A TREATISE ON HYDROSTATICS.

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ON

HYDROSTATICS

BY

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PREFACE.

THE aim of the present Treatise on Hydrostatics is to develop the subject from the outset by means of illustrations of existing problems, chosen in general on as large a scale as possible, and carried out to their numerical results; in this way it is hoped that the student will acquire a real working knowledge of the subject, while at the same time the book will prove useful to the practical engineer.

It is very important in Hydrostatics that the units employed should be kept constantly in view; and for this reason the condensed notation proposed by M. Hospitalier at the International Congress of Electricians of 1891 has been adopted. In this notation the full length expression of so many "pounds per square inch" or "kilogrammes per square centimetre" is abbreviated to lb/in^2 or kg/cm^2; and so on for other physical quantities.

The gravitation unit of force has been universally employed, except in a few problems of cosmopolitan interest, in which the variation of gravity becomes perceptible.

In accordance with modern ideas of mathematical instruction, a free use is made of the symbols and operations of the Calculus, where the treatment requires it, although an alternative demonstration by elementary methods is occasionally submitted; because, as it has well been said, "it is easier to learn the Differential Calculus than to follow a demonstration which attempts to avoid its use."

Particular attention has been given to the applications of the subject in Naval Architecture, and the *Transactions of the Institution of Naval Architects* have been ransacked for appropriate illustrations.

The diagrams, which have been drawn by Mr. A. G. Hadcock, late Royal Artillery, are intended to represent accurately to scale the objects described. No attempt has been made to rival the beautiful shaded figures of the French treatises, for fear of obscuring essential principles.

A type of uniform size has been employed throughout: although adding considerably to the bulk, it is hoped that this uniformity will prove acceptable to the eyes of the readers, and counterbalance the disadvantage of the extra size of the book.

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ERRATA.

P. 170, line 9 from the bottom, read "... $fO' = a/s$, then fO' is the depth of a vessel of box form, supposed homogeneous and of S.G. s, which will float at the draft a."

P. 436, last line, and p. 437, line 7, read $P \sin \theta (OQ/OL)$.

HYDROSTATICS.

CHAPTER I.

THE FUNDAMENTAL PRINCIPLES.

1. *Introduction.*

HYDROSTATICS is the Science of the Equilibrium of Fluids, and of the associated Mechanical Problems.

The name is derived from the compound Greek word ὑδροστατική, meaning the Science (ἐπιστήμη) of the Statics of Water; thus Hydrostatics is the Science which treats of the Equilibrium of Water, the typical liquid, and thence generally of all Fluids.

The Science of Hydrostatics is considered to originate with Archimedes (B.C. 250) in his work Περὶ ὀχουμένων, now lost, but preserved in the Latin version of Guillaume de Moerbek (1269), "*De iis quæ vehuntur in humido*"; and recently translated into French by Adrien Legrand, "*Le traité des corps flottants d'Archimède,*" 1891.

Archimedes discovered the method of determining the density and purity of metals by weighing them in water, and extended the same principles to the conditions of equilibrium of a ship or other floating body.

Ctesibius, of Alexandria, and his pupil Hero (B.C. 120), the author of a treatise on Pneumatics, are considered the inventors of the siphon and forcing pump; Vitruvius may be consulted for these and other machines known to the Romans; while the leading principles of the flow of water as required in practical hydraulics are given by Frontinus in his work *de aquæductibus urbis Romæ commentarius* (A.D. 100).

The writings of Pliny (lib. xxx. c. vi.) prove that the Romans were acquainted with the hydrostatical principle that water will rise in a pipe to the height of its source, and that lead pipes must then be employed, stone or brick conduits not being sufficiently watertight; but being ignorant of the method of casting iron pipes strong enough to stand a considerable pressure or head of water, their large aqueducts were carried on the level, while leaden pipes were used only for the distribution of the water, specimens of which pipes have recently been discovered at Bath. A long detailed edict of Augustus concerning the waterworks of Venafrum is given in Mommsen's *Corpus Inscriptionum Latinarum*, vol. 10, part i.; and allusions to the mode of water supply are found in Horace and Ovid—

> "Purior in vicis aqua tendit rumpere plumbum,
> Quam quæ per pronum trepidat cum murmure rivum?"
> (Horace, *Epist.* I. x.)

> "Cruor emicat alte,
> Non aliter quam cum vitiato fistula plumbo
> Scinditur." (Ovid, *Metamorphoses*, iv. 122.)

A great advance in the Theory of Hydraulics was made by Torricelli (1643), also the inventor of the barometer, who first enunciated the true theory of the velocity and form of a jet of water, as deduced from the experiments of Galileo and himself with the ornamental waterworks of the gardens of the Duke of Tuscany; repeated later in 1684 by Mariotte in the gardens of Versailles.

In the writings of Stevinus of Bruges (c. 1600) we find many fundamental theorems of our science clearly enunciated and explained; but the modern exact Theory of Hydrostatics is generally held to originate with Pascal (1653), in his two treatises, *Traité de l'équilibre des liqueurs* and *Traité de la pesanteur de la masse de l'air;* in which the fundamental principles are first clearly enunciated and illustrated, and the true theory and use of the barometer of Torricelli is explained.

The elastic properties of a gas were investigated by Boyle and Mariotte, about 1660, and subsequently completed by Charles and Gay Lussac; and now the fundamental principles of the equilibrium of fluids being clearly enunciated and established, the analysis was carried on and completed by Newton, Cotes, Bernoulli, d'Alembert, and other mathematicians of the 18th century; while the applications of steam in the 19th century has been the cause of the creation of the subject of Thermodynamics, first placed on a sound basis by Joule's experiments, in which the relations are investigated between the heat expended and the work produced by means of the transformations of a fluid medium.

Hydrostatics is a subject which, growing originally out of a number of isolated practical problems, satisfies the requirements of perfect accuracy in its application to the largest and smallest phenomena of the behaviour of fluids; and at the same time delights the pure theorist by the simplicity of the logic with which the fundamental theorems may be established, and by the elegance of its mathematical operations; so that the subject may be considered as the Euclidean Pure Geometry of the Mechanical Sciences.

Montucla's *Histoire des Mathématiques*, t. iii., from which the preceding historical details are chiefly derived, may be consulted for a more elaborate account of the work of the pioneers in this subject of Hydrostatics and Hydraulics.

2. *The Different States of Matter or Substance.*

A FLUID, as the name implies, is a substance which *flows*, or is capable of flowing; water and air are the two fluids most universally distributed over the surface of the Earth.

All substances in Nature fall into the two classes of SOLIDS and FLUIDS; a Solid substance (the land for instance), as contrasted with a Fluid, being a substance which does not flow, of itself.

FLUIDS are again subdivided into two classes, LIQUIDS and GASES, of which water and air are the chief examples.

A LIQUID is a fluid which is incompressible, or nearly so; that is, it does not sensibly change in volume with variations of pressure.

A GAS is a fluid which is compressible, and changes in volume with change of pressure.

Liquids again can be poured from one vessel into another, and can be kept in open vessels; but gases tend to diffuse themselves, and must be preserved in closed vessels.

The distinguishing characteristics of the three *Kinds* of Substances or States of Matter, the SOLID, LIQUID, and GAS, are summarized as follows in Lodge's *Mechanics*, p. 150:—

A SOLID has both size and shape;
A LIQUID has size, but not shape;
A GAS has neither size nor shape.

3. *The Changes of State of Matter.*

By changes of temperature (and of pressure combined) a substance can be made to pass from one of these states to another; thus, by gradually increasing the temperature, a solid piece of ICE can be melted into the liquid state as WATER, and the water again can be evaporated into the gaseous state as STEAM.

Again, by raising the temperature sufficiently, a metal in the solid state can be melted and liquefied, and poured into a mould to assume any required form, which will be retained when the metal is cooled and solidified again; while the gaseous state of metals is discerned by the spectroscope in the atmosphere of the Sun.

Thus mercury is a metal which is liquid at ordinary temperatures, and remains liquid between about −40° C. and 357° C.; the melting or freezing point being −40° C., and the vapourizing or boiling point being 357° C.

Conversely, a combination of increased pressure and of lowered temperature will if carried far enough reduce a gas to a liquid, and afterwards to the solid state.

This fact, originally the conjecture of natural philosophers, has of late years, with the improved apparatus of Cailletet and Pictet, been verified experimentally with air, oxygen, nitrogen, and even hydrogen, the last of the gases to succumb to liquefaction and solidification.

In Professor Dewar's lecture at the Royal Institution, June 1892, liquid air and oxygen were handed round in wine glasses, liquefaction in this case being produced by extreme cold, about $-192°$ C.

All three states of matter of the same substance are simultaneously observable in a burning candle; the solid state in the unmelted wax of the candle, the liquid state in the melted wax around the wick, and the gaseous state in the flame.

Although the three states are quite distinct, the change from one to the other is not quite abrupt, but gradual, during which process the substance partakes of the qualities of both of the adjacent states, as for instance the asphalte pavement in hot weather; metals and glass become plastic near the melting point, and steam is saturated with water at the boiling point.

4. *Plasticity and Viscosity.*

All solid substances are found to be plastic more or less at all temperatures, as exemplified by the phenomena of punching, shearing, and the flow of metals, investigated experimentally by Tresca (vide fig. 8); but what distinguishes the *plastic solid* from the *viscous fluid* is that the plastic solid requires a certain magnitude of stress (shear) to make it flow while the viscous fluid requires a certain length of time for any shearing stress, however small, to permanently displace the parts to an appreciable extent. (K. Pearson, *The Elastical Researches of Barré de Saint Venant*, p. 253.)

According to Maxwell (*Theory of Heat*, p. 303), "When a continuous alteration of form is only produced by stresses exceeding a certain value, the substance is called a solid, however soft (plastic) it may be.

"When the very smallest stress, if continued long enough, will cause a constantly increasing change of form, the body must be regarded as a viscous fluid, however hard it may be."

Mallet, in his *Construction of Artillery*, 1856, p. 122, and Maxwell (*Theory of Heat*, chap. XXI.) illustrate this difference between a soft solid and a hard liquid by a jelly and a block of pitch; also by the experiment of placing a candle and a stick of sealing wax on two supports; after a considerable time the sealing wax will be found bent, but the candle remains straight, at ordinary temperatures.

A quicksand behaves like a fluid, and, in opposition to the process of melting and founding metals, it requires to be artificially solidified in tunnelling operations; this is now affected either by a Freezing Process, in which pipes containing freezing mixtures are pushed into the quicksand, or else by the injection of powdered cement or lime-grouting, which solidifies in combination with the sand.

5. We are now prepared to give in a mathematical form

The Definition of a Fluid.

"A FLUID is a substance which yields continually to the slightest tangential stress in its interior; that is, it can be very easily divided along any plane (given plenty of time, if the fluid is viscous)."

Corollary. It follows that, when the fluid is at rest, the tangential stress in any plane in its interior must vanish, and *the stress must be entirely normal to the plane*—this is the mechanical axiom which is the foundation of the Mathematical Theory of Hydrostatics.

The Theorems of Hydrostatics are thus true for all stagnant fluids, however viscous they may be; it is only when we come to *Hydrodynamics*, the Science of the Motion of Fluids, that the effect of viscosity will make itself felt, and modify the phenomena; unless we begin by postulating *perfect* fluids, that is, fluids devoid of viscosity.

6. *Stress.*

We have used the word STRESS in the Definition of a Fluid above; a stress is defined as composed of two equal and opposite balancing forces, acting between two bodies or two parts of the same body.

These two forces constitute the "Action and Reaction" of Newton's Third Law of Motion, which according to this law "are equal and opposite." (Maxwell, *Matter and Motion*, p. 46.)

The Stress between two parts of a body is either (i.) of the nature of a PULL or TENSION, tending to prevent separation of the parts, or (ii.) of the nature of a THRUST or PRESSURE, tending to prevent approach, or (iii.) of the nature of a SHEARING STRESS, tending to prevent the parts from sliding on each other.

In a Solid Substance all three kinds of Stress can exist, but in a Fluid at rest the stress can only be a normal Thrust or Pressure; a tensional stress would overcome the cohesion of the fluid particles.

Nevertheless a column of mercury, many times the barometric height, may be supported in a vertical tube by its adhesion to the top of the tube, in which case the hydrostatic pressure is negative above the barometric height, or the mercury is in a state of tension; and Mr. Worthington has measured experimentally in ethyl alcohol enclosed in a glass vessel a tension up to 17 atmospheres, or 255 pounds per square inch. (*Phil. Trans.*, 1892.)

The Stress across a dividing plane in a Solid can be resolved into two components, one perpendicular to the plane, of the nature of a tension or pressure, and the other component tangential to the plane; and it is this tangential stress which is absent in a Fluid at rest.

7. *The Measurement of Fluid Pressure.*

If we consider a fluid at rest on one side of any imaginary dividing plane, the fluid is in equilibrium under the forces acting upon it and of the stress across the plane, which is of the nature of a THRUST (*poussée*), perpendicular to the plane.

Definition. "The PRESSURE (*pression*) at any point of the plane is the intensity of the Thrust estimated per unit of area of the plane."

Thus if a thrust of P pounds is uniformly distributed over a plane area of A square feet, as on the horizontal bottom of the sea or of any reservoir, the pressure at any point of the plane is P/A pounds per square foot, (but $P/144A$ pounds per square inch).

If the thrust P is not uniformly distributed over the area A, as for instance on the vertical or inclined face of a wall of a reservoir, then P/A represents the *average* pressure over the area, in pounds per square foot; and the *actual* pressure at any point is the average pressure over a small area enclosing the point.

Thus if ΔP pounds denotes the thrust on a small plane area ΔA square feet enclosing the point, the pressure there is the limit of $\Delta P/\Delta A$ ($=dP/dA$, in the notation of the Differential Calculus) pounds per square foot.

8. *Units of Length, Weight, and Force.*

As we are dealing with a Statical subject, we shall employ the statical gravitation unit of force, which is generally defined as the Attraction of the Earth on the Unit of Weight; but more strictly it is the tension of the plumb line when supporting the Unit of Weight, thus allowing for the discount in the Attraction of the Earth due to its rotation.

The British Unit of Weight is the Pound, defined by Act of Parliament, so that our unit of force is the force which is equal to the tension of a thread or plumb line supporting a Pound Weight; and we shall call this force the FORCE OF A POUND.

With a foot as Unit of Length, our pressures will be measured in *pounds per square foot;* this may be written as lb per □ foot, or □′, or ft², or as lb/ft².

The Metric Units of Length and Weight are the Metre and Kilogramme, or the Centimetre and the Gramme; and with these units, pressure will be given in *kilogrammes per square metre*, or *grammes per square centimetre*.

According to the Act of Parliament, 8th August, 1878, Schedule III.,

1 foot	=	30·47945 centims	=	·304794 metres;
1 metre	=	3·28090 feet	=	39·37079 inches;
1 pound	=	453·59265 grammes	=	·45359265 kg;
1 kilogramme	=	2·20462 lb	=	15432·3487 grains.

Therefore a pressure of one lb/ft² is equivalent to a pressure of $\cdot 4536 \times (3{\cdot}2809)^2 = 4{\cdot}8826$ kilogrammes per square metre (kg/m²); and a pressure of one kg/m² is equivalent to a pressure of

$$2{\cdot}2046 \times (\cdot 3048)^2 = 0{\cdot}2048 \text{ lb/ft}^2.$$

A pressure of one kg/cm^2 is thus 2048 lb/ft^2, or 14·2 lb/in^2; so that the normal atmospheric pressure, called an *atmosphere*, being taken as $14\frac{2}{3}$ or 14·7 lb/in^2, is the same as 1·033 kg/cm^2; and therefore, for practical purposes, the atmosphere may be taken as one kg/cm^2.

With the Gravitation Unit of Force, the weight of a body is at once the measure of the quantity of matter in the body, and also of the force with which it is apparently attracted by the Earth; and the word Weight may be used in either sense without ambiguity or confusion, when dealing with hydrostatical problems on the surface of the Earth.

We must notice however that, in consequence of the variation of g, this unit of force will vary slightly in magnitude at different points of the Earth; but the variation is so small that it makes no practical difference in engineering problems; the variation is only important when we consider tidal or astronomical phenomena, covering the Earth and extending to the Moon, Sun, and planets.

9. *The Safety Valve.*

To measure the pressure of a fluid in a vessel, and to prevent the pressure from exceeding a certain amount, the Safety Valve was invented by Papin, 1681.

It consists essentially of a spherical or conical plug C, fitting accurately into a circular orifice in the vessel, and kept closed against the pressure of the fluid by a lever AB, with fulcrum at A; carrying either a sliding weight W lb, when used on a steady fixed vessel; or else held down at the end B by a spiral spring S, which can be screwed to any desired pull of T lb, when the vessel is subject to shock and oscillation (fig. 1).

Then if the pressure of the fluid on the seat of the valve is p lb/in^2, and the orifice is d inches in diameter, the thrust on the valve is $\frac{1}{4}\pi d^2 p$ lb; so that, taking moments about the fulcrum A of the lever AB,

$$\tfrac{1}{4}\pi d^2 p \times AC = W \times AE \text{ or } T \times AB,$$

when the valve is on the point of lifting.

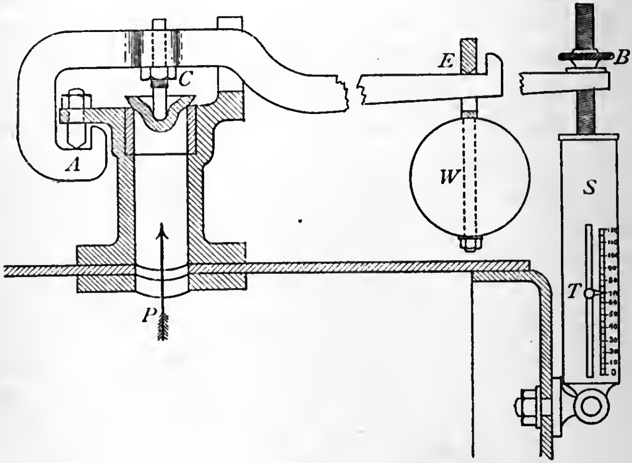


Fig. 1.

Sometimes the valve is held down by a weight (fig. 2) or by a spiral spring, superposed directly without the intervention of a lever as in fig. 3, the form used in steamers and hydraulic machinery.

The danger of the sticking of the valve in the seat is obviated in Ramsbottom's safety valve (fig. 4), consisting of two equal conical valves, held down by a bar and a spring midway between them; then one or the other valve, or both valves, will open when the thrust of the fluid on it is half the pull of the spring.

Where the pressure of a fluid is exerted over a circular area or piston, it is often convenient to estimate the pressure in pounds per circular inch, written as lb/○ in, or lb/○″; and many pressure gauges attached to hydraulic machinery are graduated in this manner; a pressure of p lb/in² being $\frac{1}{4}\pi p$ or $\cdot 7854p$ lb/○ in.

Then the thrust on a circular area d inches in diameter is obtained by multiplying this pressure in lb/○″ by d^2.

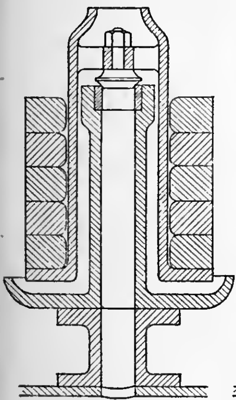


Fig. 2.

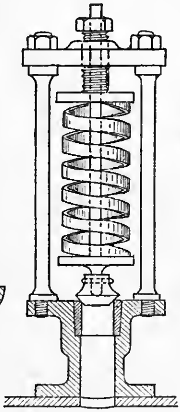


Fig. 3.

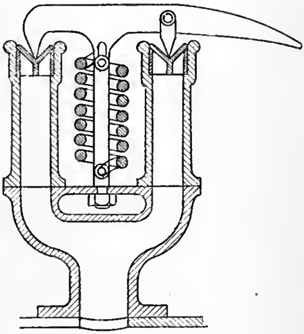


Fig. 4.

It is important in steam boilers that the area of escape from the safety valve should be sufficiently large, so as to allow the steam to escape as fast as it is generated; according to a rule given by Rankine the area of the valve in in² should be 0·006 times the number of lb of water evaporated per hour.

If the orifice of the safety valve is d ins diameter at the top and conical, the semi-vertical angle of the conical plug being a, then a lift of x ins of the valve will give an annular area of internal diameter $d-2x\tan\alpha$ ins, and therefore of area $\pi x\tan\alpha(d-x\tan\alpha)$ in^2.

But if we consider the valve as a flat disc, of d ins diameter, a lift of x ins will give πdx in^2 area of escape sideways.

10. *The Pressure Gauge.*

To measure pressures continually without blowing off at the Safety Valve, the simplest and most efficient instrument is Bourdon's Pressure Gauge (fig. 5).

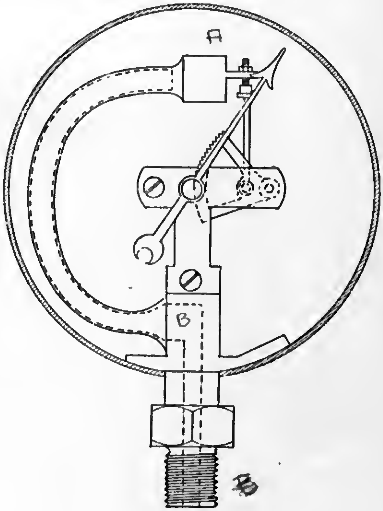


Fig. 5.

This consists essentially of a tube AB, bent into the arc of a circle, closed at one end A, and communicating at the other end B with the vessel containing the fluid whose pressure is to be measured.

The cross section of the tube AB is flattened or elliptical, the longer diameter standing at right angles to the plane of the tube AB, thus ⬭.

The working of the instrument depends upon the principle, discovered accidentally by its inventor M. Bourdon (*Proc. I. C. E.*, XI., 1851), that as the pressure in the interior increases and tends to make the elliptic cross section more circular, the tube AB tends to uncurl into an arc of smaller curvature and greater radius; and the elasticity of the tube AB brings it back again to its original shape as the pressure is removed.

The end B being fixed, the motion of the free end A is communicated by a lever and rack to a pointer on a dial, graduated empirically by the application of known test pressures.

By making the tube AB of very thin metal, and the cross section a very flattened ellipse or double segment, the instrument can be employed to register slight variations of pressure, such as those of the atmosphere; it is then called Bourdon's Aneroid Barometer.

But when required for registering steam pressures, reaching up to 150 or 200 lb/in^2, the tube is made thicker; and when employed for measuring hydraulic pressures of 750 to 1000 lb/in^2, or even in some cases to 5 or 10 tons/in^2, the tube AB must be made of steel, carefully bored out from a solid circular bar, and afterwards flattened into the elliptical cross section, and bent into a circular arc.

Pressures in artillery due to gunpowder reach up to 35,000 or 40,000 lb/in^2, and more, say up to 20 tons/in^2; or from about 2,500 to 3,000 atmospheres, or kg/cm^2; such high pressures require to be measured by special instruments called *crusher gauges*, depending on the amount of crushing of small copper cylinders by the pressure.

11. *The Equality of Fluid Pressure in all directions.*

We may now repeat the Definition of a Fluid given in Maxwell's *Theory of Heat*, chap. V.;

Definition of a Fluid.

"A fluid is a body the contiguous parts of which, when at rest, act on one another with a pressure which is perpendicular to the plane interface which separates those parts."

From the definition of a Fluid we deduce the important

THEOREM. "The pressures in any two directions at a point of a fluid are equal."

Let the plane of the paper be that of the two given directions, and draw an isosceles triangle whose sides are perpendicular to the two given directions respectively, and consider the equilibrium of a small triangular prism of fluid, of which the triangle is the cross section (fig 6).

Let P, Q be the thrusts perpendicular to the sides and R that perpendicular to the base. Then since these three forces are in equilibrium, and since R makes equal angles with P and Q, therefore P and Q must be equal.

But the faces on which P and Q act are also equal; therefore the pressures, or thrusts per unit area, on these faces are equal, which was to be proved.

Generally for any scalene triangle *abc*, the thrusts or forces P, Q, R acting through the middle points of the sides and perpendicular to the sides are in equilibrium if proportional to their respective sides, so that the pressure is the same on each face; and a similar proof will hold if a tetrahedron or polyhedron of fluid is taken.

If we consider the equilibrium of any portion of the fluid enclosed in a polyhedron when the pressure of the fluid is uniform, we are led to the theorem in Statics that

"Forces acting all inwards or all outwards through the centres of gravity of the faces of a polyhedron, each proportional to and perpendicular to the face on which it acts, are in equilibrium."

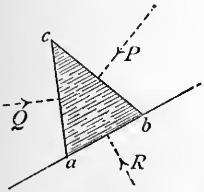


Fig. 6.

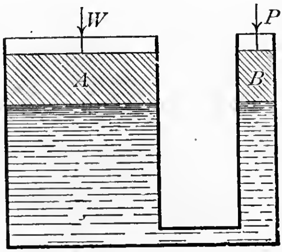


Fig. 7.

12. *The Transmissibility of Fluid Pressure. The Hydraulic Press.*

Any additional pressure applied to the fluid will, if the fluid is an incompressible liquid, be transmitted equally to every point of the liquid: this principle of the "*Transmissibility of Pressure*" was enunciated by Pascal (*Equilibre des liqueurs*, 1653), and applied by him to the invention of

The Hydraulic Press.

This machine consists essentially of two communicating cylinders, filled with liquid, and closed by pistons (fig. 7); then if a thrust P lb is applied to one piston, of area B square feet, it will be balanced by the thrust W lb applied to the other piston of area A square feet such that

$$P/B = W/A,$$

the pressure of the liquid being supposed uniform and equal to P/B or W/A, lb/ft^2; and by making the ratio of A/B sufficiently large, the mechanical advantage W/P can be increased to any desired amount.

The difficulty of keeping the pistons tight against the leakage of the liquid prevented the practical application of Pascal's invention, until Bramah (in 1796) replaced the pistons by plungers (fig. 8) and made a water-tight joint by his invention of the cupped collar CC, pressed into U shape in cross section from an annular sheet of leather, which effectually prevents the escape of the water.

The applied thrust P can be applied, directly or by a lever, to the plunger of a *force pump*, provided with a stuffing box, the invention of Sir Samuel Morland, 1675; and then repeated strokes of the pump will cause the thrust W exerted by the head of the ram to act through any required distance.

In some portable forms, required for instance for punching or rail bending, the pressure is produced and kept up by a plunger P which advances on a screw thread.

For testing gauges Messrs. Schaffer and Budenberg employ an instrument consisting of a small ram working in a horizontal barrel full of water, the traverse of the ram being effected by its revolution in a screw. The gauge to be tested or graduated and the standard gauge are attached to the barrel and each registers the pressure of the water. The machine can even be used for testing vacuum gauges by turning the ram the reverse way, so as to diminish the pressure of the water below the atmospheric pressure.

The Hydrostatic Bellows was devised by Pascal as a mere lecture experiment to illustrate his Principle of the Transmissibility of Pressure; the large cylinder in fig. 7 is replaced by leather fastened to W, as in bellows, while the small cylinder is prolonged upwards by a pipe to a certain vertical height; and the thrust P is produced by the head of water poured in at the top of the pipe by a man standing on a ladder. In this way a small quantity of water poured in the pipe is shown lifting a considerable weight W supported by the bellows, and leakage is

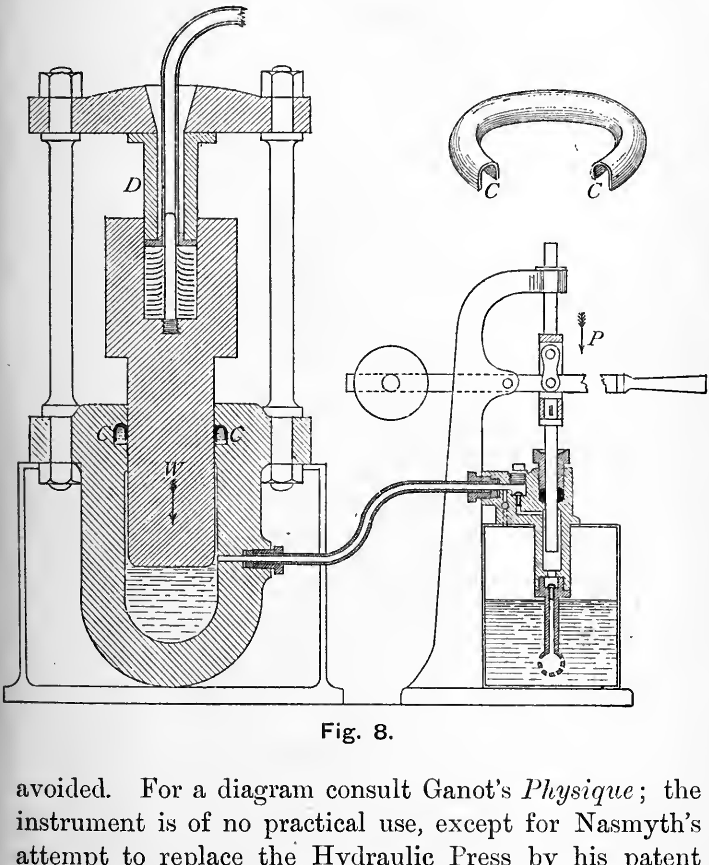


Fig. 8.

avoided. For a diagram consult Ganot's *Physique*; the instrument is of no practical use, except for Nasmyth's attempt to replace the Hydraulic Press by his patent *Steel Mattress* (*Engineer*, 23 May, 1890, p. 426).

13. *The Principle of Virtual Velocities.*

Pascal's Principle of the *Transmissibility of Pressure* was applied by him to verify the Principle of Virtual Velocities in the case of an incompressible liquid, thus showing that a liquid can be made to take the place of a complicated system of levers, in transmitting and multiplying a thrust.

For taking a closed vessel, filled with incompressible liquid, and fitted with cylindrical openings closed by pistons, of areas $A, B, C, \ldots$ ft²; then if the pistons move inwards through distances $a, b, c, \ldots$ feet respectively, the condition that the volume of liquid is unchanged requires that

$$Aa + Bb + Cc + \ldots = 0,$$

some of the quantities $a, b, c, \ldots$ being positive and some negative.

But if $P, Q, R, \ldots$ denote the thrust in lb on the pistons, then

$$P/A = Q/B = R/C = \ldots$$

$=$ the uniform pressure in lb/ft² of the liquid,

and therefore

$$Pa + Qb + Rc + \ldots = 0,$$

a verification of the Principle of Virtual Velocities.

14. *The Energy of Liquid due to Pressure.*

We have supposed the fluid employed to be incompressible liquid: for if a compressible gas had been used to transmit power, part of the energy would be used up in compressing the gas, if used to transmit power; so that a gas would behave like a machine composed of elastic levers.

But with an incompressible liquid the energy is entirely due to the pressure; and if the pressure is p lb/ft², the energy of the liquid is p ft-lb per cubic foot (or p ft-lb/ft³).

As a practical illustration of Pascal's Principle, applied to a closed vessel and a number of pistons, the Hydraulic Power Company of London supply water in mains for the purpose of lifts and domestic motors, at a pressure of 750 lb/in^2, or 108000 lb/ft^2, equivalent to an artificial head of 1728 ft, if a cubic foot of water is taken as weighing 1000 oz or 62·5 lb.

This gives an energy of 1728 ft-lb per lb of water, or 17,280 ft-lb per gallon of 10 lb; so that if water at this pressure is used at the rate of 2 gallons per minute, it furnishes energy at the rate of 34,560 ft-lb per minute, say one horse-power of 33,000 ft-lb per minute, allowing for friction in the pipe, estimated at a velocity of 5 f/s.

With a consumption of 4 million gallons, or 640,000 cubic feet per 24 hours, this gives

$$640{,}000 \times 108{,}000 = 6{\cdot}912 \times 10^{10}$$

ft-lb per 24 hours, equivalent to nearly 1500 H.P.

15. *The Hydrostatic Paradox.*

The fact that a thrust of P lb exerted on a piston of area A ft^2, fitting into a vessel filled with incompressible liquid, produces a pressure $p = P/A$ lb/ft^2 throughout the liquid and an energy of

$$p \text{ ft-lb/ft}^3, \text{ or } pv \text{ ft-lb in } v \text{ ft}^3$$

was considered paradoxical by early writers on Hydrostatics; and numerous experiments, similar in principle to the Hydraulic Press, were devised to exhibit this so-called HYDROSTATIC PARADOX (Hon. R. Boyle, *Hydrostatical paradoxes made out of new experiments, for the most part physical and easy*, 1666); and at the present time the Keeley Motor in America is a paradoxical instrument, devised with the intention of utilizing the hydrostatic energy of pressure.

But this hydrostatic energy is unavailable for continued use, unless replenished as fast as it is used, as by the force pump with the Hydraulic Press; or unless the energy is stored up by the ACCUMULATOR (figs. 9, 10), which consists of a vertical piston or ram B, loaded with weights W so as to produce the requisite pressure p.

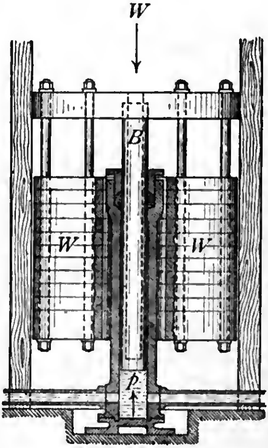


Fig. 9.

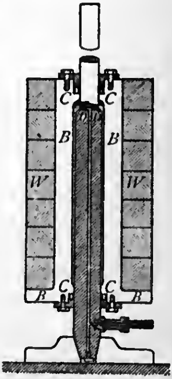


Fig. 10.

Then pv ft-lb of energy are stored up in the Accumulator when the ram is raised so as to displace v ft^3 less of water; that is, if the ram is raised v/A ft, where A is the cross section of the ram in ft^2.

In fig. 10, Mr. Tweddell's form of Accumulator, the area A must be taken as the horizontal cross section of the shoulder at DD.

The Accumulator thus acts as the flywheel of Hydraulic Power; so that an engine working continually and storing up unused energy in the Accumulator can replace a larger engine working only occasionally.

16. *The Applications of the Hydraulic Press.*

Hydraulic Power is now used to a great extent on steamers, for hoisting, steering, and working the guns; an Accumulator however cannot well be carried afloat, on account of its great weight. On land Hydraulic Power is extensively used for cranes and lifts; also on a large scale to replace steam hammers for forging steel by steady squeezing into shape, when a thrust up to 4000 tons is required, and on canals for locks.

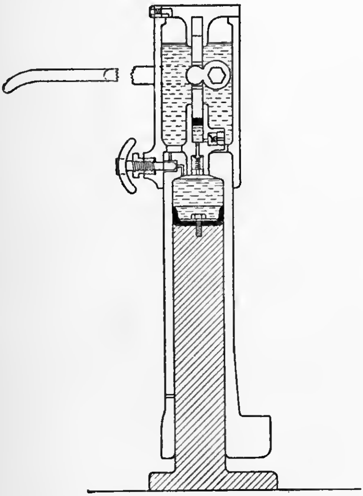


Fig. 11.

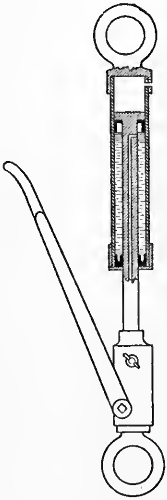


Fig. 12.

On a small scale the Hydraulic Press is useful when applied to jacks, for lifting (fig. 11) or pulling (fig. 12), as manufactured by Tangyes of Birmingham; one great advantage of the machine being that the motion in either direction can be so easily controlled. The Bramah collar in these presses is seen to be replaced by a cupped piece of leather, pressed into shape from a circular sheet.

An application to a certain form of weighing machine (Duckham's) may be mentioned here, consisting of a combination of a Bourdon Gauge and of a small Hydraulic Press, suspended from the chain of a crane (fig. 13).

The pressure of the water in the Press is read off on the Bourdon Gauge, graduated so as to show the weight of the body suspended from the ram of the Press.

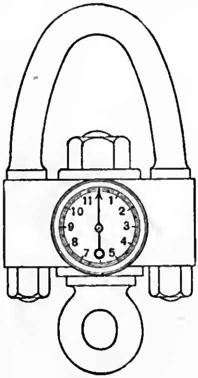


Fig. 13.

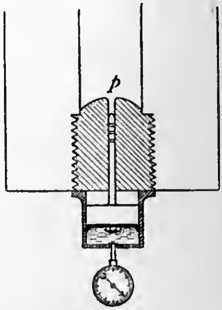


Fig. 14.

In fig. 8 the Hydraulic Press is shown as employed for making elongated rifle bullets or lead pipe; a cylindrical wire or tube of lead, in a semi-molten state, is squeezed out through the hole in the fixed plunger *D*, which fills up the cavity in the ram of the Hydraulic Press, as the ram rises; the length of wire or pipe formed will be to the length of the stroke of the ram as the cross section of the lead cavity to the cross section of the wire or pipe.

The hydrostatic pressure in the molten lead is intensified over the pressure in the water of the Press in the ratio of the cross section of the ram to the cross section of the lead cavity.

"Pressure Intensifying Apparatus" for other purposes, such as rivetting and pressing cotton, applied in a similar manner to the Hydraulic Press, is described by R. H. Tweddell in the *Proc. Inst. Mech. Engineers*, 1872, 1878.

For a general description of the applications of Hydraulic Power and of the Hydraulic Press, the reader is referred to the treatises on *Hydraulic Machinery* by Prof. Robinson and Mr. F. Colyer.

17. *The Amagat Gauge.*

In this gauge (fig. 14) devised for measuring great pressures, the principle is the reverse of the Hydraulic Press, or Pressure Intensifying Apparatus; in the Amagat Gauge (*manomètre à pistons libres*) "unequal pressures act on unequal areas, producing equal thrusts"; so that a pressure p(lb/in^2) acting over an area $a=\frac{1}{4}\pi d^2$ in^2 is measured by a balancing pressure q acting over an area $A=\frac{1}{4}\pi D^2$; and then $pa=qA$, or $p=qA/a=qD^2/d^2$.

(*Nature*, 21 Feb., 1890; *Challenger Scientific Reports on the Compressibility of Water*, by Prof. P. G. Tait.)

To allow for the friction in this gauge, or generally in a Hydraulic Press, suppose the collar is h inches high, and that μ is the coefficient of friction between the leather and the metal; then for a pressure of q lb/in^2 the total normal thrust between the leather and the metal is πDhq, and the frictional resistance to motion is

$$\mu\pi Dhq \text{ pounds.}$$

This gauge might be usefully employed either to test or even to replace the Crusher gauges used in artillery for measuring powder pressures; a mechanical fit of the pistons, if made long and provided with cannelures, is found to offer a sufficient frictional resistance to the leakage of the fluid, so that cupped leather or packing may be dispensed with.

Examples.

(1) What must be the diameter of a safety valve, the weight at the end of the lever being 60 lb, and its distance from the fulcrum 30 in, the weight of the lever 7 lb and its C.G. at 16 in from the fulcrum, the weight of the valve 3 lb and its C.G. at 3 in from the fulcrum, for the valve to blow off at 70 lb/in^2?

Find also the leverage of the weight to allow the steam to blow off at 50 lb/in^2.

(2) In a hydraulic press a thrust of 20 lb is applied at the end of a lever at 6 ft from the fulcrum, actuating the plunger of the force-pump which moves in a line 1 ft from the fulcrum; the plunger is in diameter 1 in and the ram is 10 in; find the thrust in tons exerted by the ram.

(3) The plunger of a force pump is 10 in ($8\frac{3}{4}$) diameter, the length of the stroke is 42 in (30), and the pressure of the water acted upon is 50 lb/in^2. Find the number of ft-lb and ft-tons of work performed in each stroke.

(4) The ram of a hydraulic accumulator is 10 ins in diameter, determine the load in tons requisite for a pressure of 700 lb/in^2.

Find the fall in the ram in 1 minute, if water is not being supplied, and the water is working an engine of 9 H.P.

(5) Give sketches and describe the construction of a hydraulic crane. Estimate the volume of ram necessary if a weight of 5 tons is to be lifted 20 ft, the water pressure being 700 lb/in^2 and efficiency of machine $\frac{1}{2}$.

(6) The hydraulic lifts used in the construction of the Forth Bridge had a diameter of 14 inches and a range of 12 inches; the water was supplied at a pressure of 35 cwt/in², and the lift took 5 hours. Find the lifting force of the ram and its rate of working in terms of a horse-power.

(7) Prove that in consequence of the friction of the collar, the efficiency of the hydraulic press is reduced to

$$1-4\mu n,$$

where n denotes the ratio of the height of the collar to its diameter.

18. THEOREM. "In a fluid at rest under gravity, the pressure at any two points in the same horizontal plane is the same; in other words, the surfaces of equal pressure are horizontal planes."

Suppose A and B are any two points in the same horizontal plane; draw two horizontal planes a short distance apart, one above and the other below AB; and consider the equilibrium of the stratum of fluid between these horizontal planes (fig. 15).

Draw the two vertical planes through A and B perpendicular to AB, and two vertical planes parallel to AB on each side of AB a short distance apart; and consider the equilibrium of the prism of the fluid stratum cut out by these vertical planes.

The fluid pressures being normal to the faces of the prism, and the weight acting vertically downwards, the conditions of equilibrium require the thrusts on the faces perpendicular to AB to be equal; and the faces also being equal, the pressures at A and B are equal.

A similar proof holds when the prism is replaced by any thin cylinder on AB as axis, with ends at A and B perpendicular to the axis AB.

If AC is drawn horizontally and perpendicular to AB, a similar proof shows that the pressures at A and C are equal; as also if AC is drawn in any horizontal direction; and therefore the pressure is the same at all points in the horizontal plane ABC; or in other words—

"The surfaces of equal pressure in a fluid at rest under gravity are horizontal planes."

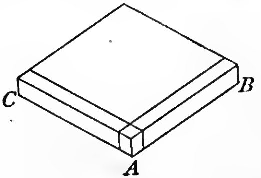


Fig. 15.

Fig. 16.

19. If the fluid is a liquid, it can have a free surface, without diffusing itself, as a gas would; and this free surface, being a surface of zero pressure or more generally of uniform atmospheric pressure, will also be a surface of equal pressure, and therefore a horizontal plane.

Hence the

THEOREM: "The free surface of a liquid at rest under gravity is a horizontal plane."

The theorem can be proved experimentally with great accuracy by noticing that the image by reflexion in the surface of the liquid of a plumb line is straight with the line itself, and not broken; and this proves that the surface is perpendicular to the vertical or plumb line, which is the definition of the horizontal plane; also by the accuracy experienced in the use of the mercurial horizon in Astronomy and Surveying.

Suppose it was possible for the free surface to be changed into a different form; for example into a series of waves at rest, like the hills and dales of dry land, or the surface of the Mer de Glace in Switzerland.

An inclined plane PQ (fig. 16) could then be drawn, cutting off the top of a wave, and the stress across PQ being normal from the definition of a fluid, the plane PQ behaves like a smooth plane, and the top of the wave would begin to slide down PQ, and equilibrium would be destroyed.

Thus the waves could not be at rest, but would move, as we see realized in nature.

These are matters of common observation, as distinguishing characteristics between a solid and a liquid; as for instance between land and water. The surface of the land has hills and valleys, but the surface of water is a horizontal plane. A road or railway has inclines, but a canal is a level road, and locks are required for a change of level.

20. We have supposed in the preceding that no distributed forces, such as those due to gravity, are acting throughout the fluid; and thus the pressure in the fluid will be uniform, and the same in all directions; and to prove this theorem we may consider the equilibrium of any finite portion of the fluid, in the form of a prism, tetrahedron, or polyhedron.

When distributed forces, gravity for instance, act throughout the fluid, the pressure will not be uniform but will vary from one point to another.

Yet the same theorem that "the pressure of a fluid is the same in all directions about a point" can still be established in exactly the same manner, by taking a prism or tetrahedron and making it indefinitely small; then the distributed forces, which are proportional to the weight or volume of the contained fluid, are indefinitely small compared with the thrusts on the faces, which are proportional to the areas of the faces, and may therefore be neglected; and the proof therefore proceeds as before.

In determining the pressure at any point of a fluid at rest, the preceding theorem shows us that we need determine it in one direction only, say in the horizontal or vertical direction when the fluid is at rest under gravity; and this we shall now proceed to investigate.

21. Theorem. "The pressure in a homogeneous liquid at rest under gravity increases uniformly with the depth."

Let p_0 denote the pressure, in lb/ft², at any point of a horizontal plane, and p the pressure at the horizontal plane z feet lower down in the liquid; and let w denote the weight in lb of a cubic foot of the liquid; w then measures the *density* or *heaviness* of the liquid.

Draw any vertical cylinder standing on a base of A square feet, and consider the equilibrium of the liquid filling the part of this cylinder cut off by the two horizontal planes (fig. 17); the liquid is acted upon by a vertical downward thrust p_0A lb on its upper end, by the vertical downward force wzA lb of its gravity, by a vertical upward thrust pA lb on the lower end; and by the thrust of the liquid on the vertical surface of the cylinder.

This last force contributes nothing to the support of the liquid; so that resolving vertically,

$$pA = p_0A + wzA,$$

or

$$p = p_0 + wz,$$

which gives the pressure at any point, and proves that it increases uniformly with the depth.

If z ft is the depth below the free surface, then p_0 denotes the atmospheric pressure on the surface; and if this atmospheric pressure is absent, then

$$p = wz,$$

obtained as before from the consideration of the equilibrium of a cylinder of liquid with zero pressure at the upper end.

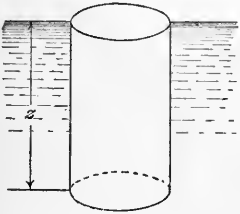


Fig. 17.

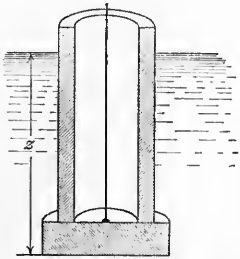


Fig. 18.

To verify this experimentally, take a glass cylinder, with the lower edge ground smooth and greased, and a metal disc of given weight and thickness and of the same diameter as the cylinder, placed on the bottom of the cylinder so as to fit watertight, and held in position by means of a thread (fig. 18).

On submerging the cylinder vertically in liquid, it will be found that the thread may be left slack and the metal disc will be supported by the pressure of the liquid, when the depth of the bottom of the disc is to the thickness of the disc in a ratio equal to or greater than the density of the disc to the density of the liquid; or algebraically, if w denotes the density of the liquid, w' of the metal disc, and a the thickness of the disc,

$$w'aA = wzA, \text{ or } z/a = w'/w.$$

The effect of the atmospheric pressure will not sensibly modify the result, provided the thickness of the glass is inconsiderable; however, in the general case, with atmospheric pressure p_0, and A denoting the area of the disc, B and C the external and internal horizontal sections of the glass cylinder, then

$$w'aA=(p_0+wz)A-(p_0+wz-wa)(A-B)-p_0C,$$

reducing, when $A=B$, to

$$w'aA=wzA+p_0(A-C).$$

In the words of one of Boyle's Hydrostatical Paradoxes, "a solid body as ponderous as any yet known (that is 20 times denser than water, such as gold or platinum), though near the top of the water, can be supported by the upward thrust of the water."

(Cotes, *Hydrostatical and Pneumatical Lectures*, p. 14.)

If we suppose the atmospheric p_0 is the pressure due to an increase of depth h in the liquid, then

$$p_0=wh,$$

and

$$p=w(h+z);$$

so that now the pressure in the liquid is the same as if the free surface of zero pressure was at a height of h feet above the horizontal plane where the pressure

$$p_0=wh.$$

Again p_0 might be the pressure due to liquid of depth h' and density w', so that

$$p_0=w'h',$$

and

$$p=w'h'+wz.$$

So also for any number of superincumbent fluids which do not mix; their surfaces of separation must be horizontal planes, for instance with air or steam on water, water on mercury, and oil on water, etc.

22. *The Head of Water or Liquid.*

The pressure wz at a depth z ft in liquid is called the pressure due to a *head* of z feet of the liquid.

Thus a head of z feet of water, of density or heaviness w lb/ft^3 produces a pressure of wz lb/ft^2 or $wz \div 144$ lb/in^2; and a head of z inches of water produces a pressure of $wz \div 1728$ lb/in^2; and on the average, $w = 62{\cdot}4$.

In round numbers a cubic foot of water weighs 1000 oz, and then $w = 1000 \div 16 = 62{\cdot}5$.

In the Metric System, taking a cubic metre of water as weighing a tonne of 1000 kilogrammes, or a cubic decimetre as weighing a kilogramme, or a cubic centimetre as weighing a gramme, a head of z metres of water gives a pressure of z tonnes per square metre (t/m^2) or 1000 z kg/m^2, or z/10 kg/cm^2, or 100 z grammes per square centimetre (g/cm^2), and a head of z centimetres of water gives a pressure of z g/cm^2; thus a great simplification in practical calculations is introduced by the Metric System of Units.

The pressure of the atmosphere, as measured by the barometer, was taken in § 8 as about $14\frac{2}{3}$ lb/in^2, or 2112 pounds (say 19 cwt, or nearly a ton) per square foot; with Metric Units the atmosphere was taken as one kg/cm^2, or 10 t/m^2); and an atmosphere is thus due, in round numbers, to a head of 30 inches or 76 centimetres of mercury, of specific gravity 13·6; a head of 10 metres or 33 to 34 feet of water; or a head of 26,400 feet or 5 miles, or 8500 metres of homogeneous air of normal density, occupying about 12·5 ft^3 to the lb, or 754 cm^3 to the g, or 0·754 m^3 to the kg.

Any discrepancy in these results is due to taking the nearest round number in each system of units.

Regnault worked with a standard barometric hei[illegible] 76 cm of mercury, and we may call this pressure du[illegible] head of 76 cm of mercury a *Regnault atmosphere*; bu[illegible] is more convenient to take 75 cm; thus a head of 300 [illegible] of mercury, in a tube up the Eiffel Tower, gives a pressure of 400 atmospheres.

The density of sea water is generally taken as 64 lb/ft^3, so that an atmosphere of $14\frac{2}{3}$ lb/in^2 is equivalent to a head of 33 ft of sea water; thus a diver at a depth in the sea of $27\frac{1}{2}$ fathoms or 165 ft experiences a pressure of 5 atmospheres over the atmospheric pressure, in all a pressure of 6 atmospheres or 88 lb/in^2.

In the previous discussions of the Hydraulic Press and Machines working by the Transmission of Pressure we supposed the pressure uniform and neglected the variations due to gravity and difference of level; but these variations are so slight compared with the great pressures employed as to be practically insensible.

Thus a pressure of 750 lb/in^2 is due to a head of 1728 feet of water, compared with which an alteration of 10 feet, or even 100 feet, is insensible.

23. *The Cornish Pumping Engine.*

Suppose M lb is added to W in fig. 7, the equilibrium is destroyed: the piston A will descend say x feet, and the piston B will be raised y feet, such that

$$Ax = By;$$

and now the pressure under the piston A will become

$$(M+W)/A \text{ lb/ft}^2,$$

while under the piston B it will still be P/B lb/ft^2; and the difference between those pressures being due to a head of $x+y$ feet of the liquid,

$$w(x+y) = (M+W)/A - P/B = M/A,$$

22. [illegible] $W/A = P/B$, when P and W balance at the same
The[illegible]; so that
pres[illegible]

$$x = \frac{MB}{wA(A+B)},$$

$$y = \frac{M}{w(A+B)}.$$

This principle is employed in the Cornish Pumping Engine; the plunger or piston A of the pump at the bottom of the mine is weighted by M sufficiently to raise the column of water and B to the surface of the ground; the action of the steam being employed merely to raise the piston A and the weight M at the end of a stroke so as to make the next stroke.

24. *A Liquid maintains its Level.*

By alternate horizontal and vertical steps of appropriate magnitude, we can make the preceding theorems apply to homogeneous liquid contained in a vessel of any irregular shape, so as to be independent of the form of the containing vessel; and thus we prove that the surfaces of equal pressure are horizontal planes and that the pressure increases uniformly with the depth, even when the liquid is divided up into irregular channels, as in water mains; and that if left to itself the water will regain its original level, the principle applied in waterworks.

It was not from ignorance of these hydrostatical principles, but of the art of making strong waterpipes that the Romans constructed high stone aqueducts to carry water to cities on the level; where nowadays iron pipes would be employed, laid in the ground, at great economy and with the additional advantage of escaping long continued frost.

Coming to more recent times, the principle that "liquids maintain their level" was doubted by our engineers when they reported a difference of level of $32\frac{1}{2}$ feet between the Mediterranean and the Red Sea, as making the Suez Canal impracticable. (*Comptes Rendus*, 1858; *British Association Report*, 1875.)

The statements that "a Liquid maintains its Level" but that "a Solid does not maintain its Level" may be taken as the fundamental distinguishing characteristics of a Liquid and a Solid; it is proved experimentally by noticing that the isolated portions of the free surface of a homogeneous liquid, filling a number of communicating vessels of arbitrary shape, all form portions of the same horizontal plane.

The principle is employed not only in the design of waterworks, but also in the theory of levelling instruments, and of the gauge glass of a boiler.

25. THEOREM. "The common surface of two liquids of different densities, which do not mix, is a horizontal plane, when at rest under gravity."

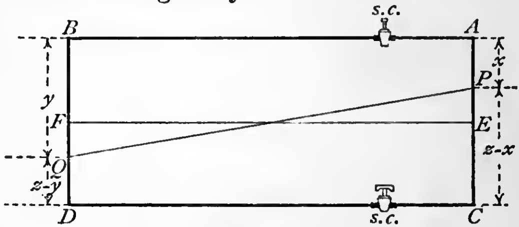


Fig. 19.

Draw any horizontal planes AB, CD in the upper and lower liquids z feet apart and draw vertical lines AC, BD cutting the surface of separation of the liquids in P and Q; and let $AP=x$, $BQ=y$; so that (fig. 19)

$$PC=z-x, QD=z-y.$$

Let w denote the density or heaviness of the upper liquid, w' of the lower. Then p_0 denoting the pressure at the level AB and p at the level CD, by considering the equilibrium of small vertical prisms or cylinders on AC and BD as axes,

$$p-p_0=wx+w'(z-x),$$
$$p-p_0=wy+w'(z-y);$$

so that, by subtraction,

$$(w'-w)(y-x)=0.$$

Now, since by supposition $w-w'$ is not zero, therefore

$$y-x=0, \text{ or } y=x;$$

and this proves that P and Q are in the same level, at E and F; and generally that the common surface EF is a horizontal plane.

26. *The Stability of Equilibrium of Superincumbent Liquids.*

If a number of liquids of different densities, such as mercury, water, and oil, are poured into a vessel, they will come to rest with their common surfaces horizontal planes; and the stability of the equilibrium requires that the densities of the liquids should increase as we go down.

For suppose a portion of two liquids to be isolated in a thin uniform endless tube $APCDQB$, EF representing the original level surface of separation; and suppose P and Q the surfaces of separation when the liquid in the tube is slightly displaced, and kept in this position by a stopcock S.C. in CD or AB.

Supposing the pressures at A and B equal, the pressure at C will exceed the pressure at D by

$$wx+w'(z-x)-wy-w'(z-y)=(w'-w)(y-x);$$

so that if $w'-w$ is positive, the liquid in $ACDB$ will tend to return to its original position of rest when the stopcock

S.C. is opened, and the equilibrium is therefore stable; *vice versa* if $w' - w$ is negative.

Various experiments have been devised for illustrating the instability of the equilibrium of two liquids, when the upper liquid has the greater density; for instance, by taking a tumbler full of water, closed by a card, and inverting it over a tumbler full of wine, so as to fit accurately; when the card is removed or slightly displaced so as to allow a communication between the two liquids, the wine will gradually rise into the upper tumbler and displace the denser water; the card may even be replaced by a handkerchief or a piece of gauze.

Ice again is less dense than water and rests in stable equilibrium on the surface; but if water is run on the surface of the ice, the horizontal plane form becomes unstable, and the ice becomes bent into waves.

By the application of heat variations of density are produced in a liquid which destroy the equilibrium, and set up *convective currents;* as seen exemplified in the Gulf Stream, and in boilers and kettles; also in the winds, and particularly the Trade Winds.

If however the heat is propagated uniformly in a downward vertical direction, the alteration of density does not interfere with the stable equilibrium of the liquid, provided the liquid expands with a rise of temperature, as is generally the case.

When the density of a fluid varies continually, the above arguments show that the fluid comes to rest under gravity so that the density increases in the downward vertical direction (exemplified in the air by the denser layers of fog), and so that

"the surfaces of equal *density* are horizontal planes."

General Exercises on Chapter I.

(1) Define a fluid, a viscous fluid, and explain how viscosity is measured.

Prove that in a perfect fluid the pressure is the same in all directions about a point, but in a viscous fluid only when the fluid is at rest.

Show how to distinguish between a soft solid and a very viscous fluid, and give examples of each.

From what property of a fluid does it follow that any portion of it may be considered solid.

(2) Show that a solid whose faces are portions of spheres is the only one possessing the property that, if immersed in any fluid whatever, the resultant pressure on each face reduces to a single force.

(3) A hollow cone, whose axis is vertical and base downwards, is filled with equal volumes of two liquids, whose densities are in the ratio of $3:1$; prove that the pressure at a point in the base is $(3-\sqrt[3]{4})$ times as great as when the vessel is filled with the lighter liquid.

(4) A vertical right circular cylinder contains portions of any number of fluids that do not mix; show that in equilibrium the fluids arrange themselves in horizontal strata and that the density cannot diminish on descending into the fluid.

Further, show that of all arrangements of the fluids in horizontal strata, geometrically possible, the one actually taken by nature corresponds to the minimum average pressure on the surface of the containing vessel.

(5) Water is poured into a vertical cylinder, whose weight is equal to that of the water which it will contain, and whose centre of gravity is at the middle point of its axis.

Find the position of the centre of gravity of the cylinder and water when the water has risen to a given height within it; show that the whole distance traversed by the centre of gravity while the cylinder is being filled is to the height of the cylinder as $3-2\sqrt{2}:1$; and that when in its lowest position it is in the surface of the water.

If mercury is poured into the water, find when the C.G. of the water and mercury is in its lowest position.

(6) A large metallic shell which is spherical and of small uniform thickness is quite full of water.

A small circular part of the shell is cut out at some distance below the top of the sphere, and provided with a hinge at the highest point of the aperture.

Given W and W' the weights of the shell and the water contained in it; prove that the water will not escape, unless the centre of the aperture and the top of the shell subtends at the centre of the shell an angle greater than

$$\cos^{-1}\frac{3W'}{W+3W'}.$$

CHAPTER II.

HYDROSTATIC THRUST.

27. Let us apply immediately the mechanical Axiom or Corollary of § 5, derived from the Definition of a Fluid, "The Stress on any plane in a Fluid at rest is a Normal Pressure," to the solution of an important Hydrostatical Question, the determination of the

Thrust of Water against a Reservoir Wall.

First, suppose the wall of the reservoir a masonry dam with a vertical face, AB representing the elevation of the face in a plane perpendicular to its length (fig. 20); and draw any plane BC through B the foot of the face AB to meet in C the surface of the water AC (which we have shown to be a horizontal plane).

Consider the equilibrium of the water in ABC; and, merely to fix the ideas, it is convenient to suppose this water solidified or frozen; we shall often make use of this supposition hereafter.

The forces maintaining the equilibrium of ABC are the force W of its weight acting vertically downwards, the thrust P of the wall acting horizontally, and the thrust R of the water on the plane BC, acting perpendicular to BC.

Then, denoting the angle *ABC* by θ, and resolving parallel to the plane *BC*,

$$P \sin\theta = W \cos\theta, \text{ or } P = W \cot\theta.$$

We notice that if $\theta = 45°$, then $P = W$, so that the thrust *P* on the wall is given by the weight of the isosceles prism *ABC*.

Now if $AB = h$, the depth of the water in feet, then $AC = h \tan\theta$; and if *l* denotes the length of the wall, and *w* the weight in lb of a cubic foot of water or the liquid, then

$$W = \tfrac{1}{2} wlh^2 \tan\theta,$$

so that

$$P = \tfrac{1}{2} wlh^2,$$

an expression independent of θ, as it should be.

The average pressure on *AB* is $P/hl = \frac{1}{2}wh$, the pressure in lb/ft² at depth $\frac{1}{2}h$ feet.

But the pressure at any point of *AB* being proportional to the depth, as represented by the ordinate of the straight line *Ab*, the thrust *P* on *AB* is not uniformly distributed over *AB*; but the distribution of thrust *P*, as represented by the pressure *p*, is uniformly varied, as represented by the ordinate of *Ab*; the thrust *P* being represented by the area *ABb*.

Resolving vertically to determine *R*,

$$R \sin\theta = W = \tfrac{1}{2} wlh^2 \tan\theta,$$

or

$$R = \tfrac{1}{2} wlh^2 \sec\theta;$$

also if the vertical through *G*, the centre of gravity of *ABC*, meets *BC* in *K*, then *K* will be the point of application of the resultant thrust *R*; and $CK = \frac{2}{3}CB$.

Therefore also *H* is the point of application of the thrust *P* on *AB*, when *HK* is horizontal, and therefore $AH = \frac{2}{3}AB$; the three forces *P*, *R*, *W* which maintain the equilibrium of *ABC* meeting in *K*.

The points H and K are called the *centres of pressure* of AB and BC.

28. Next suppose the reservoir is bounded by a parallel earthwork dam, forming a wall sloping at an angle α suppose; to determine the thrust on this sloping wall DE.

Again draw through the foot of this wall E a plane EF in any direction; but let us for simplicity choose the direction perpendicular to DE, and let $DE = b = h \operatorname{cosec} \alpha$.

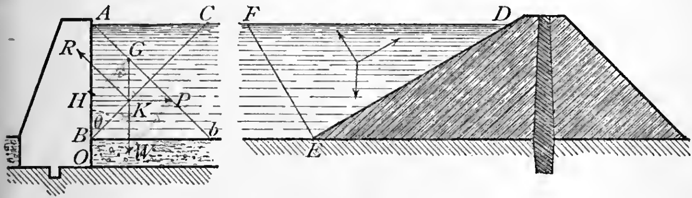


Fig. 20.

Then as before, from the conditions of equilibrium of the liquid in DEF, we find the normal thrust on DE

$$Q = \tfrac{1}{2}wlb^2 \sin \alpha = \tfrac{1}{2}wlh^2 \operatorname{cosec} \alpha;$$

so that the horizontal component of this thrust is

$$Q \sin \alpha = \tfrac{1}{2}wlh^2 \text{ or } P,$$

as it should be; and the three forces which maintain the equilibrium of DEF now intersect in its C.G.

The average pressure over DE is

$$Q/bl = \tfrac{1}{2}wb \sin \alpha = \tfrac{1}{2}wh,$$

the pressure at the depth of the middle point of DE.

The same formulas will give the thrust on the walls of the water reaching to any depth h short of the full depth; and thence we can infer the distribution of stress in the interior of the solid material of the walls, and the conditions of stability to be satisfied so that the walls shall not fail by upsetting or crushing.

This is a very important problem in practical Engineering with the high reservoir walls in existence or course of construction, such as the Vyrnwy dam of the Liverpool Water Works in North Wales, 120 feet high, or the projected Quaker Bridge Dam of the New York Water Works, to be made 260 feet high.

The height of these dams is reckoned from the foundation, which is carried down through the alluvial soil to the solid rock; it is assumed in the design of the dam that the alluvial soil is porous, so that the water pressure is propagated through it.

29. Now if the vertical wall AB in fig. 20 is continued down to the rock foundation at O, an extra depth $BO=a$ feet, then the hydrostatic thrust on the part BO under ground, being the difference of the thrusts on AO and AB, will be given by

$$\tfrac{1}{2}wl\{(a+h)^2-h^2\}=wl(\tfrac{1}{2}a^2+ah);$$

so that the average pressure over BO is $w(\frac{1}{2}a+h)$, the pressure at the depth of the C.G. of BO.

Again, by taking moments round A, the moments of the thrusts on AB and AO being $\frac{1}{3}wlh^3$ and $\frac{1}{3}wl(a+h)^3$, we find that the resultant thrust on BO will act at a depth below A, given by

$$\frac{\frac{1}{3}(a+h)^3-\frac{1}{3}h^3}{\frac{1}{2}(a+h)^2-\frac{1}{2}h^2}=\frac{\frac{1}{3}a^2+ah+h^2}{\frac{1}{2}a+h};$$

and therefore at a height above B

$$a+h-\frac{\frac{1}{3}a^2+ah+h^2}{\frac{1}{2}a+h}=\tfrac{1}{3}a\frac{a+3h}{a+2h}.$$

In earthwork dams a wall of puddled clay must be carried down to the rock foundation, to prevent percolation of water; but the great danger to avoid in an earthwork dam is water running over the crest; the water cuts a channel which increases in size and saws the dam in two, as at the Conemaugh dam failure, which caused the Johnstown flood in America; this dam was strong and safe enough till the water was allowed to overflow the crest.

*30. *The Theory of Earth Pressure.*

Substances in a finely divided, pulverised, or granular state, such as sand, loose earth, grain, meal, or a mass of spherical granules, large or small, such as lead shot or cannon balls, imitate to a certain extent the behaviour of liquids, and require to be restrained by walls; and it is important to determine the thrust which may be expected to be exerted on a retaining wall; the usual procedure is as follows.

Suppose AB (fig. 21) is an end elevation of a vertical wall, which supports a mass of the loose substance, filled up to the level AC of the top of the wall; and drawing any plane BC through the foot of the wall, consider the equilibrium of ABC, supposed solidified.

The wall AB being supposed smooth, the thrust P on it will be horizontal; and supposing the wall AB to yield ever so little horizontally, the prism ABC will slip on BC, and a stress R across BC will be called into play, which now will not be normal to BC, but will make an angle, ϵ suppose, with the normal, in the direction resisting motion.

This angle ϵ, the limiting angle of friction, is taken to be the greatest angle of slope of the loose substance at which it will stand when heaped up; it is also called the *angle of repose* of the substance.

* Articles which may be omitted at a first reading are marked with an asterisk *.

Denoting the angle ABC by θ, and by W the weight in lb of the prism ABC of length l, then

$$W=\tfrac{1}{2}wlh^2\tan\theta,$$

w denoting the heaviness of the substance; and resolving perpendicular to R,

$$\begin{aligned}P&=W\cot(\theta+\epsilon)\\&=\tfrac{1}{2}wlh^2\tan\theta\cot(\theta+\epsilon)\\&=\tfrac{1}{2}wlh^2\frac{\sin(2\theta+\epsilon)-\sin\epsilon}{\sin(2\theta+\epsilon)+\sin\epsilon}\\&=\tfrac{1}{2}wlh^2\left\{1-\frac{2\sin\epsilon}{\sin(2\theta+\epsilon)+\sin\epsilon}\right\}.\end{aligned}$$

For different directions of the plane BC, or different values of θ, P will be greatest when

$$\sin(2\theta+\epsilon)=1, \text{ or } \theta=\tfrac{1}{4}\pi-\tfrac{1}{2}\epsilon; \text{ and then}$$

$$P=\tfrac{1}{2}wlh^2\frac{1-\sin\epsilon}{1+\sin\epsilon}=\tfrac{1}{2}wlh^2\tan^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\epsilon).$$

This is the greatest thrust the wall can on this theory be called upon to support, supposing the loose substance to crack and slide along a plane BC through the foot of the wall; and it is the same as the hydrostatic thrust of a liquid of heaviness $w\tan^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\epsilon)$.

If the friction of the vertical wall AB is taken into account, the theory is more complicated.

For a substance like ice, in which we may suppose the planes of cleavage perfectly smooth, $\epsilon=0$; so that the complete hydrostatic thrust will be restored if ice, frozen up to the level AC, cracks along planes of cleavage; as, for instance, in a glacier.

*31. *Surcharged Retaining Walls.*

Suppose the substance is retained by a parallel vertical wall DE, of less height than the level AC, and that the surface is sloped down to D in the plane FD, called the *talus*; the wall DE is then said to be *surcharged* to the height of ACF above D; and the slope a of FD to the horizon cannot exceed ϵ, the angle of repose.

To determine the horizontal thrust Q on DE, suppose the wall DE to yield horizontally a slight distance, and in consequence the substance to crack along a plane of cleavage EM or EN, making an angle θ with the vertical wall DE.

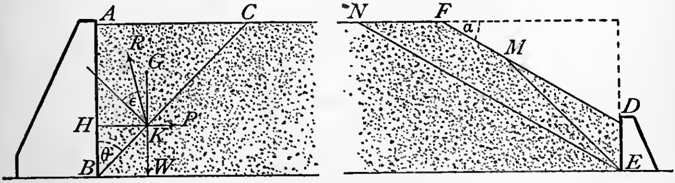


Fig. 21.

We suppose that the prism of material DEM or $DENF$ begins to slide down the plane EM or EN; and then as before, if W denotes the weight in lb of the material in the prism,
$$Q = W\cot(\theta+\epsilon).$$

If the plane EM meets the talus DF in M, and we put $DE=a$, then
$$W = \tfrac{1}{2}wla^2\frac{\sin\theta\cos a}{\cos(\theta+a)},$$
and
$$Q = \tfrac{1}{2}wla^2\frac{\sin\theta\cos a\cos(\theta+\epsilon)}{\cos(\theta+a)\sin(\theta+\epsilon)},$$
reducing to
$$Q = \tfrac{1}{2}wla^2\frac{\sin\theta\cos\epsilon}{\sin(\theta+\epsilon)}$$
$$= \tfrac{1}{4}wla^2\left\{1+\frac{\sin(\theta-\epsilon)}{\sin(\theta+\epsilon)}\right\},$$
if the slope of the talus is the angle of repose, or
$$a=\epsilon.$$

As θ increases from zero, the thrust Q also increases from zero; and when $\theta+\epsilon=\frac{1}{2}\pi$, or the plane EM is parallel to the talus DF,

$$Q=\tfrac{1}{2}wla^2\cos^2\epsilon,$$

the same as for liquid of density $w\cos^2\epsilon$.

But this implies that the wall DE is surcharged to an infinite height; but if surcharged to a finite height b, then when the plane of cleavage EN meets the horizontal level surface in N,

$$W=wl\{\tfrac{1}{2}(a+b)^2\tan\theta-\tfrac{1}{2}b^2\cot a\},$$

and the corresponding value of Q will become a maximum for a value of θ depending not only on ϵ, but also on the ratio of b to a. The determination of this maximum value must be deferred; but now it is important to notice that P and Q are not equal, the difference between them being taken up by the frictional resistance of the ground BE.

*32. *The Thrust due to an Aggregation of Cylindrical Particles or of Spherules.*

An exact Theory of Earth Pressure can be constructed if we suppose the substance which is held up by a retaining wall to be composed of individual particles or atoms of cylindrical form, such as canisters, pipes, barrels, or cylindrical projectiles, regularly stacked; or else to be composed of spherules, such as lead shot, billiard balls, or spherical shot and shell, piled in regular order, as common formerly in forts and arsenals.

It will be necessary to begin by supposing that the lowest layer of cylinders or spheres is imbedded in the ground; as otherwise a wedging action takes place, due to the slightest variation of level, and the problem is to a certain extent indeterminate, as in the preceding article on Earth Pressure.

Now in the case of cylindrical bodies, regularly packed as close as possible (fig. 22), the slope of the talus DF is 60°; and if EN is drawn through E the foot of the retaining wall DE parallel to the talus DF, the thrust between the cylinders across the plane EN will also make an angle of 60° with the horizon; so that considering the equilibrium of $DENF$, of weight W, the thrust Q on the retaining wall DE is given by

$$Q = W \cot 60°.$$

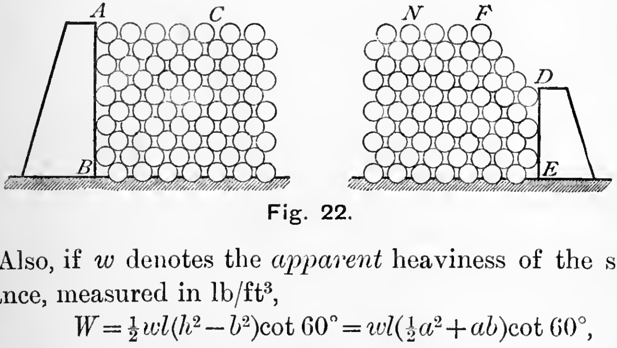


Fig. 22.

Also, if w denotes the *apparent* heaviness of the substance, measured in lb/ft³,

$$W = \tfrac{1}{2} wl(h^2 - b^2) \cot 60° = wl(\tfrac{1}{2}a^2 + ab) \cot 60°,$$

so that $\quad Q = wl(\tfrac{1}{2}a^2 + ab)\cot^2 60° = \tfrac{1}{3} wl(\tfrac{1}{2}a^2 + ab),$

the same as the hydrostatic thrust of liquid, of heaviness $\tfrac{1}{3}w$, on the portion DE of a vertical wall, of which the top edge D is submerged to a depth b in the liquid.

If the cylindrical particles are of diameter d, and composed of solid metal of density ρ, then since the triangular prism formed by the axes of three adjacent cylinders is of cross section $\tfrac{1}{4}\sqrt{3}d^2$, of which only the area $\tfrac{1}{8}\pi d^2$ is occupied by solid metal, therefore

$$\tfrac{1}{8}\pi \rho d^2 = \tfrac{1}{4}\sqrt{3} w d^2, \quad \text{or} \quad w/\rho = \tfrac{1}{6}\pi\sqrt{3};$$

so that $\quad Q = \tfrac{1}{18}\pi\sqrt{3}\rho l(\tfrac{1}{2}a^2 + ab),$

the same as for liquid of density $\tfrac{1}{18}\pi\sqrt{3}\rho$.

Now, suppose the cylinders in the lowest layer are imbedded in the ground, and regularly separated so as to be at a distance x from axis to axis; the slope a of the talus DF is given by

$$\cos a = \tfrac{1}{2}x/d,$$

and $$Q = W \cot a = wl(\tfrac{1}{2}a^2 + ab)\cot^2 a;$$

but now w, the *apparent* heaviness of the substance, is given by $$\frac{w}{\rho} = \frac{\frac{1}{8}\pi d^2}{d^2 \sin a \cos a} = \frac{\pi}{4 \sin 2a};$$

so that $$Q = \tfrac{1}{8}\pi\rho l(\tfrac{1}{2}a^2 + ab)\frac{\cos a}{\sin^3 a},$$

the same as for liquid of density

$$\tfrac{1}{8}\pi\rho\frac{\cos a}{\sin^3 a} = \tfrac{1}{2}\pi\rho\frac{xd^2}{(4d^2 - x^2)^{\frac{3}{2}}}.$$

When $x > d\sqrt{3}$, vertical planes of contact come into existence; and alternate vertical columns descend, so that

$$w/\rho = \tfrac{1}{2}\pi d/x.$$

The thrust Q will become very large when the cylinders are nearly in square order.

The thrust Q is theoretically infinite when the cylinders are in square order; but this arrangement being unstable, a seismic rearrangement takes place, and the original triangular order is regained, except that the talus now appears stepped.

Suppose the wall AB or DE to yield horizontally a slight distance; the cylinders in ABC or $DENF$ will roll and wedge down along planes of cleavage BC or EN.

The lowest layer of cylinders being imbedded, no further motion is possible; but if they were free to roll sideways, a molecular rearrangement would take place, and the cylinders would appear wedged against the walls AB and DE in close order, except along two planes of cleavage.

*33. When the substance is composed of spherules or spherical atoms, we suppose the lowest horizontal stratum is embedded in the ground and arranged

(i.) in square order: (ii.) in triangular order.

In (i.) the spheres in the talus DF are seen in triangular order, in a plane having the slope α of the face of a regular octahedron; and therefore

$$\cos\alpha = \frac{\frac{1}{2}d}{d\cos 30^\circ} = \frac{1}{\sqrt{3}}, \quad \sin\alpha = \sqrt{\tfrac{2}{3}}, \quad \tan\alpha = \sqrt{2};$$

while the thrust across the parallel plane EN is also inclined at an angle α to the horizon; so that

$$\begin{aligned} Q &= W \cot\alpha \\ &= wl(\tfrac{1}{2}a^2 + ab)\cot^2\alpha = \tfrac{1}{2}wl(\tfrac{1}{2}a^2 + ab), \end{aligned}$$

the same as for liquid of density $\frac{1}{2}w$.

In (ii.) the internal arrangement is essentially the same as in (i.), but now the talus DF may show the spheres arranged, either (ii., a) in triangular order, or (ii., b) in square order.

In case (ii., a) the slope α of the talus DF is the slope of a face of a regular tetrahedron on a horizontal base, so that

$$\cos\alpha = \tfrac{1}{3}, \ \sin\alpha = \tfrac{2}{3}\sqrt{2}, \ \tan\alpha = 2\sqrt{2};$$

while the reaction across the plane EN, parallel to DF, will be inclined to the horizon at the angle β, the slope of the edge of the regular tetahedron, so that

$$\cos\beta = \tfrac{1}{\sqrt{3}}, \quad \sin\beta = \sqrt{\tfrac{2}{3}}, \quad \tan\beta = \sqrt{2}.$$

In case (ii., b) the values of α and β are inverted; so that, in each case, (ii., a) and (ii., b),

$$Q = W\cot\alpha\cot\beta = \tfrac{1}{4}wl(\tfrac{1}{2}a^2 + ab),$$

the same as for liquid of heaviness $\frac{1}{4}w$.

If ρ denotes the density, real or apparent, of a si im- spherule, while w denotes the apparent density of be aggregation of a large number of spherules, we shall fithe

that $$\frac{w}{\rho}=\frac{\pi}{3\sqrt{2}}.$$

For if we suppose the horizontal layers in square order, and we take a volume consisting of a very large number n^3 of spheres, standing on a square base whose side is of length nd, then the height will be $\frac{1}{2}\sqrt{2}nd$, and the volume $\frac{1}{2}\sqrt{2}n^3d^3$; while the volume occupied by the n^3 spheres will be $\frac{1}{6}\pi n^3d^3$; and therefore

$$\frac{w}{\rho}=\frac{\pi}{3\sqrt{2}}.$$

So also if the horizontal layers are in triangular order, the length of the volume being nd, the breadth will be $\frac{1}{2}\sqrt{3}nd$, and the height $\frac{1}{3}\sqrt{6}nd$; so that the volume will be $\frac{1}{2}\sqrt{2}n^3d^3$, as before.

When the number of spheres is limited, the effect of the irregularity of the arrangement on the outside of the volume makes itself felt.

Thus 1000 spheres, each one inch in diameter, can be packed in cubical order in a cubical box, the interior of which is 10 inches long each way; but other arrangements are possible by which a larger number of spheres can be packed in the box; the discovery of these arrangements is left as an exercise for the student. (*Cosmos*, Sep. 1887.)

This problem of the packing of spheres is known of old as that of "the thirsty intelligent raven"; the story is given by Pliny, Plutarch, and Ælian; it is quoted by Mr. W. Walton in the *Q. J. M.*, vol. ix., p. 79, in the following form as due to Leslie Ellis:—

* "A thirsty raven flew to a pitcher and found there was water in it but so near the bottom that he could not reach it. Seeing however plenty of equal spherical pebbles near the place, he cast them one by one into the pitcher, and thus by degrees raised the water up to the very brim and satisfied his thirst. Prove that the volume of the water must have been to that of the pitcher in a ratio of $3\sqrt{2}-\pi$ to $3\sqrt{2}$, or more."

If the lead shot were melted, the density would become changed from w to ρ, and the hydrostatic thrust would be increased in the ratio of $3\sqrt{2}$ to π for the same apparent *head* of the substance.

Here again, in a substance composed of spherules, a complicated state of wedging action would take place if the lowest stratum of spheres were not imbedded, but were free to roll on a smooth horizontal floor, especially if the walls were to yield slightly.

(Osborne Reynolds *On the Dilatancy of Media composed of Rigid Particles in Contact.* Phil. Mag., Dec. 1885.

Rankine, *Stability of Loose Earth ;* Phil. Trans., 1867.

Woven Wire, Segregation, and Spherical Packing. Engineering, June 1893.)

Various simple illustrations of the thrust of spheres, leading to elegant statical theorems of the application of the Principle of Virtual Velocities, can be constructed with billiard balls, of given diameter d, placed in open canisters of various diameters D, so that the spheres arrange themselves in horizontal layers (i.) $D < 2d$, singly; (ii.) $2d < D < d(1+\frac{2}{3}\sqrt{3})$, in sets of two and two; or (iii.) $d(1+\frac{2}{3}\sqrt{3}) < D < d(1+\sqrt{2})$, three and one; or (iv.) $d(1+\sqrt{2}) < D$, four and one; etc.

34. *The Thrust on a Lock Gate.*

We have found that the thrust of water, of density w lb/ft³, and of depth h feet, on a rectangular lock gate of breadth l feet is $\frac{1}{2}wlh^2$ lb, and that it may be supposed to act as a single concentrated force at a height $\frac{1}{3}h$ from the bottom of the water (§ 27).

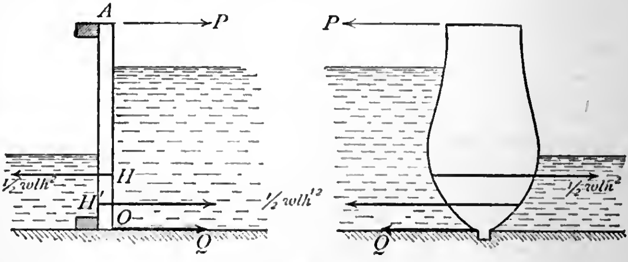


Fig. 23.

Now if there is water of depth h' on the other side of the gate (fig. 23), and consequently a thrust $\frac{1}{2}wlh'^2$ lb acting at a height $\frac{1}{3}h'$ from the bottom; and if the lock gate is supported at A and O, the top and bottom, and that P and Q are the horizontal reactions at these points, then taking moments about A and O, putting $OA = a$,

$$Pa = \tfrac{1}{2}wlh^2 . \tfrac{1}{3}h - \tfrac{1}{2}wlh'^2 . \tfrac{1}{3}h' = \tfrac{1}{6}wl(h^3 - h'^3),$$

$$Qa = \tfrac{1}{2}wlh^2(a - \tfrac{1}{3}h) - \tfrac{1}{2}wlh'^2(a - \tfrac{1}{3}h')$$

$$= \tfrac{1}{2}wal(h^2 - h'^2) - \tfrac{1}{6}wl(h^3 - h'^3),$$

which determines P and Q.

The caisson employed at the mouth of a dry dock has generally the cross section of the right hand of fig. 23, but the horizontal thrust of the water will be the same, as well as the reactions of the supports.

Representing in fig. 24 the plan of a pair of lock gates, with water of depth h feet keeping them shut, then the figure shows that the thrust between the lock gates and the thrust of each gate on its hinge post will be the same; and denoting this thrust by R,

$$2R \sin a = \tfrac{1}{2} wlh^2,$$

or

$$R = \tfrac{1}{4} wlh^2 \operatorname{cosec} a.$$

But if b is the breadth of the lock, then $b = 2l \cos a$; and therefore

$$R = \tfrac{1}{8} wbh^2 \sec a \operatorname{cosec} a = \tfrac{1}{4} wbh^2 \operatorname{cosec} 2a.$$

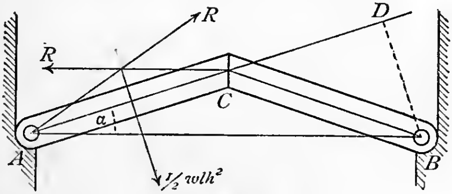


Fig. 24.

Some reservoir dams are curved in plan, for instance, the Bear Valley Dam in California; a butting thrust thereby is set up in the dam, as in a pair of lock gates.

* *Examples.*

(1) Having given the height of a rectangular revetment, find its thickness so that it shall just balance the maximum thrust of a bank of earth of the same height, the natural slope of the earth being given, as also the ratio $s' : s$, of the specific gravities of the wall and bank.

Required the thickness of the revetment that shall give a stability double that due to simple equilibrium: that is, such that the moment of resistance of the wall shall be double that of the bank.

(2) An embankment of triangular section ABC supports the pressure of water on the side AB; find the condition of its not being overturned about the angle C when the water reaches to A, the vertex

of the triangle: and show that, when the area of the triangle is reduced to the minimum consistent with stability for a given depth of water,

$$\tan B = \frac{\sqrt{s^2+2s+9}}{3-s}, \quad \tan C = \frac{\sqrt{s^2+2s+9}}{s-1},$$

where s is the specific gravity of the embankment.

Prove also that the coefficient of friction between the embankment and the ground must exceed

$$1/\{(1+s)\cot B + s\cot C\}.$$

(3) The retaining wall of a reservoir is composed of horizontal courses.

Show how to find the point of application of the resultant stress between successive courses; and prove that the locus of this point (called the *line of resistance*), when the wall is of rectangular section, is a parabola with horizontal axis and vertex at the middle of the top of the wall.

If ϕ is the angle of friction between the blocks, find a formula for the greatest height the wall can have without the blocks sliding; also find a formula for the greatest height it can have if the centre of stress is nowhere to lie in the outer $1/n$ of the breadth of the wall.

(4) Prove that in a trapezoidal retaining wall the line of resistance is a hyperbola, with a horizontal asymptote at twice the height above the top of the wall of the line of intersection of the faces of the wall.

The embankment of a reservoir is composed of thin horizontal rough slabs of stone of specific gravity s and coefficient of friction is μ. The top of the embankment is a feet wide, the face in contact with the water is vertical, and h feet deep.

Show that the slope of the outer side to the vertical must be greater than the larger of the angles

$$\tan^{-1}\left(\frac{1}{\mu s}-\frac{2a}{h}\right) \text{ and } \tan^{-1}\left\{\sqrt{\left(\frac{1}{2s}+\frac{9a^2}{4h^2}\right)}-\frac{3a}{2h}\right\}.$$

(5) At the foot of a long smooth vertical wall, fine sand is heaped so as to form a long prism, with a vertical face resting against the wall, a horizontal face resting on the rough ground, and a slant face inclined at angle 45° to the horizon, this angle being the limiting angle of friction for the sand. Prove that the pressure on the wall is to the vertical pressure on the ground as 1 to 8.

(6) Prove that if the distributed weight on the foundations of a building is p lb/ft², the foundations must be sunk to a depth, in feet,

$$\frac{p}{w}\left(\frac{1-\sin\epsilon}{1+\sin\epsilon}\right)^2=\frac{p}{w}\tan^4\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\epsilon\right),$$

in earth of density w lb/ft³ and angle of repose ϵ.

(7) A set of equal smooth cylinders, tied together by a fine thread in a bundle whose cross section is an equilateral triangle, lies on a horizontal plane. Prove that, if W be the total weight of the bundle and n the number of cylinders in a side of the triangle, the tension of the thread cannot be less than

$$\tfrac{1}{12}\sqrt{3}(1+1/n)^{-1}W \text{ or } \tfrac{1}{12}\sqrt{3}(1-1/n)W,$$

according as n is an even or an odd number; and that these values will occur when there is no thrust between the cylinders in any horizontal row above the lowest.

(8) An even number of equal smooth spheres of diameter r rest in stable equilibrium in a vertical cylinder of radius r.

Determine the thrust exerted by any sphere.

(9) A number n of equal smooth spheres (billiard balls) each of weight W and radius r are placed within a hollow vertical (or inclined) cylinder of radius a, less than $2r$, open at both ends and resting on a horizontal plane. Prove that the upsetting couple due to the thrust of the spheres is

$$(n-1)(a-r)W, \text{ or } n(a-r)W,$$

as n is odd or even; and that the least value of the weight W' of the cylinder in order that it may not upset is given by

$$aW=(n-1)(a-r)W, \text{ or } n(a-r)W.$$

(10) Prove that, if a triangular pile of four equal spheres is held close together on a horizontal table by a triangular frame, the thrust on a side of the frame is $\frac{1}{12}\sqrt{2}$ of the whole weight of the spheres.

35. *Thrust of a Liquid under Gravity against any Plane Surface.*

We have already investigated in § 27 the thrust on the plane rectangular area whose elevation is BC, by considering the vertical component of the thrust as balancing the weight of the superincumbent liquid; and a similar method of procedure will determine the thrust when BC is any plane area (fig. 25).

For consider the equilibrium of the liquid superincumbent on BC, which is bounded by the cylindrical surface $BCcb$, described by vertical lines through the perimeter of BC, cutting the horizontal surface in bc.

Resolving vertically, the thrust on the different parts of the curved cylindrical surface will be horizontal and will not help to support the liquid, so that the vertical component of the thrust on BC will balance the weight of the superincumbent liquid.

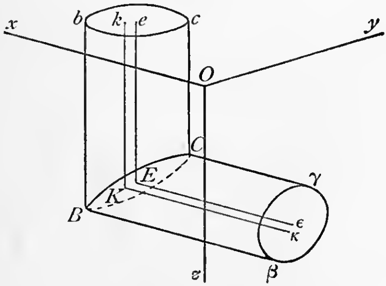


Fig. 25.

Now, if A denotes the area of BC in square feet, and θ the inclination of the plane of BC to the vertical, then the area of bc, called the *projection* of BC on a horizontal plane, is $A \sin\theta$ ft²; and by well-known geometrical theorems, if E is the C.G. of BC, then e, the projection of E, is the C.G. of bc; also the volume of the liquid superincumbent on BC will be that of a right cylinder on the base bc of height Ee.

Putting $Ee=h$, and denoting by W and R the weight of the superincumbent liquid and the thrust on BC in lb, and by w the heaviness of the liquid in lb/ft³, then resolving vertically,

$$R \sin\theta = W = wh \times \text{area } bc = whA \sin\theta,$$

or

$$R = whA;$$

and therefore the average pressure on BC

$$R/A = wh,$$

the pressure in lb/ft² at the depth of E, the C.G. of BC.

36. This theorem will still hold when the plane BC is vertical, although the preceding demonstration fails.

But by projecting the inclined plane BC on any vertical plane $\beta\gamma$ by the horizontal lines $B\beta$, $C\gamma$, $E\epsilon$, $K\kappa$, perpendicular to the plane $\beta\gamma$ (fig. 25), and by resolving perpendicular to the plane $\beta\gamma$ for the equilibrium of the liquid in $B\beta\gamma C$, then we find that the thrust on $\beta\gamma$ is equal to the component of the thrust R on BC perpendicular to the plane $\beta\gamma$.

This proves that the average pressure over $\beta\gamma$ is the pressure at ϵ, the C.G. of $\beta\gamma$; also that the resultant thrust on $\beta\gamma$ must act through κ, the projection of K on $\beta\gamma$, so that κ is the *Centre of Pressure* of $\beta\gamma$.

37. Having found G, the C.G. of the superincumbent liquid, the thrust R will act through K, the point where the vertical line through G meets the plane BC; this point K, which is at double the depth of G, is called the CENTRE OF PRESSURE of the plane area BC; and we are led to this

DEFINITION. "The CENTRE OF PRESSURE of a Plane Area is the point at which the thrust of the Liquid in contact with it may be supposed to act as a single concentrated Force."

The above method gives a simple geometrical construction for determining the C.P. (Centre of Pressure) of a Plane Area in contact with liquid at rest under gravity, by means of which we may prove the following Theorems or Examples:—

(1) The C.P. of a rectangle or parallelogram $ABCD$, with one side AB in the surface of the liquid, is at two-thirds of the depth of the opposite side CD.

This follows from the fact that the C.G. of the superincumbent triangular prism whose cross section is ACc lies in a vertical line GK at a distance from A two-thirds of the distance of the vertical line Cc.

(2) The C.P. of a triangle ABC with the vertex A in the surface and the base BC horizontal is at a depth three-quarters of the depth of BC.

This follows because the C.G. of the superincumbent pyramid $ABCcb$ lies in a vertical line GK at a distance from A three-quarters of the distance of the vertical plane $BCcb$.

(3) The C.P. of a triangle ABC with the base BC in the surface and the vertex A submerged is at a depth one half of the depth of A; because the C.G. of the superincumbent tetrahedron whose opposite edges are Aa, BC lies in a vertical plane midway between these edges.

(4) To determine the C.P. of a triangle ABC with A in the surface and B, C at depths y, z.

Produce BC to meet the surface in D, so that AD is the line of intersection of the plane of the triangle with the surface of the liquid.

Considering the triangle ABC as the difference between the triangles ABD and ACD, we know by (3) that the depths of the C.P.'s of these triangles are $\frac{1}{2}y$ and $\frac{1}{2}z$.

But the depths of their C.G.'s are $\frac{1}{3}y$ and $\frac{1}{3}z$; so that the thrusts of the liquid on the triangles are

$$w \,.\, \tfrac{1}{3}y \,.\, \Delta ABD \text{ and } w \,.\, \tfrac{1}{3}z \,.\, \Delta ACD,$$

and the moments of the thrusts about AD (supposing for an instant the plane of ABC vertical) are

$$w \,.\, \tfrac{1}{6}y^2 \,.\, \Delta ABD \text{ and } w \,.\, \tfrac{1}{6}z^2 \,.\, \Delta ACD;$$

while

$$\frac{\Delta ABD}{\Delta ACD} = \frac{y}{z}.$$

Therefore, denoting the depth of the C.P. of ABC by $\bar{z}$

$$\bar{z}=\frac{w\,.\,\frac{1}{6}y^2\,.\,\Delta ABD-w\,.\,\frac{1}{6}z^2\,.\,\Delta ACD}{w\,.\,\frac{1}{3}y\;\;.\,\Delta ABD-w\,.\,\frac{1}{3}z\;.\,\Delta ACD}$$

$$=\tfrac{1}{2}\frac{y^3-z^3}{y^2-z^2}=\tfrac{1}{2}\frac{y^2+yz+z^2}{y+z}.$$

(5) The general case of a triangle ABC completely submerged, with the corners A, B, C at depths x, y, z.

Produce BC as before to meet the surface in D, and consider the triangle ABC as the difference of the triangles ABD and ACD, the areas of which are in the ratio of BD to CD or y to z, while the depths of their C.G.'s are $\frac{1}{3}(x+y)$ and $\frac{1}{3}(x+z)$.

Therefore, $\bar{z}$ denoting the depth of the C.P. of ABC,

$$\bar{z}=\frac{w.\frac{1}{3}(x+y).\Delta ABD.\frac{1}{2}\frac{x^2+xy+y^2}{x+y}-w.\frac{1}{3}(x+z).\Delta ACD.\frac{1}{2}\frac{x^2+xz+z^2}{x+z}}{w.\frac{1}{3}(x+y).\Delta ABD-w.\frac{1}{3}(x+z).\Delta ACD}$$

$$=\tfrac{1}{2}\frac{y(x^2+xy+y^2)-z(x^2+xz+z^2)}{y(x+y)-z(x+z)}$$

$$=\tfrac{1}{2}\frac{x^2+y^2+z^2+yz+zx+xy}{x+y+z}$$

$$=\frac{\left(\frac{y+z}{2}\right)^2+\left(\frac{z+x}{2}\right)^2+\left(\frac{x+y}{2}\right)^2}{\frac{y+z}{2}+\frac{z+x}{2}+\frac{x+y}{2}},$$

the general formula, including all the preceding cases.

The form of the last result shows that the C.P. coincides with that of three equal small areas in the plane of the triangle, placed at the middle points of the sides; so that the *trilinear* coordinates of the C.P. are

$$\frac{1}{2}\frac{\Delta}{a}\frac{2x+y+z}{x+y+z},\quad \frac{1}{2}\frac{\Delta}{b}\frac{x+2y+z}{x+y+z},\quad \frac{1}{2}\frac{\Delta}{c}\frac{x+y+2z}{x+y+z}.$$

The depth of the C.G. of the triangle ABC being

$$\tfrac{1}{3}(x+y+z),$$

the C.P. is below the C.G. at a depth

$$\tfrac{1}{2}\frac{x^2+y^2+z^2+yz+zx+xy}{x+y+z}-\tfrac{1}{3}(x+y+z)$$

$$=\tfrac{1}{6}\frac{x^2+y^2+z^2-yz-zx-xy}{x+y+z}$$

$$=\tfrac{1}{12}\frac{(y-z)^2+(z-x)^2+(x-y)^2}{x+y+z}.$$

We have supposed that the surface of the liquid is a surface of zero pressure, but if the atmospheric pressure is taken into account, supposed equivalent to a head of H feet of the liquid, then x, y, z, $\bar{z}$ must be increased by H in the last formula, and now

$$\bar{z}+H=\tfrac{1}{2}\frac{(x+H)^2+\ldots+(y+H)(z+H)+\ldots}{x+y+z+3H}$$

$$\bar{z}=\tfrac{1}{2}\frac{x^2+y^2+z^2+yz+zx+xy+2H(x+y+z)}{x+y+z+3H},$$

so that the C.P. is raised thereby a vertical distance

$$\tfrac{1}{2}\frac{x^2+y^2+z^2+yz+zx+xy}{x+y+z}$$

$$-\tfrac{1}{2}\frac{x^2+y^2+z^2+yz+zx+xy+2H(x+y+z)}{x+y+z+3H}.$$

$$=\tfrac{1}{2}H\frac{x^2+y^2+z^2-yz-zx-xy}{(x+y+z)(x+y+z+3H)}$$

$$=\tfrac{1}{12}H\frac{x^2+y^2+z^2-3d^2}{d(d+H)},$$

if d denotes $\tfrac{1}{3}(x+y+z)$, the depth of the C.G. of ABC.

For instance, if the side BC is in the surface, the C.P. is raised by the addition of the atmospheric pressure

$$\tfrac{1}{2}dH/(d+H),$$

one-quarter the H.M. (harmonic mean) of d and H.

38. To find analytically the position of the C.P. of a plane area of A square feet bounded by any closed curve, when placed in a vertical plane with G, the C.G. of the area, at a depth h feet in liquid (when h is called the *mean depth* of the area), we divide up the area A into small elements ΔA, and denote by z the vertical depth of an element below the centre of gravity G, and by $\bar{z}$ the depth below G of the centre of pressure K.

Then, w denoting the heaviness of the liquid, the thrust R on the area A is given by

$$R = \Sigma w(h+z)\Delta A = whA,$$

as before; and $\Sigma z\Delta A = 0$, because G is the C.G. of the area, the symbol Σ denoting summation over the area A.

Again, taking moments about the horizontal line Gy through G, since the moment of the thrust R acting through the centre of pressure is equal to the moment of the separate thrusts on the elements of the area, therefore

$$R\bar{z} = \Sigma w(h+z)\Delta A \,.\, z = w\Sigma z^2\Delta A,$$

since w is constant, and $\Sigma z\Delta A = 0$; and therefore the moment of the resultant thrust about Gy is constant.

Also $\Sigma z^2\Delta A$ is called the *moment of inertia* of the area A about the axis Gy; and putting

$$\Sigma z^2\Delta A = Ak^2,$$

then k in feet is called the *radius of gyration* of the area A about Gy, and Ak^2 is the moment of inertia, in *biquadratic feet* (ft^4).

Then $$R\bar{z} = wAk^2;$$

and therefore $$\bar{z} = k^2/h, \quad \text{or} \quad \bar{z}h = k^2,$$

so that keeping Gy horizontal, and lowering the plane area A in the liquid, then h and $\bar{z}$ are inversely proportional, the C.P. thus tending ultimately to coincidence with G at infinite depth.

The distance HK of the C.P. K from Gz, measured parallel to Gy, and denoted by $\bar{y}$, will be given by

$$R\bar{y} = \Sigma w(h+z)\Delta A \,.\, y = w\Sigma yz\Delta A,$$

since $\Sigma y\Delta A = 0$, G being the C.G. of the area A.

The quantity $\Sigma yz\Delta A$ is called the *product of inertia* of the area A about the lines Gy and Gz, and is denoted by D; but considerations of symmetry will in general enable us to dispense with the consideration of D.

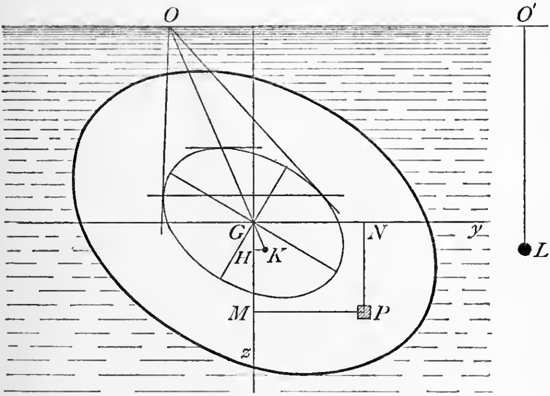


Fig. 26.

39. The student of Dynamics will notice that the C.P. of the area A coincides with the *centre of oscillation*, or the *centre of percussion* of the area, with respect to the horizontal axis OO' in which the surface is cut by the plane of the area A; also that it coincides with the C.G. of the lamina bounded by the perimeter of A, loaded so that the superficial density is proportional to the depth below the free surface of the liquid.

A bullet striking at the C.P. (centre of percussion) K of a target formed of rigid sheet metal of uniform thickness, bounded by the perimeter of the area A, will cause the target to begin rotating about the axis OO', and if suspended from the axis OO', there will be no impulsive action on the axis; so that OO' is called the *axis of spontaneous rotation* with respect to the C.P. K.

Afterwards the target will swing about OO' like a plummet at L, suspended from O' by a thread so as to hang at the level of K.

Returning to the plane area A in the liquid, as we turn the area about the axis OO', the position of K in the plane will not alter; even when coincident with the surface, for the evanescent superincumbent film of liquid will vary in thickness as the distance from OO'.

The effect of taking into account the atmospheric pressure on the surface of the liquid is equivalent to supposing that the plane area A is sunk without rotation in the liquid, so that G is submerged to an additional depth H, the head of liquid equivalent to the atmospheric pressure; h now becomes changed to $h+H$, but k^2 and D remain constant; so that

$$\bar{z}=\frac{k^2}{h+H}, \quad \bar{y}=\frac{D}{A(h+H)},$$

and

$$\frac{KH}{HG}=\frac{\bar{y}}{\bar{z}}=\frac{D}{Ak^2}, \text{ a constant,}$$

so that K will describe a straight line through G in the plane of A.

40. But if the area A is turned about G in its own plane, K will describe a curve in the plane of A.

Suppose the plane turned through the angle θ, then the new depth of the element ΔA is $h+y\sin\theta+z\cos\theta$; so that $\bar{y}, \bar{z}$ denoting the coordinates of the new C.P. with respect to the old axes Gy, Gz, fixed in the area A,

$$R = \Sigma w(h + y \sin\theta + z \cos\theta)\Delta A = whA,$$

and
$$R\bar{y} = \Sigma w(h + y \sin\theta + z \cos\theta)\Delta A \,.\, y$$
$$= wAk_z^2 \sin\theta + wD\cos\theta,$$
$$R\bar{z} = \Sigma w(h + y \sin\theta + z \cos\theta)\Delta A \,.\, z$$
$$= wD\sin\theta + wAk_y^2 \cos\theta,$$

Ak_y^2 and Ak_z^2 now denoting the moments of inertia of the area A about Gy and Gz.

These considerations show that the C.P. describes an ellipse in the plane A; and that the axes of this ellipse are those for which D vanishes; so that, changing to these axes,

$$h\bar{y} = k_z^2 \sin\theta, \quad h\bar{z} = k_y^2 \cos\theta;$$

which shows that the C.P. K is the *antipole* of the line OO' with respect to the ellipse

$$\frac{y^2}{k_z^2} + \frac{z^2}{k_y^2} = 1,$$

the *momental ellipse* or *swing conic* of the area A at G.

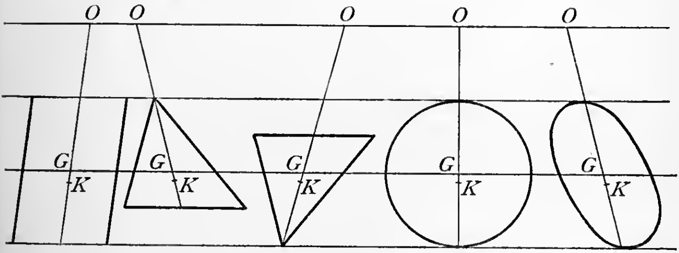


Fig. 27.

If the momental ellipse becomes a circle, as for instance in a square plate, or in a plate bounded by a regular polygon, the C.P. is always vertically below the C.G. of the area, at a depth inversely proportional to the depth of the C.G.

In fig. 27 the position of the C.P. of some simple figures is indicated, whence k^2 for the area about a horizontal axis through its C.G. can be inferred, and *vice versa.*

As drawn in fig. 27, $GK = \frac{1}{12} OG$ in the rectangle or parallelogram, having a pair of sides horizontal; and $GK = \frac{1}{16} OG$ in the triangles, having one side horizontal, and in the circle or ellipse.

Thence we infer that, about a horizontal axis through G,

$$k^2 = \tfrac{1}{12}(\text{height})^2 \text{ for the parallelogram;}$$
$$k^2 = \tfrac{1}{18}(\text{height})^2 \text{ for the triangle;}$$
$$k^2 = \tfrac{1}{16}(\text{height})^2 \text{ for the circle or ellipse.}$$

The *core* of an area (*kern, noyau*) is the name given to the limited region within which the C.P. must lie, if the area is completely immersed; the boundary of the core is therefore the locus of the C.P.'s with respect to water lines which touch the boundary of the area.

Thus the core of a circle or ellipse is a concentric circle or similar ellipse of one-quarter the size.

The C.P.'s of water lines passing through a fixed point lie in a straight line, the *antipolar* of the point; and therefore the core of a triangle is a similar triangle of one-quarter the size, and the core of a parallelogram is another parallelogram the diagonals of which are the middle third parts of the median lines.

*Examples.**

(1) A parallelogram is immersed in a fluid with one side in the surface; show how to draw a line from one extremity of this side, dividing the parallelogram into two parts, on which the pressures are equal.

* For convenience of reference the examples relating to Centre of Pressure are collected here; but the student is recommended at a first reading to attempt only a few of the simple ones, say (1) to (8).

(2) A cubical box filled with water is closed by a lid without weight which can turn freely about one edge of the cube; and a string is tied symmetrically round the box in a plane which bisects the edge. Show that, if the lid is in a vertical plane with this edge uppermost, the tension of the string is one-third of the weight of the water.

(3) A triangle is wholly immersed in a liquid with its base in the surface. Prove that a horizontal straight line drawn through the centre of pressure of the triangle divides it into two portions on which the thrusts are equal.

(4) A mill-race of triangular cross section is closed by a sluice-gate, which is supported at the three corners of the triangle; find what fraction of the pressure on it is supported at each corner.

(5) Water is flowing along a ditch of rectangular section. A horizontal bar is placed across the ditch, and the water stopped by a board fitting the ditch and leaning against the bar. How high must the water rise to force a passage by upsetting the board over the bar?

(6) Show that the centre of pressure of a square or parallelogram immersed with one angular point in the surface and one diagonal horizontal lies in the other diagonal and is at a depth equal to $\frac{7}{12}$ of the depth of its lowest point.

(7) Show that the depth of the centre of pressure of a rhombus totally immersed with one diagonal vertical and its centre at a depth h is

$$h+\tfrac{1}{24}a^2/h,$$

where a is the length of the vertical diagonal.

(8) A parallelogram whose plane is vertical, and centre at a depth h below the surface, is totally immersed in a homogeneous fluid. Show that, if a and b be the lengths of the projections of its sides on a vertical line, then the depth of its centre of pressure will be

$$h+\tfrac{1}{12}(a^2+b^2)/h.$$

(9) A cube is totally immersed in liquid with one diagonal vertical and one angle in the surface; show that the depths of the centres of pressures of its faces are $\frac{7}{18}$ and $\frac{25}{36}$ of the depth of the lowest point.

(10) A square is just immersed in liquid with one corner in the surface and a side inclined at an angle θ to the vertical; prove that the distances of the centre of pressure from the two sides of the square which meet in the surface are respectively

$$\frac{a}{6}\,\frac{4\sin\theta+3\cos\theta}{\sin\theta+\cos\theta} \text{ and } \frac{a}{6}\,\frac{4\cos\theta+3\sin\theta}{\sin\theta+\cos\theta},$$

where a is the length of a side.

(11) Prove that the depth of the centre of pressure of a trapezium immersed in water with the side a in the surface, and the parallel side b, at a depth c below the surface, is

$$\frac{a+3b}{a+2b}\cdot\frac{c}{2}.$$

(12) A lamina in the shape of a quadrilateral $ABCD$ has the side CD in the surface, and the sides AD, BC vertical and of lengths α, β respectively.

Prove that the depth of the centre of pressure is

$$\frac{1}{2}\,\frac{\alpha^4-\beta^4}{\alpha^3-\beta^3}=\frac{1}{2}\,\frac{\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3}{\alpha^2+\alpha\beta+\beta^2}.$$

(13) If a quadrilateral area be entirely immersed in water, and α, β, γ, δ be the depths of its four corners, and h the depth of its centre of gravity, show that the depth of its centre of pressure is

$$\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)-\frac{1}{6h}(\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta).$$

(14) Assuming that any quadrilateral is dynamically equivalent to six particles $+1$ at each corner, -1 at the intersection of the diagonals, and $+9$ at the C.G., find the depth of the centre of pressure in terms of the depths of the corners and of the intersection of the diagonals.

(15) If a plane regular pentagon is immersed so that one side is horizontal and the opposite vertex at double the depth of that side, prove that the depth of the centre of pressure of the pentagon is

$$a(29+3\sqrt{5})\div 48,$$

where a is the depth of the lowest vertex.

(16) If the area be a regular pentagon with one side in the surface and the plane vertical, then the vertical line through the centre of pressure, terminated at the surface, is divided by the centre of the polygon in the ratio

$$2+\sin 18^\circ : 6(1+\sin 18^\circ).$$

(17) A regular hexagon is immersed in a fluid so that its plane is vertical and its highest side in the surface of the fluid; show that the depth of the centre of pressure below the surface is $\frac{23}{36}\sqrt{3}a$, where a is the length of a side of the hexagon; so that the depth of the C.P. is to the depth of the C.G. as 23 to 18.

(18) Show that the centre of pressure of a regular polygon of n sides, wholly immersed in a uniform liquid, and in a vertical plane, is vertically below the centre, and at a distance

$$r^2(2+\cos 2\pi/n)/12h,$$

where h is the depth of the centre and r is the radius of the circumscribing circle.

(19) An elliptic lamina is just immersed in a homogeneous liquid, the major axis being vertical; prove that, if the eccentricity be $\frac{1}{4}$, the centre of pressure will coincide with the lower focus.

(20) A triangular area is immersed in liquid with one side in the surface; the ellipse of largest possible area is inscribed in it; show that the depth of the centre of pressure of the remainder of the triangle is

$$\frac{18\sqrt{3}-5\pi}{36\sqrt{3}-12\pi}$$

of the depth of its lowest point.

(21) A rectangle is immersed in n fluids of densities $\rho, 2\rho, 3\rho, \ldots, n\rho$; the top of the rectangle being in the surface of the first fluid, and the area immersed in each fluid being the same; show that the depth of the centre of pressure of the rectangle is

$$\frac{3n+1}{2n+1}\frac{a}{2},$$

where a is the depth of the lower side.

Deduce the position of the centre of pressure of the rectangle when the density of the liquid varies continuously as the depth.

(22) Prove that the depth of the centre of pressure of a parallelogram two of whose sides are horizontal and at depths a, b respectively below the surface

of a liquid whose density varies as the depth below the surface, is

$$\frac{3}{4}\cdot\frac{a^3+a^2b+ab^2+b^3}{a^2+ab+b^2}.$$

(23) If the centre of gravity G is fixed, and the centres of pressure, when a given line in the area is horizontal and vertical, are respectively K_1, K_2; then, when the line is inclined at an angle θ to the horizontal, the centre of pressure is at K, where GK meets K_1K_2 in D, so that

$$K_1D\cos\theta = K_2D\sin\theta,$$

and
$$GK = GD(\cos\theta + \sin\theta).$$

41. *Component Vertical Thrust of a Liquid under Gravity against any Curved Surface.*

Suppose BC is a portion of a curved surface, for instance, a plate on the bottom of a cistern, boiler, or ship; considering as before the equilibrium of the superincumbent liquid $BCcb$ by resolving vertically, then the *component vertical thrust on BC is equal to the weight of the superincumbent liquid BCcb, and acts vertically upwards or downwards through the C.G. of the superincumbent liquid.*

In fig. 28, BC represents a portion of a curved vessel containing the liquid, and the vertical component of the thrust on BC is downwards.

But suppose, as in fig. 29, that BC is a portion of the curved bottom of a ship; then the vertical thrust on BC is *upwards;* and now the superincumbent liquid must be taken as the fictitious quantity required to fill up the cylinder $BCcb$ to the level bc of the water outside.

So also if BC is the underside or soffit of an arched bridge; in a flood the upward hydrostatic thrust of the water may be sufficient to blow up the bridge, especially if the parapet is solid and the road is walled to prevent flooding.

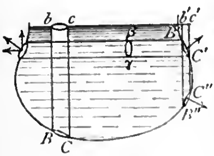


Fig. 28.

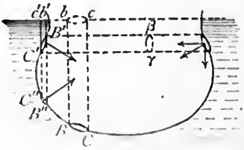


Fig. 29.

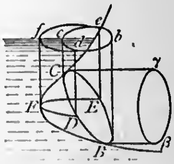


Fig 30.

It might happen, as in fig. 30, that BC was part of a curved surface like the ram of a warship, such that the vertical thrust of the liquid on it was partly upwards and partly downwards; in this case it will be necessary to cut up BC into two parts by the line DFE, along which the tangent planes to the surface BC are vertical, and then to determine the downward vertical thrust on $DCEF$, and the upward vertical thrust on $DBEF$ separately, by the preceding methods.

42. *The Hydrostatic Thrust in a Mould.*

Consider for example the upward and downward thrust on the two parts of a mould used in casting an object like a bell (fig. 31).

The bell-metal being poured in it to the level a of the top of the mould, the upward thrust on the upper half of the mould, tending to lift it and to allow the metal to escape, is equal to the weight of metal which would occupy the volume $BACcab$, and to counteract this upward thrust the mould must be fastened down, or weighted down with corresponding weights.

The downward thrust on the lower part of the mould, called the core, will be equal to the former upward thrust, plus the weight of the bell.

By making the bell very thin, we have the paradoxical result, that a very small amount of liquid metal is capable of exerting a very large upward thrust, especially with metal of high density.

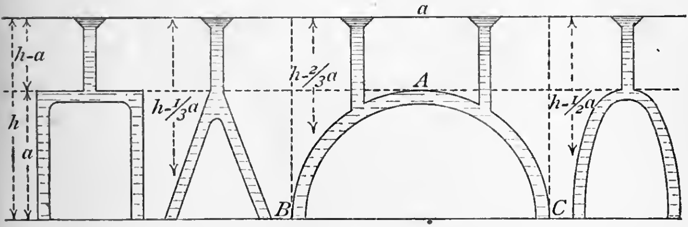


Fig. 31.

In fig. 31, the external shape of the bell is shown as a cylinder, a cone, a hemisphere or hemispheroid, and a paraboloid; if a denotes the height of the bell, and h the head of metal above the bottom of the mould, then the upward thrust on the mould is in these cases equal to the weight of a cylinder of metal whose base is the base of the bell, and altitude $h-a$, $h-\frac{1}{3}a$, $h-\frac{2}{3}a$, $h-\frac{1}{2}a$ respectively; this follows at once from the well-known theorems of Mensuration, due to Archimedes, that

"The volume of a cone is $\frac{1}{3}$, of a hemisphere or hemispheroid is $\frac{2}{3}$, and of a paraboloid is $\frac{1}{2}$ of that of a cylinder on the same base and of the same altitude."

The quasi-hydrostatic thrust of spherules may be similarly illustrated by a funnel filled with lead shot, inverted and resting on a table.

The failure of the Coney Island Stand Pipe, in December, 1886, affords an illustration of the magnitude of the upward vertical thrust of water.

This was a cistern 250 feet high, 16 feet in diameter for the first 70 feet from the ground, and diminishing to a diameter of 8 feet for the upper 160 feet by a shoulder, in the form of a frustum of a cone 20 feet high.

From the given dimensions, it will be found that the resultant upward thrust on the shoulder was over 700 tons, which acting on the lower cylindrical body was sufficient to overcome the weight of metal, 200 tons, and leave an upward force of 500 tons to rupture the seam on the ground.

A leaky tap affords another illustration; here the tension of the screw in the base of the plug, which keeps the plug in place, is practically equal to the thrust of the water at the pressure in the pipe on an annular area, the difference of the cross sections of the conical plug at the ends.

As another illustration of a so-called Paradox, suppose a ship is floated into a dry dock which the ship nearly fits (fig. 32); on readmitting the water, a very small quantity will suffice to float the ship however large, the quantity of water to be admitted being smaller the closer the ship fits the dock.

But suppose a ship or barge to settle down as the tide falls on a soft mudbank (fig. 33); it is often found to happen that the mud will fit so tight against the vessel's bottom that when the tide rises again the water cannot penetrate; and now the vessel loses nearly all buoyancy, and will not rise with the tide, unless care is taken to sway the vessel from side to side, so as to readmit the water underneath the bottom.

This apparent violation of Archimedes' Principle (§ 44) can be illustrated experimentally by means of a flat based body, say a cylinder or cone of wood, greased so as to adhere watertight to the bottom of a vessel, into which quicksilver or water is poured, as shown in fig. 32; it will now be found that the body will remain in contact with the bottom, even after the weight of water displaced exceeds the weight of the wood.

(Cotes, *Hydrostatical and Pneumatical Lectures*, p. 15; 1738.)

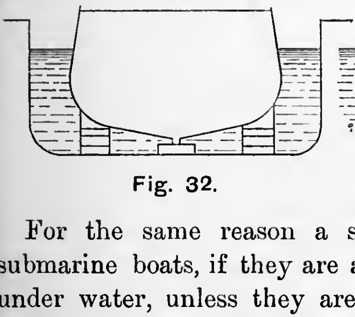


Fig. 32.

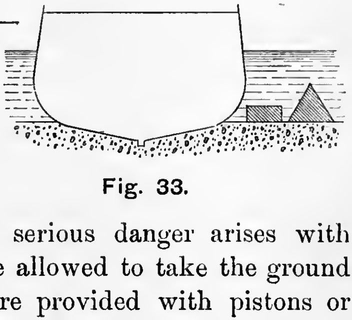


Fig. 33.

For the same reason a serious danger arises with submarine boats, if they are allowed to take the ground under water, unless they are provided with pistons or plungers, to give extra buoyancy and to act as stirrers of the mud. It is reported that one of the caissons of the Forth Bridge failed to rise with the tide, from insufficient buoyancy to overcome the adhesion of the mud and sand in which it had become imbedded.

On the other hand a tunnel, such as the Hudson River Tunnel, or the Blackwall Tunnel, driven through mud and silt, is buoyed up by the surrounding semifluid substance with a force equal to the weight of the substance cut out, and the tunnel should therefore be weighted so as to balance this buoyancy, in order to counteract its distorting and crossbreaking effect.

43. *Pascal's Vases.*

An experimental method, invented by Pascal, of illustrating the preceding principles consists in taking a number of vessels of different shape, but all standing on equal horizontal bases, as in (i), (ii), (iii), (iv), fig. 34.

When filled with the same liquid to the same height h, the thrust on the base BC is found experimentally to be the same, namely the weight of superincumbent liquid contained in a vertical cylinder standing on the same base.

In (i), the vessel being a vertical cylinder, the thrust P on the base is equal to the weight of the liquid; and the resultant thrust of the liquid on the cylinder is zero.

In (ii), the vessel enlarges, and in (iii) the vessel contracts, but the thrust on the base is the same in each case as in (i); so that the resultant thrust of the liquid on the curved surface is the difference between W the weight of the liquid and P the thrust on the base, and acts vertically; it acts vertically downwards and is equal to $W-P$ in (ii), but vertically upwards and is equal to $P-W$ in (iii).

In (iv), the vessel is a slant cylinder or pipe, and the weight of liquid is the same as in (i), and equal to P the thrust on the base; and now the resultant thrust on the curved surface is a couple, of moment $\frac{1}{2}Ph \tan a$, if a is the inclination of the axis of the cylinder to the vertical; this is seen from the consideration of the equilibrium of the liquid in (iv).

The hydrostatic thrust on a piece of straight pipe, slanting at an angle a to the vertical, cut off by two horizontal planes at a vertical distance h is therefore equivalent to a couple of moment $\frac{1}{2}Wh \tan a$, if W denotes the weight of liquid in the pipe between these two horizontal planes.

A similar theorem holds for the thrust exerted by spheres in a vertical or inclined cylinder.

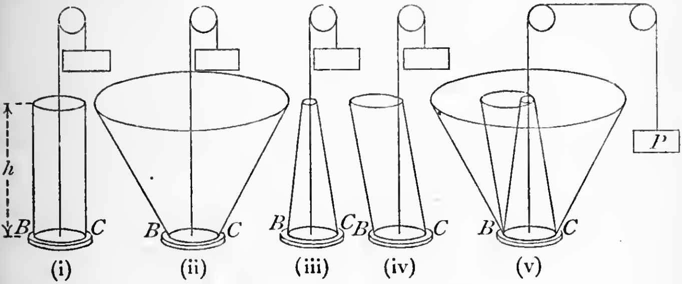


Fig. 34.

44. *A Liquid in Communicating Vessels maintains its Level.*

Communication can be established by pipes between the vessels, and the equilibrium will not be disturbed, if the vertical depth of the liquid is the same in each vessel; but if the depths are originally different, then when the communication is made, liquid will flow from one vessel to the other, till it stands at the same level in each; that is till the free surfaces are in the same horizontal plane.

This principle of § 24 that "Liquids in Communicating Vessels maintain their Level" is employed in the design of water works; it is also seen exemplified in the case of the Ocean; but isolated bodies of water, like the Caspian Sea, the Dead Sea, or large inland lakes, not in direct communication with the Ocean, can have different levels. Thus from surveys it is found that the Caspian is about 83 feet below mean sea level, and the Dead Sea about 1300 feet below mean sea level; while the Aral Sea, fed by the Oxus, is 156 feet above sea level, and the Great Salt Lake in N. America is 4200 feet above sea level.

By this it is meant that if free communication was established with the Ocean, this number of feet would be the change in the depth.

A great part of Holland is below mean sea level, and would be covered by sea water if the banks and dykes were to give way.

45. *Component Horizontal Thrust of a Liquid under Gravity against any Curved Surface.*

Project the curved surface BC on any vertical plane $\beta\gamma$ perpendicular to the given direction (figs. 25, 28, 29) by horizontal lines round BC; and consider the equilibrium of the liquid, real or fictitious, contained in $B\beta\gamma C$.

By resolving perpendicularly to BC, we find that the component horizontal thrust on BC in this direction is equal to the thrust on the plane area $\beta\gamma$, which can be found by a preceding Theorem (§ 36).

Here, again, if the horizontal cylinder through the perimeter BC cuts the surface BC under consideration (fig. 35), we must draw the cylinder formed by the tangent planes of the surface perpendicular to the vertical plane $\beta\gamma$, touching along the curve EF; and consider separately the horizontal thrust perpendicular to the plane $\beta\gamma$ on the two portions of the surface BC into which it is divided by the curve of contact EF.

But as the horizontal thrusts are in opposite directions, we see that the resultant horizontal thrust on the surface is always the same as that on $\beta\gamma$ the projection of BC, however often the cylinder $BC\beta\gamma$ may intersect the curved surface bounded by BC.

To find the *resultant* horizontal thrust on BC, we must find the component horizontal thrust in the direction perpendicular to that first found.

Corollary. If the boundary of BC vanishes, so that BC, instead of being a portion, is the whole surface of a body, immersed or partly immersed in liquid, which is in contact with the liquid, then the horizontal thrust on BC is zero; and the resultant thrust is vertical, and equal to the weight of the displaced liquid.

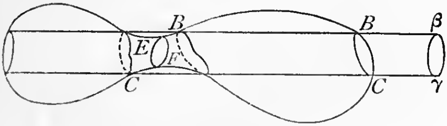


Fig. 35.

In other words "A body plunged into liquid is buoyed up by a force equal to the weight of the displaced liquid, acting vertically upwards through the C.G. of this liquid."

This Corollary is important as the first established Theorem of Hydrostatics, and it is called *Archimedes' Principle*, from the name of its discoverer; we shall return to this Theorem and its consequences in Chapter III.

46. *The Average Pressure over a Surface.*

To find the *average pressure* over a curved surface BC, we must find the sum of all the thrusts on every element of BC and then divide by the area of the surface.

Now if S denotes the area of the curved surface, and ΔS of an element of the surface at a depth z, at which the pressure is
$$p = wz,$$
the liquid being homogeneous and at rest under gravity, then the sum of all the normal thrusts
$$= \Sigma p \Delta S = \Sigma wz \Delta S = w \Sigma z \Delta S = w\bar{z}S,$$
if $\bar{z}$ denotes the depth of the C.G. of the surface S; so that the average pressure is $w\bar{z}$, the pressure at the depth of the C.G. of the surface.

The sum of all the normal thrusts on a surface is sometimes called the WHOLE NORMAL PRESSURE, on the surface; but as it has no mechanical significance for a curved surface and employs the word *pressure* in the sense of *thrust*, we shall seldom employ this expression, but use the idea of AVERAGE PRESSURE instead.

47. In the case of a plane surface, the whole pressure and the RESULTANT THRUST are the same; and we find as before (§ 35) that the AVERAGE PRESSURE is the pressure at the C.G. of the plane area.

For instance, if a rectangle or parallelogram $ABCD$, with one side AB in the surface of the liquid, is to be divided up by horizontal straight lines into n parts on which the hydrostatic thrust is the same, then if PQ is the rth line of division below AB, the thrust on $ABPQ$ must be r/n of the thrust on $ABCD$; so that if x_r and h denote the depths of PQ and CD below AB, and $AB=a$,

$$\frac{wax_r \,.\, \frac{1}{2}x_r}{wah \,.\, \frac{1}{2}h}=\frac{r}{n}, \text{ or } \frac{x_r}{h}=\sqrt{\frac{r}{n}};$$

so that the depths of the dividing lines below AB are as

$$1:\sqrt{2}:\sqrt{3}:\ldots:\sqrt{r}:\sqrt{(n-1)};$$

and from each other are as

$$1:\sqrt{2}-1:\sqrt{3}-\sqrt{2}:\ldots;$$

like the dynamical formula required for the times of falling freely through equal vertical distances, deduced from

$$s=\tfrac{1}{2}gt^2, \text{ or } t=\sqrt{(2s/g)}.$$

Similarly it can be shown that for dividing up a triangle ABC, whose vertex A lies in the surface and base BC is horizontal, by horizontal lines into parts on which the hydrostatic thrust is the same, the formula is

$$\frac{x_r}{h}=\sqrt[3]{\frac{r}{n}}.$$

Examples.

(1) A hollow sphere is filled with liquid. Prove that the average pressure on the zone of the surface between two given horizontal small circles is the pressure midway between these circles.

(2) Prove that the total normal pressure on a spherical surface, immersed to a given depth in water, is the same as that on a circumscribed cylinder.

(3) A hemispherical bowl is filled with water; show that the resultant vertical thrust is equal to two-thirds of the whole pressure on the bowl.

(4) A sphere is just immersed in a liquid; prove that the total normal pressure on the curved surface of a segment made by a plane passing through the highest point of a sphere is three times the weight of liquid which would fill the segment.

(5) A hollow hemisphere, filled with liquid, is suspended freely from a point in the rim of its base; prove that the whole pressures on the curved surface and the base are in the ratio 19 : 8.

(6) Into a vertical cylinder are put equal weights of two different liquids which do not mix; prove that the ratio of their whole pressures on the curved surface of the cylinder is equal to three times the ratio of their densities.

(7) A vertical cylinder contains equal volumes of two liquids, the density of the lower liquid being three times that of the upper liquid. Find the whole pressure on the curved surface, and prove that, if the fluids be mixed together so as to become homogeneous, the whole pressure will be increased in the ratio of 4 to 3.

(8) A cylindrical tumbler, containing water, is filled up with wine. After a time half the wine is floating on the top, half the water remains pure at the bottom, and the middle of the tumbler is occupied by wine and water completely mixed. If the weight of the wine be two-thirds of that of the water, and their densities be in the ratio of 11 to 12, prove that the whole normal pressure of the pure water on the curved surface of the tumbler is equal to the whole normal pressure of the remainder of the liquid on the tumbler.

(9) Equal volumes of n fluids are disposed in layers in a vertical cylinder, the densities of the layers commencing with the highest being as $1:2:\ldots:n$; find the average pressure on the cylinder, and deduce the corresponding expression for the case of a fluid in which the density varies as the depth. Also, if the n fluids be all mixed together, show that the average pressure on the curved surface of the cylinder will be increased in the ratio

$$3n : 2n+1.$$

(10) A vessel contains n different fluids resting in horizontal layers and of densities $\rho_1, \rho_2, \ldots, \rho_n$ respectively, starting from the highest fluid. A triangle is held with its base in the upper surface of the highest fluid, and with its vertex in the nth fluid. Prove that, if Δ be the area of the triangle and $h_1, h_2, \ldots, h_n$ be the depths of the vertex below the upper surfaces of the 1st, 2nd, ..., nth fluids respectively, the thrust on the triangle is

$$\tfrac{1}{3}\cdot\frac{\Delta}{h_1^2}\{\rho_1(h_1^3-h_2^3)+\rho_2(h_2^3-h_3^3)+\ldots+\rho_n h_n^3\}.$$

(11) A cylindrical vessel on a horizontal circular base of radius a is filled to a height h with liquid of density w. If now a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, find the increase of pressure on the base of the vessel; and show that the increase of the whole pressure on the curved surface of the vessel is

$$\frac{8\pi wc^3}{3a}\left(h+\frac{2c^3}{3a^2}\right).$$

(12) The shape of the interior of a vessel is a double cone, the ends being open, and the two portions being connected by a minute aperture at the common vertex. It is placed with one circular rim fitting close upon a horizontal plane, and is filled with water; find the whole pressure and the resultant thrust upon it, and prove that, if the latter be zero, the ratio of the axes of the two portions is 1 : 2.

If the water is on the point of escaping between the circular rim and the plane when this ratio of the axes is 2 : 1, prove that the weight of the vessel is three times the weight of the water.

(13) A circular disc moveable about its centre fits accurately into a vertical slit in the side of a vessel containing water, so that half the disc is in water and half in the air. The pressure of the water on the immersed portion acts vertically upward through the centre of gravity of that portion and will therefore tend to turn the disc about its centre. This has been proposed as a Perpetual Motion. Point out the fallacy.

(14) Determine the direction and magnitude of the resultant thrust on every foot length of either half into which a horizontal circular pipe is divided by a vertical diametral plane,

(i) when the pipe is half full of water,

(ii) when it is completely filled under a given head.

Determine also the resultant thrust on either half of a sphere, hemisphere, or vertical cone filled with liquid, and divided by a vertical diametral plane.

(The C.G. of a semicircle is at a distance from the centre $4/3\pi$ of the radius, and of a hemisphere at a distance $\frac{3}{8}$ of the radius; and the C.G. of a cone is at a distance from the vertex $\frac{3}{4}$ of the height.)

(15) A closed cylinder, whose base diameter is equal to its length, is full of water, and hangs freely from a point in its upper rim; prove that the vertical and horizontal components of the resultant thrust on its curved surface are each half the weight of the water.

(16) When the resultant fluid thrust upon any portion of a closed surface is known, show how to find the resultant thrust upon the remainder.

Prove that the resultant thrust on the curved surface of a cylinder, completely submerged, with its axis at a given angle θ to the vertical, is $W \sin \theta$, and acts at right angles to the axis through its middle point, W denoting the weight of liquid displaced by the cylinder.

Determine generally the resultant thrust on the curved surface of a body like a cask, completely submerged.

(17) Prove that the resultant thrust of the liquid in a cylindrical pipe, inclined at an angle α to the vertical, contained between two parallel planes, inclined at an angle θ to the horizon,

(i) on these plane ends is a force

$$W \cos \alpha \sec(\theta - \alpha),$$

cutting the axis of the cylinder at a depth $2h$;

(ii) on the curved surface is a force

$$W \sin \theta \sec(\theta - \alpha),$$

cutting the axis of the cylinder at right angles at a distance $h \sin(\theta - \alpha) \operatorname{cosec} \theta$, from the C.G. of this liquid; W denoting the weight of liquid, and h the depth of its C.G. below the free surface.

When the planes are horizontal, the resultant thrust on the curved surfaces reduces to a couple of moment $Wh \tan \alpha$ (§ 43).

(18) A vessel in the form of an oblique circular cylinder with a horizontal elliptic base contains homogeneous liquid.

If its curved surface is divided into two halves by any plane through its axis, prove that the resultant thrust upon either half is equivalent to a single force and a couple, the magnitude of the former being independent of the position of the dividing plane, and the magnitude of the latter being zero for one position of the plane.

(19) Prove that the thrust on an open curved surface, bounded by a plane curve, and immersed in liquid, is the resultant of a force perpendicular to the plane of the base, and of a force represented by the weight of the liquid enclosed by the surface acting through its centre of gravity.

If the curved edge is placed in contact with a rough inclined plane, and the liquid in the interior is exhausted, find when the surface is on the point of slipping up the plane; examining the cases of a cone and of a hemisphere.

(20) Show that the resultant thrust on a hemispherical surface of radius a, with its base inclined at an angle θ to the horizon and its centre at depth h, is a force acting through the centre at an inclination to the vertical

$$\cot^{-1}\left(\cot\theta + \frac{2a}{3h}\operatorname{cosec}\theta\right).$$

Deduce the position of the C.P. of the circular base.

(21) Two closely fitting hemispheres made of sheet metal of small uniform thickness are hinged together at a point on their rims, and are suspended from the hinge, their rims being greased so that they form a water-tight spherical shell; this shell is now filled with water through a small aperture near the hinge. Prove that the contact will not give way if the weight of the shell exceed three times the weight of the water it contains.

Calculate the tension of a thread tied round the hemispheres as the shell is gradually filled with water; also when the hinge is lowermost.

(22) A solid cone is just immersed with a generating line in the surface; if θ is the inclination to the vertical of the resultant thrust on the curved surface, and 2α the vertical angle of the cone, prove that

$$(1-3\sin^2\alpha)\tan\theta = 3\sin\alpha\cos\alpha.$$

If the thrust on the curved surface is horizontal, prove that the magnitude of this thrust is equal to the weight of a hemisphere of the liquid of radius equal to the radius of the base of the cone.

(23) A cone, whose vertical angle is $2a$, contains a weight W of liquid. Prove that, when the axis of the cone makes an angle θ with the vertical, the whole pressure on the curved surface is

$$W \cos\theta \operatorname{cosec} a.$$

(24) A right circular cone whose vertex is C is cut off obliquely so that the cutting plane is an ellipse with major axis AB. If it stands on its base and is surrounded with liquid to the level of its vertex, show that the whole pressure on the curved surface bears to the resultant thrust the ratio

$$\cos \tfrac{1}{2}(A - B) : \cos \tfrac{1}{2}(A + B).$$

(25) Prove that if a right circular cone with an elliptic base is held under water with its axis horizontal, the average pressure over the curved surface and over the base is the same.

(26) A closed rigid vessel is formed by half the surface of an ellipsoid cut off by any central section, and by the plane section itself. The vessel is just full of water, and stands with its plane base on a horizontal table. Prove that the resultant thrust on the curved surface is a vertical force equal to half the weight of the water, such that its line of action cuts the plane base at a distance $\frac{3}{4}\sqrt{(r^2-\varpi^2)}$ from the centre; where r is the semi-diameter conjugate to the base, and ϖ the perpendicular from the centre on the horizontal tangent plane.

General Exercises on Chapter II.

(1) A regular pyramid, of height h, has its sides connected together by hinges at the vertex, but they are otherwise free. The pyramid is inverted, and the sides fit accurately together, so that the whole may contain liquid. The vessel thus formed is suspended from a hook by equal threads attached, one to the middle of the base of each side. If the hook and the vertex be equidistant from the base of the pyramid, show that the sides will be forced apart and the liquid will escape if its depth exceed $4h\sin^2\alpha$, where α is the angle made with the vertical by each side.

(2) A box is made of uniform material in the form of a pyramid, whose base is a regular polygon of n sides, and whose slant sides are equal isosceles triangles having a common vertex; each of these triangles forms a lid which turns about a side of the polygon as a hinge, its weight is W', and its plane makes with the vertical an angle β; and when closed the box is water-tight. It is filled with a weight W of water, and is placed on a horizontal table; show that nW' must be greater than $\frac{9}{4}W\operatorname{cosec}^2\beta$, or else the water will escape.

(3) A regular hexagonal prism has its sides jointed, and is filled with fluid, the ends being closed by vertical planes. It is then held with one face horizontal, and a hole is made in this top face. Show that in the position of equilibrium the faces next the top one will be inclined at an angle $\cos^{-1}\frac{1}{3}$ to the horizon.

(4) A vessel in the form of a regular tetrahedron rests with one face on a horizontal table. The other faces are uniform plates, each of weight W, which can turn freely about their lowest edges, and when shut fit closely. Through a hole at the top water is poured in, and the sides are pressed out when the depth of the water is m times the height of the vessel. Show that, if the weight of water poured in be pW, then

$$9p(2m^2-m^3)=2(m^2-3m+3).$$

(5) $ABCD$ is a quadrilateral having the sides AB, BC in the ratio 2 : 1, the angles B, D right angles, and the angle $BAD=\alpha$. If a thin vessel of the form generated by the revolution of ABC round AD be placed with its circular opening upon a horizontal plane, and be filled with water through a small hole at A, prove that the water will be on the point of escaping by lifting the vessel if

$$3+5\tan^2\alpha=11\tan\alpha.$$

(6) A regular tetrahedron $ABCD$ is immersed with the face ABC vertical, the side AB being horizontal and in the surface of the liquid. CE is drawn perpendicular to AB, meeting it in E. Show that the line of action of the resultant thrust on the remaining faces of the tetrahedron divides CE in F so that $EF:FC=5:13$.

(7) Show that the limiting position of the centre of pressure of a crescent, formed by two circles in a plane touching at a point in the surface of a liquid, when the two circles are infinitely nearly equal, is distant from the centre two-thirds of the radius.

(8) If a square is immersed wholly in a liquid, find the C.P.; and prove that the point is not changed if the square is turned round in its own plane.

Hence prove that the C.P. is the same for all parallelograms wholly immersed, described about an ellipse at the ends of a pair of conjugate diameters.

Prove also that the C.P. of all parallelograms formed by the points of contact is the same.

(9) If a convex polygon of n sides is completely immersed, no side being parallel to the surface; and if $x_1, x_2, \dots x_n$ be the depths of the vertices $A_1, A_2, \dots A_n$, of which let A_1 and A_r be the highest and lowest; and if $\alpha_1, \alpha_2, \dots \alpha_n$ be the inclinations to the horizon of the sides $A_1A_2, A_2A_3, \dots A_nA_1$; prove that the depth of the C.P. is

$$\frac{1}{2}\,\frac{x_1^4 \sin A_1 \operatorname{cosec}\alpha_n \operatorname{cosec}\alpha_1 - x_2^4 \sin A_2 \operatorname{cosec}\alpha_1 \operatorname{cosec}\alpha_2 - \dots + x_r^4 \sin A_r \operatorname{cosec}\alpha_{r-1} \operatorname{cosec}\alpha_r - \dots}{x_1^3 \sin A_1 \operatorname{cosec}\alpha_n \operatorname{cosec}\alpha_1 - x_2^3 \sin A_2 \operatorname{cosec}\alpha_1 \operatorname{cosec}\alpha_2 - \dots + x_r^3 \sin A_r \operatorname{cosec}\alpha_{r-1} \operatorname{cosec}\alpha_r - \dots}$$

(10) A plane rectangular lamina is bent into the form of a cylindrical surface, of which the transverse section is a rectangular hyperbola. If it is now immersed in water so that, first, the transverse, secondly, the conjugate axes of the hyperbolic sections is in the surface, prove that the horizontal thrust on any the same immersed surface will be the same in the two cases.

CHAPTER III.

ARCHIMEDES' PRINCIPLE AND BUOYANCY.

EXPERIMENTAL DETERMINATION OF SPECIFIC GRAVITY, BY THE HYDROSTATIC BALANCE AND HYDROMETER.

48. The principle of Archimedes which was established as a Corollary in § 45 of the last Chapter is so important in Hydrostatics that it is advisable to restate it in a more general form, and to give an independent proof.

Archimedes' Principle.

"A body wholly or partially immersed in a Fluid or Fluids (not necessarily a single liquid), at rest under gravity, is buoyed up by a force equal to the weight of the displaced fluid, acting vertically upwards through the centre of gravity of the displaced fluid."

To prove this principle, in the manner employed by Archimedes, we suppose the body removed, and its place filled up with fluid, arranged exactly as the fluid would be when at rest; and to fix the ideas we suppose this fluid solidified or frozen; this will not alter the thrust of the surrounding fluid, which will be exactly the same as that which acted on the body.

But now this solidified fluid will remain in equilibrium of itself under two forces, the attraction of gravity on its weight and the thrust of the surrounding liquid; this thrust must therefore balance the weight of the solidified fluid and act vertically upwards through the C.G. of the solidified fluid.

Therefore also on the original body, the thrust of the surrounding liquid will be equal to the weight of the displaced fluid and act vertically upwards through its C.G.

COR. For a body to float freely at rest in a fluid or fluids, the weight of the body and of the fluid it displaces must be equal, and their C.G.'s in the same vertical line.

The principle of Archimedes is therefore true not only of a body partly immersed in liquid, like a ship, but also of a body completely submerged in liquid, like a fish, diving bell, or submarine boat; or of a body floating in air, like a balloon; and generally of a body immersed in two or more fluids, as a ship is, strictly speaking, partly in air and partly in water.

A mere increase of atmospheric pressure, due to increased temperature, will produce a uniform increase of pressure; this will not alter the draft of a ship unless accompanied by an increase of density of the air, when less water and more air would be displaced in equilibrium, and the draft of water would be diminished.

Again, a body floating in a liquid, placed in the receiver of an air-pump, sinks slightly as the air is exhausted.

The thrust of the surrounding fluid on the body is called the *buoyancy* (French *poussée*, German *auftrieb*); and Archimedes' Principle asserts that the buoyancy is equal to the weight of the displaced fluid, and acts vertically upwards through its C.G.

A similar method will show that in the general case of any applied forces (not merely gravity in parallel vertical lines), the *buoyancy* of a body in a fluid is equal and opposite to the resultant of all the applied forces on the fluid which would occupy in equilibrium the space vacated by the body.

Thus, for instance, in the case of a sponge suspended by a thread, so as to dip partially into water, capillary attraction will cause the water to rise above the level surface, and the tension of the thread will be increased by the weight of water drawn up into the sponge.

So also a feebly magnetic body, if immersed in oxygen or a liquid more magnetic than itself, will under external magnetic force appear diamagnetic, like bismuth.

According to tradition, this Principle was discovered by Archimedes in his bath from observations on the buoyancy of his own body, while pondering over a method of discovering to what extent the crown of King Hiero of Syracuse (B.C. 250) had been adulterated with baser metal.

(Vitruvius, *de Architectura*, lib. IX. c. iii.
Palæmon, *de Ponderibus et Mensuris*, 124-163.
Thurot, *Revue archéologique*, 1868, 1869.
Histoire du principe d'Archimède.
Berthelot, *Comptes Rendus*, Dec. 1890).

The Principle of Archimedes is employed not only in the determination of the density of solid and liquid substances, but also for the condition of equilibrium of floating bodies, such as ships, fishes and balloons; it may be called the fundamental Principle of Hydrostatics, as its discovery was the first step to placing the science on a sound basis.

49. *Density and Specific Gravity.*

The *density* of a homogeneous body has already been defined as the weight of the unit of volume (§ 21).

With British units the density is the weight in pounds per cubic foot; and with Metric Units the density is the weight in grammes per cubic centimetre, or kilogrammes per litre (cubic decimetre), or tonnes (of 1000 kg) per cubic metre (§ 8).

DEFINITION. "The *specific gravity* of a body is the ratio of its density to the density of water"; or "is the ratio of the weight of the body to the weight of an equal volume of water."

In the Metric System a litre of water, at or near its maximum density, was taken as the unit of weight (*poids*), and called a kilogramme; so that in this system the density and the specific gravity are the same.

Now if s denotes the metric density or specific gravity of a substance, the equation

$$W = sV \quad \text{.........................(1)}$$

gives the weight W in grammes of a volume V cm³; or the weight W in tonnes of a volume V m³ (metres cube); but the equation

$$W = 1000sV \quad \text{.......................(2)}$$

gives the weight W in kg of a volume V m³.

With British units the specific gravity s and density w (in lb/ft³) are connected by the relation

$$w = Ds \quad \text{.........................(3)}$$

where D denotes the weight in pounds of a cubic foot of water; so that the equation

$$W = DsV \quad \text{.........................(4)}$$

gives the weight W in lb of V ft³ of a substance whose S.G. (specific gravity) is denoted by s.

50. In rough numerical calculations it is usual to take a cubic foot of water as weighing 1000 oz or 62·5 lb; so that the equation (2) gives the weight W in oz of V ft^3 of a substance of S.G. s; but in equation (4) we must put

$$D=62{\cdot}5.$$

A better average value is

$$D=62{\cdot}4,$$

which is the density of water, in lb/ft^3, at a temperature of about 53° F.; while

$$D=62{\cdot}425$$

is the maximum density of water, in lb/ft^3, at a temperature of about 4° C. or 39·2° F.

A Table, due to Mendeleeff, is printed in an Appendix, giving the density of water at different temperatures; s denoting the density in g/cm^3 at the temperature $t°$ C, from which the column of D, the density in lb/ft^3, was deduced, by multiplying by

$$(30{\cdot}4794)^3 \div 453{\cdot}593 = \log^{-1}1{\cdot}7953528;$$

while v denotes the specific volume, in cm^3/g; so that s and v are reciprocal.

51. These values of s, D, and v refer to pure distilled water, at standard atmospheric pressure.

But water may contain solid matter in suspension (as mud), or in solution (as salt), by which its density is increased.

Thus in muddy water D may rise to about 75, so that a gallon of this water will weigh about 12 lb, as against the gallon of 10 lb of pure distilled water.

In ordinary sea water, we generally take

$$s=1{\cdot}025, \quad D=64,$$

and therefore weighing 10·25 lb to the gallon; but in the water of the Dead Sea, or of the Great Salt Lake, we may take

$$s=1{\cdot}25, \quad D=78,$$

so that a gallon of this water weighs 12·5 lbs.

Ice is lighter than water, and consequently forms and floats on the surface. It is found that water in freezing expands about one twelfth, so that 12 ft^3 of water forms 13 ft^3 of ice; we may thus take for ice,

$$s={\cdot}923, \quad D=58.$$

It is this expansion of water in freezing which bursts the water pipes; but the ruptures are not perceived till the water thaws again.

Cast iron behaves in a similar manner on solidification, and thereby takes a very clear impression of the mould; it is found that a piece of cast iron, if thrown into a vessel of the molten metal, will at first sink, but afterwards float on the surface.

52. If the body is not homogeneous, then W/V will represent the *average* density of the body; and to measure the actual density at any point we must find w, the limit of $\Delta W/\Delta V$ (dW/dV in the notation of the Differential Calculus) the quotient of ΔW the weight (in lb or g) of a small volume ΔV (ft^3 or cm^3) of the body, enclosing the point.

The weight W of a body can be determined with extreme accuracy by the balance, but there is great practical difficulty in measuring the volume V with any pretence to equal accuracy; so to determine the average density W/V of a body, an indirect process is adopted, depending on the use of the Hydrostatic Balance and the Hydrometer in conjunction with the Principle of Archimedes.

53. *The Hydrostatic Balance.*

This instrument differs from the ordinary Balance merely in an arrangement by which the body to be weighed can be suspended by a thread or wire so as to dip into a vessel of water.

To determine experimentally the S.G. of a solid substance, a portion of it is first weighed in air, and afterwards when immersed in pure distilled water.

Suppose a weight W lb is required to equilibrate the body in air (strictly speaking in vacuo), and a weight W' lb is required to equilibrate it when the body is suspended in the water.

Then $W-W'$, the upward buoyancy of the water, is equal to the weight of the displaced water; and therefore the S.G., "the ratio of the weight of the body to the weight of an equal volume of water," is given by

$$s=\frac{W}{W-W'}.$$

The body when weighed in water is said to have lost the weight $W-W'$, so that the S.G. is the ratio of the true weight to the lost weight; but this weight apparently lost is taken by the table supporting the vessel of water.

54. If the vessel is filled up to the level of a spout, the immersion of the body will cause an equal volume of water to overflow; and this water will be found to weigh $W-W'$; in this way Archimedes determined the density of the crown of Hiero.

For instance, if the crown, an equal weight of gold, and an equal weight of silver displaced $\frac{1}{14}$, $\frac{4}{77}$, and $\frac{2}{21}$ of their weight of water, or caused the water to rise in a cylindrical vessel through distances proportional to these numbers.

the crown will be found to be adulterated with silver in the ratio of 9 parts by weight of silver to 11 of gold.

In the experiment, attributed by tradition to Charles II., of determining the alteration in the weight of a bucket of water in one of the scale pans of a balance, due to placing a fish in it, the result will be different, according as the bucket is only partially full of water, or brim full.

If no water is spilt, the alteration of weight is exactly equal to the weight of the body placed in the bucket.

But if water flows over the brim on to the ground, the alteration in weight is equal to the weight of the body less the weight of this water, whether the body floats like a fish completely submerged, or on the surface like a dead fish, or sinks to the bottom of the bucket like a stone.

55. In using the Hydrostatic Balance to determine the density of a body which floats in water, Cotes suggests the use of a lever and fulcrum under water (*Hydrostatical Lectures*, fig. 26); but this method is not employed practically.

Instead of this, a sinker is attached to the body; and now if the body alone weighs W lb in air or vacuo, and if the sinker weighs w lb in water, while the body and sinker together weigh W' lb in water, then the S.G. s of the body is given by

$$s=\frac{W}{W+w-W'}.$$

A steel-yard is sometimes employed in weighing bodies in air and water so as to determine the S.G.; this form of instrument is known as Walker's steel-yard hydrostatic balance.

If the body is soluble in water (like soap, salt, or sugar), then some other liquid, whose S.G. is known, must be employed, such as benzine or turpentine; or the body may be varnished to prevent its dissolution.

56. To determine the S.G. of a liquid by the Hydrostatic Balance, let a metal weight of W lb be first weighed in the liquid, and afterwards in pure distilled water; and let W' and W'' denote the weights in lb which equilibrate it in these two cases; then $W-W'$ is the weight of liquid displaced, and $W-W''$ is the weight of water displaced by the weight W; so that the S.G. of the liquid is

$$\frac{W-W'}{W-W''},$$

neglecting the buoyancy of the air on the weights W' and W'' in the other scale pan.

57. In weighing a body in air and in water, we really weigh it in two fluids; so that to be accurate, denote by M lb the true weight of the body (*in vacuo*), by ρ the S.G. of the air, and by B the S.G. of the metal weights W and W' lb which equilibrate the body in air and in water, the S.G of the water being taken as unity; then allowing for the buoyancy of the air, which is $W\rho/B$, $W'\rho/B$, $M\rho/s$ on the weights W, W', M, and for the buoyancy M/s of the water on the weight M,

$$W\left(1-\frac{\rho}{B}\right)=M\left(1-\frac{\rho}{s}\right),$$

and
$$W'\left(1-\frac{\rho}{B}\right)=M\left(1-\frac{1}{s}\right);$$

so that
$$M=\frac{W-\rho W'}{1-\rho}\left(1-\frac{\rho}{B}\right),$$

the true weight of the body.

Therefore
$$\frac{W}{W'}=\frac{s-\rho}{s-1},$$
or
$$\frac{W}{W-W'}=\frac{s-\rho}{1-\rho},$$
instead of s as above (§ 53), to which this ratio reduces when we put
$$\rho=0.$$

Denoting by S this apparent S.G. obtained by putting $\rho=0$, or by neglecting the density of the air, then
$$S=\frac{s-\rho}{1-\rho},$$
the ratio of the excess of the absolute S.G. of the substance and of water over the S.G. of the air; and then
$$s-S=\rho(1-S),$$
the correction on the apparent S.G. S to obtain the true S.G. s, when the S.G. ρ of the air is taken into account.

A good average value to take of ρ is 0·00123 or 1/813, corresponding to a specific volume of 13 ft^3 to the lb.

58. When every refinement is introduced into the use of the Hydrostatic Balance, the height of the barometer h and the temperature t must be observed, in order to obtain the density of the air compared with its standard density at standard temperature T and height of barometer H; while the S.G. of the water must not be taken as unity, but from the corresponding value in Mendeleeff's Table (§ 219).

Again the value obtained for s will be the S.G. of the body at temperature t, so that the coefficient of cubical expansion of the substance must be known, to deduce from this the S.G. of the body at the standard temperature T.

In the Metric System the standard temperature is 0° C; but as this temperature is not pleasant to work in, it is preferable to adopt the standard temperature of 62° F. or $16\frac{2}{3}$ C., prescribed in the British Acts of Parliament, with a standard height of barometer of 30 inches or 76 centimetres.

For an account of the refinements required in the experimental determination of Weight and Density with the greatest possible accuracy, the reader is referred to

Jamin, *Cours de physique*, t. II.;
Stewart and Gee, *Practical Physics*, v. I.;
Chisholm, *Weighing and Measuring;*
Walker, J., *The Theory of a Physical Balance.*

59. *Weight and Weighing.*

We have used the words Weight and Weighing as familiar to all; but it will be useful at this point to examine closely the meaning of these words.

The standard British pound weight is described in the *Weights and Measures Act*, 1878, 13, and Schedule I., § 13:—"The weight in vacuo of the platinum weight (mentioned in the First Schedule to this Act) and by this Act declared to be the imperial standard for determining the imperial standard pound, shall be the legal standard measure of weight, and of measures having reference to weight, and shall be called the imperial standard pound, and shall be the only unit or standard measure of weight from which all other weights and all measures having reference to weight shall be ascertained."

§ 14. "One sixteenth part of the imperial standard pound shall be an ounce, and one sixteenth part of such ounce shall be a dram, and one seven thousandth part of the imperial standard pound shall be a grain.

"A stone shall consist of fourteen imperial standard pounds, and a hundredweight shall consist of eight such stones, and a ton shall consist of twenty such hundredweights.

"Four hundred and eighty grains shall be an ounce troy.

"All the foregoing weights except the ounce troy shall be deemed to be avoirdupois weights."

First Schedule. Imperial Standards.—"The imperial standard for determining the weight of the imperial standard pound is of platinum, the form being that of a cylinder nearly 1·35 inch in height and 1·15 inch in diameter, with a groove or channel round it, whose middle is about 0·34 inch below the top of the cylinder, for insertion of the points of the ivory fork by which it is to be lifted; the edges are carefully rounded off, and such standard pound is marked, P.S. 1844, 1 lb."

Similar precise language is used in defining the corresponding Metric standard of weight, the *kilogramme des archives*, preserved at the Conservatoire des Arts et Métiers, Paris.

The Weight of a body is the quantity which is measured out against weights, such as oz weights, lb weights, cwts, etc., by the operation of Weighing in the scales of a correct Balance, every scientific precaution for accuracy being taken.

To Weigh the body, it is placed in one of the scales, and is then equilibrated by certain standard lumps of metal called Weights (French *poids*, German *Gewichte*) stamped in this country as lb, oz, cwt, tons, etc.; and in the Metric System as grammes, kilogrammes, or tonnes, and their subdivisions.

60. Where great accuracy is required, an allowance must be made by calculation for the buoyancy of the

air (§ 219); or else the weighing should be performed in an exhausted receiver (in vacuo, in the words of the Act of Parliament).

Thus, if a pound of feathers, cork, or wood is weighed out in air against a brass or iron lb weight, and if the balance is placed in a receiver and the air exhausted, the lighter substance, being of greater volume and being more buoyed up by the air, will preponderate over the heavier substance; thus proving that the so-called lighter substance has the greater weight (we use the words *lighter* and *heavier*, as applied to a substance, to mean of *less* or *greater* density). But if carbonic acid gas was introduced into the receiver at atmospheric pressure, the smaller denser body would preponderate.

This experiment, performed originally with Boyle's *statical baroscope* (1666), illustrates the buoyancy due to the air, and proves that air has weight; this can also be demonstrated directly by weighing a flask of metal that has been exhausted of air, when on admitting the air the flask is found to preponderate to a definite amount, which is the weight of air which has entered the flask.

Air may also be forced in, as in Galileo's experiment, and the flask will preponderate still more; but Aristotle's experiment, of weighing a bladder when flaccid and when full blown, leads to no result, as the bladder weighs exactly the same, if blown out to the atmospheric pressure (Cotes, *Hydrostatical Lectures*, pp. 145, 154).

In balloon problems the buoyancy of the air is the ruling phenomenon; and to measure the weight and density of the air and of the lighter gas which fills the balloon is an operation of great experimental difficulty, depending on indirect processes.

61. In making an exact copy of the platinum standard pound weight in a different metal, say brass, a correction must be made for the buoyancy of the air.

Thus if B, P, ρ denote the S.G.'s of brass, platinum, and of air, the platinum standard pound will when weighed in air preponderate over its true copy in brass, which equilibrates it in vacuo, by

$$\frac{\rho}{B}-\frac{\rho}{P} \text{ lb} \quad \text{or} \quad 7000\rho\left(\frac{1}{B}-\frac{1}{P}\right) \text{ grains.}$$

If equilibrium is restored in air by adding a weight of x grains of a metal of S.G. G (say gold) to the brass weight, then

$$x\left(1-\frac{\rho}{G}\right)=7000\rho\left(\frac{1}{G}-\frac{1}{P}\right);$$

Taking $B=8$, $G=17{\cdot}5$, $P=21{\cdot}5$, $\rho=0{\cdot}00123$, we find $x=0{\cdot}6753$ grains (of gold), or $x=0{\cdot}6759$ grains (of brass); and this difference x must be allowed for in the construction in air of a true brass copy of the imperial standard pound weight of platinum.

Similar allowances for air buoyancy must be made in all accurate weighings; however the ratio of the apparent weights of bodies of the same substance weighed by weights of one metal will be independent of the density of the air, and will therefore be a true ratio, as in the process of "double weighing."

62. *Imperial Measures of Capacity.*

According to the *Weights and Measures Act*, 1878, 15, "The unit or standard measure of capacity from which all other measures of capacity, as well as for liquids as for dry goods, shall be derived, shall be the gallon containing ten imperial standard pounds weight of distilled water weighed in air against brass weights, with the water and the air at the temperature of sixty-two degrees of Fahrenheit's thermometer, and with the barometer at thirty inches.

"The quart shall be one fourth part of the gallon, and the pint shall be one eighth part of the gallon."

It will be noticed that the volume of the gallon, 10 lb, or 4·536 kg or litres of water, although a measure of capacity, is not defined directly; according to former measurements, the weight of a cubic inch of water was taken as 252·48 grains, when weighed in the above manner; this would make the volume of the gallon

$$70000 \div 252{\cdot}48 = 277{\cdot}25 \text{ in}^3.$$

But the measurement of the volume of a body with accuracy is one of great practical difficulty; and now, according to the latest re-determination of the Standards Department, the (apparent) weight of a cubic inch of distilled water freed from air and then weighed against brass weights of S.G. 8·143, in air at the temperature 62° F., and the barometer at 30 inches, is found to be 252·286 grains; this makes the volume of the gallon

$$277{\cdot}463 \text{ in}^3.$$

But now, if the buoyancy of the air is taken into account, and we denote by B the S.G. of the brass weights, and by ρ the S.G. of the air, referred to water at this temperature and barometric height as of unit S.G.; then we shall find that this cubic inch of water, if weighed in vacuo, will be equilibrated by W grains, given by

$$W(1-\rho) = 252{\cdot}286\left(1-\frac{\rho}{B}\right),$$

which, with $B = 8{\cdot}143$ and $\rho = 0{\cdot}0012$,

gives $W = 252{\cdot}552$,

the true weight (in vacuo) of a cubic inch of water.

Examples.

(1) Prove that an inch of rain over an acre weighs about 100 tons.

(2) Find the weight of water in a lake whose area is 5 acres and average depth $10\frac{1}{2}$ feet, and also the number of gallons it contains, supposing a cubic foot of water to weigh 1000 ounces, and a gallon to contain 277·25 cubic inches.

(3) Taking the earth as a sphere, whose girth is 40,000 kilometres, or 360×60 nautical miles of 6080 feet, and of mean S.G. 5·576, prove that the weight is about $6{\cdot}027 \times 10^{21}$ metric tonnes, or $5{\cdot}932 \times 10^{21}$ British tons.

(4) Show how the mean transverse section of a piece of fine wire may be determined by weighing it first in vacuo and then in water.

If the wire is ten yards long, find the greatest error in determining the mean transverse section if the weights are determined accurately to tenths of a grain and the weight of a cubic inch of water is 252·5 grains.

(5) Investigate the conditions of equilibrium of a body floating partially immersed in a fluid.

An iron shell one-eighth of an inch thick floats half immersed in water, the specific gravity of iron being 8; find the diameter of the shell.

(6) Prove that the calibre d, in inches, of an n bore gun, is given by the relation

$$d = \log^{-1}(0{\cdot}2226 - \tfrac{1}{3}\log n);$$

given that n spherical lead bullets, of S.G. 11·4, and diameter d inches, weigh one pound.

Hence prove that a 12 and 20 bore are 0·729 and 0·615 inches in calibre.

(7) The area of the base of a vessel with vertical sides containing water is 85 cm^2. Find how much the pressure at each point of the base is increased if 1000 grammes of lead, specific gravity 11·4, are suspended in the water by a thread.

(8) Given the S.G. s of ice and s' of sea-water, prove that the volume and weight of an iceberg, of which V cubic feet is seen above the water, is

$$Vs'/(s'-s) \text{ ft}^3 \quad \text{and} \quad DVss'/(s'-s) \text{ lb.}$$

(9) A body floats in a fluid of S.G. s with as much of its volume out of the fluid as would be immersed in a second fluid of S.G. s', if it floated in that fluid. Prove that the S.G. of the body is

$$ss'/(s+s').$$

(10) Prove that, in selling iron of S.G. 7·8 by weight, in air of S.G. ·00128, with a balance and standard brass weights of S.G. 8·4, what is sold as 100,000 tons of iron is really about 1·2 tons more.

(11) A piece of copper of S.G. 1·85 weighs 887 grains in water and 910 grains in alcohol; required the S.G. of the alcohol.

(12) Two cubic feet of cork, of S.G. 0·24, is kept below water by a rope fastened to the bottom. Prove that the tension of the rope is 95 pounds.

(13) Prove that, if volumes A and B of two different substances equilibrate in vacuo, and volumes A' and B' equilibrate when submerged in liquid, the densities of the substances and of the liquid are as

$$\frac{A'}{A}-\frac{B'}{A} : \frac{A'}{B}-\frac{B'}{B} : \frac{A'}{A}-\frac{B'}{B}.$$

(14) If the S.G. of the gold supplied by Hiero to the goldsmith was 19, and if the S.G. of the crown as debased was found to be 16, and the S.G. of the silver employed for this purpose was 11; then show that 33 parts in 128, or rather more than one-fourth part by weight, was silver.

(15) The crown used by the Stuart sovereigns, which was destroyed in the seventeenth century, is said to have been of pure gold (S.G. 19·2) and to have weighed $7\frac{1}{2}$ lbs.

How much would it have weighed in water?

If it had been of alloy, partly silver (S.G. 10·5) and partly gold, and had weighed $7\frac{1}{34}$ lbs. in water, how much of each metal would it have contained?

63. *The Hydrometer.*

For determining the density and S.G. of a liquid, an instrument called a Hydrometer (French *aréomètre*) is employed, consisting of a bulb and a uniform stem.

Hydrometers are of two kinds—

(i.) the common or Sikes's hydrometer, of variable immersion but fixed weight, for determining the density of a liquid (fig. 36);

(ii.) the Fahrenheit or Nicholson hydrometer, of fixed displacement or immersion but of variable weight, which can be used for determining the density of a liquid, or of a small solid, or to determine the weight of a small solid (fig. 37).

As these instruments are usually small, the inch and ounce are used as British units of length and weight, but better still the centimetre and gramme in questions relating to their use; and then densities will be given in oz/in^3, or with the metric units the densities and specific gravities will be the same, a great numerical simplification.

In the common hydrometer of variable immersion, let W denote its weight, in oz or g, V its volume, in in^3 or cm^3, and a the cross section of the stem, in in^2 or cm^2.

Then the liquid of smallest density w_0 in which the hydrometer will not sink, but just float with its highest point A in the surface, is given by

$$w_0 = W/V;$$

and if, when placed in a liquid of density w, the hydrometer floats immersed to the point M, then if $AM = x$,

$$w = W/(V - ax);$$

so that
$$\frac{w}{w_0} = \frac{V}{V - ax}.$$

If we put $V/a = a$, then a is the length AO of the stem which would have the total volume V; and now

$$\frac{w}{w_0} = \frac{a}{a - x};$$

so that, if we represent the density w_0 by the ordinate AC, and draw the hyperbola CP having as asymptotes the vertical side of the stem AM, and the horizontal line through O at a depth $AO = a$ below A, then to the same scale the ordinate MP will represent the density w of the liquid in which the hydrometer floats immersed up to M; and the marks of graduation M may be numbered so as to give, for instance, the density in kg per litre, or lb per gallon.

64. The hydrometer then, in its simplest form, may be considered as consisting of the rod AO, of weight W and uniform cross section a, ballasted so as to float upright.

The sensibility of the instrument, as measured by the distance between the graduations, is proportional to a or V/a, and therefore inversely proportional to the sectional area; so that the longer and thinner the rod AO is made, the greater its sensibility in indicating differences of density; but as such an instrument would be inconveniently long, the lower part of the rod is replaced by the bulb B, of equal volume and weight.

With equal increments of density, starting from w_0 and A, the graduations on the stem proceed in harmonic progression, and become closer together.

To graduate the instrument geometrically, we draw the horizontal and vertical lines AC and CE, AC representing to scale the density w_0; and now if AR represents any greater density w, and we draw the straight line OQR meeting CE in Q, then the horizontal line QM will cut the stem in the corresponding graduation M, the point to which the hydrometer will sink in liquid of density w represented by AR.

For if the vertical line RP and the horizontal line MQ meet in P, then

$$\frac{w}{w_0} = \frac{AR}{AC} = \frac{AR}{MQ} = \frac{OA}{OM},$$

so that $\quad OM \,.\, MP = OA \,.\, AC$, or rect. $OP =$ rect. OC,

and P therefore lies on the hyperbola CPD.

Incidentally we notice that this construction gives the geometrical method of inserting a given number of harmonic means between two given quantities, OA and OL.

65. The greatest density which can be measured by the common hydrometer is represented by LD, where L is the lowest division on the stem, just above the bulb B; and common hydrometers are of two kinds—

(i.) for heavy liquids (salts) denser than water, floating in water with A in the surface;

(ii.) for light liquids (spirits) less dense than water, floating in water with L in the surface.

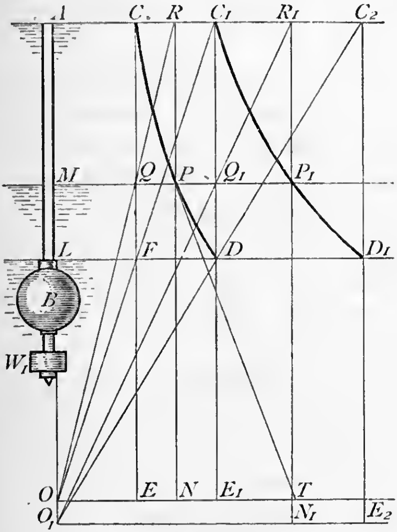


Fig. 36.

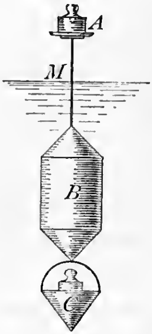


Fig. 37.

Suppose the readings of a common hydrometer are required for S.G.'s between 1 and s, and that l denotes the length of the stem AL; then

$$\frac{V}{V-al}=s, \quad \text{or} \quad \frac{1}{s},$$

according as the hydrometer is required for salts or spirits, or as s is $>$ or <1; and then

$$\frac{V-al}{al}=\frac{1}{s-1}, \quad \text{or} \quad \frac{s}{1-s},$$

giving the ratio of the volume of the bulb to the volume of the stem.

Thus the marine hydrometer or salinometer is required to register S.G.'s ranging from 1·00 to 1·04; so that the volume of the bulb must be 25 times that of the stem.

The S.G. of pure milk being 1·03125, the Lactometer requires a bulb of 32 times the volume of the stem.

66. *Sikes's Hydrometer.*

To increase the range of the instrument, a series of weights W_1, W_2, ..., are provided of known volumes, which can be fixed on the stem, below the bulb B, or above at A; this instrument is known as Sikes's hydrometer.

It is convenient to make the volumes of these weights all equal, and to use the instrument with one of these weights always attached, so that the total volume of the hydrometer may be supposed to include the volume of the additional weight.

Now when a weight W_1 is attached, the density curve becomes a similar hyperbola C_1D_1, having the same asymptotes OA and OE; and it is convenient for continuity of measurement to choose W_1 so that the initial ordinate AC_1 of the new hyperbola is equal to the final ordinate LD of the former hyperbola; the substitution of the next weight W_2 giving the new density hyperbola C_2D_2, in which $AC_2 = LD_1$, and so on.

Then, with $OA = a$, $AL = l$,

$$\frac{a}{a-l} = \frac{W+W_1}{W} = \frac{W+W_2}{W+W_1} = \dots = \frac{w_n}{w_{n-1}};$$

so that the weights W, $W+W_1$, $W+W_2$, $W+W_3$, ..., are in G.P.; and also the densities w_0, w_1, w_2, ..., at which the weights must be changed.

In this way Sikes's hydrometer is an instrument capable of measuring densities over a considerable range, obviating the necessity of an inconveniently long stem.

67. But if the weights are of equal density w, the addition of a weight W_1 will change the centre of the hyperbola which represents the density from O to O_1, where

$$AO_1 = \frac{W+W_1}{w_1 A}, \quad OO_1 = \frac{W_1}{w_1 A};$$

and so on; and now when the weight is changed from W_n to W_{n+1}, leaving $LD_n = AC_{n+1}$, then

$$W + W_n \quad = \left(V - al + \frac{W_n}{w}\quad\right) w_n,$$

$$W + W_{n+1} = \left(V \qquad + \frac{W_{n+1}}{w}\right) w_n.$$

Eliminating w_n,

$$\frac{W+W_n}{W+W_{n+1}} = \frac{w(V-al)+W_n}{w\,V \qquad + W_{n+1}} = \frac{w(V-al)-W}{wV-W};$$

so that
$$\frac{W+W_n}{W} = \left(1 - \frac{wal}{wV-W}\right)^n,$$

and therefore the weights W, $W+W_1$, $W+W_2$, ... are in G.P., as before; to which this case reduces by supposing the volume of the weights zero, or their density w infinite.

68. In a Fahrenheit or Nicholson Hydrometer of constant displacement V, in^3 or cm^3 (fig. 37), the instrument is loaded so as to bring a fixed mark M in the stem down to the surface of the liquid; and then if W is the weight of the hydrometer, w_0 the density of the liquid in which it floats immersed to M, and if W' is the weight which must be placed in the upper scale A to bring the hydrometer down to the mark M in the liquid whose density w is required,

$$\frac{w}{w_0} = \frac{W+W'}{W};$$

so that, for densities in A.P., the weights W' are in A.P. also.

69. To determine the weight, say x g, of a small body, the hydrometer is placed in a liquid, and in the scale A the weight W_1 g is observed which is required to sink the hydrometer to M; W_1 is then removed, and the body is placed in the upper scale A, and the additional weight W_2 g required to be added in A to bring the hydrometer down to M is observed; then

$$x + W_2 = W_1,$$

or

$$x = W_1 - W_2.$$

70. To determine the S.G. of the body, the liquid must be pure distilled water; and the weight x g having been determined as above, the body is placed in the lower cage C (added by Nicholson to Fahrenheit's hydrometer), being tied down if it tends to float up; and now the weight W_3 g is observed, which placed in the upper scale A brings the hydrometer down to M; then

$W_1 - W_3$ is the weight of the body in water,
$W_1 - W_2$ „ „ „ air;

and therefore the S.G. is (§ 53)

$$\frac{W_1 - W_2}{W_3 - W_2}.$$

71. When the density of the air is rigorously taken into account, and we denote by U and V the volumes of air and water displaced by the hydrometer, by D the density of water, and by s, B, σ, and ρ the S.G.'s of the body, the weights, the liquid, and the air, then

$$D(\rho U + \sigma V) = W + W_1\left(1 - \frac{\rho}{B}\right)$$

$$= W + x\left(1 - \frac{\rho}{s}\right) + W_2\left(1 - \frac{\rho}{B}\right)$$

$$= W + x\left(1 - \frac{\sigma}{s}\right) + W_3\left(1 - \frac{\rho}{B}\right).$$

Therefore $$x\left(1-\frac{\rho}{s}\right)=(W_1-W_2)\left(1-\frac{\rho}{B}\right),$$

$$x\left(1-\frac{\sigma}{s}\right)=(W_1-W_3)\left(1-\frac{\rho}{B}\right);$$

and $$\frac{s-\rho}{s-\sigma}=\frac{W_1-W_2}{W_1-W_3},$$

$$\frac{s-\rho}{\sigma-\rho}=\frac{W_1-W_2}{W_3-W_2}=S,$$

where S denotes the apparent S.G. obtained by neglecting the density of the air; so that, if the liquid employed is water, and $\sigma=1$,

$$s-S=\rho(1-S),$$

as before, with the Hydrostatic Balance (§ 57).

72. The principle of Nicholson's hydrometer is employed in weighing a large body like an elephant or a big gun, for which no balance sufficiently large is available. The body to be weighed is placed in a barge, and when all is quiet, the water-line is marked; the body is then removed and replaced by stones to bring the barge down to the same water-line; and then the aggregate weight of the stones, weighed separately, is the weight of the body.

73. *The Specific Gravity Bottle.*

This is a small glass bottle, with an accurately fitting stopper pierced with a fine hole, so that the bottle can be filled up accurately to the same point; it is also provided with a brass weight, intended to be of exactly the same weight as the bottle when empty.

The instrument is used for determining the S.G. of a liquid, or of a small fragment of a solid substance.

The bottle is weighed (i) empty, (ii) filled with pure distilled water, (iii) filled with the liquid, or (iv) with a small piece of the solid in the water; and supposing that W_1, W_2, W_3, W_4 g are the observed weights, then $W_2 - W_1$ cm³ is the volume of the bottle; thence we find

$$\rho = \frac{W_3 - W_1}{W_2 - W_1}, \quad s = \frac{x}{x + W_2 - W_4},$$

giving ρ, s, the S.G.'s of the liquid and the solid, supposing its weight is x g.

(Stewart and Gee, *Practical Physics*, I., p. 131. Guthrie, *Practical Physics*, p. 55.)

Examples.

(1) The volume between two successive graduations on the stem of a hydrometer is $\frac{1}{1000}$th part of its whole volume, and it floats in distilled water with 20 divisions, and in sea-water with 46 above the surface; prove that the S.G. of sea-water is 1·027.

(2) Prove that, if a common hydrometer sinks to the graduations a, b, c in liquids whose S.G.'s are s_1, s_2, s_3, while Fahrenheit's hydrometer requires weights W_1, W_2, W_3 in the upper cup to sink it in these liquids to the mark M; then

$$\frac{b-c}{s_1} + \frac{c-a}{s_2} + \frac{a-b}{s_3} = 0,$$

$$\frac{W_2 - W_3}{a} + \frac{W_3 - W_1}{b} + \frac{W_1 - W_2}{c} = 0.$$

(3) Prove that if lengths a and b of the stem of a common hydrometer are visible out of two given liquids in which the hydrometer floats, and a length c out of a mixture of equal volumes of these liquids, the ratio of the volume of the hydrometer to the area of the cross section of the stem is

$$\frac{(a+b)c - 2ab}{2c - a - b}.$$

(4) A common hydrometer has a piece of its bulb chipped off, so that when placed in liquids of densities α and β it registers densities α' and β'.

Find the fraction of the weight which has been chipped off; and prove that if the apparent density of any other liquid is γ', the true density γ is given by

$$\frac{1}{\gamma}-\frac{1}{\alpha} : \frac{1}{\gamma'}-\frac{1}{\alpha'} = \frac{1}{\beta}-\frac{1}{\alpha} : \frac{1}{\beta'}-\frac{1}{\alpha'}.$$

(5) If the reading of a common hydrometer when placed in liquid at the same temperature as itself be x, and if, when it is placed in the same liquid at a higher temperature than itself, its reading be at first x_1, but afterward the reading rise to x_2, the ratio of the expansions of the liquid and of the hydrometer for the same change of temperature is approximately $x-x_1 : x_2-x_1$.

(6) Nicholson's hydrometer is used to determine the weight and specific gravity of a solid and W and S are the results when the effect of the air is neglected; prove that the actual weight is

$$W\left\{1+\frac{\rho}{S(1-\rho)}\right\}\left(1-\frac{\rho}{B}\right),$$

where ρ and B are respectively the S.G.'s of air and of the material of the known weights employed.

(7) If a body of density s be weighed on a Nicholson's hydrometer in air of density ρ by means of weights of density B, shew that the correction for the density of the air to be applied to the apparent weight W to obtain the true weight is

$$W\frac{\rho}{B}\cdot\frac{B-s}{s-\rho}.$$

74. *The Density and Specific Gravity of Mixtures and Alloys.*

Denoting by $W_1, W_2, W_3, \ldots, W_n$ (lb), the weights of volumes $V_1, V_2, V_3, \ldots, V_n$ (ft³), of substances of densities $w_1, w_2, w_3, \ldots, w_n$ (lb/ft³), which are mixed or melted together to form a substance of weight W, volume V, and average density $w = W/V$; then, there being no loss of material,

$$W = W_1 + W_2 + W_3 + \ldots + W_n.$$

If there is no change of volume,

$$V = V_1 + V_2 + V_3 + \ldots + V_n;$$

but sometimes this equality does not subsist, in consequence of chemical action.

Then, since by definition (§ 21),

$$W = wV,\quad W_1 = w_1 V_1,\quad W_2 = w_2 V_2,\quad \ldots,\quad W_n = w_n V_n,$$

therefore
$$w = \frac{W_1 + W_2 + W_3 + \ldots + W_n}{V}$$
$$= \frac{w_1 V_1 + w_2 V_2 + w_3 V_3 + \ldots + w_n V_n}{V};$$

and if there is no change of volume,

$$w = \frac{w_1 V_1 + w_2 V_2 + \ldots + w_n V_n}{V_1 + V_2 + \ldots + V_n} = \frac{\Sigma w V}{\Sigma V},$$

or
$$w = \frac{\Sigma W}{\Sigma W / w}.$$

Denoting by $s_1, s_2, \ldots, s_n$, the S.G.'s of the ingredient substances, and by s the S.G. of the mixture; then since

$$w_r = D s_r,$$

D denoting the density (in lb/ft³) of water, therefore

$$s = (s_1 V_1 + s_2 V_2 + \ldots + s_n V_n)/V;$$

and if there is no change of volume,

$$s = \frac{s_1 V_1 + s_2 V_2 + \ldots + s_n V_n}{V_1 + V_2 + \ldots + V_n} = \frac{\Sigma s V}{\Sigma V}.$$

Suppose, for instance, that equal volumes of the ingredients are taken, and there is no change of volume; then

$$s=(s_1+s_2+\ldots+s_n)/n,$$

the A.M. (arithmetic mean) of the S.G.'s.

But, if equal weights are taken, then

$$s=\frac{n}{1/s_1+1/s_2+\ldots+1/s_n},$$

the H.M. (harmonic mean) of the S.G.'s.

75. *The Salinometer.*

This is an instrument consisting of a hydrometer and a thermometer, employed at sea to determine the degree of saltness in the water of the boilers.

The thermometer is required, as the reading of the hydrometer is always taken at a fixed temperature, 200 F.; and then the graduations of the hydrometer show the percentage of salt in the water, a little of which is drawn off from the boiler, and allowed to cool to 200 F.; at a lower temperature some of the salt might be thrown down.

The salinometer is graduated experimentally, because it is found that a mixture of salt and water varies capriciously in volume with the percentage of the salt, and also with its quality.

But if we assume that the solution of the salt in the water causes no addition of volume; and also take standard sea water as containing one-33rd part by weight of solid substance, so that 32 lb of fresh water are mixed with 1 lb of salt, or one gallon of pure fresh water, weighing 10 lb, is mixed with 5 oz of salt; then water n times salter is reckoned as containing n-33rds by weight of salt.

One ft³ of this water therefore contains $(33-n)/33$ by weight of pure water of equal volume, and its S.G. is therefore $33/(33-n)$; and measured on the salinometer as a hydrometer for salts, by a length x of the stem out of the water,

$$\frac{33}{33-n}=\frac{V}{V-ax}, \quad \text{or} \quad \frac{ax}{V}=\frac{n}{33};$$

so that the graduations for integral increments of n, each of which is called 10°, are equidistant.

Thus 0° on the salinometer represents pure water, 10° represents ordinary sea water, 20° represents water double as salt, and so on.

76. If one lb of salt dissolved in 32 lb of water made no addition to the volume, the S.G. would be $33/32=1{\cdot}03125$; while if the volume was the sum of the volumes of the salt and water, the S.G. of the mixture, taking the S.G. of salt as 2, would be given by

$$\frac{32+1}{32+\frac{1}{2}}=\frac{66}{65}=1{\cdot}01548.$$

But the S.G. of sea water is found to be about 1·025, so that a certain contraction must take place.

Denoting by V_1 the volume in ft³ of 32 lb of fresh water, V_2 of 1 lb of salt, and V of 33 lb of sea water of S.G. 1·025, then

$$V_1=\frac{32}{D}, \quad V_2=\frac{1}{2D}, \quad V=\frac{33}{1{\cdot}025D},$$

$$\frac{V_1+V_2}{V}=\frac{533}{528},$$

$$\frac{V_1+V_2}{V}-1=\frac{5}{528},$$

so that the percentage of diminution of volume is 500/528, nearly 1 per cent.

In 33 lb of Dead Sea water, n times salter than sea water, and of S.G. s, formed by the evaporation of sea water, in which some pure water is driven off, leaving behind the salt, of S.G. s suppose,

$$V_1=\frac{33-n}{D},\quad V_2=\frac{n}{SD},\quad V=\frac{33}{sD};$$

so that

$$\frac{V_1+V_2}{V}=\frac{33-n+nS^{-1}}{33}s.$$

If Dead Sea water is formed by the evaporation of x per cent. by weight of sea water, and contains y per cent. weight of solid matter; then 100 lb of sea water loses by evaporation x lb of pure water, leaving $\frac{3200}{33}-x$ lb of pure water and $\frac{100}{33}$ lb of salt to form $100-x$ lb of Dead Sea water; and therefore

$$n=\frac{100}{100-x}=\frac{33y}{100}.$$

The terraces round the Dead Sea, 1200 feet above its present level, show the extent to which the original sea water has been evaporated down, without driving off the salt.

Thus if $y=25$, then $x=87{\cdot}8$, and $n=8{\cdot}25$.

Also if $s=1{\cdot}25$, $S=2$,

then $(V_1+V_2)/V=1{\cdot}09375$,

implying a contraction of volume in the mixture of the salt and water of over 9 per cent.

(Page, *Physical Geography*, p. 202.)

77. Modern marine engines have surface condensers, so that fresh water can be placed in the boilers and used over and over again with little waste, and a Salinometer is not required.

But with the old fashioned jet-condenser, the steam is lost after passing through the engines, and the boilers are fed from the sea; and if the boilers were not periodically blown out, the accumulation of salt would make a strong brine solution, having such a high boiling point, that the furnaces would become red-hot and collapse.

To prevent this, a certain fraction of the feed water, say one nth, must be blown out; and then the water in the boiler, as tested by the Salinometer, will not be more than n times salter than sea water.

For let

$x =$ number of lb of pure water evaporated,
$y =$ „ brine blown out,

in a given time; then

$x + y =$ number of lb of sea water fed in, in this time.

Taking the sea water as containing one-33rd part by weight of solid substance, then

$yn/33 =$ weight in lb of solid matter blown out,
$(x+y)/33 =$ „ „ fed in;

and when these are equal, the saltness of the water in the boiler will remain stationary; so that

$$x + y = ny,$$
$$y = (x+y)/n.$$

Thus, if the Salinometer register 15°, $n = \frac{3}{2}$; and

$$y = \tfrac{2}{3}(x+y),$$

or two-thirds of the feed must be blown out, the remaining one-third being evaporated; and now it is calculated that 24 per cent. of the heat is wasted by this blowing off.

(Sennett, *The Marine Engine*, p. 209.)

It is this loss of heat by blowing out, and also the practical impossibility of carrying a high pressure with salt water, which makes the modern system of surface condensation preferable, in spite of the extra complication and expense; as in this way high pressure steam and the principle of compounding can be used, with gain of economy.

Where however steam navigation is carried on over fresh water, as on rivers or the lakes of N. America, a return is made to the old fashioned system of jet-condensing, the surface condenser not being worth the extra trouble and expense.

On the other hand, were steamers to ply on the Dead Sea, the Salt Lake of Utah, or Lake Urumia in Persia, the saltest piece of water in the world, the system of surface condensation would be unavoidable.

78. *The Spirit Hydrometer.*

This is Sikes's hydrometer applied to light liquids, employed in the excise for the proof of spirits.

The S.G. of absolute alcohol being taken as S, and of water as unity, then a mixture containing y per cent. by weight of alcohol would, if the volume of the mixture was the sum of the volumes of the alcohol and water, have a S.G. s, given by

$$(y/S+100-y)s=100;$$

so that the corresponding graduation on Sikes's hydrometer x would be given by

$$\frac{V}{V-ax}=\frac{s}{s_0}=\frac{y_0/S+100-y_0}{y/S+100-y},$$

s_0 denoting the S.G. and y_0 the percentage by weight of alcohol of the mixture in which the hydrometer sinks to 0, with one of the weights attached.

Then $$\frac{xa}{V}=\frac{(1-S)(y_0-y)}{100S+(1-S)y_0},$$

so that equal graduations of the scale give equal decrements in percentage by weight of alcohol.

But as it is found experimentally that the volume of the mixture is not the sum of the volumes of the separate alcohol and water, and that the density varies in an irregular manner with the temperature, the graduations of Sikes's hydrometer for spirits have been determined by experiment at two or three standard temperatures, and the indications recorded in a table.

(*Hydrometer*, Encyc. Britannica, p. 540, by W. Garnett.)

Proof spirit, defined by Act of Parliament, 58 G. III., so that "such spirit shall at 51° F. weigh exactly $\frac{12}{13}$ (0·9230769) of an equal measure of distilled water," may be taken as a mixture of 50 per cent. by weight of pure alcohol with 50 per cent. by weight of pure distilled water; more accurately, 49·3 per cent. by weight, or 57·09 per cent. by volume of alcohol, so that the S.G. S of pure alcohol at 51° F. is

$$S=\frac{49{\cdot}3}{50{\cdot}7}\div\frac{57{\cdot}09}{42{\cdot}91}={\cdot}7308.$$

An aqueous spirit is said to be x per cent. over proof when 100 volumes of this spirit diluted with water yields $100+x$ volumes of proof spirit; and it is said to be x per cent. under proof when it contains $100-x$ volumes of proof spirit in 100 volumes.

Let s denote the S.G. of an aqueous spirit, weighing 100 g, composed of a g of pure alcohol, of S.G. S, and $100-a$ g of water.

Then $100/s$ cm³ is the volume of the aqueous spirit, a/S and $100-a$ cm³ the volumes of the constituent alcohol and water; so that as/S and $(100-a)s$ are the percentages by volume of alcohol and water required for making this spirit, or the volumes in cm³ of alcohol and water required to make 100 cm³ of spirit.

79. *The Specific Volume of a Substance.*

The specific volume (S.V.) of a homogeneous substance is the number of units of volume occupied by the unit of weight; for instance the volume is ft^3/lb, or the volume is m^3/t, or m^3/kg, or is cm^3/g; the specific volume is thus the reciprocal of the density, it is sometimes called the *rarity* (whence the French word *aréomètre*), or *roomage* of the substance.

With very light compressible elastic substances, such as air, oxygen, hydrogen and gases generally the specific volume is better remembered and used than the density, which would be expressed by a small decimal; and it is practically more easily observed and measured when expansion takes place.

Thus the S.V. of ordinary air is about 13(ft^3/lb) meaning that 13 cubic feet of air weigh a lb, and then the S.G.

$$\rho = \frac{1}{13 \times 62{\cdot}4} = \frac{1}{811{\cdot}2} = {\cdot}00123;$$

so that, in Metric units, the S.G. of air is 1·23 kg/m^3 or g/litre, and the S.V. is 811 cm^3/g, or 0·81 m^3/kg (litres/g); while the S.V. of hydrogen is about 200 ft^3/lb, or 12·5 m^3/kg, at ordinary atmospheric pressure and temperature; and one litre of hydrogen thus weighs 0·08 g, or one m^3 weighs 0·08 kg.

For the stowage of cargo the ton is taken as the unit of weight, and the specific volume or roomage is the volume in ft^3 of a ton; thus sea-water occupies about 35 ft^3 to the ton, fresh water about 35·84, say 36.

Cargo stowing closer than 40 ft^3 to the ton is reckoned and charged by dead weight, while cargo of greater roomage is charged by volume of measurement, reckoning 40 ft^3 as a ton.

A Table in an Appendix gives the roomage of different kinds of cargo; convenient approximate rules are given by (i) roomage in ft^3/ton $= 36 \div s$,

(ii) heaviness in tons/(yard)$^3 = \frac{3}{4}s$,

where s denotes the S.G. of the substance.

80. Given the weights $W_1, W_2, \ldots, W_n$, in lb or tons, of volumes $V_1, V_2, \ldots, V_n$, ft^3, of substances of S.V. $v_1, v_2, \ldots, v_n$, so that

$$v_1 = V_1/W_1,\ v_2 = V_2/W_2,\ \ldots,\ v_n = V_n/W_n;$$

then the average S.V. of the mixture is given by

$$v = V/W,$$

where

$$W = W_1 + W_2 + \ldots + W_n,$$

the sum of the weights; and generally also

$$V = V_1 + V_2 + \ldots + V_n,$$

the sum of the volumes, supposing there is no change of volume; and then

$$v = \frac{V_1 + V_2 + \ldots + V_n}{W_1 + W_2 + \ldots + W_n}$$

$$= \frac{V_1 + V_2 + \ldots + V_n}{V_1/v_1 + V_2/v_2 + \ldots + V_n/v_n} = \frac{\Sigma V}{\Sigma V/v},$$

or

$$= \frac{W_1 v_1 + W_2 v_2 + \ldots + W_n v_n}{W_1 + W_2 + \ldots + W_n} = \frac{\Sigma W v}{\Sigma W}.$$

Thus, for example, atmospheric air of S.V. 13 being a mechanical mixture of oxygen of S.V. $11\frac{2}{3}$ and nitrogen of S.V. $13\frac{1}{3}$, it follows from the above that in a given quantity of air the weights of oxygen and nitrogen are as one to four, and the volumes as 7 to 32.

For putting $v = 13$, $v_1 = 11\frac{2}{3}$, $v_2 = 13\frac{1}{3}$ in the equations

$$v = \frac{W_1 v_1 + W_2 v_2}{W_1 + W_2} = \frac{V_1 + V_2}{V_1/v_1 + V_2/v_2},$$

we find

$$\frac{W_1}{W_2} = \frac{1}{4},\quad \frac{V_1}{V_2} = \frac{7}{32}.$$

81. Since the S.V. is the reciprocal of the density, it follows that the S.V. of a mixture of equal weights of different substances is the A.M. of the S.V.'s of the ingredients; and the S.V. of a mixture of equal volumes is the H.M. of the S.V.'s of the ingredients, supposing no change of volume to take place.

In the common hydrometer (fig. 36, p. 113), the hyperbolic curves CP, C_1P_1, ..., representing graphically the densities, would become transformed into straight lines radiating from the centres O, O_1, ..., of the hyperbolas, giving the corresponding specific volumes or rarities.

82. For a lactometer to give y the number of gallons of water to one gallon of milk, or reciprocally z the number of gallons of milk to one gallon of water, when a length x of the stem is shown above the mixture, then denoting by S and 1 the S.G. of pure milk and pure water and by s the S.G. of the mixture,

$$yS+1=(y+1)s,$$

or

$$S+z=(1+z)s,$$

so that

$$y=\frac{s-1}{S-s},$$

$$z=\frac{S-s}{s-1}.$$

But by § 63,

$$s=\frac{V}{V-ax}=\frac{a}{a-x},$$

$$S=\frac{V}{V-al}=\frac{a}{a-l},$$

so that

$$y=\frac{x(a-l)}{a(l-x)},$$

$$z=\frac{a(l-x)}{x(a-l)},$$

and the curves for y and z are equal hyperbolas.

83. *The Gravimetric Density of Gunpowder.*

Artillerists employ both the specific gravity and the specific volume in measuring the density of the powder in the cartridge of a gun.

The *gravimetric density* of the charge of powder is defined to be the ratio of its weight to the weight of the water which would fill the chamber of the bore behind the projectile in the gun.

The G.D. (gravimetric density) is therefore the S.G. of the powder, or powder gases when fired, which fill this powder chamber.

The specific volume of the powder charge, or its gases, is also given by the number of in^3 occupied by a lb; and a lb of pure distilled water having a specific volume of 27·73 in^3, the G.D. is obtained by dividing 27·73 by the number of cubic inches allotted to each lb of powder; this is equivalent to taking $D=62{\cdot}3$, the density of water at about 68° F.

Thus a gun charge expressed by

$$75P_2\frac{33{\cdot}00}{0{\cdot}84}$$

means 75 lb of P_2 powder, with 33 in^3 of space per lb of powder, and a consequent G.D. of $27{\cdot}73 \div 33 = 0{\cdot}84$.

(Mackinlay, *Text Book of Gunnery*, 1887, p. 22.)

According to §§ 32, 33 the G.D. of a charge of lead shot will be

$$\tfrac{1}{6}\pi\sqrt{2} = 0{\cdot}7403$$

of the S.G. of lead; and the G.D. of a charge of the new cordite powder, composed of cylindrical filaments, will be

$$\tfrac{1}{6}\pi\sqrt{3} = 0{\cdot}9067$$

of the S.G. of the substance of the cordite.

General Exercises on Chapter III.

(1) The diameters of two globes are as 2 : 3, and their weights as 1 : 5; compare their specific gravities.

(2) The weight of a vessel when empty is 3 oz; when filled with water, it is 9 oz; and when filled with olive oil, 8·49 oz; required the S.G. of the oil.

(3) A vessel filled with water weighs $5\frac{1}{4}$ oz, and when a piece of platinum weighing $29\frac{1}{4}$ oz is placed in it, and it is filled up with water, it weighs 33 oz; prove that the S.G. of the platinum is 19·5.

(4) The weight of a piece of cork in air is $\frac{3}{4}$ oz, the weight of a piece of lead in water is $6\frac{4}{9}$ oz, and the weight of the cork and lead together in water is 4·07 oz. Prove that the S.G. of the cork is 0·24.

(5) A piece of metal weighing 36 lb in air, and 32 lb in water, is attached to a piece of wood whose weight is 30 lb, and then the compound body is found to weigh 12 lb in water.

Prove that the S.G. of the wood is 0·6.

(6) The S.G.'s of platinum, standard gold, and silver being respectively 21, 17·5, and 10·5, and the values of an ounce of each 30s, 80s, and 5s respectively; prove that the value of a coin composed of platinum and silver, which is equal in weight and magnitude to a sovereign, is 6s 3d.

(7) A solid, whose weight is 250 grains, weighs 147 in water, and 130 in another fluid. Prove that the S.G. of the latter is 1·262.

(8) A solid, whose weight is 60 grains, weighs 40 grains in water, and 30 grains in sulphuric acid; required the S.G. of the acid.

(9) The S.G. of gold being 19·25, and of copper 8·9, what are the weights of copper and gold respectively in a compound of these metals which weighs 800 grains in air, and 750 in water?

(10) A piece of gun-metal was found to weigh 1057·9 grains in air, and 934·8 grains in water; find the proportions of copper and of tin in 100 lb of the metal, the S.G. of the copper being 8·788 and of tin 7·291.

(11) A body immersed in a liquid is balanced by a weight P, to which it is attached by a thread passing over a fixed pulley; and when half immersed, is balanced in the same manner by a weight $2P$. Prove that the densities of the body and liquid are as 3 to 2.

(12) It is found on mixing 63 pints of sulphuric acid, whose S.G. is 1·82, with 24 pints of water, that 1 pint is lost by their mutual penetration; find the S.G. of the compound.

(13) A piece of gold immersed in a cylinder of water causes it to rise a inches; a piece of silver of the same weight causes it to rise b inches; and a mixture of gold and silver of the same weight c inches; prove that the gold and silver in the compound are by weight as $b-c:c-a$.

(14) The S.G. of lead is 11·324; of cork is 0·24; of fir is 0·45; determine how much cork must be added to 60 lb of lead that the united bodies may weigh as much as an equal volume of fir.

(15) The S.G.'s of pure gold and copper are 19·3 and 8·62; required the S.G. of standard gold, which is an alloy of 11 parts pure gold and one part copper.

(16) If the liquid employed with Nicholson's Hydrometer be water, the substance a mixture of two metals whose S.G.'s are 14 and 16, and the weights used are 16 oz, 1 oz, 2 oz; find the quantity of each metal in the mixture.

(17) Show that the units may be chosen so that the specific gravity and the density of a substance are identical.

A nugget of gold mixed with quartz weighs 12 (10) ounces, and has a specific gravity 6·4 (8·6); given that the specific gravity of gold is 19·35, and of quartz is 2·15, find the quantity of gold in the nugget.

(18) Air is composed of oxygen and nitrogen mixed together in volumes which are as 21 to 79, or by weights which are as 23 to 77; compare the densities of the gases.

(19) How many gallons of water must be mixed with 10 gallons of milk to reduce its S.G. from 1·03 to 1·02?

(20) Bronze contains 91 per cent. by weight of copper, 6 of zinc, and 3 of tin. A mass of bell-metal (consisting of copper and tin only) and bronze fused together is found to contain 88 per cent. of copper, 4·875 of zinc, and 7·125 of tin. Find the proportion of copper and tin in bell-metal.

(21) Two fluids are mixed together: first, by weights in the proportion of their volumes of equal weights; secondly, by volumes in the proportion of their weights of equal volumes; compare the specific gravities of the two mixtures.

(22) A mixture of gold with n different metals contains r per cent. of gold and $r_1, r_2, r_3, \ldots, r_n$ per cent. of the other metals. After repeated processes, by which portions of the other metals are taken away, the amount of gold remaining unaltered, the mixture contains s per cent. of gold and $s_1, s_2, s_3, \ldots, s_n$ per cent. of the other metals.

Find what percentage of each metal remains.

(23) A quart vessel is filled with a saturated solution of salt. A quart of water is poured drop by drop into the vessel, causing the solution to overflow, but is poured in so slowly that it may be supposed to diffuse quickly through the solution. Show that after the operation the amount of salt left in the solution in the vessel will be $1/e$ of the original amount, where e is the base of the Naperian logarithms.

(24) From a vessel full of liquid of density ρ is removed one-nth of the contents, and it is filled up with liquid of density σ. If this operation is repeated m times, find the resulting density in the vessel.

Deduce the density in a vessel of volume V, originally filled with liquid of density ρ, after a volume U of liquid of density σ has dripped into it by infinitesimal drops.

(25) The mixture of a gallon of A with W_1 lb of B has a S.G. s_1, with W_2 lb of B a S.G. s_2, with W_3 lb of B a S.G. s_3; find the S.G.'s of A and B.

(26) Find the chance that a solid composed of three substances whose densities are ρ_1, ρ_2, ρ_3, will float in a liquid of density ρ_2.

(27) A vessel is filled with three liquids whose densities in descending order of magnitude are ρ_1, ρ_2, ρ_3. All volumes of the liquids being equally likely prove that the chance of the density of the mixture being greater than ρ is

$$\frac{(\rho_1-\rho)^2}{(\rho_1-\rho_2)(\rho_1-\rho_3)},$$

or

$$1-\frac{(\rho-\rho_3)^2}{(\rho_2-\rho_3)(\rho_1-\rho_3)},$$

according as ρ lies between ρ_1 and ρ_2 or between ρ_2 and ρ_3.

(28) Describe some method of determining the absolute expansion of a liquid.

A piece of copper is weighed in water at 16° and at 80°, the weights of water displaced being 50 g and 48·809 g; find the mean coefficient of cubical expansion of copper between those temperatures; given the S.G. of water at 16° and 80° as

0·999 and 0·972.

(29) The hydrometer is used to determine the S.G. of a liquid which is at a temperature higher than that of water.

When the hydrometer is transferred from water to the liquid the S.G. appears at first to be s, but afterwards to be s_1.

Show that, neglecting the density of the air, the true S.G. at the temperature of the water is

$$s+\frac{a'}{a}(s_1-s),$$

where a and a' are the coefficients of expansion of the hydrometer and the liquid respectively.

(30) Show that the coefficient of expansion of a body may be found as follows:—

Let s be the S.G. of the body at zero temperature compared with water at its greatest density; $1+e_1$, $1+e_2$ the volumes at temperatures t_1, t_2 of a unit volume at zero temperature; $1+E_1$, $1+E_2$ the volumes at t_1, t_2 of a unit volume of water at its greatest density; w the weight of the body in a vacuum; w_1, w_2 its apparent weights in water at temperatures t_1, t_2; then

$$e_1-e_2=E_1-E_2-s(w_1-w_2)/w \text{ very nearly.}$$

(31) Prove that, if a hydrometer of weight W sinks to certain marks on the stem in a liquid at temperatures t_1 and t_2, and to the same marks in the liquid at zero temperature, when weights w_1 and w_2 are fixed at the top of the hydrometer, the coefficients of cubical expression of the hydrometer and of the liquid are respectively

$$\frac{\frac{w_1}{t_1}-\frac{w_2}{t_2}}{w_2-w_1} \quad \text{and} \quad \frac{\frac{w_1}{t_1}(W+w_2)-\frac{w_2}{t_2}(W+w_1)}{W(w_2-w_1)}.$$

(32) Determine the S.V. in cubic feet to the ton, and the density in lb per cubic foot of lead shot, cast iron spherical shot, and cast iron spherical shells with internal radius three-quarters the outside radius, given the S.G. of lead as 11·4, and of cast iron 7·2.

Determine also the S.V. or roomage of earthenware pipes, and cylindrical barrels, of apparent density ρ.

CHAPTER IV.

THE EQUILIBRIUM AND STABILITY OF A SHIP OR FLOATING BODY.

84. *Simple Buoyancy.*

The Principle of Archimedes leads immediately, as in § 48, to the Conditions of Equilibrium of a body supported freely in fluid, like a fish in water, or a balloon in air, or like a ship floating partly immersed in water (fig. 38, p. 148).

The body is in equilibrium under two forces;

(i.) its weight W acting vertically downwards through G, the C.G. of the body; and

(ii.) the buoyancy of the fluid, equal to the weight of the displaced fluid, and acting vertically upwards through B, the C.G. of the displaced fluid;

and for equilibrium these two forces must be equal and directly opposed.

The Conditions of Equilibrium of a body, floating like a ship on the surface of a liquid, are therefore

(i.) the weight of the body must be less than the weight of the total volume of liquid it can displace, or else the body will sink to the bottom of the liquid;

(ii.) the weight of liquid which the body displaces in the position of equilibrium is equal to the weight W of the body;

(iii.) the C.G. B of the displaced liquid and G of the body must lie in the same vertical line GB.

85. In a ship the *draft* of water is a measure of the displacement and buoyancy of the water, while the *free-board*, or height of the deck above the water line, is a similar measure of the reserve of buoyancy, or of the extra cargo which the ship can carry without sinking.

The Plimsoll mark is now, by Act of Parliament, painted on all British ships; it is a mark which must not be submerged when the vessel is floating in a fresh water dock, before putting to sea; and the mark is fixed at such a height as to give the vessel a reserve of buoyancy of 25 per cent. of its total buoyancy.

The buoyancy of a pontoon or cask, employed as a support or buoy, is however generally used to mean its reserve of buoyancy, or the additional weight required to submerge it.

Thus the (reserve of) buoyancy of a body, a life buoy for instance, of weight W lb and (apparent) S.G. s, and therefore displacing W/s lb of water, is

$$\left(\frac{1}{s}-1\right) W \text{ lb.}$$

86. When a ship loses its reserve of buoyancy, and is sunk in shallow water, it can be raised by building a *caisson* on the deck so as to bring the level of the bulwarks above the surface at low water.

All leaks and orifices below water having been stopped by divers, the vessel is pumped out at low water by powerful steam pumps; and thereby soon acquires sufficient buoyancy to rise from the bottom of the sea, so as to be moved into a dock for repair; in this manner such large vessels as the Utopia, the Austral, and the Howe have been raised.

When a vessel draws too much water for entering or leaving a port, as for instance Venice, the Zuyder Zee, or Chicago through the lakes of N. America, *camels* are employed to lessen the draft of water.

These camels consist of large tanks, which are submerged by the admission of water, and then secured to the sides of the vessel by chains passing under the keel. On being pumped out the extra buoyancy of the camels raises the vessel and lessens the draft of water to the desired extent.

The same principle is employed in floating docks: the dock is submerged by the admission of water, so that the vessel can be floated on to the blocks on the bottom of the dock and be there secured: the water is then pumped out of the dock and the vessel is thereby raised above the level of the water, and can then be deposited on staging ashore, or even repaired on the floating dock itself; in this case it is convenient to secure the dock to the quay wall by pivoted bars.

The double power dock, designed by Messrs. Clark and Stansfield, consists of a central pontoon which supports the vessel, and two large side tanks or camels, which can float independently. The vessel is raised as far as possible by pumping out the central pontoon; the camels are then submerged by the admission of water, and secured to the sides of the pontoon; and now, the buoyancy of these camels, on being pumped out, is sufficient to raise the vessel completely above the water.

By this arrangement not only is economy of material secured, but the pontoon or the camel can be alternately raised completely out of the water for the purpose of examination and repair. (*Trans. I. Naval Architects,* XX.)

87. Denoting by A the *water line area* (*flottaison*) of a ship in square feet, that is, the area of the plane curve formed by the water line, then an additional load of P tons properly placed (that is, so that the C.G. of P is vertically over or under the C.G. of the water line area) will cause the ship to draw h feet more water, of density D lb/ft³ suppose, given by the equation

$$DAh = 2240P.$$

Strictly speaking this supposes either that the ship is *wall-sided*, meaning that the sides of the ship in the neighbourhood of the water line form part of a cylindrical surface; or else that the mean water line area at the mean draft is A ft²; and thus, given P/h, we can determine A, and *vice versa*.

For sea water we take $D = 64$, so that the S.V. of sea water is $2240 \div 64 = 35$ ft³/ton; and

$$Ah = 35P;$$

or if h is given in inches,

$$Ah = 420P,$$

$$\frac{P}{h} = \frac{A}{420},$$

and P/h is the number of tons required to immerse the ship one inch.

Thus in a ship loading 10 tons to the inch, the water line area is 4200 ft²; and loading or consuming 300 tons of coal will change the draught 2 ft 6 in.

For a ship L ft long and B ft broad at the water line,

$$A = cLB,$$

where c is called the *coefficient of fineness* of the area. The following rules are given by Mr. W. H. White for c the coefficient of fineness, and $n = P/h$ the number of tons per inch immersion (*Naval Architecture*) :—

	$c=$	$n=$
1. For ships with fine ends, -	0·7,	$\frac{LB}{600}$
2. For ships of ordinary form (including probably the great majority of vessels), - - - -	0·75,	$\frac{LB}{560}$
3. For ships of great beam in proportion to the length, and ships with bluff ends, - - - -	0·84,	$\frac{LB}{500}$

A Mercury Weighing Machine has been invented by Mr. Rutter (*Industries*, 16 Oct., 1891) in which the body to be weighed is placed in a scale pan which is suspended from a cylindrical plunger immersed in mercury, and the weight is read off on graduations corresponding to the weight of the extra quantity of mercury displaced.

Thus the S.G. of mercury being 13·6, the vertical graduations will correspond to kilogrammes per centimetre immersion if the cross section of the plunger is $1000 \div 13{\cdot}6 = 73{\cdot}53$ cm^2 in area, or 9·68 cm in diameter.

88. Suppose the ship's weight and displacement is W tons, and that the draft of water increases by h inches as the density of the water diminishes from w to w' tons/ft^3; then the original displacement being V ft^3,

$$W = wV = w'(V + \tfrac{1}{12}Ah),$$

the ship now acting like the common hydrometer; and these two equations are sufficient to determine V and W when A, w, and w' are known.

Or denoting the S.V.'s in ft^3/ton by v and v';

$$V = vW, \quad V + \tfrac{1}{12}Ah = v'W,$$

$$\tfrac{1}{12}Ah = (v - v')W.$$

Thus if $v=35$ for sea water, and $v'=35{\cdot}84$ for fresh water,
$$\tfrac{1}{12}Ah=0{\cdot}84\,W;$$
and if n denotes the number of tons per inch immersion,
$$h=\frac{10{\cdot}08\,W}{420n}\approx\frac{W}{40n},$$
approximately, giving the sinkage in inches of the ship in passing from salt to fresh water.

For instance if a ship of 8500 tons displacement draws 25 ft of water at sea, and if the length on the water line is 330 ft and the breadth 65 ft, the sinkage in passing into fresh water is a little over 5 inches, and draft 25 ft 5 ins.

89. Large vessels are now built in compartments separated by transverse watertight bulkheads, so as to localise and restrict the effect of a leak or perforation.

Now if one of these compartments is *bilged* and becomes filled with water, the loss of buoyancy in ft³ is the volume of water which has entered, so that the sinkage in feet
$$=\frac{\text{loss of buoyancy in ft}^3}{\text{intact water line area in ft}^2}.$$

If the compartment is fitted with a watertight deck below the water line, the water line area of the vessel may be taken as unchanged, and the sinkage will be correspondingly diminished.

If the compartment is occupied by cargo, such as coal, timber, casks, etc., the volume of water which enters is diminished by the volume of this cargo, so that the loss of buoyancy is that due to the *unoccupied* space in the compartment, up to the new water line.

Thus, according to §§ 32, 33, the unoccupied space is $1-\frac{1}{6}\pi\sqrt{3}$ or $1-\frac{1}{6}\pi\sqrt{2}$ of the volume according as the cargo in the compartment is composed of equal cylindrical or spherical bodies, such as casks or grain closely packed.

The S.G. of a lump of coal being 1·4, while a coal cargo stows at a 40 to 45 ft^3/ton, the fraction of unoccupied space is about 0·4 or 40 per cent.

90. *The Energy of Immersion.*

By the operation of plunging a body into a fluid a certain amount of energy is communicated to the fluid.

If a liquid is contained in a vessel of finite size, and a body is made to displace a volume V ft^3, the gain of energy is the work required to raise V ft^3 of the liquid from the level of B, the C.G. of the volume V of the body, to the level of B', the C.G. of the equal volume contained between the old and new surfaces of the liquid; and the gain is therefore

$$wV(z-z') \text{ ft-lb},$$

if w denotes the density of the liquid in lb/ft^3, and z, z' denote the vertical depths of B, B' below a given horizontal plane, say the new surface of the liquid.

When the vessel containing the liquid is of practically unlimited size, the level of the liquid does not sensibly change by the operation of immersion of the body, and now B' lies in the surface; so that the gain of energy is the work required to lift V ft^3 of the liquid from B to the surface.

If the body is completely submerged, the level of the surface of the liquid will not change for different positions of the body; so that, if the C.G. of the body is depressed vertically through x ft, the gain of energy or work required to depress the body is equal to that required to raise of volume V of the liquid from the second to the first position occupied by the body, supposed to displace V ft^3 of liquid, and is therefore

$$wVx \text{ ft-lb}.$$

The force required to depress the body, or the upward buoyancy of the liquid is thus a force of wV pounds, as before in § 45.

91. But if the body is only partially immersed, as at first, and is now depressed vertically through a small vertical distance x ft, the liquid will rise vertical distances on the side of the vessel, and on the side of the body,

$$\frac{ax}{\beta-a}, \text{ and } x+\frac{ax}{\beta-a} \text{ or } \frac{\beta x}{\beta-a},$$

if a, β denote the (average) areas of the horizontal cross sections of the body and of the vessel made by the horizontal planes of the surface of the liquid; so that the extra volume displaced will be

$$U=\frac{a\beta x}{\beta-a},$$

the C.G. of which will be raised through

$$\tfrac{1}{2}\frac{\beta x}{\beta-a}-\tfrac{1}{2}\frac{ax}{\beta-a} \text{ or } \tfrac{1}{2}x \text{ feet,}$$

and the consequent gain of energy is

$$\tfrac{1}{2}w\frac{a\beta x^2}{\beta-a}=\tfrac{1}{2}wUx, \text{ ft-lb.}$$

At the same time the depression through x ft of the volume V already immersed will give a gain of energy of wVx ft-lb, so that the total gain of energy is

$$wVx+\tfrac{1}{2}w\frac{a\beta x^2}{\beta-a}=w(V+\tfrac{1}{2}U)x \text{ ft-lb.}$$

The average resistance of the liquid to the depression of the body is thus a force of $w(V+\frac{1}{2}U)$ pounds, the buoyancy of a volume $V+\frac{1}{2}U$ of the liquid; reducing as before, for an infinitesimal depression, to a force of wV pounds, the buoyancy of a volume V of the liquid.

92. When α and β are constant, the body and the vessel are cylindrical, and the preceding expressions hold for finite values of the depression h.

Thus if the body is a vertical cylinder floating freely in the liquid, the weight of the cylinder and the buoyancy of the water is wV lb; and the work required to depress the cylinder vertically through x ft is the gain of energy of the liquid less the loss of energy of the body, and the work is therefore

$$\tfrac{1}{2}wUx = \tfrac{1}{2}w\frac{\alpha\beta x^2}{\beta-\alpha} \text{ ft-lb.}$$

If the cylinder is of height h and density w', the length hw'/w of the axis is submerged, and $h(w-w')/w$ stands out of the liquid; to immerse the body completely, it must be pushed down a vertical distance x, given by

$$\frac{\beta}{\beta-\alpha}x = \frac{w-w'}{w}h;$$

and the work required is therefore

$$\tfrac{1}{2}\frac{(w-w')^2}{w}\left(1-\frac{\alpha}{\beta}\right)\alpha h^2, \text{ ft-lb.}$$

93. Generally a floating body will come to rest in a position in which the energy of the system is a minimum; and the preceding considerations show that the distance between G, the C.G. of the body, and B, the C.G. of the liquid displaced, will then be a minimum; the distance being a maximum for positions of unstable equilibrium.

Examples.

(1) At low water a gallon was found to weigh 10 lbs, and at high water to weigh 10·25 lbs; and it required 25 tons to bring a vessel at high water down to the draught at low water; prove that the ship weighed 1000 tons.

(2) A steamer loading 30 (25) tons to the inch in fresh water is found after a 10 days' voyage, burning 60 (52) tons of coal a day, to have risen 2 feet (25 inches) in sea water at the end of the voyage; prove that the original displacement of the steamer was 5720 (5000) tons, taking a cubic foot of fresh water as 62·5 pounds and of sea water as 64 pounds.

(3) A steamer in going from salt into fresh water is observed to sink two inches, but after burning 50 tons of coal to rise one inch; prove that the steamer's displacement was 6500 tons, supposing the densities of sea and fresh water are as 65 to 64.

(4) A steamer of 5000 tons displacement drawing 25 feet of water has to discharge 300 tons of water ballast to lessen the draft one foot, to cross the bar into a river. Prove that after burning 50 tons of coal in going up the river the steamer will be drawing 24·2 ft in fresh water; and now the admission of 293 tons of water ballast will be sufficient to increase the draft one foot.

(5) A sphere of radius r ft and weight W lb is let gently down into a vertical cylinder of radius R, containing water of twice the density of the sphere. Show that the work done on the water before the sphere begins to float is

$$Wr(\tfrac{3}{8} - \tfrac{1}{3}r^2/R^2), \text{ ft-lb.}$$

(6) Explain why it is that if a man puts his hand and arm into a bucket partly filled with water, potential energy is imparted to the water.

A sphere of radius r is held just immersed in a cylindrical vessel of radius R containing water, and is caused to rise gently just out of the water. Prove that the gain of potential energy of the sphere and the loss of potential energy of the water are respectively

$$2Wr(1-\tfrac{2}{3}r^2/R^2),$$

and

$$W'r(1-\tfrac{2}{3}r^2/R^2), \text{ ft-lb},$$

W and W' being the weight in lb of the sphere and of the water displaced by the sphere.

If the sphere be allowed to rise until it is half out of the water, prove that the loss of potential energy is to the loss in the previous case in the ratio of

$$39R^2-24r^2 : 48R^3-32r^2.$$

If the sphere be left to itself when under water, and if we could suppose the water to come to rest on the sphere leaving it, what would be the velocity with which the sphere would shoot out.

(7) The arms of a balance are each of length a cm, and one of them at its end carries as a permanent counterpoise hanging from it a cylindrical vessel whose sectional area is α cm², containing liquid of density w, in which dips a fixed vertical solid cylinder of sectional area β cm². The beam is itself counterpoised for all inclinations, and the cylinder does not touch the vessel.

Show that, when an addition of W g is made to the load on the other arm, the sine of the inclination of the beam to the horizontal is altered by

$$\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\frac{W}{wa}.$$

94. *The Conditions of Stability of a Ship.*

In addition to satisfying the conditions of equilibrium of a floating body (§§ 45, 48, 84) it is necessary that a ship should fulfil the further condition of *stability of equilibrium,* so as not to capsize; if slightly disturbed from the position of equilibrium, the forces called into action must be such as tend to restore the ship to its original position.

Practically the stability of a ship is investigated by inclining it; weights are moved across the deck and the angle of heel thereby produced is observed, and thence an estimate of the stability can be formed.

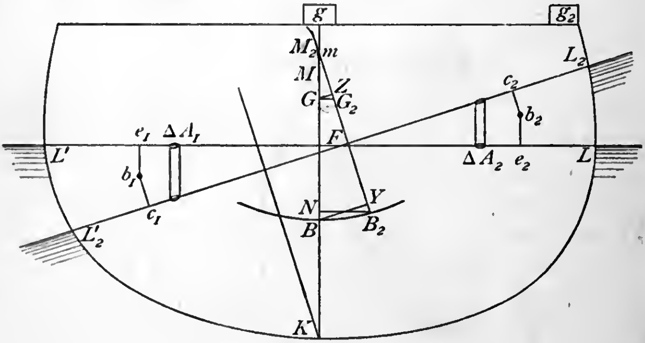


Fig. 38.

95. Let the total weight of the ship be W tons, and let it displace V ft³ of water; then

$$W = wV, \quad \text{or} \quad 2240W = DV,$$

where w denotes the density of the water in tons/ft³, and Dn lb/ft³, so that $D = 2240w$.

Now suppose a weight of P tons on board, originally amidships, is moved to one side of the deck, a distance of b ft; and that the ship, originally upright, is observed to heel through an angle θ (fig. 38).

The C.G. of the displaced water, called the *centre of buoyancy* (C.B.), will move on a curve (or surface) called the *curve (or surface) of buoyancy*, from B to B_2, such that G_2B_2 is vertical in the new position of equilibrium, G_2 being the new C.G. of the ship when P tons is moved from g to g_2, so that the ship will move as if the surface of buoyancy was supported by a horizontal plane.

As P is moved across the deck from g to g_2 a distance of b ft, so the C.G. of the body moves on a parallel line from G to G_2, such that $GG_2=bP/W$; this follows because the moments of P and W about Gg must be the same; and, if the new vertical B_2G_2 cuts the old vertical BG in m,

$$Gm=\frac{P}{W}b\cot\theta,\quad G_2m=\frac{P}{W}b\,\mathrm{cosec}\,\theta.$$

The ultimate position M of m for a small angle of heel is the point of ultimate intersection of the normals at B and at the consecutive point B_2 on the curve of buoyancy, and M is therefore the centre of curvature of the curve of buoyancy at B; the point M is called the *metacentre*, and GM is called the *metacentric height*.

In the diagram the ship is drawn for clearness in one position, and the water line is displaced; but the page can be turned so as to make the new water line horizontal.

96. For stability of equilibrium the metacentre M must be above G, for if M were below G then on bringing P back suddenly from g_2 to g, the forces acting on the ship would form a couple tending to capsize the ship; but if M is above G the forces would then form a couple, consisting of W acting vertically downwards through G, and W acting vertically upwards along B_2m, tending to restore the ship to the upright position.

The angle of heel θ is measured either by a spirit level or by the deflection of a pendulum or plummet.

If the ship is symmetrical and upright when P is amidships, and if moving the weight P tons across the deck through $2b$ ft causes the plummet to move through $2a$ ft when suspended by a thread l ft long, then

$$\sin\theta = a/l;$$

so that

$$G_2m = \frac{Pbl}{Wa},$$

and G_2m may be taken as the metacentric height GM.

Thus in H.M.S. Achilles, of 9000 tons displacement, it was found that moving 20 tons across the deck, a distance of 42 ft, caused the bob of a pendulum 20 ft long to move through 10 inches.

Here $W = 9000$, $P = 20$, $b = 21$, $l = 20$, $a = \frac{5}{12}$; and therefore $GM = 2{\cdot}24$ ft. Also $\sin\theta = 0{\cdot}02083$, $\theta = 1^\circ\,12'$.

97. The displacement W tons or V ft³ is determined by approximate calculations from the drawings of the ship, as also the C.B. B; while G is determined from the weights of the different parts of the structure, and from the distribution of the cargo and ballast.

If the weight P was hoisted vertically up the mast a distance h, B and M would not change, but G would ascend to G_1, through a height $GG_1 = hP/W$ ft.

The metacentric height would be correspondingly diminished; so that if P is now moved along a yard on the mast a distance b feet, the ship will heel through an angle θ_1, such that

$$Pb = W\,.\,GM\sin\theta = W\,.\,G_1M\sin\theta_1;$$

and to produce the original angle of heel θ, P requires to be moved through a less distance b', such that

$$b - b' = h\sin\theta.$$

98. So far we have considered as the chief practical problem the transverse metacentre M in its relation to the heeling or rolling of a ship; but a similar metacentre exists for any alteration of *trim*, caused either by change of stowage of cargo, or by press of sail and other propulsion; this *longitudinal* metacentre is found by a similar experimental process, but from the shape of a ship it is necessarily much higher than the transverse metacentre.

The *trim* of a ship is defined as the difference of draft of water at the bow and stern, and the *change of trim* is defined as the sum of the increase of draft at one end and the decrease of draft at the other.

Suppose the trim is changed x inches in a vessel L ft long at the water line by moving P tons longitudinally fore and aft through a ft, and that the ship turns through a small angle θ, a gradient of one in $12L/x$.

The moment to change the trim is Pa ft-tons, so that if M_1 denotes the longitudinal metacentre,

$$Pa = W . GM_1 . \sin\theta = W . GM_1 . x/12L;$$

thus the moment required to change the trim one inch is

$$W . GM_1/12L \text{ ft-tons.}$$

For instance, if a ship of 9200 tons, 375 ft long, has a longitudinal metacentric height of 400 ft, and a weight of 50 tons already on board is shifted longitudinally through 90 ft, the change of trim will be about $5\frac{1}{2}$ inches.

Practically it is found convenient to incline the ship by filling alternately the boats suspended on either side of the ship with a known weight of water; and to change the trim by filling and emptying water tanks at the ends of the ship.

99. As the ship heels through an angle θ, and the water line changes from LL' to L_2L_2' (fig. 38), a certain volume U of water may be supposed removed from the wedge-shaped volume $L'FL_2'$, called the *wedge of emersion*, to the volume LFL_2, called the *wedge of immersion*, so as to form the new volume of displacement L_2KL_2'.

If b_1, b_2 denote the C.G.'s of the wedges of emersion and immersion, BB_2 is parallel to b_1b_2; and if BY, c_1c_2 are the projections of BB_2, b_1b_2 on the new water line L_2L_2,

$$BB_2 = b_1b_2 \,.\, U/V, \qquad BY = c_1c_2 \,.\, U/V.$$

We notice that when θ is evanescent, the line b_1b_2 coalesces with the water line LL'; and therefore the tangent to the curve of buoyancy at B, being the ultimate direction of the line BB_2, is parallel to the water line LL', a theorem due to Bouguer.

If the C.G. of the ship is at G, and GZ is drawn perpendicular to B_2m, the moment in ft-tons of the couple tending to restore the ship to the upright position is

$$\begin{aligned} W \,.\, GZ &= W(BY - BG \sin\theta) \\ &= W\left(\frac{U}{V}c_1c_2 - BG\sin\theta\right) \\ &= w(U \,.\, c_1c_2 - V \,.\, BG \sin\theta), \end{aligned}$$

Atwood's formula (*Phil. Trans.*, 1798).

Curves are now drawn for ships by naval architects, called *cross curves of stability*, which exhibit graphically the value of the righting moment $W \,.\, GZ$ for a given inclination θ, say an angle of 30°, 45°, or 60°, and for different drafts of water of the ship and displacements W or V; the volumes U of the wedges of immersion and emersion are calculated and the corresponding values of c_1c_2; also the values of BG, and thence GZ is known for an assumed position of G.

100. In the case of a body floating completely submerged, like a fish, submarine boat, or Whitehead torpedo, the buoyancy always acts vertically upwards through B, the C.G. of the displaced liquid; or the C.B. is a fixed point in the body.

In the position of stable equilibrium, B must be above G; and if displaced through an angle θ, the righting moment will be

$$W \,.\, GB \sin \theta.$$

When the water line area is very small, as is the case of a rod or hydrometer, then BM is small, so that the body behaves as if completely submerged, and requires ballasting to bring G below B.

Stability is secured with greater facility by increase of beam on the water line area; a caisson as in fig. 23, p. 54, or a yacht of similar cross section, would require considerable ballast for stability.

In a body of revolution (fig. 45, p. 190) such as a cork, a circular pontoon, a cigar ship, a spherical buoy, or a hydrometer floating with its spherical bulb partially immersed, the curves of buoyancy are circular so long as no break in the circular cross section meets the water line; the metacentre M lies in the axis or centre, and the equilibrium is stable in the position in which G is vertically below M; the righting moment at any inclination θ being

$$W \,.\, GM \sin \theta.$$

101. To determine the metacentre theoretically, we suppose the angle θ through which the ship heels to be small, so that the C.B. moves from B to the adjacent point B_2 on the curve of buoyancy BB_2, without alteration of the displacement V.

The resultant buoyancy of W tons acting vertically upwards through B_2 in the displaced position is equivalent to an equal buoyancy W acting upwards through B, and a couple of moment $W.BY$ (fig. 38); and this couple is due to the upward buoyancy through b_2 of the wedge of immersion LFL_2, and an equal downward force through b_1, due to the loss of buoyancy of the wedge of emersion $L'FL'_2$.

Suppose the ship turns about an axis through F perpendicular to the plane of the paper; then, denoting by y the distance of an element ΔA of the water line area A from this axis of rotation, the element of volume of either wedge is ultimately

$$y \tan\theta . \Delta A.$$

The equality of the wedges of immersion and emersion leads, on dropping the factor $\tan\theta$, to the condition

$$\Sigma y \Delta A = 0,$$

so that the axis of rotation passes through the C.G. of the water line area, which we may denote by F.

The righting couple of the wedges of immersion and emersion will be

$$\Sigma wy \tan\theta . \Delta A . y = w \tan\theta \Sigma y^2 \Delta A$$
$$= w \tan\theta . Ak^2, \text{ ft-tons,}$$

where Ak^2 denotes the moment of inertia in ft^4 (biquadratic feet) of the water line area A about the axis of rotation (§ 38); so that

$$W.BY = w \tan\theta . Ak^2.$$

But $$W = wV,$$

and $$BY = BM \sin\theta, \text{ ultimately;}$$

so that, finally, with $\cos\theta = 1$, or $\sin\theta = \tan\theta$,

$$BM = Ak^2/V,$$

the radius of curvature of the curve of buoyancy at B.

The ship may thus be assimilated, in the neighbourhood of the water line in the upright position, to a surface of revolution about a horizontal axis, as in fig. 45, p. 190, when BB_2 will be the arc of a circle whose centre is M; and for an angle of heel θ the chord

$$BB_2 = 2\sin\tfrac{1}{2}\theta \,.\, BM\,;$$

and therefore the moment of the wedges of emersion and immersion will be accurately

$$2\sin\tfrac{1}{2}\theta \,.\, wAk^2.$$

102. We can prove these theorems by considering separately the wedges of immersion and emersion.

Suppose the axis of rotation through F divides the water line area into two parts, of areas A_1 and A_2, and that h_1 and h_2 denote the distances of the C.G.'s of these areas from the axis of rotation.

Then the volumes of the wedges of emersion and immersion may be taken as $A_1h_1\tan\theta$ and $A_2h_2\tan\theta$; and these volumes being equal and denoted by U,

$$U\cot\theta = A_1h_1 = A_2h_2,$$

which proves that the axis of rotation passes through the C.G. of the water line area.

Also these wedges being equivalent to laminas A_1 and A_2, loaded so that the superficial density is proportional to the distance from the axis of rotation (§ 39),

$$Fc_1 = \frac{\Sigma y^2 \Delta A_1}{A_1h_1}, \quad Fc_2 = \frac{\Sigma y^2 \Delta A_2}{A_2h_2},$$

and therefore

$$c_1c_2 = \frac{\Sigma z^2 \Delta A}{U}\tan\theta = \frac{Ak^2}{U}\tan\theta\,;$$

and $$BY = \frac{Ak^2}{V}\tan\theta, \quad BM = \frac{Ak^2}{V},$$

ultimately, as before.

103. If the vessel is "wall sided" from the upright to an angle of heel θ, then up to this limit the curve of flotation reduces to a point F; and the volume U of the wedges of emersion and immersion will be given by

$$U \cot \theta = A_1 h_1 = A_2 h_2.$$

If $b_1 e_1$, $b_2 e_2$ be the perpendiculars from b_1, b_2 on the upright water plane A, then, as before (fig. 38, p. 148),

$$e_1 e_2 = \frac{Ak^2}{U} \tan \theta, \quad NB_2 = \frac{Ak^2}{V} \tan \theta = BM \tan \theta,$$

so that the subnormal Nm is constant and equal to BM; and the curve of buoyancy BB_2 is a parabola (fig. 44, p. 190) of which BM is the semi-latus-rectum, and therefore

$$Mm = BN = \tfrac{1}{2} NB_2 \tan \theta = \tfrac{1}{2} BM \tan^2\theta.$$

Now if GZ is the arm of the righting couple $W . GZ$,

$$GZ = Gm \sin \theta = (GM + \tfrac{1}{2} BM \tan^2\theta) \sin \theta.$$

The surface of buoyancy is now given accurately by the equation of § 107, and is therefore a *paraboloid*.

For instance, if a cylinder, whose cross section is the water line area A, floats upright immersed to a depth h in liquid, $V = Ah$, and (§ 107),

$$\frac{2z}{h} = \frac{y^2}{k_z^2} + \frac{z^2}{k_y^2}$$

is the equation of the surface of buoyancy, a paraboloid.

104. For different distribution of the same weight on board, the displacement V of the ship remains constant; and drawing all the different water planes of the ship for constant displacement V (*isocarènes*), these planes all touch a certain surface F fixed in the ship, called the *surface of flotation*, and the ship moves as if this surface rolls and slides on the plane surface of the water.

We have just proved (§ 101) that the line of intersection of any two such consecutive water planes passes through the C.G. of the water line area, so that F, the point of contact of a water plane with the surface of flotation, is the C.G. of the corresponding water line area.

We have also proved that the tangent line of the curve of buoyancy is parallel to the corresponding water plane; and therefore the tangent plane of the *surface of buoyancy* B is parallel to the corresponding water plane, and the normal line is perpendicular.

105. Consequently the body can float in equilibrium wherever the normal to the surface of buoyancy passes through G the C.G. of the body, with this normal vertical; and therefore the determination of the positions of equilibrium depends on the geometrical problem of drawing normals from G to the surface of buoyancy B.

A gradually contracting liquid sphere, with centre at G, is employed as an illustration by Reech (*J. de l'école polytechnique*, 1858); the free surface of the sphere cutting the surface B in a series of spherical contour lines, like those on the Earth.

The positions of equilibrium correspond (i.) when unstable to the top of a hill on B; (ii.) stable to the bottom of a lake; (iii.) stable-unstable to a pass or bar (§ 93); and the conditions of equilibrium are the same as if the surface B was placed on a smooth horizontal plane.

106. Fig. 26, p. 65, may be taken to represent the general horizontal water line area of a ship; and now an inclining couple, due to moving a weight on board, will heel the ship about an axis perpendicular to the plane of the couple, only when this axis is a *principal axis* of the *momental ellipse* of the water line area A at its C.G. (§ 40.)

For let the area A turn through a small angle θ about the line Gy; then $b_1 b_2$, the C.G.'s of the wedges of emersion and immersion, being the C.P.'s with respect to Gy of the two parts into which the area A is divided by Gy, will, according to the methods of §§ 38-40, lie in the line OGK; and therefore the vertical plane of the inclining couple is parallel to OG, the diameter conjugate to Gy with respect to the momental ellipse; and OG, Gy are at right angles only when they are the principal axes of the momental ellipse of the water line area.

The varying directions of the inclining couple may be produced by swinging a weight of P tons suspended from a crane round in a circle of radius b, about G as centre suppose (fig. 26, p. 65); as, for instance, in a floating derrick crane, required for lifting and transporting great weights.

When the weight P is over K, the inclining couple of Pb ft-tons will turn the water plane A about Gy through a small angle θ, given by

$$\sin \theta = Pb/wAk^2,$$

a slope of one in $\qquad wAk^2/Pb$.

If the weight P was lowered on to the vessel from a crane on shore, and deposited over G, the C.G. of the water line area A, the vessel would sink bodily without heeling a distance P/n inches, n denoting the number of tons per inch of immersion.

But if the weight P was deposited over K, the vessel would be depressed and inclined; and the resultant effect will be equivalent to a heel through the same angle θ about OO', the *anti-polar* of the point K with respect to the momental ellipse.

107. The section BB_2 of the surface of buoyancy made by a vertical plane parallel to GK will have a radius of curvature Ak^2/V; so that, referred to three coordinate axes with B as origin, Bx in the normal to the surface of buoyancy, and By, Bz in the tangent plane parallel to the principal axes of the momental ellipse of the area A, the equation of the surface of buoyancy will be represented, approximately, by

$$2x=\frac{Vy^2}{Ak_z^2}+\frac{Vz^2}{Ak_y^2}+\cdots.$$

The *indicatrix* of the surface of buoyancy, or a section of the surface made by a plane parallel and close to the tangent plane, is thus an ellipse, similar and similarly situated to the momental ellipse of the water line area A; and the *lines of curvature* of the surface of buoyancy are therefore parallel to the principal axes of the corresponding water line area.

108. *Dupin's Theorems.*

I. If planes A cut off a constant volume V from a surface S, these planes touch a surface (*the surface of flotation*) such that the point of contact F is at the C.G. of the area A of section of the surface S.

II. The surface described by the C.G.'s of the volume V (*the surface of buoyancy*) has the tangent plane at any point B parallel to the corresponding plane of section A.

III. The *indicatrix* of the surface at B is similar and similarly situated to the momental ellipse at F of the plane area A of section of the surface S; and the lines of curvature at B are therefore parallel to the principal axes at F of the area A (§ 40).

It was in the preceding manner, from mechanical and hydrostatical considerations in connexion with Naval Architecture, that Dupin (*Applications de Géométrie*, 1814), was led to the discovery of these geometrical theorems, which now go by his name.

Cor. The surface of buoyancy is thus necessarily a *synclastic* or rounded surface; but the surface of flotation F may change from being *synclastic* to *anticlastic* or saddle-shaped in parts, especially where the water plane cuts the edge of the deck of a ship, or other edge of a floating body, as illustrated in fig. 39.

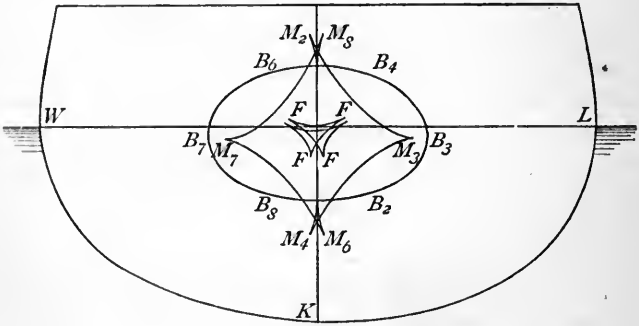


Fig. 39.

This fig. 39 is copied by permission from Mr. W. H. White's *Course of Study at the R.N. College, Greenwich* (*Trans. I.N.A.*, 1877), and represents the cross section of an actual vessel, with the corresponding sections of the surfaces of flotation and buoyancy, represented by the curves FF and BB, and also the curve MM of metacentric evolutes of the curve BB.

A cusp occurs on the curve of flotation FF in consequence of the immersion of the edge of the deck.

109. *Curves of Statical and Dynamical Stability.*

In these curves, drawn in fig. 40 for the vessel in fig. 39, the abscissa represents the angle of inclination in degrees, while the ordinate in the *curve of statical stability* represents the arm *GZ* of the corresponding righting couple at this particular displacement (fig. 38), and the ordinate in the *curve of dynamical stability* represents the work in ft-tons required to heel the vessel slowly over from the vertical to the inclined position.

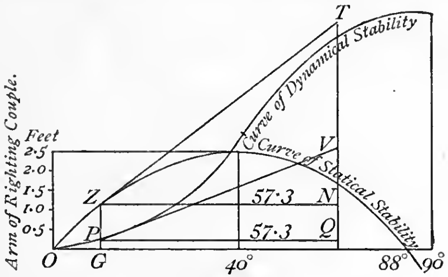


Fig. 40.

Since the work done by a constant couple is the product of the couple and of the circular measure of the angle through which it works, the ordinate of the curve of dynamical stability will be proportioned to the area of the curve of statical stability bounded by the final ordinate at the corresponding inclination.

Conversely the tangent of the inclination of the curve of dynamical stability is proportional to the ordinate of statical stability.

In a vessel of circular cross section, as in fig. 45, p. 190, the metacentre is a fixed point, and the ordinates of the curves of statical and dynamical stability are therefore proportional to the sine and versed sine of the abscissa, representing the inclination (§ 100).

110. Mr. Macfarlane Gray suggests the use of *polar curves of stability* (*Trans. I.N.A.*, 1875); the polar curve of statical stability will now be the curve described by Z in the ship (fig. 38), while the polar curve of dynamical stability will be the *pedal* of the curve of buoyancy BB_2 with respect to G, or the locus of the feet of perpendiculars drawn from G to the tangents of the curve BB_2.

By a well-known theorem of the Differential Calculus,

$$GZ = \frac{dp}{d\theta}, \quad ZM_2 = \frac{d^2p}{d\theta^2},$$

if θ denotes the inclination of the ship in radians (of $180/\pi$ or 57·3 degrees), and p denotes the length of the perpendicular, ZB_2, from G on the tangent at B_2 of the curve of buoyancy.

Thus $W(dp/d\theta)$ is the righting moment in ft-tons, and

$$\int W \frac{dp}{d\theta} d\theta = W(p_2 - p_1)$$

is the dynamical stability in ft-tons, or work done in heeling the ship from the first to the second position; so that, as in § 93, the difference of energy in the two positions is equal to the difference of vertical distances between G and B in the two positions multiplied by the weight W.

Reckoned from the upright position, the dynamical stability in ft-tons

$$= W(ZB_2 - GB) = W(B_2Y - GB \operatorname{vers} \theta),$$

Moseley's formula (*Phil. Trans.*, 1850).

If the position of G is changed, say to B, a distance $BG = a$ suppose, then the righting moment and dynamical stability are changed to

$$W(GZ + GB \sin \theta) \quad \text{and} \quad W(ZB_2 - GB \cos \theta);$$

so that the polar curves of stability with respect to the new pole B can be deduced from the curves with respect to the former pole G by describing a circle on GB as diameter, and increasing or diminishing the former lengths of GZ and $ZB_2 = p$ by the lengths of the intercepts made by this circle on these lines.

Thus in the circular pontoon of fig. 45, p. 190, the polar curves of statical stability are circles, and of dynamical stability are *limaçons* or *cardioids*, the pedals of a circle.

110. Since $$(d\,.\,GZ)/d\theta = ZM_2,$$
the tangent ZT at Z of the curve of statical stability in fig. 40 is constructed by measuring a length ZN of 57·3 graduations of degrees, to represent the radian, and erecting the perpendicular $NT = ZM_2$; and initially $ZM_2 = GM$; we are thus enabled to draw the tangent to the curve of statical stability.

Similarly, if QV in fig. 40 is drawn to represent $W\,.\,GZ$ to scale, then PV will be the tangent at P to the curve of dynamical stability (Jenkins, *Trans. I.N.A.*, 1889).

The curves of statical and dynamical stability are useful in showing how far a vessel may heel with safety; a steamer will recover the upright position if heeled to any extent short of the angle of vanishing stability; but a sailing ship, heeled over by the wind, must not be allowed to incline so far as the angle corresponding to the point of maximum value of righting arm GZ.

If the initial part OZ of the curve of statical stability is taken as a straight line, the curve OP of dynamical stability will be a parabola, the polar diagram of Z will be a *Spiral of Archimedes*, and the curve MM_2 of prometacentres the *involute* of the circle with centre G and radius GM; the curve of buoyancy BB_2 will then be an involute of the involute of a circle.

111. Any small angular displacement of the ship about an axis through F the C.G. of the water line area may be resolved into component angular displacements, θ_1 and θ_2 suppose, about the principal axes of the momental ellipse of the water line area A.

Replacing $\sin\theta_1$ and $\sin\theta_2$ by their circular measure θ_1 and θ_2, the corresponding righting couples can be written

$$W(\rho_1-h)\theta_1 \text{ and } W(\rho_2-h)\theta_2$$

ft-tons, where ρ_1, ρ_2 denote the principal radii of curvature of the surface of buoyancy, and h is the distance between the C.G. and the C.B. of the ship.

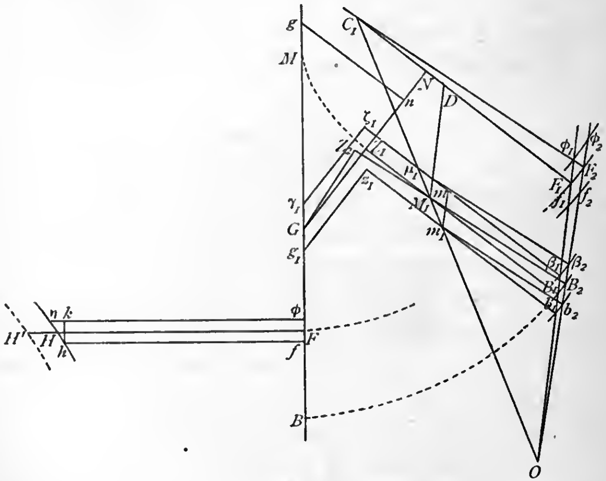


Fig. 41.

The work done by a gradually increasing couple is the product of the average value of the couple and of the circular measure of the angle through which it works; and therefore the work done in heeling the ship is, in ft-tons,

$$\tfrac{1}{2}W\{(\rho_1-h)\theta_1^2+(\rho_2-h)\theta_2^2\}.$$

112. *Leclert's Theorems.*

A simple relation connecting r, the radius of curvature of the transverse curve of buoyancy, with r_1, the radius of curvature of the corresponding parallel curve of flotation, has been discovered by M. Emile Leclert (1870).

Let B_1, B_2 denote the C.B.'s of a ship in two consecutive inclined positions, when the displacement is V, so that B_1B_2 is a small arc of the curve of buoyancy; and let F_1F_2 be the corresponding parallel arc of the curve of flotation (fig. 41).

Produce F_1B_1, F_2B_2 to meet in O; and let the normals at B_1, B_2 to the curve of buoyancy intersect in M_1, and the normals at F_1, F_2 to the curve of flotation intersect in C_1; so that B_1M_1, F_1C_1 become ultimately r and r_1.

Then since, by Dupin's Theorems, the normals at B_1 and F_1 are parallel, and also at B_2 and F_2, therefore

$$\frac{B_1M_1}{F_1C_1}=\frac{B_1B_2}{F_1F_2}=\frac{OB_1}{OF_1};$$

and therefore M_1 lies in the straight line OC_1.

Now suppose the displacement of the ship is changed from V to $V-\frac{1}{2}\Delta V$ and $V+\frac{1}{2}\Delta V$; and that in consequence B_1 changes to b_1 and β_1, B_2 to b_2 and β_2, F_1 to f_1 and ϕ_1, F_2 to f_2 and ϕ_2, and M_1 to m_1 and μ_1.

The increment ΔV which changes the displacement from $V-\frac{1}{2}\Delta V$ to $V+\frac{1}{2}\Delta V$ may be supposed concentrated at F_1, so that

$$(V-\tfrac{1}{2}\Delta V)b_1F_1=(V+\tfrac{1}{2}\Delta V)\beta_1F_1,$$

or

$$\frac{b_1\beta_1}{B_1F_1}=\frac{\Delta V}{V},$$

since B_1 may be taken as the middle point of $b_1\beta_1$.

The similarity of the small arcs b_1b_2, $\beta_1\beta_2$, F_1F_2 shows as before that m_1, μ_1 also lie on OC_1; so that, drawing m_1m, MD parallel to OF_1, we may denote $m\mu_1$, the increment in r due to the increment ΔV in V, by Δr; and

therefore $$\frac{\Delta r}{r_1 - r} = \frac{m\mu_1}{DC} = \frac{b_1\beta_1}{B_1F_1} = \frac{\Delta V}{V},$$

or $$r_1 - r = V\frac{\Delta r}{\Delta V} \qquad \text{.......................(1)}$$

Leclert's first expression for r_1.

Also, since $r = I/V$, where I denotes Ak^2, therefore in the notation of the Differential Calculus,

$$r_1 = \frac{I}{V} + V\frac{d}{dV}\left(\frac{I}{V}\right) = \frac{dI}{dV} \qquad \text{.................(2),}$$

Leclert's second expression for r_1.

Similar expressions connect the radii of curvature of the longitudinal curves of buoyancy and flotation of a ship; and the formulas may be extended to the successive evolutes of these curves of buoyancy and flotation by differentiating with respect to θ, keeping V constant.

As a simple verification the student may apply Leclert's theorems to the case of a cylindrical pontoon, a spherical buoy, or any body of revolution, floating horizontally.

We notice that an increase of displacement ΔV causes the C.B. B_1 to move towards F_1 through a distance

$$b_1\beta_1 = (\Delta V/V)B_1F_1, \qquad \text{.......................(3)}$$

and causes the metacentre M_1 to move towards C_1 through a distance $$m_1\mu_1 = (\Delta V/V)M_1C_1, \qquad \text{...................(4)}$$

so that a small increase of the load or of the draft will cause the metacentre to rise or fall in the ship according as the metacentre lies below or above the centre of curvature of the curve of flotation.

113. Suppose that the alteration of displacement ΔV is due to the subtraction or addition of a small amount of cargo at g, represented by the weight of the volume ΔV of water; and that the C.G. of the ship changes in consequence from G to g_1 and γ_1; then

$$\frac{g_1\gamma_1}{Gg}=\frac{\Delta V}{V}.$$

Dropping the perpendiculars from g_1, G, γ_1 on the lines $b_1m_1, B_1M_1, \beta_1\mu_1$, which are vertical in the corresponding inclined position of the ship, then $g_1z_1, GZ_1, \gamma_1\zeta_1$ are the righting arms for the displacements $V-\frac{1}{2}\Delta V$, V, $V+\frac{1}{2}\Delta V$; and laying off FH on the water line of the upright position of the ship to represent to scale the righting moment $V.GZ_1$, the curve of H will be the "cross curve of stability" for this inclination.

If h and η denote the positions of H for the displacements $V-\frac{1}{2}\Delta V$ and $V+\frac{1}{2}\Delta V$, and if A and $f\phi$ denote the water line area and change of draft in the upright position (fig. 41), so that

$$A=\text{lt}\,\Delta V/f\phi,$$

then

$$\begin{aligned} k\eta &=(V+\tfrac{1}{2}\Delta V)\gamma_1\zeta_1-(V-\tfrac{1}{2}\Delta V)g_1z_1 \\ &=\Delta V.GZ_1+V(\gamma_1\zeta_1-g_1z_1) \\ &=\Delta V(GZ_1+Z_1N-Gn), \end{aligned}$$

or

$$\frac{k\eta}{hk}=A\ (\text{distance of } g \text{ from } C_1F_1)\,; \quad\ldots\ldots\ldots\ldots(5)$$

by means of this theorem the tangent at H of the cross curve of stability can easily be drawn.

The righting moment FH in the diagram (fig. 41) thus increases with the draft so long as F_1 is to the right of the vertical gn, and attains a maximum when it crosses gn from right to left; and conversely it crosses from left to right, when FH is a minimum.

If the ship heels through a small angle of $\Delta\theta$ radians at the displacement V, so that the C.B. is changed from B_1 to B_2, and if GZ_2 is the perpendicular from G on the new vertical M_1B_2, and H' the corresponding point on the consecutive cross curve of stability, then, ultimately,

$$\frac{HH'}{FH}=\frac{GZ_2-GZ_1}{GZ_1}=\frac{M_1Z_1}{GZ_1}\Delta\theta=\tan M_1GZ_1 \,.\, \Delta\theta\,; \quad \ldots(6)$$

a theorem which will be found useful in interpolating cross curves of stability between calculated curves for given inclinations.

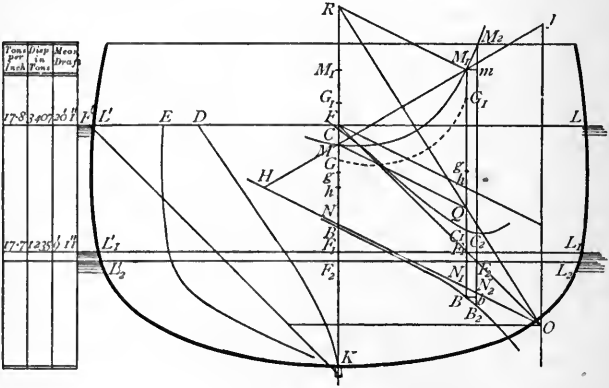


Fig. 42.

114. The stability of a ship must always be secured at the smallest draft when the hold is empty, and again at the load draft down to the Plimsoll mark.

The first condition must be the care of the naval architect, for stability at launching and during fitting out; but the second condition of stability at load draft depends on the manner of stowage of the cargo.

Denote by G, B, M the C.G., the C.B., and the metacentre of the ship at load draft LL', and by G_1, B_1, M_1 the corresponding points at light draft L_1L_1'.

A line is drawn at 45° to the vertical, either FF_1 through F, or KF' through the keel K; so that the horizontal ordinates of these lines represent either the freeboard below the load line LL', or the draft of water above the keel K (fig. 42).

A vertical line is drawn through F_1 where L_1L_1' cuts FF_1; and now G_1, B_1, M_1 are transferred horizontally on to this vertical; in this way three curves are obtained for different loads and draft, the curve GG_1 of C.G.'s, BB_1 of C.B.'s, and MM_1, the "curve of metacentres."

Curves KD and KE are also drawn, representing displacement in tons or ft³ and tons per inch immersion; also the curve CC_1 representing in the same manner the position of the centre of curvature of the curve of flotation.

Then, from § 112, if the water line changes slightly to L_2L_2', by a slight diminution ΔV of the displacement, and by the decrease of draft B_1b or M_1m (fig. 42),

$$\tan B_2B_1b = \frac{bB_2}{B_1b} = \frac{A}{V} \cdot F_1B_1, \quad \ldots\ldots\ldots\ldots\ldots (7)$$

$$\tan M_2M_1m = \frac{mM_2}{M_1m} = \frac{A}{V} \cdot M_1C_1; \quad \ldots\ldots\ldots\ldots (8)$$

theorems by means of which the tangents to the curves BB_1 and MM_1 can be drawn.

We notice that the tangent to the curve of metacentres MM_1 is horizontal where the curve CC_1 crosses it.

115. Within the limits in which the ship is "wall-sided," the curve CC_1 coalesces with the inclined line FF_1.

Also if we put $V/A = a$, as with the hydrometer (§ 63), so that a is the depth of the vessel of box form and of equal displacement; and if the inclined line FF_1 cuts the bottom of this vessel in O, then the straight line NO, where $FN=\frac{1}{2}a$, will be the curve of C.B.'s of this box-shaped vessel; and the parallel straight line Fh will be the curve for h, the C.G.'s of the volumes of emersion.

If B goes to B_1 and N to N_1 for a change of draft x,

$$(V-Ax)B_1h=V.Bh,$$

$$(V-Ax)N_1h=V.Nh;$$

and therefore

$$(V-Ax)B_1N_1=V.BN,$$

so that the curve BB_1 is a hyperbola, with ON and the vertical OJ through O as asymptotes.

Also

$$(V-Ax)B_1M_1=V.BM=Ak^2,$$

so that

$$(V-Ax)M_1N_1=VN.M,$$

and therefore the curve of metacentres MM_1 is also a hyperbola, with the same asymptotes ON and OJ.

For homogeneous cargo with a level surface, like ore or grain, the curve GG_1 of C.G.'s will also be a hyperbola, with the vertical OJ for one asymptote, and a sloping line for the other asymptote, the position of which depends on the S.G. s of the cargo.

For if the surface of the cargo at displacement V cuts the vertical GF in f, and we take a point O' in FG such that $FO'=s.fO'$, then FO' is the draft of water of a vessel of box form, supposed homogeneous and of S.G. s.

The curve of C.G.'s of these homogeneous box-shaped vessels for different drafts will be a straight line DD_1, passing through D the middle point of fO' and inclined at an angle $\tan^{-1}2s$ to the vertical; and, as before,

$$(V-Ax)G_1D_1=V.GD,$$

so that the curve GG_1 is a hyperbola, with OJ and DD_1 for asymptotes.

The point G_1 will reach its greatest depth in the ship when the horizontal surface of the cargo passes through G_1; as in this case the addition or subtraction of a small amount of cargo will not cause the C.G. of the vessel and cargo to descend.

These hyperbolas can be constructed geometrically, as in § 64; for instance the hyperbola MM_1 by drawing MQ parallel to ON to meet B_1F_1 in Q, producing OQ to meet BF in R, and drawing RM_1 parallel to ON to meet B_1F_1 in M_1.

Or drawing any straight line HMJ to meet ON and OJ in H and J, then if this cuts the hyperbola again in M_1, $HM = M_1J$, which gives another rapid construction of the hyperbola by points.

116. If W denotes the displacement in tons at load draft, and $W-P$ at light draft, so that P is the cargo put on board, then, within the limits for which the vessel is wall-sided,

$$\frac{B_1M_1}{BM} = \frac{V}{V-Ax} = \frac{W}{W-P}, \qquad \frac{B_1M_1 - BM}{BM} = \frac{P}{W-P}.$$

Also, if g denotes the C.G. of the cargo P,

$$\frac{G_1g}{Gg} = \frac{B_1h}{Bh} = \frac{B_1G_1 + gh}{BG+gh} = \frac{W}{W-P}, \qquad \frac{B_1G_1 - BG}{BG+gh} = \frac{P}{W-P}.$$

Therefore

$$G_1M_1 - GM = B_1M_1 - BM - B_1G_1 + BG$$
$$= \frac{P}{W-P}(GM - gh) = \frac{P}{W}(G_1M_1 - gh),$$

giving the difference of metacentric height for light and load draft.

If the metacentric heights at light and load draft are to be equal, then $gh = GM$, or the depth of the C.G. of the cargo below the water line at mean draft should be equal to the metacentric height.

But if the *stiffness* of the vessel, measured by $W.GM$, is to remain the same, then (§ 113) g must coincide with C; and, in the case of a wall-sided vessel, C and F coincide, and g must therefore lie in the mean water line.

Thus, for instance, if P denotes the tons of coal burnt by a steamer on a voyage, and if the C.G. of the coal in the coal bunkers is below the mean water line at a depth equal to the metacentric height, the metacentric height will remain the same; but if the C.G. of the coal is in the mean water line, the stiffness of the steamer will remain unchanged, and no water ballast need be let in to preserve stability.

For example, if a steamer consumes 2,000 tons of coal in crossing the Atlantic, and draws 4 ft less of water at the end of the voyage, then to preserve a metacentric height of 2 ft, the C.G. of the coal should be in the water line of light draft. (J. Nicholson, *Trans. I.N.A.*, 1885.)

117. If the change of draft from LL' to L_1L_1' is due to sailing from fresh water of density w (tons/ft³) to sea water of density w_1, and if the metacentre changes from M to M_1, then
$$W = wV = w_1V_1,$$
or
$$\frac{w_1}{w} = \frac{V}{V_1} = \frac{B_1M_1}{BM};$$
while
$$BB_1 = \frac{V - V_1}{V_1}Bb = \frac{w_1 - w}{w}Bb,$$
the points B_1 and M_1 being brought back to the vertical BGM.

Therefore
$$MM_1 = B_1M_1 - BM - BB_1$$
$$= \frac{w_1 - w}{w}(BM - Bb) = \frac{w_1 - w}{w}bM,$$

or the height of the metacentre above the mean water line $bM = MM_1 . w/(w_1 - w)$;
thus the metacentre rises or falls in going into sea water, according as it is above or below the water line, where the ship is wall-sided.

Similar considerations affect the stability of the Hydrometer.

Considering that F the C.G. of the water line area of a ship is not in general vertically over G the C.G. of the vessel, in the vertical longitudinal plane, a slight alteration of trim is generally observed as a vessel passes from fresh to salt water; and also when a ship heels over.

Since the increase of density Δw is small compared with w in going from fresh into salt water, the change of draft and displacement is small, and we may employ Leclert's Theorem for determining the change MM_1 of metacentric height, without the restriction of supposing that the ship is wall-sided.

The *decrease* of displacement ΔV in V will cause M to move *away* from C to M_1, so that

$$\frac{MM_1}{MC} = \frac{\Delta V}{V} = \frac{\Delta w}{w};$$

and the metacentre thus rises or falls in going from fresh into salt water, according as M is above or below C; and C is in the water line for a wall-sided ship.

So also, if a small quantity P of cargo, whose C.G. is at g, is removed from the ship, G moves to G_1 away from g, so that

$$\frac{GG_1}{G_1 g} = \frac{W}{P} = \frac{\Delta V}{V} = \frac{MM_1}{MC};$$

and therefore

$$G_1M_1 - GM = GG_1 - MM_1 = \frac{P}{W}(G_1g - MC) = \frac{P}{W}(G_1M - gC).$$

118. *Stability of a Vessel with Liquid Cargo.*

We have supposed the cargo P to be solid; but if it is liquid and free to roll about, as, for instance, petroleum carried in bulk, a considerable reduction may ensue in metacentric height and stability; also in well-decked steamers it is necessary to investigate the alteration of stability caused by shipping a wave which fills the well.

If the vessel heels through a small angle θ, the original righting couple of the outside water

$$wAk^2 \sin\theta = W \,.\, BM \,.\, \sin\theta, \text{ ft-tons,}$$

is diminished by the upsetting couple of the liquid cargo, of density w' suppose,

$$w' A'k'^2 \sin\theta = P \,.\, bm \,.\, \sin\theta, \text{ ft-tons,}$$

where m is the centre of curvature of the curve bb_2 described by b the C.G. of the liquid cargo, and $A'k'^2$ is the moment of inertia of the free surface of this liquid.

The new metacentre M' will therefore lie below M at a distance

$$MM' = \frac{P}{W} bm = \frac{P}{W}\frac{A'k'^2}{V'} = \frac{w' A'k'^2}{W}.$$

119. Examining this question closer we notice that the forces acting on the vessel when heeled through a small angle are (fig. 43)

(i.) W, the buoyancy of the outside water, acting vertically upwards through M;

(ii.) P, the weight of the liquid cargo, acting downwards through m;

(iii.) $W-P$, the remaining weight of the vessel, acting vertically downwards through H, the C.G. of the rigid part of the vessel; and

$$GH = \frac{P}{W-P} Gb.$$

The equilibrium will be stable or unstable according to the direction of the couple which is the resultant of these three parallel forces.

Taken positive as a righting couple, the moment about H is, in ft-tons,

$$(P \,.\, Hm - W \,.\, HM)\sin\theta$$
$$= (W \,.\, GM - P \,.\, bm)\sin\theta = W \,.\, GM' \,.\, \sin\theta,$$

if $$MM' = (P/W)bm,$$

and MM' is the loss of metacentric height.

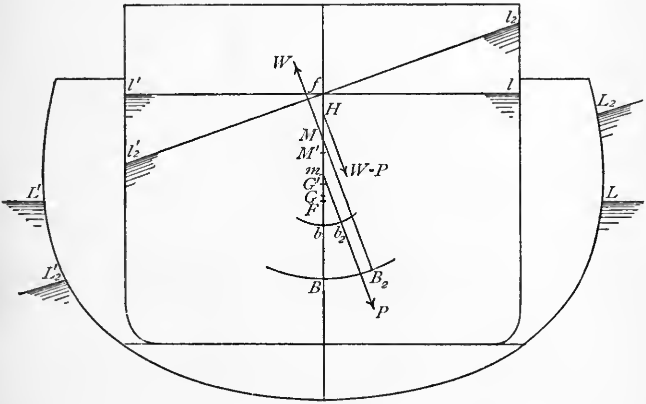


Fig. 43.

120. Otherwise, the resultant of the parallel forces of $W-P$ tons acting through H, and of P tons acting through m, is a force of W tons which cuts GM at a point G', such that

$$W \,.\, GG' = P \,.\, Gm + (W-P)GH$$
$$= P(Gm + Gb) = P \,.\, bm;$$

and $G'M$ may be taken as the metacentric height, which is less than GM by GG' or $(P/W)bm$, or MM'.

Thus for a rectangular tank of breadth b ft, filled with P tons of liquid to a depth of c ft,

$$bm = \tfrac{1}{12} b^2/c;$$

and the metacentric height is

$$\frac{1}{12} \frac{P}{W} \frac{b^2}{c} \text{ ft}$$

less than it would be if the liquid were solidified, or contained in a confined space, which it fills completely.

A longitudinal bulkhead will reduce the loss of metacentric height to one-quarter of this value. (G. H. Little, *The Marine Transport of Petroleum*, 1890.)

It is thus important that no water should be free to roll about in the hold, or in the boilers or tanks, in all the previous inclining methods for determining experimentally the metacentric height of a vessel.

When a floating vessel, a bottle, canister, or hydrometer, is ballasted by mercury or other liquid, or by cylindrical or spherical shot, which are free to roll about the bottom, the investigation of the stability proceeds in the same manner.

121. In calculations of the stability of a tumbler, kettle, or other vessel with a rounded base, containing liquid, the base may be represented by the surface of buoyancy B (§ 105).

It is proved in treatises on Statics that if the surface B, resting on the highest point of a fixed convex surface, is rolled through a small arc BB_2, the vertical at B_2 cuts the normal to the surface at B in a point O, where BO is ultimately half the harmonic mean of the radii of curvature BM, BM' of the arcs BB_2 on the surface B and on the fixed convex surface.

For ultimately the angles BMB_2, $BM'B_2$ are BB_2/BM, BB_2/BM' radians; and their sum, the angle BOB_2, is BB_2/BO radians; and therefore

$$\frac{1}{BO}=\frac{1}{BM}+\frac{1}{BM'}.$$

In the position of equilibrium of the body the point G', the C.G. of the solid part $W-P$ collected at H and of the liquid part P collected at m, must lie in the common normal vertically over B, and the equilibrium will be stable if G' is below O.

More generally, if the common normal at B of the surfaces makes an angle ϕ with the vertical, a similar argument shows that the vertical at the consecutive point of contact B_2 cuts the line BG' in a point Q, where

$$BQ=BO\cos\phi,$$

that is, where the vertical is cut by the circle on BO as diameter (called the *circle of inflexions*); so that the position of equilibrium will be stable in which the body and its liquid contents rests with G' vertically over B, if G' lies inside the circle of inflexions; provided always that slipping does not take place between the surfaces.

122. When a tipping basin is filled with water to a certain extent, the equilibrium will sometimes be noticed to become unstable.

We may suppose the interior surface spherical, with centre m; then if O denotes the axis about which the basin turns, G the C.G., and W the weight of the basin and P the weight of water when the equilibrium becomes unstable; then as W acts vertically downwards through G and P through m, taking moments about O,

$$P.Om=W.OG \quad \text{or} \quad P=W.OG/Om,$$

giving P the maximum weight of water the basin will contain without capsizing.

As the weight of water P always acts through m, we can suppose P concentrated into a spherical nucleus at m in considering the stability of the equilibrium.

Exactly the same reasoning would hold if the water was replaced by a spherical ball of the same weight.

If a body moveable about a fixed point contains a number of spherical cavities partly filled with liquid, the position of equilibrium is determined by making the C.G. of the body, and of the liquids, each concentrated in a spherical nucleus about the centre of its spherical cavity, lie in the vertical line through the fixed point; and a similar construction would hold if the liquids were replaced by spherical balls of the same weight.

A crowd of people on a deck holding themselves upright to preserve their footing when the vessel heels over, or a man standing up in a boat, would act as if their weight was concentrated in the soles of their feet.

Conversely, when a number of bodies are free to swing about fixed points of suspension, as, for instance, men in their hammocks, the weight of each body must be supposed concentrated at the point of suspension in investigating the stability of the vessel.

123. The working of the screw propeller has a tendency to heel a steamer; with a right-handed screw propeller going ahead, the reaction of the water on the propeller will cause the vessel to heel to port, and the angle of heel θ is easily calculated, knowing GM the metacentric height in feet, W the displacement in tons, I the indicated H.P. of the engines, and R the number of revolutions per minute.

For denoting by N the turning moment of the engines in foot-pounds, then $2\pi N$ is the work done in one revolution in foot-pounds, $2\pi NR$ is the work done per minute, and therefore $2\pi NR = 33000\ I$.

But N is also the righting moment, and therefore (fig. 38, p. 148)

$$N = 2240\,W.\,GZ = 2240\,W.\,GM \sin\theta\,;$$

so that
$$\sin\theta = \frac{33000\,I}{2240\,W.\,GM.\,2\pi R} = \frac{825\,I}{112\,W.\,GM.\,\pi R}.$$

To bring the vessel to the upright position a weight of P tons must be moved a distance b ft to the starboard side, such that

$$2240\,Pb = N = 33000\,I/2\pi R.$$

Thus with $W = 10000$, $I = 20000$, $R = 100$, and $GM = 2$, we find $\sin\theta = 0{\cdot}02345$, $\theta = 1^\circ\,20'$, and $Pb = 469$ ft-tons.

A similar effect exists in a paddle steamer, tending to alter the trim slightly; it is calculated in the same manner, but now GM must denote the longitudinal metacentric height.

Thus if the steamer is towing a vessel, and P denotes the pull of the tow rope in tons, b ft the distance between the line of resultant thrust of the paddles and the tow rope, we may take Pb as the couple in ft-tons, tending to alter the trim; so that if L ft denotes the length at the water line, and x inches the change of trim,

$$\frac{x}{12L} = \sin\theta = \frac{Pb}{W.\,GM}, \quad \text{or} \quad b = \frac{W}{P} GM.\sin\theta.$$

In consequence of the large value of the longitudinal metacentric height GM, this change of trim is practically insensible.

The press of sail has a similar effect in a sailing ship, not only in inclining it, but also in altering the trim so as to bury the stem and raise the stern; to counteract this effect it is found advantageous to make the masts rake aft. (Sir Edward J. Reed, *The Stability of Ships*; W. H. White, *Naval Architecture*.)

124. *The Interchange of Buoyancy and Reserve of Buoyancy.*

Prof. Elgar has pointed out (*Times*, 1st Sept., 1883), that if a homogeneous body of S.G. s floats in water in any position with LL' as the water line, the same body can float inverted with the same water line LL', provided the S.G. is changed to $1-s$, the buoyancy and reserve of buoyancy being interchanged (§ 85); and that the righting moments on the two bodies are equal, or the metacentric heights are as s to $1-s$ in the two cases.

For if G denotes the C.G. of the body, and B, B' the C.B.'s of the two volumes V, V' into which the body is divided by the plane LL', then BGB' is a straight line, and the moments of the volumes V, V' about G are equal.

But the volumes V and V' are as the S.G.'s s and $1-s$, or as the weights W and W'; and therefore the moments of the weights of the displaced water about G are equal.

Also, if M, M' denote the metacentres in a position of equilibrium, then M, M' lie in the vertical straight line BGB'; and

$$Ak^2 = V \,.\, BM = V' \,.\, B'M',$$

and

$$V \,.\, BG = V' \,.\, B'G';$$

therefore

$$V \,.\, GM = V' \,.\, GM',$$

or

$$\frac{GM}{GM'} = \frac{V'}{V} = \frac{W'}{W} = \frac{1-s}{s}.$$

Thus the stability of a vessel of deep draft and small freeboard is similar to that of a vessel of small draft and high freeboard, such as a light draft steamer for shallow waters, or a hay barge.

Again, to determine whether a regular tetrahedron can float in water with one edge just outside the surface, consider an equal tetrahedron of small density floating with an edge just submerged; this position of equilibrium is evidently unstable, and therefore the equilibrium of the first tetrahedron is also unstable.

These considerations are of great use in obviating the necessity of the re-examination of the stability of a floating body when inverted, the conditions of stability being the same in the two cases; as, for instance, with a body bounded by a plane and a spherical surface, when the plane is submerged; for a hemisphere of S.G. s the metacentric height is thus $\frac{3}{8}a(1-s)/s$.

The surface of flotation F is the same, while the surfaces of buoyancy B and B' are similar with respect to G as the centre of similitude, as also the evolutes of sections of B and B' made by planes through G.

125. More generally if a homogeneous body, of weight W and density σ, can float in two liquids of densities ρ and ρ', with displacements V and V' in these liquids, and LFL' in their plane of separation, the body can float inverted with the same plane of flotation LFL' if its weight and density are changed to W' and σ', where

$$W=\sigma(V+V')=\rho V+\rho' V',$$

$$W'=\sigma'(V+V')=\rho' V+\rho V';$$

so that

$$\frac{V}{V'}=\frac{\sigma-\rho'}{\rho-\sigma}=\frac{\rho-\sigma'}{\sigma'-\rho},$$

or

$$\rho+\rho'=\sigma+\sigma'.$$

The conditions of stability are the same in the two cases; for if turned through a small angle θ about an axis through F in the plane LL', the wedges of emersion and immersion must be taken as of densities $\rho' - \rho$ and $\rho - \rho'$; so that the righting couple is in each case,

$$(\rho - \rho')Ak^2 \sin\theta = W . CY = W' . C'Y',$$

if CY, $C'Y'$ are the arms of the righting couples; and therefore

$$W . CM = W' . C'M' = (\rho - \rho')Ak^2,$$

where C, C' now denote the C.B.'s of the two liquids displaced in the two positions, and M, M' the metacentres.

Generally, if a body of weight W is floating in equilibrium in a number of liquids, so that C is the C.B. of all the displaced liquids, and if the body is turned through a small angle θ, we shall find in the same manner that the righting couple of the liquid

$$W . CM . \sin\theta = \Sigma\Delta\rho Ak^2 \sin\theta,$$

where $\Delta\rho$ denotes the change of density in crossing a plane LL' of separation of two liquids.

The caissons employed in the construction of the foundations of a bridge, for instance, the cylindrical caissons of the Forth bridge, must be considered as immersed in air, water, and the mud or quicksand at the bottom of the water; and the corresponding calculations must be made for the stability of the equilibrium during the whole process of sinking the caisson.

When the density varies continuously, as in air, Σ and Δ must be replaced by $\int$ and d, the symbols of the Integral Calculus; but at a horizontal plane section A where the density suddenly changes from ρ' to ρ, the righting moment must be increased by the term

$$(\rho - \rho')Ak^2 \sin\theta.$$

Examples.

(1) A small quantity of mercury is placed in a hollow at one end of a uniform rod, and it is found that the rod will float half immersed and at any angle to the vertical. Show that the weight of the mercury is equal to that of the rod.

Prove that a thin uniform rod will float in a vertical position in stable equilibrium in a liquid of n times its density, if a heavy particle be attached to its lower end of weight greater than $(\sqrt{n}-1)$ times its own weight.

(2) A thin rod of uniform section is composed of two portions of S.G. s_1 and s_2; determine the ratio of the lengths when the rod can float in an inclined position in water. *E.g.* $s_1=0{\cdot}5$, $s_2=1{\cdot}5$.

(3) A solid hemisphere of radius a and weight W is floating in liquid, and at a point on the base at a distance c from the centre rests a weight w; show that the tangent of the inclination of the axis of the hemisphere to the vertical for the corresponding position of equilibrium, assuming the base of the hemisphere entirely out of the fluid, is

$$\frac{8}{3}\frac{c}{a}\frac{w}{W}.$$

(4) A right circular pontoon, 50 feet long and 16 feet in diameter, is just half immersed on an even keel. The centre of gravity is 4 feet above the bottom. Calculate, and state in degrees, the transverse heel that would be produced by shifting 10 tons 3 feet across the vessel.

State, in inches, the change of trim produced by shifting 10 tons longitudinally through 20 feet.

Trace the curves of buoyancy and of prometacentres of a raft or life-boat, supported by two parallel circular pontoons, half immersed in water.

(5) It was found that filling the boats, suspended on each side of a vessel of 5000 tons displacement, alternately with 6 tons of water caused the vessel to heel so that the bob of a pendulum 6 ft long moved through 3 inches. Given that the distance between the centre lines of the boats was 40 ft, prove that the metacentric height was 1·152 ft.

(6) A vessel of 6000 tons displacement heels over under sail through an angle of 5°; show that its metacentric height is about 2 feet: assuming that the component of the wind pressure perpendicular to the keel is a force equal to 26 tons acting at a point 25 feet above the deck, the C.G. of the ship and cargo being 15 feet below the deck.

(7) In a cargo-carrying vessel, the position of whose C.G. is known, show how the new position of the C.G. due to a portion of the cargo shifting, may be found.

A vessel of 4000 tons displacement, when fully laden with coals, has a metacentric height of $2\frac{1}{2}$ ft.

Suppose 100 tons of the coal to be shifted so that its C.G. moves 18 feet transversely, and $4\frac{1}{2}$ feet vertically; what would be the angle of the vessel if upright before the coal shifted?

(8) A ship is 220 feet long, has a longitudinal metacentric height of 252 feet, and a displacement of 1950 tons. If a weight of 20 tons (already on board) were shifted longitudinally through 60 feet, what would be the change of trim?

(9) A ship is floating at a draft of 18 ft forward, and 20 ft aft, when the following weights are placed on board in the positions named :—

Wt. in tons.	Distance from c.g. of water plane in feet.	
10	90	before ;
30	30	
70	30	abaft.
30	45	

What will be the new draft forward and aft, the "moment to change trim one inch" being 700 foot-tons, the "tons per inch" being 30 ?

(10) A bridge of boats supports a plane rigid roadway AB in a horizontal position. When a small moveable load is placed at G the bridge is depressed uniformly ; when the load is placed at a point C the end A is unaltered in level ; when at D the end B is unaltered in level ; and when at P the point Q of the roadway is unaltered in level.

Prove that $AG \, . \, GC = BG \, . \, GD = PG \, . \, GQ$;

and that the deflection produced at a point R by a load at P is equal to the deflection produced at P by the same load at R.

(11) If a plane rigid raft is supported in a horizontal position by a number of floating bodies, a weight placed on the raft vertically over the centre of inertia of the planes of flotation will sink the raft vertically, while a weight placed anywhere else will cause the raft to turn about an axis, the antipolar of the weight with respect to the momental ellipse of the planes of flotation.

Compare this with the theory of a table resting on a number of elastic supports.

(12) Find the dynamical stability in foot-tons at 30° of a rectangular pontoon 100 ft × 20 ft × 10 ft draft, having a GM. of 2 ft.

(13) The curve of statical stability of a vessel is a segment of a circle of radius twice the ordinate of maximum statical stability, which is 2500 tons-feet; estimate the total dynamical stability of the vessel, the angle of vanishing stability being 85°.

(14) The curve of stability of a vessel is a common parabola, the angle of vanishing stability is 70°, and the maximum moment of stability 4000 ft-tons. Prove that the statical and dynamical stabilities at 30° are 3918 and 1283 ft-tons.

(15) A cylindrical vessel with a flat bottom is free to turn about a horizontal axis through its C.G. Prove that if a little molten metal be poured in, the vertical position is unstable; that it does not become stable until the depth of the metal exceeds $c-\sqrt{c^2-a^2}$, where a is the radius of the cylinder, and c the height of the centre of gravity above the base; and that it is again unstable when the depth exceeds $c+\sqrt{c^2-a^2}$.

Determine the weight which must be fixed to the bottom of the vessel so as to make the equilibrium stable at first.

(16) A canister containing water floats in a liquid, with its axis vertical. Prove that its stability for angular displacements will be unaffected if a certain weight of water is removed and a spherical ball of equal weight is placed in the cylinder so as to float in the water partially immersed, even though the ball touch the cylinder.

(17) A vessel is constructed to carry petroleum in tanks formed by the sides of the vessel and a middle line and transverse bulkheads. If the water plane of the vessel be rectangular, and the tanks extend over $\frac{2}{3}$rds the length, and are not full, investigate the stability at a small angle of heel, having given —Length of vessel 240 ft, breadth 36 ft, C.G. of laden vessel 14 ft from top of keel, C.B. of laden vessel 8 ft from top of keel, displacement 2500 tons, and S.G. of petroleum 0·8.

(18) A vessel is of box form, 300 ft × 50 ft, and draws 20 ft of water when intact. A bunker, 10 ft wide, 10 ft deep (6 ft below, 4 feet above the water line), containing coal, extends a length of 100 ft amidships at each side of the vessel. The C.G. of the vessel is 18 ft above the keel, find the GM.—

(1) In the intact condition.

(2) With both bunkers riddled, the inner and end bulkheads remaining intact.

(19) A vessel in the form of a cube of side $12a$ containing liquid is placed so as to rest on the top of a fixed sphere of radius $5a$. Neglecting the weight of the vessel prove that there will be stability provided the depth of the liquid is between $4a$ and $6a$.

(20) Prove that a cylindrical kettle of radius a and height h will be in stable equilibrium on the top of a spherical surface of radius c, when the water inside occupies a height intermediate to the roots of the equation

$$x^2 - hx + \tfrac{1}{2}a^2 + n(h^2 - 2ch) = 0;$$

the weight of the kettle being n times the weight of water which fills it.

(21) A cup whose outside surface is a paraboloid of revolution of latus-rectum l, and whose thickness measured horizontally is the same at every point and very small compared with l, has a circular rim at a height h above the vertex, and rests on the highest point of a sphere of radius r.

If water be now poured in until its surface cuts the axis of the cup at a distance $\frac{3}{20}h$ from the vertex, and if the weight of water be four times that of the cup, the equilibrium will be stable, if

$$h < \frac{r-4l}{r+l} l.$$

(22) Prove that if a thin conical vessel of vertical angle 2α and weight W, whose C.G. is at a distance h from the vertex, is resting upright in a horizontal circular hole of radius c, it will become unstable when a weight P of liquid is poured into it to a depth x, so as to make

$\frac{3}{4}Px - 2(P+W)c \cot \alpha + Wh \cos^2\alpha$ positive.

(23) A cylindrical vessel, floating upright in neutral equilibrium, will really be stable if the radius of curvature at the water line of the vertical cross section is greater than the normal cut off by the medial plane.

(24) Prove that the metacentric height given by

$$(P/W)b \operatorname{cosec} \theta$$

(§ 94) can be made correct to the second order for the ship, when P is removed, by adding to it (§ 112)

$$(P/W)(c-r_1),$$

where r_1 denotes the radius of the curve of flotation, and c the height of P above the water line.

CHAPTER V.

EQUILIBRIUM OF FLOATING BODIES OF REGULAR FORM AND OF BODIES PARTLY SUPPORTED. OSCILLATIONS OF FLOATING BODIES.

126. *The Equilibrium of a floating Cylinder, Cone, Paraboloid, Ellipsoid, Hyperboloid, etc.*

When the body has the shape of one of these regular mathematical forms, the curves of flotation F and of its evolute C, of buoyancy B, and of the prometacentres M, or the metacentric evolute, can be determined by various theorems introducing interesting geometrical applications of the properties of these curves and surfaces.

For a prismatic or cylindrical body like a log, floating horizontally in water, the various surfaces are cylindrical and we need only consider their curves of cross section.

If the section is an ellipse, these curves of flotation and of buoyancy are also ellipses; and the determination of the position of equilibrium will depend on the problem of drawing normals from the C.G. of the body to the ellipse of buoyancy, or tangents to its metacentric evolute; and two or four normals or tangents can be drawn according as the C.G lies outside or inside this evolute.

If the sides of the log in the neighbourhood of the water line are parallel planes, the curve of flotation reduces to a point, and the curve of buoyancy becomes a parabola (§ 103).

If the submerged portion of the log is triangular, or more generally if the log is polygonal or if the sides in the neighbourhood of the water line are intersecting planes, the curves of flotation and of buoyancy are similar hyperbolas of which the cross section of these planes are the asymptotes.

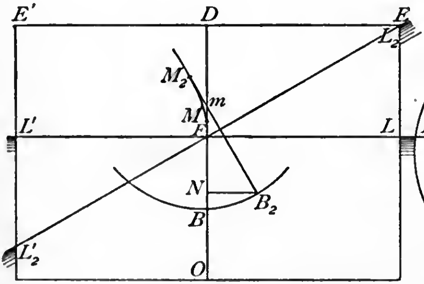


Fig. 44.

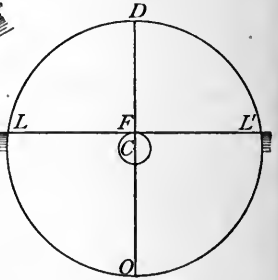


Fig. 45.

When the cross section of the log is rectangular or triangular, the curves of flotation and of buoyancy are composed of parabolic and hyperbolic arcs, interesting figures of which, by Messrs. White and John, will be found in the *Trans. Inst. Naval Architects*, March, 1871; also by M. Daymard, *I.N.A.*, 1884.

If the outside shape of the body is an ellipsoid or other quadric surface, then according to well-known theorems the surfaces of flotation and buoyancy are similar coaxial surfaces; just as in the sphere, from which the ellipsoid may be produced by homogeneous strain.

If the surface of the body is a quadric cone, the surfaces of flotation and of buoyancy will be portions of hyperboloids of two sheets, asymptotic to the cone.

127. *The Cylinder, floating upright.*

When a cylinder of S.G. s floats in water, the surface of flotation F reduces to a fixed point on the axis, so long as an end plane of the cylinder does not cut the surface of the water, and the surface of buoyancy is a paraboloid (§ 103).

If h denotes the height of the cylinder and x the length of axis immersed, then $x=sh$; and for displacements in a vertical plane from the upright position of equilibrium the curve of buoyancy is a parabola (fig. 44), and

$$BM = Ak^2/V = k^2/x = k^2/sh.$$

The equilibrium is stable in the upright position if the C.G. G lies below M, or if $BM > BG$, or

$$\frac{k^2}{sh} > \tfrac{1}{2}(h-x), \quad \frac{k^2}{h^2} > \tfrac{1}{2}s(1-s).$$

But if $\quad s^2 - s + 2\frac{k^2}{h^2} = (s-\tfrac{1}{2})^2 - \tfrac{1}{4}\left(1 - \frac{8k^2}{h^2}\right)$

is negative, or

$$\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\left(1 - \frac{8k^2}{h^2}\right)} > s > \tfrac{1}{2} - \tfrac{1}{2}\sqrt{\left(1 - \frac{8k^2}{h^2}\right)},$$

the upright position of the cylinder is unstable in the corresponding vertical plane of displacement, and the cylinder "lolls" to one side.

In this case the greatest value of k^2/h^2 is $\frac{1}{8}$, and then $s=\frac{1}{2}$; so that the cylinder will float upright in any liquid if $h^2 < 8k^2$.

When the horizontal cross section of the cylinder is a rectangle of breadth b, $k^2 = \frac{1}{12}b^2$ (§ 40); and this prismatic log cannot float upright, if

$$\frac{1}{2} + \frac{1}{2}\sqrt{\left(1 - \frac{2}{3}\frac{b^2}{h^2}\right)} > s > \frac{1}{2} - \frac{1}{2}\sqrt{\left(1 - \frac{2}{3}\frac{b^2}{h^2}\right)}.$$

But if $b^2/h^2 > \frac{3}{2}$, $b/h > \frac{1}{2}\sqrt{6}$, the log will float upright in any liquid.

In a log of square vertical section $b=h$; so that it cannot float with faces horizontal and vertical, if

$$\tfrac{1}{2}+\tfrac{1}{6}\sqrt{3}(=0{\cdot}79) > s > \tfrac{1}{2}-\tfrac{1}{6}\sqrt{3}(=0{\cdot}21).$$

When the cross section of the cylinder is a circle of radius a, $$k^2=\tfrac{1}{4}a^2 \text{ (§ 40)};$$
and this cylinder cannot float upright, if

$$\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\left(1-2\frac{a^2}{h^2}\right)} > s > \tfrac{1}{2}-\tfrac{1}{2}\sqrt{\left(1-2\frac{a^2}{h^2}\right)}.$$

But if $h/a < \sqrt{2}$, this cylinder will always float upright, like a bung.

As an exercise, the student may prove that the body in fig. 44, if floating in two liquids of S.G.'s s_1 and s_2, will be in stable equilibrium in the upright position if

$$\frac{k^2}{h^2} > \frac{1}{2}\frac{(s_1-s)(s-s_2)}{(s_1-s_2)^2}.$$

If the body comes to rest when floating in water with its axis at an inclination θ, then m must coincide with G.

But the curve BB_2 being a parabola (§ 103),

$$Om=OB+BM+Mm=\tfrac{1}{2}x+\frac{k^2}{x}+\frac{k^2}{2x}\tan^2\theta;$$

and therefore, if $Om=OG$,

$$x^2-hx+2k^2+k^2\tan^2\theta=0.$$

Thus, for instance, if fig. 44 represents the cross section of a rectangular log of breadth b, and if E just reaches the surface of the water in the position of equilibrium,

$$x=h-\tfrac{1}{2}b\tan\theta, \quad k^2=\tfrac{1}{12}b^2,$$

and $$\frac{h}{b}=\frac{1+2\tan^2\theta}{3\tan\theta}=\frac{3-\cos 2\theta}{3\sin 2\theta}.$$

128. *The Cylinder or Prism, floating horizontally.*

Now suppose that fig. 44 represents the side elevation of the same circular cylinder, when floating with its axis horizontal, and let fig. 45 represent the end elevation; so that $PP' = 2y$ is the breadth and $LL' = h$ is the length of the rectangular water line area, when the cylinder is floating at the draft $OF = x$.

The centre C is the metacentre for displacements in the plane of fig. 45, and the equilibrium is therefore stable for these displacements.

But for displacements in the plane of fig. 44, the equilibrium is stable if M lies above G, or if

$$Ak^2 = V.BM > V.BG.$$

But $$Ak^2 = 2hy.\tfrac{1}{12}h^2,$$

and by a well-known theorem,

$$V.BG = \tfrac{2}{3}hy^3;$$

and therefore the equilibrium is stable if

$$2hy.\tfrac{1}{12}h^2 > \tfrac{2}{3}hy^3, \quad \text{or} \quad h > 2y.$$

The cylinder will therefore float, like a cork, with its axis horizontal, if its length h is greater than the breadth $2y$ of the water line area; but it will float, like a bung, with its axis vertical, if

$$a/h > \sqrt{\{2s(1-s)\}},$$

the least value of a/h being $\tfrac{1}{2}\sqrt{2}$; with intermediate dimensions the cylinder will float in an inclined position.

When a cylindrical canister of thin material, whose C.G. H is at a height $OH = h$ from the bottom, floating in water when empty with a length c of the axis immersed, is ballasted with liquid of S.G. s to a depth $Of = x$, a length $OF = c + sx$ of the axis will become immersed; and as in fig. 43, p. 175, the system is equivalent to weights proportional to c, sx, and $-(c+sx)$, concentrated at H, m, and M.

The upright position of equilibrium will therefore be unstable or stable, according as

$$c \, . \, OH + sx \, . \, Om - (c+sx)OM$$
$$= ch + sx\left(\frac{x}{2} + \frac{k^2}{x}\right) - (c+sx)\left(\frac{c+sx}{2} + \frac{k^2}{c+sx}\right)$$
$$= ch + (s-1)k^2 + \tfrac{1}{2}sx^2 - \tfrac{1}{2}(c+sx)^2$$

is positive or negative; and $k^2 = \frac{1}{4}a^2$ for a circular cylinder, $k^2 = \frac{1}{12}b^2$ for a prismatic canister.

If, as in § 115 and fig. 42, for the wall-sided ship, the line of draft OF' is drawn in fig. 44, the curve of B will be the straight line OB, and the curve of metacentre M a hyperbola, with OD and OB as asymptotes.

The curve of G for homogeneous cargo will be a hyperbola, reducing to a straight line OG, if the weight of the vessel itself is insensible; and hence a graphical construction can be made for the conditions of stability in the upright position at different draft.

The stability of a cylinder floating upright in two or more liquids (§ 125) would be illustrated by a cylindrical caisson of the Forth Bridge, floating partly in air, partly in water, and partly in the mud or quicksand at the bottom of the water.

129. *The Cone, and the Triangular Prism.*

Unless otherwise stated, a homogeneous right circular cone on a circular base is intended when we speak of a cone; and we denote the S.G. of the cone by s, the altitude OD by h, the radius of the base DE by a, and the semi-vertical angle by α.

When $s = \frac{1}{2}$, the cone can evidently float with its axis horizontal and on the surface of the water; the C.B.'s B and B' of the equal immersed and unimmersed volumes

V and V' lying in a straight line BGB' perpendicular to the axis of the cone.

Now, by well-known theorems of Mensuration,

$$V = V' = \tfrac{1}{6}\pi a^2 h,$$

$$OG = \tfrac{3}{4}h, \quad BG = \frac{3}{4}\frac{4a}{3\pi} = \frac{a}{\pi};$$

and $$V.BM = Ak^2 = ah.\tfrac{1}{18}h^2 \ (\S\ 40);$$

so that this position of equilibrium is stable, if

$$V.BM > V.BG,$$

$$ah.\tfrac{1}{18}h^2 > \tfrac{1}{6}\pi a^2 h.\frac{a}{\pi}, \quad \text{or} \quad h^2 > 3a^2;$$

that is, if the vertical angle 2α of the cone is less than 60°.

130. If the cone of S.G. s floats with its base horizontal and axis vertical and vertex downwards with a length $OF = x$ of the axis immersed (fig. 46), then

$$x^3 = sh^3,$$

since the volumes of the similar cones OLL' and OEE' are as the cubes of their altitudes x and h.

Then, denoting FL by y,

$$BG = \tfrac{3}{4}(h - x) = \tfrac{3}{4}(1 - s^{\frac{1}{3}})h;$$

$$BM = \frac{Ak^2}{V} = \frac{\pi y^2.\tfrac{1}{4}y^2}{\tfrac{1}{3}\pi y^2 x} = \frac{3}{4}\frac{y^2}{x} = \tfrac{3}{4}x\tan^2\alpha;$$

so that M is found geometrically by drawing Bb horizontally to meet the cone in b, and bM perpendicular to the surface of the cone to meet the axis in M.

Also $OM = \tfrac{3}{4}x\sec^2\alpha$; and the upright position of equilibrium is stable if $OM > OG$, or $x\sec^2\alpha > h$;

or $$x > h\cos^2\alpha, \ s > \cos^6\alpha.$$

The equilibrium is neutral when

$$\tan\alpha = (s^{-\frac{1}{3}} - 1)^{\frac{1}{2}},$$

$$x = h\cos^2\alpha, \quad h - x = h\sin^2\alpha;$$

and now $$x(h - x) = x^2\tan^2\alpha = y^2,$$

or the radius of the water line circle is the geometric mean of the segments of the axis of the cone made by the water line; so that the sphere described on the axis OD as diameter will cut the cone in the circle of flotation LL' when the equilibrium in this position is neutral.

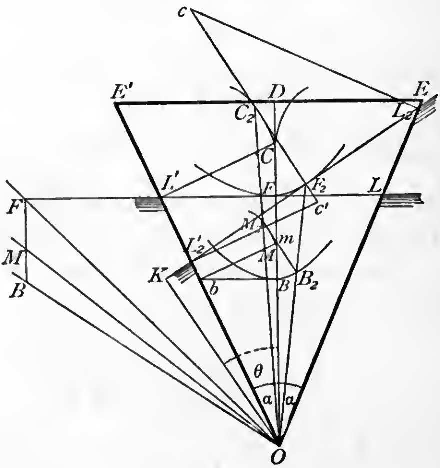


Fig. 46.

When $$\alpha = \tfrac{1}{4}\pi, \text{ then } s = \cos^6\alpha = \tfrac{1}{8};$$

so that a right-angled cone cannot float in water with its vertex downwards and axis permanently vertical, unless its S.G. s is greater than $\frac{1}{8}$.

We infer, as in § 124, that the same conditions hold for the equilibrium and stability of the cone, floating inverted with its base submerged, if the S.G. of the cone is changed to $$1 - s.$$

When $s = \frac{1}{2}$, the cone can float in both positions; and the equilibrium is neutral if

$$\cos^6\alpha = \tfrac{1}{2}.$$

131. We may also take fig. 46 to represent the vertical cross section of an isosceles triangular log, floating horizontally; and now

$$BM = \frac{Ak^2}{V} = \frac{2y \cdot \frac{1}{3}y^2}{yx} = \frac{2}{3}\frac{y^2}{x} = \frac{2}{3}x \tan^2\alpha,$$

so that M is found geometrically as before; and

$$OM = \tfrac{2}{3}x \sec^2\alpha.$$

The equilibrium is stable in this position if $x > h\cos^2\alpha$, as before; but now $s = x^2/h^2 > \cos^4\alpha$.

If the cross section of the log is an equilateral triangle

$$\alpha = \tfrac{1}{6}\pi; \text{ and } s > \tfrac{9}{16}, \text{ or } 1 - s < \tfrac{7}{16},$$

for the log to float with a face horizontal.

When $s < \frac{9}{16}$, or generally $< \cos^4\alpha$,

this upright position becomes unstable, and the triangular cylinder floats with its base inclined, until at last an edge reaches the water; and then the opposite face must be vertical, and

$$s = OL_2'/OE' = \cos 2\alpha (s = \tfrac{1}{2}, \text{ for the equilateral log}).$$

So also the vertical position of the cone becomes unstable when $s < \cos^6\alpha$, and the cone lolls over; the surface of buoyancy BB_2 being a hyperboloid, until the base EE' touches the water, when the edge OE' opposite to the point of contact must evidently be vertical, since it is parallel to GB_2 or DF_2.

132. If the cone is immersed with a generating line OE horizontal, the water line areas are parabolas, the C.G.'s of which divide the axis of the parabola in the ratio of 3 to 2; so that the locus of the C.B.'s for different immersions, with the generating line OE horizontal, is the straight line $E'H$, where

$$OH = \tfrac{3}{5}OE, \; HE = \tfrac{2}{5}OE.$$

This line $E'H$ will pass through G and be perpendicular to OE, and the cone can therefore float with a generating line horizontal, if

$$OH\,.\,OE = OG\,.\,OD, \text{ or } \tfrac{3}{5}OE^2 = \tfrac{3}{4}OD^2,$$
$$h^2 = \tfrac{4}{5}OE^2 = \tfrac{4}{5}(h^2 + a^2) = 4a^2\,;$$

so that the altitude is equal to the diameter of the base.

133. Generally for an inclined position of the cone, the axis making an angle θ with the vertical OK, the vertex O being submerged, and the base EE' out of the water, then the water line area L_2L_2' is an ellipse with centre F_2, and B_2 lies on OF_2 at a distance

$$OB_2 = \tfrac{3}{4}OF_2,$$

the curves of flotation and buoyancy being similar hyperbolas, with OE and OE' for asymptotes.

It is readily proved that C_2, the centre of curvature of the curve of flotation, is the middle point of cc', where L_2c, $L_2'c'$, are at right angles to OL_2, OL_2'.

Now if we put $OK = \lambda h$, then

$$KL_2 = \lambda h \tan(\theta + a),\quad KL_2' = \lambda h \tan(\theta - a)\,;$$
$$KF_2 = \tfrac{1}{2}(KL_2 + KL_2') = \lambda h \sin\theta \cos\theta \sec(\theta + a)\sec(\theta - a)\,;$$
$$F_2L_2 = \tfrac{1}{2}(KL_2 - KL_2') = \lambda h \sin a \cos a \sec(\theta + a)\sec(\theta - a).$$

If the line through F_2 perpendicular to the axis meets the cone in l and l', the minor axis of the water line ellipse L_2L_2' is given by

$$\sqrt{(lF_2\,.\,F_2l')} = F_2L_2\sqrt{\left\{\frac{\cos(\theta + a)}{\cos a}\;\frac{\cos(\theta - a)}{\cos a}\right\}}$$
$$= \lambda h \sin a\sqrt{\{\sec(\theta + a)\sec(\theta - a)\}}\,;$$

and therefore the volume of the cone OL_2L_2' is

$$V = \tfrac{1}{3}\lambda h\,.\,\pi\lambda^2h^2\sin^2 a\cos a\{\sec(\theta + a)\sec(\theta - a)\}^{\frac{3}{2}}.$$

If the buoyancy is equal to the weight of the cone, the S.G. s of the cone is the ratio of the volume V to $\tfrac{1}{3}\pi h^3\tan^2 a$;

or $$s = \lambda^3\cos^3 a\{\sec(\theta + a)\sec(\theta - a)\}^{\frac{3}{2}}.$$

If the cone is floating freely in this position, then B_2G and therefore also F_2D are vertical; and projecting the axis OD on the horizontal KF_2,

$$h \sin \theta = KF_2 = \lambda h \sin \theta \cos \theta \sec(\theta + \alpha)\sec(\theta - \alpha),$$

or

$$\lambda = \sec \theta \cos(\theta + \alpha)\cos(\theta - \alpha);$$

so that

$$\sqrt[3]{s} = \cos \alpha \sec \theta \sqrt{\{\cos(\theta+\alpha)\cos(\theta-\alpha)\}}.$$

Thus, when $\theta = \alpha$, the generating line OE' is vertical, the point E of the base is in the surface of the water, and

$$\sqrt[3]{s} = \sqrt{(\cos 2\alpha)}, \quad \text{or} \quad s = (\cos 2\alpha)^{\frac{3}{2}}.$$

When the cone has a slant elliptic base EE' with centre D, let h denote the altitude or perpendicular distance of the vertex O from EE', and ϕ the angle between this perpendicular and the axis of the cone; then, in fig. 46,

$$\begin{aligned} KF_2 &= \tfrac{1}{2}OE \sin(\theta+\alpha) + \tfrac{1}{2}OE'\sin(\theta-\alpha) \\ &= \tfrac{1}{2}h \sec(\phi+\alpha)\sin(\theta+\alpha) + \tfrac{1}{2}h \sec(\phi-\alpha)\sin(\theta-\alpha) \\ &= h(\cos^2\alpha \sin\theta \cos\phi - \sin^2\alpha \cos\theta \sin\phi)\sec(\phi+\alpha)\sec(\phi-\alpha). \end{aligned}$$

The volume of the cone OEE' is now

$$\tfrac{1}{3}\pi h^3 \sin^2\alpha \cos\alpha \{\sec(\phi+\alpha)\sec(\phi-\alpha)\}^{\frac{3}{2}},$$

so that, in the position of equilibrium,

$$\begin{aligned} (\cos^2\alpha \sin\theta \cos\phi - \sin^2\alpha \cos\theta \sin\phi)&\sec(\phi+\alpha)\sec(\phi-\alpha) \\ &= \lambda \sin\theta \cos\theta \sec(\theta+\alpha)\sec(\theta-\alpha), \end{aligned}$$

and

$$s = \lambda^3 \left\{\frac{\sec(\theta+\alpha)\sec(\theta-\alpha)}{\sec(\phi+\alpha)\sec(\phi-\alpha)}\right\}^{\frac{3}{2}},$$

or

$$s^{\frac{1}{3}} = \left\{\frac{\cos(\theta+\alpha)\cos(\theta-\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)}\right\}^{\frac{1}{2}} \left(\frac{\cos^2\alpha \cos\phi}{\cos\theta} - \frac{\sin^2\alpha \sin\phi}{\sin\theta}\right);$$

the minimum value of which can be shown to be

$$\frac{\cos^2\alpha(\cos\phi)^{\frac{2}{3}} - \sin^2\alpha(\sin\phi)^{\frac{2}{3}}}{\{\cos(\phi+\alpha)\cos(\phi-\alpha)\}^{\frac{1}{2}}}.$$

134. Similar considerations hold for the inclined positions of equilibrium of the triangular log.

When the angle EOE' is a right angle, we may take fig. 46, p. 196, to represent the cross section of a square log, whose centre is at D.

If the position of equilibrium in which OD is upright is stable, so that M lies above D, then $\frac{2}{3}x \sec^2 a > h$, where $a = \frac{1}{4}\pi$, while $x^2 = 2sh^2$; so that

$$s > \tfrac{9}{32}, \quad 1 - s < \tfrac{23}{32},$$

giving the limits of the S.G. for which a diagonal of the log is vertical in a stable position of equilibrium.

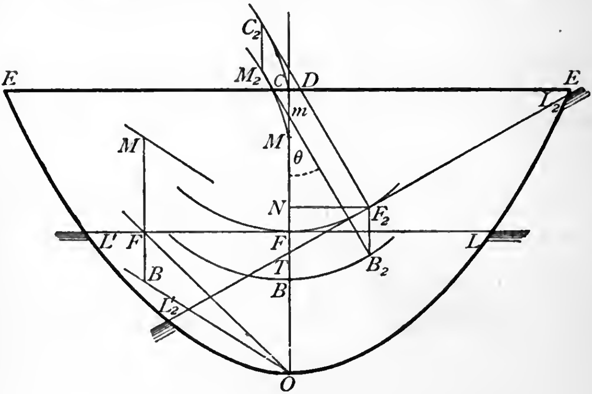


Fig. 47.

135. *The Paraboloid and Parabolic Cylinder.*

In a parabolic cylinder floating with its axis horizontal the curves of flotation F and of buoyancy B are equal parabolas (fig. 47); while in the paraboloid generated by the revolution of the parabola about its axis OD, the surfaces of flotation and buoyancy are equal paraboloids; also BM is equal to l, the semi-latus-rectum, and the equation connecting $x = OF$ and $y = FL$ is $y^2 = 2lx$.

By well-known theorems of Mensuration, the area of the parabola LOL' is $\frac{2}{3}xy=\frac{2}{3}(2lx^3)^{\frac{1}{2}}$, two-thirds of the area of the circumscribing rectangle, and its C.G. B divides OF in the ratio of 3 to 2, so that $OB=\frac{3}{5}OF$; while the volume of the paraboloid LOL' is

$$\tfrac{1}{2}\pi xy^2=\pi lx^2,$$

one-half the volume of the circumscribing cylinder, and its C.G. B is situated so that $OB=\frac{2}{3}OF$.

The equation of the surface of buoyancy of the paraboloid, in the general case when the cross sections are elliptic, is now, according to § 103,

$$\frac{4x}{h}=\frac{y^2}{k_z{}^2}+\frac{z^2}{k_y{}^2}$$

at draft h, since $V=\frac{1}{2}Ah$.

If s denotes the S.G. of the body, floating upright in water, then for the parabolic cylinder,

$$s=(\text{area } LOL')/(\text{area } EOE')=(x/h)^{\frac{3}{2}},$$

or

$$x/h=s^{\frac{2}{3}};$$

while for the paraboloid,

$$s=(\text{volume } LOL')/(\text{volume } EOE')=(x/h)^2,$$

or

$$x/h=s^{\frac{1}{2}}.$$

Then in the upright position of equilibrium of the parabolic cylinder or paraboloid, of S.G. s and C.G. G,

$$GM=BM+OB-OG=l-\tfrac{3}{5}OC(1-s^{\frac{2}{3}}),\ \text{or}\ l-\tfrac{2}{3}OC(1-s^{\frac{1}{2}});$$

so that

$$\frac{OC}{l}<\frac{5}{3}\frac{1}{1-s^{\frac{2}{3}}},\ \text{or}\ \frac{3}{2}\frac{1}{1-s^{\frac{1}{2}}},$$

$$\frac{OF}{l}<\frac{5}{3}\frac{s^{\frac{2}{3}}}{3-s^{\frac{2}{3}}},\ \text{or}\ \frac{3}{2}\frac{s^{\frac{1}{2}}}{1-s^{\frac{1}{2}}},$$

are the conditions for stability in the upright position.

This investigation of the stability of the paraboloid is originally due to Archimedes.

136. When the upright position becomes unstable, the body will loll over until the line B_2G becomes vertical; and now $NG = BM = l$, or $MG = BN = \frac{1}{2}l \tan^2\theta$.

This assumes that the base EE' is out of the water; but when the base just touches the water at E at the inclination θ, so that L_2 and E coincide,

$$\tan\theta = \frac{TD}{DE} = \frac{DF+FT}{DE} = \frac{h(1-s^{\frac{2}{3}, \text{ or } \frac{1}{2}}) + \frac{1}{2}l\tan^2\theta}{\sqrt{(2lh)}};$$

and solving this quadratic in $\tan\theta$,

$$\sqrt{l}\tan\theta - \sqrt{(2h)} = \pm\sqrt{(2h)}s^{\frac{1}{3}, \text{ or } \frac{1}{4}},$$

$$MG = \tfrac{1}{2}l\tan^2\theta = h(1-s^{\frac{1}{3}, \text{ or } \frac{1}{4}})^2;$$

this is the maximum height of the C.G. above the metacentre for the base to be out of the water in the inclined position of equilibrium.

137. *The Ellipsoid or Hyperboloid.*

Fig. 47 may be made to serve for the elliptic or hyperbolic cylinder, or for the ellipsoids or hyperboloids generated by revolving an ellipse or hyperbola about a principal axis OD; but now the curves or surfaces of flotation F and buoyancy B are concentric similar and similarly situated conics or quadric surfaces.

As in fig. 41, p. 164, the lines B_2F_2 pass through the common centre, as also the lines C_2M_2, through C_2 the centre of curvature of the curve of flotation, and M_2 the metacentre, or centre of curvature of the curve of buoyancy; thus affording illustrations of Leclert's theorem.

When the submerged portion of a floating body is a trihedral angle or triangular pyramid, the surfaces of flotation or buoyancy, referred to the three edges as coordinate axes, is readily seen to be given by the equation

$$xyz = c^3.$$

138. A similar procedure to that employed in § 128 for the determination of the stability of a cylindrical or prismatic canister of thin material, when ballasted by a given amount of liquid, will serve for the cases where the shape of the canister is a cone or wedge-shaped surface, a paraboloid or parabolic cylinder, an ellipsoid, etc.; this is left as an exercise for the student; the investigation is of importance in the determination of the gain or loss of stiffness due to "bilging" in vessels of triangular, parabolic, or circular section.

139. As in fig. 42 we can construct, with respect to the sloping line of draft OF', the curve of buoyancy B, of metacentres M, and of centres of gravity G, for varying draft in the upright position of ships of the shape of bodies represented in figs. 46, 47.

The curve described by B will be a straight line OB in both figures. In fig. 46 the curve of metacentres M will also be a straight line through O; but in fig. 47 it will be a straight line parallel to OF'.

Suppose that the draft of the vessel when empty is a, and its C.G. is at a height h above the keel O; and that the draft becomes x when loaded with cargo of S.G. s to a depth x', and that the height of the C.G. above the keel O is now y.

Then in the triangular prism or paraboloid,

$$(a^2+sx'^2)y=x^2y=a^2h+\tfrac{2}{3}sx'^3;$$

and in the cone or parabolic cylinder,

$$(a^3+sx'^3)y=x^3y=a^3h+\tfrac{3}{4}sx'^4,$$

or $$(a^{\frac{3}{2}}+sx'^{\frac{3}{2}})y=x^{\frac{3}{2}}y=a^{\frac{3}{2}}h+\tfrac{3}{5}sx'^{\frac{5}{2}};$$

representing curves of G, which are asymptotic to the straight line curves of G when $a=0$, or the weight of the vessel is neglected.

Examples.

(1) Two equal uniform thin rods (or boards), whose S.G. is s, are joined together at an angle 2α, and immersed in water with the angle downwards. Prove that the curves of buoyancy and flotation are parabolas; and prove that the rods cannot float with the line joining their ends inclined to the horizon, unless $\sin\alpha < \sqrt{\left(\frac{1-s}{1+s}\right)}$.

Determine which positions of equilibrium are stable.

(2) A solid formed of a cone and hemisphere which have a common base floats totally immersed in two liquids, the cone being wholly in the lower and the hemisphere in the upper liquid; prove that the equilibrium will be stable if the centre of gravity of the solid be above the common surface of the two liquids.

(3) A right circular cone of S.G. s, whose base is an ellipse, floats, vertex downwards, in water with the extremity of the shortest generator in the surface, and its base inclined at an angle θ to the surface.

Prove that the longest generator is vertical, and that, if α is the semi-vertical angle of the cone,

$$1 + \tan 2\alpha \tan\theta = s^{-\frac{2}{3}}.$$

(4) A right circular cone has a plane base in the form of an ellipse; the cone floats with its longest generating line horizontal; if 2α be the vertical angle of the cone, and β the angle between the plane base and the shortest generating line, show that

$$\cot\beta = \cot 4\alpha - \tfrac{1}{5}\operatorname{cosec} 4\alpha.$$

(5) A solid in the shape of a double cone bounded by two equal circular ends floats in a liquid of twice its density with its axis horizontal; prove that the equilibrium is stable or unstable according as the semi-vertical angle is less or greater than 60°.

(6) Prove that, owing to the presence of air of S.G., ρ the metacentre of a cone of S.G. s, floating upright and vertex downwards in water, will be brought nearer to the C.G. by

$$\tfrac{3}{4}h[\{(a_1-a)s+a\rho\}\sec^2\alpha-\rho]/s,$$

where ah, a_1h denote the depth of the vertex with and without air.

(7) A solid homogeneous cone of height h and semi-vertical angle α floats with its axis vertical and vertex downwards in equal thickness of four homogeneous liquids whose densities, counting from the surface, are respectively ρ, 2ρ, 3ρ, 4ρ, so that its vertex is at a depth k from the surface. Prove that the condition of stability is

$$200h\cos^2\alpha > 177k.$$

(8) A conical shell, vertex downwards, is filled to one-ninth of its depth with a fluid of density ρ', and then floats with one-third of its axis submerged in a fluid of density ρ; find a relation between ρ and ρ', in order that the equilibrium may be neutral.

(9) A frustum of a right circular cone, cut off by a plane bisecting the axis perpendicularly, floats with its smaller end in water and its axis just half immersed. Prove that the S.G. is $\frac{19}{56}$, and that the equilibrium will be neutral if the semi-vertical angle of the cone is $\tan^{-1}\dfrac{\sqrt{805}}{63}$.

(10) A frustum of a cone floats with its axis vertical in a liquid of twice its own density.

Prove that the equilibrium will be stable if

$$h^2 < \frac{(a-b)^2}{m-1}, \text{ where } m = \frac{2^{\frac{1}{3}}(a^4+b^4)}{(a^3+b^3)^{\frac{4}{3}}},$$

h being the height of the frustum, and a and b the radii of its ends.

Also if it floats with its axis horizontal, the equilibrium will be stable if

$$h^2 > \frac{(a+b)^2(a^2+b^2)}{a^2+4ab+b^2}.$$

(11) From a solid hemisphere, of radius a, a portion in the shape of a right cylinder, of radius b and height h, coaxial with the hemisphere and having the centre of its base at the centre of the hemisphere, is removed. Into this portion is fitted a thin tube which exactly fits it. The solid is placed with its vertex downwards in a fluid, and a fluid of S.G. s is poured into the tube. Find how much must be poured in, in order that the equilibrium may be neutral; and if the tube be filled to a height $2h$, show that

$$\frac{s}{s'} = \frac{a^4 - 2b^2h^2}{b^4},$$

s' being the S.G. of the solid.

(12) If a vessel be of thin material in the shape of a paraboloid of revolution, show that the equilibrium will be always stable, provided the density of the fluid inside be greater than that without; the weight of the vessel being neglected.

(13) A thin hollow shell in the form of a paraboloid of revolution floats in water with its axis vertical and its vertex at a distance $\frac{1}{6}\sqrt{3}l$ below the surface.

Prove that if the equilibrium become neutral when water whose weight is equal to three times that of the shell is poured into it, the height of the centre of gravity of the shell above the vertex is $\frac{8}{9}\sqrt{3}l$, where $2l$ is the latus rectum of the generating parabola.

(14) Prove that the work required to heel a ship of parabolic cross section (§ 135) through an angle θ from the upright position is

$$W \sin\theta(\tfrac{1}{2}l \tan\theta - b \tan\tfrac{1}{2}\theta),$$

where b denotes the distance GB.

(15) A regular solid tetrahedron floats in water with a face horizontal and not immersed; prove that the equilibrium is stable if the S.G. of the tetrahedron is greater than 0·512.

Determine the limits of the S.G. between which the tetrahedron can float permanently with two edges horizontal.

(16) A regular octahedron, S.G. $\frac{1}{2}$, floats with one diagonal or with two edges vertical; find the metacentre.

(17) To one angular point of a homogeneous regular tetrahedron of S.G. s is attached a particle of one-quarter (generally one-n^{th}) its weight.

Prove that the tetrahedron can float in water with one edge vertical and another edge horizontal,

if $s = 0·2048$, or, generally, $s = \frac{1}{2}(1+n)^{-4}$.

(18) A tetrahedron of weight W, whose base is a triangle ABC, and the angles at whose vertex are right

angles, has weights P, Q, R, whose sum is W, attached at A, B, C, respectively, and floats in a fluid with its base upwards and horizontal, the vertex being at depth h. Prove that

$$\frac{4P}{W}=2m-1+(7-6m)\cot B\cot C,$$

with two similar expressions for Q and R, where

$$h^2=4m^2r^2\cos A\cos B\cos C,$$

and r is the radius of the circle circumscribing ABC.

(19) Prove that a prism whose section is the acute angle triangle ABC cannot have three positions of equilibrium with the edge C alone immersed unless the S.G. is intermediate to

$$\frac{\{(a+b)(a+b+c)(a+b-c)\}^{\frac{2}{3}}-\{(a-b)(a-b+c)(a+b-c)\}^{\frac{2}{3}}}{2bc}$$

and
$$\frac{a^2+b^2-c^2}{2a^2 \text{ or } 2b^2}.$$

(20) Prove that the isosceles triangular log of § 134, if of density $n\rho$, can float at an inclination θ in two liquids of densities ρ and 2ρ, the upper liquid being of depth μh, with the base not immersed, provided

$$(1+\cos 2\theta)\{n(\cos 2\theta+\cos 2a)-\mu^2\}\{n(\cos 2\theta+\cos 2a)+2\mu^2\}^2$$
$$=2n^2\cos^2 a(\cos 2\theta+\cos 2a)^4.$$

(21) Investigate the equilibrium and stability for finite displacements of the cone of Ex. (22), p. 188, containing liquid and resting in a circular hole; or of a wedge-shaped prismatic vessel resting on two smooth parallel horizontal bars.

(22) A thin conical vessel, inverted over water, is depressed by a weight attached to the rim so as to be completely submerged, with a generating line vertical and the enclosed air on the point of beginning to escape.

Prove that the ratio of the weight of the cone to the weight of water it can hold is

$$\tfrac{9}{8}(\cos 2\alpha)^{\frac{3}{2}}.$$

(23) Prove that the equilibrium of a homogeneous ellipsoid of S.G. 0·5 floating in water is stable if the least axis is vertical, stable-unstable if the mean axis is vertical, and unstable altogether if the greatest axis is vertical.

(24) A homogeneous ellipsoid floats in a liquid with its least axis COC' vertical, and a weight W, $\frac{3}{5}$ of that of the ellipsoid, fixed at the upper end C, such that the plane of flotation passes through the centre.

Prove that, if it be turned about the mean axis (b) through a finite angle θ, the moment of the couple which will keep it in that position will be

$$W\{c - ae^2 \cos\theta(1 - e^2\cos^2\theta)^{-\frac{1}{2}}\}\sin\theta,$$

where e is the eccentricity of the section (a, c).

(25) A light ellipsoidal shell partly filled with water is free to rotate round its centre which is fixed, and the shell, when in its position of stable equilibrium, is turned through any angle.

Prove that the work necessary to effect the displacement varies as $a - p$, where p is the perpendicular from the centre of the ellipsoid on the tangent plane parallel to the new surface of the water, and $2a$ the longest axis.

140. *Ship Design and Calculation.*

The drawings of the outside surface of a ship give the equidistant contour lines in (fig. 48)

(i.) the side elevation (*sheer plan*);
(ii.) the end elevation (*body plan*);
(iii.) the plan (*half-breadth plan*);

these curves and planes may be supposed referred, as in Solid Geometry, to the coordinate axes, Ox vertical in the stem, Oz horizontal along the keel, and Oy perpendicular to the medial plane.

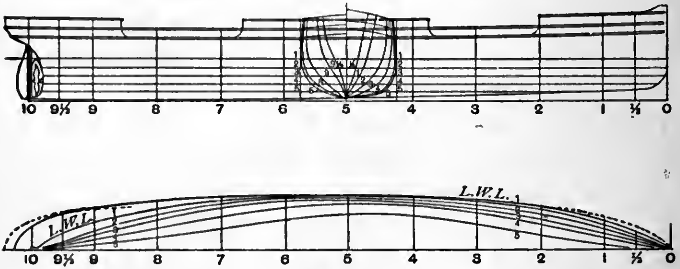


Fig. 48.

Considering that a ship is symmetrical with respect to this medial plane, a representation of one half of the ship is sufficient.

The curves and surfaces employed here are not in general such as those just investigated, which are given by analytical conditions (*curves et surfaces analytiques*); but they are fair curves and surfaces drawn through guiding points and lines (*curves et surfaces topographiques*); so that in the determination of the corresponding areas, volumes, C.G.'s, and moments of inertia, mechanical Planimeters must be employed, or else the methods of Approximate Quadrature.

(Pollard et Dudebout, *Théorie du navire.*)

Denoting the half breadths and half areas in the plane of yOz by y and $\frac{1}{2}A$, then in the notation of the Integral Calculus,

$$\tfrac{1}{2}A=\int y\,dz,\ \tfrac{1}{2}A\bar{y}=\int\tfrac{1}{2}y^2dz,\ \tfrac{1}{2}A\bar{z}=\int yz\,dz,$$

$$\tfrac{1}{2}Ak_z^2=\int\tfrac{1}{3}y^3dz,\ \tfrac{1}{2}Ak_y^2=\int yz^2dz\,;$$

and denoting by V the displacement up to any given water line,

$$V=\int A\,dx,\ V\bar{x}=\int xA\,dx,\ldots.$$

But now, according to the methods of approximate quadrature, the half area $\frac{1}{2}A$ is found by dividing it up by equidistant ordinates $y_1, y_2, y_3,\ldots y_n$, at intervals Δz; and then

(i.) by the *trapezoidal rule*, in which the boundary curve is replaced by the broken line composed of the chords joining the tops of the ordinates

$$\tfrac{1}{2}A=(\tfrac{1}{2}y_1+y_2+y_3+\ldots+y_{n-1}+\tfrac{1}{2}y_n)\Delta z\,;$$

(ii.) by *Simpson's rule*, in which an odd number of ordinates and an even number of intervals must be taken, the curve joining the tops of three adjacent ordinates is replaced by a parabola whose axis is parallel to the ordinates.

The area between the ordinates y_{2m-1} and y_{2m+1} is therefore composed of the trapezoidal part

$$\tfrac{1}{2}(y_{2m-1}+y_{2m+1})2\Delta z,$$

and of the parabolic segment, equal to $\frac{2}{3}$ of the circumscribing parallelogram, and therefore

$$\tfrac{2}{3}\{y_{2m}-\tfrac{1}{2}(y_{2m-1}+y_{2m+1})\}2\Delta z=-\tfrac{2}{3}(y_{2m-1}-2y_{2m}+y_{2m+1})\Delta z;$$

so that the whole strip is

$$(y_{2m-1}+4y_{2m}+y_{2m+1})\tfrac{1}{3}\Delta z\,;$$

and therefore, by Simpson's rule,

$$\tfrac{1}{2}A=(y_1+4y_2+2y_3+\ldots+2y_{2n-1}+4y_{2n}+y_{2n+1})\tfrac{1}{3}\Delta z.$$

The other integrals required for finding the moments, first and second (moments of inertia), and for finding the volumes and their moments, being represented graphically by the areas of curves whose ordinates are $\frac{1}{2}y^2, yz, \frac{1}{3}y^3, yz^2$, A and Ax, can be evaluated in the same manner by Simpson's rule.

Similar calculations for determining V are performed for plane sections perpendicular to the keel Oz (*square* sections), and used to check the former results; sometimes also, but rarely, for plane sections parallel to the medial plane xOz.

Thus, for instance, if the half breadths of the water plane of a vessel are 3, 4·5, 9·2, 12·4, 13·9, 14·5, 14·3, 13·4, 11·6, 8·0, 2·4 ft, at an interval of 16 ft, the area of the half plane is 1678 ft², and the C.G. of the half area is 100 ft from one end, and 4 ft from the middle line.

So also, if the areas of the water line sections, reckoned downwards, are 4000, 4000, 3200, 2500, 1500, 600, 100 ft², at an interval of 2 ft, the displacement is 27933 ft³ or 798·1 tons, and the C.B. is at a depth 3·98 ft below the load water section.

As additional exercises the student may work out the half area and its C.G. for half breadths

(1) 14, 33, 49, 57, 50, 40, 25 ft, at intervals of 30 ft;

(2) ·5, 6, 10, 12·4, 12·5, 12·5, 12·5, 12·4, 12·3, 11, 8, ·5 ft, at intervals of 12 ft.

(3) Determine the displacement in tons and the position of the C.B. for the different water planes 3 ft apart, at which the tons per inch immersion are

27, 26, 24·8, 22·8, 20·5, 17·5, 13;

the displacement below the lowest plane being 50 tons, with a C.B. 15 inches below this plane.

The Examination Papers for Science Schools and Classes, Part IV., *Naval Architecture*, may be consulted for additional exercises.

Various empirical rules have long been in existence for giving the displacement or tonnage of a vessel, given the length L, breadth B at the water line, and D the depth or draft of water (Moorsom, *Trans. I.N.A.*, vol. I.).

The simplest rule gives

$$L \times B \times D \div 100 = \text{displacement in tons of 100 ft}^3,$$

which is afterwards multiplied by an arbitrary coefficient, ranging from 0·5 to 0·8, according to the fineness of the shape of the vessel.

The ratio of the true displacement in ft³ to the volume

$$L \times B \times D$$

of the box-shaped vessel of the same dimensions is called the "block coefficient" or "coefficient of fineness"; thus for an ellipsoid half immersed this coefficient is

$$\tfrac{1}{6}\pi = 0{\cdot}5236.$$

141. *The Conditions of Equilibrium of a Floating Body partly supported.*

When a floating body is partly supported, for instance, a body suspended by a thread or fine wire and weighed in the hydrostatic balance, a ballcock in a cistern, a bucket lowered by a rope into water, a boat partly hoisted up by a rope, a ship aground or on the launching ways, or a diving bell, the body is in equilibrium under three parallel forces: (i.) the weight, (ii.) the upward buoyancy, and (iii.) the thrust of the support or pull of the rope, which must also be vertical; and then the conditions of equilibrium of three parallel forces must be applied, generally by taking moments about any point in the line of the reaction of the support.

If the body can move freely on an axis fixed at a given inclination, the resolved part of the weight and buoyancy parallel to this axis must be equal, and therefore the weight and buoyancy must be equal, as in a body floating freely; and, in addition, the moments of the weight and buoyancy about the axis must be equal.

Sometimes a body is pushed or held down, or sinks to the bottom; and when it is held completely submerged, as, for instance, a submarine mine, or a body floating up against the under side of a sheet of ice, the C.B. coincides with the centre of figure of the body. If the body weighs W lb and its S.G. is s, the buoyancy is a force of W/s pounds; the reaction $W/s-W$ of the ice therefore passes through a fixed point O in GB produced, such that

$$\frac{OB}{BG}=\frac{W}{W/s-W}=\frac{s}{1-s};$$

and the positions of equilibrium are those in which the normal from O to the surface of the body is vertical.

Thus an elliptic cylinder of S.G. s, floating horizontally against the under side of a sheet of ice, or resting on the bottom of the water, has four or two positions of equilibrium according as O lies inside or outside the evolute of the ellipse of cross section (§ 126).

142. Suppose a vessel, drawing a ft of water, to take the ground along the keel K, and that the tide falls from LL' to L_1L_1', a fall of x ft, the buoyancy thus lost being P tons (fig. 42, p. 168).

If the vessel heels through a small angle θ, an upsetting couple $P\,.\,Kh\,.\,\sin\theta$ ft-tons is introduced, by which the metacentric height GM, as in § 118, will be diminished by $(P/W)Kh$ ft.

If the vessel is wall-sided between these two water lines,

$$P/W=xA/V, \text{ and } Kh=a-\tfrac{1}{2}x,$$

so that the loss of metacentric height is

$$(ax-\tfrac{1}{2}x^2)A/V \text{ ft};$$

and when this length exceeds GM at this draft, the vessel cannot stand upright.

Precautions are necessary in launching and in docking and undocking a vessel, to prevent it from falling on one side, for the reasons given above.

Suppose, for instance, a vessel which is "trimming considerably by the stern" is being docked; at the moment before the keel takes the blocks prepared to receive it, the thrust on the keel at the stern post can be calculated from the "tons per inch of immersion" and "the moment in ft-tons to change the trim one inch"; thence the value of P can be inferred, and an estimate made of the stability of the vessel.

(F. K. Barnes, *Annual of the Royal School of Naval Architecture,* 1874; *Statics of Launching,* Proc. U.S. Naval Institute, July, 1892.)

If the ship heels over to a new position of equilibrium by turning about the keel K as a fixed horizontal axis, the displacement does not keep constant, as the water planes will touch a circular cylinder fixed in the ship; and if KF' is perpendicular to the water plane, the line of contact will pass through F'.

Supposing that B_2N denotes the depth of the new C.B. B_2 below the water plane, and GH the height of the C.G. above, and that the displacement has diminished from V to V_2 ft³, then in the position of equilibrium,

$$V_2 \,.\, F'N=V \,.\, F'H.$$

143. When a body is lowered into water by a rope, the point K must be taken as the point of attachment to the body; for instance, in lowering a bucket into water, K will lie in the axis of the pivots of the handle, say at D in fig. 44, p. 190.

Suppose the bucket is cylindrical, and of height h from the bottom to the pivots of the handle, and that it would float upright in unstable equilibrium with a length b of the axis immersed, and that D is at a height c above G.

The bucket will begin to leave the upright position when a length x of the axis is immersed, such that W downwards through G and the buoyancy Wx/b upwards through M have equal moments round D, the tension of the rope being $W(1-x/b)$; and therefore

$$W\frac{x}{b}\left(h-\tfrac{1}{2}x-\frac{k^2}{x}\right)=Wc,$$

or

$$x^2-2hx+2k^2+2bc=0,$$

$$x-h=\sqrt{(h^2-2bc-2k^2)},$$

which gives the length of axis out of the water in the position of neutral equilibrium; a greater length out of water will make the equilibrium stable, and *vice versa*; also $2k^2=\frac{1}{2}a^2$ for a circular bucket of radius a (§ 40).

In a long thin cylinder, like a spar, lowered by one end, we may put $r=0$; and now $b=sh$, if s denotes the S.G. of the spar; also $c=\frac{1}{2}h$, and then

$$x-h=h\sqrt{(1-s)};$$

so that the vertical position is unstable, and the spar will assume an inclined position if the vertical length out of water is less than $\sqrt{(1-s)}$ times the length of the spar.

144. If the body in fig. 44 is supported or pushed down at any other point, say at O, E, or E', and comes to rest at an inclination θ to the vertical, moments must be

taken about this point, remembering that the weight and buoyancy are proportional to sh and x, the weight acting through G and the buoyancy through m, where Om is given in § 127.

Similar investigations will hold for the other bodies shown in figs. 46 and 47; the reasoning is similar to that employed in § 118, and the point of support may now replace m, the metacentre of the liquid inside.

145. Suppose the cone in fig. 46 is floating in water of depth $OK=\lambda h$, with its vertex O touching the bottom; then if the axis is inclined at θ to the vertical in the position of equilibrium, equating moments round O of the weight

$$W=\tfrac{1}{3}Ds\pi h^3\tan^2\alpha,$$

acting downwards through G, and of the buoyancy DV of the volume V of the cone OL_2L_2', acting upwards through B_2 (§ 133), gives the equation

$$W.\tfrac{3}{4}h\sin\theta=DV.\tfrac{3}{4}KF_2,$$

or $$s\tan^2\alpha\sin\theta$$
$$=\lambda^4\sin\theta\cos\theta\sin^2\alpha\cos\alpha\{\sec(\theta+\alpha)\sec(\theta-\alpha)\}^{\frac{5}{2}},$$

or $$\lambda^4=s\sec\theta\sec^3\alpha\{\cos(\theta+\alpha)\cos(\theta-\alpha)\}^{\frac{5}{2}}.$$

If $\theta=0$, $\lambda^4=s\cos^2\alpha$, and the equilibrium is apparently neutral in the upright position, but really stable.

If a generating line is vertical,

$$\theta=\alpha,\text{ and }\lambda^4=s\sec^4\alpha(\cos2\alpha)^{\frac{5}{2}};$$

thus if $\alpha=30°$, and $\lambda=\frac{1}{2}$, then $s=\frac{1}{4}\surd 2$.

146. If the body in fig. 46 is pushed down at E', the condition of equilibrium is obtained by taking moments about E'; and, as in § 124, the same condition of equilibrium holds for the cone, of S.G. $1-s$, floating in water with its vertex above the surface, and its base touching the bottom at E'.

As an exercise, the student may prove that if the cone is pushed down at E' till E' is brought to the surface of the water,

$$2s\cos^{\frac{5}{2}}(\theta-a)(\sin\theta\cos a+4\cos\theta\sin a)$$
$$=\cos^{\frac{3}{2}}(\theta+a)(\sin 2\theta+4\sin 2a).$$

Similarly, if fig. 46, p. 196, represents a triangular log, touching the bottom of the water at O, with $L_2L_2'K$ as the water line, we shall find

$$\lambda^3=s\sec\theta\sec^2 a\{\cos(\theta+a)\cos(\theta-a)\}^2;$$

and so on.

If the log is pivoted about a horizontal axis through G, then the inclined position of equilibrium will be stable if B_2G, and therefore also F_2D is vertical, or $KF_2=h\sin\theta$.

If the axis lies in the surface of the water,

$$\lambda h=OK=\tfrac{2}{3}h\cos\theta,\ \lambda=\tfrac{2}{3}\cos\theta;$$

and therefore (§ 133)

$$\tfrac{2}{3}\cos^2\theta=\cos(\theta+a)\cos(\theta-a)=\cos^2\theta-\sin^2 a,$$

or

$$\cos^2\theta=3\sin^2 a.$$

Similarly for a cone

$$\cos\theta=2\sin a.$$

But if the angle θ given by this equation is not real, the stable position of equilibrium is vertical.

So also for the other bodies, if supported or submerged by the tension of a rope, or if resting on the bottom, as in the case of a ship with a rounded bottom, ashore on a sandbank or convex rock.

In this case the point of application of the lost buoyancy P in the neighbourhood of the upright position of the ship must be changed from the keel K to the point Q where the vertical through the point of contact with the rock cuts the circle of inflexions in the ship of the two surfaces in contact (§ 121).

147. *The Isobath Inkstand.*

In this ingenious instrument (fig. 49) the ink is maintained at a constant level in the dipping well A by means of a semicircular or hemispherical float F, pivoted about a diameter as axis, in the reservoir of ink C.

If the S.G. of the float is exactly $\frac{1}{2}$, the S.G. of the ink being unity, it is easily seen that the float will be in equilibrium when it has turned so as to raise the surface of the ink to the level of the axis F.

For drawing FD, equally inclined as OE at an angle θ to the horizon, the part DFE' of the float, supposed homogeneous, will balance over the axis F; so that the moment of the buoyancy of EFL must balance the moment of the weight of the part DFE of the float; and these moments are equal if the S.G. of the float is $\frac{1}{2}$.

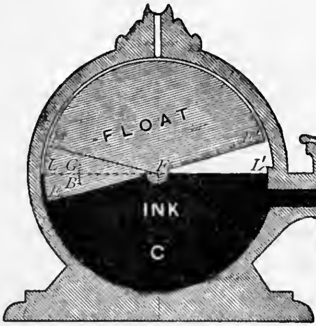


Fig. 49.

If however the S.G. s of the float is a little greater than $\frac{1}{2}$, the level of the ink will be raised slightly above F.

If the rise of level above F is x for a semicircular float of length l and radius a, the moment alx of the extra buoyancy of the ink equated to the moment of the extra weight in DFE' gives

$$Dalx \, . \, \tfrac{1}{2}a = (s - \tfrac{1}{2})Da^2l\theta \, . \, \tfrac{2}{3}a \frac{\sin\theta}{\theta},$$

or

$$x = (s - \tfrac{1}{2})\tfrac{4}{3}a \sin\theta ;$$

and if $s < \frac{1}{2}$, the level will fall slightly below the axis F; the level will also fluctuate to a slight extent.

A similar investigation will hold for a hemispherical float.

The same principle has been employed in some oil lamps, for maintaining a constant level of the oil in the wick; as in Hooke's or Milner's oil lamp, described in Young's *Lectures on Natural Philosophy,* and De Morgan's *Budget of Paradoxes,* p. 149.

Examples.

(1) A uniform rod rests in a position inclined to the vertical, with half its length immersed in water, and can turn about a point in it at a distance equal to one-sixth of the length of the rod from the extremity below the water.

Prove that the S.G. of the rod is 0·125.

(2) Three uniform rods, joined so as to form three sides of a square, have one of their free extremities attached to a hinge in the surface of water, and rests in a vertical plane with half the opposite side out of the fluid.

Prove that the S.G. of the rods is 0·775.

(3) AB and BC are two rods hinged at B, the former being heavier and the latter lighter than water. The two rods float in water with the end A freely hinged at a fixed point in the surface of the water. Prove that the S.G. of the rod AB is equal to

$$\frac{ab + 2by - 2y^2}{ab},$$

where $2a$, $2b$ are the lengths of the rods, and $2y$ the length of the immersed portion of the lighter rod.

(4) A hollow metal sphere of radius 6 inches would float freely in water with half its surface immersed. It is attached rigidly by means of a weightless rod to a tap, so that the distance of the centre of the sphere from the tap is 20 inches.

The water rises to a height of one inch above the centre of the sphere before the tap turns, and the tap is then 11 inches above the surface of the water.

Find the couple necessary to turn the tap.

(5) A rectangular parallelepiped, moveable about one edge fixed horizontally, is partly immersed in water.

If it can rest with another edge in the surface, and having one of its faces containing the fixed edge bisected by the liquid, find the S.G. of the solid; and if the section perpendicular to the fixed edge be a square, prove that the S.G. of the solid is 0·75.

Prove that, if the body rests with half its volume immersed and its faces equally inclined to the vertical, the S.G. is $\frac{3}{4} + \frac{1}{4}$ (breadth by height).

(6) A square log rests with one edge immersed in water, and partly supported along two parallel edges in the surface of the water, a distance c apart. Prove that the inclination θ of a diagonal to the vertical in a position of equilibrium is given by the equation

$$\cos 2\theta = 6s\frac{h^2}{c^2}\left(1 - \tfrac{3}{4}\sqrt{2}\frac{h}{c}\sec\theta\right),$$

where s denotes the S.G. and h the side of the square cross section of the log.

(7) A square log of density $n\rho$ floats in two liquids of densities ρ and 2ρ respectively; one edge is fixed in the surface of separation of the liquids, about which the log is capable of rotating, the whole of it being immersed.

Show that, if $2n > 3$, the angle θ which the side through the fixed axis, which lies in the upper liquid, makes with the horizon is given by

$$\tan^3\theta + (3n-4)\tan\theta + 3(n-2) = 0.$$

(8) A homogeneous log, the cross section of which is a regular hexagon *ABCDEF*, can turn freely about a horizontal edge at *A* which is in the surface of water. If in the position of equilibrium *AB* lies above the water and half of *BC* is immersed, prove that the S.G. of the log is

$$\tfrac{11}{12}.$$

(9) A regular tetrahedron has one edge fixed in the surface of water.

Show that it will be in equilibrium with the other edge inclined to the vertical at an angle $\operatorname{cosec}^{-1}3$, if the S.G. of the tetrahedron is

$$0{\cdot}294375.$$

(10) Prove that, if the bodies represented in figs. 46 and 47, floating in the upright position, are divided symmetrically by a vertical plane through *DO* into two parts, which are hinged together at *O*, the parts will not remain in contact unless

(i.) in the triangular prism or cone (fig. 46),

$$x > h\sin^2\alpha,\quad \sin\alpha < s^{\frac{1}{4}} \text{ or } s^{\frac{1}{6}},$$

(ii.) in the parabolic cylinder or paraboloid (fig. 47) $\frac{1}{3}h/l > s^{-1} - s^{-\frac{2}{3}}$, or $\frac{3}{7}h/l > s^{-\frac{3}{4}} - s^{-\frac{1}{2}}$.

(11) A spherical shell is floating in water, and is divided by a vertical plane into two halves, which are hinged at the lowest point.

Show that, if the parts remain in contact,

$$\sin\alpha(2+\cos^2\alpha)-3\alpha\cos\alpha > \pi(2+\cos\alpha)\sin^4\tfrac{1}{2}\alpha,$$

where α is the angle subtended at the centre of the sphere by any radius of the water line.

Find the condition when the sphere sinks to the bottom.

(12) Prove that half a paraboloid, cut off by a plane through its axis, just immersed in water against a rough vertical wall with its base in the surface, will be in equilibrium if the height is greater than three and a half times the semi-latus-rectum.

(13) Prove that the equilibrium of a non-homogeneous sphere of radius b, whose C.G. is at a distance h below the centre, resting partly submerged in water on the top of a fixed rough spherical surface of radius a and depth of highest point c, will be stable if its weight is less than

$$\frac{1}{4}\,\frac{c^2(3b-c)}{b(b^2-bh-ah)}$$

times the weight of an equal volume of water.

(14) On the lowest point of a rough fixed hemispherical bowl of radius c rests a prolate spheroid, of C.G. s, axis $2a$, and excentricity e, with its axis vertical. Prove that, if the bowl contains enough water to immerse half of the spheroid, the equilibrium is stable when

$$\frac{c}{a} < \frac{1-e^2}{e^2} - \frac{3(1-e^2)^2}{(16s-5)e^2}.$$

148. *The Vertical Oscillations of a Floating Body.*

When a ship, displacing V ft^3 or W tons, receives a small vertical displacement of x ft, either up or down, the change in buoyancy, tending to bring the ship back to its original position of equilibrium, is the weight of xA ft^3 of water, where A denotes the average water line area in ft^2; and this force is xA/V of the weight of the ship, or a force of

$$WxA/V \text{ tons.}$$

The ship therefore performs vertical (dipping) oscillations as if supported by a spring, of which the permanent average vertical set is V/A ft; and therefore, as proved in treatises in Dynamics, the ship will perform *simple harmonic* vertical oscillations, similar to the motion of the piston in a vertical steam engine, which will synchronize with the small oscillations of a simple pendulum of length V/A ft, or with the revolutions of a conical pendulum of the same height, V/A ft.

The *period* of the pendulum is the name now given to the time in seconds of a *double* oscillation of the simple pendulum, or to the period of revolution of the conical pendulum; this is proved in treatises on Dynamics to be

$$2\pi\sqrt{(l/g)}$$

seconds for a pendulum whose height is l ft.

This is therefore the period of a complete double vertical oscillation of the ship, if

$$l = V/A.$$

We notice that the ratio of the corresponding horizontal ordinates in fig. 42 of the curve KD of displacement in tons, and of the curve KE of tons per inch of immersion will give the length, in *inches*, of the equivalent simple pendulum for the vertical oscillations.

149. It is assumed here however that the pressure of the water is at any point that due to its *head* (§ 22), and that the changes of pressure due to the motion of the water are left out of account.

But Hydrodynamical investigations show that the motion of the water may be allowed for by adding to the inertia W of the body a certain fraction kW of the inertia W of the water displaced; provided that here again the wave motion on the surface is neglected.

For instance, in a circular cylinder floating horizontally (fig. 45) it is found that $k=1$, and in a sphere that $k=\frac{1}{2}$; so that the length of the equivalent simple pendulum is in consequence changed to $\frac{1}{2}V/A$ and $\frac{2}{3}V/A$; the preceding theory is thus not very accurate.

The hydrometer of fig. 36, p. 113, if taken as consisting of a thin stem AL and a spherical bulb B, would, if displaced through a vertical distance c and let go, perform vertical oscillations of amplitude c on each side of its position of equilibrium, which synchronize with an equivalent simple pendulum of length

$$V/a \quad \text{or} \quad a$$

on the first hydrostatical hypothesis; but when the correction is made for the inertia of the water displaced by the spherical bulb, the length of the equivalent pendulum is (§ 64)

$$a+\tfrac{1}{2}(a-l)=\tfrac{1}{2}(3a-l),$$

150. The vertical oscillations of the ship may be produced by dropping a weight of P tons, already on board; or by the vertical reciprocations of the engines.

As the weight is falling, the ship will start vertical oscillations of amplitude and equivalent pendulum length

$$\frac{P}{W}\frac{V}{A} \text{ and } \left(1-\frac{P}{W}\right)\frac{V}{A}.$$

When the weight is checked and brought to rest again on the ship, the vertical oscillations will not necessarily be stopped.

The C.G.'s, F of the water line area, G of the ship, and therefore also B of the water displaced, must lie in the same vertical line, and P also must be dropped in this vertical line; otherwise angular oscillations are generated.

151. When a body is floating in two fluids, as a ship in water and air, the additional upward buoyancy for a downward displacement x is $(w-w')Ax$, where w and w' denote the densities of the lower and upper fluids; so that the length of the equivalent pendulum is

$$\frac{W}{(w-w')A}.$$

More generally, for a body floating in a number of superincumbent fluids, in which the density increases by Δw in passing downwards across the plane of the area A of section of the body, the length of the equivalent pendulum is shown in the same manner to be

$$\frac{W}{\Sigma A\,\Delta w}, \quad \text{or} \quad \frac{W}{\int A\,dw} = \frac{\int A w\,dx}{\int A\frac{dw}{dx}dx},$$

in the notation of the Integral Calculus, when the body, like a balloon in air, is floating in stable equilibrium in a fluid arranged in horizontal strata of varying density.

Thus, for example, in the small vertical oscillations of a cone floating upright and vertex downwards, with its axis half submerged in a liquid in which the density varies as the square root of the depth, the length of the equivalent pendulum is one-seventh the height.

152. So far the ship has been supposed to be making vertical dipping oscillations on a large extent of water.

But if the surface of the water is limited, as in a dock (fig. 32, p. 77), the level of the water will be sensibly affected by the vertical motion of the ship, and the period of the oscillations will be altered.

Denote by B the area in ft² of the surface of the water in the dock, so that $B-A$ is the area of the surface when the ship is in the dock; then, as in § 91, a vertical displacement x ft with respect to the land or the bottom of the dock will change the water line on the side of the ship $xB/(B-A)$ ft, and consequently change the buoyancy by

$$\frac{wxAB}{B-A}=\frac{WxAB}{V(B-A)} \text{ tons};$$

so that the length of the equivalent simple pendulum is

$$l=\frac{V(B-A)}{AB}=V\left(\frac{1}{A}-\frac{1}{B}\right) \text{ ft};$$

reducing, as before, to V/A, when B is infinite.

Thus, (i.) for a cylinder floating upright,

$$l = \text{length of axis immersed};$$

(ii.) for a cylinder floating upright in another cylinder of m times the diameter,

$$l=(1-m^{-2}) \text{ length of axis immersed};$$

(iii.) for a sphere floating half immersed in a cylinder of twice its radius,

$$l = \text{half the radius of the sphere};$$

(iv.) for a cone of semivertical angle α, floating upright and vertex downwards in a cylinder of radius a, with a length x of the axis immersed,

$$l=\tfrac{1}{3}x\left(1-\frac{x^2}{a^2}\tan^2\alpha\right);$$

(v.) for a paraboloid, floating upright in another equal paraboloid,

$$l = \tfrac{1}{2}x\left(1 - \frac{x}{h}\right),$$

where h denotes the depth of the water, and x the immersed length of the axis of the floating paraboloid.

153. In the vertical oscillations of a ship, as represented in cross section in fig. 42, p. 168, the upright position becomes unstable for draughts of water at which the curve of metacentres MM_1 dips below the level of G.

At draught LL' the ship will lose stability as the draught diminishes, that is during the upper half of a vertical oscillation.

But at draught L_1L_1' the metacentre M_1 descends for an increase of draught, and stability is lost during the lower half of the vertical oscillation.

These considerations may help to explain the well-known liability of a sailing boat to capsize on the top of a wave (W. H. White, *On the Rolling of Sailing Ships*, Trans. I.N.A., 1881).

154. *The Angular Oscillations of a Floating Body.*

To treat the rolling oscillations of a ship in an elementary manner, it is assumed by naval architects that the ship rolls about a fixed horizontal longitudinal axis through the C.G. under the righting influence of the buoyancy W tons, acting vertically upwards through the C.B. or the metacentre; and then, denoting by WK^2 the effective moment of inertia, in tons-ft^2, of the ship about this longitudinal axis, the principles of elementary Rigid Dynamics show that the length L of the equivalent simple pendulum for small oscillations is given by

$$L = K^2/GM.$$

A similar expression gives the length of the equivalent pendulum when the ship performs small pitching oscillations about a transverse horizontal axis through G.

It can be proved by the Integral Calculus that about a horizontal axis through G perpendicular to the plane of the paper (figs. 44 to 47),

(i.) $K^2 = \frac{1}{12}(b^2 + h^2) = \frac{1}{12}(\text{diagonal})^2$,
for a rectangular prism of breadth b and height h;

(ii.) $K^2 = \frac{1}{4}(\text{radius})^2 + \frac{1}{12}(\text{height})^2$, for a cylinder;

(iii.) $K^2 = \frac{1}{24}(\text{base})^2 + \frac{1}{18}(\text{height})^2$,
for an isosceles triangular prism;

(iv.) $K^2 = \frac{3}{80}\{(\text{diameter of base})^2 + (\text{height})^2\}$, for a cone;

(v.) $K^2 = \frac{1}{20}(\text{base})^2 + \frac{12}{175}(\text{height})^2$, for a parabolic cylinder;

(vi.) $K^2 = \frac{1}{24}(\text{diameter of base})^2 + \frac{1}{18}(\text{height})^2$,
for a paraboloid;

(vii.) $K^2 = \frac{2}{5}(\text{radius})^2$, for a sphere, etc.;

(viii.) $K^2 = k^2 + \frac{1}{12}(\text{length})^2$,

for any prismatic body about an axis through G perpendicular to its length, k being the radius of gyration of the cross section through G about this axis.

The readiest way of inferring the value of K^2 for a ship is from the metacentric height GM and the period of rolling, T seconds suppose; then on the preceding hypothesis,

$$K^2 = GM(gT^2/4\pi^2) = (\tfrac{1}{2}T)^2 GM \,.\, \lambda,$$

where λ denotes the length of the seconds pendulum in ft;

$$\lambda = 39{\cdot}1932 \div 12 = 3{\cdot}2661 \text{ ft.}$$

We can now calculate the angle of heel produced by firing a broadside; for example, from guns firing 1700 lb projectiles with a velocity of 1600 f/s, from a vessel of 8000 tons displacement, whose metacentric height is 2 ft and period of rolling 20 seconds, the axes of the guns being 15 ft above the C.G. of the vessel.

The result is about 2° 33′; also

$$K^2 = 100\ GM\ .\ \lambda = 647{\cdot}22\ \text{ft}^2, \quad K = 25{\cdot}44\ \text{ft}.$$

When the period of the waves becomes commensurable with the period of rolling, there is a tendency for the oscillations to accumulate in amplitude.

155. When the ship rolls through a considerable angle, it is assumed that the dynamical stability, or work required to heel the ship through a given angle θ is the same as that required to heel the vessel slowly and steadily: in other words, the inertia of the surrounding water is neglected.

Now if a denotes the extreme inclination of the ship, and ω the angular velocity when inclined at an angle θ, the difference of dynamical stabilities at inclinations a and θ is equal to the kinetic energy of the ship, in ft-tons, $\frac{1}{2}WK^2\omega^2/g$; whence the time of rolling through any angle can be inferred by an integration, or failing that by a mechanical quadrature.

Thus, for instance, if the curve of statical stability in fig. 40, p. 161, is a curve of sines, and thence the curve of dynamical stability is a curve of versed sines; and if the angle of vanishing stability is $180/n$ degrees, the dynamical stability at an inclination θ is

$$W\ .\ GM\ .\ \text{vers}\ n\theta\ \text{ft-tons};$$

and the rolling of the ship between the extreme inclinations a will synchronize with the finite oscillations of a simple pendulum of length K^2/GM, swinging through n times this angle; and the complete solution therefore requires the *Elliptic Functions.*

156. The period of oscillation will be large, and the ship will thus be *steady* among waves, if the metacentric height GM is made small.

On the other hand the *stiffness* of the ship under sail will be improved by increasing GZ and therefore also GM.

The *stability* of a ship therefore involves the two antagonistic qualities of *steadiness* and *stiffness;* and a compromise is effected by starting with a small metacentric height GM, and making the curve of statical stability rise rapidly, as shown in fig. 40, p. 161.

A steamer will recover the upright position when rolling among waves to any inclination short of the angle of vanishing stability; but a sailing ship, heeled steadily over by the press of sail, would capsize if inclined beyond the angle of maximum righting moment.

A squall which strikes the ship during a windward roll is dangerous, because it acts through a larger angle and imparts greater energy to the ship, and is thus more likely to carry the ship beyond this critical inclination.

157. Considering that GM is small, the motion of the ship may be imitated by supposing it, or a model, to oscillate like a compound pendulum about an axis of suspension through M, and G now oscillates on the arc of a small circle; but, strictly, K^2 must be replaced by

$$K^2 + GM^2.$$

The motion of the ship can also be imitated by a cradle, rocking on the curve of buoyancy BB_2, which rolls on a horizontal plane.

A better imitation would be secured by supposing this horizontal plane smooth, so that G moves in a vertical line, and vertical oscillations come also into existence; these vertical oscillations can be allowed for by supposing the axis of suspension through M to be supported on springs; but in any case the complete solution of the oscillations of a ship leads into difficulties.

Treating the reaction of the water as acting hydrostatically in the case of the ship in § 142, aground along the keel and heeled over to a position of equilibrium, then for an additional small angular displacement θ, in which B_2N', GH' denote the perpendiculars from B_2, G on the new water plane, the additional righting moment, measured in ft³ of water multiplied into feet, due

(i.) to the wedges of immersion and emersion is

$$Ak'^2 \sin\theta,$$

where Ak'^2 denotes the moment of inertia, in ft⁴, of the plane of flotation about the axis through F';

(ii.) to the movement of B_2 is $-V_2 . B_2N . \sin\theta$;

(iii.) to the movement of G is $-V . GH . \sin\theta$.

Then $V(K^2+GK^2)$ denoting the moment of inertia in quintic feet, ft⁵, about the keel, the oscillations of the ship will synchronize with a pendulum of length

$$\frac{V(K^2+GK^2)}{Ak'^2 - V_2 . B_2N - V . GH};$$

so that the denominator must be positive for the equilibrium to be stable.

CHAPTER VI.

EQUILIBRIUM OF LIQUIDS IN A BENT TUBE. THE THERMOMETER, BAROMETER, AND SIPHON.

158. It has been proved in § 25 that "the common surface of two liquids which do not mix is a horizontal plane"; and in § 24 that "the separate parts of the free surface of a homogeneous liquid filling a number of communicating vessels all form part of one horizontal plane."

This last theorem no longer holds if two or more liquids of different densities, which do not mix, are poured into the communicating vessels.

To illustrate this difference, take a bent tube AB (fig. 50) and pour into the branches two different liquids, of densities σ and ρ, say mercury and water, or oil and water, so that the upper free surfaces stand at H and K and the plane surface of separation at the level AB.

Then if p denotes the pressure of the atmosphere, and h, k the vertical heights of H, K above AB, the pressure at A and B will be (§ 21) $p+\sigma h$ and $p+\rho k$; and these pressures being equal (§ 19)

$$\sigma h=\rho k, \quad \text{or} \quad h/k=\rho/\sigma;$$

which proves the

THEOREM.—"The vertical heights of the columns of two liquids above their common surface are inversely as the densities."

159. Suppose, for example, that the waters of the Mediterranean and the Dead Sea were in communication by a subterranean channel, reaching to a depth h below the surface of the Dead Sea, and k below the surface of the Mediterranean.

Then, according to the data of §§ 44, 76, if the waters of the two seas balance in this channel,

$$\frac{k}{h}=\frac{\sigma}{\rho}=\frac{1{\cdot}250}{1{\cdot}025}=\frac{50}{41},$$

$$k-h=1300;$$

whence $h=7200,\quad k=8500,\quad$ ft.

Fig. 50.

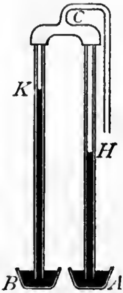


Fig. 51.

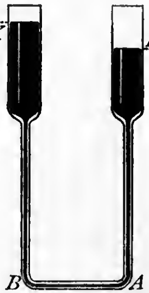


Fig. 52.

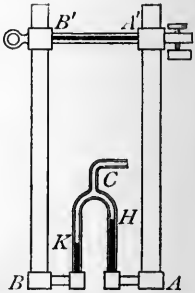


Fig. 53.

160. For the Stability of the Equilibrium of the liquids in the bent tube, it is requisite that the denser liquid should occupy the lower bend of the tube; otherwise the lighter liquid would be underneath the heavier liquid at A, and the equilibrium would be unstable, as shown in § 26.

The liquid in the bend AB may be replaced by any other liquid and the equilibrium will still subsist; but it will be unstable if the density of this liquid is less than the densities σ and ρ of the two other liquids.

Suppose then that initially a certain amount of liquid of the greater density σ is resting in a U-shaped tube, reaching to the same level in each branch.

If the lighter liquid, of density ρ, is now poured gradually into one of the branches, the equilibrium will be established in the bent tube when the heights h and k of the upper surfaces above the common surface are inversely as the densities σ and ρ.

But after a certain amount of the lighter liquid has been poured in, the denser liquid will be driven out of the bend; and now the lighter liquid where it is under the denser, will be in unstable equilibrium, and ultimately will bubble up to the upper surface, where it will form a separate column.

Provided the branches are vertical, or straight and of uniform section, the equilibrium in the other branch will be unaffected; otherwise a rearrangement takes place.

161. The equilibrium of the liquids as a whole is stable, supposing that a membrane or piston at the surface of separation prevents any instability of this surface.

For if the liquid column in the bent tube is displaced, so that H descends to H' and K rises to K', then taking for simplicity the tube as of uniform bore (fig. 50),

$$HH' = AA' = BB' = KK' = x, \text{ suppose;}$$

and now if further motion is prevented by a stop valve s.v. in the bend of the tube, and the liquid in the bend AB is supposed of density ρ', the pressure on the side B of the valve exceeds the pressure on the side A by

$$\rho k + \rho'(c+x) - \sigma h - \rho'(c-x) = 2\rho' x,$$

where c denotes the height of AB above the stop valve; and therefore the liquid column tends to return to its original position when the valve is opened.

The column will now oscillate; and, as in § 148, the force on the column tending to bring it back to its position of equilibrium when displaced through a distance x being $2\rho'\omega x$, where ω denotes the uniform cross section of the tube, the column will oscillate like a pendulum of length

$$l = \frac{W}{2\rho'\omega} = \frac{\rho k + \rho' a + \sigma h}{2\rho'},$$

where W denotes the weight of the liquids, and a the length of the filament in the bend AB.

Thus if $\rho = \rho' = \sigma$, $l = \frac{1}{2}(a+b)$.

(*Principia*, lib. II., Prop. XLIV.)

Suppose, however, that the branches of the tube are not vertical, but curved, so that the inclinations to the vertical at the points A, B, H, K are a, β, θ, ϕ.

Then to push the column through a small distance x from its position of equilibrium by a piston at H will require a thrust reaching from zero to

$$\rho\omega(x\cos\phi + k - x\cos\beta) + \rho'\omega(x\cos\beta + x\cos a)$$
$$- \sigma\omega(x\cos a + h - x\cos\theta)$$
$$= \{\rho\cos\phi + (\rho' - \rho)\cos\beta + (\rho' - \sigma)\cos a + \sigma\cos\theta\}\omega x,$$

so that, if the piston is removed, the column will oscillate like a pendulum of length

$$l = \frac{W/\omega}{\rho\cos\phi + (\rho' - \rho)\cos\beta + (\rho' - \sigma)\cos a + \sigma\cos\theta};$$

reducing for a homogeneous filament, of length c, to

$$l = \frac{c}{\cos\phi + \cos\theta}.$$

Similarly for the oscillations of any number of liquids in a uniform bent tube; but if the bore of the tube changes, as in a marine barometer, the problem is complicated by the variations of velocity in the tube.

162. *Hare's Hydrometer.*

This is an application of the principle of the Theorem of § 158; it consists of two vertical glass tubes AH and BK, dipping into vessels at A and B, containing two liquids whose densities, σ and ρ, are to be compared (fig. 51). The upper ends of the tubes are cemented into a receptacle C, from which the air can be partially exhausted by an air pump or other means; and the liquids now rise in AH and BK to heights h and k above their level at A and B, which heights are inversely as the densities, or such that

$$\sigma h = \rho k, \text{ or } \rho/\sigma = h/k.$$

An apparent *tension* draws up the liquid columns in the tubes; for this reason any small pressure below the atmospheric pressure is sometimes called a *tension*, because the difference between this pressure and that of the atmosphere is a negative pressure, or a tension thus it is usual to speak of the *tension* of aqueous and other vapours; but the word tension is sometimes improperly applied to very high pressures, such as those due to the gases of fired gunpowder.

Examples.

(1) Two equal vertical cylinders of height l stand side by side and there is free communication between their bases. Quantities of two liquids of densities ρ_1, ρ_3 which would fill lengths a and c respectively of the cylinder, are poured in, and rest in stable equilibrium, each liquid being continuous.

A given quantity of a liquid of density ρ_2, intermediate between ρ_1 and ρ_3, is poured slowly into one of the cylinders.

Find the position of equilibrium, noticing the different cases which may occur; and show that if the liquid reaches the top of both cylinders at the same time, either

$$(\rho_1-\rho_2)(2l-a-c)=(\rho_1-\rho_3)c, \text{ or } (\rho_1-\rho_2)a=(\rho_2-\rho_3)c.$$

(2) If two equal vertical cylinders in communication at the base are partly filled with mercury, and closed by pistons which are allowed to descend slowly, prove that air will not pass from one cylinder to the other if the difference of weights of the pistons is less than the weight of mercury; and find the position of equilibrium.

(3) A vessel, in the form of a cylinder with its axis vertical, is partially filled with water. The lower part of the vessel communicates with a reservoir of infinite extent; and a body, in the form of a cylinder, floats with its axis vertical, in the vessel. Fluid of less density than water is now slowly poured into the vessel.

Find the quantity which must be poured in before the floating cylinder begins to move—(i.) when the density of the second fluid is greater, and (ii.) less than that of the cylinder.

(4) A small uniform tube is bent into the form of a circle whose plane is vertical, and equal volumes of two fluids whose densities are ρ, σ, fill half the tube; show that the radius passing through the common surface makes with the vertical an angle

$$\tan^{-1}\frac{\rho-\sigma}{\rho+\sigma};$$

and find the period of a small oscillation.

(5) A circular tube contains columns of two liquids whose densities are ρ, ρ', the columns subtending angles 2θ, $2\theta'$ at the centre of the circle.

If α be the angle which the portion of the tube intercepted between its lowest point and the common surface of the liquids subtends at the centre of the tube, prove that

$$\rho \sin\theta \sin(\theta \pm \alpha) = \rho' \sin\theta' \sin(\theta' \mp \alpha);$$

and find the period of a small oscillation.

(6) A tube in the form of an equilateral triangle is filled with equal volumes of three fluids which do not mix, and whose densities are in Arithmetical Progression; prove that in equilibrium in a vertical plane the straight line joining the ends of the fluid of mean density will be vertical.

Generally, if a fine tube of uniform bore in the form of a closed regular polygon of n sides is filled with equal volumes of n liquids, and held in any vertical position, the lines joining the surfaces of separation of the liquids will form another polygon, the sides of which have fixed directions; and if the C.G. of the liquids is at the centre of the polygon, the liquids will rest in any position. (Wolstenholme, *Proc. London Math. Society*, VI.)

Find the period of a small oscillation.

(7) A fine tube ABC, of uniform bore, having the parts AB, BC straight and inclined at an angle 2γ to one another, is held in a vertical plane, and contains several liquids of different densities.

If the tube is turned about the point B in its own plane, and if α, β are the inclinations to the vertical of the straight line bisecting the angle

ABC in the two positions, prove that the weight of liquid which has passed from one branch to the other bears to the weight of the whole the ratio

$$\tfrac{1}{2}(\tan\alpha - \tan\beta)\tan\gamma \text{ to } 1.$$

(8) A fine tube bent into the form of an ellipse is held with its plane vertical, and is filled with n liquids whose densities are $\rho_1, \rho_2, \ldots, \rho_n$ taken in order round the elliptic tube.

If $r_1, r_2, \ldots, r_n$ be the distances of the points of separation from either focus, prove that

$$r_1(\rho_1-\rho_2)+r_2(\rho_2-\rho_3)+\ldots+r_n(\rho_n-\rho_1)=0.$$

State the corresponding theorem, if the fluids do not fill the tube; and find the period of a small oscillation.

(9) A cycloidal tube contains equal weights of two liquids, occupying lengths a and b; if it be placed with its axis vertical, prove that the heights of the free surfaces of the fluids above the vertex of the tube are as

$$(3a+b)^2 \text{ to } (3b+a)^2.$$

(10) An uniform tube is bent into the form of a cycloid, and held with its vertex downwards and axis vertical. It is then partly filled with mercury, S.G. 13·5, and chloroform, S.G. 1·5. Show that if the volume of the chloroform be three times that of the mercury, their common surface will be at the lowest point of the tube.

(11) Prove that the finite oscillations of a filament of liquid in a circular or cycloidal tube, or a tube of any shape, can be compared with those of a particle at the middle point of the filament.

163. *Dilatation and Coefficients of Expansion.*

The usual formulas and approximations employed in the measurement of dilatation may be explained at this stage.

By the application of heat, liquids and solid bodies expand in general, by a fraction which is sensibly proportional to the rise of temperature.

If a homogeneous solid body expands equally in all directions, so as to remain always similar to itself, and if corresponding lengths, areas, and volumes become changed from l, A, and V to $l+\Delta l$, $A+\Delta A$, and $V+\Delta V$, then $\Delta l/l$, $\Delta A/A$, and $\Delta V/V$ are called respectively the *linear extension*, the *areal expansion*, and the *cubical expansion* or *dilatation*.

Since the body is supposed to remain similar to itself, therefore

$$1+\frac{\Delta A}{A}=\left(1+\frac{\Delta l}{l}\right)^2, \quad 1+\frac{\Delta V}{V}=\left(1+\frac{\Delta l}{l}\right)^3;$$

but considering that $\Delta l/l$ is in general so small that its square is insensible to the number of decimals to which $\Delta l/l$ is given in the formulas, we may put

$$\frac{\Delta A}{A}=2\frac{\Delta l}{l}, \quad \frac{\Delta V}{V}=3\frac{\Delta l}{l};$$

so that the areal and cubical expansions are respectively twice and thrice the linear extension.

As it is found experimentally that the linear, areal, and cubical expansions are sensibly proportional to the increase of temperature τ, therefore $\Delta l/l\tau$, $\Delta A/A\tau$, and $\Delta V/V\tau$ are sensibly constant; denoting them by λ, μ, and c, and calling them the *coefficients* of *linear extension*, of *areal*, and of *cubical expansion*, then

$$\mu=2\lambda, \qquad c=3\lambda.$$

A liquid has no permanence of shape, so that its coefficients of cubical expansion c alone can be said to exist.

The lengths, areas, and volumes at a temperature τ degrees higher can be calculated from the formulas

$$l_\tau = l(1+\lambda\tau), \quad A_\tau = A(1+\mu\tau), \quad V_\tau = V(1+c\tau).$$

In the notation of the Differential Calculus

$$\lambda = \frac{dl}{ld\tau}, \qquad \mu = \frac{dA}{Ad\tau}, \qquad c = \frac{dV}{Vd\tau};$$

and when these coefficients vary sensibly,

$$\Delta l = l\int_0 \lambda d\tau, \qquad \Delta A = A\int_0 \mu d\tau, \qquad \Delta V = V\int_0 c d\tau.$$

Thus, for mercury, Regnault found

$$c = a + b\tau, \quad \text{where} \quad a = \log^{-1}\bar{4}\cdot 2529, \quad b = \log^{-1}\bar{8}\cdot 7030.$$

Similarly the S.V.'s v_τ and v of a substance at the two temperatures are connected by the formula

$$v_\tau = v(1+c\tau),$$

and therefore the densities ρ_τ and ρ by

$$\rho_\tau = \rho/(1+c\tau).$$

But as c is so small that the square of $c\tau$ is insensible, this may be written

$$\rho_\tau = \rho(1-c\tau).$$

According to Mendeleef this last formula may be taken as approximately correct for a liquid over a very large range of temperature; and he writes the formula

$$\rho = k(a\theta_1 - \theta),$$

where θ denotes the *absolute temperature* when the density is ρ (§ 197), and θ_1 denotes the absolute *critical* temperature, defined by Andrews as the highest temperature at which the substance can be liquefied from the gaseous state by pressure (§ 220); while a is a constant which it is found by experiment can be put equal to 2.

(P. T. Main, *British Association Report*, 1888, p. 505.)

164. *The Absolute Dilatation of Mercury.*

This Theorem of § 158 was employed by Dulong and Petit for the determination of the expansion of mercury with rise of temperature.

A glass vessel $HABK$ (fig. 52, p. 234) was made, consisting of a bent capillary tube and two enlarged ends at H and K, and filled with mercury; and while one branch AH was kept at the freezing point by ice and water in a surrounding vessel, the other branch BK was surrounded by oil contained in a similar vessel, which oil was heated to the desired temperature τ.

The vertical heights h and k of the free surfaces of the mercury at H and K above the horizontal filament of mercury in AB was measured by a Cathetometer (an instrument invented for this purpose, consisting of a vertical graduated metal column, supported on a tripod stand with levelling screws, on which a telescope could be moved and clamped); and now it follows from the theorem of § 158 that the ratio of the density σ_τ of the mercury in BK to the density σ in AH is inversely as k to h, or directly as h to k.

The S.V.'s v_τ and v are therefore as k to h; and therefore, e denoting the *cubical expansion* of the mercury,

$$e = \frac{v_\tau}{v_0} - 1 = \frac{k-h}{h}.$$

Putting $e = c\tau$, so that c denotes the *coefficient of cubical expansion* of mercury,

$$c = (k-h)/h\tau.$$

Dulong and Petit found that, between 0° and 100° C.,

$$c = \tfrac{1}{5550} = 0{\cdot}00018018,$$

as against Regnault's mean value

$$c = a + 50b = 0{\cdot}0001815.$$

A modified form of this apparatus, shown in fig. 53, p. 234, was employed by Regnault; in this case two vertical tubes AA', BB', filled with mercury, were connected by a fine tube $A'B'$ near the upper ends, which tube could be accurately levelled, and by an interrupted tube AB at the lower ends, the tube AB rising in the middle into a bent glass tube HCK; at C a pipe communicated with an air vessel, by the pressure in which the level of the mercury at H and K could be regulated.

Denoting by h and k the vertical heights of H and K above A and B, and by h' and k' the vertical heights AA' and BB', as measured by the Cathetometer; by σ the density of the mercury in AA', maintained at the freezing point; by σ_τ the density in BB', maintained at temperature τ; and by σ_t the density in the tube HCK at the temperature t; then the pressure in the horizontal line $A'B'$ being the same, and the pressures at H and K being equal, therefore

$$\sigma h' - \sigma_t h = \sigma_\tau k' - \sigma_t k.$$

The tube $AHCKB$ is easily maintained at the freezing point, so that $\sigma_t = \sigma$; and now

$$1 + e = \frac{v_\tau}{v} = \frac{\sigma}{\sigma_\tau} = \frac{k'}{h' - h + k}.$$

In another modification of the apparatus made by Regnault, the lower tube AB was made straight and accurately levelled, and the bent glass tube HCK was inserted in the upper tube $A'B'$; and now,

$$\sigma h' + \sigma_t h = \sigma_\tau k' + \sigma_t k;$$

or, with $\sigma_t = \sigma$,

$$1 + e = \frac{v_\tau}{v} = \frac{\sigma}{\sigma_\tau} = \frac{k'}{h' + h - k}.$$

165. The cubical expansion of mercury being thus known, the expansion of any other solid substance, say glass, can be determined by means of the *Weight Thermometer* (fig. 54, p. 247).

This consists of a glass vessel, with a thin neck, dipping into a saucer of mercury.

By alternate heating and cooling the vessel can be filled with mercury; and now if it contains a weight P g of mercury at the freezing point, and a weight p g of mercury is found to flow out into the saucer at the temperature τ, and if V, V_τ cm³ denote the volume of the vessel at these temperatures,

$$P = \sigma V, \qquad P - p = \sigma_\tau V_\tau.$$

If e denotes the cubical expansion of the mercury, E of the glass, and if ϵ denotes the *apparent* cubical expansion of the mercury, then

$$1 + \epsilon = \frac{P}{P-p} = \frac{\sigma V}{\sigma_\tau V_\tau} = \frac{1+e}{1+E},$$

or

$$1 + e = (1+\epsilon)(1+E) = 1 + \epsilon + E,$$

$$E = e - \epsilon,$$

approximately, neglecting the products ϵE; so that the *apparent* cubical expansion of the mercury is the difference between the absolute cubical expansions of the mercury and the glass; whence E is known, if e and ϵ are observed.

166. The cubical expansion of any other metal or solid can be obtained by placing a piece of it, of known weight W g, and density ρ or volume V cm³ at the freezing point, inside such a weight thermometer, and observing the same phenomena as it is heated (Jamin, *Cours de Physique*, t. II.).

Now if M g of mercury, of density σ and volume U cm^3 at the freezing point, is required to fill up the vacant space in the weight thermometer, and if K denotes the expansion of the metal and m g the weight of mercury which issues when the temperature is raised to τ; then the volume of the glass cavity being always equal to the volumes of the mercury and the metal inside,

$$(V+U)(1+E)=V(1+K)+\frac{M-m}{M}U(1+e),$$

whence K is determined, knowing E and e.

167. *The Thermometer.*

The Weight Thermometer, just described, is so called because it is employed in experiments in the laboratory for determining the temperature.

The weight of mercury expelled when the temperature is raised from the freezing to the boiling point is observed; and now the percentage of this weight which is expelled at any other temperature is taken as the measure of that temperature, in degrees Centigrade.

But for general purposes the ordinary thermometer is more convenient; this consists of a glass bulb B and a uniform stem S, filled with mercury and its vapour, and hermetically sealed (fig. 55).

The freezing and boiling points are marked on the glass, the first when the thermometer is placed in ice and water, and the second when it is held in steam from water boiling at a standard barometric height of about 30 ins or 76 cm.

In the Centigrade or Celsius scale, now universally employed in scientific work, these points are marked 0 and 100, and the stem between these points is divided into 100 equal degrees.

But in Fahrenheit's scale these points are marked 32 and 212, the degrees being thus one-180th of this distance; while in Reaumur's scale (still occasionally used in the excise and the kitchen) the points are marked 0 and 80.

Fig. 54.

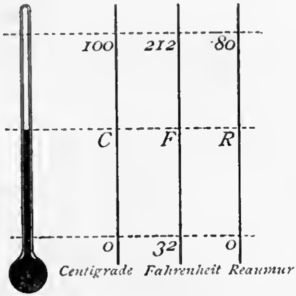


Fig. 55.

If C, F, R denote the reading of the same temperature in degrees of the thermometer on the Centigrade, Fahrenheit, and Reaumur scale,

$$\frac{C}{100} = \frac{F - 32}{180} = \frac{R}{80};$$

so that five degrees Centigrade, nine degrees Fahrenheit, and four degrees Reaumur are equivalent.

Thus $4\,C = 39{\cdot}2\,F,$

the temperature of water of the maximum density;

$$15\,C = 59\,F = 12\,R, \quad 62\,F = 16\tfrac{2}{3}\,C = 13\tfrac{1}{3}\,R,$$

ordinary normal temperatures;

$$-273\,C = -459{\cdot}4\,F, \text{ say } -460\,F,$$

the absolute zero of temperature; and so on.

168. Suppose that at the freezing point the mercury in a centigrade thermometer occupies a volume V cm³, and that an additional U cm³ would fill the stem at this temperature from the freezing to the boiling point.

Denoting by c and C the coefficients of cubical expansion of mercury and glass per degree centigrade, then at the boiling point the volume of mercury becomes

$$V(1+100c), \text{ cm}^3,$$

and it occupies a volume

$$(V+U)(1+100C)$$

of the cavity in the glass; so that

$$V(1+100c)=(V+U)(1+100C),$$

$$\frac{V+U}{V}=\frac{1+100c}{1+100C}=1+100(c-C),$$

since the product of c and C is insensible; or

$$U/V=100(c-C).$$

Dulong and Petit found that

$$c=\tfrac{1}{5550}=0{\cdot}00018018, \text{ per degree Centigrade,}$$

$$(\text{or } \tfrac{1}{9990}=0{\cdot}0001001, \text{ per degree Fahrenheit}),$$

while $C=\frac{1}{7}c$ on the average; therefore

$$V/U=64{\cdot}75.$$

169. The sensibility of the thermometer, as measured by the length of a degree on the scale, is increased as in the Hydrometer (§ 64) by making the bore of the stem small; but now it is not always possible to keep the bulb and stem at the same temperature.

Suppose then that the bulb is placed in a medium of which the temperature x is required; and that the reading of the stem is τ, when the mercury in the part of the stem which is outside this medium, extending over n degrees, is maintained at a temperature t'.

The apparent temperature τ must be increased by the elongation which n degrees would take on raising its temperature from t' to x, and therefore,

$$x=\tau+n(c-C)(x-t'),$$

whence x is found.

170. We have tacitly assumed that the ratio of c to C is constant at all temperatures, which is very nearly true; and so far equal increments of temperature are defined by the thermometer as those in which the mercury in the glass expands by equal amounts.

But when we compare very carefully thermometers filled with alcohol, water, or air, slight discrepancies arise, which require to be explained and corrected from Thermodynamical principles.

Thus it is found experimentally that water dilates in an abnormal manner, and has a maximum density at 4 C, and that the volume V_τ cm³ at any other temperature τ C of V_4 cm³ at 4 C is given very accurately by the formula

$$V_\tau = V_4\{1+a(t-4)^2\}, \quad a=0{\cdot}000008,$$

so that the coefficient of cubical expansion of water is

$$2a(t-4).$$

The cubical expansion of glass being given, as before, by the formula

$$V_\tau = V_0(1+C\tau),$$

the glass and water expand at the same rate when

$$2a(\tau-4)=C, \quad \tau=4+\tfrac{1}{2}C/a, \quad \text{and} \quad C/a=5, \text{ about.}$$

A thermometer made up of glass filled with water will thus be stationary at a temperature of about 6·5 C.

Suppose this thermometer is graduated between the freezing and boiling points in the same way as the mercury thermometer; if x, τ denote corresponding readings of the water and mercury thermometer, and if we put $C=ma$, then as before in § 168,

$$V_\tau = V\frac{1+a(\tau-4)^2}{1+16a} = \left(V+\frac{xU}{100}\right)(1+ma\tau),$$

$$V_{100} = V\,\frac{1+9216a}{1+16a} = (V+U)(1+100ma);$$

so that $$x = \frac{(1+100ma)\tau(\tau-8-m-16ma)}{(92-m-16ma)(1+ma\tau)},$$

$$\tau - x = \frac{a(1+8m+m^2+16m^2a)\tau(100-\tau)}{(92-m-16ma)(1+ma\tau)};$$

and the error $\tau - x$ is a maximum, when

$$ma\tau^2 + 2\tau = 100.$$

Taking $m=5$, we see that x is negative from 0 to a little over 13 C.

Alcohol is used in thermometers required for registering low temperatures; but if an alcohol thermometer is graduated by comparison with a mercury thermometer, its degrees will not be of equal length, but will become longer in ascending the scale.

Examples.

(1) A hollow copper spherical shell is floating just immersed in water at 0°C.

Prove that as the temperature rises the shell will again be just immersed at a temperature $8+3k/a$, k being the coefficient of linear expansion of copper, and the law of density of water being

$$\rho_t = \rho_4\{1 - a(t-4)^2\}.$$

Prove also that the shell will be highest out of the water at a temperature half-way between those for which it is just immersed.

(2) A thermometer is plunged into a liquid, and its rate of rising or falling is proportional to the difference of temperature between it and the liquid: show that if x, y, z be three readings of the thermometer at equal intervals of time, the true temperature of the liquid is

$$\frac{xz-y^2}{x+z-2y}.$$

(3) A compensating measuring instrument is made by two parallel rods AB, CD of materials whose coefficients of expansion are as e to e', their extremities being joined by two rods PAC, QBD of equal length, and of the same material, so that PQ is parallel to AB or CD: show that in order that PQ may be constant,

$$PC : PA :: e' . CD : e . AB.$$

(4) A homogeneous solid floats in liquid; if, when the temperature of both is raised by the same amount, the depth of the lowest point of the solid remains unaltered, then the coefficients of cubical expansion of the solid and liquid are in the ratio of

$$3W \text{ to } 3W - W',$$

where W is the weight of the solid, and W' of a cylindrical volume of liquid of base equal to the area of the plane of flotation and height the constant depth of the lowest point.

171. *The Barometer.*

Torricelli found (1643) that if a glass tube, something over 30 inches or 76 cm long, was sealed at one end and filled with mercury (nowadays freed from air by boiling in a vacuum) and then carefully inverted in a vessel of mercury so that no air enters the tube, the mercury inside the tube will not wholly subside but only in part, till it rests at an altitude of about 30 inches or 76 cm above the level of the mercury in the vessel; being supported in this position by the pressure of the air, and leaving an empty space above called the Torricellian vacuum (fig. 56); this arrangement is called a *Barometer*, and it serves to measure the atmospheric pressure.

The barometer may also be constructed, though not so easily, by filling with mercury the bent tube closed at O of fig. 57 when it is in a nearly horizontal position; and now when placed upright, the difference of level of the mercury in the two branches is as before about 30 ins or 76 cm; this form is called a *siphon barometer*.

Fig. 56.

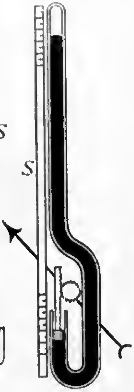


Fig. 57.

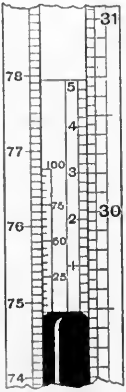


Fig. 58.

Fig. 59.

Fig. 60.

The column AH of mercury in this bent tube now balances a column of air, of which the containing tube is absent; so that, as in fig. 50, the end O being closed and the space OH free of air, the column AH of mercury, of height h and density σ, balances a column of air, of pressure p and density ρ, reaching if homogeneous to a height k, such that

$$p = k\rho = \sigma h;$$

and h is then called the *barometric height* or *height of the mercury barometer*, and k is called the *height of the homogeneous atmosphere* (ἀτμοσφαῖρα = vapour sphere).

The pressure due to the head h of mercury or k of air is thus the same (§ 22).

172. The gravitation measure of force is here employed (§ 8); so that with British units, the pound and the foot or inch, h and k are given in feet or inches, p in lb/ft² or lb/in², and σ and ρ in lb/ft³ or lb/in³.

With Metric units, the kilogramme and the metre, or the gramme and the centimetre, h and k are given in m or cm, σ and ρ in kg/m³ or g/cm³, and p in kg/m² or kg/cm² or g/cm².

Thus, from the numerical results of §§ 8, 22, we may take as average results in British units, at a standard temperature of 62 F,

$$p = 14\tfrac{2}{3} \text{ lb/in}^2 = 2112 \text{ lb/ft}^2,$$
$$h = 30 \text{ ins} = 2{\cdot}5 \text{ ft}, \quad k = 27800 \text{ ft};$$

so that k is about the height of Mount Everest, the highest mountain on the Earth.

$$\sigma = 13{\cdot}6 \times 62{\cdot}4 = 848{\cdot}64 \text{ lb/ft}^3,$$
$$\rho = 1 \div 13 \qquad = 0{\cdot}0769 \text{ lb/ft}^3.$$

With Metric units, at about 15 C,

$$p = 1 \text{ kg/cm}^2 = 1000 \text{ g/cm}^2,$$
$$h = 76 \text{ cm}, \quad k = 8400 \text{ m},$$
$$\sigma = 13{\cdot}6, \qquad \rho = 0{\cdot}00123 \text{ g/cm}^3.$$

173. Treating the air as homogeneous for small variations of height, an ascent of x m should make the column of mercury fall y cm, such that, near the ground,

$$\frac{x}{y} = \frac{\sigma}{100\rho} = \frac{k}{h} = \frac{8400}{76} \approx 110;$$

so that the mercury should fall about 1 mm for every 11 m ascended, or 1 inch for every 900 ft.

This was first verified by Pascal in 1648, on the tower of Saint Jacques in Paris, 30 m high, on the top of which the barometer was found to fall about 3 mm; and with $\sigma = 13{\cdot}6$, this makes $1/\rho = 800$, about.

The fall in the barometer is smaller as the air is more rarefied; hence the apparent increase of pressure with the height in a gas main, the pressure in the lighter gas diminishing more slowly than the pressure of the air.

174. The Barometer in its simplest form for scientific purposes consists of a vertical tube of glass and a cistern (fig. 56) both sufficiently large to eliminate the capillarity effect and to enable the mercury to move quickly in response to changes of atmospheric pressure.

The vertical height h between the two surfaces of the mercury at H and in the cistern at AB is read by the Cathetometer, or else on a scale and vernier (§ 177) parallel to the tube, the scale being screwed down until the point C at its lower end just touches the mercury in the cistern, when the point and its image by reflexion in the surface of the mercury will be seen in coincidence.

The cistern screw, required for filling up the vacant spaces of the barometer with the mercury during transport, is sometimes turned to bring the level AB in the cistern to the point C; but this method is not considered desirable, as the mercury takes some time to resettle.

Any irregularity in the shape of the cistern or tube does not now affect the reading, nor does the presence of dross or floating bodies on the mercury, provided that the free surface can be observed.

But it is important that the tube and the scale should be vertical; for if inclined at an angle a to the vertical, its reading would have to be reduced by the factor $\cos a$.

In Sir Samuel Morland's *diagonal barometer* (1670) the upper end of the tube is purposely bent over from the vertical at an angle a, in a straight or spiral form, so as to multiply the travel of the mercury by $\sec a$.

175. Corrections for temperature must, however, be applied, one for the cubical expansion of the mercury, and the other for the linear expansion of the scale, to reduce the barometric height to the standard temperature of 0 C or 32 F.

The linear expansion of h the barometric height is e, the cubical expansion of mercury; for if h_τ denotes the height of the barometer and σ_τ the density of the mercury at the temperature τ, and h, σ the height and density at 0 C for the same pressure p, then

$$p = \sigma_\tau h_\tau = \sigma h,$$

or

$$\frac{h_\tau}{h} = \frac{\sigma}{\sigma_\tau} = \frac{v_\tau}{v} = 1 + e.$$

Thus if the mercury were to expand to double its volume, the density would be halved, and the barometer would stand at double the height; and so on, in proportion.

So also the coefficient of expansion of k, the height of the homogeneous atmosphere, is the coefficient of cubical expansion of air at constant pressure; and this is found to be about $\frac{1}{273}$ per degree Centigrade, so that the height is reduced from 8400 m at 15 C to

$$8400 \times 273 \div 288 \approx 8000 \text{ m at } 0 \text{ C},$$

or about 26200 ft.

If b denotes the coefficient of linear expansion of the scale, and if the scale is graduated in centimetres and millimetres at 0 C, then if l denotes the reading on the scale at a temperature τ,

$$l(1 + b\tau) = h_\tau = h(1 + c\tau),$$

or

$$l - h = (c - b)\frac{l\tau}{1 + c\tau} \approx (c - b)l\tau.$$

The correction $l-h$ is to be subtracted from the apparent reading l to obtain the true corrected barometric height h; this correction, connecting l and τ, is given graphically by hyperbolic arcs of equal correction in a diagram. (*Annuaire du bureau des longitudes.*)

But if the scale is, as in British barometers, graduated in inches and decimals, which are true inches at a standard temperature T° F; then at any other temperature t° F, if the reading is l,

$$l\{1+\beta(t-T)\}=h_\tau=h\{1+\gamma(t-32)\},$$

β and γ denoting the values of the coefficients b and c when reduced from the Centigrade to the Fahrenheit scale, so that

$$\beta=\tfrac{5}{9}b, \qquad \gamma=\tfrac{5}{9}c.$$

Then, approximately,

$$h=l\{1+\beta(t-T)-\gamma(t-32)\}$$

$$l-h=l(\gamma-\beta)\left(t-\frac{32\gamma-\beta T}{\gamma-\beta}\right).$$

The correction therefore vanishes when the temperature

$$t=\frac{32\gamma-\beta T}{\gamma-\beta};$$

it is subtractive for higher and additive for lower temperatures.

Taking $T=62$, and the coefficients of cubical expansion of mercury c or γ, and of linear expansion of brass b or β, as given by

$$c=0{\cdot}00018018, \quad \gamma=0{\cdot}0001001,$$
$$b=0{\cdot}00001878, \quad \beta=0{\cdot}0000104,$$
$$c/b=\gamma/\beta=9{\cdot}6,$$

then the correction vanishes when $t=28{\cdot}3$ F, and is subtractive for higher temperatures.

176. *Temperature Correction in Standards of Length.* No temperature correction is required in Standards of Weight, but an accurate scale of Length has engraved upon it the temperature at which its indications are correct.

Thus the British Standard Yard is correct at 62° F, and the French Mètre des Archives at 0° C; and if the brass scale of the barometer is engraved with true inches at 62° F or $16\frac{2}{3}$° C, and with true millimetres at 0° C, then at the higher temperature 62 F a scale millimetre division has stretched by

$$(62-32)\beta = 0{\cdot}000312 \text{ mm},$$

$$= 0{\cdot}312\,\mu \text{ (microns) or } 312\,\mu\mu \text{ (micromillimetres).}$$

In scientific work the relation

1 m = 39·370432 (39·37) ins, 1 in = 2·539977 (2·54) cm,

is generally employed; but if, in accordance with the Act of Parliament, 1878 (§ 8), we take

1 metre = 39·37079 ins, 1 inch = 2·53995 cm,

then, at 62° F,

the scale metre = 39·37079 ÷ 1·000312 = 39·35851 ins;

and, as the scale elongates or contracts uniformly with the temperature, this will be the invariable relation connecting the scale metres and inches; thus

30 scale ins = 762·22 scale mm.

For instance, in fig. 58, a simultaneous barometric reading in the two graduations gave

29·482 ins, and 748·70 mm.

Now

29·482 scale ins = 29·482 ÷ 0·0393585 = 749·06 scale mm.

748·70 scale mm = 748·70 × 0·0393585 = 29·468 scale ins;

so that the mm scale reads 0·014 ins or 0·36 mm below the inch scale, and is therefore this distance too high, if the inch scale is correct; but of this discrepancy, 0·22 mm may be accounted for by the appearance of the graduation of 30 ins at the level of 762 mm, instead of 762·22 mm.

(*Standards of Length*, Pratt and Whitney Co., 1887; *Unités et étalons*, C. E. Guillaume, 1893.)

177. *The Vernier.*

To make the scale reading l to fractions of a scale division, the *Vernier* is employed (fig. 58, p. 252).

This consists of a sliding piece of metal, the zero of which is brought to the level of the top of the mercury column; and now if the vernier is to read upwards to one nth of a scale division, the length of vernier is made equal to $n-1$ scale divisions, and it is then divided into n equal parts.

Each scale division is therefore greater than a vernier division by one n-th of a scale division; so that if the r-th vernier division coincides with a scale division, the extra fractional part of l is r/n of a scale division; the number r is called the *least count* of the vernier.

In fig. 58 the scale is shown to full size, divided into inches and twentieths of an inch on the right and into centimetres and millimetres on the left; and 24 parts on the right are taken and divided into 25 parts to form the right hand vernier, while 19 parts on the left are divided into 20 for the left hand vernier; the verniers therefore read to two-thousandths of an inch, or five-thousandths of a centimetre; and the reading of the verniers is

29·482 inches, or 748·70 mm.

The vernier might also be made to read downwards to one n-th of a scale division, by taking $n+1$ scale divisions and dividing them into n equal parts to form the vernier; and now each scale division is less than a vernier division by one nth of a scale division.

178. It will be noticed that while in a thermometer the mercury in the stem appears like a filament, the spherical bulb of the thermometer and the tube of the barometer appear like solid mercury; so also the tube of the gauge glass of a boiler appears like solid water for the part occupied by water; and this water will appear coloured if a thin line of red glass is fused into the back of the tube.

This appearance is due to the Law of Refraction in Optics; from which it is easily seen that the magnification of the mercury column is equal to the index of refraction; so that the tube will have the appearance of a thermometer or barometer according as the ratio of the external diameter to the diameter of the bore is greater or less than the index of refraction of the glass; the value of this index is about 1·5.

So also for a spherical shell of glass, filled with mercury or liquid, like the bulb of a thermometer.

179. The siphon barometer (Gay-Lussac's) can also be employed for scientific purposes if the level of each surface at B and H is measured on the scale S, which is made vertical; and now there is no need to take account of any variation in the cross section of the tube, although the double correction for capillarity at the two surfaces B and H is troublesome.

The ordinary Weather Glass is a siphon barometer in which a float in the mercury at B turns a dial hand by means of a vertical rack or by a thread and counterpoise, actuating the axle of the dial (fig. 57, p. 252); but this instrument is not capable of scientific accuracy, because of the influence of the varying cross section of the tube and cistern.

Thus if a denotes the cross section of the tube and β of the cistern and if the float B descends a distance x while the surface H rises a distance y, then, the volume of the mercury remaining unchanged,

$$\beta x = ay,$$

and the change in barometric height

$$x+y=\left(1+\frac{\beta}{a}\right)x=\left(1+\frac{a}{\beta}\right)y\,;$$

and x and y thus depend on β/a, which may change with the shape and temperature of the tube.

The Marine Barometer is constructed on this system, and suspended from gimbals so as to hang vertically; the graduations for inches on the scale being shortened by the factor $\beta/(a+\beta)$; and the intermediate portion of the tube AH is contracted to a small bore, so as to make the oscillations of the mercury sluggish, and to prevent the so-called "pumping," due to the motion of the ship.

With a uniform bore the oscillations of the mercury would synchronize with a pendulum of half the length of the mercury column (§ 161).

180. In Sir Samuel Morland's *steelyard* or *balance* barometer (1670) the tube is suspended freely in the cistern from the arm of a balance (fig. 59, p. 252); and it is found necessary to counterpoise not only the weight of the tube, less its buoyancy in the cistern, but also the barometric column of mercury in the tube, or its equivalent thrust of air on the top of the tube.

Measuring upwards from the bottom of the cistern, let x cm denote the height of the lower end of the tube, y cm the height of the surface of the mercury in the cistern, and $y+h$ cm the height of the mercury in the tube, corresponding to a barometric height h cm.

Reckoning pressure in cm of barometric height, and weight in cm^3 of mercury, let the glass tube weigh M cm^3 of mercury and contain V cm^3 of mercury when full.

Then, if the internal length of the tube is l cm, and its internal cross section at the top is a cm^2, the tube now contains $V-(x+l-y-h)a$ cm^3 of mercury; and therefore the equilibrium of the tube, if counterpoised by a weight of W cm^3 of mercury, requires

$$W=M+h\beta+V-(x+l-y-h)a-(h+y-x)\beta, \quad (1)$$

if β denotes in cm^2 the external cross section of the lower submerged part of the tube.

But if the total quantity of mercury is U cm^3, and the cross section of the cistern is γ cm^2,

$$U=y\gamma-(y-x)\beta+V-(x+l-y-h)a, \quad \text{.........}(2)$$

and therefore

$$W=M+U-y\gamma. \quad \text{.................................}(3)$$

We may suppose the counterpoise W to be constant, and then y is also constant, so that the level of the mercury in the cistern does not change.

But if Δx denotes the change in x due to a change of Δh in h, then, from (1) or (2),

$$a\Delta h-(a-\beta)\Delta x=0,$$

or

$$\frac{\Delta x}{\Delta h}=\frac{a}{a-\beta};$$

and in this way a continuous magnified mechanical register of the fluctuations of barometric height can be obtained by a pen attached to the counterpoise, tracing a line on a uniformly revolving drum.

By sufficiently increasing the length $y-x$ of the submerged part of the tube, the buoyancy of the mercury could be made sufficiently large to dispense with the counterpoise, and the tube would now float freely; but the stability would now require separate attention.

181. A barometer consisting of a column of mercury in a straight vertical tube, slightly conical in the bore, was suggested by Amontons (1695).

As the atmospheric pressure increases the column of mercury rises slightly in the tube, and at the same time elongates, so as to come to a new position of equilibrium; but the instrument is not of practical utility, as a shake is liable to spill out the mercury.

We may notice here that, as the pressure in the mercury is greater than the atmospheric pressure in the cistern, a crack or leak in the cistern if below the level of the mercury will allow the mercury to escape till the level is lowered below the crack; but as the pressure in the tube is less than the atmospheric pressure, a crack in the tube will admit the air, and destroy the column.

It may happen that the air which entered will drive the mercury above the leak to the top of the tube, thus filling up the Torricellian vacuum space.

If membranes are used to cover these leaks, they will be found correspondingly bulged outwards or inwards.

182. *The Water and Glycerine Barometer.*

If the mercury is replaced by some lighter liquid, say water or glycerine, the height of the barometer and its fluctuations are correspondingly increased, in the ratio of the S.V. of the liquid employed.

A water barometer, constructed by Prof. Daniell (*Phil. Trans.*, 1832) stood formerly in the hall of the Royal Society, and was afterwards placed in the Crystal Palace; the mean height of the water column was

$$33\frac{1}{3} \text{ ft or } 400 \text{ ins.}$$

Glycerine is now employed instead of water, as less liable to vitiate the Torricellian vacuum.

In the Jordan glycerine barometer, at the *Times* office, employed for the meteorological records, a height of about 329·2 ins of glycerine corresponds to 30·61 of mercury; the S.G. of glycerine is thus about 1·262, the S.G. of mercury being 13·569.

183. *Huygens's Barometer.*

Huygens made a suggestion (1672) for magnifying the fluctuations of the barometer without greatly increasing its height by the combination of a column of mercury and of water superposed, the common surface of the mercury and the water being placed at an enlarged part of the tube, so that a small increase of the height of the mercury column should cause a magnified motion of the upper surface of the water.

Suppose then that A_1 denotes the cross section of the upper tube of water, A_2 of the middle enlargement where the mercury and water are in contact, A_3 of the lower tube of mercury, and A of the free surface of the cistern (fig. 60, p. 252).

Then if x_1 and x_2 denote the heights above the top of the cistern of the upper surfaces of the water (or glycerine) and the mercury, and x denotes the depth below the top of the cistern of the free surface of the mercury; if h denotes the height of the standard barometer, σ the density of mercury, and ρ the density of the water (or glycerine),

$$\sigma h = \rho(x_1 - x_2) + \sigma(x_2 + x).$$

Denoting by Δh any change in h, and by Δx the corresponding change in x,

$$A_1\Delta x_1 = A_2\Delta x_2 = A\Delta x;$$

and therefore

$$\Delta x = \Delta h \frac{\sigma}{A} \Big/ \left\{ \rho \left(\frac{1}{A_1} - \frac{1}{A_2} \right) + \sigma \left(\frac{1}{A_2} + \frac{1}{A} \right) \right\},$$

giving the fluctuation of level in the cistern, which would be recorded by a float; while

$$\Delta x_1 = \Delta h \frac{\sigma}{A_1} \Big/ \left\{ \rho \left(\frac{1}{A_1} - \frac{1}{A_2} \right) + \sigma \left(\frac{1}{A_2} + \frac{1}{A} \right) \right\},$$

the fluctuation of the top of the barometric column.

Thus for example, if the cistern is sufficiently large for $1/A$ to be neglected, and if we take $\sigma/\rho = 13$, then we shall find that the ratio $A_2 = 10A_1$ will make $\Delta x_1 = 6\Delta h$ about, so that the fluctuations of the top of the column are magnified six times.

184. Generally in a barometric column composed of n superincumbent liquids of densities

$$\rho_1, \rho_2, \ldots \rho_n,$$

reckoned from the top; if

$$x_1, x_2, \ldots x_n$$

denote the heights above the top of the cistern of their upper surfaces, and

$$A_1, A_2, \ldots A_n,$$

the cross section of the column at these levels; if x denotes the depth below the top of the cistern of the free surface of the liquid, of density ρ_n, and A the area of this free surface, then compared with a standard mercury barometer, of height h and density σ, the corresponding fluctuations of level are given by the relations

$$A_1 \Delta x_1 = A_2 \Delta x_2 = \ldots = A_n \Delta x_n = A \Delta x$$
$$= \frac{\sigma \Delta h}{\rho_1 \left(\frac{1}{A_1} - \frac{1}{A_2} \right) + \ldots + \rho_{n-1} \left(\frac{1}{A_{n-1}} - \frac{1}{A_n} \right) + \rho_n \left(\frac{1}{A_n} + \frac{1}{A} \right)};$$

this is left as an exercise.

185. *The Aneroid Barometer.*

This is an application of the principle of Bourdon's Pressure Gauge, described in § 10, to the measurement of small variations of external atmospheric pressure; to make the instrument sufficiently sensitive the flattened curved tube must now be made very thin; but in other respects the arrangement remains the same.

The Aneroid Barometer is now more often made with a corrugated box, exhausted of air.

A *Brief Historical Account of the Barometer*, by William Ellis, Q. J. Meteorological Society, July 1886, may be consulted for further details concerning the Barometer; also the *Smithsonian Meteorological Tables*.

186. *The Weight of the Atmosphere.*

If the two branches of the bent tube in fig. 50, p. 234, are vertical and of the same uniform bore, we notice that the weight of each liquid above the horizontal plane of separation AB is the same, as in Hare's Hydrometer (fig. 51, p. 234).

Thus if the bore of the siphon barometer (fig. 57, p. 252) is uniform, the weight of superincumbent air in the upward prolongation of the cistern as an imaginary vertical tube reaching to the limit of the atmosphere will be practically equal to the weight of the mercury in the column AH; consequently the "weight of the atmosphere" is practically the same as that of an ocean of mercury covering the Earth, of uniform depth h, about 30 ins. or 76 cm, the average barometric height, and hence the name *barometer*, as measuring the weight of the air.

This weight is the same as that of an ocean of fresh water about 34 ft or 10·33 m deep, or of sea water about 33 ft or 10 m deep.

Professor Dewar remarks in his Royal Institution Lecture, June 1892, that if the Earth were cooled down to about -200 C, the atmosphere would form a liquid ocean about 35 ft deep, of which about 7 ft at the top would be oxygen.

Taking the atmospheric pressure as 1 kg/cm^2, there will be about one kg of air per cm^2 of the Earth's surface; with a quadrant of 10^9 cm, or a radius of $10^9 \div \frac{1}{2}\pi$ cm, the surface of the Earth will be

$$4\pi \times 10^{18} \div \tfrac{1}{4}\pi^2 = 10^{18} \times 16 \div \pi = 10^{18} \times 5{\cdot}093 \text{ cm}^2;$$

and this number will be practically the number of kg of air in the atmosphere.

More accurately, with an average barometric height of 76 cm, and a density of mercury 13·6 g/cm^3, the atmospheric pressure is 1·0336 kg/cm^2; and the weight of the atmosphere is about $10^{15} \times 5{\cdot}264$ t.

According to Cotes (p. 94) this is the weight of a sphere of lead about 60 miles in diameter.

If R cm denotes the radius and ρ the mean density of the Earth,

$$\frac{\text{the weight of the Earth}}{\text{the weight of the atmosphere}} = \frac{\frac{4}{3}\pi\rho R^3}{4\pi\sigma R^2 h} = \frac{\rho R}{3\sigma h} = 10^6 \times 1{\cdot}129,$$

on putting $R = 10^9 \div \frac{1}{2}\pi$, $h = 76$, $\sigma = 13{\cdot}6$, $\rho = 5{\cdot}5$.

187. A calculation is given in the *Traité de l'équilibre des liqueurs et de la pesanteur de la masse de l'air*, Blaise Pascal, 1653; he takes the 90° of the quadrant of the meridian as 1800 leagues, a degree = 50,000 toises, a toise = 6 ft, a ft^3 of water = 62 livres, and the mean height of the water barometer as 31 Paris feet; and thence finds that the atmosphere weighs $10^{18} \times 8{\cdot}284$ livres.

Mascart asserts (*Comptes Rendus*, 18 Jan. 1892) that allowing for the curvature of the Earth, and for *adiabatic* expansion, the atmosphere really weighs about one-sixth more than a sea of mercury 76 cm deep, or of water about 10 m deep.

188. *Meteorology of the Barometer.*

By simultaneous observations at a number of widely scattered meteorological observatories the height of the barometer, corrected by the thermometer, is registered; and thence the *isobars*, or *isobaric lines*, lines on the Earth along which the barometer stands at the same height at a given time, can be plotted, as in the weather charts issued from the Meteorological Office.

The gradient of the barometer, that is its most rapid rate of rise or fall in the direction perpendicular to the isobar, usually measured in hundredths of an inch per 15 miles, can thence be calculated; and from these observations much weather lore and prophecy valuable in Navigation can be deduced. (*A Barometric Manual for the use of Seamen, issued by Authority of the Meteorological Council*, 1890.)

Given the height of the barometer a, b, c at three different stations A, B, C, not too far apart, the gradient of the barometer is the same as that of the inclined plane which passes through the top of three posts of height a, b, c erected at sea level at A, B, C; and the isobars will be the lines of intersection of parallel planes with the horizontal plane.

The geometrical construction of these isobars depends on the problem of dividing an angle into two parts whose sines are in a given ratio; for instance to draw the isobar $\beta A \gamma$ through A, where the barometer is highest suppose, we must determine $\beta A \gamma$ so that the perpendiculars on it $B\beta, C\gamma$ from B, C are in the ratio $a-b : a-c$.

Otherwise, P and Q being the points on the lines AB and AC where the height of the barometer is x, then PQ is an isobar, and the parallel isobar $\beta\gamma$ through A is such that

$$\frac{\sin PA\beta}{\sin QA\gamma}=\frac{AP}{AQ}=\frac{AB}{AC}\frac{a-c}{a-b},$$

because
$$\frac{AP}{AB}=\frac{a-x}{a-b},\quad \frac{AQ}{AC}=\frac{a-x}{a-c}.$$

189. If the gradient of the barometer exists over a sheet of water like an inland lake, then since the surfaces of equal pressure in the water are horizontal planes (§ 19) it follows that the free surface will no longer be horizontal, but will have an opposite gradient 13·6 times the gradient of the barometer, 13·6 being the S.G. of mercury.

To find the rise and fall of the water in the lake at any point we must draw the isobar through the C.G. of the surface of the lake; and now the free surface will be a plane passing through this nodal isobar, at the incline of the enlarged gradient opposite to that of the barometer; for in this way, according to the theorem of § 101, the total quantity of water in the lake will be preserved unchanged.

Thus if the gradient of the barometer is two-hundredths of an inch per 15 miles, the surface of the water will have an opposing gradient of 27·2 hundredths; so that in a lake 180 miles in diameter this will amount to a rise at one end and a fall at the other of 1·044 in.

The sheet of water over which there is a variation of barometric pressure must be of considerable size, like the American Lakes, for an appreciable rise and fall of the water to be produced.

Over the Ocean it is the variations of the gravity gradient due to the perturbation of the Moon and Sun, which produce such marked phenomena as the Tides, although insensible to the most delicate plumb-line observations; and when the tidal current is constrained in narrow waters, the average gradient of the sea at any instant between two places may become easily perceptible, being the difference of height of water above mean sea level divided by the distance between the places.

Examples.

(1) A straight pipe 40 feet long 6 inches in diameter, closed at the top and full of ice, is inverted in a barrel a yard in diameter. Given the specific gravity of ice 0·9, and the height of the water barometer, 30 feet, show that when the ice melts the water will rise 2 inches in the barrel.

(2) A barometer has a dial attached to it, and if the tube were cylindrical the markings on the dial would be at equal distances, but the small arm is really a cone of small angle. If a_p, a_q, a_r be three consecutive angular intervals on the dial, show that

$$a_p^2(a_q-a_r)+a_r^2(a_r-a_q)+2a_pa_qa_r=a_q^2(a_p+a_r).$$

Verify this result when the tube is cylindrical.

(3) From the following data obtain the true reading, the barometer being placed at a height of 20 feet above sea level. Barometer reading, 29·5 in; attached thermometer, 20°C; ratio of area of section of tube to section of cistern = 1 : 41.

Capillary action +·04 in; fall in barometric height for each foot above sea-level, ·001 in; coefficient of expansion of mercury for 1°C = ·00018.

When the mercury in the cistern is at the zero of the scale, supposed marked on the tube, the mercury in the tube stands at 30 in.

(4) A thin weightless cylindrical shell of radius a, closed at the top, stands in a large basin which contains a depth b of mercury; the mercury stands in the cylinder at the barometric height h. Prove that, if the cylinder be turned about a point in the rim of its base, it will tend to return to its original position so long as the inclination of the axis to the vertical is less than

$$\sin^{-1}\frac{2a}{h+2b},$$

provided that no part of the rim has reached the level of the mercury in the basin, and that the mercury in the cylinder has not reached the top.

(5) A very wide cylindrical tube, closed at the upper end, rests on the bottom of a flat dish of mercury, with the air inside the tube partially exhausted; find the condition that it be not lifted up by buoyancy. Show also that when the tube is bent over from the vertical it will tend to come back again so long as the centre of gravity of the tube, together with the portion of the mercury above the open level in the dish, less the mercury below that level which is displaced by the tubes, does not fall outside the point of support.

(6) A barometer consists of a vertical tube closed at the top, the diameter of which changes at the middle, so that the area of the transverse section of the upper portion of the tube is A and that of the lower portion B.

The tube contains a volume V of mercury, which is supported by the pressure of the air on its lower surface, the space above the mercury being free from air, and V being greater than hA and less than hB, where h is the height of the mercurial barometer. Show that the height of the upper surface of the mercury above the point where the diameter of the tube changes is

$$(Bh - V)/(B - A);$$

and that if the barometer falls an inch, this surface will fall through $B/(B-A)$ inches.

If the mercury be slightly displaced, the oscillations will synchronize with a pendulum of length

$$hA/(B-A).$$

190. *The Siphon.*

The siphon (Greek, σίφων) is a bent tube ABC, originally designed for drawing off liquid from a cask or jar, and now employed on a large scale for going over or under an obstacle in carrying liquid to a lower level.

As the action of the siphon depends essentially on the same principle as that of the barometer, the discussion of it is introduced here, and first in connection with drawing off mercury.

The tube is first filled with mercury, and then the ends A and C being closed by the fingers or by corks, or the stopcock s.c. being closed, the tube is inverted with the ends in the vessels of mercury at A and C, at different levels, and the fingers or corks removed.

The pressure at A and C being that of the atmosphere, the pressure in the mercury above the stopcock, when closed, exceeds the pressure below by σz, where z denotes the difference of level of A and C; so that, on opening the stopcock, the mercury flows through the tube ABC from A to C in a continuous stream (fig. 61).

If the density ρ of the surrounding medium is taken into account, the pressure of the liquid above the stop valve exceeds the pressure below the valve by

$$(\sigma - \rho)z.$$

Thus if $\rho > \sigma$, the action of the siphon is reversed, as, for instance, in transferring hydrogen by a siphon tube; and now the siphon and the vessels at A and C must be inverted, and the fluid will be transferred from the lower to the upper level.

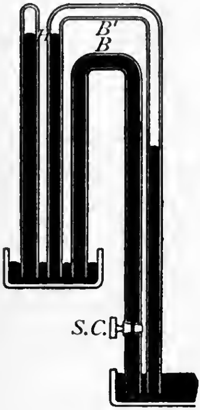


Fig 61.

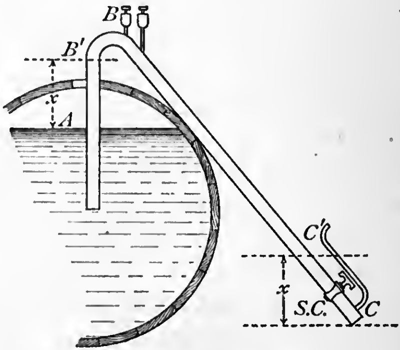


Fig 62.

The vertical height of the highest point B of the tube above A must not, however, exceed h, the height of the barometer; otherwise, as in $AB'C$, the mercury in the branch AB' will subside to the barometric height AH, when the finger is removed from A; and now, when the stopcock in $B'C$ is opened, the mercury in the branch $B'C$ will subside to the barometric height CH'.

This supposes that the mercury column divides where the pressure vanishes or becomes negative; but if, as in Mr. Worthington's experiments (§ 6), we suppose that the mercury column can support a tension of a certain amount, σk suppose, without breaking, the siphon can still work, so long as the height of B' above H does not exceed k.

191. In its dynamical action the siphon may be assimilated to a chain coiled up at A, and led over a pulley at B so that the end hangs at C; the preponderating length z will set the chain in motion, so that the coil at A will become gradually transferred into a coil at C.

If x cm of chain have passed over in t seconds, and the moving part ABC, of length l cm suppose, has then acquired a velocity v cm/sec, if w denotes the weight in g/cm of the chain, and T g denotes the tension of the chain at A, the equation of motion is

$$\frac{wl}{g}\frac{dv}{dt}=wz-T,$$

and $$T=\frac{wv^2}{g}$$

the momentum in second-grammes generated per second;

so that $$l\frac{dv}{dt}=gz-v^2; \qquad \text{(1)}$$

and the chain therefore starts with an initial acceleration gz/l, and tends to a *terminal* velocity $\sqrt{(gz)}$, just like a body falling under gravity in a medium in which the resistance varies as the square of the velocity.

In the siphon there will be no loss of energy at A due to the continuous series of impacts, so that we may halve the above value of T; and now the equation of motion in the siphon becomes

$$l\frac{dv}{dt}=gz-\tfrac{1}{2}v^2, \quad \text{.........................(2)}$$

so that the terminal velocity in the siphon is $\sqrt{(2gz)}$.

By integration of equation (2),

$$t=\int_0 \frac{ldv}{gz-\frac{1}{2}v^2}=\frac{l}{\sqrt{(\frac{1}{2}gz)}}\tanh^{-1}\frac{v}{\sqrt{(2gz)}},$$

or
$$v=\sqrt{(2gz)}\tanh\frac{\sqrt{(\frac{1}{2}gz)}}{l}t\,;$$

$$x=\int_0 \frac{lvdv}{gz-\frac{1}{2}v^2}=l\log\frac{gz}{gz-\frac{1}{2}v^2};$$

or
$$\tfrac{1}{2}v^2=gz(1-e^{-x/l}).$$

192. A leak in the tube ABC, if above the level of A, will admit air, and vitiate the action of the siphon, even to the extent of stopping the flow if the leak is sufficiently large; but liquid will escape from a leak in the tube below the level of A.

The large siphons or standpipes of waterworks are designed to reach the altitude of the service reservoirs, so that the water in passing through may be cleared of air, which tends to accumulate in the mains.

In the distiller's siphon (fig. 62, p. 272) the action is started by opening the stopcock S.C., closing the end C with the hand, and sucking the air out by the curved mouthpiece at C'; as soon as the spirit passes the highest point of the bend at B, the action of the siphon commences, and it can be stopped and restarted by closing and opening the stopcock.

The preceding methods are not desirable with noxious liquids, such as acids, which cannot be handled or tasted with impunity; the siphon is then started by first closing the stopcock S.C., and filling the branch BC

through one of two small funnels at B, the other funnel permitting the escape of air; on shutting off these funnels by plugs or stopcocks and opening s.c., the liquid is made to flow through the siphon.

193. If however the length of the longer branch BC is insufficient, the liquid will not be drawn up to the level of the bend B in the branch AB, but will rest at a lower level B' a certain vertical height x above A, and at a certain distance y from A if the branch AB is curved; and the siphon will not start.

The vertical height of the liquid CC' left in the longer branch BC will also be x, in consequence of the equality of the pressures at A and C, and at B' and C'; and the pressures at A and C being due to a head h of the liquid, the pressure in $B'C'$ will be due to a head $h-x$.

Denoting by a, b the lengths of the branches AB, BC, and supposing for simplicity that BC is inclined at an angle α to the vertical, the air which originally occupied the length a of the shorter arm AB now occupies the length

$$a+b-y-x\sec\alpha.$$

Therefore by Boyle's Law (Chap. VII.), which asserts that the product of the volume and pressure of a given quantity of air remains the same at the same temperature,

$$ah=(a+b-y-x\sec\alpha)(h-x),$$

or

$$a+b=\frac{ah}{h-x}+y+x\sec\alpha.$$

In the critical case when the liquid just reaches the bend B, $y=a$, and x denotes the vertical height of B above A; so that

$$b=\frac{ah}{h-x}+x\sec\alpha;$$

and a greater value of b will start the siphon.

In the siphons of fig. 61 the branches are vertical; and now if a, b denote the lengths of the vertical branches, and c the length of the horizontal part, then in the critical case

$$(a+c)h=(b+c-a)(h-a),$$

or

$$b=\frac{(a+c)h}{h-a}+a-c.$$

194. As employed for drawing off water over an embankment, the siphon is shown in fig. 63; for example, over the reservoir dam of water works (fig. 20), or in draining a fen or inundation.

(*Proc. Inst. Civil Engineers*, XXII.)

An automatic valve, opening inwards, is placed at A and a stop valve at C.

The siphon is filled either through a funnel by means of a hand pump, or else by exhausting the air by an air pump at B. On opening the stop valve C, the water flows through the siphon; and on closing the stop valve, the siphon remains filled for an indefinite time, the valve at A preventing the return of the water in AB.

In this, as in all other cases, the height of B above the upper level of the liquid must be kept below the head of liquid corresponding to the atmospheric pressure.

Sometimes the siphon is *inverted*, as required for carrying a water main across the bed of a river; and now there is no limitation of depth to its working.

A water main, or a pipe line for conveying oil, carried in an undulating line in the ground, may be considered as a series of erect and inverted siphons; and on an emergency, the pipe may be carried over an obstacle, which is higher than the supply source or *hydraulic gradient* by something under the *atmospheric head* of the liquid.

195. An *intermittent* siphon is shown in fig. 64; the vessel is gradually filled up to the level of B, when the action of the siphon suddenly commences, and the vessel is rapidly emptied; and so the operation goes on periodically.

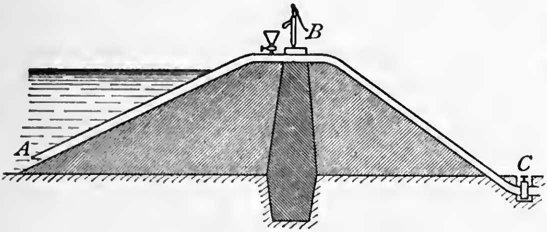


Fig. 63.

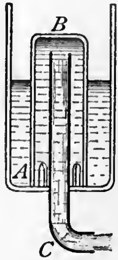


Fig. 64.

The *Cup of Tantalus*, invented by Hero, depends on this principle; and it is also used for securing an intermittent scouring flow of water. The action of natural intermittent springs and geysers is explained in this manner; and the underground flow of certain rivers, such as the Mole, by subterranean inverted siphons.

Examples.

(1) If a vessel contains liquids of various densities, will the action of the siphon be impeded?

Two equal cylindrical pails of horizontal section A are placed, one on the ground, and the other on a stand of height h; the former is empty, and the latter contains masses m_1, m_2 of two different homogeneous liquids; a fine siphon tube of negligible volume has its two ends at the bottoms of the two pails and through it flows liquid until equilibrium is attained, a mass m_3 of density ρ remaining in the upper pail; prove that

$$m_1+m_2-2m_3=Ah\rho.$$

(2) A siphon tube with vertical arms filled with mercury, of S.G. σ, and closed at both ends is inserted into a basin of water.

When the stoppers are removed, examine what will ensue, and prove the following results if the barometer is sufficiently high :—

(1) If b, the whole length of the outside arm, exceeds a, the whole length of the immersed arm, the mercury will flow outwards and the water will follow it.

(2) If $a > b$, the end of the immersed tube must be at a depth below the free surface of the water exceeding

$$(a-b)\sigma$$

in order that the mercury may not flow back into the basin.

(3) Two equal cylinders side by side contain mercury, one quite full and open at the top, the other full to 20 inches from the top and closed, the 20 inches being occupied by air at the atmospheric pressure, which is 30 inches of the barometric column.

If the two vessels are connected by a siphon dipping into the two liquids, prove that, when the siphon is put in action, 5 inches of mercury will flow from one of the cylinders into the other.

What takes place when the leg of the siphon which is in the closed cylinder is not long enough to reach the mercury in that cylinder?

CHAPTER VII.

PNEUMATICS. THE GASEOUS LAWS.

196. Hitherto we have dealt with the properties of Liquids or Incompressible Fluids like Water; and now we proceed to consider Air and Gases, or Compressible Fluids, and their properties, a branch of Hydrostatics sometimes called Pneumatics, from the Greek word πνευματική, meaning the science which concerns πνεῦμα, air or gas.

A given quantity of a Gas ("a parcel of gas" in Boyle's words) requires to be kept in a closed vessel, to prevent diffusion; and by changing the volume of the vessel and the temperature, the pressure of the gas is altered.

Given the volume and the temperature, the pressure of a given quantity of a gas is determinate; so that the pressure p is a function of the volume v or density ρ, and of the temperature τ.

Expressed analytically

$$p = f(v, \tau),$$

or

$$F(p, v, \tau) = 0;$$

and to determine this function, two new Laws, based upon experiment, are required, which are called

197. *The Gaseous Laws.*

LAW I.—BOYLE'S LAW.

"At constant temperature the pressure of a given quantity of a Gas is inversely proportional to the volume, or directly to the density."

This law was enunciated by Boyle in his *Defence of the Doctrine touching the Spring and Weight of the Air in answer to Linus*, 1662; abroad it is attributed to Mariotte, who did not however publish it till 1676.

Thus if p denotes the pressure and v the volume of unit quantity of the gas, one gramme suppose, and ρ denotes the density, so that $\rho = 1/v$, then

$$p = k\rho, \text{ or } pv = k,$$

where k depends only on the temperature: so that, on the (p, v) diagram, an *isothermal* is a hyperbola (fig. 65), along which the hydrostatic energy pv (§ 14) is constant.

For instance, a gunner, who can push with a force of P pounds, can, under an atmospheric pressure of p lb/in², introduce an airtight sponge into a closed cannon, d ins in calibre and l ins long in the bore, a distance x ins, given by

$$P + \tfrac{1}{4}\pi d^2 p = \tfrac{1}{4}\pi d^2 p \frac{l}{l-x}, \text{ or } \frac{x}{l} = \frac{P}{P + \tfrac{1}{4}\pi d^2 p}.$$

Thus, if $P = 100$, $p = 15$, $d = 5$, $l = 120$, we find that $x = 30{\cdot}42$.

LAW II.—CHARLES'S OR GAY-LUSSAC'S LAW.

"At constant pressure the volume of the Gas increases uniformly with the temperature, and at the same rate for all gases."

Combining this with Boyle's Law we find that the product of the pressure and volume of a given quantity of any Gas increases uniformly with the temperature at the same rate; so that we may write

$$pv = k = k_0(1 + \alpha\tau),$$

where α is a constant coefficient of expansion, the same for all gases.

On the Centigrade scale of temperature

$$\alpha = 0{\cdot}003665 = \tfrac{1}{273};$$

and now putting $\quad k_0 = R/\alpha$

(the height of the homogenous atmosphere at 0 C).

$$pv = R\left(\frac{1}{\alpha} + \tau\right) = R\theta, \text{ where } \theta = \frac{1}{\alpha} + \tau;$$

and θ is called the *absolute* temperature, and $-1/\alpha$ the *absolute zero*; this is therefore -273 C, or about -460 F, since $1/\alpha = \frac{9}{5} \times 273 = 492$ on the Fahrenheit scale.

But -274 C, or -461 F is sometimes taken as nearer to the correct value of the absolute zero of temperature.

At this absolute zero the pressure of a given quantity of gas would be zero, whatever the volume.

In an experiment by Robins (*New Principles of Gunnery*, Prop. V., p. 70) a gun barrel, which would contain about 800 grains of water, was raised to a white heat, and plunged into water, when it was found that about 600 grains of water had entered the barrel.

This proves that the air left in the barrel had been expanded to four times its volume; so that, if the water was at 15° C, or 288 absolute, the temperature of the white heat was about 1152 absolute, or 880° C, or 1552° F.

198. The equation

$$pv = R\theta \quad\text{.........................}\quad \text{(A)}$$

connecting p the pressure, v the volume, and θ the absolute temperature of a given quantity, say one g, is called the *Characteristic Equation of a Perfect Gas.*

It may be illustrated geometrically by the surface shown in fig. 65, in which the *isothermals*, along which θ is constant, are hyperbolas, while the *isometrics*, v constant, and the *isobars*, p constant, are straight lines.

This model surface can be constructed of pieces of card-board, as made by Brill of Darmstadt.

Denoting by P, V, Θ the pressure volume, and temperature in any given initial state, then (A) may be written

$$R=\frac{pv}{\theta}=\frac{PV}{\Theta},$$

embodying the Laws of Boyle and Charles in a form suitable for calculation from experiments.

Regnault found that at Paris a litre of dry air at 0 C and a barometric height 76 cm is 1·293187 g; so that measuring pressure in millimetres of mercury head, we find, for a gramme of air at Paris,

$$P=760, \quad \rho=1/V=\cdot 001293187, \quad \Theta=273,$$

and $$R=PV/\Theta=215\cdot 3.$$

Taking the density of mercury as 13·59, this makes the height at Paris of the homogeneous atmosphere at 0 C

$$k_0=h\sigma/\rho=798676\cdot 5 \text{ cm, say } 8000 \text{ m}.$$

The weight in g of V litres of dry air at a temperature τ C and a pressure of h mm of mercury is therefore

$$1\cdot 293187\times\frac{h}{760}\times\frac{273}{273+\tau}\times V,$$

a formula required in exact weighings, in allowing for the buoyancy of the air.

199. It must be noticed that the gravitation measure of force is employed in these formulas, so that in accurate comparisons the local value of g must be allowed for.

Thus if g changes to g' in going from Paris to any other locality, Greenwich for instance, the absolute pressure due to a given head of mercury changes in the ratio of g to g', and therefore also the density of a litre of dry air defined by these conditions.

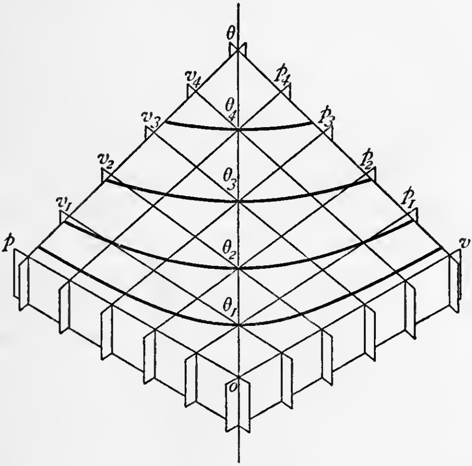


Fig. 65.

Thus, with the centimetre and second as units of length and time, $g=980{\cdot}94$ at Paris, $g'=981{\cdot}17$ at Greenwich, $G=980{\cdot}61$ at sea level in latitude 45°; so that the weight of a litre of dry air at 0°C and under 760 mm of mercury head changes from 1·293187 g at Paris to

$$1{\cdot}293187(g'/g \text{ or } G/g)=1{\cdot}293559 \text{ g}$$

at Greenwich, or to 1·292752 g in latitude 45°.

But the head h of mercury or k of homogeneous air which will produce the same absolute pressure as 760 mm of mercury or 8000 m of air at Paris is

at Greenwich (760, or 8000)g/g'

$=759{\cdot}82$ mm of mercury, or 7998·2 m of air;

and at sea level in latitude 45° is (760, or 8000)g/G

$=760{\cdot}26$ mm of mercury, or 8002·7 m of air.

Suppose, for instance, that g is doubled or halved, as might appear to be the case in a lift, or the cage of a mine; the pressure due to a given head of liquid is doubled or halved, but the head corresponding to a given pressure is halved or doubled; and generally the pressure due to a head h of mercury or k of air is proportional to g; but, for a given pressure, gh or gk remains constant.

These variations are due to the employment of the gravitation unit of force; but as the variations on the surface of the Earth do not amount to 0·3 per cent, they are insensible in most practical problems.

All physical measurements of force are primarily made in gravitation units, from their convenience, intelligibility, and precision; and these measurements can afterwards be converted into absolute units by multiplying by the local value of g, when it is required to compare delicate measurements made in different localities.

200. With British units, the foot and the pound, and with the Fahrenheit scale, a ft^3 of air at 55 F and a barometric height of 30 ins, equivalent to a pressure of $14\frac{2}{3}$ lb/in² or 2112 lb/ft², is found to weigh about 1·25 oz, so that about 13 ft³ weigh one lb; and therefore, putting

$$P=2112, \quad V=13, \quad \Theta=460+55=515,$$

$$R=PV/\Theta=53{\cdot}3, \text{ for one lb of air;}$$

then $$k_0=R/a=53{\cdot}3\times 492=26{,}224 \text{ ft.}$$

The work required to compress the air at constant temperature from volume V to v is represented by the area of the hyperbolic isothermal

$$pv=PV$$

on the (p, v) diagram, cut off by the abscissas V and v; it is therefore, by a well known formula (§ 233),

$$PV \log V/v,$$

expressed in ft-lb, if V is given in ft³ and P in lb/ft².

Thus if a cubic yard of atmospheric air is compressed to a cubic foot,

$$P=2112, \qquad V=27, \qquad v=1;$$

and the work required is

$$2112 \times 27 \times \log_e 27 = 188,000 \text{ ft-lb}.$$

201. A third law, sometimes called Dalton's law, is added for a mixture of gases which do not act chemically on each other; thus atmospheric air is a mechanical mixture of oxygen and nitrogen.

LAW III.—DALTON'S LAW.

The pressure of a mechanical mixture (not a chemical mixture) of gases all at the same temperature in a closed vessel is the sum of the separate pressures each gas would have if it alone was present in the vessel.

This law again must be accepted as based upon experimental proof.

Granted Dalton's law, Boyle's law is seen to follow immediately; for if a pound of a gas is introduced into an exhausted chamber and produces a certain pressure, the introduction of a second pound of the gas will by Dalton's law double the pressure and will also double the density; a third pound of the gas will treble the pressure and density, and so on; so that the pressure is proportional to the density.

Take $$W_1, W_2, \dots W_n,$$

lb or g, of n perfect gases, having pressures

$$p_1, p_2, \cdots p_n,$$

when the volumes are $v_1, v_2, \dots v_n$,

and the absolute temperatures are

$$\theta_1, \theta_2, \dots \theta_n.$$

Now if all these gases are brought to the same volume V and temperature Θ, the new pressures

$$\varpi_1, \varpi_2, \dots \varpi_n,$$

will be given by

$$\frac{\varpi_n V}{\Theta} = \frac{p_n v_n}{\theta_n} = R_n, \text{ suppose.}$$

If these gases are now mixed together mechanically in a closed vessel of volume V, the pressure P of the mixture will, by Dalton's law, be given by

$$P = \varpi_1 + \varpi_2 + \dots + \varpi_n;$$

so that
$$\frac{PV}{\Theta} = \frac{p_1 v_1}{\theta_1} + \frac{p_2 v_2}{\theta_2} + \dots + \frac{p_n v_n}{\theta_n};$$

or, if PV/Θ is denoted by S,

$$S = R_1 + R_2 + \dots + R_n.$$

If $\rho_1, \rho_2, \dots \rho_n$ denote the original density of each gas, and $k_1, k_2, \dots k_n$ the head of each gas which produces its original pressure; then

$$k_n = \frac{p_n}{\rho_n} = \frac{p_n v_n}{W_n} = \frac{R_n \theta_n}{W_n};$$

and if ρ denotes the density, and K the pressure-head of the mixture,

$$K = \frac{P}{\rho} = \frac{PV}{W_1 + W_2 + \dots + W_n} = \frac{S\Theta}{\Sigma W};$$

or
$$S = \frac{K}{\Theta} \Sigma W, \text{ and } R_n = W_n \frac{k_n}{\theta_n},$$

so that
$$K = \Sigma k_n W_n \frac{\Theta}{\theta_n} \Big/ \Sigma W_n.$$

202. *The Experimental Verification of Boyle's Law.*

Boyle took a bent tube $OABK$ with vertical branches (fig. 66, p. 295) and filled the bend with mercury so that it stood at the same level AB in the two branches; the end O being closed, the air in OA is at the atmospheric pressure, measured by the height h of the mercurial barometer DC.

Now if mercury is poured into the branch BK up to the level K, then mercury will rise in OA to a point H, such that the head of mercury due to the difference of level of H and K measures the excess of pressure of the air in OH over the atmospheric pressure; or the head $h+KH$ of mercury measures the pressure in OH.

Boyle found that, if the tube OA is cylindrical,

$$\frac{AH}{OH}=\frac{HK}{h}, \text{ or } \frac{OA}{OH}=\frac{h+KH}{h}=\frac{HL}{h},$$

if $KL=h$; so that the pressure in OH is inversely as the volume, and therefore proportional to the density.

Thus if $HK=h$, it is found that $OH=\frac{1}{2}OA$.

This apparatus of Boyle is not susceptible of very great accuracy, as the graduations for H become very close together when the pressure is great; and this militates against the use of this arrangement as a pressure gauge, as has been suggested.

203. An apparatus was devised by Regnault (Jamin, *Cours de Physique*, t. I.) in which the tube BK was carried to a height of 30 metres up a tower and mast, and the mercury was forced in by a pump from below; at the same time a known quantity of air, carefully dried, was forced into OA, so as to keep the level of the mercury nearly constant at A, the tube OA being surrounded by water to keep it at a constant temperature.

More recently Amagat has obtained great pressures by the head of a column of mercury in narrow flexible steel tubes, carried down the shaft of a mine some 400 m in depth, or up the Eiffel tower, 300 m high.

These experiments were recorded by plotting the value of pv, the hydrostatic energy of a given quantity of the gas, corresponding to values of p; and if Boyle's Law was accurately true, the points plotted at constant temperature should range themselves in a straight line parallel to the axis of p; this was found to be approximately the case.

Examined more closely the points were found, for great values of p, to lie very nearly in a straight line slightly inclined to the axis of p, indicating the law

$$pv = bp + k,$$

or

$$p(v-b) = k;$$

so that the isothermals on the (p, v) diagram are still hyperbolas and now b is called the *co-volume.*

A full account of the experiments upon which the Gaseous Laws are based will be found in two Reports to the British Association, 1886 and 1888, *Experimental Knowledge of the Properties of Matter with respect to Volume, Pressure, Temperature, and Specific Heat.* By P. T. Main.

Boyle's Law will be found to hold when the air in OA is expanded, by drawing off mercury in the bend AB through a stopcock at the lowest point; and generally a convenient mode of varying the level of K is by means of a large vessel of mercury which can be raised or lowered by a winch, and which communicates by a flexible tube with the stopcock (fig. 66, p. 295).

204. The Law for rarefaction of the air in OA can also be shown by the apparatus of fig. 67, p. 295, also devised by Boyle.

A cylindrical glass tube OA, closed at O, is partly filled with mercury, and sunk vertically in a deep vessel of mercury; the point A is marked where the mercury stands at the same level inside and outside the tube, and now the air in OA is at atmospheric pressure.

On raising the tube vertically, the pressure of the air in OA will be diminished and the air will expand, so that the mercury will be drawn up to a level H, such that the pressure of the air in OH is due to a head $h-AH$ or DH of mercury, if CD is the true barometric height; and it is found experimentally that

$$OH \,.\, HD = OA \,.\, CD,$$

so that Boyle's law is verified.

The same apparatus could be employed to compress the air by depressing the tube; but now the level H of the mercury inside the tube OA would be below the level of the mercury in the vessel, and its position would be difficult to observe, except by the mark left by the liquid in rising in the tube, unless transparent water and glass were employed.

It was in this way that Charles II. won his wager that he would demonstrate the compression of air in a hollow cane, the Royal Society being appointed referees (*Phil. Trans.*, Jan. 1671); the other story of him concerning a fish in a bucket of water appears spurious, as no record of it is to be found.

If the hollow cane was 21 in long, closed at one end, and depressed vertically in water till it was just submerged, the water would rise one inch in the cane, if the water barometer stood at 400 ins; or, if the cane was 44 ins long, and just submerged vertically, the water would rise 4 ins in the cane.

The principle is employed in the Deep Sea Sounding Machine; a tube of length a, closed at the top, is lowered vertically into the sea, and the length y of the interior marked by the entrance of the water being measured, the depth x reached by the lower end of the tube is given by the equation of the two expressions of the pressure in atmospheres of the imprisoned air,

$$\frac{H+x-y}{H}=\frac{a}{a-y},$$

where H denotes the height of the water barometer; and

$$x=y(H+a-y)/(a-y),$$

so that the depths corresponding to the graduations of the tube are the ordinates of a hyperbola, which can be constructed geometrically as in § 115; and the graduations can be made uniform by giving a hyperbolic shape to the tube.

205. A simple experimental illustration is described in Weinhold's *Experimental Physics;* in this the tube OA, closed at O, is of fine uniform bore, and contains a filament AB of mercury of length k suppose, less than h, the barometric height.

The tube may be suspended from a fixed point at O, and now, when held in the horizontal position, the pressure in OA is equal to the atmospheric pressure.

When the tube is held vertically upwards, the air in OA is compressed by a head $h+k$ of mercury, and will therefore occupy a length $ah/(h+k)$, if Boyle's Law is true, the original length OA being denoted by a.

When the tube hangs vertically downwards, the air in OA will expand to a length $ah/(h-k)$; and generally, if OP is the length of the air column when the tube makes an angle θ with the vertical, the pressure in OP will be due to a head $h+k\cos\theta$ of mercury; so that

$$OP=ah/(h+k\cos\theta),$$

and P therefore describes a conic section with focus at O, excentricity k/h, and semi-latus-rectum a.

This arrangement may be used as a barometer, for the determination of h, and it is then called a *Sympiezometer*.

If read in the two vertical positions, the effect of temperature is eliminated; for if x, y denote the lengths of the air column in the two positions

$$ah = x(h-k) = y(h+k),$$

or

$$\frac{h}{k} = \frac{x+y}{x-y};$$

and x and y expand at the same rate, and their ratio is therefore independent of the temperature, while the mercury columns h and k have the same coefficient of expansion.

A similar theory holds when both ends of the tube are closed; now if a, b denote the lengths of the air columns, in the horizontal position, and k the length of the separating mercury filament, then when inclined at an angle θ to the vertical the filament will travel through a distance r given by the equation

$$k \cos \theta = \frac{ah}{a-r} - \frac{bh}{b+r}.$$

Thus if $k = h = 30$ ins, and the lengths of the air filaments when the tube is held vertical are 10 and 5 ins, then in the horizontal position their lengths will be $6\frac{2}{3}$ and $8\frac{1}{3}$ ins.

206. If the open tube is of insufficient length, some of the mercury will run out as the tube is inclined.

Thus if the tube, of length l, is brim full of mercury when in the upright position, the mercury occupying a length k, and the air underneath a length $l-k$ under a mercury head $h+k$; then at an inclination θ the air column will expand to a length r, and the fraction of mercury spilt is $1-(l-r)/k$, given by

$$r\{h+(l-r)\cos\theta\}=(l-k)(h+k),$$

$$r=\tfrac{1}{2}(h\sec\theta+l)+\sqrt{\{\tfrac{1}{4}(h\sec\theta-l)^2+k\sec\theta(h+k-l)\}},$$

$$l-r=\tfrac{1}{2}(l-h\sec\theta)-\sqrt{\{\tfrac{1}{4}(l-h\sec\theta)^2+k\sec\theta(h+k-l)\}}.$$

If c denotes the length of the mercury filament in the horizontal position, when the contained air is at atmospheric pressure,

$$h(l-c)=(l-k)(h+k), \quad \text{or} \quad ch=k(h+k-l).$$

When the tube hangs vertically downwards, $\cos\theta=-1$, and

$$l-r=\tfrac{1}{2}(h+l)-\sqrt{\{\tfrac{1}{4}(h+l)^2-ch\}}.$$

This can be illustrated experimentally with a tumbler of height l, filled with water to a depth c, and closed by a card; on carefully inverting the tumbler, a certain fraction of the water will leak out, so that the depth of the water is reduced to

$$\tfrac{1}{2}(h+l)-\sqrt{\{\tfrac{1}{4}(h+l)^2-ch\}},$$

where h now denotes the height of the water barometer.

The same principle applies to the *pipette*; if it is dipped into liquid to a depth c cm, and a volume V cm^3 of air is imprisoned by the finger, while the area of the surface of the liquid inside is a cm^2; then, on raising the pipette out of the liquid, the surface will sink x cm, given by the positive root of the equation

$$\frac{h-c+x}{h}=\frac{V}{V+ax}.$$

207. *Say's Stereometer.*

This instrument, the invention of a French officer Say, is intended for the measurement of the density of gunpowder and other substances, which must not be contaminated with moisture or contact with liquids.

As the action of the instrument is an application of Boyle's Law, the description is introduced here, and illustrated by reference to the arrangement in fig. 66, p. 295.

The stop-cock is replaced by a three-way cock, which can establish communication between the branches AO and BK, between either branch and the reservoir of mercury, or between all three.

Two fixed marks are made on the tube AO, at A and H suppose, and the mercury filling AH is drawn off into the reservoir and weighed, and thence the volume, U cm³, of AH is inferred.

The mercury is now brought to the same level at A and B in the two branches, and in the reservoir, and a globe is screwed on at O, so that the air in AO and the globe is at atmospheric pressure.

The reservoir is now raised till the level of the mercury in the branches rises to H and K; and now denoting the vertical height HK by k, and the volume of the globe, reaching to H, by V cm³, then by Boyle's Law

$$\frac{V+U}{V}=\frac{h+k}{h}, \text{ or } \frac{V}{U}=\frac{h}{k};$$

which determines V.

The globe is now unscrewed, and a known weight W g of the substance, whose volume x cm³ and density W/x is required, is placed in it; the globe is again screwed on, and the operation is repeated.

Now if k' is the difference of level of the mercury in the two branches when the level in the branch AO rises to H,

$$\frac{V-x}{U}=\frac{h}{k'}, \text{ or } \frac{x}{U}=\frac{h}{k}-\frac{h}{k'}.$$

As an increase of pressure tends to make the gunpowder absorb air, the process may be reversed by starting with the level of the mercury in the two branches at H, and drawing off mercury by lowering the reservoir till the mercury stands at the level of A in AO.

208. *Graphical Representations of Boyle's Law.*

Putting $OA=a$, $OH=x$, $OK=y$, $KL=OO_1=OO_2=h$, in fig. 66, then according to Boyle's Law

$$OH \,.\, HL = OA \,.\, OO_2, \quad \text{or} \quad x(x+y+h)=ah;$$

a relation of the second degree in x and of the first in y.

Describe a circle on AO_2 as diameter, cutting the horizontal line through O in Q.

Now if the level of K is given, take C the middle point of KO_2, and with centre C and radius CQ describe a circle cutting the outside line OA in H and H'; then the point H in OA will be the corresponding level of the mercury.

For $OO_2=KL$, and therefore $OC=CL$, or $OH'=HL$;

and $$OA \,.\, OO_2 = OQ^2 = OH \,.\, OH' = OH \,.\, HL.$$

The other point H' is such that the air which occupies a length OA of the tube at atmospheric pressure, or under a head OO_2 of mercury, will occupy a length OH' under a head LH' or OH of mercury.

To determine K geometrically when H is given, take OH' the third proportional to OH and OQ, and mark off $H'K$ equal to HO_2.

209. The geometrical method of § 114 may be employed to give a graphical representation of the relative levels of H and K in the branches of the tube in fig. 66, and of the relative heights of O and H in fig. 67.

Draw through O, O_1, O_2 straight lines sloping at 45° to the horizon.

If H is projected horizontally to H_1 on the sloping line through O, and the vertical line $K_1H_1M_1$ is drawn to meet the horizontal line through K in K_1 and the sloping line through O_1 in M_1, then $x+y+h=M_1K_1$, so that $O_1M_1 \,.\, M_1K_1$ is constant; and the curve described by K_1, while H_1 describes the sloping line OH_1, is a hyperbola with the vertical and the sloping line through O_1 for asymptotes.

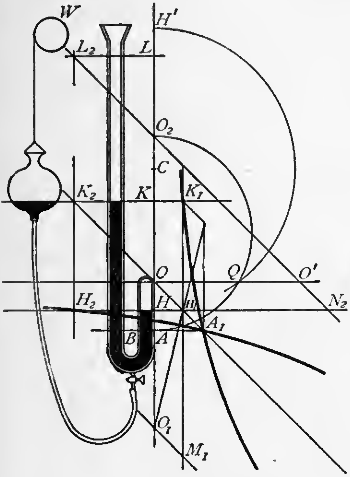


Fig. 66.

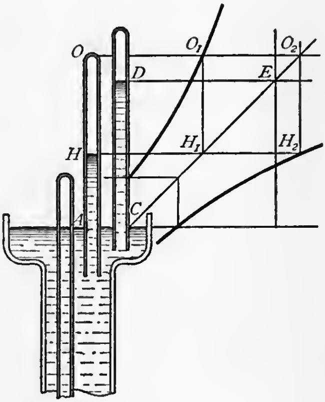


Fig. 67.

But projecting K horizontally to K_2 on the sloping line through O, drawing the vertical $L_2K_2H_2$ to meet the sloping line through O_2 in L_2, and drawing the horizontal line through H to meet this vertical in H_2 and the sloping line through O_2 in N_2, then, as before,

$$x+y+h=L_2H_2=H_2N_2,$$

so that $$OH \,.\, H_2N_2 \text{ is constant,}$$

and as K_2 describes the sloping line OK_2, H_2 describes a hyperbola with the horizontal line through O and the sloping line through O_2 for asymptotes.

These hyperbolas can be constructed geometrically in the manner previously explained in §§ 64, 115.

210. Similarly in fig. 67, assuming Boyle's Law to hold,

$$OH \,.\, HD = OA \,.\, CD,$$

or

$$(x-y)(h-y)=ah,$$

if x and y denote the heights of O and H above the level of the mercury in the cistern; so that if the sloping line CE is drawn through C, meeting the horizontal line through D in E, and if the points O and H are projected as before, then as H_1 describes CE, O_1 will describe a hyperbola with the vertical and sloping line through E for asymptotes; and as O_2 describes CE, H_2 will describe a hyperbola with the horizontal and the sloping line through E for asymptotes.

The value of h can be inferred from the observation of two positions of the tube OA; for if l, l' denote the lengths of the air column OH corresponding to heights y, y' of the mercury column AH, then

$$l(y-h)=l'(y'-h)-ah,$$

or

$$h=\frac{ly-l'y'}{l-l'}.$$

211. *Vitiated Vacuum of a Barometer.*

If the tube OA is fixed in position with O at the same level as the top of the barometric tube CD, the relative fluctuations of H and D will represent the barometric fluctuations of a barometer in which a small quantity of air has vitiated the Torricellian vacuum, compared with the indications of a standard barometer.

Now if b denotes the constant height of O above the mercury in the cistern,

$$(b-y)(h-y)=\text{a constant}=ab \text{ suppose,}$$

where h and y fluctuate, assuming that the temperature is constant.

Thus if the vitiated barometer reads 29·5 ins when the standard barometer reads 30 ins, the correction to be added to the reading y of the vitiated barometer is

$$\frac{1·25}{32-y}.$$

Also, if a vitiated siphon barometer reads 31 ins when the true reading is 32, and the length of the vacuum is then one inch, the correction to be added to the observed height of 29 ins is 0·5 inch; and generally to an observed height y is $$2/(23-y).$$

Referred as before to the sloping line through O (fig. 68) D_2 will describe a hyperbola with the vertical and sloping line through O for asymptotes, as H_2 describes the line OH_2, whence the barometric correction D_2H_2 for the height y in the vitiated barometer can be measured off from the diagram; and conversely, as D_1 describes the sloping line, H_1 will describe a hyperbola with the horizontal and sloping line through O for asymptotes.

212. Denoting by y_1, y_2 the readings of this vitiated barometer when the true barometric heights are h_1, h_2,

$$(b-y)(h-y)=(b-y_1)(h_1-y_1)=(b-y_2)(h_2-y_2);$$

$$b=\frac{y_1(h_1-y_1)-y_2(h_2-y_2)}{(h_1-y_1)-(h_2-y_2)},$$

and the correction to be added to the height y is

$$h-y=\frac{(h_1-y_1)(h_2-y_2)(y_1-y_2)}{(y_1-y)(h_1-y_1)-(y_2-y)(h_2-y_2)}.$$

Thus if the readings of a vitiated barometer are 29·9 and 29·4 ins when the true barometric heights are 30·4 and 29·8, then $b=31·9$, so that the length of the vitiated vacuum when the reading is 29 is 2·9 ins; and the correction to be added to the reading y is $1/(31·9-y)$.

But to deduce the barometric correction from the comparison with another vitiated barometer, of height c, in which the corresponding readings are z, z_1, z_2, the additional equations

$$(c-z)(h-z)=(c-z_1)(h_1-z_1)=(c-z_2)(h-z_2)$$

will determine h_1 and h_2; and thence we find that the corrections to the observed heights y, z are

$$h-y=\frac{(b-y_1)(b-y_2)\{(c-z_1)(y_1-z_1)-(c-z_2)(y_2-z_2)\}}{(b-y)\{(b-y_1)(c-z_2)-(b-y_2)(c-z_1)\}},$$

$$h-z=\frac{(c-z_1)(c-z_2)\{(b-y_1)(y_1-z_1)-(b-y_2)(y_2-z_2)\}}{(c-z)\{(b-y_1)(c-z_2)-(b-y_2)(c-z_1)\}}.$$

If the temperature varies, and is denoted by τ, τ_1, τ_2 C at the three observations, then

$$\frac{(b-y)(h-y)}{273+\tau}=\frac{(b-y_1)(h_1-y_1)}{273+\tau_1}=\frac{(b-y_2)(h_2-y_2)}{273+\tau_2};$$

so that, eliminating b,

$$\frac{(y_1-y_2)(273+\tau)}{h-y}+\frac{(y_2-y)(273+\tau_1)}{h_1-y_1}+\frac{(y-y_1)(273+\tau_2)}{h_2-y_2}=0.$$

Merely inclining the tubes would cause the true and vitiated barometers to fluctuate differently, and the arrangement has been suggested as a *gravimeter* for measuring a change in g (*Comptes Rendus*, June–July, 1893); for if h' and y' denote the barometric heights when g has changed to g', and the temperature does not change,

$$g'(h'-y')(b-y')=g(h-y)(b-y).$$

Work out, for example, the value of g' when $g=981$,

$$h=h'=76,\ b=100,\ y=38 \text{ cm},\ y-y'=10\mu\mu.$$

213. Suppose the vacuum of the barometer OA was originally perfect, and that the air which vitiates the vacuum was introduced as a small bubble, which at atmospheric pressure occupied a length AB of the tube.

As the bubble PQ rises slowly and steadily in the tube, by the gradual transfer of mercury from above to below the bubble, the height of the superincumbent mercury QR is the mercury head of the pressure in the bubble, and is therefore equal to the difference of level of P and D, and inversely proportional to PQ.

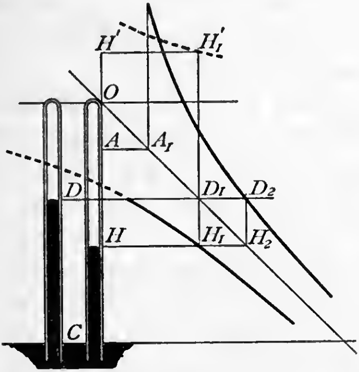


Fig. 68.

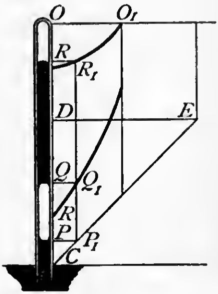


Fig. 69.

If therefore P is projected horizontally to P_1, on to the sloping line CE (fig. 69), the corresponding point Q_1 will describe a hyperbola with the vertical and sloping line through E for asymptotes; and the corresponding point R_1 will describe a rectangular hyperbola, with horizontal and vertical asymptotes through E.

When R reaches O, the column QR may remain fixed; but practically the mercury will gradually trickle down the sides of the tube, or the bubble will otherwise make its way to the top, until the air bubble is entirely above the mercury, and we have the vitiated barometer.

Suppose for instance that a barometer stands at 30 ins, and that the length of the Torricellian vacuum is then 2 ins, so that $b=32$.

If a bubble of air, which at atmospheric pressure would occupy half an inch of the tube, is now introduced, the barometric column will fall 3 inches; and the correction to be added to a reading y is $15/(32-y)$.

214. The behaviour of rising bubbles can be studied in soda-water, champagne, or water boiled in a glass vessel; the pressure diminishes and the bubbles expand as they rise, until they reach the surface and burst.

Denoting by V the volume of a bubble at the surface and by h the height of the water barometer (about 400 ins or 10 m), then at a depth h the pressure is doubled; and therefore, by Boyle's Law, the volume of the bubble is halved and the density of the air in it is doubled; at a depth $(n-1)h$ the pressure will be increased n fold and the volume will become V/n; and generally at a depth z the pressure will be $1+z/h$ atmospheres, and the volume of the bubble will be $Vh/(h+z)$.

Denoting by k the height of the homogeneous atmosphere (or of the air barometer) so that h/k is the S.G. of air, then when

$$h+z=k \quad \text{or} \quad z=k-h,$$

the density of the bubble will be the same as that of water, and the bubble will be in equilibrium.

Taking $\qquad h=10 \text{ m}, \qquad k=8000 \text{ m},$

then $\qquad k-h=7990 \text{ m},$

the depth in the sea at which a bubble of air will be in equilibrium; also

$$k/h=800,$$

the pressure in the bubble in atmospheres.

The density of hydrogen being about one-fourteenth of that of air at the same pressure, a bubble of hydrogen will be in equilibrium at about 14 times this depth.

In Metric units, if the volume of the bubble at the surface is V cm³, its weight will be practically Vh/k g; and at a height x cm above the position of equilibrium, the volume or buoyancy will be $Vh/(k-x)$ cm³ or g; so that the upward moving force on the bubble will be

$$V\left(\frac{h}{k-x}-\frac{h}{k}\right)=\frac{Vhx}{k(k-x)}\text{ g}.$$

215. Allowing, as in § 149, for the effective inertia of the displaced water, spherical in form, as half the weight of water displaced, the upward acceleration of the bubble is

$$\frac{\dfrac{Vhx}{k(k-x)}}{\dfrac{Vh}{k}+\frac{1}{2}\dfrac{Vh}{k-x}}g=\frac{x}{\frac{3}{2}k-x}g.$$

The equilibrium of the bubble at a depth $k-h$ is consequently unstable; if slightly depressed, the bubble is compressed and sinks to the bottom; but if it rises the bubble expands and reaches the surface.

216. By imprisoning the bubble in a glass bottle which is ballasted so as to be on the point of sinking, we can study the equilibrium and stability near the surface, and this arrangement is now called a *Cartesian Diver;* it illustrates the action of the bladder in a fish, and incidentally the dangerous instability of a submarine boat.

Denoting by W g the weight of the bottle, its S.G. by s, and by V cm³ the volume at atmospheric pressure of the imprisoned air, or the weight of water in g the bottle will hold, then at a depth z cm the volume and buoyancy of the bubble is $Vh/(h+z)$; and this is equal to $W-W/s$, and the bottle is in equilibrium, when

$$z=h\left(\frac{V}{W}\frac{s}{s-1}-1\right).$$

When z is greater than the depth of water, the Cartesian Diver will float on the surface, and to depress it, we must have means for increasing h; this is effected by covering the vessel of water with a bladder, which can be pressed in by the hand; and now if h is increased to h', the volume of the bubble at a depth z' becomes $Vh/(h'+z')$; so that in the position of equilibrium

$$h'+z'=h+z,$$

or z' diminishes as h' increases, by an equal amount; and when z' is negative, the diver will sink.

A Cartesian Diver is readily constructed of a glass bottle, ballasted by lead wire wound round the neck; thus a pint bottle can be so weighted that it just floats in water when one quarter of an ounce is placed in it; and now if inverted neck downwards over water, it is in unstable equilibrium when the level of the water inside it is at a depth of a little over 5 inches, the height of the water barometer being 400 inches; and increasing this to over 405 inches will cause the diver to sink.

217. *Ebullition and the Laws of Vapour Pressure.*

When water or a liquid boils, bubbles of vapour are formed in the interior, which rise to the surface and burst; and the pressure of the vapour in a bubble may be taken as that of the surrounding liquid.

Any cause which tends to the formation and disentanglement of bubbles, as for instance the shaking of a locomotive boiler, is of assistance to ebullition, and improves the performance of the boiler.

At the surface the pressure of the vapour given off during ebullition is equal to that of the external air or vapour.

A *saturated vapour* is the name given to a gas formed in contact with the liquid from which it is derived by ebullition; and the "Law of Saturated Vapours" asserts that "the pressure of a saturated vapour is a function of the temperature alone, and not of the volume"; for if the volume of the vapour in contact with its liquid is increased or diminished, more vapour is evaporated or condensed, so that the pressure is unaltered, if the temperature does not change.

The pressure of the vapour is determined experimentally by observing the depression of the column CH in fig. 68 at different temperatures when a small bubble of the liquid just small enough to evaporate completely is introduced into the tube.

A Table has been constructed and a curve drawn out from experiments by Regnault, giving the pressure of aqueous vapour or steam, expressed in millimetres of mercury head for temperatures below the standard boiling point, reaching from about 4·6 mm at 0° C to 760 mm at 100° C; and expressed in atmospheres for temperatures above the standard boiling point, reaching to about 5 atmospheres at 200 C.

In determining the boiling point of a thermometer it is thus important to observe the barometric height, as it is found that the boiling point varies about 1° C for 27 mm of mercury head; and as the pressure and therefore the temperature of the bubbles increase with the depth in the water, the bulb of the thermometer should be in the steam, and not immersed in the water.

From Regnault's Table we can calculate the depth at which water of given temperature can be made to boil, when the pressure on the surface is suddenly diminished, as for instance in the receiver of an air pump.

Thus at 100 C the vapour pressure is one atmosphere of 760 mm; so that if the pressure is suddenly reduced to a fraction m of an atmosphere, the water will boil to a depth $(1-m)$ of the height of the water barometer.

218. According to Dalton's Law (§ 201), if h denotes the total pressure of the air, in mm of mercury head, and f mm the pressure of the aqueous vapour present, then $h-f$ will be the pressure of the dry air alone.

The density of the aqueous vapour is found experimentally to be about $\frac{5}{8}$ of the density of dry air at the same pressure; so that, if a litre of dry air weighs 1·2932 g at 0° C and 760 mm of mercury head, then at a temperature of τ C, a barometric height h mm, and an aqueous pressure of f mm, a litre of air weighs

$$\frac{1{\cdot}2932}{1+\alpha\tau}\left(\frac{h-f}{760}+\frac{5}{8}\frac{f}{760}\right)\text{ (g)};$$

and the density ρ of this damp air, in g/cm³, is given by

$$\rho=\frac{0{\cdot}0012932}{1+\alpha\tau}\,\frac{h-\frac{3}{8}f}{760};$$

so that damp air is perceptibly lighter than dry air.

It is important therefore that the air should be carefully dried in all experiments on Boyle's Law; so that the mode of experimenting with a gas in a vessel inverted over water is unsuitable for accurate measurements.

219. *Corrections for Weighing in Air.*

When a weight is marked, say P g, this means the *absolute* weight, as determined *in vacuo*.

But if B denotes the density of the metal of the weight, and ρ the density of the air, the *apparent* weight is diminished, by the buoyancy of the air $P\rho/B$, to

$$P\left(1-\frac{\rho}{B}\right).$$

At the temperature τ

$$B = B_0(1 - c\tau),$$

where c denotes the coefficient of cubical expansion of the metal, and B_0 the density at 0° C; also ρ is given by the formula in the last article.

When therefore, as in § 57, a body of some other density σ is being weighed in air, and its true weight is W g when equilibrated by the weight marked P g,

$$P\left(1 - \frac{\rho}{B}\right) = W\left(1 - \frac{\rho}{\sigma}\right),$$

or

$$W - P = P\left(\frac{\rho}{\sigma} - \frac{\rho}{B}\right) \Big/ \left(1 - \frac{\rho}{\sigma}\right),$$

and

$$\sigma = \sigma_0/(1 + C\tau),$$

if C denotes the coefficient of expansion of the body.

We might, for instance, be weighing W g of hydrogen, in which case the negative value obtained for P would mean that the weight P would have to be placed in the same scale as the "parcel" of hydrogen.

If the body is now weighed by the Hydrostatic Balance in water, of density D_τ at τ° C, and equilibrated by a weight marked P' g, then

$$P'\left(1 - \frac{\rho}{B}\right) = W\left(1 - \frac{D_\tau}{\sigma}\right);$$

and therefore

$$\frac{P'}{P} = \frac{\sigma - D_\tau}{\sigma - \rho},$$

$$\sigma = \frac{PD_\tau - P'\rho}{P - P'},$$

$$\sigma_T = \sigma\{1 + C(\tau - T)\},$$

the density at standard temperature T.

Here D_τ may be calculated from the formula of § 170, or preferably taken from Table I., in consequence of the abnormal dilatation of water.

220. *Vapours and their Critical Temperature.*

When a saturated vapour is isolated from contact with the liquid from which it is formed, and the temperature is raised, the vapour is said to be *superheated*; and the vapour now begins to obey approximately the Gaseous Laws of § 197.

If not too much superheated, the vapour can by compression be brought again to the *saturated* state; and further compression liquefies the vapour in part.

But Andrews found (1871) that beyond a certain *critical temperature* of the vapour liquefaction by compression became impossible; and the Gaseous Laws were now obeyed very closely.

For this reason it is advisable to divide the gaseous state of matter (§ 2) into two classes:

(i.) *true gases*, which cannot be liquefied, because heated above the critical temperature, and which obey the Gaseous Laws of § 197 very closely;

(ii.) *vapours*, which can be ultimately liquefied by compression, the temperature being below the critical point; and which show increasing divergence from these Gaseous Laws.

Thus, for instance, the Critical Temperatures of water, chlorine, oxygen, and nitrogen are, respectively,

370, 141, −113, and −146 C.

The Characteristic Equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=R\theta,$$

where a and b are small constants, has been proposed as a generalised form of the Characteristic Equation of a Perfect Gas, (A) § 198, by Van der Waals in his *Essay on the Continuity of the Liquid and Gaseous State of Matter*, 1873, a translation of which is published by the Physical Society of London.

This equation is intended to serve for large variations above and below the critical temperature of a substance.

The pressure p is infinite when $v=b$, so that according to the formula b, the *covolume* (§ 203), would appear to be the *ultimate* volume of the substance, below which it cannot be compressed; but Van der Waals assigns reasons for supposing that b is about four times the ultimate volume. Thus there is evidence that liquids cannot be compressed below 0·2 to 0·3 of their volume at ordinary atmospheric pressure.

221. *The Air Thermometer.*

Charles's Law is really equivalent to the definition of equal degrees of temperature as being those for which a perfect gas (say hydrogen) receives equal increments of volume under constant pressure.

The Air Thermometer differs only from the ordinary mercury thermometer in having the bulb filled with air instead of mercury, the air being confined by a small filament of mercury in the stem.

As with the hydrometer (§ 64) the bulb may be supposed replaced by a length of the stem of equivalent volume, sealed at one end; and now if the positions of the mercury at the freezing and boiling points are marked, the air being at the same pressure, and this interval is divided in 100 degrees Centigrade, then it will be found that the distances of these points from the sealed end contain 273 and 373 degrees, called *absolute degrees;* and, according to Charles's Law, the Air Thermometer will give the same records when filled with any other so-called perfect gas (hydrogen for instance).

Heating the air from 0 to 200 C would cause the volume to expand from 273 to 473, so that the density of the air would be about that of coal gas, as employed in balloons.

The mercury thermometer is preferred for ordinary purposes, as its indications are independent of the height of the barometer, which requires to be observed and allowed for in a reading of the Air Thermometer.

For if the barometer changes from the standard height h to h', the reading of absolute temperature θ on the air thermometer corresponds to a true temperature $\theta h'/h$.

Thus if the barometer falls from 30 to 29 ins, the reading 300 corresponds to a true absolute temperature 290.

If, however, the open end is made to dip into a vessel of mercury, like the tube *OA* in fig. 67, p. 295, in which the level of the mercury can be raised or lowered, the air in the thermometer can be easily brought to a standard atmospheric pressure, measured by a fixed head *HD* of mercury.

222. The *Differential Air Thermometer* consists of two equal glass bulbs containing air, connected by a horizontal uniform tube in which a small filament of mercury or liquid separates the air in the two bulbs. A small difference of temperature of the two bulbs is recorded by the displacement of the filament.

Denoting by V_1 and V_2 cm^3 the volumes of the two portions of air when the bulbs are at the same temperature τ, and by ω cm^2 the cross section of the tube, then when the bulbs are raised to temperature τ_1 and τ_2 C, the filament will be displaced from zero through a distance x cm, such that, the pressure in each bulb being changed from P to p,

$$\frac{p(V_1-\omega x)}{273+\tau_1}=\frac{PV_1}{273+\tau}, \quad \frac{p(V_2+\omega x)}{273+\tau_2}=\frac{PV_2}{273+\tau};$$

or
$$\frac{V_1-\omega x}{V_2+\omega x}=\frac{273+\tau_1}{273+\tau_2}\frac{V_1}{V_2}=\frac{\theta_1}{\theta_2}\frac{V_1}{V_2},$$

if θ_1, θ_2 denote the corresponding absolute temperatures.

If, as usual, $V_1=V_2=V$ suppose,

$$\frac{\omega x}{V}=\frac{\frac{1}{2}(\tau_2-\tau_1)}{273+\frac{1}{2}(\tau_2+\tau_1)}=\frac{\theta_2-\theta_1}{\theta_2+\theta_1};$$

and now if the temperature of each bulb is increased by τ', the filament will reach a distance y, given by

$$\frac{\omega(x-y)}{V}=\frac{\frac{1}{2}(\tau_2-\tau_1)}{273+\frac{1}{2}(\tau_2+\tau_1+2\tau')}=\frac{\theta_2-\theta_1}{\theta_2+\theta_1+2\tau'},$$

or
$$\frac{x-y}{x}=\frac{\theta_2+\theta_1}{\theta_2+\theta_1+2\tau'},$$

so that the graduations are in H.P. for equal increments in τ'.

223. *Isothermal Equilibrium of the Atmosphere.*

In ascending or descending in a quiescent atmosphere of uniform temperature it is found that the logarithm of the barometric height or of the pressure changes uniformly with the height; or expressed analytically, the barometric height h, the pressure p, and the density ρ at any height z are connected with the corresponding quantities, h_0, p_0, ρ_0 at the ground by the formula

$$\frac{h}{h_0}=\frac{p}{p_0}=\frac{\rho}{\rho_0}=e^{-\frac{z}{k}},$$

where
$$k=\frac{p}{\rho}=\frac{p_0}{\rho_0}$$

is the height of the homogeneous atmosphere; so that at a height k the barometer stands at $1/e$ of its height on the ground.

Taking logarithms,

$$z = k \log_e(h_0/h) = k \log(p_0/p)$$
$$= \mu k(\log_{10} h_0 - \log_{10} h),$$
$$z_1 - z_2 = \mu k(\log_{10} h_2 - \log_{10} h_1),$$

where $\mu = \log_e 10 = 2{\cdot}30258509 \approx 2{\cdot}3,$

the modulus which converts common into Naperian logarithms.

Expanded in powers of $(h_0 - h)/(h_0 + h)$,

$$z = k \log_e \frac{h_0}{h} = 2k \tanh^{-1} \frac{h_0 - h}{h_0 + h}$$
$$= 2k\left\{\frac{h_0 - h}{h_0 + h} + \frac{1}{3}\left(\frac{h_0 - h}{h_0 + h}\right)^3 + \ldots\right\},$$

of which the first term only is retained in *Babinet's barometric formula*, employed when the barometric change is small.

Thus if the barometer falls from 30 to 27 inches in ascending 2800 ft, we find that $k = 26{,}600$ ft; and the barometer will fall to 21·87 inches at a height of 8400 ft.

With British units, at a temperature 32 F,

$$k = 26{,}214 \text{ ft},$$

and $\mu k = 60{,}300 \text{ ft} \approx 10{,}000$ fathom;

so that the difference of the common logarithms of the barometric heights multiplied by 10,000 gives the difference of level in fathoms.

For a height of 7 miles or $7 \times 880 = 6160$ fathom,

$$\log(h_0/h) = {\cdot}6160, \quad h_0/h \approx 4.$$

On Snowdon, 3720 ft high,

$$\log(h_0/h) = 0{\cdot}062, \quad \log(h/h_0) = \bar{1}{\cdot}938, \quad h/h_0 = 0{\cdot}867;$$

so that if $h_0 = 30$, $h = 26$, $h_0 - h = 4$, a fall of four inches.

Down a mine the pressure will be n atmospheres at a depth $k \log_e n$; thus for $n = 2$, the depth would have to be about 18,720 ft.

With Metric units, at a temperature τC,

$$k = 8000(1 + a\tau), \text{ where } a = 0{\cdot}003665,$$

$$z_1 - z_2 = 18{,}400(1 + a\tau)\log_{10}(h_2/h_1), \text{ metres.}$$

224. *Laplace's Barometric Formula.*

The atmosphere is supposed of uniform temperature τ, the arithmetic mean $\frac{1}{2}(\tau_1 + \tau_2)$ of the temperatures τ_1 and τ_2 at the two stations; and since a is slightly increased by the presence of aqueous vapour, Laplace puts

$$a = 0{\cdot}004,$$

so that
$$1 + a\tau = 1 + \frac{2(\tau_1 + \tau_2)}{1000}.$$

To allow for the variation in g, given in latitude λ and at height h, a fraction h/R of the Earth's radius R, by Bouguer's and Clairaut's formula

$$g = G\left(1 - \frac{2h}{R} - 0{\cdot}00266 \cos 2\lambda\right),$$

Laplace introduces the factor

$$\frac{G}{g} = 1 + \frac{2h}{R} + 0{\cdot}00266 \cos 2\lambda,$$

because k is inversely proportional to g (§ 199); and now he writes the formula

$$z_1 - z_2 = 18400\left(1 + \frac{2h}{R} + 0{\cdot}00266 \cos 2\lambda\right)\left\{1 + \frac{2(\tau_1 + \tau_2)}{1000}\right\}\log_{10}\frac{h_2}{h_1},$$

in which h may be replaced by the mean height $\frac{1}{2}(z_1 + z_2)$, supposed known approximately.

This gives the height in metres; and to give it in feet, 18400 must be replaced by 60369, the equivalent number of feet. Also on the Fahrenheit scale, the temperature factor becomes

$$1 + \frac{\tau_1 + \tau_2 - 64}{900}.$$

225. To prove the formula of § 223, employing the Integral Calculus, suppose that $-dp$ denotes the diminution of pressure in ascending a height dz in a stratum of density ρ; then

$$dp = -\rho dz.$$

But if

$$p = k\rho,$$

$$\frac{dp}{p} = -\frac{dz}{k},$$

and, integrating, $\log p - \log p_0 = -\frac{z}{k}$, or $\frac{p}{p_0} = e^{-\frac{z}{k}}$.

The proof of this theorem is not easy by elementary methods, not involving the use of the Calculus; the following elementary proof is submitted :—

Suppose the air between two horizontal planes at a distance z, at which the pressures are p and p_0, is divided into a large number n of strata, of equal depth z/n; and that the density in each stratum is taken as uniform, the density in the rth stratum being denoted by ρ_r, and the pressure being denoted by p_r at the bottom and by p_{r+1} at the top of the stratum.

Then considering the equilibrium of the vertical column of these strata, standing on a base of unit area,

$$p_r - p_{r+1} = \rho_r \frac{z}{n} = p_r \frac{z}{nk},$$

or

$$\frac{p_{r+1}}{p_r} = 1 - \frac{z}{nk};$$

so that $\frac{p_1}{p_0} = \frac{p_2}{p_1} = \frac{p_3}{p_2} = \ldots = \frac{p_{r+1}}{p_r} = \ldots = \frac{p_n}{p_{n-1}} = 1 - \frac{z}{nk}$, the pressure decreasing in G.P. as we ascend in A.P. by equal vertical steps z/nk; and therefore

$$\frac{p_n}{p_0} = \left(1 - \frac{z}{nk}\right)^n.$$

Now making n indefinitely great, and putting $p_n = p$,

$$\frac{p}{p_0} = \operatorname{lt}\left(1 - \frac{z}{nk}\right)^n$$

$$= 1 - \frac{z}{k} + \frac{1}{2!}\left(\frac{z}{k}\right)^2 - \frac{1}{3!}\left(\frac{z}{k}\right)^3 + \dots$$

$$= e^{-\frac{z}{k}}, \text{ or } \exp\left(-\frac{z}{k}\right).$$

We notice that p and ρ are zero when z is infinite; so that on our theoretical assumptions the height of the atmosphere of uniform temperature is infinite.

226. *Convective Equilibrium of the Atmosphere.*

According to the experience of mountaineers and aeronauts, the temperature of a quiescent atmosphere is not uniform, but diminishes slowly with the height at a rate which may be supposed uniform; so that we put

$$\theta = \theta_0\left(1 - \frac{z}{c}\right),$$

connecting θ the absolute temperature at the height z with θ_0 the absolute temperature at the ground; and c will now represent the theoretical height of the atmosphere, the absolute temperature and the pressure being zero at this height.

Then the characteristic equation of the air becomes

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}\frac{\theta}{\theta_0} = k\left(1 - \frac{z}{c}\right);$$

so that in gravitation units, as in § 225,

$$\frac{dp}{p} = -\frac{\rho dz}{p} = -\frac{1}{1 - \frac{z}{c}}\frac{dz}{k};$$

and integrating,

$$\log p - \log p_0 = \frac{c}{k}\log\left(1 - \frac{z}{c}\right),$$

or
$$\frac{p}{p_0}=\left(1-\frac{z}{c}\right)^{\frac{c}{k}}=\left(\frac{\theta}{\theta_0}\right)^{\frac{c}{k}},$$

$$\frac{\rho}{\rho_0}=\frac{p}{p_0}\frac{\theta_0}{\theta}=\left(1-\frac{z}{c}\right)^{\frac{c}{k}-1}=\left(\frac{\theta}{\theta_0}\right)^{\frac{c}{k}-1}=\left(\frac{p}{p_0}\right)^{1-\frac{k}{c}}.$$

Then for two stations, at altitude z_1 and z_2,

$$\frac{z_1-z_2}{c}=\frac{\theta_2-\theta_1}{\theta_0}=\left(\frac{p_2}{p_0}\right)^{\frac{k}{c}}-\left(\frac{p_1}{p_0}\right)^{\frac{k}{c}}=\left(\frac{\rho_2}{\rho_0}\right)^{\frac{k}{c-k}}-\left(\frac{\rho_1}{\rho_0}\right)^{\frac{k}{c-k}}.$$

If we put $\qquad \dfrac{c}{c-k}=\gamma, \quad$ or $\quad \dfrac{c}{k}=\dfrac{\gamma}{\gamma-1},$

then $\qquad p \propto \rho^\gamma$, or pv^γ is constant,

if v denotes the S.V. of the air; and, as will be shown hereafter, this is the relation connecting p and v when the air expands *adiabatically*, that is without parting with any of its heat, if γ denotes the ratio of the specific heats of air at constant pressure and constant volume.

227. By Conduction of Heat a quiescent atmosphere tends to Isothermal Equilibrium; but considering the extreme slowness of conduction in air compared with the rapidity with which the air is carried to different heights by the wind, Sir W. Thomson has proposed the *Convective Equilibrium* as more representative of normal conditions (*Manchester Memoirs*, 1865).

According to his hypothesis the interchange of the same weight W lb of gas at any two places A and B without loss or gain of heat by conduction (*adiabatically*) would simply interchange the pressure, density, and temperature, so that no real change would ensue; W lb of gas moving from A to B through an imaginary non-conducting pipe ACB, and an equal weight W lb from B replacing it at A through another imaginary pipe BDA.

In this manner the condition $p=\lambda\rho^\gamma$, or $pv^\gamma=\text{constant}$, is realised; and, as above, the temperature diminishes uniformly with the height, and the theoretical height of the atmosphere is $\gamma/(\gamma-1)$ times the height of the homogeneous atmosphere at the ground.

As the result of experiment we may put $\gamma=1{\cdot}4$, so that
$$\gamma/(\gamma-1)=3{\cdot}5.$$

This makes the height of the atmosphere about 28,000 m, 28 km, or 17 miles.

228. The best experimental determination of γ is from the observed velocity of sound; denoting it by a f/s, then theory shows that the velocity of sound is that acquired in falling vertically through $\frac{1}{2}\gamma k$ ft; so that
$$a^2=g\gamma k=g\gamma p/\rho, \quad \gamma=a^2/gk.$$

At the freezing temperature, $a=1093$, $k=26214$; so that, with $g=32{\cdot}19$,
$$\begin{aligned}\log a\ &=3{\cdot}0386,\\ \log a^2&=6{\cdot}0772,\\ \log g\ &=1{\cdot}5077,\\ \log k\ &=4{\cdot}4185,\\ \log gk&=5{\cdot}9262,\\ \log\gamma\ &=0{\cdot}1510,\\ \gamma\ &=1{\cdot}416, \text{ say } 1{\cdot}4.\end{aligned}$$

Now, if zero suffixes refer to the ground level, and θ denotes the absolute temperature at a height z, then the differential equation of equilibrium is
$$\frac{dp}{p}=-\frac{\rho dz}{p}=-\frac{\rho_0}{p_0}\frac{\theta_0}{\theta}dz=-\frac{\theta_0}{\theta}\frac{dz}{k_0}=-\frac{g\gamma\theta_0}{a_0{}^2\theta}dz,$$

This leads, on integration, to
$$\log\frac{p_0}{p}=\frac{g\gamma\theta_0}{a_0{}^2}\int_0^z\frac{dz}{\theta}.$$

229. Starting with the experimental relation

$$p=\lambda\rho^\gamma,$$

we can employ the elementary method of § 225 to determine the pressure, density, and temperature, at any height z; and now the equation

$$p_r-p_{r+1}=\rho_r\frac{z}{n}$$

becomes

$$\rho^\gamma{}_r-\rho^\gamma{}_{r+1}=\rho_r\frac{z}{n\lambda},$$

$$\left(\frac{\rho_{r+1}}{\rho_r}\right)^\gamma=1-\frac{1}{\rho_r{}^{\gamma-1}}\frac{z}{n\lambda},$$

$$\left(\frac{\rho_{r+1}}{\rho_r}\right)^{\gamma-1}=\left(1-\frac{1}{\rho_r{}^{\gamma-1}}\frac{z}{n\lambda}\right)^{\frac{\gamma-1}{\gamma}}$$

$$=1-\frac{\gamma-1}{\gamma}\frac{1}{\rho_r{}^{\gamma-1}}\frac{z}{n\lambda},$$

neglecting squares and higher powers of z/n;

$$\rho_r{}^{\gamma-1}-\rho_{r+1}{}^{\gamma-1}=\frac{\gamma-1}{\gamma}\frac{z}{n\lambda};$$

so that the $(\gamma-1)$th powers of the density, and therefore the $(\gamma-1)/\gamma$th powers of the pressure diminish uniformly with the height; and finally

$$\rho_0{}^{\gamma-1}-\rho^{\gamma-1}=\frac{\gamma-1}{\gamma}\frac{z}{\lambda},\quad p_0{}^{\frac{\gamma-1}{\gamma}}-p^{\frac{\gamma-1}{\gamma}}=\frac{\gamma-1}{\gamma}\frac{z}{\lambda^{\frac{1}{\gamma}}}.$$

Also

$$\frac{\theta}{\theta_0}=\frac{p}{\rho}\frac{\rho_0}{p_0}=\frac{\lambda}{k}\rho^{\gamma-1},$$

so that

$$\theta_0-\theta=\frac{\gamma-1}{\gamma}\frac{z}{k}\theta_0=\frac{z}{c}\theta_0;$$

and p, ρ, θ vanish when

$$z=c=\frac{\gamma k}{\gamma-1},$$

the theoretical height of the atmosphere.

230. Denoting the depth $c-z$ below the free surface of the atmosphere by x,

$$\rho^{\gamma-1}=\frac{\gamma-1}{\gamma}\frac{x}{\lambda},\quad p^{\frac{\gamma-1}{\gamma}}=\frac{\gamma-1}{\gamma}\frac{x}{\lambda^{\frac{1}{\gamma}}};$$

or putting $\gamma-1=1/n$,

$$\rho=\frac{x^n}{(n+1)^n\lambda^n}=\mu x^n, \text{ suppose};$$

$$p=\frac{x^{n+1}}{(n+1)^{n+1}\lambda^n}=\frac{\mu x^{n+1}}{n+1}=\frac{\rho x}{n+1}.$$

We thus obtain the expression for the pressure p when the density ρ varies as some power n of the depth below the free surface; and if the fluid is now supposed incompressible, a pressure P applied at the free surface is transmitted without change throughout the fluid; and

$$p=P+\frac{\rho x}{n+1}.$$

Suppose for instance that $\gamma=2$, so that the density is proportional to the square root of the pressure, or the excess of the pressure over a standard pressure P; then

$$p=P+\lambda\rho^2,$$

the law employed by Laplace in calculating the density of the strata of the Earth; then $n=1$, and the density varies as the depth.

231. When the compressibility of water is taken into account in determining the pressure and compression in a deep ocean, we employ the experimental law

$$\frac{v}{v_0}=\frac{\rho_0}{\rho}=1-\lambda p,$$

connecting v_0, the volume of a lb of water at atmospheric pressure and ρ_0 the density, with v the volume and ρ the density at a pressure of p atmospheres more; also

λ is *the coefficient of cubical compression per atmosphere*, and we may put

$$\lambda = 0{\cdot}00005.$$

Then if h denotes the height of the water barometer, and x denotes the depth to which water of depth x_0 is reduced by the compressibility,

$$p = \frac{x_0}{h} = \frac{\rho x}{\rho_0 h},$$

where ρ denotes the average density, which may be taken as the density at the mean depth $\frac{1}{2}x_0$.

Then $$\frac{x}{x_0} = \frac{\rho_0}{\rho} = 1 - \lambda\frac{\frac{1}{2}x_0}{h};$$

so that the surface is lowered by the compression

$$x_0 - x = \tfrac{1}{2}\lambda x_0^2/h.$$

Thus if $x_0 = 100\,h$, and $h = 33$ ft, the surface is lowered about 8 ft.

Similarly it is calculated that the depth of an ocean 6 miles deep is lowered about 620 ft by compression, corresponding to $\lambda = 0{\cdot}00004$; and that, if incompressible, the Ocean would have its surface 116 ft higher, and cover two million square miles of land.

So also, in allowing for the compression of the mercury in his experiments on Boyle's Law (§ 203) Regnault took $\lambda = 0{\cdot}000004628$ per metre head of mercury; and now

$$p = x_0 \quad \text{and} \quad x_0 - x = \tfrac{1}{2}\lambda x_0^2.$$

Thus a column of mercury 25 m high is shortened 1·45 mm, which is negligible.

232. Reckoning x downwards from the free surface, and using the gravitation unit of force,

$$dp = \rho dx$$

or $$p = P + \int \rho dx,$$

so that if the density ρ is represented on a diagram by the horizontal ordinates of a vertical axis Ox, the pressure p will be represented by the area of the curve of density; and a separate curve of pressure can be plotted from this condition, just as curves of tons per inch immersion and curves of displacement in fig. 42, p. 168.

Thus if the density is uniform, the curve of pressure is a straight line, as in fig. 20, p. 43; if the density increases uniformly with the depth, the curve of pressure is a parabola, and the density varies as the square root of the pressure; and so on.

Take the case of the curve of pressure in going from water into air; at a depth x in water

$$p=P+Dz,$$

represented by a straight line; at a height z in an atmosphere in Thermal Equilibrium,

$$p=P\exp(-z/k),$$

represented by the Exponential Curve, in which the subtangent is constant and equal to k; and in an atmosphere in Convective Equilibrium,

$$p=P\left(1-\frac{\gamma-1}{\gamma}\frac{z}{k}\right)^{\frac{\gamma}{\gamma-1}}.$$

233. The work required to compress a substance from volume $v+\Delta v$ to v ft³ by the application of an average external pressure of p lb/ft² is $p\Delta v$ ft-lb.

Thus if Δv ft³ of atmospheric air is forced into a receiver of volume v, filled with air at atmospheric pressure p and density ρ, the increase of density is $\rho(\Delta v/v)$, and of pressure is $p(\Delta v/v)$, when thermal equilibrium is established; and the average increase of pressure being $\frac{1}{2}p(\Delta v/v)$ lb/ft², the energy is increased by

$$\tfrac{1}{2}p(\Delta v)^2/v \text{ ft-lb}.$$

For a finite range of compression, from v_1 to v_2 ft³, the work is therefore, in the notation of the Integral Calculus,

$$\int_{v_2}^{v_1} p\, dv,$$

where p is given as some function of v from the physical properties of the substance.

Thus in the *adiabatic* compression (§ 226) of a given quantity, say one lb, of air,

$$pv^\gamma = PV^\gamma;$$

so that the work required to compress it from v_1 to v_2 ft³ is, in ft-lb,

$$\int_{v_2}^{v_1} P(V/v)^\gamma dv = \frac{p_2 v_2 - p_1 v_1}{\gamma - 1} = R\frac{\theta_2 - \theta_1}{\gamma - 1},$$

the difference of the hydrostatic energies divided by $\gamma - 1$.

This result follows geometrically from the graphical representation on the (p, v) diagram of the adiabatic curve $Q_1Q'QQ_2$. The tangent at Q, the limit of the chord QQ' through the consecutive point Q', cuts the axis Op in T, where $pT = \gamma . Op$; and therefore the elementary rectangles Qp' and Qv' are in the ratio of γ to 1; and therefore also the whole areas $p_1Q_1Q_2p_2$ and $v_1Q_1Q_2v_2$; while their difference is $p_2v_2 - p_1v_1$.

For isothermal compression we must put $\gamma = 1$, and the work required is

$$PV \log(v_1/v_2);$$

this may be obtained either by integration, or by the Exponential Theorem, as the limit, when $\gamma = 1$, of

$$\frac{PV}{\gamma - 1}\left\{\left(\frac{v_1}{v_2}\right)^{\gamma - 1} - 1\right\}$$

$$= \text{lt}\frac{PV}{\gamma - 1}\left\{(\gamma - 1)\log\frac{v_1}{v_2} + \frac{(\gamma - 1)^2}{2}\left(\log\frac{v_1}{v_2}\right)^2 + \dots\right\}.$$

In an atmosphere in Convective Equilibrium the work required to compress 1 lb of air at altitude z_1 to its density at a lower level z_2 is, in ft-lb,

$$\frac{p_2v_2-p_1v_1}{\gamma-1}=\frac{k}{\theta_0}\frac{\theta_2-\theta_1}{\gamma-1}=\frac{k(z_1-z_2)}{c(\gamma-1)}=\frac{z_1-z_2}{\gamma},$$

or $1/\gamma$ times the work required to raise 1 lb of air from the level z_2 to the level z_1; and when $\gamma=1$, as in an Isothermal Atmosphere, the work is the same.

234. *The Draught of a Chimney.*

The currents of air in the atmosphere are primarily due to inequalities of temperature and thence of density; a familiar instance of the artificial production of a current of air is seen in the draught of a chimney.

Considering the draught through the closed furnace of a steam engine boiler, the air makes its way through the grate bars and the fire, as through a porous plug, and acquires with the gases of combustion a certain average temperature, which we shall denote by τ' C or θ' absolute, τ or θ denoting the temperature of the outside cold air.

It is calculated that about 20 lb of air is required to burn 1 lb of coal; and denoting by ρ the density of the cold air, then the density of the hot air issuing from the top of the chimney at the same pressure may be taken to be $\rho\theta/\theta'$; so that h ft denoting the vertical height of the top of the chimney above the fire, the pressure of the cold air outside will exceed the pressure of the hot air inside the furnace, taking their densities as uniform, by

$$\left(1-\frac{\theta}{\theta'}\right)\rho h;$$

and this will be felt as a pressure on the furnace door.

This will also be the upward pressure on a lid at the top of the chimney, if the furnace door is opened.

To measure the draught a glass inverted siphon gauge filled with water (fig. 71, p. 345) is placed in the side of the chimney, and now if z inches is the difference of level of the surface of the water in the two branches, and D denotes the density of water,

$$\frac{Dz}{12}=\left(1-\frac{\theta}{\theta'}\right)\rho h.$$

In round numbers, $D/\rho=800$; so that

$$\frac{z}{h}=\frac{3}{200}\left(1-\frac{\theta}{\theta'}\right).$$

If the horizontal cross section of the chimney is A ft², then the weight of cold air which fills the chimney is $Ah\rho$ lb; and the height of the column of hot air of equal weight is $h\theta'/\theta$, and their difference of height,

$$\left(\frac{\theta'}{\theta}-1\right)h=\frac{\tau'-\tau}{273+\tau}h$$

is taken as the head producing the velocity v of the hot air up the chimney.

Or, otherwise, if x ft denotes the head of hot air equivalent to z inches of water,

$$12x\rho\frac{\theta}{\theta'}=Dz=12\left(1-\frac{\theta}{\theta'}\right)h\rho,$$

or

$$\frac{x}{h}=\frac{\theta'}{\theta}-1.$$

The rate of flow of the air through the chimney depends very much on the state of the fire; it is assumed that the average velocity v of the hot air up the chimney is either due to this head x, or to some fraction of it, depending on the resistance of the fire and the friction of the flues; but putting

$$\tfrac{1}{2}v^2=gx=\left(\frac{\theta'}{\theta}-1\right)gh,$$

the weight of air in lb which flows up the chimney per second is

$$Av\rho\frac{\theta}{\theta'} = A\rho\surd(2gh)\surd\left(\frac{\theta}{\theta'} - \frac{\theta^2}{\theta'^2}\right)$$

$$= A\rho\surd(2gh)\surd\left\{\frac{1}{4} - \left(\frac{\theta}{\theta'} - \frac{1}{2}\right)^2\right\},$$

a maximum $\frac{1}{2}A\rho\surd(2gh)$, when $\theta' = 2\theta$.

Thus if the outside air is at 17 C, $\theta = 290$, and $\theta' = 580$, $\tau' = 307$ C, nearly the temperature of melting lead; and

$$\frac{z}{h} = \frac{3}{400},$$

so that a chimney 100 ft high produces a draught of $\frac{3}{4}$ inch of water, when the flow through it is a maximum.

In Metric units, a litre of air at 0 C weighs 1·29 g; so that a chimney h m high will produce a draught of z cm of water, given by

$$\frac{z}{h} = 0{\cdot}129 \times 273\left(\frac{1}{\theta} - \frac{1}{\theta'}\right) = 35\left(\frac{1}{\theta} - \frac{1}{\theta'}\right);$$

for instance, if $\theta' = 2\theta$, and $\tau = 0$, $z = 0{\cdot}0645\ h$.

Variations of barometric height or of temperature will cause air to enter or leave a given space, such as a room or a mine.

This is illustrated by a "whistling well," in which a whistle placed in the lid is blown by the current of air which enters or leaves the well; also by the liberation of gas in a coal mine when the barometer falls.

Thus if the barometer falls from h to h', or the temperature rises from θ to θ', the density of the air falls from ρ to ρ', where

$$\frac{\rho'}{\rho} = \frac{h'\theta}{h\theta'};$$

and the percentage of air which leaves a room is

$$100\left(1-\frac{\rho'}{\rho}\right)=100\left(1-\frac{h'\theta}{h\theta'}\right).$$

A room 10 m by 6 m by 5 m will, with a barometric height 76 cm and a temperature 0 C, contain

$$10 \times 6 \times 5 \times 1{\cdot}2932 = 388 \text{ kg of air};$$

and now, if the barometer falls to 75 cm and the temperature rises to 15° C, the room will lose about $6\frac{1}{2}$ per cent or 25 kg of air.

Examples.

(1) Prove that if volumes V_1 and V_2 of atmospheric air are forced into vessels of volume U_1 and U_2, and if communication is established between them, a quantity of air of volume

$$(U_1V_2 - U_2V_1)/(U_1 + U_2)$$

at atmospheric pressure will pass from one to the other.

(2) Prove that, if a partially exhausted siphon with equal arms is dipped into mercury; and if the sum of the heights to which the mercury rises in an arm, when first this and then the other arm is unstopped, is equal to the height of the barometer; then the original pressure of the air in the siphon is due to a head of mercury whose height is equal to the difference of the lengths of the mercury in the siphon in the intermediate and final stages.

(3) Prove that, if a piston of weight W lb is in equilibrium in a vertical cylinder with a ft of air beneath it, and if it is depressed a small distance

x ft, the energy of the system when the temperature is unaltered is increased by about $\frac{1}{2} Wx^2/a$ ft-lb.

(4) The mouth of an inverted cup is submerged to a depth of 6 ins in warm mercury, and it is found that no air escapes. Prove that, if the barometer stands at 30 ins, the mercury cannot be more than 100° F. warmer than the atmosphere.

(5) A gas saturated with vapour, originally at a pressure p, is compressed without change of temperature to one-nth of its volume, and the pressure is then found to be p_n.

Prove that the pressure of the vapour and of the gas in its original state is respectively

$$\frac{np-p_n}{n-1} \text{ and } \frac{p_n-p}{n-1}.$$

(6) Prove that, if the height of the column of mercury in § 211 is reduced from y to y' by the introduction of a bubble of water into the vitiated Torricellian vacuum, just small enough to evaporate completely, the pressure of the vapour, in mercury head f, is given by

$$f=(b+h-y-y')\frac{y-y'}{b-y'};$$

reducing to $y-y'$ when the vacuum is perfect, or $y=h$; and that now

$$(b-y)(h-y-f) \text{ is constant,}$$

when b, h, and y change, the temperature remaining constant.

E.g., If $h=29{\cdot}81$ ins; and $y=29$ when $b=32\frac{3}{16}$, $y'=28{\cdot}5$ when $b'=30\frac{1}{8}$; then $f=0{\cdot}47$ in, and the dry air would occupy a length $0{\cdot}04$ in of the tube at atmospheric pressure.

(7) A piston, of weight ϖA, in a closed vertical cylinder of height a and section A is in equilibrium at a height a/n from the base, the pressure of the air underneath it being p.

Prove that a small rise τ C of the temperature of the air underneath will raise the piston through a height approximately equal to

$$\frac{n-1}{n} \cdot \frac{p}{np-\varpi} \frac{\tau}{273} a.$$

(8) An air thermometer is made of a bulb and tube inverted vertically in a reservoir of mercury of depth b so that the tube rests on the bottom.

Prove that if the volume of the bulb and tube is equal to a length c of the tube, and h is the height of the barometer, the graduation for the temperature θ is at a height above the bottom of the tube

$$\tfrac{1}{2}(h+b+c)-\sqrt{\left\{\tfrac{1}{4}(h+b-c)^2+(h+b)c\frac{\theta}{\theta_0}\right\}},$$

where θ_0 is the absolute temperature at which the enclosed air begins to escape.

(9) Assuming that the relative distribution of oxygen and nitrogen at different heights in an atmosphere in equilibrium follows the law that one is not affected by the other, find at what height in an isothermal atmosphere the proportion of oxygen would be reduced to half what it is at sea level, where the proportions by weight may be taken to be 80 parts of nitrogen to 20 of oxygen, and where the densities are in the ratio of 14 to 16.

CHAPTER VIII.

PNEUMATIC MACHINES

235. *The Montgolfier Hot-Air Balloon.*

This balloon, invented by the Montgolfiers in 1783, is historically interesting as the first employed by the aeronauts Pilâtre de Rozier and d'Arlandes to make an ascent in the atmosphere.

The principle is the same as that of the ordinary hot-air toy balloon; the air in the balloon is rarefied by heat to such an extent that the total weight of the balloon, of the hot air it contains, of the car and of the aeronauts is equal to or less than the weight of the external cold air displaced, when the balloon begins to rise.

Denote by W lb the weight of the balloon, car, and aeronauts, as weighed in vacuo, or corrected for the buoyancy of the air; and denote by W' lb the weight of cold air they displace, so that $W-W'$ lb is the apparent weight when weighed in air; denote also by V ft^3 the capacity of the balloon, so that $M=V\rho$ lb denotes the weight of cold air which fills the balloon, ρ denoting the density, in lb/ft^3, of the surrounding cold air.

Then when the air inside is raised in temperature from θ to θ' degrees absolute, part of the air will flow out, leaving the remainder at the same pressure but at density $\rho\theta/\theta'$, and therefore of weight

$$V\rho\frac{\theta}{\theta'}=M\frac{\theta}{\theta'}\text{ lb.}$$

By Archimedes' principle the balloon will float in equilibrium when the weight of the balloon and the hot air it contains is equal to the weight of cold air displaced; that is, when

$$W+M\frac{\theta}{\theta'}=W'+M,$$

or
$$\frac{\theta}{\theta'}=\frac{M-W+W'}{M},\quad \frac{\theta'-\theta}{\theta}=\frac{W-W'}{M-W+W'}, \quad \text{.......(1)}$$

giving $\theta'-\theta$, the requisite increase of temperature.

The balloon will now be in unstable equilibrium, like a bubble of air in water, and will begin to rise, as it cannot descend.

236. The balloon will continue to rise and the hot air to escape, till another stratum of air is reached, of height z suppose, and of density ρ_z and absolute temperature θ_z, and therefore of pressure p_z, given by

$$\frac{p_z}{p}=\frac{\rho_z}{\rho}\frac{\theta_z}{\theta},$$

p denoting the pressure at the ground.

The pressure of the hot air in the balloon being also p_z, the quantity of hot air left in the balloon, supposed always at the absolute temperature θ', is

$$V\rho_z\frac{\theta_z}{\theta'}=M\frac{\rho_z}{\rho}\frac{\theta_z}{\theta'}\text{ lb},$$

while the weight of cold air displaced is

$$(M+W')\frac{\rho_z}{\rho}\text{ lb};$$

so that, for equilibrium,

$$W + M\frac{\rho_z}{\rho}\frac{\theta_z}{\theta'} = (M + W')\frac{\rho_z}{\rho},$$

or

$$\frac{\rho_z}{\rho} = \frac{W}{W' + M(1 - \theta_z/\theta')}.$$

But the barometer and thermometer carried by the aeronauts give p_z the pressure and θ_z the absolute temperature, compared with p and θ, the pressure and temperature at the ground; and by the Gaseous Laws

$$\frac{\rho}{\rho_z} = \frac{p_z}{p}\frac{\theta}{\theta_z} = \frac{W}{W' + M(1 - \theta_z/\theta')}. \qquad \text{............(2)}$$

If θ' denotes the temperature which is just sufficient for levitation, as given by equation (1), then in (2)

$$\frac{\rho}{\rho_z} = 1 + \frac{M - W + W'}{W}\left(1 - \frac{\theta_z}{\theta}\right),$$

or

$$\frac{\rho - \rho_z}{\rho_z} = \frac{(M - W + W')\left(1 - \dfrac{\theta_z}{\theta}\right)}{W}, \qquad \text{................(3)}$$

so that with this temperature θ' the balloon will not rise unless θ_z/θ is less than unity, or unless the temperature of the air diminishes as we ascend in the atmosphere.

Thus in an atmosphere in thermal equilibrium of uniform temperature, θ' must be increased beyond the value given by (1) for the balloon to ascend.

In such an atmosphere it has been shown that, with

$$\theta_z = \theta, \quad \frac{p_z}{p} = \frac{\rho_z}{\rho} = e^{-\frac{z}{k}},$$

where k denotes the height of the homogeneous atmosphere at the temperature θ; so that taking this height at the freezing temperature as 26,214 ft,

$$k = 26214\frac{\theta}{273} \approx 96\,\theta.$$

The temperature θ' of the hot air required to ascend to a height z ft is now given by equation (2),

$$\theta' = \frac{M\theta_z}{M + W' - W\rho/\rho_z}, \quad\text{.......................(4)}$$

where $\theta_z = \theta$, and $\rho_z = \rho\exp(-z/k)$,

in an atmosphere of uniform temperature; but

$$\frac{\theta_z}{\theta} = 1 - \frac{\gamma - 1}{\gamma}\frac{z}{k}, \quad \frac{\rho_z}{\rho} = \left(1 - \frac{\gamma-1}{\gamma}\frac{z}{k}\right)^{\frac{1}{\gamma-1}},$$

in an atmosphere in Convective Equilibrium (§ 226).

237. *The Hydrogen or Gas Balloon.*

In this balloon the requisite levitation is secured by filling the balloon with hydrogen, as first carried out by Charles and Robert in 1783, a few months after the first ascent in the Montgolfier balloon; or nowadays with coal gas, which is specifically lighter than air.

With hydrogen the balloon can be made of much smaller dimensions; but this advantage is counterbalanced by the difficulty of the manufacture of the gas and its great speed of diffusion; so that a balloon is now generally made of larger dimensions and filled with coal gas from the nearest gasworks.

For military purposes, however, where the balloon is required to be held captive at a height of about 1000 ft, it is important to keep down the size, so as to reduce the effect of the wind; so that military balloons are now filled with hydrogen, carried highly compressed in steel flasks, at a pressure of about 100 atmospheres.

With the same notation as for the Montgolfier balloon, suppose the gas employed has a specific volume nv and a density ρ/n, n times and one-nth that of air at the same pressure and temperature.

Then the balloon will be in unstable equilibrium on the ground when U ft³ of the gas, weighing $P=U\rho/n$ lb, has been placed in the balloon, given by the equation, derived from Archimedes' principle,

$$W+U\rho/n=W'+U\rho,$$

or

$$W+P=W'+nP,$$

$$P=\frac{W-W'}{n-1},\quad U=\frac{W-W'}{\rho}\,\frac{n}{n-1}. \quad \text{.........(1)}$$

Thus to lift a ton with hydrogen, 14 times lighter than air, one-13th of a ton of hydrogen is required, occupying 31,360 ft³, taking the specific volume of air as 13 ft³/lb.

The great Captive Balloon of Chelsea had a capacity of 424,000 ft³; so that the gross weight lifted by hydrogen could be about 13½ tons.

The aeronaut's practical rule is that "1000 ft³ of coal gas will lift 40 lb"; so that putting

$$W-W'=40,\quad U=1000,\quad v=1/\rho=12{\cdot}5,$$

we find

$$\frac{n}{n-1}=\frac{1000}{40\times 12{\cdot}5}=2,\quad n=2.$$

Denoting generally by A the *lift* or *ascensional force*, in kg/m³, of a gas n times lighter than the air, or of density ρ' kg/m³, ρ denoting the density of the air, then

$$A=\rho-\rho'=\rho\left(1-\frac{1}{n}\right), \quad \text{.................(2)}$$

where $\rho=1{\cdot}293$; and thus for hydrogen, $A=1{\cdot}2$.

Now if the balloon is on the point of rising when inflated with U m³ of gas, weighing P kg,

$$W-W'=UA=(n-1)P. \quad \text{.................(3)}$$

A m³ of air at 0 C and 76 cm of barometer weighs 1·293 kg, of hydrogen 0·088 kg, of coal gas at 0 C or of air at 200 C about 0·78 kg; so that a m^3 of hydrogen will lift $1·293-0·088=1·205$ kg, and a cm^3 of air at 200 C, or of coal gas will lift 0·513 kg; this makes $n=14·7$ for hydrogen, and $n=1·8$ for the hot air or coal gas.

But coal gas, especially from the latest products of distillation, can be made of density from 0·33 to 0·37 of the density of the air, so that now n is about 3.

Gas of this lightness was employed in the celebrated ascent by Coxwell and Glaisher on 5th September, 1862, from Wolverhampton, when an altitude was attained at which the barometer stood at 7 inches and the thermometer at about -12 F.

238. The balloon in unstable equilibrium on the ground is like a bubble of air compressed to the density of water at a great depth in the ocean (§ 214); and being unable to descend, it will rise when let go.

To carry the balloon rapidly clear of the neighbouring obstacles it is advisable that the volume U or quantity P of gas should be increased so as to give an ascensional force, which at starting will be a force of

$$(n-1)P-(W-W') \text{ pounds.}$$

As the balloon rises, the gas contained in it will expand until the balloon is completely inflated, and will now occupy V ft³; and this will take place where the density of the air is $\rho U/V$, and the density of the gas is $\rho U/nV$, the temperature being supposed unaltered.

The ascensional force will now be

$$(n-1)P-\left(W-W'\frac{U}{V}\right) \text{ pounds, ...(4)}$$

or $W'-W'U/V$ pounds less than at starting; and therefore practically unaltered, since W' is small.

239. The balloon will still continue rising; but now it is very important that the neck of the balloon should be left open, to allow gas to escape as the balloon rises into the more rarefied air, and thus to equalize the pressure of the gas and the air; otherwise the pressure of the imprisoned air might burst the balloon, especially if the rays of the sun should suddenly strike upon it.

At a height z the ascensional force will now be

$$V\rho_z\left(1-\frac{1}{n}\right)-\left(W-W'\frac{\rho_z}{\rho}\right)$$

$$=\{(n-1)Q+W'\}\frac{\rho_z}{\rho}-W \text{ pounds, } \quad \text{.....(5)}$$

on putting $\quad Q=V\rho/n=M/n, \quad$ ....(6)

where Q denotes the weight in lb of the gas and M of air which would fill the balloon at the ground; also

$$VA=(n-1)Q, \quad \text{.....................(7)}$$

A denoting the *lift* of the gas, in lb/ft³.

This ascensional force is zero, and the balloon comes to rest, when

$$\frac{\rho_z}{\rho}=\frac{p_z\theta}{p\theta_z}=\frac{W}{(n-1)Q+W'}. \quad \text{................(8)}$$

The quantity of gas now left in the balloon is, in lb,

$$V\frac{\rho_z}{n}=Q\frac{\rho_z}{\rho}=\frac{QW}{(n-1)Q+W'}$$

$$=Q-\frac{(n-1)Q-(W-W')}{(n-1)Q+W'}Q, \quad \text{.........(9)}$$

so that the number of lb of gas lost is

$$\frac{(n-1)Q-(W-W')}{(n-1)Q+W'}Q+P-Q, \quad \text{..........(10)}$$

or, if the balloon started full, with $P=Q$, the gas lost is

$$\frac{(n-1)Q-(W-W')}{(n-1)Q+W'}Q \text{ lb.} \quad \text{...............(11)}$$

To pull the balloon down to the ground without any further loss of gas will require a force, gradually increasing to

$$(n-1)Q\frac{\rho_z}{\rho}-(W-W')=\frac{(n-1)QW}{(n-1)Q+W'}-(W-W')$$
$$=W'\left(1-\frac{\rho_z}{\rho}\right) \text{ pounds } \ldots\ldots.(12)$$

at the ground; this is such a very small quantity that a very slight loss of gas is sufficient to cause the balloon to descend, and ballast must be thrown out to restore equilibrium.

In consequence of the balloon losing gas in the ascent, and collapsing in the descent, a captive balloon, raised and lowered slowly, requires less work to pull it down than the work required to resist its ascent.

The gradual loss of gas by diffusion brings the balloon down again, generally in about two hours' time; and if it is required to descend rapidly, a valve is opened at the top of the balloon, to let the gas escape quickly.

On the other hand, by throwing out ballast the height lost can be recovered and even exceeded, or the balloon can be guided into a favourable current of air.

A free balloon is always rising or falling, and it must be steered in a vertical plane either by throwing out ballast or letting off gas; but it can be kept at a moderate average elevation by the areonaut Green's invention of a rope trailing on the ground, acting as a spring.

240. In the absence of knowledge of the distribution of pressure, density, and temperature in the upper strata of the atmosphere, we must suppose that the true state is something intermediate to the states of Thermal and of

Convective Equilibrium, and calculate on these two hypotheses; these calculations are required, not only in ballooning, but also in gunnery with high angle fire at long ranges, to allow for the *tenuity* of the atmosphere and consequent reduction in the resistance of the air.

In an atmosphere of uniform temperature

$$z = k \log_e \frac{\rho}{\rho_z} = \mu k \log_{10} \frac{(n-1)Q + W'}{W}, \quad \text{.........(13)}$$

with modulus $\mu = 2{\cdot}3$, giving the height z attainable.

But with Convective Equilibrium

$$z = \frac{\gamma k}{\gamma - 1}\left\{1 - \left(\frac{\rho_z}{\rho}\right)^{\gamma-1}\right\}$$

$$= \frac{\gamma k}{\gamma - 1}\left[1 - \left\{\frac{W}{(n-1)Q + W'}\right\}^{\gamma-1}\right]; \quad \text{......(14)}$$

also $$z = \frac{\gamma k}{\gamma - 1}\left(1 - \frac{\theta_z}{\theta}\right) = 336(\theta - \theta_z), \quad \text{..............(15)}$$

with $\gamma = 1{\cdot}4$, and $k = 96\,\theta$;

so that the thermometer falls 1° C for every 336 ft ascended, or 1° F for every 186 ft.

241. Suppose the balloon is completely inflated on the ground, and prevented from rising by B kg of ballast; then, neglecting W',

$$W + B = VA. \quad \text{......................(16)}$$

The removal of the ballast B (*delestage*) will allow the balloon to rise to a stratum of density ρ_z, where

$$\frac{\rho_z}{\rho} = \frac{W}{VA} = 1 - \frac{B}{VA}; \quad \text{.................(17)}$$

and to reach an additional height Δz, where the density has diminished $\Delta\rho_z$, additional ballast ΔB must be thrown out, given by

$$\Delta B = VA(\Delta\rho_z/\rho). \quad \text{.....................(18)}$$

In an isothermal atmosphere,

$$\Delta z = k\frac{\Delta \rho_z}{\rho_z} = k\frac{\rho}{\rho_z}\frac{\Delta B}{VA}. \quad \text{.................(19)}$$

Thus if $V=500$ m³, and if hydrogen is used, $A=1{\cdot}2$, $VA=600$ kg; and to rise to a level where $\rho_z=\frac{3}{4}\rho$, or

$$z=18400\log_{10}(\rho/\rho_z)=2300 \text{ m},$$

the ballast removed must be

$$B=\tfrac{1}{4}VA=150 \text{ kg};$$

and throwing out 10 kg more ballast will make the balloon rise an additional height of 178 m.

242. Neglecting the weight of everything except the envelope of the balloon, supposed a sphere of volume V m³, and diameter $d=\sqrt[3]{(6V/\pi)}$ m, and of superficial density m kg/m², then if filled with gas of ascensional force A kg/m³, the *lift* at the ground is

$$VA-\pi d^2 m \text{ kg}.$$

A small hole being left open for the escape of the gas, the lift is reduced to

$$(VA/q)-\pi d^2 m \text{ kg}$$

in a stratum where the density is reduced to one-qth of the density on the ground; and the balloon will just rise to this stratum if

$$\frac{VA}{q}=\pi d^2 m=\pi\left(\frac{6V}{\pi}\right)^{\frac{2}{3}} m,$$

or

$$V=\frac{36\pi m^3 q^3}{A^3}, \quad \text{..........................(20)}$$

called "the equation of the three cubes."

In an isothermal atmosphere

$$z=8000\log_e q=18400\log_{10} q=6133\log V-M,$$

where $M=12600+18400(\log m-\log A)$; ....(21)

so that z increases by 6133 m when V is multiplied tenfold.

For hydrogen $A=1\cdot2$; and the superficial density of the envelope is found to range from 300 down to 50 g/m²; so that putting, on the average, $m=0\cdot1$, we find

$$V=0\cdot01q^3;$$

and hence the table (*Cosmos*, April, 1893, *La pratique des ascensions aérostatiques*).

q	5	10	40	200	500
V	1·25	10	640	80,000	1,250,000 m³
z	12,900	18,400	29,500	42,300	49,700 m

For copper one 100th of an inch or $\frac{1}{4}$ mm thick, $m=2\cdot2$, so that if a copper sphere of this thickness, 100 ft or 30·5 m in diameter, is filled with hydrogen, the ascensional force at the ground is about 11,400 kg or 25,000 lb; also $\log q=0\cdot4429$, so that $z=8150$ m, or over 5 miles.

By similar calculations the Jesuit Francis Lana (1670) first demonstrated the possibility of aeronautics; but the copper sphere exhausted of air which he considered would not bear the external pressure without collapse.

Valuable information concerning the state of the atmosphere has been obtained by free balloons, carrying self-recording meteorological instruments.

(*Comptes Rendus*, April, 1893, G. Hermite, *L'exploration de la partie atmosphérique au moyen des ballons.*)

243. To attain a height of 30,000 to 40,000 ft, as proposed by Bixio and Barral in 1850, let us put $z=35{,}000$; also $k=27{,}800$, corresponding to 15 C or 60 F, and a barometric height of 30 ins at the ground.

On the isothermal theory

$$p/p_z=\rho/\rho_z=\exp z/k=3\cdot5,$$

giving at the altitude z a barometric height of 8·6 inches.

On the convective equilibrium theory,

$\theta_z/\theta = \cdot640$, so that $\tau_z = -88$ C;

$\rho_z/\rho = \cdot331$, or $\rho/\rho_z = 3$;

$p_z/p = \cdot213$, a barometric height of 6·4 inches.

Now, from the condition of equilibrium (8), p. 333,

$$\frac{(n-1)Q+W'}{W} = \frac{\rho}{\rho_z} = 3\cdot5, \text{ or } 3.$$

according as we assume the isothermal or convective equilibrium state.

The quantity W'/W is so small that it may be neglected; and taking $n=3$ for very light coal gas, we have $Q/W = 1\cdot75$ or $1\cdot5$, a mean value being $1\cdot625$, for a height of 35,000 feet to be attained; and $W/Q = \cdot6154$.

An ordinary balloon of 680 cubic yards capacity will lift about 750 lb gross; and Coxwell employs a gross weight $W = 1254$ lb of a balloon with a capacity of 32,000 cubic feet, to be filled with gas of specific gravity 0·440 of that of the air, so that $n = 2\cdot27$ about.

Then at the maximum height attainable with this balloon

$$\rho/\rho_z = (n-1)Q/W = 1\cdot1 \text{ about};$$

and in an isothermal atmosphere

$$z = k \log_e 1\cdot1 = 2650;$$

and when there is convective equilibrium,

$$z = \frac{\gamma k}{\gamma - 1}\left\{1 - \left(\frac{\rho_z}{\rho}\right)^{\gamma-1}\right\} = 3650.$$

244. In Coxwell's balloon $V = 90{,}000$ ft³; and supposing the coal gas to have a specific volume of 40 cubic feet to the lb, $Q = 90{,}000 \div 40 = 2250$ lb, about a ton; and then $W = 1385$ lb, the gross weight of balloon, car, ropes, ballast, instruments and aeronauts the balloon can take to a height of 35,000 feet.

The lowest barometric height recorded by Glaisher in the ascent of 5th September, 1862, was 7 inches; this would give, in an atmosphere of uniform temperature 15 C or 60 F, a height

$$z = k \log_e 30/7 = 41{,}000 \text{ feet};$$

and, when in convective equilibrium, from (8),

$$z = \frac{\gamma k}{\gamma - 1}\left\{1 - \left(\frac{p_z}{p}\right)^{\frac{\gamma-1}{\gamma}}\right\} = 32{,}670 \text{ feet},$$

with a temperature $\theta_z = 191$, $\tau_z = -82$ C, or -116 F; the mean of 60 F and -116 F being -28 F.

245. If Coxwell's balloon had been used as a Montgolfier fire balloon, with a capacity $V = 90{,}000$ ft^3, we must employ equation (4) to determine the requisite temperature to ascend to a height 35,000 ft; and then from (2), p. 329, neglecting W'/M,

$$\frac{\theta_z}{\theta'} = 1 - \frac{W}{M}\frac{\rho}{\rho_z}.$$

Here M is the weight of 90,000 ft^3 of air at 60 F; and, supposing 13 ft^3 to weigh a lb,

$$M = 90{,}000 \div 13.$$

Supposing $W = 1400$ lb, and putting $\tau = 15$, $\theta = 288$, $\rho/\rho_z = 3{\cdot}5$ in the isothermal atmosphere, we find $\theta'/\theta = {\cdot}3$, and therefore $\theta' = 960$, $\tau' = 687$ C.

In convective equilibrium $\theta_z = 181$, $\rho/\rho_z = 3$; and (§ 236)

$$\theta/\theta' = 0{\cdot}39, \quad \theta' = 460, \quad \tau' = 187° \text{ C},$$

not as excessive temperature.

Similarly it can be shown that for a balloon of 15,000 ft^3 capacity, and gross weight 500 lb, to rise to a height of 1000 ft, the temperature of the air inside must be raised to about 250° C, on either hypothesis of Thermal or Convective Equilibrium.

The fire balloon increases in efficiency as the temperature diminishes in the upper strata of the atmosphere, and it can keep the air for as long as the fuel lasts, so that it is suitable for long voyages, especially now that the former risk of catching fire, by which Pilatre de Rozier lost his life, is obviated by Mr. Percival Spenser's adoption of asbestos for the material (*Times*, 31 Jan. 1890).

On the other hand, its great size compared with the hydrogen balloon is a disadvantage for military purposes, where the wind is the great difficulty to be encountered in filling and in holding captive the balloon.

For the History and Practice of Aeronautics the following works may be consulted :—

Faujas de Saint Fond. *Description des experiences de la machine aérostatique de Mm. de Montgolfier.* Paris, 1783-4.

Vincent Lunardi. *Aerial Voyages.* 1784.

Tiberius Cavallo, F.R.S. *The History and Practice of Aerostation.* London, 1785.

Thomas Baldwin, M.A. *Airopaidia.* 1786.

Monck Mason. *Aeronautica.* 1838.

Hatton Turnor. *Astra Castra.* 1865.

Glaisher. *Travels in the Air.* 1871.

Tissandier. *Histoire des ballons.* 1887.

Stevens. *The History of Aeronautics.* Scientific American Supplement, March, 1890.

The Scientific American. *Hot air ballooning,* p. 143, 5 Sep., 1891.

Cleveland Abbe. *The Mechanics of the Earth's Atmosphere.* Smithsonian Institution, 1891.

Revue maritime et coloniale. *Ballons.* May–Nov., 1892.

Examples.

(1) What must be the dimensions of a balloon the whole weight of which, with its appendages, is 700 pounds, that it may just rise half a mile high; supposing air to have 14 times the specific gravity of the gas under the same pressure, that 5 cubic feet of air at the earth's surface weighs 6 ounces, and that the density at the earth's surface is 4 times as great as at the height of 7 miles?

(2) A balloon ascends 1000 feet above the height at which it is fully inflated; determine the fraction of gas which escapes, with the temperature uniform.

(3) A weightless balloon, filled with hydrogen, is held by a string, and has a small hole on the lower side. The tension of the string is 3·79 g; while after the barometer has fallen 8 mm the tension becomes 3·75 g.

Find the height of the barometer and the volume of the balloon, given that the temperature of the air is 15° C, and that at 0° C and 760 mm pressure a litre of air weighs 1·293 g, and of hydrogen ·089 g.

246. *The Gasholder.*

Gasholders, formerly called Gasometers, are vertical cylindrical vessels of sheet iron, used for storing gas as it is manufactured.

The cylinder is closed at the top, but open at the bottom and floats inverted over water contained in a circular tank, in which the cylinder can rise or fall (fig. 70.)

The buoyancy of the water displaced by the sheet iron may be considered inappreciable; and now if W lb is the weight of the gasholder and r ft the radius, the gas inside must be at a pressure $W/\pi r^2$ lb/ft² (over atmospheric pressure); and this can be measured in a siphon gauge (fig. 71) by a column of water

$$W/D\pi r^2 \text{ ft or } 12\ W/D\pi r^2 \text{ ins high,}$$

the difference of level of the water inside and outside the gasholder.

Denoting the weight of the iron plate in lb/ft² by m, and the height of the cylinder by a ft, then

$$W = m(2\pi ra + \pi r^2).$$

This supposes that the top of the cylinder is flat; but it is generally slightly dome-shaped, for strength.

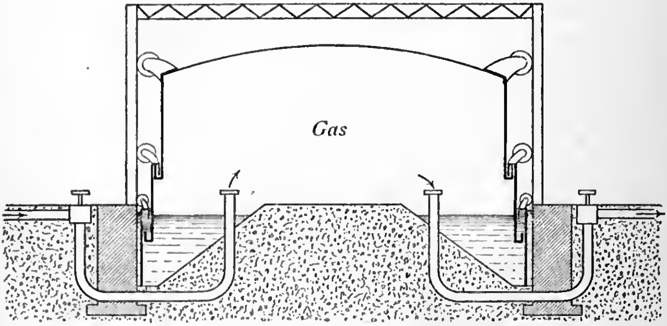


Fig. 70.

To keep down the pressure of the gas, W must be made as small as possible for a given volume

$$V = \pi r^2 a;$$

and this is secured by making $a = r$, or the height equal to half the diameter; the proof of this is a simple exercise in the Differential Calculus.

Then, with these proportions,

$$W = 3\pi r^2 m, \quad \text{or} \quad W/\pi r^2 = 3m,$$

so that the pressure of the gas is three times the weight of the sheet iron, in lb/ft².

For sheet iron plates one 16th inch thick, $m = 2{\cdot}5$; and then the minimum pressure of the gas is

$$7{\cdot}5 \text{ lb/ft}^2,$$

indicated on the siphon gauge by

$$12 \times 7{\cdot}5 \div 62{\cdot}4 = 7{\cdot}5 \div 5{\cdot}2 = 1{\cdot}5 \text{ ins of water;}$$

one inch of water giving a pressure of 5·2 lb/ft^2.

If the pressure is too great the weight W is partly relieved by counterbalance weights or by buoyancy chambers (fig. 73).

247. The vertical stability of the Gasholder against wind is secured by rollers running against outside guiding columns, braced together.

Large gasholders are made telescopic, with two or more *lifts*, to avoid a great depth in the tank; and as the gasholder is filled, the innermost section rises first, and when fully inflated it raises the next outside section by a joint, made gastight by a channel of water (fig. 72), and so on.

The largest gasholders attain a capacity of over 12 million ft^3; the one at Greenwich, 300 ft in diameter and 180 ft high in six lifts of 30 ft, will contain 12,730,000 ft^3 or 220 tons of gas, of specific volume 25 ft^3/lb, the produce of about 1500 tons of coal.

With sheet iron one 16th inch thick, the pressure will be 9·5 lb/ft^2 or about 1·8 ins of water; and the weight of the sheet iron will be about 300 tons.

If the gasholders are placed in communication, the gas will flow into the one of lowest pressure, generally the largest; just as soapbubbles in communication gradually exhaust into the largest bubble.

It is important that a gasholder should not be filled too full by day, as the Sun's rays will cause the gas to expand and escape, and the gasholder is said to *blow;*

the increase of volume or pressure for a given rise of temperature can be calculated by the Gaseous Laws of § 197.

248. *The Governor of a Gasholder.*

Denoting by ρ and ρ' the density of the air and of the gas, then in ascending a moderate height h ft vertical along a gas pipe, the pressure of the air has diminished by ρh and of the gas by $\rho' h$, treating their densities as uniform (§ 173).

The pressure of the gas over atmospheric pressure has therefore increased by

$$(\rho - \rho')h = Dz \div 12,$$

if measured by z ins of water, of density D, in a siphon gauge; and therefore

$$\frac{z}{h} = 12\frac{\rho}{D}\left(1 - \frac{\rho'}{\rho}\right).$$

It is estimated that a difference of level of 20 ft corresponds to a change of pressure of one 10th of an inch of water, so that $h/z = 200$; and putting $D/\rho = 800$, this makes

$$\rho' = \tfrac{2}{3}\rho.$$

It is important then that the gas from the gasholder should be delivered into the gas mains leading to different levels at an appropriate pressure, and this is effected by the *Governor* (fig. 73).

It consists of a miniature gasholder, provided with counterbalance weights W and buoyancy chambers B; and the flow of the gas is regulated by a conical plug C in a circular hole, which rises and falls with a governor according to the increase or diminution of the pressure, and thereby checks or stimulates the flow.

The pressure in the main is adjusted by alteration of the counterbalance weight.

249. The theory of the governor is therefore the same

as that of the old fashioned wooden reservoirs of gas, in which the buoyancy of the water was appreciable.

Starting with a volume of U ft³ of gas at atmospheric pressure, when the upward buoyancy of the water is equal to the weight of the gasholder less the counter-balance weights; suppose that V ft³ of gas at atmospheric pressure is introduced into the gasholder, and that it rises x ft in consequence, with respect to the outside level of the water.

Fig. 71.

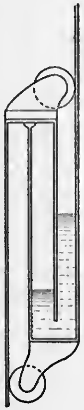


Fig. 72.

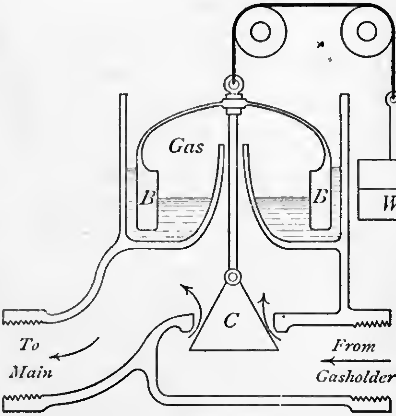


Fig. 73.

Denoting by β ft² the cross section of the buoyancy chamber, the loss of buoyancy is $D\beta x$ lb; so that if r is the internal radius of the vessel, the gas is at a pressure

$$D\beta x/\pi r^2 \text{ lb/ft}^2,$$

over atmospheric pressure.

If the siphon gauge records y ft of water, then

$$y = \beta x/\pi r^2 ;$$

and y will be the difference of the levels of the water inside and outside the gasholder.

The volume of the gas will now be $U+\pi r^2(x+y)$ ft³, under a head of $H+y$ ft of water, H denoting the height of the water barometer, about 34 ft.

Therefore by Boyle's Law,

$$\frac{U+V}{U+\pi r^2(x+y)}=\frac{H+y}{H},$$

$$V=U\frac{y}{H}+\pi r^2(x+y)\left(1+\frac{y}{H}\right);$$

and the gasholder in its descent can give out V ft³ of gas at atmospheric pressure.

250. If this gasholder is depressed from its position of equilibrium through a small vertical distance z ft, and if the level of the water inside and outside rises through p and q ft, then a and γ denoting in ft² the interior and exterior water line areas

$$\beta z=ap+\gamma q, \quad\text{.........................(1)}$$

expressing the condition that the quantity of water in the tank is unchanged.

The pressure of the gas inside rises from

$$D(H+y) \text{ to } D(H+y-p+q),$$

while the volume of the gas diminishes from

$$aa \text{ to } a(a-z-p),$$

if a denotes the length of the cylinder occupied by the gas.

The increase of upward buoyancy and air thrust is therefore

$$P=D\beta(z+q)+Da(q-p), \quad\text{.................(2)}$$

and this is the force in lbs required to depress the vessel through a distance z.

If no gas can escape, then, by Boyle's law,

$$a(H+y)=(a-z-p)(H+y-p+q),$$

or

$$q=p+(H+y)\frac{z+p}{a-z-p}.$$

But as z, p, q are small, we may put

$$q=p+(H+y)\frac{z+p}{a}, \quad q-p=(H+y)\frac{z+p}{a},$$

Therefore, from equation (1),

$$P=D(\alpha+\beta+\gamma)\frac{\beta a+(\alpha+\beta)(H+y)}{(\alpha+\gamma)a+\gamma(H+y)}z;$$

and, according to the usual theory of § 148, if the gasholder and its counterpoise weigh W lb, the oscillations will synchronize with a pendulum of length

$$\frac{Wz}{P}=\frac{W}{D(\alpha+\beta+\gamma)}\frac{(\alpha+\gamma)a+\gamma(H+y)}{\beta a+(\alpha+\beta)(H+y)}.$$

251. But if the counterbalance weights are replaced by an equal gasholder, and there is free communication between their interiors, so that there is no change of volume or pressure in the gas, then $p=q$; and the length of the equivalent pendulum is

$$\frac{Wz}{D\beta(z+q)}=\frac{W}{D}\left(\frac{1}{\beta}-\frac{1}{\alpha+\beta+\gamma}\right),$$

as in the vertical oscillations of a ship in a dock (§ 152).

When β is small, the oscillations are slow, and the force P is small.

252. *The Diving Bell and Diving Dress.*

With the aid of the Balloon we are able to mount up in the air, and the Diving Bell is an instrument by which we can descend in water and make a prolonged stay.

The Diving Bell is a bell-shaped body made of cast iron (or lead; Evelyn's *Diary*, 19th July, 1661) sufficiently thick for it to sink in water, even when full of air, on being lowered mouth downwards by a chain (fig. 74, p. 355).

Denoting by s the S.G. of the material (cast iron) by U ft^3 the volume of air which fills the bell, and by k^3U ft^3 the total volume of water displaced by the bell when submerged and full of air, so that $(k^3-1)U$ is the volume of the metal; then for the bell to sink, neglecting the weight of the air inside the bell,

$$(k^3-1)sU > k^3U, \text{ or } k > \sqrt[3]{\frac{s}{s-1}},$$

determining the linear dimensions of the interior of the bell, supposing it similar to the exterior.

Taking $s=7{\cdot}2$ for cast iron, then $k>1{\cdot}05$, so that the thickness of the bell should exceed one 40th the external diameter, for the bell to sink in water when full of air.

As the bell descends the pressure in the interior increases, and the air would be compressed and the water would rise in the bell; but by pumping in air from above the water is kept out, and fresh air is introduced, the foul air escaping under the lower edge of the bell.

Taking $D=64$ for sea water, and an atmospheric pressure of $14\frac{2}{3}$ lb/in², the head H of water which produces the pressure of an atmosphere is given by

$$H=14\tfrac{2}{3}\times 144 \div 64 = 33 \text{ feet} = 5\tfrac{1}{2} \text{ fathoms};$$

and 9 ft or $1\frac{1}{2}$ fathoms of depth gives an increased pressure of 4 lb/in².

When the bell is lowered to a depth of 33 ft or 10 m in sea water, air must be pumped in against a pressure of 2 atmospheres, and a barometer in the bell would stand at double its ordinary height; at a depth of 66 ft the pressure rises to 3 atmospheres, and so on in proportion, up to a depth of 165 feet or $27\frac{1}{2}$ fathoms, when the pressure would be 6 atmospheres, which is about the extreme limit endurable, and then only for a few minutes by the most practised divers.

253. If a diving bell is lowered without pumping in air, as in the Sounding Machine of § 204, then when the

depth of the base is x ft, and the water has risen y ft in the bell, so that $x-y$ is the depth of its surface, the pressure of the air is

$$(H+x-y)/H \text{ atmospheres};$$

and therefore the volume of the air is

$$UH/(H+x-y) \text{ ft}^3.$$

But if air is pumped in so that the volume in the bell is V ft^3, and if y still denotes the height of the water in the bell, then at atmospheric pressure this air would occupy a volume

$$V(H+x-y)/H \text{ ft}^3;$$

so that

$$V\frac{H+x-y}{H}-U \text{ ft}^3$$

of atmospheric air (that is, of air at atmospheric density) has been pumped in.

Thus, putting $y=0$, and $V=U$, we find that Ux/H ft^3 of atmospheric air must be pumped in to clear the bell of water at a depth of x ft; and if the bell is lowered at a uniform rate v ft/sec, the pressure and density of the air increase uniformly, so that atmospheric air must be pumped in at a uniform rate, Vv/H ft^3/sec, to keep the water from rising in the bell.

If the bell is made of metal of S.G. s and weighs W lb, the tension of the chain by which it is lowered will be T lb, where

$$T=W-\frac{W}{s}-DV=W-\frac{W}{s}-\frac{W'V}{U},$$

W' denoting the weight of water which will fill the bell.

254. Suppose now that the water barometer rises from H to $H+\Delta H$, while the temperature of the air in the bell changes from τ to τ' C, and that the water rises an additional distance Δy in the bell; then A ft^2 denoting

the water line area inside, the new pressure, expressed in the former atmospheres, is given by

$$\frac{H+\Delta H+x-y-\Delta y}{H}=\frac{H+x-y}{H}\frac{V}{V-A\Delta y}\frac{273+\tau'}{273+\tau},$$

or, denoting the volume of atmospheric air in the bell by V_0, so that

$$\frac{V_0}{V}=1+\frac{x-y}{H},$$

$$\left(1-\frac{A\Delta y}{V}\right)\left(1+\frac{V}{V_0}\frac{\Delta H-\Delta y}{H}\right)=1-\frac{\tau-\tau'}{273+\tau}.$$

The diminution ΔT in the tension of the chain is given by

$$\Delta T=W'A\Delta y/U;$$

and the tension is therefore unaltered if $\Delta y=0$, or

$$\frac{\Delta H}{H}=\left(1+\frac{x-y}{H}\right)\frac{\tau'-\tau}{273+\tau}.$$

Neglecting squares and products of Δy and ΔH,

$$\frac{A\Delta y}{V}-\frac{V}{V_0}\frac{\Delta H-\Delta y}{H}=\frac{\tau-\tau'}{273+\tau};$$

so that, if $\Delta H=0$,

$$\Delta y=\frac{VV_0H}{V^2+V_0AH}\frac{\tau-\tau'}{273+\tau};$$

but if there is no change of temperature,

$$\frac{\Delta y}{\Delta H}=\frac{V^2}{V^2+V_0AH}.$$

Since $\Delta V_0=\frac{V_0}{V}\Delta V-\frac{V}{H}\Delta y$, and $\Delta V=-A\Delta y$,

a volume ΔV_0 of atmospheric air pumped into the bell will lower the water a distance $-\Delta y$, given by

$$-\frac{\Delta y}{\Delta V_0}=\frac{VH}{V^2+V_0AH}.$$

We may suppose that ΔV_0 is the volume of atmospheric air liberated by opening a bottle of aerated water; conversely, a flask which has been opened and screwed up again inside the bell will contain the condensed air, and be liable to burst on reaching the surface.

Suppose the bell is lowered an additional depth Δx, without pumping in air; then the new pressure in the former atmospheres

$$\frac{H+x+\Delta x-y-\Delta y}{H}=\frac{H+x-y}{H}\frac{V}{V-A\Delta y},$$

so that $$\left(1-\frac{A\Delta y}{V}\right)\left(1+\frac{V_0}{V}\frac{\Delta x-\Delta y}{H}\right)=1;$$

or, approximately, $$\frac{\Delta x}{\Delta y}=\frac{V^2}{V^2+V_0AH}.$$

255. If the bell is cylindrical and of height a, then

$$U=Aa, \qquad V=A(a-y);$$

and if the mercurial barometer in the bell has risen from h to h' in descending to a depth x, and no air has been pumped in, $V_0=U$, and

$$\frac{h'}{h}=\frac{V_0}{V}=\frac{a}{a-y}=1+\frac{x-y}{H};$$

and then $$y=a\left(1-\frac{h}{h'}\right),$$

$$x=y\left(\frac{H}{a-y}+1\right)$$

$$=(h'-h)\left(\frac{H}{h}+\frac{a}{h'}\right)$$

$$=(h'-h)\left(\frac{\sigma}{D}+\frac{a}{h'}\right),$$

if σ denotes the density of mercury and D of the (sea) water.

Also $$\frac{aH}{a-y}-a+y=H+x-a,$$

$$\frac{aH}{a-y}+a-y=\surd\{(H+x-a)^2+4aH\},$$

so that if the temperature alone varies

$$\Delta y=\frac{(a-y)aH}{(a-y)^2+aH}\,\frac{\tau-\tau'}{273+\tau}$$

$$=\frac{aH}{\surd\{(H+x-a)^2+4aH\}}\,\frac{\tau-\tau'}{273+\tau}.$$

If the bell is stationary and a volume ΔV_0 of atmospheric air is pumped in,

$$-\frac{A\Delta y}{\Delta V_0}=\frac{(a-y)H}{(a-y)^2+aH},$$

and therefore, if air is pumped in at a uniform rate, the water descends in the bell with velocity inversely proportional to

$$\frac{aH}{a-y}+a-y=H+x+a-2y.$$

If the bell descends a small distance Δx,

$$\frac{\Delta y}{\Delta x}=\frac{(a-y)^2}{(a-y)^2+aH};$$

and if the velocity v of descent is constant,

$$x=vt=y\left(\frac{H}{a-y}+1\right).$$

256. *The Stability of the Diving Bell.*

Although it is convenient to make the apparent weight of the bell in water, as measured by the tension of the chain, as small as possible for ease of manipulation, still a certain preponderance is requisite to ensure the *stability* of the bell in its vertical position hanging downwards, so as to prevent it from turning mouth upwards when the air escapes under the lower edge.

The investigation of the stability is similar to that required for a ship aground (§ 142); the bell is supposed slightly displaced through an angle θ from the vertical position, represented in fig. 74, p. 355, by drawing the surface of the water and the vertical forces in their displaced position relative to the bell.

Now, if K denotes the point of attachment of the chain, G the C.G. of the metal of the bell, supposed homogeneous, B and B_2 the C.G.'s of the volume occupied by the air in the upright and inclined positions, and M the metacentric centre of curvature of the curve of buoyancy BB_2; then, as in § 101,

$$BM = Ak^2/V,$$

where Ak^2 denotes the moment of inertia, in ft^4, of the water area A about its C.G. F, and V denotes the volume, in ft^3, of the air in the bell; but now the metacentre M lies below the centre of buoyancy B.

The forces acting upon the bell in the displaced position

(i.) $W - \frac{W}{s}$ lb, acting downwards through G;

(ii.) DV lb, acting upwards through M;

(iii.) $W - \frac{W}{s} - DV$ lb, the tension of the chain, acting upwards through K;

form a couple, whose moment round K, dropping the factor $\sin\theta$, is in ft-lb,

$$W\left(1 - \frac{1}{s}\right)KG - DV \,.\, KM$$
$$= W\left(1 - \frac{1}{s}\right)KG - DV \,.\, KB - DAk^2;$$

and the equilibrium of the bell is stable if this moment is positive.

Suppose the bell is initially full of water, and that a small volume of air is pumped in; the bell will not continue to hang vertically unless

$$W\left(1-\frac{1}{s}\right)KG-DAk^2 \text{ is positive.}$$

But in working the bell the water is expelled, and the air escapes under the lower edge; although the air would force its way out through a hole at the top of the bell, but then the level of water in the bell could not be kept steady.

The stability of the bell, as measured by the above righting moment, diminishes as air is pumped in and V increases, Ak^2 remaining constant if the interior of the bell is cylindrical; so that stability must be secured when the bell is full of air.

At a great depth, where the density ρ of the air in the bell becomes appreciable, D in the formula must be replaced by $D-\rho$.

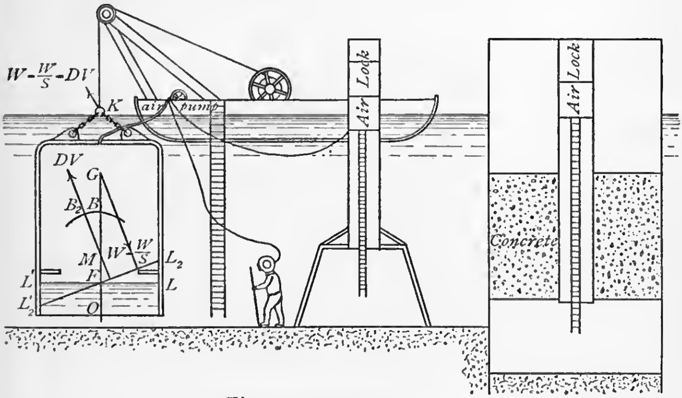
257. The original idea of the Diving Bell is of great antiquity (Berthelot, *Annales de Chimie et de Physique*, XXIV., 1891); but Smeaton was the first to use it for Civil Engineering operations in 1779; and it was extensively employed on the wreck of the Royal George, in 1782 and 1817; and in Ramsgate Harbour by Rennie in 1788.

But the Diving Bell is now generally superseded by the Diving Dress (fig. 75), which is an india-rubber suit for the diver, provided with a copper helmet fitted with small circular windows and an air valve.

The diver has thick lead soles on his boots, and carries leaden weights round his neck, so adjusted that his apparent weight in water and stability are nearly the same as ordinarily on land.

Fresh air is pumped down to him through a tube as in a diving bell, the air escaping through the valve at the back of his head; if the diver wishes to rise he partly closes the valve, which causes the dress to be inflated and increases his buoyancy; so also sunken vessels are raised nowadays by large india-rubber bags placed in the hold and pumped full of air.

In the Fleuss system the diver carries with him a vessel of compressed air, and he is thus independent of the pipe and can travel long distances; as was required, for instance, during the construction of the Severn tunnel, when the water burst in and flooded the workings.



Figs. 74, 75, 76, 77.

A form of diving bell is shown in fig. 76 which is useful for the construction of harbours; the neck of the bell reaches the surface of the water, and entrance is made to the bell through an *air lock;* a large diving bell of this nature was employed at the construction of the New Port of La Rochelle (*Cosmos*, 26th April, 1890).

The upward thrust of the air being equal to the weight of a cylindrical column of the water on an equal base, the weight of this bell must exceed the weight of water it displaces.

The same principles are employed in sinking caissons for underwater foundations, as in the Forth Bridge (fig. 77); or in driving a tunnel under a river through muddy soft soil, as in the Hudson River and the Blackwall tunnels, now in progress. Air is forced in to equalize the pressure of the head of water, and to prevent its entrance, being retained by air locks through which the workmen and materials can pass; a slight diminution of air pressure allows the water to percolate sufficiently to loosen the ground, but an increase of pressure is apt to allow the air to blow out in a large bubble.

This system, due to Mr. Greathead, has overcome the difficulties of subaqueous tunnelling; but if employed in the projected Channel Tunnel, a pressure of about 10 atmospheres would be required, to which the workmen are not yet accustomed.

(*The Diving Bell and Dress*, J. W. Heinke, Proc. Inst. Civil Eng. XV.;

Diving Apparatus, W. A. Gorman, Proc. Inst. Mechanical Engineers, 1882;

The Forth Bridge, Engineering, Feb., 1890.)

Examples.

(1) Two thin cylindrical gasholders which will hold four times their weight of water, and one of which just fits over the other, will float mouth downwards half immersed in water.

If the larger one is now placed over the smaller, determine the position of equilibrium.

(2) Two equal cylindrical gasholders, of weight W and height a, float with a length ma occupied by gas, which at atmospheric pressure would occupy a length a. If a weight P is placed upon one of them, and gas is transferred to the other till the top of the first just reaches the water, prove that the other rises a height

$$ma - \left(\frac{1}{m} - 1\right)\left(1 + \frac{P}{W}\right)\left\{1 + (1-m)\frac{P}{W}\right\}H.$$

(3) Prove that gas of constant pressure, measured by a height h of water, can be delivered by a gasholder in the form of a truncated cone, whose sides are inclined at an angle a to the vertical, if the thickness of the sides is $h \sin a$.

(4) Coal gas, of density 0·6 of that of the air, is delivered to the pipes at a pressure of 2 ins of water; prove that 300 ft higher the pressure will be given by 3·8 ins; the temperature being 10° C.

(5) Prove that the small vertical oscillations of a cylindrical solid, closed at the top and inverted over mercury in a wide basin, will synchronize with a pendulum of length

$$\frac{M}{\sigma\beta} \frac{1}{1 + \dfrac{a}{\beta} \dfrac{h+z}{h+z+V/a}} \text{ cm},$$

where Mg denotes the weight of the body, a and β cm² the horizontal cross sections of the interior and of the material of the body, σ and h the density of mercury and the height of the barometer, V cm³ the volume of air in the cavity, and z cm the difference of level of the mercury inside and outside.

(6) A caisson, closed at the top and divided in the middle by a horizontal diaphragm, whose weight is half that of the water it will contain, is floating over water. Prove that the draft of the caisson will be doubled when a hole is opened in the diaphragm.

(7) A diving bell with a capacity of 125 ft^3 is sunk in salt water to a depth of 100 ft. If the S.G. of salt water is 1·025, and the height of the fresh water barometer 34 ft, find the volume of atmospheric air required to clear the bell of water.

(8) Two cylindrical caissons closed at the top, of equal cross section and heights $\frac{1}{4}H$ and $\frac{1}{2}H$, are placed in water so that the first is just submerged, and the second at a depth such that the air occupies the same volume in each; prove that $h\sqrt{2}$ is the depth of the water surface in the second.

What will happen if communication is made by a pipe between the air spaces in the two caissons?

(9) Find how deep a cylindrical diving bell of height a and radius c, with a hemispherical top, must be sunk so that the water rises inside to the base of the hemisphere; and prove that the volume of atmospheric air now required to clear the bell of water is

$$\frac{a}{H}+\frac{3}{2}\frac{a}{c}$$

times the volume of the bell.

(10) Prove that if two equal cylindrical diving bells of height a, whose air spaces communicate by a pipe, are sunk so that their tops are at depths z_1 and z_2, and if a volume of atmospheric air is forced in, which would occupy a length b of either bell, the

surface of the water in each bell is lowered

$$\tfrac{1}{2}\sqrt{[\{H+\tfrac{1}{2}(z_1+z_2)\}^2+2H(2a+b)]}$$
$$-\tfrac{1}{2}\sqrt{[\{H+\tfrac{1}{2}(z_1+z_2)\}^2+4aH]}.$$

(11) Determine the effect on the level of the water in a diving bell, on the pressure of the air, and on the tension of the chain, due to a floating body inside, according as it has come from the exterior, or has been detached from the interior; or due to a workman leaving his seat in the bell to work on the bottom of the water.

Prove that if a bucket of water weighing P lb is drawn up into the bell, then (§ 254), (i) the fall of water level in the bell, (ii) the diminution of volume of the air, (iii) the increase of tension of the chain are respectively

$$\text{(i)}\ \frac{PV_0H}{D(V^2+V_0AH)},\quad \text{(ii)}\ \frac{PV^2}{D(V^2+V_0AH)},\quad \text{(iii)}\ \frac{PV^2}{V^2+V_0AH}.$$

Write down the values of these expressions for a cylindrical bell.

258. *Pumps.* The simplest form of water pump is the common *syringe,* consisting of a piston rod and piston, working in a cylinder, which is dipped into water.

If in contact with the lower surface of the piston, the water will, in consequence of the atmospheric pressure, follow the piston to a height which is only limited by the barometric head of water; the cylinder thus becomes filled with water, which is ejected on reversing the motion of the piston; this is the earliest form of fire-engine.

By the addition of valves, as in fig. 8, p. 19, the cylinder may be fixed in position, and the piston with its packing may be replaced by a plunger working through a stuffing box, as easier of manufacture and of adjustment in working, and now this machine is called a *force pump* (§ 12); it is used on a large scale in Cornish pumping engines (§ 23) for driving water to a high reservoir in water works, and in draining mines; the water lifted being often 30 times the weight of the coal raised.

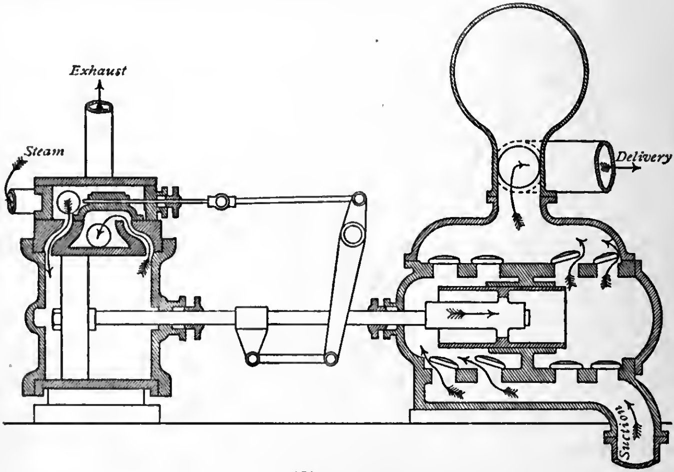


Fig. 78.

Two such force pumps, placed side by side, and worked in alternate opposite directions by a lever, constitute the modern *manual fire engine*, which does not, however, differ essentially from the machine invented by Ctesibius, described in Hero's Πνευματικά, B.C. 120; the pumps discharge into an air vessel, in which the cushion of air preserves a steady continuous stream of water in the hose.

In a steam fire engine the piston rod of the steam cylinder actuates the piston of a double acting force pump (fig. 78), by which the continuous stream is produced.

The Worthington pumping engine is of similar design; the ratio of the piston area of the steam cylinder to that of the pump being made somewhat greater than the inverse ratio of the steam and water pressures, according to the speed at which the pump is to be worked.

Water may be used instead of steam to actuate the pump; and now the product (Ah) of its head (h ft) and of the area of piston on which it acts (A ft^2) must exceed the product (Bk) of the height to which the water is forced (k ft) and of the area of the pump plunger (B ft^2); the delivery (Q ft^3/sec) depending on this excess.

For, denoting the length of stroke by l ft, the moving force $D(Ah-Bk)$ lb on the piston, acting through x ft suppose, will for $l-x$ ft, the remainder of the stroke, be changed to a resisting force of DBk lb; and therefore if we take W lb as the inertia of the piston and the moving water in the pump, and v f/s as the maximum velocity acquired,

$$\tfrac{1}{2}Wv^2/g = D(Ah-Bk)x = DBk(l-x).$$

The average velocity of the piston being $\frac{1}{2}v$, the delivery

$$Q=\tfrac{1}{2}vB;$$

and hence we find

$$Q=\sqrt{\left\{\tfrac{1}{2}glB^3\frac{Dk}{W}\left(1-\frac{Bk}{Ah}\right)\right\}};$$

which increases from zero to $\sqrt{(\frac{1}{2}glB^3Dk/W)}$, as Ah increases from Bk to infinity.

If the water pressure is used to check the motion of the piston, then

$$\tfrac{1}{2}Wv^2/g = D(Ah-Bk)x = D(Ah+Bk)(l-x);$$

and therefore, as before,

$$Q=\sqrt{\left\{\tfrac{1}{4}glB\frac{DAh}{W}\left(1-\frac{B^2k^2}{A^2h^2}\right)\right\}}.$$

The adjustment of the valves must be very accurate to secure this motion without shock; or else a connecting rod, crank, and fly-wheel must be added, as in the ordinary steam engine.

In the inventions of the Marquis of Worcester (1663) and of Savery (1696) steam acted directly upon the surface of the water, without the intervention of pistons; and considerable waste by condensation took place.

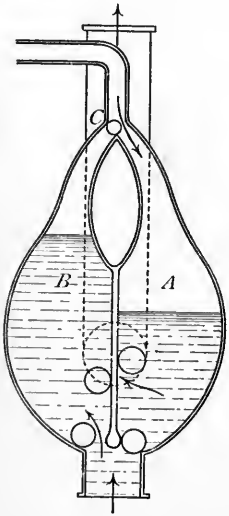


Fig. 79.

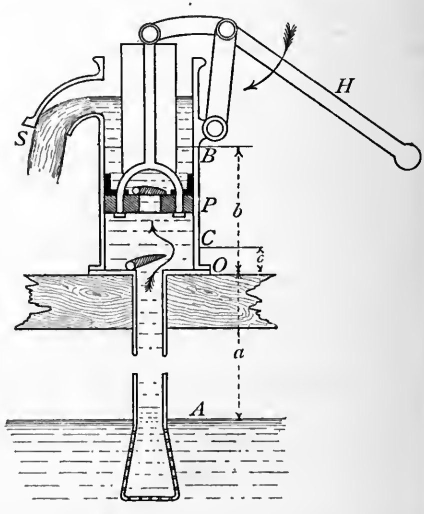


Fig. 80.

This method is nevertheless employed nowadays in the *Pulsator* or *Pulsometer* pump (fig. 79) where economy is not of so much importance as rapidity and certainty of action; an automatic spherical valve *C* admits steam to act alternately on the surface of the water in the vessels *A* and *B*, by which the water is forced to the required level; entrance and exit valves being provided to each chamber, as in a pump.

259. In the common *suction pump* of domestic use the upper valve is placed in the bucket or piston, so that water passes through the bucket and is lifted by it, when the upper valve closes, to the desired level (fig. 80).

If the piston rod is thickened so that its cross section is about half that of the pump barrel, half the water will be ejected during the down stroke of the bucket, and a more equable flow is thereby secured.

The *suction* of a pump is the height reckoned from the surface of the water supply to the lower valve, but the height of the discharge above the lower valve is the height to which the water is *forced* or *lifted*.

When the *lift* of the pump is considerable, a relief valve, opening upwards, is placed in the discharge pipe, and the barrel is closed with a cover and stuffing box, through which the piston rod works; the lower fixed valve may now be dispensed with, and this arrangement is called a *Lifting Pump;* but if the bucket valve is suppressed, the water is raised in the down stroke, and this is called a *Forcing Pump* (fig. 8).

The *suction* is limited theoretically by the barometric head of water, about 33 ft or 10 m, but water can be lifted or *forced* to an indefinite height; the *suction and forcing* pumps of a mine or of the pumping engines of water works must therefore be placed at a low level, very nearly that of the water supply.

Instances are recorded in which the suction of a pump has reached even 40 ft; but in such cases the water must be highly aerated; so that we may consider the column in the suction pipe as composed of alternate strata of liquid and air, as in the Sprengel pump (§ 275), instead of continuous solid water.

Denote by α and β the cross section, in ft^2, of the suction pipe AO and barrel OB of a vertical suction pump, by a the height of the suction pipe, and by b and c the greatest and least height, OB and OC, of the lower side of the bucket P above the lower fixed valve O (fig. 80).

Then in the nth stroke, while the pump is sucking, the water rises in the suction pipe AO from a height x_{n-1} to x_n ft above the level A of the supply, and the air above the column changes in density from ρ_{n-1} to ρ_n, where

$$\frac{\rho_n}{\rho_{n-1}} = \frac{H-x_n}{H-x_{n-1}},$$

and the tension of the pump rod increases from

$$D\beta x_{n-1} \text{ to } D\beta x_n \text{ lb.}$$

Also by Boyle's law,

$$\alpha(a-x_n)\rho_n + \beta b\rho_n = \alpha(a-x_{n-1})\rho_{n-1} + \beta c\rho;$$

so that

$$\alpha\{(a-x_{n-1})(H-x_{n-1})-(a-x_n)(H-x_n)\} = \beta\{b(H-x_n)-cH\},$$

a quadratic equation for x_n in terms of x_{n-1}, of which the positive root must be taken, the negative root corresponding to a different physical problem.

This equation may be written

$$\beta(b-c) - \alpha(x_n - x_{n-1}) = \beta\frac{(b-c)(a-x_{n-1})+cx_n}{H+a-x_n-x_{n-1}};$$

and the second member being positive, it follows that $\alpha(x_n - x_{n-1})$, the volume which enters the suction pipe in the nth stroke, is less than $\beta(b-c)$, the volume swept out by the bucket.

The water will reach the barrel in the first stroke if $x_1 = a$, x_0 being zero; and therefore if

$$\frac{a}{H} = \frac{\beta(b-c)}{\alpha H + \beta b}.$$

It will just reach the barrel in the second stroke, if $x_2 = a$; and the condition is obtained by eliminating x_1 between the equations

$$a(a-x_1)(H-x_1) = \beta\{b(H-a)-cH\},$$
$$a\{aH-(a-x_1)(H-x_1) = \beta\{b(H-x_1)-cH\};$$

and so on.

If the pump will not *suck*, then x_n is always less than a; and the greatest height x to which the water rises in the suction pipe is obtained by putting

$$\rho_n = \rho_{n-1}, \quad x_n = x_{n-1} = x;$$
$$a(a-x)(H-x)+\beta cH = a(a-x)(H-x)+\beta b(H-x),$$
$$\frac{x}{H} = 1-\frac{c}{b}.$$

The first requisite for the pump to work is therefore

$$a < \left(1-\frac{c}{b}\right)H;$$

so that water can be drawn through the lower valve.

260. Now if x_{m-1} and x_m denote the height above the level of the supply of the water in the barrel at the beginning and end of the mth stroke, the air which occupied a length $a+c-x_{m-1}$ of the barrel under a head H at the beginning of the stroke will at the end occupy a length $a+b-x_m$ under a head $H-x_m$; so that,

$$(a+b-x_m)(H-x_m) = (a+c-x_{m-1})H,$$

a quadratic for determining x_m in terms of x_{m-1}.

Also the tension of the pump rod, due to the pressure of the air, increases from zero to $D\beta x_m$ during the stroke; and if the lower valve O opens when the pressures above and below are equal, the bucket has then risen a distance z, given by

$$\frac{cH}{c+z}+x_m-a = H-a, \text{ or } \frac{z}{c} = \frac{x_m}{H-x_m}.$$

Similarly the position of the bucket in the down stroke when its valve opens can be determined.

The greatest height to which the water can be drawn in the barrel, provided it does not reach the bucket, is obtained by putting $x_{m-1}=x_m=x$; and therefore

$$x=\tfrac{1}{2}(a+b)+\sqrt{\{\tfrac{1}{4}(a+b)^2-(b-c)H\}};$$

and therefore a requisite condition is

$$(b-c)H<\tfrac{1}{4}(a+b)^2, \text{ or } BC<\tfrac{1}{4}AB^2/H;$$

otherwise the water would sink during the successive strokes; and the least value of x is thus $\frac{1}{2}(a+b)$.

Finally for the pump to draw, C must be below this level; and now, in full working order with the passages full of water, if h denotes the height of the discharge above the lower valve, and y the height of the bucket at any part of the stroke, the tension of the pump rod is

$$D\beta(H+h-y)-D\beta(H-a-y)=D\beta(a+h) \text{ lb};$$

so that the work done in one stroke is

$$D\beta(a+h)(b-c) \text{ ft lb};$$

the work required to lift the volume of water $\beta(b-c)$ ft^3 through $a+h$ ft.

261. *Air Pumps.*

In the ancient method of producing a vacuum, as invented by Otto von Guericke, 1650, the vessel to be exhausted (the Magdeburg hemispheres, for instance) was first filled with water, which was afterwards pumped out by a water pump.

The mechanical improvements of the pump made by Boyle, Hooke, and Hauksbee enabled them to dispense with the water, and to construct the true air pump, as we have it nowadays.

Two suction pumps, side by side, actuated in opposite directions by racks on the piston rods engaging in a toothed wheel between them, worked in a reciprocating motion by a handle, constitute *Hauksbee's air pump;* and two pumps are used, so that the atmospheric pressure on the tops of the pistons should balance them in any position.

The pumps draw the air through a pipe which terminates in the centre of a horizontal brass plate, upon which the glass jar or *receiver*, which is to be exhausted, has been placed, the lower edge of the glass having been ground and greased so as to make an air-tight contact with the brass plate.

262. Smeaton's air pump is essentially the *lifting pump;* he formed it by closing the top of Hauksbee's air pump with a cover, provided with a stuffing box for the piston rod and a valve opening outwards; the piston is thereby relieved from the pressure of the air during the greater part of the stroke, so that two pumps, balancing each other, are not required.

The lower fixed valve may also be dispensed with; and the piston valve too, if the pipe communicating with the receiver enters the side of the barrel at a distance from the bottom a little over the thickness of the piston.

These principles are illustrated in Tate's air pump, consisting of a double acting pump and the receiver (fig. 81); the piston is made long and provided with cannelures, by which leakage of air past it is prevented, in spite of the absence of packing; which may however be supplied by cupped leathers, as in figs. 11, 12, p. 23.

A valve at each end, consisting of a small flap of oiled silk covering from the outside a narrow slit, permits the escape of the air when compressed to the atmospheric pressure; no valve is required in the middle, as the piston is worked just past the communication with the receiver.

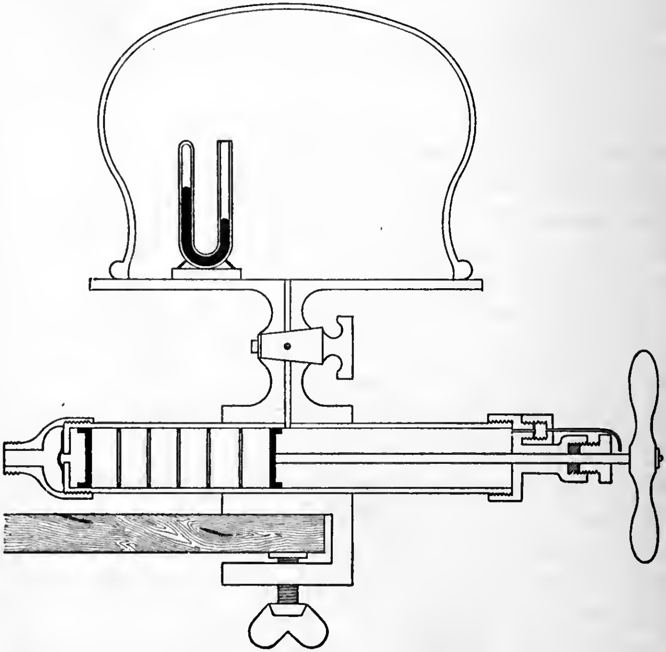


Fig. 81.

263. Denoting by A the volume of the receiver and by B the volume of the barrel of the pump swept out by the passage of the piston in a single stroke; then, in the absence of any *clearance*, the air which occupied the volume A at the beginning will occupy the volume $A+B$ at the end of the stroke; or denoting by ρ_{n-1} and ρ_n the densities of the air in the receiver at the beginning and end of this the nth stroke

$$(A+B)\rho_n = A\rho_{n-1},$$

or $$\rho_n = \frac{A}{A+B}\rho_{n-1} = \left(\frac{A}{A+B}\right)^2 \rho_{n-2} = \dots = \left(\frac{A}{A+B}\right)^n \rho,$$

if ρ denotes the density of atmospheric air.

264. Supposing the temperature constant, the piston valve opens in the return stroke when the air of density ρ_n occupying the volume B of the barrel is compressed to atmospheric density ρ; and therefore at a fraction ρ_n/ρ of the stroke, distances diminishing in G.P.

If β denotes the cross section of the pump barrel in ft², and $k\rho$ the atmospheric pressure in lb/ft², the force in lb required to move the pistons in Hauksbee's air pump, measured by the difference of the tensions of the rods, is

$$k\rho\beta - k\rho_{n-1}\beta\frac{A}{A+Bx} - k\rho\beta + k\rho_{n-1}\beta\frac{A}{B-Bx},$$

at a fraction x of the nth stroke, until $x = \rho_{n-1}/\rho$; after which the valve in the descending piston opens, and the effective tension of this rod is zero.

These tensions can be represented graphically by hyperbolas, as in the hydrometer of fig. 36, p. 113.

265. *Work required to exhaust the receiver.*

In the nth stroke, the ascending piston in Hauksbee's pump pushes back the atmosphere through a volume B, while the air underneath, originally at pressure $k\rho_{n-1}$ and volume A, expands to volume $A+B$.

Therefore the work done by the ascending piston is, in ft-lb (§ 233),

$$k\rho B - k\rho_{n-1}A \log\left(1+\frac{B}{A}\right).$$

The descending piston yields to the atmospheric pressure $k\rho$ through a volume $B - B\rho_{n-1}/\rho$, and at the same time compresses a volume B of air at pressure $k\rho_{n-1}$ to a volume $B\rho_{n-1}$.

Therefore the work done *on* the descending piston is

$$k(\rho-\rho_{n-1})B-k\rho_{n-1}B\log\rho/\rho_{n-1}.$$

The work done in the nth stroke is the difference,

$$k\rho_{n-1}\left\{B+B\log\frac{\rho}{\rho_{n-1}}-A\log\left(1+\frac{B}{A}\right)\right\}.$$

Putting $\quad \dfrac{A}{A+B}=p,\;$ and $\;\dfrac{\rho_n}{\rho}=p^n,$

the work done in the n strokes is

$$k\rho(A+B)\Sigma\{p^{n-1}(1-p+\log p)-(1-p)np^{n-1}\log p\}$$
$$=k\rho(A+B)(1-p^n+p^n\log p^n)$$
$$=k\rho(A+B)\left(1-\frac{1}{q}-\log\sqrt[q]{}/q\right),$$

if the air is exhausted to one-qth of the atmospheric density; this reduces, when the exhaustion is complete, to $k\rho(A+B)$ ft-lb, the work required to force back the atmospheric pressure $k\rho$ lb/ft² through a volume $A+B$ ft³, as is evident *à priori*.

A similar result holds for Smeaton's and Tate's air pump, where there is no clearance; the investigation when the effect of clearance is taken into account is left as an exercise.

266. *Clearance.*

Suppose the piston does not completely sweep out the cylinder, but leaves an untraversed space C, at the bottom of the barrel in Hauksbee's air pump; this space C is called the *clearance* (*espace nuisible, schädlicher Raum*).

A volume C of atmospheric air is now left in the barrel at the end of each stroke; and therefore the volume $A+B$ of air at density ρ_n is equivalent to a volume A at density ρ_{n-1} and a volume C at density ρ; so that

$$(A+B)\rho_n=A\rho_{n-1}+C\rho.$$

Writing this equation in the form

$$\rho_n - \frac{C}{B}\rho = \frac{A}{A+B}\left(\rho_{n-1} - \frac{C}{B}\rho\right)$$

shows that, according to the laws of Geometrical Progression,

$$\rho_n - \frac{C}{B}\rho = \left(\frac{A}{A+B}\right)^n \left(1 - \frac{C}{B}\right)\rho.$$

Thus, after an infinite number of strokes, when $n = \infty$,

$$\rho_\infty = \frac{C}{B}\rho,$$

which gives the ultimate exhaustion when there is a clearance C; and it is only when $C=0$ that $\rho_\infty = 0$, or the theoretical exhaustion is complete.

267. With a clearance C at the bottom and C' at the top of the stroke in Smeaton's Air Pump, and denoting the density of the air in the barrel in the $(n-1)$th down stroke by σ_{n-1},

$$B\sigma_{n-1} = (B - C')\rho_{n-1} + C'\rho,$$

while in the nth up stroke

$$(A + B - C')\rho_n = A\rho_{n-1} + C\sigma_{n-1};$$

so that, eliminating σ_{n-1},

$$B(A + B - C')\rho_n = AB\rho_{n-1} + C(B - C')\rho_{n-1} + CC'\rho,$$

or

$$\rho_n - \mu\rho = \lambda(\rho_{n-1} - \mu\rho) = \lambda^n(1 - \mu)\rho,$$

where $\lambda = \dfrac{B(A+C) - CC'}{B(A+B-C')}$, $\mu = \dfrac{CC'}{(B-C)(B-C')}$;

and $\mu\rho$ is the ultimate density in the receiver, after a large number of strokes.

Similarly, the elimination of ρ_n leads to

$$\sigma_n - \nu\rho = \lambda(\sigma_{n-1} - \nu\rho) = \lambda^n(1 - \nu)\rho,$$

where $\nu = \dfrac{C'}{B - C'}.$

In the $(n-1)$th down stroke the piston valve opens when the volume C' of atmospheric air has expanded to a density σ_{n-1} and therefore to a volume $C'\rho/\sigma_{n-1}$.

In the nth up stroke the lower fixed valve opens when the volume C of air of density σ_{n-1} has expanded to density ρ_{n-1} and therefore to volume $C\sigma_{n-1}/\rho_{n-1}$; and the upper fixed valve opens when the air of volume $B-C$ and density σ_{n-1} has been compressed to atmospheric density ρ, and therefore to volume

$$(B-C)\sigma_{n-1}/\rho.$$

268. The weight of the valves is another cause tending to limit the rarefaction; suppose then that ϖ and ϖ' denote the pressures required to lift the fixed and piston valve in Hauksbee's pump.

So long as the valves operate, the pressures in the barrels at the end of the nth in and out stroke are respectively

$$k\rho+\varpi' \text{ and } k\rho_n-\varpi.$$

During the nth stroke the air which occupied the receiver A at pressure $k\rho_{n-1}$ and the clearance C at pressure $k\rho+\varpi'$ has expanded to air of volume A and pressure $k\rho_n$ and of volume B and pressure $k\rho_n-\varpi$; and therefore

$$Ak\rho_n+B(k\rho_n-\varpi)=Ak\rho_{n-1}+C(k\rho+\varpi');$$

or $$\rho_n-\frac{\varpi}{k}-\frac{C}{B}\left(\rho+\frac{\varpi'}{k}\right)=\left(\frac{A}{A+B}\right)\left\{\rho_{n-1}-\frac{\varpi}{k}-\frac{C}{B}\left(\rho+\frac{\varpi'}{k}\right)\right\}$$

$$=\left(\frac{A}{A+B}\right)^n\left\{\left(1-\frac{C}{B}\right)\rho-\frac{\varpi}{k}-\frac{C}{B}\frac{\varpi'}{k}\right\}.$$

The piston valve is lifted in the down stroke when the volume B of air of pressure $k\rho_{n-1}-\varpi$ is compressed to pressure $k\rho+\varpi'$ and volume to xB, given by

$$k\rho_{n-1}-\varpi=x(k\rho+\varpi');$$

and the fixed valve is opened in the up stroke when the volume C of air of pressure $k\rho+\varpi'$ is expanded to volume yC and pressure $k\rho_{n-1}-\varpi$, given by

$$k\rho+\varpi'=y(k\rho_{n-1}-\varpi);$$

so that

$$xy=1.$$

Generally

$$x=\frac{1}{y}=\frac{k\rho_{n-1}-\varpi}{k\rho+\varpi'}$$
$$=\frac{k\rho+\varpi}{k\rho+\varpi'}\left(\frac{A}{A+B}\right)^{n-1}+\frac{C}{B}\left\{1-\left(\frac{A}{A+B}\right)^{n-1}\right\}.$$

The piston valve opens after the fixed valve, if

$$B+C-yC>xB,$$

or if $B(1-x)-C(y-1)$

$$=\frac{1-x}{x}\left(B\frac{k\rho+\varpi}{k\rho+\varpi'}-C\right)\left(\frac{A}{A-B}\right)^{n-1}$$

is positive, as is generally the case; and the valves cease to act when

$$x=\frac{C}{B},\quad y=\frac{B}{C}.$$

269. To measure the rarefaction, a glass tube may be led down from the bottom of the receiver to a cistern of mercury; and the height of the column, drawn up as in Hare's hydrometer (§ 162) will measure the rarefaction, the difference of the height of this column and of the barometric height being the mercury head of the pressure in the receiver.

The pressure may also be measured by a barometer inside the receiver; and, to keep down the height, a shortened siphon barometer is employed, in which the Torricellian vacuum does not begin to appear till the pressure is considerably reduced, say to a head of 2 ins of mercury.

270. *The Condensing Pump.*

If the direction of motion of the valves of an air pump is reversed, air will be forced into the receiver by the motion of the piston, and the pump is called a *compressing* or *condensing pump* or a *condenser.*

Thus Smeaton's pump can be used as a condenser if the receiver is fixed over the top valve, air being drawn from the atmosphere through the lower valve, which may be dispensed with.

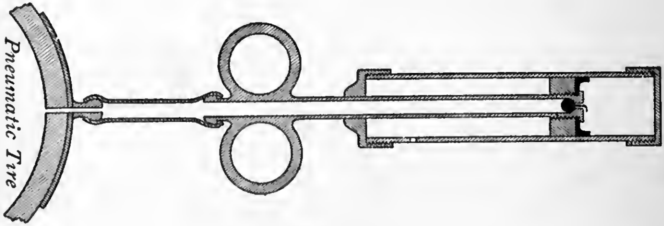


Fig. 82.

A simple form of condensing pump, employed for inflating pneumatic tires of bicycles, is shown in fig. 82; and fig. 78 may be taken to represent the condensing or exhausting pump required for the Westinghouse or vacuum brake on railway carriages.

Condensing pumps are required to supply fresh air, condensed to the requisite pressure, to submarine divers and diving bells, to caissons and other subaqueous operations; also in the transmission of power to drive machinery (Popp's system), to the Whitehead torpedo, and to boring machines in mines, and in the construction of long tunnels such as the St. Gothard, the air in its return assisting the ventilation.

With no clearance, a volume B of atmospheric air is forced every stroke into the volume A of the receiver, so that after n complete strokes the density ρ_n is given by

$$A\rho_n = (A + nB)\rho,$$

or

$$\rho_n = \left(1 + n\frac{B}{A}\right)\rho,$$

so that the density (and pressure) mounts up in A.P.

The thrust P_n lb required to make the nth stroke complete is given by

$$P_n = (p_n - p)\beta = np\beta B/A.$$

A gauge for the condenser may be made of a horizontal glass tube OA, closed at O, and containing a filament of mercury AB, exposed to the pressure in the receiver.

Now as the pressure mounts up in n strokes from p to p_n, the filament will move from A towards O, to A_n suppose, such that, by Boyle's law,

$$\frac{OA_n}{OA} = \frac{p}{p_n} = \frac{A}{A + nB} = \frac{m}{m + n}, \quad \frac{AA_n}{OA_n} = \frac{n}{m},$$

if

$$m = A/B.$$

Sometimes this gauge is held vertical, dipping into a cistern of mercury; and now, if $OA = a$, at atmospheric pressure, and $OA_n = y_n$, the pressure in atmospheres after n strokes is

$$\frac{a}{y_n} - \frac{a - y_n}{h} = 1 + n\frac{B}{A},$$

so that the graduations are given graphically by equidistant ordinates of a hyperbola.

271. With clearance C, the volume B of atmospheric air of density ρ and the volume A of air of density ρ_{n-1} becomes condensed in the nth stroke to the volume $A + C$ of air of density ρ_n, so that

$$(A + C)\rho_n = A\rho_{n-1} + B\rho.$$

Writing this equation as a Geometrical Progression,

$$\rho_n - \frac{B}{C}\rho = \frac{A}{A+C}\left(\rho_{n-1} - \frac{B}{C}\rho\right),$$

$$\rho_n - \frac{B}{C}\rho = \left(\frac{A}{A+C}\right)^n\left(\rho - \frac{B}{C}\rho\right),$$

or

$$\frac{\rho_n}{\rho} = \frac{B}{C} - \left(\frac{B}{C} - 1\right)\left(\frac{A}{A+C}\right)^n.$$

The ultimate compression, when $n=\infty$, is B/C atmospheres; and this is otherwise evident, if we notice that air will cease to enter the receiver A when its pressure is that of the air compressed to the volume C from the volume B of atmospheric air.

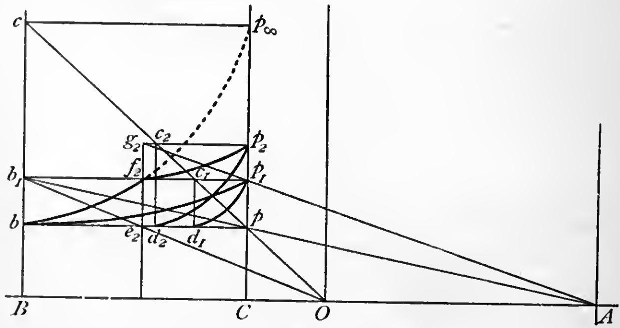


Fig. 83.

Writing this relation

$$\frac{\rho_n}{\rho} = \frac{B}{C} - \left(\frac{B}{C} - 1\right)\left(1 - \frac{C}{A+C}\right)^n,$$

expanding by the Binomial Theorem, reducing, and lastly putting $C=0$, we obtain, as before,

$$\frac{\rho_n}{\rho} = 1 + n\frac{B}{A}.$$

But practically, as in diving operations, the pumps are set to compress the air to the requisite number of atmospheres B/C, by an adjustment which gives the clearance C the appropriate magnitude.

If ϖ and ϖ' denote the pressures required to lift the fixed and moving valves, and p_∞ the limiting pressure; then air of pressure $p_\infty + \varpi$ in the clearance C will have been compressed from volume B and pressure $p - \varpi'$; so that

$$C(p_\infty + \varpi) = B(p - \varpi'),$$

$$p_\infty = \frac{B}{C}(p - \varpi') - \varpi.$$

If it is assumed that in a diving bell or dress at a depth z, and under a pressure of $\frac{B}{C}$ or $1 + \frac{z}{H}$ atmospheres, the air escapes at a rate of

$$V \log \frac{B}{C} \text{ or } V \log\left(1 + \frac{z}{H}\right) \text{ ft}^3\text{/minute},$$

the pump must make n strokes a minute, given by

$$n = \frac{V}{A} \log \frac{B}{C} = \frac{V}{A} \log\left(1 + \frac{z}{H}\right).$$

272. A graphical construction of the working of the condenser is given in fig. 83; here OA represents the volume of the receiver, OB of the barrel, OC of the clearance, while ordinates represent the pressure of the air.

In the first stroke the atmospheric air filling AB is compressed from atmospheric pressure Bb or Cp along the isothermal hyperbola bp_1, centre A, until it cuts the ordinate Cp in p_1, and then Cp_1 represents the pressure in the receiver at the end of the first stroke.

In the return stroke the pressure in the barrel OB falls along the hyperbola p_1d_1, centre O, until it reaches atmospheric pressure; after which the piston valve opens.

The work done in the first complete stroke is therefore represented by the area bp_1d_1.

In the second stroke the atmospheric air in OB is compressed along the hyperbola bf_2, centre O, until the pressure becomes p_1; and then the valve at O opens, and the pressure in OA mounts up along the hyperbola f_2p_2, centre A.

In the return stroke the pressure in OC falls along the hyperbola p_2d_2, centre O, and the work done in the second stroke is represented by the area $bf_2p_2d_2$; and so on.

To construct these points geometrically, draw Ap to meet Bb in b_1, then b_1p_1 is parallel to AB; draw Op to meet b_1p_1 in c_1, then c_1d_1 is parallel to Cp; draw Ob_1 cutting bp in e_2, then e_2f_2 is parallel to Cp; draw Ap_1 to meet e_2f_2 in g_2, then g_2p_2 is parallel to AB; and if g_2p_2 meets Op in c_2, then c_2d_2 is parallel to Cp; and so on.

The ultimate compression in the receiver, represented by the pressure Cp_∞, is obtained by producing Op to meet Bb in c, and drawing cp_∞ parallel to AB.

A similar construction can be employed for the air-pump, or for the combined condenser and air-pump.

273. *The Air Pump and Condensing Pump combined.* Tate's air pump can be made to act as a condensing pump by screwing the receiver which is to be filled on the end of the barrel.

Suppose then, as the most general case, that air is pumped from a vessel of volume A and forced into a vessel of volume A' by means of a Smeaton pump of volume B, leaving clearances C and C' at the ends.

Starting with all the air at atmospheric density ρ and the piston close to A, and denoting by ρ_{n-1} and ρ'_{n-1} the densities in the vessels A and A' at the end of the $n-1$th stroke towards A'; then

$$(A+B-C')\rho_{n-1}+(A'+C')\rho'_{n-1}=(A+A'+B)\rho, \quad (1)$$

an equation expressing the constancy of the total quantity of air enclosed.

During the nth stroke towards A the fixed valves are closed and the piston valve opens, when the air in the barrel B assumes the uniform density σ_{n-1}, given by

$$B\sigma_{n-1}=(B-C')\rho_{n-1}+C'\rho'_{n-1}, \quad \ldots\ldots\ldots\ldots(2)$$

the valve opening when the piston divides the volume B into two parts P and P', such that

$$\left.\begin{aligned} P\,\sigma_{n-1}&=(B-C')\rho_{n-1}\\ P'\sigma_{n-1}&=C'\rho'_{n-1}\end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots(3)$$

During the nth return stroke towards A' the piston valve is closed, and the fixed valves open; and at the end of this stroke

$$(A+B-C')\rho_n=A\rho_{n-1}+C\sigma_{n-1}, \quad \ldots\ldots\ldots\ldots(4)$$

$$(A'+C')\rho'_n=A'\rho'_{n-1}+(B-C)\sigma_{n-1}; \quad \ldots\ldots\ldots(5)$$

and the addition of equations (2), (4), and (5) leads as a verification to equation (1).

Eliminating σ_{n-1} between (2) and (4),

$$\begin{aligned} B(A+B-C')\rho_n&=AB\rho_{n-1}+C(B-C')\rho_{n-1}+CC'\rho'_{n-1}\\ &=(AB+BC-CC')\rho_{n-1}\\ &\quad+\frac{CC'}{A'+C'}\{(A+A'+B)\rho-(A+B-C')\rho_{n-1}\},\end{aligned}$$

or $B(A'+C')(A+B-C')\rho_n$

$$\begin{aligned} &=\{B(A+C)(A'+C')-CC'(A+A'+B)\}\rho_{n-1}\\ &\quad+CC'(A+A'+B)\rho,\end{aligned}$$

which may be written in the form

$$\rho_n-\mu\rho=\lambda(\rho_{n-1}-\mu\rho)=\lambda^n(1-\mu)\rho, \quad \ldots\ldots\ldots(6)$$

where $\lambda=\dfrac{B(A+C)(A'+C')-CC'(A+A'+B)}{B(A'+C')(A+B-C')}$,

$$\mu=\frac{CC'(A+A'+B)}{(A'+C')(B-C)(B-C')+CC'(A+B-C')};$$

and the ultimate exhaustion of A is to a density $\mu\rho$.

Similarly $\rho'_n - \mu'\rho = \lambda(\rho'_{n-1} - \mu'\rho) = \lambda^n(1-\mu')\rho$,

where $\mu' = \dfrac{(B-C)(B-C')(A+A'+B)}{(A'+C')(B-C)(B-C')+CC'(A+B-C')}$,

and the ultimate compression in A' is to a density $\mu'\rho$.

In Smeaton's air pump, $A' = \infty$; and in Hauksbee's air pump, B and C' are infinite, but $B - C'$ is finite; so that, putting $B - C' = B'$,

$$\lambda = \frac{A}{A+B'}, \quad \mu = \frac{C}{B'}, \text{ as before (§ 266).}$$

In the condensing pump, $A = \infty$; and now, putting $B - C = B''$, so that B'' is the volume swept out by the piston, and making B and C infinite, gives the result for the ordinary condensing pump, with clearance C' (§ 271).

274. In the preceding investigations the temperature of the air has been assumed constant; but if the pumps are worked rapidly, the adiabatic laws employed in § 226 show that the temperature rises and falls with the density and pressure.

Refrigerating machinery depends to a great extent in its action on this lowering of temperature with rarefaction; while on the other hand the compressed air supplied to the diver is warmed to an appreciable extent.

The expressions obtained for the density of the air will not be altered; and the change in pressure will only affect the points at which the valves operate and the work required for exhaustion or compression; and when after a lapse of time thermal equilibrium has been restored by conduction of heat, the pressure will assume the value that has been employed.

275. *Mercurial Air Pumps.*

The rarefaction is limited by the leakage of the piston and valves, and by the air absorbed and given off by the oil; so that the vacuum which can be produced by an ordinary air pump is not sufficiently good for incandescent electric lamps.

In the Fleuss air pump the passages of the pump are filled with oil which circulates and fills up the clearance, and the oil is freed from air in a duplicate pump alongside.

Sometimes mercury is employed to fill up the clearance, as in Kravogl's air pump; and one of the earliest and best methods of exhausting a vessel is to make it into a Torricellian vacuum, as in Torricelli's original method, during which he discovered the barometer (§ 171); but this method requires a large quantity of mercury.

In Sprengel's mercurial pump (fig. 84) the exhaustion of the air from a globe *C* is performed automatically by an intermittent flow of mercury by drops from the reservoir *A*, which gradually sweep out the air, and discharge it in bubbles in the cistern *B*; a pinch cock *E* on a short length of india-rubber tube controlling the flow of the mercury.

This is the essential part of the instrument; but a duplicate arrangement *FG* is now generally placed alongside, the vessel *G* to serve as an air trap for the bubbles in the mercury (fig. 85).

To make a joint perfectly airtight, it is sealed by mercury surrounding it, as shown in the joint above *D* in fig. 85.

If the bubbles at *B* are discharged into a receiver, partially exhausted of air by a pump, the apparatus can be considerably shortened below its normal height of about 40 ins; three or four fall tubes may be employed, to make the exhaustion more rapidly.

The barometric column HK measures the rarefaction; this is also measured by the *M^cLeod gauge*, consisting of a graduated tube closed at the top which can be filled up with mercury to a given head, as in fig. 66; the compression of the rarefied air imprisoned in the gauge measures the rarefaction, to a millionth of an atmosphere, which would be quite insensible on the barometric column HK.

If the drops of mercury occupy equal lengths a of the fall tube, and if c denotes the length of the air bubble when at the level of the cistern B, and therefore at atmospheric pressure, then the lengths of the successive bubbles above the cistern are

$$\frac{ch}{h-a}, \quad \frac{ch}{h-2a}, \quad \frac{ch}{h-3a}, \ldots,$$

where h denotes the height of the mercury barometer; these lengths of air increasing in H.P. till the junction D is reached, so that they can be represented by the ordinates of a rectangular hyperbola.

An inverted Sprengel tube LMA, driven by compressed air at L, can be employed to raise the mercury again from the cistern B to the reservoir A. (Rev. F. J. Smith, *Phil. Mag.*, 1892; *Nature*, Aug. 1893; *Mercurial Air Pumps*, S. P. Thompson: Journal Soc. of Arts, 1887.)

Examples.

(1) Prove that if the height of the spout of a suction pump above the water supply is the height of the water barometer, and if at the commencement of any stroke the water in the suction pipe is m and n ft below the spout and the fixed valve, the water will rise $\sqrt{m}(\sqrt{m}-\sqrt{n})$ ft in the next stroke, if there is no clearance and if the pump is of uniform section throughout.

(2) Examine the effect of taking alternate strokes of an air pump and of a condenser attached to a receiver.

Prove that if the barrel of each pump is one twentieth of the receiver, and the condenser be worked for 20 strokes and then the air pump for 14 strokes, the density of the air will be practically unaltered.

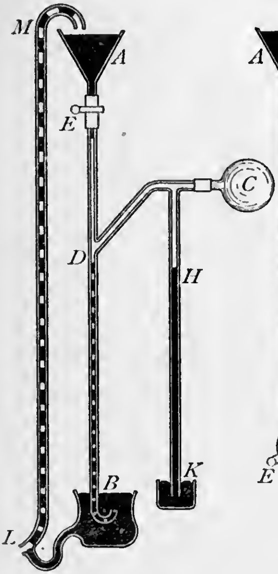


Fig. 84.

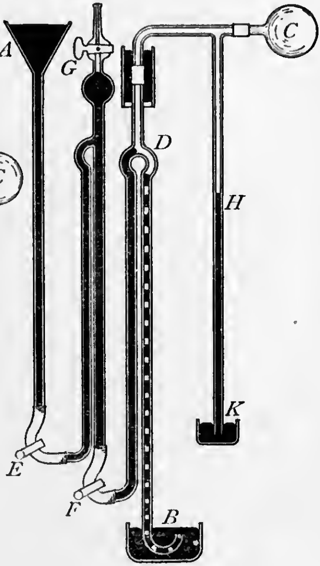


Fig. 85.

(3) Prove that if a bladder occupies one nth of the volume of the receiver of an air pump, and if it bursts when the pressure is reduced to one mth of an atmosphere, the mercurial gauge will fall

$$h(m-1)/mn,$$

when h is the height of the barometer.

(4) Prove that if the temperature is constant the work required to increase q fold, or to diminish to one-qth, the density of atmospheric air of pressure P in a receiver of volume V is, respectively,

$$PV(\log q^q - q + 1) \quad \text{and} \quad PV\left(1 - \frac{1}{q} - \log \sqrt[q]{q}\right).$$

Calculate the work required and the change of temperature in these cases when the compression or rarefaction takes place *adiabatically* (§ 233).

Work out $V = 1\,\text{m}^3$, $P = 10^4\,\text{kg/m}^2$, $q = 100$.

(5) Examine the change in the indications of the siphon barometer of § 179, placed in the receiver of an air pump or condenser, when the dimensions of the barometer are taken into account; and prove that, in one stroke of the air pump, the barometric column falls a distance, approximately,

$$\frac{Bh}{A+B}\left\{1 - \frac{A}{(A+B)^2}\,\frac{\alpha\beta h}{\alpha+\beta}\right\},$$

while, in n strokes of the condenser, it rises, approximately,

$$n\frac{Bh}{A}\left\{1 - \frac{\alpha\beta h}{\alpha+\beta}\,\frac{1}{A}\left(1 + n\frac{B}{A}\right)\right\},$$

where A denotes the original volume of atmospheric air in the receiver.

Prove also that if v denotes the volume and a the height of the air pump gauge in § 269, the mercury will rise in one stroke (neglecting the square of v),

$$\frac{Bh}{A+B} - \frac{Bhv}{(A+B)^2}\left(1 + \frac{A}{A+B}\,\frac{h}{a}\right).$$

CHAPTER IX.

THE TENSION OF VESSELS. CAPILLARITY.

276. The vessels employed for containing a fluid under great pressure are generally made cylindrical or spherical for strength; and it is important to determine the stress in the material for given fluid pressure, or the maximum pressure allowable for given strength of material, for the purpose of calculating the requisite thickness.

The simplest case of a vessel in tension is a circular pipe or cylindrical boiler, exposed to uniform internal pressure, so that there is no tendency to distortion from the circular cross section.

With an internal pressure p, a circumferential pull will be set up in the material of magnitude T per unit of length suppose, acting across a longitudinal section or seam of the cylinder; and to determine T we suppose a length l of the cylinder to be divided into two halves by a diametral plane, and consider the equilibrium of either half.

Denoting by d or $2r$ the internal diameter of the tube, the resultant fluid thrust on the curved semicircular surface is equal to the thrust across the plane base, and is therefore pld; and this thrust being balanced by the pull Tl on each side of the diametral plane, therefore

$$2Tl = pld, \quad \text{or} \quad T = \tfrac{1}{2}pd = pr. \qquad \text{.............(1)}$$

277. When the cylindrical vessel is closed there is in addition a longitudinal tension in the material; denoting by T' the longitudinal pull per unit length across a circumferential seam or section, the fluid thrust $\frac{1}{4}\pi d^2 p$ on the end of the cylinder must be balanced by the longitudinal pull $\pi dT'$ across a circumferential seam, and therefore

$$\pi dT' = \tfrac{1}{4}\pi d^2 p, \quad \text{or} \quad T' = \tfrac{1}{4}pd = \tfrac{1}{2}pr = \tfrac{1}{2}T. \qquad \text{.......(2)}$$

Thus the longitudinal tension is half the circumferential tension; or in the cylindrical shell of a boiler the circumferential joints and rows of rivets which resist the longitudinal pull need be only half the strength of the longitudinal joint, which resists the circumferential pull.

278. A spherical surface in tension is beautifully illustrated by a soap bubble as a complete sphere in air, or as a hemisphere on the surface; denoting the internal diameter of a spherical vessel by d or $2r$, the tension per unit length across a diametral section, due to internal pressure p, is also T' or $\frac{1}{2}pr$; for considering the equilibrium of either hemisphere into which the sphere is divided, the fluid thrust on the hemisphere is equal to the thrust on the base, or $\frac{1}{4}\pi d^2 p$; and this is balanced by $\pi dT'$, the resultant pull round the circumference; so that, as before,

$$\pi dT' = \tfrac{1}{4}\pi d^2 p, \quad \text{or} \quad T' = \tfrac{1}{4}pd = \tfrac{1}{2}pr.$$

Thus if a cylindrical boiler is made with hemispherical ends, these ends need have only half the thickness of the cylindrical shell; but they will weigh the same as flat ends of the same thickness as the shell.

The same results are obtained by considering the equilibrium of the part cut off by any plane parallel to the axis of the cylinder; if this part subtends an angle 2θ at the axis, the fluid thrust on it, $2lr \sin\theta \,.\, p$, is balanced by the components of the pull on each side, perpendicular to this plane, $2lT \sin\theta$; and $\sin\theta$ divides out; so also for the spherical surface.

279. If the cylinder has a conical end or shoulder, of vertical angle $2a$, the stresses in the surface will be no longer uniform; taking a circular cross section PMQ, of centre M and diameter $2y$, cutting off the conical end POQ, and denoting by T_1 and T_2 the tensions per unit length across this section and across the straight section of the surface, then from the equilibrium of POQ,

$$2\pi y T_1 \cos a = \pi y^2 p, \quad \text{or} \quad T_1 = \tfrac{1}{2} py \sec a.$$

To determine T_2, consider the equilibrium of either half of the surface cut off by two adjacent circular sections pmq, $p'm'q'$, equidistant from PMQ; therefore

$$2T_2 \,.\, pp' = p \,.\, PQ \,.\, mm' + \tfrac{1}{2} p \tan a \,.\, pm \,.\, pq - \tfrac{1}{2} p \tan a \,.\, p'm' \,.\, p'q',$$

or $\quad T_2 = py \cos a + py \tan a \sin a = py \sec a = 2T_1.$

Thus T_1 and T_2 become large when the conical end is nearly flat; so that the ends require strengthening with longitudinal stays.

Suppose however that there is a conical shoulder, as in the Coney Island Stand Pipe (§ 42); then if $2a$ denotes the diameter of the upper small end, and p the average pressure at the shoulder,

$$2\pi y T_1 \cos a = \text{upward thrust} = \pi (y^2 - a^2) p.$$

280. If the thickness of the material is e, then T/e and T'/e are the *average* circumferential and longitudinal tensions, per unit of area, in the cylinder; denoting them by t and t',

$$t = 2t' = pr/e.$$

When the thickness e is small compared with r, these average values will differ only slightly from their maximum or minimum values; t may be supposed limited by the working tension or tenacity of the material, as determined in the testing machine; and the requisite thickness e is given by

$$e = rp/t.$$

Thus a locomotive boiler 4 ft in diameter, of steel of working tenacity 6 tons/in², should be 0·27 inch thick to carry a pressure of 150 lb/in²; and water mains 6 ft in diameter, of cast iron of tenacity 1 ton/in², to carry water under a head of 200 ft, should be 1·4 inches thick.

281. If l denotes the length of a cylindrical vessel of radius r and thickness e, required to contain a volume v of gas at a pressure p, the volume of metal of tenacity t required in the cylindrical part is

$$2\pi rle = 2\pi r^2 lp/t = 2vp/t,$$

which is independent of the proportions of the cylinder.

Thus in the Herresschoff boiler, composed of a long spiral copper tube, or in vessels required to carry gas like oxygen or hydrogen at a great pressure, the proportions may be varied without altering the weight.

Vessels for holding compressed air for pneumatic guns are now made spherical; and if a volume v at pressure p is to be carried in n spherical vessels of radius r and thickness e,

$$e = \tfrac{1}{2}pr/t,$$

for tenacity t; and the volume of metal

$$4\pi nr^2 e = 2\pi nr^3 p/t = \tfrac{3}{2}vp/t,$$

which again is independent of the radius of the vessels, and shows a saving of 25 per cent. of material over the cylindrical shape.

282. For an external collapsing pressure p, such as is experienced by the tubes and flues of a boiler, we take d or $2r$ to denote the external diameter; and the tensions t and t' become changed into equal pressures.

Thus in the Severn tunnel, a brick cylinder 30 ft external diameter and 2 ft thick, the average crushing pressure in the brickwork due to a head of 100 ft of water outside would be 325 lb/in²; this excessive pressure would crush the mortar and cause the bricks to fly, and requires to be kept down by incessant pumping of the water in the neighbouring ground.

The longitudinal thrust and average pressure in a curved dam can be calculated in the same way; for example, in a concrete dam in Australia, with a radius of 1400 ft, 110 ft high and wide at the bottom, and 14 ft wide at the top.

It is asserted that the Exeter canal, one of the earliest in this country, was purposely made winding, from the supposed extra stability of the curved banks.

So also with spherical surfaces; thus the Magdeburg hemispheres, a Magdeburg ell or 2 ft in diameter, would require a force of about 3400 lb to pull them apart, when half exhausted of air; and the thrust at the joint would be about 45 lb/in².

283. *The Stress Ellipse.*

Sometimes a tube is made with a winding spiral seam; and to determine the stress across this seam, or generally across any oblique section, we must investigate the distribution of stress in the material, due to given circumferential and longitudinal tensions (or pressures), t and t': and this introduces the theory of the *stress ellipse.*

Taking OA in the direction of the circumference and OB of the axis at any point O of the surface (fig. 86), the stress, represented by the vector OP, across a short length $\alpha O\beta$ of an inclined section will be balanced by the stress t across $\beta\gamma$, represented by OA, and by the stress t' across $\alpha\gamma$, represented by OB suppose, where $\alpha\gamma$ and $\beta\gamma$ are drawn parallel to OA and OB.

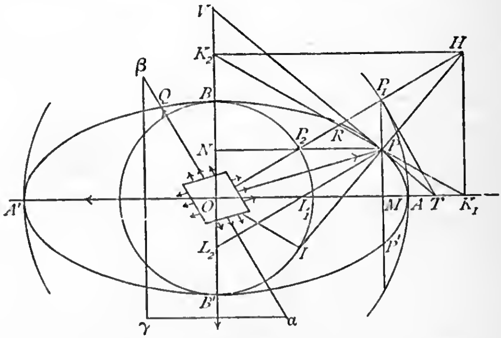


Fig. 86.

Therefore, if OM and ON are the projections of OP on OA and OB,

$$OM \,.\, \alpha\beta = OA \,.\, \beta\gamma, \quad ON \,.\, \alpha\beta = OB \,.\, \alpha\gamma;$$

or, denoting the angle between $\alpha\beta$ and OB by θ, and OA, OB by a, b,

$$OM = a\cos\theta, \qquad ON = b\sin\theta;$$

and therefore P describes an ellipse, with a and b as semi-axes, and θ is the excentric angle of P; this is called the *stress ellipse*.

In fig. 86, $OP_1 = a$, $OP_2 = b$, $OR = \frac{1}{2}(a+b)$,
$RP = \frac{1}{2}(a-b)$, $PRP_1 = 2\theta$;

TPV is the tangent and HPI the normal at P; also

$$K_2P=a,\quad PK_1=b,\qquad PL_2=a,$$
$$PL_1=b,\quad OH=a+b,\quad OI=a-b, \text{ etc.};$$

thus various mechanical descriptions of the ellipse are inferred (Prof. T. Alexander, *Trans. R. Irish Academy*, 29).

284. *Conjugate Stresses.*

It is easy to see that if OP is the stress across the plane OQ, then OQ is the stress across OP, by considering the equilibrium of the parallelogram $pqq'p'$, of which OP, OQ are the median lines; for the resultant forces across pq and $p'q'$ balance; and therefore also the resultant forces across qq' and pp', which cannot be the case unless they pass through the centre of the parallelogram, and are therefore parallel to pq.

The stresses OP and OQ are called *conjugate stresses;* but OP and OQ are not conjugate diameters; and denoting the angle BOQ by ϕ, then

$$OP\sin\phi=a\cos\theta,\quad OP\cos\phi=b\sin\theta;$$

and, by symmetry,

$$OQ\sin\theta=a\cos\phi,\quad OQ\cos\theta=b\sin\phi;$$

so that $\quad OP\,.\,OQ=ab,\qquad \tan\theta\tan\phi=a/b.$

285. The stress OP can be resolved into the shearing component OF, tangential to the plane pq, and the normal component, or tension FP (§ 6); and

$$OF=(a-b)\sin\theta\cos\theta,$$
$$FP=a\cos^2\theta+b\sin^2\theta.$$

Therefore OF attains its maximum value $\frac{1}{2}(a-b)$ when ORP is a right angle or $\theta=\frac{1}{4}\pi$; and OF vanishes only when $\theta=0$ or $\frac{1}{2}\pi$, unless $a=b$, when the stress ellipse becomes a circle, and the stress is *hydrostatic*, as in § 11.

The angle POR is called the *obliquity of the stress*; it attains its maximum value, ϵ suppose, when OPR is a right angle, and then

$$\sin\epsilon = \frac{a-b}{a+b}, \text{ or } \frac{b}{a} = \frac{1-\sin\epsilon}{1+\sin\epsilon} = \tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\epsilon),$$

and now the angles AOR, BOP or θ, ϕ being equal, each to $\frac{1}{4}\pi + \frac{1}{2}\epsilon$, the axes of the stress ellipse bisect the angles between OP and OQ; and $OP = \surd(ab)$.

286. In the theory of Earth Pressure (§ 30) ϵ denotes the angle of repose; and thence the ellipse of stress can be constructed at any point when the substance is on the point of moving.

Thus if, in fig. 21, p. 47, two consecutive planes $pq, p'q'$ are drawn parallel to the talus FD, and two consecutive vertical planes pp', qq', cutting out an elementary prism of the earth, the stresses on the pairs of opposite faces are equal conjugate stresses, of magnitude $wz\cos\epsilon$, if z denotes the vertical depth of the element below the surface; and w the *heaviness* or *density* of the earth; and the axes of the stress ellipse bisect the angles between the vertical and the slope ϵ, its semiaxes being given by

$$a = wz(1+\sin\epsilon), \quad b = wz(1-\sin\epsilon).$$

In foundations in level ground, the ultimate stress the earth can bear at a depth z under the adjacent ground is given by a stress ellipse whose vertical and horizontal semi-axes are

$$wz \text{ and } wz\tan^2(\tfrac{1}{4}\pi + \tfrac{1}{2}\epsilon),$$

the major axis being horizontal; but under the foundations the stress ellipse changes to one with horizontal and vertical semi-axes

$$wz\tan^2(\tfrac{1}{4}\pi + \tfrac{1}{2}\epsilon) \text{ and } wz\tan^4(\tfrac{1}{4}\pi + \tfrac{1}{2}\epsilon);$$

so that the depth z of the foundations required to support a distributed pressure p is given by

$$z = \frac{p}{w}\tan^4(\tfrac{1}{4}\pi - \tfrac{1}{2}\epsilon) = \frac{p}{w}\left(\frac{1-\sin\epsilon}{1+\sin\epsilon}\right)^2.$$

Thus the foundations of the Tower of Pisa, 22 ft deep in earth of density $w = 0{\cdot}828$ cwt/ft³, could carry with safety a pressure of 75 or 162 cwt/ft², according as ϵ is taken as 20° or 30°, the actual pressure being about 150 cwt/ft² (*Builder*, Jan. 1890.)

287. Putting $b=0$ gives the state of stress across oblique sections of a body transmitting a simple pull or thrust, such as a rope or pillar, a tie or strut.

Sometimes, as in the cross section of a tube, the principal stresses are of opposite sign; in this case the smaller stress b is taken as negative, and now OP' in fig. 86 will represent the stress across pq.

The normal component of the stress can now vanish, and the stress becomes tangential or shearing, for two conjugate positions of pq.

By twisting the tube the lines of principal stress become spirals, but the investigation of the stress ellipse remains the same; in this way the tension in the spiral strands of a rope can be investigated.

288. *The Stresses in a Thick Tube.*

When the thickness e of the tube is considerable, as in a gun, the *average* circumferential tension T/e or pr/e may fall considerably below the maximum tension, which must be kept below the working tenacity of the material; so that it is important to determine the radial pressure p and circumferential tension t at any radius r in a thick tube of internal and external radii r_0 and r_1 (ins), due to an internal pressure p_0 (tons/in²).

The curves P_0PP_1 and T_0TT_1 are drawn in fig. 87, representing to scale the radial pressure p by RP and the circumferential tension t by RT at any radius r or OR; and first supposing the external pressure p_1 insensible, the equilibrium of the quadrant AR of unit length of the tube requires the equality of the area of the rectangle OP_0, representing the thrust of the internal pressure in the direction OA, and of the area $T_0R_0R_1T_1$, the total pull across the section R_0R_1 of the tube.

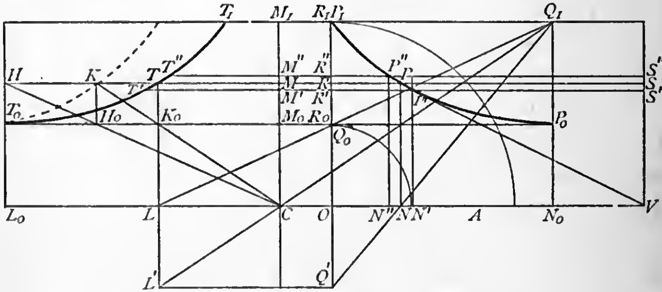


Fig. 87.

So also the equilibrium of a quadrant bounded externally by any intermediate radius r requires that the area T_0R_0RT should equal the difference of the rectangles OP_0 and OP, or the area T_0PT equal the rectangle NP_0.

Expressed in the notation of the Integral Calculus,

$$\int_{r_0}^{r} t\,dr = p_0r_0 - pr; \quad\text{.......................(3)}$$

and therefore, by differentiation,

$$t = -p - r\frac{dp}{dr}, \quad t + p = -r\frac{dp}{dr}, \quad \text{or} \quad TP = NV \text{ or } PS,$$

if PV the tangent at P cuts OA in V.

This can be proved by elementary geometry by drawing two consecutive lines $P'R'T'$, $P''R''T''$, equidistant from PRT, and considering the equilibrium of the tube bounded by the radii OR' and OR''; then

$$T'R'R''T'' = \text{rect. } OP' - \text{rect. } OP''$$
$$= \text{rect. } P'N'' - \text{rect. } P''R'$$
$$= \text{rect. } P'S'' - \text{rect. } P''R'$$

or $$T'p'P''T'' = \text{rect. } P'S'',$$

if the chord $P''P'$ cuts OA in V, and $VS'SS''$ is drawn parallel to OR, cutting $T'P'$, TP, $T''P''$ in S', S, S''.

Therefore, ultimately, $TP = PS$, and PV is the tangent at P.

Hence the curve of t can be drawn when the curve of p is arbitrarily assigned, and *vice versa;* thus, for instance, if t is assumed constant, as in the wire-gun, the curve of p is a hyperbola, with OA and T_0T as asymptotes.

289. Supposing the metal of the tube is homogeneous, two particular solutions can be obtained from elementary considerations, by which the general case can be built up.

First assume $p = t$ (Barlow's hypothesis); then PS or $NV = 2RP$, and, with $\gamma = 2$ in § 233,

$$pr^2 = a, \text{ a constant, } \quad \text{.......................(4)}$$

along the curve P_0PP_1, called in consequence a *Barlow curve;* the curve T_0TT_1 being an equal reflexion.

Now, by § 233, the area

$$T_0R_0R_1T_1 = t_0r_0 - t_1r_1,$$

and therefore the *average* tension is

$$\frac{t_0r_0 - t_1r_1}{r_1 - r_0} = a\frac{r_0^{-1} - r_1^{-1}}{r_1 - r_0} = \frac{a}{r_0r_1},$$

the G.M. of the tensions t_0 and t_1, and the actual tension at a radius the G.M. of r_0 and r_1.

Next assume a state of *hydrostatic* stress, in which

$$p = -t = b \text{ (Rankine's hypothesis)};$$

then $NV=0$, and the curves P_0P and T_0T coalesce in a straight line parallel to OR.

In the general case, by the superposition of Barlow's and Rankine's states of stress in varying proportions,

$$t = ar^{-2} + b, \qquad p = ar^{-2} - b,$$

given by Barlow curves of appropriate magnitude a, moved a distance b from right to left, represented in fig. 87 by OC; and the values of the arbitrary constants a and b can be determined from two conditions.

Thus with given internal and external pressures p_0, p_1,

$$p_0 = ar_0^{-2} - b, \qquad p_1 = ar_1^{-2} - b,$$

$$a = \frac{p_0 - p_1}{r_0^{-2} - r_1^{-2}}, \qquad b = \frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2}.$$

290. Representing in fig. 87 the given applied pressures by R_0P_0 and R_1P_1, and drawing the diagonal Q_1Q_0 to meet ON in L, then OL will represent the average tension, and this will be the actual tension RT at the radius OR, the G.M. of OR_0 and OR_1.

Produce TL to L', making $LL' = OR_0$; join Q_1L', cutting ON in C; then C will be the centre of the Barlow curves, which can now be readily constructed by a repetition of the geometrical method of § 64.

Thus to find the point T where the Barlow curve, with centre C, starting from T_0, cuts the line MT (fig. 87), produce L_0T_0 to meet MT in H; join CH, cutting M_0T_0 in H_0; draw H_0K parallel to CM, cutting MT in K; join CK, cutting M_0T_0 in K_0; then LK_0 parallel to CM will cut MT in T; for

$$\frac{MT}{MK} = \frac{CM_0}{CM}, \quad \frac{MK}{M_0T_0} = \frac{CM_0}{CM}, \quad \text{and} \quad \frac{MT}{M_0T_0} = \frac{CM_0^2}{CM^2}.$$

To determine geometrically the thickness of a tube of given bore $2r_0$, to carry a pressure p_0, with a working tenacity t_0, bisect L_0N_0 in C (fig. 87); C will be the centre of the Barlow curve which can now be constructed.

The external radius OR_1 will be determined by the point where the pressure curve cuts OR; take M_0Q the G.M. of M_0P_0 and M_1R_1, and draw CQ cutting N_0P_0 in Q_1; then P_0Q_1 or R_0R_1 will be the required thickness.

Now
$$\frac{r_1^2}{r_0^2}=\frac{t_0+p_0}{t_0-p_0};$$
and, as in § 281,
$$\frac{\text{the volume of metal}}{\text{the volume of gas}}=\frac{r_1^2-r_0^2}{r_0^2}=\frac{2p_0}{t_0-p_0}.$$

A similar procedure with spherical shells will give
$$p=ar^{-3}-b, \qquad t'=\tfrac{1}{2}ar^{-3}+b;$$
and show that to hold a given volume of gas at a given pressure in spherical shells the volume of metal required is again independent of the radius of the vessels (§ 281).

291. In a flue of external and internal radii ρ_0 and ρ_1, the circumferential pressure τ and radial pressure ϖ at a given radius ρ will be given by
$$\tau=\beta+a\rho^{-2}, \qquad \varpi=\beta-a\rho^{-2};$$
where β and a are determined by the external conditions, for instance, an applied pressure ϖ_0 at the outer surface.

By making $\rho_0=r_0$, $\varpi_0=p_0$, we thus determine the state of initial stress when a jacket is shrunk over a tube, so as to produce a given pressure p_0 or ϖ_0 at their surface of contact; in this way the tube is rendered capable of withstanding a greatly increased internal pressure.

So too hose or steam piping can be strengthened by wire wound spirally round the exterior.

292. *Capillarity.*

On looking at the edge of contact of a liquid with the containing vessel, of water in a bottle for instance, a slight violation is observable of the Theorem of § 20, "The free surface of a liquid at rest under gravity is a horizontal plane"; as the surface of the liquid is seen to be perceptibly curved in the immediate neighbourhood of the edge of contact of the liquid with the vessel.

If the vessel is moderately wide all trace of curvature of the surface disappears at a short distance from the edge, so that the surfaces of separation of superincumbent liquids are sensibly horizontal planes again (§ 21); insomuch that Lord Rayleigh finds that the most accurate method of forming a parallel plate of an optical medium for interference experiments is by means of a layer of water poured on the top of mercury in a wide vessel.

The curvature of the surface is well illustrated by filling a wineglass brimfull of water, and then carefully dropping coins into the glass; the surface will become more and more convex, until at last the water runs over the edge.

This experiment seems to be referred to in certain versions of the story of Charles II. (§§ 54, 204), in which it is asserted that a dead fish will spill no water, even if the bucket in which it is placed is brimfull.

293. The theory of the laws observable is called Capillarity or Capillary Attraction, from the Latin word *capillaris*, because it was first noticed that if a very fine tube, so fine as only to admit a hair, is dipped into liquid and partly withdrawn, the liquid is found to rise in the tube, so that there is a violation of the Theorem in § 24, that "the separate parts of the free surface of a

homogeneous liquid, filling a number of communicating vessels, all form part of one horizontal plane."

Following Young, the phenomena of Capillarity are now explained, without entering into the complicated molecular theories inaugurated by Laplace, by supposing that the surface of separation of two fluids forms a kind of skin or vesicle, having a certain tension or pull, T units of force per unit length, the same in all directions, depending on the nature of the two fluids.

The floating of a needle on water, and the progression of a water-beetle on the surface, can be accounted for by this skin, which is seen slightly indented.

So also the small ripples on water are explained by the capillary tension of the surface; and the walking of a fly on the ceiling is supposed to be due to the capillarity of the air.

Part of the energy of the fluids is thus due to their surface of separation; and the energy will be T units of work per unit area; because the work required to draw out by unit length an element of the surface of unit breadth is T.

As capillary tensions are small, the grain and the inch, or the gramme and the metre (or the milligramme and millimetre) are employed as units of force and length; and now T will denote the superficial tension in grains/inch or g/m, or the superficial energy in inch-grains/m^2, or gramme-metre/m^2.

Thus for water and air,

$$T = 3{\cdot}23 \text{ grains/inch} = 8{\cdot}24 \text{ g/m},$$

and for alcohol and air,

$$T = 1{\cdot}02 \text{ grains/inch} = 2{\cdot}60 \text{ g/m};$$

since $1 \text{ g} = 15{\cdot}432$ grains, $1 \text{ m} = 39{\cdot}37$ inches (§ 8).

294. A mass of liquid under no external disturbing forces will assume a spherical shape under the influence of the superficial tension; for instance, as in Plateau's experiments, when floating immersed in another liquid of equal density; or if falling freely.

Thus lead shot is made by allowing drops of melted lead to fall and solidify in the air; and the optical phenomena of the Rainbow prove that the falling raindrops are spherical.

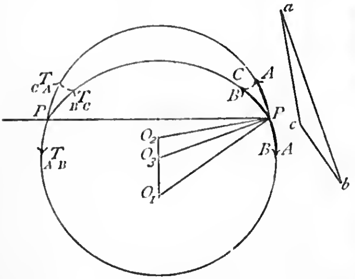


Fig. 88.

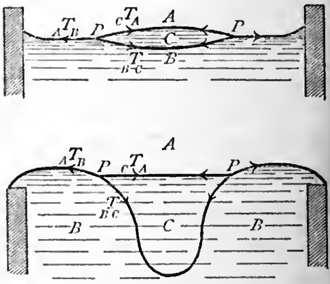


Fig. 89.

The spherical shape is approximately realised with small drops of mercury on a table, the deviation from sphericity being slighter as the drops are smaller; because the effect of gravity, which depends on the volume, becomes insensible compared with the capillary energy, which depends on the surface.

Sir W. Thomson in his lecture on Capillarity begins by supposing that gravity may be left out of account, as for instance in a laboratory at the centre of the Earth; and now the pressure p in a spherical mass of liquid of radius r, due to the tension T of the surface, will be given by (§ 278)

$$p = 2T/r.$$

So also for a spherical soap bubble of radius r; but now p denotes the excess of the pressure in the interior over atmospheric pressure (the gauge-pressure), and the tension T is that due to *both* sides of the liquid film.

295. If two liquid spheres B and C are now brought together, and the liquids do not mingle, the compound mass will come to rest in a configuration, consisting of two intersecting segments of spheres, constituting the outside surface, and a third spherical segment as the interface of the liquids, the body thus forming a sort of compound lens or meniscus (fig. 88).

The angles α, β, γ, at which the spherical surfaces meet, are the same as the angles which three balancing forces make with each other, when their magnitudes are the surface tensions of the interface and of the exterior surfaces of the liquids, denoted by ${}_BT_C$, ${}_CT_A$, ${}_AT_B$, A denoting the exterior medium, air suppose; and represented by the triangle of forces abc, the angles of which are the supplements of α, β, γ.

In the illustration given by Sir W. Thomson, B is formed of sulphate of zinc and C of carbon bisulphide, and the angles of contact are 165°, 25°, and 170°.

According to Lord Rayleigh, however (Maxwell, *Heat*, p. 287), this triangle of tensions can never be observed, as a thin film of the liquid of intermediate tension always spreads on the interface of the two other fluids.

Denoting by r_1, r_2, r_3 the radii of the spherical surfaces, and by r the radius of the circle in which they intersect; then for the equilibrium of the surface between B and C,

$$\frac{2\,{}_BT_C}{r_1} = \frac{2\,{}_AT_B}{r_3} - \frac{2\,{}_AT_C}{r_2}, \quad \text{or} \quad \frac{\sin\alpha}{r_1} + \frac{\sin\beta}{r_2} = \frac{\sin\gamma}{r_3}.$$

Also, as an exercise, the student may prove that

$$\frac{\sin^2\alpha}{r^2}=\frac{1}{r_2^{\,2}}+\frac{2\cos\alpha}{r_2r_3}+\frac{1}{r_3^{\,2}},$$

$$\frac{\sin^2\beta}{r^2}=\frac{1}{r_3^{\,2}}+\frac{2\cos\beta}{r_3r_1}+\frac{1}{r_1^{\,2}},$$

$$\frac{\sin^2\gamma}{r^2}=\frac{1}{r_1^{\,2}}-\frac{2\cos\gamma}{r_1r_2}+\frac{1}{r_2^{\,2}}.$$

For instance with liquid films, $\alpha=\beta=\gamma=\frac{2}{3}\pi$, and

$$\frac{3}{4r^2}=\frac{1}{r_2^{\,2}}-\frac{1}{r_2r_3}+\frac{1}{r_3^{\,2}}=\frac{1}{r_3^{\,2}}-\frac{1}{r_3r_1}+\frac{1}{r_1^{\,2}}=\frac{1}{r_1^{\,2}}+\frac{1}{r_1r_2}+\frac{1}{r_2^{\,2}}.$$

296. If a third sphere of another liquid was brought into contact, a compound body would be formed, bounded by portions of spherical surfaces; and the same conditions would have to be satisfied at the interfaces and edges of intersection, with the additional conditions of equilibrium of the points of intersection; and so on for an aggregation of any number of liquids.

The equilibrium of the edges and of their points of intersection can be studied in the arrangement of the liquid films in froth, especially when imprisoned in a glass vessel; the tension being the same everywhere, three films meet in an edge at angles of 120°; and four edges and six faces meet in a point, at equal angles.

According to Maxwell (*Mathematical Tripos*, 1869), the number of regions and edges is equal to the number of faces and points.

297. Suppose the regular arrangement of the spheres in § 33 was subjected to a uniform squeezing pressure; the spheres if plastic would be flattened into *rhombic dodecahedrons,* in which any two adjacent faces are in-

clined at 120°, as with the liquid films, and also in the honeycomb.

The corners of the rhombic dodecahedron are of two kinds, where (i) three, (ii) four edges and faces meet; and these corners can be constructed by planes through the edges of a tetrahedron or cube, meeting in the centre.

This can be realised practically with liquid films proceeding from the edges of a wire tetrahedron or cube, which has been dipped in soapy water; but whereas the arrangement is stable in the tetrahedron, a small cubelet, with curved edges and faces, is generally formed at the centre of the cube; and on examination the interior arrangement of the liquid film in froth will be found to be composed of these elementary arrangements.

If a right prism whose ends are equilateral triangles is dipped into the liquid, two corners of the first kind can be formed if the height is greater than $\frac{1}{6}\sqrt{6}$ times a side of the triangle.

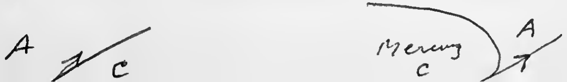
298. If the substance B is made solid, with a plane face, the liquid C will form a drop on it in the shape of a spherical segment or meniscus, meeting the plane at an angle a, called the *angle of contact*, given by

$$T_{AC}\cos a + T_{BC} = T_{AB}, \quad \text{or} \quad \cos a = (T_{AB} - T_{BC})/T_{AC},$$

the condition of equilibrium of the edge of contact; the normal component $T_{AC}\sin a$ being balanced by the reaction of the plane; thus with mercury on glass it is found that a is about 140°.

The air bubble in a spirit level is another illustration of a drop in contact with a solid.

But if T_{AB} exceeds the sum of T_{AC} and T_{BC}, the liquid C cannot stand as a drop, but will be drawn out into an attenuated film over the surface, like oil on water.



In a V-shaped groove in B, straight or circular, the liquid C will gather in a cylindrical or ring shape, having the same angle of contact.

299. Suppose a volume V of the liquid C is between two parallel planes B and B', a small distance d apart; it will meet these planes at the angle of contact α and form a film, of area A suppose.

Then, neglecting the curvature of the outline of the film, the pressure in the liquid C will be less than the pressure outside by $2T\cos\alpha/d$, T denoting the surface tension of C; and the planes will be pressed together in consequence by a thrust

$$2AT\cos\alpha/d = 2VT\cos\alpha/d^2,$$

which becomes considerable when d is small.

A parallel plane, dividing the liquid into two parts of volumes V_1 and V_2, will have a position of equilibrium, at distances x and y from the fixed planes, where

$$V_1/x^2 = V_2/y^2;$$

but this position of equilibrium will be unstable, and the plane will stick to one or other of the two fixed planes.

The Regelation of Ice may be explained in this manner; and also the sticking together of two accurate plane surfaces, in consequence of the capillarity of the air.

300. When a large number n of rain drops of radius r coalesce into a single drop of radius R, then the volume of water being unchanged,

$$\tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi r^3 n, \quad \text{or} \quad R = n^{\frac{1}{3}}r.$$

The diminution of surface is thus

$$4\pi(nr^2 - R^2) = 4\pi(n - n^{\frac{2}{3}})r^2 = 4\pi(n^{\frac{1}{3}} - 1)R^2;$$

so that surface energy has been liberated amounting to

$$4\pi(n - n^{\frac{2}{3}})r^2T = 4\pi(n^{\frac{1}{3}} - 1)R^2T;$$

and it is supposed that this is the source of the electric energy in a thunderstorm.

Thus if a thousand million rain drops coalesce to form a single drop 0·1 inch in diameter, $n=10^9$ and $4R^2=10^{-2}$; and, with $T=3·23$ grains/inch, this energy amounts to

$$\pi(10^3-1)10^{-2}\times 3·23 \text{ inch-grains, or } 0·00123 \text{ ft-lb.}$$

A cubic foot of water will make about 330 millions of such large drops, so that the corresponding energy would be about 400 thousand ft-lb.

301. When we return to the surface of the earth, and restore gravity, the shape of the liquids will be considerably altered, as shown for instance in fig. 89, p. 400; but the conditions of equilibrium of the edges will remain the same as before.

To determine the height h (ins) which a liquid of density w (grains/in³), and surface tension T (grains/inch), will ascend in a capillary tube of internal bore d (ins), when a is the angle of contact of the liquid with the solid of the tube; take h to denote the *mean* height of the column above the level of the liquid outside, so that $\frac{1}{4}\pi d^2 hw$ is the weight of the column in grains.

Then resolving vertically,

$$\pi dT\cos a=\tfrac{1}{4}\pi d^2hw,$$

or

$$h=\frac{4T}{w}\frac{\cos a}{d}=\frac{4c^2\cos a}{d},$$

on putting $T/w=c^2$; so that h is inversely as d.

302. If the tube is slightly conical, β denoting the semi-vertical angle, and the vertex is at a height a above the outside surface, then

$$2\pi(a-h)\tan\beta\,.\,T\cos(a-\beta)=w\pi h(a-h)^2\tan^2\beta,$$

$$ah-h^2=2c^2\cos(a-\beta)\cot\beta,$$

a quadratic for h, of which the smaller root gives the stable position of equilibrium; the larger root will give a position of equilibrium which will tend to fill the cone.

The *potential energy* of the liquid raised is

$$\tfrac{1}{3}w\pi a^3\tan^2\beta \,.\, \tfrac{1}{4}a - \tfrac{1}{3}w\pi(a-h)^3\tan^2\beta\{h+\tfrac{1}{4}(a-h)\}$$
$$=\tfrac{1}{12}w\pi h^2a^2\tan^2\beta\left(6-8\frac{h}{a}+3\frac{h^2}{a^2}\right);$$

reducing to a constant, $2w\pi c^4\cos^2 a$, for a cylindrical tube.

The height h to which the liquid ascends is independent of the shape of the vessel, except at the part near the upper surface.

In this way the rise of sap in trees may be explained.

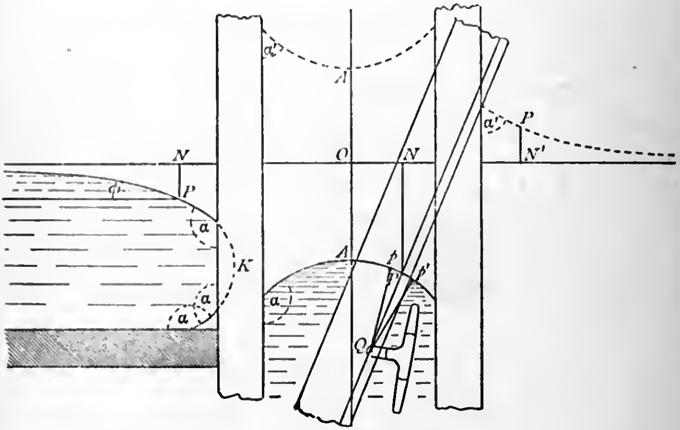


Fig. 90.

303. For the rise h between two parallel, vertical, plane plates, a distance d apart, we have

$$2T\cos a = wdh, \quad \text{or} \quad h = 2T\cos a/wd,$$

half the rise in a circular tube, as is easily observed.

If the plates are vertical, but not quite parallel, then d varies as x the distance from the line of intersection of the plates; so that the elevation y is inversely as x, showing that the upper surface of the liquid between the plates will be a curve in the form of a hyperbola, with vertical and horizontal asymptotes, one the line of intersection of the planes, and the other the line of intersection with the free liquid; this is easily verified experimentally.

304. *The Capillary Curve.*

The vertical cross section of the cylindrical surface formed by the free surface of a liquid in contact with a plane boundary, or of a broad drop on a horizontal plane, is called the *capillary curve.*

Supposing the angle of contact is obtuse, as with mercury, the free surface is depressed below the asymptotic horizontal plane with which it coalesces at a distance from the edge; and the depression x and the slope ϕ of the capillary curve are connected by the simple relation,

$$x = 2c \sin \tfrac{1}{2}\phi,$$

where $c = \sqrt{(T/w)}$, T denoting the surface tension, and w the density of the liquid.

For considering the equilibrium of the liquid in unit length of the capillary cylindrical surface cut off by a horizontal plane through a point P on the capillary curve, the horizontal hydrostatic thrust $\frac{1}{2}wy^2$ must be balanced by the tension T of the asymptotic horizontal surface, and by the horizontal component, $T \cos \phi$, of the tension at P; so that

$$\tfrac{1}{2}wy^2 = T(1 - \cos \phi) = 2T \sin^2 \tfrac{1}{2}\phi.$$

305. Thus if, in a large drop of mercury on a horizontal plate, the depth k below the flat top of the point K where the tangent plane is vertical is measured (fig. 90), then

$$T = \tfrac{1}{2}wk^2, \quad \text{or} \quad k = c\sqrt{2}.$$

The exact position of K is determined by Lippmann by observing the position of the reflexion at K of a spot of light approximately at the same level; he finds $T=48$ g/m for mercury and air.

If h denotes the height of the drop, and a the angle of contact,

$$h=2c\sin\tfrac{1}{2}a;$$

and in some barometers the level of the mercury in the cistern is kept constant by allowing the mercury to overflow as a large flat drop over a horizontal plate.

If the angle of contact is acute, as with water, the capillary surface is elevated above the asymptotic horizontal plane; so that the point K does not exist on a drop of water on a horizontal plate.

306. If a plane plate of glass is placed vertically in water or mercury, the liquid will be raised or depressed to equal distances on each side; but now if the plate is inclined at an angle β to the vertical, the slope of the capillary curves at their contact will be changed from

$$\tfrac{1}{2}\pi-a \text{ to } \tfrac{1}{2}\pi-a-\beta \text{ and } \tfrac{1}{2}\pi-a+\beta;$$

so that the difference of elevation of the edges of the liquid will be

$$2c\{\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}a+\tfrac{1}{2}\beta)-\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}a-\tfrac{1}{2}\beta)\}=4c\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}a)\sin\tfrac{1}{2}\beta;$$

or the distance between the edges, measured parallel to the plate, of thickness b suppose, will be

$$x=b\tan\beta+4c\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}a)\sin\tfrac{1}{2}\beta\sec\beta,$$

When $\beta=\tfrac{1}{2}\pi-a$, the free surface on one side of the plate is undisturbed from the horizontal plane, and this can be observed with precision; and now

$$x=(b+2c)\tan\beta.$$

307. Between two parallel vertical planes the liquid will be raised or depressed to a greater extent; the

vertical cross section is shown in the curve AP of fig. 90, and this curve is found to be the same as Bernoulli's *Lintearia* or *Elastica*, of which the capillary curve is a particular case.

Taking Oy in the undisturbed horizontal free surface of the liquid, the pressure at a depth x below it will exceed the atmospheric pressure by wx; so that, resolving horizontally and vertically, the equilibrium of unit length of the liquid, of cross section $OAPN$, gives, with $OA = a$,

$$\tfrac{1}{2}w(x^2 - a^2) = T(1 - \cos\phi),$$
$$w \, . \, OAPN = T \sin\phi.$$

These are also the conditions of equilibrium of a flexible watertight cylindrical surface (a tarpaulin) distended by water, under a head equal to the depth below the level of O; the tension T being constant for the same reason that the tension of a rope round a smooth surface is constant; the curve AP is called the *Lintearia* (the sail curve) in consequence.

If the water pressure acted on the upper side of AP, the tension T would become changed into a thrust or pressure; this arrangement would be unstable unless the flexibility of AP was destroyed; and now Rankine's *Hydrostatic Arch* is realised, in which the thrust is uniform, when the load is due to material of uniform density reaching to the level Ox.

308. The differentiation of these equations gives

$$wx\frac{dx}{ds} = T\sin\phi\frac{d\phi}{ds}, \quad \text{or} \quad x\rho = c^2.$$

This can be proved directly by considering the equilibrium of the elementary arc pp', whose middle point is P and centre of curvature Q; the hydrostatic thrust $wx.pp'$ on the chord pp' is balanced by $2T \sin PQp$, the component of the tensions along the normal, so that

$$wx = T\,\mathrm{lt}\,\frac{2\sin PQp}{pp'} = \frac{T}{\rho}.$$

The curvature $1/\rho$ is thus proportional to the distance x from Oy, so that the curve AP is also the *Elastica*, the curve assumed by a bow, of uniform *flexural rigidity* B, bent by a tension F in Oy; the bending moment Fx at P is then equal to the moment of resilience B/ρ, and

$$F/B = T/w = c^2.$$

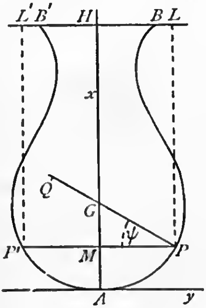


Fig. 91.

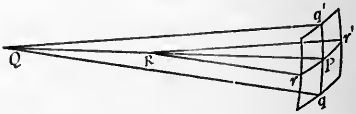


Fig. 92.

309. If fig. 90 represents the vertical section of a circular tube, and of the corresponding capillary surfaces of revolution, the curves are of a much more complicated analytical nature.

The general property of all such capillary surfaces, separating two fluids at different pressures, is expressed by the fact that "the difference of pressure on the two sides is equal to the product of the surface tension and of the total curvature of the surface," the *total curvature* being defined as the sum of the reciprocals of the two *principal radii of curvature* of the surface.

To prove this, take a small element of the surface at P cut out by lines equidistant from P and parallel to the *lines of curvature* qPq', rPr', of which Q, R are the centres of curvature (fig. 91).

Then if p denotes the difference of pressure on the two sides and T, T' the surface tensions in the directions Pq, Pr, resolving along the normal PQR,

$$p \,.\, 2Pq \,.\, 2Pr = 2T \sin PQq \,.\, 2Pr + 2T' \sin PRr \,.\, 2Pq,$$

or
$$p = T \operatorname{lt} \frac{\sin PQq}{Pq} + T' \operatorname{lt} \frac{\sin PRr}{Rr} = \frac{T}{PQ} + \frac{T'}{PR}.$$

This reduces to the above when $T = T'$; but a tangential stress U will exist in the lines of curvature, if they do not coincide with the axes of the stress ellipse at P.

310. Thus if $T = T'$, a constant, then p is proportional to the total curvature; so that a pressure p varying in this manner over an open sheet of a surface can be balanced by a uniform tension T round the edge at right angles to it; and a closed surface is in equilibrium.

Again, if p is proportional to $(PQ \,.\, PR)^{-1}$, the *Gaussian measure of curvature*, then

$$T \,.\, PR + T' . PQ = \text{constant};$$

and this can be satisfied by making T and T' inversely proportional to PR and PQ; in this case the variable pressure p acting over an open sheet will be balanced by the tensions T and T' acting on a serrated edge consisting of elements of the lines of curvature; and the closed surface is in equilibrium.

If $p = 0$, the total curvature is zero, the characteristic property of *minimum* surfaces; these are realised experimentally by liquid films, sticking to various boundaries, straight, circular, helical, or twisted.

311. In cylindrical surfaces one of the radii of curvature, PR, is infinite and its reciprocal zero, and we obtain the preceding relations for the Elastica (§ 308).

In a conical surface, PQ is infinite, and the value of T_2 in § 279 is obtained immediately.

In a surface of revolution, AP (fig. 91) about Ox as axis, the principal radii of curvature are PQ and PG, the radius of curvature and normal to Ox of the meridian curve AP; so that, if filled to the level LL',

$$w \,.\, LP = T\left(\frac{1}{PQ} + \frac{1}{PG}\right),$$

or

$$\frac{1}{PQ} = \frac{LP}{c^2} - \frac{1}{PG}.$$

The complete integration of this intrinsic relation is intractable; but the curve AP can be drawn, as Young pointed out in 1804, by means of successive small arcs, struck with Q as centre; this method was first put into operation by Prof. John Perry, acting under the instructions of Sir W. Thomson, in 1874; and now Mr. C. V. Boys has constructed a celluloid scale with reciprocal graduations (fig. 90), by means of which the curves can be drawn with ease and rapidity.

The scale carries a glass pen at P, and is pivoted instantaneously at Q by means of a brass tripod, provided with three needle points, two of which stick in the paper, and the third acts as the centre at Q.

312. Sir W. Thomson illustrates the form of a liquid drop, and generally of a flexible elastic surface, by means of a sheet of indiarubber fastened to a horizontal circular ring; water is poured into the sheet, by which it is distended and assumes a variety of forms of revolution about a vertical axis (fig. 91).

Denoting by T and T' the tension per unit length of the surface in the direction of the meridian AP and perpendicular to it, then at the section PP', of diameter $2y$, where the surface makes an angle ψ with the vertical,

$$2\pi y T \cos\psi = w \,.\, \text{volume } LPAP'L'$$

$$= w\int_0^x \pi y^2 dx + w\pi y^2(h-x),$$

if the vessel is filled to a height h above the lowest point A with liquid of density w; and the value of T' is then given by the characteristic equation (§ 309),

$$\frac{T}{PQ} + \frac{T'}{PG} = p = w(h-x).$$

As an exercise the student may prove that

$$\frac{dT}{ds} = \frac{T'-T}{y}\sin\psi;$$

so that T is a max. or min., either when $\psi=0$, that is at the widest or narrowest part of the surface; and then

$$\frac{d^2T}{ds^2} = \frac{T-T'}{y\rho} \text{ or } \frac{T'-T}{y\rho};$$

or else when $T=T'$.

If, however, T' should turn out negative, the surface would tend to pucker.

Thus, as exercises, the student may prove that if the surface is a paraboloid, generated by $y^2=4ax$, then

$$T' = w\sqrt{\left(\frac{a}{a+x}\right)}\{\tfrac{1}{8}\sqrt{(16h^2+8ah+9a^2)} - (x-\tfrac{1}{2}h+\tfrac{3}{8}a)^2\},$$

so that T' is negative and the surface will pucker where

$$x > \tfrac{1}{2}h - \tfrac{3}{8}a + \tfrac{1}{8}\sqrt{(16h^2+8ah+9a^2)};$$

while in a sphere, of radius a,

$$T' = \tfrac{2}{3}wa\frac{\{x-\tfrac{3}{4}(a+h)\}^2 - \tfrac{9}{16}(3a-h)(\tfrac{1}{3}a-h)}{2a-x},$$

which is always positive if $3a > h > \tfrac{1}{3}a$; but if $h < \tfrac{1}{3}a$, T' is negative where

$$x < \tfrac{3}{4}(a+h) - \tfrac{3}{4}\sqrt{\{(3a-h)(\tfrac{1}{3}a-h)\}}.$$

We may consider also the stresses in a Catenoid (§ 319) due to a constant pressure difference p on its two sides.

A closed surface cannot be deformed without stretching or contraction in the material; so that pouring in liquid cannot alter the shape if the material is inextensible.

But if fig. 93 represents the cross section of a cylinder with horizontal generating lines, the surface if flexible will assume the form of the Lintearia; but if it possesses *flexural rigidity*, the *bending moment* at any point P can be calculated for a given cross section.

Suppose, for example, that the cross section is circular, as in a boiler filled up to a depth h, or in the pontoon of fig. 45, p. 190, floating to a draft h; bending moments and shearing stresses are called into play, which can be calculated as an exercise.

313. *Modification of Archimedes' Principle by Capillarity.*

When the surface of the liquid is depressed by a floating body, as a needle on water or a platinum ball on mercury, the upward buoyancy, originally due to the displaced liquid when the surface is undisturbed from the level plane, is increased by the weight of the volume of liquid depressed below this plane, that is, the volume required to fill up the liquid to its original level.

This follows immediately if we suppose the body is provided with a weightless projection, fitting close against the capillary surface, which may now be supposed deprived of its capillary tension.

So also in the case of an ordinary body, floating in liquid which wets it, like water, the buoyancy is diminished by the weight of the volume of liquid raised above the level plane; that is, the volume which must be removed to reduce the surface to this plane.

When the body is floating in a vessel of limited extent, so that the capillarity of the edge is appreciable, these volumes must be supposed limited by a vertical cylinder whose generating lines pass through the curve on the capillary surface where the tangent plane is horizontal.

As an exercise, take the needle floating on water; then

$$(\pi s - \theta)\cos\tfrac{1}{2}(\theta+a)$$
$$=(\cos\theta - 2\cos\beta)\sin\tfrac{1}{2}(\theta-a)-\cos^2\beta\sin\tfrac{1}{2}(\theta+a).$$

where s denotes the S.G. of the needle, a the angle of contact (obtuse), and 2θ, 2β the angles subtended at the centre by the chords formed by the water lines, and by the undisturbed plane surface of the water.

Similarly the prism in fig. 44, p. 190, of length a and breadth b, will in consequence of capillarity have its apparent draft increased by

$$2c\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}a) + 2\cos a\left(\frac{1}{a}+\frac{1}{b}\right).$$

314. *Liquid Films.*

A soap bubble, blown out of a mixture of soap and water, or a mixture invented by Plateau, assumes a spherical shape; and now if T denotes the surface tension (reckoned for both sides of the film) and r the radius of the sphere, the pressure ϖ inside will exceed the atmospheric pressure p outside by $2T/r$ (§ 294), and this excess is greater the smaller the value of r; small bubbles thus tend to exhaust the air into larger ones.

The quantity of air in a bubble of radius r, surface s, and volume v is

$$\frac{4}{3}\pi r^3\frac{\varpi}{k} = \tfrac{4}{3}\pi r^3\left(\frac{p}{k}+\frac{2T}{kr}\right) = \frac{pv}{k} + \frac{2}{3}\frac{Ts}{k}.$$

Thus if a number of bubbles coalesce into a single bubble of radius R, surface S, and volume V,

$$p\Sigma v+\tfrac{2}{3}T\Sigma s=pV+\tfrac{2}{3}TS,$$

$$\frac{V-\Sigma v}{\Sigma s-S}=\frac{2}{3}\frac{T}{p},$$

or the increase of volume bears a constant ratio to the decrease of surface.

The air inside a bubble of radius r at volume v and pressure $p+2T/r$, would occupy a volume $v(1+2T/pr)$ at atmospheric pressure; for instance, when the bubble bursts.

Thus a volume $\Delta v=2Tv/pr$ of atmospheric air has been forced into the volume v; and the work required is (§ 233)

$$\frac{1}{2}p\frac{(\Delta v)^2}{v}=\frac{2T^2v}{pr^2}=\frac{2}{3}\frac{T^2s}{pr};$$

while the work required to form the surface is Ts; the ratio is thus $\tfrac{2}{3}T/pr$.

If the barometric height h falls a small distance Δh, the radius of the bubble will increase Δr, given by

$$p=\varpi-\frac{2T}{r},\ p\left(1-\frac{\Delta h}{h}\right)\doteq\varpi\left(1+\frac{\Delta r}{r}\right)^{-3}-\frac{2T}{r}\left(1+\frac{\Delta r}{r}\right)^{-1}$$

or
$$p\frac{\Delta h}{h}=\left(3\varpi-\frac{2T}{r}\right)\frac{\Delta r}{r}=(2\varpi+p)\frac{\Delta r}{r}.$$

315. When a bubble is electrified, the radius is increased in consequence of the electrical repulsion, $2\pi\sigma^2$ per unit area, if σ denotes the surface electrification (Maxwell, *Electricity and Magnetism*, I., Chap. VIII.)

If the radius increases from r to a when the bubble is electrified to potential A, then the charge $E=Aa$, and

$$\sigma=\frac{E}{4\pi a^2}=\frac{A}{4\pi a}.$$

If p' now denotes the pressure of the air,

$$p+\frac{2T}{a}=p'+2\pi\sigma^2, \text{ and } p'=\left(p+\frac{2T}{r}\right)\frac{r^3}{a^3},$$

by Boyle's law, and therefore

$$\frac{A^2 a}{8\pi}=p(a^3-r^3)+2T(a^2-r^2).$$

If $p=p'$, then $A^2=16\pi aT$, and the electric energy is

$$\tfrac{1}{2}EA=\tfrac{1}{2}A^2 a=8\pi a^2 T=2ST,$$

or double the surface energy of the bubble.

316. If a circle of wire is dipped into a basin of soapy water, and raised gently in a horizontal position, a surface of revolution is formed by the film sticking to the wire, orthogonal to the surface of the water, and the meridian curve is a *catenary*.

For neglecting the weight of the film, the condition of equilibrium of the zone cut off by an upper horizontal circle PP', of radius y, and by the circle CC' on the water, of radius c suppose, is

$$2\pi yT\cos\psi=2\pi cT, \text{ or } y\cos\psi=c,$$

where ψ denotes the angle MPG between the ordinate PM and the normal PG.

Dropping the perpendicular MH on the normal, then $PH=c$, a constant, a property of the catenary.

317. The film always breaks when the height of the wire above the water exceeds a certain amount, about one-third the diameter of the wire circle; this may be accounted for as follows—

While the wire is at a moderate distance from the water, two *catenoids* can theoretically be drawn, satisfying the conditions, and corresponding to the two loops or festoons in which an endless chain will hang over two smooth pullies at the same level, not too far apart.

These catenoids are similar surfaces, and their common tangent cone will have its vertex on the surface of the water; and thus the tangent cones along the junction BB' with the wire will have their vertices, one below and the other above the water, the second surface being unstable and therefore non-existent.

At a certain distance of the wire from the water, these two catenoids coalesce, just as the two festoons of the endless chain coalesce when the pullies exceed a certain distance apart; and the liquid film always breaks at this distance, that is, when the vertex of the tangent cone round the edge of the wire reaches the surface. In fact, the plane film formed on the wire circle is now found to have a smaller area than the corresponding catenoid.

318. If air is blown into the space bounded by the catenoid and the plane film, sticking to the wire circle, or if air is removed; the pressure in this space exceeds or falls below the atmospheric pressure by a certain amount, which we can denote by T/a; so that $2a$ is a length, the radius of the sphere into which the plane film is bulged; and the meridian curve CPB of the surface will change, still however cutting the surface of the water at right angles at C (figs. 93, 94).

Considering the equilibrium of the zone of the film, bounded by the horizontal circles PP' and CC',

$$2\pi yT\cos\psi - 2\pi cT = \pi(y^2-c^2)\frac{T}{a} \text{ or } \pi(c^2-y^2)\frac{T}{a},$$

or
$$2a(y\cos\psi - c) = y^2-c^2, \text{ or } c^2-y^2,$$

according as the pressure inside is increased or diminished; therefore, writing n for the length of the normal PG,

$$\frac{2ac-c^2}{y^2} = \frac{2a}{n}-1, \text{ or } \frac{2ac+c^2}{y^2} = \frac{2a}{n}+1.$$

319. The meridian curve CPB defined by this relation is of transcendental nature, and the surface is that which contains a given volume with minimum area; but the above relation proves that the curve CPB can be generated by the focus P of an ellipse or hyperbola AA', rolling on the axis OG.

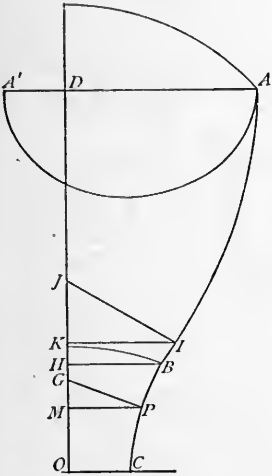


Fig. 93.

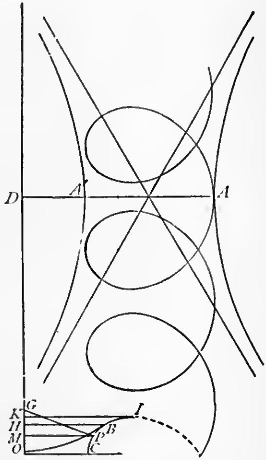


Fig. 94.

For writing p and r for the lengths PM and PG, then a well-known relation gives, with the usual notation,

$$\frac{b^2}{p^2} = \frac{2a}{r} \pm 1$$

in the ellipse or hyperbola; and here

$$b^2 = 2ac\beta^2.$$

When a is infinite, then, as at first,

$$y = c\sec\psi, \quad \text{or} \quad y^2 = cn$$

in the curve CP; and therefore $p^2 = cr$ in the rolling curve AG, which is therefore a parabola; and therefore the roulette of the focus of a parabola is a catenary, as is well known.

The surface formed by the revolution of the roulette of the focus of an ellipse is called the *unduloid,* and by the roulette of the focus of a hyperbola the *nodoid,* the surface formed by the catenary has already been called the *catenoid.*

The unduloid becomes a cylinder when the rolling ellipse is a circle; and it becomes a sphere, or a series of contiguous spheres, when the ellipse degenerates into a finite straight line.

The portion of the nodoid formed by the revolution of a loop must be taken as having the internal pressure T/a over atmospheric pressure; this can be produced by blowing a bubble between two plates of glass, propped open at a small angle by a piece of wood at the node.

If the portion of the catenoid bounded by the planes BB' and CC' could be solidified without losing its flexibility, then on cutting it open along a meridian CPB, and pulling CC' out straight, the surface will take the form of the *helicoid,* the surface assumed by a film sticking to a uniform helix and its axis.

If any section PP' of the Catenoid is replaced by a thread or wire, the tension due to the film above, or the thrust due to the film below will be $yT\sin\psi$ or sT, if s denotes the arc CP; and the stress in the corresponding spiral on the Helicoid will be found to be

$$T\left(s+\frac{c^2}{s}\right),$$

where s is the radius of the cylinder on which the spiral is wound (C. V. Boys, *Soap Bubbles.*)

320. In these investigations the weight of the liquid film has been taken as insensible; otherwise the weight would influence the result, as in tents and marquees.

But the thickness of the liquid films, as determined by Reinold and Rucker from optical measurements, may be as small as $12\mu\mu$; and half a litre of oil poured on the surface of the sea has been found to cover 10^8 cm^2, or 100 m square, without losing its continuity, implying a thickness of $50\mu\mu$; but this thin film is still effective for checking the ripples and small waves and calming the surface of the sea in a storm.

Many interesting maximum and minimum problems can be solved in a simple manner from the mechanical considerations involved in the theory of flexible surfaces under tension; as for instance:—

The circle has the greatest area for given perimeter; illustrated by an endless thread in a plane film, when the film in its interior is broken.

The sphere has the greatest volume for given surface; illustrated by the soap bubble.

Two segments on given bases and of given perimeter will enclose a maximum area when they are arcs of equal circles, realised by passing the endless thread through rings at the ends of the bases; and so also for spherical segments on given circles, realised by blowing bubbles on the ends of a frustum of a cone; etc.

Examples.

(1) Prove that the height of a flat drop of mercury is a mean proportional between the diameter of a capillary tube, and the depth to which mercury is depressed in it, supposing the angle of contact 180°.

(2) Investigate the coefficient of expansion of the radius of a soap bubble, supposing that the surface tension diminishes uniformly with the temperature.

(3) A soap bubble of radius a is blown inside another of radius b; and the radii change to a' and b' when the atmospheric pressure changes from p to p'.

Prove that

$$\frac{p}{p'}=\frac{b'}{b}\frac{b(a^2-a'^2)(b'^3-a'^3)+a^3b'^3-a'^3b^3}{b'(a^2-a'^2)(b^3-a^3)+a^3b'^3-a'^3b^3}.$$

(4) Prove that a flexible surface, of superficial density w lb/ft², hanging as a horizontal cylinder the vertical cross section of which is a catenary, is changed by a pressure difference p lb/ft² on its sides into a cylinder in which the tension across a generating line is still wy, where y is the height above a fixed horizontal plane, and in which the radius of curvature is changed to

$$\frac{w}{p+w}\frac{y^2}{a};$$

if $y=a$ where the slope $\psi=0$; also that

$$(p+w\cos\psi)y=(p+w)a.$$

Determine the equation of this curve.

(5) Prove that if r, r' denote the radii of curvature of a pair of perpendicular normal sections of the surface in § 309, making an angle ϕ with the lines of curvature; and if t, t', and u denote the corresponding normal tensions and tangential stress, due to a pressure difference p; then

$$p=\frac{t}{r}+\frac{t'}{r'}-\left(\frac{1}{r}-\frac{1}{r'}\right)u\tan 2\phi=\frac{t}{r}+\frac{t'}{r'}-\left(\frac{1}{R}-\frac{1}{R'}\right)u\sin 2\phi,$$

where R, R' denote the principal radii of curvature.

CHAPTER X.

PRESSURE OF LIQUID IN MOVING VESSELS.

321. When a vessel containing liquid is moving steadily, as for instance a locomotive engine, with given acceleration a (ft/sec^2), an attached plumb line is deviated from the vertical; and the surfaces of equal pressure and the free surface will be perpendicular to this plumb line, when the liquid is moving bodily with the vessel.

If the liquid fills the vessel completely so that there is no free surface, the liquid will move bodily with the vessel, provided the vessel has no rotation.

If however a vacant space is left, which may be supposed filled with some other liquid of a different density, oscillations will be set up in the free surface or surface of separation; but these oscillations die out rapidly in consequence of viscosity (§ 4), until the liquid and vessel move together bodily.

322. No oscillations however need be set up in the free surface by a vertical motion of the vessel (although Lord Rayleigh asserts that the horizontal free surface may become unstable), nor will the plumb line be deviated; this we may suppose realised in Atwood's machine, or else, initially, in the scales of a common balance, when equilibrium is destroyed.

Suppose then that a bucket A and a counterpoise B, or else two buckets A and B, are suspended by a rope over a pulley, and that equilibrium is destroyed and motion takes place, in consequence of the inequality of the weights of A and B.

Denoting these weights in lb by W and W', by T pounds the tension of the rope, by a the vertical acceleration of A and B, and by g the acceleration of gravity, in ft/sec²; then by the principles of Elementary Dynamics and by Newton's Second Law of Motion,

$$\frac{a}{g}=\frac{W-T}{W}=\frac{T-W'}{W'}=\frac{W-W'}{W+W'};$$

and therefore
$$T=\frac{2WW'}{W+W'},$$

the H.M. of W and W'.

We suppose the preponderating bucket A to be reduced to rest by applying to it an upward acceleration a; so that now the pressure at any depth z in the water in the bucket becomes changed from

$$Dz \text{ to } Dz\left(1-\frac{a}{g}\right).$$

If B was also a bucket of water, the pressure at a depth z in it would be changed from

$$Dz \text{ to } Dz\left(1+\frac{a}{g}\right).$$

If the buckets are cylindrical and of weight negligible compared with the water they contain, then the hydrostatic thrust on the bottom of the buckets is

$$W\left(1-\frac{a}{g}\right) \text{ or } W'\left(1+\frac{a}{g}\right),$$

each equal to T, as is otherwise evident.

Barometers attached to A and B, standing at a height h when at rest, would now have heights

$$\frac{h}{1-a/g} \text{ and } \frac{h}{1+a/g}.$$

So also with a bucket attached to a spring, performing vertical, simple harmonic oscillations, or placed on a vessel performing dipping oscillations; or with the water on the top of the piston of a vertical engine; a horizontal plane of cleavage may make its appearance when the amplitude and speed of the oscillations is sufficiently increased.

323. Suppose now that in each bucket a part of the weight, W or W', consists of a piece of cork of S.G. s.

If the corks are floating freely no change will take place in consequence of the motion.

But if completely submerged by a thread attached to the bottom of the bucket, then denoting the tensions of the thread in A by P lb, and the weight of the cork by M lb, the buoyancy of the cork at rest will be M/s lb; and therefore in motion will be

$$\frac{M}{s}\left(1-\frac{a}{g}\right) \text{ lb.}$$

Therefore

$$P=M\left(1-\frac{a}{g}\right)\left(\frac{1}{s}-1\right)=\frac{2MW'}{W+W'}\left(\frac{1}{s}-1\right)=\frac{M}{W}\left(\frac{1}{s}-1\right)T;$$

and if $s>1$, P becomes negative and the body must be supposed suspended by a thread from the top of the bucket.

For the tension of the thread in B the sign of a must be reversed.

Suppose $W=W'$, so that the buckets balance; then if the thread holding down the cork M in A is cut, the equilibrium is destroyed; and the student may prove as an exercise that the bucket A will descend, and the cork will rise through the water, with accelerations respectively

$$g\left(\frac{1}{s}-1\right)\frac{M}{2W+M\left(\frac{1}{s}-1\right)} \quad \text{and} \quad g\left(\frac{1}{s}-1\right)\frac{2W}{2W+M\left(\frac{1}{s}-1\right)}.$$

This treatment, as in § 148, ignores the motion of the water due to the passage of the cork; but, as in § 149, the result can be corrected for a small spherical or cylindrical cork in a large vessel.

324. If the bucket A strikes the ground with velocity v and is suddenly reduced to rest, an impulsive pressure is set up for an instant in the water.

Suppose, however, that the impact takes an appreciable time, t seconds.

To stop the body A weighing W lb, moving with velocity v ft/sec, in a short time t seconds, requires an average resistance R of the ground, given in pounds by

$$R=\frac{Wv}{gt}.$$

The product Rt of the force of R pounds and of the t seconds for which it acts is called the *impulse*, in *second-pounds;* and its mechanical equivalent Wv/g is called the momentum, also in sec-lb, of W lb moving with velocity v ft/sec.

To stop the water in the bucket reaching to a depth z ft requires therefore a force $Dzav/gt$ pounds, a ft^2 denoting the cross section of the bucket, or a pressure Dzv/gt lb/ft^2, compared with which the pressure due to gravity is insensible when t becomes small, so that the bucket runs the risk of bursting when it strikes the ground, a tension $Drzv/gt$ pounds per unit depth being set up in the circumferential hoops, if of radius r.

As for the bucket B, the rope becoming slack, it moves freely under gravity, and the pressure of the water in it is reduced throughout to zero, or atmospheric pressure.

325. When a closed vessel, completely filled with water, and moving bodily in a given direction with velocity v, is suddenly stopped, the water comes to rest simultaneously; so that the impulse on any portion of the water is equal and opposite to the momentum of this liquid.

Suppose this portion of water is removed and replaced by an equal solid, of S.G. s and weight M lb; the momentum of this solid, moving bodily with the liquid with momentum Mv/g sec-lb, is changed by the impulse Mv/gs of the surrounding liquid into

$$\frac{Mv}{g}\left(\frac{1}{s}-1\right) \text{ sec-lb}$$

in the direction opposite to the original motion; so that if the body is lighter than the surrounding water, the body will recoil with this momentum.

If the body is like the cork attached by a thread to the bottom of the bucket A, the impulsive tension of the thread will be

$$\frac{Mv}{g}\left(\frac{1}{s}-1\right) \text{ sec-lb.}$$

The *impulsive pressure* at depth z in the water on impact of the bucket is therefore (in sec-lb/ft^2)

$$Dzv/g;$$

or, if the velocity is suddenly reduced from v to v', the impulsive pressure is

$$Dz(v-v')/g.$$

326. So also in shutting off quickly, in t seconds, a stop valve in a water pipe or main, l ft long, filled with water flowing with velocity v ft/sec, the pressure in the neighbourhood of the valve is increased by

$$Dlv/gt \text{ lb/ft}^2;$$

this may become excessive if t is made too small.

The increase of pressure due to a sudden check of motion in a pipe is called *water ram;* so also in the impact of sea waves, the spray is sent to a great height.

To diminish the shock of water ram, an air vessel must be provided, as in fig. 78, p. 360; and if p denotes the average pressure of the cushion of air, the water is stopped in

$$t = Dlv/gp \text{ seconds,}$$

during which the average velocity is $\frac{1}{2}v$, so that

$$q = A \,.\, \tfrac{1}{2}vt = \tfrac{1}{2}ADlv^2/gp \text{ ft}^3$$

of water enters the air vessel, if the cross section of the main is A ft^2; and this water can be delivered at a pressure p, or to a head $h = p/D$ ft, if its return into the main is prevented by a valve.

This is the principle of Montgolfier's *hydraulic ram,* in which the main is laid at a slope, with a fall of H feet suppose, and a valve at the lower end opens and shuts automatically, to start and check the flow.

If the valve is open T seconds for the column of water in the main to acquire the velocity v, then

$$v = gTH/l;$$

and the water which flows out of the valve is, in ft^3,

$$Q = A \,.\, \tfrac{1}{2}vT = \tfrac{1}{2}Alv^2/gH.$$

Since q ft^3 is thereby lifted to an effective height $h - H$ ft, the efficiency is

$$\frac{q(h-H)}{QH} = 1 - \frac{H}{h}.$$

327. Consider now the pressure in liquid which is moving bodily in a vessel with given acceleration a in a fixed direction, like the water in the boiler or tender of a locomotive engine.

It is convenient to reduce Dynamical problems to a question of Statics by the application of D'Alembert's principle, which asserts that "the reversed effective forces and impressed forces of a system are in equilibrium," the *effective force* of any particle or body being defined as the force required to give it the acceleration which it actually takes.

Thus if the weight of a particle is m lb, and if it has an acceleration a ft/sec² in a given direction, its effective force is ma/g pounds (ma poundals) in that direction; in C.G.S. units the effective force of m g moving with acceleration a spouds (cm/sec²) is ma dynes.

If the particle is carried along steadily by the vessel as a plumb bob at the end of a short thread, this thread will therefore assume the direction of the resultant of g and of a reversed.

The surfaces of equal pressure in the liquid will be parallel planes perpendicular to this direction of the plumb line; and therefore the free surface, if it exists, will also have this direction, when the liquid moves bodily with the vessel.

Any floating body will occupy the same position as before, relative to this new free surface, all the forces being changed in the same ratio.

328. The level of the surface of the water in the boiler is marked in the gauge glass (§ 24); and now if the engine, originally on the level, is standing or running steadily with uniform velocity on an incline a, the mean

water level will still be a horizontal plane FL_1, making an angle α with its original plane FL, the nodal line of the two surfaces passing through F, the C.G. of the water line area (§ 108); so that if c is the distance of the glass tube from this nodal line, the change of level in the gauge glass is $c \tan \alpha$ (fig. 95).

But if the engine and train is moving freely with acceleration $g \sin \alpha$ down the incline, the direction of the attached plumb line and of the normal to the surfaces of equal pressure will be that of the resultant of g and $g \sin \alpha$ reversed, and will therefore be perpendicular to the rails; so that the water in the gauge glass will return to its normal position.

329. When steam is turned on, the engine will receive an additional acceleration a, which we may suppose constant and represented by fa, so that the reversed acceleration up the plane is

$$a + g \sin \alpha;$$

and the plumb line will now make an angle θ with the perpendicular to the rails, given by

$$\tan \theta = a/g \cos \alpha.$$

The water level FL_2 will be perpendicular to the plumb line, so that the height of the water in the glass will change by $$c \tan \theta = ca/g \cos \alpha,$$
and will be the same whether the engine is going up or down the incline.

Conversely when steam is shut off and the brakes fully applied, the retardation produced with a coefficient of friction μ will be $\mu g \cos \alpha$; and now the plumb line and surfaces of equal pressure will be deviated with respect to the gauge glass through the angle of friction

$$\phi = \tan^{-1} \mu;$$

and the water in the gauge will fall $c \tan \phi$ or $c\mu$ below its normal level.

The tension of the plumb line, and therefore also the pressure in the water at a given depth below the free surface, will be altered in the ratio of Fa to Fg (fig. 95) or of $\sec \theta$ to $\sec \alpha$; and therefore the pressure will be $\sec \theta \cos \alpha$ times its value when the system is at rest or in uniform motion, at full speed.

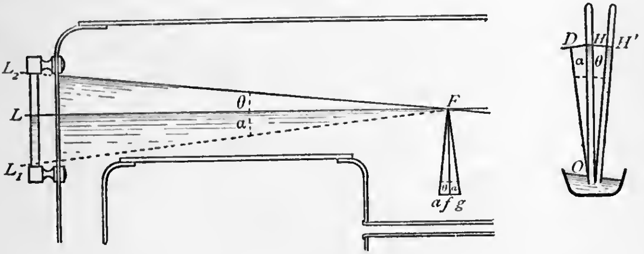


Fig. 95.

330. If a barometer is fixed parallel to the gauge glass, the length of the column of mercury will, when it registers atmospheric pressure, change from h to x, given by

$$\sigma h = \sigma x \cos \theta \,.\, \sec \theta \cos \alpha = \sigma x \cos \alpha\,;$$

so that

$$x = h \sec \alpha,$$

which is independent of the acceleration.

But if the barometer can swing freely in gimbals, like a marine barometer, then it hangs parallel to the plumb line, and the column will have a length y given by

$$\sigma h = \sigma y (Fa/Fg) = \sigma y \sec \theta \cos \alpha,$$

or

$$y = h \cos \theta \sec \alpha.$$

331. As another illustration, consider the problem in the waiter's mind, to carry as quickly as possible tumblers which are nearly full on a tray, which he holds level.

Then if the tumblers will spill when the level of the water makes an angle α with the horizon, his acceleration or retardation a is restricted to $g \tan \alpha$; so that the shortest time in seconds he can move a distance l feet from rest to rest is given by

$$t = 2l/v,$$

where $$\tfrac{1}{2}v^2 = a \,.\, \tfrac{1}{2}l,$$

or $$v = \surd(al),$$

thus $$t = 2\surd(l \cot \alpha / g).$$

By judiciously inclining the tray he can, however, keep the tumblers from slipping and the water from spilling.

332. Again, suppose a tumbler of water is placed on a table in a railway train on a level railway; then the mean level of the water is changed when the train is getting up speed or is coming to rest, as on the waiter's tray, but will be level when the train is going at full speed on the straight.

But on coming to a curve of radius r feet, then if the speed is v f/s, the acceleration of the tumbler is v^2/r f/s^2 to the centre of the curve; so that the slope of the mean level of the water in the tumbler will be at an angle whose tangent is v^2/gr; and the mean surface of the water will be at right angles to the mean direction of the plumb line of a plummet suspended by a short cord in the railway carriage.

If, however, the tumbler is free to slide on a smooth horizontal table, then on entering a curve the tumbler will proceed to describe the *involute of the curve* on the table, because of containing in space the straight line of its original motion, and now the mean level of the water will remain a horizontal plane.

333. *Pressure of Liquid in a Rotating Vessel.*

Suppose a closed vessel filled with liquid is attached to a wheel, rotating about a horizontal axis with angular velocity ω radians/sec.

If the cavity is a smooth sphere or horizontal cylinder, the liquid will not turn, but every particle will describe a circle of radius c with velocity $c\omega$, where c denotes the distance of the centre of the cavity from the axis.

The resultant of g and of the reversed acceleration $c\omega^2$ will therefore be parallel and proportional to CO', the line joining C the centre of the cavity with the point O', vertically above O the axis of rotation at a height g/ω^2 (fig. 96); and therefore the surfaces of equal pressure will be parallel planes, perpendicular to CO'; and if AB is parallel to CO', the pressure at B exceeds that at A by

$$w \,.\, AB(O'C/OO').$$

334. But if the cavity is of any arbitrary form, the liquid will be stirred up by internal motion; but this internal motion is soon destroyed by viscosity (§ 4), so that the liquid finally moves bodily with the vessel.

(An exceptional case is pointed out in Thomson and Tait's *Natural Philosophy*, § 759, in which the liquid is heterogeneous, and arranged in coaxial cylindrical strata in which the density is inversely proportional to the square of the distance from an external axis; in this case, if rotated about a parallel axis, the liquid will move bodily with the vessel, whatever the shape,

335. When the liquid is carried round bodily, any particle Q moves in a circle of radius $r = OQ$, with velocity $r\omega$; and the resultant of g and the reversed acceleration $r\omega^2$ along OQ is given in magnitude and direction by

$$g(O'Q/OO').$$

The surfaces of equal pressure are now coaxial cylinders round O' as axis; and if AB passes through O', and C is its middle point, the average force per unit volume from A to B is $w(O'C/OO')$, so that the pressure at B exceeds the pressure at A by

$$w \cdot AB(O'C/OO').$$

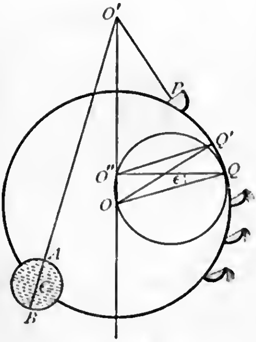


Fig 96.

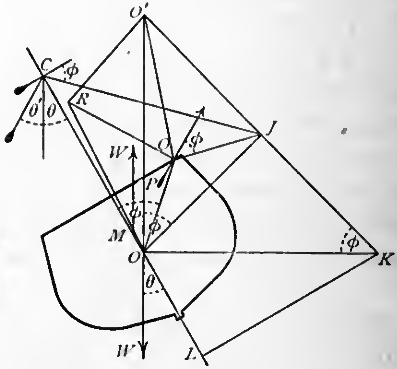


Fig 97.

Thus if the cavity is spherical or cylindrical and AB is the diameter through O', the pressure at A will be zero or a minimum, and at B the pressure will be a maximum, and the same whether the liquid does or does not rotate with the vessel.

In these cases the surfaces of equal pressure are constantly changing in the vessel, so that a fixed free surface is not possible; and the liquid or liquids should fill the vessel completely.

336. But for many practical purposes, as in the design of water wheels, it is assumed that the free surface may be taken as coincident with the surface of equal pressure,

in consequence of the viscosity; and thus the amount of water spilt out of a bucket of the wheel at any point may be determined.

As the point P describes the circle uniformly, then M, the foot of the perpendicular from P on a horizontal or inclined diameter, will oscillate in a *simple harmonic motion*, like the piston of a steam engine; and here again the line MO' is the normal to the free surface of the liquid in a vessel at M; this property is useful in determining the shape of oil cups on the moving parts of machinery, so that the oil shall not be spilt.

337. When the angular velocity is considerable, the point O' comes down to O'', close to O, so that the surfaces of equal pressure are nearly coaxial with the axis of rotation of the wheel.

Thus if a bucket of water is swung round rapidly in a vertical circle, the water will not fall out; and this property is utilized in keeping a fly wheel cool, which is working in a friction brake for testing the indicated horse power of an engine; the fly wheel is made with interior flanges so as to form a trough on the inner circumference, and into this a stream of water is directed, which is either skimmed off by a pipe from its surface, where it is hottest and lightest, or is allowed to evaporate with the heat; and the angular velocity of the wheel is made sufficiently great to prevent the water running out at any point (Druitt Halpin, *Proc. I. M. E.*, 1886).

338. If the water is replaced by sand, forming a layer of uniform thickness in the interior trough, the sand will not slip in the wheel if the angle between the normal of the wheel OQ and the line of force $O''Q$ to any point Q of the surface of the sand is less than the angle

of repose of the sand; and we may call OQO'' the virtual slope or gradient of the sand.

This angle OQO'' is a maximum where the circle OQO'' touches the circle OQ, and therefore where $OO''Q$ is a right angle; and if OQO'' is ϵ, the angle of repose of the sand, and $OQ=r$,

$$\sin \epsilon = g/\omega^2 r.$$

339. *Surfaces of Equal Pressure in a Swinging Body.* Suppose a bucket of water or mercurial horizon, a marine barometer, a box of sand, and a plummet, P lb, at the end of a short thread are fastened to any point Q of a large body, oscillating like a pendulum through an angle $2a$ about a fixed horizontal axis through O (fig. 97); consider, for instance, a ship, when O may be taken at the C.G., and the length of the equivalent pendulum (§ 154)

$$OL = K^2/OM.$$

Then the angular acceleration at an inclination θ to the vertical will be

$$\frac{d^2\theta}{dt^2} = -\frac{g}{OL}\sin\theta = -\frac{g}{OK}, \quad \text{.................(1)}$$

if LK drawn perpendicular to OL meets the horizontal through O in K; while the angular velocity ω is from the Principle of Energy (§ 155) given by

$$\tfrac{1}{2}OL\,.\,\omega^2 = g(\cos\theta - \cos a), \quad \text{................(2)}$$

The plumb line QP will assume the direction of the resultant of gravity and of the reversed effective force of P, denoted in gravitation units of force by P and F pounds.

Then if F makes an angle ϕ with OQ, the effective force F can be resolved into the two components

$$F\cos\phi = P\,.\,OQ\,.\,\omega^2/g, \quad \text{the } \textit{centrifugal} \text{ force,}$$

$$F\sin\phi = P\sin\theta, \quad \text{the } \textit{transversal} \text{ force.}$$

As in § 333 a point O' is taken vertically above O at a height g/ω^2; so that the resultant of gravity and the centrifugal force of P is a force

$$P(O'Q/OO') \text{ pounds, in the direction } O'Q.$$

If QR is drawn perpendicular to OQ of length $OQ \tan \phi$, so that the angle $QOR = \phi$, the tangential force

$$P \sin \theta = P \,.\, OQ \tan \phi \,.\, \omega^2/g = P(QR/OO');$$

so that the resultant of gravity and the reversed effective force at Q is a force T pounds, in the direction $O'R$, where

$$T = P(O'R/OO').$$

If OJ is drawn perpendicular to $O'K$, then

$$O'OJ = OKO' = \phi;$$

and the triangles OJO', OQR are similar, as also the triangles OQJ, ORO', homologous sides being inclined at an angle ϕ.

The tension of the plumb line is thus

$$T = P(JQ/OJ)$$

pounds, in a direction making an angle ϕ with JQ; and thus the rolling of the ship converts the steady vertical lines of force due to gravity alone into variable equiangular spirals round J as pole, of radial angle

$$\phi = \tan^{-1} \frac{\sin \theta}{2(\cos \theta - \cos \alpha)}.$$

The surfaces of equal pressure in the bucket of water will be cylinders, the sections of which are orthogonal equiangular spirals; and the free surface, if so small that the waves on it die out rapidly, will be perpendicular to the plumb line PQ.

340. When the angle between the plumb line PQ and the normal to the surface of the sand in the box exceeds ϵ the angle of repose (§ 30), the sand will slip; and thus the preceding investigation will enable us to determine

the tendency of a grain cargo to shift, of the water in the boilers, or of petroleum cargo to wash about, and to a certain extent the tendency to produce sea-sickness at any point of the ship.

Hence the necessity of loading grain cargo up to the beams, and of filling petroleum tanks up to a height in the expansion chamber; while the disturbing effect of the rolling is seen to diminish in descending towards the keel.

A marine barometer at Q, if free to swing in gimbals, will hang in the direction QP, and the mercury column, if mobile, will stand at a height y instead of h, and register the pressure

$$\sigma h = \sigma y(JQ/OJ),$$

so that

$$y = h(OJ/JQ);$$

to prevent this *pumping* action, the tube of the marine barometer is contracted to a fine bore for part of its length (§ 179).

341. At the end of a roll $\phi = \frac{1}{2}\pi$, and J coincides with K; the plumb line PQ now sets itself at right angles to QK, and the surface of water in the bucket will pass through K; while a grain cargo will slip if ϵ the angle of repose is less than the angle between QK and the surface at Q (P. Jenkins, *Trans. I.N.A.*, 1887).

Thus if the bucket is taken up the mast to a point C, the water will be spilt at the end of roll through an angle $2a$ which would require a steady inclination ϵ (the angle of repose of the sand) given by

$$\tan \epsilon = \frac{CL}{KL} = \frac{CL}{OL} \tan a.$$

The tension T of the plumb line is now given by

$$T = P(KC/OK).$$

Thus a yard weighing P tons, attached to the mast at C, will at the end of a roll call up a stress of

$$P(KC/OK) \text{ tons.}$$

For instance, if the time of a single roll is 5 seconds, $OL = 25 \times 3{\cdot}2661 = 81{\cdot}6$ ft; and if $\alpha = 20°$, $OC = 75$ ft, and $P = 4$ tons, then $T = 4{\cdot}6$ tons, about.

Similarly for 2 tons at a height of 100 ft, $T = 2{\cdot}4$; and for 1 ton at a height of 125 ft, $T = 1{\cdot}3$.

As the ship is rolling through the upright position, $\phi = 0$, J coincides with O', and the plumb line PQ sets itself in the direction QO', and the water in the bucket will be perpendicular to QO'.

342. When the weight P of the plummet becomes sensible compared with W the weight of the rest of the ship, and when the length l of the plumb line is considerable, the oscillations of the ship will be modified thereby; but now the treatment becomes intractable, except for small oscillations.

Suppose then that the plumb line is suspended from the mast at a point C at a height $OC = h$, and that it swings through a small angle θ' from the vertical as the ship rolls through a small angle θ; the ship will also move sideways a small distance, y suppose.

The approximate equations of motion of the ship and of P are now, putting $OM = a$,

$$\frac{W}{g}\frac{d^2y}{dt^2} = -P\theta',$$

$$\frac{WK^2}{g}\frac{d^2\theta}{dt^2} = -(W+P)a\theta + Ph(\theta+\theta'),$$

$$\frac{P}{g}\left(\frac{d^2y}{dt^2} - h\frac{d^2\theta}{dt^2} - l\frac{d^2\theta'}{dt^2}\right) = P\theta'.$$

The combined motion will be compounded of small oscillations of simple pendulum type, of equivalent length λ, given by

$$\frac{d^2y}{dt^2}=-\frac{gy}{\lambda},\quad \frac{d^2\theta}{dt^2}=-\frac{g\theta}{\lambda},\quad \frac{d^2\theta'}{dt^2}=-\frac{g\theta'}{\lambda};$$

so that we find, putting $P/W=p$,

$$\frac{y}{\theta}=p\lambda\frac{\theta'}{\theta}=\frac{(1+p)a-ph-\frac{K^2}{\lambda}}{\frac{h}{\lambda}}=\frac{ph}{1+p-\frac{l}{\lambda}},$$

a quadratic for λ; and the two values of y/θ are the heights above O of the parallel axes about which the ship rolls in the independent simple oscillations.

Neglecting p^2, the values of λ are given by

$$\frac{2K^2l}{\lambda}=(1+p)(K^2+al)+p(h^2-lh)$$
$$\pm\left\{(1+p)(K^2-al)+p\frac{(K^2+al)(h^2-lh)+2K^2lh}{K^2-al}\right\},$$

and, λ_0 denoting the value of λ when $l=0$,

$$\frac{K^2}{\lambda_0}=(1+p)a-ph\left(1+\frac{ah}{K^2}\right).$$

Thus, taking the larger value of λ, we find

$$1-\frac{\lambda_0}{\lambda}=\frac{pah^2l}{K^2(K^2-al)},$$

so that, by winding the weight up close to the point C, the period of rolling is shortened by the fraction

$$\tfrac{1}{2}pah^2l/(K^4-alK^2).$$

In a similar manner the effect of a sphere or cask rolling across the bottom of the hold, or of the motion of the water in the cylindrical boilers, can be investigated.

343. The pitching, or rolling, and tossing oscillations may now be investigated, which must coexist in a ship

when, as in § 150, F and G are not in the same vertical line; as for instance when the ship is not on an even keel, or is heeled over by the sails.

If the ship is depressed bodily a small distance x ft, and turned through a small angle θ about a horizontal transverse axis through G, then F will be depressed through $x+b\theta$ ft, if b denotes the horizontal distance between F and G, so that the additional buoyancy will be

$$wA(x+b\theta) \text{ tons};$$

while the righting couple called up by the heeling is

(i.) $wAK^2\theta$ ft-tons due to the change of trim,
(ii.) $wA(x+b\theta)b$ ft-tons due to the extra immersion, and
(iii.) $-Wc\theta$ ft-tons due to the change of position of the C.B., if at a vertical height c above G.

The equations of motion are

$$\frac{W}{g}\frac{d^2x}{dt^2} = -wA(x+b\theta),$$

$$\frac{WK^2}{g}\frac{d^2\theta}{dt^2} = -wAk^2\theta - wA(x+b\theta)b + Wc\theta;$$

and putting $W=wV$, and

$$\frac{d^2x}{dt^2} = -\frac{gx}{\lambda}, \quad \frac{d^2\theta}{dt^2} = -\frac{g\theta}{\lambda},$$

$$\frac{x}{\theta} = \frac{\dfrac{Ab}{V}}{\dfrac{1}{\lambda} - \dfrac{A}{V}} = \frac{\dfrac{K^2}{\lambda} + c - \dfrac{A}{V}(b^2+k^2)}{\dfrac{Ab}{V}},$$

a quadratic for λ, the lengths of the pendulums which synchronize with the normal oscillations; and the two values of x/θ are the distances from G of the axes about which the independent simple oscillations can exist.

Examples.

(1) Prove that if pieces of lead and of cork, of S.G. s_1 and s_2, connected by a thread of length a, are in equilibrium in a closed vessel filled with water, and if the vessel is moved downwards with acceleration ng, the lead and cork will change places in

$$\sqrt{\left\{\frac{2a}{g}\;\frac{s_1 s_2}{(n-1)(s_1-s_2)}\right\}} \text{ seconds.}$$

(2) Two tanks on wheels, filled with liquid, move on inclines under gravity, being connected by a rope passing over a pulley at the top of the inclines.

Prove that the surfaces of equal pressure will be parallel, and bisect the angle between the inclines, if the gross weight of a tank is proportional to the secant of the slope of the incline.

Determine the hydrostatic thrust on the sides of a tank, supposing it cubical and open at the top.

(3) Prove that if an open hemispherical bowl of radius a and weight W, filled with liquid of density w, slips down an inclined plane of which the angle of friction is ϕ, then

$$\sin\phi = \tfrac{1}{2} W \cos\beta / \{ W + \tfrac{1}{3}\pi w a^3 (1-\cos\beta)^2 (2+\cos\beta)\},$$

where 2β is the angle which the surface of the liquid left in the bowl subtends at its centre.

(3) Prove that in liquid filling a closed box, which is suspended by equal parallel chains of length l and oscillating through an angle 2α, the surfaces of equal pressure are parallel planes perpendicular to the chains; and that when the chains make an angle θ with the vertical, the rate of increase of pressure is

$$w(3\cos\theta - 2\cos\alpha).$$

344. *Liquid in a Vessel rotating about a Vertical Axis.* When the axis of rotation of the vessel is vertical, the surfaces of equal pressure are surfaces of revolution about this axis and therefore do not move in the vessel, so that a free surface is possible; and when the liquid moves bodily with the vessel this surface is a *paraboloid.*

For the plumb line supporting a particle at P, at a distance $PM = y$ from the axis, will assume the direction of the resultant of gravity g and the reversed acceleration $y\omega^2$, acting along MP; so that if the plumb line cuts the axis in G (fig. 99),

$$\frac{GM}{MP} = \frac{g}{y\omega^2};$$

and therefore the surface of equal pressure AP through P, which cuts the plumb line at right angles, will have the subnormal $GM = g/\omega^2$, a constant, a characteristic property of the paraboloid, generated by the revolution of a parabola of semi-latus-rectum $l = g/\omega^2$.

The velocity at the end of the latus-rectum is $l\omega = \surd(gl)$, the velocity due to the *head* $\frac{1}{2}l$; and l is the height of the conical pendulum which has the same period of revolution as the liquid.

If the angular velocity is expressed by R, the revolutions per second, then $\omega = 2\pi R$, and (§ 154)

$$l = \frac{g}{\omega^2} = \frac{g}{4\pi^2}R^{-2} = \tfrac{1}{4}\lambda R^{-2} = 0{\cdot}8165 \times R^{-2} \text{ ft.}$$

345. Taking any point P_1 vertically below P in the surface, and considering the equilibrium of a thin prism about PP_1 as axis, then in liquid of density w, the pressure at P_1 must exceed the pressure at P by $w \,.\, PP_1$, exactly as when at rest; so that the surface of equal pressure through P_1 will be another paraboloid.

If the vessel is filled with liquids of different densities, for instance, air, oil, water, and mercury, then, as in a state of rest, the liquids will distribute themselves in strata of densities increasing as they go down; but the surfaces of separation of any two liquids and the free surface will be equal paraboloids, of latus-rectum $2g/\omega^2$, instead of horizontal planes, the vertical depth of each stratum remaining the same as when at rest.

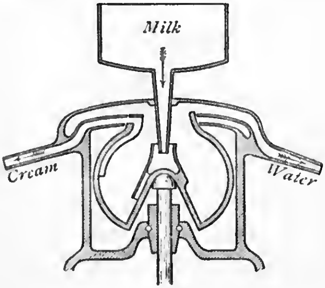


Fig. 98.

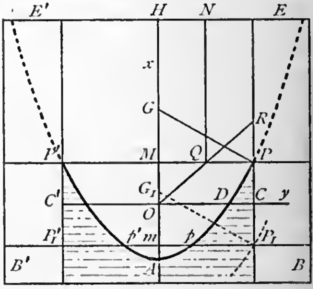


Fig. 99.

Plaster of Paris on the top of mercury can thus be made to assume the form of a paraboloid; in this way Mendeleef proposes to form the speculum of a telescope.

346. If the free surface AP is in contact with air, the air must also be supposed to rotate bodily with the liquid when its density ρ is taken into account; for if the air is at rest, the free surface AP is no longer a surface of equal pressure, as the pressure at A exceeds the pressure at P by ρx, if $AM=x$.

Generally, if it was possible for two liquids of different densities ρ and σ, water over mercury for instance, to rotate bodily with different angular velocities ω and Ω, slipping smoothly over each other at their surface of separation, then along this surface

$$\frac{\rho y^2\omega^2}{2g} - \rho x = \frac{\sigma y^2\Omega^2}{2g} - \sigma x,$$

so that the surface is a paraboloid of latus-rectum

$$2g\frac{\sigma - \rho}{\sigma\Omega^2 - \rho\omega^2}.$$

347. The surfaces of separation of liquids in rotation can be shown experimentally in a glass vessel swung round at the end of a thread, like a conical pendulum; also in a glass spinning top, filled with water and mercury, a bright silver belt being formed by the mercury.

Cream is now separated from milk in an iron vessel, poised on the rounded top of a rapidly rotating vertical axle (fig. 98), the cream rising through the side pipe reaching to the broadest part of the interior, while the refuse milk escapes past the upper edge; cloth, etc., is dried in the same way in *Hydroextractors.*

The C.G. of the vessel is placed as close as possible under the centre of the support, by means of adjusting screws; this tends to diminish the vibration of the spindle, which is held in position at the upper end by an india-rubber ring; for if the depth of the C.G. below the centre of the support is h, the angular velocity must exceed $\sqrt{(g/h)}$ before the vessel will tend to wobble (§ 349); thus if $h = 0{\cdot}02$ inch, the critical number of revolutions is about 22 per second.

The rise of the liquid round the inside of the vessel will also diminish h, and tend to increase the dynamical stability.

348. When the revolutions are very great, the stresses due to the rotation will exceed the *elastic limit,* and the metal will be deformed; in this manner teapots are spun out of ductile pewter.

A ring of metal of radius r ft, cross section a ft^2, and density w lb/ft^3, moving with peripheral speed v f/s, will experience a centrifugal force wav^2/gr lb/ft; and this will (§ 276) set up a circumferential tension T lb/ft, given by

$$2Ta = \frac{wav^2}{gr} \times 2r, \quad \text{or} \quad T = \frac{wv^2}{g};$$

so that T is independent of the radius; and if T denotes the tenacity, the ring will burst if

$$v = \sqrt{(gT/w)}.$$

Thus, in a steel flask a foot in diameter, making 16,000 revs/min, and taking $w=500$, we find T is about

11 million lb/ft^2, or 34 tons/in^2.

349. If the plummet is sunk to P_1, its plumb line will take the direction G_1P_1, normal to A_1P_1 the surface of equal pressure through P_1, but not normal to the free surface.

The plumb line GP can also hang vertically from G, but the equilibrium will be unstable until its length is shortened to less than g/ω^2, the radius of curvature of the paraboloid at the vertex; and if the plummet is lighter than the liquid it displaces, like a cork, the plumb line must be in the prolongation of GP, and fastened at a point beyond P; and the equilibrium in the axis will be stable.

The resultant thrust of the liquid on a finite portion whose C.G. is P_1, at a distance y from the axis, has a vertical component W', the weight of this portion of liquid, and a horizontal component towards the axis

$$W'y\omega^2/g \text{ or } W'y/l.$$

Thus, if the submerged plummet is of finite size, of weight W lb and S.G. s, the vertical and horizontal effective forces on it are

$$W\left(1-\frac{1}{s}\right) \text{ downwards, and } W\left(1-\frac{1}{s}\right)\frac{y}{l}$$

away from the axis, where P_1 now denotes the C.G. of the plummet; and these forces must be balanced by the tension of the thread.

If $s < 1$, the body will be urged towards the axis of this whirlpool; and if it floats freely it cannot be in equilibrium unless the C.G. is in the axis, or coincides with the C.G. of the displaced liquid.

A floating body thus tends to the central depression of a whirlpool as a position of stable equilibrium.

350. If the floating body has a vertical axis of symmetry, like a cylinder, cone, etc., the water line is given by the condition that the weight is equal to the weight of liquid displaced, bounded by the parabolic free surface, continued through the body.

Consider, for instance, a vertical cylinder of radius a, height h, and S.G. s, and let v denote the circumferential velocity of its curved surface.

For moderate values of v the top of the cylinder will be out of water, and the bottom will be covered by water; the water line will rise on the side, and the vertex A of the free surface will descend through equal distances; also the vertical oscillations will be the same as in the liquid at rest.

As v is increased the water will first flow over the top, or the vertex A will first come below the bottom of the cylinder, according as s is $>$ or $< \frac{1}{2}$; and finally, for greater values of v, when both the top is partly covered and the bottom uncovered, we shall find that the depth of A below O, the centre of the cylinder, is

$$(1-s)\tfrac{1}{2}v^2/g.$$

For if this depth OA is denoted by x, the volume of water displaced is (§ 135)

$$\pi a^2 h - \pi l\{(x+\tfrac{1}{2}h)^2-(x-\tfrac{1}{2}h)^2\} = \pi(a^2-2lx)h;$$

and this is equal to $\pi s a^2 h$ in the position of equilibrium.

Also the increase of buoyancy due to a small additional vertical displacement x' is $2\pi l h x'$, so that (§ 148) the vertical oscillations synchronize with a pendulum of length $\tfrac{1}{2}sa^2/l$.

If the water is contained in a coaxial cylinder of radius b, the cylinder sinks $\tfrac{1}{4}(b^2-a^2)\omega^2/g$, until the water reaches the top; the modification when the angular velocity ω is still further increased is left as an exercise.

So also for a cone, of height h and radius of base a, floating vertex downwards; it can be shown that the water line reaches the base when

$$\omega^2 = \tfrac{4}{3}(1-s)gh/a^2.$$

351. *The Parabolic Speed Measurer.*

This consists essentially of a glass cylinder or cell, partly filled with mercury, the remaining volume being occupied by air or water; it is placed on a vertical shaft Ox, the angular velocity ω of which is to be measured by noting the rise of the mercury on a graduated scale.

Suppose Oy is the level of the mercury when at rest; the mercury will rise on the walls to the level PMP', while the vertex of its paraboloidal surface will sink to A suppose, where (fig. 99)

$$\frac{PM^2}{AM} = \frac{2g}{\omega^2}.$$

Putting $AM = x$, then in a cylindrical vessel

$$AO = OM = \tfrac{1}{2}x,$$

from the fact that the volume of the paraboloid swept out by AMP is half the volume of the circumscribing cylinder (§ 135); so that, denoting the radius by a,

$$\frac{x}{a^2}=\frac{\omega^2}{2g}, \quad \text{or} \quad \omega^2=\frac{2gx}{a^2};$$

and therefore the angular velocity is proportional to the square root of OM, the rise of the mercury.

The free surface of the mercury will cut the horizontal plane OC in a circle at D, such that $OD=\frac{1}{2}\sqrt{2}a$; and this is therefore a fixed circle.

In another form of speed indicator the rotating vessel is closed at the top by a fixed cover CC', provided with a cylindrical part $CP_1P_1'C'$ dipping into the mercury; and the mercury can rise through holes in the circumference P_1P_1' into a fine glass tube Ox in the vertical axis.

The pressure in the rotating mercury gliding round P_1P_1' is measured by the column at rest in Ox, and the column will therefore rise to M at the same level as would the free surface PP', while the vertex A will now remain near O, if the closed vessel is filled with mercury; and the graduations for M are the same as before.

The graduations for equal increments of angular velocity can be made geometrically by dividing HE and CE into the same number of equal parts n; and now, if N and R are corresponding divisions, such that

$$\frac{HN}{HE}=\frac{CR}{CE}=\frac{r}{n},$$

the vertical NQ and OR will meet in a point Q at the level of P, such that

$$\frac{CP}{CE}=\frac{CP}{CR}\frac{CR}{CE}=\frac{r^2}{n^2},$$

so that the angular velocity registered at P is r/n of the angular velocity registered at the level of E.

The depth of the directrix plane below O

$$=\tfrac{1}{2}x+\frac{1}{2}\frac{g}{\omega^2}=\frac{a^2\omega^2}{4g}+\frac{g}{2\omega^2}=\frac{a}{\sqrt{2}}+\left(\frac{a\omega}{2\sqrt{g}}-\frac{\sqrt{g}}{\sqrt{2}\omega}\right)^2,$$

a minimum, $\tfrac{1}{2}\sqrt{2}a$, when $\omega=\sqrt{(\sqrt{2}g/a)}$, and then O is the focus of the paraboloid.

352. As the speed ω is increased, P will meet the top EE' (and the liquid will spill out unless the top is closed) or A will meet the bottom BB' first, according as the cylinder is more or less than half full; and finally, for still greater values of ω, the top EE' will be partly covered with mercury, and the bottom BB' partly uncovered (along pp'), as in a cream separator.

Taking for simplicity the cylinder as half full of mercury, and of height h, the depth x of the vertex A below O will now be given by the equation (§ 135)

$$\pi l\{(x+\tfrac{1}{2}h)^2-(x-\tfrac{1}{2}h)^2\}=\tfrac{1}{2}\pi a^2 h,$$

or

$$x=\tfrac{1}{4}a^2/l, \quad \text{where} \quad l=g/\omega^2;$$

and the areas of the circular edges of the mercury are

$$2\pi l(x\pm\tfrac{1}{2}h).$$

If the bottom BB' is still covered while the mercury meets the top EE' in a circle PP' of radius y, and if $OH=b$, $AH=x$, then the volume of the paraboloid PAP

$$\tfrac{1}{2}\pi x y^2=\pi a^2 b,$$

so that

$$\frac{2g}{\omega^2}=\frac{y^2}{x}=\frac{2a^2b}{x^2}=\frac{y^4}{2a^2b},$$

and y could be observed on a transparent cover to the cylinder; while x, if it could be observed, would be proportional to ω.

The depth of the directrix below O is now

$$=x+\frac{g}{2\omega^2}=x+\frac{a^2b}{2x^2}=a\omega\sqrt{\frac{b}{g}}+\frac{g}{2\omega^2},$$

a minimum when $\omega=g^{\frac{1}{2}}a^{-\frac{1}{3}}b^{-\frac{1}{6}}$.

So also when the bottom BB' is uncovered before the mercury reaches the top.

353. A parabolic speed measurer, showing the form of the free surface, can be made by replacing the glass cylinder by a rectangular box or cell, of which two vertical sides are sheets of plate glass a small distance apart.

The mercury will now spread out into a film, bounded above by a parabola, or parabolic cylinder as we may consider it, when the thickness of the mercury film is small.

In this case we shall find, since the area of the parabola AMP is two-thirds of the rectangle AP, that if $AM=x$, then $AO=\frac{1}{3}x$, $OM=\frac{2}{3}x$; so that the graduations for P and A are proportionately the same as before; and now $OD=\frac{1}{3}\sqrt{3a}$, and this is the minimum depth of the directrix below O.

When the speed is sufficiently increased for the parabola AP to meet the top EE', then as before,

$$\tfrac{2}{3}xy=ab, \quad \text{and} \quad \frac{2g}{\omega^2}=\frac{y^2}{x}=\frac{9a^2b^2}{4x^3},$$

or ω varies as $x^{\frac{3}{2}}$.

The cases may also be worked out when, as before, the bottom is partly uncovered, and when the top is also partly covered.

354. Even when the thickness of the mercury film and the distance between the glass plates is appreciable, we shall find no material difference in the results; for the vertex A of the paraboloid will sink below O half the distance which MP rises above O (fig. 99), so that the graduations for P will be proportionately the same as before, increasing as the square of the angular velocity.

This follows from the geometrical theorem that the volume of mercury above the horizontal tangent plane at A is one-third of the volume of the vessel included between the horizontal planes through A and MP, when the cross section of the vessel is rectangular.

355. In finding the thrust on a plane wall of a rotating vessel of liquid and its C.P., we notice that the pressure at a point at a distance y from the axis exceeds the pressure wz at the same level on the axis, at a depth z below the vertex of the free surface, by $\frac{1}{2}w\omega^2y^2/g$, or $\frac{1}{2}wy^2/l$; and it is convenient to calculate the thrust and C.P. due to wz and $\frac{1}{2}wy^2/l$ separately, the thrust due to the latter being $\frac{1}{2}wAk^2/l$, where Ak^2 denotes the moment of inertia of the wetted area A about the axis of rotation. Consider as an exercise the side of a rectangular vessel.

So also the average pressure over the surface S of a vessel filled with liquid, due to the rotation, is $\frac{1}{2}wk^2/l$, where k denotes the radius of gyration of the surface S about the axis.

But if the axis of rotation does not pass through the highest point of S, the small bubble left at the top moves to the point where the highest surface of equal pressure touches S, and where a small drop of liquid can be removed without altering the pressure, and the average pressure due to gravity is diminished by the head equivalent to the vertical descent of the bubble.

356. If the vessel is composed of a tube of small bore, the paraboloids of equal pressure must be drawn through the ends and through the points of separation of the liquids; and now the pressures are equal at the points where one of these paraboloids cuts the tube; so that is now no longer generally true that

"the heights of the free surfaces above their common surface of two liquids in a bent tube are inversely as their densities" (§ 158).

When ω is sufficiently increased a free surface will reach an end of the tube, and liquid will be spilt, unless the end is closed; and now the pressure of the imprisoned air modifies the result.

If, as in § 161, a filament of length c makes angles θ, ϕ, with the normals to the free surface through the ends, and if these normals make angles θ', ϕ' with the vertical, the same reasoning will show that the small oscillations of the filament will synchronize with a pendulum of length

$$c/(\sec\theta' \cos\theta + \sec\phi' \cos\phi).$$

The same principles will apply to the liquid in a rotating vessel of any shape, by means of which the student can prove the results of the following theorems or exercises.

Examples.

(1) A fine tube bent into three sides of a square, each of length a, is filled with liquid and rotated about a vertical axis bisecting the middle side at right angles. Prove that no mercury will escape until the angular velocity exceeds

$$\sqrt{\{8g(a+h)/a^2\}},$$

where h denotes the height of the barometer.

If the horizontal part of the tube alone contains mercury and the vertical parts atmospheric air, the tube being closed at the upper ends, determine the angular velocity required to make the mercury rise a height c in the vertical branches.

(2) A straight tube closed at the lower end is rotated about a vertical axis through this end at a constant angle a to the vertical; prove that if $l = g/\omega^2$, the greatest length of mercury it can contain is

$$\{l \cot a + \sqrt{(2lh)}\} \operatorname{cosec} a.$$

(3) If a circular tube of radius a rotates about a vertical diameter, and contains a filament of mercury subtending an angle $2a$ at the centre, the filament divides at the lowest point for angular velocity

$$\Omega = \sqrt{(g/a)} \sec \tfrac{1}{2}a.$$

For a greater angular velocity ω the filament divides into two equal halves, the centres of which subtend at the centre of the circle an angle

$$2 \cos^{-1}\{(\Omega/\omega)^2 \cos \tfrac{1}{2}a\}.$$

(4) If the tube rotates about a vertical tangent, a filament of length $(\pi + \theta)a$ just reaches the highest point if

$$\omega^2 = \frac{g}{a} \Big/ (\tan \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta).$$

(5) If an open vertical cylinder of height h and radius a, full of liquid, is set spinning about its axis with peripheral velocity v, the vertex of the free surface will sink to a depth $\frac{1}{2}v^2/g$ below the rim, and a volume $\frac{1}{4}\pi a^2 v^2/g$ of liquid is spilt; so that if the cylinder is stopped, the surface will be at a depth $\frac{1}{4}v^2/g$ below the rim.

When $\frac{1}{2}v^2/g$ is greater than h, the bottom of the cylinder will be uncovered in a circle of area

$$\pi a^2\left(1 - \frac{2gh}{v^2}\right);$$

and when the cylinder is stopped, the liquid stands at a depth $2gh^2/v^2$.

(6) If the cylinder is closed at the top by a piston, the upward thrust on the piston is

$$\tfrac{1}{4}w\pi a^2v^2/g=\tfrac{1}{2}w\pi a^2h',$$

if $h'=\tfrac{1}{2}v^2/g$, the *head* of the velocity v; and the downward thrust on the bottom is

$$w\pi a^2(h+\tfrac{1}{2}h').$$

The average pressure over the curved surface will be due to a head $h'+\tfrac{1}{2}h$, and will therefore be $1+2h'/h$ times its value when the liquid is at rest.

(7) The piston will be lifted when the upward thrust on it is greater than its weight P; or, denoting the weight of liquid by W, when

$$\frac{h'}{2h}>\frac{P}{W};$$

and the piston will rise a distance z, given by

$$\sqrt{\frac{z}{h}}=\sqrt{\frac{h'}{2h}}-\sqrt{\frac{P}{W}}.$$

(8) Prove that if the circular cylinder is rotated about a parallel vertical axis, the edge of the free surface will be a plane ellipse.

(9) If the vessel is a rectangular box rotated about a vertical axis through its centre, so that v is the velocity of its vertical edges, the volume spilt will be

$$\tfrac{1}{12}abv^2/g=\tfrac{1}{6}abh',$$

provided the bottom is not uncovered, which will happen when $v^2=8gh$, or $h'=4h$; and now $\tfrac{2}{3}$ of the liquid is spilt.

(10) If the vessel is a closed sphere rotated about a vertical diameter, the greatest pressure is at the lowest point so long as $\omega<\surd(g/a)$.

For greater values of ω prove that the greatest pressure is along the circle where the paraboloidal surface of equal pressure touches the sphere, at a depth g/ω^2 below the centre; and determine the paraboloid orthogonal to the sphere.

If W denotes the weight of the liquid filling the sphere, the liquid thrusts on the upper and lower hemispheres are

$$\tfrac{1}{4}W\left(1+\frac{3}{4}\frac{a\omega^2}{g}\right) \quad \text{and} \quad \tfrac{1}{4}W\left(5+\frac{3}{4}\frac{a\omega^2}{g}\right).$$

(11) When the vessel is an open hemisphere, the bottom is uncovered when $\omega > \sqrt{(g/a)}$; and the volume of liquid left in the vessel is $\frac{4}{3}\pi l^3 = \frac{4}{3}\pi(g/\omega^2)^3$, even with a hole at the lowest point.

The average pressure over the surface, before the bottom is uncovered, is diminished from

$$\tfrac{1}{2}wa \text{ to } \tfrac{1}{2}wa\left(1-\frac{a\omega^2}{3g}\right).$$

(12) In an open paraboloid, one-nth of the liquid is spilt if $g/\omega^2 = nL$, where L denotes the semi-latus-rectum; and all is spilt if $g/\omega^2 = L$, the surfaces of equal pressure being now similar to the paraboloid; and thus if a hole is made anywhere in this vessel, all the liquid must flow out.

If the horizontal cross sections of the paraboloid at a depth z below the rim are the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z}{c},$$

the water line of the liquid will sink to a depth

$$(a^2-b^2)\bigg/\left(\frac{2g}{\omega^2}-\frac{b^2}{c}\right).$$

(13) If the paraboloid is closed and filled with liquids of various densities, originally in horizontal strata, the arrangement of the strata is inverted when the angular velocity exceeds $\sqrt{(g/L)}$.

(14) In an open cone, the upper part of the surface becomes uncovered when the depth of the vertex of the free surface exceeds half the depth of the cone; and the volume the cone can now hold is

$$\tfrac{1}{12}\pi\left(\frac{g}{\omega^2}\right)^3\cot^4\alpha = \tfrac{1}{12}\pi l^3\cot^4\alpha.$$

Consider also an inverted cone on a whirling table, and find when the liquid will escape.

(15) If a closed surface filled with liquid is rotated with angular velocity ω about a fixed axis at an angle α to the vertical, the surfaces of equal pressure are paraboloids of revolution round a parallel axis at a distance $g\sin\alpha/\omega^2$, of latus-rectum $2g\cos\alpha/\omega^2$, the same as for liquid under the diminished gravity in the direction of the axis.

357. *The Free Surface of the Ocean.*

Careful measurement shows that the surface of the Ocean, which we may take as the mean surface of the Earth, is not exactly spherical, as drawn by Archimedes, but of the spheroidal shape the surface would assume in consequence of the diurnal rotation.

The normal to the free surface of the Ocean is at any point in the direction of the plumb line; and now if we assume that the solid part of the Earth is spherical or rather *centrobaric*, the plumb line will take the direction of the resultant g of the attraction of pure gravitation G to the centre of the Earth, and of the centrifugal force due to the whirling motion of rotation.

It is more convenient now to employ the absolute unit of force; so that the attraction of pure gravitation on a plummet weighing W g is WG dynes, where G denotes the acceleration of gravity on a projectile or freely falling body; and the centrifugal force at a distance y cm from the polar axis is $Wy\omega^2$ dynes, where ω denotes the angular velocity of the Earth, in radians per second.

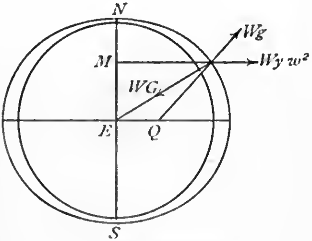


Fig. 100.

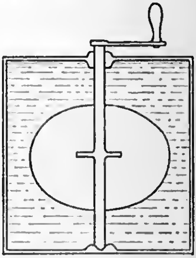


Fig. 101.

Producing the plumb line to meet the equator EQ in Q, then by the triangle of force EQP (fig. 100)

$$\frac{EQ}{EP}=\frac{Wy\omega^2}{WG}, \quad \text{or} \quad \frac{EQ}{MP}=\frac{\omega^2}{G}.EP.$$

Now in the case of the surface of the Earth, G is so nearly constant and equal to g (980) and EP is so nearly equal to R the mean radius of the Earth, $10^9 \div \frac{1}{2}\pi$ cm, that we may, as a close approximation, put

$$EQ=e^2.MP, \text{ where } e^2=R\omega^2/g,$$

so that PQ is the normal to an ellipse of excentricity e passing through P; and the surface of the Ocean may be taken as the spheroid generated by the revolution of this ellipse about the polar axis.

This would be accurately true if the acceleration of pure gravity at a distance r from the centre was $G(r/R)$; and g at any point P would now vary as the normal PQ.

The time of a sidereal revolution of the Earth being

$$T = 23\text{ h } 56\text{ m } 4\text{ s} = 86164\text{ s},$$

$$\omega = 2\pi/T = \log^{-1}\bar{5}\cdot 8629.$$

$$\log R = 8\cdot 8039,$$

$$\log \omega^2 = \bar{9}\cdot 7258,$$

$$\log R\omega^2 = 0\cdot 5297,$$

$$\log g = 2\cdot 9912,$$

$$\log g/R\omega^2 = 2\cdot 4615,$$

$$g/R\omega^2 = 289\cdot 4 \approx 17^2,$$

$$e = 1/17.$$

358. Denoting the equatorial and polar semi-axes of the spheroidal surface by a and b, then $(a-b)/a$ is called the *ellipticity* and denoted by ϵ; and

$$\epsilon = 1 - \sqrt{(1-e^2)} \approx \tfrac{1}{2}e^2.$$

The above value of e^2 gives $\epsilon = 1/578$; but geodetic measurements make $\epsilon = 1/300$ about; the discrepancy is due to the fact that the solid Earth is not spherical or centrobaric, but that its shape partakes of the ellipticity of the Ocean, and to precisely the same amount; showing that the solid Earth was once in a molten viscous condition, during which time it took the present shape.

Suppose we increase ω to Ω, so that $R\Omega^2/g = 1$; then with the above value of e^2, $\Omega = 17\omega$; and now the plumb line would be parallel to the Earth's axis, and would point to the Pole Star; at the equator water would fly off into space and bodies too, unless fastened down to the ground; and the water would collect in lakes with the free surface always parallel to the equator.

This implies however that the solid part of the Earth is rigid and does not change its spherical shape; but practically the solid form would be deformed into a spheroid, to a much greater extent than at present.

359. Plateau has devised an apparatus by which these phenomena may be imitated; a vertical axis is fixed in a vessel of water, and oil, of equal density, is placed on a solid nucleus fixed to the axis; the spheroidal form is closely imitated when the axis is revolved (fig. 101).

Here the constraining cause is the capillarity tension of the surface of separation, T dynes/cm suppose; and it can be proved as an exercise that if the mean radius is R cm and the density σ g/cm^3, the ellipticity due to a small angular velocity ω is $\frac{1}{8}\sigma\omega^2 R^3/T$.

360. According to astronomical definitions the angle PGQ is called the *latitude* of the place; not the angle PEQ, which is distinguished as the *geocentric* latitude, the angle EPG being called the *angle of the centre;* the tangent plane at P perpendicular to the plumb line GP is called the *sensible horizon*, and the parallel plane through the centre E the *rational* horizon of P.

The angle of the centre EPG is the gradient of the free level surface with respect to the mean spherical surface through P; denoting it by ϕ, and the geocentric latitude by θ,

$$\sin\phi = (EG/EP)\sin\theta = e^2\cos\theta\sin\theta = \epsilon\sin 2\theta.$$

Thus in latitude 45°, where ϕ is greatest, a river flowing south is running away from the centre of the Earth on an apparent gradient of about one in 300.

The Mississippi rises in latitude 75°, and flows nearly due south into the sea in latitude 30°, a distance of 900 geographical miles, at an average gradient with respect to the Earth's centre of one in 320; so that if the source is one-quarter of a mile above sea level, the mouth will be about $2\frac{1}{2}$ miles farther from the centre of the Earth.

CHAPTER XI.

HYDRAULICS.

361. The word *Hydraulics* means primarily the science of the Motion of Water in Pipes; but it is now extended to cover the elementary parts of the practical science of the Motion of Fluids.

This includes the Discharge from Orifices, the Theory of Hydraulic Machinery, such as Water Wheels, Turbines, Paddle Wheels and Screw Propellers, Injectors, etc., which can be treated by the aid of Torricelli's and Bernoulli's Theorems; and the Motion in Pipes, Canals, and Rivers, taking into account the effect of Fluid Friction so far as it can be treated in an elementary manner.

362. *Torricelli's Theorem.*

The velocity v of discharge of water from a small orifice a depth h below the free surface was given by Torricelli (1643) as the velocity v acquired in falling from the level of the free surface, so that

$$\tfrac{1}{2}v^2 = gh, \quad \text{or} \quad v = \sqrt{(2gh)};$$

and v is then called the velocity due to the head h.

This is argued by asserting that the hydrostatic energy of the water, Dh ft-lb per ft^3, or h ft-lb per lb, becomes converted on opening the orifice into the kinetic energy

$$\tfrac{1}{2}Dv^2/g \text{ ft-lb/ft}^3, \quad \text{or} \quad \tfrac{1}{2}v^2/g \text{ ft-lb/lb}.$$

Thus the jet of water, if directed nearly vertically upwards, would nearly reach the level of the free surface; and if directed in any other direction will form a parabolic jet, of which the directrix lies in the free surface of the still liquid.

The cross section of the jet OVR, while continuous and not shattered into drops, will be inversely as the velocity; and the horizontal component of the velocity being constant, equidistant vertical planes will intercept equal quantities of water, so that G the C.G. of the water will coincide with the C.G. of the parabolic area cut off by the chord; and the height of the C.G. of the jet cut off by a horizontal chord OR will be two-thirds of the height of the vertex (fig. 104, p. 469).

If the jet could be instantaneously reduced to rest and frozen, it could stand as an arch, without shearing stress across normal sections.

If the vessel is in motion, the velocity of efflux v is still taken as due to the head of the pressure p; in this way the efflux from an orifice in a rotating vessel (Barker's Mill or a Turbine) is given (§ 345) by $v=\surd(2gz+y^2\omega^2)$, or from an orifice in an ascending or descending bucket; balanced by a counterpoise at the end of a rope over a pulley by $v=\surd\{2(g\pm a)z\}$ (§ 322); the student may work out the motion of the buckets completely as an exercise.

363. The velocity of efflux v must be reckoned not exactly at the orifice, but a little in front at the point where the jet is seen to contract to its smallest cross section; this part is called the *vena contracta*, and the ratio of the cross section of the vena contracta to that of the orifice is called the *coefficient of contraction*, and denoted by c_1.

Practically, in consequence of friction, the velocity v at the vena contracta is a little less than $\sqrt{(2gh)}$, and the ratio of v to $\sqrt{(2gh)}$ is called the *coefficient of velocity*, and denoted by c_2.

Now if the area of the vena contracta is A ft², and of the orifice is B ft², $A = c_1 B$; and the flow of water is

$$Av = c_1 Bv = c_1 c_2 B\sqrt{(2gh)} \text{ ft}^3/\text{sec};$$

and the product $c_1 c_2$ is denoted by c, and called *the coefficient of discharge.*

The flow of water through a standard vertical orifice one in² in section, under a head of 6½ ins, is called the *miner's inch;* since $B = 1 \div 144$ ft², $h = 0{\cdot}54$ ft, and we may put on the average $c = 0{\cdot}62$, this flow is about

$$1{\cdot}5 \text{ ft}^3/\text{minute}.$$

364. Torricelli's Theorem is still employed when the head varies, as in filling or emptying a reservoir or lock, in sinking a ship by a hole under water, or in pouring out water from a vessel through a spout; and now, if X denotes the area of the surface of the water at a height x above the orifice B,

$$X\frac{dx}{dt} = -cBv = -cB\sqrt{(2gx)},$$

$$t = \int_x^h \frac{X\,dx}{cB\sqrt{(2gx)}},$$

giving the time t of filling or emptying the vessel between the levels x and h; this may be worked out for vessels of various form, as the cylinder, cone, sphere, etc.

Thus, for example, if an orifice of one ft² be opened in the bottom of a sheet iron tank, 30 ft long, 20 ft broad, and 9 ft deep, drawing 4 ft of water, the tank will sink in about 26 minutes, taking $c = 0{\cdot}6$.

If the orifice in a vertical wall is large, and the variations of head over its area is taken into account, and if y denotes the breadth of the orifice at a depth x below the surface, the efflux in ft³/sec is, with $c=1$,

$$Q=\int y\sqrt{(2gx)}dx \quad \text{and} \quad t=\int X dx/Q.$$

Thus if h, h' denote the depth of the top and bottom of a rectangular orifice of breadth b,

$$Q=b\sqrt{(2g)}\int_{h'}^{h} x^{\frac{1}{2}}dx=\tfrac{2}{3}b\sqrt{(2g)}(h^{\frac{3}{2}}-h'^{\frac{3}{2}});$$

so that the average velocity of efflux is due to the head

$$\frac{4}{9}\left(\frac{h^{\frac{3}{2}}-h'^{\frac{3}{2}}}{h-h'}\right)^2;$$

and this is $\frac{4}{9}h$, if $h'=0$.

For example, the time of draining to a depth of 3 inches the ditch of a fortress, one mile long, 30 ft broad, and 9 ft deep, by a vertical cut 2 ft broad, is $13\frac{3}{4}$ hours; and to lower the depth to one inch will take 12 hours more.

365. The flow of water is $DAv=DA\sqrt{(2gh)}$ lb/sec, possessing momentum $DAv^2/g = 2DAh$ second-lb/sec; this will therefore be the thrust in lbs of the jet against a fixed plane perpendicular to its direction.

This thrust is double the hydrostatic thrust due to the head h; thus, for instance, the water of Niagara, falling 162 ft, can balance a column of water 324 ft high in a J-shaped tube, with its lower mouth under the fall.

The energy of the jet is $DAv\,.\,\frac{1}{2}v^2/g=DAvh$ ft-lb/sec; and therefore the H.P. (horse-power) is

$$\tfrac{1}{2}DAv^3/550g=DAvh/550.$$

With a metre and kilogramme as units, $D=1000$, $g=9{\cdot}81$; and 75 kg-m/sec is the *cheval-vapeur*.

Thus a jet of water 10 ins in diameter, issuing under a head of 600 ft has 7300 H.P.; these large jets are used for hydraulic mining in California, the nozzle being controlled by an apparatus called a *hydraulic giant.*

366. Denoting by p the hydrostatic pressure Dh lb/ft^2 due to the head h, then $v = \sqrt{(2gp/D)}$; and the jet discharges

$$A\sqrt{(2gp/D)}\ \text{ft}^3/\text{sec}, \quad \text{or} \quad A\sqrt{(2gpD)}\ \text{lb/sec},$$

possessing momentum $2Ap$ sec-lb, and energy and H.P.

$$\frac{A(2gp)^{\frac{3}{2}}}{2gD^{\frac{1}{2}}}\ \text{ft-lb}, \quad \text{and} \quad \frac{A(2gp)^{\frac{3}{2}}}{1100gD^{\frac{1}{2}}}.$$

Thus the velocity of efflux from the Hydraulic Power main (§ 14) would be 333 f/s, and the flow through a hole $\frac{1}{4}$ inch diameter, with $c = 0{\cdot}65$, would be 40,000 gallons in 24 hours.

Again the pressure in the air vessel in fig. 78, for a steady flow of V ft^3/sec of water through a delivery pipe A ft^2 in section, is $\frac{1}{2}DV^2/gA^2$ lb/ft^2.

367. Suppose that two fluids, water and steam for instance, are issuing by two nearly equal nozzles, of cross sections A and a ft^2, from a vessel in which the (gauge) pressure is p; denoting the density of the steam by δ, then, according to Torricelli's theorem,

$$\frac{\text{the velocity of the steam jet}}{\text{the velocity of the water jet}} = \sqrt{\frac{D}{\delta}},$$

$$\frac{\text{the delivery in lb of the steam jet}}{\text{the delivery in lb of the water jet}} = \frac{a}{A}\sqrt{\frac{\delta}{D}},$$

$$\frac{\text{the momentum of the steam jet}}{\text{the momentum of the water jet}} = \frac{a}{A},$$

$$\frac{\text{the energy or H.P. of the steam jet}}{\text{the energy or H.P. of the water jet}} = \frac{a}{A}\sqrt{\frac{D}{\delta}}.$$

If the area of the water surface in the boiler is C ft², the time required to lower the surface one inch is

$$\frac{1}{12}\frac{C}{A}\sqrt{\left(\frac{D}{2gp}\right)} \text{ seconds.}$$

For instance, if a piston of weight W lb is placed in a vertical cylinder of cross section C ft², resting on the surface of water, and a small vertical nozzle of area A ft² is opened in the piston, the water will flow through the orifice with velocity $\surd(2Wg/DC)$ f/s; and the piston will descend with velocity $\surd(2A^2Wg/DC^3)$.

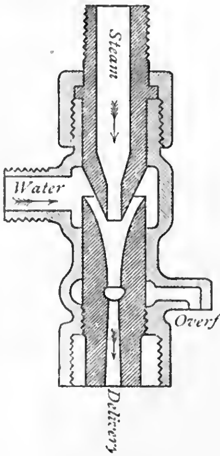


Fig. 102.

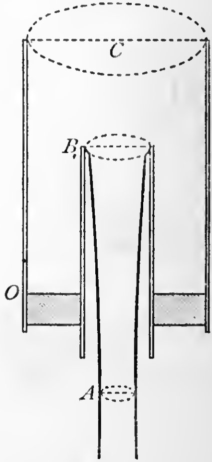


Fig. 103.

368. The superior energy of the steam jet enables it, even when mixed and condensed with water, to overcome the water jet and to enter the boiler; the maximum water fed in being the difference between the quantities of water and steam blown out, and therefore

$$A\surd(2gpD) - a\surd(2gp\delta) \text{ lb/sec.}$$

In this way the action of *Giffard's Injector* may be popularly explained; also of the jet pump, working with two liquids, say water and mercury; the injector is shown in fig. 102, in the form patented by Hall.

Thus for a pressure of 100 lb/in², and a jet $\frac{1}{4}$ inch in diameter, taking $\delta = 0{\cdot}23$, $D = 62{\cdot}4$ lb/ft³, the water fed in is 2·4 lb/sec or 14·4 gallons/minute.

A good average value of $\sqrt{(D/\delta)}$ is 16, so that one lb of steam forces 15 lb of water; but in a steam pump (fig. 78, p. 360) one ft³ of steam will force nearly one ft³ of water, or 380 times its weight; the injector has therefore a mechanical efficiency much inferior to that of the pump, but it has the advantage of working when the engine is still, and of heating the feed water.

If the requisite diameter of the vena contracta A at the throat of the injector is d inches, so that $A = \frac{1}{4}\pi d^2 \div 144$ ft², when Q denotes the given number of ft³ of water and condensed steam to be injected in one hour against a gauge pressure of p atmospheres, of 14·7 lb/in², then

$$Q = 3600 \frac{\frac{1}{4}\pi d^2}{144} \sqrt{\left(\frac{2 \times 32 \times 14{\cdot}7 \times 144}{62{\cdot}4}\, p\right)},$$

or $$\tfrac{1}{4}\pi d^2 = \frac{Q}{M\sqrt{p}}, \text{ where } M = 1165.$$

Rankine replaces M by 800, to allow for the steam used in the injector, for the air sucked in, and for the friction in the pipes; the discrepancy is also partly due to the erroneous hypothesis concerning the flow of the steam jet, and to thermodynamic influences left out of account.

369. *Bernoulli's Theorem.*

In Bernoulli's Theorem the gradual interchange of the energies due to pressure, head, and velocity in a stream line filament in the interior of the liquid, or in a smooth pipe of gradually varying section is, expressed by the equation

$$p + Dx + \tfrac{1}{2}Dv^2/g = Dh, \text{ a constant,}$$

or

$$\frac{p}{D} + x + \frac{v^2}{2g} = h, \text{ a constant,}$$

where p denotes the pressure, D the density, v the velocity and x the height above a fixed horizontal plane.

Thus with British units, the total constant energy Dh along a stream line is in ft-lb/ft^3, and composed of p due to the pressure, Dz to the head, and $\frac{1}{2}Dv^2/g$ due to the velocity; or in ft-lb/lb, the energy or head h is composed of p/D due to the pressure, z to the head, and $\frac{1}{2}v^2/g$ to the velocity.

370. Bernoulli's Theorem is illustrated experimentally in fig. 104 by an apparatus devised by Froude (*British Association Report*, 1875); a tube of varying section carries a current of water between two cisterns filled with water to nearly the same level, and the pressure is measured by the height of water in small vertical glass tubes called *piezometer* tubes; and it is found, in accordance with Bernoulli's Theorem, that the water stands higher where the cross section of the current is greater, and the velocity consequently less.

If the velocity at the throat E is that given by Torricelli's Theorem, the pressure there is reduced to atmospheric pressure, and the tube can be removed in the neighbourhood of E, as at the throat of the injector jet.

At M the cross section is less than at E and the pressure is below atmospheric pressure, so that water will be drawn up in a curved piezometer tube like a siphon.

By the observation of the heights in piezometer, at L and N as well, the velocity of flow can be inferred, knowing the cross section of the current; this is the principle of the Venturi Water Meter, invented by Mr. Clemens Herschel; also of the *aspirator*.

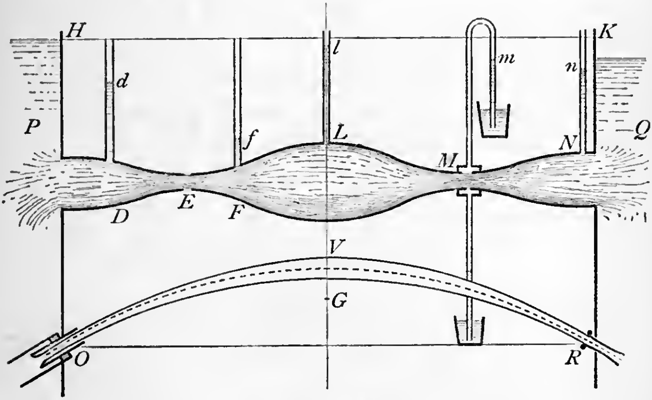


Fig. 104.

371. As an application of Bernoulli's Theorem, Lord Rayleigh (*Phil. Mag.*, 1876) determines the area A of the vena contracta of a jet, issuing from a re-entrant pipe at O (figs. 103, 104), of cross section B, inserted in a pipe of cross section C; this ajutage is called a ring nozzle, and it is employed sometimes with a fire engine jet.

Then if V f/s denotes the velocity, and p lb/ft² denotes the gauge pressure in the pipe C, the *equation of continuity* gives

$$Av = CV,$$

and Bernoulli's equation gives

$$\frac{p}{D} = \frac{v^2 - V^2}{2g}.$$

A third equation is given by the principle of momentum; taking the momentum which enters and leaves the space cut off by two planes A and C (fig. 103, p. 466) the momentum which leaves this part, in sec-lb/sec, is

$$D(Av^2 - CV^2)/g.$$

But if p' denotes the (average) pressure in lb/ft² over the annular end of the tube C,

$$pC - p'(C - B) \text{ sec-lb}$$

is the momentum per second due to the pressure; and therefore, in the state of steady motion,

$$pC - p'(C - B) = D(Av^2 - CV^2)/g.$$

If we assume $p = p'$, then

$$pB = \frac{Av^2 - CV^2}{g},$$

and therefore

$$\frac{2}{B} = \frac{v^2 - V^2}{Av^2 - CV^2} = \frac{1}{A} + \frac{1}{C};$$

so that B is the H.M. of A and C.

In particular, if C is infinite, $A = \frac{1}{2}B$; so that the inferior limit of the coefficient of contraction c_1 is 0·5.

By the addition of mouthpieces or *ajutages* of various shapes, c_1 can be increased, and even made greater than unity, a fact known to the Romans and prohibited by the law (*Calix devexus amplius rapit*, Frontinus).

In this case a partial vacuum, of head h', is formed in the throat of the ajutage, so that the effective head becomes $h + h'$; and the jet now acts as an *aspirator*, for creating a partial vacuum, or as a *lifting* injector.

372. The principle of momentum shows that if water is flowing with velocity v through a tube of cross section, the axis of the tube being in the form of a circle, the tension of the tube is thereby increased by DAv^2/g lb.

As this is independent of the curvature of the tube, it follows that the flow of the water has no tendency to move the tube; so that if the tube assumes a definite shape under external forces when the water is at rest, it will preserve the same form when the water is flowing with constant velocity v; but the tension is increased by DAv^2/g lb, or by the weight of water which will fill twice the length of tube which is equal to the head of the velocity.

For this reason the siphon tubes of figs. 61, 62 are liable to be lifted by the flow of the liquid, if the weight of the tube is small.

Similar results hold if the tube and water are replaced by a uniform chain (§ 191).

373. As another application of Bernoulli's theorem, consider plane motion symmetrical about a central axis Ox, between two parallel horizontal planes, a distance a ft apart (Rankine, *Applied Mechanics*, §§ 629–633).

I. Suppose the water to be flowing radially from the axis, so that the velocity is v f/s at a distance r ft; then the volume Q which crosses any circle of radius r is

$$Q = 2\pi a r v \text{ ft}^3/\text{sec};$$

and since Q is constant, therefore

$$v = \frac{Q}{2\pi a r} \propto \frac{1}{r}.$$

Then Bernoulli's equation gives

$$\frac{p}{D} = h - x - \frac{Q^2}{8\pi^2 a^2 r^2 g},$$

so that the surfaces of equal pressure are given by

$$r^2(h - x) = \text{a constant};$$

and these surfaces are thus generated by the revolution of Barlow's Curve (§ 289).

II. Turn the direction of motion at each point through a right angle; the surfaces of equal pressure are now unchanged, and we obtain what is called a *free circular vortex*, in which $\frac{1}{2}rv$, the area swept out by the radius of a particle in one second, is the same for all particles.

Since the circular lines of flow now lie on the surfaces of equal pressure, a free surface can exist; this state of motion is easily produced in a hemispherical basin when the plug at the bottom is removed.

Generally in any state of vortical motion about an axis, if two circular filaments of radii r and r', of velocities v and v', and of equal weight W, are made to change place, their new velocities will become u and u', given by

$$rv = r'u \quad \text{and} \quad r'v' = ru';$$

so that the work required is equal to the increase of kinetic energy,

$$\frac{1}{2}\frac{W}{g}(u^2 + u'^2 - v^2 - v'^2) = \frac{1}{2}\frac{W}{g}\left(\frac{1}{r^2} - \frac{1}{r'^2}\right)(r'^2 v'^2 - r^2 v^2);$$

and this is positive, so that the motion is stable, if

$$r'v' > rv \text{ when } r' > r.$$

Thus if v^2/r is constant, we obtain a stable vortex, in which the surfaces of equal pressure are cones.

III. By superposing the states of motion in I. and II. in given proportions we obtain Rankine's *free spiral* vortex (*Applied Mechanics*, § 631) in which the lines of flow are equiangular spirals; this is useful in the discussion of certain turbines and centrifugal pumps.

The surfaces of equal pressure remain the same as before; but in this case the lines of flow cross the surfaces of equal pressure, so that a steady free surface is not possible; this is observable when the basin is nearly emptied.

374. Whirlpools and cyclones are vortices of the nature of II. and III.; but the central part, where the velocity would be very great, soon assumes in consequence of viscosity a bodily rotation, as in § 344, called a *forced vortex;* thus a comparative calm is found in the centre of a cyclone, but the barometer is very low.

The combination of a free circular vortex with a central forced vortex is called a *compound vortex;* the pressure is due to the head from a point up to the free surface, which is formed by the revolution of the combination of a Barlow curve and of a parabola.

Thus the *centrifugal pump* consists of a wheel with curved blades, which draws water in at its axis, forms the water into a forced vortex, and delivers it into an external whirlpool chamber as a free spiral vortex, where its dynamic head enables the water to overcome a certain head and rise to the additional height.

375. *The Water Wheel and Paddle Wheel.*

If the jet of water impinges normally on a series of plane plates, following in regular procession with velocity u on the circumference of a wheel (the *Undershot* Water Wheel), then in t seconds $DAvt$ lb of water is reduced in velocity from v to u, and the change of momentum is $DAvt(v-u)/g$ sec-lb; so that the thrust of the jet is

$$DAv(v-u)/g \text{ lb},$$

and the wheel is working with horse-power

$$\text{H.P.} = DAvu(v-u)/550g.$$

The ratio of this H.P. to the H.P. of the jet is called the efficiency of the wheel, and denoted by e; so that

$$e = \frac{vu(v-u)}{\frac{1}{2}v^3} = \tfrac{1}{2} - 2\left(\frac{u}{v} - \frac{1}{2}\right)^2,$$

a maximum $\frac{1}{2}$ or 50 per cent., when $u = \frac{1}{2}v$.

376. By replacing the plane plates by cupped vanes, as in the Pelton wheel, the efficiency is greatly increased; and it is claimed, may reach 90 per cent.

The velocity of the water relatively to the vane is supposed constant and equal to $v-u$; so that if the cupped vane is hemispherical, the water leaves the lip of the cup with relative velocity $v-u$ backwards, and therefore with velocity

$$u-(v-u) \quad \text{or} \quad 2u-v$$

in the forward direction relatively to the ground.

The H.P. of the wheel is now doubled; so that the efficiency

$$e=1-4\left(\frac{u}{v}-\frac{1}{2}\right)^2,$$

a maximum 1, when $u=\frac{1}{2}v$.

The water now drops out of the wheel with no velocity, and there is theoretically no loss of energy; but practically a small amount of divergence must be given by the vanes to make the water run away clear of the wheel, so as not to be carried over.

For instance, under a head of 1100 ft of water a Pelton wheel should run with a peripheral speed of 183 f/s; and would give 3 H.P. for every *miner's inch*, of 1·6 ft³ per minute, with an efficiency of 90 per cent.

377. The Paddle Wheel may be assimilated to the undershot wheel with flat floats or plates, working faster than the jet; and now if the circumference of the wheel measured through the centre of the floats is p ft, and the wheel makes n revolutions per second, the water which was flowing past the sides of the ship with velocity v will be driven by the paddles with velocity np, v denoting the velocity of the ship through the water.

Then if the area of a pair of floats is A ft^2 (or of a single float in a stern wheel), every second a volume Anp ft^3 of water has its velocity changed by $np-v$; so that the thrust in pounds of the paddles is given by

$$T=DAnp(np-v)/g.$$

The *effective horse power* (E.H.P.) is

$$Tv\ /550=DAnpv(np-v)/550g,$$

while the *indicated horse power* (I.H.P.) of the engines is

$$Tnp/550=DAn^2p^2(np-v)/550g;$$

and therefore the efficiency e, the ratio of the E.H.P. to the I.H.P., is given by

$$e=v/np.$$

The steamer advances as if a toothed wheel of circumference ep engaged in a horizontal rack, so that the velocity of advance is v; the paddle wheel is thus always slipping a certain percentage, and not merely at starting, as in a locomotive engine.

The velocity $np-v$ is called the *slip velocity*, and the ratio $(np-v)/np$ is called the *slip ratio*, and denoted by s, while $100s$ is called the *slip percentage*; so that

$$s=1-e, \quad \text{or} \quad e+s=1;$$

378. *Oblique impact on a Sail. The Windmill and Screw Propeller.*

The previous method will serve to determine the thrust and power of wind on a sail moving obliquely, if we assume that the air behaves like an incompressible fluid, or like a dust cloud of inelastic particles.

Let α and β denote the angles between the normal of the sail and the directions of the wind and of the ship; let v denote the velocity of the ship and u of the wind, in f/s; and let the density of the air be δ lb/ft^3, and the area of the sail A ft^2.

Then, resolving perpendicular to the sail, a weight

$$\delta A(u \cos \alpha - v \cos \beta)t \text{ lb}$$

of air strikes the sail in t seconds, and is reduced in velocity in that direction by $u \cos \alpha - v \cos \beta$; the normal thrust R on the sail is given in pounds by

$$\delta A(u \cos \alpha - v \cos \beta)^2/g,$$

and the propulsive thrust on the ship is

$$T = \delta A \cos \beta(u \cos \alpha - v \cos \beta)^2/g;$$

so that the E.H.P. of the sail is

$$Tv/550 = \delta A v \cos \beta(u \cos \alpha - v \cos \beta)^2/550g.$$

Thus the propulsive H.P. of the sails of a ship, spreading 30,000 ft² of canvas, set at an angle $\beta = 60°$, and sailing S.W. at 10 knots in a trade wind blowing S.E. at 12 knots is about 322·4, taking $\delta = 0{\cdot}078$ lb/ft³.

379. In an ordinary windmill the sail moves at right angles to the wind, so that $\alpha + \beta = \frac{1}{2}\pi$; and now the

$$\text{E.H.P.} = \delta A v \cos \beta(u \sin \beta - v \cos \beta)^2/550g;$$

which is a maximum when $v = \frac{1}{3}u \tan \beta$, or one-third the speed at which the wheel would revolve without load.

Here A denotes the area of an element of the sail at a given distance r from the axis; so that $v = 2\pi rn$, if the wheel revolves n time a second; and it is assumed that the sails do not interfere with each other.

But when the sails nearly overlap, as in the Canadian windmill, A must be taken to represent an annular strip of mean radius r, intercepting $\delta A u$ lb/sec of air; so that now the H.P. of the element A is

$$\delta A uv \cos \beta(u \sin \beta - v \cos \beta)/550g,$$

a maximum when $v = \frac{1}{2}u \tan \beta$, or one-half the speed of the wheel when unloaded.

(Smeaton, *Experimental Enquiry on the Power of Water and Wind to turn Mills,* Phil. Trans., 1759.)

380. Suppose this wheel was enclosed in an annular pipe, of external and internal radii r_1, r_2, and areas A_1, A_2; and suppose the wheel was driven by water, as a *Pressure Turbine;* then if the axial flow u is supposed to remain unchanged, the water after passing through the wheel will be rotating bodily with $n-u/p$ revs/sec, or with angular velocity $\omega=2\pi(n-u/p)$ radians/sec.

The turning moment L of the wheel is equal to the angular momentum generated per second; and therefore,

$$L=Wk^2\omega/g \qquad \text{(ft-lb)},$$

$$=\tfrac{1}{2}\pi\frac{D}{g}(r_1^4-r_2^4)u\,.\,2\pi\left(n-\frac{u}{p}\right)$$

$$=\frac{D}{g}(A_1^2-A_2^2)u\left(n-\frac{u}{p}\right);$$

and the H.P. of the wheel is

$$\frac{2\pi nL}{550}=\frac{D}{550g}(A_1^2-A_2^2)2\pi nu\left(n-\frac{u}{p}\right);$$

a maximum when $n=\tfrac{1}{2}u/p$.

381. By supposing the water at rest and the wheel to advance with velocity u, we can thus obtain a preliminary idea of the action of the Screw Propeller; but now np, the speed of advance of the screw in a solid nut with n revs/sec must be greater than u, for the screw propeller to exert a forward propulsive thrust T; so that

$$L=\frac{D}{g}(A_1^2-A_2^2)u\left(\frac{u}{p}-n\right);$$

while, the reaction being normal on a smooth screw,

$$Tp=2\pi L.$$

Now the efficiency

$$e=\frac{\text{E.H.P.}}{\text{I.H.P.}}=\frac{Tu}{2\pi nL}=\frac{u}{np}=1-s,$$

as in the paddle wheel.

Of the work wasted by the propeller, one half is carried away by the energy of the wake, rotating as a forced vortex, and the other half is lost by the shock at the leading edge of the blades.

The first loss can be minimised by making the pitch p small and correspondingly increasing n the revolutions, provided fluid friction is left out of account; but the second loss can be suppressed by employing a screw of gaining pitch, increasing from u/n to any final value p; and the effective pitch will now be found to be the H.M. of u/n and p, while

$$e=1-\tfrac{1}{2}s.$$

382. A second screw on the same axis, of opposite pitch and revolutions, can be employed to utilise the energy of rotation of the wake of the first screw, as in the Whitehead torpedo; and now the efficiency is perfect.

Such an arrangement of propellers enclosed in a tube, and running in reverse order, could be employed as an axial flow Pressure Turbine, one screw being fixed to act as guide blades.

The Theory of the Turbine is perfect because the water is compelled by the guide blades to take the best course; the general principle to be followed is to make the water enter without shock and leave with little or no velocity.

But the screw propeller works in water already set in motion by the ship, and the water is free to follow its easiest course, so that the vortex formed may be anything intermediate to the *free* vortex and the *forced* vortex assumed above, while the axial flow may be accelerated; and the theory is correspondingly complicated (Rankine, *Trans. I. N. A.*, 1865).

383. In the *Impulse* Turbine jets of water are received without shock on appropriately curved blades, and guided by them so as to leave the wheel radially; and if Q ft^3/sec of water moving with velocity v enter the wheel at an angle β on a circumference of radius r, moving with velocity V, the H.P. of the wheel is

$$DQVv \cos \beta / 550g;$$

this follows from the mechanical principle that the rate of change of angular momentum about an axis is equal to the impressed couple.

384. *Fluid Friction in Pipes. The Hydraulic Gradient.*

So long as the water in a main is at rest, the hydrostatical theorems laid down in §§ 21–24 hold good; and if stand pipes are erected at different points, the water will rise to the level of the supply reservoir in all of them, while the pressure in the main will be that due to the head in the adjacent stand pipe.

Small stand pipes of this nature, inserted in a tube for measuring the pressure, are called *piezometers*.

But when the water is in motion, fluid friction absorbs the energy of the water at a uniform rate; so that the water levels in the piezometers will no longer lie in a horizontal line, but will slope downwards in the direction of motion at an incline called the *hydraulic gradient;* this gradient is observable in the strata of the earth and in the sands of the desert.

In contradistinction to Morin's Laws of Friction for Solids, Fluid Friction is found to be

(i.) independent of the pressure;
(ii.) proportional to the surface;
(iii.) proportional to the square of the sliding velocity.

If water is flowing bodily with velocity v f/s through a main of diameter d ft, the frictional drag on a length l ft is

$$\pi dlkv^2 \text{ pounds},$$

where k is a coefficient found by experiment; a good average value in iron pipes is 0·008.

Thus the frictional drag in a pipe line for the conveyance of petroleum, 30 miles long and 6 inches in diameter, with a delivery one ft³ per second would be about 26,000 pounds, requiring a pumping pressure of

$$920 \text{ lb/in}^2.$$

385. The loss of pressure on the hydrostatic pressure is

$$\frac{\pi dlkv^2}{\frac{1}{4}\pi d^2} = \frac{4l}{d}kv^2;$$

and the loss of head is

$$x = \frac{4lk}{Dd}v^2 = f\frac{4l}{d}\frac{v^2}{2g}, \text{ if } f = \frac{2gk}{D}.$$

Thus if $d/l = 4f$, all the energy due to the head $\frac{1}{2}v^2/g$ is wasted by frictional drag; for instance, if $f = 0·01$,

$$\frac{l}{d} = \frac{1}{4f} = 25.$$

But if the water enters under a head h ft, the H.P. given out at the end of the pipe is

$$\frac{\frac{1}{4}\pi d^2 D(h-x)v}{550} = \frac{\frac{1}{4}\pi d^2 D}{550}\sqrt{\left(\frac{gd}{2fl}\right)}(hx^{\frac{1}{2}} - x^{\frac{3}{2}});$$

a maximum when $x = \frac{1}{3}h$, and the efficiency is $\frac{2}{3}$.

The loss of head is proportional to the length, so that for a straight pipe the hydraulic gradient through the upper levels in the piezometers is a straight line, sloping at an angle θ, given for a level main by

$$\tan\theta = \frac{4f}{d}\frac{v^2}{2g}.$$

If the pipe is required to deliver Q ft³/sec, then

$$Q=\tfrac{1}{4}\pi d^2 v,$$

and $$\tan\theta=\frac{32fQ^2}{g\pi^2 d^5}, \quad \text{or} \quad d=\left(\frac{32fQ^2}{g\pi^2\tan\theta}\right)^{\frac{1}{5}},$$

giving the requisite diameter d for a given delivery Q and hydraulic gradient θ in water works; for instance, between a reservoir and a cistern at a lower level.

The slight deviations of a main from a straight level line do not sensibly affect the results of this formula, unless the pipe rises above the hydraulic gradient, when it acts in the manner of a *siphon* (§ 194).

386. *The Resistance of Ships.*

The resistance to the motion of a vessel through the water is initially zero, but the resistance mounts up as the velocity increases.

Taking the knot as a speed of 100 ft a minute, one H.P. is equivalent to 330 knot-pounds, or $33 \div 224$ knot-tons; so that if a steamer of W tons displacement is propelled at a speed of V knots, it experiences a resistance the same as that of a smooth gradient of one in $224WV \div 33$ H.P.; in the *Paris* and *New York*, for instance, if $W=10000$, $V=20$, and H.P. $=20000$, the gradient is about one in 68.

Although no theory is in existence which will enable us to predict with certainty the resistance at a given speed of a vessel of given design, still the experiments of Froude enable us to assign this resistance from the measured resistance of a model of the vessel run at a correspondingly reduced speed.

According to Froude's Law, "The resistance of *similar* vessels at speeds as the square root of the length, or as the sixth root of the displacement, is as the displacement or as the cube of the length."

Then if L, n^2L are the lengths, and D, n^6D the displacements of a vessel and its model, the resistances at speeds V, nV will be R, n^6R; and therefore the H.P.'s will be as 1 to n^7.

If the resistance is supposed to be due to skin friction, and this again is supposed to be proportional to the wetted surface, or as 1 to n^4, then the remaining factors of the resistance are as 1 to n^2, or proportional to the square of the velocity, as in fluid friction.

Thus, for instance, we may take $n=0{\cdot}1$; and if a vessel is designed to have a length L and a speed of V knots, a reduced model of 100th the length is run at one-tenth the speed, and the resistance r pounds is measured; then Froude's Law asserts that the full-sized vessel will experience a resistance 10^6r pounds at V knots, and the effective H.P. required will be $10^6rV/330$.

The coal required per hour is proportional to the H.P. or to n^7, but the coal per mile is as n^6 or as the displacement; so that over the same length of voyage the coal endurance is the same.

In popular language, to increase the speed one per cent. over a given voyage, we must increase the length two per cent. or the tonnage and coal capacity 6 per cent., and the horse power, boiler capacity, and daily consumption of fuel by 7 per cent.

Thus taking the *Paris* and *New York* as the model for a new design of a steamer to cross the Atlantic, 2800 miles, at a speed of 21 knots, then

$$n=1{\cdot}05, \quad n^6=1{\cdot}34, \quad n^7=1{\cdot}40;$$

so that the new steamer would have about 13,400 tons displacement and require 28,000 H.P.; this is approximately the case in the *Campania* and *Lucania*.

The voyage would take $2800 \div 21 = 133\frac{1}{3}$ hours; but if 5 hours is deducted for longitude difference on the westward voyage, when running before the sun, the apparent time is $128\frac{1}{3}$ hours, so that the apparent speed is raised to 21·82 knots.

Examples.

(1) A bucket of water in a balance discharges 4 lb of water per minute through an orifice in its base at 45° to the vertical, and is kept constantly full by a vertical stream which issues from an orifice 8 ft above the surface with velocity 30 f/s.

Prove that the bucket must be counterpoised by about 0·066 lb more than its weight.

(2) The bucket valve in fig. 80 (p. 362) has a small leak, one-800th of the cross section of the barrel, and the height of the water barometer is taken as 32 ft, the height AO as 16 ft, and the specific volume of the air 800 times that of water.

Prove that the pump will not suck unless the bucket is moved with a velocity greater than

$$\tfrac{4}{5}\sqrt{2} = 1\cdot13 \text{ f/s};$$

but that afterwards water will be lifted if the velocity is greater than 0·04 f/s.

(3) Prove that a hydraulic engine (fig. 78), in which water under pressure is admitted through small orifices to actuate the piston, will do most work when the speed is $\frac{1}{3}\sqrt{3}$ of the unloaded speed, and the load is $\frac{2}{3}$ of the maximum load, and that the efficiency is then $\frac{2}{3}$.

(4) Discuss the influence of inertia and of fluid friction in the pipe, when the Hydraulic Press (§ 12) is actuated by the Accumulator (§ 15). (Cotterill, *Applied Mechanics*, § 256.)

(5) Prove that the H.P. of the feed pump of a boiler, which evaporates W lb/min of water at a gauge pressure p lb/in², must exceed

$$144\, Wp \div 33000.$$

(6) Prove that if the jet of § 371 delivers Q ft³/sec, and the hose is l ft long, the pumping H.P. of the fire engine is

$$\frac{DQ^3}{1100\, g}\left(\frac{1}{A^2}-\frac{1}{C'^2}+\frac{2fl\sqrt{\pi}}{C'^{\frac{5}{2}}}\right).$$

(7) If water is scooped up from a trough between the rails into a locomotive tender to a height of h ft, determine the minimum velocity required, and the delivery at a given extra speed, taking the frictional losses as represented by a given fraction of the head.

(8) Show how liquid may be raised through a siphon tube, made to revolve about its longer branch, which is held vertical; and determine the delivery and the mechanical efficiency for a given angular velocity.

(9) Show how to determine the elements of a cyclone from observations at three points.

What is the direction of rotation in the N. and S. hemispheres?

CHAPTER XII.

GENERAL EQUATIONS OF EQUILIBRIUM.

387. It was proved in Chapter I., §§ 19, 20, that the surfaces of equal pressure and the free surface of a liquid at rest under gravity are horizontal planes; but this assumes that gravity acts in parallel vertical lines.

When we examine more closely the surface of a large sheet of water like the open sea, we find it uniformly curved, so that the surface is spherical; showing that the lines of force of gravity converge to the centre of the Earth; and Archimedes in his diagrams of floating bodies represents them immersed in a spherical ocean.

If three posts are set up, a mile apart in a straight canal, to the same vertical height out of the water, the visual line joining the two extreme posts will, in the absence of curvature by refraction, cut the middle post 8 ins lower; hence it is inferred that the diameter of the Earth in miles is the number of 8 ins in 1 mile, or 7920.

If l miles apart, the visual line cuts at a depth $8l^2$ ins; for instance, the Channel tunnel 20 miles long, if made *level*, would rise in the middle 800 ins from the straight chord; but if made *straight*, it would have a gradient of about one in 400 at the ends, and water reaching to the ends would have a head of 800 inches in the middle.

388. Careful measurements of a degree of the meridian in different latitudes reveal that the mean level surface of the Ocean is not exactly spherical, but slightly ellipsoidal and bulging at the Equator; an effect attributable to the Earth's rotation, and investigated in the theory of the Figure of the Earth.

Lastly, the imperceptible deflections of the plumb line, due to the perturbative attraction of the Moon and Sun, are rendered very manifest by the phenomena of the Tides, due to the same cause of perturbation.

389. All these manifestations are examples of the general principle, enunciated in § 24 as

"Liquids tend to maintain their Level,"

but now the level surface must be taken to mean the surface which is everywhere perpendicular to the resultant force of gravity at the point, as indicated by the plumb line.

To prove the theorem that

"The surfaces of equal pressure in a fluid at rest under given forces are at every point perpendicular to the line of resultant force";

draw two consecutive surfaces of equal pressure PP', QQ', on which the pressures are p and $p+\Delta p$ suppose; and consider the equilibrium of a cylindrical element of cross section a, the axis PR of which is normal to the surfaces of equal pressure.

The resultant thrust on the curved side of the cylinder of the surrounding liquid, of uniform pressure in planes perpendicular to the axis, being zero, the resultant thrust on the ends must be balanced by the resultant impressed force, which must therefore act along the normal PR to the surface of equal pressure.

Denoting this force per unit volume by F, and the element PR of the normal by $\Delta\nu$,

$$Fa\,.\,\Delta\nu = a\Delta p,$$

$$F = \text{lt}\,\frac{\Delta p}{\Delta\nu} = \frac{dp}{d\nu}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or the resultant force per unit volume is the space variation, or gradient, of the pressure p in its direction.

This is true also for any other direction PQ, making an angle θ with the normal PR; for if it meets the consecutive surface QQ' in Q, and $PQ = \Delta s$,

$$\cos\theta = \text{lt}\,\frac{PR}{PQ} = \frac{d\nu}{ds},$$

and
$$F\cos\theta = \frac{dp}{d\nu}\,\frac{d\nu}{ds} = \frac{dp}{ds}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

so that "the component force in any direction is the space variation of the pressure in that direction; and the resultant force is the greatest space variation, and therefore normal to the surface of equal pressure."

Thus if the force F has components X, Y, Z parallel to three fixed rectangular axes Ox, Oy, Oz,

$$\frac{dp}{dx} = X, \quad \frac{dp}{dy} = Y, \quad \frac{dp}{dz} = Z;$$

or in the notation of *differentials*,

$$dp = Xdx + Ydy + Zdz; \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and the lines of force must be capable of being cut orthogonally by a system of surfaces, of equal pressure.

390. The impressed forces of gravity and inertia (but not of electricity or magnetism) are proportional at any point to the density ρ; so that it is usual to multiply F by ρ, and thus measure F per unit of *mass*, lb or g, instead of per unit volume, ft^3 or cm^3; and now we write

$$dp = \rho(Xdx + Ydy + Zdz). \quad\ldots\ldots\ldots\ldots\ldots(4)$$

This equation may be deduced from the equilibrium of the fluid filling a fixed closed surface S; denoting by l, m, n the direction cosines of the outward drawn normal at a point of the surface, and resolving parallel to Ox,

$$\iint lp\,dS = \iiint \rho X\,dx\,dy\,dz,$$

the integrations extending over the surface and throughout the volume of S.

But, by Green's transformation,

$$\iint lp\,dS = \iiint \frac{dp}{dx}\,dx\,dy\,dz;$$

and therefore $\dfrac{dp}{dx} = \rho X, \quad \dfrac{dp}{dy} = \rho Y, \quad \dfrac{dp}{dz} = \rho Z.$ ....(5)

Also taking moments round the axes,

$$\iint p(ny - mz)\,dS = \iiint \rho(yZ - zY)\,dx\,dy\,dz;$$

and these equations are now satisfied identically.

391. From equations (4) and (5),

$$\frac{d^2p}{dy\,dz} = \frac{d\rho}{dz}Y + \rho\frac{dY}{dz} = \frac{d\rho}{dy}Z + \rho\frac{dZ}{dy},$$

$$\rho\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) = Y\frac{d\rho}{dz} - Z\frac{d\rho}{dy},$$

with two similar equations; so that, eliminating ρ,

$$X\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + Y\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + Z\left(\frac{dY}{dx} - \frac{dX}{dy}\right) = 0, \quad (6)$$

this is the analytical expression of the fact that the lines of force are perpendicular to a system of surfaces.

The pressure p, density ρ, and temperature τ of a fluid are connected by a characteristic equation (§ 196); and if the temperature is variable, the surfaces of equal pressure and of equal density must intersect on the surfaces of equal temperature.

392. If the work done by the forces X, Y, Z on a particle carried round a closed curve in the fluid, namely,

$$\int(Xdx+Ydy+Zdz),$$

vanishes when the particle has completed a circuit, then

$$Xdx+Ydy+Zdz$$

must be the total differential of some function $-V$, called the *potential;* and now

$$\frac{1}{\rho}\frac{dp}{dx}=-\frac{dV}{dx},\quad \frac{1}{\rho}\frac{dp}{dy}=-\frac{dV}{dy},\quad \frac{1}{\rho}\frac{dp}{dz}=-\frac{dV}{dz};$$

so that dp/ρ must be the differential of some other function P, and, as in the case with physical substances at uniform temperature, ρ must be a function of p only, given by the relation $\rho=dp/dP$; and now

$$dP+dV=0,$$

or

$$P+V=H, \text{ a constant}, \quad \text{..............(7)}$$

is the condition of equilibrium.

Thus if $\quad X=y+z, \quad Y=z+x, \quad Z=x+y,$

then

$$V=-yz-zx-xy$$
$$=\tfrac{1}{2}(x^2+y^2+z^2)-\tfrac{1}{2}(x+y+z)^2,$$

and the surfaces of equal pressure and density are hyperboloids of revolution.

But if $\quad X=y(a-z), \quad Y=x(a-z), \quad Z=xy,$

equation (4) becomes integrable if ρ is treated as an integrating factor, made proportional to $(a-z)^{-2}$, so that surfaces of equal density are parallel planes; or more generally, by putting

$$1/\rho=(a-z)^2f'(p).$$

Now, by integration,

$$f(p)=\frac{xy}{a-z},$$

so that the surfaces of equal pressure are hyperbolic paraboloids (fig. 65, p. 283).

But if the forces can do work in a closed circuit, currents will be set up; this generally happens in consequence of inequalities of temperature, illustrations of which may be seen in the Trade Winds and the Gulf Stream, which can be imitated by water in a tank, heated by a lamp at one end, and cooled by ice at the other end.

393. We can now utilise the methods of the Calculus, and obtain without difficulty the solution of more complicated problems, of the nature of those in §§ 223–232.

Thus if, as in § 230, the density ρ at a depth x in liquid at rest under gravity is μz^n, and if we use the gravitation unit of force,

$$dp=\mu x^n dx, \quad p=P+\frac{\mu x^{n+1}}{n+1}=P+\frac{\rho x}{n+1}.$$

394. Suppose the variations of gravity are taken into account in the equilibrium of the atmosphere; we must now employ the absolute unit of force, the poundal or dyne, and measure pressure in poundals/ft^2, or in dynes/cm^2 (*barads*); so that the former gravitation measure of p must be multiplied by g, and

$$p=gk\rho,$$

gk being constant over the Earth at the same temperature (§ 199).

If a denotes the radius of the Earth, then at a height z from the surface, or at a distance r from the centre, g becomes changed to

$$\frac{ga^2}{(a+z)^2}, \quad \text{or} \quad \frac{ga^2}{r^2};$$

so that equation (4) becomes

$$\frac{dp}{\rho dz}=-\frac{ga^2}{(a+z)^2}, \quad \text{or} \quad \frac{dp}{\rho dr}=-\frac{ga^2}{r^2}.$$

395. When the atmosphere is in isothermal equilibrium (§ 223), $p=gk\rho$, where k is constant, and

$$\frac{kdp}{pdz}=-\frac{a^2}{(a+z)^2},$$

$$\log\frac{p}{p_0}=\frac{a^2}{k}\left(\frac{1}{a+z}-\frac{1}{a}\right)=-\frac{az}{k(a+z)}.$$

At this altitude z the barometer will stand at a height h, given by

$$p=\int_0^h\frac{g\sigma a^2dh}{(a+z+h)^2}=\frac{g\sigma a^2h}{(a+z)(a+z+h)}.$$

Differentiating logarithmically,

$$\left(\frac{1}{h}-\frac{1}{a+z+h}\right)\frac{dh}{dz}-\frac{1}{a+z}-\frac{1}{a+z+h}$$

$$=\frac{dp}{pdz}=-\frac{a^2}{k(a+z)^2},$$

and therefore h is a minimum when, approximately,

$$r=a+z=\tfrac{1}{2}a^2/k,$$

which is about 400 times the Earth's radius, taking

$$a=10^7\div\tfrac{1}{2}\pi\text{ m},\quad\text{and}\quad k=8000\text{ m};$$

or taking $a=4000$ miles, and $k=5$ miles.

Putting $z=\infty$,

$$p/p_0=\rho/\rho_0=e^{-a/k}\approx e^{-800}=10^{-348};$$

so that a kg of air occupying at ordinary atmospheric pressure a volume 0·773 m³, would now occupy 10^{348} times this volume.

396. In convective equilibrium (§ 226) equation (1) becomes

$$\frac{d\theta}{\theta_0dz}=-\frac{\gamma-1}{\gamma k}\frac{a^2}{(a+z)^2},$$

$$\theta=\theta_0\left(1-\frac{\gamma-1}{\gamma k}\frac{az}{a+z}\right).$$

With absolute temperature θ inversely as r,

$$\theta = \theta_0(a/r);$$

and equation (1) becomes

$$\frac{dp}{pdr} = -\frac{a}{kr}, \quad \log\frac{p}{p_0} = -\frac{a}{k}\log\frac{r}{a}, \quad \text{or} \quad \frac{p}{p_0} = \left(\frac{a}{r}\right)^{\frac{a}{k}}.$$

Similarly we shall find, if

$$\theta = \theta_0(a/r)^n,$$

$$\log\frac{p}{p_0} = -\frac{a}{k}\,\frac{(r/a)^{n-1} - 1}{n-1}.$$

If the Centigrade temperature τ varies inversely as r^2, we shall find

$$\log\frac{p}{p_0} = -m\left(\tan^{-1}\frac{r}{c} - \tan^{-1}\frac{a}{c}\right),$$

where m and c are constants.

397. If the rotation of the Earth is taken into account, the atmosphere being supposed to rotate bodily with angular velocity ω, equation (1) must be written

$$\frac{dp}{\rho} = -g\left(\frac{a}{r}\right)^2 dr + \omega^2 y dy,$$

where y denotes the distance from the polar axis.

In isothermal equilibrium this becomes

$$\frac{kdp}{p} = -\left(\frac{a}{r}\right)^2 dr + \frac{\omega^2}{g} y dy,$$

$$\log\frac{p}{p_0} = -\frac{a}{k}\left(1 - \frac{a}{r}\right) + \frac{\omega^2 y^2}{2gk},$$

if p_0 denotes the atmospheric pressure on the ground at the pole.

Thus at the equator, where $y = a$,

$$\log_e\frac{p}{p_0} = \frac{a^2\omega^2}{2gk} = 1{\cdot}385, \quad \frac{p}{p_0} = 4,$$

on putting $\quad g/a\omega^2 = 289, \qquad a/k = 800.$

This proves that the Earth cannot be spherical; for here the surface of equal pressure p_0 is given by

$$1-\frac{a}{r}=\frac{\omega^2y^2}{2ga};$$

so that, if the equatorial radius of this surface is b,

$$\frac{a}{b}=1-\frac{\omega^2b^2}{2ga}\approx 1-\frac{\omega^2a}{2g}=\frac{577}{578}.$$

In convective equilibrium

$$\frac{\gamma k}{\gamma-1}\frac{d\theta}{\theta_0}=-\left(\frac{a}{r}\right)^2dr+\frac{\omega^2}{g}ydy,$$

$$\frac{\gamma}{\gamma-1}\left(1-\frac{\theta}{\theta_0}\right)=\frac{a}{k}\left(1-\frac{a}{r}\right)-\frac{\omega^2y^2}{2gk}.$$

398. Equation (4) is employed to determine the pressure in the interior of a spherical liquid mass due to its own gravitation, for instance in the Earth when in a molten condition.

We assume the well-known theorems that the attractions of a homogeneous spherical shell (i.) is zero in the interior cavity, and (ii.) is the same as if the matter is condensed at the centre for an exterior point.

With C.G.S. units, denoting by γ the *constant of gravitation*, that is the attraction in dynes between two homogeneous spheres each weighing one g when their centres are one cm apart, and first supposing the liquid of uniform density ρ g/cm³, then (1) becomes

$$\frac{dp}{dr}=-\tfrac{4}{3}\pi\gamma\rho^2r,$$

since the attraction at a distance r from the centre in the interior of the liquid sphere is the same as that of the mass $\frac{4}{3}\pi\rho r^3$ condensed at the centre, the attraction of the shell exterior to the radius r being zero.

Denoting by V the *gravitation potential* of the liquid, the *rate of increase* of which in any direction is the force per unit mass in that direction, then in the interior of a sphere of radius a,

$$\frac{dV}{dr} = -\tfrac{4}{3}\pi\gamma\rho r, \quad V = 2\pi\gamma\rho(a^2 - \tfrac{1}{3}r^2),$$

if a constant is added, so that there is no abrupt change in V in passing into exterior space, where

$$V = \tfrac{4}{3}\pi\gamma\rho a^3/r,$$

the same as for the whole mass collected at the centre of the sphere.

Supposing the pressure zero at the exterior radius a, where $$V = V_0 = \tfrac{4}{3}\pi\gamma\rho a^2,$$
then $$p = \rho(V - V_0) = \tfrac{2}{3}\pi\gamma\rho^2(a^2 - r^2).$$

Hence the thrust across a diametral plane, or the attraction between the two hemispheres, is

$$\int_0^a p\,.\,2\pi r dr = \tfrac{1}{3}\pi^2\gamma\rho^2 a^4 = \tfrac{3}{4}\gamma M^2/a^2,$$

if M g denotes the mass of the liquid sphere.

399. The gravitation constant γ is determined by the Cavendish experiment; it is found that, in C.G.S. units,

$$\gamma = 10^{-8} \times 6{\cdot}69.$$

Now if a denotes the radius ($10^9 \div \tfrac{1}{2}\pi$ cm), ρ g/cm³ the mean density of the Earth, and g *sponds* (cm/sec²) the mean acceleration of gravity on the surface,

$$g = \gamma M/a^2 = \tfrac{4}{3}\pi\gamma\rho a = \tfrac{8}{3}\gamma\rho \times 10^9;$$

so that, with $g = 981$ sponds,

$$\gamma\rho = 10^{-7} \times 3{\cdot}68, \quad \rho = 5{\cdot}5.$$

Thus at the centre of the Earth, if of uniform density ρ, the pressure

$$p = \tfrac{2}{3}\pi\gamma\rho^2 a^2 = \tfrac{1}{2}g\rho a.$$

This is the pressure due to a head $\frac{1}{2}a$, with uniform surface gravity g; also

$$p = 10^9 \times g\rho/\pi = 10^{12} \times 1{\cdot}718 \text{ *barads* (dynes/cm}^2)$$

$= 1{,}718{,}000$ atmospheres, or *megabarads*, of a million barads.

400. Considering that the surface density of the Earth is about 2 only, the density cannot be uniform, but must increase towards the centre.

Equation (1) of § 389 now becomes

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{\gamma}{r^2}\int_0^r 4\pi\rho r^2 dr,$$

so that ρ must be some known function of r, or else the relation connecting p and ρ must be known, for this equation to be integrated.

Thus, if $\rho \propto r^{-2}$, $p - p' \propto \rho - \rho'$.

Laplace assumed that the density ρ was proportional to the square root of the pressure p (or of $p + p_0$), equivalent to assuming that the *cubical elasticity* (§ 422) is double the pressure, or $\quad \rho dp/d\rho = 2p$;

and now, putting $\quad p + p_0 = 2\pi\gamma\rho^2\kappa^2$,

the student may prove, as an exercise, that, at any radius r,

$$\rho = \sigma\frac{\kappa}{r}\sin\frac{r}{\kappa},$$

where σ denotes the density at the centre.

The mean density of the Earth being denoted by $\bar{\rho}$, and its mass in g by E,

$$E = \tfrac{4}{3}\pi\bar{\rho}a^3 = \int_0^a 4\pi\rho r^2 dr$$

$$= 4\pi\sigma\kappa\int_0^a r\sin\frac{r}{\kappa}dr = 4\pi\sigma\kappa^3\left(\sin\frac{a}{\kappa} - \frac{a}{\kappa}\cos\frac{a}{\kappa}\right).$$

Denoting the surface density by ρ_0, then

$$\rho_0 = \sigma\frac{\kappa}{a}\sin\frac{a}{\kappa};$$

so that
$$\frac{\bar\rho}{\rho_0} = 3\left(\frac{\kappa}{a}\right)^2\left(1 - \frac{a}{\kappa}\cot\frac{a}{\kappa}\right),$$
whence κ is determined: for instance, by putting

$$\bar\rho = 5{\cdot}5, \qquad \rho_0 = 2.$$

With $p_0 \doteq 0$, and surface density $\rho_0 = 0$, we must put $a/\kappa = \pi$; and now we find

$$\sigma = \tfrac{1}{3}\pi^2\bar\rho.$$

401. If homogeneous liquid fills a spherical case of radius a, and if a spherical cavity of radius b cm is made anywhere in the interior of the liquid, with its centre B in Ox at a distance c from O the centre of the liquid sphere, the cavity will collapse unless kept distended by a rigid spherical surface; and now the potential at a point P in the interior of the cavity is

$$2\pi\gamma\rho(a^2 - \tfrac{1}{3}OP^2) - 2\pi\gamma\rho(b^2 - \tfrac{1}{3}BP^2) = \text{constant} - \tfrac{4}{3}\pi\gamma\rho c x;$$

so that the field of force in the cavity is of uniform intensity $\frac{4}{3}\pi\gamma\rho c$, parallel to xO.

If the cavity is filled by a solid spherical nucleus of density σ, the attraction on this sphere is

$$\tfrac{4}{3}\pi\gamma\rho c \cdot \tfrac{4}{3}\pi\sigma b^3 = \tfrac{16}{9}\gamma\rho\sigma b^3 c \text{ dynes},$$

in the direction BO.

The potential in the interior of the liquid is now due to a complete sphere of centre O, radius a, and density ρ, and to a sphere of centre B, radius b, and density $\sigma - \rho$; and therefore in the liquid

$$\frac{p}{\rho} = \tfrac{2}{3}\pi\gamma\rho(a^2 - r^2) + \tfrac{4}{3}\pi\gamma(\sigma - \rho)b^3\left(\frac{1}{BP} - \frac{1}{BA}\right),$$

the pressure vanishing at the end A of the diameter Ox remote from B.

402. The hydrostatic thrust on the nucleus, over which BP is constant, is in the direction OB; and integrating over the surface and throughout the volume of the nucleus, this thrust in dynes is given by

$$\iint -lpdS = \iiint -\frac{dp}{dx}dxdydz$$
$$= \iiint \tfrac{4}{3}\pi\gamma\rho^2 x dxdydz = \tfrac{4}{3}\pi\gamma\rho^2 . \tfrac{4}{3}\pi b^3 c.$$

The resultant of gravitation and hydrostatic thrust is therefore

$$\tfrac{16}{9}\pi^2\gamma\rho(\sigma-\rho)b^3 c \text{ dynes}$$

in the direction BO; so that, if $\sigma > \rho$, the nucleus will be in stable equilibrium at the centre of the sphere, with B and O in coincidence; but, if $\sigma < \rho$, the nucleus will tend to float up to the surface; so also the Earth would float up to one side out of the Ocean, if its mean density were less than the density of the water of the Ocean.

(*Principia*, lib. iii., prop. x.)

403. The same result can be proved by calculating the variable part of the potential energy of the liquid and the solid nucleus; this is the same as that of a complete liquid sphere of density ρ and of a solid nucleus of density $\sigma - \rho$, and is therefore

$$\text{a constant} - \tfrac{8}{9}\pi^2\gamma\rho(\sigma-\rho)b^3c^2 \text{ ergs};$$

and the variable part is the work required to displace the nucleus to a distance c from the centre O.

The same method holds when the external vessel and the nucleus are of any shape; and now the pressure in the liquid is $\gamma\rho(V-V_0)$, where V_0 is the minimum value of the potential V on the equipotential surface which touches the vessel; and the nucleus is in equilibrium when the potential energy of the system is a maximum.

404. The potential at a point P external to the case is

$$V = \tfrac{4}{3}\pi\gamma\rho a^3 \frac{1}{OP} + \tfrac{4}{3}\pi\gamma(\sigma-\rho)b^3 \frac{1}{BP};$$

and drawing the equipotential surfaces in the neighbourhood of the surface, these will represent the surfaces of equal pressure and the free surface of an Ocean composed of a small quantity of liquid poured over the surface.

Suppose, for instance, that the Ocean just covers the surface; it will be shallowest near A, and its greatest depth h at the point A' nearest to B will be given by

$$\tfrac{4}{3}\pi\gamma\frac{\rho a^3}{a+h} + \tfrac{4}{3}\pi\gamma\frac{(\sigma-\rho)b^3}{a-b+h} = \tfrac{4}{3}\pi\gamma\rho a^2 + \tfrac{4}{3}\pi\gamma\frac{(\sigma-\rho)b^3}{a+b},$$

or

$$\frac{\rho a^2 h}{a+h} = \frac{(\sigma-\rho)b^3(2b-h)}{(a+b)(a-b+h)}.$$

405. If the nucleus B is liquefied, but kept spherical in shape by a fixed rigid case, and if a solid sphere of radius b' and density σ' is placed in this interior liquid with its centre B' in OB at a distance x from B, the potential energy of the system is

$$\text{a constant} - \tfrac{16}{9}\pi^2\gamma\rho\sigma' b'^3 cx - \tfrac{8}{9}\pi^2\gamma\sigma\sigma' b'^3 x^2 \text{ ergs};$$

and this is a maximum when

$$\rho c + \sigma x = 0, \quad x = -c\rho/\sigma,$$

provided however that $x < b - d$, so that there is no contact between the surfaces.

If this solid sphere is placed anywhere in the liquid ρ, with its centre B' at a distance c' from O, the potential energy is the same as that of a complete sphere of density ρ and of two spheres of densities $\sigma-\rho$ and $\sigma'-\rho$; and therefore, denoting BB' by d, its variable part is in ergs

$$\tfrac{16}{9}\pi^2\gamma(\sigma-\rho)(\sigma'-\rho)\frac{b^3 b'^3}{d} - \tfrac{8}{9}\pi^2\gamma\rho(\sigma-\rho)b^3c^2 - \tfrac{8}{9}\pi^2\gamma\rho(\sigma'-\rho)b'^3c'^2.$$

The nucleus B consequently experiences an attraction

$$\tfrac{16}{9}\pi^2\gamma\rho(\sigma-\rho)b^3c \text{ dynes to the centre } O,$$

and $$\tfrac{16}{9}\pi^2\gamma(\sigma-\rho)(\sigma'-\rho)b^3b'^3/d^2 \text{ dynes to } B'.$$

Thus the two spheres B and B' will repel if the density ρ of the medium is intermediate to their densities.

If the two spheres are in contact and in equilibrium, and P dynes is the thrust between them,

$$P+\tfrac{16}{9}\pi^2\gamma\rho(\sigma-\rho)b^3c=P-\tfrac{16}{9}\pi^2\gamma\rho(\sigma'-\rho)b'^3c'$$

$$=\tfrac{16}{9}\pi^2\gamma(\sigma-\rho)(\sigma'-\rho)\frac{b^3b'^3}{(b+b')^2};$$

where $$b+b=c'-c;$$

so that $$(\sigma-\rho)b^3c+(\sigma'-\rho)b'^3c'=0;$$

or the C.G. of the system is at O; and thence

$$P=\tfrac{16}{9}\pi^2\gamma(\sigma-\rho)(\sigma'-\rho)b^3b'^3\left\{\frac{1}{(b+b')^2}+\frac{\rho(b+b')}{(\sigma-\rho)b^3+(\sigma'-\rho)b'^3}\right\}.$$

Similarly for any number of spheres in the liquid.

406. If gravitating liquid of density ρ, filling a spherical case of radius a, is set rotating bodily with angular velocity ω about a diameter Ox, then equation (1) becomes

$$dp=-\tfrac{4}{3}\pi\gamma\rho^2rdr+\rho\omega^2(ydy+zdz),$$

$$p=\tfrac{2}{3}\pi\gamma\rho^2(a^2-r^2)+\tfrac{1}{2}\rho\omega^2(y^2+z^2)$$

$$=\tfrac{2}{3}\pi\gamma\rho^2(a^2-x^2)-(\tfrac{2}{3}\pi\gamma\rho^2a^2-\tfrac{1}{2}\rho\omega^2)(y^2+z^2),$$

the pressure being zero at the poles of the axis of revolution; and now the surfaces of equal pressure are similar quadric surfaces of revolution, ellipsoids or hyperboloids as $\omega^2 \lesseqgtr \tfrac{4}{3}\pi\gamma\rho$, and parallel planes if $\omega^2=\tfrac{4}{3}\pi\gamma\rho$.

Over the spherical surface,

$$p=\tfrac{1}{2}\rho\omega^2(y^2+z^2),$$

so that the force tending to split the shell across a meridian circle is

$$\iint \tfrac{1}{2}\rho\omega^2(y^2+z^2)mdS=\iiint \rho\omega^2ydxdydz=\tfrac{1}{4}\pi\rho\omega^2a^4.$$

407. If the case is removed, the rotating liquid will lose the spherical form; and as the angular velocity ω is gradually increased from zero, the free surface of the liquid when rotating bodily will first assume the form of an oblate spheroid (Maclaurin's), and afterwards of an ellipsoid (Jacobi's).

We assume that the potential V of the homogeneous ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

of density ρ, is given by (Minchin, *Statics*, II., 308),

$$V=\int_0^\infty\left(1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}-\frac{z^2}{c^2+\lambda}\right)\frac{\pi\gamma\rho abc d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}}$$

$$=P-\tfrac{1}{2}Ax^2-\tfrac{1}{2}By^2-\tfrac{1}{2}Cz^2$$

or $$2\pi\gamma\rho abc(P'-\tfrac{1}{2}A'x^2-\tfrac{1}{2}B'y^2-\tfrac{1}{2}C'z^2)$$

suppose, where

$$P'=\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}},\quad \text{and}\quad \tfrac{1}{2}A'=-4\frac{dP'}{da^2},\ \ldots;$$

and then $$A'+B'+C'=\frac{2}{abc},\quad A+B+C=4\pi\gamma\rho.$$

Then in the rotating liquid

$$\frac{p}{\rho}=V+\tfrac{1}{2}\omega^2(y^2+z^2)+\text{a constant}$$

$$=\text{a constant}-\tfrac{1}{2}Ax^2-\tfrac{1}{2}(B-\omega^2)y^2-\tfrac{1}{2}(C-\omega^2)z^2;$$

so that the surfaces of equal pressure are similar to the exterior surface, if

$$Aa^2=(B-\omega^2)b^2=(C-\omega^2)c^2$$

or $$\frac{\omega^2}{\pi\gamma\rho abc}=\frac{B'b^2-A'a^2}{b^2}=\frac{C'c^2-A'a^2}{c^2};$$

$$(b^2-c^2)\int_0^\infty\left(\frac{b^2c^2}{b^2+\lambda\,.\,c^2+\lambda}-\frac{a^2}{a^2+\lambda}\right)\frac{d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}}=0.\quad\text{(A)}$$

408. One solution is obviously $b^2=c^2$ (Maclaurin's spheroid); and then

$$P'=\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)}=\frac{2}{\sqrt{(c^2-a^2)}}\cos^{-1}\frac{a}{c},$$

$$\tfrac{1}{2}A'=-\frac{dP'}{ada}=\quad\frac{2}{a(c^2-a^2)}-\frac{2}{(c^2-a^2)^{\frac{3}{2}}}\cos^{-1}\frac{a}{c},$$

$$\tfrac{1}{2}C'=-\frac{dP'}{cdc}=-\frac{a}{(c^2-a^2)c^2}+\frac{1}{(c^2-a^2)^{\frac{3}{2}}}\cos^{-1}\frac{a}{c},$$

$$\tfrac{1}{2}A'+C'=\frac{2}{ac^2};$$

$$\frac{\omega^2}{2\pi\gamma\rho}=\tfrac{1}{2}C'ac^2-\tfrac{1}{2}A'a^3$$

$$=\frac{2a^2+c^2}{(c^2-a^2)^{\frac{3}{2}}}\cos^{-1}\frac{a}{c}-\frac{3a^2}{c^2-a^2}=\frac{3+f^2}{f^3}\tan^{-1}f-\frac{3}{f^2},$$

on putting $c^2/a^2=1+f^2$.

409. The *ellipticity* (§ 358) of the spheroid is given by

$$\epsilon=\frac{c}{a}-1=\sqrt{(1+f^2)}-1\approx\tfrac{1}{2}f^2;$$

and then expanding in powers of f,

$$\frac{\omega^2}{2\pi\gamma\rho}=\tfrac{4}{15}f^2-\tfrac{8}{35}f^4\ldots\approx\tfrac{8}{15}\epsilon.$$

Thus if we assume that the Earth is homogeneous, and of mean radius R, then

$$\tfrac{4}{3}\pi\gamma\rho R=g,$$

and
$$\frac{1}{\epsilon}=\frac{16\pi\gamma\rho}{15\omega^2}=\frac{4g}{5R\omega^2}=\frac{4}{5}\times 289=231\cdot 2,$$

agreeing very nearly with Newton's estimate of 230.

(*Principia*, lib. iii., prop. xix.)

But as the true ellipticity of the Earth is about 1/300, the density must be greater in the interior; for if we assume that there is a concentric centrobaric nucleus of density $\mu\rho$ and radius k, the additional potential at the pole and the equator is

$$\tfrac{4}{3}\pi\gamma\rho(\mu-1)\frac{k^3}{a} \quad \text{and} \quad \tfrac{4}{3}\pi\gamma\rho(\mu-1)\frac{k^3}{c};$$

so that equation (A) of surface equilibrium becomes

$$\tfrac{4}{3}\pi\gamma\rho(\mu-1)\frac{k^3}{a}-\tfrac{1}{2}Aa^2=\tfrac{4}{3}\pi\gamma\rho(\mu-1)\frac{k^3}{c}-\tfrac{1}{2}Cc^2+\tfrac{1}{2}\omega^2c^2,$$

or
$$\frac{\omega^2}{2\pi\gamma\rho}=\tfrac{1}{2}C'ac^2-\tfrac{1}{2}A'a^3+\tfrac{4}{3}(\mu-1)\frac{k^3}{c^2}\left(\frac{1}{a}-\frac{1}{c}\right)$$
$$\approx \tfrac{8}{15}\epsilon+\tfrac{4}{3}(\mu-1)\frac{k^3}{R^3}\epsilon;$$

and now
$$g=\tfrac{4}{3}\pi\gamma\rho R+\tfrac{4}{3}\pi\gamma\rho(\mu-1)\frac{k^3}{R^2},$$

so that
$$\frac{1}{\epsilon}=\frac{4g}{5R\omega^2}\cdot\frac{1+\tfrac{5}{2}(\mu-1)(k/R)^3}{1+\ (\mu-1)(k/R)^3}.$$

410. The integral which is the remaining factor of (A), § 407, gives the relation connecting b/a and c/a for a Jacobian ellipsoid.

Putting $b=c$ in this integral gives

$$\int_0^\infty \frac{c^4d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^3}-\int\frac{a^2d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(c^2+\lambda)}=0,$$

or
$$\tfrac{1}{2}c^4\frac{d^2P'}{(dc^2)^2}+a\frac{dP'}{da}=0,$$

a transcendental relation which becomes

$$(3+14f^2+3f^4)\tan^{-1}f=3f+13f^3,$$

and gives, approximately,

$$f=1{\cdot}395, \quad \frac{a}{c}=0{\cdot}584, \quad \frac{\omega^2}{2\pi\gamma\rho}=0{\cdot}187.$$

At this critical angular velocity the stable figures of equilibrium of the rotating liquid will pass from Maclaurin's spheroids into Jacobi's ellipsoids (Thomson and Tait, *Natural Philosophy*, §§ 771-778).

411. A plummet, weighing W g, at the end of a plumb line on the surface of Jacobi's ellipsoid, will experience an apparent attraction of gravitation, having components

$$WAx, \quad W(B-\omega^2)y, \quad W(C-\omega^2)z \text{ dynes};$$

and these may be written

$$\frac{WAa^2}{p}\left(\frac{px}{a^2}, \frac{py}{b^2}, \frac{pz^2}{c^2}\right),$$

where p denotes the length of the perpendicular from the centre on the tangent plane; so that the plumb line will take the direction of the normal to the ellipsoid; and denoting the polar gravity by G, and the length of the normal to the equatorial plane by ν, the tension in dynes of the plumb line,

$$Wg = WAa^2/p = WA\nu = WG\nu/a.$$

An ocean of small depth would spread itself over this ellipsoid, so that the depth at any point is inversely as g, and therefore directly as p.

412. If this Jacobian ellipsoid is enclosed in a rigid case, and rotated with new angular velocity Ω, then

$$p = \text{constant} - \tfrac{1}{2}\rho A x^2 - \tfrac{1}{2}\rho(B-\Omega^2)y^2 - \tfrac{1}{2}\rho(C-\Omega^2)z^2;$$

so that at the surface the change of pressure is

$$\tfrac{1}{2}\rho(\omega^2-\Omega^2)(y^2+z^2).$$

If there is a liquid nucleus of density $\rho+\rho_1$, it can assume the form of the coaxial ellipsoid of semi-axes a_1, b_1, c_1, determined by the condition that

$$\pi\gamma\rho\, abc \quad (P' - \tfrac{1}{2}A'x^2 - \tfrac{1}{2}B'y^2 - \tfrac{1}{2}C'z^2)$$
$$+\pi\gamma\rho_1 a_1 b_1 c_1 (P'_1 - \tfrac{1}{2}A'_1 x^2 - \tfrac{1}{2}B'_1 y^2 - \tfrac{1}{2}C'_1 z^2) + \tfrac{1}{2}\omega^2(y^2+z^2)$$

is constant over its surface, the suffixes referring to this interior ellipsoid; and therefore

$$a_1^2(\pi\gamma\rho abcA' + \pi\gamma\rho_1 a_1 b_1 c_1 A'_1)$$
$$= b_1^2(\pi\gamma\rho abcB' + \pi\gamma\rho_1 a_1 b_1 c_1 B'_1 - \omega^2)$$
$$= c_1^2(\pi\gamma\rho abcC' + \pi\gamma\rho_1 a_1 b_1 c_1 C'_1 - \omega^2),$$

equations for determining A_1', B_1', C_1', etc., when A', B', C' and ω^2 are given.

Thus if the outer case is spherical,

$$A' = B' = C', \quad \text{and} \quad abcA' = \tfrac{2}{3}.$$

It might even be possible for the interior nucleus to rotate bodily as a concentric but not coaxial ellipsoid, when the outer case is made to rotate about an axis not a principal axis.

413. When $a = \infty$, the ellipsoidal case becomes an elliptic cylinder; and now

$$A = 0,$$

$$B = \int_0^\infty \frac{2\pi\gamma\rho bc\,d\lambda}{(b^2+\lambda)^{\frac{3}{2}}(c^2+\lambda)^{\frac{1}{2}}} = \frac{4\pi\gamma\rho c}{c+b},$$

$$C = \int_0^\infty \frac{2\pi\gamma\rho bc\,d\lambda}{(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{3}{2}}} = \frac{4\pi\gamma\rho b}{c+b};$$

so that if filled with one liquid rotating bodily, the surfaces of equal pressure are the quadric cylinders given by

$$\left(\frac{c}{c+b} - \frac{\omega^2}{4\pi\gamma\rho}\right)y^2 + \left(\frac{b}{c+b} - \frac{\omega^2}{4\pi\gamma\rho}\right)z^2 = \text{constant};$$

and if there is a central nucleus of density $\rho + \rho_1$, bounded by the coaxial elliptic cylinder of semi-axes a_1, b_1, the condition of equilibrium of the surface is

$$b_1^2\left(\frac{4\pi\gamma\rho c}{c+b} + \frac{4\pi\gamma\rho_1 c_1}{c_1+b_1} - \omega^2\right) = c_1^2\left(\frac{4\pi\gamma\rho b}{c+b} + \frac{4\pi\gamma\rho_1 b_1}{c_1+b_1} - \omega^2\right),$$

or

$$(c_1^2 - b_1^2)\omega^2 = 4\pi\gamma\left(\rho\,\frac{bc_1^2 - b_1^2 c}{c+b} + \rho_1\,\frac{b_1 c_1^2 - b_1^2 c_1}{c_1+b_1}\right).$$

CHAPTER XIII.

THE MECHANICAL THEORY OF HEAT.

414. When work is done by the expansion of a gas, as, for instance, by the powder gases in the bore of a gun, or by the steam in the cylinder of a steam engine, a certain amount of heat is found to disappear; and according to the First Law of Thermodynamics, the heat which disappears bears a constant ratio to the work done by the expansion.

Thermodynamics is the science which investigates the relations between the quantities of heat expended and work given out in the Conversion of Heat into Work, and *vice versa;* and for a complete exposition of the subject, the reader is referred to the treatises of Clausius, Tait, Verdet, Maxwell, Shann, Baynes, Parker, Alexander, Anderson, etc.; also to the Smithsonian Index to the Literature of Thermodynamics.

In measuring quantities of heat, the unit employed is either the *British Thermal Unit* (B.T.U.) or the *calorie.*

The B.T.U. is the quantity of heat required to raise the temperature of one lb of water through 1° F.

The *calorie* is the quantity of heat required to raise the temperature of one g of water through 1° C.

This is the *small* calorie, also called the *therm*; as the calorie is sometimes defined as the quantity required to raise one kg of water through 1° C; this is 1000 therms.

To be precise the water should be at or near its maximum density, or at a temperature of 4° C.

415. Different substances require different quantities of heat to raise their temperatures through the same number of degrees; and are thus distinguished by their *specific heat.*

The *specific heat* (S.H.) of a substance is the number of B.T.U. required to raise the temperature of one lb of the substance through 1° F, or of calories required to heat one g through 1° C.

In other words, the specific heat is the ratio of the quantity of heat required to heat the substance to the quantity required to heat an equal weight of water through the same number of degrees; the specific heat is thus the same in any system of units.

With solid or liquid substances the specific heat is practically independent of the pressure or temperature, so that the above definition is sufficient for them; and now if weights

$$W_1, W_2, \ldots, W_n \text{ (lb or g)},$$

of substances (solid or liquid) of S.H.'s

$$c_1, c_2, \ldots, c_n,$$

at temperatures $\tau_1, \tau_2, \ldots, \tau_n$ (F or C)

are placed in a vessel impervious to heat, the final uniform temperature T assumed by conduction is given by

$$(W_1c_1 + W_2c_2 + \ldots + W_nc_n)T = W_1c_1\tau_1 + W_2c_2\tau_2 + \ldots + W_nc_n\tau_n,$$

or

$$T = \Sigma Wc\tau / \Sigma Wc.$$

But substances in the gaseous state absorb or give out heat in a manner depending on the relation between the volume, pressure, and temperature, and the specific heat may be made to assume any value by a properly assigned relation, which must therefore be specified in defining the specific heat; for instance, the assigned relation may be of constant volume, or of constant pressure.

416. In melting a lb or g of a solid substance, although the temperature does not vary, a certain number of units of heat disappear, called the *latent heat of fusion;* and again, in converting the substance into vapour, the number of units of heat required is called the *latent heat of vaporisation.*

The latent heat of fusion of ice into water is found to be 144 B.T.U. or 80 calories; and of vaporisation into steam at 212 F or 100 C is found to be about 966 B.T.U. or 537 calories.

Suppose for instance that a meteor weighing 3 tons, of S.H. 0·2, heated to 3,000 F, fell into a pond containing 10 tons of water at 60 F; then x tons of water would be boiled away, given by

$$966x + 10(212-60) = 3 \times 0{\cdot}2 \times (3000-212), \quad x = 0{\cdot}158.$$

If the water was at the freezing point, and one ton was frozen into ice, the temperature would be raised by the meteor to 210 F; and if the meteor weighed 4 tons, about 0·3 tons of water would be boiled away.

According to Regnault's experiments, the latent heat of steam at any other temperature F or C is

$1091{\cdot}7 - 0{\cdot}695(F-32)$, B.T.U., or $606{\cdot}5 - 0{\cdot}695\ C$, calories; so that to heat one lb or g, respectively, of water from the freezing point, and to evaporate it into steam at temperature F or C requires

$1091{\cdot}7 + 0{\cdot}305(F-32)$, B.T.U., or $606{\cdot}5 + 0{\cdot}305\ C$, calories; this is called the *total heat of steam* at that temperature.

417. The constant factor which, according to the First Law of Thermodynamics, converts units of heat which disappear into the equivalent units of work performed is called the *Mechanical Equivalent of Heat*, and is denoted by J.

According to Joule's experiments, as revised recently by Rowland and Griffiths,

1 B.T.U. $=779$ ft-lb (at Manchester);

1 large calorie $=427$ kg-m (at Paris);

1 small calorie $=4{\cdot}19\times 10^7$ ergs $=4{\cdot}19$ joules.

The reciprocal of J is the *Heat Equivalent of Work*; it is generally denoted by A.

In the Thermodynamical equations, unless expressly stated otherwise, we adopt one of two systems of units:—

(i.) The British (F.P.S) system of the foot, pound, second, and Fahrenheit scale, and the gravitation measure of force; measuring volume v in ft³, pressure p in lb/ft², work in ft-lb, heat H in B.T.U.; and thus take $J=779$.

(ii.) The C.G.S. system of the centimetre, gramme, second, and Centigrade scale, and the absolute measure of force; measuring volume v in cm³, pressure p in *barads* (dynes/cm²), work in ergs, heat in small calories or therms; and take $J=4{\cdot}19\times 10^7$, $A=2{\cdot}386\times 10^{-8}$.

As an application, consider the theory of the Injector on Thermodynamical Principles; then if W lb of water is injected by S lb of steam against a pressure head of h ft of water, and if the water injected is raised in temperature from F_1 to F_2, and if H denotes the *total heat* of one lb of steam at the boiler temperature F; then the heat which disappears in the Injector is, in B.T.U.,

$$SH-S(F_2-32)-W(F_2-F_1),$$

where $$H=1091{\cdot}7+0{\cdot}305(F-32);$$

and if the water is lifted h_0 ft, and the frictional losses are denoted by L ft-lb, the work done is

$$(W+S)h+Wh_0+L.$$

Therefore, by the First Law of Thermodynamics,

$$(W+S)h+Wh_0+L=J\{SH-S(F_2-32)-W(F_2-F_1)\}.$$

Suppose for example that the boiler pressure is 100 lb/in², and $F=328$; suppose also $F_1=50$, $F_2=120$; then, neglecting h_0 and L, we find $W/S=16$, about.

418. The simplest thermodynamic machine is a gun or cannon; it is a single-acting engine which completes its work in one stroke, and does not work in a continuous series of cycles like most steam engines.

When the gun is fired, the shot is expelled by the pressure of the powder gases; the pressure is represented on a (p, v) diagram (§ 197) by the ordinate MP of the curve CPD, OM representing to scale the volume of the powder gases when the base of the shot has advanced from A to M; the curve CPD starts from a point C, such that the ordinate AC represents the pressure when the shot begins to move (fig. 105).

The area $AMPC$ then represents the work done by the powder (per unit area of cross section of the bore) when the base of the shot has advanced from A to M, the area $ABDC$ representing the total work done by the powder as the base of the shot is leaving the muzzle B.

If OM represents cubic inches and MP represents tons per square inch, then the areas represent inch-tons of work, reducible to foot-tons by dividing by 12.

Suppose the calibre of the gun is d inches and the shot weighs W lb; and that it acquires velocity v f/s at M; then equating the kinetic energy and the work done,

$$\tfrac{1}{2}Wv^2/g=2240\times\tfrac{1}{4}\pi d^2\times\text{area } AMPC\div 12.$$

This supposes the bore is smooth; but if it is rifled with a pitch of b feet, the angular velocity at M is $2\pi v/b$; so that if the radius of gyration of the shot about its axis is k feet, the kinetic energy is replaced by

$$\frac{Wv^2}{2g}\left(1+\frac{4\pi^2k^2}{b^2}\right).$$

To allow for the friction of the bore an empirical deduction, say of $\frac{1}{2}$ ton/in², is made from the pressure represented by MP.

419. Such a diagram is called the Indicator Diagram of the shot; and if the gun is free to recoil, there is a similar indicator diagram for the gun, representing the pressure on the base of the bore at corresponding points of the length of recoil.

The recoil can be measured at any instant by Sebert's velocimeter; the travel of the shot is measured by electric contacts at equal intervals along the bore, and the corresponding pressures are recorded by crusher gauges (§ 10) fixed in the side of the gun; the muzzle velocity is found from electric records outside the gun, and thence is inferred the *average* pressure in the bore, represented by the ordinate AH, such that the rectangle AB, AH is equal to the area $ABDC$.

A comparison of these different records affords an independent check on the work done by the powder gases, inferred from the experiments of Noble and Abel, and enables us to assign the pressure deduction due to the friction of the bore.

As in fig. 42, the curve AQE is drawn, such that its ordinate MQ represents to scale the work done by the powder or the kinetic energy acquired by the shot, each proportional to the area $AMPC$; and therefore the velocity at M may be represented by the ordinate Mv of the curve AvV, where Mv is proportional to $\sqrt{MQ}$.

Thus if, as in the pneumatic gun, we may take the pressure as uniform and represented by the line HK of average pressure, then AQE will be a straight line, and AvV a parabola; in this case the gun may be made of uniform thickness, calculated by § 290, and great economy in weight is secured.

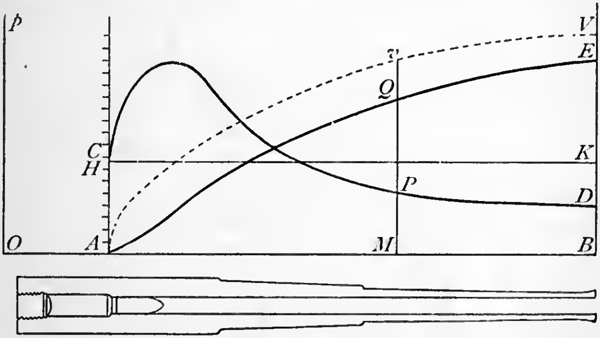


Fig. 105.

If the curve CPD is taken as a straight line sloping downwards, then AQE is a parabola and AvV an ellipse; if sloping upwards, AvV is a hyperbola.

If the pressure curve is assumed to be an adiabatic, $pv^\gamma = p_0 v_0{}^\gamma$, the work done on the shot is (§ 233)

$$\frac{p_0 v_0 - pv}{\gamma - 1} = \frac{p_0}{\gamma - 1}\left(v_0 - \frac{v_0{}^\gamma}{v^{\gamma-1}}\right), \text{ inch-tons,}$$

where v_0 in³ denotes the volume of the powder chamber, and v in³ the total volume of the bore.

Thus if p_0 and v are given, the work is a maximum when

$$\gamma(v_0/v)^{\gamma-1} = 1, \text{ or } v_0 = v(1/\gamma)^{\frac{1}{\gamma-1}};$$

reducing, when $\gamma = 1$, to $v_0 = v/e$.

Fig. 105 represents a 6 inch gun, firing a projectile weighing 100 lb, with a charge of 13 lb of cordite, giving a muzzle velocity of about 2200 f/s.

The length of travel of the shot being 16 ft, and the pitch of the rifling 15 ft, this implies an average pressure of $7\frac{1}{2}$ tons/in^2.

The initial pressure AC is found to be about 8 tons/in^2, rising to a maximum of 16 tons/in^2, and falling to 4 tons/in^2 at the muzzle.

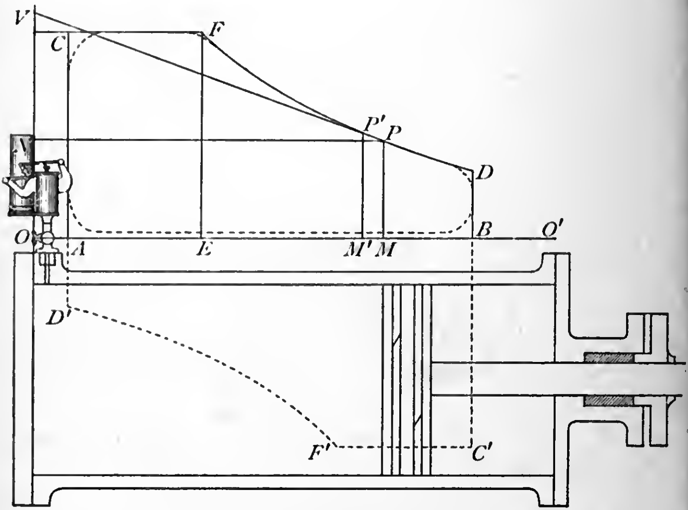


Fig. 106.

420. In the cylinder of a steam engine (fig. 106) the steam is admitted alternately to act on each side of the piston, and the operations continue periodically in cycles, when the piston actuates the crank of a revolving shaft by means of a connecting rod, as in an ordinary or locomotive steam engine.

We represent, as in the gun, the volume of the steam and its pressure by *OM* and *MP*, and now *OA* represents the *clearance* at the end of the cylinder (§ 266).

As the piston moves from *A* to *E*, the point of cut off of the steam, the pressure is supposed to be equal to the full boiler pressure, represented by the line *CF*, so that the work done on the piston from *A* to *E* is represented by the rectangle *AE*, *AC*.

Communication with the boiler is now cut off; and the steam, which at boiler pressure filled the length *OE* of the cylinder, does work by expansion; so that if *FPD* is the pressure curve, the area *EBDF* represents the work done by the expansion of the steam.

Steam is now being admitted in the same manner to the other end of the cylinder (of which the indicator diagram *C'F'D'* may be drawn below *OB*), while the steam first admitted is allowed to escape; either into the atmosphere, as in a locomotive engine, when the pressure *MP* is taken as the gauge-pressure; or else into a condenser, which may be supposed at nearly zero pressure absolute, or at a negative gauge pressure of one atmosphere; and these operations continue periodically.

421. An instrument, called Watt's Indicator, is employed to record the pressure at any point of the stroke, and thence the work done by the steam; it consists of a small cylinder communicating with one end of the engine cylinder (fig. 106).

The Indicator cylinder is closed by a light piston held down by a light spiral spring, the piston rod actuating a pencil which draws a line on a piece of paper wrapped round a brass drum; the moving parts are made as light as possible, to diminish the effect of inertia.

As the engine moves round and the piston reciprocates, the drum is made to revolve through a proportionately reduced distance of the piston travel by a thread attached to a point of the reciprocating machinery, the thread being kept tight by a spiral spring in the drum.

The spring of the Indicator piston gives a displacement proportional to the pressure, so that the pencil traces on the paper a reduced copy of the curve CFD, giving in addition the curve of diminished pressure as the steam is being exhausted, and drawing periodically the same closed cycle, the area of which represents the work done by the steam on one side of the piston in a single stroke; a similar Indicator giving the work done on the other side of the piston.

We may take fig. 106 to represent the diagram drawn by the Indicator, the pencil moving in a fixed line ON, while the paper moves in a perpendicular direction through a proportionally reduced distance; and the ideal Indicator Diagram would be the closed curve $ACFDB$; the real diagram has the corners more or less rounded, as shown by the dotted line.

The area is either read off by a Planimeter, or else calculated by Simpson's Rule, and the mean pressure P lb/in² is thence inferred; and now the indicated horse-power of the one end of the cylinder is given by

$$PLAN \div 33{,}000,$$

where L ft denotes the length of stroke, A in² the piston area, and N the revs/min.

The Indicator may be made double acting, like the engine, and now the indicator diagram $ACFDC'F'D'$ will be made up by the superposition of the two separate diagrams, end for end.

422. The behaviour of a given quantity of gas with varying volume, pressure, and temperature may be studied by supposing it to fill the space OE behind the piston at pressure EF, as in a gas engine; and then determining the curve of pressure FPD as the gas expands from the volume $OE=v_0$ and pressure $EF=p_0$ to any other volume $OM=v$, at which the pressure is $MP=p$; we thus discuss the (p, v) diagram, connecting p or MP and v or OM.

If the gas is compressed from OM to OM', or from v to $v-\Delta v$ by reversing the motion of the piston, then Δv is the diminution of volume, and the ratio $\Delta v/v$ is called the *cubical compression*.

DEFINITION. The elasticity of a fluid under given conditions is the (limiting) ratio of the small increment of pressure to the cubical compression produced.

Thus if the pressure rises from MP to $M'P'$, or from p to $p-\Delta p$, then the elasticity is

$$\text{lt}\ \frac{-\Delta p}{\Delta v/v} = -v\frac{dp}{dv},$$

a positive quantity because p increases as v diminishes.

On the diagram,

$$PR=\Delta v,\ RP'=-\Delta p,$$

and the elasticity is

$$\text{lt}\ \frac{RP'}{PR/NP} = NP\tan NPV = NV,$$

if the tangent at P meets ON in V.

Thus along an isothermal hyperbola (§ 198)

$NV=ON$, so that the elasticity is p.

Along an *adiabatic*

$$pv^\gamma = \text{constant},$$

$NV=\gamma\,.\,ON$, and the elasticity is γp (§ 233).

The work done in expanding to the volume v or OM is represented by the area $EMPF$; and if FP is the adiabatic curve of a perfect gas, this work is (§ 233)

$$\int_{v_0}^{v} pdv = \frac{p_0 v_0 - pv}{\gamma - 1} = R\frac{\theta_0 - \theta}{\gamma - 1},$$

reducing for an isothermal curve along which $\gamma = 1$ to

$$pv \log v/v_0 = R\theta \log p_0/p.$$

423. As the piston moves from E to M, and the point F follows along the curve FP, a certain quantity H units of heat is absorbed (or given out) by the gas, which depends upon the shape of the curve FP, upon the characteristic equation of the gas (§ 198), and upon the change of the *internal energy* of the gas.

As the piston moves from M' to M, suppose that dH units of heat are absorbed, and that the change of internal energy is dE heat units; then, according to the First Law of Thermodynamics,

$$dH = dE + AdW, \qquad \text{(1)}$$

where dW denotes the number of units of work performed in the motion from M' to M; in this case $dW = pdv$.

It is beyond the scope of the present treatise to discuss the Second Law of Thermodynamics and Thomson's Absolute Scale of Temperature; it will be sufficient for our purposes to assume that the absolute scale of temperature is given practically by the indications of an Air or Hydrogen Thermometer (§ 221) obeying the Characteristic Equation (§ 198),

$$pv = R\theta.$$

Thus in the experiments on the Absolute Dilatation of Mercury (§ 164) an air thermometer must be employed, as the mercury thermometer could not detect variations in the coefficient of expansion.

It is assumed also that the Second Law of Thermodynamics is embodied in the equation

$$dH = \theta d\phi, \quad \text{.........................(2)}$$

where ϕ is a certain function, called the *entropy*; then

$$dE = dH - Apdv = \theta d\phi - Apdv, \quad \text{............(3)}$$

embodying the First and Second Laws.

The internal energy E depends only on the state of the gas as given by p, v, θ, its pressure, volume, and temperature, connected by the Characteristic Equation,

$$F(p, v, \theta) = 0;$$

so that a change in E is independent of the intermediate states; or, in other words, dE is a *perfect differential*, and so also is $d\phi$, according to the Second Law.

The First and Second Laws of Thermodynamics are thus expressed by the relations

$$\int dH = AW, \qquad \int dH/\theta = 0,$$

W denoting the work done, and the integrals being taken round a closed cycle in which there is no escape of heat by conduction; the quantity dH/θ is sometimes called the *heat-weight* of the heat dH.

If H units of heat pass from a body at a temperature θ_2 to another body at a lower temperature θ_1, the entropy of the first body falls H/θ_2 and of the second rises H/θ_1; so that the entropy of the system rises

$$H\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right).$$

The entropy is thus unchanged if no heat passes except between bodies at the same temperature; but *conduction* of heat between bodies of different temperature raises the entropy, and the entropy thus tends to a maximum.

424. Of the four quantities p, v, θ, ϕ, two only are independent; and any pair may be taken as independent variables.

Prof. Willard Gibbs selects v and ϕ as variables, so that from (3), with Clausius's notation for partial differential coefficients,

$$\theta=\frac{\partial_v E}{\partial\phi}, \quad Ap=\frac{\partial_\phi E}{\partial v}.$$

But denoting by x, y any pair of independent variables

$$dE=\theta\left(\frac{\partial_y\phi}{\partial x}dx+\frac{\partial_x\phi}{\partial y}dy\right)-Ap\left(\frac{\partial_y v}{\partial x}dx+\frac{\partial_x v}{\partial y}dy\right),$$

so that $$\frac{\partial E}{\partial x}=\theta\frac{\partial_y\phi}{\partial x}-Ap\frac{\partial_y v}{\partial x}, \quad \frac{\partial E}{\partial y}=\theta\frac{\partial_x\phi}{\partial y}-Ap\frac{\partial_x v}{\partial y};$$

and $$\frac{\partial^2 E}{\partial x\partial y}=\frac{\partial\theta}{\partial y}\frac{\partial\phi}{\partial x}+\theta\frac{\partial^2\phi}{\partial x\partial y}-A\frac{\partial p}{\partial y}\frac{\partial v}{\partial x}-Ap\frac{\partial^2 v}{\partial x\partial y}$$

$$=\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial y}+\theta\frac{\partial^2\phi}{\partial x\partial y}-A\frac{\partial p}{\partial x}\frac{\partial v}{\partial y}-Ap\frac{\partial^2 v}{\partial x\partial y},$$

or $$\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\theta}{\partial y}\frac{\partial\phi}{\partial x}=A\left(\frac{\partial p}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial v}{\partial x}\right),$$

$$\frac{\partial(\theta,\phi)}{\partial(x,y)}=A\frac{\partial(p,v)}{\partial(x,y)}. \qquad\qquad (4)$$

This proves that if the plane of the (p, v) diagram is covered by *isothermal* lines, for which θ is constant, and *isentropic* or *adiabatic* lines, for which ϕ is constant, then integrating round any closed cycle,

$\iint d\theta d\phi=A\iint dp dv=A$ times the area of the cycle;

or the area of the cycle is $J\iint d\theta d\phi$.

425. A cycle $\alpha\beta\gamma\delta$ which is bounded by two isothermals θ_1 and θ_2, and two adiabatics ϕ_1 and ϕ_2, is called a *Carnot cycle*, fig. 107; and it thus encloses an area

$$J(\theta_2-\theta_1)(\phi_2-\phi_1). \qquad\qquad (5)$$

With θ and ϕ as variables, the Carnot cycle on the (θ, ϕ) diagram is a rectangle.

Starting from the point $a(\theta_1, \phi_1)$ and moving along the side $a\beta$, the entropy ϕ_1 is constant, and no heat is absorbed or given out in this compression.

In expanding from β (θ_2, ϕ_1) along $\beta\gamma$, the temperature θ_2 is constant, and the heat absorbed is thus

$$H_2 = \theta_2(\phi_2 - \phi_1).$$

In expanding from γ (θ_2, ϕ_2) to δ along $\gamma\delta$, the entropy ϕ_2 is constant, and no heat is gained or lost.

From δ (θ_1, ϕ_2) back to a along δa, the temperature θ_1 is constant, and the entropy changes from ϕ_2 to ϕ_1, so that the heat given out is

$$H_1 = \theta_1(\phi_2 - \phi_1).$$

The heat which has disappeared in completing the Carnot cycle is

$$H_2 - H_1 = (\theta_2 - \theta_1)(\phi_2 - \phi_1)$$

$= A$ times the area of the cycle, or the work done, in accordance with the First Law of Thermodynamics; also the heat-weights,

$$\frac{H_1}{\theta_1} = \frac{H_2}{\theta_2} = \frac{H_2 - H_1}{\theta_2 - \theta_1}.$$

The *efficiency* of the cycle, defined as the ratio of the heat converted into work to the heat absorbed, is thus,

$$\frac{H_2 - H_1}{H_1} = \frac{\theta_2 - \theta_1}{\theta_1}. \quad \text{.....................(6)}$$

Carnot assumed that the efficiency of an engine working in this cycle between the temperatures θ_1 and θ_2, was

$$C(\theta_2 - \theta_1);$$

and he supposed that C was constant; but we see now that C, called *Carnot's function*, is the reciprocal of the absolute temperature of the source of heat.

The Carnot cycle $\alpha\beta\gamma\delta$ is *reversible*; that is, if described in the reverse direction $\alpha\delta\gamma\beta$, as in a *refrigerating* machine, the heat H_1 absorbed at temperature θ_1 is given out as heat H_2 at a higher temperature θ_2, at the expense of the work represented by the area of the cycle.

Carnot's principle asserts that the efficiency of a reversible cycle is a maximum; for if it were possible to obtain a greater efficiency by another arrangement, this could be made to drive the Carnot cycle backwards and thus create energy, and realise "Perpetual Motion."

Thus a thermodynamic engine, for instance a low pressure engine, working between the extreme temperatures of the freezing and boiling points, 0° C and 100° C, gives away at least 273 out of 373 units of heat to the condenser; so that its efficiency falls short of 0·27.

426. By taking x, y to represent any pair of the variables p, v, θ, ϕ, we obtain various thermodynamical relations; thus with independent variables

$$\text{(i.)}\quad p,\ v;\qquad \frac{\partial(\theta,\phi)}{\partial(p,v)}=A;$$

$$\text{(ii.)}\quad \theta,\ \phi;\qquad \frac{\partial(p,v)}{\partial(\theta,\phi)}=J;$$

$$\text{(iii.)}\quad \theta,\ p;\qquad \frac{\partial_\theta\phi}{\partial p}=-A\frac{\partial_p v}{\partial\theta},\ \text{ or }\ \frac{\partial_p v}{\partial\theta}=-J\frac{\partial_\theta\phi}{\partial p};$$

$$\text{(iv.)}\quad p,\ \phi;\qquad \frac{\partial_\phi\theta}{\partial p}=\ \ A\frac{\partial_p v}{\partial\phi},\ \text{ or }\ \frac{\partial_p v}{\partial\phi}=\ \ J\frac{\partial_\phi\theta}{\partial p};$$

$$\text{(v.)}\quad \theta,\ v;\qquad \frac{\partial_\theta\phi}{\partial v}=\ \ A\frac{\partial_v p}{\partial\theta},\ \text{ or }\ \frac{\partial_v p}{\partial\theta}=\ \ J\frac{\partial_\theta\phi}{\partial v};$$

$$\text{(vi.)}\quad v,\ \phi;\qquad \frac{\partial_\phi\theta}{\partial v}=-A\frac{\partial_v p}{\partial\phi},\ \text{ or }\ \frac{\partial_v p}{\partial\phi}=-J\frac{\partial_\phi\theta}{\partial v}.$$

These relations are proved geometrically in Maxwell's *Theory of Heat*, by taking the Carnot cycle $\alpha\beta\gamma\delta$ so small that it may be considered a parallelogram $ABCD$; and now an inspection of fig. 108 shows that the area of the parallelogram, or

$$J\Delta\theta\Delta\phi = AK \,.\, Ak = AL \,.\, Al = AM \,.\, Am = AN \,.\, An.$$

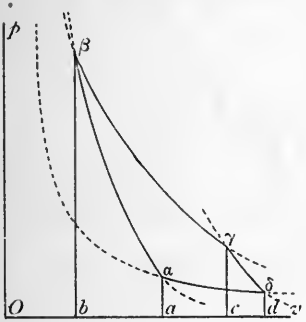


Fig. 107.

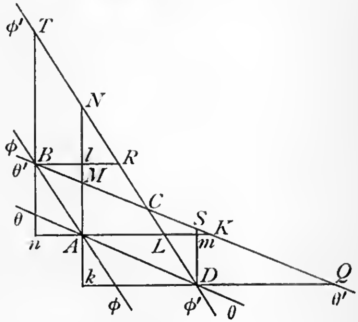


Fig. 108.

Then the relation (iii.) is equivalent to

$$\text{(iii.)} \quad \frac{AK}{\Delta\theta} = J\frac{\Delta\phi}{Ak};$$

for AK is the dilation of v at constant pressure, while Ak is $-\Delta_\theta p$, is the diminution of pressure corresponding to the increment $\Delta\phi$ along the isothermal AD.

Similarly the relations (iv.), (v.), (vi.) are equivalent to

$$\text{(iv.)} \quad \frac{AL}{\Delta\phi} = J\frac{\Delta\theta}{Al}, \quad \text{with } AL = \Delta_p v, \; Al = \Delta_\phi p;$$

$$\text{(v.)} \quad \frac{AM}{\Delta\theta} = J\frac{\Delta\phi}{Am}, \quad \text{with } AM = \Delta_v p, \; Am = \Delta_\theta v;$$

$$\text{(vi.)} \quad \frac{AN}{\Delta\phi} = J\frac{\Delta\theta}{An}, \quad \text{with } AN = \Delta_v p, \; An = -\Delta_\phi v.$$

427. The specific heat c for any given change of state is given by

$$c=\frac{dH}{d\theta}=\theta\frac{d\phi}{d\theta}=\theta\frac{\partial\phi}{\partial x}\frac{dx}{d\theta}+\theta\frac{\partial\phi}{\partial y}\frac{dy}{d\theta};$$

and if the change of state is given by the relation

$$f(x, y)=0,$$

then

$$\frac{\partial f}{\partial x}\frac{dx}{d\theta}+\frac{\partial f}{\partial y}\frac{dy}{d\theta}=0,$$

and

$$\frac{\partial \theta}{\partial x}\frac{dx}{d\theta}+\frac{\partial \theta}{\partial y}\frac{dy}{d\theta}=1;$$

so that

$$c=\theta\frac{\partial(\phi, f)}{\partial(x, y)}\Big/\frac{\partial(\theta, f)}{\partial(x, y)}. \qquad (7)$$

Taking θ and v as variables, and denoting the S.H. at constant volume by c_v, then

$$c_v=\theta\frac{\partial_v\phi}{\partial\theta},$$

$$c-c_v=\theta\frac{\partial_\theta\phi}{\partial v}\frac{dv}{d\theta}=A\theta\frac{\partial_v p}{\partial\theta}\frac{dv}{d\theta}.$$

Thus if c_p denotes the S.H. at constant pressure, when

$$\frac{\partial_v p}{\partial\theta}+\frac{\partial_\theta p}{\partial v}\frac{dv}{d\theta}=0,$$

$$c_p-c_v=-A\theta\left(\frac{\partial_v p}{\partial\theta}\right)^2\Big/\frac{\partial_\theta p}{\partial v}. \qquad (8)$$

Now the elasticity at constant temperature is (§ 422)

$$-v\frac{\partial_\theta p}{\partial v},$$

so that, denoting it by E_θ,

$$E_\theta(c_p-c_v)=Av\theta\left(\frac{\partial_v p}{\partial\theta}\right)^2. \qquad (9)$$

The elasticity when no heat is allowed to escape is given by

$$E_\phi = -v\frac{\partial_\phi p}{\partial v};$$

and

$$\frac{E_\phi}{E_\theta} = \frac{v\dfrac{\partial_\phi p}{\partial v}}{v\dfrac{\partial_\theta p}{\partial v}} = \frac{\dfrac{Al}{An}}{\dfrac{Ak}{Am}} = \frac{\dfrac{AN}{AL}}{\dfrac{AM}{AK}}.$$

Again referring to fig. 107, AM is the increase of pressure at constant volume due to a rise of temperature $\Delta\theta$ or a quantity of heat $c_v\Delta\theta$, and AN is the increase due to a quantity of heat $\theta\Delta\phi$; so that

$$\frac{c_v\Delta\theta}{\theta\Delta\phi} = \frac{AM}{AN}.$$

Also at constant pressure, AK is the increase of volume due to the heat $c_p\Delta\theta$ and AL to the heat $\theta\Delta\phi$; so that

$$\frac{c_p\Delta\theta}{\theta\Delta\phi} = \frac{AK}{AL}.$$

Therefore, for all substances,

$$\frac{c_p}{c_v} = \frac{AK}{AL} \cdot \frac{AN}{AM} = \frac{E_\phi}{E_\theta} \qquad (10)$$

Maxwell proves equations (8) and (9) geometrically from fig. 107, as follows:—

$$E_\phi - E_\theta = v\left(\frac{Al}{An} - \frac{Ak}{Am}\right) = v\frac{\text{area } ABCD}{Am \,.\, An} = v\frac{AN}{Am},$$

$$c_v = \theta\frac{AM}{AN}\frac{\Delta\phi}{\Delta\theta},$$

$$c_v(E_\phi - E_\theta) = v\theta\frac{AM}{Am}\frac{\Delta\phi}{\Delta\theta} = v\theta\frac{AM^2}{J\Delta\theta\Delta\phi}\frac{\Delta\phi}{\Delta\theta}$$

$$= Av\theta\left(\frac{AM}{\Delta\theta}\right)^2 = Av\theta\left(\frac{\partial_v p}{\partial\theta}\right)^2 = E_\theta(c_p - c_v),$$

since $\quad c_v E_\phi = c_p E_\theta.$

428. Applying these formulas to air, for which

$$pv = R\theta,$$

then
$$\frac{\partial_v p}{\partial \theta} = \frac{R}{v} = \frac{p}{\theta}, \quad \frac{\partial_\theta p}{\partial v} = -\frac{R\theta}{v^2} = -\frac{p}{v};$$

and
$$c_p - c_v = AR = R/J.$$

Also $c_p = \gamma c_v$, where γ is 1·4, about (§ 228); so that

$$c_p = \frac{\gamma AR}{\gamma - 1}, \quad c_v = \frac{AR}{\gamma - 1}.$$

With British units, $J = 779$, $R = 53{\cdot}3$ (§ 200); so that

$$AR = 0{\cdot}068, \quad c_p = 0{\cdot}238, \quad c_v = 0{\cdot}170;$$

and the numbers are the same with metric units and the Centigrade scale; these numbers were obtained in this manner by Rankine in 1850, before they had been determined experimentally.

If we divide the S.H. c_v by the S.V. v of the gas, we obtain the *thermal capacity per unit volume;* this is found to be very nearly the same number for all gases at the same temperature.

The numerical value of γ is determined most accurately from the observed velocity of sound (§ 228); another mode of determination, due to Clement and Desormes, is to compress air into a closed vessel, and to observe the pressure p, when the temperature θ is the same as that of the atmosphere.

A stopcock is then opened, and suddenly closed when the air ceases to rush out; and it is assumed that the enclosed air has expanded adiabatically to atmospheric pressure p.

After a time the air inside will regain the surrounding temperature θ, and its pressure p_2 is again observed; so that θ_2, the temperature at the instant of closing the stopcock, is given by $\theta_2 = \theta p/p_2$.

If V denotes the volume of the vessel, then the air left inside, at pressure p_2 and temperature θ, originally occupied a volume Vp_2/p_1 at pressure p_1; and in expanding adiabatically to volume V it assumed the atmospheric pressure p; so that

$$pV^\gamma = p_1(Vp_2/p_1)^\gamma, \quad \text{or} \quad p_1/p = (p_1/p_2)^\gamma,$$

$$\gamma = \frac{\log p_1 - \log p}{\log p_1 - \log p_2}.$$

429. Taking θ and v as variables with a perfect gas,

$$dH = \theta d\phi = c_v d\theta + \theta \frac{\partial_\theta \phi}{\partial v} dv$$

$$= c_v d\theta + A\theta \frac{\partial_v p}{\partial \theta} dv = c_v d\theta + A p dv,$$

so that we may put

$$E = c_v\theta = AR\theta/(\gamma - 1).$$

Thus the internal energy of the gas, in heat units E, in the state represented by the point a in the diagram of fig. 107, is A times the area of the indefinitely extended adiabatic curve $aa\delta v$, cut off by the ordinate aa.

The increase in internal energy E in passing from the state a to the state β by any path $a\beta$ is thus A times the area $vaab\beta v$; and this area is made up of $aab\beta$, representing the work done in compressing the gas from a to β, and of $va\beta v$, representing the mechanical equivalent of the heat supplied in going from a to β.

Also $$d\phi = c_v \frac{d\theta}{\theta} + AR\frac{dv}{v};$$

and, integrating,

$$\phi = c_v \log \theta + (c_p - c_v)\log v + \text{a constant}$$

$$= c_v \log \theta v^{\gamma-1} + \text{a constant},$$

$$\phi - \phi_0 = c_v \log \frac{\theta}{\theta_0}\left(\frac{v}{v_0}\right)^{\gamma-1} = c_v \log \frac{p}{p_0}\left(\frac{v}{v_0}\right)^\gamma.$$

With θ and ϕ as variables,

$$(c_p - c_v)\log\frac{v}{v_0} = \phi - \phi_0 - c_v\log\frac{\theta}{\theta_0},$$

$$(c_p - c_v)\log\frac{p_0}{p} = \phi - \phi_0 - c_p\log\frac{\theta}{\theta_0};$$

so that the *isometrics* and *isobars* of a perfect gas are logarithmic curves on the (θ, ϕ) diagram.

In the Carnot cycle $\alpha\beta\gamma\delta$ for a perfect gas, for instance in an ideal gas engine, the work done in compressing the gas adiabatically from α to β is

$$R(\theta_2 - \theta_1)/(\gamma - 1),$$

and this work is therefore given out again in the adiabatic expansion from γ to δ, so that the areas $\alpha ab\beta$ and $\gamma cd\delta$ are equal (fig. 107); and the above equations also show that

$$\frac{Oa}{Ob} = \frac{Od}{Oc} = \left(\frac{\theta_2}{\theta_1}\right)^{\frac{1}{\gamma-1}}, \quad \frac{b\beta}{a\alpha} = \frac{c\gamma}{d\delta} = \left(\frac{\theta_2}{\theta_1}\right)^{\frac{\gamma}{\gamma-1}};$$

$$\frac{Od}{Oa} = \frac{a\alpha}{d\delta} = \frac{Oc}{Ob} = \frac{b\beta}{c\gamma} = e^{\frac{\phi_2-\phi_1}{c_p-c_v}}.$$

The work done, per lb or g of the gas, by the isothermal expansion from β to γ is

$$R\theta_2\log\frac{v_3}{v_2} = \frac{R\theta_2}{c_p - v_v}(\phi_2 - \phi_1) = J\theta_2(\phi_2 - \phi_1),$$

while the work consumed by the isothermal compression from δ to α is $\quad J\theta_1(\phi_2 - \phi_1)$;

the difference, as before, being

$$J(\theta_2 - \theta_1)(\phi_2 - \phi_1).$$

Examples.

(1) Prove that the orthogonal curves of the adiabatics on the (p, v) diagram are the similar hyperbolas

$$p^2 - \gamma v^2 = \text{constant}.$$

Prove that the isothermals and adiabatics cut at a maximum angle $\cot^{-1} 2\sqrt{\gamma}$ on the line

$$p\sqrt{\gamma} - v = 0.$$

Discuss the same problem for the *isometrics* and *isobars* on the (θ, ϕ) diagram.

(2) Prove that, if a perfect gas expands along the curve $pv^k = \text{constant}$, the work done by expansion is $(\gamma - 1)/(\gamma - k)$ of the mechanical equivalent of the heat absorbed.

(3) Prove that the specific heat of a perfect gas, expanding along the curve $f(p, v) = 0$, is

$$\left(p\frac{\partial f}{\partial p}c_p - v\frac{\partial f}{\partial v}c_v\right)\Big/\left(p\frac{\partial f}{\partial p} - v\frac{\partial f}{\partial v}\right).$$

(4) Prove that, if $pv = R\theta^n$,

$$c_p - c_v = n^2 R\theta^{n-1}.$$

(5) Determine the heat equivalent of the kinetic energy of rotation of the Earth, supposed homogeneous and of S.H. c; and determine the number of degrees which this heat would raise the temperature of the Earth, taking $c = 0{\cdot}2$.

(6) Find what fraction of the coal raised from a mine 500 fathoms deep is used in the engine raising the coal, and 30 times its weight of water, supposing the heat of combustion of 1 lb of coal is 14,000 B.T.U., and the efficiency of the engine is $\frac{1}{7}$.

(7) Compare the work done and the work given out when V ft³ of atmospheric air is compressed adiabatically to n atmospheres, cooled down to the original temperature, and expanded adiabatically to atmospheric pressure; for instance, in a Whitehead torpedo.

TABLE I.—DENSITY OF WATER (MENDELEEF).

C.	s(g/cm³).	D(lb/ft³).	v(cm³/g).	C.	s(g/cm³).	D(lb/ft³).	v(cm³/g).
0°	0·999873	62·4162	1·000127	40°	0·992334	61·9456	1·007725
5°	0·999992	62·4237	1·000008	50°	0·988174	61·6860	1·011967
10°	0·999738	62·4078	1·000262	60°	0·983356	61·3852	1·016926
15°	0·999152	62·3712	1·000849	70°	0·977948	61·0476	1·022549
20°	0·998272	62·3163	1·001731	80°	0·971996	60·6760	1·028811
25°	0·997128	62·2449	1·002881	90°	0·965537	60·2729	1·035693
30°	0·995743	62·1584	1·004275	100°	0·958595	59·8395	1·043193

TABLE II.—SPECIFIC GRAVITY.

Platinum, - - -	22	Aluminium, - - -	2·6
Pure Gold, - - -	19·4	Stone, Brickwork, or Earth,	2
Standard Gold, - -	17·5	Glycerine, - - -	1·26
Mercury, - - -	13·6	Sea Water, - - -	1·026
Lead, - - - -	11·4	Pure Distilled Water, -	1
Silver, - - - -	10·5	Ice, - - - -	0·92
Copper, - - -	8·8	Oak, - - - -	0·93
Brass, - - - -	8	Petroleum, - - -	0·88
Wrought Iron or Steel, -	7·8	Pure Alcohol, - -	0·79
Cast Iron, - - -	7·2	Cork, - - - -	0·24

TABLE III.—ROOMAGE.

Salt Water, - -	35 ft³/ton.	Cast Iron, - -	4·6 ft³/ton.
Fresh Water, -	36 ,,	Wheat or Grain, -	45
Coal, - - -	40 to 46	Timber, - - -	66
Pig Iron, - -	9	Tea, - - -	90

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