadth of the inlet-orifices is 1.25 ft. Find the efficiency. *Ans.* .98.

88. An I. F. turbine, of 4 ft. external diameter, works under an effective head of 250 ft. Find the speed of the wheel in revolutions per minute. *Ans.* 427.

89. An I. F. turbine of 4 ft. external and 3 ft. internal diameter, makes 360 revolutions per minute. The sectional area of flow is 3 sq. ft. and is the same in every part of the turbine. The direction of the inflowing water makes an angle of 30° with the wheel's periphery. Assuming that the whirling velocity at the outlet-surface is nil, find (*a*) the efficiency, (*b*) the H.P., and (*c*) the delivery in cubic feet per minute. The total head is 200 ft. *Ans.* (*a*) .86; (*b*) 2476.8; (*c*) 7593.

90. An inward-flow turbine being required for an available head of 20 ft. and a discharge of 800 cub. ft. per minute, determine (*a*) the size and (*b*) the speed of the wheel, (*c*) the inclinations of the guide and wheel-vanes, and (*d*) the efficiency of the turbine, assuming $r_2 = \frac{1}{2}r_1 =$ depth of wheel; $v_r' = \frac{1}{8}\sqrt{2gH}$; $v_w'' = 0$ and $\alpha = 90°$.

Ans. (*a*) $r_2 = .487$ ft., $r_1 = .974$ ft.;
(*b*) 240 revolutions per minute;
(*c*) $\gamma = 10° 21'$, $\beta = 36° 8'$; (*d*) $93\frac{3}{4}$ per cent.

91. A vortex turbine passes Q cub. ft. of water per second under an effective head of H ft. The inlet-lip of the vanes is radial and the direction of the entering water makes an angle of 30° with the wheel's periphery. The areas of the inlet- and outlet-orifices are $\frac{\pi D_1^2}{8}$ and $\frac{\pi D_2^2}{5}$

respectively, and the width of the wheel is $\frac{D_1}{10}$, D being the diameter of the inlet-surface. If the whirling velocity at the outlet-surface is nil, find (a) the efficiency, (b) the direction of the outlet edge of the vane, (c) the velocity with which the water enters and leaves the wheel; (d) the speed of the wheel in revolutions per minute, (e) the diameters of the inlet- and outlet-surfaces.

Ans. (a) .938; (b) $\beta = 24° \ 17'$; (c) $6.3291H^{\frac{1}{2}}$; $1.977H^{\frac{1}{2}}$; (d) $116.7\frac{H^{\frac{3}{4}}}{Q^{\frac{1}{2}}}$; ($e$) $.896\frac{Q^{\frac{1}{2}}}{H^{\frac{1}{4}}}$; $.717\frac{Q^{\frac{1}{2}}}{H^{\frac{1}{4}}}$.

92. A vortex turbine passes 11 cub. ft. of water per second under a head of 35 ft.; the diameter of the outlet-surface is 2 ft. and its breadth 6 in. Find the power of the turbine, disregarding friction and assuming that the whirling velocity at the outlet-surface is nil.

Ans. 43.5 H.P.

93. An inward-flow turbine has an internal radius of 12 in. and an external radius of 24 in.; the water enters at 15° with the tangent to the circumference, and is discharged radially; the velocity of outer periphery of wheel is 16 ft. per second. Find the angles of the vanes at the inner and outer circumferences, and the useful work done per pound of fluid. *Ans.* $\beta = 32°$, $\alpha = 118° \ 1'$; 9.33 ft.-lbs.

94. A radial impulse turbine passes $8\frac{1}{2}$ cub. ft. of water under an effective head of 560 ft. The direction of the entering water is inclined at 17° to the wheel's periphery. The linear speed of the inlet-surface is 87 ft. per second. Assuming that the velocity of whirl at the outlet is nil, and disregarding hydraulic resistances, find (a) the efficiency, (b) the velocity with which the water enters the wheel, (c) the velocity of the water as it leaves the wheel, (d) the sectional areas of the inflowing and outflowing stream, (e) the direction of the vane-lip at inlet, (f) the power of the turbine.

The radii of the inlet- and outlet-surfaces are $4\frac{19}{24}$ ft. and $4\frac{1}{8}$ ft. respectively. Find (g) the direction of the vane edge at outlet.

Ans. (a) .879; (b) 189.31 ft. per sec.; (c) 65.86 ft. per sec.; (d) .15356 sq. ft.; .129 sq. ft.; (e) $\alpha = 149° \ 34'$; (f) 475.43 H.P.; (g) $\beta = 33° \ 21'$.

95. In the preceding example show how the results are affected by taking .94 as the coefficient of velocity in calculating the velocity with which the water enters the wheel.

Ans. (a) .828; (b) 178.49 ft. per sec.; (c) 78.36 ft. per sec.; (d) .163 sq. ft.; .1085 sq. ft.; (e) $\alpha = 148° \ 3'$; (f) 441.25 H.P.; (g) $\beta = 38° \ 4'$.

96. In an I. F. turbine the radius to the inlet-surface is twice that to the outlet-surface; the linear velocity of the inlet-surface is one half that due to the head; the water enters the wheel with a velocity of flow

(v_r') equal to one eighth that due to the head, and the sectional area of the water-way is constant from inlet to outlet. Find the angle between the discharging-lip of the vane and the wheel's periphery, the whirling velocity at the outlet-surface being nil. *Ans.* $\text{Cot}^{-1}\sqrt{8}$.

97. In a vortex turbine the depth of the inlet-orifices is one eighth of the diameter of the wheel $\left(=\dfrac{D_1}{8}\right)$ and $\dfrac{25}{32}$ of the depth of the outlet-orifices. The width of the wheel is one tenth of the diameter $\left(=\dfrac{D_1}{10}\right)$. The inlet-lip of the vanes is radial, and the water enters at an angle of 30° with the inlet periphery. Find the size, speed, and efficiency of the turbine in terms of the supply of water Q and the effective head H. Also find the direction of the outlet edge of the vanes.

Ans. I. Assume $v_w'' = 0$. Then $r_1 = .448\dfrac{Q^{\frac{1}{2}}}{H^{\frac{1}{4}}}$;

No. of revolutions per minute $= 116.7\dfrac{H^{\frac{3}{4}}}{Q^{\frac{1}{2}}}$;

$\eta = .938$; $\beta = 24°\ 17'$.

II. Assume $u_2 = V_2$. Then $r_1 = .45\dfrac{Q^{\frac{1}{2}}}{H^{\frac{1}{4}}}$;

No. of revolutions per minute $= 116.3\dfrac{H^{\frac{3}{4}}}{Q^{\frac{1}{2}}}$;

$\eta = .935$; $\beta = 26°\ 49'$.

98. A vortex turbine with a wheel of 2 ft. diameter and 6 in. breadth passes 10 cubic ft. of water per second under a head of 32 feet. Find the inclination of the guides and the power of the turbine. Assume $u_2 = V_2$ and $\alpha = 90°$. *Ans.* 5° 41′, $36\frac{4}{11}$ H.P.

99. An inward-flow turbine has an internal radius of 12 in. and an external radius of 24 in.; the water enters at 15° with the tangent to the circumference, and is discharged radially; the velocity of radial flow is 5 ft. at both circumferences; the velocity of outer periphery of wheel is 16 ft. per second. Find the angles of the vanes at the inner and outer circumferences, and the useful work done per pound of fluid.

Ans. $\beta = 32°$; $d = 118°\ 1'$; 9.33 ft.-lbs.

100. A radial I. F. reaction turbine, with or without draught-pipe, passes 113 cub. ft. of water under an effective head of 13 ft. The radius to the inlet-surface is 1.169 times the radius to the outlet-surface, and the ratio of the outlet to the inlet area is .92. The vane-lip at outlet makes an angle of 15° with the wheel's periphery, and the water enters at an angle of 12° with the wheel's periphery. The sectional area of the draught-tube (if there is one) at the point of discharge is 1.035 times the sectional area of the outlet-orifice. Show that the effective work per pound of

water is 13 ft.-lbs., and that the work consumed in hydraulic resistance (Art. 28, page 303) is nearly 1.96 ft.-lbs.; also find A_1, A_2, v_2, and the efficiency.

Ans. (*a*) 28.26 sq. ft.; 26.12 sq. ft.; (*b*) 4.4 ft. per sec.; .977.

101. In the preceding example, if the radius to the inlet-surface is 4 ft., find (*a*) the speed of the wheel in revolutions per minute. Also find (*b*) the depth of the wheel at inlet and outlet, the guide-vanes being 40 and the wheel-vanes 41 in number, and the thickness of the former being $\frac{3}{16}$ inch and of the latter $\frac{1}{4}$ inch. *Ans.* (*a*) 38.95; (*b*) 1.23 ft.; 1.32 ft.

102. In example 38 find the efficiency if the diameter of the draught-tube is made the same as the diameter of the outlet-surface, the lower edge of the tube being rounded. What will be the "loss in shock" in the tube per pound of water? *Ans.* .861; .071 ft.-lbs.

103. An axial-flow turbine is to be used for raising water. Explain how the vanes should be arranged, write down the resulting equations, and determine the efficiency.

104. Write down the equations for a Jonval modification of Euler's turbine.

105. An inward-flow turbine has an external diameter of 3 ft. and an internal diameter of 2 ft. It passes 12 cub. ft. of water per second under an effective head of 40 ft. The water enters the wheel at an angle of 30° with the wheel's periphery, and the depth of the outlet-orifices is twice the depth of the inlet-orifices. The efficiency of the turbine is .9. Disregarding friction, find (*a*) the vane-angles at inlet and outlet, (*b*) the velocity with which the water leaves the wheel, (*c*) the speed of the turbine in revolutions per minute, (*d*) the velocity with which the water enters the wheel, (*e*) the areas of the outlet- and inlet-orifices, (*f*) the power of the turbine.

Ans. (*a*) $\alpha = 105.09'$, $\beta = 35° \, 35'$ (*b*) 16 ft. per sec.; (*c*) 198.39; (*d*) $42\frac{2}{3}$ ft. per sec.; (*e*) .5625 sq. ft.; .75 sq. ft.; (*f*) $49\frac{1}{11}$ H.P.

106. A centrifugal pump with a 12-in. fan delivers 1000 gallons per minute, the actual lift being 20 ft. and the *gross* lift (allowing for friction, etc.) 30 ft. The velocity of whirl at the outlet-surface is reduced one half. Find the revolutions of the pump per minute. *Ans.* 623.1.

107. In a centrifugal pump the external diameter of the fan is 2 ft., the internal 1 ft., and the width 6 in. Determine the speed and efficiency of the pump when delivering 2000 cub. ft. per minute against a pressure head of 64 ft., the inclination of the wheel-vanes at outlet-surface being 90°. *Ans.* 643.36 revols. per min.; .656.

108. A centrifugal pump delivers 1500 gallons per minute. Fan, 16 in. diameter; lift, 25 ft.; inclination of vanes at outer periphery to the tangent, 30°. Find the breadth at the outer periphery that the velocity of whirl may be reduced one half, and also the revolutions per minute, assuming the *gross* lift to be $1\frac{1}{2}$ times the actual lift.

Also find the proper sectional area of the chamber surrounding the fan for the proposed delivery and lift. Examine the working of the pump at a lift of 15 ft. *Ans.* Breadth, $\frac{3}{4}$ in.; revolutions, 700; 24 sq. in.

109. For a given discharge (Q) and head (H), and considering only the losses of head due to flow and to the resistance in the wheel, show that the maximum efficiency of a centrifugal pump of chamber D is

$$1 - A\frac{D^2H^{\frac{1}{2}}}{Q},$$

A being a constant depending on the size of the wheel.

110. A centrifugal pump lifts 35 cub. ft. of water per second a height of 20 ft. At the outer periphery the vane-angle (β) is 15° and the radial velocity is 5 ft. per second. If the wheel makes 140 revolutions per minute, find (a) its diameter. If the diameter of the outer periphery of the wheel is three times that of the inner periphery and if the radial velocity at the latter is 8 ft. per second, find (b) the vane-angle at the inner periphery and (c) the depths of the wheel at the inner and outer peripheries. *Ans.* (a) $5\frac{1}{2}$ ft.; (b) 30° 58′; (c) 11.1 in.; 5.76 in.

111. The pump in the preceding example is supplied with a vortex chamber of $6\frac{7}{8}$ ft. diameter. Show that the "gain of head" is a maximum when the velocity of flow in the volute is 8.46 ft. per second. Also show that the frictional loss of head is 4.1785 ft.

112. In a centrifugal pump the diameter of the fan = 16 in., the depth = 2 in., the lift = 25 ft., and the delivery = 300 cub. ft. per minute. Determine (a) the speed, (b) the efficiency, and (c) the power expended when the vane-angle (β) at the outer periphery is (1) 90°, (2) 45°, and (3) 30°.

Ans. (1) (a) 785 revols. per min.; (b) .47; (c) 30 H.P.;
(2) (a) 805.8 " " " (b) .58; (c) 24.4 H.P.;
(3) (a) 846.1 " " " (b) .68; (c) 22.9 H.P.

113. An Appold pump delivers 10,000 gallons per minute. The gross lift is 50 ft. The radial velocity at the outlet-surface is one eighth of that due to the gross lift. and the velocity of whirl and the peripheral velocity are reduced one half. Find (a) the radius of the wheel, (b) the vane-angles, (c) the speed of the wheel, (d) the efficiency.

Take the breadth of the wheel at outlet equal to one sixth of the radius, and $g = 32$.

Ans. (a) 1.9 ft.; (b) 56° 16′; 23° 16; (c) 331 revols per min.; (d) .74.

114. The internal and external diameters of the fan of a centrifugal pump are 9 in. and 18 in., respectively; the depth is 6 in., and it passes 400 cub. ft. per minute against a pressure head of 16 ft. The inclination (β) of the discharging-lips of the fan being 30°, determine (a) the speed, (b) the efficiency, (c) the power expended, and (d) the inclination of the receiving-lips of the fan.

Ans. (a) 413.58 revols. per min.; (b) .571; (c) 21.23 H.P.; (d) 19° 48′.

Find the efficiency when a vortex chamber 36 in. in diameter surrounds the fan. *Ans.* .581.

115. A centrifugal pump with a gross lift of 17 ft. delivers 25 cub. ft. of water per second. At the outer periphery the vane-angle is 80° and the radial velocity is 5 ft. per second. The diameters of the outer and inner peripheries of the disc are 54 in. and 18 in., respectively, and the hydraulic efficiency is .75. Find (*a*) the speed of the fan, (*b*) the vane angle at the outlet periphery, (*c*) the velocity of flow in the volute, (*d*) the diameter of the volute, (*e*) the diameter of the suction-pipe.

If there are six $\frac{1}{2}$-in. vanes, find (*f*) the width of the disc at the outer and inner peripheries.

Assuming the discharge-pipe to be 4 ft. per second, show that there is a loss of 5.026 ft. of head due to hydraulic friction.

Ans. (*a*) 116 revols. per min.; (*b*) 41° 14′; (*c*) 26.0 ft. per second; (*d*) 14.7 in.; (*e*) 33.8 in.; (*f*) 9.64 in.; 4.8 in.

116. The vane of a centrifugal pump or turbine is the involute of a circle concentric with the pump circumference. Show that $V_1 = V_2$ in an I. F. or O. F., and $\dfrac{V_1}{V_2} = \dfrac{r_1}{r_2}$ in a D. F.

117. A race is straight and close fitting so that the loss of effect due to escape of water may be disregarded. A single undershot wheel with plane floats is replaced by four similar tandem wheels. If the delivery of each of the four wheels is the same, and if it is assumed that the water reaches each wheel with the same velocity with which it leaves the preceding wheel, find the total maximum velocity due to impact.

Ans. $1\frac{1}{2}$ times the delivery of the single wheel.

118. Discuss the preceding example, assuming that the delivery of each wheel is not the same, but that the total delivery is a maximum.

Ans. 1.6 times the delivery of the single wheel.

119. If n wheels of the same type are substituted for the single wheel in example 117, and if the assumptions are the same as those in example 118, show that the total delivery of the n wheels is to the delivery of the single wheel in the ratio of $2n$ to $2n + 1$, and that, theoretically, if the number is made very large, they will approximately give the entire work of the fall.

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