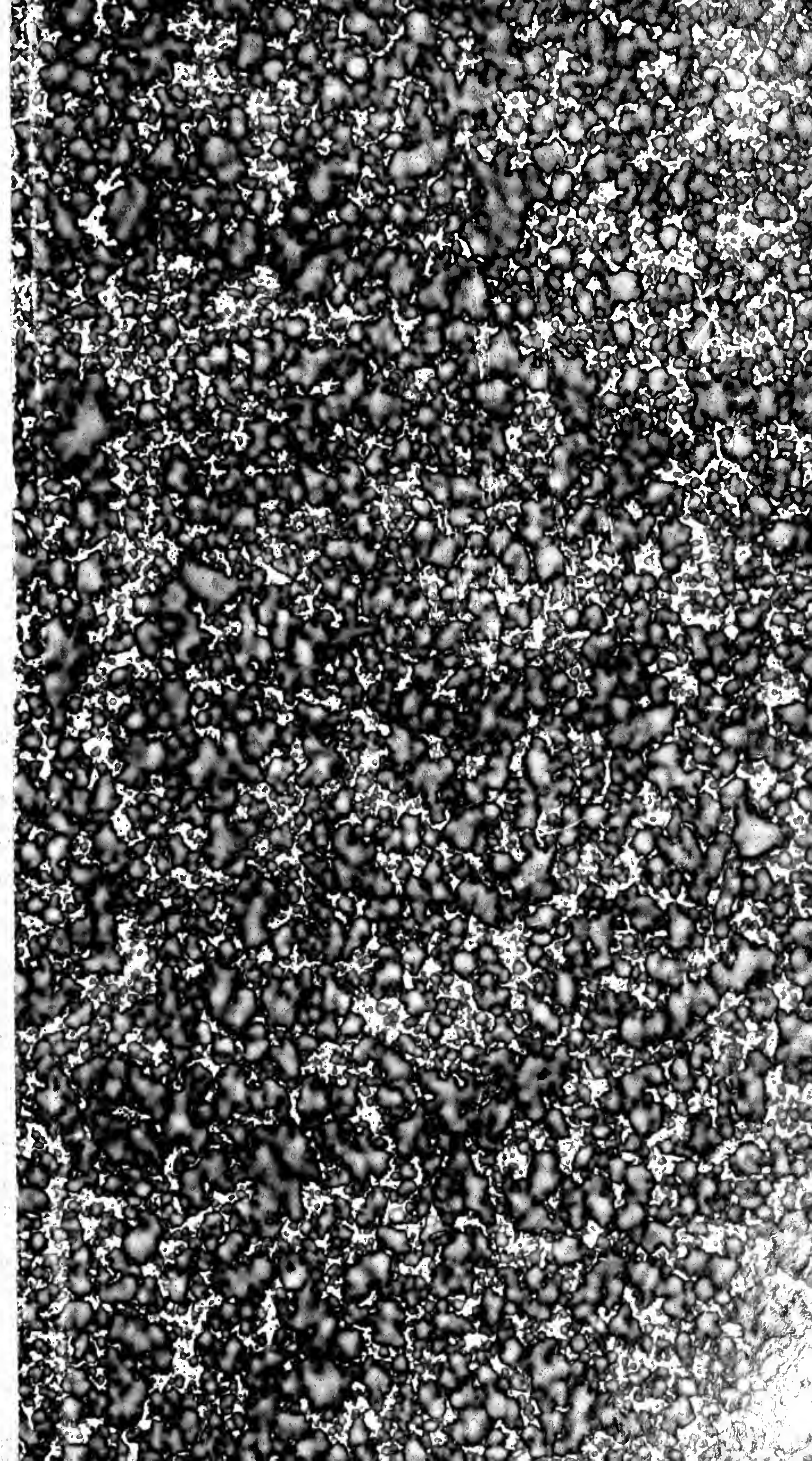

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May 10, 1902 – April 28, 1903.

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# UNIVERSITY OF KANSAS.

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AT LAWRENCE.

# SCIENCE BULLETIN,
## Vol. I, Nos. 1, 2, 3, and 4.

(Continuation of Kansas University Quarterly.)

FEBRUARY 1902.

Entered at the Post-office as Second-class Matter.

# NEW ANNOUNCEMENT.

For ten years the University of Kansas has maintained the *Kansas University Quarterly*, a journal devoted to the publication of the results of research by members of the University of Kansas. At the opening of the second decade of the journal's existence it has been thought best to begin a new series, with some important changes in the name, manner, and time of publication. It has been decided to abandon the quarterly form of issue, and instead to publish the separate papers as soon as they are ready for the printer. By this means the results of research will be more promptly given to the public, and in a form more satisfactory both to the authors and subscribers.

Hereafter this journal will be called the *Kansas University Science Bulletin*, and will be issued at irregular intervals. Each volume will contain from 300 to 400 pages of reading matter, with the necessary illustrations; and approximately one volume a year will be issued. The amount of scientific matter thus published will be not less, and probably more, than that formerly appearing in four issues of the *Quarterly*. The subscription price will be hereafter $3 per volume; the price of each part will vary with the cost of publication. We hope our present large exchange list will remain intact, and new exchanges are solicited. All exchanges should be addressed to the Library of the University of Kansas.

Communications should be addressed to

H. B. Newson,
**Managing Editor of Science Bulletin,**
Lawrence, Kan.

THE

# KANSAS UNIVERSITY SCIENCE BULLETIN.

(Vol. I, No. 1—February, 1902. Whole Series, Vol. XI, No. 1.)

CONTENTS:

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KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. I, No. 1. FEBRUARY, 1902. { WHOLE SERIES, VOL. XI, No. 1.

# DISTRIBUTION OF KANSAS CRAYFISHES.

BY J. ARTHUR HARRIS.

With map.

IN the brief paper here presented, it is my purpose to bring up to date my catalogue of the crayfishes of Kansas[1] by the addition of such localities and notes as have been collected since its appearance; to represent by means of a map the distribution by counties of the different species, and to show, so far as is possible at the present time, the distribution of the different species by river systems. The distribution by river systems has been included, since I feel that a thorough knowledge of this phase of the subject will be of interest in the determination of the phylogenetic relationship of the different species. Of course, any conclusions as to the distribution of the species by water systems can be only provisional, since more systematic collecting will surely change any such conclusions. It is my desire to put the data available at present into such form that the addition of new data and the deduction of more certain conclusions will be possible with the least amount of labor.

The form of the annotated catalogue has, as far as possible, been retained. No new species have been found in the state, and there has been practically no new literature of a taxonomic nature since the appearance of the catalogue. The synonomy has, therefore, been omitted. In referring to localities reported in the previous paper, I shall designate them by numbers enclosed in parentheses, *C. virilis*, (3), the number being that of the locality as given in that paper—the example given being: *C. virilis*, Wabaunsee county (coll. Washb. Coll.), J. B. Fields, coll. (Faxon, '85, b.)

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[1] Harris, J. Arthur: Annotated catalogue of the crayfishes of Kansas. Kans. Univ. Quart., vol. IX, No. 4, October, 1900.

I wish to express here my gratitude to my sister, Nellie Harris, without whose kind assistance in this and other work the appearance of this material at the present time would have been impossible. My thanks are also due those who have collected material in various parts of the state.

The greater part of the material belongs to the private collection of the writer, but is deposited at the present time in the museum of the University of Kansas.

### 1. **Cambarus simulans** Faxon.

I have not seen either of the two lots of material assigned to this species. It will be seen that the territory from which it is reported, while much the same as that from which is taken the material provisionally assigned to *C. gallinas*, lies a little to the west and extends north beyond the Smoky Hill river, while the material provisionally reported as *gallinas* is confined, so far, to the territory drained by the Arkansas.

### 2. **Cambarus gallinas** Cockerell and Porter.

As in my catalogue, I assign only provisionally to this species material from:

3. A stream near Wichita, Sedgwick county; Mr. Willis Henderson, coll.

4. A slough northeast of Caldwell, Sumner county; T. J. Kinnear, coll.

Mr. Kinnear's material was taken at a small slough four miles northeast of Caldwell. The slough had been dry all summer. It will be remembered that the drought of the summer of 1901 was very severe, but a spring a little distance from the edge still contained a little pool of water, perhaps three feet in diameter, although it had ceased to run into the slough. In this little pool of water a few small crayfish, about one inch in length, were noticed; while none of the small specimens were taken, they undoubtedly belong to the same species as the eight adults secured at the same place.

The specimens were secured in digging a well in the old spring. They had burrowed down through the loose surface soil for from six to thirty-six inches, depending upon whether the burrows were at the center or on the edge of the old basin of the spring. The burrows, which were about two inches in diameter, went down almost perpendicularly until they came to the surface of a stratum of Wellington shale. Here they were enlarged into almost round chambers, about ten inches in diameter and not more than three inches in height. The burrows were supplied with "chimneys" above. In these chambers the crayfish were found. They were not very active or pugnacious. The whole burrow was, of course, filled with water. The crayfish had burrowed down a little ways into the rather disintegrated shale. The excavations into the shale were conical, about four inches in diameter at the top and four inches deep. Mr. Kinnear thought that, as the shale was somewhat softened by the water, the crayfish had removed it bit by bit. There were about three or four of the main burrows coming from the upper surface terminating in the large chambers as described above. These chambers were then connected by passageways running from one to another.

Two of the specimens were taken August 1, and the other six August 25–27. Two were males and the other six females. All the females were, with one exception, well loaded with eggs, which appear, from an examination with a hand lens, to be in a very early stage of development, and have probably been only comparatively recently laid.

So far as reported, this species is confined to a narrow strip of territory running north for about eighty miles from the southern boundary of the state and drained by the Arkansas river.

### 3. **Cambarus gracilis** Bundy.

Specimens of this species are hard to obtain, and this doubtless accounts for its few localities. It is found in the territory drained by the Arkansas (1) as well as that drained by the Kansas river (2).

In August, 1901, I found an adult female of *C. gracilis* in a stagnant pond near Lawrence—the only time I have ever taken an adult specimen in open water during the summer.

### 4. **Cambarus diogenes** Girard.

Reported so far only from a limited territory along the Kansas and Missouri rivers, in the northeastern part of the state. As with *C. gracilis*, the difficulty of obtaining material probably accounts for the rarity of the reports on this species.

### 5. **Cambarus immunis** Hagen.

Mr. Crevecœur collected *C. immunis* in a stagnant pond on the prairie near Onaga, Pottawatomie county, April 1, 1901.

The pond had been in existence about six years, and had never been known to go dry. The nearest creek was about a quarter of a mile away. They were probably never connected when the water was high in the creek. No fish had ever been taken in the pond, but specimens of *Amblystoma tigrinum* (green) were found. (*A. tigrinum* and *C. immunis* are sometimes found in the same ponds in Douglas county.)

Among a dozen specimens given to me, some of the females were carrying eggs but none were noticed with young.

Mr. Crevecœur drained the pond in obtaining the material, and was careful to secure a representative collection. If *C. gracilis* were common in the region, it would not be improbable that females would be found in the pond at this time.

Reports so far would indicate a distribution of this species from the Missouri river west along the Kansas and its tributaries nearly two-thirds the distance across the state.

### 5a. **Cambarus immunis** Hagen, var. **spinorostris** Faxon.

The limits of distribution are embraced within those given for *C. immunis*.

### 6. **Cambarus nais** Faxon.

So far as reported, this species is confined to the southeastern portion of the state, drained by the tributaries of the Arkansas river.

### 7. **Cambarus virilis** Hogue.

23. Wakarusa river, Douglas county.

24. Bull Foot creek, Lincoln county. Taken under stones, in about six inches of running water. Miss Ella Weeks, coll.

25. Spillman creek, Lincoln county. Under stones, in shallow running water. Miss Ella Weeks, coll.

26. Wildcat creek, about two miles west of Manhattan, Riley county.

27. Crayfish are not at all common in the lower part of the Kansas river, at least near Lawrence, where I have had opportunity to observe it. The fishermen, as a rule, say there are

none in the river. I have, however, seen the casts of *C. virilis*, and think it hardly probable that they could have washed in from any of the small tributaries.

No. 18 was collected in a small stream.

### 8. **Cambarus rusticus** Girard.

Reported only from Osage river.

### 9. **Cambarus pilosus** Hay.

This somewhat questionable species has been reported from two localities comparatively close together in the north-central portion of the state.

Mr. Sutton informs me that his material (2) was collected in Kelos Fork, a "wet weather" stream of fresh water which flows into Salt creek, which empties into the Saline river about four or five miles from where the specimens were taken. At the time the material was taken there was no water flowing from the pools into the creek. During the summer of 1901, Mr. Sutton took material from a well about five feet in depth, near the above region. The water from the pools would overflow into the well when the water was high.

### 10. **Cambarus neglectus** Faxon.

4. Wildcat creek, about two miles west of Manhattan, Riley county. Collected with the specimens of *C. virilis* mentioned above. Presented by J. N. Westgate.

*C. neglectus*, so far as reported, is quite closely confined to the Republican river valley. The Republican river drains Cheyenne (2) and Decatur (3) counties. Mill creek (1), in Wabaunsee county, empties into the Kansas river about fifty miles east of the Republican. Cat creek (4) empties into the Kansas river about twelve miles from the Republican. Tributaries of the Republican approach to within six miles of the upper part of Cat creek, but I know nothing of the nature of the country separating these streams.

With the exception of the Republican river, in Cheyenne county (2), *C. virilis* has also been taken from all the above localities.

The Republican river, in Cheyenne county, wherever I have seen it, is a shallow stream, perhaps 50 to 100 feet wide, with a bed of loose sand. It sometimes, though rarely, goes dry in places so far as the surface is concerned, but it is said that water can always be found by digging a few inches into the sand of the bed.

During the early part of June, 1901, while near Springfield, Greene county, Missouri, I had the opportunity of making a few observations on the habits of *C. neglectus*. In the James river, near Galloway, about eight miles southeast of Springfield, this was the only species observed, although probably not the only one occurring in the river. At this place the James river is a rather swift-flowing stream, with a rocky bed and with rather high wooded hills along the sides. The stream is quite shallow in the swiftly running places. The crayfish were quite abundant, being found under the loose stones and resting in the strands of the rich vegetation, which stood almost horizontal in the swiftly flowing water.

The specimens were very plentiful around Boiling Springs, a place where one of the cold, underground rivulets of the region breaks through the rocks in the bottom of the stream.

In a clear, rocky stream,[2] shallow in most places, flowing between high hills, about four miles northwest of Springfield, crayfish were found in abundance. The smaller and by far the more numerous species was *C. neglectus*. The animals were very active, darting from stone to stone when disturbed, but usually remaining under cover but a short time.

In a stream flowing from Galloway Cave, at Galloway, Greene county, *C. neglectus* and *C. rusticus* were taken. At the mouth of the cave, *C. neglectus* was by far the more abundant, if not the only species, being found in great abundance under loose stones at the very mouth. The water here has practically the same temperature as that on the inside. The temperature on the inside of the cave is said to remain at fifty-seven degrees F. winter and summer. The animals were very inactive, the cold water, apparently, numbing them to such an extent that it was not at all difficult to take them with the hand.[3] A little distance down the stream, where the water was much warmer, the animals were noticed to be as active as ever.

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[2] I am not sure, in a trip across country, which of the two creeks, which flow together in this vicinity, I examined.

[3] The water here is probably not more than fifteen degrees above that in which *C. virilis* was found to be so numb as to be almost incapable of movement. See Harris, Annotated Catalogue.

A striking effect of the low temperature was noticed in the effect on the hatching of the eggs. Many of the females taken at the mouth of the cave carried eggs or recently hatched young, while none of those taken in the other localities were found with young at all. I believe I found young crayfish which had but recently left the female in the vegetation near Boiling Springs, in the James river.

In the table following, the distribution by river systems of the different species is given. For convenience, the rivers of the state from which material has been reported have been arranged as follows:

- MISSOURI RIVER
- KANSAS RIVER
  - Big Blue River
  - Republican River
  - Solomon River
  - Saline River
  - Smoky Hill River
- OSAGE RIVER
- ARKANSAS RIVER
  - Medicine Lodge River
  - Chikaskia River
  - Little Arkansas River

The tributaries of any stream are arranged in order, beginning with the lower and passing towards the upper portion of the stream. Those tributaries emptying outside the state are designated by an asterisk.

When it is impossible to determine from which of two or more streams a lot of material is reported, as is often the case when the localities given is a country traversed or drained by two rivers, or a town situated on some large stream, or where two streams join, it is reported from each, with the catalogue number followed by a question mark. Of course, in the greater number of these cases, the species will be found to occur in greater or less numbers in each locality. The one thing to be desired is, that collectors would furnish full data with their material.

Whenever possible, the name of the stream from which material was taken is given. When this is not possible, the term "tributary" is used. As a general rule, the tributaries are streams emptying directly into the river under which they are placed, and the only exceptions to this, I believe, are the tributaries of the Arkansas arising in the southeastern corner of the state. Stagnant ponds in the region drained by a stream have been classed as tributaries, even through they have no direct connection.

The above method of classifying the water systems of the state will, very possibly, be found not the best for a final arrangement, but for a preliminary classification—and nothing more than a preliminary arrangement can be hoped for at present—it seems quite satisfactory.

MISSOURI RIVER, *C. virilis*, (5?).
  Tributaries, *C. virilis*, (5?), (22), (20?); *C. diogenes*, (1); *C. immunis*, (1).

KANSAS RIVER, *C. virilis*, (3?), (6?), (11?).
  Tributaries, *C. virilis*, (1), (3?), (6?), (11), (18), (20); *C. gracilis*, (2); *C. diogenes*, (2); *C. immunis*, (2); *C. immunis*, var. *spinorostris*, (2).
  WAKARUSA RIVER, *C. virilis*, (23).
    Rock creek, *C. virilis*, (14).
    Washington creek, *C. virilis*, (15).
    Coon creek, *C. virilis*, (16).
    Wildhorse creek, *C. virilis*, (17).
    Ward's creek, *C. virilis*, (2); *C. immunis*, var. *spinorostris*, (1).
    Mill creek, *C. neglectus*, (1).
    Wildcat creek, *C. virilis*, (26); *C. neglectus*, (4).
  REPUBLICAN RIVER, *C. virilis*, (7); *C. neglectus*, (2).
    Sappa creek, *C. virilis*, (9); *C. neglectus*, (3).
  SOLOMON RIVER, *C. pilosus*, (1 ?).
    Tributaries, *C. pilosus*, (1 ?).
  SMOKY HILL RIVER.
    Big creek, or tributary to it, *C. simulans*, (2); *C. immunis*, (2); *C. virilis*, (8).
  SALINE RIVER.
    Bullfoot creek, *C. virilis*, (24).
    Spillman creek, *C. virilis*, (25).
    A tributary of Salt creek, *C. pilosus*, (2).

OSAGE RIVER, *C. virilis*, (10); *C. rusticus*, (1).

ARKANSAS RIVER, *C. virilis*, (4 ?).

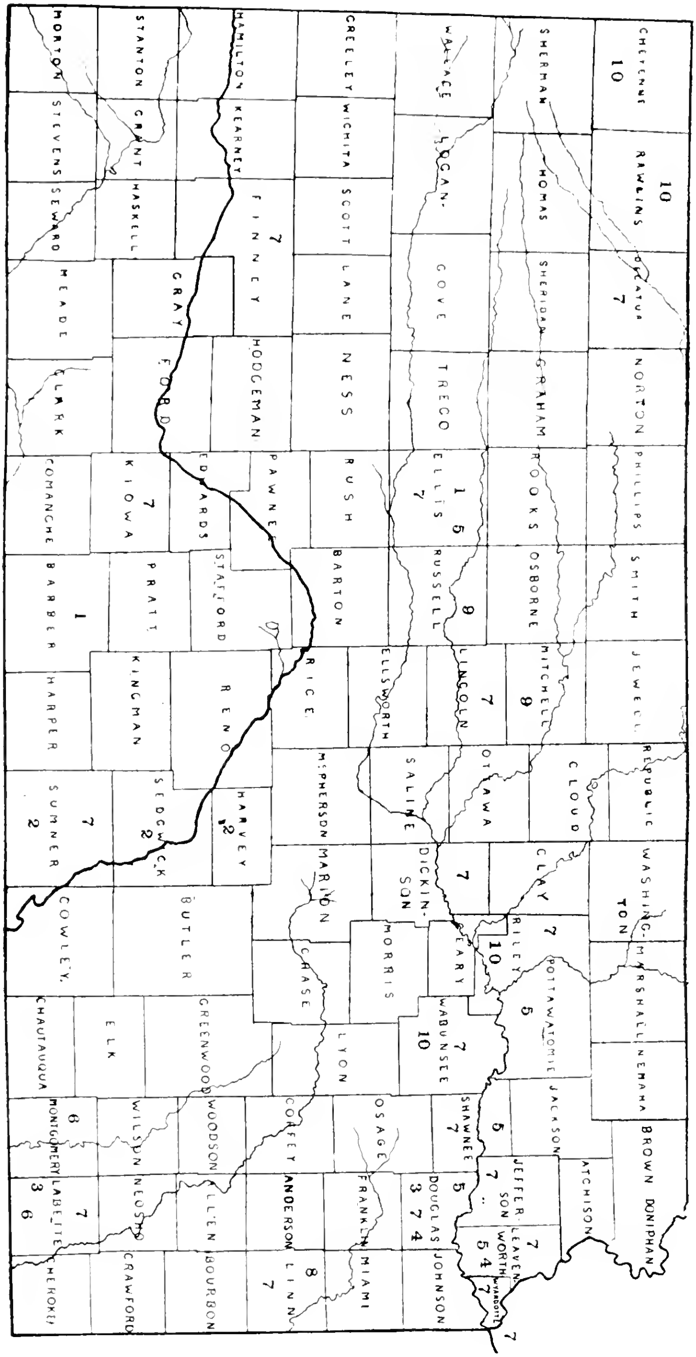
*Tributaries, *C. nais*, (1); *C. gracilis*, (1).
Tributaries, *C. virilis*, (4 ?); *C. gallinas*, (3).
*Coal creek, *C. nais*, (**2**).
*Labette creek, *C. virilis*, (21).

CHIKASKIA RIVER.
Tributary, *C. virilis*, (9); *C. gallinas*, (2), (4).

LITTLE ARKANSAS RIVER.
Tributary, *C. gallinas*, (1).

MEDICINE LODGE RIVER.
Tributary, *C. simulans*, (1).

LABORATORY OF ZOOLOGY AND HISTOLOGY,
UNIVERSITY OF KANSAS.



CHEYENNE 10
RAWLINS 10
DECATUR 7
NORTON
PHILLIPS
SMITH
JEWELL
REPUBLIC
WASHINGTON
MARSHALL
NEMAHA
BROWN
DONIPHAN
SHERMAN
THOMAS
SHERIDAN
GRAHAM
ROOKS
OSBORNE
MITCHELL 9
CLOUD
CLAY 7
RILEY 10
POTTAWATOMIE 5
JACKSON
ATCHISON
JEFFERSON 7
LEAVENWORTH 7 5 4
WYANDOTTE 7
7
WALLACE
LOGAN
GOVE
TREGO
ELLIS 1 5 7
RUSSELL 9
LINCOLN 7
OTTAWA
DICKINSON 7
GEARY
WABUNSEE 7 10
SHAWNEE 7 5
DOUGLAS 3 7 4
JOHNSON
GREELEY
WICHITA
SCOTT
LANE
NESS
RUSH
BARTON
ELLSWORTH
SALINE
MORRIS
LYON
OSAGE
FRANKLIN
MIAMI
HAMILTON
KEARNEY
FINNEY 7
HODGEMAN
PAWNEE
STAFFORD
RICE
McPHERSON
MARION
CHASE
COFFEY
ANDERSON
LINN 8 7
STANTON
GRANT
HASKELL
GRAY
FORD
EDWARDS
KIOWA 7
PRATT
RENO
HARVEY 2
BUTLER
GREENWOOD
WOODSON
ALLEN
BOURBON
MORTON
STEVENS
SEWARD
MEADE
CLARK
COMANCHE
BARBER 1
HARPER
KINGMAN
SEDGWICK 2
SUMNER 7 2
COWLEY
ELK
CHAUTAUQUA
WILSON
NEOSHO
CRAWFORD
MONTGOMERY 6 3
LABETTE 7 6
CHEROKEE
5

# OBSERVATIONS ON THE POLLINATION

## Of Solanum rostratum Dunal and Cassia chamæcrista L.

BY J. ARTHUR HARRIS AND OSCAR M. KUCHS.

With Plate I.

IN 1882 Professor Todd published his interesting observations[1] on the pollination of *Solanum rostratum* and *Cassia chamæcrista*. Since that time, so far as the writers are aware, nothing has appeared upon this subject. During the months of August and September, 1901, opportunity was afforded the writers for making more extended observations on these species. The notes here given are the result of these observations. In some respects, these observations, or the conclusions drawn from them, differ essentially from those made by Professor Todd; in others they are practically the same. The writers feel that, even where observations or conclusions are the same, the confirmation of Professor Todd's results is of value, since the data have been collected in a different locality and a different year.

It must be borne in mind that the lack of agreement between the observations in the present paper with those made by Professor Todd is probably largely due to the inferior quality of his material. *S. rostratum* had been but recently introduced into southern Iowa when Professor Todd's article was written. It apparently did not thrive very well, the greatest number of flowers mentioned as appearing on any one plant being ten—a very small number to be produced by a plant of any considerable size. While also an introduced plant in eastern Kansas, it has been long and well established, and grows luxuriantly.[2]

The data upon which the conclusions here given are based have been given largely in tabulated form. For the present purpose, it might have been sufficient to give only the summarized results of some of the tables. They have, however,

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[1] Todd Prof. J. E.: On the Flowers of *Solanum rostratum* and *Cassia chamæcrista*, Amer. Nat., vol. XVI, pp. 281-287, 1882. A brief review of Todd's paper is given by Dr. P. Knuth, Handbuch der Blutenbiologie, Leipzig, 1898.

[2] According to Carruth—Carruth, J. H., Catalogue of Plants seen in Kansas, with additions by Prof. F. H. Snow and Prof. E. Hall—*S. rostratum* first appeared in Kansas in 1864. This date probably refers to eastern Kansas. Dr. S. W. Williston informs the writers that it appeared around Manhattan in 1860 or 1861.

been inserted in full, since the writers hope that they may be useful in future work on these plants, and since they believe that the collection of carefully prepared statistical data of this kind is very valuable for the decision of some biological questions.

The writers wish to express their gratitude to W. C. Stevens, professor of botany, for suggestions on the work, and to Hugo Kahl, entomologist of the University of Kansas, for the identification of the insects. The drawings were made by Miss Marguerite E. Wise.

*S. rostratum* is a low, spreading, bushy annual, sometimes attaining a diameter of four or five feet and a height of one and one-half feet.[3] The pinnately lobed leaves, as well as other parts of the plant, are beset with strong prickles. It seems to be especially adapted to arid regions, thriving on the dry plains of the Southwest.[4]

The material studied by the writers grew, for the most part, in clayey soil, around old stone-quarries on Mount Oread, a projection of the Kaw river bluffs. A brief examination was made of material growing in waste places in St. Joseph, Mo.

During the very severe drought, which extended up to August, *S. rostratum* was one of the few plants which were apparently uninjured and blossomed with any considerable vigor. The most of the observations were made after the drought was broken by the rain of August 9, when the plants were in the height of their flowering season.

The flower has a somewhat irregular, wheel-shaped, gamopetalous corolla, bright yellow in color. Four of the stamens are normal in their structure, but the fifth, which is on the lower side of the flower, has attained a length almost twice that of the others. Its anther is large and tapering. At about the middle it is crooked a little toward the outside, and its slender, tapering apex is curved upward. The filaments of all the stamens are very short, bringing the anthers close up to the base of the corolla. The small anthers are of about the same color as the

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[3] One specimen observed growing in rich soil back of a feed-store in St. Joseph, Mo., in early September had a diameter of over seven feet and a height of three feet. The plant might be considered as normally developed, having produced apparently the normal number of seed pods, and so would not be classed with the rank vegetative development which plants sometimes show when grown in very rich soil.

[4] *S. rostratum* appears to be better adapted to xerophytic conditions by its extensive root system than by any adaptation for the prevention of evaporation of water. When cut down on a warm day, the plants wilt in a very few minutes. Roots extend down sometimes for more than three feet, so that the plants generally appear perfectly fresh when others around are wilted and drying up.

corolla, varying sometimes to a greenish yellow. The large anther, however, is quite different; the proximal half being of a greenish yellow, while the distal half has a more or less pronounced purple color. Professor Todd, in his paper, does not speak of the color of the anthers, but Fritz Mueller,[5] in writing of *S. rostratum*, says: "All the anthers, as I am informed by Professor Todd, are of the same dull yellow color." All the material examined by the writers from this locality shows a decidedly different color for the distal half of the large stamen. It seems hardly probable that material growing in Iowa should show such a marked difference, but in case this statement is not the result of an oversight on the part of Professor Todd, it is of considerable interest. The anthers dehisce by terminal pores, as is common in the genus to which the plant belongs.

The two lower lobes of the corolla are produced into short wings, which in the bud enfold the pistil and the large stamen, which is clearly differentiated as such in the youngest buds in which the stamens may be discerned by careful dissection. In the bud the pistil lies immediately above the large stamen, but upon the opening of the flower extends between the filaments of the large stamen and that of the small stamen either to the right or to the left.

Professor Todd's statement is: "The pistil in any flower turns toward the axis of the raceme." While in a general way this is true, the statement might be more clearly expressed, since it is only in the general direction of the pistil as a whole that it points toward the axis of the raceme.

The style is not inserted perfectly perpendicularly upon the top of the ovary, but bends slightly downward from the longitudinal axis of the flower. Professor Todd has overlooked this point in his figure. Throughout the remainder of its course until near the tip it is almost straight. Thus it will be seen that the large stamen and the pistil are placed almost opposite each other on the lower side of the flower. The angle between their incurved ends, which approach within about three mm. of each other, is about seventy degrees, thus causing them to point toward opposite sides of the flower. Thus it will be seen that, since the flowers are arranged alternately on the opposite sides

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5. Mueller, Fritz: Two Kinds of Stamens with Different Functions in the same Flower, Nature, vol. XXVII, pp. 364, 365, 1883.

of a simple, bractless raceme, and the tip of the large stamen always points toward the axis of this raceme, the flowers on the opposite sides of the raceme have both the stigma and the pores of the large stamen turned in opposite directions.

Professor Todd says: "The flowers are arranged on simple, bractless racemes which extend in a horizontal position." The material examined by the writers does not quite agreee with this observation, the most of the racemes extending upward at a considerable angle. Ten racemes from different plants were selected at random and their angle above the horizontal taken. From the table, it will be noted that the nearest approach to the horizontal is fifteen degrees above, one raceme is vertical, and the average of the ten is fifty-seven degrees above the horizontal.

TABLE A.

| | | | |
|---|---|---|---|
| I | 65 | VII | 60 |
| II | 75 | VIII | 80 |
| III | 45 | IX | 50 |
| IV | 90 | X | 45 |
| V | 15 | | |
| VI | 45 | Average | 57 |

The terminal portion of the raceme, bearing the buds, is strongly decurved, so that unopened buds obstruct in no way a clear view of the conspicuous flowers, which thus appear to be terminal. The racemes, when in flower, are so far to the outside that the flowers are very little screened by the foliage, whose dark green background renders them more conspicuous.

The fact that the racemes extend upward at some angle from the horizontal, by bringing the flowers above the foliage, renders them more conspicuous.

The terminology used throughout this paper is the same as that suggested by Professor Todd. Those flowers in which the pistil as a whole extends towards the right hand, facing in the same direction as the flower, will be called right-handed, and those in which the pistil as a whole extends toward the left, left-handed. It will be seen that, since the tips of pistil and large stamen approach each other, as above described, the tip of the pistil in a right-handed flower turns considerably toward the left, and *vice versa*. The flowers on the right-hand side of the raceme, as we pass out from the central axis of the plant, are always left-handed, and those on the left side, right-handed.

Professor Todd found from the examination of a small series of material that about an equal number of right- and left-handed flowers is produced. He also says: "It is also a fact of observation that the flowers of a cluster on any one branch and opening about the same time are either all right-handed or all left-handed. Any plant, however, if it is at all large, exhibits right- and left-handed flowers in about equal numbers."

The regularity with which the flowers are divided into the two classes is very striking. Table B shows the condition of ten plants observed at the same time.

TABLE B.

| | | | | | | |
|---|---|---|---|---|---|---|
| Plant..... | I........ ..... | 7 | pistils | right-handed, | 7 | left-handed. |
| “ | II............ | 6 | “ | “ | 6 | “ |
| “ | III..... ..... | 8 | “ | “ | 9 | “ |
| “ | IV............ | 29 | “ | “ | 31 | “ |
| “ | V............ | 11 | “ | “ | 7 | “ |
| “ | VI.......... | 10 | “ | “ | 7 | “ |
| “ | VII............ | 10 | “ | “ | 13 | “ |
| “ | VIII ..... ..... | 3 | “ | “ | 3 | “ |
| “ | IX............ | 3 | “ | “ | 2 | “ |
| “ | X............ | 6 | “ | “ | 9 | “ |
| Total.. | 10 | 93 | pistils | right-handed, | 94 | left-handed. |

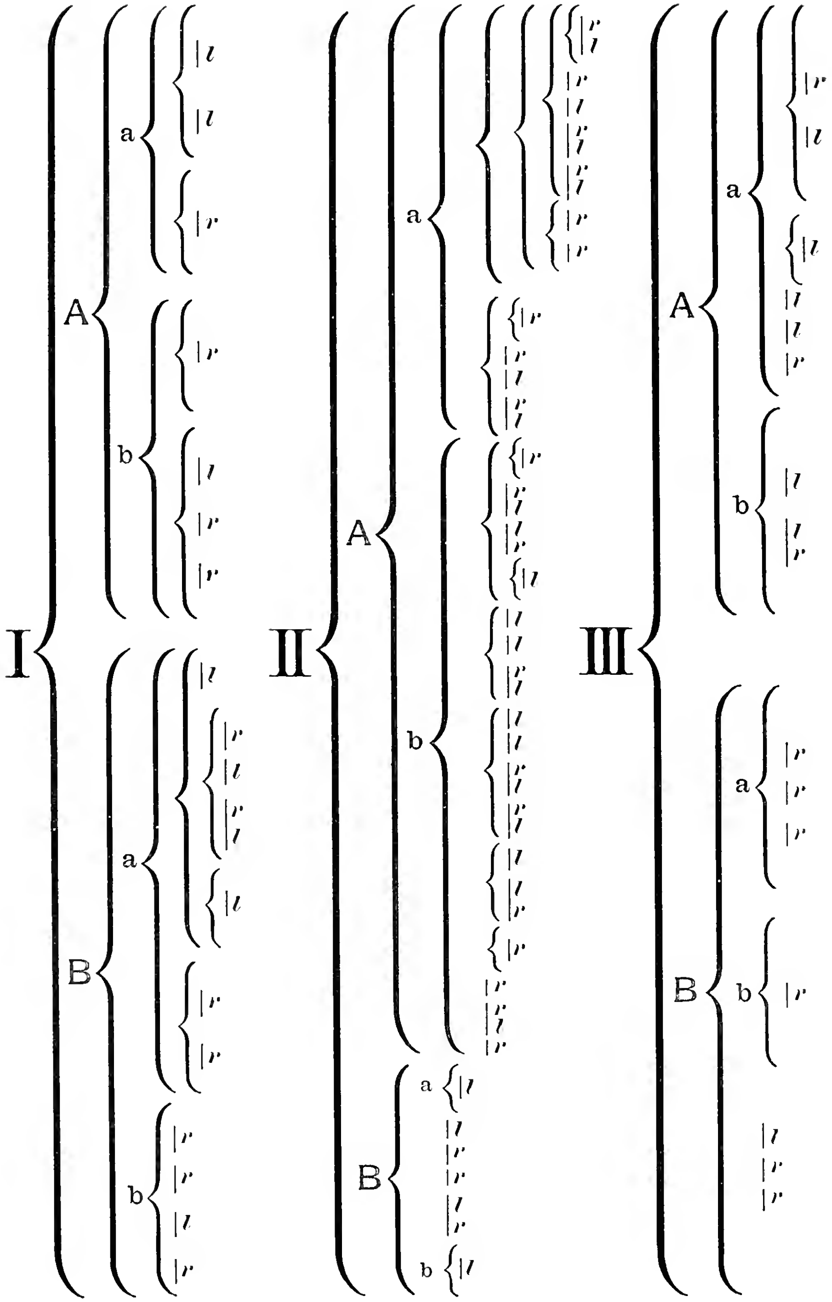
So in these ten plants the number of right- and left-handed flowers is practically equal. The greatest difference in the number of the two kinds is seen in number X, where forty per cent. are right-handed and sixty per cent. left-handed.

Considerable care was exercised in determining the number of right- and left-handed flowers opening on the racemes of different branches at the same time.

Only those flowers were considered which had opened simultaneously. In order to effect this, all the flowers were removed from the plant the evening before and note was made of the condition of those opening the next day.

The following diagram shows the conditions of flowers opening on three plants on the morning of August 20, braces indicating the branches of the plant, and the straight lines the racemes; the numbers of right- and left-handed flowers being indicated under the raceme by *r* and *l*. (See page 20.)

From the table, it will be seen that there are on the first plant 8 left- and 11 right-handed flowers; on the second, 24 left- and 27 right-handed; on the third, 7 left- and 9 right-handed flowers. The numbers of right- and left-handed flowers occurring on the divisions a and b of the main branches, A and B, of the three plants, are as follows:


I
A
a
l
l
r
b
r
l
r
r
B
a
l
r
l
r
l
l
r
r
b
r
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II
A
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B
a
l
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r
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l
r
b
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III
A
a
r
l
l
l
l
r
b
l
l
r
B
a
r
r
r
b
r
l
r
r

I.—Aa, 2*l*, 1*r*; Ab, 1*l*, 3*r*; Ba, 4*l*, 4*r*; Bb, 1*l*, 3*r*.
II.—Aa, 8*l*, 13*r*; Ab, 13*l*, 11*r*; Ba, 1*l*, Bb, 1*l*; B, 2*l*, 3*r*.
III.—Aa, 4*l*, 2*r*; Ab, 2*l*, 1*r*; Ba, 3*r*; Bb, 1*r*; B, 2*r*, 1*l*.

On the three plants, with 36 racemes bearing branches, there were 18 branches which produced only one kind of flowers. Of these branches, however, 15 bore only 1 flower each. From this it will be seen that the flowers opening at the same time on any one branch are not all either right- or left-handed. In the large branches, A and B, the number of the two kinds is quite evenly distributed; in only one case—branch B of plant III—is a large per cent. of the flowers alike. Even in branches of the second denomination—Aa, Bb—flowers of one kind occur exclusively, where more than one flower is found, only in Ba of plant III.

In addition to the above table, observations were made on three plants to determine the regularity with which they bore right- and left-handed flowers. On three successive mornings the plants had produced:

| | I. | | II. | | III. | |
|---|---|---|---|---|---|---|
| | Right. | Left. | Right. | Left. | Right. | Left. |
| First morning | 7 | 7 | 6 | 6 | 8 | 9 |
| Second morning | 7 | 11 | 10 | 7 | 14 | 17 |
| Third morning | 16 | 10 | 8 | 12 | 13 | 10 |
| Total | 30 | 28 | 24 | 25 | 35 | 36 |

It will be noticed that when a marked excess of flowers of one kind occurs one morning, a somewhat proportionate excess of the other type occurs the following morning. This is of course necessary if an equal number of the two types of flowers are to be produced and, to a certain extent, to be maintained on the same plant; and is to be expected from the alternate occurrence of the two types on opposite sides of the raceme.

The flowers open early in the morning and remain open from three to four days, depending somewhat upon the condition of the weather. Some which were covered with cheese-cloth "tents" were noticed to remain open almost a week. At the end of this period the corolla wilts and falls off, as does also the pistil. The flowers seem to partially close at night.

A limited series of experiments were made to determine if self-fertilization and cross-fertilization between flowers of the opposite type opening simultaneously on the same raceme are possible.[6]

The writers have not made sufficiently extensive observations to arrive at any general conclusions of value as to the comparative fertility of cross- and self-pollination, either between flowers on the same or different racemes, or between the flowers of different plants, but they have been able to obtain a limited series of definite results which may be of interest.

In making experiments to determine these points, all old flowers were removed from the plants in the afternoon or evening and the plants covered with a small "tent" of cheese-cloth. The cheese-cloth was of a mesh sufficiently small to prevent the access of any insects large enough to effect pollination, while large enough to allow a ready circulation of air and good illumination. The following morning pollination was effected between the flowers which had opened by tapping pollen from the large anther onto a clean glass slip and transferring it to the stigma of the same or another flower. The plant was then again covered and allowed to remain so, except when examined from time to time, until the corolla and pistil had fallen off. The following results were obtained from three plants upon which observations were made:

### Plant I.

August 20. (*a*) Twelve stigmas pollinated with pollen from large stamen of the same flower. (*b*) Cross-pollination effected between two flowers which had opened on a raceme at the same time.

August 22. (*a*) Five of the twelve flowers had fallen off. (*b*) One flower had fallen off. The other seemed to be developing.

August 24. (*a*) Five ovaries with their corollas fallen off appeared fresh and healthy and seemed to be developing. Two more of the twelve had dropped off. (*b*) Remaining pod seemed to be thriving.

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[6] Experiments for the determination of the fertility of close and cross-fertilization are always interesting, but are of especial interest in the case of a plant such as *S. rostratum*, in which, if the method of pollination described by Professor Todd is the one actually depended upon, cross-pollination is sometimes possible and sometimes impossible on the same raceme. Of course, if, as suggsted in the latter portion of this paper, the method of pollination suggested by Professor Todd is not the only one, these experiments do not have the interest which they otherwise would.

August 26. (*a*) Four of the five ovaries were clearly developing. The fifth appeared doubtful.

September 13. (*a*) Three fully developed pods remained. (*b*) Development of pod arrested when about half grown.

### PLANT II.

August 20. (*a*) Four flowers self-pollinated as with plant I. (*b*) Cross-pollination effected on seven racemes between flowers which had opened simultaneously on the racemes.

August 22. (*a*) All yet on. (*b*) One flower of a pair had fallen off.

August 24. (*a*) Two ovaries remained and looked as though they might develop. (*b*) Three pairs fallen off at raceme. The single pod of one pair still remained and looked as though it might develop. All remaining pairs seemed to be thriving.

August 26. (*a*) One pod developing; the other doubtful. (*b*) Single pod of pair developing. Two pairs were thriving; one pair was almost grown. The third pair looked doubtful.

September 13. (*a*) One pod fully developed; the other fallen off. (*b*) One pair fully developed. One each of two other pairs were fully developed.

### PLANT III.

August 20. Thirteen flowers marked to see if autogamy takes place.

August 22. All flowers still on the plant.

August 24. All but two flowers had fallen off. One of these seemed to be developing; the other looked wilted.

August 26. One pod was thriving; the other had wilted and fallen off.

September 13. One pod fully developed.

From the structure of the flowers it would seem that self-pollination would be impossible. When the flower is open, the stigma has never been observed to be in contact with the terminal portion of the large stamen. The stamens do not dehisce until after the flower has opened, nor does the stigma come in contact with the tip of the anther in the bud; thus, clistogamy would be out of the question. It appears from the results obtained from plant III that spontaneous self-pollination is possible. Of course, however, the positive result in this one case should by no means be taken as conclusive evidence of self-pollination. At the present, the most logical explanation to be suggested seems to be that, when the flowers partially close at night, the tips of the pistil and the large stamen are brought into contact. This might occasionally occur, but it is by no means always the case. At the time of the writing of this paper, material for the determination of this point is not available, but two or three flowers examined at night during the summer, before the results of the above experiments had suggested the importance of a careful examination of a large series of material, did not show the stamen and pistil in contact. Of course, note will be taken of the fact that in only one out of thirteen flowers on the plant did spontaneous pollination take place. Another suggestion might be that, approaching so near as they do to each other, a puff of pollen might be thrown from the large stamen and fall upon the pistil when the plant is shaken.

In plants I and II, it will be seen that, in the first case, three fully developed seed pods were obtained from twelve flowers the stigmas of which were supplied with pollen from the large stamen of the same flower. In the second case, one fully developed seed pod was obtained from four pollinated flowers—just twenty-five per cent. in each case.

In the cases in which cross-pollination was effected between right- and left-handed flowers opening simultaneously on the same raceme, we find that, in the first, one pod of the two was only half developed at the end of twenty days. Since the pods are normally fully developed in somewhat less than this length of time, and this undeveloped pod appears somewhat dried, its development seems doubtful. In the second case, one pair of seed pods out of seven pairs of flowers crossed were fully developed, and one seed pod from each of two other pairs were fully and normally developed, making four out of fourteen flowers which yielded seed pods—28.5 per cent.

Professor Todd observed only a small humblebee visiting the flowers of this plant. Owing, probably, to more favorable opportunities for observation, the writers have been able to secure other insects collecting pollen.

The following is a list of the species:

*Agapostemon texanus* Cress. Two specimens collected August 5, at two P. M. The insects were collecting pollen from the small stamens, to which they clung while they forced the pollen out by pinching the anthers between their fore legs. Pollen was stored on the hind legs. The insect was not seen to come in contact with the tip of the large stamen or the stigma.

*Apis mellifica* Linn. Taken at two P. M., August 5. They sometimes came in contact with large stamen and pistil, but more often did not touch them at all. Occasionally both stamen and pistil would come in contact with the same side of the insect's body. Short stamens were sometimes approached from above, the large stamen and pistil remaining untouched.

*Anglochora pura* Say. Taken at 10:30 A. M., August 6. Obtained pollen from the large stamen by alighting on it, crawling to the tips, and collecting it from the terminal pores.

*Halictus* sp. A smaller insect than the preceding one, but obtained pollen in the same manner.[7]

No humblebees were taken around Lawrence, although many were noticed working on the plants; consequently the names of the species noticed cannot be given. In St. Joseph, Mo., there were taken at three P. M., when bees were not generally seen working on the plant:

*Bombus virginicus.* One specimen.

*Bombus pennsylvanicus.* One specimen.

An examination of fifty flowers taken at this time showed from the dented condition of the small stamens that they had all been visited.

Between eight and nine A. M., September 3, when bees were numerous, there were taken:

*Bombus virginicus.* Two specimens.

*Bombus pennsylvanicus.* Twelve specimens.

*Bombus scutellaris.* One specimen.

The writers found that the humblebees were the principal agents effecting cross-pollination. It was observed that the bee in visiting the flower allowed itself to rest on the tips of the extending stamen and pistil, which, being of the same length, came in contact with both sides of the body just in front of the hind legs, these being left perfectly free. The weight of the bee springs down both stamen and pistil.

Professor Todd's theory in regard to the pollination of this plant is as follows: "The weight of the bee so springs down the

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[7] As will be remarked, the above insects were all taken August 5 and 6. Careful collecting extending over a considerable period of time would doubtless secure many other forms which visit the plant more or less frequently.

flower, that it is quite difficult, on account of the large, flexible corolla, to see just what is done, but repeated observations led me, quite satisfactorily, to this conclusion. The bee seeks the pollen—for the flowers have neither nectar nor odor—and this she uniformly gets from the four shorter stamens; never, so far as I could determine, from the larger one. This she does by seizing each one, near its base, between her mandibles, and with a sort of milking motion crowds the pollen out of the terminal pores; meanwhile, by the movements of her feet, the larger stamen is repeatedly sprung backwards, and as often throws a cloud of pollen on one side of her body; this in a right-handed flower. When she passes to a left-handed flower, which, as was explained above, is very likely not to be on the same plant, the pollen is carried directly to the pistil of that flower, and so on. We have here, therefore, a novel apparatus for cross-fertilization, quite distinct from those that have been most commonly noticed."

A considerable quantity of pollen may be thrown from the terminal pores of the large stamen upon tapping it. It thus seems quite possible that some pollen is thrown upon the side of the insect, as described by Professor Todd. All the meaning of Mr. Meehan's[8] statement is not clear to the writers, but he says, in speaking of Professor Todd's results: "In regard to the manner in which the pollen is extracted, he found that 'this she does by seizing each anther near its base between her mandibles, and, with a sort of milking motion, crowds the pollen out of the terminal pores.' If this were the general way, there would be no necessity for any pollen being ejected from the long stamens, for the stigma would surely receive some during the 'milking' process; and the pore at the apex in the long anther is beyond the line of the stigma, so that on ejection from the pore the pollen would go still farther beyond."

It seems that this statement is of considerable importance for *S. rostratum* as well as for *C. marilandica*. Professor Todd very evidently overlooked the fact that, in securing the pollen from the small stamens and transferring it to the hind legs, the sides of the insect are sure to be well dusted with pollen from these stamens. In the case of *Apis mellifica*, as noted above, there is no certainty that in visiting the flower the same side will be

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[8] Meehan, Thomas: On the Fertilization of *Cassia marilandica*, Proc. Acad. Nat. Sci. Phila., 1886, pp. 314-318.

turned toward the stamen or pistil. Even in the case of large insects, such as *Bombus*, it would seem that the probability that the stigma will be supplied with pollen from the large stamen exclusively is very small. It seems improbable that *S. rostratum* should depend exclusively upon such an uncertain method of pollination as the projection, by the jarring of a stamen, of a puff of pollen upon the side of an insect, and the subsequent transfer of this pollen to the stigma of a flower of a different type. Of course, it is not improbable that a part of the pollen is furnished by the large stamen, as suggested by Professor Todd, but that fertilization should be effected exclusively by this means seems highly improbable.

The pollen from the large stamen has been shown to be fertile in a certain number of cases, but unfortunately opportunity was not offered for experiments on the fertility of pollen from the small stamens. A rather hasty microscopic examination of fresh, unstained pollen from the large and small stamens reveals no very striking difference in form.

In *C. marilandica*, Meehan[9] found that the large, strong stamens on each side of the pistil served only as a platform upon which the insect could rest while procuring the pollen from the small stamens. He found that the lower stamens, while filled with pollen, did not dehisce of their own account, nor were they opened by the insect.[10]

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[9] Meehan, Thomas, *loc. cit.*

[10] It must be stated that in a later paper [Robertson, Charles: Flowers and Insects, V. Bot. Gaz., vol. XV, No. 8, pp. 199–204], Charles Robertson does not give the same results as those found by Thomas Meehan. Robertson says: "Two long stamens, one on each side of the style, furnish pollen for cross-fertilization. They have inflated anthers which probably have a bellows-like action, like the long stamen of *Solanun rostratum* and *Rhexia virginica*." Meehan states expressly in his paper that in case of *C. marilandica* he was sure no pollen was ejected, as Todd found for *S. rostratum*, since in the flowers, which were covered with a gauze bag, the membrane at the apex was never ruptured when the stamens were ready to fall. Rebertson describes the method of extracting the pollen in *C. chamæcrista* in a way which is essentially the same as Todd gives for *S. rostratum*. He then says, in speaking of *C. marilandica:* "Four small stamens furnish pollen for the visitors. Bumblebees milk the pollen out of these, using their jaws, as in the case of *chamæcrista*." Meehan says: "Nor was there any draw-out of the pollen, as observed by Professor Todd. It is abstracted solely through the pores; and, although I could see no evidence that such was actually the case, I suspect that fertilization could only occur through some of this extracted pollen escaping from the insect to the stigma." It must be noted here that the method which Meehan describes for the method of opening of the anthers, the pollen being "abstracted solely through the pores," does not agree with the method described by other observers. Leclerc du Sablon, in a paper, "Recherches sur la Structure et la Dehiscence des Anthers," in vol. 1 of the seventh series of Annales des Sciences Nouvelles, discusses the anatomical modifications of the anther walls, by which dehiscence is secured. His observations cover *Cassia cremophilia* and *Solanum* (sp.?) His observations do not cover a sufficient number of species to make them of the greatest value in deciding the present points. The author presents, in a condensed form, his results in: Note sur la Dehiscence des Anthers, La Belgique Horticole, vol. XXXIV, pp. 148–150, 1884. Robertson says, in speaking of the central of the three long stamens: "Bees, no doubt, force the pollen out of this as they do from the short stamens." Meehan says: "I watched a mass of plants containing eighty-eight flower-stems on the 30th of July, and the same lot for an hour on the 6th of August, but saw no attempt to get pollen from the longer anthers or to use them in any way but as a platform. It would indeed be hardly possible for the bee to stand anywhere so as to get power to pierce the apical membranes of the longer stamens. When the flower matured and the anthers were ready to fall they were examined — the four short ones were empty sacs — the three lower ones proved that they had not served any purpose to the bees, for they were full of pollen."

The lower stamens and the pistil of the *Solanum* under consideration serve the purpose of a platform when the flowers are visited by the larger bees. It seems to the writers that this is not improbably the function of the greatest importance of the observed arrangements of the stamen and pistil in *S. rostratum*. In *C. marilandica*, the pollen for fertilization, as well as for the attraction of the insect visitor, is furnished by the small stamens, while the pollen produced by the large stamens appears to have no function.[11] The condition is not so specialized in the species of *Solanum* under consideration. Here the pollen produced by the small anthers serves for the attraction of insects and, as it seems to the writers, for fertilization, while the large stamen, in connection with the pistil, serves as a support for the visiting insect, and possibly furnishes some pollen for cross-fertilization.[12]

In reference to the relative amount of pollen produced by a large and small stamen, Halstead has given a note, in his paper in the Botanical Gazette.[13] The material in the hands of the writers at the time of the writing of this paper is not suitable for a verification of Mr. Halstead's results; consequently they are simply quoted on his authority. Even if the amount of pollen produced by the large anther is no greater than that produced by one of the smaller, it is still very considerable, as may be readily seen by tapping it out on a glass slip. He says:

"The single large stamen of *Solanum rostratum*, with its beak-like appearance, is a giant among its fellows, but does not exceed them in the production of pollen, for, while three or four times larger than the others, its thecæ are reduced to narrow, curved lines of mother-cells. The ordinary stamens, on the the other hand, possess unusually large cavities in which the pollen is borne. The giant stamen, in cross-section, is shown at *a*, in fig. 3, while a similar section of an ordinary stamen is shown at *b*. The almost infertile condition of the large stamen reminds one of the structure of the stamens of the cultivated

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[11] This, of course, in case, as Meehan states, the large anthers do not dehisce. Of course the statement loses entirely its significance if, as Robertson states, the large stamens furnish pollen for cross-fertilization.

[12] While the experiments made upon artificial pollination were very limited, it will be seen that the pollen from the large stamen in no case fertilized over twenty-nine per cent. of the flowers pollinated from it. These flowers, however, were on the same raceme; so the low per cent. might be due to this, or to the mechanical manipulation. The suggestion that the pollen of the large stamen is less fertile than that of the smaller ones is at least interesting as a working hypothesis.

[13] Notes on Stamens of Solanaceæ, Bot. Gaz., vol. XV, pp. 103-106, 1890.

potatoes. In these, while large and plump, there is almost no pollen-bearing layer, and usually no apical pore opens for the discharge of pollen."

In *C. marilandica*, as Meehan has shown, autogamy is impossible, while in *S. rostratum* autogamy may possibly sometimes take place.

The bee visits the flower for pollen; contrary, however, to the statement of Professor Todd, that "the flowers have neither nectar nor odor," the writers observed that, especially in the early morning, the odor was decidedly pronounced. It was observed that the bee collected no pollen from the large stamen, but took it regularly from the four smaller. This it did by grasping the anthers, one at a time, near the base, and forcing the pollen out through the terminal pores, by pinching it throughout the length between its mandibles. An exception to this in the case of *Agapostemon texanus* Cress. is already noted in the list of species. It will be remarked that our observations on this point correspond in general to those of Professor Todd.

Of course the statement of Professor Todd, that the next flower of the opposite type which is visited by the bee is very apt to be on another plant, loses entirely its significance, since it has been shown that the flowers on a branch are not at all likely to be all right- or left-handed. In visiting the flowers, the humblebees, as a general rule, simply pass to the flower most conveniently at hand, and this flower is very apt to be on the same plant, especially where the plants are at all large. The humblebees especially work vigorously in the early morning. In a patch of *S. rostratum* examined between eight and nine o'clock, in St. Joseph, Mo., nearly all the flowers had already been visited. At this time fifteen specimens of humblebees were taken. A great many flowers would be visited by the bee before it found one which had not already been despoiled of its pollen. In visiting such flowers, the bee would alight for a moment on the pistil and large stamens, as described above, and then pass on to the next flower when it had ascertained that there was no pollen present. In this way over twenty flowers may be visited in a minute. It will be seen that, when the bees are at all numerous and as well dusted with pollen as they usually are, the pistil is almost certain to receive pollen, and fertilization to be effected, especially if the pollen from the small stamens is functional.

Among other insects found visiting the plant, the honey-bee was most frequent.

As will be noticed from our list, some insects visit the plants without effecting cross-pollination. Those insects which obtain pollen in an illegitimate manner do not secure it from the small stamens exclusively, but almost invariably visit the large stamen as well.

The adaptation of the plant to propagation by the production of seeds is of considerable significance.[14]

A normal plant will produce in the neighborhood of 7000 seeds. In making observations on this point, it was found from five pods examined there was an average of fifty-six seeds.

| | | | | |
|---|---|---|---|---|
| Pod | 1 | contained | 66 | seeds. |
| " | 2 | " | 53 | " |
| " | 3 | " | 51 | " |
| " | 4 | " | 53 | " |
| " | 5 | " | 58 | " |
| | | | 281 | seeds; av., 56. |

Pods 4 and 5 were from the same plant but separate racemes; the others were from different plants. In determining the average number of seeds produced by the plant, five plants growing normally and in different localities were observed, with the following results:

| | | | |
|---|---|---|---|
| Plant | 1...... | 192 | pods. |
| " | 2 ..... | 50 | " |
| " | 3...... | 66 | " |
| " | 4..... | 113 | " |
| " | 5 ..... | 210 | " |

Taking the average of fifty-six seeds per pod obtained above, we see that the plant producing 122.5 pods, the average from the preceding table, would produce about 7000 seeds.

One plant was observed upon which occurred fifty-five to sixty racemes. Allowing the low average of six pods to the raceme, the plant will produce in the neighborhood of 20,000 seeds. Occasionally a very large plant is observed which pro-

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[14] Observations on the number of seeds produced and the surety of fertilization may be of especial interest, when the wonderful distribution which this plant has attained in recent years is taken into consideration. The original habitat of *S. rostratum* was the southwestern portion of the United States. It has since spread over a large part of the United States, in many places being recognized as a very noxious weed. It is also reported from several European localities. Reports on the destructiveness of the plant as a weed may be found in publications of the agricultural departments, as: Dewey, L. H., A Weed Bulletin, Farmers' Series, No. 28, U. S. Dept. Agr.; Pammel, L. H., Two Noxious Weeds, Bull. Iowa Exp. Sta., 1895. L. H. Pammel, —Distribution of Some Weeds in the United States, especially *Iva xanthifolia*, *Lactuca scariola*, *Solanum carolinense*, and *Solanum rostratum*, Proc. Iowa Acad. Sci., 1895, vol. II, pp. 103-127—gives the eastward migration of this weed up to 1895.

duces as many as 125 racemes. Allowing the same low average of six pods to the raceme, it will be seen that on a plant of this size there will be produced in the neighborhood of 40,000 seeds.

Only a very small proportion of the ovaries fail to develop. Out of the forty-one racemes observed in five plants, taken at random in different localities, results were obtained as follows:

| | | | |
|---|---|---|---|
| Plant 1, | 5 racemes, | 53 pods, | 4 failed. |
| " 2, | 5 " | 42 " | 3 " |
| " 3, | 9 " | 55 " | 5 " |
| " 4, | 11 " | 91 " | 6 " |
| " 5, | 11 " | 121 " | 5 " |
| | 41 racemes, | 367 pods, | 23 failed, or 6.2 per cent. |

According to these figures, not more than 6.2 per cent. of the ovaries failed to be fertilized.

While *Cassia chamæcrista* is usually abundant in Douglas county, owing, probably, largely to the severe drought, opportunities for study were not nearly so favorable as for *Solanum*. The material studied was found growing, for the most part, in somewhat shaded localities on the banks of Lake View.

Professor Todd has given very well the points in the structure of the flower of this species. He says: "The points that are of interest to us are the sickle-shaped pistil, the stamens with long, rigid anthers opening by terminal pores, and the most of them pointed toward the incurved petal, which is always on the opposite side from the pistil."

The flowers are arranged in small clusters a little above the axils of the leaves. In some cases the axillary bud also develops into a flower cluster. The axillary clusters have been considered separately in the calculations made upon the conditions of the flowers.

Owing to the lack of material, Professor Todd was unable to determine any definite law governing the arrangement of the flowers in *C. chamæcrista*. This the writers have attempted to do. The determination of any law governing the order of development of the flowers in a plant like *C. chamæcrista*, where they are arranged in clusters developed from buds produced on the main axis, and the development of which is probably accelerated or retarded by various conditions, is much more difficult than in *S. rostratum*, where they are produced on a definite raceme, which is early differentiated from the terminal growing

TABLE D.

| *Cluster..* | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| PLANT. | | | | | | | | | | | | | |
| I | 1. | A | 2b | b, 5p | b, l, p | A | A | A | A | A | b, 2p | 2b, 2p | 2b, 2p |
| | 2.. | l, p | 2p | A | 3p | p | A | A | 2b, 1p | b | A | 3b, p | 2b<br>1b |
| | 3.. | p | 3p | 4p | b, 3p | br, 4p | b, r, p | b, 4p | 2b, bl, 3p | 2b, br, 2p, a | 2b, p, a | b, l, 2p | 2b, bl, p |
| II | —1.. | 2b | b, 3p, a | b, 2p, a | 2b, 3p | 2b, 3p | r | 2b, br | 2b, br | b, r, p | 2b, bl | 3b | 2b |
| III | —1.. | 3b | 2b, p | b, 2p | 2b, 2p | b, br, p | b, 3p | b, 3p | 4p | b, r, 3p | b, bl, 3p | b, 3p | 3b, 2p |
| IV | 1.. | b | 2b, bl | 2b, r | 2b, bl | 3b | 3b, br | 2b, l | 3b | 3b | 3b | 3b | 3b |
| | 2.. | A | b, bl | 3b, bl | 3b, p | 3b | 3b, r, p | 3b, br | 3b, l | 4b | 4b | 4b | ......... |
| V | —1.. | A | 2b, p | b, l, p | b, p | b, r, p | 2b, 2p | b, bl, p | b, r, p | 2b, r | 2b | 2b | 2b |
| VI | 1.. | A | A | A | A | p | 2p, a | b, 3p | b, 3p, a | b, r, 3p | 2b, 2p | 2b, 3p | 3b, p |
| | 2.. | A | p | b, 3p | b, 2p | 2b, 3p | 2b, p | b, r, p | 2b, p, a | 2b, l, p | 3b, p | 3b, r | 3b |
| VII | —1.. | b, bl, 2p | 2b, 2p | b, l, p | 2b, bl, 2p | b, r, 2p | 2b, br, p | 2b, l, p<br>2b | 3b, bl | 3b, r<br>b | 3b | 3b | 3b |
| VIII | 1.. | 2b | 2b | 2b | 2b | 2b | ......... | ... .... | ......... | ......... | ......... | ...... | .......... |
| | 2.. | b | 2b, 1p | b, l, p | 2b, r, p | 3b, br | 2b, l<br>b | 3b<br>2b | 3b<br>b | ......... | ......... | ...... | .......... |
| IX | 1.. | 2p | b, br, 2p | absent | 2b, 3p | b, r, 2p | 2b, a | 2b, bl | 2b, r, p | 2b, a | 2b, l | 3b | 2b |
| | 2.. | b, 3p | 2b, 4p | b, l, 3p | 2b, 2p | b, 3p | 2b, bl, 2p | 2b, r | 2b, p | b, l, p | 2b, bl | 3b | 3b |
| | 3.. | A | 2b, 3p | 2b, p | 2b, p, a | 2b, 3p | 2b, p | b, 2p | l | 2b, 2p | b, br, 2p | 2b, 2p | 2b |
| | 4.. | A | b, 2p | b, p | b, r, 2p | b, 3p | b, p, a | b, r, 3p | b, 3p | ......... | ......... | ....... | .......... |
| X | 1.. | 2b, 1p | 2b | 2b, p | b, 2p | br, p | b, br, p | b, bl, r | 2b, br | 2b | 2b | 2b | b, a |
| | 2.. | A | A | b | b, a | b, p | 2b, p | b, r | 2b, a | l | 2b | 2b | ........ |

point, and at first develops more rapidly than the bud which is to continue the main axis of the branch.[15]

Abundant material in apparently the best condition was found growing around Lake View. Ten plants from this locality were examined, and their condition is here given in tabulated form. In the table following, the number of the plant is given in Roman numerals, the numbers of the branches following it in Arabic numerals. Beginning with the lower portion of the branch and passing upward, the flower clusters are numbered consecutively. These numbers, designated by "cluster," are given in the first line at the top of the table. In the column beneath each of these numbers is shown the condition of the flowers of that cluster on the different branches of the different plants. The table was arranged in this form, not because a comparison of the condition of clusters of the

---

[15] The racemes of *S. rostratum* are produced by a scorpoid sympodial dichotomy of the branch, in which the racemes represent the alternate branches. At first the raceme develops much more rapidly than the bud which is to continue the main stem, and so the racemes, when flowering, are always well towards the outside of the plant.

TABLE D.

| 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | b, 2p | 2b, 2p | 3b, 2p | 3b, p<br>b | br, 2b, p<br>b | 3b, p | 2b,br<br>b | 3b | 3b<br>b | .... | .... | 6A, 35b, 2br, 21, 21p. |
| 3b, p | 2b, 1, p | 2b, bl | 2b, r | 3b<br>b | 2b | 2b | ...... | ....... | ..... | ..... | .... | 4A, 26b, 1bl, 21, 1r, 11p. |
| b, br, a | 2b,br,a<br>b | 2b, 1a | 2b | b | 2b | ....... | ....... | ....... | .... | .... | .... | 23b, 2bl, 4br, 11, 1r, 31p, 5a. |
| 2b | ......... | ....... | ....... | ....... | ....... | ....... | ....... | ....... | .... | .... | .... | 22b, 2br, 1bl, 2r, 12p, 2a. |
| b, 3p | 2b, bl | 2b, p | 2b, r | b | b | 2b,2p<br>b | b, 1, p | 2b, 1<br>b | 3b, a | 2b, a | 2b | 41b, 2bl, 1br, 21, 2r, 34p, 2a. |
| ....... | ......... | ....... | ...... | ....... | ....... | ....... | ....... | ...... | .... | .... | .... | 30b, 2bl, 1br, 11, 1r. |
| ....... | ..... ... | ....... | ...... | ....... | ....... | ....... | ...... | ...... | .... | ... | .... | 1A, 31b, 2bl, 1br, 1l, 1r, 2p. |
| ....... | ......... | ....... | ...... | ....... | ....... | ....... | ...... | ...... | .... | .... | .... | 1A, 17b, 1bl, 11, 3r, 8p. |
| 3b, a | 3b, a | 2b, 1 | 3b | 3b | 3b | 3b | ....... | ....... | .... | ...... | .... | 4A, 30b, 11, 1r, 18p, 4a. |
| 3b | 3b | 2b | ...... | ....... | ....... | ....... | ....... | ... .. | .... | .... | ...... | 1A, 28b, 11, 2r, 14p, 1a. |
| 2b | 2b | ....... | ....... | ....... | ....... | ....... | ...... | ....... | .... | ...... | ...... | 33b, 3bl, 1br, 21, 2r, 11p. |
| ...... | ......... | ....... | ....... | ....... | ....... | ....... | ...... | ....... | .... | .... | .... | 10b. |
| ...... | ......... | ...... | ...... | ...... | ....... | ....... | ....... | ....... | ..... | .... | ..... | 21b, 1br, 21, 1r, 3p. |
| ...... | ......... | ...... | ....... | ....... | ....... | ...... | ...... | ....... | ..... | .... | .... | 19b, 1bl, 1br, 11, 2r, 10p, 2a. |
| 3b | ......... | ...... | ...... | ....... | ....... | ....... | ...... | ....... | .... | .... | .... | 25b, 2bl, 21, 1r, 19p. |
| 2b, p | 2b, a | 3b | 2b | ....... | ........ | ....... | ...... | ....... | .... | .... | .... | 1A, 27b, 1br, 11, 18p, 2a. |
| ...... | ......... | ....... | ...... | ....... | ....... | ....... | ....... | ....... | .... | .... | .... | 1A, 7b, 2r, 15p, 1a. |
| ...... | ......... | ....... | ....... | ...... | ....... | ....... | ....... | ....... | .... | .... | .... | 18b, 1bl, 3br, 1r, 6p, 1a. |
| ...... | ......... | ....... | ....... | ....... | ....... | ....... | ....... | ...... | .... | .... | .... | 2A, 12b, 11, 1r, 2p, 2a. |
| | | | | | | | | | | | | 21A, 455b, 18br, 18bl, 211, 21r, 231p, 22a. |

same number is especially desired, but because this seemed the most compact form in which it could be arranged. In the columns under the different clusters, the condition of the flowers is designated as follows: *r*=right-, *l*=left-handed flower; *b*= bud; *br* and *bl* designate buds which are so well developed that it is possible to determine whether they are right-handed or left-handed—these buds will probably open the following morning; a=a bud or flower which has fallen off or failed to develop; A, indicates that the whole cluster has failed to develop. When an axillary cluster is developed it is included in a brace, with the cluster occurring immediately above it, the axillary cluster always being placed below. A seed pod is designated by *p*.

In the last column to the right the condition of each branch is summarized, and finally the grand total is given at the foot of the column.

In table D we have taken into account 241 flower clusters, and 21 which are either abortive or injured. The number of abortive clusters might be somewhat increased if great care had been exercised in looking for the accessory buds just above the axils of the lowest leaves on the branches. As a rule, however, the first internode or so, if questionable, was omitted. From this it would seem that about eight per cent. of the clusters fail to develop, a percentage which would probably be somewhat increased if care had been exercised in noting the buds where development had been arrested at a very early stage.

On the 10 plants, 14 axillary clusters were produced, being 5.5 per cent. of all the developed clusters. Of these 14 clusters, 2 produced 2 buds each, the others only 1; an average of 1.14 flowers per cluster. The 241 normally developed clusters produced 773 buds, an average of 3.27 flowers per cluster. Of the 773 buds produced on the 10 plants, 22 are found to be injured and fail to develop normally; a percentage of 2.84.

While the series of material is too limited to permit of indulging in generalizations, it might be of interest to note that on 40 plants bearing 332 seed pods, taken from two square feet of ground, September 4, and 3 other plants producing 130 seed pods, taken at the same time, not a single pod developed from an axillary cluster was found. These plants, however, with the exception of the last three, represent all produced on a definite area. It might not be improbable that the smaller, crowded plants would not be so likely to produce axillary clusters as the larger plants growing under more favorable conditions. We may at least conclude from this that the axillary flowers are of little consequence in the seed-producing capacity of the plant.

For the sake of convenience, it has been deemed advisable to summarize in table E the conditions of the flowers and buds which will probably open the day following, as given in table D. From this table, it will be seen that on the day the plants were examined 42 flowers were open—5.4 per cent. of the 773 buds produced on the 10 plants. These flowers as well as the buds, 36 in number, which were to open the next day, are equally divided into right- and left-handed.

The buds which are next to open do not, in any of the cases noted in the above tables, occur on a cluster with flowers which are already open.

TABLE E.

| | | | | |
|---|---|---|---|---|
| Plant I, | 6*r* buds, | 3*l* buds, | 2*r* flower, | 5*l* flowers. |
| II, | 2*r* " | 1*l* " | 2*r* " | 0*l* " |
| III, | 1*r* " | 2*l* " | 2*r* " | 2*l* " |
| IV, | 2*r* " | 4*l* " | 2*r* " | 2*l* " |
| V, | 0*r* " | 1*l* " | 3*r* " | 1*l* " |
| VI, | 0*r* " | 0*l* " | 3*r* " | 2*l* " |
| VII, | 1*r* " | 3*l* " | 2*r* " | 2*l* " |
| VIII, | 1*r* " | 0*l* " | 1*r* " | 2*l* " |
| IX, | 2*r* " | 3*l* " | 5*r* " | 4*l* " |
| X, | 3*r* " | 1*l* " | 2*r* " | 1*l* " |
| | 18*r* buds, | 18*l* buds, | 24*r* flowers, | 21*l* flowers. |

There seems to be no law governing the production of right- and left-handed flowers on the opposite sides of the main axis of the plant. Sometimes two right- or left-handed flowers will be produced in succession on one side of the raceme, and sometimes right- and left-handed alternate on the same side.

Concerning the method of pollination in *C. chamaecrista*, the writers have not been able to thoroughly satisfy themselves. Todd says: "I consider the following explanation most probable: In getting the pollen, some grains are dropped on the incurved petal, and by it made to adhere to points of the bee, and to such points in a right-handed flower as will carry it to the stigma of a left-handed flower, and *vice versa*." Robertson[16] says: "The pollen, being thus forced out of the terminal anther pores, falls either directly upon the bee or upon the lateral petal which is pressed close against the bee's side. In this way the side of the bee which is to the incurved petal receives the most pollen. . . . A bee visiting a left-hand flower receives pollen upon the right side and then flying to a right-hand flower strikes the same side against the stigma."

It is very difficult to see just what takes place when the flowers are visited by a large insect, but the writers have observed that when they are visited by honey-bees, for instance, the insect supports itself by hooking his left hind leg over the terminal, upturned portion of the stigma in a right-handed flower, and the right leg in a left-handed flower. The pistil then would serve the function of support for the insect visitor. It was noticed that sometimes bees would attempt to get the pollen by approaching the flower from some direction other than that described above. The insect usually failed in this, and after one or two unsuccessful endeavors would give up the

---

[16] Robertson, Charles, *loc. cit.*

attempt and support itself by placing the leg over the terminal portion of the pistil while it secured the pollen. The function of the incurved petal is not perfectly clear. With an insect well dusted over with pollen from both right- and left-handed flowers, it seems improbable that cross-fertilization in any considerable number of cases should occur from some grains dropped on the incurved pistil.

The writers are not sure that the insect in flying to another flower strikes the tip of the pistil against the side, as stated by Robertson. Certainly, in many cases, the insect, while collecting the pollen, supports itself by placing one leg over the tip of the pistil. When the leg bears a large mass of pollen, which is being stored there, it seems hardly possible that the flowers could fail to be pollinated. It might be suggested that, since the stamens for the most part point in the direction of the incurved petal, the function of this petal is to prevent access to the stamens, except in the cases in which the insect supports itself by means of the pistil. While this seems to the writers, at the present time, the most logical of the two functions so far suggested, much more careful observation work must be done before this point is finally decided. The petal may to a certain extent, in connection with the pistil, serve as support for the insect. Todd and Robertson observed only humblebees visiting the flowers. The writers obtained:

*Apis mellifica* Linn. Lake View, August 7. Seven specimens.
*Agapostemon texanus* Cress. Lake View, August 7.
*Mellisoides bimaculata* (St. Farg) Lepl. Lake View, August 7.
*Megachile petulans* Cress. Lake View, August 7.
*Bombus separatus* Cress. Lake View, August 7.

As in the case of *Solanum*, it will be seen that the collecting period extended over a very short period of time. More search would doubtless greatly increase the list.

Robertson reports the following species as collecting pollen: *Bombus virginicus* Oliv., *B. separatus* Cress., *B. americanorum* F., and *B. scutellaris* Cress.

August 28, when the blossoming season for *C. chamæcrista* was almost over, an examination of material from the above-named region was made for the purpose of determining the number of seeds produced by a single plant. Fifteen pods were selected at random from different plants and the number of ovules counted. It was impossible to tell about the number in each pod which were fully and normally developed seeds or which would become such; consequently this factor is not taken into consideration. The percentage of ovules which fail to develop is, however, small. The number of seeds found to the pod is shown by the following:

TABLE F.

| | |
|---|---|
| Pod I, 13 seeds. | Pod X, 14 seeds. |
| II, 14 " | XI, 17 " |
| III, 11 " | XII, 18 " |
| IV, 11 " | XIII, 17 " |
| V, 10 " | XIV, 15 " |
| VI, 12 " | XV, 14 " |
| VII, 16 " | |
| VIII, 12 " | Total XV, 202 seeds, or 13.4 per pod. |
| IX, 8 " | |

From this it will be seen that the minimum number of seeds found was 8, the maximum, with an average of 13.4. Since the pods were simply gathered at random, there is no certainty of gaining the maximum or minimum number of seeds, but a fair average of the number produced may be expected. September 4 three plants were examined to determine something about the range of variation in the number of ovules produced in the pods of a single plant. The results are given as follows:

Plant 1 varies from 5 to 11.

Plant 2 varies from 8 to 18.

Plant 3 varies from 9 to 14.

Plant 1 had 35 pods, plant 2 had 64, and plant 3 had 27.

Plant 1 was selected on account of the small number of seeds produced per pod.

It will be seen from table D that an average of nearly 3.3 flower buds per cluster is produced. These were moderate-sized, healthy plants, producing on the whole probably more than the average number of clusters per plant. On the ten plants, there were produced 342 clusters, which bore 344 seed pods, instead of about 1120, the number of flowers which might be expected, thus giving less than thirty-three per cent. of the buds which produce mature seed pods.

It will be seen that, while in the observations made on *S. rostratum* the flowers which failed to produce seed did not reach much over six per cent., in *C. chamæcrista* it is over sixty per cent. In addition to this fact, it is rare to see a seed pod of *S. rostratum* which has been destroyed by insects or other destructive agencies, while in 460 pods of *C. chamæcrista* which were examined at Lake View, September 4, not one was found which did not have some of the ovules destroyed by the larvæ of some insect, and probably this would amount on the average to fifty per cent. of all the seeds produced, being in the case of some plants as high as seventy-five per cent.

A convenient method of approaching the question of the production of seeds might be to determine the number of seed pods produced on a given area of ground. A general idea may be obtained from the examination of the plants growing upon two square feet of ground. In the first case, the plants were much crowded; in the second, not nearly so much so; in fact, it may be said they were growing under "normal" conditions. It might be interesting to compare the results. The material for the two tables was taken September 4.

FIRST SQUARE FOOT.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Plant | 1, | pods | 0 | Plant | 16, | pods | 10 |
| " | 2, | " | 10 | " | 17, | " | 2 |
| " | 3, | " | 4 | " | 18, | " | 0 |
| " | 4, | " | 13 | " | 19, | " | 0 |
| " | 5, | " | 0 | " | 20, | " | 0 |
| " | 6, | " | 12 | " | 21, | " | 0 |
| " | 7, | " | 3 | " | 22, | " | 6 |
| " | 8, | " | 1 | " | 23, | " | 1 |
| " | 9, | " | 4 | " | 24, | " | 5 |
| " | 10, | " | 3 | " | 25, | " | 2 |
| " | 11, | " | 0 | " | 26, | " | 3 |
| " | 12, | " | 25 | " | 27, | " | 0 |
| " | 13, | " | 2 | " | 28, | " | 3 |
| " | 14, | " | 0 | | | | |
| " | 15, | " | 0 | Total, | 28, | pods | 109 |

SECOND SQUARE FOOT.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Plant | 1, | pod | 1 | Plant | 8, | pods | 11 |
| " | 2, | " | 50 | " | 9, | " | 9 |
| " | 3, | " | 36 | " | 10, | " | 3 |
| " | 4, | " | 15 | " | 11, | " | 0 |
| " | 5, | " | 48 | " | 12, | " | 7 |
| " | 6, | " | 34 | | | | |
| " | 7, | " | 9 | Total, | 12, | pods | 223 |

In the first square foot of ground, where the plants were much crowded, of the twenty-eight plants, ten produced no seed pods at all, and of the remaining eighteen only six produced over five each. On these plants an average of a little less than four pods per plant was produced. In the second lot, where, evidently, the plants were not nearly so crowded, only four produced fewer than five seed pods, and there was a general average of 18.7 pods per plant.

On the first foot of ground, then, there might be produced in the neighborhood of 1300 seeds; on the second, 2600. The large *Solanum* upon which 40,000 seeds were estimated would probably cover an area of 12.5 square feet, giving 3200 seeds per square foot. Of course, these figures represent only certain isolated cases, which in a way are typical, but must not be taken to represent the average condition.

The largest plant noted September 4 had produced 100 pods, with an average of thirteen seeds per pod; this plant might show 1300 seeds.

Professor Todd discusses in his paper the occurrence of similar divergences from the typical form in other Solanaceæ and Leguminosæ, and tries to discover some hint as to their origin. Lack of material for observation precludes any present discussion of these points.

The results of these observations may be briefly summarized as follows:

### Solanum rostratum.

1. As Professor Todd observed, the numbers of right- and left-handed flowers on a plant of any considerable size are about equal.
2. As a general rule, only one flower opens at a time on a raceme, but very commonly two will open on the raceme the same morning, giving a right- and left-handed flower opening simultaneously, and thus permitting in a considerable number of cases pollination between flowers on the same raceme, even if Professor Todd's theory of the method of pollination be the correct one.
3. Even on the smaller branches of the plant, the flowers are almost always approximately divided into the two types.
4. The flower has a distinct odor.
5. Various species of insects visit the flowers for pollen. Many insects secure pollen without effecting pollination.
6. In a rather hasty microscopic examination, no very apparent difference was detected between the pollen from large and small stamens.
7. A very important function of the observed arrangement of

stamen and pistil in *S. rostratum* seems to the writers to be that of support for the visiting insect.

8. It might seem that the pollen from the small stamens is of much more importance in the process of fertilization than Professor Todd suspected, especially since it seems that there is much more certainty of the pollen from the small stamens reaching the pistil than there is of that from the large stamen. The fact that there is some question as to the fertility of the pollen from the large stamen in all cases, and that in the case of another plant stamens of somewhat similar arrangement seem to have lost entirely their direct reproductive function, would indicate the same.

9. In a limited number of cases the pollen from the large stamen of a flower seems to be fertile on its own stigma, as well as upon the stigma of a flower opening simultaneously on the opposite side of raceme.

10. Spontaneous self-pollination seems sometimes to occur.

11. The percentage of cases in which seeds develop in those flowers in which artificial pollination is effected in the same flower or in two flowers of the same raceme is much smaller than when cross-pollination is effected by insects, reaching, in the case of the somewhat limited experiments of the writers, only as high as 28.5 per cent. Whether this is partially due to the method of applying the pollen or not has not been determined; whether the seeds produced by these cases of pollination of the same flower or flowers on the same raceme are capable of germination or not has not yet been determined. It might be suggested that the low percentage of cases is due to a lack of fertility in the pollen of the large stamen.

12. Estimated from the number of seed pods which normally develop, the number of flowers in which pollination is not effected is very small, not reaching, in the observations of the writers, much over six per cent.

### Cassia chamæcrista.

1. Right- and left-handed flowers are produced at the same time on the plant. When several plants are taken, the number of right- and left-handed flowers produced is practically the same.

2. So far as observed, two flowers were never seen open at

the same time on a cluster, nor was a bud ready to open the following morning ever found on a cluster with an open flower. Thus, cross-pollination between flowers on the same cluster would not be possible, as it frequently is in *S. rostratum*.

3. So far as the writers have been able to ascertain, there is no law governing the producing of right- and left-handed flowers on the opposite sides of the main axis.

4. Various species of insects visit the flowers for pollen.

5. It seems that pollination is effected in many cases by the transfusal of pollen from the leg of the insect, where it is being carried, to the stigma of the stamen upon which it is supporting itself. The function suggested by Professor Todd for the incurved petal seems to the writers entirely improbable.

Botanical Laboratory, University of Kansas,
September 28, 1901.

## EXPLANATION OF PLATE I.

### **Solanum rostratum** Dunal.

FIG. 1. Right-handed flower from the front and a little to one side, showing large and small stamens, pistil, and wings of the corolla, which enfold pistil and large stamen in the bud. × 2.

FIG. 2. Tip of a raceme from the front, showing one left- and two right-handed flowers: also the decurved end of the raceme, with the buds.

FIG. 3. Lateral view of decurved tip of raceme, showing the buds.

FIG. 4. Lateral view of bud ready to open the following morning, showing the two lower lobes of the corolla, enfolding pistil and large stamen.

FIG. 5. Ovary and pistil. × 5.

FIGS. 6 and 7. Lateral and dorsal view of large stamen. × 5.

FIGS. 8 and 9. Lateral and dorsal view of small stamen. × 5.

FIGS. 10 and 11. Hind leg of *Bombus*, with and without mass of pollen.

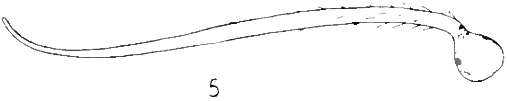
# PLATE I.


9


6


8


7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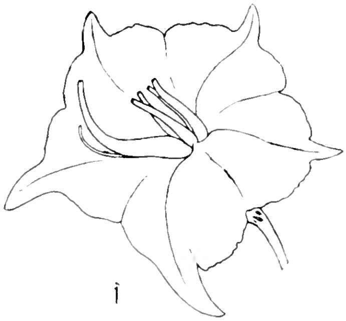
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THE

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# SPERMATID TRANSFORMATIONS

## In Gryllus assimilis, with Special Reference to the Nebenkern.

(Thesis for the degree of Master of Arts.)

BY W. J. BAUMGARTNER.

With Plates II and III.

IT is the object of these studies to follow carefully the spermatid transformations, and, if possible, to get a complete series of changes occurring therein. I do not pretend that I have as yet found all the stages, but in the present paper will publish observations made on *Gryllus assimilis* pertaining chiefly to the nebenkern, of which I have found some stages not previously described, as far as I know. I shall first describe my own findings, and then compare them with the results of others.

## I. Methods.

While other fixing agents were used, Flemming's chromo-acetic-osmic mixture (strong) proved the best. Its fixation was satisfactory in all but a few stages, where the chemical changes going on in the cell are such that it seems difficult, probably impossible, to secure good, clear results; at least, neither Gilson's nor Zenker's mixture did any better on cells in these stages.

Heidenhain's iron-hæmatoxylin stain gave the best results, although Flemming's three-color method was very satisfactory.

Most of the observations are made from sectioned material, but smear preparations were used for advanced spermatozoa.

## II. Material.

The first observations were made in January of the present year on some material prepared by Professor McClung, to whom I wish here to make most grateful acknowledgment for proposing the line of investigation, and for many helpful suggestions during the progress of the same. The testes were those of adult insects, and showed mostly only mature or almost mature spermatozoa. Enough spermatids were seen to indicate that their transformation into spermatozoa was somewhat peculiar.

The material for the drawings was taken from specimens collected under stones and boards on and near the University campus, at Lawrence, Kan. Young nymphs of *Gryllus* were caught as early as March 1, but the cells of the testes were all in the spermatogonia and spermatocyte stages. During the early part of May, after the nymphs had passed their third molt, crickets were secured whose testes showed all stages of development and transformation.

*Gryllus assimilis*, the common black field cricket, has paired testes lying in the anterior dorsal part of the abdomen. These have a whitish transparent appearance, which becomes duller in the adult, showing sometimes a slight yellowish tinge. The shape of each is that of a somewhat conically rounded body, not unlike a flattened strawberry. Each testis consists of a central rachis about 3 mm. long, from which extend a large number of curved follicles varying in length from 1 to 3 mm.

The follicles are larger toward the blind end and taper toward the rachis. Each follicle is divided into cysts, but more often transversely than longitudinally, for frequently one cyst occupies the whole follicle in cross-section. The cysts toward the rachis end of the follicle are much longer and narrower than those of the blind end. The cells are not very large and contain twenty-four chromosomes in the spermatogonial generations; eleven and twelve were most frequently found after the reduction.

The follicles of the cricket testis show the different cell generations and the same relative arrangement of them as McClung ('00) found in *Hippiscus*, and Sutton ('00) in *Brachystola*. The spermatogonia are nearest the blind end of the follicles, spermatocytes next, and the spermatids following the latter. The cells of a cyst are not, as in *Anasa* (Paulmier, '99), "in the same stage of development," but only approximately so, for some cysts show cells in the metaphase, while others have reached the telophase. Successive cysts, as in *Brachystola*, do not show successive stages; for, frequently, growing spermatocytes and late spermatids, or even young spermatozoa, were observed side by side. The individual cells in division pass through prophases, metaphases, anaphases, and telophases—these terms being used according to their usually accepted meanings. (See McClung, '00.) In this paper I shall begin with the cell when it has reached the telophase of the second spermatocyte division; that is, just after the chromosomes have reached the poles of the spindle.

## III. Observations.

### (*a*) THE NUCLEUS.

The chromatin in *Gryllus* behaves very much as Henking ('91) has described for *Pyrrhocoris*, and Paulmier ('99) for *Anasa*. At the end of the anaphase the chromosomes are crowded together at the poles of the spindle. (Fig. 1.) A clear space begins to be formed around them, especially on the equatorial side. (Fig. 2.) They then separate and are scattered somewhat evenly over the nuclear membrane when it is formed, as seen in figs. 4, 5, and 6. Soon protuberances appear on the surface of the individual chromosomes, making their outlines irregular. At about this time they begin to show a granular structure. The chromosomes break up more and more, and soon the chromatin appears in irregularly arranged patches or blotches of granules of various sizes. (Fig. 8.)

Meanwhile the nucleus has slowly increased in size. It reaches its maximum, which is probably about treble its original diameter, before the cell begins to elongate. The granules have been breaking up into finer and finer pieces, until, a little after the cell has reached its maximum, the chromatin has largely disappeared, or, to be more exact, has lost its affinity for stains.

(Fig. 16.) As the cell begins to elongate, the nucleus goes to one end, namely, that toward the periphery of the cyst. (Figs. 14 and 15.) The chromatin again frequently appears in patches as the cell decreases in size. (Figs. 33 and 36.) In many cells the patches never disappear. When it has diminished to less than half its maximum diameter, the nucleus begins to elongate, becoming elliptical, and finally forms the long, tube-like head of the spermatozoon. (Figs. 43 and 45.)

I found here that the nucleus was hollow, or, better, that it showed a clear space within. For a long time it contains the many chromatin granules, as seen in figs. 39, 40, and 41; but at last the walls seem to attract all these granular masses, and the center is entirely clear, as in fig. 45. In fig. 44 are shown cross-sections of spermatozoa heads, of the stage of fig. 43. In fig. 45 we have a mature spermatozoon head.

The *accessory chromosome* in *Gryllus* lags behind during the spermatogonial divisions, just as in *Xiphidium* (McClung, '99), and retains its identity all through the growth period, in many cases becoming quite large. At first it cannot be distinguished in the spermatid, but soon its stronger stain and exemption from disintegration make it apparent. (Fig. 4.) It is flattened against the nuclear wall and, unlike Paulmier's "small chromosome," it does not break up but remains intact, as Henking and McClung have described. (Figs. 14, 18, 32, 37, and others.) It gets larger and then smaller, and is finally lost in the rest of the chromatin, as the latter condenses to form the spermatozoon head. I was not able to see that it occupied one side of the elongating head, as McClung describes it in *Xiphidium*. The darker stain and regular contour, and sometimes its greater size, made it in most stages quite evident.

### (*b*) CYTOPLASMIC STRUCTURES.

It is in the cytoplasmic structures that *Gryllus* shows some things that have not been described, so far as I know; yet I almost hesitate to enter the discussion concerning the nebenkern, the mitosome, the idiozome, the archoplasm, the attraction sphere, the acrosome, the "mitochondrion körper" and other bodies which have been described in the extra-nuclear parts of the germ-cell. But because I have been unable to find in any of the papers a description of a structure like the one in *Gryllus*, I shall proceed to describe the nebenkern as I find it. I shall interpret the appearances as far as possible, leaving to others, or to later efforts, the complete harmonizing of this element with the structures of other species.

Before I proceed, I would add my protest against the name "nebenkern," as voiced by Calkins ('95), Erlanger ('96), Meves ('00), and others. It is inapt and not at all descriptive or distinctive. But since we have the word, and investigators will use it to mean something, I think it would be best to restrict the word, as Calkins and Erlanger suggest, to the remaining spindle fibers and connective fibers which go to each spermatid and which have an important part in forming the tail membrane. I shall retain the name for a further reason, which St. George ('97) gives for first using it, "weil sie die Frage nach der Herkunft dieses Gebildes offen lässt." A comparison of the literature will convince any one that the origin of this element is still a much-disputed question.

### 1.—*Nebenkern.*

There are, in *Gryllus* spermatids, two quite permanent and prominent extra-nuclear bodies. The larger of these I shall call *nebenkern*, because I believe it originates from the spindle remains, and goes to form the tail covering. Its behavior is as follows: At the beginning of the telophase the fibers which had connected the centrosomes with the chromosomes bulge out like barrel staves, as Henking ('91) has described. These break loose from the chromosomes and centrosomes as the clear space begins to form around the chromatin mass (fig. 2); but they are not separated into two rings, as in *Anasa* and *Pyrrhocoris*. As the dividing cell membrane is formed, the middle portions of the fibers are drawn together and the so-called "mid-body" is formed. Some of the fibers unite, producing thicker ones, as seen in figs. 1, 2, and 3. In the last-named figure some of the central fibers seem to be still united with those of the companion spermatids.

As the daughter-cell shortens and the nucleus takes a more central position these fibers unite still more, shorten, bend together (figs. 4, 5, and 6), and finally assume the shape shown in fig. 7. I shall call it for the present the "striated condition," or simply "striated nebenkern." The shape resembles that of an egg with rather sharply drawn-out ends. The surface is occupied at intervals with deeply staining lines which look much like hoops. In most cases the long axis of the nebenkern is at right angles to the axis of the spindle from which it was formed, while the striæ, or dark lines, are in planes parallel to it, as is seen in figs. 7 and 8*a*. In a few cases the lines are parallel to the long axis of the nebenkern, as shown in fig. 8*b*. Fig. 9 exhibits a partial end view of the nebenkern, while fig. 26 represents a polar view in all planes.

The fibers in figs. 4 and 5 are not all in the same plane, giving frequently, with changing focus, the impression that the lines are the edges of plates. Even in figs. 7 and 8, the appearances shown made me hesitate for some time before I dared say that the lines represent striæ on the surface and not plates extending through the body. But the end views (figs. 9 and 26) and the two drawings in fig. 19 show clearly that we are dealing with encircling hoops and not dividing plates. Just how or why the fibers bend and assume the shape of figs. 7 and 8 I am not able to say, but I am sure it is not an artefact, as might be supposed, because I have found them occupying whole cysts of several testes, and nearly all of them were as regular and definite as fig. 7.[1]

I have found the nebenkern in its perfect condition (fig. 7) only in material fixed in Flemming; although Zenker's fixation showed some of the breaking-up stages. This may be due partly to poorer fixation in other mixtures and partly to chance, as not so many testes fixed in other agents were examined, and the appearance evidently represents a very brief stage. These fibers now begin to break up, and we get figures that remind one of Meves's "mitochondrion körper"; that is, a darker staining center and radiating lines to the outer ring. (Fig. 11.) Sometimes stages a little later recall Paulmier's "blackberry stage." (Figs. 12 and 25.) Fig. 19*a* is a surface view and fig. 19*b* an optical section of a nebenkern that is in the process of breaking up. Figs. 20 and 21 have a part of one end cut off, while fig. 23 is a polar view. Fig. 22 is probably a forming blackberry stage.

All the above are brief stages, and we soon get the appearance of figs. 13–16. Here we have the darkly staining central ball surrounded by a clear space, enclosed by a ring, as Meves

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[1] Professor McClung, Mr. Sutton, and Mr. Blackman, of Kansas University, and Doctor Child and Mr. Harper, of Chicago University, have seen my preparations, and all confirm my statement that the appearance is *not* an artefact.

('00) found it in *Pygæra*. The central ball sometimes shows vacuoles (fig. 16), but most frequently stains as one mass. In some cysts with cells resembling fig. 16, it is difficult to distinguish between nucleus and nebenkern, yet the latter's stain is never as intense as that of chromatin. The stages shown in figs. 13 and 14 persist for a long time, and it is probable that this is the end of the telophase and the beginning of the spermatid transformations. I have treated the above more in detail and shown more drawings than its importance may deserve, but it is in these stages that *Gryllus* seems to differ from other species, and I have tried to describe and illustrate the apperances and changes fully.

As the cell elongates, the axial filament grows out, the nebenkern approaches it, and a junction is effected in such a way that the axial filament runs over the surface of the nebenkern. (Figs. 18, 27, 28, and 30.) At first the nebenkern is still almost round, but it begins to elongate and the dark inner ball sends out a protrusion to the nucleus, and sometimes back to the point of separation between the external envelope and the axial filament. The outside ring disappears and the dark mass moves down the axial filament. (Fig. 31.) In many cases it breaks up into several small drops, which appear at intervals on the axial filaments. (Figs. 34 and 36.) In fig. 38 is seen such a hanging drop. In this way the nebenkern material is distributed over the axial filament and forms a sheath around it.

### 2.—*The Acrosome.*

The second extra-nuclear body I shall call the acrosome, as it forms the point of the spermatozoon head. I could not trace its origin, because the fixation was not definite in some of the early stages. Fig. 4 shows a small, darkly staining body, which may be a centrosome, but more likely is the beginning of the acrosome, of which figs. 7, 9 and 10 show developing stages, and figs. 11, 13 and 15 more advanced ones. Sometimes the central portion stains darkest (figs. 11, 32, and 36), but more frequently there is a small clearer space in the center surrounded by a ring or band of darker material. (Figs. 10, 13, 27, and 30.) This ring usually stains darker on one side, very often on the side toward the nucleus.

The position, which is, with few exceptions, in the angle between the nucleus and the nebenkern, induces me to consider the differently staining bodies as the same organ of the cell. For some time after the nebenkern has disappeared the acrosome keeps its position; then it approaches the nuclear wall, flattens against it, and later wanders to the apical end, where it forms the tip of the spermatozoon. (See figs. 32–36 and 39–42.) Fig. 16 shows the acrosome back of the nebenkern. It is not an isolated example, still, I think, it is an abnormality. Fig. 37 shows an apparent division of the acrosome. I did not see enough instances to consider it a regular occurrence.

### 3.—*The Axial Filament.*

As the cell is elongating, the axial filament is seen apparently growing out of the nucleus. I do not mean to say that it grows out of the nuclear substance, but in almost every cell observed, where the axial filament was incomplete, there was rather a large mass of chromatin gathered at the place where the axial filament was attached to the nucleus. It may be that the smaller one of the extra-nuclear bodies in figs. 4, 7 and 9 is a centrosome which is passing to the equatorial region of the nucleus, where it later develops the axial filament. As already noted, the axial filament does not pass through the nebenkern, but only over its surface. Fig. 29, *a*, *b*, and *c*, shows different cross-sections through the elongating cell. In *b* the axial filament was cut at an angle.

### 4.—*The Centrosome.*

I have not followed the centrosome through its migrations. In fig. 2 one of the two bodies is the centrosome, but I did not trace its changes farther.

### 5.—*Cell Body.*

The cells, up to the stage of figs. 13 or 14, have been scattered promiscously through the cyst; but as the cell begins to elongate, the one end becomes the anterior-nuclear end, and it shifts to the periphery of the cyst. The central part of the cyst now shows the so-called central lumen. As the lengthening goes on the heads of the forming spermatozoa are all turned toward the distal end of the cyst. The cyst becomes very narrow and long, apparently preparing to contain the long, slender spermatozoa. Near the rachis the spermatozoa seem much twisted, and I surmise that they turn so as to have the head foremost when set free into the *vas deferens*. Fig. 42 may indicate some such behavior.

### (*c*) THE SPERMATOZOON.

The mature spermatozoon consists: (1) of a sharp, spear-like point; (2) a tube-like head about .02 mm. long, composed of a densely staining outer wall and a clearer central cavity; and (3) a filamentous tail about .5 mm. long. The point comes from the acrosome, the head from the nucleus, and the tail from the axial filament and nebenkern. While I do not deny the presence of a middle piece, my observations do not warrant me in describing one, although fig. 40 would suggest it.

## IV. Comparison of Results.

Dr. R. von Erlanger ('97, 1, and '97, 2) and Dr. F. Meves ('00) have given excellent discussions of the literature on the nebenkern. I cannot do better than direct the attention of other investigators to them. Hence, I shall discuss only such authors and such points as have special bearing on my own results. The spermatid changes in the Gryllidæ have been studied, so far as I know, only by St. George ('67) and by vom Rath ('92).

St. George, in his early paper, described the transformation process in the "Hausgrille" along with other insects. He found the "Nebenkörper," and saw that it took a part in forming the envelope of the axial filament, a small particle remaining near the nucleus and other drops of it appearing at intervals on the tail. In his paper on *Blatta* ('86, 1) he accepted Bütchli's name, "nebenkern," and traced its origin to the spindle remains. In his paper on *Phratora* ('86, 2) he takes up the description of the nebenkern in the spermatocytes and spermatogonia. In the spermatid, he describes the blackberry stage in these words: "Während das andere (not nucleus) Körperchen weniger lebhaft gefärbt als ein kleiner Fadenknäuel mit unregelmässigem Contour erscheint."

There can be no true nebenkern in the spermatogonia and spermatocytes, as St. George first used the word, and as I have limited its application. But St. George found the object, traced its origin and fate, and described several of the stages as they appear in *Gryllus*.

Vom Rath ('92) found *Gryllotalpa* very poor material for following the spermatid changes. However, he described an oval body which he found varied in appearance, depending on the method of fixation. He calls it the "nebenkern" and thinks it goes to help form the tail.

The description is too incomplete for comparison. Judging from *Gryllus*, I question its being very poor material.

Bütschli ('71) studied the spermatid transformations in Acrididæ and Locustidæ along with that of other animals. He saw the object formerly described by St. George and named it "nebenkern." He saw it divide into halves, elongate, and form the spermatozoon tail.

The dividing of the nebenkern into halves seems to be an appearance quite common in insect spermatogenesis. I have myself seen it in several genera of Acrididæ—*Hippiscus*, *Arphia*, *Melanoplus*, and *Brachystola*; besides Bütschli, St. George, Henneguy, Platner, Paulmier and others have described it.

Platner in his studies has given special attention to the nebenkern. In his first paper on "Pulmonates" ('85) he did not trace the origin of the nebenkern, but described it as consisting of four to six rods of different lengths and irregularly bent. These were connected, forming an irregular polygon. In its later stages he saw it with a mass of protoplasm pass down along the primary tail—an early protrusion of protoplasm. Finally it is lost.

In his succeeding paper ('86, 1) he studied the "nebenkern" spindle remains—in the spermatogonia and spermatocytes of pulmonates. In the spermatids, as in the former generations of cells, the nebenkern grows out of the nucleus, where it, with the chromatin, had formed the spireme. It appears as a loop, which becomes larger, twisted, and entangled, and finally breaks loose from the nucleus. Later it goes to form the spiral covering of the primary tail, changing it to the axial filament and true tail.

In his next paper ('86, 2) he describes the changes when the dividing of the protoplasm lags behind in the spermatocyte divisions. His description agrees in so many points with my own, that I shall quote his exact words:

"Die Spindelfasern hingegen contrahiren sich mehr und mehr nach dem Equator hin, wobei sie mit ein ander verschmelzen und merkwürdiger Weise je weiter dieser Verdichtungsprocess fort schreitet um so mehr an Tinctions fähigkeit speciell gegenüber dem Hämatoxylin gewinnen. Sie stellen jetzt zwei dreieckige oder hakenförmige Gebilde dar, die mit der Spitze noch im Equator zusammenhängen mit der breiten Seite sind sie den zugehörigen Zellen zugewendet. Hier sind ihre Grenzen undeutlicher, verwaschen und zeigen hier auch noch häufig eine streifige Beschaffenheit, welche auf ihren Ursprung hin weist. Zuweilen lassen sich einzelne Fäden noch eine beträchtliche Strecke weit in das Protoplasma hinein verfolgen, welches zwischen ihnen und den sich ausbildenden Zellkernen liegt."

The "häkenformige Gebilde" moves away from the periphery, its sides elongate, break, and unite at the nuclear end, thus form the polygonal nebenkern. "Derselbe geht also in diesem Falle direckt aus den Spindelfasern hervor." When the protoplasm does divide a similar process takes place. The spindle remains divide at an equatorial line and each half forms a nebenkern, and, as he says: "Also auch hier geht der Nebenkern direckt aus den Spindelfasern hervor. Vielleicht geht in den Spermatiden der Nebenkern überhaupt immer aus den Spindelfasern hervor, in dem die langfädige Verbindüngsbrücke, die ihn oft mit dem Kern verbindet, sich wohl als ein noch einige zeit persisttirender Rest der esteren deuten lässt."

In his paper on *Limax* ('89, 1) he followed the nebenkern through all the divisions and thinks it a constant organ of the cell. In the second part of the paper on *Helix* and *Paludina* the nebenkern was considered as formed from the remains of the spindlepole and the centrosome. Later Platner ('89, 2) found the nebenkern in the pancreas cells. In reports upon *Pygæra* and *Sphinx* he changed the name nebenkern to mitosome. The centrosome lies in front of the nucleus and forms the point of the head. This he calls the nebenkern. From the spindle remains arise two bodies. A large, fibrous one from the equatorial end has a clear space around it and the axial filament passes through it. It soon elongates and forms the tail. This is the large mitosome. The other is much smaller and arises from the polar end of the spindle fibers. It takes its place in the angle between the large mitosome and the nucleus. Here it persists till the nucleus begins to elongate, when it lengthens and surrounds the basal end of the spermatozoon tail. This is the small mitosome.

Platner ('85) saw a true nebenkern. I have already criticized the use of the word "nebenkern," as the name for the spindle remains in the spermatocytes. Platner himself later denied the nuclear origin of the nebenkern. His results ('86, 2) agree with mine concerning the uniting, converging, staining and bending of the spindle-fiber remains and their passing to the tail. Again, Platner ('89, 1) probably followed the centrosome and attraction sphere, as well as the nebenkern, in the second part of the paper on *Helix* and *Paludina*. In the pancreas the body is a result of secretion, and is not a nebenkern. In *Pygæra* Platner's large mitosome is the real nebenkern, as shown by its fibrous structure, its surrounding clear space and destiny. His small mitosome is what I have described as the acrosome, and he is mistaken as to its final use. The body he called centrosome in the nucleus is the persisting accessory chromosome. Such would be my interpretation of his figures.

Henking ('91) has followed the spermatid changes in *Pyrrhocoris* quite carefully. He finds that the fibers between the chromatin masses are separated into peripheral fibers and central spindle fibers. The first, a part of the second and the yolk mass, forms the nebenkern, which passes down over the axial filament. The rest of the central fibers form the mitosome. This takes its position at the angle between the nebenkern and the nucleus. On the surface of the nucleus it passes to the anterior end, then back to its original position. A piece now is constricted off and disappears, while the rest, increasing its affinity for stains, again wanders to the anterior pole of the nucleus, and becomes the acrosome.

The large amount of yolk substance is a disturbing element in *Pyrrhocoris*, but the nebenkern agrees with that of *Gryllus* in having the same origin and destiny, as does also the acrosome in fate and position. Besides, there is much similarity in the stages, as seen by comparing Henking's fig. 63 with my fig. 25, and his fig. 85 with my fig. 32.

Wilcox ('95 and '96) described the spermatid metamorphosis in *Caloptenus*. The interzonal fibers, a long, striated body composed very plainly of distinct fibers, contracts longitudinally, and the corners round themselves off, and this forms the nebenkern. It loses its fibrous structure, comes close to the nucleus, and then grows out into the axial filament. The centrosome moves half way around the nucleus and lies between the latter and the nebenkern. Later the centrosome becomes the middle piece.

This description does not give much detail nor do the drawings show the stages clearly. As far as given, the formation of the nebenkern is the same as in *Gryllus;* but in *Gryllus* the axial filament is not formed from the nebenkern, as can be plainly seen from fig. 17.

Erlanger ('96), in a short paper, discusses the use of the term "nebenkern," and suggests limiting it as is done in this paper. He opposes St. George's opinion, that the nebenkern comes from the cytomicrosomes. In *Blatta* the cytomicrosomes are preserved during the whole process of mitosis, and have no connection with the spindle fibers, but during the telophase they collect in reduced numbers around the daughter nuclei.

In 1897 he called the collection of granules around the centrosome, the centrodeutoplasm. He considers them to be identical with St. George's cytomicrosomes and the archoplasm (or attraction sphere) of other writers. In order to harmonize results, he suggests that, since the centrosome sometimes wanders around the nucleus, the centrodeutoplasm (or sphere) may unite with the spindle remains in some cases to form one body, as shown by the descriptions of Henking, Henneguy, Meves, and others. He later ('97, 2) describes the so called "sphere," and distinguishes between it and the true nebenkern.

I would strongly commend his excellent discussion of literature and his careful comparison of the results of investigators. He has shown clearly that the nebenkern comes from the spindle remains.

Calkins ('96) finds that the nebenkern comes from the spindle fibers and is useless in the cell. But *Lumbricus* is peculiar in having the nebenkern simply disintegrate, for, in many cases, he admits that the nebenkern has an important function. I do not have access to Henneguy's or Bolles-Lee's or Toyama's works, yet I should judge from Erlanger's and Meves's criticism that all of these have the nebenkern originate from the spindle remains, and Henneguy describes it as having a "fibrillar appearance," and Bolles-Lee as "fibrillar structure." Accordingly, I think that each of these has discovered the correct origin of the structure, and I do not doubt that there is, at least in the first two, a more or less direct change from the spindle remains to the nebenkern.

Paulmier ('99) finds that in *Anasa* the nebenkern comes from the yolk mass and remains of the spindle fibers. A part of this mass separates off, while the whole is still in a confused condition, and forms the acrosome. The nebenkern forms the tail sheath, while the acrosome forms the point to the head.

A comparison of Paulmier's fig. 42 with my fig. 4 suggests the thought that they are the same stage, and his fibers are remains of the spindle. His fig. 43, of course, agrees with my fig. 25; and fig. 43 may correspond to figs. 26 or 14, only that his stain is weaker.

Meves's investigations are the most extensive of the recent ones on the nebenkern. He has used *Paludina* and *Pygæra* as his objects. The consideration of the mitochondrion in the spermatocytes I shall pass over, as it is not within the bounds of this paper; yet I hope to study the earlier generations of my material, and shall then compare the results. In Meves's description of *Paludina*, I find but one point in which it agrees with *Gryllus*. The head of the young spermatozoon in *Paludina* has a clear space in the center filled with nuclear fluid, which remains till the head begins to stretch. In *Gryllus* it is hollow until maturity.

In *Paludina* the nebenkern in one kind of spermatids is formed from threads made up of mitochondria—small, round bodies identical with St. George's cytomicrosomes and Erlanger's centrodeutoplasm. These threads change to vesicles, which, reduced to four, closely surround the centrosome as it lengthens into the middle piece. At first they form a four-cleft cylinder, but finally a single sheath. An idiozome and spindle remains are seen in the spermatid. They persist for awhile, and the former changes into the acrosome. In *Pygæra* Meves finds two kinds of spermatids, distinguished by a small difference in size. The larger forms the typical spermatozoon. The spindle remains form a "Spindelrest körper," which is soon lost. The ends of the mitochondrion chains form a ring of dark mass surrounding a clear space. The ring is broken by radiating clear spaces. These spaces collect and unite into larger vacuoles, which surround the darker center. Finally there is only a dark ball with a surrounding clear space shut in by a ring. The centrosome with attached axial filament fastens itself to the nucleus; then both begin to grow longer. The axial filament passes over the surface of the ring surrounding the ball—the "mitochondrion körper" or nebenkern. This body elongates a great deal; then the darker mass puts forth numerous threads which surround the axial filament.

As to *Pygæra*, the peculiarity is the complete agreement of his "mitrochondrion körper" and my nebenkern in appearance and behavior for a part of the transformation and their complete disagreement in the other part. The question with me is, Do the two bodies whose final stages are so similar originate so differently, or has one of us mistaken the origin of the body?

Since reading Meves's paper I have carefully reexamined my material, and I am positive that I am right as to the origin of the nebenkern; but, on the other hand, I would not say that Meves is wrong in his observations, as in doing so I should fall into the same error which I think Meves himself has made. In his discussion of the literature he has forced every description to agree with his ideas, or has declared that the author has described some extra-nuclear organ as a nebenkern which is *not* a nebenkern. Thus by implication, if not by direct statement, he says that a nebenkern never comes from the spindle remains. With due respect to his ability and long experience as an investigator, I must say that Meves is mistaken in this. St. George himself traced the cytomicrosomes back to the spindle remains. Besides, the many investigators whom I have cited above cannot be mistaken as to the origin of the nebenkern. In my own material I am positive that there is a direct passing of the interzonal fibers over into the earlier stages of the nebenkern.

That the body which forms the tail covering does not come from the spindle remains in all cases, I am willing to admit. Meves has cited many investigators, especially on vertebrates, whose results favor such an opinion. The spindle remains do not change into a nebenkern even in all Arthropods, as Blackman ('01) finds no nebenkern, nor anything in anywise resembling it, in *Scolopendra.*

From my study of the results of other investigators, it is evident to me that there are at least two general methods for the formation of the covering for the spermatozoon tail. One of these plans will harmonize Meves's mitochondrion körper, Erlanger's centrodeutoplasm, Heidenhain's pseudo-chromosomes, and other similar structures. The other will show that Platner's large mitosome, Paulmier's blackberry stage and my striated nebenkern are only different stages of the spindle remains changing into the tail covering.

## V.—Summary.

1. The chromosomes of the second spermatocytes break up and the chromatin becomes diffused all through the nucleus. Later the chromatin collects in granules again and finally forms the walls of the tube-like spermatozoon head.

2. The spindle fibers break loose as the clear space is formed around the chromatin mass. They unite and contract, becoming fewer, thicker, and shorter. These bend and form the "striated nebenkern." The fibers break up and sometimes show a blackberry appearance. Soon there is a collection of darker material at the center, surrounded by a clear space,

which is shut off from the cytoplasm by a darkly staining membrane which in sections appears as a ring. This stage persists for some time; then the nebenkern moves against the axial filament, elongates, loses the ring, and the dark mass passes down along the axial filament. Often it appears in several small drops.

3. The axial filament does not come from the nebenkern nor from the acrosome. It comes apparently from the nucleus; probably from the centrosome closely attached to the nucleus. It never passes through the nebenkern—only over its surface.

4. The acrosome occupies a position in the angle between the nebenkern and the nucleus. It shows a central clear space surrounded by a darker mass which stains more intensely on one side. Later it passes to the front of the nucleus and forms the point of the spermatozoon head.

5. The mature spermatozoon consists of a sharp point, a slender, tube-like head filled with a clear fluid, and a long, thread-like tail.

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September 27, 1901.

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*Nos. 12 and 26 were used when describing the material only and not the spermatid transformations.

## Explanation of Plate II.

All drawings were made by the author with the aid of a camera lucida. All figures except 2, 3, 4 and 45 were made with a B. & L. $\frac{1}{16}$ objective and one-inch eye-piece, producing a magnification of 1340 diameters. Figs. 2, 3, 4 and 45 were drawn with a Leitz $\frac{1}{16}$ objective and a Zeiss compensating ocular No. 12, giving a magnification of 2325 diameters. The drawings were not reduced in the photo-mechanical reproduction.

Fig. 1. An early telophase, in which the fibers appear like barrel staves. Some of the fibers have thickened.

Fig. 2. A later stage, in which the fibers are uniting, the clear space is forming, and the chromosones are beginning to separate.

Fig. 3. A daughter-cell, showing the chromosomes separating and the fibers thickening.

Fig. 4. The chromosomes are scattered over the nuclear membrane: the fibers appear as a few thick rods. The beginning of the acrosome is seen. The accessory chromosome can be distinguished.

Fig. 5. Same as fig. 4, with some of the fibers curved and no acrosome.

Fig. 6. Same as fig. 5, but with the acrosome visible.

Fig. 7. The chromosomes are granular: the fibers have bent and formed the round "striated nebenkern." The acrosome is present.

Fig. 8*a*. Same as fig. 7, but the acrosome is not present.

Fig. 8*b*. The fibers run parallel with the long axis of the nebenkern.

Fig. 9. Here is shown a partial end view of the nebenkern. The accessory chromosome and the acrosome are both prominent.

Fig. 10. The nebenkern shows some of the fibers on the under side. It is beginning to break up.

Fig. 11. The nebenkern shows the beginning of the dark center; the fibers extend from it to the ring. The acrosome has a peculiar appearance.

Fig. 12. The fibers have broken up and the whole has assumed a vesicular appearance, resembling a blackberry. The acrosome is present.

Fig. 13. This shows a persisting spermatid stage. The chromatin appears in patches. The nebenkern is in the ball-and-ring stage. The acrosome shows its characteristic clear center surrounded by a ring darker on one side. The two dark bodies on the nuclear surface are probably artefacts.

Fig. 14. A stage a little later than 13. The dark central ball shows some vacuoles. The cell is beginning to elongate.

Fig. 15. Same as fig. 14, with the ball denser.

Fig. 16. The nucleus has become almost clear. The acrosome is behind the nebenkern.

Fig. 17. This shows the axial filament apparently growing out from the nucleus. There is an aggregation of chromatin at its base.

Fig. 18. The axial filament passes over the nebenkern, which has slightly elongated. The accessory chromosome is very plain.

Fig. 19. *a* shows a surface view, and *b* an optical section, of a nebenkern which is in process of breaking up.

Figs. 20 and 21 have part of the surface fibers cut off and show the dark mass forming within.

Fig. 22. Shows a nebenkern passing from the blackberry stage into the ring stage.

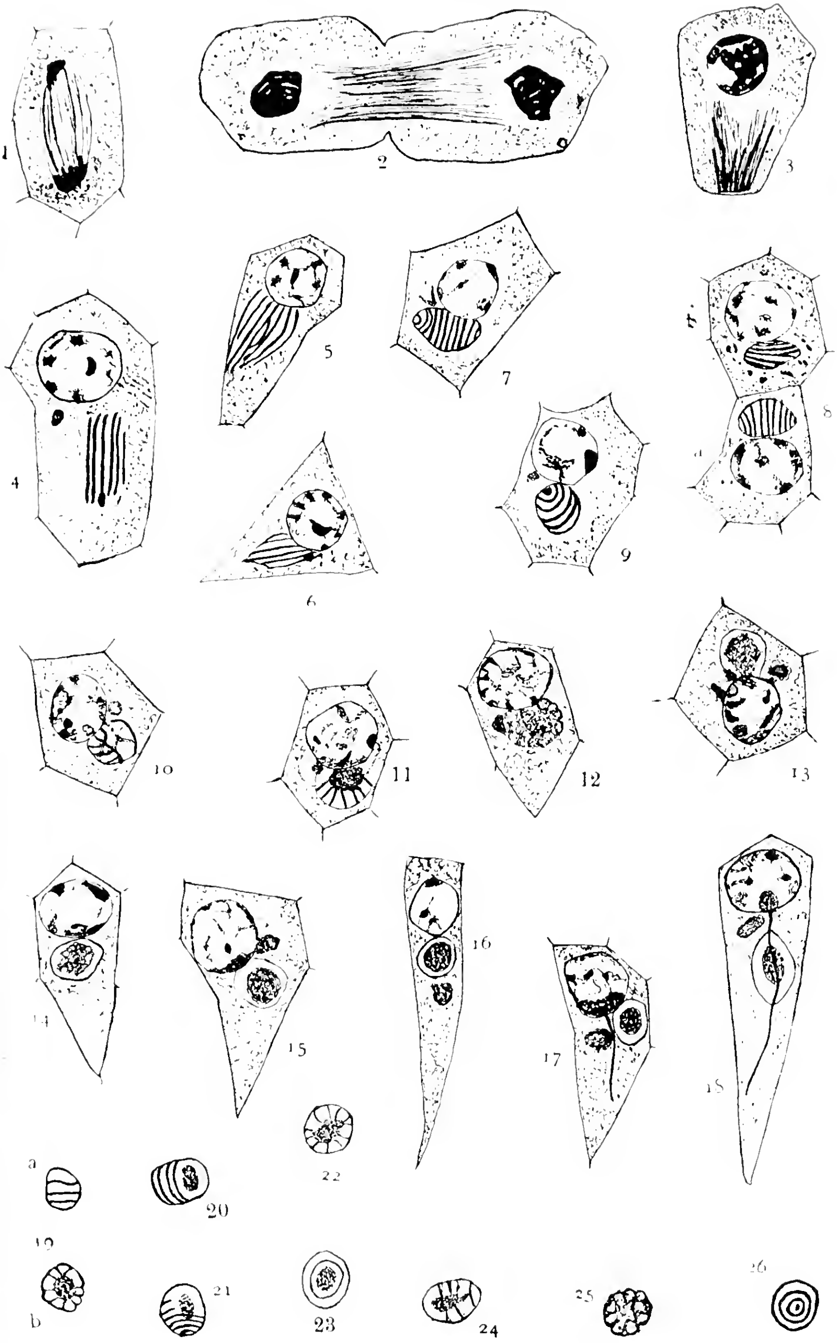
Fig. 23. This is an end view of the stage shown in figs. 20 and 21.

Fig. 24. The fainter lines are the fibers on the under side of the nebenkern.

Fig. 25. The blackberry stage, showing the mass of vesicles.

Fig. 26. This represents a nebenkern of the stage shown in fig. 7, drawn from a polar view, with adjusting focus.

# PLATE II.



## Explanation of Plate III.

Fig. 27. This shows the nebenkern elongating. The axial filament passes over its surface.

Fig. 28. A little later than fig. 27.

Fig. 29. Shows cross-sections of elongating nebenkern: *a* shows axial filament, nebenkern and acrosome: in *b* the axial filament is cut at an angle.

Fig. 30. A little later than fig. 28: the axial filament not so plainly shown.

Fig. 31. The ring has disappeared, and the dark mass is passing down along the filament. The acrosome is indistinct.

Fig. 32. The nebenkern has disappeared, but the acrosome has kept its position.

Fig. 33. The acrosome has flattened against the nucleus. Some remains of the nebenkern are seen.

Fig. 34. The acrosome has moved to the side of the nucleus. Some nebenkern remains are shown.

Fig. 35. The acrosome is in front of the nucleus.

Fig. 36. A little later stage than fig. 35.

Fig. 37. Shows a divided acrosome. It is not a frequent appearance.

Fig. 38. The nebenkern appears as a hanging drop on the side of the axial filament.

Fig. 39. The nucleus has condensed, the walls have thickened, and the chromatin appears in granules. The acrosome is pointed.

Fig. 40. A little later stage than 39.

Fig. 41. A little later stage than 40.

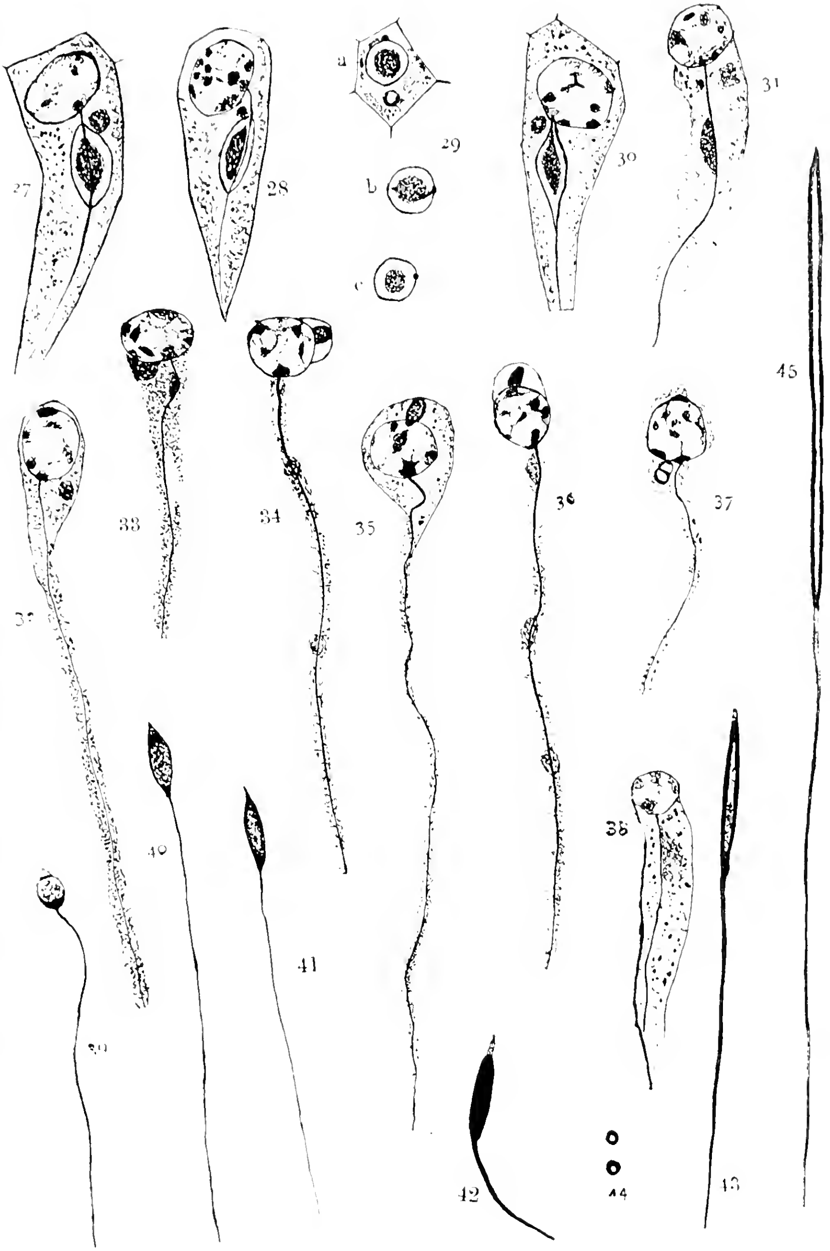
Fig. 42. The head is stained intensely, while the acrosome is lighter. Probably a middle piece indicated.

Fig. 43. The head shows the clear space.

Fig. 44. Cross-sections of the head, of the stage shown in fig. 43.

Fig. 45. The head and part of the tail of a mature spermatozoon.

# PLATE III.



# THE

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# DOLICHOPODIDÆ OF GRENADA, W. I.

BY J. M. ALDRICH.

THIS family is represented by fifty-five species, of which fourteen are herein described as new, and twenty-seven were described in my previous paper on the Diptera of St. Vincent. Twelve species in the lot from St. Vincent were not found in this collection. Only one of these was known prior to the publication of the St. Vincent paper.

I have made a consolidated list of the Dolichopodidæ of these two islands for comparison of their fauna. *V.*, after the species, indicates its occurrence on St. Vincent; *G.*, on Grenada.[1] A star indicates that the species was previously known. All unstarred are described in the two papers. New genera are indicated.

*List of Dolichopodidæ from St. Vincent and Grenada.*

GYMNOPTERNUS.
- ruficornis. V.

PELASTONEURUS.
- argentiferus. V. G.
- unguiculatus. V.

PARACLIUS.
- nigripes. G.
- fuscicornis. G.
- discifer. G.

PARACLIUS.
- quadrinotatus. G.
- *venustus. G.
- bellus. G.
- filiferus. V. G.
- abdominalis. G.
- *arcuatus. G.
- *claviculatus. G.

---

[1] NOTE.—Grenada and St. Vincent are two small islands of the Windward group of the Lesser Antilles, distant, respectively, about 75 and 150 miles from the Venezuela coast. Extensive collections were made in the two islands a few years ago, by Mr. H. H. Smith, for the Entomological Society of London. The material from St. Vincent was worked up by Dr. S. W. Williston, except the families *Dolichopodidæ* and *Phoridæ*, which were undertaken by me, at Doctor Williston's request. The report was published in Trans. Ent. Soc. Lond., 1896, pp. 253-446, 7 pl. It was intended that the Grenada material should be published in the same way, and the present paper was prepared some time ago: but pressure of work has prevented the completion of the report on the other families, and it has been thought best not to delay the publication of the present paper any longer.

*List of Dolichopodidæ from St. Vincent and Grenada.*

- Sarcionus.
  - lineatus. V. G.
- Hercostomus.
  - latipes. V. G.
- Polymedon.
  - superbus. V.
- Neurigona.
  - decora. G.
  - signifera. V. G.
- Anepsius.
  - linearis. V. G.
- Sympyonus.
  - frater. G.
  - falco. V. G.
  - similis. V. G.
- Leptocorypha, n. gen.
  - pavo. V.
- Eutarsus.
  - sinuatus. V. G.
- Diaphorus.
  - amœnus. G.
  - dimidiatus. V. G.
  - flavipes. V. G.
  - contiguus. V. G.
  - simplex. V. G.
  - dubius. V. G.
  - *opacus. V. G.
  - *spectabilis. V. G.
  - parvulus. V.
- Asyndetus.
  - *syntormoides. G.
  - fratellus. V. G.
- Chrysotus.
  - paradoxus. G.
  - albipalpus. V. G.
  - niger. V. G.
  - excisus. V. G.
  - hirsutus. V. G.
- Chrysotus.
  - longipalpus. V. G.
  - proximus. V. G.
  - flavus. V.
  - acutus. V.
  - inermis. V.
  - *picticornis. V. G.
  - *barbatus. V. G.
- Thrypticus.
  - cupuliferus. V. G.
  - minor. V. G.
  - singularis. V.
- Achalcus.
  - caudatus. G.
  - sordidus. V. G.
- Xanthina, n. gen.
  - plumicauda. G.
- Thinophilus.
  - sp. G.
- Cœloglutus, n. gen.
  - concavus. V.
- Medeterus.
  - exiguus. G.
- Plagioneurus.
  - *univittatus. G.
- Leptorhethum, n. gen.
  - angustatum. V.
- Psilopus.
  - bellulus. V. G.
  - *jucundus. G.
  - *chrysoprasius. V. G.
  - *caudatulus. V. G.
  - insularis. V. G.
- Gnamptopsilopus.
  - flavicornis. V. G.
  - flavidus. V. G.
  - *bicolor. V.

The most striking things about the table are the preponderance of *Paraclius* in Grenada, near the mainland, and the larger number of new genera in St. Vincent, farther off shore.

The mounting of these collections is exceptionally fine, and deserves mention. Out of several hundred cases where a male and a female were mounted on the same pin, there was only one where they were not of the same species. This is an astonishing fact—a record to be proud of, considering that the collecting was general and these species all minute.

## PARACLIUS.

Loew, Monogr., 97, 1864 (*non* Bigot).

The relationships of *Paraclius*, *Pelastoneurus* and *Sarcionus* are somewhat puzzling. I have considered as belonging to *Pelastoneurus* those species only which have the face broad in both sexes, with a slight concavity in the upper part and a marked convexity below. The arista is distinctly plumose. The fourth vein in its last part converges toward the third with more or less abruptness, but I do not regard this as the essential character. *Sarcionus* includes species with narrow face, comprising in the male about a tenth the width of the head; it is but little, if any, convex below; the arista is plumose. In the female the face is about a sixth the width of the head, and has more resemblance to that of *Pelastoneurus*. The fourth vein converges strongly toward the third in the latter part of its course. In *Paraclius* the face is like *Sarcionus*, but the arista is bare or only moderately pubescent. The fourth vein near its end makes a somewhat sudden curve toward the third, then gradually resumes its former course, forming a curve with its concavity outward; this is quite variable in shape, and in at least one species there is no marked concavity, as in *Pelastoneurus vagans*. The characters of the face and arista seem most substantial.

*Table of North American and West Indian Species of* PARACLIUS.

| | | |
|---|---|---|
| 1. | Front femora black | 2 |
| | Front femora yellow | 5 |
| 2. | Tibiæ all black, costa not thickened | *nigripes*, n. sp. |
| | Tibiæ not all black, or else costa conspicuously thickened | 3 |
| 3. | First section of costa greatly thickened in male, moderately in female | *albonotatus* Loew. |
| | First section of costa not thickened | 4 |
| 4. | Hind tibiæ yellow | *fuscicornis*, n. sp. |
| | Hind tibiæ mostly black | *pumilio* Loew. |
| 5. | Antennæ black | *arcuatus* Loew. |
| | Antennæ chiefly yellow | 6 |
| 6. | Arista in the male enlarged at tip, disk-like | 7 |
| | Arista in the male not enlarged at tip | 8 |
| 7. | Arista long, tapering, thin | *discifer*, n. sp. |
| | Arista quite short, thick | *claviculatus* Loew. |
| 8. | Two velvet-black spots between the root of the wing and the humerus | *quadrinotatus*, n. sp. |
| | Only one spot | 9 |
| 9. | Dorsum of thorax wholly deep green on anterior part | 10 |
| | Dorsum of thorax with a bronze stripe between acrostichal bristles | *filiferus* Ald. |
| 10. | The curve of fourth vein occurs half-way between the posterior cross-vein and the apex | *venustus*, n. sp. |
| | The curve occurs two-thirds of the way to the tip | *bellus*, n. sp. |

**Paraclius nigripes**, n. sp.

Male: Face with dense whitish dust, rather wide in the lowest part, antennæ of ordinary size, wholly black, arista bare, front with gray dust, orbital cilia, except a few above, snow-white. Thorax steel-blue above; the same color overlaid with thin white dust on the pleuræ. Halteres yellow, tegulæ whitish, with black cilia. Abdomen dark blue-green, rather dull, with some white dust along the sides; hypopygium large, the appendages small, brown; the lamellæ are only rudimentary. Legs altogether greenish black, with thin white dust, from the knees down less green. Fore tarsus as long as its tibia; hind metatarsus as long as the next joint. Wings narrow, infuscated except at base, the fourth vein ending considerably before the apex, only a little recurved near the tip.

Female: Face but little wider, wings less infuscated.

Length, 3.5 to 3.8 mm.; of wing, 2.5 mm. Five males, five females. The shape of the face and the venation are strikingly like the genus *Macellocerus*. The head is not nearly so lengthened vertically, however, as in that genus. It must be regarded as a connecting link between the two genera.

**Paraclius abdominalis**, n. sp.

Male: Face moderately narrow, extending down to the lower edge of the eye, white on its lower two-thirds, somewhat brownish above; front covered with white pollen, except the ocellar triangle and the root of the large vertical bristles; antennæ black, of moderate size, with plain, tapering arista; orbital bristles white, except about one-third of them on the upper part. Thorax bright, shining bluish green above, very slightly inclined towards violet posteriorly; a broad, black, velvet-like spot above and before the root of the wing; a small, yellow spot just over the humeral spiracle; pleuræ black, thinly dusted with pale gray or whitish; halteres and tegulæ yellow, cilia of the latter black. Abdomen crossed with nearly equal bands of green and violet, the latter occupying the last half of each segment; the green bands are covered with white dust on the sides, and gradually widened; hypopygium thick, large, black, with small, short lamellæ of a sordid white color, black on the edges, and bearing some stout curved hairs on the free basal angle, which is prominent. Legs black, the tips of all coxæ, the trochanters, extreme tips of femora, all the tibiæ and the bases of all tarsi yellow. Wings subhyaline, only faint traces of brown along the veins; the bend of the fourth vein occurs just past the middle of the last section, and is quite abrupt; the curve back towards its former course is also rather short, so that the vein is nearly parallel with the third for some distance at the end; costa with a long and striking swelling, beginning just after the humeral cross-vein, and continuing almost to the tip of the first longitudinal. Length, 4.4 mm.

Female: Face as in male; front with pollen a little grayish; abdomen with same bands; thickening of costa perceptible.

Length, 4.2 mm. Four males, two females. The following label on one: "Lights generally on stones in streams; commonly seen in pairs, the two sexes lighting near to each other; very wary, and difficult to capture."

The species is closely related to *albonotatus* Loew, but the latter has darker wings, third and fourth veins convergent to the tips, swelling of costa larger in female, etc.

**Paraclius fuscicornis**, n. sp.

Male: Face very narrow in the middle, with brownish pollen, sometimes more whitish; antennæ varying from light to dark brown, of ordinary structure; front with light brown dust. Cilia of lateral and inferior orbit white. Thorax deep blue, varying to purple, shining; pleuræ with thin whitish dust; halteres yellow; tegulæ yellow, their cilia black, abdomen shining blue, the posterior margins of the segments deep purple in the best-developed specimens, the sides of the segments with white dust; hypopygium large, thick, the lamellæ short, brown. Coxæ and tarsi black; the trochanters, knees and tibiæ yellow. The femora are not so dark as in the related species, more deep brown than black. Wings rather slender, slightly infuscated, bend of fourth vein midway between great cross-vein and apex, last part of fourth parallel with third. Costa not thickened.

Female: Face a little wider than in male.

Length, 3.5 to 4.3 mm.; of wing, 3.3 to 4 mm. Five males, five females, one bearing label: "Balthazar (Windward), 250 feet, April 7. Lighting on stones in bed of shady stream."

**Paraclius arcuatus.**

Loew, Neue Beitr., VIII, 39; Monogr., 101.

Twenty-three specimens, both sexes, one labeled: "Lights on wet mud, generally near water."

These specimens differ quite materially from Loew's description in having the fore coxæ scarcely infuscated at base, hind femora only a little infuscated toward the tip, hind tibiæ yellow except the tip, and the wings only a little brownish; in other words, a light variety as compared with the types. I should have regarded these as a distinct species had I not the undoubted *arcuatus* from Jamaica (by the kindness of Mr. C. W. Johnson) for comparison.

**Paraclius discifer**, n. sp.

Male: Face moderately wide, with silvery dust; front bright green; antennæ yellow, third joint slightly infuscated at tip, with arista a little longer than the front femur, thick at base, very slender toward the tip and ending in a small oval lamella, just large enough to be visible to the naked eye when held against a white background. Lateral and inferior orbital cilia white. Thorax bright green, a little dusted, the space between the acrostichal bristles, and a roundish spot on each side above the transverse suture, bronze; pleuræ rather dark with gray dust, halteres and tegulæ yellow, cilia of the latter black. Abdomen bright green, but little dusted; hypopygium large, somewhat greenish, the lamellæ brown, with a long upper corner curving backward, so that they might be called crescent-shaped, attached by the lower angle. Front coxæ yellow, the others infuscated; femora and tibiæ yellow; all the hairs infuscated from the tip of first joint. Wings subhyaline, the posterior cross-vein forming a right angle with the axis of the wing, the curve of the fourth vein half way from the cross-vein to the tip, making a very obtuse angle. Third and fourth veins parallel at the apex for a short distance.

Female: Face a litle wider, arista normal, with moderate pubesence. Thorax without the bronze spots.

Length, 3 mm. One male, one female.

**Paraclius claviculatus.**
Loew, Centuries, VII, 83.

Two males. The arista is rather thick, two-thirds the length of the front femur; the wings are infuscated and the dorsum darker in one specimen. There is a very close relation with *discifer*, yet they are undoubtedly distinct.

**Paraclius quadrinotatus**, n. sp.

Male: Face moderately wide, the pollen whitish gray; front with brownish pollen; antennæ red, of ordinary appearance, the arista slightly pubescent, cilia of the lateral and inferior orbit white, thorax above somewhat mottled; when viewed from the side, the light coming from above, there is a large, velvet-black spot above the root of the wing and another still larger in the transverse suture. These are separated from each other by a narrow metallic area, and present a very distinctive appearance. Pleuræ black, with gray dust, tegulæ and halteres yellow; tegular cilia black. Abdomen green, the segments dusted with white on the sides; hypopygium of moderate size, the lamellæ small, black, triangular. Fore coxæ yellow, the others brown; femora, tibiæ and first joint of tarsi yellow; tips of tarsi moderately infuscated. Wings more or less infuscated on the costal part, usually but little; curve of fourth vein more than half way from cross-vein to apex, not at all angular, the return curve just before the tip of the same shape reversed.

Female: Differs only in having the face a little wider.

Length, 2.4 mm.; of wing, 2.2 mm. Twenty-five specimens.

**Paraclius filiferus.**
Aldrich, Trans. Lond. Entom. Soc., 1896, 314.

Numerous specimens.

**Paraclius venustus.**
Aldrich, Biologia. Diptera, Supplement.

Numerous specimens.

**Paraclius bellus**, n. sp.

Male: Face narrow, silvery; front green, antennæ red; the arista slightly pubescent; cilia of the lateral and inferior orbit white. Thorax dark blue, shining; the silvery spot on each side at the suture rather large; pleuræ rather heavily coated with bluish-white dust; halteres and tegulæ yellow; cilia of the latter black. Abdomen with a ringed appearance, the anterior half of each segment bright green, on the sides whitish-dusted, the posterior half blackish, dull; hypopygium large, whitish-dusted; the lamellae very small, yellow, with the apical third blackish. Front coxæ and the tips of the others yellow; femora and tibiæ yellow, fore tarsi infuscated at the tip only, middle ones from the tip of the first joint, hind ones from the base; hind femora and tibiæ both distinctly darkened at the apex. Wings with a yellowish tinge, the fourth vein bent forward near its tip and thence nearly straight, as in *Pelastoneurus*. The hind femora have a row of bristles above of which the last four or five toward the end are quite conspicuous.

Length, 3.1 to 3.6 mm. Four males, seven females.

## PELASTONEURUS.

Loew, Neue Beitr., VIII, 36, 1861; Monogr., 103.

**Pelastoneurus argentiferus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 313.

Thirty-five specimens, both sexes.

## SARCIONUS.

Aldrich, Biologia, Dipt., Suppl., 1902.

**Sarcionus lineatus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 312 (*Pelastoneurus*).

Numerous specimens, both sexes. The width of the face in the male is one-twelfth, in the female one-sixth, the width of the head; therefore the species cannot be a *Pelastoneurus*.

## HERCOSTOMUS.

Loew, Neue Beitr., V, 9, 1857.

**Hercostomus latipes.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 311.

Numerous specimens, both sexes.

## NEURIGONA.

Rondani, Prod. Dipt. Ital., 142, 1856.

**Neurigona signifera.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 337.

One male.

**Neurigona decora,** n. sp.

Male: Eyes barely contiguous on the upper part of the face, slightly separated above and below; front broad, opaque, dark; antennæ small, red, the tip of the third joint brownish; orbital cilia pale. Thorax bright, shining blue, the concavity before the scutellum more bronze; pleuræ green, with thin dust, and the hind margin yellow; tegular cilia yellowish. Abdomen rather short, the first two joints yellow, the rest dark green, shining above. Hypopygium rather prominent, exserted, yellow. Coxæ yellow, the middle ones dark at base; remainder of legs and the tarsi yellow; a slender hair on the outer side of the second joint of the fore tarsus at its apex. Wings a little yellowish; the fourth vein converges toward the third at the end, terminating before the apex of the wing.

Female: Face linear; eyes not contiguous.

Length, 2.8 to 3 mm. Two males, two females. One of the latter is from St. Vincent, but was not mentioned in the previous paper.

**Anepsius linearis.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 317.

Nineteen specimens.

**Eutarsus sinuatus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 334.

Nineteen specimens.

## PLAGIONEURUS.

Loew, Wien Ent. Monatsch., I, 43.

**Plagioneurus univittatus.**

Loew, loc. cit.; Neue Beitr., VIII, 69; Monogr., 196.
Williston, Trans. Amer. Ent. Soc., XIII, 295.

One female. This species has a very striking distribution, considering that it is everywhere uncommon. It is found from Brazil as far north as South Dakota, and east to Massachusetts.

## SYMPYONUS.

Loew, Neue Beitr., V, 42, 1857; Monogr., 185.

**Sympyonus frater,** n. sp.

Male: Front blackish with gray dust; face narrow below, of the same color; antennæ of moderate length, first two joints yellow, the third black, rather pointed, arista inserted near the base; palpi rather pointed, yellow; occiput with gray dust, the orbital bristles small, at the sides and below whitish. Thorax dark green above, with thin dust and rather large bristles; pleuræ black, with dense, almost white, dust; tegular cilia dark, in some lights almost russet, in others blackish; halteres yellow; abdomen not elongated, scarcely equaling the head and thorax in length, not compressed, dull green in color, with row of longer hairs at the margin of each segment; hypopygium black, thick, rounded, protruding behind and below on the ventral side, making up nearly half the length of the abdomen, with some partially concealed slender yellow parts at the front end. Middle coxæ infuscated on the outer side, the other coxæ and all the feet yellow; the tarsi scarcely infuscated; fore tarsi as long as the tibia, the last three joints of equal length, the fifth black, the inner claw stout, black, curved back upon the joint to form a clasping organ, very much as in *S. falco;* the outer claw normal. On the front side of the hind femur below is a row of black bristles, longer toward the tip; wings grayish, the last section of the fourth vein almost perfectly straight, the third converging toward it at tip.

Female: Like the male, but the face not narrowed below; the fore tarsi normal; no bristles on hind tibiæ.

Length, 2.2 to 2.4 mm.; of wing, 2 mm. Numerous specimens, both sexes. This species, with *falco* and *similis* and the Mexican *coxalis*, forms a group of close relationship, yet sufficiently distinct from each other.

**Sympyonus falco.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 336.

Two females. The black antennæ and plain fore tibiæ sufficiently distinguish the females from those of the related species.

**Sympyonus similis.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 336.

Two males, two females. The row of bristles on the fore tibia occurs also in the female, making an easy distinguishing mark.

### DIAPHORUS.

Meigen, Syst. Beschr., IV, 1824.
Loew, Mon. N. A. Dipt., II, 156, 1864.

In attempting to separate this genus from *Chrysotus*, which it often greatly resembles, I have followed these rules: First, to assign to *Chrysotus* all species in which the eyes of the male are approximated below the antennæ, and to *Diaphorus* all in which they are approximated above; second, of the considerable number of species in which there is no approximation, to refer to *Chrysotus* all in which the male does not have a group of large bristles at the apex of the abdomen.

In my preceding paper on the Dolichopodidæ of St. Vincent, I made a bad blunder in failing to recognize *Diaphorus spectabilis* Loew, which I redescribed as *D. approximatus*. I have since found further that the numerous females which would almost inevitably accompany "numerous males" in a collection were mixed up with the females of *Lyroneurus simplex* Ald., and they are there yet, as I am unable to find any characters for their separation. The two species being nearly, if not entirely, indistinguishable in the female sex, it seems evident that the genus *Lyroneurus* is insufficiently characterized, and I have accordingly abandoned it. Loew himself (Mon. N. A. Dipt., II, 169) was a little staggered when he came to compare the genus with some North American species of *Diaphorus;* but with the material then known it still seemed tenable. It would certainly be an anomaly, however, if two females not distinguishable from each other would have to be placed in separate genera.

*Table of West Indian Species of* Diaphorus.

| | | |
|---|---|---|
| 1. | Femora yellow | 2 |
| | Femora black or brown | 5 |
| 2. | Fore coxæ with a row of three long bristles | *dimidiatus* Ald. |
| | Fore coxæ with only small bristles | 3 |
| 3. | Male with very large pendent white palpi | *amœnus*, n. sp. |
| | Male with only ordinary palpi | 4 |
| 4. | Eyes of male not approximated | *parvulus* Ald. |
| | Eyes of male approximated, tarsi infuscated (Cuba) | *subsejunctus* Loew. |
| | Eyes of male contiguous, tarsi not or scarcely infuscated | *flavipes* Ald. |
| 5. | Color green, shining | 6 |
| | Color black, but little shining, wings deeply infuscated | *opacus* Loew. |
| | Color black, dorsum gray pollinose, opaque, wings but little clouded | *contiguus* Ald. |
| 6. | Eyes of male approximated | *spectabilis* Loew. |
| | Eyes of male not approximated | 7 |
| 7. | Male with a long erect bristle on the under side of the hind metatarsus | *simplex* Ald. |
| | Male without such bristle | *dubius* Ald. |

(*Parvulus* and *subsejunctus* are not represented.)

**Diaphorus dimidiatus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 322.

Thirteen males, four females. The latter are readily distinguishable by the bristles of the fore coxæ, although they do not have the yellow color on the abdomen that characterizes the males.

**Diaphorus amœnus,** n. sp.

Male: Face rather wide, green, in an oblique direction yellow pollinose; front wide, bright green; antennæ black, the first joint slightly elongate, third pointed, arista completely apical; palpi very large, yellowish white, hanging down about a third the height of the head, about two-thirds as wide as long; cilia of inferior orbit white; thorax and abdomen bright green above, somewhat golden, the former a little dusted; pleuræ blackish green, slightly pruinose with white; tegular cilia few, yellowish white; halteres yellow, abdomen slender, the bristles at apex distinct; fore coxæ yellow, the others brown; femora and tibiæ yellow, the hind femora slightly infuscated at tip; fore and middle tarsi slightly exceeding their tibiæ in length; all tarsi gradually infuscated toward the tip; front pulvilli elongated, the others also a little conspicuous; wings of medium size, tinged with gray.

Female: The only female I have is almost exactly like that of *flavipes*. The antennæ being more distinctly black, and the coxæ with noticeably longer and more numerous brown hairs, are the only differences I can detect. The male *amœnus* has but a few long, brownish hairs in a row down the fore coxæ.

Length, 2.5 mm.; of wing, 2 mm. Three males, one female.

**Diaphorus flavipes.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 323.

Numerous males and females.

**Diaphorus opacus.**

Loew, Neue Beitr., VIII, 56; Mon. N. A. Dipt., II, 160.
Aldrich, Trans. Lond. Entom. Soc., 1896, 320.

Six males. A comparison with specimens from the United States establishes their identity, as also that of the St. Vincent specimens referred to in the previous article.

Length of the present specimens, 2.8 mm.

**Diaphorus contiguus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 323.

Eight males, three females.

**Diaphorus spectabilis.**

Loew, Neue Beitr., VIII, 57; Mon. N. A. Dol., II, 162.
Aldrich, Trans. Lond. Entom. Soc., 1896, 321 (*D. approximatus*, n. sp.)

Numerous males and females, one labeled "Windsor estate (Windward), 1500 feet, March 28. Cocoa orchard, on herbage." Comparison with specimens from the United States fully establishes the identity. The only difference between the females of this and *simplex* is that the latter are, on the average, slightly larger, but doubtless there are individual cases the other way.

**Diaphorus simplex.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 333 (*Lyroneurus*).

Numerous males and females. I have explained the change of generic reference at the beginning of the discussion of this genus.

**Diaphorus dubius.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 324.

Five males, seven females.

## ASYNDETUS.

Loew, Centur., VIII, 58, 1869.

In this genus the essential character is in the costal vein reaching only to the tip of the third vein, rather than in the fracture, or even the evanescence, of the fourth vein. The latter characters vary somewhat, but in all the species there is at least a marked thinness of the last part of the fourth vein.

Loew considered that the elongated fore pulvilli of the male would keep *interruptus* in the genus *Diaphorus*, under which it was described. The shortened costal vein, applying to both sexes conspicuously, seems to me to outrank the size of the male pulvilli. I therefore place the species in *Asyndetus*.

*Table of North American Species of* ASYNDETUS.

1. Last part of fourth vein making approximately two right angles .......... 2
   Last part of fourth vein only gently curved in the middle .......... *syntormoides* Wheeler.
2. Fourth vein at the end straight, ending exactly in apex .......... 3
   Fourth vein at the end curved back, ending distinctly behind apex .......... 4

*Table of North American Species of* ASYNDETUS.

| | | |
|---|---|---|
| 3. | Dorsum of thorax with a broad, brown-dusted middle stripe, enclosed by two bluish-gray ones | *fratellus* Ald. |
| | Dorsum not striped | *interruptus* Loew. |
| 4. | Second joint of fore tarsi in the male with a large spine, | *ammophilus* Loew. |
| | Second joint of fore tarsi in the male with two yellow processes | *appendiculatus* Loew. |

NOTE.—The last two are known only from the United States; *interruptus* is Cuban.

**Asyndetus fratellus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 332.

Two males, two females.

**Asyndetus syntomoides.**

Wheeler, Proc. Cal. Acad. Sci., 1899, 32.

One male, two females. The species ranges as far north as Massachusetts, and west to Kansas.

## CHRYSOTUS.

Meigen, Syst. Beschr., IV, 40, 1824.

*Table of Species of* CHRYOTUS.

North America, including West Indies.

| | | |
|---|---|---|
| 1. | Femora chiefly black or brown | 2 |
| | Femora yellow, tips of hind ones sometimes infuscated | 16 |
| | Hind femora black, middle ones yellow, front ones usually with a dark stripe above (U. S. and W. I.) | *barbatus* Loew. |
| 2. | Cilia of tegulæ black or brown | 3 |
| | Cilia of tegulæ pale | 10 |
| 3. | Third antennal joint in the male large, the tip drawn out in a slender point (W. I.) | *acutus* Ald. |
| | Third antennal joint in the male large, with a blunt tip, upper basal angle rising prominently above the second joint, lower edge straight, eyes of male contiguous | *obliquus* Loew. |
| | Third antennal joint in the male elongate, blunt, symmetrical | *cornutus* Loew. |
| | Third joint not conspicuously elongate | 4 |
| 4. | Eyes of male not approximated on the face | *vividus* Loew. |
| | Eyes of male approximated or contiguous | 5 |
| 5. | Palpi of male large, projecting, snow white (W. I.) | *albipalpus* Ald. |
| | Palpi of male of ordinary structure | 6 |
| 6. | Femora metallic greenish black | 7 |
| | Femora brownish black | 8 |
| 7. | Middle tibiæ wholly yellow | *obliquus* Loew. |
| | Middle tibiæ largely infuscated | *affinis* Loew. |
| 8. | Third antennal joint of male small, plain (W. I.) | *niger* Ald. |
| | Third joint of male kidney-shaped, arista inserted in an apical notch | 9 |
| 9. | Tibiæ black or brown (W. I.) | *excisus* Ald. |
| | Tibiæ wholly yellow (W. I.) | *proximus* Ald. |
| 10. | Eyes of male approximated | *choricus* Wheeler. |
| | Eyes of male not approximated | 11 |
| 11. | Palpi white | *longimanus* Loew. |
| | Palpi blackish | 12 |

*Table of Species of* CHRYOTUS.

| | | |
|---|---|---|
| 12. | Wing of male with thickened costa | 13 |
| | Wing of male without thickened costa | 15 |
| 13. | Costa with a strong enlargement | *costalis* Loew. |
| | Costa with a slight enlargement | 14 |
| 14. | Abdomen with pale hairs | *pratincola* Wheeler. |
| | Abdomen with black hairs | *subcostatus* Loew. |
| 15. | Femora dark green, shining | *discolor* Loew. |
| | Femora black | *auratus* Loew. |
| 16. | Cilia of tegulæ pale | 17 |
| | Cilia of tegulæ black | 21 |
| 17. | Antennæ black, first joint yellow (U. S. and W. I.) | *picticornis* Loew. |
| | Antennæ black | 18 |
| 18. | Eyes of male approximated, not contiguous | 19 |
| | Eyes of male contiguous | 20 |
| 19. | Abdomen with rather abundant black hair above | *pallipes* Loew. |
| | Abdomen with very few hairs (W. I.) | *paradoxus*, n. sp. |
| 20. | Male with dense brown hair down the outer side of the hind tibiæ and tarsi (W. I.) | *hirsutus* Ald. |
| | Male without such hair | *wisconsinensis* Wheeler. |
| 21. | Antennæ yellow (W. I.) | *longipalpus* Ald. |
| | Antennæ black | 22 |
| 22. | Eyes of male not approximated (W. I.) | *inermis* Ald. |
| | Eyes of male contiguous | 23 |
| 23. | Third antennal joint of the male drawn out in a fine point (W. I.), | *acutus* Ald. |
| | Third antennal joint not drawn out (W. I.) | *hirsutus* Ald. |

**Chrysotus albipalpus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 327.

Numerous males and females, one labled "Mt. Gay (Leeward), 300 feet, Sept. 10. Flying at sunset."

**Chrysotus niger.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 327.

Numerous males and females.

**Chrysotus excisus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 325.

Numerous specimens.

**Chrysotus hirsutus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 328.

Numerous males and females.

**Chrysotus longipalpus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 329.

Numerous males and females.

**Chrysotus picticornis.**

Loew, Monogr., II, 184; Wheeler, Psyche, June, 1890, p. 328; Aldrich, Trans. Lond. Entom. Soc., 1896, 329.

Numerous males and females. I have seen it also in abundance from Georgia. An easily recognized species.

**Chrysotus proximus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 326.

Numerous specimens. The lighter color, as compared with *excisus*, applies principally to the fore and middle tibiæ; the thoracic dorsum also is less blue and more bright green.

**Chrysotus barbatus.**

*Synarthrus barbatus* Loew, Neue Beitr., VIII, 48; Mon. N. A. Dipt., II, 138 (see also p. 134) (male only).

*Chrysotus validus* Loew, Neue Beitr., VIII, 63; Mon. N. A. Dipt., II, 175 (female only).

*Xiphandrium americanum* Wheeler, Ent. News, VII, 152 (both sexes).

*Chrysotus apicalis* Aldrich, Trans. Lond. Entom. Soc., 1896, pt. II, 330 (both sexes); Ent. News, XI, 533, synonymy.

Numerous specimens of both sexes.

As will be seen, this species has had peculiar misfortunes at the hands of the describers, in which I have indicated my share. If the insect could deceive Loew so completely, certainly the later writers may be pardoned. Professor Wheeler called my attention to the identity of his *X. americanum* with *S. barbatus*. The remainder of the synonymy is my own. The peculiar coloring of the femora has been the main guide in associating these descriptions: Hind ones green, middle yellow, fore yellow, with more or less of a dark stripe above, making a very unusual combination. I see no great impropriety in referring the species to this genus.

**Chrysotus paradoxus, n. sp.**

Male: Face narrow, the eyes decidedly approximated, front bright green; antennæ small, blackish, with minute pale hairs; palpi minute, dark; dorsum of thorax bright green, the pleuræ blackish; halteres yellow; tegulæ yellow, with a few hairs which are decidedly pale in the proper light, but otherwise indistinctly brownish, especially at base. Abdomen bright above, the hairs yellowish or brownish, not distinctly black; venter slightly pale anteriorly; hypopygium mainly concealed, black, with minute yellowish appendages. I notice two rather strong hairs, somewhat like those of *Diaphorus*, but they are not blunt at tip. Legs slender, yellow, with minute yellowish hairs; tarsi rather elongate, last joint but little infuscated, the pulvilli all rather large; middle coxæ considerably infuscated. Wings rather slender.

Length, 1.4 mm. One male.

The name is intended to refer to the distinct interblending of the characters of *Diaphorus* with those of *Chrysotus*.

## MEDETERUS.

Fischer de Waldheim, Notice, etc., 1819.

**Medeterus exiguus**, n. sp.

Male: Minute, greenish black, the tibiæ, tarsi and tips of femora yellow, hind metatarsi less than half the length of the following joint and noticeably thicker than it; antennæ black, minute; face and the minute palpi black. Hypopygium black, elongate, slender, reaching the hind coxæ. I cannot make out the lamellæ. Only the apical joint of the tarsi infuscated. Wings hyaline, posterior cross-vein more than its length from the margin, third and fourth veins distinctly convergent. Halteres pure yellow. Bristles of the thorax brownish, almost yellow in some lights.

Female: Hind metatarsi as in the male, but slightly infuscated at tip. Face rather blue above.

Length, 1.4 mm. One male, one female.

## CHRYSOTIMUS.

Loew, Neue Beitr., V, 48 (1824).

**Chrysotimus barbatus**, n. sp.

Male: Face broad, gray, with a distinct transverse angle or suture, just above which are two distinct hairs nearly as large as those of the orbit; palpi small, dark; front gray; antennæ short, rounded, rather dark yellow, the arista dorsal. Dorsum of thorax, pleuræ and abdomen of a dull cinereous cast, with rather pale hairs and bristles. Feet, including coxae and posterior margin of pleuræ, yellow. Hind metatarsus less than half the length of the following joint, with a fine, dense row of pale hairs forming a brush in the shape of an arc of a circle on the inner side at the tip. This is a cleaning or scraping organ for the wings, body, and other legs, I take it.

Length, 1.6 to 1.7 mm. Two females.

## THRYPTICUS.

Gerstaecker, Stett. Ent. Zeit., 1866, 43.
Wheeler, Psyche, 1890, 375 (*Aphantotimus*); Proc. Cal. Acad. Sci., 1899.
Aldrich, Trans. Lond. Entom. Soc., 1896, 339 (*Xanthotricha*); Ent. News, 1900, 533.

**Thrypticus cupuliferus.**
Aldrich, loc. cit. (*Xanthotricha*).

Eight specimens.

**Thrypticus minor.**
Aldrich, loc. cit., 340 (*Xanthotricha*).

Nine specimens.

## XANTHINA.

New genus (*Xanthos*, yellow).

Similar to *Xanthochlorus;* in the described species without metallic color; bristles of legs brown, of body dark brown, almost black; face very narrow in both sexes, eyes almost contiguous in the male; prescutellar depression very distinct. Antennæ short, first joint minute, third rounded, hairy; arista with moderate pubescence, almost or completely apical; acrostichal bristles in two rows. Hypopygium concealed, with two hairy filaments nearly as long as the abdomen. Palpi and legs hairy, hind tibiæ especially so. Venation as in *Chrysotus*.

**Xanthina plumicauda**, n. sp. (Plate IV, fig. 1).

Male: Front and very narrow face, dorsum of abdomen except the base, and a spot below the tegula, black; elsewhere yellow; palpi decidedly large and hairy; hind tibiæ with two rows of pale bristles on opposite sides toward the tip; wings hyaline.

Female: Abdomen dark brown on the middle of the dorsum; hind tibiæ with ordinary but coarse hairs.

Length, 1.4 to 2.6 mm.; wing about the same. One male, three females.

## THINOPHILUS.

Wahlberg, Oefv. K. Vet. Akad., 1844, 37.
Wheeler, Ent. News, VII, 152-156, 1896.

A single specimen of an undescribed species. As Professor Wheeler is engaged on this genus, I prefer not to describe the present species.

## ACHALCUS.

Loew, Neue Beitr., V, 30, 1857.

**Achalcus sordidus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 339.

Six specimens.

**Achalcus caudatus**, n. sp.

Male: Face but little narrowed, blackish; palpi and antennæ small, dark brown, the latter with small, rather pointed third joint, the arista almost completely apical, finely pubescent; hairs of the orbit few, seeming to be paler below. Front and thorax black, overlaid with brown dust, which becomes somewhat bluish gray on the pleuræ. Abdomen rather dull black. Cilia of tegulæ yellowish; halteres yellowish, with brown knob. Hypopygium rather disengaged, brown, rounded, in some specimens showing parts of the slender yellow organs inside. Coxæ and all the legs and feet sordid yellow; the fore coxæ and femoræ specially rather dark. Hind metatarsi distinctly shortened. Wings uniform grayish, somewhat slender, fourth vein ending scarcely past the apex; first vein reaching one-fourth the length of the wing; sixth vein represented only by a slight fold.

Female: Except for the absence of the hypopygium, and a slightly stouter form, the same as the male.

Length, 1 to 1.1 mm. Fifteen specimens, both sexes; one labeled "Windsor estate (Windward), 500 feet, March 28. Cocoa orchard, on foliage."

## PSILOPUS.

Meigen, Syst. Beschr., VI, 1824.
Loew, Monogr. N. A. Dolichopodidæ, 229, 1864.
Aldrich, Kans. Univ. Quart., II, 47, 1893.

**Psilopus bellulus.**

Aldrich, Trans. Lond. Entom. Soc., 1896, 343.

Twelve specimens, both sexes.

**Psilopus chrysoprasius.**

Walker, List, etc., III, 646.
Loew, Neue Beitr., VIII, 90; Monogr., II, 266.
Aldrich, Trans. Lond. Entom. Soc., 1896, 343.

Numerous specimens, both sexes.

**Psilopus caudatulus.**
Loew, Neue Beitr., VIII, 93: Monogr., II, 271.
Aldrich, loc. cit., 343.

Thirteen specimens, both sexes.

**Psilopus insularis.**
Aldrich, loc. cit., 344.

Two males, two females.

**Psilopus jucundus.**
Loew, Neue Beitr., VIII, 87; Monogr., II, 258.

## GNAMPTOPSILOPUS.

Aldrich, Kans. Univ. Quart., II, 48, 1893.

**Gnamptopsilopus flavicornis.**
Aldrich, Trans. Lond. Entom. Soc., 1896, 342.

One female.

**Gnamptopsilopus flavidus.**
Aldrich, loc. cit., 341.

Two males, four females.

# PLATE IV.

FIG. 1. *Xanthina plumicauda.*
New genus and species. × 20. The head had become shriveled in drying.

THE

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# REPORT OF SANITARY CHEMICAL WATER ANALYSES.

BY EDWARD BARTOW.

IN the spring of 1898, there was begun at the University a series of systematic analyses of the waters of the rivers, wells and water-supplies of the state. The work has been carried on with more or less continuity till the close of the school year in June, 1900. At that time, owing to the removal of the department of chemistry to the new building, and the fact that the rooms set apart for such work could not be equipped, it was necessary to stop the work. It is hoped, however, that we will soon be able to take it up again.

The results set forth in this article have been obtained by the help of several of the students. The first analyses* were made by C. E. McKinnie, Ph. C., 1898, and G. M. Smith, Ph. C., 1898. The analyses were continued by D. F. McFarland, B. A., 1900, W. J. Rothrock, B. A., 1900, B. C. Winslow, B. A., 1899, E. A. Sweet, Ph. C., 1900, and F. W. Fogwell, Ph. C., 1900.

As far as possible, each student has made a specialty of some problem, at the same time carrying on the work in such a way as to give a systematic comparison of all the work. There have thus been made comparative analyses of the water of the Kansas river at Lawrence at different seasons, and at different stages of the river. There are comparative analyses of the river water, filtered and unfiltered; comparative analyses of the river water, above and below the city, at the middle of the river, and along the bank. There are analyses of well-waters from city wells and from wells in the country near Lawrence; analyses of water from wells suspected of contamination, and of

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* Proceedings of the Kansas Pharmaceutical Association, 1898, page 53.

those which were supposed to be all right. There were but few analyses of waters away from Lawrence, owing to the difficulty of obtaining samples. Through the kindness of Prof. E. C. Murphy, a number of samples were obtained from the Neosho, the Verdigris, and the Marais des Cygnes. The comparison of these with the waters of the Kansas is quite interesting. It would be still more interesting, however, if the analyses of the river waters could be made more frequently, and at different stages of the rivers.

The methods of analysis are those in use in the laboratory of the state board of health of Massachusetts. Analyses have usually been made on the day following the date of collection, or within twenty-four hours from the time the water was obtained. The records are given in parts per million, according to the recommendation of the American Chemical Society.

We realize that the work has been merely begun, and that but little has been done, due partly to the lack of a well-equipped water-analysis laboratory, and partly to the fact that the students who assisted in the work were carrying on the regular university courses at the same time. Nearly 250 analyses have been made, the reports of which are arranged in the twelve tables of this report.

Table I.—Analyses of the water from the Kansas river, collected about 100 feet above the dam at Lawrence. (Parts per 1,000,000.)

| Number. | Date of collection. | Gauge. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 1 | 1898.<br>Feb. 4 | 0.0 | Slight. | Slight. | 0.0 | 644 | 42 | 602 | 114 | .019 | .094 | .334 | .000 | ..... | .... |
| 3 | " 11 | 1.1 | V. D. | V. H. | 0.0 | 736 | 204 | 532 | 86 | .171 | .612 | .208 | Trace. | ..... | .... |
| 5 | " 18 | 1.4 | " | " | 0.0 | 1,410 | ...... | ...... | 41 | .400 | 2.290 | 1.070 | ...... | ..... | .... |
| 9 | " 26 | 0.8 | Slight. | Slight. | 0.0 | 607 | ...... | ...... | 69 | .253 | .930 | .450 | ...... | ..... | .... |
| 13 | Mar. 5 | 0.6 | " | " | 0.0 | 844 | ...... | ...... | 69 | .066 | .312 | .439 | ...... | ..... | .... |
| 17 | " 11 | 0.4 | " | " | 0.1 | 574 | ...... | ...... | 77 | .006 | .299 | .323 | .000 | 215 | .... |
| 22 | " 18 | 0.8 | " | " | 0.0 | 566 | ...... | ...... | 67 | .020 | .510 | .230 | ...... | 220 | .... |
| 24 | " 25 | 0.9 | " | Heavy. | 0.1 | 656 | ...... | ...... | 58 | .036 | .700 | .638 | ...... | 230 | 10.8 |
| 31 | Apr. 1 | 0.6 | " | Slight. | 0.0 | 706 | 152 | 554 | 67 | .060 | .560 | .286 | ..... | 240 | 5.7 |
| 35 | " 8 | 1.0 | " | " | 0.0 | 744 | 176 | 558 | 59 | .085 | .645 | .243 | ...... | 230 | 8.0 |
| 45 | " 15 | 1.0 | " | " | 0.0 | 446 | 70 | 370 | 56 | .039 | .596 | .379 | .000 | 205 | 7.4 |

TABLE I.—*Concluded.*

| Number. | Date of collection. | Gauge. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 50 | " 22 | 1.8 | V. D. | V. H. | 0.2 | 3,080 | 582 | 2,498 | 37 | .310 | 5.990 | .150 | .002 | 170 | 16.0 |
| 55 | " 29 | 1.2 | ........ | ........ | 0.0 | 750 | 198 | 552 | 46 | .125 | .895 | .370 | .001 | 210 | 11.5 |
| 59 | May 6 | 2.7 | V. D. | V. H. | 0.0 | 2,426 | ..... | ..... | 41 | .340 | 4.810 | ..... | ..... | ..... | .... |
| 65 | " 13 | 1.6 | Decided. | Heavy. | 0.0 | 868 | 208 | 660 | 26 | .110 | 1.110 | .024 | ..... | 204 | 10.7 |
| 66 | " 16 | 3.9 | V. D. | V. H. | 0.0 | 7,348 | 778 | 6,570 | 18 | .660 | 10.290 | ..... | ..... | ..... | .... |
| 70 | " 20 | 3.1 | " | Heavy. | 0.0 | 4,172 | 458 | 3,714 | 14 | 1.800 | 3.575 | ..... | ..... | 187 | 46.6 |
| 77 | Jun. 10 | 5.2 | " | V. H. | 0.0 | 7,132 | ..... | ..... | 13 | 1.000 | 15.010 | .000 | .025 | ..... | 79.2 |
| 79 | " 17 | 3.7 | Decided. | Heavy. | ..... | 3,248 | ..... | .... | 9 | .280 | 4.720 | .181 | .005 | ..... | 33.5 |
| 83 | " 24 | 2.4 | ....... | ........ | .... | 1,058 | ..... | ..... | 19 | .150 | 1.280 | ...... | ..... | 166 | 14 1 |
| 88 | Jul. 1 | 1.4 | ........ | ........ | .... | 618 | ...... | ..... | 18 | .170 | .920 | .057 | .008 | 214 | 4.4 |
| 93 | " 15 | 1.0 | ........ | ........ | ...... | 528 | ..... | ..... | 46 | .255 | 1.000 | .995 | .003 | 280 | 2.9 |
| 105 | Aug. 1 | 0.5 | Slight. | Slight. | ...... | 536 | ..... | ..... | 56 | .054 | .530 | .058 | .000 | 240 | 1.6 |
| 117 | " 15 | 0.2 | " | " | ..... | 612 | ..... | ..... | 82 | .048 | .372 | .165 | .004 | 240 | 3.2 |
| 147 | 1899.<br>May 4 | ..... | " | Heavy. | ..... | 740 | 64 | 676 | 80 | .020 | .490 | .618 | .000 | 220 | 5.7 |
| 162 | " 17 | ..... | Heavy. | " | ...... | 670 | 280 | 390 | 68 | .053 | .928 | .774 | .000 | 240 | 1.8 |
| 181 | Dec. 15 | 0.0 | Slight. | Slight. | ...... | 698 | 128 | 570 | 124 | .027 | .241 | ..... | ..... | ..... | .... |
| 191 | 1900.<br>Apr. 6 | 0.6 | ........ | ........ | ..... | 520 | 200 | 320 | 65 | .040 | .240 | ...... | ..... | 278 | 5.5 |
| 203 | " 16 | 0.9 | ........ | ....... | ..... | 540 | 260 | 280 | 77 | .025 | .435 | ...... | ..... | 270 | 5.3 |
| 212 | " 23 | 2.6 | ........ | ........ | ..... | 3,736 | ..... | ...... | 80 | .015 | 2.280 | .... | ..... | 144 | 78.0 |
| 218 | " 30 | 2.6 | ........ | ........ | .... | 2,440 | 332 | 2,108 | .... | .095 | 4.305 | .... | ...... | 126 | 61.0 |
| 225 | May 7 | 1.7 | ........ | ........ | ...... | 810 | 128 | 682 | 39 | .039 | 1.000 | ...... | ..... | 168 | 21.0 |
| 230 | " 14 | 2.2 | ........ | ........ | .... | 4,000 | 240 | 3,760 | 32 | .070 | 3.140 | ..... | ..... | 146 | 63.0 |
| 235 | " 21 | 2.6 | ........ | ....... | .... | 1,438 | ...... | .... | 30 | .080 | 5.300 | ..... | ...... | 126 | 42.0 |

In table I the variation of the constituents with the height of the water is quite striking. When the first analysis was made, February 4, 1898, no water was flowing over the dam. This stage was again reached on August 15 and December 15 of the same year. Analyses made in the interim show the soluble solids, chlorine and hardness to be lower, and insoluble solids to be higher. Nitrogen in all its forms increases with the increase in the height of the water. The variation of the chlorine is especially noticeable. With the gauge at zero, there were 114 parts of chlorine; with the gauge at five feet, there were only 14 parts of chlorine; and the following week, though the water had gone down, there were only 9 parts of chlorine. As the water recedes the chlorine increases again, reaching 80 in August, when again there is no water flowing over the dam, and 124 in December.

TABLE II.—Analyses of water from the Kansas river, taken about 300 feet below the Lawrence sewers. (Parts per 1,000,000.)

| Number. | Date of collection. | Gauge. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 2 | 1898.<br>Feb. 4 | 0.0 | Slight. | Slight. | 0.0 | 672 | 134 | 538 | 114 | .160 | .224 | .377 | Trace. | ..... | ..... |
| 4 | " 11 | 1.1 | Decided. | Large. | 0.0 | 882 | ..... | ...... | 73 | .205 | 1.290 | .573 | Trace. | ..... | ..... |
| 6 | " 19 | 1.4 | " | " | 0.0 | 1,578 | ...... | ...... | 37 | .333 | 2.810 | 1.372 | ..... | ..... | ..... |
| 10 | " 26 | 0.8 | " | " | 0.0 | 678 | ...... | ...... | 69 | .186 | .950 | .341 | ..... | ..... | ..... |
| 14 | Mar. 5 | 0.6 | " | " | 0.0 | 642 | ...... | ..... | 65 | .072 | .474 | .434 | ...... | ..... | ..... |
| 18 | " 11 | 0.4 | Slight. | Slight. | 0.1 | 790 | ...... | ...... | 74 | .026 | .395 | .307 | .000 | 210 | ..... |
| 23 | " 18 | 0.8 | " | " | 0.0 | 566 | ...... | ...... | 67 | .026 | .535 | .142 | ...... | 225 | ..... |
| 25 | " 25 | 0.9 | " | Decided. | 0.1 | 850 | ...... | ...... | 57 | .033 | 1.200 | .472 | ...... | 230 | 13.8 |
| 30 | Apr. 1 | 0.6 | ........ | ........ | 0.2 | 604 | 80 | 524 | 68 | .085 | .511 | .349 | ...... | 230 | 5.8 |
| 38 | " 8 | 1.0 | Dccided. | Heavy. | 0.1 | 756 | 228 | 538 | 59 | .081 | .634 | .243 | ...... | 220 | 9.2 |
| 44 | " 15 | 1.0 | Slight. | Slight. | 0.1 | 732 | 310 | 422 | 54 | .088 | .602 | 2.562 | .000 | 225 | 10.7 |
| 51 | " 22 | 1.8 | Decided. | Decided. | 0.2 | 4,366 | 634 | 3,732 | 35 | .500 | 5.600 | ...... | .007 | 170 | ..... |
| 56 | " 29 | 1.2 | ........ | ....... | 0.0 | 982 | 334 | 648 | 41 | .150 | 1.340 | .411 | .000 | 210 | 15.0 |
| 58 | May 6 | 2.7 | Decided. | Decided. | 0.2 | 2,752 | ...... | ..... | 41 | .346 | 4.034 | ...... | ...... | ..... | ..... |

This series shows the same variation in the constituents with the height of the river as is noted in the analyses made above the dam. Solids, free ammonia, albuminoid ammonia, and oxygen-consuming capacity are higher than above the dam. Nitrites were sometimes present. Chlorine was slightly lower, showing a similar variation noted by Bailey and Franklin* in examination of the water of the river above and below Topeka. The sewage of Lawrence and Topeka is hence lower in chlorine than the water of the Kansas river, fed by rivers rich in chlorine.

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* Kan. Univ. Quart. III, 91 (1894.)

TABLE III.—Analyses of water from the Kansas river. Samples collected one and one-half miles above the city of Lawrence. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 11 | 1898.<br>Mar. 5 | Slight. | Slight. | 0.0 | 634 | ...... | ...... | 70 | .080 | .480 | .461 | ...... | ..... | ...... |
| 15 | " 11 | " | " | 0.0 | 612 | ...... | ...... | 76 | .013 | .375 | .291 | ..... | 215 | ...... |
| 34 | Apr. 9 | " | " | 0.0 | 818 | 182 | 636 | 59 | .104 | .656 | ..... | .000 | 230 | 9.8 |

TABLE IV.—Analyses of water from the Kansas river. Samples taken from the middle of the river at the foot bridge, Lawrence. Collected 320 feet from the south approach. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 8 | 1898.<br>Feb. 26 | Slight. | Slight. | 0.0 | 768 | ...... | ...... | 69 | .200 | .800 | .494 | ...... | ...... | ...... |
| 12 | Mar. 5 | " | " | 0.0 | 636 | ...... | ...... | 69 | .066 | .384 | .439 | ...... | ...... | ..... |
| 16 | " 11 | " | " | 0.0 | 594 | ...... | ...... | 77 | .013 | .355 | .316 | .000 | 220 | ..... |
| 21 | " 18 | " | " | 0.0 | 564 | ...... | ...... | 68 | .013 | .355 | .318 | ...... | 220 | ...... |
| 32 | Apr. 1 | " | " | 0.1 | 628 | 98 | 530 | 67 | .075 | .640 | .273 | ...... | 245 | 6.3 |
| 39 | " 8 | " | Heavy. | 0.0 | 790 | 176 | 614 | 59 | .081 | .709 | .243 | .000 | 225 | 10.0 |

The analyses shown in tables III and IV do not show a sufficient variation to form any definite conclusions.

TABLE V.—Analyses of water from the Kansas river, collected about 100 feet above the dam at Lawrence, filtered through paper before making tests. (Parts per 1,000,000.)

| Number. | Date of collection. | Gauge. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 67 | 1898.<br>May 16 | 3.9 | ...... | ...... | ...... | 360 | 182 | 178 | 18 | .775 | ...... | ...... | ...... | ..... | ..... |
| 71 | " 20 | 3.1 | Decided. | None. | 0.0 | 282 | 126 | 156 | 14 | ...... | ...... | ..... | ...... | 151 | 3.6 |
| 78 | June 10 | 5.2 | None. | " | 0.1 | 268 | ...... | ...... | 13 | .178 | .133 | .000 | .025 | ..... | 6.2 |
| 80 | " 17 | 3.7 | Slight. | " | ...... | 230 | ...... | ...... | 9 | .104 | .287 | .326 | .005 | ..... | 3.7 |
| 84 | " 24 | 2.4 | ...... | ...... | ...... | 284 | ..... | ..... | 19 | .126 | .260 | ...... | ...... | 162 | 3.1 |
| 89 | July 1 | 1.4 | ...... | ...... | ..... | 392 | ...... | ...... | 18 | .099 | .245 | .322 | .008 | 220 | 2.4 |
| 94 | " 15 | 1.0 | ...... | ...... | ...... | 456 | ..... | ...... | 45 | .020 | .500 | .395 | .003 | 250 | 1.9 |
| 106 | Aug. 1 | 0.5 | ...... | ...... | ...... | 490 | ...... | ...... | 56 | .053 | .240 | .007 | .000 | 120 | 1.3 |
| 118 | " 15 | 0.0 | ...... | ...... | ...... | 508 | ...... | ..... | 82 | .042 | .239 | .006 | .004 | 120 | 2.5 |
| 192 | 1900.<br>Apr. 6 | 0.6 | Slight. | ...... | ...... | 520 | 200 | 320 | 65 | .036 | .240 | ...... | .000 | ..... | 3.0 |
| 204 | " 16 | 0.9 | ...... | ...... | ...... | 490 | 148 | 342 | 74 | .040 | .265 | ...... | ...... | ..... | 3.8 |
| 213 | " 23 | 2.6 | ...... | ...... | ..... | 578 | 164 | 414 | 68 | .520 | 1.160 | ...... | ...... | 144 | 7.4 |
| 219 | " 30 | 2.6 | ...... | ...... | ...... | 376 | 100 | 276 | 23 | .320 | .555 | ..... | ...... | 126 | 6.1 |
| 226 | May 7 | 1.7 | .... .. | ...... | ...... | 496 | 132 | 364 | 39 | .370 | .181 | ...... | .007 | 160 | 4.7 |
| 231 | " 14 | 2.2 | ...... | ...... | ...... | ...... | ...... | ...... | 32 | ...... | .340 | ...... | ...... | 146 | 4.8 |
| 236 | " 21 | 2 6 | ...... | ...... | ...... | ...... | ...... | ...... | 30 | ..... | .355 | ...... | ...... | 126 | 5.0 |

These analyses were made on dates corresponding to those of table I. It will be noted that the same variations occur as in the unfiltered water, but the solids do not show such a wide range. It is easy to find the amount held in suspension, if the solids in the filtered water be subtracted from the amount in the unfiltered. The decrease in the amount of oxygen consumed and in the amount of nitrogenous constituents shows that a large amount of the nitrogen is in the suspended matter.

TABLE VI.—Analyses of water from wells of the city of Lawrence. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Fixed. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 36 | 1898.<br>Apr. 8 | None. | None. | 0.0 | 1,092 | 562 | 526 | 104 | .015 | .126 | 7.893 | ...... | 250 | 1 15 |
| 37 | " 8 | " | " | 0.0 | 661 | 408 | 256 | 60 | .020 | .075 | 5.250 | ...... | 305 | 1.30 |
| 40 | " 15 | " | " | 0.0 | 788 | 518 | 270 | 66 | .015 | .074 | 7.890 | .000 | 290 | 2.10 |
| 41 | " 15 | " | " | 0.0 | 552 | 324 | 228 | 36 | .010 | .056 | 3.280 | .000 | 230 | .95 |
| 42 | " 15 | " | Slight. | 0.0 | 908 | 310 | 598 | 85 | .040 | .145 | 1.120 | .000 | 375 | 2.75 |
| 43 | " 15 | " | None. | 0.0 | 856 | 442 | 414 | 100 | .049 | .103 | 6.700 | .000 | 305 | 1 50 |
| 46 | " 22 | " | Slight. | 0.0 | 752 | 388 | 364 | 74 | .013 | .126 | 5.259 | .000 | 250 | 1.65 |
| 86 | June 25 | ........ | ........ | ...... | 1,678 | ...... | ...... | 160 | .019 | .112 | 10.522 | .190 | 434 | 1.58 |
| 102 | July 22 | ........ | ........ | ...... | 324 | ...... | ...... | 10 | .028 | .115 | .306 | .001 | 95 | .40 |
| 91 | " 2 | ........ | ........ | ...... | ...... | ...... | ...... | 153 | ...... | .... | 8.211 | .240 | ..... | ..... |
| 114 | Aug. 8 | ........ | ........ | ...... | 2,408 | ...... | ..... | 236 | .200 | .158 | 3.952 | .010 | 90 | 1.00 |
| 133 | 1899.<br>Apr. 21 | None. | None. | 0.0 | 808 | 198 | 610 | 150 | .034 | .083 | ..... | .001 | 250 | .55 |
| 134 | " 21 | " | " | 0.0 | 228 | 1,250 | 1,030 | 120 | .075 | .217 | ...... | .000 | 480 | 1.50 |
| 135 | " 21 | " | " | 0.0 | 1,060 | ...... | ...... | 83 | .044 | .116 | .164 | ..... | 365 | .85 |
| 136 | " 28 | " | " | 0.0 | 490 | 132 | 368 | 20 | .006 | .170 | .323 | .001 | 180 | 2.00 |
| 137 | " 28 | " | " | 0.0 | 1,322 | 614 | 708 | 44 | .015 | .116 | .132 | .000 | 320 | 1.50 |
| 139 | " 28 | " | " | 0.1 | 1,460 | 570 | 890 | 161 | .110 | .134 | 2.164 | .205 | 410 | 1.00 |
| 140 | " 29 | " | " | 0 0 | 3,198 | ...... | ..... | 400 | .080 | .385 | .000 | .000 | 240 | 1.60 |
| 141 | " 29 | " | " | 0.0 | 660 | ...... | ...... | 80 | .145 | .060 | ...... | .000 | 110 | .50 |
| 142 | " 28 | " | " | 0.0 | 1,550 | 780 | 770 | 260 | .023 | .130 | ..... | .000 | 200 | .75 |
| 143 | " 29 | " | " | 0.0 | 1,220 | ...... | ...... | 36 | .030 | .140 | ..... | .000 | 150 | 1.00 |
| 144 | May 5 | " | " | 0.0 | 410 | 140 | 300 | 52 | .018 | .162 | 1.634 | .000 | 260 | .75 |
| 145 | " 5 | " | " | 0.0 | 1,000 | 320 | 680 | 108 | .010 | .086 | 2.050 | .000 | 380 | .75 |
| 146 | " 5 | " | " | 0.0 | 1,100 | 100 | 1,000 | 160 | .220 | .123 | 2.701 | .000 | 240 | .70 |
| 148 | " 5 | " | " | 0.0 | 1,140 | 200 | 940 | 166 | .090 | .230 | 2.396 | .000 | 110 | 1.00 |
| 152 | " 10 | Slight. | " | 0.0 | 664 | ...... | ...... | 78 | .015 | .071 | .000 | .001 | 260 | 1.80 |
| 153 | " 10 | " | Slight. | 0.0 | 396 | ...... | ...... | 92 | .020 | .295 | .178 | .001 | 370 | 1.60 |
| 154 | " 11 | " | " | 0.0 | 1,052 | ...... | ..... | 116 | .028 | .140 | .203 | .000 | 290 | 1.10 |
| 157 | " 13 | None. | None. | 0.0 | 750 | 260 | 490 | 36 | .043 | .149 | 1.611 | .000 | 250 | 1.50 |
| 158 | " 13 | " | " | 0.0 | 800 | 180 | 620 | 38 | .060 | .190 | .959 | .000 | 250 | 1.00 |
| 159 | " 13 | " | " | 0.0 | 910 | 280 | 630 | 42 | .030 | .170 | .387 | .001 | 230 | 1.00 |
| 160 | " 17 | " | " | 0.0 | 870 | 220 | 650 | 104 | .042 | .186 | .656 | .000 | 390 | 1.00 |
| 168 | " 17 | " | " | 0.0 | 1,266 | 266 | 1,000 | 372 | .023 | .240 | 2.661 | .016 | 370 | 1.60 |
| 182 | Dec. 16 | ........ | ........ | ...... | ..... | ..... | ...... | 72 | .015 | .038 | .... | ...... | ..... | ...... |
| 188 | 1900.<br>Jan. 26 | None. | None. | 0.0 | ..... | ...... | ...... | 60 | .011 | .043 | ...... | .000 | ..... | 2.30 |
| 189 | Feb. 16 | " | " | 0.0 | 1,398 | 338 | 1,060 | 90 | .448 | .281 | ...... | .000 | 900 | 5.60 |

TABLE VI.—*Concluded.*

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Fixed. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 194 | Apr. 6 | " | " | 0.0 | 942 | 404 | 538 | 120 | .172 | .215 | .023 | .002 | 416 | 1.60 |
| 195 | " 9 | " | " | 0.0 | 634 | 294 | 340 | 56 | .002 | .011 | .163 | .000 | 256 | 1.40 |
| 197 | " 10 | " | " | 0.0 | 718 | 248 | 470 | 91 | .024 | .060 | .968 | .000 | 264 | 1.40 |
| 198 | " 10 | " | " | 0.0 | 966 | 342 | 624 | 146 | .031 | .051 | .963 | .... | 500 | 1.70 |
| 199 | " 10 | " | " | 0.0 | 1,864 | 602 | 1,262 | 252 | .079 | .081 | 1.390 | .012 | 600 | 2.80 |
| 206 | " 17 | " | " | 0.0 | 1,156 | ...... | ...... | 123 | .008 | .065 | 1.732 | .001 | 536 | 1.30 |
| 207 | " 17 | " | " | 0.0 | 708 | 206 | 502 | 80 | .010 | .042 | .815 | .001 | 320 | 1.50 |
| 214 | " 24 | ........ | ........ | ...... | 474 | 84 | 390 | 26 | .176 | .240 | .061 | ...... | 166 | .80 |
| 221 | May 1 | ........ | ........ | ...... | 368 | 110 | 258 | 41 | .014 | .081 | .077 | ...... | 160 | 3.60 |
| 222 | " 1 | ........ | ........ | ...... | 572 | 266 | 316 | 24 | .036 | .185 | .093 | ...... | 262 | 2.30 |
| 223 | " 1 | ........ | ........ | ...... | 600 | 354 | 246 | 42 | .020 | .040 | .801 | ...... | 331 | 1.10 |
| 232 | " 17 | ........ | ........ | ...... | 1,180 | 680 | 500 | 116 | .055 | .045 | ...... | ...... | 432 | 2.70 |

LOCATION OF THE WELLS OF TABLE VI.

36.—933 Vermont street. In house.
37.—1340 Ohio street.
40.—1407 Kentucky street.
41.—Chancellor's residence, Louisiana street; 17 feet deep.
42.—Well at boat-house.
43.—1310 Kentucky street.
46.—1144 New Jersey street.
86, 91, and 206.—1335 Vermont street. Suspected contamination.
102.—Well of A. E. Tweed, East Lee street; 22 feet deep.
114.—925 Connecticut street; 25 feet deep.
133.—Well north of M. E. church.
134.—920 Vermont street.
135.—1113 Rhode Island street.
136.—Well at Quaker church.
137.—1344 Luisiana street.
139.—1339 Kentucky street.
140.—1132 New York street; 24 feet deep.
141.—1203 Rhode Island street; 41 feet deep.
142.—1025 New Jersey street; 42 feet deep.
143.—1345 Vermont street; 43 feet deep.
144.—1211 Rhode Island street.
145.—1200 Kentucky street.
146.—1317 Kentucky street.
148.—1043 Quincy street.
152.—185 Locust street, North Lawrence; shallow.
153.—192 Locust street, North Lawrence; 36 feet deep.
154.—1300 Massachusetts street.
157.—Well at barn of B. W. Woodward.
158.—Well at house of B. W. Woodward.
160.—920 Vermont street; 30 feet deep.
163.—920 Massachusetts street; 30 feet deep.
182.—1211 Tennessee street.
188.—1346 Massachusetts street.
189 and 232.—800 Louisiana street.
194.—1202 Tennessee street.
195.—1326 Tennessee street.
197.—1215 Tennessee street.
198.—1214 Tennessee street.
199.—1325 Vermont street; 29 feet deep.
207.—1324 Vermont street; 37 feet deep.
214.—946 Ohio street.
221.—1018 Vermont street.
222.—1633 Vermont street.
223.—1238 Tennessee street.

These analyses show that we cannot rely on the amount of chlorine as an indication of sewage. Wells less than 500 feet apart and of the same elevation, shown to be pure by complete analyses, have a variation of chlorine from 10 to 308 parts per million, the deeper well having the most chlorine.

Wells of the same depth but of different elevation show more chlorine the less the elevation.

TABLE VII.—Analyses of water from wells in the vicinity of Lawrence. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as — | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 48 | 1898.<br>Apr. 22 | None. | Slight. | 0.0 | 814 | 428 | 386 | 46 | .012 | .122 | 5.990 | .000 | 170 | 0.50 |
| 49 | " 22 | " | " | 0.0 | 340 | 302 | 38 | 8 | .027 | .137 | 3.213 | .000 | 190 | 1.75 |
| 52 | " 29 | " | None. | 0.0 | 632 | 200 | 432 | 10 | .025 | .059 | 1.668 | .000 | 110 | 7.75 |
| 53 | " 29 | " | " | 0.0 | 2,010 | 608 | 1,402 | 308 | .012 | .118 | 22.070 | .001 | 125 | 2.50 |
| 54 | " 29 | " | " | 0.0 | 574 | 260 | 314 | 10 | .004 | .111 | 2.014 | .000 | 335 | 2.20 |
| 63 | May 13 | " | " | 0.0 | 200 | 130 | 70 | 3 | .095 | .050 | 2.027 | .025 | 170 | 1.50 |
| 75 | June 10 | Decided. | Slight. | 0.0 | 368 | ...... | ..... | 31 | .063 | .068 | .362 | .025 | ..... | ..... |
| 81 | " 17 | Slight. | " | ...... | 4,054 | ...... | ..... | 1,720 | .670 | .094 | .271 | Present | 328 | 3.80 |
| 82 | " 17 | " | " | ...... | 2,504 | ...... | ..... | 880 | .690 | .071 | ........ | ...... | ..... | .96 |
| 85 | " 24 | ........ | ........ | ...... | 760 | ...... | ...... | 141 | .020 | .130 | 9.057 | ...... | 289 | .90 |
| 87 | July 1 | ........ | ........ | ...... | 1,578 | ...... | ...... | 174 | .033 | .228 | 20.561 | ...... | 84 | .46 |
| 95 | " 15 | ........ | ........ | ...... | 864 | ...... | ...... | 28 | .010 | .115 | .637 | .013 | 193 | .37 |
| 96 | " 15 | ........ | ........ | ...... | 2,816 | ...... | ...... | 426 | .036 | .129 | 26.309 | .014 | 155 | 7.70 |
| 101 | " 22 | Decided. | Heavy. | ...... | 608 | ...... | ...... | 15 | .204 | .255 | .038 | .001 | 132 | 2.50 |
| 103 | " 22 | ........ | ........ | ...... | 524 | ...... | ...... | 26 | .019 | .084 | .280 | .000 | 144 | .60 |
| 104 | " 22 | ........ | ........ | ...... | 800 | ...... | ...... | 30 | .063 | .510 | 1.430 | .000 | 140 | 1.85 |
| 109 | " 31 | Decided. | Heavy. | ...... | 672 | ...... | ...... | 18 | .156 | .400 | .005 | .001 | 180 | .40 |
| 110 | " 31 | ........ | ........ | ...... | 730 | ...... | ..... | 66 | .240 | .206 | .049 | .001 | 72 | .90 |
| 115 | Aug. 14 | ........ | ........ | ...... | 724 | ...... | ...... | 24 | .032 | .147 | 5.738 | .004 | 150 | .93 |
| 116 | " 14 | ........ | ........ | ...... | 596 | ...... | ..... | 15 | .044 | .166 | 4.905 | .000 | 170 | .40 |
| 227 | 1900.<br>May 8 | ........ | ........ | ...... | 870 | 300 | 570 | 60 | .030 | .295 | .380 | ..... | 376 | 6.30 |
| 228 | " 8 | ........ | ........ | ..... | 368 | 134 | 234 | 10 | .027 | .630 | .348 | ...... | 238 | 5.90 |

LOCATION OF THE WELLS OF TABLE VII.

48 and 227.— Well at the residence of D. H. Spencer, one-half mile northwest of the University.
49 and 228.— Well in a pasture three-quarters of a mile northwest of the University.
52.— Well in rear of lot south of the first road south of the University.
53.— Well north of the first road south of the University. Deep well.
54.— Well at farmhouse northeast of Lawrence.
63.— Well northwest of Lawrence, on property of Mrs. Richardson. Depth, fifty-seven feet.
77.— Well on land of J. G. Haskell. Deep well.
81.— Well in the southwest corner of Oak Hill Cemetery. Depth, 130 feet.
82.— Well at farm of Mr. Koehring, east of Oak Hill cemetery, 125 feet deep.
85.— Well on farm of Mr. Smith, south of Oak Hill cemetery, about thirty feet deep.
87.— Well on farm of Mr. Gowan, southwest of Oak Hill cemetery, ninety-five feet deep.
95.— Well on J. Russell's farm, north of Oak Hill cemetery.
96.— Well on farm of Mrs. A. Russell, one mile east of Lawrence, twenty-eight feet deep.
101.— Well of W. Kappis, three miles north of Lawrence, twenty-four feet deep.
103.— Well on farm of A. W. Garvin, four and one-half miles north of Lawrence, forty feet deep.
104.— Well on farm of Phil. Heck, four and one-half miles north of Lawrence, seventy-five feet deep.
109.— Well on farm of L. M. Spray, four miles northwest of Lawrence.
110.— Well on farm of J. Wilson, six miles northwest of Lawrence, twenty-four feet deep.
115.— Well on lot of Mr. Roberts, west of McCook field.
116.— Spring near Mr. Duncan's residence, west of Lawrence.

We noted that the solids in solution and the chlorine were lowest to the west and north of the city, while to the south and east they are much higher and in the deeper wells are very high.

The normal chlorine maps of Massachusetts and Connecticut are so helpful in those states, giving a standard for comparison, that we would like one here. The analyses of these country wells from all sides of Lawrence, together with the city wells, show that such a map of this district would be difficult to make, and, if made, valueless, for chlorine alone cannot be used as a test for contamination.

TABLE VIII.—Analyses of water from the Lawrence Water Company's system. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 26 | 1898.<br>Mar. 26 | None. | None. | 0.0 | 564 | ...... | ...... | 65 | .033 | .225 | ...... | ..... | 220 | ...... |
| 33 | Apr. 1 | ........ | ........ | ..... | 562 | 121 | 441 | 66 | .069 | .125 | .447 | ...... | 160 | 1.52 |
| 57 | May 6 | ........ | ........ | ...... | 470 | 224 | 246 | 60 | .088 | .438 | ...... | ...... | ...... | ...... |
| 90 | July 2 | ........ | ........ | ...... | ...... | ...... | ...... | 60 | .350 | .230 | .206 | .000 | 140 | 1.18 |
| 111 | Aug. 8 | Decided. | Heavy. | ..... | 484 | ..... | ...... | 45 | .270 | .410 | .261 | ...... | 125 | 3.30 |
| 112 | " 8 | ........ | ........ | ..... | 476 | ...... | ..... | 45 | .460 | .332 | .031 | ...... | 125 | 2.45 |
| 113 | " 8 | Decided. | Heavy. | ...... | 508 | ...... | ..... | 45 | .515 | .291 | .136 | ...... | 125 | 3.00 |
| 186 | 1900.<br>Jan. 15 | ........ | ........ | ...... | 598 | 142 | 456 | 62 | .262 | .074 | .278 | .000 | 245 | 2.20 |
| 190 | Feb. 17 | None. | None. | 0.0 | ..... | ...... | ..... | .... | .060 | .135 | ..... | ...... | ...... | ...... |
| 193 | Apr. 6 | ........ | ........ | ...... | 700 | 300 | 400 | 79 | 1.021 | .160 | ...... | ...... | 303 | 4.00 |
| 196 | " 9 | None. | None. | 0.0 | 610 | 106 | 494 | 81 | .225 | .262 | ...... | ...... | 302 | 4.34 |
| 200 | " 12 | " | " | 0.0 | 626 | 120 | 506 | 82 | 1.710 | .138 | ...... | ..... | 284 | 3.23 |
| 201 | " 12 | Slight. | Slight. | 0.0 | 642 | ...... | ...... | 80 | .960 | .336 | ...... | ...... | 284 | ...... |
| 202 | " 12 | " | ........ | 0.0 | 620 | 220 | 400 | 82 | .850 | .316 | ...... | ...... | 274 | ...... |
| 205 | " 16 | " | Slight. | 0.0 | 600 | 212 | 358 | 80 | .105 | .130 | ...... | ..... | 268 | 2.85 |
| 209 | " 19 | ........ | ........ | ...... | 684 | 154 | 530 | 80 | 1.570 | .155 | ...... | ...... | 298 | 3.41 |
| 210 | " 19 | ........ | ........ | ..... | 602 | 230 | 372 | 80 | 1.000 | .347 | ..... | ..... | 295 | 3.90 |
| 211 | " 19 | ........ | ........ | ...... | 608 | 234 | 374 | 80 | 1.000 | .265 | ..... | ...... | .... | 4.35 |
| 215 | " 26 | ........ | ........ | ..... | 656 | 144 | 512 | 84 | 1.910 | .252 | ..... | ..... | 295 | 3.90 |
| 216 | " 26 | ........ | ........ | ...... | 656 | 212 | 444 | 82 | .490 | 1.850 | ...... | ...... | 300 | 1.40 |
| 217 | " 26 | ........ | ........ | ..... | 576 | 110 | 466 | 81 | .060 | .404 | ...... | ..... | 280 | 4.43 |
| 220 | " 30 | ........ | ....... | ...... | 632 | 200 | 432 | 82 | .490 | .235 | ..... | ...... | 299 | 2.62 |
| 233 | May 17 | ........ | ....... | ...... | ...... | ...... | ...... | 86 | .720 | .210 | ...... | ...... | 270 | 3.30 |
| 234 | " 17 | ........ | ........ | ..... | ..... | ..... | ..... | 88 | .800 | .200 | ..... | ...... | 270 | 3.30 |

The water for analyses 26, 33, 57, 90, 186 and 196 was drawn from a tap at 1344 Ohio street; for 190, 205, and 220, from a tap at the University; for 111, from a tap at the water-works; 112, from a tap at 920 Tennessee street, and 113, from a hydrant at Oak Hill cemetery. These three (111–113) formed a comparative analysis. 193, 200, 209 and 215 were analyses of water from the fountain. The water was clear when first collected, but on standing would become turbid, and a sediment would settle out. 201, 210 and 216 are from the second reservoir, and 202, 211 and 217 are from the third reservoir. Three sets of comparative analyses of the water from the fountain and the second and third reservoirs were made. The most remarkable item noted was the very high free ammonia in the fountain, with the gradual decrease in the reservoirs, and the still further decrease noted in the water from the taps.

Numbers 233 and 234 were comparative analyses, on the same day, of water from different parts of the city.

TABLE IX.—Analyses of water from springs in or near the city of Lawrence. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 47 | 1898.<br>Apr. 22 | None. | None. | 0.0 | 330 | 164 | 166 | 28 | .008 | .087 | 1.475 | .000 | 240 | 1.1 |
| 116 | Aug. 14 | ........ | ........ | ..... | 596 | ...... | .... | 15 | .044 | .166 | 4.905 | .000 | 170 | .4 |
| 138 | 1899.<br>Apr. 28 | Decided. | Slight. | .... | 788 | 260 | 528 | 48 | .001 | .136 | .013 | ..... | 260 | 4.5 |
| 156 | May 13 | None. | None. | 0.0 | 640 | 200 | 440 | 40 | .035 | .200 | 1.614 | .000 | 235 | 1.5 |

Analyses 47 and 156 are of spring-water from the residence of B. W. Woodward.
116, a spring at Mrs. Duncan's, west of Lawrence.
138, a spring in the 1400 block on Ohio street.

TABLE X.—Analyses of water from deep wells. (Parts per 1000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 76 | 1898.<br>June 10 | ........ | ........ | ...... | ..... | .... | ..... | 21.38 | ...... | ..... | .025 | ..... | ..... | ..... |
| 149 | 1899.<br>May 5 | Decided. | Heavy. | 0.0 | 86.84 | ...... | ...... | 36.00 | .00195 | .0009 | .0007 | .000 | .20 | ..... |
| 170 | " 25 | None. | None. | 0 0 | .78 | .23 | .55 | .38 | .00004 | .0022 | .0006 | .000 | .26 | ..... |
| 171 | " 25 | " | " | 0.0 | ...... | ..... | ...... | .05 | .00002 | .0002 | .0006 | .000 | .24 | ..... |

Analyses 76 and 149 are from the deep well near the Santa Fe freight station at Lawrence.
Analyses 170 and 171 are from a well at Tonganoxie.

TABLE XI.—Analyses of water from cisterns. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 132 | 1899. Apr. 22 | None. | None. | 0.0 | 178 | ...... | ...... | 12 | .049 | .242 | .000 | .001 | 60 | .45 |
| 155 | May 11 | “ | “ | 0.0 | 666 | ...... | ...... | 24 | .120 | .200 | .041 | .000 | 135 | ...... |
| 161 | “ 17 | “ | “ | 0.0 | 480 | 100 | 380 | 34 | .158 | .518 | .446 | .005 | 240 | 4.50 |
| 168 | “ 27 | Slight. | “ | Brown | ...... | ...... | ...... | 20 | 8.500 | 7.300 | ..... | .000 | 60 | 630.00 |
| 187 | 1900. Jan. 26 | None. | “ | 0.2 | 200 | 80 | 120 | 14 | .041 | .260 | ...... | .000 | ..... | 9.44 |
| 208 | Apr. 18 | ........ | ........ | ...... | 14 | 4 | 9 | 8 | .150 | .230 | .... | ..... | ..... | 9.64 |
| 224 | May 7 | ........ | ........ | ...... | 138 | 42 | 96 | 8 | .024 | .079 | ...... | ...... | 103 | 1.93 |

Analyses 132 and 224 are of water from the University cistern; 155 is from 1300 Louisiana street; 161 is from 920 Vermont street; 168 is from 1345 Kentucky street; 187 is from Beal's, north of Warren street; 201 is from 946 Ohio street.

With the exception of the University cistern water, these analyses are of water from cisterns supposed to contain contamination. Number 187 was from a cistern the water from which had been used to wash milk-cans. The milk from these cans is supposed to have aided in spreading typhoid fever to several families in the city.

TABLE XII.—Analyses of water from some of the Kansas rivers. (Parts per 1,000,000.)

| Number. | Date of collection. | Appearance. | | | Residue on evaporation. | | | Chlorine. | Ammonia. | | Nitrogen as— | | Hardness. | Oxygen consumed. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Turbidity. | Sediment. | Color. | Total. | Loss on ignition. | Residue. | | Free. | Albuminoid. | Nitrates. | Nitrites. | | |
| 27 | 1898. Mar. 24 | None. | None. | 0.7 | 568 | 324 | 244 | 6.5 | .160 | .279 | 1.449 | .001 | 155 | 6.1 |
| 29 | Apr. 1 | “ | “ | 0.0 | 702 | 278 | 424 | 67.1 | .073 | .247 | .107 | .000 | 185 | 3.5 |
| 64 | May 13 | Slight. | Heavy. | ...... | 524 | 232 | 292 | 22.0 | .065 | .375 | .039 | .000 | 225 | 4.5 |
| 68 | “ 18 | ........ | ........ | ...... | 874 | 186 | 688 | 2.6 | .200 | 1.570 | .598 | .000 | 124 | 16.8 |
| 69 | “ 18 | ........ | None. | ...... | 260 | 112 | 148 | 2.6 | ..... | .630 | ..... | ..... | 124 | 4.4 |
| 72 | “ 21 | Decided. | Heavy. | ...... | ...... | ...... | ...... | 2.8 | .130 | 3.870 | .546 | .005 | 160 | 8.7 |
| 74 | “ 20 | “ | “ | ...... | ...... | ...... | ...... | 0.5 | .110 | 3.040 | .486 | .002 | 134 | 7.4 |
| 173 | 1899. Oct. 27 | “ | Slight. | 0.0 | ...... | ...... | ...... | ...... | .254 | .336 | .734 | .000 | 252 | ..... |

27, 68 and 69 are from the Neosho river, at Iola, collected by Prof. E. C. Murphy.
69 is 68 filtered through paper.
29 and 64 are from the Solomon, at Beloit, collected by Mr. A. D. Rodgers.
72 is from the Marais des Cygnes, at Ottawa, collected by Prof. E. C. Murphy.
74 is from the Verdigris, at Liberty, collected by Prof. E. C. Murphy.
173 is from the Neosho, at Iola, collected by Mr. C. G. Waite.

It is interesting to note the small amount of chlorine in the southern part of the state, out of the valley of the Kansas river.

### SUMMARY.

The analyses of the river water given in tables I–V show a direct variation of solids in suspension, and a direct variation of organic material with the height of the water, and an inverse variation of soluble ingredients with the height of the water.

A comparison of tables I and II shows that the Lawrence sewers have a little effect on the river. This is scarcely observable in the comparative tests we have made, as most of them were made at high stages of the river.

Tables I–V show that very little difference could be observed in samples taken from different places in the river near Lawrence.

A comparison of tables I and V shows that filtration removes a great part of the solids and very much decreases the organic matter.

In tables VI, VII, IX, and X, analyses of water from wells and springs in the city of Lawrence and vicinity show that chlorine cannot be taken as indicating contamination. No chlorine map of the district could be made. The chlorine and solids vary directly with the depth of the wells, and in wells of equal depth increase toward the east. Softest waters and those with least chlorine are found on the hills southwest of the city. Nitrates are sometimes very high in what are apparently pure waters. (See especially table VII.)

Analyses of water from the Lawrence Water Company's system (table VIII) compared with the analyses of the river water (tables I and V), samples taken on the same days, show wide differences in the soluble constituents, thus showing that the water from the wells is very different from river water. In this connection, we especially note that the chlorine in the water company's water, though varying in the analyses made, is much less variable than that in the river water. Abnormal free ammonia in the water taken from the fountain is remarkable and worthy of further investigation.

The few analyses of water from rivers in different parts of the state (table XII) show variations due to the geological formation in their respective localities, and a further study of them promises interesting results.

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VOL. III. No. 6.

# BULLETIN
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## Vol. I, Nos. 5, 6, 7, 8. and 9.

(Continuation of Kansas University Quarterly.)

SEPTEMBER 1902.

Entered at the Post-office as Second-class Matter.

# ANNOUNCEMENT.

For ten years the University of Kansas has maintained the *Kansas University Quarterly*, a journal devoted to the publication of the results of research by members of the University of Kansas. At the opening of the second decade of the journal's existence it has been thought best to begin a new series, with some important changes in the name, manner and time of publication. It has been decided to abandon the quarterly form of issue, and instead to publish the separate papers as soon as they are ready for the printer. By this means the results of research will be more promptly given to the public, and in a form more satisfactory both to the authors and subscribers.

Hereafter this journal will be called the *Kansas University Science Bulletin*, and will be issued at irregular intervals. Each volume will contain from 300 to 400 pages of reading-matter, with the necessary illustrations; and approximately one volume a year will be issued. The amount of scientific matter thus published will be not less, and probably more, than that formerly appearing in four issues of the *Quarterly*. The subscription price will be hereafter $3 per volume; the price of each part will vary with the cost of publication. We hope our present large exchange list will remain intact, and new exchanges are solicited. All exchanges should be addressed to the Library of the University of Kansas.

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H. B. Newson,
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Lawrence, Kan.

# THE

# KANSAS UNIVERSITY SCIENCE BULLETIN.

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KANSAS UNIVERSITY SCIENCE BULLETIN.

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# PROJECTIVE TRANSFORMATIONS IN ONE DIMENSION AND THEIR CONTINUOUS GROUPS.

BY H. B. NEWSON.

## INTRODUCTION.

THE writer of this paper is engaged in establishing and developing a new theory of collineations and their Lie groups. A recent memoir in the *American Journal of Mathematics*, Vol. XXIV, p. 109, entitled "A New Theory of Collineations and their Lie Groups," treats of collineations in a plane. A series of papers applying the theory to collineations in space has been published in the Kansas University Quarterly, Vol. X. This theory of collineations in two, three and higher dimensions is based upon, and presupposes, a corresponding theory of one-dimensional projective transformations.

In the present paper the theory is developed for the range of points on a line, but it applies equally well to all three one-dimensional primary forms of projective geometry, viz., the range of points on a line, the pencil of lines through a point, and the pencil of planes through a line. The object of the paper is to collect and set forth the principal facts of one-dimensional transformations, and to build thereon a comprehensive theory of their continuous groups. The paper is sufficiently complete to serve as a foundation on which to build a consistent theory of collineations in two, three and higher dimensions. In form and content it is suitable for the beginner, and will serve as an introduction to the general theory.

The chief sources are as follows: In his "Vorlesungen über Continuierliche Gruppen," Kapitel 5, Lie gives a detailed account of his theory of one-dimensional projective transformations and their continuous groups. His equation is,

$$x_1 = \frac{ax+b}{cx+d}, \qquad (1)$$

where all the variables and coefficients are assumed to be complex numbers. His geometrical interpretation is by points on a line, both points and line being, in general, imaginary.

Klein,[*] Poincaré[†] and others have extensively investigated the same transformation, but chiefly with respect to its discontinuous groups. They have interpreted their results geometrically by points in the complex plane. In addition to the results of Lie and Klein, the principal results of two former papers by the writer are incorporated in the present memoir. These are entitled "Continuous Groups of Projective Transformations Treated Synthetically,"[‡] and "Continuous Groups of Circular Transformations."[§] In the first of these papers is developed geometrically the elements of this new theory of projective transformations in one dimension. In the second paper the continuous groups of the transformations given by (1) are determined and interpreted as transformations of the points of a complex plane.

## PART I.

### § 1. GENERAL PROPERTIES OF ONE-DIMENSIONAL TRANSFORMATIONS.

*The resultant of two transformations.*—A projective transformation should be looked upon as an operation which, when applied to a finite set of points on a line or to the totality of all points on the line, has the effect of rearranging and redistributing them in such a way that the new set or range is projective with the old. If we carry out in succession two projective transformations on a set of points, the result is equivalent to a single projective transformation. This may be shown as follows:

Let T and $T_1$ be two transformations whose equations are respectively

$$x_1 = \frac{ax+b}{cx+d} \text{ and } x_2 = \frac{a_1x_1+b_1}{c_1x_1+d_1}.$$

The first transforms the point x into $x_1$, and the second transforms $x_1$ into $x_2$. We suppose the operations are carried out in the order in which the equations are written. If we eliminate $x_1$ from the above, we get

$$x_2 = \frac{(a_1a+b_1c)\,x+(a_1b+b_1d)}{(c_1a+d_1c)\,x+(c_1b+d_1d)}. \qquad (2)$$

It should be observed that (2) is of the same form as (1) and differs from it only in the value of the coefficients. Equation (2) therefore expresses a projective transformation $T_2$, which transforms the point x directly to $x_2$, and is equivalent to the successive applications of T and $T_1$ in the order named. The transformation $T_2$ is

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[*] See Modulfunctionen, Band I, S. 163-207.

[†] See Acta Mathematica, Tome 1, pp. 1-62.

[‡] Kansas University Quarterly, Vol. IV, pp. 71-93.

[§] Bulletin Amer. Math. Soc., 2d Series, Vol. IV, pp. 107-121.

called the resultant of the transformations $T$ and $T_1$, which are called the component transformations. The operation is symbolized thus: $TT_1 = T_2$. If the two component transformations $T$ and $T_1$ are taken in the inverse order, the resultant, $T_1T = T'_2$, is not the same as $T_2$. Thus:

$$T'_2: \quad x_2 = \frac{(aa_1 + bc_1)x + (ab_1 + bd_1)}{(ca_1 + dc_1)x + (cb_1 + dd_1)}, \qquad (3)$$

which is not the same as $T_2$. The two projective transformations $T_2$ and $T'_2$ are called *conjugate* transformations.

*Determinant of a transformation.*—The determinant

$$\triangle = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called the determinant or modulus of the transformation (1), and it is assumed, for the present at least, that $\triangle$ does not vanish.

By referring to the transformations lettered $T$, $T_1$, $T_2$, we see that the determinant of $T_2$ is

$$\begin{vmatrix} a_1a + b_1c & a_1b + b_1d \\ c_1a + d_1c & c_1b + d_1d \end{vmatrix};$$

but this is the product of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ by } \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix};$$

these determinants are respectively the determinants of $T$ and $T_1$, the components of $T_2$. Hence the determinant of a transformation, $T_2$, which is the resultant of the transformations $T$ and $T_1$, is equal to the product of the determinants of $T$ and $T_1$.

This result is capable of immediate extension; for let $T_a$, $T_b$ and $T_c$ denote three transformations, the result of whose successive applications is equivalent to $T_d$; the compounding of $T_a$ and $T_b$ is equivalent to a third transformation, $T_{ab}$. The resultant of $T_{ab}$ and $T_c$ is $T_d$, and the determinant of $T_d$ is equal to the product of the determinants of $T_{ab}$ and $T_c$; hence the determinant of $T_d$ is equal to the product of those of $T_a$, $T_b$, and $T_c$. This mode of reasoning is applicable to the resultant of any number of transformations; hence by induction we infer the following theorem:

THEOREM 1. The resultant $T_n$ of $n$ projective transformations $T_i$ ($i = 0, 1, 2, \ldots\ldots n-1$) is a projective transformation, and the determinant of the resultant is equal to the product of the determinants of the components.

*Inverse and identical transformations.*—The transformation $T$ expressed by

$$x_1 = \frac{ax + b}{cx + d}$$

transforms the point x into $x_1$. This equation may be solved for x, giving us

$$x = \frac{-dx_1 + b}{cx_1 - a}. \qquad (4)$$

The transformation expressed by this equation is called the *inverse* of T, and is symbolized by $T^{-1}$. $T^{-1}$ transforms a point $x_1$ into x. The resultant of T and $T^{-1}$ is geometrically evident; it leaves every point on the line unchanged, and is therefore called an *identical* transformation. The resultant of T and $T^{-1}$ reduces to $x_2 = x$, which shows that every point on the line is transformed into itself.

*Invariant points.*—When the points of a line are shifted into new positions by a projective transformation T, does it ever happen that one or more of the points are unaltered in position? To answer this question, we reason as follows: The coordinate of a point x, which remains at rest or unaltered in position, *i. e.*, which is transformed into itself, must satisfy the equation

$$x = \frac{ax + b}{cx + d}.$$

Clearing of fraction, we see that the coordinates of all such points satisfy the quadratic equation

$$cx^2 + (d - a)x - b = o;$$

whence we conclude that a projective transformation T leaves unaltered two points on the line, and their coordinates are given by the roots of equation (5). These two points are generally distinct, but for special values of a, b, c, d, they may coincide. They are called the invariant points of the transformation. Two transformations will not generally have the same invariant points, but, as we shall learn, an unlimited number of transformations may have one or both invariant points in common.

THEOREM 2. Every projective transformation of the points on a line, except the identical transformation, leaves invariant two points which are distinct or coincident.

*Pseudo-transformations.*—If the determinant of T vanishes, the transformation is called a pseudo-transformation. In defining the transformation, it was expressly stated that the determinant must not be zero. This condition excludes just these transformations called pseudo-transformations. The equation of the transformation is written

$$x_1 = \frac{ax + b}{cx + d};$$

if the determinant $ad - bc = o$, then $d = \frac{bc}{a}$. Substituting this value of d in the equation, we have

$$x_1 = \frac{a(ax + b)}{c(ax + b)} = \frac{a}{c}; \qquad (6)$$

which shows that every point on the line is transformed into the fixed point $\frac{a}{c}$. The inverse of the transformation T is written

$$x = \frac{-dx_1 + b}{cx_1 - a}.$$

The determinant of this is also ad — bc, which equated to zero also gives $d = \frac{bc}{a}$. Substituting this value of d in the last equation, we have

$$x = -\frac{b(cx - a)}{a(cx - a)} = -\frac{b}{a}; \tag{7}$$

which shows that every point on the line is transformed by (7) into the fixed point $-\frac{b}{a}$.

The invariant points of a pseudo-transformation are also given by equation (5). Putting $d = \frac{bc}{a}$ in this equation, it breaks up into

$$\left(x - \frac{a}{c}\right) \left(x + \frac{b}{a}\right) = 0; \tag{8}$$

thus showing that $\frac{a}{c}$ and $-\frac{b}{a}$ are the invariant points of the pseudo-transformation.

THEOREM 3. A pseudo-transformation transforms every point on the line into one or the other of its invariant points.

*Three conditions determine a projective transformation.*—The equation of a projective transformation T contains three independent constants, viz., a : b : c : d. We infer, therefore, that three conditions determine such a transformation. In particular, three points and their corresponding points determine uniquely and completely a projective transformation.

Let $x'$, $x''$, $x'''$ be any three points on a line, and $x_1'$, $x_1''$, $x_1'''$ their corresponding points, respectively. Substituting successively in (1) the coordinates of each pair of corresponding points, we have three equations, viz.:

$$\begin{aligned} cx'x'_1 + dx'_1 - ax' - b &= 0, \\ cx''x''_1 + dx''_1 - ax'' - b &= 0, \\ cx'''x'''_1 + dx'''_1 - ax''' - b &= 0. \end{aligned} \tag{9}$$

These equations are linear and homogeneous in a, b, c, d, and determine the ratios of these quantities uniquely and completely, provided no two of these equations are identical or have their coefficients proportional.

THEOREM 4. There is one and only one projective transformation that transforms three given points on a line into three other given points.

*The identical transformation.*—Suppose that the transformation (1) leaves three points of the line invariant. If we put $x'_1 = x'$, $x''_1 = x''$ and $x'''_1 = x'''$ in equations (9), these reduce to the following:

$$\begin{aligned} cx'^2 + (d - a)x' - b &= o, \\ cx''^2 + (d - a)x'' - b &= o, \\ cx'''_2 + (d - a)x''' - b &= o. \end{aligned} \qquad (10)$$

The determinant of these equations,

$$\begin{vmatrix} x'^2 & x' & 1 \\ x''^2 & x'' & 1 \\ x'''^2 & x''' & 1 \end{vmatrix} \equiv (x' - x''')\,(x'' - x''')\,(x' - x''), \qquad (11)$$

does not vanish so long as the three points are distinct; consequently, the coefficients of the above equations must vanish identically. Thus, $c = o$, $b = o$, $d = a$. Putting these values in (1) we get $x_1 = x$, which is the identical transformation. The identical transformation we know transforms every point of the line into itself.

THEOREM 5. A projective transformation which leaves three points of a line invariant is the identical transformation and leaves all points of the line invariant.

## § 2. TYPES AND NORMAL FORMS OF PROJECTIVE TRANSFORMATIONS.

*Two types of projective transformations.*—The invariant points of a transformation T are given by the roots of the quadratic equation (5). The roots of this equation are:

$$(A, A') = \frac{a - d \pm \sqrt{(a + d)^2 - 4\,(ad - bc)}}{2\,c}. \qquad (12)$$

These two roots are distinct or coincident, according as

$$(a + d)^2 - (ad - bc)$$

is not or is equal to zero. Thus there are two distinct types of transformation. The first type is characterized by the fact that it has two invariant points, while the second type has only one. Every transformation not identical belongs to one or the other of these types.

*Normal form of type I.*—A transformation T of type I, whose invariant points are A and A′, may be written in the form:

$$\frac{x_1 - A'}{x_1 - A} = k\,\frac{x - A'}{x - A}, \qquad (13)$$

where the constant *k* is expressible in the terms of the coefficients a, b, c, d, as follows:

$$k = \frac{\left(a + d + \sqrt{(a + d)^2 - 4(ad - bc)}\right)^2}{4(ad - bc)}. \qquad (14)$$

In order to verify this, we must solve equation (13) for $x_1$; this gives us

$$x_1 = \frac{(A - kA')x - AA'(1 - k)}{(1 - k)x + (kA - A')}, \qquad (15)$$

which is of the same form as

$$x_1 = \frac{ax+b}{cx+d}.$$

Comparing the coefficients of these forms, we have

$$\frac{A-kA'}{1-k} = \frac{a}{c},\quad AA' = -\frac{b}{c},\quad \frac{kA-A'}{1-k} = \frac{d}{c};$$

solving for A, A′ and k, we find

$$A = \frac{a-d+\sqrt{(a+d)^2-4(ad-bc)}}{2c},$$
$$A' = \frac{a-d-\sqrt{(a+d)^2-4(ad-bc)}}{2c}, \qquad (16)$$
$$k = \frac{(a+d+\sqrt{(a+d)^2-4(ad-bc)})^2}{4(ad-bc)}.$$

The values of A and A′ thus obtained are the same as the roots of equation (5). Equation (13) is called the *implicit* normal form of type I.

*Normal form of type II.*—A transformation of type II, whose single invariant point is A, is reducible to the form

$$\frac{1}{x_1-A} = \frac{1}{x-A} + t. \qquad (17)$$

To verify this, solve for $x_1$; thus,

$$x_1 = \frac{(1+tA)x - t^2A}{tx + (1-tA)}. \qquad (18)$$

This is the same form as (1). A and t are found in terms of a, b, c, d, as before, by comparing coefficients and solving for A and t; thus,

$$A = \frac{a-d}{2c} \text{ and } t = \frac{2c}{a+d}. \qquad (19)$$

Equation (17) is called the implicit normal form of type II.

THEOREM 6. Every transformation of the form $x_1 = \frac{ax+b}{cx+d}$ is reducible to one or the other of the normal forms

$$\frac{x_1-A'}{x_1-A} = k\frac{x-A'}{x-A} \text{ or } \frac{1}{x_1-A} = \frac{1}{x-A} + t.$$

*Geometrical interpretation of the normal forms.*—The normal form of type I may be written:

$$k = \frac{x-A}{x-A'} : \frac{x_1-A}{x_1-A'} = (AA'xx_1); \qquad (20)$$

*i. e.*, k is the cross-ratio of the four points A, A′, x, $x_1$, where A and A′ are the invariant points, and x and $x_1$ a pair of corresponding points. Here x and $x_1$ are any pair of corresponding points, and k is a constant quantity.

In the normal form of type II the expressions $x-A$ and $x_1-A$ are the distances of a pair of corresponding points from the invariant point. The normal form of type II may be written:

$$\frac{1}{x_1-A} - \frac{1}{x-A} = t, \qquad (21)$$

which shows that the difference of the reciprocals of the distances of a pair of corresponding points from the invariant point is constant for all pairs of corresponding points. Let x be the point at infinity on the line; t is thus seen to be the reciprocal of the segment $Ax_1$ where $x_1$ is the point into which the point at infinity is transformed.

THEOREM 7. In a transformation of type I, k, the cross-ratio of the invariant points and a pair of corresponding points, is constant for all pairs of corresponding points; in a transformation of type II, t, the difference of the reciprocals of the distances of a pair of corresponding points, is constant for all pairs of corresponding points.

*The natural parameters.*—When the transformation is written in the form of equation (1), we see that there are three independent parameters viz., $\frac{a}{d}$, $\frac{b}{d}$, $\frac{c}{d}$, when it is of type I; in the case of a transformation of type II, the relation, $(a+d)^2 = 4(ad - bc)$, is satisfied, and there are but two independent parameters. The coefficients, a, b, c, d, have no simple geometric meanings; but in the normal forms A, A′, k and A, t have definite important geometric meanings. The parameters A, A′, k and A, t are called the *natural* parameters of the transformation.

*Explicit normal forms.*—Equations (15) and (18) may be put into the forms:

$$x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ A' & 1 & kA' \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ A' & 1 & k \end{vmatrix}}, \text{ and } x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ 1 & 0 & tA+1 \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ 1 & 0 & t \end{vmatrix}}. \qquad (22)$$

These are called the explicit normal forms of types I and II, respectively.

*Type II as the limiting form of type I.*—It is evident that type II is the limiting form of type I when the two invariant points coincide. From equation (16) we see that $k = 1$ when $A = A'$. The fraction $\frac{k-1}{A'-A}$ becomes indeterminate when $A = A'$. Putting for A, A′ and k their values from (16), we have:

$$\lim_{A'=A} \frac{k-1}{A'-A} = \frac{2c}{a+d}.$$

But from (19) $\frac{2c}{a+d} = t$; hence, $t = \lim_{A'=A} \frac{k-1}{A'-A}$. (23)

By means of this relation the normal form of type II can be deduced directly from that of type I. Dividing both numerator and denominator of (15) by $A - A'$, we get:

$$x_1 = \frac{\frac{(A-kA')}{(A-A')}x - \frac{AA'(1-k)}{(A-A')}}{\frac{(1-k)}{(A-A')}x + \frac{kA-A'}{(A-A')}}.$$

Putting $A'=A$ and $\lim\limits_{A'=A}\frac{1-k}{A-A'}=t$, this reduces to (18).

In the explicit normal form of type I, (22), subtract the second row from the last, divide through by $A'-A$, and pass to the limit. In this way we get the explicit normal form of type II.

*Resultant of T and $T_1$.*—We next consider the resultant of two transformations T and $T_1$, both of type I, which have no invariant point in common, and which are given in their explicit normal forms. T and $T_1$ are as follows:

$$x_1=\frac{(A-kA')x-AA'(1-k)}{(1-k)x-(A'-kA)} \text{ and } x_2=\frac{(A_1-k_1A'_1)x_1-A_1A'_1(1-k_1)}{(1-k_1)x_1-(A'_1-k_1A_1)}.$$

Eliminating $x_1$ we have:

$$x_2=\frac{\left\{(A-kA')\ (A_1-k_1A'_1)-AA'(1-k)\ (1-k_1)\right\}x}{\left\{\ (1-k)\ (A-kA')-(1-k_1)\ (A'_1-k_1A_1)\ \right\}x}$$
$$\frac{-\left\{(AA'(1-k)\ (A_1-k_1A'_1)-A_1A'_1(1-k_1)\ (A'-kA')\right\}}{-\left\{\ AA'(1-k)\ (1-k_1)-(A'-kA)\ (A'_1-k_1A_1)\ \right\}}$$
$$=\frac{px-q}{rx-s}=\frac{(A_2-k_2A'_2)x-A_2A'_2(1-k_2)}{(1-k_2)x-(A'_2-k_2A_2)}. \tag{24}$$

We readily find:

$$A_2A'_2=\frac{q}{r},\ A_2+A'_2=\frac{p+s}{r},\ A_2-A'_2=\frac{1-k_2}{1+k_2}\cdot\frac{p-s}{r},$$
$$A_2-A'_2=\frac{\sqrt{(p+s)^2-4\,qr}}{r}.$$

Hence,

$$\frac{1-k_2}{1+k_2}(p-s)=\sqrt{(p+s)^2-4\,qr}.$$

From this we find:

$$\frac{1+k_2}{\sqrt{k_2}}=\frac{p-s}{\sqrt{qr-ps}}=\frac{(1-kk_1)\ (A-A'_1)\ (A'-A_1)+(k+k_1)\ (A-A_1)\ (A'-A'_1)}{\sqrt{kk_1}\ (A-A')\ (A_1-A'_1)}$$
$$=\frac{1-k-k_1+kk_1}{\sqrt{kk_1}}\,l+\frac{k+k_1}{\sqrt{kk_1}}, \tag{25}$$

where

$$l=\frac{A_1-A'}{A_1-A'_1}:\frac{A-A'}{A-A'_1}=(A'A'_1)A_1A,$$

one of the cross-ratios of the four points $A$, $A'_1$, $A_1$, $A'$.

We also find

$$A_2=\frac{p+k_2s}{r(1+k_2)} \text{ and } A'_2=\frac{s+k_2p}{r(1+k_2)}. \tag{26}$$

If one or both of the transformations T, $T_1$ are of type II, the resultant is obtained by putting $A'=A$ and $\lim\frac{1-k}{A-A'}=t$, or $A'_1=A'$ and $\lim\frac{1-k_1}{A_1-A'_1}=t_1$, or both, as the case may be, in (24), (25), and (26.)

## § 3. ONE-PARAMETER GROUPS OF PROJECTIVE TRANSFORMATIONS.

*Resultant of T and $T_1$ with common invariant points.*—Let T and $T_1$ be two transformations of type I having the same invariant points A and A′, and let T transform the point x to $x_1$, and let $T_1$ transform $x_1$ to $x_2$. The resultant of T and $T_1$ also leaves A and A′ invariant and transforms x directly to $x_2$. Let T and $T_1$ be given in the implicit normal forms:

$$\frac{x_1 - A'}{x_1 - A} = k\frac{x - A'}{x - A} \text{ and } \frac{x_2 - A'}{x_2 - A} = k_1\frac{x_1 - A'}{x_1 - A}.$$

We eliminate $x_1$ from these equations by multiplication, and obtain:

$$\frac{x_2 - A'}{x_2 - A} = kk_1 \frac{x - A'}{x - A}.$$

The cross-ratio of $T_2$ is therefore $k_2 = kk_1$.

In the same way it may be shown that the resultant of any number of transformations with the same invariant points has its cross-ratio equal to the continued product of the cross-ratios of the components.

The cross-ratio k may have an infinite number of values, and hence there are an infinite number of transformations leaving both A and A′ invariant. The transformations of this system have the property that the resultant of any two of them is another of the same system. Hence they form a continuous group, the parameter of the group being the cross-ratio k, which may be made to vary continuously. We shall designate the group by $G_1(AA')$.

THEOREM 8. The totality of transformations which leave the same two points of a line invariant form a continuous group; the cross-ratio of the resultant of any two transformations of this group is equal to the product of the cross-ratio of the components.

*Properties of the group $G_1(AA')$.*—The fundamental property of the group $G_1(AA')$ is that the resultant of the two transformations of the group is another of the same group. Other properties of the group will now be developed.

The inverse of T, any transformation in $G_1(AA')$, is also to be found in $G_1(AA')$. For if T transforms x into $x_1$, then

$$T: \frac{x_1 - A'}{x_1 - A} = k\frac{x - A'}{x - A}.$$

If $T^{-1}$ be the inverse transformation which transforms $x_1$ back to x, then

$$T^{-1}: \frac{x - A'}{x - A} = \frac{1}{k}\frac{x_1 - A'}{x_1 - A}.$$

Hence the cross-ratios of a pair of inverse transformations have reciprocal values.

The resultant of a pair of inverse transformations is the identical transformation. The product of the cross-ratios of a pair of inverse transformations is unity; hence the cross-ratio of the identical transformation is unity, and the group $G_1(AA')$ always contains the identical transformation.

The group $G_1(AA')$ contains one transformation which is identical with it own inverse. In this case we have the condition $k = \frac{1}{k}$, or $k^2 = 1$; whence $k = \pm 1$. The value $k = 1$ gives the identical transformation of the group. That this is its own inverse is self-evident. The value $k = -1$ gives the involutoric transformation of the group. This transformation has the effect of interchanging every pair of corresponding points on the line, since its second power is the identical transformation; thus this transformation gives rise to an involution, whence its name.

The group $G_1(AA')$ contains two very noteworthy transformations whose cross-ratios are $o$ and $\infty$, respectively. The first transforms all points of the line except A into A'; the second transforms all points of the line except A' into A. These are pseudo-transformations and form an inverse pair.

The cross-ratio of the identical transformation is unity, and this transformation leaves every point of the line invariant. The transformation of the group whose cross-ratio is $1 + d$, when d is an infinitesimal number, moves every point on the line an infinitesimal distance, and is hence called an infinitesimal transformation. d has an infinite number of different values, viz., $|\delta| \exp. i\theta$. If an infinitesimal transformation be repeated n times, the cross-ratio of the resultant is $(1 + d)^n$. By a proper choice of d, *i. e.*, of $\theta$ and n, this cross-ratio may be made any number we please; hence every transformation in $G_1(AA')$ may be generated from an infinitesimal transformation of the group. The chief properties of the group $G_1(AA')$ may be summed up as follows:

THEOREM 9. The transformations of the group $G_1(AA')$ can be arranged in inverse pairs; it contains the identical transformation, one involutoric, two pseudo and an infinite number of infinitesimal transformations; every transformation of the group may be generated from an infinitesimal transformation of the group.

*One-parameter group $G_1(A)$*.—Let T and $T_1$ be two transformations of type II having the same invariant point A. They may be written:

$$\frac{1}{x_1 - A} = \frac{1}{x - A} + t \text{ and } \frac{1}{x_2 - A} = \frac{1}{x_1 - A} + t_1.$$

T transforms x to $x_1$, and $T_1$ transforms $x_1$ to $x_2$. Their resultant, $T_2$, is obtained by eliminating $x_1$ from these two equations by addition, giving us:

$$\frac{1}{x_2 - A} = \frac{1}{x - A} + t + t_1.$$

Thus, $t_2 = t + t$. The resultant, $T_2$, is of type II, has the same invariant point A, and its constant, $t_2$, is equal to the sum of the constants of T and $T_1$.

The constant, t, may have an infinite number of values, and hence there are an infinite number of transformations of type II having the same invariant point. These possess the same fundamental group property and form a continuous group $G_1(A)$ of type II.

THEOREM 10. The totality of transformations of type II, which leave the same point invariant, form a continuous group; the constant of the resultant of any two transformations of the group is equal to the sum of the constants of the components.

*Properties of the group $G_1(A)$.*—The properties of the group $G_1(A)$ are not identical with those of the group $G_1(AA')$ but very similar. If T be the transformation

$$\frac{1}{x_1 - A} = \frac{1}{x - A} + t,$$

its inverse, $T^{-1}$, which transforms x back to $x_1$, is

$$\frac{1}{x - A} = \frac{1}{x_1 - A} - t.$$

Hence two transformations in $G_1(A)$ whose constants are numerically equal with opposite signs form an inverse pair. All transformations in the group may be arranged in inverse pairs.

The resultant of a pair of inverse transformations is the identical transformation whose constant is $t_2 = t - t = o$. The group $G_1(A)$ therefore contains the identical transformation.

The only transformation in the group which is its own inverse is the identical transformation, *i. e.*, the group contains no involutoric transformation. It contains one pseudo-transformation for which $t = \infty$. This transforms every point on the line to the invariant point.

A transformation of the group whose constant $t = \delta \exp. i\theta$ is infinitesimally near to zero is an infinitesimal transformation. If an infinitesimal transformation is repeated *n* times, the resultant has the constant nt. By a proper choice of n and $\theta$ this may be made any number we please; hence every transformation in the group $G_1(A)$ can be generated from an infinitesimal transformation of the group.

THEOREM 11. The transformation of the group $G_1(A)$ can be arranged in inverse pairs; it contains the identical transformation, one pseudo but no involuntoric transformation; it contains an infinite number of infinitesimal transformations, and every transformation of the group can be generated from an infinitesimal transformation of the group.

*Number of one-parameter groups.*—We have thus found two types of one-parameter groups of transformations of the points on a line, viz., $G_1(AA')$ and $G_1(A)$. Evidently there are as many groups of the first type as there are pairs of points on a line, viz., $\infty^2$. Also, there is a group of type II for every point on a line; therefore, $\infty^1$ in number. It is also evident that every transformation of the points on the line belongs to one and only one of these one-parameter groups (except the identical transformation which is common to all).

## §4. Two- and Three-parameter Groups.

*The group $G_2(A)$.*—Let us take two transformations of type I, T and $T_1$, having one, but only one, invariant point, A′, in common, and find their resultant. The point A′ may be taken for the origin, and the two equations then reduce to the form

$$\frac{x_1}{x_1 - A} = k\frac{x}{x - A} \text{ and } \frac{x_2}{x_2 - A_1} = k_1\frac{x_1}{x_1 - A_1}. \tag{27}$$

Eliminating $x_1$ from these equations, we have the resultant

$$x_2 = \frac{kk_1 AA_1 x}{(kk_1 A - kA + kA_1 - A_1)x + AA_1}. \tag{28}$$

The invariant points of this transformation are found by making $x_2 = x$ and solving the resulting quadratic. The invariant points are thus found to be $A = o$, and

$$A_2 = \frac{(kk_1 - 1)AA_1}{kk_1 A - kA + kA_1 - A_1}. \tag{29}$$

Putting $kk_1 A - kA + kA_1 - A_1 = \frac{(kk_1 - 1)AA_1}{A_2}$ in equation (28), we have:

$$\frac{x_2}{x_2 - A_2} = kk_1\frac{x}{x - A_2}. \tag{30}$$

From this we see that the resultant of two transformations of type I having one invariant point in common has for one of its invariant points the common invariant point of the components; also, we learn that the cross-ratio of the resultant in this case is equal to the product of the cross-ratios of the components, viz., $k_2 = kk_1$. Since (30) is of the same form as (27), we infer that all transformations leaving a single point invariant form a continuous group of two parameters, $G_2(A)$. The two parameters are the cross-ratio k and the abscissa of the other invariant point $A_2$.

Theorem 12. All transformations which leave a single point invariant form a two-parameter group; the cross-ratio of the resultant of any two transformations of the group is equal to the product of the cross-ratios of the components.

*Structure and properties of $G_2(A)$.*—The structure of the group $G_2(A)$ is evident from the above discussion. It contains $\infty^1$ one-parameter subgroups of type I and one of type II. The point $A_2$ may be taken in turn with every other point on the line to form the invariant points of a group $G_1(AA')$ and once with itself to form the invariant point of $G_1(A)$. From the continuity of the point system on a line and from the known continuity of each subgroup, we infer the continuity of the group $G_2(A)$. The transformations of the group $G_2(A)$ are not commutative. Since $k_2 = kk_1$, it is evident that the cross-ratio of the resultant is independent of the order of the components; but the position of the second invariant point of $T_2$ is not independent of the order of T and $T_1$. For if k and $k_1$ be interchanged in equation (29), the value of $A_2$ is not changed; but if A and $A_1$ are interchanged in (29), the value of $A_2$ is changed, thus showing that T and $T_1$ are not commutative in $G_2(A)$.

When T and $T_1$ have both invariant points in common and $k_1 = \frac{1}{k}$, their resultant is the identical transformation; but when T and $T_1$ have only one invariant point in common and $k_1 = \frac{1}{k}$, the resultant is of type II. For, putting $k_1 = \frac{1}{k}$ in (29), we find $A_2 = 0$; thus the two invariant points of $T_2$ coincide, and it is of type II. The value of the constant t of $T_2$ is found as follows:

$$t = \lim_{k_2 = 1} \frac{k_2 - 1}{A_2} = \lim \frac{kk_1 A - kA + kA_1 - A_1}{AA_1} =$$
$$\left(\frac{1}{A_1} - \frac{1}{A}\right)(1 - k). \qquad (31)$$

THEOREM 13. The group $G_2(A)$ contains $\infty^1$ subgroups $G_1(AA')$ and one subgroup $G_1(A)$. The transformations in $G_2(A)$ are not commutative. The resultant of two transformations of type I in $G_2(A)$, for which $k_1 = \frac{1}{k}$ and $A'_1$ not equal to $A'$ is of type II.

*The three-parameter group $G_3$.*—It was shown (Theorem 1) that the resultant of T and $T_1$, any two projective transformations of the points on a line, is again a projective transformation. From this we infer that all projective transformations of the points on a line form a group. This is called the general projective group $G_3$. It is a group of three parameters; for the equation of T contains three independent parameters, viz., a : b : c : d. If these coefficients, a, b, c, d, be made to vary continuously, all the resulting transformations belong to the group $G_3$; and conversely all transformations belonging to the above group are obtained by continuously varying the coefficients in T. Such a group is evidently continuous. If the equation of T be put into the normal form,

$$\frac{x_1 - A'}{x_1 - A} = k \frac{x - A'}{x - A},$$

the three natural parameters, A, A', k, may be made to vary continuously, thus generating the group $G_3$. The group $G_3$ contains $\infty^1$ two-parameter groups $G_2(A)$, one for each point on the line. It contains, as we have already shown, $\infty^2$ groups $G_1(AA')$ and $\infty^1$ groups $G_1(A)$.

## § 5. TRANSFORMATIONS OF PENCILS OF LINES AND PLANES.

The theory sketched in the foregoing pages applies equally well to the one-dimensional transformations of the lines of a flat pencil or the planes of an axial pencil. There are two varieties of such transformations, viz., those with two invariant elements and those with only one invariant element.

In the first case let O be the vertex of a flat pencil, A and A' the two invariant lines of the pencil, and x and $x_1$ any pair of corresponding lines in the transformation. Then we have the cross-ratio $O(AA'xx_1) = k$, and the theory requires no further development. The second case, with one invariant element, may be deduced as the limiting form of the first case in the following manner:

Let $O(AA'xx_1) = k$; whence $O(AxA'x_1) = 1 - k$. Writing out the last cross-ratio in full, we have:

$$\frac{\sin(A'OA)}{\sin(A'Ox)} : \frac{\sin(x_1OA)}{\sin(x_1Ox)} = 1 - k.$$

Whence

$$\frac{\sin(x_1Ox)}{\sin(A'Ox) \,.\, \sin(x_1OA)} = \frac{1-k}{\sin(A'OA)}.$$

But $(xOx_1) = (AOx) - (AOx_1)$; therefore,

$$\lim_{A'=A} \frac{\sin(AOx)\cos(AOx_1) - \cos(AOx)\sin(AOx_1)}{\sin(A'Ox) \,.\, \sin(x_1OA)} = \lim_{A'=A} \frac{1-k}{\sin(A'OA)} = t.$$

Hence

$$\cot(x_1OA) - \cot(xOA) = t,$$

$$\text{or} \quad \cot\theta_1 = \cot\theta + t. \qquad (32)$$

THEOREM 14. In a transformation of a pencil of lines (or planes) of type II, the difference of the cotangents of the angles made with the invariant line (or plane) by a pair of corresponding lines (or planes) is constant for all pairs of corresponding lines (or planes).

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## PART II.

### §6. THE GENERAL PROJECTIVE GROUP $H_6$ AND ITS SUBGROUPS.

*The complex double parameter.*—Hitherto we have considered the natural parameters, A, A′, k of T to be complex numbers, but have regarded them as single indivisible parameters. However, a complex variable is of the nature of a double parameter and may assume $\infty^2$ different values. From this point of view, the groups which we have designated by $G_1(AA')$ and $G_1(A)$, whose parameters are respectively k and t, are two-parameter groups. In like manner, $G_2(A)$ and $G_3$ have, respectively, four and six parameters. The first point of view is that of Lie and his followers, while the second point of view is more that of Klein and other exponents of Riemann's function theory.

According to Lie's theory of continuous groups, there are but four varieties of continuous groups of transformations in one dimension, viz., $G_3$, $G_2(A)$, $G_1(AA')$, $G_1(A)$. But as soon as we regard a complex parameter as a double parameter, we shall find several other varieties of continuous groups within the general projective group of points on the line. These will be determined in the next section.

Hereafter we shall use H instead of G to designate a group whose parameters are regarded as double parameters, and shall write $H_6$, $H_4(A)$, $H_2(AA')$, $H_2(A)$ instead of $G_3$, $G_2(A)$, $G_1(AA)$, $G_1(A)$.

These two sets of symbols represent exactly the same things, but from different points of view.

*Loxodromic, hyperbolic, elliptic and parabolic transformations.*—We have already shown that the transformation $T: x_1 = \frac{ax+b}{cx+d}$ is reducible to one or the other of the normal forms:

$$\frac{x_1 - A'}{x_1 - A} = k\,\frac{x - A'}{x - A} \quad\text{or}\quad \frac{1}{x_1 - A} = \frac{1}{x - A} + t.$$

Transformations of the first type are subdivided into three kinds, according to the values of k. When k is real,* the transformation is called hyperbolic; when k is of the form exp $i\theta$, *i. e.*, when $|k| = 1$, it is called elliptic; for all the other values of k it is called loxodromic. All transformations of the second type are called parabolic. These distinctions will be of much use to us.

*Known subgroups of $H_6$.*—From § 4 we know that $H_6$ contains $\infty^2$ subgroups $H_4(A)$, one for each value of A. $H_6$ also contains $\infty^4$ subgroups $H_2(AA')$, one for each pair of values of A and A′. It also contains $\infty^2$ groups $H_2(A)$, one for each value of A. The $\infty^4$ parabolic transformations in $H_6$ do not form a group. This follows from the fact that the resultant of two parabolic transformations is not generally a parabolic transformation.

*One-parameter subgroups of $H_2(A)$.*—In the group $H_2(A)$ the law of combination of parameters is $t_2 = t + t_1$; the parameter t may be written in the form, $r \exp i\theta$. Thus we have

$$r_2 \exp i\theta_2 = r \exp i\theta + r_1 \exp i\theta_1.$$

If $\theta_1 = \theta$, then $\theta_2 = \theta$. Keeping $\theta$ constant and letting r vary, we get $r_2 = r + r_1$.

We have here the conditions for a one-parameter subgroup of $H_2(A)$. All transformations in $H_2(A)$ for which the t's have the same amplitude form a subgroup of $H_2(A)$. It is clear that there is one such subgroup for each value of $\theta$.

If $t = r \exp i\theta$ be represented in the usual manner by points in the complex plane, we see that there is a transformation in $H_2(A)$ corresponding to each point in the complex plane. The transformations in $H_2(A)$, corresponding to the points of the complex plane on a line through the origin and making an angle $\theta$ with the axis of reals, constitute a subgroup $H_1(A)\theta$.

*Properties of $H_1(A)\theta$.*—The group $H_1(A)\theta$ contains the identical transformation corresponding to $r = o$. Two transformations corresponding to values of r numerically equal, but with opposite signs, are inverse to one another. $H_1(A)\theta$ also contains the pseudo-

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*Klein (Modulfunctionen, Band I, S. 164), and Poincare (Acta Mathematica, Tome I, p. 5), call the transformation hyperbolic only when k is real and positive. The change here made is ustified by the simplification it effects in the statement of the results.

transformation of $H_2(A)$, which is given by $r = \infty$. This group contains two subdivisions corresponding to positive and negative values of $r$. These are separated by the identical and the pseudo-transformations.

The group $H_1(A)\theta$ contains two infinitesimal transformations corresponding to $+\delta r$ and $-\delta r$. Each infinitesimal transformation generates its corresponding subdivision of the group. It is evident that each transformation of the group $H_2(A)$ can be generated from one and only one infinitesimal transformation.

One may, if one chooses, call the aggregate of the transformations in $H_2(A)$, which are represented by points on a half ray from the origin to the infinity point, a group. Every such group contains the identical transformation, the pseudo-transformation, one infinitesimal transformation, and every transformation in the group is generated from the infinitesimal transformation of the group. But the inverse of every transformation in such a group $H_1(A)\theta$ would be found in group $H_1(A)(\theta+\pi)$. It is better, however, to consider the group as made up of two subdivisions than to regard each subdivision as a separate group. The group $H_6$ contains $\infty^2$ subgroups $H_2(A)$, one for each value of the complex number $A$. Each of these two-parameter groups $H_2(A)$ contains $\infty^1$ one-parameter subgroups, one for each real value $\theta$ between $0$ and $\pi$.

THEOREM 15. There are two varieties of subgroups of parabolic transformations in $H_6$, viz., $H_2(A)$ and $H_1(A)\theta$.

*One-parameter subgroups of $H_2(AA')$.*—In the group $H_2(AA')$ the law of combination of parameters is $k_2 = kk_1$. The parameter $k$ may be written in the form $\exp(c+i)\theta$; then we have:

$$\exp(c_2+i)\theta_2 = \exp\left\{(c+i)\theta + (c_1+i)\theta_1\right\}.$$

If $c_1 = c$, then $c_2 = c_1 = c$, and $\theta_2 = \theta + \theta_1$. Keeping $c$ fixed and making $\theta$ vary, we get $\exp(c+i)\theta_2 = \exp(c+i)(\theta+\theta_1)$. We have here the conditions for a one-parameter subgroup of $H_2(AA')$; all transformations in $H_2(AA')$, for which the $k$'s have the same $c$ in the formula $k = \exp(c+i)\theta$, form a subgroup $H_1(AA')c$. There is one such subgroup for each real value of $c$.

There is a transformation in $H_2(AA')$ corresponding to each value of $k$, which is a complex number. Let $k = \exp(c+i)\theta$ be represented by points in a complex plane. We wish to see how the transformations in the one-parameter group $H_1(AA')c$ are distributed in the plane. We have the equation $k = \exp(c+i)\theta$, where $c$ is a constant and $\theta$ a variable. Let $k = x + iy$. Then we have:

$$x + iy = \exp c\theta(\cos\theta + i\sin\theta); \qquad (33)$$

$$x = \exp c\theta \cos\theta;$$

hence, $$y = \exp c\theta \sin\theta;$$

therefore, $$x^2 + y^2 = \exp 2c\theta;$$

putting $x^2 + y^2 = r^2$, we have $r = \exp c\theta$. But this is the polar equation of a logarithmic spiral about the origin cutting the axis of real numbers at an angle whose cotangent is c. Hence the locus of points in the complex plane which correspond to the transformations in the one parameter group H(AA')c, is a logarithmic spiral about the zero point cutting the axis of reals at an angle $\phi$, such that $\cot \phi = c$.

Different values of c give different spirals, each of which corresponds to a one-parameter subgroup of $H_2(AA')$. The real number c may assume in turn all values from $-\infty$ to $+\infty$, so that these spirals lie infinitely close to one another. These spirals all pass through the unit point. For $c = o$, the corresponding spiral becomes the unit circle; for $c = \infty$, the spiral reduces to a straight line, the axis of real numbers.

The family of spirals for which c is positive fills the entire plane, and no two of them intersect except in the unit point. The same is true of the family of spirals for which c is negative. But every spiral of one family intersects an infinite number of times every spiral of the other family; thus these spirals cover twice over the entire plane. Every point in the plane not on the unit circle lies on two of these spirals, from which we infer that every loxodromic transformation in the group $H_2(AA')$ belongs to two distinct loxodromic one-parameter subgroups.

Every hyperbolic transformation in $H_2(AA')$, except the involutoric transformation for which $k = -1$, belongs to three one-parameter subgroups; for two spirals and the axis of real numbers pass through every point for which k is real. The elliptic transformations in $H_2(AA')$ belong only to the elliptic subgroup. The involutoric transformation is common to the elliptic and the hyperbolic subgroups. The identical transformation is common to all subgroups, and the two pseudo-transformations, for which $k = o$ and $k = \infty$, are common to all subgroups except the elliptic. Two loxodromic subgroups for which the c's have the same sign have no transformations in common other than the identical and the pseudo-transformations, while two loxodromic subgroups for which the c's have opposite signs have in common an infinite number of discrete transformations.

THEOREM 16. Every elliptic transformation in $H_2(AA')$ belongs to one and only one subgroup; every loxodromic transformation in $H_2(AA')$ belongs to two distinct subgroups; every hyperbolic transformation in $H_2(AA')$, except the involutoric transformation, belongs to three distinct subgroups.

*Generation of $H_2(AA')$ from infinitesimal transformations.*—The same geometric representation enables us to discuss intuitively the generation of finite transformations in $H_2(AA')$ by the repetition of an infinitesimal transformation. Every spiral passes through the unit point, and, corresponding to the two points on a spiral adjacent to the unit point, we have two infinitesimal transformations belonging to a one-parameter group. These are given by $k = \exp(c + i)\delta\theta$ and $k = \exp - (c + i)\delta\theta$. The indentical transformation divides the group $H_1(AA')c$ into two portions, each of which contains an infinitesimal transformation. Every finite transformation in each portion of $H_1(AA)c$ can be generated by the repetition of the corresponding infinitesimal transformation.

Since every loxodromic transformation in $H_2(AA')$ belongs to two distinct subgroups, it follows that every such loxodromic transformation can be generated from either of two distinct infinitesimal transformations. In the elliptic group, for which the spiral reduces to the unit circle, every finite transformation can be generated from either elliptic infinitesimal transformation. Every hyperbolic transformation for which k is positive can be generated from three infinitesimal transformations, while every hyperbolic transformation in $H_1(AA')$ for which k is negative, except the involutoric transformation, can be generated from two distinct loxodromic infinitesimal transformations, but not from either hyperbolic infinitesimal transformation.

THEOREM 17. Every hyperbolic transformation in $H_2(AA')$, for which k is positive, can be generated from three distinct infinitesimal transformations belonging to $H_2(AA')$; every other finite transformation in $H_2(AA')$ can be generated from two distinct infinitesimal transformations of the group.

*Three-parameter subgroups of $H_4(A)$.*—The four-parameter group $H_4(A)$ breaks up, as we already know, into $\infty^2$ two parameter subgroups $H_2(AA')$. We shall now show that the transformations in $H_4(A)$ may be distributed into $\infty^1$ three-parameter subgroups. The law of combination of parameters k within the group $H_4(A)$ is expressed (Theorem 12) by $kk_1 = k_2$. Written in another form, this is

$$\exp(c_2 + i)\theta_2 = \exp\left\{ (c + i)\theta + (c_1 + i)\theta \right\}.$$

If $c_1 = c$, we have $\exp(c_2 + i)\theta_2 = \exp(c + i)(\theta + \theta_1)$; whence $c_2 = c$ and $\theta_2 = \theta + \theta_1$. Hence we see that if we choose from each of the two-parameter groups, $H_2(AA')$, in $H_1(A)$, the one-parameter group characterized by a certain constant value of c, the aggregate of the transformations contained in these one-parameter groups forms a three-parameter group $H_3(A)c$. It is clear that there is one such group in $H_4(A)$ for every real value of c.

In particular, all elliptic transformations in $H_1(A)$ form a group $eH_3(A)$; likewise, all hyperbolic transformations in $H_4(A)$ form a group $hH_3(A)$.

THEOREM 18. All transformations having a common invariant point A, and for which the value of c in the formula $k = \exp(c + i)\theta$ is the same, form a three-parameter subgroup $H_3(A)c$ of $H_4(A)$; $eH_3(A)$ and $hH_3(A)$ are among the subgroups $H_4(A)$.

## § 7. INTERPRETATION OF $H_6$ IN COMPLEX PLANE.

The linear fractional transformation,

$$T : z_1 = \frac{az + b}{cz + d}, \tag{34}$$

in which the coefficients and variables are complex quantities, is capable of interpretation as a point transformation of the complex plane. We proceed to the development of the transformation from this point of view.

*T is a circular transformation.*—We shall first show that the transformation T transforms circles into circles. The equation of a real circle in Cartesian coordinates is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \tag{35}$$

Let $x_1 + iy_1 = z$, $x_1 - iy_1 = \bar{z}$, $g + if = \alpha$, $g - if = \beta$. Making these substitutions in (34), we get as the equation of the circle:

$$z_1\bar{z}_1 + \beta z_1 + \alpha \bar{z}_1 + \gamma = 0. \tag{36}$$

$\alpha$ and $\beta$ are conjugate complex quantities; so, also, are $z_1$ and $\bar{z}_1$, but $\gamma$ is real.

If we change the sign of i throughout the equation of T, we have:

$$\bar{z}_1 = \frac{\bar{a}\,\bar{z} + \bar{b}}{\bar{c}\,\bar{z} + \bar{d}}, \tag{37}$$

where the quantities under the dash are the conjugates of those without the dash. Substituting (34) and (37) in (36) and clearing, we have:

$$Kz\bar{z} + Lz + M\bar{z} + N = 0, \tag{38}$$

where

$$\begin{aligned} K &= a\bar{a} + \beta a\bar{c} + \alpha \bar{a}c + \gamma c\bar{c}, \\ L &= a\bar{b} + \beta a\bar{d} + \alpha c\bar{b} + \gamma c\bar{d}, \\ M &= \bar{a}b + \beta \bar{c}b + \alpha \bar{a}d + \gamma \bar{c}d, \\ N &= \bar{b}b + \beta b\bar{d} + \alpha \bar{b}d + \gamma \bar{d}d. \end{aligned}$$

On examination, K and N are found to be real and L and M conjugate imaginary. These are the necessary and sufficient conditions that equation (38) represents a circle. Thus circles are transformed into circles by T, which is, therefore, called a circular transformation.

*Straight lines are transformed into circles.*—It may happen for certain values of a, b, c, d that K in equation (38) vanishes. In this case (38) represents a straight line. If we substitute (34) and (37) in $Ax + By + C = 0$, we find that this line is transformed into a circle.

Hence, circles are sometimes transformed into a straight line, and straight lines usually into circles.

In the geometry of the complex plane, a straight line is regarded as a special case of a circle, and every circle passing through the one point at infinity of the complex plane is a straight line. Regarding a straight line as a particular case of a circle, our transformation always transforms circles into circles.

THEOREM 19. A linear fractional transformation in complex variables is a circular transformation of the complex plane.

*T is a conformal transformation.*—Another fundamental property of our transformation T is that angles are transformed into equal angles. To show this, consider a small triangle formed by the points $z, z', z''$, and compare it with the triangle formed by their corresponding points $z_1, z'_1, x''_1$. In $\frac{z'_1 - z_1}{z''_1 - z_1}$ substitute their values from (34); we get:

$$\frac{z'_1 - z_1}{z''_1 - z_1} = \frac{cz'' + d}{cz' + d} \cdot \frac{z' - z}{z'' - z}. \qquad (39)$$

The ratio $\frac{cz'' + d}{cz' + d}$ tends to unity as the triangles become very small. The quantities $z'_1 - z_1$, $z' - z$, etc., are the corresponding sides of the two triangles; hence, when the corresponding triangles are very small, the corresponding sides are proportional and the triangles are similar. Angles are therefore transformed into equal angles. A transformation which leaves angular magnitudes unaltered is called a *conformal* transformation.

THEOREM 20. A linear fractional transformation of the complex plane is a conformal transformation.

*T is a direct circular transformation.*—The two triangles, $z, z', z''$ and $z_1, z'_1, z''_1$, are not only in the limit similar triangles, but they are also congruent triangles. The homologous sides are arranged in the same order around the two triangles, and if they were of the same size the two triangles could be made to coincide.

Suppose that the three sides of the triangle $z, z', z''$ are colored red, green, and blue, and that an observer at $z$, facing the opposite side, has on his right the red, on his left the blue, and before him the green. The transformation T changes the triangle $z, z', z''$ into the new triangle $z_1, z'_1, z''_1$, and the observer at his new position at $z_1$ still finds the red on his right and the blue on his left. Such a transformation is a direct circular transformation.

There are other transformations which are both circular and conformal, but which interchange the parts of the triangle, so that in his new position the observer finds the blue on his right and the red on his left. These are called indirect circular transformations.

THEOREM 21. A linear fractional transformation of the complex plane is a direct circular transformation.

*Path curves of the group $H_1(AA')c$.*—The effect of the successive transformations of the group $H_1(AA)c$ upon a point P of the complex plane is to move it to new positions, $P_1$, $P_2$, $P_3$, etc., which lie on the path curve of the point P. By a transformation $T(AA')c$ every point of the plane is moved along its path curve. The character and properties of these path curves are now to be determined.

Starting with normal form,

$$\frac{z_1 - A'}{z_1 - A} = \exp(c + i)\theta \frac{z - A'}{z - A},$$

for a given value of $\theta$, z is transformed into $z_1$; for another value of $\theta$, z is transformed into $z_2$, etc. The point z may thus be regarded as a fixed point, and $\frac{z - A'}{z - A} = C$, a constant. Let $z'$ denote the moving point $z_1$, $z_2$, etc. We thus get:

$$z' - A' = C \exp(c + i)\theta\, z' - A.$$

Let $z'_1 - A = r' \exp i\phi'$, $C = \rho \exp i\psi$, and $z' - A = r \exp i\phi$; we then have

$$r' \exp i\phi' = \rho \exp i\psi \,.\, \exp(c + i)\theta \,.\, r \exp i\phi.$$

Whence, $\quad r' = \rho \exp i\theta\, r \text{ and } \phi' = \psi + \theta + \phi. \qquad (40)$

These two equations enable us to determine the path curves of the group $H_1(AA')c$.

In the first place, let $c = o$; we then have $r' = \rho\, r$, where $\rho$ is a constant. The locus of a point z, which moves so that the ratio of its distances from two fixed points A and A′ is a constant, is a circle having A and A′ for a pair of inverse points. Hence, the path curves of the elliptic group $eH_1(AA')$ form an elliptic system of coaxial circles having A and A′ for vanishing points.

In the second place, let $\exp(c + i)\theta \equiv \rho \exp i\theta$ be real; whence $\theta = 0$. We now have $\phi' - \phi = \psi$, where $\psi$ is a constant angle. The locus of a point z, which moves so that the angle AZA′ is constant, is a circle passing through A and A′. Hence, the path curves of the hyperbolic group $hH_1(AA')$ form a hyperbolic system of coaxial circles intersecting in A and A′.

In the third place, let us consider the most general case where c in $\exp(c + i)\theta$ is any real constant. In the equation $r' = \rho \exp c\theta\, r$, the quantity $\rho \exp c\theta$ is not constant, but increases as $\theta$ increases from o to $2\pi$, from $2\pi$ to $4\pi$, etc. Thus the locus of z is not a circle returning into itself, but a spiral about both A and A′. It is known as the double spiral of Holzmüller. The equation, $\phi' - \phi = \psi + \theta$, shows that the spiral cuts the hyperbolic system of circles through A and A′ at a constant angle $\psi$. Hence the path curves of the loxodromic group $H_1(AA')c$ form a system of equiangular double spirals about A and

A'. There is a different system of spirals for each value of c. For $c = 0$ and $c = \infty$ this system of spirals reduces, respectively, to elliptic and hyperbolic systems of circles.

THEOREM 22. The path curves of a one-parameter loxodromic group $H_1(AA')c$ form a system of double spirals about the invariant points A and A'; the path curves of $eH_1(AA')$ and $hH_1(AA')$ are, respectively, elliptic and hyperbolic systems of circles about and through A and A'.

*Path curves of $H_1(A)\theta$.*—The path curves of the one-parameter parabolic group $H_1(A)\theta$ may be determined in the following manner: The normal form of a parabolic transformation is

$$\frac{1}{z_1 - A} = \frac{1}{z - A} + t. \qquad (41)$$

Suppose that z is a fixed point and $z_1$ a movable point whose locus we wish to find. Equation (41) may be written:

$$z_1 - z = -t\,(z - A)\,(z_1 - A).$$

Putting

$$(z_1 - z) = R' \exp i\phi, \quad -t = \rho \exp i(\theta + \pi), \quad (z - A) = r \exp i\psi,$$
$$(z_1 - A) = R \exp i\phi,$$

this becomes

$$R' \exp i\phi' = \rho \exp i(\theta + \pi)\,.\,r \exp i\psi\,.\,R \exp i\phi.$$

Whence we have $\quad R' = \rho r R$ and $\phi' = \theta + \pi + \phi + \psi. \qquad (42)$

Since $\psi$, $\theta$ and $\pi$ are constants, we have $\phi' - \phi = \text{const.}$ Thus we see that the locus of $z_1$ is a circle passing through A and z, two fixed points.

If now z be given different positions in the plane, we have a system of circles all passing through A. The angle $\psi$ varies with the position of z, but $\theta$ is a constant; consequently, all circles of the system are tangent at A, and $\theta$ is the angle which the common tangent line at A makes with the axis of real numbers. Hence the path curves of the group $H_1(A)\theta$ consist of a parabolic system of circles tangent to each other at A and to the line through A which makes with the axis of reals an angle $\theta$.

THEOREM 23. The path curves of the one-parameter group $H_1(A)\theta$ consist of a parabolic system of circles through A and having in common the lineal element $A\theta$.

## § 8. OTHER SUBGROUPS OF $H_6$.

*The group $hH_2(AC)$.*—The developments of § 6 enable us to find some other subgroups of great importance in $H_6$. Let us consider the hyperbolic group $hH_3(A)$; it contains, as we know, $\infty^2$ subgroups $hH_1(AA')$. The resultant of any two transformations, $hT(AA')$ and $hT_1(AA'_1)$, in $hH_3(A)$ is a hyperbolic transformation $hT_2(AA'_2)$. We know that $k_2 = kk_1$. We wish to find the position of the point $A'_2$. The path curves of the two groups $hH_1(AA')$ and $hH_1(AA'_1)$ have one circle in common, viz., C, the circle through the points A, A′, $A'_1$. This circle is therefore invariant under the resultant transformation $T_2(AA'_2)$. Hence, $A'_2$ is somewhere on the circle C. The system of transformations which leave A and C invariant have the group property and form a two-parameter group $hH_2(AC)$. This group contains $\infty^1$ one-parameter hyperbolic subgroups, one for each point on C, and one parabolic subgroup, $H_1(A)\theta$, which has C among its path curves.

*The group $H_3(C)$.*—There is a two-parameter group $hH_2(AC)$ corresponding to each point on C. These $\infty^3$ transformations all leave C invariant, but they do not form a group, as we shall show. There are also $\infty^3$ elliptic transformations which leave C invariant. Let A be any point within C, and A′ its inverse point with respect to C. The one-parameter group $eH_1(AA')$ has C among its system of path curves. In like manner all one-parameter groups of elliptic transformations, whose invariant points are a pair of invariant points with respect to C, leave C invariant. There are $\infty^2$ such pairs of points, and hence there are $\infty^3$ elliptic transformations leaving C invariant.

The aggregate of all transformations leaving C invariant forms a three-parameter group $H_3(C)$. This group contains $\infty^3$ hyperbolic, $\infty^3$ elliptic and $\infty^2$ parabolic transformations. It breaks up into subgroups as follows:

$$H_3(C) \equiv \infty^1\ H_2(AC) + \infty^2\ eH_1(AA') \equiv \infty^2\ hH_1(AA') +$$
$$\infty^2\ eH_1(AA') + \infty^1\ H'_1(A),$$

Evidently the group $H_6$ contains $\infty^3$ subgroups of the kind $H_3(C)$, one for each circle in the complex plane.

THEOREM 24. All transformations of the complex plane, which leave a circle C invariant, form a three-parameter group $H_3(C)$. This group is composed of all hyperbolic transformations whose invariant points are on the circle, of all elliptic transformations whose invariant points are a pair of inverse points with respect to the circle, and of all parabolic transformations whose invariant point is on the circle and whose invariant line is tangent to the circle at the invariant point.

*The group $H_3(iC)$.*—There is another type of the three-parameter group consisting entirely of elliptic transformations, which is closely related to the group $H_3(C)$. This group of transformations of the complex plane leaves invariant an imaginary circle (iC). The relation of this group $H_3(iC)$ to the group $H_3(C)$ with real invariant circle is as follows:

The elliptic system of coaxial circles which form the path curves of a one-parameter elliptic group contains imaginary as well as real circles. Let us select from the path curves of the group $eH_1(AA')$ an imaginary circle (iC), with its center at a real point O and radius equal to iR. The center O is on the line AA', between A and A', and the points A and A' are a pair of inverse points with respect to iC; whence we have $OA \,.\, OA' = -R^2$. In the case of a real circle C, we have $OA \,.\, OA' = R^2$. All transformations in the group $eH_1(AA')$ leave invariant the circle iC and also $\infty^1$ other circles.

There are $\infty^2$ pairs of points in the plane which are inverse points with respect to iC. These all satisfy the relation $OA \,.\, OA' = -R^2$. Each of these pairs of points are the invariant points of a one-parameter group of elliptic transformations, and iC is one of the path curves of each of these groups. Thus we see that there are $\infty^3$ transformations which leave iC invariant; these form a three-parameter group $H_3(iC)$, which contains only elliptic transformations. Evidently the group $H_6$ contains $\infty^3$ subgroups $H_3(iC)$, one for each imaginary circle in the complex plane.

THEOREM 25. There are $\infty^3$ transformations of the complex plane which leave invariant any given imaginary circle iC; these form a group $H_3(iC)$. This group is composed entirely of elliptic transformations whose invariant points are a pair of inverse points with respect to (iC).

*No other subgroups of $H_6$.*—A circular transformation transforms points into points and circles into circles. We have considered all possible groups which leave one or two points invariant; a transformation leaving invariant more than two points is identical. We have also considered all possible groups of transformations leaving a circle invariant. If there be a continuous group characterized by the invariance of some curve other than a circle, such a curve must be the path curve of a one-parameter group. The only other path curve besides the circle is the double spiral of Holzmüller. This has two singular points, and is invariant only under those transformations whose invariant points are these two singular points; hence, there is only one one-parameter group leaving invariant such a double spiral. These considerations indicate that there are no other subgroups of $H_6$.

## § 9. THE REAL SUBGROUP OF $H_6$.

*$RH_3$ a special case of $H_3(C)$.*— In the geometry of the complex plane a straight line is regarded as a special case of a circle, and since every circle in the plane is the invariant circle of a three-parameter group $H_3(C)$, it follows that every line in the plane is also invariant under a group $H_3(L)$, isomorphic with $H_3(C)$. One of these groups $RH_3(L)$ leaves invariant the axis of real numbers in the complex

plane. It is called the real subgroup of $H_6$. The equation of $RH_3(L)$ is

$$x_1 = \frac{ax + b}{cx + d}, \tag{43}$$

in which the variable and coefficients are all real quantities. The structure and properties of $RH_3(L)$ are identical with those of $H_3(C)$.

THEOREM 26. The three-parameter group of real projective transformations of points on a real line is a special subgroup of the six-parameter group of circular transformations of the complex plane.

*Properties of $RH_3(L)$.*—The properties of the real projective group in one dimension, which we shall now designate by $RG_3$, may be determined in two ways. They may be deduced from the properties of the group $H_3(C)$, or they may be derived directly from equation (43). The analytical developments of §§ 1, 2, 3, 4 apply equally well to equation (1) whether the coefficients a, b, c, d are real or complex. From the known properties of $H_3(C)$, given in § 8, we may deduce the following for $RG_3$:

The real group of projective transformations $RG_3$ contains three kinds of transformations, viz., hyperbolic, elliptic, and parabolic. A hyperbolic transformation has two real invariant points; an elliptic, two conjugate imaginary invariant points; a parabolic, one real invariant point. $RG_3$ breaks up into $\infty^1$ subgroups $RG_2(A)$, one for each real point on the line. It further breaks up into one-parameter subgroups, as follows: $\infty^2 hRG_1(AA')$, $\infty^2 eRG_1(AA')$, and $\infty^1 pRG_1(A)$. The natural parameters of $hRG_1(AA')$ are A, A′ and k, which are all real. Those of $eRG_1(AA')$ are A, A′ and k where A and A′ are conjugate imaginary and k is of the form $\exp i\theta$. The parameters of $pRG_1(A)$ are A and t, both real.

*Generation of real groups from infinitesimal transformations.*—The generation of the real groups of one-dimensional transformations requires special notice. We consider first the group $hRG_1(AA')$. The natural parameter of this group is k, which is always a real number. The identical transformation of the group is given by $k = 1$. The two transformations of the group given by $k = \pm d$, where d is a real, positive, infinitesimal quantity, are infinitesimal transformations. The transformations of $hRG_1(AA')$, for which k is between 1 and $\infty$, may be generated from the infinitesimal transformation for which $k = 1 + d$; for $(1 + d)^n$ may be made any positive number greater than unity by proper choice of n. In like manner we see that the transformations of the group for which k is between 1 and 0 may be generated by a repetition of the infinitesimal transformation for which $k = 1 - d$. The transformations in $hRG_1(AA')$ for which k is negative cannot be generated from either infinitesimal transformation of the group. They may be generated from complex infinitesimal transformations but not from real.

Hence, we distinguish three subdivisions of $hRG_1(AA')$: those transformations of the group for which k is positive and less than 1 constitute subdivision I; those for which k is between 1 and $\infty$ constitute subdivision II; those for which k is negative constitute subdivision III.

THEOREM 27. The group $hRG_1(AA')$ is composed of three subdivisions characterized by $0 < k < 1$, $1 < k < \infty$, $k < 0$; subdivisions I and II contain each a generating infinitesimal transformation: the transformations in subdivision III cannot be generated from either infinitesimal transformation of the group.

*Properties of the group $eGH_1(AA')$.*—In an elliptic subgroup of $RG_3$ the natural parameter k has the form of $\exp i\theta$. The value of $\theta$ may be restricted to the interval $-\pi < \theta < \pi$. The identical and involutoric transformations of the group for which $\theta = 0, \pi$, *i. e.*, $k = -1$, divide the group into two subdivisions. Each subdivision contains an infinitesimal transformation given by $\theta = \pm\delta$ where $\delta$ is an infinitesimal real number; every transformation in the group may be generated from either infinitesimal transformation of the group.

THEOREM 28. The group $eRG_1(AA')$ contains two infinitesimal transformations given by $\theta = \pm\delta$, each of which may generate the entire group.

*Properties of the group $pRG_1(A)$.*—The parabolic group $pRG_1(A)$ has the real parameter t, and the law of combination of parameters is $t_2 = t + t_1$. Since the identical transformation is given by $t = o$, it follows that the group has two infinitesimal transformations given by $+\delta$ and $-\delta$. Each infinitestimal transformation generates its corresponding portion of the group, which, therefore, is composed of two subdivisions. For one subdivision t is positive; for the other, negative.

THEOREM 29. The group $pRG_1(A)$ is composed of two subdivisions, each of which contains its generating infinitesimal transformation.

## TABLE OF CONTINUOUS GROUPS IN ONE DIMENSION.

In the following table the fourteen varieties of continuous groups of one-dimensional transformations are classified according to their types, and the structure of each is shown.

### LOXODROMIC GROUPS.

*First class.*

| Symbol. | Structure. | Invariant figure. |
|---|---|---|
| $H_2(AA')$ | $= H_2(AA') + hH_1(AA') + eH_1(AA')$. | Two points. |
| $H_4(A)$ | $= \infty^2 H_2(A\underline{A}' + pH_2(A) + hH_3(A) + eH_3(A)$. | One point. |
| $H_6$ | $= \infty^4 H_2(\underline{A}\,\underline{A}') + \infty^2 pH_2(\underline{A}) + \infty^2 hH_3(\underline{A}) + \infty^2 eH_3(\underline{A})$. | No invariant figure. |

### *Second class.*

| | |
|---|---|
| $H_1(AA')_c$. | Two points and a system of double spirals. |
| $H_3(A)_c = \infty^2 H_1(A\underline{A'})_c + pH_2(A)$. | One point. |

### HYPERBOLIC GROUPS.

| | |
|---|---|
| $hH_1(AA')$. | Two points and a hyperbolic system of circles. |
| $H_2(AC) = \infty^1 hH_1(A\underline{A'}) + pH_1(A)_0$. | Circle and point on it. |
| $hH_3(A) = \infty^2 hH_1(A\underline{A'}) + pH_2(A)$. | One point. |
| $H_3(C) = \infty^2 hH_1(\underline{A}\,\underline{A'}) + \infty^2 eH_1(\underline{A}\,\underline{A'}) + \infty^1 pH_1(A)_0$. | Circle. |

### ELLIPTIC GROUPS.

| | |
|---|---|
| $eH_1(AA')$ | Two points and an elliptic system of circles. |
| $eH_3(A) = \infty^2 eH_1(A\underline{A'}) + pH_2(A)$. | One point. |
| $H_3(iC) = \infty^2 eH_1(\underline{A}\,\underline{A'})$. | Imaginary circle. |

### PARABOLIC GROUPS.

| | |
|---|---|
| $pH_1(A)_0$. | One point and a parabolic system of circles. |
| $pH_2(A) = \infty^1 pH_2(A)_0$. | One point. |

# THE

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# NEW FOSSILS FROM THE UPPER CARBONIFEROUS OF KANSAS.

BY J. W. BEEDE.

With Plate V.

THE following will be of interest and value in completing the faunal succession of the uppermost part of the Carboniferous, as well as for the biologic interest of one of them.

**Ceriocrinus harshbargeri,** n. sp. Plate V, figs. 1–1*e*.

The distinctive features of this species are: Surface ornamented with both pustules and granules, large interradial supporting plate above calyx, ten short, massive arms.

Description: Calyx basin-shaped, shallow, quite concave below. Infrabasals partly covered, but located entirely within the body cavity. Stem small and apparently round. The five basals are large, recurved below the middle, concave in the center of the more depressed part, four apparently hexagonal and one heptagonal, and larger than the others supporting the interradial: all higher than wide, sutures not much depressed. There are five large, massive radials a trifle more than twice as wide as high, well beveled, faceted, and apparently sagging a little on the upper articular surface. Interradial large, fully half within the calyx, higher than broad, and the upper portion very strongly curved inward. This plate supports another entirely without the calyx, which is comparatively large and appears to be pentagonal when seen from without, the two upper sides being much longer than the rest, making the plate appear triangular at first sight, fitting closely between the arms on either side. The five brachial plates are large, contiguous save on the posterior side, and produced into an obtuse spinous process. Much of the upper surface of these processes is faceted, supporting the costals, and in this manner giving the animal its greater lateral diameter above the calyx, and affording more room for the massive arms which, when closed, form a box around the inner part. The costals are ten in number, two to each brachial, large, nearly three times as broad as high, and convex. The distichals are sometimes present and sometimes wanting, very broad and thin when present. Arms ten, broad, outer surface of each nearly flat massive, composed of two series of interlocking plates, each one of which bears a pinnule. Pinnulæ long and narrow. The articular surface of the arm plates is minutely crenulated. These plates decrease in length though but little in height as they approach the tips of the arms. The surface of almost the entire animal was ornamented with granules, and the lower part with both granules and pustules.

| Measurements: | height. | breadth. | length. |
|---|---|---|---|
| Calyx | 11 mm. | 30 mm. | |
| Basals (vertical) | 5½ " | 12 " | |
| Radials | 9 " | 18+" | |
| Brachials | 7 " | 17 " | 8 mm. beyond calyx. |
| Costals | 4 " | 9 to 11 mm. | |
| Distichals | 2 " | 9 " | |
| Arm plates | 2 " | 3 to 5 " | |
| Interradial | 7 " | 6 " | |

Aside from the markings mentioned, the calyx is also ornamented with depressions which look at first like borings of some kind, but which have the same granular character as the rest of the surface.

Position and locality: Carboniferous, Upper Coal Measures, Topeka, Kan., from the Osage City shales, over the Osage coal.

The type was collected by Prof. W. A. Harshbarger, in honor of whom it is named. Type now in the collection of Washburn College.

In general appearance this species is strikingly like *C. craigii* Worthen, but differs in the following respects: The body is not smooth but highly ornamented, and the anal piece is much larger. In all specimens of *C. craigii* that I have seen the body is smooth and glossy, and shows (even when apparently unworn) no indications of former surface-markings, while this species possesses both pustules and granules, the latter covering about the entire specimen. In mature individuals the interradial is much larger. Two smaller specimens, probably younger ones of this species, are figured in the accompanying plate, showing the relative growth of the calyx and interradial. In both the ornamentation is the same as described in this species. *C. monticulatus* Beede is from the same locality and horizon, but has long, slender arms, brachials not spinous, and the calyx is deeper. The most fundamental difference, however, is the number of arms possessed by the latter species, the number being sixteen or eighteen, while the one here described has but ten.

*C. nodulifera* Butts differs from this species in having a node at the upper part of each basal and also on the radials. It also has more acute and apparently longer spines. Mr. Butts makes no mention of finer surface-markings. The Topeka specimen is from a much higher horizon.

**Aviculopecten subequivalvus**, n. sp. Plate V, figs. 3, 3*a*.

Shell thick, moderately large, subequivalvular, rather convex, quite oblique, ears well developed. The hinge is nearly straight, the beak does not project, the angle of divergence of its sides is about eighty to ninety degrees. The left valve, exclusive of the ears, is ovate; anterior ear well developed, obtusely angular, marked only by strong lines of growth; the rise from the ear to the body of the shell is abrupt; the marginal sinus separating the ear from the rest of the shell broad, shallow, and ill-defined. The posterior ear is unknown. The anterior margin below the ear forms an ovate curve, which is probably continued on the ventral and postero-ventral margins. The surface of this valve is apparently marked only by stronger and fainter concentric lines except on the front and back sides, where there are radiating rows of vaulted lamellæ. It is entirely probable that these marks once extended over the entire surface, but have been worn off from the more convex portions. Judging from another specimen, the right valve is somewhat flatter than the left and quite as oblique. Posterior ear very small and obtuse; anterior ear quite large, marked by obscure, large, radiating ribs and probably vaulted lamellæ, as well as strong concentric markings; separated from the shell by a deep sulcus. Margin from the beak around the posterior to near the middle of the shell is a regular ovate curve, antero-ventral margin somewhat produced but rounded, extending obliquely toward the beak until the deep byssal sinus is reached. Ornamentation as in the other valve. In this specimen it seems that the radiating rows of scales covered the entire surface before being worn away. Length, 36 mm.; height, 32 mm.; hinge, 17 mm.; thickness, about 5 mm.

Position and locality: Thin limestone, south of Dover, Kan., in Upper Coal Measures. Type in author's collection.

This shell may prove to be a *Pseudomonotis*, as the critical characters are not well known. It is not liable to be confused with any other shell from the Coal Measures.

**Pinna lata**, n. sp. Plate V, fig. 4.

Shell small for this genus, not very convex, probably plain except the usual growth marks, acutely pointed at the beaks, which are terminal. The angle of divergence of the shell is thirty degrees. This species is based on three casts. The type is not distorted, but the posterior end is broken away. There are two other specimens from the same locality, but compressed dorso-ventrically, which show the same characters as the type. The size is about that of *Aviculopinna americana* Meek, but the form and markings as shown on the cast are very different, and the beaks are terminal. Length of type along hinge, 27 mm.; height (20 mm. back of beak), 11 mm.

Position and locality: Carboniferous, Upper Coal Measures, Howard limestone, Topeka, Kan.

The small size, great angle of divergence at the beak and sharp-pointed beaks easily distinguish this species from other members of the genus from the Coal Measures.

**Pleurophorus whitei**, n. sp. Plate V, figs. 5, 5*b*.
Pleurophorus —? White, Bull. U. S. Geol. Surv. 77, p. 27, pl. IV, figs. 5–10.

Shell of medium to large size, transversely ovate, with the depth of the shell greater at the anterior than at the posterior end. Beaks moderately prominent, subterminal; umbonal ridge prominent, extending obliquely to postero-ventral margin. Hinge straight or arcuate, usually about three-fourths the length of the specimen. The right posterior lamina is well shown on good casts. The anterior adductor impression is usually prominent, being set apart by the depression produced by the ridge behind it. The impression of the two cardinal teeth is also visible. The anterior margin drops obliquely downward from the beak nearly to the middle of the shell, when it rounds, rapidly at first, into the ventral margin, which may be straight, somewhat sinuous or even convex on approaching the postero-ventral region: at the end of the umbonal ridge it rounds rapidly into the posterior, oblique truncation, reaching the hinge at an oblique angle. The lines formed by the hinge and ventral margins converge toward the posterior, thus leaving the greatest depth of the shell in front. Above the umbonal ridge the shell is obliquely flattened to the hinge. The surface, as shown in excellent molds, is ornamented only by fine lines of growth, which are sometimes a little stronger beneath the beak than elsewhere, and weaken on and above the umbonal slope.

Position and locality: This species is found in the Upper Coal Measures, upper Wabaunsee stage, east of Barrett, Kan. This is the only locality known to the writer where this shell, abundant in the Permian, is found below the Wreford limestone. Types from Permian west of Stockdale, Kan., "about 100 feet above Cottonwood limestone." Collected by Prof. E. A. Popenoe.

This species is extremely variable, as shown in Professor White's figures, above cited. The short shell with the posterior shallower than the anterior end is in sharp contrast to most of the species of the genus. In surface ornamentation it resembles *P. tropidophorus* more closely than any other species. It is distinguished from *P. subcuneatus* Meek by its short, thick form and larger size. This species is the more abundant of the two in the Permian rocks of the Big Blue series, while *P. subcuneata* only has been identified from the Cimarron series.

**Allorisma kansasensis,** n. sp. Plate V, figs. 6, 6*d*.

Shell small, rather gibbous, transversely subovate. Beaks prominent, incurved, approximate, located about one-third the length of the shell from the anterior end. Anterior outline descending obliquely from the beaks nearly to the ventral margin, where it turns rather abruptly backward along the nearly straight basal edge to the posterior extremity, where it rounds off regularly upward and then forward to meet the hinge. The hinge is apparently straight and more than half as long as the shell. The umbonal ridge is prominent, rounded, ill-defined, fading away at the extremity of the shell. Above this ridge is a depression which, with the smooth, elevated hinge, would cause a strong keel on the shell back of the beaks. The concentric ribs are prominent, fading out at the upper part of the umbonal ridge and near the anterior border. These ribs are quite as prominent on the casts as on the shell itself. On the central and postero-central regions are the characteristic distant, radiating rows of fine, closely set granules. The shell is less convex below and back of the beak than on either side of this region, though it is not concave. The length varies from 1.7 to 1.4 the height.

Position and locality: Upper Coal Measures, Howard limestone, Topeka, Kan.

This species is very closely related to *A. curta* Swallow, but differs from it in having a straighter hinge and a more remote beak. It is from a lower horizon. In the latter respect it differs from McChesney's species also.

PALEONTOLOGICAL LABORATORY, INDIANA UNIVERSITY,
MAY 22, 1902.

## Explanation of Plate V.

All figures about natural size. Drawings by C. McK. Beede.

*Ceriocrinus harshbargeri.*

Fig. 1. Right posterior view of type.
Fig. 1*a*. Diagram of anal plates of same.
Fig. 1*b*. Surface detail enlarged.
Fig. 1*c*. Diagram of basal view.
Fig. 2. Calyx of another specimen, probably of this species.
Fig. 2*a*. Diagram of a portion of posterior view of another slightly smaller specimen, showing variation in size of anal plate.

*Aviculopecten subequivalvus.*

Fig. 3. View of type. One side incomplete.
Fig. 3*a*. Opposite valve of another specimen.

*Pinna lata.*

Fig. 4. View of type, a left valve.

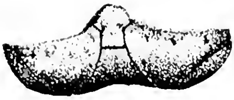
*Pleurophorus whitei.*

Fig. 5. Type specimen; cast of a left valve on the same slab as the two following and the cephalothorax of a large limuloid.
Fig. 5*a*. Cast of a broken specimen, showing the anterior muscular impressions.
Fig. 5*b*. Specimen of different outline.

*Allorisma kansasensis.*

Fig. 6. Imperfect cast, showing strong concentric undulations impressed upon it.
Fig. 6*a*. Lateral view of type. Shell exfoliated in front.

# PLATE V.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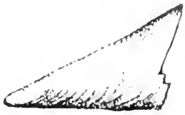
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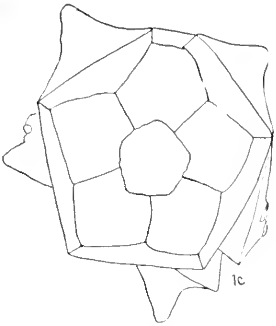
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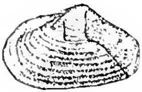
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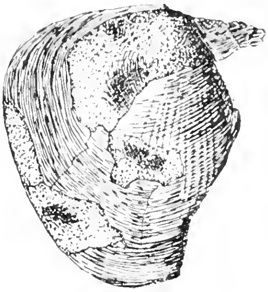
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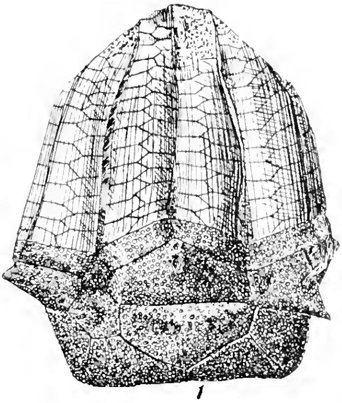
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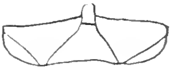


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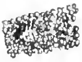
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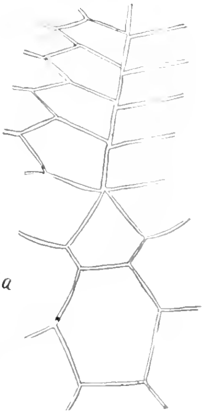
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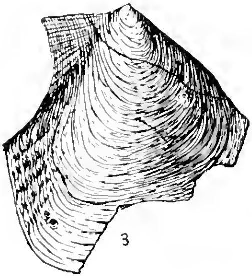
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5a


5

# VARIATION OF THE SPIRALIA IN SEMINULA ARGENTIA (Shepard) Hall.

BY J. W. BEEDE.

With Plate VI.

OWING to the systematic importance and the rarity of good material of the brachial framework of the brachiopods, any light on the extent of individual variation of these parts is of considerable importance. In the spring of 1899 the University of Kansas received from Prof. C. N. Gould a set of ten specimens of *Seminula argentia* (Shepard) Hall that show the position of the spires. These specimens were all from the same horizon in the Lower Permian. The University also had another specimen showing these characters. as did a specimen in the writer's collection. Recently the study of these specimens was taken up and some remarkable results developed.

Both valves of this species are quite convex and not infrequently as broad as long. The older specimens are quite ventricose. However, the shell is subject to a considerable variation in form. Four of the twelve specimens studied were somewhat compressed, but it so happens that three of these approach the normal type very closely, while the fourth does not vary from it greatly. Those showing greatest variation have not been subject to any visible external deformation. The specimens under discussion are of about the average size and form.

The normal position of the spire is with its apex pointing to the side, near the line where the valves meet, at, or a little in front of, the middle of the shell, which is also the widest point. In the central part of the shell cavity the edges of the spires nearly meet. In front they flare apart, leaving a large, subcircular opening. For convenience, in this paper, this opening will be spoken of as the frontal aperture of the spiralia. It will also be necessary to orient the specimens so that definite positions may be referred to. For this purpose we will consider the specimens as front toward (back away from) the observer, with the brachial valve uppermost.

A specimen from the Topeka limestone, Upper Coal Measures, at Topeka, Kan., shows the spires with the apex of one of them pointing almost directly forward toward the anterior end of the shell, turned through a horizontal angle of about ninety degrees from its normal position, while, as nearly as may be determined from the specimen as cut, the apex of the other one is directed toward the median line of the pedicle valve just in front of the hinge. This specimen was selected to be ground. because it was a good specimen, of normal form. Another specimen (No. 3), from the Permian of Cowley county, Kansas, has the apices of the spires turned at an angle of about forty-five degrees or more in a vertical direction, causing the point of the spire to be located near the middle of the right side of the brachial valve, while the other points to the opposite of the pedicle valve. The spire is quite flattened, approaching disk shape, with the apex quite obtuse and the frontal aperture very narrow and almost slit-like (this may be partially due to a very slight compression, but the compression, if present, is so slight as to modify it very little), owing to the position of the spires in the shell. which prevents their flaring much at the front. The shell is not an old one, and, for this reason, is somewhat less ventricose than many adults. The young specimens are much flatter than the old ones in this species.

The spiralia of No. 10 are turned in a similar manner, but through a much smaller angle. The frontal aperture is typical, as is also the general form of the shell. The spiralia are conical, and the tips probably acute. In No. 6 the position of the spiralia, their form and that of the aperture are normal. Nos. 6 and 9 show spines on the spires The spines are closely set, thick, blunt, and nearly twice as long as broad. In fact, some appear to be nearly as large at the tips as at the base. In No. 9 the spires are normal, except that, instead of being flaring conical, they are more in the shape of a folded shield shape with acute apex. In other words. the frontal aperture is produced by the dorso-ventral compression of the entire cone rather than the flaring of the frontal portion. The apex of the left spire is bent somewhat downward. The position of the spiralia in No. 7 is about normal, the tips obtuse. the spires almost perfectly depressed conical, frontal aperture only slightly wider than the space farther back on the side next the brachial valve. No. 5 has apparently been compressed laterally, through this compression did not affect the positions of the spiralia, for they are normal. They appear to have been of the typical form in every respect. In No. 8 the spires seem to have been turned through a slight vertical angle, though the incompleteness of the specimen prevents a close study of position. The form was apparently normal, except that the lower edges may have been somewhat pressed inward. Nos. 1 and 4 are about normal throughout. No. 2 is normal as far as can be seen. except that the apices are turned through a small vertical angle. The anterior portion of the pedicle valve is crushed in. No. 12 is normal throughout.

It is unfortunate that none of the specimens show the crural attachment of the spiralia. Such structures must certainly vary in order to support the spires in their various positions.

The above facts would seem to indicate the following conclusions: First, in those spire-bearing brachiopods in which the form of the shell does not govern the position of the spires, the Athyridæ in particular, the spires may be subject to a considerable variation in both position and form. Second, that the crural supports are probably so modified as to accommodate the spires in their various positions.

In the light of the foregoing, it will be seen that in the future it will be necessary to study the structure of several specimens before using small variations of internal structure in these shells as bases for group divisions.

One might expect greater individual variation among the Flint Hills specimens, owing to the physical changes that were taking place during the close of the Carboniferous and the early Permian: changes which soon caused this species, together with many others, to become extinct. However, this cannot be said of the most striking case (No. 11) from the Topeka limestone. It would seem that such variations as are found in Nos. 3 and 11 would be decidedly detrimental to the well-being of the animal, though the latter of the two seems to have been a vigorous individual.

INDIANA UNIVERSITY, BLOOMINGTON,
April 8, 1902.

## Explanation of Plate VI.

Drawings by Sydney Prentice, except No. 12, which is by C. McK. Beede.
All specimens natural size.

*Seminula argentia.* Variation of spires.

FIG. 1. Specimen brachial side up.

FIG. 2. This specimen shows spires viewed with pedicle side partly removed.

FIG. 3. Pedicle view, showing part of spire.

FIG. 3*a*. Above individual, brachial side up.

FIG. 4. Brachial side of fig. 4*a*.

FIG. 4*a*. Pedicle view of specimen, showing position of spires.

FIGS. 5 and 5*a* are the brachial and pedicle views of a somewhat laterally compressed specimen.

FIG. 6. Pedicle view of specimen.

FIG. 6*a*. Brachial view of fig. 6.

FIGS. 7 and 7*a* are the brachial and pedicle views of a specimen.

FIG. 8. This figure probably shows the brachial side of specimen, while fig. 8*a* is the opposite side.

FIG. 9. Brachial view.

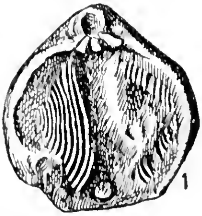
FIG. 10. The pedicle view of a specimen, and fig. 10*a* the brachial side of the same.

FIG. 11. Brachial valve nearly ground away, to show spire. The sickle-shaped structure is probably the base of the opposite spire. Beede's collection.

FIG. 12. Specimen with brachial valve largely broken away.

All specimens but No. 11 in collection of University of Kansas.

# PLATE VI.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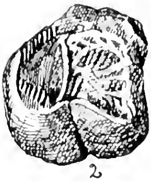
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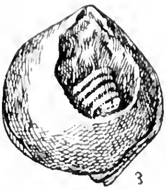
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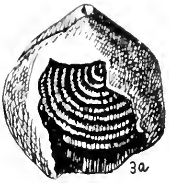
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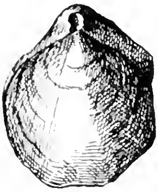
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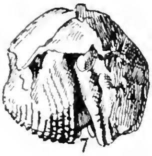
4a


5


5a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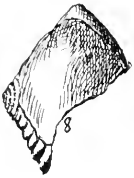
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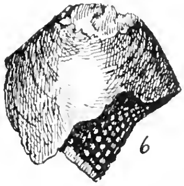
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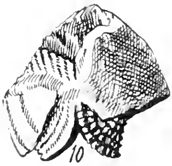
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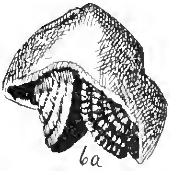
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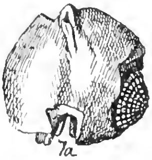
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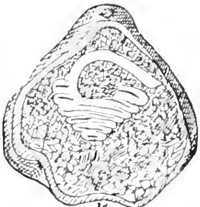
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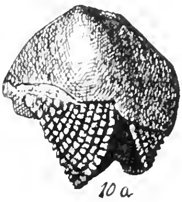
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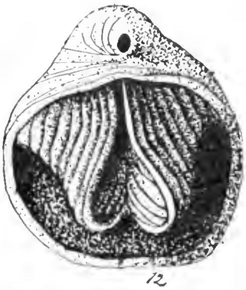
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THE

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# COAL MEASURES FAUNAL STUDIES, II.

(BY J. W. BEEDE AND AUSTIN F. ROGERS.)

## FAUNA OF THE SHAWNEE FORMATION (HAWORTH), THE WABAUNSEE FORMATION (PROSSER), THE COTTONWOOD LIMESTONE.

BY J. W. BEEDE.

BEING a continuation of the foregoing lists of Rogers, published in this journal, vol. IX, No. 4, pp. 232–254.[1]

The present paper deals with the fauna of the rocks, beginning at the base of what Haworth has called the "Shawnee formation"[2] and continuing upward to the top of the Cottonwood limestone. The strata are treated in ascending order. It must be borne in mind that it is the object of these papers to bring out the fauna of the rocks of the Kansas Coal Measures in sufficient detail to establish time divisions on a paleontological basis. These lists are, of course, incomplete for the horizons *in toto*, but it is hoped that they do give the characteristic fossils of the rocks of the Kansas river section in sufficient fulness to warrant some deductions of value. The fauna of the Lower Coal Measures still remains to be completed.

**20.** KANWAKA SHALES (Adams MSS., by permission U. S. Geological Survey). Bennett's description:[3] "Above the last (Oread) limestone lies a heavy shale deposit, at least ninety-seven feet thick at Lecompton. The lower sixty-five feet of this is a clay shale, then sixteen feet of arenaceous shale, then five feet

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[1] In justice to Mr. Rogers, it should be stated that he did not see the proof of the first article and is not to blame for the very bad errors which it contains.

[2] Univ. Geol. Surv. Kans., III, pp. 93, 94, 1898.

[3] Univ. Geol. Surv. Kans., I, p. 116, 1896.

of sand rock, above which lies eleven feet of sandy buff shales." No fossils known from this horizon.

**21.** LECOMPTON LIMESTONE. Bennett's description: "Capping the hills around Lecompton is a five-foot limestone in two layers, which we will provisionally name the 'Fusulina' limestone; not that it alone bears that fossil, but because of the abundance of *Fusulina* in it. It is the lower of another triple system [formation] of limestones, the members of which are separated by a few feet of shale, and which retain this order as far as observed to the west. Above the 'Fusulina' stratum are five and one-half feet of clay shales, then one and one-fourth feet of blue limestone, which weathers dark buff, like all its associate strata. Above this are four feet of shales having a bituminous streak in the middle; then ten feet of light gray, easily disintegrated limestone. This group [formation] may be called the Lecompton limestone, on account of their outcropping being near Lecompton. At Spencer, six miles west of Lecompton, the upper part of the series [formation] finally disappears below the alluvial soils of the valley."

From near the horizon of the above formation, from a well in the road in the bed of Deer creek (the one emptying into Wakarusa creek), near the Shawnee-Douglas county line, the following species have been collected:

*Fusulina secalica* (Say).

*Campophyllum torquium* Owen.

*Archaeocidaris* cf. *agassizi* Hall.

*Fenestella limbata* Foerste.

*Fenestella limbata remota* Foerste.

*Pinnatopora* sp.

*Polypora distincta* Ulrich?

*Polypora elliptica* Rogers? It is quite probable that this is a distinct species. It has fenestrules arranged in two intersecting series when viewed on the reverse side, and the branches are not striated. The fenestrules appear nearly square on account of their arrangement, but in reality they are subcircular. On the obverse side the nodes are very prominent and some of them appear to possess acanthopores.

*Polypora* cf. *nodocarinata* Ulrich.

*Rhombopora lepidodendroides* Meek.

*Chonetes granulifer* Owen.

*Productus nebraskensis* Owen.
*Productus pertenuis* Meek.
*Reticularia perplexa* (McChesney) Schuchert.
*Seminula argentia* (Shepard) Hall.
*Spirifer cameratus* Morton.
*Aviculopecten carboniferus* (Stevens) Meek.
*Aviculopecten maccoyi* Meek and Hayden.
*Aviculopecten occidentalis* (Shumard) White.
*Limopteria gibbosa* (Meek and Worthen).

**22.** TECUMSEH SHALES. A stratum of shales about seventy-five feet in thickness, nearly non-fossiliferous, of fine texture, containing abundant ferruginous concretions and occasional layers of soft shaly sandstone. According to Bennett, these may represent the Kanwaka shales.[4]

**23.** DEER CREEK LIMESTONE.[4] Three layers of limestone separated by layers of shale. The total thickness of the formation is fifteen to twenty-five feet. The principal stratum is the uppermost, which is from seven to twelve feet thick. Most of the fossils below are collected from this upper layer. It is a massive, light gray limestone, tinged with yellow. The texture often varies much in a short distance, grading into inclusions or banks of blue argillaceous limestone in which the fossils are excellently preserved. It is best exposed at Calhoun's Bluffs, in the cut of the Union Pacific railroad three miles northeast of Topeka.

*Fusulina secalica* (Say).
*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.
*Archæocidaris agassizi* Hall.
*Oligoporus minutus* Beede.
*Crinoid* stems.
*Fenestella hexagonalis* Rogers?
*Fenestella* sp.
*Pinnatopora pyriformis* Rogers.

*Polypora distincta* Ulrich? Probably a new species. It differs from this species in having fenestrules broader and shorter and slightly more slender branches. However, this may be due in a large part to the particular portion of the zoarium of which our specimens are fragments.

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[4] "Deer Creek Limestone," Univ. Geol. Surv. Kans., II, p. 117.

*Polypora nodocarinata* Ulrich.
*Septopora pinnata* Ulrich.
*Ambocœlia planoconvexa* (Shumard) Hall and Clarke.
*Chonetes granulifer* Owen.
*Crania modesta* White?
*Derbya crassa* (Meek and Hayden) Waagen.
*Derbya keokuk* (Hall and Clarke).
*Dielasma bovidens* (Morton) White.
*Enteletes hemiplicata* (Hall) Hall and Clarke.
*Meekella striaticostata* (Cox) White and St. John.
*Productus cora* d'Orbigny.
*Productus costatus* Sowerby? de Koninck.
*Productus longispinus* Sowerby.
*Productus nebraskensis* Owen.
*Productus pertenuis* Meek.
*Productus punctatus* (Martin) Morton.
*Productus semireticulatus* (Martin) d'Orbigny.
*Pugnax utah* (Marcou) Hall and Clarke.
*Reticularia perplexa* (McChesney) Schuchert.
*Seminula argintia* (Shepard) Hall and Clarke.
*Spirifer cameratus* Morton.
*Spiriferina cristata* (Schlotheim) Dawson.
*Allorisma granosum* (Shumard) Keyes.
*Allorisma subcuneatum* Meek and Hayden.
*Aviculopecten occidentalis* (Shumard) Meek and Hayden.
*Aviculopecten* sp.
*Entolium aviculatum* (Swallow) Meek.
*Myalina kansasensis* Shumard?
*Myalina swallowi* McChesney.
*Pinna peracuta* Shumard.
*Pinna subspatulata* Worthen?
*Bellerophon crassus* Meek and Worthen.
*Trachydomia wheeleri* (Swallow) Keyes.
*Pleurotomaria tabulata* Hall.
*Conularia crustula* White? (Very large for this species.)

**24.** CALHOUN SHALES. This formation is fifty to sixty-five feet in thickness. The lower part is a layer of soft argillaceous sandstone from twelve to twenty feet thick. The upper portion of the formation is a bluish shale, arenaceous below, clayey above, and of comparatively fine texture. No fossils have been collected from this formation, except a few fragments of *Calamites* and *Cordaites* in a soft sandstone immediately beneath the Topeka limestone, in the eastern part of that city. The type exposure is at Calhoun's Bluffs, three miles northeast of Topeka.

**25.** HARTFORD (TOPEKA) LIMESTONE. (Adams's MSS., by permission of U. S. Geological Survey.) The following section of the rocks is about as given by Doctor Bennett when he first described them:

| | ft. | in. |
|---|---|---|
| *g.* Limestone weathering buff | 2 | 0 |
| *f.* Drab shales | 3 | 0 |
| *e.* Limestone weathering buff | 1 | 6 |
| *d.* Buff calcareous shales with abundant fossils | 2 | 0 |
| *c.* Blue to brown limestone, always weathering to a brown buff, cherty near the top, fossiliferous | 5 | 8 |
| *b.* Blue shales | 1 | 6 |
| *a.* Blue limestone, weathering dark buff | 6 | 0 |
| Total thickness of formation | 21 | 8 |

Most of the fossils enumerated below were taken from layers *c* and *d*, though the entire set is more or less fossiliferous, with about the same species running through them.

*Fusulina secalica* (Say).

*Amblysiphonella prosseri* Clarke.

*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.

*Crinoid* stems and plates, referred by Bennett to *Zeacrinus mucrospinus* and *Z. acanthophorus*.

*Archæocidaris agassizi* Hall.

*Archæocidaris trudifer* White.

*Worm.*

*Fenestella shumardi* Prout.

*Fenestella remota* Foerste.

*Polypora elliptica* Rogers.

*Polypora submarginata* Meek.

*Rhombopora lepidodendroides* Meek.

*Septopora biserialis* (Swallow) Foerste.

*Septopora* sp.

*Ambocœlia planoconvexa* (Shumard) Hall and Clarke.

*Chonetes granulifer* Owen.

*Derbya crassa* (Meek and Hayden) Waagen.

*Derbya keokuk* (Hall) Hall and Clarke.

*Dielasma bovidens* (Morton) White.

*Hustedia mormoni* (Marcou) Hall and Clarke.
*Meekella striaticostata* (Cox) White and St. John.
*Orbiculoidea missouriensis* (Shumard) Hall and Clarke.
*Productus cora* d'Orbigny.
*Productus costatus* Sowerby? de Koninck.
*Productus longispinus* Sowerby?
*Productus nebraskensis* Owen.
*Productus pertenuis* Meek.
*Productus punctatus* (Martin) Morton.
*Productus semireticulatus* (Martin) d'Orbigny.
*Reticularia perplexa* (McChesney) Schuchert.
*Seminula argentia* (Shepard) Hall and Clarke.
*Spirifer cameratus* Morton.
*Spiriferina cristata* (Schlotheim) Dawson.
*Allorisma granosum* (Shumard) Keyes.
*Allorisma subcuneatum* Meek and Hayden.
*Astartella vera* Hall.
*Aviculopecten occidentalis* (Shumard) Meek and Hayden.
*Entolium aviculatum* (Swallow) Meek.
*Limopteria marian* (White) Beede?
*Macrodon* Cf. *tenuistriatus* Meek and Worthen.
*Macrodon* sp.
*Myalina swallowi* McChesney.
*Pinna peracuta* Shumard.
*Schizodus curtus* Meek and Worthen.
*Schizodus curtiformis* Wolcott.
*Schizodus rossicus* de Verneuil.
*Aclisina swalloviana* (Geinitz) Meek, and other gastropods too poorly preserved to be identified.
*Bellerophon carbonarius* Cox.
*Orthoceras* sp.
*Griffithides scitula* (Meek and Worthen) Vogdes.

**26.** SEVERY SHALES. These shales are blue below, varying through yellow to black in places above the coal. They are fifty to seventy-five feet in thickness. The texture varies, and in the upper part they contain the Osage coal, which is mined at Topeka, Burlingame, Osage City, and other places.

In correlating isolated sections the rocks between the Topeka limestone and the Barclay (= Burlingame = Wyckoff, etc.) limestone have been somewhat confused. Under the title, "Stratigraphy of the Kansas Coal Measures,"[5] Haworth describes them as follows: "Above them (Topeka limestones) lies another shale bed fifty feet thick, at the top of which lies the Topeka coal, a seam about eleven inches thick, which has been mined in different places. The coal is immediately overlaid by two thin limestone beds, separated by less than three feet of shales. Above the limestone is the Osage City shale, more than 100 feet thick, at the top of which is the Osage coal, averaging eighteen or twenty inches thick. . . . Above the Osage coal is a thin limestone system [formation], superseded in turn by the Burlingame shales, a body about 150 feet thick in the vicinity of Burlingame, and possibly more in places. Both the Burlingame and Osage City shales extend for long distances to the southwest and northeast, and are important landmarks in stratigraphy." Bennett[6] describes the succession at Topeka correctly, but supposes the coal above the Topeka coal corresponds to the Osage horizon, instead of to the one already indicated in this paper as its equivalent.

Haworth's statement in Vol. I, p. 162, of the Kansas Survey, is practically a repetition of the one just quoted, but he corrects the correlation of the coals in a foot-note at the bottom of page 161. In volume III of the same reports (p. 94) he uses the term "Osage shales" for all the shales between the Topeka limestone and the Barclay limestone. The section is correctly given by Hall in his "Section from Boicourt to Alma,"[7] though it is not clear just what is meant by his "Osage City Shales, Coal, and Limestone."

From the foregoing, it will be seen that the terms "Osage City" and "Burlingame," when strictly applied, are proposed for one and the same set of rocks, namely, those above the Osage-Topeka coal, while the shales below the coal and above the Hartford limestone are not designated at all. Later, in Vol. III of the University Survey (p. 66), in quoting Doctor Adams's notes, Professor Haworth gives the following: "*Severy Shales.*—'Above the Elk Falls limestone is a bed of shales averaging fifty to seventy-five feet in thickness, which, with the protected limestones above, forms a light escarpment that may be

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[5] Kans. Univ. Quart., III, p. 278, 1895.

[6] Univ. Geol. Surv. Kans., I, pp. 118, 119, 1896.

Ibid., p. 394.

traced from a few miles below Eureka to Cedar valley, forming a line from two to five miles west of the Elk Falls escarpment. This shale bed is therefore sufficiently prominent to be recognized in the field, and to be of considerable local and stratigraphic importance. The town of Severy lies within it, and therefore it may be called the Severy shales.'" Dr. George I. Adams, of the United States Geological Survey, under whose direction the work of correlating the Coal Measures rocks of Kansas was done last summer, informs me that the names used in this paper and accredited to him have been passed upon by the committee on nomenclature, and he has kindly permitted me to use them in advance.

So far as known these shales are not fossiliferous, save for a few fragments of fern leaves, below the coal, but are very fossiliferous locally just above it.

*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.

*Ceriocrinus craigi* (Worthen) Wachsmuth and Springer.

*Ceriocrinus harshbargeri* Beede.

*Ceriocrinus hemisphericus* (Shumard) Wachsmuth and Springer. These three species of crinoids are from the dump, and may be from the shales between the two layers of the Howard limestone above.

*Spirorbis* sp.

*Fenestella dentata* Rogers.

*Fenestella limbata* Foerste.

*Fenestella mimica* Ulrich.

*Pinnatopora elliptica* Rogers.

*Polypora whitei* Foerste.

*Ambocœlia planoconvexa* (Shumard) Hall and Clarke.

*Chonetes glaber* Geinitz.

*Chonetes granulifer* Owen.

*Derbya crassa* (Meek and Hayden) Waagen.

*Dielasma bovidens* (Morton) White.

*Hustedia mormoni* (Marcou) Hall and Clarke.

*Lingula umbonata* Cox. If this species is considered separate from *L. mytiloides*, the specimens here referred to would probably be classed with the latter.

*Productus cora* d'Orbigny.

*Productus longispinus* Sowerby.

*Productus nebraskensis* Owen.

*Productus pertenuis* Meek.
*Productus semireticulatus* (Martin) d'Orbigny.
*Productus symmetricus* McChesney.
*Pugnax utah* (Marcou) Hall and Clarke.
*Reticularia perplexa* (McChesney) Schuchert.
*Seminula argentia* (Shepard) Hall and Clarke.
*Spirifer cameratus* Morton.
*Spiriferina cristata* (Schlotheim) Dawson.
*Astartella vera* Hall.
*Aviculopecten occidentalis* (Shumard) Meek and Hayden.
?*Aviculopecten whitei* Meek?
*Cardiomorpha missouriensis* Shumard?
*Edmondia* sp.
?*Myalina exasperata* Beede.
*Myalina perattenuata* Meek and Hayden.
*Myalina swallowi* McChesney.
*Nucula ventricosa* Hall.
*Pleurophorus tropidopherus* Meek.
*Schizodus curtus* Meek and Worthen.
*Schizodus* sp.
*Solenomya radiata* Meek and Worthen.
*Sedgwickia topekensis* (Shumard) Meek and Hayden.
*Bellerophon* Cf. *bellus* Keyes.
*Euomphalus subrugosus* (Meek and Worthen) Meek.
*Pleurotomaria perhumerosa* Meek.
*Pleurotomaria sphærulata* Conrad.
*Pleurotomaria subdecussata* Geinitz.
*Orthoceras cribosum* Geinitz.
*Griffithides scitula* (Meek and Worthen) Vogdes.

**27.** HOWARD LIMESTONE. This consists of two thin layers of limestone separated by two to ten feet of shales. The lower of these is a hard, blue limestone from twenty inches to two feet in thickness, quite fossiliferous in places, and sometimes quite full of crinoid stems and of fish teeth. The upper layer is usually coarser and more shaly. The clay between them is often very fossiliferous.

From Doctor Adams's notes, published by Professor Haworth,[8] it will be seen that the Howard limestone is the same as the

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[8] Ibid., pp. 66, 67.

rock over the Osage coal, and that his Severy shale is the same as the Osage City shales.

Some of the fossils here listed were collected from ballast near Lawrence, on the old Carbondale railroad, which was taken from this layer of rock at Carbondale. These references are marked with an asterisk.

* *Fusulina secalica* (Say).
* *Campophyllum torquium* Owen.
*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.
* *Lophophyllum westii* Beede.
*Ceriocrinus monticulatus* Beede.
*Erisocrinus megalobrachius* Beede.
*Scaphiocrinus washburni* Beede.
*Spirorbis* sp.
* *Chainodictyon laxum* Foerste.
*Fenestella dentata* Rogers.
*Fenestella remota* Foerste.
* *Fenestella shumardi* Prout.
*Fistulipora nodulifera* Meek.
*Pinnatopora pyriformipora* Rogers ?
*Pinnatopora tenuilineata* (Meek) Ulrich.
*Polypora aspera* Rogers.
*Polypora elliptica* Rogers.
* *Polypora* sp.
*Rhombopora lepidodendroides* Meek.
*Streblotrypa prisca* Gabb and Horn.
*Ambocœlia planoconvexa* (Shumard) Hall and Clarke.
*Chonetes glaber* Geinitz.
*Chonetes granulifer* Owen.
*Derbya crassa* (Meek and Hayden) Waagen.
*Derbya keokuk* (Hall) Hall and Clarke.
*Dielasma bovidens* (Morton) White.
*Enteletes hemiplicata* (Hall) Hall and Clarke.
*Productus cora* d'Orbigny.
*Productus costatus* Sowerby ? de Koninck.
*Productus longispinus* Sowerby.
*Productus nebraskensis* Owen.
*Productus pertenuis* Meek.
*Productus punctatus* (Martin) Morton.
*Productus semireticulatus* (Martin) d'Orbigny.

*Pugnax utah* (Marcou) Hall and Clarke.
*Seminula argentia* (Shepard) Hall and Clarke.
*Spirifer cameratus* Morton.
*Spiriferina cristata* (Schlotheim) Dawson.
*Allorisma costatum* Meek and Worthen.
*Allorisma geinitzi* Meek.
*Allorisma granosum* (Shumard) Keyes.
*Allorisma kansasensis* Beede.
*Aviculopecten carboniferus* (Stevens) Meek.
*Aviculopecten hertzeri* Meek.
*Aviculopecten maccoyi* Meek and Hayden.
*Aviculopinna americana* Meek.
*Edmondia aspenwallensis* Meek.
*Edmondia nebraskensis* (Geinitz) Meek.
*Entolium aviculatum* (Swallow) Meek.
*Lima retifera* Shumard.
*Limopteria gibbosa* (Meek and Worthen).
*Macrodon* cf. *tenuistriata* (Geinitz) Meek and Worthen.
*Modiola subelliptica* Meek.
*Myalina kansasensis* Shumard.
*Myalina swallowi* McChesney.
*Nucula?* sp. (cast, very small).
*Nuculana bellistriata attenuata* Meek var.
*Pinna lata* Beede.
*Placunopsis carbonaria* Meek and Worthen.
?*Pseudomonotis hawni equistriata* Beede var.?
**Pteria longa* (Geinitz).
*Schizodus alpina* (Hall) Keyes.
*Schizodus circulus* Worthen.
*Schizodus wheeleri* (Swallow) Meek.
*Yoldia subscitula* Meek and Hayden.
*Bellerophon carbonaria* Cox.
*Bellerophon crassus* Meek and Worthen.
*Bellerophon montfortianus* Norwood and Pratten.
**Bellerophon percarinatus* Conrad.
*Bulimorpha nitidula* (Meek and Worthen) Keyes.
*Capulus parvus* Swallow.

*Pleurotomaria illinoiensis* Worthen? Differs from this species in being much smaller and in having a larger number of nodes to the whorl. The notch in the lip extends back some distance as a thin, almost linear slit.

*Pleurotomaria tabulata* Conrad.

*Soleniscus* sp.

*Sphærodoma medialis* (Meek and Worthen) Keyes.

*Sphærodoma ponderosa* (Swallow) Keyes?

*Sphærodoma* sp.

*Strophostylus nana* (Meek and Worthen) Keyes.

? *Glyphæoceras* sp. (Two specimens, very small and not so preserved as to be well identified.)

* *Goniatites subcavus* Miller and Gurley.

* *Metacoceras sangamonensis* (Meek and Worthen) Hyatt.

* *Nautilus planovolvis* Shumard.

*Orthoceras* Cf. *rushensis* McChesney. Pittings on part of the surface.

*Solenocheilus* Cf. *collectus* Hyatt.

*Stearoceras gibbosum* Hyatt? Perhaps a young specimen of this species, though it will probably prove to be different. The cast preserves a fine line down the center of the ventral surface which disappears before reaching the deep sinus. The sinus is somewhat shallower than that figured by Hyatt, and it may be an *Endolobus*.

*Griffithides scitula* (Meek and Worthen) Vogdes.

**28.** BURLINGAME SHALES. Olive shales, generally very argillaceous, though arenaceous in streaks, and in places even contain sandstone. These shales are 120 feet or more in thickness, and for the most part are not fossiliferous, though in places fossils are very abundant. In the upper part is the Dover-Silver Lake coal, and above it usually a layer of impure limestone.

*Chonetes granulifer* Owen.

*Hustedia mormoni* (Marcou) Hall and Clarke.

*Lingula umbonata* Cox. (See previous note to this species.)

*Orbiculoidea missouriensis* (Shumard) Hall and Clarke.

*Productus nebraskensis* Owen.

*Seminula argentia* (Shepard) Hall.

*Aviculopecten occidentalis* (Shumard) Meek and Hayden.

*Aviculopecten whitei* Meek?

*Myalina congeneris* Walcott?

*Myalina perattenuata* Meek and Hayden.

*Myalina swallowi* McChesney.
*Nucula?* sp.
*Nuculana bellistriata attenuata* Meek var.
*Pinna peracuta* Shumard.
*Pleurophorus* Cf. *angulatus* Meek and Worthen.
*Pleurophorus tropidopherus* Meek.
*Pseudomonotis hawni* Meek.
*Sedgwickia topekensis* (Shumard) Meek and Hayden.
*Bellerophon carbonarius* Cox.
*Bellerophon marcouianus* Geinitz.
*Dentalium meekianum* Geinitz.
*Euomphalus subrugosus* (Meek and Worthen) Meek.
*Gastropod*, minute, undetermined.
*Pleurotomaria subdecussata* Geinitz.
*Ostracoda*.

**29.** Barclay Limestone. (Adams, MSS., by permission of U. S. Geological Survey.) Base of the Wabaunsee formation. For the present it seems best to group several strata in this formation.[9] They cannot well be mapped on the scale of the U. S. folios, and they are also intimately connected faunally, and usually all assist in producing a high escarpment with the upper part retreating. The rocks aggregate about seventy feet in thickness. The following section will serve to give an idea of these rocks:

| | | ft. | in. |
|---|---|---|---|
| *g*. | Gray, argillaceous, fossiliferous limestone, from 1 foot to | 2 | 0 |
| *f*. | Bluish calcareous shales with *Enteletes*, *Myalina*, *Allorisma* and *Bellerophon* fauna, from 8 feet to | 20 | 0 |
| *e*. | Hard, shelly, bluish limestone | 4 | 0 |
| *d*. | Shales with thin limestones, varying from 21 feet to | 40 | 0 |
| *c*. | Hard gray limestone weathering light yellow, from 1½ feet to | 1 | 0 |
| *b*. | Thin layer of shale | 0 | 6 |
| *a*. | Massive yellowish-gray limestone, very hard, with but few fossils, from 4 to 7 feet thick, averaging about | 6 | 0 |

Most all of the fossils listed below are from layers *e*, *f*, and *g*, by far the greater part ranging through all three.

*Somphospongia multiformis* Beede (from Robinson, Brown county).

*Aulacorhynchus millepunctatus* (Meek and Worthen) Hall and Clarke?

*Derbya crassa* (Meek and Hayden) Waagen.

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[9] For a more detailed description of these rocks, see Trans. Kans. Acad. Sci., XV, p. 30.

*Dielasma bovidens* (Morton) White.
*Entcletes hemiplicata* (Hall) Hall and Clarke.
*Meekella striaticostata* (Cox) White and St. John.
*Productus cora* d'Orbigny.
*Productus punctatus* (Martin) Morton.
*Seminula argentia* (Shepard) Hall.
*Allorisma costatum* Meek and Worthen.
*Allorisma geinitzi* Meek.
*Allorisma granosum* (Shumard) Meek.
*Allorisma subcuneatum* Meek and Hayden.
*Allorisma* sp.
*Aviculopecten occidentalis* (Shumard) Meek and Hayden.
*Edmondia aspenwallensis* Meek.
*Edmondia ovata* Meek and Worthen.
*Edmondia* cf. *nebraskensis* Meek.
*Macrodon* sp.
*Myalina perattenuata* Meek and Hayden.
*Myalina subquadrata* Shumard.
*Pinna peracuta* Shumard.
*Pleurophorus* sp.
*Pseudomonotis kansasensis* Beede.
*Pseudomonotis hawni* Meek and Hayden.
*Pseudomonotis* cf. *robusta* Beede.
?*Sedgwickia altirostrata* Meek and Hayden.
*Bellerophon carbonarius* Cox.
*Bellerophon marcouianus* Geinitz.
*Bellerophon* cf. *montfortianus* Norwood and Pratten.
*Bellerophon percarinatus* Conrad.
*Bulimorpha nitidula* (Meek and Worthen) Keyes.
*Capulus* sp.
*Soleniscus paludinæformis* (Hall) White.
*Sphærodoma texana* (Shumard) Keyes.
*Orthoceras* sp.
*Tainoceras occidentalis* (Swallow) Hyatt ?

**30.** WILLARD SHALES. These shales are fifty-five feet thick in their thinnest exposure, and are thicker in some places. Bennett gives their thickness at Willard at from seventy-five to eighty-five feet, including a thin stratum of limestone.[10] The

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[10] Univ. Geol. Surv. Kans., I, pp. 119, 120, 1896.

Bennett collection contains the following specimens taken from this thin limestone:

*Fusulina secalica* (Say) Fischer.
*Meekella striaticostata* (Cox) White and St. John.
*Productus semireticulatus* (Martin) d'Orbigny.
*Bellerophon* sp.

**31.** CHOCOLATE LIMESTONE. This name, as well as the one preceding and the one following, are used merely for convenience here, as they have been used before for the designation of these rocks, knowing that with further study and careful tracing they will be found to be the equivalents of similar rocks on the Neosho river section. This limestone is buff brown in color, varying from seven to ten feet in thickness, composed principally of two layers of massive stone, the upper of which is composed largely of the large variety of *Fusulina secalica*. In the Kansas river region it always forms high escarpments with rocky edges. It is not rich in any fossils except the *Fusulinas*.

*Fusulina secalica* (Say).
*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.
*Chonetes granulifer* Owen.
*Enteletes hemiplicata* (Hall) Hall and Clarke.
*Meekella striaticostata* (Cox) White and St. John.
*Productus cora* d'Orbigny.
*Productus nebraskensis* Owen.
*Seminula argentia* (Shepard) Hall.

**32.** Shales and sandstones shown near Dover, eighty-five feet in thickness, varying from light yellow to brownish red.

**33.** DOVER LIMESTONE. A limestone about four feet in thickness and of grayish color.

**34.** Somewhere from forty to seventy feet of shales, with occasional thin limestones with *Myalina perattenuata*.

**35.** Ten to twelve inches of very fossiliferous limestone in thin layers. Numbers 32 to 35 are as exposed on Mission creek and its tributaries near Dover. Number 35 is shown in ravines southwest of Dover, in the high region east and south of Mission creek.

*Fenestella* sp.
*Productus nebraskensis* Owen.

*Aviculopecten maccoyi* Meek and Hayden.
*Aviculopecten occidentalis* (Shumard) Meek and Hayden.
*Edmondia* sp.
*Limopteria marian* (White) Beede.
*Myalina perattenuata* Meek and Hayden.
*Myalina swallowi* McChesney.
*Pseudomonotis hawni* Meek and Hayden.
*Pseudomonotis* cf. *kansasensis* Beede.
*Schizodus* sp.
*Schizodus* sp.
*Gastropod* cast.
*Loxonema* sp.
*Pleurotomaria perhumerosa* Meek.

**36.** Owing to lack of detailed study of the rocks from the last described to the Cottonwood limestone, it will be necessary to combine the less important strata into groups and mention only the more important. It is a matter of regret that the rocks of this part of the Kansas river section cannot be referred with certainty to the corresponding rocks of the Neosho river and Cottonwood river sections, which have been studied by various geologists. As near as I am able to judge, No. 13 of Bennett's Buffalo Mound section[11] corresponds to the Americus limestone near Emporia.[12] All the rocks between No. 35 and No. 13 of Bennett's Buffalo Mound section are put under No. 36. They consist of an alternation of thin limestones and shales. These shales form a part, at least, of the "Olpe" shales of Dr. Geo. I. Adams (by permission, from his MSS.)

**37.** No. 13 of Bennett's Buffalo Mound section. Probably is the equivalent of the Americus limestone before mentioned.

**38.** ELMDALE FORMATION. Prosser and Beede, MSS. Shales with occasional thin limestones, quite fossiliferous in the lower portion. This is probably the same horizon as No. 2 and No. 3, except the limestone at the top, of Prosser's Manhattan section.[13] It is also, in all probability, the same horizon as No. 8 of my section on the South Fork of the Black Vermillion river.[14]

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[11] Ibid., p. 120.

[12] Ibid., p. 80. See, also, Smith, Bull. Geol. Lyon Co. (Kans.), Emporia, 1901.

[13] Bull. Geol. Soc. Amer., VI, p. 33.

[14] Kans. Univ. Quart., IX, p. 193, 1901. (For July, 1900.)

However, the fossils from the latter place are given in a separate list. The thickness of these shales on the Kansas river and Mill creek are from 111 to 118 feet. Grouping the Mill creek and the Manhattan equivalents, we have the fossils of this horizon as follows:

*Fusulina secalica* (Say).
*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.
*Chætetes?* sp.
*Crinoid* stems and plates.
*Archæocidaris* sp.
*Archæocidaris* sp.
*Dichotrypa subramosa* Rogers, MSS.
*Fistulipora nodulifera* Meek.
*Rhombopora lepidodendroides* Meek.
*Septopora biserialis* (Swallow) Foerste.
*Ambocœlia planoconvexa* (Shumard) Hall and Clarke.
*Chonetes glaber* Geinitz.
*Chonetes granulifer* Owen.
*Crania modesta* White?
*Derbya crassa* (Meek and Hayden) Waagen.
*Dielasma bovidens* (Morton) White.
*Enteletes hemiplicata* (Hall) Hall and Clarke.
*Hustedia mormoni* (Marcou) Hall and Clarke.
*Lingula umbonata* Cox.
*Meekella striaticostata* (Cox) White and St. John.
*Orbiculoidea manhattanensis* (Meek and Hayden) Hall and Clarke.
*Productus cora* d' Orbigny.
*Productus longispinus* Sowerby.
*Productus nebraskensis* Owen.
*Productus semireticulatus* (Martin) d' Orbigny.
*Productus symmetricus* McChesney.
*Pugnax utah* (Marcou) Hall and Clarke.
*Rhipidomella pecosi* (Marcou) Hall and Clarke.
*Seminula argentia* (Shepard) Hall and Clarke.
*Spirifer cameratus* Morton.
*Spiriferina cristata* (Schlotheim) Dawson.
*Allorisma subcuneatum* Meek and Hayden.
*Aviculopecten occidentalis* (Swallow) Meek and Hayden.
*Nuculana bellistriata attenuata* Meek var.

*Dawsonella meeki* Brady? (Prosser's identification.)
*Griffithides* sp.

From about the same horizon on the South Fork of the Black Vermillion:

*Fusulina secalica* (Say).
*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.
*Dichotrypa subramosa* Rogers, MSS.
*Fenestella limbata* Foerste.
*Fistulipora nodulifera* Meek.
*Septopora* sp.
*Thamniscus octonarius* Ulrich.
*Crinoid* stems.
*Archaeocidaris* sp.
*Archaeocidaris* sp.
*Ambocœlia planoconvexa* (Shumard) Hall and Clarke.
*Chonetes granulifer* Owen.
*Derbya crassa* (Meek and Hayden) Waagen.
*Hustedia mormoni* (Marcou) Hall and Clarke.
*Meekella striaticostata* (Cox) White and St. John.
*Productus cora* d' Orbigny.
*Productus longispinus* Sowerby.
*Productus nebraskensis* Owen.
*Productus semireticulatus* (Martin) d' Orbigny.
*Pugnax utah* (Marcou) Hall and Clarke.
*Rhipidomella pecosi* (Marcou) Hall and Clarke.
*Spirifer cameratus* Morton.
*Allorisma subcuneata* Meek and Hayden.
*Aviculopecten occidentalis* (Shumard) Meek and Hayden.
*Chænomya* sp.
*Pleurophorus whitei* Beede.
*? Sedgwickia altirostrata* Meek and Hayden.
*Euomphalus subrugosus* (Meek and Worthen) Meek.
*Griffithides scitula* (Meek and Hayden) Vogdes.

**39.** NEVA LIMESTONE. Prosser and Beede, MSS. This is the "dry bone" limestone of Swallow.[15] It is a gray limestone six or eight feet in thickness, in two layers, separated by a layer of shale. It weathers very rough, from which fact Swallow called it the "dry bone" limestone.

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[15] See discussion in Bull. Geol. Soc. Amer., VI, p. 33 and p. 36.

**40.** ESKRIDGE SHALES. Prosser, MSS. About thirty feet of shales between the above and the base of the Cottonwood limestone.

**41.** COTTONWOOD LIMESTONE. Six feet of light buff-gray limestone, in two layers, the upper usually somewhat cherty, and filled with a small form of *Fusulina secalica* (Say). An excellent dimension stone, of wide distribution. Fossils rare.

*Fusulina secalica* (Say).
*Lophophyllum profundum* (Milne-Edwards and Haime) Foerste.
*Archæocidaris* sp.
*Cystodictia inequimarginata* Rogers.
*Fenestella limbata* Foerste.
*Fenestella remota* Foerste.
*Fistulipora nodulifera* Meek.
*Pinnatopora* sp.
*Rhabdomeson americanus* Rogers.
*Rhombopora lepidodendroides* Meek.
*Septopora biserialis* (Swallow) Foerste.
*Streblotrypa prisca* Gabb and Horn.
*Chonetes granulifer* Owen.
*Pinna* sp.
*Griffithides scitula* (Meek and Hayden) Vogdes.

GEOLOGICAL LABORATORY, INDIANA UNIVERSITY,
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THE

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# THE SPERMATOCYTE DIVISIONS OF THE LOCUSTIDÆ.

BY C. E. M'CLUNG.

With Plates VII, VIII, IX, and X.

## I. INTRODUCTION.

UNDER the title, "A Peculiar Nuclear Element in the Male Reproductive Cells of Insects" (**16**), I published a preliminary account of the process characterizing the maturation divisions of the Locustidæ. This was of a general character and served merely as a basis for a description of the accessory chromosome in these cells. It is my present intention to give a detailed history of the spermatocyte divisions occurring in this family, after the manner followed previously in considering corresponding stages in the Acrididæ (**17**). Besides giving this account of processes, however, I shall be able to draw some comparisons between the two families. Eventually I hope to complete such a comparative study of all the Orthopteran families. Material for this larger investigation is now partially on hand, and is being added to as circumstances permit, so that it may be possible to carry through a study of the maturation stages in this order of insects within a few years.

The value of comparative cytological study was urged by Vom Rath (**25**), and its importance in relation to the accessory chromosome and the maturation mitoses received recognition in both my earlier papers (**16**, **17**). Recently Montgomery (**15**) has added his influence to the movement.

The observations upon which the present paper are based were originally made upon *Xiphidium*. The cells in this genus are, unfortunately, small in size, and much difficulty was experienced during the early work in getting clear images. This embarrassment was further increased by the large number of chromosomes and their habit of compact arrangement. Later it was found that species of *Anabrus*, *Orchesticus*, *Microcentra* and *Scudderia* have cells much better adapted for study, and because of this they have been largely utilized. The account which follows is therefore based upon a study of all these genera, and is considered representative of the family. The accompanying figures are principally from *Orchesticus*, since the number of stages represented exceeded those in material derived from other genera. I am indebted to a friend and former student, Mr. W. S. Sutton, now of Columbia University, for a generous supply of *Orchesticus* and *Anabrus* testes from his collections.

## II. TECHNICAL METHODS.

For the fixation of material used in these studies, it has been found that the osmic acid mixtures of Flemming and Hermann are the most generally applicable and are productive of the best results. In connection with these, however, Gilson's aceto-nitric-sublimate mixture has been tried, and frequently affords an excellent fixation. Extensive shrinkage in the melted paraffin is sure to follow the use of sublimate mixtures unless celloidin is used to support the soft tissue. This double infiltration of celloidin, followed by paraffin, has been found the best method of securing clear and accurate figures, for, because of the lessened shrinkage, the elements are not crowded together and rendered indistinct. This circumstance is particularly fortunate in the case of the Locustid cells, where the nuclear elements are so numerous and crowded.

The stains employed are the iron-hæmatoxylin of Heidenhain and the safranin-gentian violet-orange combination of Flemming. For general purposes, nothing excels the hæmatoxylin stain, but it is frequently advantageous to trace the chemical changes undergone by the different cell elements in the process of mitosis, and the aniline stain above mentioned serves excellently for this. Kernschwarz has also been found a valuable stain for some purposes.

## III. NOMENCLATURE.

The terminology as outlined in a former paper (**17**) will be followed in the present one.

## IV. OBSERVATIONS.

### (*a*) *General Form and Structure of the Testes.*

The testes of the Locustidæ are paired structures lying in the anterior dorsal portion of the abdomen. Each organ is made up of numerous short follicles, which are bound together by a connective tissue investment. In adult animals the testes are a bright yellow color, while in nymphs the color varies from white in the youngest to yellow in the oldest. The pigment is lodged in the connective tissue sheath about the testis, and is seen in sections as irregularly rounded masses in the cytoplasm.

### (*b*) *The Spermatogonia.*

No further discussion of the spermatogonia will be given here than is necessary for an understanding of the derivation of the first spermatocytes. As appears to be universally the case, the second spermatogonia, in their last generation at least, are much reduced in size as compared with the primary spermatogonia that preceded them and with the first spermatocytes that arise from them. The entire cell stains dark with almost all stains and, as the nucleus occupies nearly the whole cell body, the chromatin appears relatively large in amount. A cyst of spermatogonia, therefore, looks as if composed almost entirely of chromatin aggregated into rounded masses—the nuclei.

The chromosomes are of the rod type, and divide longitudinally in each mitosis. The number of chromosomes is large and could not be determined with absolute certainty, but a number of careful enumerations makes it evident that there are most probably thirty-three. In most species of Locustids, one chromosome is easily distinguished from the others by its larger size and tardy division in the act of metakinesis. This is the element as described for *Xiphidium*, which passes into the first spermatocyte as a formed chromosome, while its fellows break up into the spireme.

In the anaphase the chromosomes are drawn away from the equator, and extend lengthwise of the spindle as long rods. During the telophase the disintegration of the chromosomes takes place rapidly, and, for a time, the individual chromosomes may be distinguished in the loose masses of chromomeres. This distinction, however, is soon lost, and the nuclear vesicle becomes covered with fine and apparently unrelated chromomeres. It is at this point that the transformation of the cells from second spermatogonia to first spermatocytes takes place. So long as the chromosomes are present in the somatic number, we have to deal with spermatogonia, but when the disintegrating process comes upon them and they are lost to view as distinct entities, then is reached the end of destructive spermatogonial changes, and upon their reconstruction they are chromosomes of the spermatocytes.

### (*c*) *The First Spermatocytes.*

The main features characterizing the next steps in the process are the rapid increase in size of the cell and nucleus, and the arrangement of the chromomeres into a fine thread or threads (figs. 2–4). This is well called the growth stage, for all parts of the cell engage in the work of regaining the ground lost during the period of multiplication in the secondary spermatogonia. As a result of this metabolic activity, the first spermatocytes at the end of the prophase have reached a volume often as much as ten times that possessed by the last generation of the secondary spermatogonia from which they were derived. Nucleus and cytoplasm, in about an equal degree, participate in this enlargement, and, at the end of the period, present an appearance much different from that of the spermatogonia. This consists most strikingly in the greater clearness of all the parts, due to the increased amount of hyaloplasm which separates by greater distances the more solid structures of the cell.

In the nucleus, for instance, the chromatin aggregates are now definitely apparent, and each stands free and clear except for connecting threads of linin. The cytoplasm, likewise, instead of showing a coarsely granular aspect, exhibits a clearly reticular structure, with such large intervening hyaloplasmic areas as to suggest an almost alveolar structure, especially in the later stages (figs. 3–9). This increased amount of fluid becomes evident by an examination of sections under even a low power of the microscope, principally by the lessened density of the general stain in the cell.

A peculiarity of the archoplasm in these early prophases is the persistence manifested by the spindle fibers of the previous generations. Often connecting fibers may be seen, joining cell to cell, as has been described by many writers, but, in addition to this, the spindle remains of more remote ancestral mitoses show themselves. In figure 3 is represented a cross-section through three persisting spindles of as many generations. Their age is suggested by size and intensity of stain, both factors being least marked in the oldest structure.

Centrosomes and astral radiation do not present themselves with the prominence and frequency of such structures in corresponding cells in *Hippiscus*.

The main interest of these studies, however, attaches to the movements of the chromatin granules. As was suggested in an earlier paper (**17**), it is only by an understanding of the constructive processes in the prophase that we can appreciate the structure and changes of the chromosomes in the metaphase. It is to this period in the history of the chromosomes that I have given the most attention and to which I will devote the most space in the record of observations.

Apparently the chromomeres resulting from the disintegration of the spermatogonial chromosomes are loosely scattered through the nucleus, so that no formed structure is to be seen. With the increase in size of the cell, however, a linear arrangement of the elements becomes apparent, so that it seems as if a thread is formed. Whether this is continuous or segmented it is not possible to determine. The large amount of chromatin and the tortuous course of the filaments put a solution of the problem beyond the range of assured observation. It is with much regret that this fact is recognized, for one of the most important questions connected with the maturation mitoses hinges upon the method by which the chromosomes, as such, are derived from those of the spermatogonia. Upon this point the evidence of the ordinary chromosomes of these cells would, if anything, tend to confirm the view that there is a possibility of complete rearrangement of the chromomeres in the different chromosomes. Concerning this, however, the accessory chromosome is much more conclusive and convincing, as will be shown later.

Disregarding the relations of the chromosomes of the two generations, it is evident that from the material of the spermatogonial elements there is formed the thread of the spermatocyte prophase. As indicated in figures 3 and 4, this is at first composed of a single series of chromomeres. But in a slightly later stage, represented by figure 5, it becomes plain that the thread is wider and at the same time double. A careful investigation will show that the halves of the thread are exact duplicates of each other, each granule of the one having its mate in the other. There is but one conclusion to be derived from the appearances just described, which is that the double thread is formed by a longitudinal division, granule by granule, of the original filament. The evidence afforded, not only by the Locustids, but by all the Orthoptera, is unequivocal on this point. The cleavage of the thread is not exaggerated in the accompanying figures, and is distinctly in evidence even under ordinary conditions of illumination and magnification.

Much controversy has recently arisen among both botanists and zoologists concerning an appearance of the chromatin in the prophase, which has received the common designation "synapsis," by which is meant, usually, a one-sided contraction of the chromatin in the nuclear vesicle. No such stage in the nucleus could be found in *Hippiscus*, and it is likewise absent in the Locustid cells. I therefore repeat the assertion made in the previous paper (**17**), that in properly fixed material derived from Orthopteran sources the first spermatocyte prophase shows no unilateral massing of the chromatin.

Shortly after the formation of the double spireme, it is to be seen that the thread is no longer—even if it was previously—continuous, but is composed of segments (figs. 5–10). So early as this it is possible to observe that the segments are of very unequal lengths. The extent of this inequality may be gathered by consulting figures 6 and 7. Even in this early stage the real structure of the segments may be determined, and in those favorably situated the quadripartite nature of the future chromosomes manifests itself very distinctly.

This important stage in the history of the first spermatocyte chromosomes first received attention at the hands of Paulmier in his studies upon *Anasa*. Almost at the same time I found structures in the Orthopteran spermatocytes so nearly identical that it would be impossible to distinguish any marked difference between them. The Locustid material, equally with the Acridian, permits an exact determination of the chromosome structures, which later become so masked as to be indeterminate.

The interest attaching to the construction of the spermatocyte chromosomes is so great as to warrant an account of the process, although, in general, it is largely a repetition of what has been given for *Anasa* and *Hippiscus*. As early as the stage represented in figure 6, it becomes noticeable that the chromatids near the middle of the thread tend to diverge from each other, leaving a diamond-shaped space. This becomes more pronounced, and it is soon seen that each half of the thread is broken across at the same level, resulting in the production of a chromosome of four parts. Still retaining their general shape, these segments shorten and broaden until they are almost the size of the metaphase chromosome.

All variations conceivable upon the wider separation of the halves along the longitudinal split, the movement of the parts upon the line of separation at right angles to the original cleft, or of approximation and rotation of the free segmented ends are found. Thus do we get the cross-shaped, the double-V, the figure-of-8, the Y-shaped and ring figures, in figure 11. Many of the rings give the impression, upon superficial examination, of loops with their free ends crossed. A careful examination will always reveal the fact, however, that what appears to be the crossed ends is really the middle portion of the segment, with the chromatids drawn out along the plane of the cross-division. In segments that are favorably placed, there is never any difficulty in correlating the structures with the typical one of a cross-split lengthwise of each arm.

The quadripartite nature of the chromatin segments may be determined, as already indicated, almost as soon as the longitudinal split occurs. From this time on until the chromosomes are divided in the metaphase, it is possible to trace the formation of the tetrad chromosomes and to be sure of the relation existing between the longitudinal and cross planes of separation. As evidence of the existence of a longitudinal division of the chromatin thread and of the sequence of the two divisions, I do not see how more could be asked of any material. In the early prophase the greatly elongated and granular thread becomes twice split, once along its length and once across it. As the cell ages, a continuously closer approximation of the chromomeres occurs, without obliterating the lines of separation between the four parts of the segment; accompanying this, the segment becomes shorter and thicker, and the previously existing linear arrangement of the chromomeres is superseded. When the segments have reached approximately the size of the definitive chromosomes of the metaphase, the nuclear membrane disappears and distinction between cytosome and nucleus is lost. As a coincident step, the formerly granular segments become homogeneous in structure by the disappearance of the chromomeres as individual structures; all lines of separation between parts are lost to view, so that an examination of the formed element would betray no indication of composite structure. But, having traced the formation of the chromosomes in this way, one is at no loss to identify each part of the preexisting quadripartite chromatin segment. This is possible because, while all trace of internal structure is gone, the general outline is retained and the crosses and rings of the early stages are still, even up to the metaphase, crosses and rings.

Having traced the formation of the ordinary chromosomes through the various stages of the prophase, I should like to return to the beginning again and bring up to a like degree of development the aberrant element which I have called the accessory chromosome. This has already been given in general outline in my first paper upon *Xiphidium* (**16**), but a number of important observations since made render a general discussion desirable.

I have not yet found it possible to make a detailed study of the spermatogonia of the Locustids, as was done for the Acridid by Sutton in this laboratory, but sufficient observations have been made to be assured that the accessory chromosome participates normally in the mitoses of the secondary spermatogonia. It is here distinctly visible because of its large size, which causes it to extend down to the equatorial plate, while the other chromosomes are in a late anaphase.

At the close of the spermatogonial divisions, when the disruptive processes reduce the other chromosomes to masses of chromomeres in which chromosome identities are not apparent, the accessory chromosome, with apparently more cohesive vigor than the others, retains its general form and is at all times distinguishable. It is marked off from the others, not only by persistence of form, but also by the difference in staining reaction, this being such as is usually exhibited by chromatin when concentrated into homogeneous masses. While studying the cells of *Xiphidium*, I noticed that, at one stage, this color reaction changed somewhat and more nearly approached that of the diffused chromatin. At this time the accessory chromosome had the form of a flattened, apparently fenestrated, plate. I have been fortunate enough, in preparations of *Orchesticus*, to discover that the accessory is really at this time in the form of a long, coiled thread (fig. 5). It is thus seen that, even in respect to the spireme stage, the accessory chromosome is comparable to the others, the only difference being that the diffusion of the chromomeres is less, and the independence of the element greater, than is the case with the other chromosomes.

As the chromatin segments shorten and thicken, the thread of the accessory likewise increases in diameter at the expense of its length, and is finally observable in various degrees of contortion, as shown in figure 12. By the time the chromosomes are ready for division, the accessory has assumed a form very similar to that it shows in the spermatogonia. With the establishment of the equatorial plate, *the accessory moves to one pole of the spindle and there remains undivided during the first spermatocyte mitosis.* It is accordingly a member of only one second spermatocyte resulting from the division of each first spermatocyte.

Returning to the group of chromosomes preparing for metakinesis, we find that in their earlier stages they lie so that their longer diameter is in the equatorial plate, while attached to the enlargement in the center of each, representing the point of separation laid out for the second spermatocyte division, are the mantle fibers running to the centrosomes. The changes now ensuing are easily decipherable, because the chromosomes do not all undergo division at the same time. Since the main differences at present existing between insect spermatologists relate to the sequence of the divisions in the spermatocyte mitoses, I shall again describe the process, although it is identical with that already given for *Hippiscus*.

The necessity for a thorough understanding of the chromosome construction here becomes evident. Knowing how the chromatids were associated in the chromosomes, one can follow understandingly their movements during metakinesis.

It is first to be noted that the chromosomes lie with their longer axis in the equatorial plate. This, as we have seen, is the plane along which the longitudinal cleft occurred, so that a separation in this way means the longitudinal division of the chromosomes in the first spermatocyte. This is, in reality, what occurs. The contracting mantle fibers attached to the middle of the segments drag the adhering chromatids apart without at any time exposing a separating space. It is in this way that in the beginning the longer axes are at right angles to the spindle axis and at the end parallel with it, while during intermediate periods crosses with arms of varying length exist (figs. 13, 14).

The previously disguised lines of separation become at once visible in the daughter chromosomes, for, instead of remaining closely apposed, as formerly, the chromatids spring apart at the free ends and the chromosomes pass through the anaphase as V-shaped bodies instead of as simple rods. The space thus disclosed represents that which separates what would be the ancestral spermatogonial chromosomes, assuming that the reduced number occurs by the end-to-end union of chromosomes of the secondary spermatogonia. As already stated, the accessory chromosome does not divide at this time.

At the end of the anaphase we find the ordinary chromosomes massed at the poles of the cell, and, in addition, at one the undivided accessory chromosome. The second spermatocytes are therefore of two kinds, one possessing the accessory chromosome and the other not. One additional feature of interest that becomes apparent during the migration of the daughter chromosomes to the poles is the retarded division of one of the elements (figs. 22–24). Some cysts contain cells that almost invariably exhibit this peculiarity. The lagging chromosome is always one of the small ones, but whether the same in each case could not be determined.

In the telophase, the main interest is centered in the question as to whether there is a loss of identity of the chromosomes or not. The evidence afforded by the Locustid cells is strongly in favor of the conception of persisting elements. As is usually the case, I believe, the chromosomes, when not under the active influence of the archoplasm, loosen up, and their homogeneous structure gives way to the granular appearance noticeable in the prophase. Although the chromosomes become closely massed and granular, their outlines can usually be distinguished (figs. 23–27). The accessory chromosome does not change its form and structure at this time (figs. 25, 27). The telophase ends with the ingrowth of the dividing cell-wall, and the second spermatocyte mitotic figure is established without any real prophase. Between the two generations it is evident that there exists no such thing as a "rest stage."

### (*d*) *The Second Spermatocytes.*

In the metaphase of the second spermatocyte are formed exact duplicates of the chromosomes seen in the anaphase of the first spermatocyte. These arrange themselves radially in the equatorial plate, one chromatid immediately above the other, so that the plane separating the halves is at right angles to the spindle axis. Mantle fibers attach to the inner ends of the chromatids at the point at which, in all probability, the fibers of the first spermatocyte were connected. I am inclined to regard this as true because the opposite ends, during the anaphase, seemed to be mutually repulsive.

The spindle itself is small and weak as compared with that of the first spermatocyte, and does not long survive the anaphase condition. The material composing it, however, persists as the nebenkern of the spermatid.

A marked difference between the second spermatocytes that contain the accessory chromosome and those which do not is observable. In the metaphase, the element, already longitudinally split in the prophase of the first spermatocyte, projects from the equatorial plate for some distance into the cytoplasm.

It is very much larger than most of the other chromosomes, as may be seen in figure 28. It divides readily in metakinesis, and its chromatids travel to the poles with those of the other chromosomes, but, on account of their greater length, project downward from the mass (fig. 31). Here, as always, the accessory stubbornly maintains its independence, and can be seen extending out from the mass of other chromosomes at each end of the mother cell (fig. 32).

The division of the other class of second spermatocytes is, of course, unaccompanied by modifications due to the presence of the accessory chromosome. Aside from this, no difference between cells of the two classes is noticeable.

To summarize, we may say, that resulting from the division of each first spermatocyte are two second spermatocytes, one of which contains an accessory chromosome while the other does not. The second spermatocyte containing the accessory divides, and with it the accessory, so that each of the spermatids derived from it contains a chromatid from the accessory. The other second spermatocyte, not containing the accessory, also divides, producing two spermatids in which the accessory is absent. Thus half of the spermatids contain accessory chromosomes while the other half does not.

### (*e*) *Number of Chromosomes.*

The enumeration of the chromatic elements, while a very important part of any study upon the nucleus, is unsatisfactory at the best. If there is any great number of chromosomes in the cell, it is impossible to secure a determination of it in a lateral view of the metaphase, because the elements overlie one another so as to render their distinction very uncertain. A polar view is much more desirable, but even here one is never certain that all the elements are represented, or that only entire chromosomes of one cell are present. The first of these contingencies arises from the fact that, in the event of a cell being cut in two, some of the chromosomes may drop out and not appear in the sections; or, if still on the slide, and in a small group, they may lie so close to a mass of chromosomes in another cell as to be confused with them. An excess in number may be found if a portion of the chromosomes have already divided in the equatorial plate, while the remainder are still united (*cf.* fig. 19), or if one or two from the fragment of another cell are in the neighborhood. All these embarrassments are increased when an independent structure like the accessory chromosome is present. These difficulties exist when the conditions are most favorable, *i. e.*, when the chromosomes are arranged in the equatorial plate; they become practically insurmountable during any other stage of mitosis by the intertwining of the chromatic segments or by fusion of chromosomes in later stages.

Because of these considerations, I do not put implicit confidence in conclusions drawn from numerical relations when they involve the question of whether or not there is a difference of one chromosome between two cells. What I have to say, therefore, concerning the numbers of chromosomes in the different cell generations of the Locustid testis, I must state as my best judgment in the matter, based upon the most careful observations I could make upon cells showing the elements with the greatest clearness. While I regard them as in all probability correct, I do not rely so thoroughly upon them as I do upon observations of structural details, and have therefore based no conclusions upon numerical relations alone.

As is stated elsewhere, the number of chromosomes in the spermatogonia appears to be thirty-three. This was ascertained by selecting the clearest possible cases of the metaphase that could be found and drawing them under the *camera lucida*. Subsequent countings were made, and in most of the cells thirty-three chromosomes were found. An inspection of figure 1 will show that there is a characteristic arrangement of the chromatin bodies, the larger ones being on the outside of the group, the smaller within. Amongst the large ones, it was impossible to distinguish the accessory chromosome, but a lateral view of the anaphase shows it clearly. From the fact that it was a single element in the spermatogonia, it was to be expected that an uneven number of chromosomes would appear in this cell generation.

In the spermatocytes, as in the spermatogonia, the polar view of the metaphase was the stage selected for use in counting the chromatin elements. A large number of cases showed that sixteen and seventeen were the prevailing numbers. The smaller of these is easily accounted for when it is recalled that the accessory chromosome is at one pole of the spindle, and would very often lie in another section, where it would not be possible to be sure of its relations. I am convinced from these counts that seventeen is the reduced number in the first spermatocyte, sixteen of the elements being ordinary chromosomes, the other one being the accessory chromosome which has come over unaltered from the spermatogonia. This coincides with the theoretically expected number, deduced from the independently determined number of spermatogonial elements.

In view of the divergences found in insect spermatogenesis, the established theory that the reduced number of chromosomes is exactly half the normal or somatic number is not a strictly accurate one, for in this case the reduction is from thirty-three to seventeen. Similar instances may be found in the forms investigated by Montgomery and de Sinéty.

When we come to consider the second spermatocytes, spermatids, and spermatozoa, it is necessary to divide them into two classes, because of the unequal apportionment of the accessory chromosome consequent upon its remaining undivided in the first spermatocyte mitosis. There are formed, accordingly, two numerically equal classes of second spermatocytes—those containing sixteen chromosomes plus the accessory chromosome, and those with merely the sixteen chromosomes. The members of each of these classes divide and double their kind, forming spermatids marked as were the second spermatocytes—one class with seventeen chromatic elements, and the other with sixteen. From these, by the usual transformations, are derived the mature male elements, which are thus of two distinct kinds.

### (*f*) *Spermatids.*

The limits set to this paper preclude anything more than passing mention of the spermatids. As stated above, cells at this stage of development are of two classes, depending upon the presence or absence of the accessory chromosome. The distinction thus set up continues to exist visibly far through the transformation stages of the spermatid, by reason of the persisting independence of the accessory chromosome. Of the dual nature of the spermatids I was very early convinced, because the accessory chromosome is so strikingly displayed by the nuclei in which it exists that it is impossible to overlook its absence in a large proportion of the cells. As to the certainty of this partial distribution in the transforming spermatozoa, I am rendered positive by the most careful and painstaking study. This is valuable corroboration of the observed fact that the accessory chromosome remains undivided in one of the spermatocyte mitoses.

## V. COMPARISONS AND CONCLUSIONS.

The literature relating to the spermatocytes of insects was reviewed at some length in my previous paper upon the history of these cells in the Acrididæ (**17**). It is not my purpose to go over this same ground again except in so far as increased knowledge makes it necessary. More recent papers by Montgomery, Wilcox and others will, however, be discussed in detail. The policy previously announced, of restricting comparisons to results derived from insects, will again be adhered to. I believe that the main features of the maturation divisions are essentially the same in all insects, and I desire to see this belief either well established or overthrown. If it can be demonstrated that so large a class as the insects are characterized by a common process, it will be a firm basis upon which to conduct further comparative studies into more comprehensive groups. On the contrary, if it is shown that there is no type, even in the class, then it is useless to seek agreements between widely removed species.

### (*a*) *Nomenclature.*

A necessary basis for any comparative work is a common terminology. Confusion inevitably follows the loose application of names to the structures compared. This is perhaps unavoidable in the early stages of an investigation, but should be overcome as soon as possible. There is surely no reason for continuing uncertainty after terms have received general acceptance. Believing this, I feel called upon to repeat my criticisms of Montgomery's application of the well-accepted terms "prophase," "metaphase," "anaphase," and "telophase."

In reply to my previous objection directed against this part of his work, Montgomery acknowledges the validity of the criticism so far as it relates to the metaphase, but denies the application to the other phases, particularly to the anaphase. He alleges in support of his position that the introduction of an unusual condition, the "synapsis," makes it impossible to correlate strictly the stages of the germ-cells with those of ordinary divisions. Upon this point I must again disagree with him. It is impossible for any known modification of the prophase to change the essential character of the anaphase, so as to make it precede instead of follow the metaphase. This stage marks the movements of the chromosomes from the equatorial plate to the poles, and terminates when they are massed around the centrosomes. How can the "synapsis" in the least affect the duration or character of this process? It is apparent enough, I think, that Montgomery's subphases of the "anaphase" do not belong to this portion of the mitotic cycle at all, but are really portions of the telophase of the spermatogonia and prophase of the first spermatocyte. Further, it may be noted that, even were these subphases properly included in the anaphase, they would belong to the spermatogonia and not to the spermatocytes.

Montgomery himself seems to be rather uncertain of the position of his "anaphase." In the first paper, upon *Euchistus* (**12**), it was put down as the anaphase of the first spermatocyte; in his later paper (**14**), upon *Peripatus*, it is recorded as the anaphase of the spermatogonia. Still more confusing is his use of the "telophases," for in the article upon *Peripatus* (**14**) it is, in the "Contents," placed as a substage of the spermatogonial anaphase, and in the body of the work, page 307, as the telophase of the spermatocyte! Neither the anaphase nor the telophase can, by any possible construction of their proper meanings, be made to apply to the "growth period" of the germ cycle, as Montgomery insists; they are the last stages of the "division period," in reality. The prophase of the first spermatocyte is the initial stage in the constructive process marking the growth period.

Montgomery's translocation of the terms makes the "synapsis" occur in the anaphase. This is manifestly an impossible condition of the chromatin at this time, and his figures show definitely enough that it is a prophase, or, at the earliest, a spermatogonial telophase, that witnesses the contraction of the chromatin. The objection urged in my earlier paper (**17**) to the use of the term as a designation for the mere contracted condition of the chromatin cannot apply to Montgomery's latest use of it; for he here recognizes the justice of my contention that it was primarily designed to indicate the fusion of the spermatogonial chromosome to produce the chromosomes of the spermatocyte. He states this clearly in the following words: "Moore (1895) first gave the name 'synaptic phase' to that stage in the growth period of Elasmobranchs when the reduction in the number of chromosomes takes place. Accordingly, the criterion of the synapsis stage is, first of all, the combination of univalent chromosomes to form bivalent ones; whether the chromosomes are then densely grouped or not is of secondary importance."

### (*b*) *The Spermatocytes of the Locustidæ and Acrididæ.*

The formation of the first spermatocyte chromosome gives us an insight into the later changes undergone by these elements such as cannot be obtained in any other way. The great importance attaching to this part of the spermatogonial process renders it desirable to exhaust every effort in obtaining a knowledge of the actual changes here taking place. This thought has been held constantly in mind during the progress of these investigations, and every point of resemblance or of difference between the various species studied has received careful attention. Despite variations in details, however, I must state that the essential features of the maturation divisions are the same in all species of the Orthoptera examined. It is true that as yet only two families, the Acrididæ and the Locustidæ, have been worked out in a detailed way, but the close agreement between these raises a strong presumption in favor of the general prevalence of the type. The processes of the two families have already been described in detail, but it will perhaps be well to call particular attention to some points worthy of mention.

The general appearance of the material derived from the two families is quite different in sections. Even the hastiest observation will show this. The spermatocytes of the Locustid testis are much smaller, denser and more deeply staining than those of the Acrididæ. The relative quantity of chromatin is greater, so that it is possible by microscopical examination of a section to tell whether it was prepared from Locustid or Acrididan material.

The transformation from the telophase of the last spermatogonial division to the prophase of the first spermatocyte is marked by practically the same changes in both families. It is to be observed, however, that the derivation of the spireme from the disintegrating chromosomes of the previous generation is not so clearly indicated in the Locustid cells, and it was for this reason that in the examination of *Xiphidium* I was not able to determine certainly that the accessory chromosome came over from the spermatogonia into the spermatocytes as a formed element. Upon this point, as upon others, my later material is clearer, and I was able to reconcile the appearances in the two families. In both, unfortunately, it has been found impossible to determine the exact origin of the first spermatocyte chromosomes.

In connection with the transformation of the chromatin from the spermatogonial condition to that of the spermatocyte, we must take notice of that stage which is commonly denominated the "synapsis." The evidence afforded by the Orthopteran cells is entirely negative regarding this. In properly fixed material there is no distortion of the chromatin in the nucleus at any time. It would, if present, be particularly easy to observe, as was stated in my previous paper, for during the entire winter the spermatocytes exist in the spireme stage, and in a longitudinal section of a follicle all stages may be discerned. On the other hand, in poorly fixed or hastily prepared material the synapsis is present, and always in such a form as to indicate its artificial character. What is here said regarding the synapsis refer to the appearance commonly thus designated, but, as has already been stated, such an application of the term does not meet the spirit of the definition as intended by Moore (**20**). A fusion of the spermatogonial chromosomes of some sort must certainly occur, but that it is always marked by a unilateral massing of chromosomes, I deny.

During the prophase the chromatin segments in the cells of *Orchesticus* and other species of the Locustids are heavier, more granular and denser than they are in *Hippiscus*. It is to be observed, also, that there is a greater variation in the size of the elements. This fact is observable from the earliest appearance of definite segments down through both the spermatocyte mitoses. This disproportion may be such that one chromosome will exceed another in the same cell by twenty or thirty times its volume. We have here, as is pointed out in another place, a strong proof concerning the individuality of the chromosomes, for in some species it is possible to distinguish a particular chromosome in all the spermatocytes. This is strikingly the case in *Anabrus*, where there is always one chromosome very much larger than any of the others. It exceeds in size even the accessory chromosome, and might be mistaken for it were it not for the difference in form. It is, however, typically a tetrad, and shows the four chromatids, while the accessory chromosome exhibits the usual spermatogonial condition.

As was indicated under the head of "Observations," the prophase tetrad characteristic of *Anasa* and *Hippiscus* is again exemplified in the Locustid cells. So close is the resemblance of the maturation chromosomes of these various insect cells in their early stages, that I now regard it as practically established that they are commonly present in all insect spermatocytes. No more important evidence regarding chromosome structure and behavior can be obtained than that afforded by these elements. Particularly are the ring figures of value in the determination of the sequence of the longitudinal and cross divisions, and upon this point the material from the two families is equally convincing and positive in demonstrating that the first spermatocyte mitosis witnesses a separation along the longitudinal cleft of the spireme thread.

I should like to emphasize the fact that the chromosomes in both the Orthopteran families studied have been carefully traced from their earlier appearance down to the time of their dissolution in the spermatid through such a gradual series of changes that there can be no reasonable doubt of the accuracy of the conclusion set forth in these papers. The Orthopteran material possesses one distinct advantage over the Hemipteran, in that the point of cross-division is always marked by the same sort of a protuberance as is to be distinguished in the early chromatin segments. When the two free ends of the element are brought around to form a closed ring, the last particle of doubt regarding the position of the planes of separation marked out for the two spermatocyte divisions is dispelled.

This diagnostic character seems to be lacking in the chromosomes of the Hemiptera, and Paulmier, in his work on *Anasa*, depends for his criteria of orientation upon the relative lengths of the chromosome axes. Such a feature would be valueless in Orthopteran cells, because, as has been shown, the chromatids move upon each other in such a way as to exactly reverse the preexisting relation between the axes. How applicable this observation may be to conditions in the Hemipteran cells, I do not know; but, judging from the great resemblance of the elements in the prophase, it would seem most reasonable to expect a similarity of the divisions.

Paulmier (**22**) advances the suggestion that in the double-V figures we may find a structure that will serve to reconcile the divergent accounts concerning the longitudinal and cross divisions of the tetrads. The only way in which this might be accomplished would be to suppose that each of the interspaces represents a longitudinal cleavage of the thread, the first being at right angles to the second. I have given this suggestion careful consideration, and find no evidence to support it. The double Vs are only of rare occurrence, the common element being a straight rod, in the center of which is a diamond-shaped clear spot representing the two planes of division laid out for the spermatocyte mitoses. If two longitudinal divisions occur, one must precede the other considerably and the resulting halves become mutually repulsive, so that they move apart and lie in one plane with only a slight connection at the point of final separation. Moreover, the second cleavage must begin at the opposite end of the segment and proceed in a reverse direction from the first. Not only this, but the first spermatocyte mitosis divides the elements along what is generally conceded to be the longitudinal split, and this must necessarily succeed the supposititious first longitudinal cleavage by some time. Without going into a consideration of these points, I may say that they suggest such deviation from normal processes that only extensive and acccurate observations would make Paulmier's suggestion worthy of further consideration.

### (*c*) *Formation of the Tetrads.*

In my former paper I reviewed the results obtained by Montgomery upon the Hemiptera, but further notice of his work will now be necessary, since on almost every important point relating to chromosome structure he has changed his opinion. His late extensive comparative study upon the Hemipteran cells, as well as that upon *Peripatus*, will at the same time receive consideration.

It appears from Montgomery's account that at the point where the Orthoptera are least valuable in demonstrating chromosomal relations the Hemiptera and *Peripatus* are most convincing. I refer here to the derivation of the first spermatocyte chromosomes from the chromatin of the spermatogonia. He claims to have observed the union by pairs of the secondary spermatogonial chromosomes during the anaphase (his synapsis) so clearly as to be positive of this fusion. I hope this may be verified, for it offers a logical explanation of the process of reduction, and is a confirmation of what has previously been assumed true without sufficient basis in observed fact, as was suggested in my paper on *Hippiscus*. This, if established, would also be a strong support of the theory relating to the constancy of the chromosomes. If this true synapsis is accomplished at this time, however, it must be noted that it occurs during the last phase of the final spermatogonial mitosis, and is not an act of the spermatocyte prophase. But as to the exact location of this point no contention need be made, for it is conceivable that the time of its occurrence might vary considerably without affecting the essential nature of the process.

With regard to such an origin of the first spermatocyte chromosomes, there is an important difference to be noted between the earlier and later work of Montgomery, and one which he fails to mention. In his paper (**12**) upon *Euchistus* he states the matter as follows: "But in the post synapsis we do not find seven chromosomes, the definitive number present in the spermatocyte divisions, but a smaller number; hence, in the synapsis the true (*i. e.*, exactly half) reduction of the chromosomes does not take place, but the number is reduced to less than one-half." This statement is based, he says, upon a most careful and painstaking enumeration of the chromatic segments in a number of nuclei, and is unhesitatingly declared correct.

In his later paper, on the contrary, he is just as positive that the definitive reduction is here accomplished, for he says: "Since then I have been able to demonstrate that this numerical reduction is effected in the synapsis by the union into seven pairs of the fourteen chromosomes, each of the seven bivalent chromosomes (pairs) being composed of two univalent chromosomes joined end to end." This statement is made without adducing any specific proof, as was formerly done. By what means we are to reconcile these diametrically opposite statements Montgomery does not say. He, however, insists that he has always known that the fusion by pairs takes place. How this was to be brought about under his previous assumption that one of the fourteen spermatogonial chromosomes became removed from participation in the usual processes of the cell to form a "chromatin nucleolus," he fails to state. Until the confusion is cleared up by corroborative evidence on one side or the other, a most important part of Montgomery's work must still be regarded as uncertain.

Despite his recognition of the fusion of the chromosomes in the synapsis as the essential feature of this stage, Montgomery is insistent upon the concentration of the chromatin as its distinguishing characteristic. Regarding this he says: "McClung considers the appearance of the synapsis stage as artefacts. It is hardly necessary to reply to this criticism, since in all *Metazoa* where the spermatogenesis has been carefully examined, with the exception of certain *Amphibia*, the dense massing of the chromosomes (?) in the synapsis stage has been shown to be a perfectly normal phenomenon."

Concerning two points in this statement I wish to take exception. First, as was suggested in my previous paper (**17**), the term synapsis is usually applied to a condition of the prophase in which the apparently unsegmented spireme exists. It must be remembered that most investigators consider that the reduction of the chromosomal number takes place by the segmentation of a spireme into half the usual number of segments. In the second place, I must resent the implication that the work done in this laboratory is not "carefully" conducted. Many "*Metazoa*" have been examined "carefully," and in none has the "synapsis" occurred when the material was well fixed and prepared. It has, moreover, been found possible to produce the appearance at will. One case of this kind is sufficient to raise the presumption that it may not be normal even when constantly found in certain preparations. I have not, however, absolutely denied the possibility of such an occurrence, because it is conceivable that from the telophase of the preceding division the massing of the chromosomes may persist during their elongation. My contention is that the appearance is not a constant or necessary condition in "all the *Metazoa*," and this I have proven.

In rather striking contrast to the work of Montgomery, in which an effort is made to formulate a typical process for the entire *Metazoa* from the study of a single order, is that of Wilcox, wherein a general denial of any apparent system in the maturation divisions of animals is based practically upon the study of a single species. As was stated in my former paper, I regard Wilcox entirely in error upon the vital point of his theory of tetrad formation, not by "forced interpretation" of his own views, but by an actual examination of the object upon which he worked. There is no point upon which Orthopteran material affords more indisputable evidence than upon the occurrence of the longitudinal division of the chromatin thread in the early prophase. My statement regarding Wilcox's position on this subject was in no sense "misdirected criticism," but an actual statement of fact; it was not an attempt to explain away "abundant and evident cases which cannot be made to fit into the scheme," but simply the presentation of proof that *one* case was wrongly interpreted.

Wilcox claims the distinction of being the first and only investigator to doubt the hypothesis that longitudinal and cross divisions of the chromatic thread produce chromosomes of a different character. It is perhaps well that this is so, in view of the reasoning by which such a distinction is secured. Upon his own unconfirmed and disputed statement that there is no longitudinal division of the spireme, Wilcox presumes to disparage the accepted view of practically all cytologists. The constructive thought of the last two decades is summarily disposed of by this author in the following language: "The whole question, therefore, whether a certain division is longitudinal or transverse loses its practical significance, since the theoretical interpretation which has long been placed upon these divisions is shown to be impossible and absurd!" The showing alluded to consists in the statement that the chromosomes consist of an indefinite number of granules, which cannot be expected to arrange themselves in any order, and which, therefore, may be divided in any way without affecting the results.

Laying aside for a moment the question as to the occurrence of a longitudinal division, we may well inquire whether the belief that, "In view of this manner of the formation of the chromosomes (by the aggregation of the chromomeres), it seems absurd to assume that the separation of an individual chromosome by one plane could be quantitative while the separation by another plane was qualitative,'' is well founded. At the basis of such an assumption lies the implication that any definite arrangement of chromomeres is impossible ; for if any definite order were possible, then the supposed argument against the longitudinal disposition of the chromomeres would be invalid.

The argument of Wilcox is therefore directed against order in general, and not against order in any one particular, as he would have it appear. For it must be admitted that if it is possible for the scattered chromatic granules of the early prophase to arrange themselves at all (and this even Wilcox does not deny), it is equally possible for them to come together in a definite order. That they do this is amply evidenced by the fact that later they appear in definite groups or chromosomes. It is to be noted, moreover, that the later investigations tend to suggest that the apparently unorganized chromatic granules in the first spermatocyte prophase are really bound together and represent merely a diffuse condition of the spermatogonial chromosomes.

Wilcox's chief error, however, is not to be sought in speculative theories, but rather in his faulty observations. He repeatedly denies the occurrence of any longitudinal split in the chromatic thread of the first spermatocyte prophase. That he is mistaken here I am thoroughly convinced, both from a study of his own object and from investigations upon many other species of the same family. At the present time, also, practically every spermatologist is aligned in support of the view denounced by Wilcox. For a while Wilcox had some backing, but most of those who advocated only cross-divisions of the thread have later been able to demonstrate the longitudinal cleavage in better prepared material.

There is general acceptance of the opinion that the chromomeres of the last secondary spermatogonia appear in a linear arrangement to form what is commonly known as the ''spireme.'' Wilcox declared that while in a very fine condition this thread breaks across into segments, which unite by pairs to form the chromosomes of the first spermatocyte. The great majority of other investigators are unanimous in the opinion that this fine thread, made up of granules, becomes double by the division of each granule individually, thus producing a double thread. Thus it is that the two halves of a longitudinally divided chromosome are made equivalent, not by the sifting apart of preexisting granules, but by the division of these after they are arranged in a linear series. It need hardly be mentioned that the formation of the thread has here a reason for existence which is entirely lacking according to Wilcox's scheme.

This much space has been devoted to Wilcox's statements, not because they present any arguments against the generally accepted views of his fellow workers, but because he represents a rapidly lessening minority which is content to work in a very limited field and to resort for the explanation of diverse results to the very convenient theory that great differences may be expected in the normal processes of even closely related forms. One needs only to glance at the work of all insect spermatologists to see how closely the agreement now is upon the important points of the process. This accordance of results Wilcox notes, but interprets in his own way, which may be regarded as not exactly complimentary to the skill and judgment of his colaborers. "It is only necessary," he says, "to refer to any recent publication on the subject to find examples of this attempt to force the divergent processes in different species to fit the same formula." This is certainly a very easy and convenient way to dispose of the accumulated observations of the many careful investigators who have come to an agreement upon the important questions under discussion, but I venture to think will hardly satisfy any one except its sponsor.

After handing in this article for publication, I fortunately secured a copy of the paper by R. de Sinéty (**37**) in which the spermatogenesis of various Orthopteran species is described. I regret that the available time is so short that I shall not be able to bestow upon this contribution to insect spermatogenesis the attention it deserves, but I shall try at least to consider the principal points wherein a difference exists between the results of de Sinéty and of myself.

It is unfortunate that we have here a further complication of the problem concerning the character of the two maturation divisions in insects. At this time it had begun to appear as if there was every possibility of insect spermatologists coming to an agreement with regard to the maturation processes. Indeed, with the exception of Wilcox, who occupies a unique and solitary position in the field, workers upon the subject are committed to a belief in the occurrence of a cross and a longitudinal division of the chromosomes in the spermatocyte mitoses. The sole difference of opinion relates to the sequence of the divisions. We have now to consider in connection with insects the remaining possibility in tetrad formation—that of two longitudinal divisions—which finds an advocate in de Sinéty.

Because of a thorough acquaintance with the forms upon which this author has worked, I do not hesitate to say that he is entirely mistaken with regard to the character of the second spermatocyte division. I am convinced of this because of the fact that in the early period of my work upon Orthopteran spermatogenesis I was inclined to place just such an interpretation upon the phenomena encountered in the spermatocytes of the Acrididæ as does de Sinéty. I soon became convinced, however, that I was proceeding upon a wrong assumption, and abandoned it in favor of the one which more extended observation taught me is correct. I hope to demonstrate here the ground for my plain statement that de Sinéty is in error upon the question of a double longitudinal division of the chromatin thread during the formation of the tetrads in insect spermatocytes.

It is fortunate that our author has properly appreciated the value of the early prophase in the determination of the structure of the first spermatocyte chromosomes, for we are here upon common ground, and need only compare like stages in order to reach our conclusions. As will be recalled, the statement is made elsewhere in this paper that the typical chromosome of the first spermatocyte is an approximately straight rod, split longitudinally, and again cleft in its middle by a second fissure at right angles to the first. Such an element is represented in figures 15*a*, 17, D and E of my paper upon the Acrididæ, and in figures 7, 9, 11 and 38 of the present one. Although this is extremely common, and, as the photomicrographs show, undeniably present, de Sinéty does not figure it at all. The nearest approach to such a structure is found in figure 123*c*, where a cross with two nearly equal arms is represented. My interpretation of this figure, based upon a great number of careful observations, is that this represents merely an extension of the shorter arms at the expense of the longer ones. In support of this, I have stated that all intermediate stages between a rod with a mere enlargement at the center and a cross with equal arms could be found. How, according to de Sinéty's conception of overlying free elements, could these structures be explained?

It is not necessary, however, to have these gradations in order to disprove the theory under discussion. One needs only to carefully examine one of these crosses to be convinced that the two arms lie in one plane where they intersect, and are not superimposed one upon the other as de Sinéty shows in his figure 123. Our author clearly realizes the importance of the cross, as may be judged by the following quotation:

> "La croix est de toutes ces figures celle dont la genèse peut le plus facilement donner lieu à des interprétations en sens contraire.—C'est précisément pour cette raison que nous croyons devoir l'étudier spécialement au point de vue critique, persuadé que, cette figure une fois rattachée à une théorie, les autres doivent en suivre le sort."

It is unfortunate, therefore, that he was not able to trace the formation of the element in its very early stages and through the various modifications which connect it with the typical rod already described.

As the simplest modification of this basic form, we find the one where it is evident that the change consists merely in a flexure of the rod at the weak spot in its center. Such forms are shown in figure 14 of my former paper (**17**) and in figures 8, 9 and 11 of this one, but are not illustrated by de Sinéty. It occasionally happens that in chromosomes of this character the halves diverge widely at the center, producing the double-Vs of Paulmier, as is represented in figure 14 of my paper upon the Acrididæ (**17**) and in figure 8 of the present one. These structures are not shown by de Sinéty and would be difficult to explain in agreement with his conception of the tetrad.

I have consistently placed great reliance upon the frequent ring-shaped chromosomes in determining the structure of the first spermatocyte elements, and have no occasion to change my opinion of them since examining the work of de Sinéty.

This investigator joins issue with me upon my interpretation of these structures, and states his attitude in the following language:

> "McClung fait grand fond, pour appuyer son interprétation, sur une forme spéciale, la forme en anneau, qui pour lui dérive du bâtonnet $\frac{a'\,b'}{a'\,b'}$, supposé placé transversalement sur le fuseau, inséré par son milieu et incurvé en dehors jusqu' à rapprochement et soudure de ses extrémités.
>
> "Le chromosome en anneau est en effet trés fréquent chez les acridiens; mais il nous a été possible d'en reconstituer l'histoire, grâce à des détails qui ne semblent pas s'être rencontrés dans les figures de McClung. On se souvient que nous avons établi les deux points suivants en complet désaccord avec la théorie de l'auteur américain:
>
> "1. Les deux moitiés de l'anneau proviennent de la première division longitudinale.
>
> "2. L'insertion est terminale."

With equal emphasis, I must deny that the enclosed space in the ring represents any plane of division in the chromatin thread; and that the insertion of the spindle fibers is at any place except at the center of what would be the typical rod-shaped chromosome were the ring straightened out. We encounter in de Sinéty's interpretation of these rings the very error against which I was careful to caution elsewhere in this paper, *i. e., of regarding the points where the fibers are attached as the crossed ends of a simple segment.* This mistake de Sinéty has made, and has thereby vitiated all his conclusions concerning the structure of the tetrads. It is not necessary to repeat here the proof which I have brought forward in support of my views. No one, I am sure, will find difficulty in reducing the various forms of chromosomes found in the first spermatocytes to the type of a doubly split rod, in which one plane of division is parallel to the long axis and the other at right angles to it. The explanation offered by de Sinéty requires us to conceive a doubly split rod in which one separating space may vary indefinitely while the other is constant. There is here no common type, but an infinitely variable one, which differs with every modification of the interspace between the first pair of chromatids in each chromosome.

As a constructive basis for the foundation of his theory of a double longitudinal division, de Sinéty uses particularly the chromosomes of *Œdipoda* (*Hippiscus*) *miniata*, represented in figures 129 and 130, concerning which he says:

> "Survient le phénomène exceptionnellement important de la seconde division

> longitudinale; nous regardons comme un point capital dans notre travail d'en mettre l'existence hors de doute et pour cela nous désirons ne faire appel qu'à des images extrêmement claires. Nous considérons comme telles les fig. 129 et 130 rapprochées l'une de l'autre.
>
> "Il est de toute évidence que le chromosome *a*, fig. 130, n'est que le chromosome de même désignation, fig. 129, dont les deux anses jumelles se sont clivées. De même, le chromosome en forme de boucle, *c*, fig. 129, dont les deux branches représentent, comme nous l'avons fait remarquer, deux anses jumelles, se retrouve avec un clivage très évident en *d*, fig. 123. On pourrait faire les mêmes rapprochements entre *b*, fig. 105, et *a*, fig. 107; ici, le clivage est moins avancé, mais les granules sont nettement divisés."

I am obliged to confess that I have never seen in other species of this genus any appearances that would incline me to place an interpretation upon them such as does our author upon these. I would venture to suggest, on the contrary, that the chromosomes represented in figure 129 have not as yet demonstrated any division, but show merely irregular spaces between chromosomes. At even an earlier stage (figs. 5, 37, and 38), I have shown the formation of the tetrads by means of simultaneous cross and longitudinal divisions so clearly that presumed successive divisions, as represented by de Sinéty, cannot be regarded as occurring.

Finally, I would emphasize the fact mentioned in connection with the discussion of the cross-shaped chromosomes, that where the elements of one of these compound chromosomes intersect *they lie in one plane, and are not superimposed upon each other*, as de Sinéty's theory demands and as his figures represent. This was shown clearly in Paulmier's figures as well as in my own, and is even more clearly demonstrated, if posssible, in the very long, slender chromosomes of the myriapods, which I have observed in Mr. Blackman's preparations. This, and the continuity of the chromatin in contiguous arms of the cross, is alone sufficient to disprove de Sinéty's theory, and, fortunately, is easily demonstrated. This same fault of de Sinéty's is encountered, in another form, in his discussion of the ring figures. He asserts that the halves of the rings are pulled past each other while they lie in the plane of the spindle axis. Herein my observations fail entirely to agree with his. The rings lie in the plane of the equator, and no elements of the mitotic figure show a lateral displacement of the separating halves equal to the width of the chromosome when viewed in this plane.

### (*d*) *The Spermatocyte Divisions.*

I approach a discussion of Montgomery's conclusions regarding the form of the chromosomes in the first spermatocyte, and the sequence of their divisions, with considerable hesitation, because of the difficulty I experience in appreciating his exact position. This is due, not to any lack of positive statements on his part, but to the partial contradictions that result from his frequent changes of opinion. The most important statement in his first paper upon *Euchistus* reads as follows: "From the resting stage of the first spermatocyte to the formation of the spermatid, there is absolutely no longitudinal division of the chromosomes. I have studied hundreds of nuclei in these stages, and at the first with a hope of finding a trace of such a process, but observation shows that all divisions of the chromatin elements are transverse divisions."

This would certainly seem to be as strong a stand as one could take upon the subject, but in later papers Montgomery assumes with equal assurance the opposing position, which holds for a longitudinal division. Regarding this he says: "During the synapsis stage the chromosomes become split longitudinally, as was first shown by Paulmier (1898, 1899) for *Anasa*—a process that I had overlooked (!) in my former paper (1898)." Throughout his later investigations this hypothesis serves as the basis of all his theories, and the careful longitudinal division of the thread is assigned an important *rôle* in the maturation process. So far as positive assertions to the contrary are concerned, a general acceptance of the theoretical importance attaching to this act is to be supposed.

Notwithstanding this, I find nowhere in his later writings any statement that he abandons the conception formerly entertained regarding the non-importance of the longitudinal cleavage. This attitude is indicated in the following language: "If it can be proved that the mode of division of a chromosome, *i. e.*, the axis of the line of division, is merely a function of its chromomeres, then it would be of no theoretical value whether the division be longitudinal (equation) or transverse (reduction). But it happens that the postulated difference forms one of the main premises of Weismann's theoretical superstructure. On account of the differences observed in different objects in regard to the modes of division of the chromosomes, it would appear that the differences have no theoretical value, but that the halving of the mass of chromatin is the process of importance—the standpoint taken by Hertwig.

"In the two reduction divisions the chromosomes may split by two longitudinal divisions, by two transverse divisions, by one longitudinal and one transverse division, or by one division (longitudinal or transverse) preceded or followed by an elimination division. The facts show already that there is no general uniformity in the mode of division of the chromosomes in the reduction mitoses. The long line of observations on different objects show this to be the case, and demonstrates that the expected uniformity does not occur."

Herein lies the essential conclusion of the work upon *Pentatoma*, which, so far as a specific retraction is concerned, stands yet. If this be abandoned, then the first work upon the chromatin structure of *Pentatoma* is practically discredited, for Montgomery has definitely retreated from his positions concerning the absence of the "chromatin nucleolus" in the spermatogonia, the non-occurrence of a longitudinal cleft in the spireme thread, the lack of an equational division of the chromatin in the spermatocyte, the origin of the "chromatin nucleolus," and the fragmentation of the "chromatin nucleolus." In addition to these specifically acknowledged errors, we may infer that Montgomery (**12**) considers himself at fault in his views upon the production of chromosomes from the "three to six chromatin loops" by breaking apart in the prophase, and upon the occurrence of both longitudinal and cross divisions of ordinary chromosomes in the same mitosis. The observations recorded in his last paper (**15**) upon the production of the spermatocyte chromosomes by the end-to-end union of those in the last spermatogonial division warrant this assumption.

It follows from all this that we may practically disregard Montgomery's earlier work upon chromosomal structure and take his views as expressed in the later papers (**14**, **15**) as representing his opinions upon the subject. These later theories are largely the result of his investigations upon *Peripatus*, but they seem to be carried over and applied to the Hemiptera without essential modifications, and we may regard this concept as applicable to the forms studied by him.

I called attention in my previous paper to the fact that, by many investigators, the definitive form of the chromosome is used as the basis for determining the direction and sequence of the chromosome divisions. This fact and the danger attending the practice was partly realized by Montgomery in his work upon *Euchistus* (**12**), for he devotes considerable space to a consideration of the prophase segments, but in determining the character of the second spermatocyte division he regards only the formed element. With respect to this he says: "And now a fact may be determined which is of the greatest importance in estimating the morphological value of the second division of the chromosomes. While the latter are still parallel to the axis of the spindle, there may be clearly seen in some cases a transverse constriction on some of the chromosomes, so that they already acquire a dumb-bell shape." This constriction is not correlated with any similar one on the prophase elements, and is here observed for the first time.

In his paper upon *Peripatus*, however, he definitely supports the contention that it is only in the prophase of the first spermatocyte that we can learn the construction of the chromosomes, for he says: "The early stages in the prophase are of the greatest importance in determining the exact constitution of the chromosomes of the first maturation division. . . . Since, then, as has been shown in another section of the present paper, the split of the univalent chromosome of the second spermatocyte is a true longitudinal split, corresponding perfectly in position with the longitudinal split of the early prophase, it follows that the univalent chromosome does not become turned upon its axis to take its place on the equator of the spindle." Orientation is in both spermatocytes based, accordingly, upon planes determined in the prophase. Upon this point Paulmier and Montgomery, as students of Hemipteran spermatogenesis, are now agreed, and their results correspond with observations made upon Orthopteran cells.

It is upon the sequence of divisions in the spermatocyte that differences now exist between these investigators and myself. In my previous paper I took occasion to elaborate the proof in support of my position regarding the early occurrences of the longitudinal division in the Orthopteran spermatocytes. Montgomery follows Paulmier in ascribing the reduction division to the first spermatocyte, and takes no account of my results upon

*Hippiscus.* The objections that I previously urged against Paulmier's conclusions apply equally well to Montgomery's. Until the chromosomes are traced in a more detailed way through the prophase to the metaphase, I shall consider the presumption against the occurrence of the cross-division in the first spermatocyte mitosis. In this I believe that I am justified by the definite proof of my position brought forward in the work upon *Hippiscus.* Here, it may be recalled, I observed and photographed in the same mitosis all stages of movement by the chromatids along the plane of the longitudinal split. In addition, I was able to locate definitely the position of the future cross-division in the ring figures, so that it is impossible to mistake the character of the first division in them. These two proofs I consider incontrovertible so far as they apply to the Orthopteran families studied.

Paulmier judged the planes of the division by the relative lengths of the chromosome axes, but, as I pointed out, this is not conclusive unless it can be shown that they have not shifted, as it is possible for them to do, during the prophase. The value of the ring figure, which is formed at such an early stage that it would be impossible for the shifting of the axis to occur, is here evident.

Montgomery finds these rings in *Peripatus,* and realizes the importance of their evidence in determining the planes of division, but places his conclusions upon a much more insecure footing than those founded upon the Orthopteran cells, because of the criterion used in determining which point represents the junction of the paired chromosomes. The diagnostic feature he uses is the linin connection persisting between the "central ends" of the chromosome, which holds them together until the "distal fibers" connect with the centrosomes and cause the rupture of the "central" fiber. Since the whole of his elaborate theory regarding the continuance of the linin spireme is practically a theoretical conception with little basis in observed fact, the value of such proof cannot compare with that furnished by the definitely formed chromosomes themselves in the Orthopteran cells.

In view of all these facts, I think it must still be held an open question as to which is the reduction and which the equation division in the Hemipteran spermatocytes, although it is not to be doubted that the probability of the first spermatocyte being witness of the reduction division is much increased when thus interpreted by two independent observers.

### (*e*) *The Accessory Chromosome.*

I have already, in another paper (**19**), taken up a comparative study of the accessory chromosome in different insect spermatocytes, and shall not be obliged, for that reason, to enter into a very lengthy discussion of the subject here. The great interest attaching to this structure, however, compels me to consider the work that has been done since the manuscript of the earlier article was sent in for publication. This review will concern, very largely, the investigations of Montgomery upon a considerable number of Hemipteran species, which are set forth in his paper under the pretentious title "A Study of the Chromosomes in the Germ Cells of Metazoa."

In his first work upon *Euchistus*, Montgomery describes a cell element under the name "chromatin nucleolus" which corresponded so closely to my accessory chromosome that I concluded the two structures were identical. These similarities were, the origin from a spermatogonial chromosome, the integrity and constancy of staining power and position during the spermatocyte prophase, and participation in the division act during metakinesis of a spermatocyte.

Among the numerous changes of opinion recorded by Montgomery in his latest work, there are several relating to his "chromatin nucleolus" that materially alter the aspect of the question. Perhaps the most important of these concerns the origin of the element. I was some time in determining that the accessory chromosome is a spermatogonial chromosome which divides in the spermatogonia with the other chromatin elements and comes over into the first spermatocyte as a formed structure. The work of Sutton upon the early history of the element in *Brachystola*, however, was convincing in this respect and confirmed me in the opinion I had already formed. I therefore gave Montgomery the credit for this discovery, and set it down as strong confirmation of the assumption that we were dealing with similar structures in the two orders of insects.

Upon this point Montgomery now completely reverses himself, and declares that his "chromatin nucleolus" is not a spermatogonial chromosome, but may be noted in the earlier generations as a nucleolar structure, which, however, divides in metakinesis. The most important feature to be noted in this connection is the fact that the structure does not exist as a simple element, but is observed as a number of granules, and that this number varies considerably in different species. These granules fuse during the "synapsis stage," as do the chromosomes, to produce in the spermatocyte half the number of "chromatin nucleoli" that were present in the spermatogonia. In this respect the "chromatin nucleolus" differs radically from the accessory chromosome, which has the same valence in both cell generations. The indefinite number and insignificant size of Montgomery's structures are other characters that point to extensive differences between them and the accessory chromosome.

In his work upon *Peripatus*, Montgomery states that in restudying his preparations of *Euchistus* he observes a continuous linin spireme which involves the "chromatin nucleolus" as well as the chromosomes. Here, again, there is a difference between the Hemipteran element and the accessory chromosome; for the latter is entirely free from linin connections in the prophase and is usually surrounded by a hyaloplasmic investment.

According to Montgomery, also, his "chromatin nucleolus" usually takes part in both spermatocyte mitoses. In this respect there exists an essential difference between his element and that found in the Orthoptera, for, after extended and most critical studies, I have become convinced that only one division takes place in the spermatocytes. In those cases where Montgomery admits but a single division, it is stated to occur in the first spermatocyte, while in the Orthoptera the accessory chromosome remains undivided here and is halved in the second spermatocyte.

If, therefore, Montgomery's recent observations are correct, it must follow, I think, that his "chromatin nucleolus" and the accessory chromosome are different structures. I am free to admit, however, that his statements are far from convincing. So much dependence is placed upon the numerical relationships of elements that are admittedly very minute, and so little corroborative proof is given, that I entertain serious doubts as to the accuracy of the observations. In this connection I would suggest a comparison between the figures of the "chromatin nucleolus" in the first paper upon *Euchistus* (figs. 55–68) (**12**) and those in the last one (figs. 1–17) (**15**). The showing here made would alone be sufficient to raise a question as to the nature of the "chromatin nucleolus," and until further evidence is forthcoming the character of the peculiarly modified chromosomes in the spermatocyte of the Hemiptera must remain in doubt.

Aside from definite retractions that Montgomery has made regarding his earlier views on the character of the "chromatin nucleolus," there are noticable different attitudes toward it in his earlier and later works. Thus, in his lecture at Woods Holl (**13***a*), we find the following: "These remarkable 'nucleolar' structures which stain like chromatin have been observed by numerous writers, but as yet no satisfactory description has been given of their mode of origin. They have been observed by me in spermatocytes of various insects, in hypodermal and other cells of *Carpocapsa*, and in follicle cells of the testicles of *Plethodon* and *Mus*." At this early stage of Montgomery's investigations it is apparent that he views his "chromatin nucleolus" primarily as a nucleolus with chromatic origin and characters, but the fact is equally apparent that he now regards it primarily as a "chromosome" with nucleolar attributes. This is made evident in his recent definition, which reads: "The chromatin nucleoli are morphologically chromosomes, undergoing division in mitosis like the other chromosomes, but differing from them in the rest stage by preserving a definite (usually rounded) form."

What has here been said regarding the "chromatin nucleolus" applies to those structures in *Euchistus* and other Hemiptera to which Montgomery has given the name without qualification. According to his definition, however, there is present in the cells of *Protenor* and other species another form, the "chromosome x." Not only by inference is this classification operative, but by direct statement we learn that Montgomery regards this element as a member of the class of bodies which he calls "chromatin nucleoli." In speaking of *Protenor* chromosomes, he says: "This is the only case in the Hemiptera where one chromosome becomes differentiated into a 'chromatin nucleolus' for the first time in the spermatocyte generation."

The noteworthy thing about this "chromosome x" is the fact that in every essential detail it corresponds to the accessory chromosome of the Orthoptera. It is a spermatogonial chromosome that comes over intact into the spermatocyte; it retains its form and staining power unchanged through the prophase of the spermatocyte; it divides in only one of the spermatocyte mitoses; and is a large and conspicuous element of the cell at all times.

This "chromosome x" agrees just as closely in its description to the accessory chromosome as do the ordinary ones of the two orders, and, if Montgomery's account is correct, there would seem to be no reason for doubting their identity. In two respects, however, there are differences between these structures. First, it is to be noted that the "chromosome x" divides in the first spermatocyte, while the accessory chromosome undergoes separation in the second spermatocyte. Should Montgomery's observations prove correct, it would yet indicate no fundamental difference in the character of the element, for the result is the same whether division takes place in the first or second mitosis. In either event, one-half the spermatozoa are provided with the odd chromosome while the remaining half are not.

The second point of difference would seem to be a more serious one. Montgomery states that during the spermatogonial mitosis the "chromosome x" regularly divides as do all the other chromosomes, *i. e.*, longitudinally. In the spermatocyte mitosis, however, the element is broken across, and the longitudinal split, which is apparent in the early stages, disappears and is not utilized in division. We have here the remarkable occurrence of a chromosome entirely unchanged in its structure, but merely differing in its surroundings, which, instead of dividing along the plane marked out for it, as it has done in all preceding mitoses, breaks across after it is a formed element. An occurrence of this kind, so different from the usual method of division, would require strong proof to establish it, and this, in my opinion, Montgomery has not brought forward.

A criticism of the degeneration theory as advocated by Paulmier and Montgomery has already been given (**17**), so that it would not be necessary to consider it here except in so far as it has been modified since its promulgation. As a rule, Montgomery refers to his "chromatin nucleoli" throughout his late paper (**15**) as degenerating chromosomes, but in discussing their function specifically he makes important changes in this conception. These are stated as follows: "When we find, accordingly, the mutual apposition of them (true nucleoli) to chromatin nucleoli, it would be permissible to conclude that the chromatin nucleoli are chromosomes which are especially concerned with nucleolar metabolism. And this, I think, would be the correct interpretation. The chromatin nucleoli are in that sense degenerate that they no longer behave like the other chromosomes in the rest stages, but they would be specialized for a metabolic function; and from this point of view they would certainly seem to be much more than degenerate organs."

It is difficult to comment upon a contradictory statement like this; but, fortunately, it is not necessary to do so, since it carries with it its own refutation. The conception of a chromosome specialized in the direction of increased metabolic activity as being in the process of disappearing from the species can hardly be regarded seriously.

Taking everything into consideration, it may be said that Montgomery's work upon the Hemiptera has left the subject in a very disturbed condition, and any prospect of a complete agreement between the accessory chromosome of the Orthoptera and the "chromatin nucleolus" of the Hemiptera is made more remote than was previously the case. This, I think, is largely due to the inferior character of the Hemipteran material, which has lead to misconception of phenomena that are clearly marked in Orthopteran cells.

It is gratifying to note that the recent work of de Sinéty (**37**) practically corroborates the conclusions herein set forth regarding the history of the accessory chromosome. Aside from failure to observe the important spireme condition of this element in the first spermatocyte prophase, de Sinéty describes the same series of processes with scarcely an exception. His summary contains the following account of the accessory chromosome:

"Le 'chromosome accessoire,' découvert par McClung chez *Xiphidium fasciatum*, se retrouve chez les locustiens que nous avons étudiés. Chez *Orphania*, il se divise dans les spermatogonies en deux masses volumineuses et allongés, que l'on reconnait dans les nucléoles, également volumineux et allongés, des spermatocytes de premier ordre en prophase. A la métaphase de la première cinèse, on le trouve situé excentriquement et plus près de l'un des pôles; *il va tout entier a l'une des cellules-filles.* Dans celle-ci, il se divise comme un chromosome ordinaire, d'où il suit que *sur quatre spermatides formant la descendance d'un spermatocyte, deux se trouvent privilegiées.* Par ce partage inégal, non réalisé dans *Xiphidium fasciatum*, d'après McClung, le chromosome spécial d'Orphania rappelle celui des hémiptères."

A like series of processes is recognized in the Phasmids.

As is elsewhere explained in this paper, the occurrence of two divisions of the accessory chromosome in *Xiphidium*, which was mentioned as a possible occurrence in my preliminary paper, is shown not to take place. While it is much more difficult to demonstrate the undivided condition of the accessory chromosome in one of the spermatocyte mitoses of *Xiphidium* than it is in the cells of *Orchesticus*, *Anabrus*, *Scudderia*, and *Microcentrum*, I am convinced that it does not differ from the other Locustids in this respect.

We may therefore feel assured that our knowledge of the morphological character of the accessory chromosome in the Orthoptera is fairly well established. This gives us a good base from which to conduct further comparative studies into other groups, and it is to be hoped that our knowledge of this element will rapidly increase.

Unfortunately, de Sinéty has chosen to add another name to the already overburdened list of synonyms, and "chromosome spécial" now takes its place in the literature of insect spermatogenesis. The reason for adding this name—

"Il reçu successivement les noms de 'accessory chromosome' (McClung), 'small chromosome' (Paulmier), 'chromatin nucleolus' (?), 'chromosome *x*' (Montgomery). Nous avons préféré éviter ces appellations, qui semblent toutes supposer une signification qui n'a jamais été définie ou s'appuyer sur des caractères plus ou moins secondaires, pour adopter un nom indifférent, celui de 'chromosome spécial,' nous conformant à l'idée de Wilson, pour qui c'est un 'extra chromosome,'"

would seem to be at least insufficient, since "accessory chromosome" can scarcely be regarded as implying any more primary or secondary function than can "chromosome spécial."

### (*f*) *Individuality of the Chromosomes.*

In each of my preceding papers I took the opportunity to point out the fact that, even were the accessory chromosome of no other value, it would certainly be worthy of study for the light it throws upon the question of the individuality of the chromosomes. On this point Montgomery has much to say in his late paper (**15**). I think it cannot be questioned that we have here indisputable proof that at least one chromosome may be identified through all the cell generations of the testis. While this does not prove that chromosomes are persisting and independent structures, it does evidence the fact that they may be, and greatly strengthens the hypothesis that they are.

In addition to the evidence here offered by the accessory chromosome, there must be noted that derived from a study of spermatocytes in which there is always present one ordinary chromosome that greatly exceeds the others in size. Such a condition is found in the cells of *Anabrus*. The disproportion in size of the elements is here so striking that it would be impossible to fail in distinguishing the giant chromosome. In each of the spermatocytes of *Anabrus* there are therefore two chromosomes which are plainly recognizable. It may be observed further that the remaining chromosomes are quite different in size, and it may be possible within reasonable limits of certainty to pick out one or more other chromosomes in each cell. Unless this could be done for each element, however, it would not definitely prove that all the chromosomes are distinct and recognizable structures. The actual recognition of two elements in each cell of the same generation and its ancestors or descendants in other generations goes far, however, to render probable the individuality of each chromosome.

Beyond this point studies upon the Orthopteran cells will not permit me to go; but Montgomery has been fortunate enough to find in *Peripatus* an object in which he considers it possible to demonstrate the continuity of the chromosomes from one generation to another, and their fusion by pairs in the early history of the spermatocyte to bring about the reduced number. This is, in the main, a logical conclusion to my own work, and I am therefore bound to regard his results as probably correct. While doing this, however, I recognize that the absolute proof he brings forward in support of his hypothesis is very slight. I consider any deductions based upon observations of linin structures as very insecure, and it is upon these that Montgomery principally relies to demonstrate his theory. Further observations upon the behavior of the chromosomes between the spermatogonia and the spermatocytes in objects favorable for study will be awaited with interest. In the meantime it must be conceded that the work upon insect spermatogenesis has at least lent strong support to the theory of the individuality of the chromosomes in general and has definitely shown that there is such a thing in some instances.

### (*g*) *Nucleoli.*

Considerable importance is attached by some investigators to the nuclear structures, properly called plasmasomes, that occur in the spermatocytes. It is probable that there are marked differences between the cells of various species in regard to the occurrence of these bodies, for in the Orthoptera they either do not appear at all, or, if present, they are minute and inconspicuous. This fact would tend to disprove any theory which would attach a fundamental importance to these structures, such as is conceived for the chromatin. The Orthopteran cells do not allow any observations which would add to our positive knowledge of the nucleoli, and I include this brief statement merely for the negative value it may possess.

### (*h*) *Rest Stage.*

In his first paper upon *Euchistus*, Montgomery assigns an important and conspicuous place to the "rest stage" among his numerous subphases preceding the first spermatocyte mitosis. As a result of his later comparative work upon the Hemiptera, however, we learn that in certain families no trace of such a condition of diffusion on the part of the chromatin is observable, from which we conclude that "accordingly such a stage would appear to have no broad significance." It has already been announced that nothing like a rest stage intervenes between the spermatogonia and spermatocytes of the Orthoptera, and the work of most investigators would tend to indicate that it is the exception rather than the rule. In those cases where such a condition of the nucleus exists, it would seem to be true that nothing more unusual than an excessive diffusion of the spermatogonial chromosomes occurs, and this is of hardly sufficient importance to receive a special designation.

The existence of a rest stage between the first and second spermatocytes is also negatived by the conditions found in the

Orthopteran cells. The formation of chromosomes in the prophase of the first spermatocyte that are already prepared for two divisions would *a priori* render improbable the intervention of a rest stage here; and the actual observed persistence of the chromosomes, as such, through the telophase of the first spermatocyte and through the modified prophase of the second spermatocyte gives actual proof in support of the view that commonly prevails regarding the suppression of the second spermatocyte rest stage.

Observations upon numerous species tend to show that the behavior of the chromatin during the period between the two spermatocyte mitoses varies considerably with the species and even within the species itself. The amount of diffusion would, in some measure, seem to be related to the form of the chromosomes and to vary correspondingly in those individuals where the chromosomes are of diverse forms. Thus, where the elements of the second spermatocyte metaphase appear as short double rods, the amount of diffusion is slight, and the individual chromosomes may be distinguished throughout the telophase of the first spermatocyte; but in those cases where the members of the mitotic figure are much elongated the diffusion is more extensive and the distinction between elements is made difficult or impossible. Since these two conditions may prevail in the same testis, it is probably only a question as to the extent of elongation on the part of each chromosome. In those cases where the elements become very much extended the appearance of the resting condition would be simulated closely, while, on the contrary, chromosomes consisting of spherical or short cylindrical chromatids would never give a suggestion of such a stage. In this we may find, I think, an explanation for those cases in which a rest stage is described as occurring between the spermatocyte generations.

## VI. SUMMARY.

1. The secondary spermatogonia are much reduced in size at the end of their divisions and the cytoplasm is very small in amount. The rod-shaped chromosomes number thirty-three, and, of these, one is to be distinguished from its fellows by greater size and slower division.

2. From the substance of the disintegrated spermatogonial

chromosomes, the tetrads of the first spermatocytes are formed. It was impossible to determine the relation of the elements of the two generations, but the changes are rapid and there is no intervening resting condition of the nucleus.

3. It could not be determined whether or not the spireme is continuous. A longitudinal split appears very early, and shortly after the chromatin segments may be seen. These soon betray at their centers an indication of the cross-division, producing crosses with arms that may vary considerably in relative lengths. No reason was found for considering both divisions longitudinal.

4. The typical element is granular and more or less rod-shaped, with the longitudinal division merely indicated by a narrow line, and with but slight elongation of the chromatids along the plane of the cross-division. Various modifications of this occur, by which the longitudinal cleft is much increased in width at the center, the cross-arms are greatly extended, or approximation of the ends of the rod brought about, producing a ring.

5. The definitive chromosomes of the metaphase are produced by a concentration of the prophase elements, whereby they become shorter, heavier, and entirely homogeneous in structure. Distinct lines of division between the chromatids are not visible, but the tetrad character of the elements is readily established by observing the steps in their formation.

6. The accessory chromosome early becomes distinguishable because of its peripheral position and strong tendency to stain with safranin, while the remaining chromatin takes the gentian violet by Flemming's three-color method. At first it appears as a homogeneous plate, but later this is seen to be a closely coiled thread. As the chromatin segments shorten and broaden to form the chromosomes of the mitotic figure, this thread also grows shorter and heavier until it forms an element of essentially the same character as that of the spermatogonial chromosome from which it was derived.

7. Upon the establishment of the mitotic figure, the chromosomes arrange themselves in the equatorial plate with their longer axis perpendicular to the spindle axis. Division of the elements is not synchronous, so that all stages of the chromatid movements may be observed in one nucleus. By this means it

is possible to determine that separation of the chromosomes takes place along the plane which marked the longitudinal division of the prophase thread in such a way that the chromatids show no clear interspaces. The individual chromosome near the end of its division has the same form as that with which it started, except for the difference that the chromatids are now in contact for the greater part of their length along the plane of their cross-division. As the daughter chromosomes separate, this line of division comes into evidence through the springing apart of the two chromatids now composing each chromosome. The result is the formation of two V-shaped chromosomes with mantle fibers attached to their apices. The accessory chromosome does not participate in this division, but passes unchanged to one pole of the spindle.

8. By reason of the action of the accessory chromosome in the first spermatocyte mitosis, there are produced two numerically equal classes of second spermatocytes—(*a*) those containing sixteen dyad chromosomes and an undivided accessory chromosome, and (*b*) those with merely the sixteen dyad elements. In both cases the mitotic figure quickly reforms without an intervening rest stage in which the chromosomes lose their identity. There is a loosening up of the chromomeres in all the elements except the accessory chromosome, so that they have a structure and staining reaction similar to that of the first spermatocyte chromosomes just before they enter the metaphase. The dyads of the first spermatocyte telophase, and of the succeeding and greatly abbreviated second spermatocyte prophase, are quite as definite structures as are the chromosomes of the first spermatocyte prophase.

9. All the chromosomes of the second spermatocyte are paired structures and divide in a similar way. The spindle is small and weak as compared with that of the first spermatocyte, and the chromosomes arrange themselves radially on its periphery in such a way that the pairs lie in the plane of the spindle axis with their joined ends inward. The space between the chromatids represents the line of cross-division observable in the prophase segments of the first spermatocyte, and their separation accordingly represents a reduction division. The accessory chromosome, on the contrary, divides along the plane marking the longitudinal cleft of the spermatogonial spireme.

10. From each first spermatocyte there are formed, by two divisions, four spermatids, of which two are distinguished from the remaining pair by the possession of an extra chromosome in addition to the number—sixteen—common to them all. Both classes undergo a like series of transformations by which they become mature spermatozoa. These are necessarily of two kinds; and it is believed that those containing the accessory chromosome, in the act of fertilizing the egg, determine that the germ-cells of the embryo shall be sexually male, or like themselves, while those from which it is absent are unable to impress their sex upon the egg and assist in producing female embryos.

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## DESCRIPTION OF FIGURES.

Drawings were made with a *camera lucida*, the optical combination being a 1–16 B. & L. objective and a Watson "Holoscopic" ocular No. 7. Details were studied with a Zeiss 2-mm. apochromat, N. A. 1.30. As reduced in reproduction, an enlargement of 1500 diameters exists. Photomicrographs, excepting those of figures 37 and 38, were made by the use of the arc light and horizontal camera. The exceptions represent illumination by ordinary diffuse daylight. In all cases the lenses used were the Zeiss 2 mm., N. A. 1.30 objective and projection oculars. A Watson "Parachromatic" oil-immersion condenser of 1.30 N. A. was employed to illuminate the objects. In use it was stopped down to between .75 N. A. and 1.0 N. A.

## Explanation of Plate VII.

FIG. 1. Pole view of spermatogonial metaphase, showing the thirty-three chromosomes. It will be observed that the chromosomes are of unequal sizes, and that the large ones arrange themselves in a circle on the outside of the figure.

FIG. 2. Very young spermatocyte. The chromatin derived from the breaking down of the spermatogonial chromosomes in a diffuse condition, with no trace of a linear arrangement. The accessory chromosome $x$ on the periphery of the nucleus, darkly staining and homogeneous.

FIG. 3. Early stage in the formation of the spireme. In the cytoplasm the remains of the spermatogonial spindle. The cell has entered upon the growth period.

FIG. 4. A later stage in the spireme formation. The accessory chromosome larger and more flattened. A surface view shows it as an apparently fenestrated plate. The remains of the two spermatogonial spindles still persisting.

FIG. 5. First appearance of definite chromosomes. One shown entire with longitudinal and cross-divisions marked. The accessory chromosome is here seen to be in a spireme condition.

FIG. 6. Condition of the chromosomes after further contraction of the early segments. As here shown, they are more granular than is usually the case.

FIG. 7. Common types of the prophase chromosomes.

FIG. 8. A cell in which one of the chromosomes has its halves widely separated along the longitudinal division, forming Paulmier's double-V figure.

FIG. 9. In this cell may be seen the variation in form and size of the early spermatocyte chromosomes.

FIG. 10. Two cells of the late prophase, with the chromosomes at almost the extreme degree of concentration.

FIG. 11. Chromosomes of cells in the stage shown in figure 10. These represent the different types of rings, crosses, etc., commonly observed in first spermatocytes just before the formation of the mitotic figure.

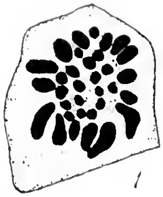
FIG. 12. Different forms assumed by the accessory chromosome in the prophase of the first spermatocytes of *Xiphidium*.

FIG. 13. Metaphase of the first spermatocyte. The accessory chromosome is seen at one pole of the spindle, to which it has moved before the separation of the chromatids of the remaining chromosomes.

FIG. 14. Another cell in about the same stage as that represented in the preceding figure.

FIG. 15. A first spermatocyte metaphase in which the accessory chromosome has not as yet moved to the pole of the spindle. This is uncommon in *Orchesticus*, but frequent in *Anabrus*.

FIG. 16. Pole view of a first spermatocyte metaphase, showing seventeen chromosomes. The variation in size of the elements, so marked in the spermatogonia, is even more pronounced here. This is a cell similar to that of figure 15, in which the accessory chromosome lies in the equatorial plate.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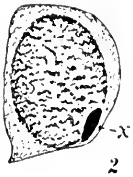
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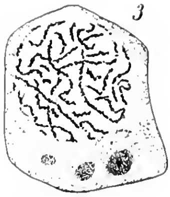
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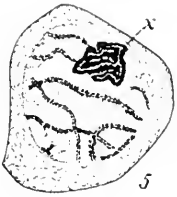
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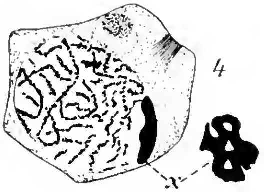
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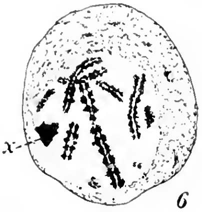
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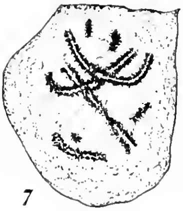
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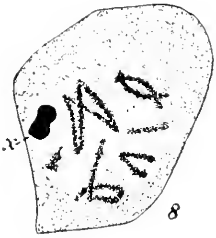
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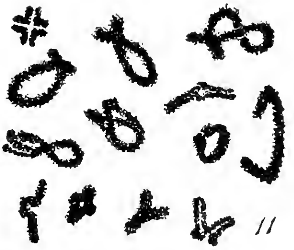
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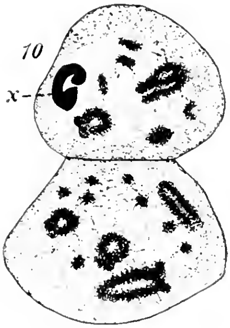
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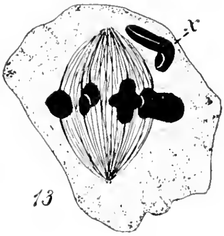
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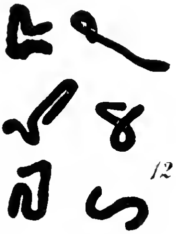
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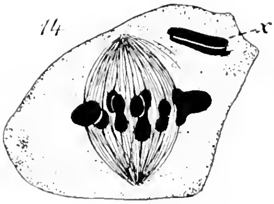
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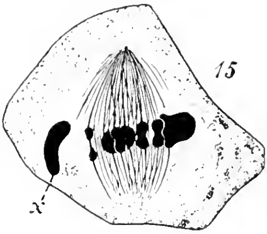
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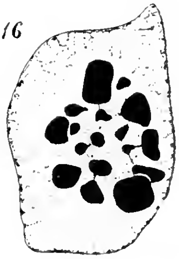
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## Explanation of Plate VIII.

FIG. 17. Two cells in metaphase—a pole view of one and an oblique view of the other. The accessory chromosome does not show in the former, the cell being such a one as is represented in figures 14 and 15.

FIG. 18. Pole view of another cell, showing but sixteen chromosomes.

FIG. 19. Early anaphase of the first spermatocyte, with the accessory chromosome already at one pole.

FIG. 20. Mid-anaphase, with the giant chromosome still undivided.

FIG. 21. Later anaphase, in which the accessory chromosome is seen at the lower pole. This figure shows, also, the character and extent of the intercellular material.

FIG. 22. Later anaphase. The accessory chromosome at the upper pole. An undivided chromosome lying between the groups of daughter chromosomes.

FIG. 23. About the stage of figure 22, but the lagging chromosome has divided.

FIG. 24. Very late anaphase. Here, again, the lagging chromosome is divided.

FIG. 25. Pole view of first spermatocyte telophase, showing the accessory chromosome at one side of the daughter chromosomes.

FIG. 26. Pole view of a cell in the same stage as that represented in figure 25. Here, however, the accessory chromosome is not present.

FIG. 27. Lateral view of telophase, with the accessory chromosome in the lower daughter-cell.

FIG. 28. Fragment of second spermatocyte, showing the chromosomes in metaphase. The relative sizes of the accessory chromosome and the remaining chromosomes is well shown.

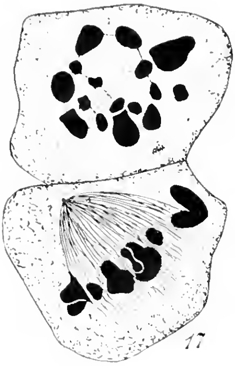
FIG. 29. Metaphase of a second spermatocyte, in which the accessory chromosome is not present.

FIG. 30. Anaphase of second spermatocyte, in which there is no accessory chromosome.

FIG. 31. Anaphase of second spermatocyte, where the accessory chromosome is present—$x^1$ and $x^2$.

FIG. 32. Telophase of the same class of second spermatocytes. The accessory chromosome extends out from the mass of chromosomes at each pole—$x^1$ and $x^2$.

FIG. 33. Telophase of the class of second spermatocytes from which the accessory chromosome is absent.



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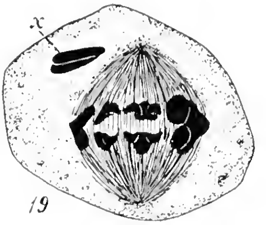

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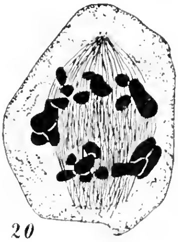

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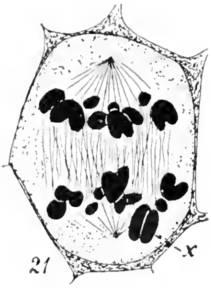

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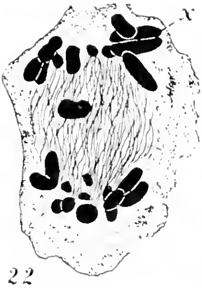

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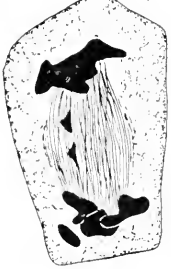

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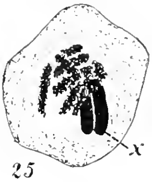

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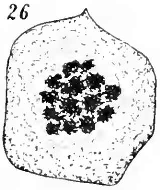

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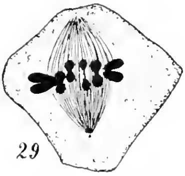

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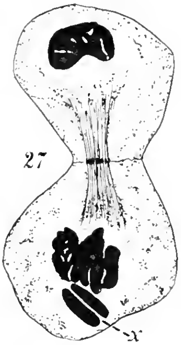

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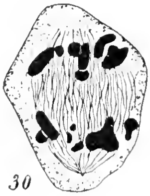

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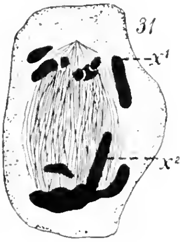

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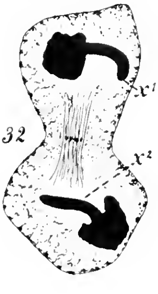

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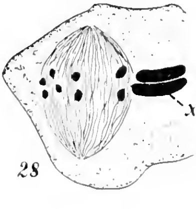

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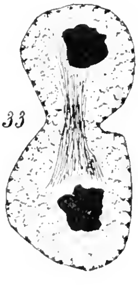

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## Explanation of Plate IX.

Fig. 34. Photomicrograph of early spireme stage of first spermatocyte, showing peripheral position of the accessory chromosome *x*. At the left, secondary spermatogonia, last generation. × 1300.

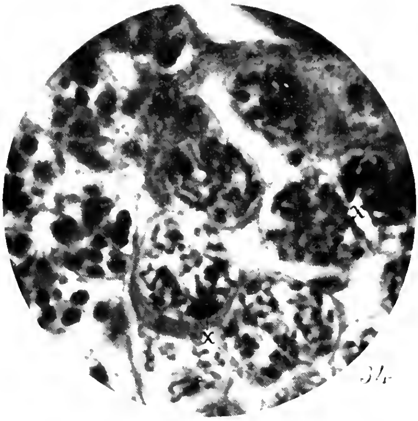
Fig. 35. A late prophase, showing accessory chromosome *x*, and spindle remains *s* (*cf.* figs. 3 and 4). × 1300.

Fig. 36. Coarse spireme of first spermatocyte. × 1300.

Fig. 37. Prophase, with chromosomes in the form of long segments. At *a*, the cell drawn in figure 9. In the cyst at the left are spermatocytes in a later stage, with the chromosomes homogeneous. × 1000.

Fig. 38. Prophase with segments divided longitudinally and across. At *a* is is one shown *en face*. Accessory chromosome at *x*. × 1000.

Fig. 39. Metaphase and anaphase of first spermatocyte. The accessory chromosome *x* at one pole of the spindle. Lagging chromosome at *c*. × 1300.



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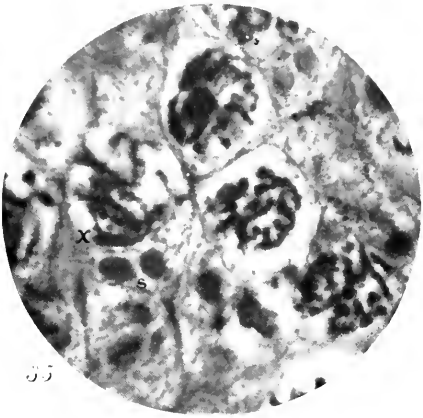

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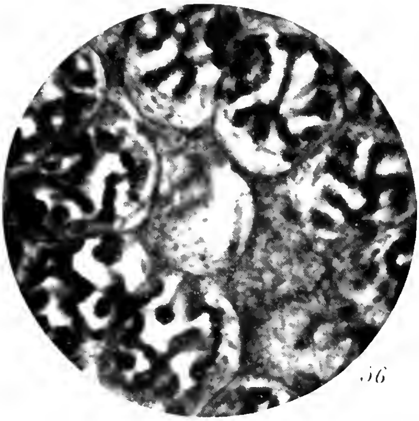

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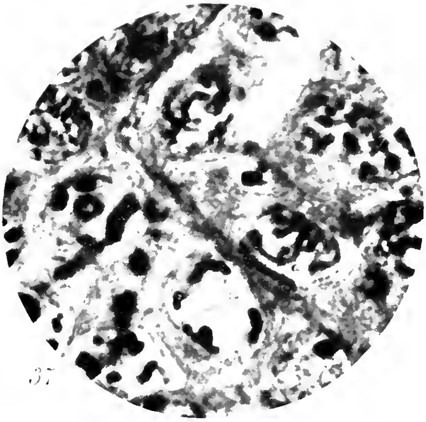

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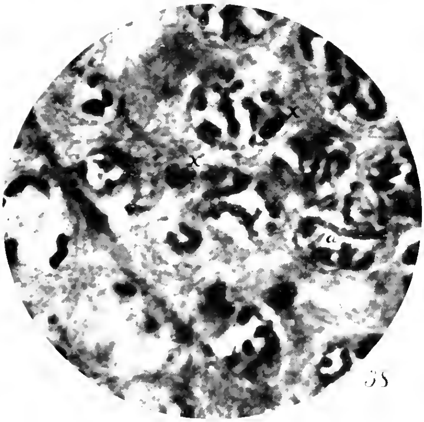

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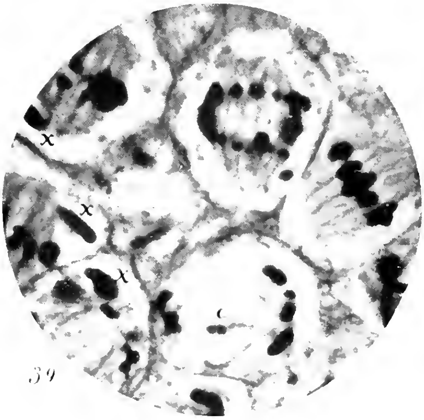

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## Explanation of Plate X.

FIG. 40. Anaphase of first spermatocyte. Accessory chromosome *x* at one pole. The form of chromosome in the anaphase well shown. The lagging chromosome *c* seen in two cells. × 1300.

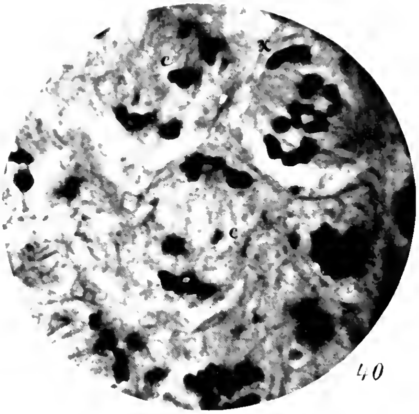
FIG. 41. Anaphase of the first spermatocyte, showing the longitudinally divided condition of the accessory chromosome *x* in the cell near the center. Compare with the accessory chromosome in the metaphase of second spermatocyte, figure 43. × 1300.

FIG. 42. Second spermatocyte in metaphase. In most of the cells the focus is upon the ends of the chromosomes, but in one a side view is obtainable. Compare with the chromosome of the upper cell in figure 40. No accessory chromosome in most of the cells in focus. × 1300.

FIG. 43. Second spermatocyte metaphase and spermatids. Note the relative sizes of the accessory chromosome and the other chromosomes. In the spermatids the accessory chromosome has taken its place on the periphery of the nucleus in the same way that it does in the prophase of the first spermatocyte. × 1300.

FIG. 44. Anaphase of the second spermatocyte, showing the accessory chromosome *x* separated. Other cells in metaphase. × 1300.

FIG. 45. Telophase of the second spermatocyte. Two daughter-cells with persisting spindle between, showing the accessory chromosome *x* in each. Other nuclei in focus show no accessory chromosomes. × 1300.



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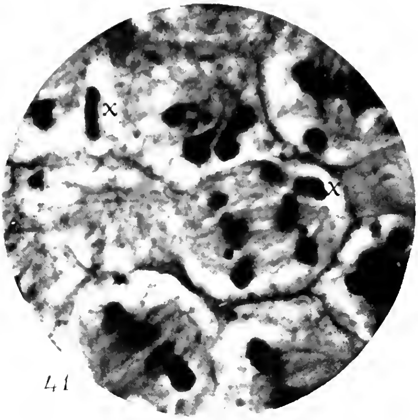

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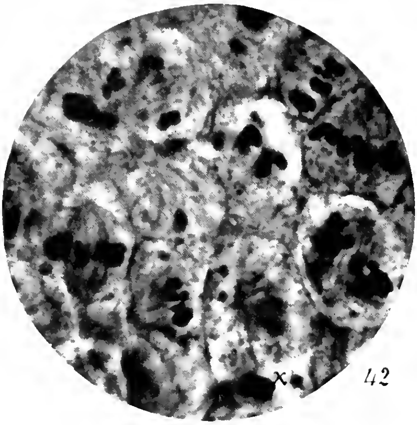

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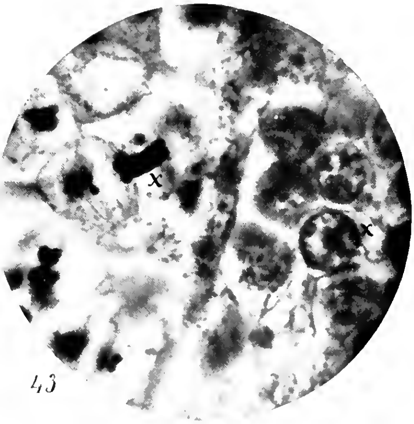

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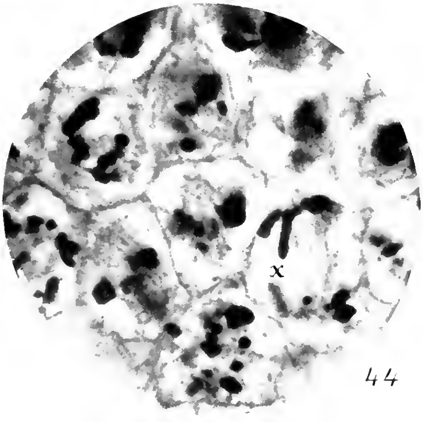

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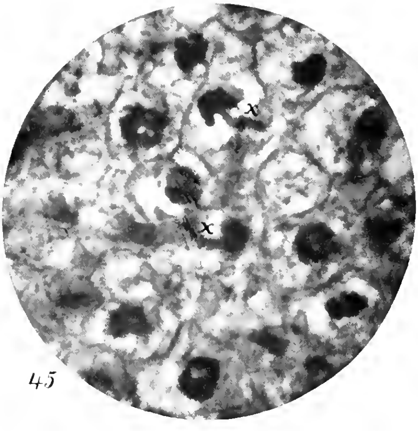

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KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. I, No. 9. SEPTEMBER, 1902. { WHOLE SERIES VOL. XI, NO. 9.

# RESTORATION OF DOLICHORHYNCHOPS OSBORNI, A NEW CRETACEOUS PLESIOSAUR.

BY S. W. WILLISTON.

With Plate XI.

No less than twenty-seven species and fourteen genera of plesiosaurs have been described from the mesozoic deposits of North America, with but few exceptions based upon very scanty material. The earliest of these are from the Baptanodon beds of the Jurasssic of Wyoming, and the latest is reputed to be from the Laramie Cretaceous. Every intervening epoch, unless it be the Dakota, has furnished one or more species, but, notwithstanding all this great diversity and wide geological range, we have hitherto known but little concerning this order of reptiles in America—less perhaps than of any other equally extensive group. Enough, however, is already known to make it certain that we have to do with a great diversity of types. The following is a preliminary list of the American forms of this order hitherto described :

**Plesiosaurus.**

*lockwoodi* Cope, Ext. Batrach., etc., 1869, 40. Cretaceous, New Jersey.

*brevifemur* Cope, Cret. Vert. 256. Cretaceous, New Jersey.

*gulo* Cope, Proc. Acad. Nat. Sci. Phil., 1872, 128. Niobrara Cretaceous, Kansas.

*occiduus* Leidy, Proc. Acad. Nat. Sci. Phil., 1870, 74; Rep. Geol. Surv. Terr., 1873, I, 345, pl. XV, ff. 11-13; Cope, Cret. Vert. Laramie Cretaceous, Dakota.

*gouldii* Williston, Kans. Univ. Quart, VI, 57. Comanche Cretaceous, Kansas.

*mudgei* Cragin, Fifth Publ. Colo. College Sci., 1894. Comanche Cretaceous, Kansas.

*shirleyensis* Knight, Amer. Jour. Sci., X, 1900, 115. Baptanodon beds, Jurassic, Wyoming.

**Discosaurus** Leidy, Proc. Acad. Nat. Sci. Phil., 1851, 326 (1852).

*vetustus* Leidy, l. c.; Cope, Ext. Batrach. Alabama.

**Brimosaurus** Leidy, Proc. Acad. Nat. Sci. Phil., 1854, 73.

*grandis* Leidy, l. c., pl., I, ff. 1-3; Cope, Ext. Batrach., 1869, 43. Clark county, Arkansas.

**Polycotylus** Cope, Ext. Batrach., 1869, 39.

*latipinnis* Cope, l. c.; Cret. Vert. 60. Niobrara Cretaceous, Kansas.

**Cimoliasaurus** Leidy, Proc. Acad. Nat. Sci. Phil., 1851, 325 (1852).

*magnus* Leidy, l. c.; ibid., 1854, 72, pl. II, ff. 4-6; Cret. Rept., 1865, 25, pl. V, ff. 13-19, pl. VI; Cope, Ext. Batrach., 1869, 42; Lydekker, Cat. Fos. Rept., II, 211. Cretaceous, New Jersey.

*planior* Leidy, Proc. Acad. Nat. Sci. Phil., 1870, 22. Cretaceous, New Jersey.

*snowii* Williston, Trans. Kans. Acad. Sci., 1891; Cope, Proc. Amer. Phil. Soc., 1894, 199. Niobrara Cretaceous, Kansas.

*laramiensis* Knight, Amer. Jour. Sci., X, 1900. Baptanodon beds, Jurassic, Wyoming.

**Uronautes** Cope, Proc. Acad. Nat. Sci. Phil., 1876, 345.

*cetiformis* Cope, l. c., 346. Fort Pierre Cretaceous, Montana.

**Embaphias** Cope, Proc. Amer. Phil. Soc., 1894, 112.

*circulosus* Cope, l. c., 112. Fox Hills Cretaceous, South Dakota.

**Oligosimus** Leidy, Proc. Acad. Nat. Sci. Phil., 1872, 39 (1873).

*grandævus* Leidy, l. c., 40. Green River, Wyoming (Cretaceous).

**Orophosaurus** Cope, Amer. Nat., 1887, 564.

*pauciporus* Cope, l. c. Fox Hills Cretaceous, New Mexico.

**Piptomerus** Cope, Amer. Nat., 1887, 564.

*megaloporus* Cope, l. c. Fox Hills Cretaceous, New Mexico.

*microporus* Cope, l. c. Ibid.

*hexagonus* Cope, l. c. Ibid.

**Megalneusaurus** Knight, Amer. Jour. Sci., V, 1898, 375.

*rex* Knight, Science, 1895, 449 (*Cimoliasaurus*). Jurassic, Wyoming.

**Trinacromerum** Cragin, Amer. Geologist, Dec. 1888.

*bentonianum* Cragin, l. c.; ibid., 1891, 171. Benton Cretaceous, Kansas.

**Elasmosaurus** Cope, Notes of Geology Ul Pacific, Leconte, 1868.

*platyurus* Cope, l. c., 68; Ext. Batrach., 1869, 44; Hayden, Geol. Surv. Terr., 1870, 393. Fort Pierre Cretaceous, Kansas.

*serpentinus* Cope, Bull, U. S. Geol. Surv., III, 578, 1877. Nebraska Cretaceous, Nebraska.

*intermedius* Cope, Proc. Amer. Phil. Soc., 1894, 112. Fort Pierre Cretaceous, South Dakota.

*orientalis* Leidy, Proc. Acad. Nat. Sci. Phil., 1870, 22.
(*Discosaurus*) Cope, Ext. Batrach., 1869, 55. Cretaceous, New Jersey.

**Pantosaurus** Marsh, Rep. Geol. Congress, 1891, 155; Amer. Jour. Sci., 1895, 408; ibid., XLII, 1892, 388 (*Parasaurus*).

*striatus* Marsh, Amer. Jour. Sci., XLII, 338 (1891); ibid., I, 406, 1895, ff. Baptanodon beds, Jurassic, Wyoming.

These known American plesiosaurs vary in length from that of *Dolichorhynchops*, described in the present paper, to that of *Megalneusaurus rex* Knight, or of one whose remains are in the University of Kansas museum, forty or more feet in length.

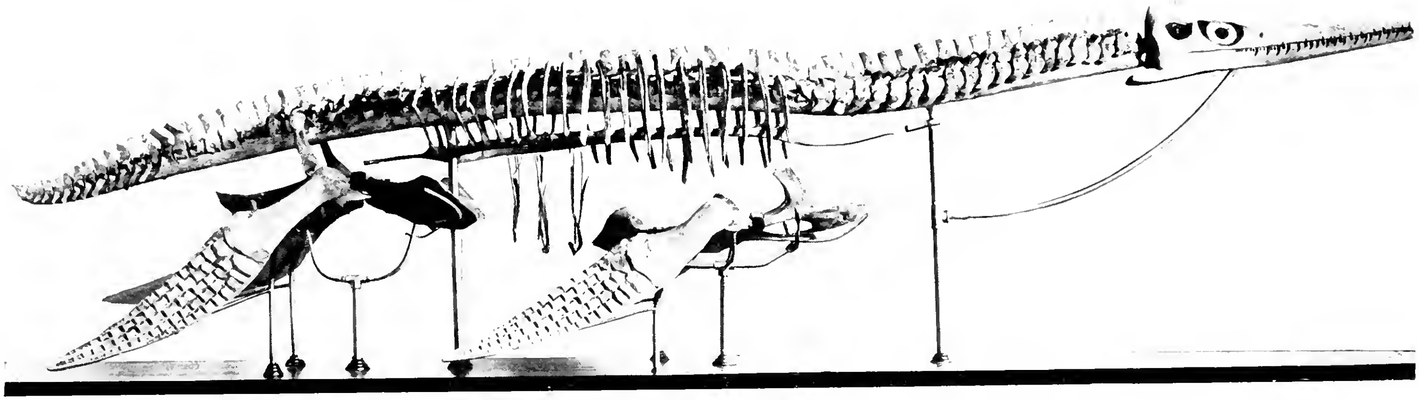
The specimen of *Dolichorhynchops osborni* herewith figured was collected in 1901 in the chalk of western Kansas by Mr. Charles Sternberg. When received at the museum the skeleton was almost wholly contained in a large chalk slab, with all the bones disassociated and entangled together closely. Owing to their extreme softness, a characteristic of plesiosaur bones in general, the labor of removing the different parts of the skeleton and hardening them so that they could be handled has been exceedingly slow and painful. The task of removing and mounting the bones has required the skilled labor of Mr. Martin, the preparator, the larger part of a year, under my constant supervision. The skeleton as now mounted is an example of great skill on Mr. Martin's part and of much study.

As mounted there is little conjecture. It is possible that some of the vertebræ may be missing, but I do not think so, unless a very few from the tail. Three or four vertebræ and a few ribs were so badly decomposed that they have been modeled in plaster, and a number of the dorsal spines could not be recovered. There is also doubt about the proper length of some of the posterior ribs. The skull, after its complete removal from the chalk, was found to be so very fragile that it was not thought to be expedient to mount it with the skeleton. A model, therefore, has been made, under my careful supervision, and mounted in its stead. The many bones of the limbs were so intermingled that only a careful study permitted their collocation, a task, aided, however, by a nearly complete paddle of an allied form, about which there was less doubt. There are nineteen vertebræ in the neck, thirty in the trunk, and twenty-five preserved in the tail, which tapers very abruptly at the extremity. There may have been a few more vertebræ in the tail, though this is doubtful. The skeleton, as mounted, is just ten feet in length.

There are not a few new or strange characters presented by this skeleton, especially in the skull, nearly all of the details of which have been made out, such as the presence of two separate and disconnected supraoccipital bones, peculiar frontal bones, the presence of a quadratojugal, and of a well-ossified sclerotic ring, etc. The pectoral girdle, which, with the pelvic girdle, is marvelously complete and undistorted, has a large interclavicular foramen, never before observed in these animals, though present in other forms. A full description of the skeleton, with illustrations, is already prepared for publication.

It is possible that the present species may prove to belong to some one of the previously described genera, but from a prolonged study of the descriptions extant, and of the nine or ten species actually known to me, I do not think so. I have therefore given it the name of *Dolichorhynchops osborni* in honor of Prof. H. F. Osborn, of Columbia University, the well-known paleontologist.

PLATE XI.



Restoration of *Dolichorhynchops osborni* Willist. One-sixteenth natural size.

# NOTES ON SOME NEW OR LITTLE-KNOWN EXTINCT REPTILES.

BY S. W. WILLISTON.

With Plates XII and XIII.

### Lower Cretaceous Dinosaur.

Among some material collected from the Comanche Cretaceous of Clark county, Kansas, several years ago, by Prof. C. N. Gould, there is a nearly complete centrum of a carnivorous dinosaur allied to *Creosaurus* or *Allosaurus*. It measures 100 mm. in length, 87 in transverse diameter at the extremities, and about 100 in height. The ends are vertically oval and nearly flat; the sides and below are deeply concave in outline. The specimen doubtless represents a new form, but the material preserved is too incomplete to furnish even generic characters.

This is, I believe, the second specimen of a dinosaur that has been discovered in the Cretaceous of Kansas, the other being the type specimen of *Claosaurus agilis* Marsh, discovered by Marsh many years ago in the Niobrara Cretaceous of the Smoky Hill valley, in western Kansas.

### New Benton Turtle. (Protostega ?)

The turtles of the genus *Protostega* have hitherto been known only from the Niobrara Cretaceous of western Kansas, and those of the nearly allied genus *Archelon* from the Pierre of South Dakota. Several years ago a specimen of a large turtle, belonging either to the genus *Protostega* or an allied new one, was collected from a lower stratum of the Benton Cretaceous of Kansas by Prof. W. N. Logan. The specimen consists of some fragmentary ribs and a part of a large humerus, apparently larger than that bone in *P. gigas*. The material does not permit a close comparison, however.

### Mosasaurs.

At different times I have expressed a doubt of the validity of the genus *Holosaurus* Marsh as distinct from *Platecarpus*. An examination of the type specimen of the genus has not wholly dispelled this doubt, but has strengthened my opinion of its invalidity. There is less difference, I believe, between *Holosaurus abruptus* and *Platecarpus coryphaeus* than between *Clidastes velox* and *C. dispar*, and especially between the latter and *C. stenops*, mentioned below, forms presenting similar differences to the chief one separating *Holosaurus* from *Platecarpus*.

*Holosaurus* has no zygosphenes, as I at one time suspected it possessed, nor can I distinguish any difference in the skull as it lies in the matrix with the dorsal surface concealed. It is to be hoped that the specimen may be removed from its matrix at no distant day, and so preserved that all parts may be available for comparison.

There are, however, certain differences in the girdles and limb bones which distinctly remove this species from those of *Platecarpus* hitherto made known. Plate XLI, figs. 2 and 2*a*, of my report upon the Kansas mosasaurs, show the distinctive forms of the pubis and ischium, easily distinguishable from those bones of *Platecarpus coryphaeus*. The specimens there figured I found isolated near Monument Rock, and I was never sure where they belonged, though, from their general resemblance to the bones of *Platecarpus*, I referred them to that genus. These differences are seen in the greater expansion of the distal extremity of the ischium, and in the shape of the proximal end of the pubis. The radius figured in plate LVII, figs. 1 and 2, also belongs with this species, distinguishable by the differences in form from that of *Platecarpus coryphaeus*. I know of no other bones which may be confidently referred to that species, nor are the differences in other bones at all conspicuous or even perceptible. There is, of course, a possibility that the species *H. abruptus* may prove to be identical with some species of *Platecarpus* described by Cope. Evidence of such identity can only be ascertained by a study of the types.

### **Clidastes stenops** Cope.

At the time of the publication of my report upon the mosasaurs of Kansas, I had never recognized a specimen of this species, of which nothing has been published since its original meager description by Cope. Recently a considerable part of a skeleton, which I refer to this species, has been obtained from Mr. Chas. Sternberg, who collected it in western Kansas. The specimen consists of the premaxillo-nasals, maxillary, dentary, splenial, prefrontal, palatine, pterygoids, transverse bones, orbitosphenoids, atlas, axis, and a number of the bones of the anterior girdle and limbs. The bones are in good preservation, and throw not a little light, not only upon this species, but also upon the genus *Clidastes* as well. The specimen is large —larger, indeed, than any other Kansas mosasaur except *Tylosaurus dyspelor*. It differs from the type specimen in size, but I cannot consider this of specific value yet. The prefrontal bone, while not complete, is of a very different type from that found in the other species of this genus. It has no horizontal plate extending over the orbits, resembling in this respect the same bone in *Platecarpus* and *Tylosaurus*. The teeth in the mandible and maxilla are remarkable for their large and conspicuous facets, unequaled in any other mosasaur known to me. The pterygoid has but eight teeth, as in species of *Mosasaurus*, while the number in all the other known species of *Clidastes* is from twelve to sixteen. The palatine bone is remarkably small, while the ectopterygoid is large and stout.

The axis shows scarcely a trace of the zygosphene, but Cope figures it as well developed in the posterior vertebræ. The bones of the paddles are characteristic, as they always are in species of this genus. The coracoid, though incomplete, is evidently emarginate, as in *C. velox*. This character was unexpected, and tends still further to involve in doubt its value in the separation of genera. The radius is very broad proximally, and has its two concave borders of nearly equal length, whereas in other forms the radial border is always much the shorter. The ulna is remarkable for its slenderness, resembling a phalanx rather than an epipodial bone. The first bone in the radial side of the carpus, the radiale, has a very narrow free border; the mediale thus is separated from both the radius and the ulna, which is not the case in any other species of this genus known to me. The first metacarpal also, as usual, is distinctive. The sides are less deeply concave than is usual, and the proximal border is widely expanded.

*Clidastes stenops* brings up the question of the generic characters of the genus in contrast with those of *Mosasaurus*. I have been somewhat skeptical of the value of the characters for several years, and the present species makes me more so. There seems to be absolutely nothing left except the zygosphenes. The number of pterygoid teeth varies between eight and sixteen. The absence of the coracoid emargination in *C. tortor* prevents its use as a generic character.

So long as there are no indications of a zygosphene in certain species and a well-developed one in others, the character may remain as the sole difference between the two genera; but I suspect that, as in *Platecarpus*, species will be discovered in which this character is rudimentary or incomplete. In such case the genus *Clidastes* must be merged into *Mosasaurus*.

An examination of the type specimen of *Edestosaurus wymani* Marsh discloses the fact that the species is not at all distinct from *Clidastes velox*.

The known Kansas species of *Clidastes* are the following:

*C. velox* Marsh.
*C. dispar* Cope.
*C. stenops* Cope.
*C. westii* Williston.
*C. cinerianus* Cope.
*C. liodontus* Merriam.

The last two are yet very imperfectly known and may be identical with each other; or, it is possible that one or the other is identical with some one of the others.

### Orbitosphenoid of the Mosasaurs.

In plate IV, fig. 5, of my report upon the mosasaurs of Kansas (Univ. Geol. Surv., vol. IV), there is given an enlarged picture of an undetermined bone of *Platecarpus*. At the time I could reach no certain conclusion as to its position in the skeleton, and preferred to leave it without comment rather than to guess at its position. In the specimen of *Clidastes stenops* just described, two similar bones, differing only in being a little more slender, were found in the matrix lying upon the axis, and another attempt was made to determine them. That they could not be hyoid bones was certain, since they differed too much from the real hyoid of *Tylosaurus;* nor could they be epipterygoids, bones described by Marsh, but of which I had been very uncertain, since there are no places on the pterygoids and petrosals to admit of their attachment. An examination, aided by Mr. Martin, of the material in the University of Kansas museum, disclosed them in four different skulls. In one the bone lies transversely in the bottom of the supratemporal fossa anteriorly; in another, one was found impressed upon the under surface of the parietal, also transversely. A third was found by the side of the basisphenoid. In the fourth specimen, fortunately, both bones were discovered lying side by side on the under surface of a frontal bone of *Platecarpus*, with the broad or flattened end lying over the two pits or depressions for their attachment. These pits are toward the posterior end of the frontal bone, on the sides of the median line, and narrowly separated by the olfactory groove. They are clearly shown in fig. 3, plate XXVI, as also in plates XII and XV of the above-mentioned work. The other extremity of the bone has not been found in position, but, by placing a well-formed bone in position, this extremity is found to approach or meet the rounded, conspicuous pit on either side of the base of the presphenoid, at the anterior end of the sphenoid, and here is doubtless where it belongs. The bone, as seen in the enlarged figure, terminates below in a cylindrical and truncated extremity, which is either flat or cupped. The other extremity is broad and flattened, with a transverse, curved sutural surface, for lodgment in the frontal pits. The posterior border is deeply concave; the anterior border convex and subangulated near the middle. The inner surface is more flattened than the outer (the one shown in the figure). The two bones lie parallel with each other, almost touching, with only a narrow space between them above for the passage of the olfactory bulb.

Because of this attachment to the frontal bone, it would seem certain that the bone is the orbitosphenoid and not the alisphenoid, especially so if the lower attachment is, as I believe it to be, in front of the optic foramen. However, the same or a similar bone is found among some lizards, though less completely ossified, and with feeble or no attachments. It is called by Cope the "postoptic," provisionally:

"The membranous wall of the brain-case anterior to the petrosal contains an ossification which is of uncertain homology. It reaches or approaches by its superior extremity the frontal, and might hence be supposed to be the orbitosphenoid, but this homology is vitiated by the fact that its inferior portion passes behind the optic foramen. The latter position is that of the alisphenoid, and so the bone is named by Parker. (Trans. Roy. Soc., 1879, p. 615.) But there is another element, the epipterygoid, posterior to it, and immediately anterior to the petrosal, which has been supposed to be the true alisphenoid.[1] Leaving this question, and adopting for the bone in question the provisional name of postoptic, I remark that it is typically triradiate, sending two branches up and one downward," etc.[2]

In one or two instances I have seen a slender rod of bone in *Clidastes*, which I have thought to be the epipterygoid, but I have never been sure. Marsh[3] described the epipterygoid as a rod of bone in *Platecarpus*, and Osborn also identified it as a flattened rod in *Tylosaurus*. It is strange, however, that it does not appear more frequently in the many skulls of these animals discovered in the Kansas Cretaceous.

Lying by the side of an excellent skull of *Clidastes tortor* in the museum there is a slender rod, with a sharply truncated extremity, whose position in the skeleton I do not know, unless it belongs in the hyoidean apparatus. It is too large and too long to be an epipterygoid. The bone is cylindrical, gradually tapering, with the truncated surface oval.

### Platecarpus Species from North Dakota.

In plate XIII is shown an ischium and some paddle bones of a species of *Platecarpus*, recently sent me for examination by Professor Sardeson of the University of Minnesota. The bones were obtained from near Milton, in the northeastern part of North Dakota, enclosed in a blue shale very much like that of western Kansas. Associated with the specimen were others of *Hesperornis*, etc., all evidently belonging to the upper part of the Niobrara Cretaceous. The present specimen differs from any species of *Platecarpus* that I know, in the shape of certain bones, and indicates a distinct species. But I do not know all the species that have been described from the Kansas Cretaceous in this genus, and will leave its determination until further information is forthcoming.

The following notes were made by me after an examination of the type specimens at the Yale museum, two years ago; I

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[1] This is, however, contested by Baur, because of the presence of both the alisphenoid and epipterygoid in *Sphenodon*.

[2] Cope, An. Rep. U. S. Nat. Mus. for 1898, p. 187.

[3] Marsh, Amer. Journ. Sci.

have to thank Professor Beecher for permission to examine them:

*Lestosaurus gracilis* Marsh. The type consists of fragments of the skull and cervical vertebræ. The vertebræ show much compression, as do also the bones of the skull. The fragment of the parietal is normal, the opening rather large. Quadrate somewhat distorted. The specimen is a little smaller than usual. It is apparently identical with *Platecarpus coryphæus*. No. 1264. Three miles east of Fort Wallace, north side of Smoky Hill river, August 31, 1871.

*Lestosaurus simus* Marsh (*Platecarpus ictericus*). No. 1352. Seventeen miles east of Fort Wallace, south side of Smoky Hill, in blue shale, July 26, 1871.

*Rhinosaurus micromus* Marsh. South side of Smoky Hill river, November 2, 1872.

*Edestosaurus dispar* Marsh. North fork (?) Smoky Hill river.

*Edestosaurus velox* Marsh. "Fossil from Sheridan, Kan., collected by Savage, November, 1870." Marsh's handwriting.

*Edestosaurus rex* Marsh. Three miles southeast of Fort Wallace, eight feet below top of blue shale, July 10. Is *Clidastes tortor*.

*Lestosaurus felix* Marsh. Fourteen miles east of Fort Wallace, July 15, E. S. Lane. Is *Platecarpus coryphæus*, apparently, though the sides of the frontal bone show only a moderate emargination.

*Hadrosaurus agilis* Marsh. Twenty-six miles east of Fort Wallace, north side of Smoky Hill river. Is from the Niobrara, the bones showing the characteristic compression.

Dr. M. Janischewsky[4] figures and describes a bone which he believes to be a mosasaur coracoid from "den untertertiaeren Ablagerungen des Gouv. Saratow." I regret my inability to read the Russian text of his article, but, from the resume and the figures, I hardly think the bone can pertain to a mosasaur. The size would be enormous for a mosasaur—fully twice that of any known species. Though incomplete, the shape of the bone is somewhat unlike that of any mosasaur coracoid known, though the resemblance is evident. It seems to be a reptilian

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[4] In Ann. Geol. et Min. de la Russe, v, p. 94, 1902.

coracoid, and its discovery in the "grunlich grauen eocenen Glaukenitsande" is remarkable. It apparently does not have the characteristic coracoid foramen. Further information concerning this Tertiary animal will be looked forward to with interest.

In conclusion, I desire to correct an error made by myself. That which I considered the nuchal fringe in the mosasaurs is evidently only the slender cartilaginous rings of the trachea, first described and figured by Professor Osborn. I have no excuse to make for the mistake, which I recognized when too late to correct.

## Explanation of Plates XII and XIII.

PLATE XII. Upper figure—Part of hind paddle of *Plateearpus coryphæus* Cope, in matrix one; natural size. Lower figure—Part of front paddle of *Clidastes stenops* Cope.

PLATE XIII. Ischium and part of hind paddle of *Platecarpus* sp., from Milton, N. Dak.

# PLATE XII.

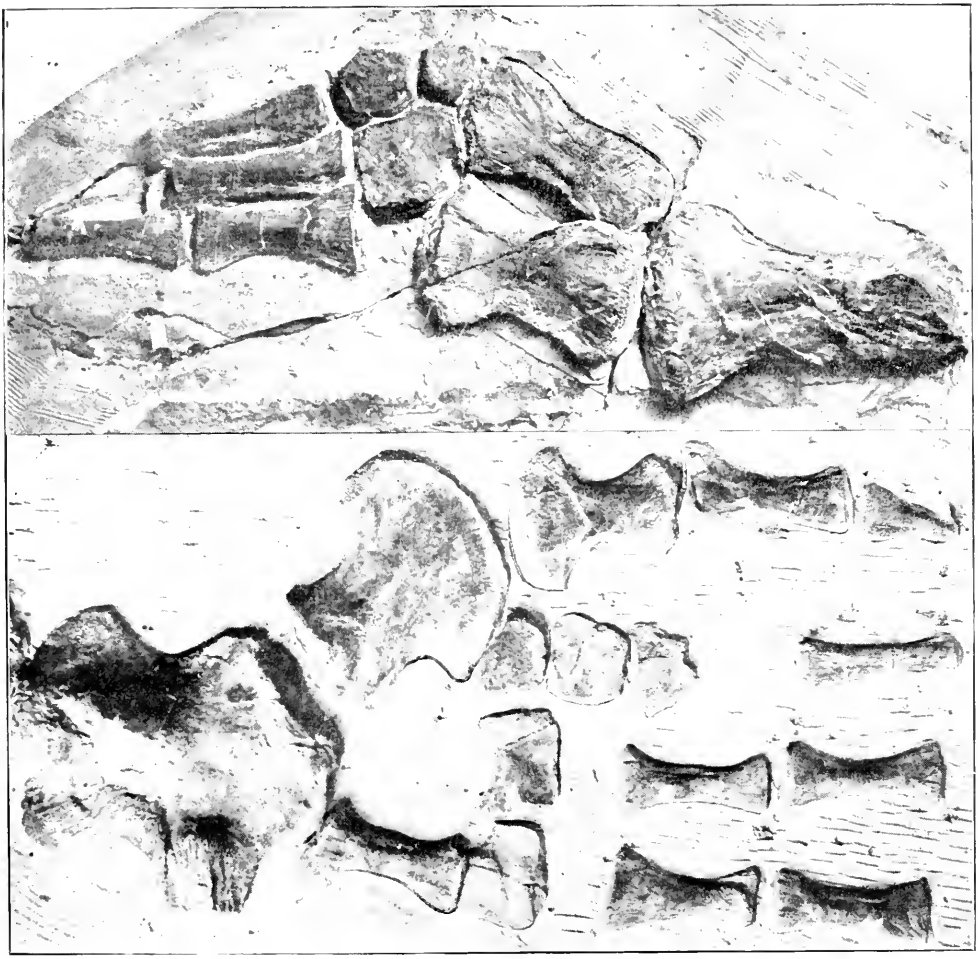
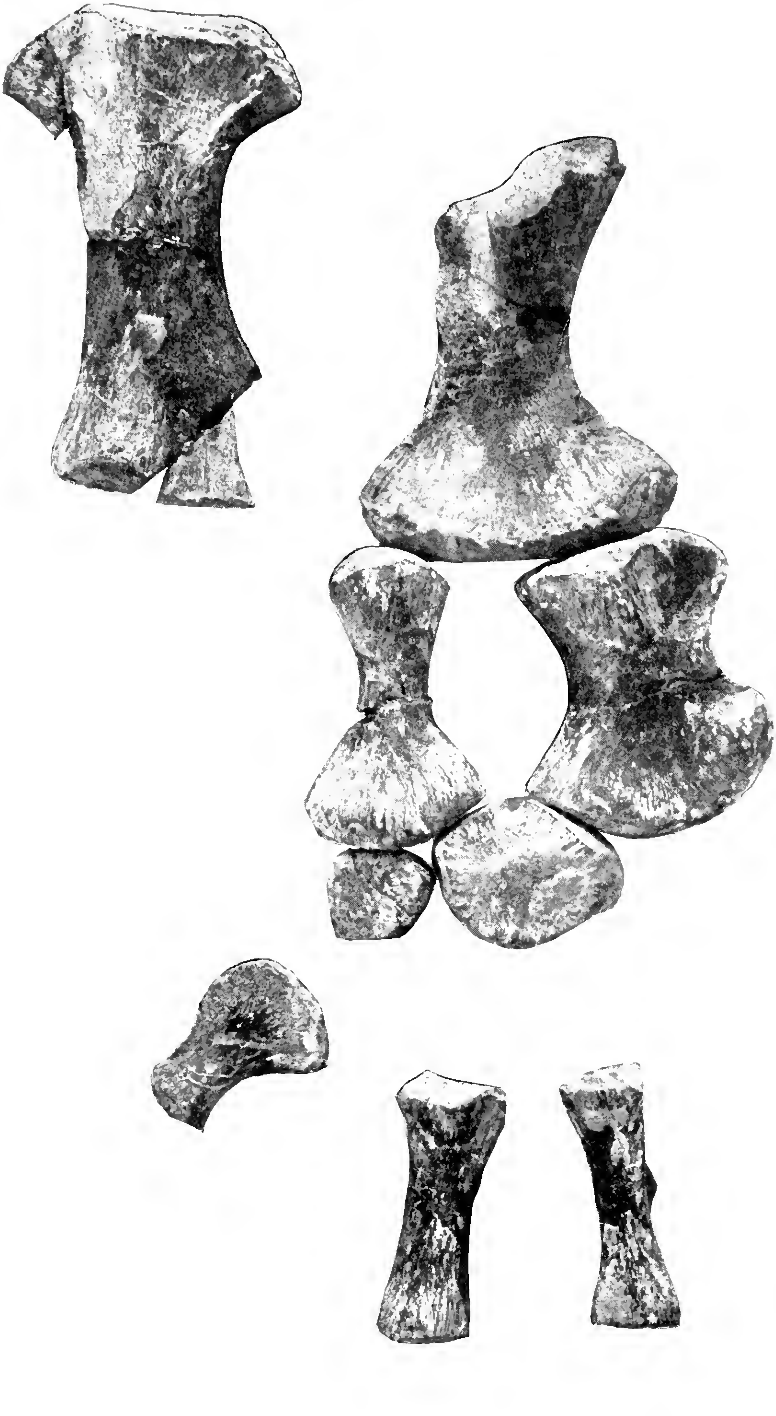


PLATE XIII.



# ON CERTAIN HOMOPLASTIC CHARACTERS IN AQUATIC AIR-BREATHING VERTEBRATES.

BY S. W. WILLISTON.

In the endeavor to reach more satisfactory conclusions regarding the life habits of the plesiosaurs, the following article was written nearly a year ago, since which time I have received the very interesting and valuable paper by Professor Fraas on the thalattosuchian crocodiles,[1] in which I was pleased to find a similar discussion, in large part anticipating what I have to say here. I venture to publish my observations as extending and corroborating his statements, and have added further details gathered from his paper—details bearing materially on the subject-matter.

Among living and extinct vertebrates there are at least six more or less unrelated groups which have become in part or entirely adapted from terrestrial to aquatic existence—the cetaceans, sirenians, ichthyosaurs, plesiosaurs, and certain crocodiles, to which might be added certain testudinates and rhynchocephalians. The pinniped carnivores show a much less complete transformation from the terrestrial carnassidentia, and are of interest here chiefly by reason of the modifications which the limbs have undergone.

The many structural and phylogenetic differences which these diverse orders present render their many "convergent" or homoplastic features of peculiar interest. These features may be for the most part explained by a progressive adaptation to environment. Wherever food habits and environment have concurred the homomorphic results have often been striking. In other forms, however, where the food habits have been dissimilar, modifications have occurred. And it is in these modifications that I have sought some hints as to the probable habits of the plesiosaurs. The more important adaptive characters of these air-breathing vertebrates seem to be the following. Of course they are not all concomitant, but they are all,

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[1] Paleontographica, xlix, p. 3, 1902.

I believe, the result of adaptive changes from terrestrial to aquatic life:

Elongation of the head, with attenuation of the facial region.

Elongation of trunk and tail, but especially of the latter, with progressive weakness of the zygapophysial articulations posteriorly.

Shortening of the neck.

The acquirement of a caudal fin.

The presence of sclerotic plates.

Recession of the external nares.

Absence of sacrum and the obsolescence or absence of the sternum.

Greater slenderness and small size or loss of the hind limbs.

Broadening and shortening of the limb bones.

Hyperphalangism and hyperdactylism.

Smoothness of the skin.

Sponginess of the bones of the skeleton, with a larger proportion of organic matter.

Increase in number and decrease in size or loss of the teeth.

In the fish-eating swimming vertebrate, speed and the ability to quickly change the direction of movement must be essential for the welfare of the individual. These have been obtained by such changes in the form of the body as decrease resistance to the water, and by the development of organs to give strong propulsion and rapid evolution. Propelling power may be furnished by large and strong paddles or flippers, but more effective control and power are given by a long and flexible tail with a fin-like termination. In reality, it is found that, as the tail is large and powerful, the limbs, especially the hind ones, lose their propelling power and become organs of equilibration. A large tail and large paddles are not found together. When, as is sometimes the case, the tail is small, the hind limbs take on more the character of guiding or sculling organs than of oar-like paddles, as is probably the case in many pinnipeds.

The elongation of the head and attenuation of the facial portion are best exemplified by the ichthyosaurs, some plesiosaurs, the semiaquatic gavial, and the more aquatic geosaurs, and perhaps, also, by some of the rhyncocephalians—all, probably, with like food habits. Among some of the odontoceti only is there an approach to the reptilian slenderness of the face; an example of such is the dolphin of the Ganges, a fish-eating cetacean. This type of head is best illustrated in the gar-pike, a surface-feeding fish, whose food consists almost exclusively of small fishes, which are caught by quick movements of the body and jaws rather than by rapid swimming. I doubt very much whether any of the reptiles before mentioned had the ability to move with any great degree of speed through the water.

In the long-necked plesiosaurs the head is broader, flatter, and less elongate, with the teeth larger and more unequal in size. In the mosasaurs the skull is larger, less elongate, though still of a distinctly elongate type, especially in the more specialized forms, such as *Tylosaurus*. The teeth are larger, are less firmly implanted in the jaws, and there are additional teeth upon the pterygoids, wanting in the other aquatic vertebrates. Furthermore, the mandibles have a peculiar lateral articulation; all of which characters indicate larger and more powerful prey. Indeed, remains of fishes of considerable size have been found in the fossilized stomach contents of these animals.

Plesiosaurs, or at least some of the long-necked forms, had the strange habit of swallowing siliceous pebbles, sometimes of considerable size and in considerable quantities; as many as a peck have been found among the remains of a single animal. Seals and crocodiles are said to have similar habits; the reason therefor is hard to conjecture. These pebbles, moreover, were carried in the plesiosaur's stomach until reduced to very small dimensions, indicating clearly the smallness of the pyloric orifice and the complete digestion of the food material. Coprolites, evidently of mosasaurs, are frequently found, with large, undigested fish bones, and fragments of fish bones, and sepia shells have been found in reputed ichthyosaurian coprolites.

The posterior position of the external nares, not only in the cetaceans, but also in the ichthyosaurs, plesiosaurs and parasuchian "crocodiles," is doubtless due to environmental causes. In the mosasaurs the nares are large, but are located anteriorly, though not at the extremity of the snout. Why they should be so small and situated so far back in the ichthyosaurs, plesiosaurs and *Belodon* is not evident.

In all the aquatic reptiles sclerotic plates are well developed. They are present in the plesiosaurs, a fact hitherto published, though modern text-books still continue to affirm their absence in these reptiles. They doubtless furnished better protection for the eyes under varying pressure, and perhaps aided vision, both of which would indicate that these animals were in the habit of going to considerable depths.

The shortening of the neck is carried to the extreme in the ichthyosaurs and cetaceans, the most specialized of aquatic air-breathing vertebrates. In the mosasaurs there has been a material reduction in the length of the neck and in the number of the vertebræ from the ancestral Dolichosaur-like squamate. A similar reduction in length and in number (seven) is also found in the thalattosuchian crocodiles, according to Fraas. Among the plesiosaurs there is extraordinary variability in the length of the neck and in the number of the vertebræ. In such forms as *Elasmosaurus*, from the Pierre Cretaceous, as many as forty vertebræ are credited to the cervical region, and the most of them of an elongated shape. In *Dolichorhynchops*, on the other hand, there are not more than twenty vertebræ in the same region, and all are very short. Both types may have been contemporaneous in the latter time of the plesiosaur's existence. How far the correlation of short neck and elongated skull obtains in this order I do not know, but I believe that such is the rule. I can conceive of no terrestrial form with so extraordinary a neck as have some of the plesiosaurs, and the conclusion is unavoidable that there has been an actual increase in the number, even as it is certain that there has been an increase in the number of vertebræ in the more modern snakes. It is also evident that there has been a decrease in number from the long-necked to the short-necked plesiosaurs. Perhaps in no other order of vertebrates has there been such great variability in the number of cervical vertebræ as in the plesiosaurs.

The elongation of the trunk is constant in the more specialized aquatic types, and there is in such a progressive weakness or obsolescence of the zygapophysial articulations posteriorly, giving greater freedom of movement laterally. The elongation of the tail and the development of a distal fin are correlated with the weakness or absence of the hind limbs, as fins or paddles. In the mammals the fin is placed horizontally, more for the purpose of deep-sea diving than for speed or quickness of evolution, which are needed in but few of these animals. In

*Ichthyosaurus* and *Geosaurus* the downward curvature of the distal caudal vertebræ gave support to the lower side of the caudal fin. In the mosasurs the tail is not decurved, and the dilatation was evidently not terminal but along the distal part. In the plesiosaurs the tail is never greatly elongated, and in some was probably short. There was, however, a terminal, fin-like dilatation, and, in some forms at least, the distal caudal vertebræ were curved upward.

The tendency toward a metamorphosis of the ambulatory into natatory limbs is constant; it is always more pronounced in the anterior than in the posterior extremities. In the cetaceans and sirenians the hind limbs have disappeared entirely in the living forms, though the earlier, extinct sirenians were also provided with hind extremities. In the ichthyosaurs the hind limbs are always decidedly smaller than are the front ones, while in the mosasaurs a similar, though less pronounced, weakening of the hind limbs is evident, particularly in the more specialized groups like the Tylosaurinæ. Among the plesiosaurs the hind extremities are always powerful and well developed, though in all the forms known to me they are less powerful than the front ones. Here the relative shortness of the tail has imposed a distinct propelling or sculling function upon the hind limbs, as is the case to a much greater degree among some of the pinnipeds. In *Geosaurus*, on the contrary, the hind limbs are larger than the fore limbs, while the tail is elongated and provided with a terminal fin, evidently of no small size. But these legs in *Geosaurus* were distinctly ambulatory and of little use in the water; they were doubtless carried close to the tail in swimming, as in the partly metamorphosed tadpole.

Hyperphalangism is best exhibited in the cetaceans, ichthyosauria, and plesiosauria; and in the ichthyosaurs there is hyperdactylism also. With our present knowledge, it would seem remarkable that serious argument was needed to controvert Gegenbaur's views as to the ichthyic affinities of the paddles in these animals. Among the mosasaurs hyperphalangism is more pronounced in the more specialized *Tylosaurus* than in the other groups. While in *Mosasaurus* the fifth digit has been lost in the hind foot, and in other forms is reduced in size, inherited terrestrial characters, in *Tylosaurus* the fifth toe has become greatly elongated, and its phalanges increased in number commensurate with those of the other digits.

With hyperphalangism has occurred also a shortening of the limb bones, especially those of the front pair, which are the first to show natatorial characters. There has also occurred a restriction of articular movement, with imperfect joints. Those ichthyosaurs (*Mixosaurus*) and plesiosaurs (*Plesiosaurus*) which have preserved an elongated form of the epipodial bones are clearly more generalized than those in which the bones have become mere polygonal platelets, a contestation long ago made by Baur in support of the terrestrial derivation of the ichthyosaurs. Concurrent with such changes is the presence often of three or even four elements in the epipodial series. The attempt to correlate these with mesopodial elements of the terrestrial vertebrate is, I believe, futile. Accessory, sesamoid, mesopodial, or whatever they may be called, they are in reality new growths, like the sixth and seventh digits of the ichthyosaur paddle, or the supernumerary digits of the human hand or foot, and not misplaced carpals and tarsals. It would not surprise me greatly to learn of plesiosaurs with additional digits.

In *Geosaurus* there is a distinct beginning of the aquatic paddle in the fore limb, although there is scarcely any change from the ambulatory limb behind. As an example of a stage through which probably all other aquatic vertebrates have passed, the thalattosuchia are very instructive. In those aquatic lizards, the mosasauria, we find a more advanced stage, with shortening of the bones and hyperphalangism more or less pronounced, but the widely spreading digits, the relatively large size of the hind limbs, the presence of a sternum in the more generalized forms at least, and the less elongated head, are all indications of a lesser specialized stage than is seen in the cetaceans, ichthyosauria, and plesiosauria.

In the cetacea and sirenia, the sternum, though reduced, is not entirely lost, nor is it lost in the mosasaurs, at least in the more generalized groups. It has not suffered in the aquatic crocodiles, but in the ichthyosaurs and plesiosaurs it has vanished utterly. The sacrum is wanting in all the true aquatic types. No longer needed for the support of the body in ambulation, the united vertebræ, with rigid or close union with the ilium, must have interfered with free movements of the body, and its loss was to be expected In *Geosaurus*, with its ambulatory hind limbs, the sacrum has not disappeared. The ilium, where present, is small, merely a connecting rod, united by ligament to some transverse process. In the mosasaurs it is directed forward from the acetabulum; in the plesiosaurs, on the contrary, its position is upward and backward.

Why the pectoral and pelvic girdles should have been so remarkably developed on the under side of the body in the plesiosaurs is not clear, forming, as they do, a nearly complete, convex, bony armor on the under side of the trunk, supplemented by the abdominal ribs. In the ichthyosaurs, with the less broad expansion of their girdles, the more numerous ventral ribs may have served a similar end. Abdominal ribs are found in the crocodiles, but there are none in the mosasaurs.

It is a remarkable fact that in all aquatic vertebrates, except the bottom-feeding sirenia, and the ichthyosaurs, of which I can get no information, the bones of the skeleton have become more cancellated and spongy. The oily character of the pinniped bones is well known, as also the great sponginess of the cetacean bones. Among the plesiosaurs a similar soft and friable structure of the bones is painfully evident, and in the mosasaurs there is a very distinctly greater softness and sponginess of the bones in the more specialized types.

Nakedness of the skin is of course to be expected among swimming animals; hair has almost disappeared in the mammals, and scales or plates in the reptiles; in the mosasaurs alone scales are still persistent, but they are relatively small and smooth.

It is very evident that the plesiosaurs were not quick-swimming animals; the body was relatively broad, flat, and thick-set, very unfish-like in form; the paddles were small as compared with the size of the body, and the tail was not of sufficient capacity to compensate for their incapacity. The more or less elongated neck—for in none was the neck short in the sense of the ichthyosaurs—did not give any advantages in movement. The movements of these animals must have been more like those of the turtles than of the mosasaurs or crocodiles. The head was relatively small, and the narrow gullet, between unyielding bones, permitted the deglutition of small animals only—very small indeed as compared with the size of the body. The spongy structure of the bones clearly indicates that they were not bottom feeders, like the sirenians. At the same time they must have traveled or migrated long distances in the water. The siliceous pebbles, so often found with their remains in the Kansas chalk, could only have been obtained hundreds of miles away, for they are absolutely unknown away from the remains of these animals.

The ichthyosaurs are known to have fed upon cephalopods and other invertebrates, if the coprolites ascribed to them are really ichthyosaurian. I believe that the plesiosaurs, or many of them, fed largely upon such food, living along the shores in comparatively shallow water, and were not accustomed to go to considerable depths; that is, they lived for the most part near the surface of the water. Evidence presented by Seeley years ago seems to indicate that they were viviparous in habit like the ichthyosaurs, though such evidence has not been forthcoming in America. It is a singular fact that embryonic propodial bones of plesiosaurs are not at all rare in the Kansas chalk. There are five such bones in the Kansas museum collection, and I have seen perhaps a score of others. These bones are always found isolated, and were for a long time very problematical. In some, the peculiar epiphysial structure is beautifully shown. In all, there are several foramina perforating the middle, with a groove leading therefrom corresponding to the ent- or ectepicondylar groove of the testudinates and mosasaurs. The conical 'epiphyses' terminate at these canals.

"Entsprachen die soeben gegebenen Ausführungen den Thatsachen, so kann man sich die Plesiosaurier auch nicht länger als auf der Oberfäche des Meeres, etwa wie Schuimmvögel lebende Geschöpfe vorstallan, sondern sie haben, wie die Ichthysaurier, unter der Oberfläche im Meera gelebt."[2]

This conclusion of Dames I have no doubt is correct. His restoration (l. c., p. 79), I believe, is somewhat faulty in the two angular caudal expansions and in the pedunculation of the limbs. I believe the latter were broader proximally, as was certainly the case in both the ichthyosaurs and mosasaurs.

It is evident from the great variations in form among the plesiosaurs that they varied not a little in habits. Though exceeding in size the largest mosasaurs, I do not think that they were ever a match for them in prowess or voracity. Like the gavials, they were comparatively harmless creatures.

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2. Dames, Abh. Konigl, Preuss. Akad. S. Wissensch., 1895, p. 79.

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VOL. III. No. 8.

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# UNIVERSITY OF KANSAS.

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# SCIENCE BULLETIN,
## Vol. I, Nos. 10, 11, and 12.

(Continuation of Kansas University Quarterly.)

DECEMBER 1902.

Entered at the Post-office as Second-class Matter.

# ANNOUNCEMENT.

For ten years the University of Kansas has maintained the *Kansas University Quarterly*, a journal devoted to the publication of the results of research by members of the University of Kansas. At the opening of the second decade of the journal's existence it has been thought best to begin a new series, with some important changes in the name, manner and time of publication. It has been decided to abandon the quarterly form of issue, and instead to publish the separate papers as soon as they are ready for the printer. By this means the results of research will be more promptly given to the public, and in a form more satisfactory both to the authors and subscribers.

Hereafter this journal will be called the *Kansas University Science Bulletin*, and will be issued at irregular intervals. Each volume will contain from 300 to 400 pages of reading-matter, with the necessary illustrations; and approximately one volume a year will be issued. The amount of scientific matter thus published will be not less, and probably more, than that formerly appearing in four issues of the *Quarterly*. The subscription price will be hereafter $3 per volume; the price of each part will vary with the cost of publication. We hope our present large exchange list will remain intact, and new exchanges are solicited. All exchanges should be addressed to the Library of the University of Kansas.

Communications should be addressed to

H. B. Newson,
Managing Editor of Science Bulletin,
Lawrence, Kan.

# THE

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KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. I, No. 10. NOVEMBER, 1902. { WHOLE SERIES, VOL. XI, NO. 10.

# SOME NEW AMERICAN SPECIES OF CYCLUS FROM THE COAL MEASURES.

BY AUSTIN F. ROGERS.

With Plate XIV.

The xiphosuran genus *Cyclus*, founded upon *Agnostus? radialis* Phillips, by De Koninck, in 1841, is now represented by fourteen European species—eleven from the Coal Measures of Great Britain, (one species of which is also known from Belgium), one from the Permian of Sicily, and two from the Muschelkalk (Triassic) of Germany. A single American species, *C. americanus* Packard, has been described from the Lower Coal Measures of Mazon creek, Illinois, and as far as can be learned this is the only occurrence mentioned of the genus in this country.

During the past few years the writer has collected from the Upper Coal Measure limestones of western Missouri and eastern Kansas a considerable number of specimens of this interesting genus which are referred to five new species herein described. They occur for the most part in the Iola limestone* at Kansas City, Mo., associated with typical Coal Measure fossils.†

Of especial interest is the discovery of the compound eye in *Cyclus*. This I have found in several specimens of one of the species, *Cyclus communis*, n. sp., *postea*. It is rather large, oval or elliptical in shape, and is situated near the anterior lateral border (see figure 1*a*). In one specimen the facets are, with the aid of a compound microscope, plainly seen to be hexagonal in outline and greatly resembling those of a trilobite. The simple eye (sometimes stalked) has been noted by Woodward and other observers, but as far as can be learned this is the first mention of a compound eye.

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*No. 98, Broadhead's Gen. Sect. Coal Meas. Rocks of Missouri.

†For a full list of the fossils of this limestone, see a paper by the author, "Coal Measure Faunal Studies, I," this journal, vol. IX, pp. 245-247, 1900.

### Generic Diagnosis of Cyclus De Koninck, 1841.

Cephalothorax small, disk-shaped to subhemispherical, bilaterally symmetrical, usually with a median ridge posteriorly, circular to elliptical or oval in outline; ornamentation consisting of lobes, ribs, "forks," or nodes, often with granules occupying more or less of the surface of the carapace; border entire, or with posterior projection or indentation; margin usually present, varying in width and character; compound eyes oval, situated at the antero-lateral border, much resembling those of a trilobite; appendages nearly as in those of embryonic *Limulus*.

**Cyclus communis**, n. sp. Plate XIV, figs. 1, 1*a*.

Carapace subhemispherical, circular in outline. Lateral profile view a semioval, the highest point being a little back of the center of the carapace. Border entire, with a narrow, nearly vertical margin, which is separated from the main part of the carapace by a shallow furrow. A narrow, inconspicuous median ridge extends from the posterior margin to the center of the carapace. On each side of its anterior half is a narrow furrow, each of which on reaching the end of the ridge continues in an antero-lateral direction for a short distance and then again becomes parallel to the median ridge. The diverging furrows thus form a polygonal area, open anteriorly. Within this area are situated three inconspicuous very slight elevations, which are symmetrically disposed with reference to the median line of the carapace, a single unpaired one posterior to the paired anterior ones, all approximately equal in size. The surface of the carapace is thickly covered with rather coarse granulations, with the exception of the anterior portion, which is almost smooth. In one specimen the coarse granulations and their interspaces are occupied by still finer granulations. On the antero-lateral border is a prominent oval or elliptical-shaped compound eye with distinct facets.

| Measurements. | Eudora specimen. | Kansas City specimen. |
|---|---|---|
| Length | 6 mm. | 4½ mm. |
| Width | 6 mm. | 4½ mm. |
| Height | 2½ mm. | 2¼ mm. |
| Width of margin | | ½ mm. |
| Compound eye | | ½ mm. × ¾ mm. |

Position and locality: Upper Coal Measures, Iola limestone, Kansas City, Mo.; Lower Garnett limestone,* Eudora, Douglas county, Kansas. Many specimens.

This species resembles in several respects *C. jonesianus* H. Woodw. but presents the following differences: It is a little smaller; is circular in outline instead of elliptical; the anterior part of the carapace

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*For the other fossils occurring in this limestone, see paper by the author, l. c.

is smooth, and the granulations are more prominent than in *C. jonesianus*.

*C. communis* is much more abundant than the other species here described, though good specimens are rare. About eighty specimens, most of them very imperfect, were obtained from the Iola limestone, whereas from only one to three specimens of the other species were obtained. Some of the specimens are quite small and not nearly so tumid as the types. These may possibly represent another species, but it is more probable that they are young or stunted forms.

**Cyclus packardi**, n. sp. Plate XIV, figs. 2, 2*a*.

Carapace rather flat, decidedly shield-like, transversely subelliptical in outline. The edge of the carapace is concealed by the matrix, so that it cannot be said whether or not a margin is present. A well-defined median ridge reaches from the posterior border to the center of the carapace, where it expands into a small subcircular prominence, with a slight horizontal carina. Directly in front of the expansion is a quinquelobate prominence, consisting of a median anterior lobe and two paired posterior ones. Of these lobes, the middle or anterior paired ones are the largest and project laterally over the others. The posterior pair are occupied with several transverse ribs. Arising on each side of the median ridge, near its middle, is a ridge extending toward the antero-lateral border, where it expands to form a prominent oval-shaped lobe. Distad of this last-mentioned ridge and separated from it by a narrow furrow is a sharp, narrow ridge also extending antero-laterally, but after a short distance it curves forward, crossing the other ridge, and then expands to form a subtriangular lobe, which is but a short distance from the lateral surface of the quinquelobate prominence. These ridges are the "inner fork" and "outer fork" of Reed.*

The more elevated parts of the carapace, with the exception of the quinquelobate prominence and the median ridge, are covered with small tubercles which are sometimes rather prominent.

| Measurement. | Largest specimen. | Smallest specimen. |
|---|---|---|
| Length | 7½+ mm. | 5½ mm. |
| Breadth | 11 mm. | 8 mm. |
| Height | (cannot be determined.) | |

Position and locality: Upper Coal Measures, Iola limestone, Kansas City, Mo. Three specimens.

*C. packardi* does not resemble any other described species of the genus. In the disposition of the "inner" and "outer forks" it is a

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*Geol. Mag., vol. X, p. 64, 1893.

little like *C. woodwardi* Reed, but the ribs on the latter species give it an entirely different appearance. The lobes on the anterior part of the carapace are quite different in number and character.

**Cyclus permarginatus**, n. sp. Plate XIV, fig. 3.

Carapace rather flat, disk-shaped, subcircular in outline, with a wide margin. A faint, interrupted median ridge extends along in a shallow furrow from the posterior border to a point near the center of the carapace. At the latter point the furrow divides dichotomously and the branches proceed in an antero-lateral direction, the whole forming an indistinct, Y-shaped depression. On each side of the median ridge or furrow are situated six or seven ribs reaching to the margin. The posterior ribs have an oblique direction, and are more prominent than the anterior ones, which can scarcely be recognized except near the border.

The anterior surface of the carapace bears several slight elevations, but nothing definite concerning them can be made out. A wide, flat, smooth margin extends around the lateral and posterior borders. The anterior border is hidden by the matrix. Only small portions of the outside surface remain. It is apparently occupied by small pits.

Measurements: Length, 7 mm. + ; width, 8 mm.; height, 1.5 mm.; width of margin, 1 mm.

Position and locality: Upper Coal Measures, Iola limestone, Kansas City, Mo. Two specimens.

The margin of this species is relatively wider than that of any other described species of the genus. It is more like *C. americanus* Packard than any of the other species herein described, but that species is larger, its margin is narrower, and it is apparently not ribbed.

The radiating ribs give our species an appearance somewhat like that of *C. radialis* Phillips, but it is flatter than that species, and its ribs are not nearly so prominent. There is no flattened margin in *C. radialis*. *C. permarginatus* bears some resemblance to a specimen from Ayrshire, figured by Woodward* and referred by him to *C. radialis*. He states that the specimen is imperfect around the border. If we add a border of one millimeter, the measurements will be quite near those of our species, but as no profile view is given its height cannot be judged. At any rate, our species is not *C. radialis*.

*C. harknessi* H. Woodw. and *C. woodwardi* Reed are also ribbed, but the prominent lobes on those species give them an aspect entirely different from that of our species. Moreover, they are quite gibbous. There is also some resemblance to *C. johnsoni* H. Woodw., but there

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*Geol. Mag., pl. XV, fig. 2 (opposite p. 530), 1894. Measurements: Length, 6 mm.; breadth, 5 mm.

are no ribs, and the margin is not as wide as it is in our species. Besides, in our species the margin is not separated from the body of the carapace by a furrow.

**Cyclus limbatus**, n. sp. Plate XIV, fig. 4.

Carapace flat, disk-shaped, short transversely elliptical in outline. Margin quite narrow, with a fringe of slender spines which vary in size and length. About twelve can be counted on one side of the carapace, but this does not include the anterior border which is partially embedded in the matrix. The spines are flat and pointed at the end. The largest and longest one occurs at the middle of the lateral border. There is a slight posterior median projection of the carapace, and from the appearance it seems that a spine was broken off here. If such be the case we have here in *Cyclus* the embryotype of *Limulus*, suggestions of some of the characters of the latter genus, namely, the telson and the genal spines.

The ornamentation of the carapace consists of a series of small, inconspicuous nodes surrounding the postero-lateral borders. Centrad of these is another row, made up of ill-defined elongate nodes slightly larger than the last-mentioned ones. Next is a pair of low ridges or a "fork," near the anterior end of each of which is a rather large oval lobe, perhaps the most conspicuous feature of the carapace. At the anterior border, near its middle, there is on each side a smaller lobe, and in the V-shaped space, between these and the fork, are several very inconspicuous elevations.

Measurements: Length, 3 mm.; width, 3.5 mm.; height, 0.75 mm.; length of longest spine, 0.5 mm.

Position and locality: Upper Coal Measures, Iola limestone, Kansas City, Mo. One specimen.

This species is unique in the possession of a fringed margin.

**Cyclus minutus**, n. sp. Plate XIV, fig. 5.

Carapace very small, longitudinally elliptical in outline, rather flat, and apparently without a margin. A narrow, rather sharp median ridge extends from the posterior border nearly one-half the length of the carapace. Distad of the anterior part of the median ridge there are on each side two other ridges wider than the median one. There the "inner fork" and the "outer fork" of Reed and their expanded anterior ends are much like those of *C. packardi*, n. sp., *antea* (the description of which see). Partly enclosed by the inner fork and in front of the median ridge are slightly elevated nodes, a posterior median one and two paired anterior ones. Around the anterior border are three inconspicuous nodes smaller than those just mentioned. There is a median anterior one and two paired posterior ones.

Measurements: Length, 1.8 mm.; width, 1.25 mm.; height, 0.5 mm.

Position and locality: Upper Coal Measures, Iola limestone, Kansas City, Mo. Three specimens.

This is the smallest described species of the genus. In some respects it resembles *C. packardi*, n. sp., *antea*, and may possibly be a young form of that species. But in the absence of intermediate forms, and as the specimens seem to be perfectly formed, it is designated as a distinct species.

Our species also resembles *C. jonesianus* H. Woodw., but it is much smaller, less tumid, the arrangement of the anterior lobes is different, and the furrow separating the inner and outer forks is much more prominent.

### Analytic Key to the American Species of the Genus *Cyclus*.

| | | |
|---|---|---|
| 1. | With ribs. Wide, flat margin | *permarginatus*. |
| | Without ribs | 2 |
| 2. | Circular or subcircular in outline | 3 |
| | Transversely elliptical in outline | 4 |
| | Longitudinally elliptical in outline, carapace very small | *minutus*. |
| 3. | Subhemispherical, with narrow, nearly vertical margin | *communis*. |
| | Disk-shaped, with wide, horizontal margin | *americanus*. |
| 4. | Carapace rather large, surface with lobes and small tubercles | *packardi*. |
| | Carapace small: surface with lobes but no tubercles: fringed margin | *limbatus*. |

### Alphabetical List of Described Species of the Genus *Cyclus*, with Synonymy.

**Cyclus agnotus** von Meyer.
*Halicyne agnotus* von Meyer, 1847.
*Cyclus agnotus* H. Woodw., 1870.

**Cyclus americanus** Packard.
*Cyclus americanus* Packard, 1885, 1886.

**Cyclus bilobatus** H. Woodw.
*Cyclus bilobatus* H. Woodw., 1870, 1878.

**Cyclus harknessi** H. Woodw.
*Cyclus harknessi* H. Woodw., 1870, 1878.

**Cyclus johnsoni** H. Woodw.
*Cyclus johnsoni* H. Woodw., 1894.

**Cyclus jonesianus** H. Woodw.
*Cyclus jonesianus* H. Woodw., 1870, 1878, 1891.

**Cyclus laxus** von Meyer.
*Halicyne laxus* von Meyer, 1847.
*Cyclus laxus* H. Woodw., 1870.

**Cyclus radialis** Phillips.
*Agnostus? radialis* Phillips, 1836.
*Cyclus radialis* De Koninck, 1841, 1842: H. Woodw. and Salter, 1865; H. Woodw., 1868, 1870, 1894.

**Cyclus rankini** H. Woodw.
*Cyclus rankini* H. Woodw., 1868, 1870, 1878.

**Cyclus reussi** Gemmellaro.
*Paraprosopon reussi* Gemmellaro, 1890.
*Cyclus reussi* Frech, 1900.

**Cyclus scotti** H. Woodw.
*Cyclus scotti* H. Woodw., 1893, 1894.

**Cyclus testudo** Peach.
*Cyclus testudo* Peach, 1883; H. Woodw., 1894.

**Cyclus torosus** H. Woodw.
*Cyclus torosus* H. Woodw., 1870, 1878.

**Cyclus woodwardi** Reed.
*Cyclus woodwardi* Reed, 1893; H. Woodw., 1894.

**Cyclus wrighti** H. Woodw.
*Cyclus wrighti* H. Woodw., 1870, 1878.

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Woodward, H., 1868.—British Association Reports, Norwich, p. 72.
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Woodward, H. (and Salter, J. W.), 1865.—Cat. and Chart. Foss. Crustacea, p. 15, f. 14.

In conclusion, I wish to thank Mr. Gilbert Van Ingen for aid in the preparation of this paper.

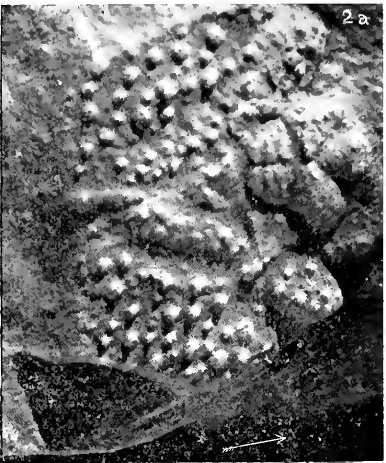
COLUMBIA UNIVERSITY, IN THE CITY OF NEW YORK,
FEBRUARY 10, 1902.

## Explanation of Plate.

(The pointed end of the arrows on the plate indicates the anterior ends of the carapace.)

FIG. 1. *Cyclus communis* Rogers.
  1. Dorsal view of carapace. × 6.
  1*a*. Profile view of carapace (diagrammatic), showing position of facetted eye and slight groove separating margin from carapace proper. × 7.
FIG. 2. *Cyclus packardi* Rogers.
  2. Dorsal view of carapace. × 3.
  2*a*. Dorsal view of portion of carapace of another specimen, showing surface detail; the broken parts of the carapace overlap. × 6.
FIG. 3. *Cyclus permarginatus* Rogers.
  3. Dorsal view of carapace. × 6.
FIG. 4. *Cyclus limbatus* Rogers.
  4. Dorsal view of carapace. × 7.
FIG. 5. *Cyclus minutus* Rogers.
  5. Dorsal view of carapace. × 7.

# PLATE XIV.



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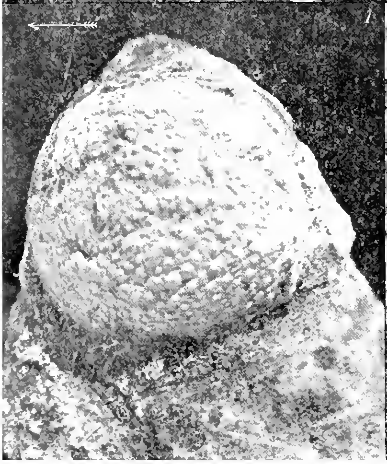

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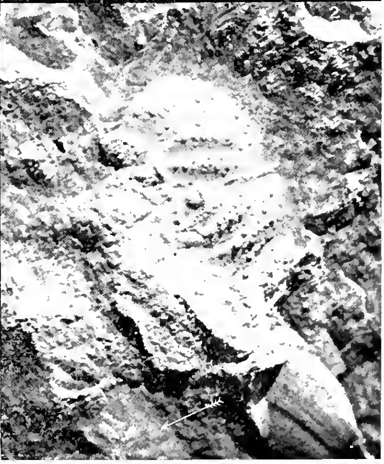

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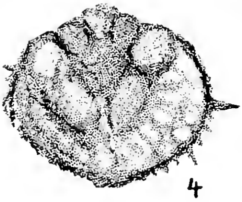

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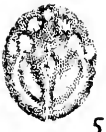

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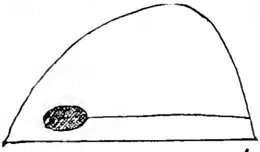

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# COLLINEATIONS OF SPACE WHICH LEAVE INVARIANT A QUADRIC SURFACE.

BY HELEN B. BREWSTER.

1. In an article published in the "Annals of Mathematics," second series, Vol. II, No. 4. Dr. Ruth G. Wood has discussed "Collineations of Space which Transform a Non-degenerate Quadric Surface into Itself." In his "Theorie der Transformationsgruppen," Vol. III, pp. 251–254, Lie has given a table of groups of transformations having the above property, these groups having been developed analytically. Prof. H. B. Newson has developed a theory of collineations of space, his work being synthetical.

The object of this paper is to follow the last-named method, and thereby determine a table of all groups of collineations of space which leave invariant a non-degenerate quadric surface. This method has been found to admit more thorough classification than that of Doctor Wood.

An acquaintance of the reader with Professor Newson's work has been assumed, the language used being practically in complete agreement with that of Newson's writings; but it has been found to be impracticable to attempt the same symbolic notation. No effort has been made to repeat proofs given by Newson, and references to his paper are frequent. My sincere thanks are due to Professor Newson for the careful consideration he has given this work and the time he has spent in the correction of errors.

A close reading of the following table of symbols will obviate the danger of vagueness in the language of the remaining work. The pre-exponent and subscript of a given group denote, respectively, the type and number of parameters of the group; the invariant figure is denoted by the symbols enclosed in the parentheses, while constants of the transformation are given as subscripts to the parentheses. Thus, ${}^1G_3(\mathrm{lmF})_a$ denotes a three-parameter group of type I, having for invariant figure a quadric surface and one generator of each system on that surface, together with a constant relation between the cross-ratios of the one-dimensional transformations along these generators. We have indicated the various parts of the invariant figure as follows:

T = transformation.
$^{ex}G_{sub}$ = group of type (ex.); parameters (sub.)
F = "Fläche," a non-degenerate quadric surface.
l, l′ . . . . : m, m′ . . . ; = generators of first and second systems of generators on the surface, respectively.
$\lambda$; $\mu$ = all generators of first and second systems on the surface, respectively.
n, n′ = reciprocal polar lines with regard to the quadric.
t, t′ . . . . = lines tangent to surface.
*l*, *m* . . . . = lines of invariant points in their respective positions as generators, etc.
A, B, C, D = points on the surface.
P = point without the surface.
p = tangent plane at *A* to surface.
s = polar plane to P with regard to surface.
K, K′ . . . . = conics on surface.
$\bar{x}$ = variation of the portion indicated as x.
$\overline{\underline{\quad}}$ = contains.

## **A.—Transformations of Six Types. Number and Groups.**

### § 1. TYPE XI.

2. *Fundamental group.*—If we consider the invariant figure of a transformation of space of type XI (Newson's classification), we find that this consists of a straight line, together with all points on, and all planes through, that line; and a pencil of lines in each plane, the vertices of which are distinct, but all lying on the axis of the pencil of the invariant planes. These lines, then, form the generators of a system of quadric surfaces, on each of which are left invariant all the generators of one system, and the generator, common to all, of the other system; therefore, also, the common tangent plane. In this work we shall choose one of the quadric surfaces and neglect any changes that may occur on the other quadric surfaces of the system. This group we shall call $^{11}G_1(l\mu F)$—the only parameter being the constant of the parabolic, one-dimensional transformation along each of the generators of system $\mu$. The one-dimensional transformation along *l* is identical.

3. *Number of transformations.*—Of these groups there are double infinity leaving invariant the chosen quadric surface, each group

leaving invariant, in addition, all the generators of one system and one generator of the other system; *i. e.*, there is one such group for each generator of the given quadric surface. Hence, there are $2\infty^2$ transformations of type XI, of space, leaving a quadric surface invariant.

4. *No higher groups.*—These $2\infty^2$ transformations do not form a two-parameter group; for if we take the resultant of two transformations, ${}^{11}T(l\mu F)$ and ${}^{11}T_1(l_1\mu F)$, in which the invariant figure of the one differs from that of the other only in the position of the generator $l$, we have along each generator of the system $\mu$ two one-dimensional parabolic transformations without common invariant point, which generally do not result in a parabolic transformation; hence, the resultant collineation of space cannot be of type XI. Again, if we consider the only other possible combination, ${}^{11}T(l\mu F)$ and ${}^{11}T_1(m\lambda F)$, one taken on each of the systems of generators, along $m$ we have identical and parabolic one-dimensional transformations whose resultant is parabolic; along $l$ we have again the resultant of identical and parabolic one-dimensional transformation, which, being parabolic, makes the resultant a collineation of type XIII.

THEOREM. There are $2\infty^2$ transformations of space of type XI, skew elations, which leave invariant a given quadric surface, each one of which leaves also invariant all of one system, and one of the second system of generators. These $2\infty^2$ transformations form $2\infty^1$ one-parameter groups, one for each generator of the surface, but they do not combine to form two-parameter groups.

## § 2. TYPE X.

5. *Fundamental group.*—In the fundamental group of type X, the invariant figure consists of two non-intersecting lines and the congruence meeting those lines, *i. e.*, a family of quadric surfaces, together with all the generators of one system on each surface, and two generators of the other system which are common to all the surfaces of the family. One of this family of quadrics we shall choose as our invariant surface. Therefore, from previous work,* we have a one-parameter group which we shall designate as ${}^{10}G_1(ll'\mu F)$, neglecting all the surfaces of the system, save the one we call $F$. The one-dimensional transformations are identical along $l$ and $l'$, and loxodromic along each of the generators of $\mu$.

6. Of these one-parameter groups we shall find $2\infty^2$, one for each pair of generators of the same system. Hence, there are $2\infty^3$ transformations of space of type X which leave invariant a given quadric

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*This reference is to unpublished work by Professor Newson.

surface, each of which leaves also invariant all the generators of one system and two generators of the other.

7. *Higher groups.*—Let us take the resultant of two transformations, ${}^{10}T(ll'\mu F)$ and ${}^{10}T_1(ll'_1\mu F)$, in which the invariant figures are the same with the exception that $l'_1$ is a distinct generator from $l'$, though still of the same system. Along each of the generators $l$, $l'$, $l'_1$ the resultant of two identical transformation is identical, while along each of the generators of $\mu$, the resultant of two loxodromic transformations with one common invariant point is loxodromic. The remainder of the invariant figure is common and the resultant space collineation is of type X. The generator $l'$ has been given one degree of freedom without destroying the group property. Therefore, there is a two-parameter group ${}^{10}G_2(l\mu F)$ in which $l'$ varies over the invariant surface in the system $\lambda$.

Further, let us consider two transformations ${}^{10}T(ll'\mu F)$ and ${}^{10}T(l_1l'_1\mu F)$, where $l$ and $l'$ are generators of the system $\lambda$ but distinct from $l_1$, $l'_1$. Each of these transformations leaves invariant every generator of $\mu$, and must therefore transform each generator of $\lambda$ into another of the same system, *i. e.*, changes a straight line cutting all the generators of $\mu$ into another straight line cutting all the generators of $\mu$. The resultant of two loxodromic one-dimensional transformations is generally loxodromic. Moreover, since by each component the generators of $\lambda$ as a system remain unchanged, the resultant also leaves the system $\lambda$ unchanged, and the two new invariant lines are generators of the system $\lambda$. Hence, the resultant space collineation is of type X, and we have established the group property for a three-parameter group, ${}^{10}G_3(\mu F)$.

Furthermore, it is readily seen that there can exist no other groups of type X which leave invariant a non-degenerate quadric surface.

THEOREM. There are $2\,\infty^3$ transformations of space of type X which leave invariant a given quadric surface, each of which leaves also invariant all generators of one and two generators of the other system on the surface. These combine to form one-, two- and three-parameter groups but no groups of higher order.

## § 3. TYPE IX.

8. *Fundamental group.*—The fundamental invariant figure of type IX, third class, is a pencil of planes, a pencil of conics in each plane, together with a line of invariant points, and a pencil of invariant lines in one of the invariant planes. This can mean only a system of quadric surfaces to each of which the plane determined by the line of invariant points is a tangent. There are $\infty^1$ transformations which

leave this same figure invariant and form a one-parameter group. We shall choose as our invariant surface one of those quadrics which is tangent to p, the plane of the invariant points, and has this line as a tangent line. This group we call ${}^{9}\mathrm{G}_1(\mathrm{lmp}t\mathrm{F})$.

The generators l and m, in which the tangent plane cuts the quadric, are rays of the invariant pencil in that plane. Therefore, along each of them there is a parabolic transformation, differing from each other only by a constant factor. Moreover, these two generators are harmonically separated by the axis of the pencil of invariant points; for, in the general case, two reciprocal polars with regard to a quadric surface are harmonically separated by that surface. But should those two polars, as a special case, intersect, they must do so in a tangent plane, and their point of intersection is on the surface. But the two axes referred to above form just this special case of reciprocal polars; and the lines harmonically separating them are the generators cut from the quadric by the tangent plane. Hence, these two axes are harmonically separated by the two invariant generators of the surface.

9. For each pair of generators we may have $\infty^1$ positions of the tangent line of invariant points, *i. e.*, $\infty^1$ of the one-parameter groups just discussed. Hence, there are of space $\infty^4$ collineations of type IX which leave invariant a given quadric surface, each of which leaves also invariant one generator of each system on the quadric and a line of invariant points tangent to the quadric at the intersection of these two generators.

10. *Higher groups.*—Consider next the resultant of two transformations ${}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F})$ and ${}^{9}\mathrm{T}_1(\mathrm{lmp}t_1\mathrm{F})$, where $t$ and $t_1$ denote different positions of the line of invariant points in the tangent plane. Since the generators l and m are always to be harmonically divided by the axes of invariant planes and points, as $t$ shifts in the tangent plane, the axis of the pencil of planes must shift correspondingly in the same plane. We have left in each case a system of quadrics through a point tangent to a given plane. Therefore, in each component, the quadric of which l and m are generators remains unchanged. The resultant space collineation leaves invariant the given quadric and its two generators, a new pencil of planes, a new line of points, and is again of type IX. But the line $t$ may take $\infty^1$ positions in the plane and through the given point. For each position there exists, as has been shown (art. 8), a one-parameter group leaving invariant a system of quadric surfaces. Each of these one-parameter groups has in common the quadric tangent to the invariant plane along l and m. Therefore, the resultant of two transformations ${}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F})$ and

${}^9T(lmpt_1F)$ is again of type IX, and we have established a two-parameter group whose symbol is ${}^9G_2(lmpF)$.

For any given quadric we have $\infty^2$ such two-parameter groups, one group for each pair of generators chosen, one generator from each system. It may be easily shown that these do not combine to form three- or four-parameter groups.

THEOREM. There are $\infty^4$ transformations of type IX which leave invariant a given quadric surface. each of which, in addition, leaves invariant one generator of each system and a line of invariant points through their intersection and in the plane thus determined. These combine to form one- and two-parameter groups, but not to form three- or four-parameter groups.

## § 4. TYPE VI.

11. *Fundamental group.*—The fundamental group of third class of type VI leaves invariant a pencil of planes, two points on the axis, a line of points cutting that pencil, and two planes through that line, together with a system of quadric surfaces. Of this system we choose one surface to which the two invariant axes will be reciprocal polars. We have then a one-parameter group, ${}^6G_1(ll'mm'nF) = {}^6G_1(APpsnkF)$, for each line in space taken with its reciprocal polar.

There are $\infty^5$ transformations of space of type VI having the required property, $\infty^1$ for each line in space taken with its reciprocal polar with regard to the given quadric.

13. *Higher groups.*—Take two transformations ${}^6T(ll'mm'nF)$ and ${}^6T_1(ll'mm'_1n_1F)$ as m′ shifts along the quadric, always remaining a generator of the system $\mu$. The resultant has been shown to be ${}^6T_2(ll'mm'_2n_2F)$, *i.e.*, a transformation of type VI leaving the quadric invariant. Therefore, there exists a two-parameter group ${}^6G_2(ll'm\overline{n}F)$. In like manner the three-parameter group ${}^6G_3(lm\overline{n}F)$ has been determined, and it is readily seen that there can exist no three-parameter group ${}^6G_3(ll'\overline{n}F)$, and no groups of higher order.

Next, considering again our one-parameter group ${}^6G_1(APpsnF)$, allow the line of invariant points, $n$, to shift through P, the pole of an invariant plane, s, which cuts the surface, and to shift in a plane tangent to the surface. Let us examine the resultant of two transformations taken one from each of two one-parameter groups with different axes of invariant points. In the invariant tangent plane we have, as plane transformations, two perspective collineations with common vertex and intersecting axes of invariant points. These necessarily result in a perspective collineation having the same vertex and axis of invariant points through the intersection of the axes of the two components. Other portions of the invariant figures of the two space collineations correspond, and hence it is readily seen that their resultant is of type VI, with a new line of invariant points still in the tangent plane and through the fixed point P. We have, then, a new two-parameter group ${}^6G_2(\mathrm{APpsF})$. Again, letting the line of invariant points vary through the invariant pole in space, we have a three-parameter group ${}^6G_3(\mathrm{PsF})$ leaving invariant pole and polar plane with regard to the invariant surface. There are $\infty^3$ such three-parameter groups, one for each point in space.

This would indicate $\infty^6$ transformations of type VI leaving invariant a quadric surface. But if we examine more closely, we see that all three-parameter groups whose poles are collinear contain one common ${}^6G_1(\mathrm{APps}n\mathrm{F})$. Hence, there are only $\infty^4$ distinct one-parameter groups, or $\infty^5$ distinct trnsformations of type VI, leaving a quadric invariant, as before stated.

Hence, there are of type VI two varieties of two- and two varieties of three-parameter groups, containing, in all, $\infty^5$ collineations of space which transform a quadric surface into itself.

Theorem. There are $\infty^5$ collineations of space of type VI leaving invariant a given quadric surface, each of which also leaves invariant two generators of each system on the surface, and a line of invariant points cutting the surface. These $\infty^5$ collineations combine to form one-parameter groups, two varieties of two- and two varieties of three-parameter groups, but no groups of higher orders.

## § 5. Type III.

14. *Fundamental group of first class.*—The group $G_2(\mathrm{ABll'})_{\mathrm{II}}$ of type III leaves invariant a family of quadric surfaces, and, in each, two generators of one and one generator of the other system.* In all of our work we take into consideration but one quadric surface, and shall therefore designate this group as ${}^3G_2(\mathrm{ll'mF})$. The one-dimensional transformations of this type are loxodromic along m, and parabolic along l and l′, while the plane collineations are of type II in each of the invariant planes.

15. Of these two-parameter groups in space we shall find $\infty^3$, since l, l′ and m have each one possible degree of freedom. Hence, there are $\infty^5$ such collineations of type III.

16. *Higher groups.*—If we consider the resultant of two transformations ${}^3T(\mathrm{ll'mF})$ and ${}^3T_1(\mathrm{ll'mF})$, along m the resultant of two loxodromic transformations with one common invariant point is loxodromic, along l the resultant of two parabolic transformations

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* Kansas University Quarterly, vol. X, p. 95.

with common invariant point is parabolic. The resultant is of type III, and we have a three-parameter group ${}^3G_3(lmF)$ established. If, however, we allow m to vary, along l and l′, the resultant of two parabolic transformations without common invariant point is in general loxodromic. Hence, we can have no group ${}^3G_3(ll'mF)$.

Next consider ${}^3T(ll'mF)$ and ${}^3T_1(l_1l'_1mF)$. Along m the resultant of two loxodromic, one-dimensional transformations without common invariant point is loxodromic, while along the new l and l′ the two parabolic transformations give a parabolic transformation. The entire resultant is of type III, and we have established the four-parameter group ${}^3G_4(mF)$.

The resultant of any two collineations of type III, taken in more general position, is of type I. Therefore, there are no groups of higher orders.

THEOREM. There are of space $\infty^4$ collineations of type III which leave invariant a given quadric, and each of which leaves invariant, in addition, two generators of one and one of the other system on the surface. These collineations combine to form two-, three- and four-parameter groups of the first class.

17. *Second class.*—In the collineations of our two-parameter group ${}^3G_2(ll'mF)$, let us assume the relation between the t along l and the k along m to be $k = a^t$, where a is a constant; and take the resultant of two transformations having different t's. Along l and l′ these combine by the law $t_2 = t + t_1$, and along m according to the law $k_2 = kk_1 = a^t a^{t_1} = a^{t+t_1}$. Therefore, the a of the resultant transformation is the same as in each of its components. There is a one-parameter subgroup of ${}^3G_2(ll'mF)$ for each value of a, the symbol being ${}^3G_1(ll'mF)_a$.

18. *Higher groups.*—Take two collineations, ${}^3T(ll'mF)_a$ and ${}^3T_1(ll'_1mF)_a$, and form their resultant. In the plane lm the plane transformations are of type II, and have in common two lines and their point of intersection. But this is the case in which the plane collineations with constant a combine according to the law $t_2 = t + t_1$. Hence, the resultant of the space transformations is of type III with constant a, and there is determined a group ${}^3G_2(lmF)_a$. It may be easily seen that there are no higher groups of type III of this class.

THEOREM. There are $\infty^5$ collineations of type III which transform a given quadric surface into itself. These combine to form one- and two-parameter groups of the second class; but there are no groups of this class of higher orders.

## § 6. TYPE I.

19. *Fundamental group of first class.*—The group $G_2(ABCD) r + r' = 1$, of type I, second class, leaves invariant a tetrahedron and a system of quadric surfaces, of which AC, BD, AD and BC are generators, while CD and AB are reciprocal polars.* The one-dimensional transformations are all loxodromic, while the plane transformations are of type I. Here, again, we concern ourselves with but one of the quadric surfaces, and shall therefore designate this group as $^1G_2(ll'mm'F)$. This group has as its independent parameters the cross-ratios k and k'. Hence, it is a two-parameter group and contains $\infty^2$ collineations.

There exists one such two-parameter group for each set of four generators, chosen two from each system on the quadric. Therefore, there are $\infty^6$ collineations of space of type I which leave a given quadric invariant.

20. *Higher groups.*—Let us form the resultant of two transformations $^1T(ll'mm'F)$ and $^1T_1(ll'mm'_1F)$, where m' varies so as to remain always a generator of the system $\mu$.

Express the two tetrahedrons as ABCD and $AB_1CD_1$, respectively. Substituting the relation of cross-ratios necessary for the presence of an invariant quadric, viz., $r + r' = 1$, we have our cross-ratios as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| $^1T =$ | AB | BC | CD | DB | AC | AD, |
| | $k$ | $k^{-r}$ | $k^{2r-1}$ | $k^{1-r}$ | $k^{1-r}$ | $k^r$ |
| $^1T_1 =$ | $AB_1$ | $B_1C$ | $CD_1$ | $D_1B_1$ | AC | AD, |
| | $k_1$ | $k_1^{-r}$ | $k_1^{2r-1}$ | $k_1^{1-r}$ | $k_1^{1-r}$ | $k_1^r$ |
| $^1T_2 = {}^1T \cdot {}^1T_1 =$ | $AB_2$ | $B_2C$ | $CD_2$ | $D_2B_2$ | $A_2C$ | $A_2D_2$. |
| | $k_2$ | $k_2^{-r}$ | $k_2^{2r-1}$ | $k_2^{1-r}$ | $k_2^{1-r}$ | $k_2^r$ |

The above resultants are obtained as follows: Along AC, two loxodromic transformations with two common invariant points combine according to the law $k_2 = kk_1$. Along $AD_2$ and $AB_2$, where the transformations have but one common invariant point, the same law holds true. Moreover, through A, in the plane ADC, the loxodromic one-dimensional collineations have two invariant rays in common, and hence, combine according to the above law, and along any line cutting that pencil of rays the new cross-ratio is $k_2^{2r-1}$. The two remaining cross-ratios may be determined by taking the product of cross-ratios around the various triangles, this product around any given triangle always equaling 1. From this it is readily seen that the group property holds throughout. Hence, there is a three-parameter group $^1G_3(ll'mF)$.

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*Kansas University Quarterly, vol. X, p. 38.

This method of proof is a very general one, and is made use of in showing the existence of higher groups of this class. There exist four such groups, ${}^1G_4(\mathrm{ll'F})$, ${}^1G_4(\mathrm{lmF})$, ${}^1G_5(\mathrm{lF})$, and ${}^1G_6(\mathrm{F})$. The proof of the group property in these last cases is left to the reader.

21. *Groups of second class.*—If, to the conditions already placed on the relations of cross-ratios in the preceding groups, we add that of a constant r, we obtain several new groups.

The first of these is a one-parameter group ${}^1G_1(\mathrm{ll'mm'F})_r$, whose existence as a subgroup of ${}^1G_2(\mathrm{ll'mm'F})$ is evident.* The higher groups are as easily shown, the existence or non-existence of the group depending upon the corresponding group of the plane.† The details of the work are omitted for the two remaining groups, ${}^1G_2(\mathrm{lmF})_r$ and ${}^1G_3(\mathrm{lmF})_r$.

THEOREM. There are $\infty^6$ collineations of space of type I which leave invariant a given quadric surface, each of which, in addition, leaves invariant a pair of generators from each system on the quadric. These combine to form groups of two classes. Of the first class there are two-, three-, two varieties of four-, five- and six-parameter groups. Of the second class there are one-, two- and three-parameter groups, but none of the higher order.

## B.—Combinations of Various Types.

### § 1. TYPE XI.

22. The resultant of two transformations of the same type we have often found to be of a different type. We shall proceed next to examine not only the resultant of two transformations of one type, but also the resultant of a collineation of one type taken with one from each of the other types. Generally, this resultant is at once evident and will be stated without proof.

The resultant of two transformations of type XI with common invariant figure was shown to be of type XI. If, however, the invariant figures differ in the generators of the second system, the resultant one-dimensional transformation is not parabolic but loxodromic, and the space collineation is of type X.

Similarly, taken with different invariant systems of generators, the resultant is of type IX.

$$\begin{gathered} {}^{11}T(l\mu F) + {}^{11}T_1(l\mu F) = {}^{11}T_2(l\mu F) \\ {}^{11}T(l\mu F) + {}^{11}T_1(l_1\mu F) = {}^{10}T(l_2 l'_2 \mu F) \\ {}^{11}T(l\mu F) + {}^{11}T_1(m\lambda F) = {}^{9}T(\mathrm{lm}tF). \end{gathered}$$

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* Kansas University Quarterly, vol. X, p. 33.

† American Journal of Mathematics, vol. XXIV, No. 2, p. 134.

## § 2. TYPE X.

23. For combinations with collineations of the same type in the case of type X, the work has been given, save that in which the invariant systems are different. This evidently gives a collineation of type I.

In combination with transformations of type XI, only one is worthy of special mention. ${}^{10}T(ll'\mu F)$ and ${}^{11}T(m\lambda F)$ give along $l$ and $l'$ the combination of identical and parabolic one-dimensional transformations, evidently a parabolic resultant, while along $m$ the resultant of loxodromic and identical gives a loxodromic transformation. The space collineation is of type III.

The following statement is given for reference:

$$
\begin{gathered}
{}^{10}T(ll'\mu F) + {}^{10}T_1(mm'\lambda F) = {}^{1}T_2(ll'\mathrm{mm}'F) \\
{}^{10}T(ll'\mu F) + {}^{11}T(l\mu F) = {}^{10}T_1(ll'_1\mu F) \\
{}^{10}T(ll'\mu F) + {}^{11}T(l_1\mu F) = {}^{10}T_1(l_2l'_2\mu F) \\
{}^{10}T(ll'\mu F) + {}^{11}T(m\lambda F) = {}^{3}T(\mathrm{ll'm}F).
\end{gathered}
$$

## § 3. TYPE IX.

24. It is worthy of notice that a transformation of type IX, in combination with those of other types, may result in one of type III. The resultants of type I are not very frequent in this case.

$$
\begin{gathered}
{}^{9}T(\mathrm{lmp}tF) + {}^{9}T_1(\mathrm{lmp}tF) = {}^{9}T_2(\mathrm{lmp}tF) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{9}T_1(\mathrm{lmp}t_1F) = {}^{9}T_2(\mathrm{lmp}t_2F) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{9}T_1(\mathrm{lm_1p_1}t_1F) = {}^{1}T(\mathrm{ll'_2m_2m'_2}F) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{9}T_1(\mathrm{l_1m_1p_1}t_1F) = {}^{1}T(\mathrm{l_2l'_2m_2m'_2}F) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{11}T(\mathrm{l}\mu F) = {}^{9}T_1(\mathrm{lmp}t_1F) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{11}T(\mathrm{l_1}\mu F) = {}^{3}T(\mathrm{l_2l'_2m}F) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{10}T(\mathrm{ll'}\mu F) = {}^{3}T(\mathrm{ll'_2m}F) \\
{}^{9}T(\mathrm{lmp}tF) + {}^{10}T(\mathrm{l_1l'_1}\mu F) = {}^{3}T(\mathrm{l_2l'_2m}F).
\end{gathered}
$$

## § 4. TYPE VI.

25. The resultant of a transformation of type VI with another of the same type has been discussed. Combined with another of any other type, the resultant is easily seen to be always of type I, since no transformation of another type leaves invariant all points on a polar line, save a special transformation of type X, which will be treated under the head of singular transformations,* or when combined with one of type IX. This last case we will explain more fully. Take two transformations, ${}^{6}T(\mathrm{ll'mm'}nF)$ and ${}^{9}T(\mathrm{l'mp}tF)$. On the quadric

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* Art. 35.

surface the resultant transformation evidently leaves invariant the figure $(\mathrm{l_2l'mm'_2})$. In the plane of l′m we have the resultant of an elation and perspective collineation, the axis of the elation passing through the vertex of the prospective collineation. This is a perspective collineation having the same vertex and axis as the first. The complete invariant figure of the space collineation is $(\mathrm{l_1l'm_1m'}n\mathrm{F})$, which is that of type VI. Taken in any other position, the two component transformations result in one of type I.

## § 5. Type III.

26. The combinations of type III, first class, are comparatively simple. But one thing is to be noted: the resultant of a transformation with one of any other type is, in general, never anything other than type II or type I; but type III may be combined with any other type to produce a transformation of type I.

$$
\begin{aligned}
{}^{3}\mathrm{T(ll'mF)} + {}^{3}\mathrm{T_1(ll'mF)} &= {}^{3}\mathrm{T_2(ll'mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{3}\mathrm{T_1(ll'_1mF)} &= {}^{3}\mathrm{T_2(ll'_2mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{3}\mathrm{T_1(l_1l'_1mF)} &= {}^{3}\mathrm{T_2(l_2l'_2mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{3}\mathrm{T_1(l_1l'_1m_1F)} &= {}^{1}\mathrm{T(l_2l'_2m_2m'_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{3}\mathrm{T_1(ll'm_1F)} &= {}^{1}\mathrm{T(ll'm_2m'_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{3}\mathrm{T_1(ll'_1m_1F)} &= {}^{1}\mathrm{T(ll'_2m_2m'_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{11}\mathrm{T}(\mathrm{l}\,\mu\,\mathrm{F}) &= {}^{3}\mathrm{T_1(ll'_1mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{11}\mathrm{T}(\mathrm{l_1}\,\mu\,\mathrm{F}) &= {}^{3}\mathrm{T_1(l_2l'_2mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{11}\mathrm{T}(\mathrm{l}m\,\lambda\,\mathrm{F}) &= {}^{3}\mathrm{T_1(ll'mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{11}\mathrm{T}(m'\,\lambda\,\mathrm{F}) &= {}^{1}\mathrm{T(ll'm_2m'_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{10}\mathrm{T}(ll'\,\mu\,\mathrm{F}) &= {}^{3}\mathrm{T_1(ll'mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{10}\mathrm{T}(ll'_1\,\mu\,\mathrm{F}) &= {}^{3}\mathrm{T_1(ll'_2mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{10}\mathrm{T}(l_1l'_1\,\mu\,\mathrm{F}) &= {}^{3}\mathrm{T_1(l_2l'_2mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{10}\mathrm{T}(mm'\,\lambda\,\mathrm{F}) &= {}^{1}\mathrm{T(ll'mm'_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{10}\mathrm{T}(m_1m'\,\lambda\,\mathrm{F}) &= {}^{1}\mathrm{T(ll'm_2m'_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) &= {}^{3}\mathrm{T_1(ll'_2mF)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) &= {}^{3}\mathrm{T_1(ll'_2m_2F)}\\
{}^{3}\mathrm{T(ll'mF)} + {}^{9}\mathrm{T}(\mathrm{l_1mp_1}t_1\mathrm{F}) &= {}^{1}\mathrm{T(l_2l'_2mm'_2F)}.
\end{aligned}
$$

## § 6. Type I.

27. The resultant of a transformation of type I with any other transformation is, in general, of type I. Moreover, a transformation of type I may be resolved into components of any one of the five types discussed in this paper.

In all the work of the preceding articles, it must be borne in mind that the values of the cross-ratios of the one-dimensional transformations in each case are taken perfectly general and with no fixed relation to each other. Our next task will be to investigate the result of choosing space collineations in which the one-dimensional transformations bear definite relations to each other. By proper choice of parameters, we may resolve a collineation of any type into components of almost any other type. This is the foundation principle of group structure; for it is only by this means that we may obtain groups of one type within groups of another type.

## C.—Group Structure.

### § 1. TYPE XI.

28. For special values or specially related values of the parameters of a group, the resulting transformation may degenerate into one of a lower type. For the most part these will be only indicated where they are easily seen. It has been shown already that the group ${}^{11}G_1(l\,\mu\,F)$ contains only transformations of that type. Hence we may state in symbolic language as its structure,

$$G_6(F) \mathrel{\overline{\underline{\cdot}}} 2\ \infty^1\ {}^{11}G_1(l\,\mu\,F).$$

### § 2. TYPE X.

29. Two transformations of type X, having reciprocal values of k, the same invariant system, but only one other common invariant generator, result in a collineation of type XI; since along a line two loxodromic transformations, with reciprocal values of k and but one invariant point in common, result in a parabolic transformation. Hence, by proper choice of components within the group ${}^{10}G_2(l\,\mu\,F)$ we may get all the collineations of the group ${}^{11}G_1(l\,\mu\,F)$. Combining $\infty^1$ of the groups ${}^{10}G_2(l\,\mu\,F)$, we get ${}^{10}G_3(\mu\,F)$; but the $\infty^1\ {}^{11}G_1(l\,\mu\,F)$ do not form a two-parameter group of type XI. Symbolically expressed, we have:

$${}^{10}G_1(ll'\,\mu\,F) = {}^{10}G_1(ll'\,\mu\,F)$$

$${}^{10}G_2(ll'\,\mu\,F) = \infty^1\ {}^{10}G_1(l\,\bar{l}'\,\mu\,F) + {}^{11}G_1(l\,\mu\,F)$$

$${}^{10}G_3(\mu\,F) = \infty^2\ {}^{10}G_1(\bar{l}\,\bar{l}'\,\mu\,F) + \infty^1\ {}^{11}G_1(\bar{l}\,\mu F)$$

$$G_6(F) \mathrel{\overline{\underline{\cdot}}} 2\,{}^{10}G_3(\bar{\mu}\,F) = 2\ \infty^2\ {}^{10}G_1(\bar{l}\,\bar{l}'\,\bar{\mu}\,F) + 2\ \infty^1\ {}^{11}G_1(\bar{l}\,\bar{\mu}\,F).$$

### § 3. TYPE IX.

30. In article (10) it was shown that the invariant generators of a transformation of type IX divided harmonically the line of invariant points and the axis of invariant planes. Therefore, as *t* varies, the axis of planes must also vary, so that when *t*, the axis of invariant

points, coincides with one of the invariant generators, it must also coincide with the axis of planes. In this case, however, the transformation of type IX degenerates to one of type XI. The line of invariant points may coincide with either generator giving two groups, ${}^{11}G_1(l\,\mu\,F)$ and ${}^{11}G_1(m\,\lambda\,F)$, one on each system of generators of the quadric surface.

The resultant of two transformations of type XI, taken one from each of these groups, is of type IX (art. 22). From the $2\,\infty^1$ transformations included within these two groups may be found $\infty^2$ combinations of the kind discussed, *i. e.*, $\infty^2$ transformations of type IX which combine to form a two-parameter group, and every transformation of type IX which leaves invariant a quadric surface may be resolved into components of type XI.

$$
\begin{aligned}
{}^{9}G_1(\mathrm{lmp}\,t\,F) &= {}^{9}G_1(\mathrm{lmp}\,t\,F)\\
{}^{9}G_2(\mathrm{lmp}F) &= \infty^1\ {}^{9}G_1(\mathrm{lmp}\bar{t}F) + {}^{11}G_1(l\,\mu\,F) + {}^{11}G_1(m\,\lambda\,F)\\
G_6(F) \equiv \infty^2\ {}^{9}G_2(\mathrm{lmp}F) &= \infty^3\ {}^{9}G_1(\mathrm{lmp}tF) + 2\,\infty^1\ {}^{11}G_1(l\,\mu\,F).
\end{aligned}
$$

## § 4. TYPE VI.

31. The presence of a group ${}^{11}G_1(\mathrm{l}\,\mu\,F)$ in the two-parameter group ${}^{6}G_2(\mathrm{ll'm}\bar{n}F)$ is determined as in the case of type X (art. 29). The only other groups of different type are those of type IX. In the group ${}^{6}G_3(\mathrm{PsF})$ the line of invariant points shifts through the pole P in space. It may take $\infty^1$ positions tangent to the quadric surface; but in these cases the transformation degenerates to type IX. Hence there are $\infty^1$ one-parameter groups of type IX included in the group ${}^{6}G_3(\mathrm{PsF})$. These do not form a two-parameter group ${}^{9}G_2(\mathrm{lmpF})$, since the line of invariant points shifts, not in a plane and through a point on the surface, but about the surface of a cone with vertex not on the quadric. Therefore, the resultant of two transformations taken one from each of two one-parameter groups is not of type IX.

$$
\begin{aligned}
{}^{6}G_1(\mathrm{ll'mm'}\,nF) &= {}^{6}G_1(\mathrm{APps}nF) + \mathrm{S.\,T.}\\
{}^{6}G_2(\mathrm{ll'm}\,nF) &= \infty^1\ {}^{6}G_1(\mathrm{ll'm}\overline{\mathrm{m}}'\,\bar{n}F) + {}^{11}G_1(m\,\lambda\,F) + \mathrm{S.\,T.}\\
{}^{6}G_3(\mathrm{lm}\bar{n}F) &= \infty^2\ {}^{6}G_1(\mathrm{ll'm}\,\overline{\mathrm{m}}'\,\bar{n}\,F) + {}^{9}G_2(\mathrm{lmpF}) + {}^{11}G_1(l\,\mu\,F)\\
&\qquad + {}^{11}G_1(m\,\lambda\,F) + \mathrm{S.\,T.}\\
{}^{6}G_2(\mathrm{APpsF}) &= \infty^1\ {}^{6}G_1(\mathrm{APps}\bar{n}F) + {}^{9}G_1(\mathrm{lmp}\,t\,F) + \mathrm{S.\,T.}\\
{}^{6}G_3(\mathrm{PsF}) &= \infty^2\ {}^{6}G_1(\mathrm{AP}\bar{\mathrm{p}}\mathrm{s}\,\bar{n}\,F) + \infty^1\ {}^{9}G_1(\mathrm{l}\,\bar{\mathrm{m}}\,\bar{\mathrm{p}}\,\bar{t}\,F).
\end{aligned}
$$

## §5. TYPE III.

32. If from the group ${}^3G_2(ll'mF)$ we choose two transformations with reciprocal values of k, then along m the result is identical, and we have the invariant figure $(m\lambda F)$, that of type XI. For every pair of reciprocal values of k, we have a new transformation of type XI with this same invariant figure; hence the group ${}^{11}G_1(m\lambda F)$. Similarly we determine the presence of the group ${}^{10}G_1(ll'\mu F)$.

In combining $\infty^1$ of these two-parameter groups of type III, we find that we have not $\infty^1$ distinct groups ${}^{11}G_1(m\lambda F)$, but only the same one occurring each time. There now appears another group ${}^{11}G_1(l\mu F)$, each tranformation of which is obtained by choosing two transformations from different two-parameter groups ${}^3G_2(ll'mF)$ and ${}^3G_2(ll''mF)$, with $t = -t_1$, and $k = \frac{1}{k_1}$, when our transformation along l becomes identical, and that along m parabolic. Having these two groups, ${}^{11}G_1(l\mu F)$ and ${}^{11}G_1(m\lambda F)$, on different systems of generators, the presence of ${}^9G_2(lmpF)$ is recognized from the previous work (art. 30), and we have, finally:

$${}^3G_2(ll'mF) = {}^3G_2(ll'mF) + {}^{10}G_1(ll'\mu F) + {}^{11}G_1(m\lambda F).$$

$${}^3G_3(lmF) = \infty^1\ {}^3G_2(l\bar{l}'mF) + {}^{10}G_2(l\mu F) + {}^9G_2(lmpF) + {}^{11}G_1(l\mu F) + {}^{11}G_1(m\lambda F).$$

$${}^3G_4(mF) = \infty^2\ {}^3G_2(\bar{l}\bar{l}'mF) + {}^{10}G_3(\mu F) + \infty^1\ {}^9G_2(\bar{l}m\bar{p}F) + \infty^1\ {}^{11}G_1(\bar{l}\mu F) + {}^{11}G_1(m\lambda F).$$

It is to be noted that in ${}^3G_4(mF)$, while there occurs $\infty^1\ {}^{11}G_1(l\mu F)$, the group ${}^{11}G_1(m\lambda F)$ occurs only once.

*Second class.*—The structure of the second class involves but one type other than itself, save where transformations of such types occur as singular transformations. If we consider the resultant of two transformations of type III, with l and m as the only common invariant generators, we have the resultant one-dimensional transformations parabolic; *i. e.*, the resultant space collineation is of type IX.

$${}^3G_1(ll'mF)_a = {}^3G_1(ll'mF)_a + \text{S. T.}$$

$${}^3G_2(lmF)_a = \infty^1\ {}^3G_1(l\bar{l}'mF)_a + {}^9G_1(lmptF) + \text{S. T.}$$

## §6. TYPE I.

33. *First class.*—The groups of this most general type include all of the groups of the types heretofore considered. The method of determining the presence of the various types is the same as that employed before. The list, while long, is very simple, if we bear in mind that, having once determined the presence of one group, we have necessarily determined at the same time the presence of all of its component parts. Thus, for example, in ${}^1G_3(ll'mF)$, we may de-

termine the existence of ${}^3G_2(ll'm)$, by taking two tranformations having on l and l′ one common invariant point with reciprocal values of k, the resultant is at once a parabolic transformation, and the space collineation is of type III. If we combine $\infty^1$ of these, in order to produce ${}^1G_4(lmF)$, we have also combined $\infty^1$ ${}^3G_2(ll'mF)$, giving us the group ${}^3G_3(lmF)$. But this group we have already proved to contain ${}^9G_2(lmpF)$, ${}^{11}G_1(l\mu F)$, and ${}^{11}G_1(m\lambda F)$. Hence, we may state without further comment that these same three groups exist in ${}^1G_4(lmF)$.

The existence in the fundamental group ${}^1G_2(ll'mm'F)$ of both of the dualistic groups ${}^6G_1(ll'mm'nF)$ and ${}^6G_1(ll'mm'n'F)$ of type VI, and of the two one-parameter groups of type X, one on each system of generators of the invariant surface, is apparent, the conditions necessary to produce them being merely the proper choice of the parameters r and r′ of the group. The following table gives the group structure of type I, first class, in symbolic language:

$$\begin{aligned}{}^1G_2(ll'mm'F) \equiv {}^1G_2(ll'mm'F) &+ {}^{10}G_1(ll'\mu F) + {}^{10}G_1(mm'\mu F)\\ &+ {}^6G_1(ll'mm'nF) + {}^6G_1(ll'mm'n'F).\end{aligned}$$

$$\begin{aligned}{}^1G_3(ll'mF) \equiv \infty^1\ {}^1G_2(ll'm\overline{m}'F) &+ {}^3G_2(ll'mF) + {}^6G_2(ll'm\overline{n}F)\\ &+ {}^6G_2(ll'm\overline{n}'F) + {}^{10}G_1(ll'\mu F) + {}^{10}G_2(m\lambda F)\\ &+ {}^{11}G_1(m\lambda F).\end{aligned}$$

$$\begin{aligned}{}^1G_4(lmF) = \infty^2\ {}^1G_2(ll'mm'F) &+ {}^3G_3(lmF) + {}^6G_3(lm\overline{n}F)\\ &+ {}^6G_3(ApF) + {}^{10}G_2(l\mu F) + {}^{10}G_2(m\lambda F)\\ &+ {}^{11}G_1(m\lambda F) + {}^{11}G_1(l\mu F) + {}^9G_2(lmpF).\end{aligned}$$

$$\begin{aligned}{}^1G_4(ll'F) = \infty^2\ {}^1G_2(ll'\overline{m}\,\overline{m}'F) &+ \infty^1\ {}^3G_2(ll'\overline{m}F) + \infty^1\ {}^6G_2(ll'\overline{m}\,\overline{n}F)\\ &+ \infty^1\ {}^6G_2(ll'\overline{m}\,\overline{n}'F) + {}^{10}G_1(ll'\mu F) + {}^{10}G_3(\lambda F)\\ &+ \infty^1\ {}^{11}G_1(\overline{m}\lambda F).\end{aligned}$$

$$\begin{aligned}{}^1G_5(lF) = \infty^3\ {}^1G_2(l\overline{l}'\overline{m}\,\overline{m}'F) &+ {}^3G_4(lF) + \infty^1\ {}^6G_3(l\overline{m}\,\overline{n}F)\\ &+ \infty^1\ {}^6G_3(A\overline{p}F) + {}^{10}G_2(l\mu F) + {}^{10}G_3(\lambda F)\\ &+ {}^{11}G_1(l\mu F) + \infty^1\ {}^{11}G_1(m\lambda F) + \infty^1\ {}^9G_2(l\overline{m}\,\overline{p}F).\end{aligned}$$

$$\begin{aligned}{}^1G_6(F) = \infty^4\ {}^1G_2(\overline{l}\,\overline{l}'\overline{m}\,\overline{m}'F) &+ 2\,\infty^1\,{}^3G_4(\overline{l}F) + \infty^2\ {}^6G_3(\overline{l}\,\overline{m}\,\overline{n}F)\\ &+ \infty^2\ {}^6G_3(\overline{A}\,\overline{p}F) + \infty^3\ {}^6G_3(\overline{P}\,\overline{s}F) + 2\ {}^{10}G_3(\overline{\lambda}F)\\ &+ 2\,\infty^1\ {}^{11}G_1(\overline{m}\overline{\lambda}F) + \infty^2\ {}^9G_2(\overline{l}\,\overline{m}\,\overline{p}F).\end{aligned}$$

34. *Second class.*—We found (art. 21) that for any definite value of r there existed a one-parameter subgroup of ${}^1G_2(ll'mm'F)$. But with r a rational number, we may have three cases, viz., r may be odd over odd, even over odd, odd over even; the fourth possible case, even over even, degenerating always to some one of the other three. The group structure in each of these three cases must be separately examined. For all continuous groups the three cases are identical; but we shall see that, with regard to singular transformations, the three are distinct. Hence, in our present statement, we shall merely indicate

the structure in a general way, leaving the fuller discussion of the three varieties as distinct groups for the next article.

The general group structure will be seen to be identical with the first three groups of type VI, which is in fact only the special case of type I, for $r = \infty$.

$$
\begin{aligned}
{}^{1}G_1(ll'mm'F)_r &= {}^{1}G_1(ll'mm'F)_r + S.\ T. \\
{}^{1}G_2(ll'mF)_r &= \infty^1\ {}^{1}G_1(ll'm\overline{m}'F)_r + {}^{11}G_1(m\lambda F) + S.\ T. \\
{}^{1}G_3(lmF)_r &= \infty^2\ {}^{1}G_1(l\bar{l}m\overline{m}'F)_r + {}^{11}G_1(m\lambda F) + {}^{11}G_1(l\mu F) \\
&\quad + {}^{9}G_2(lmpF) + S.\ T.
\end{aligned}
$$

## D.—Singular Transformations.

### § 1. TYPE I.

35. We shall define as a singular transformation one of a discrete set of transformations of a lower type occurring in a continuous group of a higher type. Such transformations are not found in the groups of the types IX, X, or XI, nor in the groups of the first class of types I or III.

If in ${}^{1}G_2(ll'mm'F)$ we choose our two parameters k and r, such that r is any rational number and $k = \pm 1$, we have a transformation of type X; for if we choose our tetrahedron to be ABCD, then the cross-ratios are:

| AB | BC | CD | BD | AC | AD |
|---|---|---|---|---|---|
| $k$ | $k^{-r}$ | $k^{r-r'}$ | $k^{-r'}$ | $k^{1-r}$ | $k^{1-r'}$, |

and since $r - r' = 1$, the condition for invariant quadric surface, we have, after making $k = -1$ and $r = \frac{o}{o}$, then $r = \frac{e}{o}$.

| AB | BC | CD | BD | AC | AD |
|---|---|---|---|---|---|
| $-1$ | $-1$ | $+1$ | $+1$ | $+1$ | $-1$ for $r = \frac{o}{o}$ |
| $-1$ | $+1$ | $-1$ | $-1$ | $-1$ | $+1$ for $r = \frac{e}{o}$, |

If we choose $k = -1$ when $r = \frac{o}{e}$ we have imaginary cross-ratios. But if $r = \frac{e}{o}$ and $k = +1$, we have the following set of ratios:

| AB | BC | CD | BD | AC | AD |
|---|---|---|---|---|---|
| $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$. |

The only interpretation of these ratios, however, is a transformation of type X with the axes of invariant points, reciprocal polars with regard to the invariant quadric. A transformation of type X with general cross-ratios cannot leave invariant a quadric surface, with the condition just stated; but, for every pair of reciprocal polars with regard to the given quadric, there exists one involutoric collineation which does leave that quadric invariant. Hence, there are $\infty^4$ involutoric transformations of space of type X having the axes of invariant points reciprocal polars with regard to the quadric surface which they leave invariant.

In each ${}^1G_1(ll'mm'F)_r$, where r is a rational number, there exists one of these collineations of type X, which we shall therefore term a singular transformation. The value of k and the position of the axes in each case depends upon the value of r—*i. e.*, whether it be odd over odd, even over odd, odd over even. If we combine $\infty^1$ of the one-parameter groups of type I, where r is odd over odd, we obtain no group of type X; nor, if we combine each of the singular transformations with the transformations of type I of the group, or with those of type XI occurring in this group, do we obtain any new singular transformations. However, if r be even over odd, we see (art. 35) that the group of type XI and the singular transformation of type X have their invariant systems lying in opposite directions. But a combination of a collineation of type X and one of type XI in this position we have shown (art. 23) to be of type III. Each singular transformation of type X combined with each of the group of collineations of type XI gives such a resultant. These resultants do not form a continuous group, but there exists a discrete set of them, and they must be classed as singular transformations of type III. In the third case, r being odd over even, the axes of invariant points are reciprocal polars with regard to the given quadric, and the combination of this transformation with those of types I, IX, XI, gives only type I.

For singular transformations in type I, we may then state in symbolic language:

$${}^1G_1(ll'mm'F)_{r=\frac{o}{o}}\ \overline{\underline{\cdot}}\ {}^{10}S.T.(mm'\lambda F)_{k=-1}$$

$${}^1G_2(ll'mmF)_{r=\frac{o}{o}}\ \overline{\underline{\cdot}}\ \infty^1\ {}^{10}S.T.(m\overline{m}'\lambda F)_{k=-1}$$

$${}^1G_3(lmF)_{r=\frac{o}{o}}\ \overline{\underline{\cdot}}\ \infty^1\ {}^{10}S.T.(m\overline{m}'\lambda F)_{k=-1}$$

$${}^1G_1(ll'mm'F)_{r=\frac{e}{o}}\ \overline{\underline{\cdot}}\ {}^{10}S.T.(ll'\mu F)_{k=-1}$$

$${}^1G_2(ll'mF)_{r=\frac{e}{o}}\ \overline{\underline{\cdot}}\ \infty^1\ {}^{10}S.T.(ll'\mu F)_{k=-1}+{}^3S.T.(ll'mF).$$

$${}^1G_3(lmF)_{r=\frac{e}{o}}\ \overline{\underline{\cdot}}\ \infty^1\ {}^{10}S.T.(l\overline{l}\mu F)_{k=-1}+\infty^1\ {}^3S.T.(l\overline{l}'mF)$$

$${}^1G_1(ll'mm'F)_{r=\frac{o}{e}}\ \overline{\underline{\cdot}}\ {}^{10}S.T.(ll'mm'nn'F)_{k=+1}$$

$${}^1G_2(ll'mF)_{r=\frac{o}{e}}\ \overline{\underline{\cdot}}\ \infty^1\ {}^{10}S.T.(ll'm\overline{m}'\overline{n}\overline{n}'F)_{k=+1}$$

$${}^1G_3(ll'F)_{r=\frac{o}{e}}\ \overline{\underline{\cdot}}\ \infty^2\ {}^{10}S.T.(l\overline{l}'m\overline{m}'\overline{n}\overline{n}'F)_{k=+1}.$$

It should be carefully noted that these transformations of type X cannot combine to form a group, since the resultant of two of them, with the same invariant figure, would give an identical transformation, while if their invariant figures differed in any part the resultant would necessarily be of type I.

## § 2. Type VI.

36. The groups of type VI of the first variety correspond very closely to those of type I, second class. If we choose $k = -1$, the cross-ratios become:

| AB | BC | CD | BD | AC | AD |
|---|---|---|---|---|---|
| $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$, |

which we see to be an involutoric transformation of type X, with the invariant axes reciprocal polars with regard to the quadric surface. Such transformations, we have seen, cannot form a group. Moreover, in each ${}^6G_1(ll'mm'nn'F)$ there exists but one such transformation — *i. e.*, the one for $k = -1$.

In building up the higher groups of type VI of the first variety these singular transformations remain throughout, but when combined with other types which may occur in the group of type VI, they produce no new singular transformations. But in ${}^6G_2(APpsF)$ we have a group of collineations of type IX, each of which, when combined with the singular transformations of type X, gives a resultant of type IX, but with a new line of invariant points, therefore, not belonging to the group ${}^9G_1(lmptF)$ — *i. e.*, we have singular transformations. Moreover, these will remain as singular transformations in ${}^6G_3(PsF)$, since the $\infty^1$ axes of invariant points included in the $\infty^1$ ${}^9G(lmptF)$ necessarily lie in a cone, while those of the singular transformation may take any position in space.

## § 3. Type III.

37. The only singular transformations left for consideration are those of type XI and type IX occurring in groups of type III, second class. Our fundamental group here is ${}^3G_1(ll'mF)_a$, in which there exists the relation of parameters, $k = a^t$ where a is a constant. It has been proved* that we may so choose two transformations with cross-ratios $k = a^t$ and $k_1 = a^t$, that the resultant may be $k_2 = a^{t+t_1} = 1$, where $t + t_1$ is not 0. Such a resultant would give an identity along m, and a parabolic transformation along each generator of system $\lambda$. This is a collineation of type XI. These singular transformations of type XI in the group ${}^3G_2(lmF)_a$ may combine with the collineation of ${}^9G_1(lmptF)$, resulting in singular transformations of type IX.

Therefore, we find:

$$ {}^3G_1(ll'mF)_a \ \overline{\underline{\cdot}}\ {}^{11}S.T.(m\,\lambda\,F) $$

$$ {}^3G_2(lmF)_a \ \overline{\underline{\cdot}}\ {}^{11}S.T.(m\,\lambda\,F) + {}^9S.T.(lmp\,\overline{t}\,F). $$

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*American Journal of Mathematics, vol. XXIV, No. 2, p. 159.

## **E.—Mixed Groups.**

### § 1. TYPE XII.

38. We now come to the consideration of what Doctor Wood terms* "symmetry transformations," *i. e.*, transformations which leave a given quadric surface invariant, but interchange the two systems of generators. The discussion of such transformations brings to our notice collineations of a type not yet considered. Choose as the invariant figure of a transformation of type XII the vertex and axial plane as pole and polar with regard to the given quadric surface, and next choose one cross-ratio along each invariant line equal $-1$. Then each invariant line cuts the surface in two points which are harmonically divided by the two invariant points on that line, since a quadric surface divides every line through a pole and polar plane harmonically. But this is merely stating, in a different way, that the cross-ratio of the four points is $-1$; *i. e.*, in the given transformation, the two points on the quadric surface are interchanged. An effect of the same kind is produced along each of the invariant lines; *i. e.*, every point on the surface of the quadric is interchanged with another, but the surface as a whole is transformed into itself. There are $\infty^3$ such transformations which leave the given quadric invariant, one for each point in space taken with its polar with regard to a given surface. The resultant of these transformations with those of the types heretofore discussed give rise to the mixed groups—*i. e.*, those groups which leave a given figure as a whole invariant, but interchange its parts in such a manner that a repetition of the transformation will bring it back to the original invariant figure.

A combination of this involutoric transformation of type XII with any one of types XI or X cannot result in a transformation of either of these types. Therefore, we can have no mixed groups of these types.

### § 2. TYPE IX.

39. If we consider the group ${}^9G_2(\text{lmpF})$, we see that the invariant figure consists of one generator of each system and the plane of those generators. We may choose an involutoric collineation of type XII which will interchange these two generators. Their point of intersection must be an invariant point of the involutoric transformation. For any plane through that point as axial plane and its pole as vertex of an involutoric collineation, the point must remain invariant. Thus there are $\infty^2$ involutoric collineations of type XII which leave the figure of ${}^9G_2(\text{lmpF})$ invariant as a whole, though interchanging parts of it. We see at once that our mixed group must

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* Annals of Mathematics, second series, vol. II, No. 4, p. 167.

contain *all* of the continuous subgroups included under the head of the continuous groups of like symbol. Hence, in writing out these groups, we shall not express in full detail the structure of the continuous groups, leaving that to be understood from previous work.

$$m^9G_2(\mathrm{lmpF}) = {}^9G_2(\mathrm{lmpF}) + \infty^2\ {}^{12}T(\bar{P}\,\bar{s}\,F)_{k=-1}.$$

## § 3. TYPE VI.

40. The continuous group ${}^6G_1(ll'mm'nF)$ may be combined with $\infty^1$ involutoric collineations of type XII and the invariant figure as a whole be left unchanged; *i. e.*, there will exist one involutoric ${}^6T(PsF)$, which will interchange l and m, l′ and m′ for each plane passing through the two invariant points, the intersection of l and m, l′ and m′. These combine to form a one-parameter mixed group $m^6G_1(ll'mm'n'F)$. There can exist no $m^6G_2(ll'mF)$, since, if we combine with this a transformation which interchanges the systems of generators, l′ will be thrown into a position m′, thus giving a new invariant figure and destroying one of the essentials of the group property.

The group ${}^6G_3(lmF)$ may be combined with $\infty^2$ of these involutoric collineations, exactly as in the case of the group ${}^9G_2(lmpF)$, and there exists the three-parameter mixed group $m^6G_3(lmF)$.

There can exist no other mixed group of type VI, since the combination of ${}^6T(PsF)$ with ${}^{12}T(PsF)$ does not alter any part of the invariant figure.

THEOREM. There are two mixed groups of transformations of type VI, each composed of a continuous group of the same symbol and involutoric transformations of type XII. Expressed symbolically, they are:

$$m^6G_1(ll'mm'nF) = {}^6G_1(ll'mm'nF) + \infty^1\ {}^{12}T(\bar{P}\,\bar{s}\,F)_{k=-1}$$
$$m^6G_3(lm\,\bar{n}\,F) = {}^6G_3(lm\,\bar{n}\,F) + \infty^2\ {}^{12}T(\bar{P}\,\bar{s}\,F)_{k=-1}.$$

## § 4. TYPE III.

41. *First class.*—It is readily seen that there can exist but one mixed group of this type and class, there being but one corresponding continuous group having similar invariant parts on the two systems of generators. This is the continuous group ${}^3G_3(lmF)$, which, if combined with $\infty^2\ {}^{12}T(PsF)_{k=-1}$, gives us the mixed group

$$m^3G_3(lmF) = {}^3G_3(lmF) + \infty^2\ {}^{12}T(P\,\bar{s}\,F)_{k=-1}.$$

42. *Second class.*—There exists in this class a continuous group ${}^3G_2(lmF)_a$, which if combined with $\infty^2\ {}^{12}T(PsF)_{k=-1}$, results in the mixed group $m^3G_2(lmF)_a$.

$$m^3G_2(lmF)_a = {}^3G_2(lmF)_a + \infty^2\ {}^{12}T(\bar{P}\,\bar{s}\,F)_{k=-1}.$$

## § 5. TYPE I.

43. *First class.*—There are three continuous groups of type I, first class, having corresponding parts on each set of generators. Each of these may be combined with $^{12}\mathrm{T}(\mathrm{PsF})_{\mathrm{k}=-1}$, giving us a corresponding mixed group. In the first of these groups there are necessarily two invariant points, and therefore but $\propto^1\ ^{12}\mathrm{T}(\mathrm{PsF})_{\mathrm{k}=-1}$. This gives us the mixed group $\mathrm{m}^1\mathrm{G}_2(\mathrm{ll'mm'F})$. In the second case, there is but one invariant point; hence, $\propto^2\ ^{12}\mathrm{T}(\mathrm{PsF})_{\mathrm{k}=-1}$, and the third case drops to a continuous group.

In symbolic language:

$$\mathrm{m}^1\mathrm{G}_2(\mathrm{ll'mm'F}) = {}^1\mathrm{G}_2(\mathrm{ll'mm'F}) + \propto^1\ {}^{12}\mathrm{T}(\overline{\mathrm{P}}\,\overline{s}\,\mathrm{F})_{\mathrm{k}=-1}$$

$$\mathrm{m}^1\mathrm{G}_4(\mathrm{lmF}) = {}^1\mathrm{G}_4(\mathrm{lmF}) + \propto^2\ {}^{12}\mathrm{T}(\overline{\mathrm{P}}.\,\overline{s}.\,\mathrm{F}.)_{\mathrm{k}=-1}.$$

44. *Second class.*—The second-class groups are derived from those of the first class by making the parameter $r$ any constant value, and we have:

$$\mathrm{m}^1\mathrm{G}_1(\mathrm{ll'mm'F})_{\mathrm{a}} = {}^1\mathrm{G}_1(\mathrm{ll'mm'F})_{\mathrm{a}} + \propto^1\ {}^{12}\mathrm{T}(\overline{\mathrm{P}}\,\overline{s}\,\mathrm{F})_{\mathrm{k}=-1}$$

$$\mathrm{m}^1\mathrm{G}_3(\mathrm{lmF})_{\mathrm{a}} = {}^1\mathrm{G}_3(\mathrm{lmF})_{\mathrm{a}} + \propto^2\ {}^{12}\mathrm{T}(\overline{\mathrm{P}}\,\overline{s}\,\mathrm{F})_{\mathrm{k}=-1}.$$

### F.—Table of Groups of Collineations of Space Leaving Invariant a Quadric Surface.

TYPE XI.

| | *Symbol.* | *Additional invariant figure.* | *Lie.** |
|---|---|---|---|
| (1) | $^{11}\mathrm{G}_1(l\,\mu\,\mathrm{F})$. | All generators of one system, and one generator of other system. | (VI. A) |
| | | TYPE X. | |
| (2) | $^{10}\mathrm{G}_1(ll'\,\mu\,\mathrm{F})$. | All generators of one system, and two generators of second system. | (VI. D) |
| (3) | $^{10}\mathrm{G}_2(l\,\mu\,\mathrm{F})$. | All generators of one system, and one generator of other system. | (V. A) |
| (4) | $^{10}\mathrm{G}_3(\mu\,\mathrm{F})$. | All generators of one system. | (IV. A) |
| | | TYPE IX. | |
| (5) | $^{9}\mathrm{G}_1(\mathrm{lmp}t\mathrm{F})$. | One generator of each system, their common tangent plane, a line of invariant points in that plane tangent to the quadric. | (VI. B) |
| (6) | $^{9}\mathrm{G}_2(\mathrm{lmpF})$. | One generator of each system and their common tangent plane. | (V. E) |

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* Theorie der Transformationsgruppen, vol. III, pp. 251-254.

## TYPE VI.

| | | | |
|---|---|---|---|
| (7) | ${}^6G_1(ll'mm'nF)$,<br>or<br>${}^6G_1(APpsmF)$. | Two generators of each system, a pair of reciprocal polar lines, and all points on one of those lines. | (VI. $E_{c=1}$) |
| (8) | ${}^6G_2(ll'mF)$. | Two generators of one system, and one of other system. | (V. $C_{c=1}$) |
| (9) | ${}^6G_3(lmF)$. | One generator of each system. | (IV. $D_{c=1}$) |
| (10) | ${}^6G_2(APpsF)$. | A point on the surface, tangent plane at that point; a point without surface and its polar plane. | (V. D) |
| (11) | ${}^6G_3(PsF)$. | Point and polar plane. | (IV. E) |

## TYPE III.

*A.—Groups of first class.*

| | | | |
|---|---|---|---|
| (12) | ${}^3G_2(ll'mF)$. | Two generators of one system, one generator of other system. | (V. F) |
| (13) | ${}^3G_3(lmF)$. | One generator of each system. | (IV. B) |
| (14) | ${}^3G_4(lF)$. | One generator of one system. | (III. A) |

*B.—Groups of second class.*

| | | | |
|---|---|---|---|
| (15) | ${}^3G_1(ll'mF)_a$. | Same as ${}^3G_2(ll'mF)$ and constant $a$. | (VI. C) |
| (16) | ${}^3G_2(lmF)_a$. | Same as ${}^3G_3(lmF)$ and constant $a$. | (V. B) |

## TYPE I.

*A.—Groups of first class.*

| | | | |
|---|---|---|---|
| (17) | ${}^1G_2(ll'mm'F)$. | Two generators of each system. | (V. G) |
| (18) | ${}^1G_3(ll'mF)$. | Two generators of one system, and one of the other system. | (IV. C) |
| (19) | ${}^1G_4(ll'F)$. | Two generators of one system. | (III. B) |
| (20) | ${}^1G_4(lmF)$. | One generator of each system. | (III. C) |
| (21) | ${}^1G_5(lF)$. | One generator of one system. | (II) |
| (22) | ${}^1G_6(F)$ | None. | (I) |

*B.—Groups of second class.*

| | | | |
|---|---|---|---|
| (23) | ${}^1G_1(ll'mm'F)_r$. | Same as ${}^1G_2(ll'mm'F)$ and constant r. | (VI. E) |
| (24) | ${}^1G_2(ll'mF)_r$. | Same as ${}^1G_3(ll'mF)$ and constant r. | (V. C) |
| (25) | ${}^1G_3(lmF)_r$. | Same as ${}^1G_4(lmF)$ and constant r. | (IV. D) |
| (26) | | Identical transformation. | (VII) |

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# ON MONOIDS.

BY JOHN N. VAN DER VRIES.

With Plate XV.

The object of this paper is, first, to consider the general monoid. and, second, to classify the quartic monoids in detail.

A monoid $M_a$ of order $a$ is a surface of order $a$ that has an $(a-1)$-tuple point. If the point (0, 0, 0, 1) is taken as this $(a-1)$-tuple point, the equation of the monoid can be written

$$u_a + u_{a-1} s = 0, \qquad (1)$$

where $u_a$ and $u_{a-1}$ are homogeneous functions of x, y and z of degrees $a$ and $a-1$, respectively. The point (0, 0, 0, 1) is called the vertex, and the cones $u_a$ and $u_{a-1}$ the superior and the inferior cones, respectively, of the monoid. There are $a(a-1)$ lines on $M_a$ passing through its vertex, viz., the lines common to the superior and inferior cones. These lines are commonly called "the lines of the monoid."* It is evident that if a line is a $k$-tuple line on $M_a$, it is a line of multiplicity $k$ on one cone and of multiplicity not less than $k$ on the other cone. Four kinds of lines are thus seen to be possible on the monoid, according to the relative multiplicities of the lines on the two cones of $M_a$. We can represent them by the following equations of monoids containing them:

$$\text{I.} \quad (w_k z^{p+1} + w_{k+1} z^p + \ldots\ldots) + s(v_{k+1} z^{p-1} + v_{k+2} z^{p-2} + \ldots\ldots) = 0.$$

where $w_k$ is not identically equal to 0; $v_{k+1}, \ldots\ldots v_{k+g-1}$ may be zero;

$$\text{II.} \quad (w_k z^{p+1} + w_{k+1} z^p + \ldots\ldots) + s(v_k z^p + v_{k+1} z^{p-1} + \ldots\ldots) = 0,$$

where $w_k$ is not identically equal to 0, and $v_k$ does not contain $w_k$ as a factor;

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*This nomenclature is due to Cayley, who uses this surface in his consideration of twisted curves.
Comptes Rendus, t. LIV (1862), pp. 55, 396, 672.

III. $$(w_{k+1}z^p + w_{k+2}z^{p-1} + \dots\dots) + s(v_k z^p + v_{k+1}z^{p-1} + \dots\dots\dots) = 0,$$
where $v_k$ is not identically equal to 0, and $w_{k-1}$ does not contain $v_k$ as a factor;

IV. $$(w_{k+2}z^{p-1} + w_{k+3}z^{p-2} + \dots\dots) + s(v_k z^p + v_{k+1}z^{p-1} + \dots\dots\dots) = 0,$$
where $v_k$ is not identically equal to 0: $w_{k+2}, w_{k+3}, \dots . w_{k+g-1}$ may be zero.

In these equations, $v_k$, $v_{k-1}$, $w_k$, etc., represent functions of x and y of degrees $k$, $k+1$, $k$, etc., respectively; also, $k+p+1=a$. The line xy is a $k$-tuple line on each of these monoids, and differs in the four cases only in its relative multiplicities on the two cones. We shall hereafter distinguish between the different $k$-tuple lines and designate them as lines of kinds I, II, III, and IV, respectively.* No $k$-tuple line of kind I or II has a point on it of higher multiplicity, whereas every line of kind III or IV has on it a point of multiplicity $k+1$ (*e. g.*, the point (0, 0, 1, 0) on the monoids III and IV above). Every line of kind I, III or IV is torsal on every sheet of $M_a$ that contains it, whereas a line of kind II is scrolar on at least one sheet of the monoid that contains it.† In the case of a line of kind II the scrolar sheets are inseparably connected and the tangent plane to them revolves through 180° as we pass along the line. The lines of kinds I, II, III and IV count as $k(k+g)$, $k^2$, $k(k+1)$, and $k(k+g)$ lines, respectively, of the monoid. Lines of kind III differ from lines of kind IV only in the fact that the tangent cone at a $(k+1)$-tuple point on a line of kind IV breaks up entirely into planes.

It is evident that all multiple lines on $M_a$ pass through the vertex. The only lines that $M_a$ can have not passing through the vertex are simple lines. For, a plane passing through the vertex and such a line of multiplicity greater than one would meet the monoid, in addition to this line, in a curve of order less than $a-1$ having an $(a-1)$-tuple point at the vertex; this is impossible. A plane through the vertex and a simple line not passing through the vertex meets the monoid, in addition to this line, in a curve of order $a-1$ having an $(a-1)$-tuple point at the vertex; that is, in $a-1$ lines passing through the vertex. Such simple lines on the monoid that do not pass through the vertex are called transversals by Cayley.‡ Thus, for every transversal on $M_a$ it is necessary to have $a-1$ of the $a(a-1)$ lines of $M_a$ in one plane.

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* For a fuller explanation of these lines, see a paper published in the proceedings of the American Academy of Arts and Sciences, 1902 and 1903.

† A line is torsal or scrolar on a sheet of the surface containing it according as the tangent plane to that sheet of the surface is or is not the same at every point of the line. See Cayley, Collected Papers, vol. VI, p. 334.

‡ Comptes Rendus, t. LVIII (1864), pp. 994-1000.

For the same reason, it is possible to place some limits on the multiplicities of points on the monoid. We have seen that the line joining a $(k+1)$-tuple point to the vertex is at least a $k$-tuple line on the monoid. Suppose the monoid whose equation is (1), with vertex A (0, 0, 0, 1), has a $k$-tuple point B (0, 0, 1, 0) and a $k'$-tuple point C (0, 1, 0, 0), and the lines AB and AC as lines of kind III. Assume $2 \leqq k' \leqq k$. The terms of lowest degree in x, y and s will be of the form $v_{k-1}s + v_k$, multipled by $z^{a-k}$, where $v_{k-1}$ and $v_k$ are homogeneous functions of x and y of degrees $k-1$ and $k$, respectively. Similarly, the terms of lowest degree in x, z and s will be of the form $v_{k'-1}s + v_{k'}$, multiplied by $y^{a-k'}$, where $v_{k'-1}$ and $v_{k'}$ are homogeneous functions of x and z of degrees $k'-1$ and $k'$, respectively. The terms of lowest degree in x and s will be of degree $k-k'$ more in y than in z, and will be of the form

$$ax^{k+k'-a}y^{a-k'}z^{a-k} + bx^{k+k'-a-1}y^{a-k'}z^{a-k}s\,;$$

the line $xs \equiv BC$ thus being a line of multiplicity $k+k'-a$ on $M_a$. As this cannot be a line of multiplicity greater than one, we must have

$$k+k'-a \leqq 1, \text{ i. e., } k+k' \leqq a+1\,;$$

that is, the monoid cannot have two points on it, excluding the vertex, the sum of whose multiplicities exceeds $a+1$. Thus, if $4 \leqq a$, $M_a$ can have at most two vertices, excluding the case where it has an infinite number and is a ruled surface. As we know, the cubic may have four, a quadric has a double infinity, and the plane a triple infinity of vertices.

If $M_a$ has an $(a-1)$-tuple point B in addition to its vertex A, it can have in addition no point of multiplicity greater than two. The line AB is a line of multiplicity $a-2$, at least, on the monoid. (If the line is an $(a-1)$-tuple line, any point of it may be considered as the vertex of $M_a$. It must then be a line of kind II, and counts as $(a-1)^2$ lines of $M_a$. Therefore, through any point of the line taken as vertex will there pass $a(a-1)-(a-1)^2 \equiv a-1$ other lines of $M_a$. These are the additional lines in which the tangent planes to the $a-1$ sheets of $M_a$ at any point of the line meet $M_a$.) If the line AB is an $(a-2)$-tuple line on $M_a$, the monoid can have no other multiple line and, therefore, no point of multiplicity greater than two. It can have a point of multiplicity two, the lines joining this point to A and B being simple lines on the surface. A plane through the $(a-1)$-tuple points A and B and a double point C intersects the surface in a curve having $(a-1)$-tuple points at A and B and a double point at C, that is, in the $(a-2)$-tuple line AB and the two simple lines AC and BC. If the $(a-2)$-tuple line AB is of kind III, it counts as $(a-2)(a-1)$ lines of $M_a$. The monoid can then have only $a(a-1)-(a-1)(a-2)$ or $2(a-1)$ other lines. If it has double points, the lines joining these to the vertex must be simple lines of kind III or IV. If they are of kind III, each counts as 2 lines of $M_a$. The monoid can thus have at most $a-1$ double points. (If the lines are all of kind IV, each counts as $\beta$ lines of the monoid; where $3 \gtreqless \beta$. $M_a$ can have at most $\frac{2(a-1)}{\beta}$ double points.) This is also evident from the fact that a cone of order a can have at most $\frac{a(a-1)}{2}$ double edges. The $(a-2)$-tuple line on the monoid, being an $(a-1)$-tuple line on the superior cone, counts for $\frac{(a-1)(a-2)}{2}$ double edges of this cone. This cone can thus have at most $a-1$ double edges in addition to the $(a-1)$-tuple edge AB; that is, $M_a$ can have at most $a-1$ double points in addition to the $(a-1)$-tuple point B.*

*If a monoid having such an $(a-1)$-tuple point B has its maximum number of double points, these $a-1$ points all lie on one line.* For the superior cone breaks up into $a$ planes, of which $a-1$ pass through one line, the $a-1$ double edges being the intersections of these $a-1$ planes with the $a$-th plane of the cone. The $a-1$ double points of $M_a$ thus lie in one plane passing through A. Similarly, considering B as the vertex, it is evident that they all lie in one plane passing through B. They must, therefore, lie on the line CD, which is the intersection of the two planes. This line CD is a simple line on $M_a$. A monoid, therefore, with a second $(a-1)$-tuple point and $a-1$ double points has just $a$ transversals, $a-1$ of them being the lines from the double points to the $(a-1)$-tuple point, and the remaining one the line joining all the double points.†

If a monoid having a second $(a-1)$-tuple point, and the line from this point to the vertex as a line of kind III has $a-2$ double points in addition (that is, all but one of the maximum number of additional double points), these will in general lie on a twisted cubic: for, the superior cone, in order to have an $(a-1)$-tuple edge and $a-2$ double edges, must break up into at least $a-1$ components. If it breaks up into this number, $a-2$ of them will be planes and the remaining one a quadric cone. The $a-2$ planes will all pass through one edge of this quadric cone, the remaining lines of intersection being the $a-2$ double edges of the superior cone; *i. e.*, the lines to the $a-2$ double points of $M_a$. The double points will thus in general lie on two quadric cones that have the two $(a-1)$-tuple points of $M_a$ as their vertices and the line AB as a common edge; that is, they will lie on the cubic curve that is the residual intersection of these quadric

---

*If the $(a-2)$-tuple line AB is of kind IV, it counts as $a(a-2)$ lines of $M_a$; the monoid can, therefore, have at most $a$ other lines. It can have, however, no double points, for the superior cone contains the line AB as an $a$-tuple edge and can, therefore, have no double edge.

†The points will then lie as in figure (1) in the accompanying plate.

cones. In special cases, however, the superior cone from one or both vertices may break up entirely into planes. In these cases the $a-2$ double points will lie either on a conic or on a straight line.

*It is thus evident that, if a monoid* $M_a$ *having a second* $(a-1)$*-tuple point and an* $(a-2)$*-tuple point of kind III joining this point to the vertex has* $a-m$ *double points in addition, these points will in general lie on a curve of order* $2m-1 \left\{ viz., m^2-(m-1)^2 \right\}$, *which is the residual intersection of two cones of order* $m$ *having different vertices but having an* $(m-1)$*-tuple line in common.*

A plane through the $(a-2)$-tuple line and a simple line (of which there can be at most $2(a-1)$ ) cuts $M_a$ in a curve of order $a$ having $(a-1)$-tuple points at A and B. This curve consists of the $(a-2)$-tuple line AB, an ordinary line through A, and a transversal through B. This transversal intersects the simple line, if of kind II, at the point where the cutting plane coincides with the scrolar tangent plane to $M_a$ along this line, or at the double point, if the simple line has a double point on it. A monoid with an $(a-1)$-tuple point and no double points has at least $2(a-1)$ transversals, whereas every double point reduces this minimum number by one. As we have seen, for every additional transversal (*i.e.*, additional to the $2(a-1)$ if $M_a$ has no double points, to the $2(a-1)-\beta$ if it has $\beta$ double points) $a-1$ of the lines of the monoid, excluding the $(a-2)$-tuple line, must lie in one plane. If the cones of the monoid are general cones, this does not in general occur. There are, however, special cases in which the cones break up. As both cones have a line on them of multiplicity one less than the order of the cone, at least all but one of the components of each cone are planes. If the superior cone breaks up into $a$ planes, of which $a-1$ pass through one line, the $a$-th plane meets the inferior cone in $a-1$ lines, which are thus $a-1$ lines of $M_a$ in one plane. The inferior cone can never break up entirely into planes, for the plane not passing through the multiple line would meet the superior cone in $a$ lines; *i. e.*, we would have $a$ lines of the monoid lying in one plane. This is impossible. A monoid thus with an $(a-1)$-tuple point B and no double points has, in general, $2(a-1)$ transversals; it may have one more.

A plane through the $(a-2)$-tuple line and not through any other line of $M_a$ meets the monoid in the $(a-2)$-tuple line and a conic passing through both $(a-1)$-tuple points.

If $M_a$ has an $(a-2)$-tuple point B, and if the line joining this point to the vertex is an $(a-3)$-tuple line of kind III, the monoid can have at most $2a-3$ double points in addition. The line from each of these points to B is then a simple line of kind III. If $M_a$ has this maximum number of double points, the superior cone breaks up entirely in planes, of which $a-2$ pass through the line AB. The double points will thus lie in two planes through the vertex, $a-1$ lying in each plane. As a plane through $a-1$ lines to $a-1$ double points intersects the monoid in addition to these lines in one other line, viz., a transversal, and as the curve of intersection of the monoid by this plane has a double point wherever the plane meets the monoid in a double point, it is evident that this transversal passes through all the double points in that plane. The $2a-3$ double points thus lie on two intersecting lines, the point of intersection being one of these $2a-3$ double points. It can be shown by similar reasoning that the $(a-2)$-tuple point lies in the same plane as these $2a-3$ double points. Thus, if $a=7$, $2a-3=11$, the points lie as in figure (2).

A monoid having an $(a-2)$-tuple point B and $2a-4$ double points will, in general, be one of two kinds. Thus, if $a=7$, $2a-4=10$, the points will lie either as in figure (3) or as in figure (4). If, as in figure (3), where the superior cone breaks up into a quadric cone, $a-3$ planes that pass through an ordinary edge of the quadric cone and a plane that does not pass through this edge, the double points all lie in one plane. If, as in figure (4), where the superior cone breaks up into a quadric cone and $a-2$ planes that have a line in common, the double points lie by twos on lines that pass through the point B.

By this method of considering the possible ways in which the cones of the monoid may break up, we can determine the possible positions of the double points on the monoid. We shall not consider any more general cases, but shall make use of this method in the consideration of the quartic monoids.

A monoid with an $(a-2)$-tuple point may have in addition a certain number of triple points. It can clearly have no points of multiplicity greater than three. A triple line on the superior cone counts for three double edges of this cone; a line to a triple point on the monoid, if it is of kind III, counts for six lines of the monoid, whereas a line to a double point counts only for two of these lines; nevertheless, we cannot always substitute a triple point for three double points. We can say that a monoid with an $(a-2)$-tuple point can never have more than $\frac{2a-3}{3}$ triple points, but the exact number must be investigated in each case.

A monoid having an $(a-k)$-tuple point on a line of kind III can have in addition points of multiplicity $k+1$ at most, where $2 \leqq a-k$. If it has a $(k+1)$-tuple point C in addition to an $(a-k)$-tuple point B, the line BC is a transversal on the monoid; but if the point C is a point of any multiplicity less than $k+1$, the line BC will not in general be a transversal. If the line AB is a line of kind III, it counts for $(a-k)(a-k-1)$ of the $a(a-1)$ lines of the monoid. The monoid with such an $(a-k)$-tuple point may thus have in addition $k(2a-k-1)$ simple lines passing through the vertex. If the line to a $k$-tuple point is a line of kind III, it counts as $k(k+1)$ lines of the monoid. The monoid with an $(a-k)$-tuple point can thus never have more than $\frac{2a-k-1}{k+1}$ $(k+1)$-tuple points.

A monoid can have at most $\frac{a(a-1)}{2}$ double points, for every line of kind III joining a double point of a monoid to its vertex counts for two lines of the monoid. This is also evident from the fact that this is the maximum number of double lines possible on the superior cone; for we can always pass a cone of order $a-1$ through this number of lines, and can therefore always have these double lines of the superior cone as simple lines on the inferior cone, and, therefore, as simple lines of kind III on the monoid. If the monoid has this maximum number of double points, the superior cone breaks up entirely into planes through the vertex. Each of these $a$ planes meets each of the remaining $a-1$ planes in a line that is a line of the monoid to a double point. Thus $a-1$ double points lie in each of these $a$ planes. Each plane thus intersects the monoid in $a-1$ lines through the vertex and in a transversal which passes through all the double points in the cutting plane. The $\frac{a(a-1)}{2}$ double points thus lie on $a$ lines, $a-1$ of them on each line, *and therefore lie in one plane.* If $a=7$, the points will lie as in figure (5).

No monoid can have on it a multiple curve of an order higher than the first, and then, as we have seen, it passes through the vertex; for, any line drawn from the vertex to a $k$-tuple curve, say of order $\beta$, where $2 \overline{\gtrless} k$, would meet the monoid in at least $a+1$ points, and would therefore lie on the monoid. The monoid would then break up into a cone of order $\beta$, having the vertex of the monoid as vertex and the curve as base, and a monoid of order $a-\beta$.

## Classification of Quartic Monoids.

1. The equation of the general quartic monoid with vertex at the point A(0, 0, 0, 1) may be written

$$u_4 + u_3 s = 0,$$

where $u_4$ and $u_3$ are homogeneous functions of x, y and z of degrees 4 and 3, respectively. This monoid has twelve lines on it passing through the vertex; it will, in addition, have a transversal whenever three of these twelve lines lie in one plane.

2. If $M_4$ has another triple point (say the point B(0, 0, 1, 0), and if

the line AB is a line of kind III, the equation of the monoid may be written

$$u_4 + (az + bs)u_3 + u_2zs = 0, \qquad (2)$$

where $u_4$, $u_3$ and $u_2$ are homogeneous functions of x and y of degrees 4, 3, and 2, respectively. This line $AB \equiv xy$ is a double line on $M_4$, but counts as six lines of the monoid. This monoid thus has in general six other lines. A plane through the double line AB and one of these six lines meets the monoid in addition in a transversal that intersects AB at B and the single line at the point where the tangent plane to the surface along this line coincides with the cutting plane. There will thus be six transversals on the monoid, all passing through the point B. For every additional transversal it is necessary for the two cones of the monoid to have three of their six simple intersections in one plane. If the cones are proper cones—that is, in the general case of the monoid whose equation is (2)—this does not occur. The monoid then has six lines in addition to AB passing through each vertex, each line through one vertex meeting one line through the other vertex. There will be no other lines on this monoid.

If, however, the superior cone breaks up into four planes, of which three pass through the line xy, while the inferior cone does not break up, there will be three lines of the monoid in one plane, viz., the intersections of the inferior cone by the fourth plane of the superior cone. A cross section of the two cones by a plane not passing through their common vertex will show this. See figure (6).* Taking x, y, ax + by and z as the four planes of the superior cone, the equation of the monoid will be of the form

$$(ax + by)xyz + s(u_3 + u_2z) = 0,$$

where $u_3$ and $u_2$ are homogeneous functions of x and y of degrees 3 and 2, respectively. This monoid will thus have seven transversals, or thirteen lines in all, in addition to the double line AB. In figure (6), 1, 2 and 3 are the traces of lines that lie in one plane. It is, however, possible that the lines whose traces are 4, 5 and 6 also lie in one plane; the monoid then has eight transversals. Similarly, when the superior cone breaks up into planes and the inferior cone breaks up into a plane and a quadric cone, the monoid has either seven or eight transversals. The equation may then be written

$$(ax + by)xyz + su_1u_2 = 0,$$

where $u_1$ and $u_2$ are homogeneous functions of x, y and z of degrees 1 and 2, respectively. The traces of the lines will then be as in figure (7). The lines whose traces are 1, 2 and 3 lie in one plane; the line

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*In the figures representing the traces of the cones of the monoid on a plane, we use broken lines to distinguish the inferior from the superior cone.

whose trace is 2 may or may not lie in the same plane with the lines whose traces are 4 and 5.

A plane passing through the line AB but not through one of these six ordinary* lines of the monoid cuts the surface, in addition to the double line AB, in a conic that passes through both points A and B.†

3. A quartic monoid may have a double point, say the point C(0, 1, 0, 0), in addition to the two triple points A and B. If the line AC is a simple line of kind III, it will count for two lines of the monoid. The equation may then be put into the form

$$x^2u_2 + xzv_2 + xsw_2 + zst_2 = 0,$$

where $u_2$, $v_2$, $w_2$ and $t_2$ are homogeneous functions of x and y of degree 2. The superior cone breaks up into the plane x and a cubic cone that contains the line xy as a double line. In addition to the line xy and the simple line xz, there are four ordinary lines on the monoid. A plane through the double line and one of the five simple lines cuts the monoid in addition in a transversal that intersects the double line in the triple point and the simple line either at the double point or at the point where the tangent plane coincides with this cutting plane. This monoid thus has in general five transversals. There may be cases, as in the previous monoid, where three of the five simple lines lie in one plane, thus causing an additional transversal.

4. If $M_4$ has a second double point, say the point D(1, 0, 0, 0), and if the line from this point to the vertex is a line of kind III, the equation may be written

$$x^2y^2 + xyzu_1 + xysv_1 + u_2zs = 0,$$

where $u_1$, $v_1$ and $v_2$ are homogeneous functions of x and y of degrees 1, 1, and 2, respectively. The superior cone breaks up into two planes x and y and a quadric cone that contains the line xy as a simple edge. There are through each triple point two ordinary lines in addition to the two lines to the double points. This monoid thus has in general four transversals passing through the second triple point. As in the previous cases, three of the four simple lines may lie in one plane, thus causing an additional transversal. It is evident that such a line may be considered as a transversal with respect to both vertices.

5. A quartic monoid can have at most three double points in addition to a second triple point. The lines from the three double points to either triple point are simple lines of the monoid, and are lines of kind III; each one, therefore, counts as two lines of the monoid, considering the triple point through which it is drawn as vertex. Thus,

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* Hereafter, when we speak of ordinary lines, we mean simple lines of kind II.

† In every case where the monoid has two triple points, either point may be taken as the vertex; whatever is true for the lines passing through one triple point is likewise true for the lines passing through the other triple point.

through either vertex there is a double line to the other vertex and three simple lines to the three double points. The superior cone, in order to have a triple edge and three double edges, must break up into four planes, three of which pass through the line xy. It is thus evident that the three double points must lie on one straight line. The monoid thus has four transversals; it can have no more. If the four planes of the superior cone are the planes x, y, z, and $ax + by$, the equation of the monoid may be written

$$(ax + by)(cz + ds)xy + zsu_2 = 0,$$

where $x = z = s = 0$, $y = z = s = 0$ and $ax + by = z = s = 0$ are the three double points.

6. If, in addition to the triple point B on the line AB of kind III, the monoid has a double point C(0, 1, 0, 0) lying on a line AC of kind IV, where the line AC is a simple line on the inferior cone and a triple line on the superior cone, the equation of the monoid may be written

$$x^3u_1 + x^2zv_1 + xsw_2 + zst_2 = 0,$$

where $u_1$, $v_1$, $w_2$ and $t_2$ are homogeneous functions of x and y of degrees 1, 1, 2, and 2, respectively. The superior cone breaks up into the plane x doubled and a quadric cone that contains the lines xy and xz as simple lines. The line xz thus counts for three of the six lines of the monoid. The line BC also counts for three lines of the monoid when B is considered as the vertex. The monoid has in general three transversals, the transversals being the additional lines in which the planes through the line AB and the three lines through the vertex cut the monoid. It has an additional transversal only if the three lines lie in one plane, or if two of them lie in one plane with the line AC; in the latter case the transversal passes through the point C.

7. If the monoid has in addition a second double point, say the point D(1, 0, 0, 0), the line AD will be a simple line of kind III on the monoid. The equation of the monoid may then be written

$$x^2yz + xysu_1 + zsu_2 = 0,$$

where $u_1$ and $u_2$ are homogeneous functions of x and y of degrees 1 and 2, respectively. There is thus one ordinary line passing through A in addition to the lines AB, AC, and AD; it lies in the same plane with AC and AD. For the plane $z = 0$ meets the monoid in the lines AC, AD, CD, and $z = u_1 = 0$. Call this last line AE, where E is the point in which it meets CD. This quartic monoid therefore has on it four lines, viz., AB, AC, AD, and AE, and four transversals, viz., BC, BD, BE, and CD. It can have no other line.

8. If the triple point B is connected with the vertex A by a line of kind IV, the equation of the monoid must be of the form

$$u_4 + u_3s + u_2zs = 0,$$

where $u_4$, $u_3$ and $u_2$ are homogeneous functions of x and y of degrees 4, 3, and 2, respectively. The line xy is a quadruple line on the superior and a double line on the inferior cone; it therefore counts as eight of the twelve lines of the monoid. The monoid, therefore, always has four lines on it passing through each triple point, every line passing through one of the two triple points meeting one line through the other triple point. If three of the lines through one triple point lie in one plane, the three corresponding lines through the other triple point lie in one plane; the monoid then has an additional transversal. A plane through the double line AB, and not through an ordinary line of this $M_4$, meets the monoid in addition in a conic that passes through the two triple points.

9. If the monoid has in addition a double point C(0, 1, 0, 0) on a line AC of kind III, the equation may be written

$$x^2u_2 + xsv_2 + zsw_2 = 0,$$

where $u_2$, $v_2$ and $w_2$ are homogeneous functions of x and y of degree 2. Since the line AC counts for two lines of $M_4$, there are only two other lines on $M_4$ through A. The monoid has in general three transversals —one the line BC, and the other two lines that pass through the triple point and intersect the two ordinary lines through A at the points where the tangent planes to the monoid along these lines coincide with the cutting planes. If we take these two ordinary lines to be the lines yz and (ax + by)z, the equation of the monoid may be written

$$xy(ex + fs)(ax + by) + xzs(cx + dy) + y^2zs = 0.$$

The line $z = ex + fs = 0$ lies on this monoid; $M_4$ thus has four transversals in this case.

10. If the monoid has in addition a second double point, say the point D(1, 0, 0, 0), that lies on a line AD of kind III, the equation may be written

$$x^2y^2 + xysu_1 + zsu_2 = 0,$$

where $u_1$ and $u_2$ are homogeneous functions of degrees 1 and 2, respectively. The monoid has no lines in addition to the lines AB, AC, and AD. It has two transversals, viz., the lines BC and BD; it can have no other. A plane through the two lines AC and AD intersects the monoid in addition in a conic that passes through the vertex A and the two double points C and D.

11. If the monoid that has a triple point B lying on a line AB of kind IV has, in addition, a double point C on a line AC of kind IV, it may be of two kinds. The line AC may be a triple line on the superior and a simple line on the inferior cone. The equation may then be written

$$x^3u_1 + xsu_2 + zsv_2 = 0,$$

where $u_1$, $u_2$ and $v_2$ are homogeneous functions of x and y of degrees 1, 2, and 2, respectively. The line AC counts for three lines of $M_4$; the monoid can therefore have but one additional line through A. There are, therefore, only two transversals, one being the line BC and the other the additional line in which a plane through AC and the ordinary line through A intersects the monoid. If, however, the line AC is a quadruple line on the superior cone and a simple line on the inferior cone, the equation may be written

$$x^4 + xsu_2 + zsv_2 = 0,$$

where $u_2$ and $v_2$ are homogeneous functions of x and y of degree 2. The line AC then counts as four lines of the monoid; the monoid has, therefore, no more lines through A. This monoid has thus but one transversal, viz., the line BC.

12. A monoid $M_4$ that has no second triple point can have at most six double points; for every double point causes a line on the monoid that counts for at least two lines of the monoid. It is thus evident that, if a quartic monoid has six double points, the lines joining these points to the vertex are all lines of kind III on the monoid. If the monoid has $\beta$ double points (where $\beta \overline{\gtrless} 6$), and if the lines from these to the vertex are all lines of kind III, the monoid may have $\beta$ lines from the vertex to the $\beta$ double points and $12 - 2\beta$ ordinary lines through the vertex, or $12 - \beta$ lines in all. There are thus six different quartic monoids that have only double points and that have these joined to the vertex by lines of kind III.

In general, a plane passing through two lines on a monoid intersects the monoid, in addition, in a conic that passes through the vertex and that crosses the lines at the double points on them, if there are any, otherwise at the points at which the tangent plane coincides with the cutting plane. In special cases, however, this conic may break up into a line through the vertex and a transversal.

If the quartic monoid has a double point, say the point B(0, 0, 1, 0), the equation of the monoid may be written

$$u_4 + u_3z + u_2z^2 + v_3s + v_2zs + v_1z^2s = 0,$$

where $u_4$, $u_3$, $u_2$, $v_3$, $v_2$ and $v_1$ are homogeneous functions of x and y of degrees 4, 3, 2, 3, 2, and 1, respectively. The line AB then counts for two of the twelve lines of the monoid; the monoid thus has, in addition, ten ordinary lines passing through the vertex. It will not in general have any transversals.

If the quartic monoid has a second double point, say the point C(0, 1, 0, 0), the equation may be written

$$x^2u_2 + xzv_2 + z^2w_2 + xst_2 + zsr_2 + z^2su_1 = 0,$$

where $u_2$, $v_2$, $w_2$, $t_2$, $r_2$ and $u_1$ are homogeneous functions of x and y of degrees 2, 2, 2, 2, 2, and 1, respectively. The line AC also counts for two lines of the monoid; the monoid, therefore, has eight ordinary lines through the vertex in addition to the lines AB and AC; that is, ten lines in all. It will not in general have any transversals.

If the quartic monoid has a third double point, say the point D(1, 0, 0, 0), the equation of the monoid may be written

$$x^2y^2 + xyzu_1 + z^2u_2 + xysv_1 + zsv_2 + z^2sw_1 = 0,$$

where $u_1$, $u_2$, $v_1$, $v_2$ and $w_1$ are homogeneous functions of x and y of degrees 1, 2, 1, 2, and 1, respectively. This monoid has six ordinary lines through the vertex in addition to the lines AB, AC, and AD; that is, nine lines in all. This monoid does not in general have the three double points lying in one plane. If, however, we take these three points to be the points $x = y = s = 0$, $x = z = s = 0$, and $x = y + z = s = 0$, the superior cone breaks up into the plane x and a cubic cone. The plane x meets the inferior cone in three lines, which are the lines of kind III, to the three double points. The equation of the monoid may then be written

$$x^2u_2 + x^2zu_1 + xsv_2 + xzsv_1 + xz^2t_1 + yz(y + z)w_1 = 0,$$

where $u_2$, $u_1$, $v_2$ and $v_1$ are homogeneous functions of x and y of degrees 2, 1, 2, and 1, respectively, and where $t_1$ and $w_1$ are homogeneous functions of x and s of degree 1. The plane x intersects the monoid in the four lines $x = y = 0$, $x = z = 0$, $x = y + z = 0$, $x = s = 0$; the monoid thus has a transversal passing through the three double points.

The equation of the monoid that has four double points, and that has these points joined to the vertex by lines of kind III, is of the form

$$u_2v_2 + u_3s = 0,$$

where $u_2$, $v_2$ and $u_3$ are homogeneous functions of x, y and z of degrees 2, 2, and 3, respectively. The superior cone breaks up into two quadric cones, the lines of intersection of these cones being simple lines on the inferior cone. If these cones are general quadric cones, no three of the four lines to double points will lie in one plane. Two of them may, however, lie in the same plane as one of the four additional lines through the vertex; the monoid will then have an additional transversal through the two double points in that plane. It will also have an additional transversal if three of these four lines lie in one plane.

If a quartic monoid has five double points, and has the lines from these points to the vertex as lines of kind III, its equation may be written

$$u_1v_1w_2 + u_3s = 0,$$

where $u_1$, $v_1$, $w_2$ and $u_3$ are homogeneous functions of x, y and z of degrees 1, 1, 2, and 3, respectively. The superior cone thus breaks up into a quadric cone and two planes; it therefore has five double edges which are simple lines on the inferior cone. The monoid contains only two ordinary lines in addition to the lines to these five double points. It is evident, from the way in which the superior cone breaks up, that there are always two planes that contain three lines to double points. There are, therefore, always two transversals on a quartic monoid of this kind, and each of these two lines passes through three double points. The five double points therefore lie in one plane, as in figure (8). There may be additional cases in which one of the two ordinary lines of kind II lies in a plane with two of the lines to double points, or in which the two lines lie in a plane with one of the lines to a double point, an additional transversal being the result in each case.

The equation of the quartic monoid that has six double points, and has these points joined to the vertex by lines of kind III, may be written in the form

$$u_1v_1w_1t_1 + u_3s = 0,$$

where $u_1$. $v_1$, $w_1$, $t_1$, and $u_3$ are homogeneous functions of x, y and z of grees 1, 1, 1, 1, and 3, respectively. The superior cone breaks up into four planes, for in no other way can it have six double edges; the inferior cone contains these lines as simple edges. These six lines lie by threes in four planes. There are, therefore, four transversals, each one passing through three double points. The six double points therefore lie in one plane. The monoid can have no line on it in addition to the six to the double points and no transversal in addition to the four joining the six double points.

13. If the quartic monoid has a double point, say the point B(0, 0, 1, 0), and has the line AB as a particular line of kind IV (viz., as a line of multiplicity three on the superior cone and of multiplicity one on the inferior cone), its equation may be written

$$u_4 + u_3z + v_3s + u_2zs + u_1z^2s = 0, \qquad (3)$$

where $u_4$, $u_3$, $v_3$, $u_2$ and $u_1$ are homogeneous functions of x and y of degrees 4, 3, 3, 2, and 1, respectively. The monoid may have nine ordinary lines in addition to the line AB. It will not in general have a transversal.

If the monoid whose equation is (3) has a second double point, say the point C(0, 1, 0, 0), and has the line AC as a simple line of kind III, its equation may be written in the form

$$x^2u_2 + xzv_2 + xsw_2 + zst_2 + z^2su_1 = 0,$$

where $u_2$, $v_2$, $w_2$, $t_2$ and $u_1$ are homogeneous functions of x and y of degrees 2, 2, 2, 2, and 1, respectively. The monoid has seven lines passing through the vertex in addition to the lines AB and AC. The superior cone breaks up into the plane x and a cubic cone. The plane x intersects the monoid in the lines AB, AC, BC, and $(x = ay + bz = 0)$. The monoid thus has in general one transversal; it may in special cases have more, as in some of the monoids already considered.

If this quartic monoid has a third double point, say the point D(1, 0, 0, 0), and if the line AD is a line of kind III on the monoid, the equation of the monoid may be written

$$x^2y^2 + xyzu_1 + xysv_1 + zsu_2 + z^2sw_1 = 0,$$

where $u_1$, $v_1$, $u_2$ and $w_1$ are homogeneous functions of x and y of degrees 1, 1, 2, and 1, respectively. The superior cone breaks up into two planes and a quadric cone. Each of these two planes intersects the inferior cone of the monoid in three lines that are therefore three lines of the monoid lying in one plane. The monoid therefore has, in general, two transversals, each one passing through two double points. The monoid has on it five ordinary lines in addition to the lines AB, AC, and AD.

If this monoid has a fourth double point, say the point E, and if the line AE is a line of kind III on the monoid, the superior cone must break up entirely into planes, of which three pass through the line $AB \equiv xy$. The three lines AC, AD and AE are therefore in one plane. We can therefore take the line AE to be the line $ax + by = z = 0$. The plane z intersects the monoid in the three lines AC, AD, and AE, and in a transversal that passes through the three double points C, D, and E. The equation of the monoid may then be written

$$xyz(ax + by) + xys(ax + by) + zsu_2 + z^2su_1 = 0,$$

where $u_2$ and $u_1$ are homogeneous functions of x and y of degrees 2 and 1, respectively. Each of the three planes through the triple line of the superior cone intersects the inferior cone in three lines that are lines of the monoid lying in one plane. This monoid, therefore, always has four transversals, one of which passes through three double points and the other three each through two double points. This monoid has on it three ordinary lines in addition to the four lines to the double points.*

This monoid cannot have any more double points, as the superior cone cannot break up into any more components, three double edges being the maximum number of double edges that a quartic cone can have in addition to a triple edge.

14. If the quartic monoid whose equation is (3) has a second double point, say the point C(0, 1, 0, 0), and has the line AC as a

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*There is a special monoid with four double points that has only three transversals, viz., the monoid whose superior cone breaks up into a double plane and two single planes that do not have a line of the double plane in common, where three of the lines to double points are lines of the double plane of the superior cone, as in figure (9).

triple line on the superior cone and a simple line on the inferior cone, the equation of the monoid may be written

$$x^3u_1 + x^2zv_1 + xsu_2 + zsv_2 + z^2sw_1 = 0, \qquad (4)$$

where $u_1$, $v_1$, $u_2$, $v_2$ and $w_1$ are homogeneous functions of $x$ and $y$ of degrees 1, 1, 2, 2, and 1, respectively. The superior cone breaks up into a double plane and two single planes, the lines common to the double plane and the single planes being simple lines on the inferior cone. Each of the three planes of the superior cone meets the inferior cone in three lines, which are, therefore, lines on the monoid; this monoid therefore has, in general, three transversals. There are six ordinary lines on the monoid in addition to the lines AB and AC.

The monoid whose equation is (4) cannot have another double point on it and have the line from this point to the vertex as a triple line on the superior cone and a simple line on the inferior cone. It may, however, have a double point, say the point D(1, 0, 0, 0), and the line AD as a simple line of kind III; its equation may then be written

$$x^2yz + xysu_1 + zsu_2 + z^2sv_1 = 0, \qquad (5)$$

where $u_1$, $u_2$ and $v_1$ are homogeneous functions of $x$ and $y$ of degrees 1, 2, and 1, respectively. The superior cone breaks up into a double plane and two single planes, as in the previous case; the intersection of the two single planes being, moreover, an edge of the inferior cone of this monoid. The double plane of the superior cone intersects the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a double line on the superior cone and a single line on the inferior cone, is also a line to a double point. Therefore, if the monoid whose equation is (4) has one double point in addition it must have a second double point, and each of these must be connected with the vertex by a line of kind III. This monoid whose equation is (5) has therefore, in general, three transversals, two of which pass through two and the third of which passes through three double points. This monoid has two ordinary lines in addition to the lines to the double points, and they lie in the two planes determined by the vertex and the two transversals that pass through only two double points. The double points will lie as in figure (10).

The monoid cannot have any more double points, as the superior cone cannot break up into any more components, one double edge being the maximum number of double edges that a quartic cone can have in addition to two triple edges.

15. If the quartic monoid has a double point, say the point B(0, 0, 1, 0), and has the line AB as a quadruple line on the superior cone and a simple line on the inferior cone, it is evident that the su-

perior cone must break up entirely into planes that pass through the line AB. The equation of the monoid may then be written

$$u_4 + u_3 s + u_2 z s + u_1 z^2 s + z^3 s = 0, \qquad (6)$$

where $u_4$, $u_3$, $u_2$ and $u_1$ are homogeneous functions of x and y of degrees 4, 3, 2, and 1, respectively. Each of the planes of the superior cone meets the inferior cone in two lines in addition to the line to the double point; the monoid, therefore, always has four transversals passing through the double point. There are eight ordinary lines on the monoid in addition to the line AB; they lie by twos in four planes that pass through the line AB. If three lines of which no two lie in one of these four planes lie in one plane, an additional transversal will lie on the monoid.

If this monoid has in addition a double point, say the point C(0, 1, 0, 0), and has the line AC as a simple line of kind III, the equation of the monoid may be written

$$x^2 u_2 + x s v_2 + z s w_2 + z^2 s u_1 + z^3 s = 0,$$

where $u_2$, $v_2$, $w_2$ and $u_1$ are homogeneous functions of x and y of degrees 2, 2, 2, and 1, respectively. The superior cone breaks up into a double plane and a quadric cone; the double plane intersects the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a double line on the superior cone and a single line on the inferior cone, is also a line to a double point. Therefore, if the monoid whose equation is (6) has one double point in addition, it must have a second double point, and each of these must be connected with the vertex by a line of kind III. There are four ordinary lines on the monoid in addition to the lines to the double points, and they lie by twos in the two planes which are single components of the superior cone. This monoid always has three transversals, of which one passes through all the double points.

If the monoid whose equation is (6) has in addition a double point C(0, 1, 0, 0), and has the line AC as a line of kind IV that is a triple line on the superior cone and a simple line on the inferior cone, its equation may be written

$$x^3 u_1 + x s v_2 + z s w_2 + z^2 s v_1 + z^3 s = 0,$$

where $u_1$, $v_2$, $w_2$ and $v_1$ are homogeneous functions of x and y of degrees 1, 2, 2, and 1, respectively. The superior cone breaks up into the triple plane x and another plane through the line AB; the triple plane meets the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a triple line on the superior cone and a simple line on the inferior cone, is a line to a double point that counts for three lines of the monoid. The monoid thus has on it two transversals, of which one passes through all three double points and the other one through only one double point. The monoid has on it two ordinary lines in addition to the lines to the three double points, and these two lines lie in the plane that is the simple component of the superior cone.

If the monoid whose equation is (6) has in addition a double point C(0, 1, 0, 0), and has the line AC as a line of kind IV that is a quadruple line on the superior cone and a simple line on the inferior cone, its equation may be written

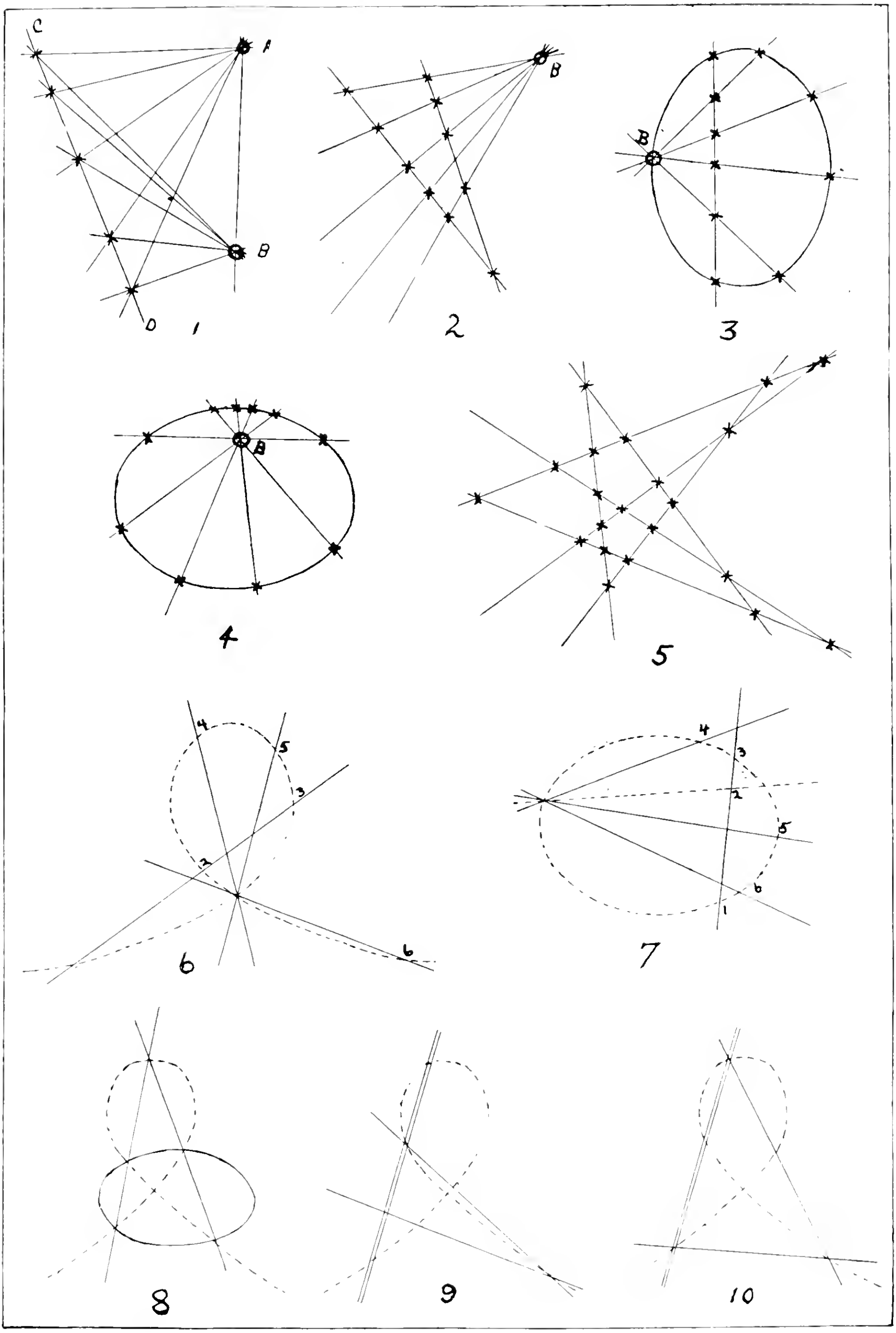
$$x^4 + xsu_2 + zsv_2 + z^2su_1 + z^3s = 0,$$

where $u_2$, $v_2$ and $u_1$ are homogeneous functions of x and y of degrees 2, 2, and 1, respectively. The superior cone breaks up into the quadruple plane x. This plane intersects the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a quadruple line on the superior cone and an ordinary line on the inferior cone, is a line to a double point that counts for four lines of the monoid. The monoid has on it no lines passing through the vertex in addition to the lines to the three double points. It has on it only one transversal, and this passes through the three double points.

16. We have finished all possible cases of quartic monoids having on them only multiple points that lie on lines of multiplicity one less than the multiplicity of the lines; *i. e.*, all possible cases of monoids that have on them only lines of kinds III and IV in addition to simple lines of kind II. We shall now consider the possible cases of quartic monoids that have lines of kind I and multiple lines of kind II in addition to these multiple points that lie on lines of kinds III and IV. It is evident that a quartic monoid that has a second triple point can have on it in addition no multiple line of kind I or II.

(*To be continued.*)

# PLATE XV.



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Alfalfa, Grasshoppers, Bees—their Relationship; supply small. Postage, 7 cents.
Scale-insects Injurious to Orchards; supply large. Postage, 2 cents.
The More Injurious Grasshoppers of Kansas; supply large. Postage, 2 cents.

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