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WIND, SEA, AND SWELL: THEORY OF RELATIONS FOR FORECASTING

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H. U. SVERDRUP AND W. H. MUNK

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NOTATION

x Horizontal coordinate.
z Vertical coordinate, positive upward from undisturbed sea surface.
t Time.
p Pressure.
ρ Density of water (1.025 g/cm^3).
ρ' Density of air (1.25×10^{-3} g/cm^3).
u Horizontal component of particle velocity.
w Vertical component of particle velocity.
u' Mass transport velocity.
U Wind velocity at about 8m above sea surface.
C Velocity of progress of wave (phase velocity).
V' Group velocity.
V Velocity of energy flow associated with wave motion.
L Wave length.
k $\frac{2\pi}{L}$
T Wave period.
σ $\frac{2\pi}{T}$
a Wave amplitude.
H Wave height (from crest to trough).
h Depth of water.
η Elevation of sea surface relative to undisturbed level.
μ Viscosity of water (0.018 at 0° C.).
τ Stress of wind at sea surface.
E_p Mean potential energy of wave per unit area.
E_k Mean kinetic energy of wave per unit area.
E Mean total energy of wave per unit area.
R_N Mean rate of energy transfer to wave due to normal wind pressure.
R_T Mean rate of energy transfer to wave due to tangential wind stress.
R_U Rate at which wind energy is dissipated below level to which wind velocity is referred.
R_μ Mean rate of energy dissipation due to viscosity.
R_H Mean rate at which energy is used to change wave height.
R_C Mean rate at which energy is used to change wave velocity.
ϵ $\frac{R_N+R_T}{R_U}$
F Length of fetch.
D Distance of decay.
g Acceleration of gravity (980 cm/sec ^{2}).
r Coefficient of energy partition (0.580).
s Sheltering coefficient (0.013).
β C/U (Wave age).
δ H/L (Wave steepness).
γ^2 Resistance coefficient applicable to wind ($\gamma^2=2.6\times10^{-3}$ for wind velocities >5 m. per sec).
α $\frac{s}{2\gamma^2}$ (2.50).
A $2\gamma^2\rho'/\rho$ (6.35×10^{-5}).

Subscripts:
F refers to value at end of the fetch.
D refers to values at end of distance of decay.

For all numerical computation, c. g. s. units are employed.

ERRATA

Page v—Right hand column, 6th line from bottom:

For "$A\ 2\gamma^2\rho'/\rho(6.35\times10^{-5})$", read "$A\ 2\gamma^2\rho'/\rho(6.35\times10^{-6})$"

Page 15—Right hand column, 3rd line of equations from top:

For "$\frac{d\varrho}{d\beta}$", read "$\frac{d\delta}{d\beta}$"

Page 17—Right hand column, 5th line of equations from top:

For "$\frac{d\delta}{d\beta}+2\beta\delta=0$", read "$\frac{d\delta}{d\beta}+\frac{2\delta}{\beta}=0$"

Page 18—Left hand column, last line of equations:

For "$\frac{\sqrt{\beta}}{1+\alpha(1-\beta)}$", read "$\frac{\beta\sqrt{\alpha}}{1+\alpha(1-\beta)}$"

Page 29—Left hand column, 30th line from top:

For "2 feet", read "2 meters".

Left hand column, 34th line from top:

For "0.9 feet", read "0.9 meters"

INTRODUCTION

IN ORDER to forecast sea and swell from weather data it is necessary to know the character of the waves produced by a given wind that blows for a known length of time over a known stretch of water, the fetch. Prior to 1942 such knowledge was based on empirical relationships many of which were inconsistent among themselves. In the fall of 1942 a need for sea and swell forecasts arose in connection with the planned invasion of North Africa. In order to improve the basis for forecasting, the authors began their studies at the request of the Oceanographic Division of the Directorate of Weather, Army Air Forces.

Preliminary conclusions were helpful in the African and Mediterranean operations but consistent results were not achieved until the summer of 1943. By that time all oceanographic work had been transferred to the United States Navy. Under contracts with the United States Hydrographic Office and the Bureau of Ships, the studies of sea, swell, and surf were carried on at an accelerated pace.

In this paper a close combination of theoretical conclusion and empirical knowledge is attempted. From a study of the processes by which energy is transmitted from wind to waves, certain energy equations are derived which relate wave height and velocity to wind velocity, duration of wind, and fetch. In order to solve these equations it is necessary to make certain formal assumptions. The physical significance of these assumptions are as yet obscure but the solutions obtained are in some cases in agreement with empirical relationships and in other cases they remove inconsistencies which have arisen because the relative importance of the different variables had not been clearly recognized.

The present paper is a revision of a report, Wind Waves and Swell; A Basic Theory for Forecasting, which was submitted to the Hydrographic Office in September 1943, but which could not be released because of wartime restrictions. In the revision a few changes in the theory have been made, partly because some of the original assumptions were too arbitrary and partly because some new approaches have been stimulated by C.-G. Rossby's recent studies of wave motion.

When the original paper was prepared it was believed that the results would apply to "the larger waves present" but no attempt was made to describe these larger waves more specifically. It is now proposed to introduce a statistical term and to define "the larger waves" as waves having "average height and period of the one-third highest waves." The waves described by these averages are called "significant waves," because experience gained so far indicates that a careful observer who attempts to establish the character of the higher waves will tend to record the significant waves as defined here. The concept of "significant waves" is important because only the significant waves are known empirically, and because for these waves the classical requirement that crests are conserved is not fulfilled. Therefore, the growth and decay of significant waves do not obey the laws that would apply to the waves of the classical theory, but take place according to other laws that will be developed in this paper.

All solutions which relate waves to wind can be represented as relations between nondimensional parameters, but because the empirical data available in 1943 were too incomplete to be shown in this form, nondimensional presentations could not be used for comparison between theory and observations. Through the efforts of agencies partaking in wave research new and more comprehensive material has now made it possible to check the nondimensional relationships against observations.

As a whole, the changes are few considering the intensity with which wave research has been carried out since 1943. The basic numerical relationships between wave height and period as functions of wind velocity, fetch, duration, and distance of decay differ only slightly from those proposed in 1943 and used subsequently with considerable success in sea and swell forecasting during the war.

In 1942 work on sea and swell forecasting had been started independently in Great Britain, and some of the ideas developed there were of value to the authors, although the numerical relationships differed. Subsequently Commander C. T. Suthons, British Admiralty, developed relationships on an empirical basis which, with a few important exceptions, are similar to the ones presented here.

This paper does not deal with the forecasting technique, but a practical manual in forecasting was prepared in September 1943 on the basis of the 1943 report and published by the Hydrographic Office (1944a) in March 1944. The relation between swell and surf is not covered in the present paper nor in the forecasting manual, but studies of the transformation of waves in shallow water at the Woods Hole Oceanographic Institution; the Beach Erosion Board; the Department of Mechanical Engineering, University of California, Berkeley; and the Scripps Institution led to the joint preparation of a second manual on the forecasting of breakers and surf, published by the Hydrographic Office (1944b) in November 1944.

The authors are indebted to the Army Air Forces for initial encouragement and to the United States Navy, which gave the program its wholehearted support. They are also indebted to the Oceanographic Research Group, Admiralty Research Laboratory, Teddington, England, for having made available unpublished records of swell at Pendeen near Land's end. In the preparation of this manuscript the authors have been greatly aided by Lt. R. S. Arthur, USNR; Capt. J. C. Burke, AUS, AC; Lt. (jg) J. F. Munch, USNR; Sgt. R. E. Jentoft, AUS, AC; and Capt. M. A. Traylor, USMCR; all of whom were assigned to the project at the Scripps Institution.

The authors are also indebted to about 200 weather officers of the Army Air Forces and of the Navy, who attended the courses in sea and surf forecasting at the Scripps Institution and whose questions and interest have contributed much to improve the understanding of the problems.

THEORY OF SURFACE WAVES

Waves of Infinitely Small Amplitude

Equations required for subsequent developments will be summarized here. Derivations and discussions of these equations are not repeated since they are readily available (Lamb, 1932, Sverdrup et al., 1942). In water of constant depth the wave velocity can be represented by means of the equation of classical hydrodynamics:

$$C^2=\frac{gL}{2\pi}\tanh\frac{2\pi}{L}h \qquad (1)$$

where h is the depth to the bottom and L is the wave length.

Defining deep water waves and shallow water waves as:

Deep water waves:

$$h>\frac{1}{2}L$$

Shallow water waves: (2)

$$h<\frac{1}{25}L$$

equation (1) becomes with sufficient accuracy for these special cases:

Deep water waves:

$$C^2=\frac{gL}{2\pi} \qquad (3a)$$

Shallow water waves:

$$C^2=gh \qquad (3b)$$

This paper deals only with deep water waves' for which length, period, and velocity are interrelated as follows:

$$C=\frac{L}{T}=\sqrt{\frac{g}{2\pi}L}=\frac{g}{2\pi}T$$

$$L=\frac{2\pi}{g}C^2=\frac{g}{2\pi}T^2 \qquad (4)$$

$$T=\sqrt{\frac{2\pi}{g}L}=\frac{2\pi}{g}C$$

The water particles move in circles, the radii of which decrease exponentially with depth according to the relation:

Radius of particle orbit

$$=\frac{1}{2}He^{2\pi z/L} \qquad (5a)$$

where z is taken positive upwards; consequently the particle velocity is uniform and has the scalar value

Particle velocity

$$=\frac{\pi}{T}He^{2\pi z/L} \qquad (5b)$$

The mean energy per unit area of a wave equals (Lamb, 1932, p. 370)

$$E=\frac{1}{2}\rho g a^2 \qquad (6)$$

The rate at which this energy is changed by dissipation equals (Lamb, 1932, p. 624)

$$R_\mu=-2\mu k^3a^2C^2 \qquad (7)$$

With every wave there is associated a flow of energy in the direction of propagation of the wave. Let V be the velocity at which energy is transmitted. Then $V\times E$, the rate at which energy flows across a vertical plane of unit width, equals

$$VE=\int_{\infty}^{0}pu\ dz$$

where p is pressure and u is the horizontal component of the particle velocity. For deep water waves (Lamb, 1932, p. 383)

$$VE=\frac{1}{2}\rho g a^2C\sin^2 k(x-Ct) \qquad (8a)$$

with a mean value

$$VE=\frac{1}{4}\rho g a^2C=\frac{EC}{2} \qquad (8b)$$

Equation (8b) can be interpreted to mean that the entire energy is propagated at half the wave velocity or half the energy at full wave velocity. The significance of these interpretations will be discussed later (p. 6).

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Waves of Finite Amplitude

Two theories have been developed for deep water waves of finite amplitude: the Stokes theory dealing with irrotational waves, and the Gerstner theory dealing with a specific type of rotational waves. For the Stokes' waves the velocity of progress depends also upon the steepness of the wave, as expressed by the ratio $\delta=H/L$ (Lamb, 1932, p. 420)

$$C^2=\frac{g}{2\pi}L\left[1+\pi^2\delta^2+\frac{5}{4}\pi^4\delta^4+\ \ldots\ldots\ldots\right] \qquad (9a)$$

For moderate values of δ the wave form is very nearly trochoidal, but for larger values the troughs become wider and flatter, and the crests steeper. According to Mitchell (Lamb, 1932, p. 418), the greatest possible value of δ is equal to 1/7:

$$\delta_{max}=1/7 \qquad (9b)$$

When this value is reached the wave profile becomes unstable and the wave breaks. The velocity of progress of a wave for which $\delta = 1/7$ is 1.12 times the velocity of a low wave of the same length.

Generally waves in the ocean are much less steep, and equations 1 to 8 are sufficiently accurate for waves of finite height. The latter differ, however, from waves of very small amplitude in one important respect: the particle velocity is not uniform but is at a maximum when the particles are at the highest point in their orbit and moving in the direction of the wave. Upon the completion of each nearly circular motion the particles have advanced a short distance in the direction of progress of the wave and have brought about a small transport of mass (figure 1). The average velocity of this forward motion during one wave period, the mass transport velocity, is denoted by u'. In deep water, at a depth z (Lamb, 1932, p. 419),

$$u'=\pi^2\delta^2Ce^{4\pi z/L} \qquad (10)$$

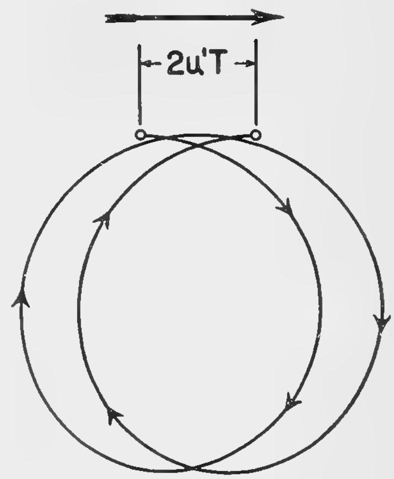

Figure 1.—Orbital motion during two wave periods of a water particle in a deep water wave of finite height.

Rossby (1945) has shown the Stokes wave to be a special case of an infinite number of possible irrotational waves. His very general treatment has no bearing upon the following discussion of the growth of waves, but may modify some of the conclusions regarding the propagation of swell.

The Gerstner waves are exactly trochoidal and equations 1 to 8 are valid for them without approximation. The Gerstner waves are without mass transport velocity.

PROPAGATION OF A DISTURBANCE THROUGH A REGION PREVIOUSLY UNDISTURBED

General Considerations and Observations

It has long been a controversial question whether a disturbance of the sea surface in a storm area advances into an area of calm with the velocity of the individual waves, or at half the wave velocity (group velocity). A theoretical investigation, which is presented in this section, leads to the conclusion that for practical purposes the disturbance shall be taken to advance at half the wave velocity. Some waves travel faster and arrive earlier, but their height is probably small.

As a simple form of wave motion, consider a train of waves represented by $\sin k(x—Ct)$ where x is positive in the direction of propagation and C is the velocity of transmission of phase, termed wave velocity in this report. If the medium in which the motion occurs is such that C is the same for all wave lengths the velocity C has additional physical significance, it equals V, the rate of transmission of energy. If waves are emitted from a source in such a manner that full intensity is reached with the emission of the very first wave, any subsequent position of this first wave defines a wave front. As an effect of viscosity the height of the wave decreases with increasing distance from the source, but at any given distance from the source all waves arrive with the same height as the initial wave. A steady state is reached with the passage of the wave front.

If, however, the phase velocity C is a function of the wave length, a fact which is expressed by describing the medium as dispersive, this ideal simplicity no longer exists. In general, C and V are no longer equal. Radio signals in hollow guides, seismic waves in the interior of the earth, and surface waves in deep water, are examples of waves traveling through dispersive media. In these only a portion of the energy travels along with the wave form, the remaining portion being left behind. Focusing attention on a single wave in deep water, one should expect this wave to gain the portion of the energy left behind by the preceding one and, in turn, leave a portion of its energy for the wave following it. Dealing with a group of, say, 10 waves, the first wave, which loses energy to the wave behind and gains none from the front, soon becomes very small, but the last wave, which leaves energy behind, will form a new wave behind it and become the second from the last.

This is exactly what has been observed and reported by numerous observers. In the Admiralty Navigation Manual (1939), Vol. III, page 389, it is stated that:

> if motion of the first wave of the group is followed, it will be found that this motion dies out and that the wave next behind takes the lead. If, on the other hand, the last wave of the group is watched, another wave will be seen to appear behind it. The new waves constantly rise in the rear as rapidly and as constantly as those in the front die out, so that the general appearance of a group of waves remains unchanged. The group as a whole has a definite velocity of propagation, which has been found to be half of that of the individual waves comprising the group * * *

Krümmel (1911, page 95), also states that:

> * * * the wave, which at any instant is in front, flattens so much as it travels over the surface, that it becomes invisible after traveling 2–4 meters, whereupon the next one becomes the first, again goes through the appearance of flattening, etc.

The same experience in tanks is reported in the Technical Report No. 2 of the Beach Erosion Board (1942).

> In wave tanks, wave groups may be generated by operating the wave machine through only a few strokes * * * the observer can follow a particular wave crest only a finite distance before it disappears. Close observation reveals that the wave group maintains its identity, that individual waves pass through the group, rising out of comparatively calm water at the rear, reaching a maximum at the center, and then disappearing at the front of the group.

When a long group reaches an observer stationed at a given distance from the source, the initial waves will be very low, but the wave height will increase with time. It will reach a maximum as the center of the group passes the observer and will then decrease. If waves of constant height are continuously generated at the source the observer will find that, after a transient stage of wave growth, a steady state condition involving constant wave height will be approached. The problem is *to find how long it takes to reach, for example, 50 or 90 percent of this constant height.* The problem can also be stated: *to find the rate at which an appreciable portion of the energy of the disturbance is propagated through the area of calm.*

It will be shown that there exists an actual "wave front" which advances with the velocity C of the initial wave but the magnitude of the disturbance so propagated is negligible compared with that of the main group which travels at lower speed.

Group Velocity and Energy Flow

The slow rate at which a disturbance is transmitted through an area of calm, as compared to the rate of travel of the individual waves, is generally explained by stating that the disturbance travels with the group velocity V'. An expression for group velocity is derived by considering the combined effect of two trains of waves of equal wave height whose lengths differ by a small amount dL. The resulting interference pattern will travel with a velocity equal to

$$V' = C - L\frac{dC}{dL}. \qquad (11a)$$

Since $C^2 \sim L$, it follows:

$$V' = \frac{C}{2} \qquad (11b)$$

(Beach Erosion Board, 1942, p. 32; Cornish, 1934, p. 137; Krümmel, 1911, p. 95.)

This derivation has led to much confusion because, unless further qualified, it would indicate that group velocity is important only under very special conditions as, for example, in the case of two wave trains of equal height and slightly different lengths. But equation (11a) is much more general and can also be derived by making use of methods of summation or integration if the simultaneous existence of an infinite number of wave trains involving a frequency spectrum is assumed (Havelock, 1914, p. 5). If the group velocity V' is defined in such a manner that the wave length L does not vary in the neighborhood of a geometrical point traveling with velocity V',

$$\frac{dL}{dt}=\frac{\partial L}{\partial t}+\frac{\partial L}{\partial x}V'=0 \qquad (12)$$

then equation (11a) is valid for any *limited initial disturbance provided the wave crests maintain their identity* (Lamb, 1932, p. 381). This initial disturbance can be expressed in terms of a Fourier integral and the resulting interference pattern derived as a function of time and distance.

As a simple case, assume again that waves of constant height and length are emitted at the source and that their wave length remains constant thereafter. It might appear that under these assumptions only one single wave length is present and that group velocity is not involved. Since, however, the wave train is of finite size the wave length must either equal a constant value L' (within the train), or be zero (outside the train) and this distribution can be represented by a spectrum of wave lengths in a Fourier integral. For long trains the various wave lengths in this spectrum cluster closely about L' but the use of one single wave length is justified only in the case of an infinitely long train.

A study of the propagation of a disturbance into an area of calm could be based on these considerations but we prefer to use a method based on consideration of energy, which appears to be simpler and in better accord with the point of view from which this paper is prepared.

From a comparison of equations (11b) and (8b) follows

$$V'=V$$

This identification of the group velocity with the mean rate of transmission of energy, here shown for deep water waves, can be extended to shallow water waves, indeed to all kinds of waves (Havelock, 1914, p. 55 and p. 61). However, the physical significance does not appear obvious, as evident from explanations attempted by Lamb (1932, p. 383) and Rayleigh (1877, p. 21).

To the present problem it is of particular importance to know whether equation (8b) shall be interpreted to mean that

(1) *all* the energy advances with group velocity, or

(2) *half* the energy advances with wave velocity.

To answer this question consider the flow of energy through a parallelepiped of unit width, length dx, and extending to a depth below which wave motion is negligible (fig. 4, p. 14). The time rate of change of energy within the parallelepiped must equal $-\partial(VE)/\partial x$, the net inflow in the direction of the x-axis; and, therefore:

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}(VE)=0 \qquad (13)$$

In the first case (13) becomes

$$\frac{\partial E}{\partial t}+\frac{C}{2}\frac{\partial E}{\partial x}=0 \qquad (14a)$$

with the solution

$$E=f\left(x-\frac{C}{2}t\right) \qquad (14\ )$$

which gives no information as to the manner in which E varies with x.

In the second case equation (13) becomes

$$\frac{\partial E}{\partial t}+C\frac{\partial(E/2)}{\partial x}=0 \qquad (15)$$

leading, as will be shown, to a solution of the transient state which is consistent with necessary physical boundary conditions.

Transmission of Energy by Wave Motion

The difference in form between equations (14a) and (15) has been given physical significance: equation (15) has been interpreted to mean that the ratio "group velocity to wave velocity" denotes the fraction of energy, E', which advances with wave velocity.*

$$\frac{E'}{E}=\frac{V}{C} \qquad (16)$$

This physical significance follows from a consideration of the distribution of potential and kinetic energy along a wave. Consider a sinusoidal wave, for which the surface elevation is given by

$$\eta=\frac{1}{2}H\sin k(x-Ct) \qquad (17)$$

The potential energy is computed from the elevation or depression relative to the still water surface as:

$$E_P=g\rho\int^{\eta}zdz=\frac{1}{8}g\rho H^2\sin^2 k(x-Ct) \qquad (18)$$

Substituting in equation (8a) we find for the rate at which energy is transmitted

$$VE=CE_P$$

*This interpretation is, according to Rossby (1945), not generally valid, but it is applicable to the type of waves examined here.

indicating that the *potential energy is transmitted with wave velocity.* The kinetic energy is obtained from the horizontal and vertical components of particle velocity:

$$E_K=\frac{1}{2}\rho\int_{-h}^{\eta}(u^2+w^2)dz$$

where h is the depth. For deep water waves,

$$u=\frac{1}{2}H\frac{g}{C}e^{kz}\sin k(x-Ct)$$

$$w=-\frac{1}{2}H\frac{g}{C}e^{kz}\cos k(x-Ct)\text{ and} \qquad (19)$$

$$E_K=\frac{1}{16}g\rho H^2 \qquad (20)$$

In contrast to the potential energy, which is a periodic function and advances in phase with the deformation of the surface, the kinetic energy is evenly distributed along the entire wave and is independent of the position or the velocity with which the surface deformation advances. (See fig. 2.)

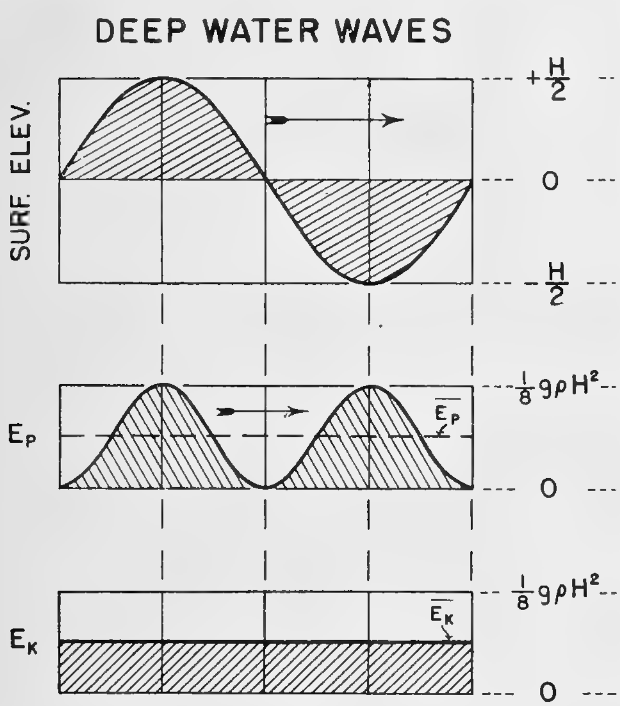

Figure 2.—Variation of surface elevation, potential energy, E_P, and kinetic energy, E_K, along one wave length.

From (18) we find the mean potential energy over a wave length, $\overline{E_P}$,

$$\overline{E_P}=\frac{1}{16}g\rho H^2=E_K=\frac{E}{2} \qquad (21)$$

and equation (16) is satisfied, since

$$\frac{E'}{E}=\frac{\overline{E_P}}{E}=\frac{1}{2}=\frac{V}{C}$$

The following interpretation can now be given to observations of wave motion in deep water. Again quoting the Technical Report No. 2 of the Beach Erosion Board (1942):

> As the first wave in the group advances one wave length, its form induces corresponding velocities in the previously undisturbed water and the kinetic energy corresponding to these velocities must be drawn from the energy flowing ahead with the form. If there is equipartition of energy in the wave, half of the potential energy which advanced with the wave must be given over to the kinetic form and the wave loses height. Advancing another wave length another half of the potential energy is used to supply kinetic energy to the undisturbed liquid. The process continues until the first wave is too small to identify. The second, third, and subsequent waves move into water already disturbed and the rate at which they lose height is less than for the first wave. At the rear of the group, the potential energy might be imagined as moving ahead, leaving a flat surface and half of the total energy behind as kinetic energy. But the velocity pattern is such that flow converges toward one section thus developing a crest and diverges from another section forming a trough. Thus the kinetic energy is converted into potential and a wave develops in the rear of the group.

This concept can be interpreted in a quantitative manner, by taking the following example from R. Gatewood (Gaillard 1935, p. 34). Suppose that in a very long trough containing water originally at rest, a plunger at one end is suddenly set into harmonic motion and starts generating waves by periodically imparting an energy $E/2$ to the water. After a time interval of n periods there are n waves present. Let m be the position of a particular wave in this group such that $m=1$ refers to the wave which has just been generated by the plunger, $m=(n+1)/2$ to the center wave, and $m=n$ to the wave furthest advanced. Let the waves travel with constant velocity C, and neglect friction.

After the first complete stroke one wave will be present and its energy is $1/2E$. One period later this wave has advanced one wave length but has left one-half of its energy or $1/4E$ behind. It now occupies a previously undisturbed area to which it has brought energy $1/4E$. In the meantime, a second wave has been generated, occupying the position next to the plunger where $1/4E$ was left behind by the first wave. The energy of this second wave equals $1/4E+1/2E=3/4E$. Repeated applications of this reasoning lead to the results shown in table 1.

The series number n gives the total number of waves present and equals the time in periods since the first wave entered the area of calm; the wave number m gives the position of the wave measured from the plunger and equals the distance from the plunger expressed in wave lengths. In any series, n, the deviation of the energy from the value $E/2$ is symmetrical about the center wave. Relative to the center wave all waves nearer the plunger show an excess of energy and all waves beyond the center wave show a deficit. For any two waves at equal distances from the center wave the excess equals the deficiency. In every series, n, the energy first decreases slowly with increasing distance from the plunger, but in the vicinity of the center wave it decreases rapidly. Thus, there develops an "energy front" which advances with the speed of the central part of the wave system, that is, with half the wave velocity.

Table 1

Distribution of Wave Heights in a Short Train of Waves

Series number n	Wave number, m							Total energy of group
	1	2	3	4	5	6	7	
1	1/2E	------	------	------	------	------	------	1/2E
2	3/4	1/4E	------	------	------	------	------	2/2
3	7/8	4/8	1/8E	------	------	------	------	3/2
4	15/16	11/16	5/16	1/16E	------	------	------	4/2
5	31/32	26/32	16/32	6/32	1/32E	------	------	5/2
6	63/64	57/64	42/64	22/64	7/64	1/64E	------	6/2

According to the last line in table 1 a definite pattern develops after a few strokes: the wave closest to the plunger has an energy $E\ (2^n-1)/2^n$ which approaches the full amount E, the center wave has an energy $E/2$, and the wave which has traveled the greatest distance has very little energy $(E/2^n)$.

Approximate Solution to Equation (15)

Let ${}^nR_m={}^nE_mE$, where ${}^nE_m$ denotes the energy of the mth wave in a group of n waves. Then

$$ {}^nR_m=\frac{n!}{2^n}\sum_{r=0}^{r=n-m}\frac{1}{r!(n-r)!} \qquad (22) $$

gives the value of any term in table 1. This table and, therefore, equation (22) are in agreement with observations. It remains to be shown that (22) satisfies the differential equation (15).

$$ \text{Let } t=nT,\ x=mL \qquad (23) $$

where, as usual, T and L denote wave period and length. Then

$$ \frac{\partial R}{\partial n}+\frac{1}{2}\frac{\partial R}{\partial m}=0,\ (R\equiv{}^nR_m) \qquad (24) $$

which can be written as the following difference equation

$$ \frac{{}^{n+1}R_m-{}^nR_m}{(n+1)-(n)}+\frac{1}{2}\,\frac{{}^nR_m-{}^nR_{m-1}}{(m)-(m-1)}=0 $$

or

$$ {}^{n+1}R_m-\frac{1}{2}{}^nR_m-\frac{1}{2}{}^nR_{m-1}=0 \qquad (25) $$

The general expression for ${}^nR_m$, given by (22), satisfies equation (25). The proof, based on the method of mathematical induction, is given in appendix 1.

Equation (22) is not practicable since we are dealing with such large distances that m and n have values up to 10^4. The process of summation would be very cumbersome even if tables of binominal coefficients were available. From an analogy with the probability theory it is possible, however, to find approximate values of ${}^nR_m$ directly, no matter how large m and n happen to be. Since the binominal coefficnt ${}^nC_r$ is defined as

$$ {}^nC_r=\frac{n!}{r!(n-r)!} $$

equation (22) can be written

$$ {}^nR_m=\frac{1}{2^n}\sum_{r=0}^{r=n-m}{}^nC_r \qquad (26) $$

The binominal distribution can be closely approximated by the normal frequency distribution curve because the binominal summation corresponds nearly to the area under the normal curve which is found in tables of the probability integral. From an analogy with the probability problem we can write at once

$$ {}^nR_m=\frac{1}{2}-\frac{1}{\sqrt{2\pi}}\int_0^u e^{-\frac{u^2}{2}}du,\quad u=\frac{2m-n-1}{\sqrt{n}} \qquad (27) $$

To find the degree of approximation the approximate solution (27) is substituted directly in the differential equation (24).

$$ \frac{\partial R}{\partial n}=\frac{\partial R}{\partial u}\frac{du}{dn}=+\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\right)\left(\frac{2m+n-1}{2n^{3/2}}\right) $$

$$ \frac{1}{2}\frac{\partial R}{\partial m}=\frac{1}{2}\frac{\partial R}{\partial u}\frac{du}{dm}=-\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\right)\left(\frac{1}{\sqrt{n}}\right) $$

Values of $\frac{\partial R}{\partial n}, \frac{1}{2}\frac{\partial R}{\partial m}$, and their differences for $n=900$ are given in columns 4–6 of table 2.

For the center wave ($m=450\frac{1}{2}$) the agreement is exact, but for all other waves there are found small differences. The approximation is satisfactory.

Table 2

Degree of Approximation Involved in Equation (27)

(1)	(2)	(3)	(4)	(5)	(6)
m	R	$\frac{\partial R}{\partial u}$	$\frac{\partial R}{\partial n}$	$\frac{1}{2}\frac{\partial R}{\partial m}$	$\frac{\partial R}{\partial n}+\frac{1}{2}\frac{\partial R}{\partial m}$
300	1.0000	−.0000	$+0.00\times10^{-4}$	-0.00×10^{-4}	-0.00×10^{-4}
400	.9996	−.0014	$+0.44\times10^{-4}$	-0.47×10^{-4}	-0.03×10^{-4}
420	.9788	−.0508	$+16.36\times10^{-4}$	-16.93×10^{-4}	-0.57×10^{-4}
430	.9147	−.1561	$+50.84\times10^{-4}$	-52.03×10^{-4}	-1.19×10^{-4}
435	.8485	−.2347	$+76.89\times10^{-4}$	-78.23×10^{-4}	-1.34×10^{-4}
440	.7590	−.3123	$+102.87\times10^{-4}$	-104.10×10^{-4}	-1.23×10^{-4}
445	.6430	−.3730	$+123.57\times10^{-4}$	-124.33×10^{-4}	-0.76×10^{-4}
447	.5920	−.3882	$+128.90\times10^{-4}$	-129.40×10^{-4}	-0.50×10^{-4}
449	.5398	−.3970	$+132.11\times10^{-4}$	-132.33×10^{-4}	-0.22×10^{-4}
450	.5033	−.3988	$+132.86\times10^{-4}$	-132.93×10^{-4}	-0.07×10^{-4}
450½	.5000	−.3989	$+132.98\times10^{-4}$	-132.98×10^{-4}	-0.00×10^{-4}
451	.4967	−.3988	$+132.86\times10^{-4}$	-132.93×10^{-4}	-0.07×10^{-4}
601	.0000	−.0000	$+0.00\times10^{-4}$	-0.00×10^{-4}	-0.00×10^{-4}

Significance of the Solution

Table 2 also illustrates an important point: within a very short distance the ratio R decreases from very nearly 100 percent to a minute percentage. In figure 3, R has been plotted against m, the distance in wave lengths from the generating area. The scale to the left gives percent of maximum wave energy, the scale to the right gives percent of maximum wave height; the height ratio equals the square root of the energy ratio. The center wave is halfway between the wave furthest advanced and the one at the very rear. Its energy is exactly one-half the maximum energy, its height 70.7 percent of the maximum height. In this example the total waves present number 900, hence the center wave is 450.5 wave lengths from the generating area. Let the "region of sharp decrease in wave height," the shaded portion of the figure, be defined as the region within which the wave heights decrease from 90 to 10 percent of their maximum value. These two limits correspond to $m=485$, and $m=435$, and the area of sharp wave decrease is only 50 wave lengths wide.

The "region of sharp decrease in wave height" has at any instant traveled only half as far as the leading wave; its velocity is half the velocity of the leading wave. Therein lies the answer to a controversial question: how fast does a disturbance travel from the storm area, at wave velocity or at half the wave velocity? The answer is that each wave travels with its own wave velocity and arrives at a distance D from the storm area at a time $t'=D/\bar{C}$, where $\bar{C}$ denotes the mean wave velocity. However, these early waves are extremely low. From a practical point of view appreciable waves will appear at D only slightly before $2t'$, the time of arrival of the "center wave." *Hence for purposes of forecasting, the velocity of the disturbance should be taken at one-half the wave velocity.*

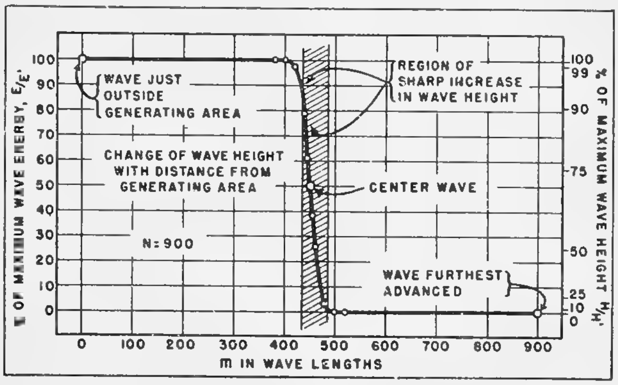

Figure 3.—Change of wave height with distance from source region, assuming that uniform waves produced at the source region advance through water originally at rest. Deep water waves.

ENERGY TRANSFER FROM WIND TO WAVES

Energy Transfer by Normal Pressure

The average rate at which energy is transferred to a wave by normal pressure equals

$$R_N=\frac{1}{L}\int_0^L p_{zz}w_o dx \qquad (28a)$$

where

$$w_o=-kaC\cos k(x-Ct) \qquad (28b)$$

is the vertical component of the particle velocity at the surface, and p_{zz} the normal tension acting on the sea surface.

When a wind blows, the tension p_{zz}, which is directed opposite to the pressure exerted on the sea surface, can, according to Jeffreys (Jeffreys, 1925 and Lamb, 1932, p. 625), be taken as the sum of two negative terms, one representing the constant atmospheric pressure and the other, the wind pressure against different portions of the wave:

$$p_{zz}=-p-\Delta p$$

The wind pressure can be considered composed of a series of harmonic terms of wave lengths L,

$2L$, $3L$, ... , but of these terms only the one in phase with w_o does a net amount of work. Jeffreys assumes this term to be proportional to the product of the density of the air, the square of the wind velocity, and the slope of the surface:

$$\Delta p = s\rho'(U-C)^2 \frac{\partial \eta}{\partial x} = s\rho'(U-C)^2 ka \cos k(x-Ct) \tag{29}$$

and he calls the coefficient of proportionality, s, the "sheltering coefficient." It might be more descriptive to use the term "streamlining coefficient" because the coefficient is a measure of the form resistance offered by the wave.

From (28a), (28b), and (29) it follows:

$$R_N = +\frac{1}{2}s\rho'(U-C)^2k^2a^2C \text{ for } C<U \tag{30a}$$

Equation (30a) holds for waves traveling more slowly than the wind. Should the wave velocity exceed that of the wind, and the relative wind velocity be directed against the direction of wave motion, then Δp is 180° out of phase with the slope of the wave, and the sign in equations (29) and (30a) must be reversed. Hence

$$R_N = -\frac{1}{2}s\rho'(U-C)^2k^2a^2C \text{ for } C>U \tag{30b}$$

Jeffreys assumes R_N to be the only important source of wave energy. On this assumption the energy of a wave can increase only if R_N exceeds $R\mu$, the rate at which energy is dissipated by viscosity, or if, according to (7) and (30), using (3a),

$$s\rho'(U-C)^2C > 4\mu g \tag{31a}$$

This is Jeffreys' criterion for the growth of deep water surface waves. According to (31a), waves cannot grow if, approximately,

$$C > U - \sqrt{\frac{4\mu g}{s\rho' U}} \tag{31b}$$

and the wave velocity cannot exceed the wind velocity during the stage of growth. Jeffreys evaluated s from equation (31a) by noting that for a given wind velocity the term on the left-hand side is at a maximum when

$$C = \frac{1}{3}U \tag{32a}$$

The least wind which can maintain waves is, therefore,

$$U_{min} = 3\left(\frac{\mu g}{s\rho'}\right)^{\frac{1}{3}} \tag{32b}$$

Observations by Jeffreys gave U_{min} approximately 110 cm/sec. With this value and with $\mu=0.018$, $g=980$, and $\rho'=1.25\times10^{-3}$, one obtains $s=0.27$.

The corresponding wave velocity is $U_{min}/3$, or about 35 cm/sec, the corresponding wave length is 8 cm, and the wave period

$$T_{min} \doteq 0.22 \text{ seconds} \tag{32c}$$

Observations by others agree as to the order of magnitude of the least wind velocity and the smallest wave length.

The mechanism described by Jeffreys appears, therefore, to give a satisfactory explanation for the initial formation of waves. *However, it is not possible to apply Jeffreys' concept and his numerical value of s when studying the growth of waves after their initial formation because the observed increase in wave height indicates that the transfer of wind energy is only about one-tenth of that demand by Jeffreys.* This conclusion is substantiated by experiments conducted with small wooden models of waves placed in a wind tunnel (Stanton, 1937). The pressure distribution along the wave profile was measured and the amplitude of the component in phase with w_o (28b) determined from a harmonic analysis of the pressure distribution. The sheltering coefficient can be evaluated from these measurements, using equation (29),

$$s = \frac{\Delta pL}{\rho' U^2 \pi H}$$

The results are summarized in table 3.

Table 3

Determination of Sheltering Coefficient From Experiments by Sir Thomas Stanton (1937)

Diameter of wind tunnel (inches)	Wave length (cm)	Wave height (cm)	Wind velocity (cm/sec)	s
12	10.8	.055	325	0.036
12	10.8	.55	470	.047
12	21.6	1.1	330	.068
12	21.6	1.1	580	.090
36	7.6	0.75	1,400	.006
Average				.049

Although the measurements were subject to large experimental errors as indicated by the wide variation in s, the average shows clearly that Jeffreys' value of 0.27 is too high.

Energy Transfer by Tangential Stress

Jeffreys did not take into account a transfer of energy by tangential stress because he considered this process as negligible compared to the transfer by normal pressure, but the following considerations show that tangential stress cannot be neglected.

The average rate at which energy is transmitted to the wave by tangential stress equals

$$R_T = \frac{1}{L}\int_0^L \tau u_0 dx \qquad (33)$$

where u_0 denotes the horizontal component of particle velocity at the sea surface and where τ is the stress which the wind exerts on the sea surface.

At wind velocities above 500 cm/sec the stress of the wind equals (Rossby, 1936)

$$\tau = \gamma^2 \rho' U^2, \qquad (34a)$$

where ρ' is the density of the air, U is the wind velocity at a height of 8 to 10 m and γ^2 is the resistance coefficient.

Various types of observations have consistently led to the value

$$\gamma^2 = 2.6 \times 10^{-3} \qquad (34b)$$

provided that the wave velocity does not differ too much from the wind velocity. If this condition is not fulfilled the value of γ^2 is probably greater. Introducing (34a) in (33), assuming τ to be independent of x:

$$R_T = \gamma^2 \rho' U^2 \int_0^L \frac{1}{L} u_0 dx \qquad (35)$$

For waves of small amplitude

$$u_0 = \pi\delta C \sin k(x - Ct)$$

and the integral in (35) vanishes. This, apparently, led Jeffreys to assume that energy transmitted by tangential stress does not play an important part in the generation of waves. For Stokes' waves of finite amplitude, which are accompanied by mass transport (10), the integral in (35) has the value

$$u'_0 = \pi^2 \delta^2 C \qquad (36)$$

and, therefore,

$$R_T = \gamma^2 \pi^2 \rho' \delta^2 C U^2 \text{ (for } U > 500 \text{ cm/sec)} \qquad (37)$$

It would be more correct to write $(U - u'_0)^2$ for U^2 in equation (37), since stress is caused by the wind velocity relative to the water surface, but since u'_0 is small compared to U, (37) gives a satisfactory approximation.

The energy of waves can increase only if $(R_N + R_T)$ the rate at which energy is added by both normal and tangential stresses of the wind exceeds R_μ, the rate at which energy is dissipated by viscosity, or if, according to (7), (30), and (37),

$$\pm s\rho'(U - C)^2 C + 2\gamma^2 \rho' U^2 C > 4\mu g \qquad (38)$$

where + refers to $C < U$.

Equation (38) now takes the place of (31), the Jeffreys criterion for the growth of waves. According to the latter, waves cannot attain velocities exceeding the wind velocity, but equation (38) does not place this restriction upon the development. Take, for example, $C = U$ and let $U = 500$ cm/sec, the lowest wind velocity for which (34) and hence (37) are valid. Then $s\rho'(U - C)^2 C = 0$, $2\gamma^2 \rho' U^2 C = 812$, and $4\mu g = 71$; hence, according to (38), waves continue to grow even after their velocities exceed that of the wind, in agreement with observations. Since it must be assumed that the wave velocity increases the longer the waves travel, the ratio $\beta = C/U$ will indicate the state of development of the wave and can appropriately be considered a parameter which describes the age of the wave.

Equation (38) is valid for $U > 500$ cm/sec only and therefore cannot be applied to the problem of the first formation of waves, which takes place when U is about 100 cm/sec. At wind velocities less than about 500 cm/sec the sea surface is hydrodynamically smooth (Rossby, 1936) and the relation between the stress and the wind velocity differs from that expressed by (34a). The problem of the initial formation of waves must therefore be approached in a different manner. It deserves further attention but lies outside the scope of this paper which deals with the growth of wind waves at wind velocities above 500 cm/sec.

It is of interest to compare the accuracy of R_T and R_N, as defined by equations (37) and (30). R_T depends mainly upon the accuracy with which u' and γ^2 (17b) are known. The expression for u' has been checked experimentally and found to be in good agreement with theory. The numerical value of the resistance coefficient γ^2 has been arrived at by several different methods but is, as already stated, applicable only to wind velocities exceeding 5 m/sec, for which the sea surface can be considered hydrodynamically rough. Otherwise it is independent of wind velocity. One might estimate roughly that R_T can be obtained from equation (37) with an accuracy of ±25 percent. The accuracy to which R_N can

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be obtained from (30) depends mainly upon the knowledge of the "sheltering coefficient" s, and upon the extent to which it remains constant. Especially during the early stages of growth, a variation in s might be expected. A detailed theoretical and experimental investigation of energy transfer by normal stress would be highly desirable because the accuracy to which R_N can be evaluated is less than the corresponding accuracy for R_T. Fortunately, the term R_T plays a more important part in the development of waves than R_N (see p. 23).

Friction

The effect of molecular viscosity is small compared to the wind effects. Collecting constants, by putting $A=2\gamma^2\rho'/\rho$ and $\alpha=s/2\gamma^2$, equations (30), (37) and (7) become:

$$R_T=EAgU^{-1}\beta^{-3} \tag{39}$$

$$|R_N|=EAgU^{-1}\beta^{-3}\alpha(1-\beta)^2 \tag{40}$$

$$R_\mu=\frac{-4\mu g^2}{\rho}EU^{-4}\beta^{-4} \tag{41}$$

From equations (39), (40), and (41):

$$\frac{R_\mu}{R_T\pm R_N}=-\frac{4\mu g}{\rho A[1\pm\alpha(1-\beta)^2]}U^{-3}\beta^{-1}$$

With $\mu=1.8\times10^{-2}$, $g=980$, $\rho=1$, $A=6.5\times10^{-6}$

and $\alpha=2.5$, as will be shown later:

$$\frac{R_\mu}{R_T\pm R_N}=-\frac{1.09\times10^7}{1\pm2.5(1-\beta)^2}U^{-3}\beta^{-1}$$

For $U=500$ and $\beta=0.1$ ($C=50$):

$$\frac{R_\mu}{R_T+R_N}=-0.29$$

For $U=1{,}000$ and $\beta=0.1$ ($C=100$):

$$\frac{R_\mu}{R_T+R_N}=-0.036$$

For all but very small values of β and for moderate and large values of U, R_μ is small compared to R_T+R_N and can be neglected when dealing with the growth of waves.

After waves have left the storm area and travel through regions of calm, the effect of ordinary viscosity is still negligible. An 8 second wave, for example, would have to travel more than 2 years and could complete 10 "equatorial round trips" before its height would decrease by 63 percent ($1/e=0.37$).

It may be argued that the rapid decay of waves could be explained by introducing an eddy viscosity, as is done in order to account for the low velocities of wind driven currents. When dealing with wind currents the eddy viscosity has been found to be from 1,000 to 100,000 times as large as the ordinary viscosity, but when dealing with waves the introduction of an eddy viscosity seems undesirable for the following reasons:

1. The decrease of particle velocity with depth would be much more rapid than is shown by equation (5), which has been verified by observations.
2. The eddy viscosity applicable to wind currents gives much too rapid a decrease of wave height, and it would be necessary to introduce a smaller coefficient applicable only to wave motion.
3. The observed decrease of wave height can be explained as the effect of air resistance against the advancing wave.

Assuming, therefore, that dissipation takes place by ordinary viscosity only, the effect of friction is neglected.

THEORY FOR THE GROWTH OF WAVES

"Significant" and "Conservative" Waves

In the last sections we have tacitly assumed the existence of infinite trains of waves, but in the oceans we have to deal with trains of finite length. Within the generating area there always exist a large number of such trains of waves of different lengths, traveling with the wind or at small angles with the wind direction. From interference and criss-crossing there results an extremely irregular appearance of the sea surface, but the larger waves can be recognized and the theoretical relationships between period, length, and velocity apply to these (Krümmel 1911, Sverdrup et al, 1942).

Because of the simultaneous presence of many trains the wave characteristics have to be described by some statistical terms. For that purpose it has been found convenient to introduce "the average height and period of the one-third highest waves." The waves defined in this manner are called "the significant waves," but the definition requires further refinement because the composition of the "one-third highest waves" depends upon the extent to which the lower waves have been considered. Experience so far indicates that a careful observer who attempts to establish the character of the higher waves will record values which approximately fit the definition. It is also found that the concept of "significant waves" is essential for the purpose of forecasting.

The significant waves behave differently compared to the classical waves in a single finite train. The wave crests in such a train maintain their identity, that is, the waves are conservative, and according to (12):

$$\frac{\partial C}{\partial t}+\frac{C}{2}\frac{\partial C}{\partial x}=0 \qquad (42)$$

A steady state ($\partial C/\partial t=0$) cannot exist simultaneously with an increase of wave velocity (or period) with distance in fetch; nor can a transient state exist during which the wave velocity (or period) increases with the time but remains uniform over an area ($\partial C/\partial x=0$).

These conclusions are in contrast with ordinary experience as to the behavior of the significant waves. When a wind of constant velocity has blown for a long time over a limited stretch of water, such as a lake, a steady state is established. At any fixed locality the significant waves do not change with time, but on the downwind side of the lake they are higher and longer than on the upwind side. If, on the other hand, a uniform wind blows over a wide ocean, waves grow just as fast in one region as in any other region and the significant waves change with time but do not vary in a horizontal direction.

The discrepancy between the behavior of significant waves and individual waves must lie in the fact that the crests of significant waves do not maintain their identity: in the storm area significant waves are not conservative. The implications of this conclusion are as yet not clear but it is possible that the significant waves represent interference patterns which in a given locality are formed by ever-changing combinations of wave trains. In all events, relationships between waves and wind, fetch, and duration which shall agree with empirical results must be based on a study of significant waves. Such a study represents a radical departure from the study of the conservative waves of the classical theory.

Energy Budget of Conservative and of Significant Waves

The transient (or unsteady) state will be discussed first. The total energy per unit crest width of a wave equals EL, where E is the mean energy per unit surface area. The energy added each second by the normal pressure of the wind equals $\pm R_N L$ (30) and that added by the tangential stress equals $R_T L$ (37).

Only half the energy, the potential energy, travels with the wave (18). Kinetic energy is constantly gathered at the forward edge of the wave, and left behind at the rear edge. This feature can be illustrated by considering a parallelepiped of unit width, extending to a depth below which wave motion is negligible, and whose forward and rear edges travel beneath two adjacent crests (fig. 4). At the forward edge of the moving parallelepiped energy is gained at the rate

$$C\frac{E}{2}+L\frac{\partial}{\partial x}\left(C\frac{E}{2}\right)$$

and at the rear edge energy is lost at the rate $CE/2$.

The total energy budget can therefore be written:

$$\frac{d(EL)}{dt}=\left\{R_T\pm R_N+\frac{\partial}{\partial x}\left(C\frac{E}{2}\right)\right\}L \qquad (43)$$

The rate at which the wave length increases and, therefore, at which the parallelepiped "stretches" is determined by the difference in speed between the adjacent wave crests:

$$\frac{dL}{dt}=\frac{\partial C}{\partial x}L \text{ or } \frac{\partial C}{\partial x}=\frac{1}{L}\frac{dL}{dt}$$

Since $C^2=gL/2\pi$, this equation can also be written

$$\frac{\partial C}{\partial x}=\frac{2}{C}\frac{dC}{dt} \qquad (44)$$

and (43) takes the form:

$$\frac{dE}{dt}+\frac{E}{C}\frac{dC}{dt}-\frac{C}{2}\frac{\partial E}{\partial x}=R_T\pm R_N \qquad (45)$$

In this form the above equation applies to a train of conservative waves, but not to significant waves because experience shows that under the stated conditions the energy of the significant waves is independent of x:

$$\frac{\partial E}{\partial x}=0 \qquad (46)$$

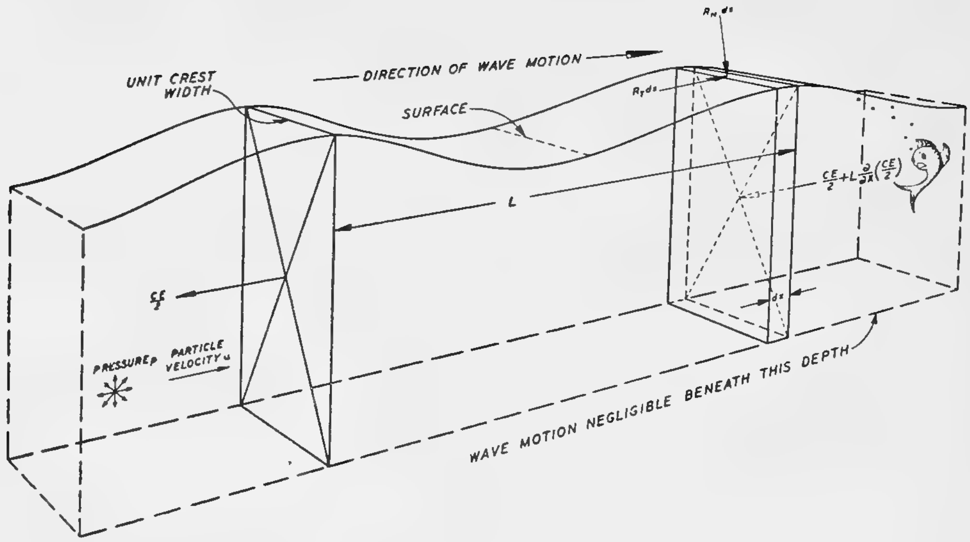

Figure 4.—Energy changes of an individual wave of length L traveling from left to right with velocity C.

With this condition (45) takes the form

$$\frac{dE}{dt}+\frac{E}{C}\frac{dC}{dt}=R_T\pm R_N \tag{47}$$

which will be further examined. It is not immediately apparent that this equation can be applied to nonconservative systems, but when the constants in the solution are evaluated for the significant waves the results check with observations.

Integration of (47) gives the change with time of the significant waves at any locality in the storm area. Initially the significant waves will have originated in the immediate neighborhood of the locality considered. As the time increases, the waves reaching this locality will have travelled a longer time and originated at larger distances. In practice the distance from which waves can come is limited by the dimensions of the storm system or by a shore line. This distance is called the fetch.

The time necessary for the waves to travel from the beginning of the fetch to the locality in question is called the minimum duration, t_{min}. If the duration of the wind exceeds t_{min} the character of the significant waves that are present in the fetch remains constant in time: a steady state is established.

To examine the steady state, consider a parallelepiped fixed in space of unit width and length δx, but otherwise similar to the one considered above. Since the parallelepiped is fixed in space, potential energy flows into the volume at the rear edge at the rate $CE/2$ and leaves at the forward edge at the rate

$$C\frac{E}{2}+\frac{\partial}{\partial x}\left(C\frac{E}{2}\right)\delta x$$

The local change in energy must equal the sum of the amounts which enter or leave the parallelepiped:

$$\frac{\partial E}{\partial t}\delta x=-\frac{\partial}{\partial x}\left(C\frac{E}{2}\right)\delta x+(R_T\pm R_N)\delta x$$

or, rearranging,

$$\frac{\partial E}{\partial t}+\frac{C}{2}\frac{\partial E}{\partial x}+\frac{E}{2}\frac{\partial C}{\partial x}=R_T\pm R_N \tag{48a}$$

This equation corresponds to (45) and applies to conservative waves. In order to apply it to the significant waves which are present over a limited fetch after a steady state has been reached, we write

$$\frac{\partial E}{\partial t}=0$$

and obtain

$$\frac{C}{2}\frac{dE}{dx}+\frac{E}{2}\frac{dC}{dx}=R_T\pm R_N \tag{48b}$$

This equation is equivalent to (47) and the comments on the applicability of (47) apply.

The solution of equation (48b) gives the height and velocity as function of fetch after a steady state has been reached, that is, for $t\geqq t_{min}$. It will be shown later how t_{min} depends upon fetch and wind velocity. The value of t_{min} determines whether equation (47), the duration equation, or (48), the fetch equation, is to be used for determining the characteristics of the significant waves.

The meaning of equations (47) and (48) can be further illustrated. Let the subscript "x" denote steady state conditions and the subscript "t" transient state conditions. Let

$$R_{Hx}=\frac{C}{2}\frac{dE}{dx} \quad \Big| \quad R_{Ht}=\frac{dE}{dt} \qquad (49a, b)$$

denote the increments of energy going into the increase of wave height, and let

$$R_{Cx}=\frac{E}{2}\frac{dC}{dx} \quad \Big| \quad R_{Ct}=\frac{E}{C}\frac{dC}{dt} \qquad (50a, b)$$

be the increments of energy going into the increase of wave velocity for the steady and the transient state, respectively. Equations (48) and (47) can then be written:

$$R_{Cx}+R_{Hx}=R_T\pm R_N \mid R_{Ct}+R_{Ht}=R_T\pm R_N \qquad (51a, b)$$

The Fundamental Equation

According to equation (4)

$$H=\delta L=\frac{2\pi}{g}\delta C^2=\frac{2\pi}{g}U^2\delta\beta^2 \qquad (52)$$

where

$$\delta=\frac{H}{L}\ ,\quad \beta=\frac{C}{U} \qquad (53a, b)$$

are two nondimensional parameters to be called wave steepness and wave age.

According to (6) and (52)

$$\frac{1}{E}\frac{dE}{dx}=\frac{2}{H}\frac{dH}{dx}=\frac{2}{\delta}\frac{d\delta}{dx}+\frac{4}{\beta}\frac{d\beta}{dx} \quad \Big| \quad \frac{1}{E}\frac{dE}{dt}=\frac{2}{\delta}\frac{d}{dt}+\frac{4}{\beta}\frac{d\beta}{dt} \qquad (54a, b)$$

Assuming that δ is a function of β only:

$$\frac{d\delta}{dx}=\frac{d\delta}{d\beta}\frac{d\beta}{dx} \quad \Big| \quad \frac{d\delta}{dt}=\frac{d\delta}{d\beta}\frac{d\beta}{dt}$$

With these substitutions and the expressions for R_T and R_N already given in (39) and (40), the equations for the steady state and the transient state can be written in the form

$$\frac{d\beta}{dx}=2AgU^{-2}\beta^{-3}\frac{1\pm\alpha(1-\beta)^2}{5+2\frac{\beta}{\delta}\frac{d\varrho}{d\beta}} \quad \Big| \quad \frac{d\beta}{dt}=AgU^{-1}\beta^{-2}\frac{1\pm\alpha(1-\beta)^2}{5+2\frac{\beta}{\delta}\frac{d\delta}{d\beta}} \qquad (55a, b)$$

where the upper sign (+) refers to $\beta\leq 1$.

Equations (55) relate the dependent variables δ and β to the independent variables x and t. Complete solutions can be found in one of two ways:

1. Eliminate either δ or β (or the equivalent dimensional terms H, T) from the equations by deriving solutions of the type $\delta=\delta\ (x,t)$ on the basis of completely independent considerations. The study of wave dispersion, for example, does indicate solutions $T=T\ (x,t)$ which, when substituted

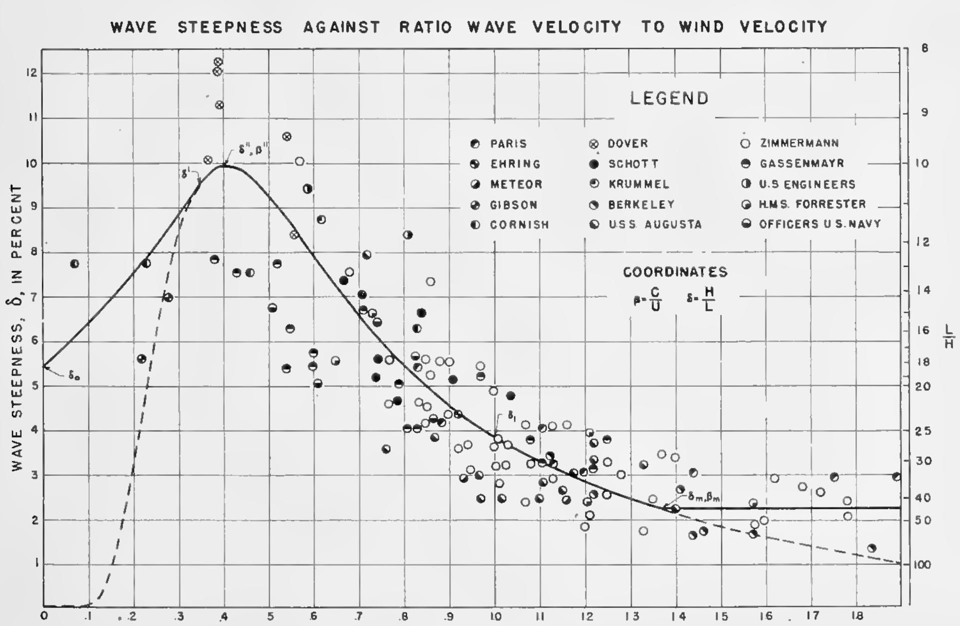

Figure 5.—Relation between wave steepness and wave age. Observed values shown by symbols, assumed relationship shown by full drawn line.

into equations (55) may make it possible to solve for $H=H(x,t)$.

2. Determine a relationship, $\delta=f(\beta)$, between the dependent variables from empirical evidence. The latter procedure has the disadvantage of being semiempirical, but it is the only one found possible so far.

Observations of Wave Age and Wave Steepness

It has been assumed that δ is a function of β only. Such a relationship has already been suggested by Krümmel (1911, p. 82) who writes that "the ratio of wave length to wave height depends upon the stage of development of the waves; in case of a 'young' sea the ratio L/H $(1/\delta)$ equals 10 or may be even smaller, but this ratio increases for a more advanced stage. * * *"

In the past many fruitless attempts have been made to relate wave steepness to wind velocity or other variables, but it has not been attempted to relate steepness, H/L, to wave age, C/U. Such a relationship is suggested by dimensional* considerations and, if existing, has the advantage of being independent of fetch and duration. The question can be examined by means of the 128 sets of observations entered in table I of appendix II. The corresponding values of the two nondimensional parameters δ and β are plotted in figure 5, which clearly demonstrates that the two variables are related.

The data have been collected from many different sources and from many localities, varying from a pond at Kensington Park, London, to the trade-wind belt of the North Altantic. Observations listed under Gassenmayr, Officers United States Navy, Paris, and Schott were taken before the turn of the century. Many of these old observations from small vessels are more reliable than recent observations.

Wave data listed under Gibson, Berkeley, and Ehring represent values based upon a statistical analysis of instrumental records. Gibson recorded waves against a pole in Buzzards Bay by means of a moving-picture camera, and the wave characteristics entered in the table apply to the average of the one-third highest of 20 or 30 waves. The Berkeley observations were taken from a United States Navy Weather Patrol ship by means of an automatic pressure-recording instrument and the wave characteristics apply to the one-third highest of a small number of waves. The single point in figure 5, which is based on the observations discussed by Ehring, deserves particular attention. It gives the significant wave age and steepness derived from a statistical analysis of 579 waves from a 30-minute instrumental record in the North Sea. Observations marked U. S. S. *Augusta* were taken by a weather officer during the invasion of Normandy those marked H. M. S. *Forrester* during an Atlantic crossing in February 1944. Some casual observations at Dover prior to the invasion of the continent are also included. The observations mentioned in this paragraph were obtained from unpublished reports.

According to figure 5 the steepness reaches a maximum value of about 0.10 as stated by Krümmel and as has been mentioned in reports from Great Britain. The maximum steepness occurs for β approximately 0.4, drops off somewhat for younger waves and diminishes rapidly for older waves.

Derivation of an Equation for Wave Age and Steepness

An empirical curve could be fitted to the empirical data shown in figure 5, and equations (55) and (54) could be integrated numerically. Instead, an analytical relationship between δ and β will be chosen and the integration will be carried out analytically. The choice of the relationship can be guided by physical considerations. It is common experience that the wave period increases continuously* with x or t. Evidence will be given later but here it will be assumed that

$$R_C>0 \tag{56}$$

If it is assumed that the energy transmitted to the waves by tangential and normal pressures is divided in a certain fixed manner between the energy required to increase wave height (R_H) and wave velocity (R_C), then either of equations (51) can be split up in the following manner:

$$R_H=(1-r)R_T\pm\left(1+\frac{r}{\alpha}\right)R_N \tag{57a}$$

$$R_C=rR_T\mp\frac{r}{\alpha}R_N \tag{57b}$$

where the signs are determined by (56). Addition of equations (57a) and (57b) leads to (51).

*Leonardo da Vinci was familiar with this when he wrote: "L'onda quanto piu si muove piu si abassa, e piu si dilata a piu fa veloce" (the further a wave moves, the more its height decreases and its length and velocity increase).

Introducing the definitions of R_C, R_T, and R_N into (57b) gives

$$\frac{d\beta}{dx}=2AgU^{-2}\beta^{-3}r[1\mp(1-\beta)^2] \tag{58a}$$

$$\frac{d\beta}{dt}=AgU^{-1}\beta^{-2}r[1\mp(1-\beta)^2] \tag{58b}$$

and by comparison with (55) leads to the following analytical relationship between δ and β:

$$\frac{1\pm\alpha(1-\beta)^2}{5+2\frac{\beta}{\delta}\frac{d\delta}{d\beta}}=r[1\mp(1-\beta)^2] \tag{59}$$

Agreement between equation (59) and the empirical data in figure 5 for all but the youngest waves justifies the assumptions made regarding the "energy split-up." It would have been possible to establish an equation involving δ and β by applying certain technique of curve fitting, but in view of the scatter of the observations it seemed preferable to find the relationship from physical assumptions and to use the observations to evaluate the constants.

Equation (59) can be written:

$$\frac{d\ln\delta}{d\beta}=\frac{1+\alpha-2(r+\alpha)\beta+(r+\alpha)\beta^2}{2r\beta^2(2-\beta)}-\frac{2}{\beta},\quad 0\leq\beta\leq1 \tag{60a}$$

$$\frac{d\ln\delta}{d\beta}=\frac{1-r-(r+\alpha)(\beta-1)^2}{2r\beta[1+(\beta-1)^2]}-\frac{2}{\beta},\quad 1\leq\beta \tag{60b}$$

whose integrals are

$$\ln\frac{\delta}{\delta_1}=-\frac{1+\alpha}{4r}\frac{1-\beta}{\beta}-\frac{1+\alpha}{8r}\ln(2-\beta)+\frac{3\alpha+20r-1}{8r}\ln\frac{1}{\beta},\quad 0\leq\beta\leq1 \tag{61a}$$

$$\ln\frac{\delta}{\delta_1}=-\frac{\alpha+10r-1}{4r}\ln\beta-\frac{1+\alpha}{8r}\ln[1+(\beta-1)^2]+\frac{1+\alpha}{4r}\tan^{-1}(\beta-1),\quad 1\leq\beta \tag{61b}$$

where δ_1 refers to the wave steepness when $\beta=1$.

According to (61a) $\delta\rightarrow0$ for $\beta\rightarrow0$. This is not in agreement with experience because very young waves are known to have considerable steepness and even the smallest waves generated by wind gusts have finite velocities and heights (equation 32c). For that reason it will be assumed that

$$\left.\begin{aligned}\frac{d\ln\delta}{d\beta}&=m \quad (62)\\ \ln\delta&=\ln\delta_0+m\beta \quad (63)\end{aligned}\right\}\beta\leq\beta'$$

where δ_0 and m are constants to be determined by the conditions that δ and $d\delta/d\beta$ must be continuous at $\beta=\beta'$. Since β cannot be zero, the logarithmic curve should not be drawn all the way to the y-axis but the initial development of the wave is so rapid that the exact form of the logarithmic relationship is of very little consequence to the later development of the wave.

Evaluation of Numerical Constants

The logarithmic relationship (62) is assumed applicable from $\beta=0$, $\delta=\delta_0$ to $\beta=\beta'$, $\delta=\delta'$. Assuming continuity in slope at β', equations (60a) and (62) give

$$m=\frac{1+\alpha-2(r+\alpha)\beta'+(r+\alpha)\beta'^2}{2r\beta'^2(2-\beta')}-\frac{2}{\beta'} \tag{64a}$$

while if δ is to be continuous

$$\ln\delta_0=\ln\delta_1-\frac{1+\alpha}{4r}\frac{1-\beta'}{\beta'}-\frac{1+\alpha}{8r}\ln(2-\beta')+\frac{3\alpha+20r-1}{8r}\ln\frac{1}{\beta'}-m\beta' \tag{64b}$$

according to (61a) and (63).

The steepness reaches a maximum, to be denoted by δ'', for $\beta=\beta''$, which is found by setting $d\ln\delta/d\beta=0$ in (60a):

$$\beta''=1-\sqrt{\frac{5r-1}{5r+\alpha}} \tag{64c}$$

The corresponding value of δ'' is given by (61a) for $\beta=\beta''$. At $\beta=1$, $\delta=\delta_1$ by definition.

The wave height reaches a maximum, H_m, for $\beta=\beta_m$, where according to (52)

$$\frac{d\delta}{d\beta}+2\beta\delta=0$$

and, from (60b)

$$\beta_m=1+\sqrt{\frac{1-r}{r+\alpha}} \tag{64d}$$

The corresponding value of the steepness, δ_m, can be determined from (61b) by substituting $\beta=\beta_m$.

In choosing numerical values for the constants, particular attention was paid to the point δ'', β''. The theoretical maximum value of δ is 1/7 (9b) but according to Krümmel, British reports, and the data in figure 5, δ does not exceed 1/10. The value of β'' must be approximately 0.4 and that of δ_1, 0.04. These considerations are sufficient to determine the chief constants, r, α, and δ_1. The fourth constant, β', was chosen to give δ_0 approximately 0.05, but its value is only of secondary importance.

In this manner the values of the constants can be determined within fairly narrow limits. In choosing the exact values adjustments were made on the basis of other empirical relationships, particularly the ones dealing with the decay of waves, but by far the greatest emphasis has been placed on the relationship between wave steepness and wave age. The curves in figure 5 are based upon the following exact values of the numerical constants:

Symbol	Value	Source	Symbol	Value	Source
r	0.580		β''	.407	Equation 64c.
α	2.500		δ''	.0990	Equation 61a.
m	1.627	Equation 64a.	δ_1	.038	
δ_0	.0537	Equation 64b.	β_m	1.369	Equation 64d.
β'	.350		δ_m	.0219	Equation 61b.
δ'	.095	Equation 61a.			

Wave heights decrease for $\beta>\beta_m$ but younger and shorter waves which must also be present will be most significant. For that reason the section of the curve to the right of β_m is dotted and will be neglected in practical forecasting.

The solid line in figure 5 is assumed to represent the relationship between significant wave steepness and age. Other relationships, such as wave height and velocity as functions of fetch or duration, and the decay of waves, follow directly and can be compared to observational evidence as check of the validity of our assumptions.

Wave Velocity, Wave Height, Fetch, and Duration

The wave velocity will be expressed in nondimensional form by means of the wave age, $\beta=C/U$.

(a): $0\leq\beta\leq\beta'$

Substituting (62) in (55):

$$\frac{d\beta}{dx}=2AgU^{-2}\beta^{-3}\,\frac{1+\alpha(1-\beta)^2}{5+2m\beta} \qquad (65a)$$

$$\frac{d\beta}{dt}=AgU^{-1}\beta^{-2}\,\frac{1+\alpha(1-\beta)^2}{5+2m\beta} \qquad (65b)$$

with the solutions

$$\frac{gx}{U^2}=\frac{m}{A\alpha}\left\{\frac{\beta^3}{3}+\frac{K_1}{2}\,\beta^2+K_2\beta+\frac{1}{2}\,(2K_2-K_1K_3).\right.$$
$$\left.\ln\left(\frac{\beta^2-2\beta+K_3}{K_3}\right)+K_4\tan^{-1}\left[\frac{\sqrt{\alpha}\beta}{1+\alpha(1-\beta)}\right]\right\} \qquad (66a)$$

$$\frac{gt}{U}=\frac{2m}{A\alpha}\left\{\frac{\beta^2}{2}+K_1\beta+\frac{K_2}{2}\ln\left(\frac{\beta^2-2\beta+K_3}{K_3}\right)+\right.$$
$$\left.(K_2-K_1K_3)\sqrt{\alpha}\tan^{-1}\left[\frac{\sqrt{\beta}}{1+\alpha(1-\beta)}\right]\right\} \qquad (66b)$$

where

$$K_1=\frac{5}{2m}+2=3.536$$

$$K_2=\frac{5}{m}+3-\frac{1}{\alpha}=5.673$$

$$K_3=1+\frac{1}{\alpha}=1.400$$

$$K_4=\sqrt{\alpha}(2K_2-K_1K_3-K_2K_3)=-2.446$$

are combinations of known constants.

(b): $\beta'\leq\beta\leq1$

using the upper signs of (58):

$$\frac{d\beta}{dx}=2AgrU^{-2}\beta^{-2}(2-\beta)$$

$$\frac{d\beta}{dt}=AgrU^{-1}\beta^{-1}(2-\beta) \qquad (67a, b)$$

with the solutions

$$\frac{gx}{U^2}=\frac{2}{Ar}\left[\ln\frac{2}{2-\beta}-\frac{\beta^2}{8}-\frac{\beta}{2}\right]-K_x$$

$$\frac{gt}{U}=\frac{1}{Ar}[(2-\beta)-2\ln(2-\beta)]-K_t \qquad (68a, b)$$

where

$$K_x=3.539\times10^2$$

$$K_t=1.705\times10^5 \qquad (69a, b)$$

are constants of integration, chosen to make solutions (66) and (68) continuous at $\beta=\beta'$.

(c): $1\leq\beta$.

Using the lower signs of (58):

$$\frac{d\beta}{dx}=2AgrU^{-2}\beta^{-3}[1+(\beta-1)^2]$$

$$\frac{d\beta}{dt}=AgrU^{-1}\beta^{-2}[1+(\beta-1)^2] \qquad (70a, b)$$

with the solutions

$$\frac{gx}{U^2}=\frac{1}{2Ar}\left\{\ln[16(\beta^2-2\beta+2)]+\frac{\beta^2}{2}+\right.$$
$$\left.2\beta-2\tan^{-1}(\beta-1)-5\right\}-K_x \qquad (71a)$$

$$\frac{gt}{U}=\frac{1}{Ar}[\beta+\ln(\beta^2-2\beta+2)]-K_t \qquad (71b)$$

where K_x and K_t are the constants in equations (69).

A graph of β against the nondimensional fetch parameter gx/U^2 (equations 66a, 68a, 71a) is shown in figure 6; against the non-dimensional duration parameter gt/U (equations 66b, 68b, 71b) in figure 7. The continuity of the solutions throughout the entire range is apparent, but in accordance with the assumption that the character of the sea surface is determined by the highest waves present, β remains constant after reaching a value of β_m (64). Comparison with observations, which are indicated by points, will be made later.

Wave height will be represented by the nondimensional parameter gH/U^2. According to (52):

$$\frac{gH}{U^2}=2\pi\delta\beta^2=f\left(\frac{gx}{U^2}\right) \text{ or } f\left(\frac{gt}{U}\right) \qquad (72)$$

because $\delta=f(\beta)$ (fig. 5) and $\beta=f(gx/U^2)$ (fig. 6) or $\beta=f(gt/U)$ (fig. 7). The wave heights are plotted nondimensionally in figures 6 and 7. Comparison with observations will be made later.

General Case

So far, the growth of wind waves has been examined in two special cases; the growth with fetch, assuming constant wind of unlimited duration, and the growth in time, assuming a constant wind blowing over an unlimited fetch. Under actual conditions both fetch and duration are limited, and for any given situation the wave height and age from the fetch graph (fig. 6) will differ, in general, from height and age from the duration graph (fig. 7). The smaller of the two values is considered valid for the following reasons.

If a wind of constant speed has blown for many hours over a small lake the height of the waves depends entirely upon the distance from the upwind shore. If, on the other hand, a wind has blown for just a few hours over a fetch of several thousand miles the limitation of the fetch can be of no consequence and the wave height must depend upon the duration only. Therefore, for any given wind velocity the wave height is determined by either the fetch (fig. 6) or the duration (fig. 7), depending upon which of the two factors imposes the greater limitation to the full development of the waves. This can also be stated by saying that for any given fetch there exists a "minimum" duration, t_{min}, for which the fetch and duration graphs give the same wave height and age. If the duration is less than t_{min}, the waves are determined from the duration graph; if the duration is greater than t_{min}, the waves are determined from the fetch graph.

The minimum duration can be expressed by means of the nondimensional parameter $t_{min}U/x$ which is found by reading off corresponding values of gx/U^2 and gt/U for various values of β from figures 6 and 7, and dividing:

$$\frac{t_{min}U}{x}=\left(\frac{gt}{U}\Big/\frac{gx}{U^2}\right)_\beta \qquad (73)$$

As a numerical example, assume that a 20 m/sec wind blows in an offshore direction. When the wind first starts to blow, $t=0$ and $H=0$ over the entire fetch (fig. 8A). Five hours later, according to the duration graph, the wave height would equal 4 meters as shown by the straight line in figure 8B, but in the immediate vicinity of the coast the waves will be lower, and at the very beginning of the fetch, at $x=0$, the wave height remains zero. The point marked "P" is placed at the distance from the coast at which a steady state has been established. To the left, upwind from point P, the significant wave height depends upon the fetch only. To the right of P the significant wave height is constant at any given time and its value depends only on the duration of the wind.

As the wind continues to blow, the point P travels downwind. For $t=5$ hours P lies 60 km. from the beginning of the fetch (fig. 8B). For $t=24$ hours P lies 500 km. from the beginning of the fetch (fig. 8C), and after 130 hours when P has advanced 4,500 km. the waves have attained the maximum height and the wave height is everywhere determined by the fetch graph alone (fig. 8D). A similar reasoning applies to the wave velocity or the wave age.

Let x and t be fetch and duration for a chosen value of β, and $x+dx$, $t+dt$, correspond to $\beta+d\beta$. Then the rate at which the influence of the limiting fetch advances, that is, the velocity of the point P, equals

$$\frac{dx}{dt}=\frac{d\beta/dt}{d\beta/dx}=\frac{C}{2} \qquad (74)$$

according to equations (55a, b). Hence the region in which the limiting fetch is dominant expands at group velocity. The same conclusion was reached by Commander Suthons (verbal communication).

When developing the theory of the growth of significant waves, the wind velocity in the generating area has been assumed to be constant in time and space. The limited experience gained so far indicates that satisfactory forecasts can be made on this assumption, especially for wind systems separated by well-defined fronts. It might

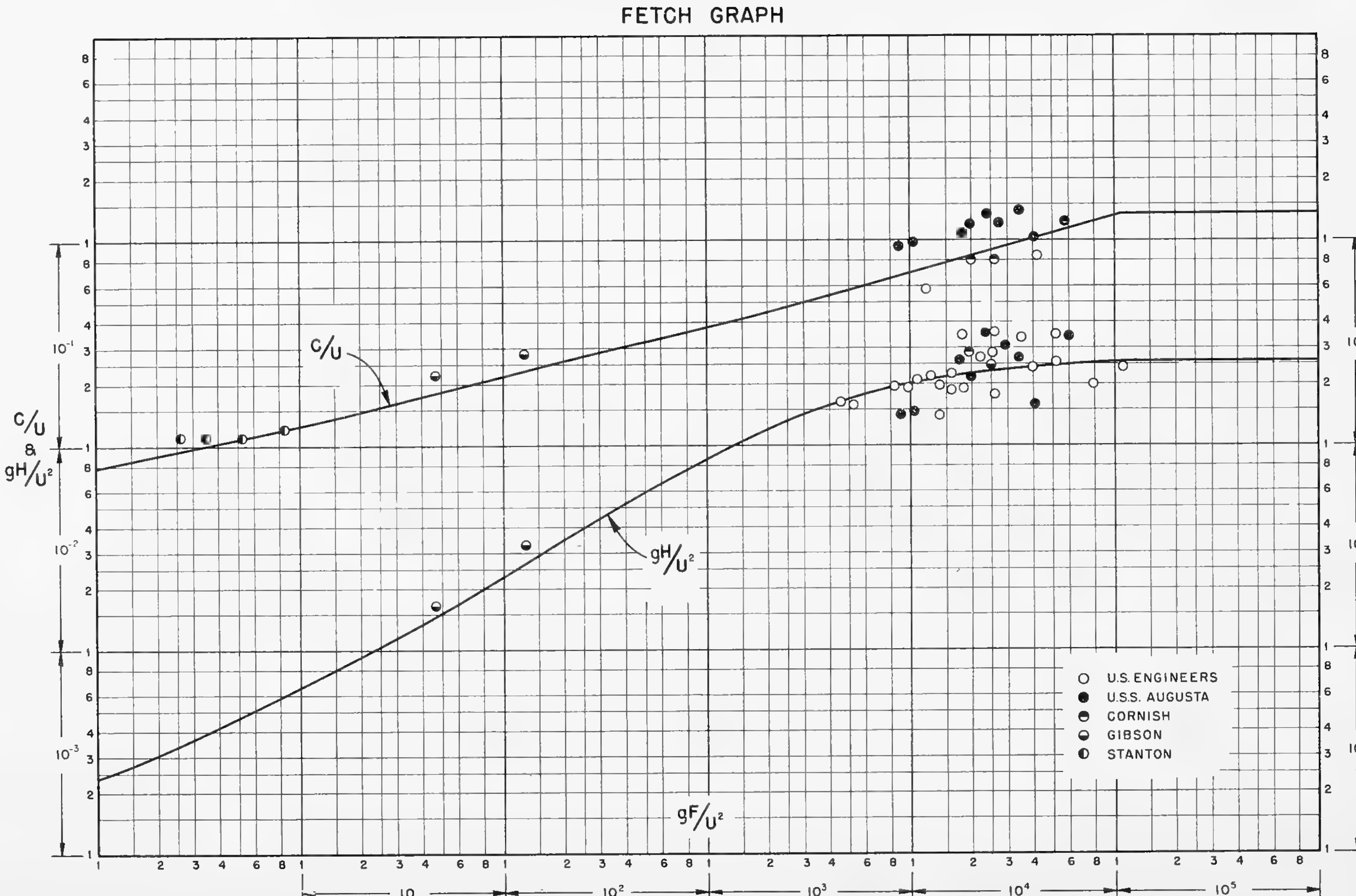

Figure 6.—Wave height and velocity as functions of fetch, using nondimensional parameters. Theoretical relationships shown by curves; observations by symbols.

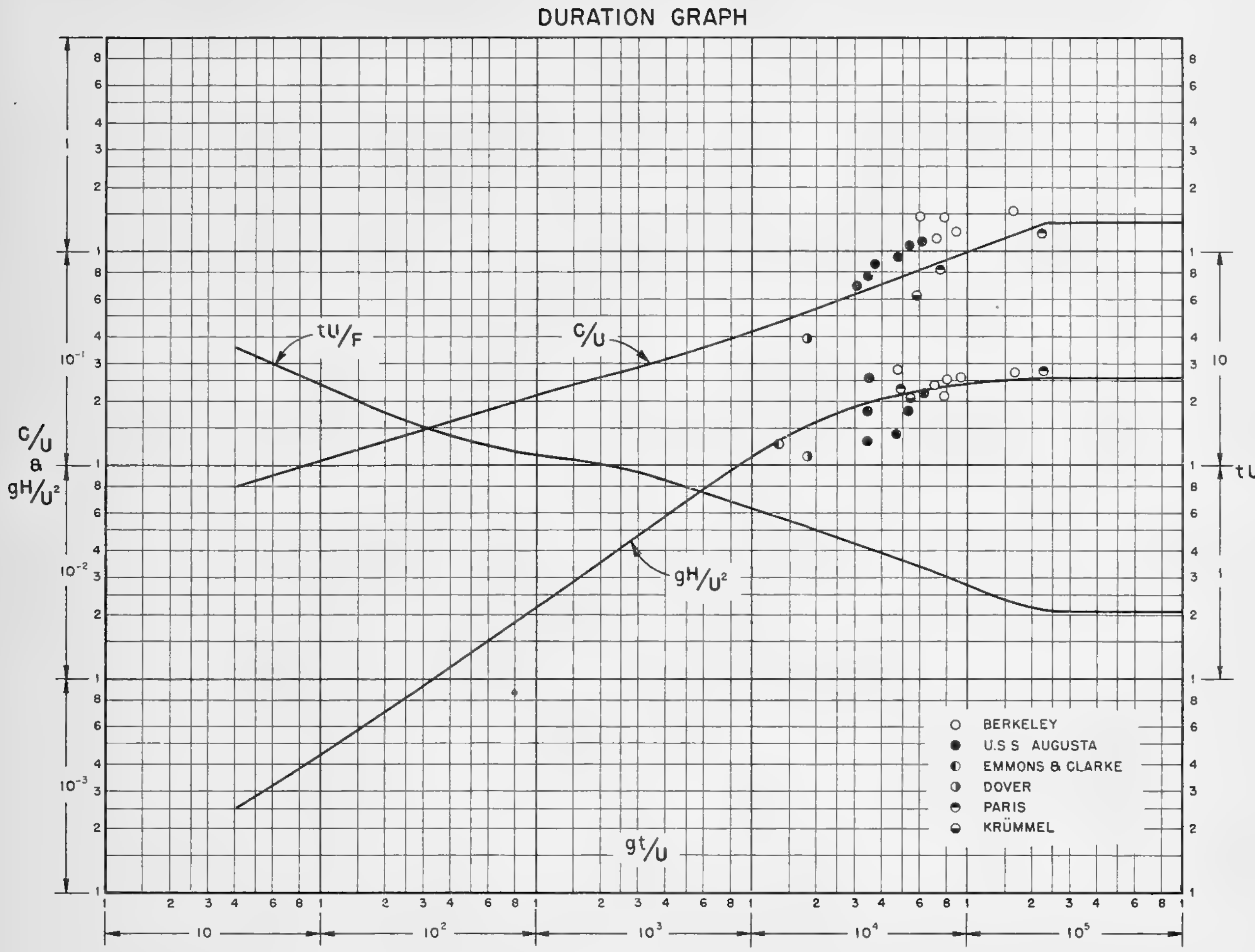

Figure 7.—Wave height and velocity as functions of duration, using nondimensional parameters and relation between minimum duration, fetch and wind velocity. Theoretical relationships shown by curves; observations by symbols.

be possible to find solutions for the case $U=f(x, t)$ by using numerical methods, but such attempts would involve a great deal of time and labor and would probably not be justified in view of the limited accuracy with which the fundamental variables, wind velocity, fetch, and duration, can be determined.

Wind and Wave Energy

In the theoretical development dealing with the growth of waves solutions to equation (55a, b) were found under two assumptions: the wave period increases continuously (equation 56), and the fundamental equation can be split up into two equations (57a and 57b). The latter assumption specified the manner in which the contributions of energy from the two "sources" R_T and R_N, are distributed to the two "energy sinks" R_H and R_C. The energy budget during the generation of waves is further illustrated in figure 9 where the light solid curves refer to the energy sources, the dashed curves to the sinks, all expressed as percentage of R_U,

$$R_U=\tau U=\gamma^2\rho' U^3 \tag{75}$$

the total energy dissipated by the wind in the lowest 8 to 10 meters. The significant wave has an energy income, ϵ', which must equal its total expenditure of energy:

$$\epsilon'=\frac{R_T}{R_U}\pm\frac{R_N}{R_U}=\frac{R_H}{R_U}+\frac{R_C}{R_U} \tag{76}$$

During the very early stages of wave development most of the energy is transmitted by normal stress, but for β larger than 0.37 transmission by tangential stress is dominant. With a 10 m/sec wind a wave age of 0.37 is reached in 1.88 hours and with a 20 m/sec wind, in 3.75 hours. Therefore, the effect of the normal stresses dominates for a short time only, and during the greater part of the time during which waves grow the effect

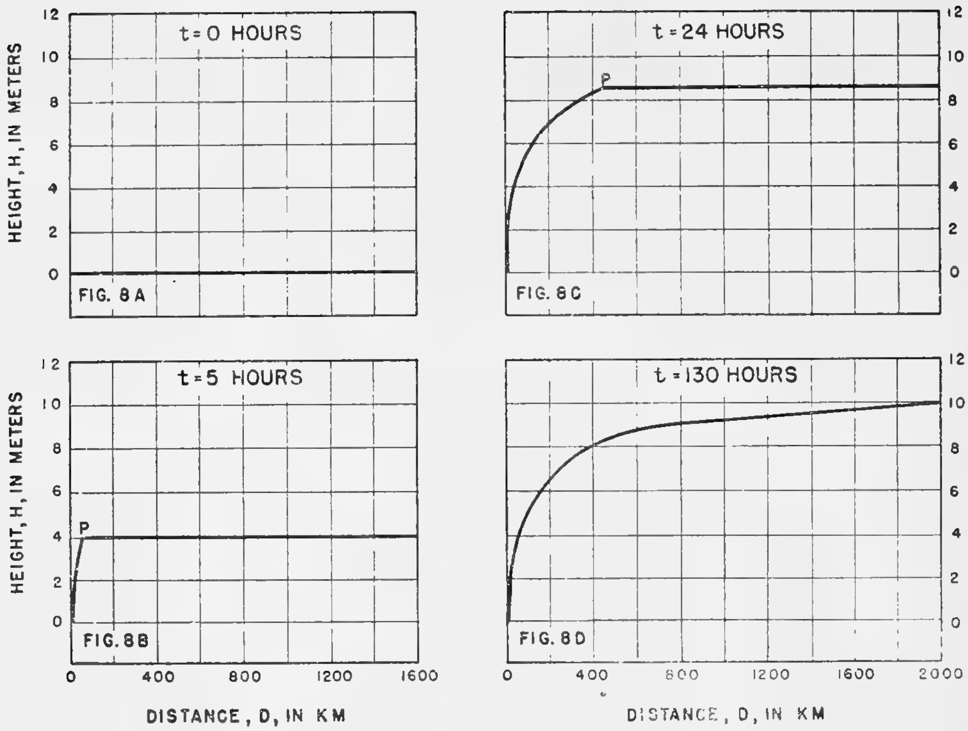

Figure 8.—Growth of wave height with time and distance from beginning of fetch.

of the tangential stress is more important. The ratio between the total amounts of energy transferred by tangential and normal stresses depends upon β only:

β value	0. 37	0. 7	1. 0	1. 369
Ratio between total amounts of energy transferred by tangential and normal stresses	. 87	1. 54	2. 84	4. 43

For $\beta<1$ the larger portion of the energy goes toward increasing the wave height, for $\beta=1$ the energy consumed in the increase of wave height and wave velocity are nearly equal, but for very old waves the larger portion of energy is required for the increase of wave velocity.

Again it must be emphasized that as yet no physical significance can be attached to the split-up of the fundamental energy equation. This split-up leads, in what appears to be the simplest mathematical manner, to solutions consistent with empirical evidence, and therein lies its chief justification. Solutions to (55) can perhaps be based on other considerations such as the increase of wave period, but the results obtained here are in such good agreement with observations that the main features in figure 9 can be regarded as significant. Any other attempts to find solutions must lead to similar results.

A very small portion, about 1 percent, of the wind energy dissipated in the lowest layer is transferred to the sea for maintaining the pure wind current. Another fraction goes into formation and maintenance of the significant waves. For any given value of β this fraction equals [(76), (39), and (40)]:

$$\epsilon'=\pi^2\delta^2\beta^2[1\pm\alpha(1-\beta)^2] \tag{77}$$

where

$$\alpha=\frac{s}{2\gamma^2}=2.5 \tag{78}$$

Actually, a number of waves of different velocities and directions are present simultaneously, so that the total percentage, ϵ, of the wind energy going into the formation of waves may be two or three times the energy needed for the formation of the significant waves, that is, 10 to 15 percent of the wind energy dissipated in the lowest layer may go toward increasing and maintaining the energy of waves.

With $\alpha=2.50$ and $\gamma^2=2.6\times10^{-3}$, one obtains

$$s=0.013 \tag{79}$$

This value is lower than that derived from Sir Thomas Stanton's experiments (table 3), and it is very much lower than Jeffrey's value of 0.27. If the latter value were valid, it would imply that all the dissipated energy would be required for raising the significant waves alone, but no support can be found for such a contention.

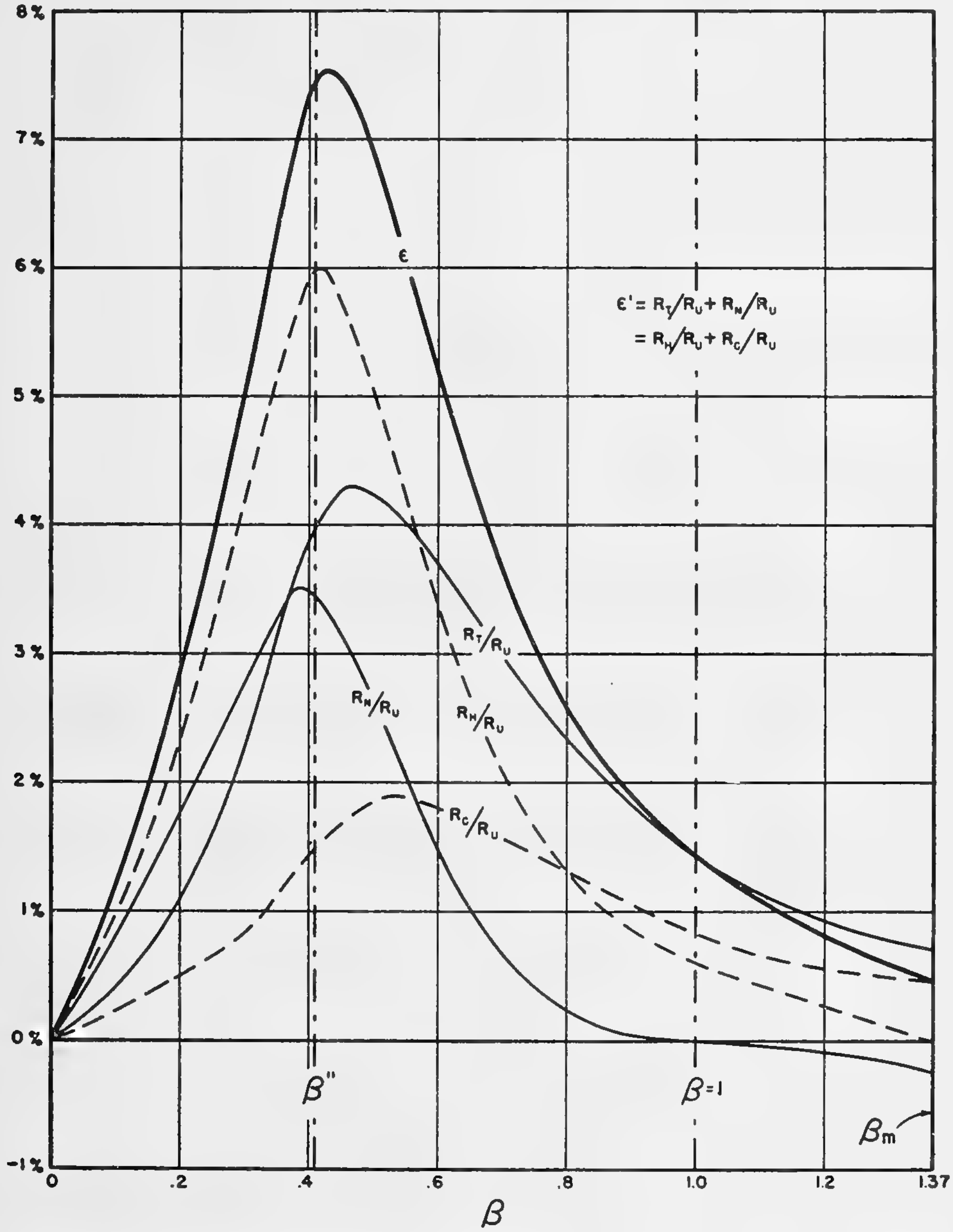

Figure 9.—Energy budget for growing waves. Solid lines show amounts of energy transmitted by normal and tangential wind stresses; dashed lines show amounts needed for changing wave height and wave velocity. All amounts are expressed as percentages of the energy dissipated in the lowest layer of the atmosphere and are shown as functions of wave age.

OBSERVATIONS OF THE GROWTH OF WAVES

The sea surface does not appear as a sequence of rhythmic waves but in spite of its irregular appearance it is possible to apply the terms wave period, wave length, and wave velocity, because some of the waves are more conspicuous than others and their characteristics can be observed. Simultaneous measurements of two or three of the wave characteristics, C, L, and T, afford checks on the possibility of applying the wave theory since they are so interrelated that if one is measured the other two can be computed (equation 4). Numerous measurements (Krümmel, 1911, or Sverdrup et al., 1942, p. 526) are in fair agreement with theoretical expectation. The theory also requires the ratio H/L not to exceed 1/7, a fact fully confirmed by measurements which rarely give values as high as 1/10. Therefore, observations show that the wave formulae are applicable to wind waves.

The empirical knowledge of the relation between wind and waves, however, presents a picture of confusion because the variables which influence the development of waves have not been properly recognized and separated. A complete observation should consist of at least the following five variables:

a. Significant wave height, H.

b. Significant wave period, T, or velocity, C, or length, L.

c. Wind velocity, U.

d. Wind fetch, x.

e. Wind duration, t.

It must furthermore be borne in mind that the waves in any one region are the result not only of local winds but also of winds in other areas, and that the relation between wind and waves during the periods of growth or of decay (sea or swell) should be considered separately.

Although numerous wave observations have been taken, only relatively few are complete in the above sense. Even these observations are unreliable, partly because the measured quantities are poorly defined. For ready comparison with theory, observations of growing waves have been grouped into the following five classes:

	Tabulation	Graph
1. δ as function of β	Appendix 2, table I	Fig. 5.
2a. gH/U^2 as function of gx/U.	Appendix 2, table II.	Fig. 6.
2b. β as function of gx/U	Appendix 2, table II.	Fig. 6.
3a. gH/U^2 as function of gt/U.	Appendix 2, table III.	Fig. 7.
3b. β as function of gt/U	Appendix 2, table III.	Fig. 7.

These dimensionally correct relations will be discussed first, to be followed by a consideration of certain well-known empirical relationships:

4. $H=f(x)$	Stevenson's law.
5. $H=f(t)$	Boergens law.
6. $H=f(U)$	Zimmerman, Cornish, Rossby, and Montgomery, miscellaneous data.
7. $C=f(U)$	Zimmerman, Cornish, Schott, miscellaneous data.

These latter relationships are incomplete. Stevenson's law, for example, is applicable to "maximum wind velocities" only, and objections can be raised to most of the other relationships. It will be shown, however, that the established empirical laws, although dimensionally erroneous, give numerical values which appear reasonable in the light of their nondimensional relations listed above.

1. *Wave age and steepness.*—This relationship is based on 128 sets of observations, tabulated in table I, appendix 2. The individual observations have been numbered because they also appear in the other tables where they can be recognized by their number. The observations are plotted in figure 5, where the solid curve is assumed to represent the true relationship. Various constants have already been determined on the basis of this curve, and the observations have been previously discussed (p. 16).

2a. *Wave height against fetch and wind velocity.*—The observational material consists of 39 observations from various sources. Those marked "U. S. ENGINEERS" were taken from the Milwaukee lightship in Lake Michigan (U. S. Engineers, 1932). The wave height was determined by measuring the rise and fall of the water surface along a graduated line. According to the original report, "sufficient readings were taken at each observation for a representative wave height to be obtained." Wind velocity, fetch, and duration were determined from Daily Synoptic Series Northern Hemisphere Sea Level Weather Maps, but the direct wind observations on the lightship were also taken into consideration. According to the minimum duration graph in figure 7, all observations were limited by fetch.

Observations marked "U. S. S. *Augusta*" were taken visually by Commander R. C. Steere, USN, off the Normandy Beach Head (unpublished reports), and the fetch, duration, and wind velocity were determined from available weather data. Cornish (1934) has compiled some of his own very careful observations. Gibson's observations have already been discussed when dealing with wave age and steepness (p. 16).

All observations are expressed by the nondimensional parameters gH/U^2, gx/U and plotted on figure 6. They show good agreement with theory.

2b. *Wave velocity against fetch and wind velocity.*—This relationship can be tested by means of 19 observations. The wave velocity is expressed in nondimensional form by the ratio C/U, the wave age. Four observations were obtained by Stanton (1937) in an experimental 50-foot tank under closely controlled conditions. The other sources of observations have been discussed already. All observations have been tabulated (table II, appendix 2) and plotted on figure 6, and they are in fair agreement with theory. For large values of gx/U^2 the observed values lie slightly above the theoretical values. When examining the agreement between theory and observation (fig. 6) it should be considered that observed values of fetch vary between 50 feet and more than a thousand kilometers, and that other parameters also vary over wide limits.

3a. *Wave height against duration and wind velocity.*—Seventeen observations could be found to test this relationship. Observations marked "BERKELEY" were taken by means of instruments from weather ships in the Pacific (unpublished reports). Fetch, duration, and wind velocity were determined from weather maps. Durations were corrected for initial wave heights according to the method given in H. O. Misc. 11275. One observation off the coast of Massachusetts by Emmons and Clarke (unpublished report) was adjusted in a similar manner, but the wave height was determined by a photographic method. For the latter observation the duration time was unusually well defined since the waves were generated by a storm which formed very suddenly. One observation marked "DOVER," England, was also taken during a period when the wind rose abruptly from zero to Beaufort 4. According to Krümmel (1911, p. 75), an old observation by Paris is of particular interest: "He noticed waves in the southern Indian Ocean rising to an over-all height of 6 to 7 meters as a result of a strong storm which lasted 4 days or, roughly, 100 hours, with remarkable uniformity. The fetch can undoubtedly be set at infinite* * *." This wave height was considered by Krümmel to represent the maximum wave height for the existing wind of 16 m/sec, a conclusion which, according to figure 7, agrees remarkably well with theory. The other observation was taken during the same sequence after a duration of only 24 hours. One observation by Krümmel himself in the equatorial Pacific is also included (Krümmel, 1911, p. 17). The observations available at this time confirm the theoretical relationship between wave height, gH/U^2, and duration, gt/U, but further observations, particularly for short durations and at very high wind velocities, are very desirable.

3b. *Wave velocity against duration and wind velocity.*—The manner in which all 15 observations plotted in figure 7 were taken has already been discussed. Here the agreement between theory and observations is poorer than it has been for all other relationships. All observations by Berkeley and U. S. S. *Augusta* give high values for the wave velocity; those marked "Dover, Krümmel, and Paris" are too low. Further observations are particularly desirable to check the relationship between wave velocity and duration.

4. *Wave height against fetch.*—Stevenson has established an empirical formula giving the "greatest" wave height, H, in cm. as function of the fetch, x, in cm. (Krümmel, 1911, p. 68), according to which

$$H=0.105\sqrt{x} \tag{80}$$

The formula was established by means of data from lakes where the value of x ranged from a few kilometers up to about 250 km. Stevenson pointed out that for small values of x the wave heights were greater than those given by the above simple equation. For the Mediterranean, Cornish (1934 p. 33) has verified the relation for fetches up to 830 km., and it is generally assumed that the relationship holds for values of x up to 1,000 km.

The formula is incomplete since it does not take the wind velocity into account but it is intended to apply at the highest wind velocity that can be expected to occur. A few of the observations which served as basis for Stevenson's law did include wind velocity and those, at least, were made during unusually strong winds.

The dimensionless relationship between H, x, and U shown in figure 6 can be made to agree with equation 80 if the following relation existed between fetch and wind velocity:

Fetch (km.)	10	50	100	250	500	1,000
Wind velocity (cm./sec.)	1,230	1,260	1,370	1,600	1,870	2,100

Such a relationship may exist because the wind velocities are higher over water than over land and the greater the body of water, the greater is the "maximum wind velocity" over the water.

5. *Wave height and wind duration.*—According to common experience at sea a high wind can generate a rough sea in a few hours. C. Boergen pointed out (Krümmel, 1911, p. 73) that the height of waves does not increase linearly with the duration of the wind, but that the increase is rapid in the beginning and becomes slower later on. For that reason Boergen suggested a relationship

between wave height and duration of the form

$$H=\frac{H_m}{1+\nu/t} \tag{81}$$

where H_m is the maximum wave height for any given wind velocity, and ν a coefficient which must be determined from observations. Although a close comparison with the relations given in this report is not possible, the chief features of (81) are in agreement with the present theory.

6. *Wave height and wind velocity.*—According to equation (72) the wave height equals

$$H=\frac{2\pi}{g}U^2\delta\beta^2$$

and depends, therefore, not only upon wind velocity but also upon fetch and duration, since δ and β are functions of these variables. The maximum wave height is found by setting $\beta=\beta_m$, $\delta=\delta_m$ (p. 18) and therefore depends upon the wind velocity only:

$$H_m=\frac{0.26}{g}U^2 \tag{82}$$

Equation (82) is a dimensionally correct expression and is in fair agreement with the equation

$$H_m=\frac{0.3}{g}U^2 \tag{83}$$

proposed by Rossby and Montgomery (1935) from entirely different considerations.

Linear relations between wind velocity and the general (not the maximum) wave height have been proposed by Cornish (1934) and Zimmerman (Patton and Marmer, 1932). These relations, together with equation (82) (marked "∞") and (83), are shown in figure 10. Several lines of equal fetch and equal duration have been computed according to the nondimensional relations of this paper and are also shown in figure 10. The points represent the 128 observations of table I, appendix 2, which were plotted on figure 5. Where observations were too crowded for individual plotting they have been combined and are shown by large circles with the number of observations indicated in the center.

Equation (82) gives a fair indication of the maximum wave heights but the two linear relationships, which relate wave height to wind velocity only, do not fit well. Some measure of success of these linear relationships may be explained by the fact that the authors have been concerned mainly with the highest waves observed

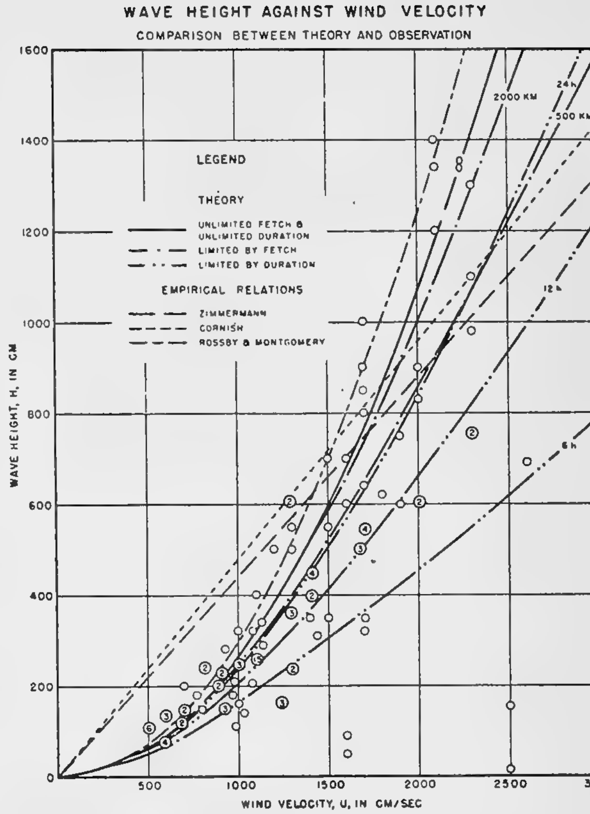

Figure 10.—Wave height as function of wind velocity only.

at different wind velocities. These wave heights will be too high at low wind velocities when it is likely that waves are present which were previously generated by stronger winds or which come from nearby areas. On the other hand, observed wave heights will be too low at very high wind velocities, where the limitations imposed by the fetch and the duration may not have permitted a full development of the wave.

7. *Wave velocity and wind velocity.*—This relationship is shown in figure 11, the legend of which corresponds closely to that of figure 10. The line marked infinity follows from the definition of $\beta_m=C_m/U$ (p. 17). The same 128 sets of observations have again been entered. Three empirical laws, shown by the thin dashed curves, give the empirical relationship between velocity of the largest waves and the wind velocity. According to Zimmermann (Patton and Marmer, 1932), $C=2.35\ U^{2/3}$, and hence C exceeds U for wind velocities above 1331 cm/sec, but according to Cornish (1934), $C=0.8\ U$, and to Schott (1893), $C=0.76\ U$, that is, the wave velocity is less than the wind velocity. Observations in figure 8, however, show that the wave velocity frequently exceeds that of the wind.

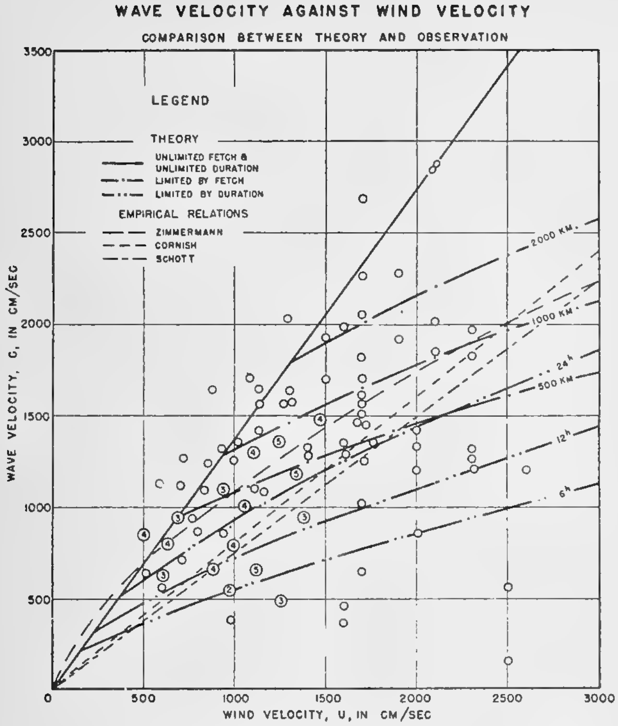

Figure 11.—Wave velocity as function of wind velocity only.

Cornish has discussed the relation between wind and wave velocity in great detail, and his conclusions are based on a lifetime of careful observations at sea. It is possible that his conclusions were somewhat influenced by Jeffreys, according to whom waves can grow only if their velocity is less than that of the wind (31), but this limitation no longer holds when the transmission of energy by tangential stress is taken into account (41).

Observations from the trade-wind regions where the wind blows with nearly uniform and moderate velocity over large areas are highly instructive. Some of these observations, which have been plotted in figure 5, are collected in table 4. They were obtained by Paris (Krümmel, 1911, p. 52, 80) and Schumacher (1939) and show clearly that the wave velocities may exceed wind velocities in the generating area.

Table 4
Average Wave Characteristics in Trade Wind Regions

Number table I, appendix 2	Locality	Observations made by—	U cm/sec	C cm/sec	β C/U	δ H/L
32	Trade winds, North Atlantic	Meteor	935	858	0.93	0.029
17	do	do	670	780	1.16	.024
33	do	Paris	590	1,120	1.90	.029
34	Trade winds, Indian Ocean	do	720	1,260	1.75	.029
39	Western Pacific Ocean	do	860	1,240	1.44	.030
47	Westerlies, South Atlantic	do	1,240	1,400	1.13	.032
75	Westerlies, Indian Ocean	do	1,490	1,500	1.01	.045
71	China Sea	do	1,300	1,140	.88	.042

THEORY FOR THE DECAY OF WAVES

Energy Budget

After the waves have left the generating area they travel through a region of calm where the wind velocity is small compared to the wave velocity. The waves receive no energy by normal pressure but, on the contrary, they meet an air resistance. The loss of energy due to the air resistance, according to (30b), equals

$$R_N=-\frac{1}{2}s\rho'k^2a^2C^3=-\frac{1}{8}s\rho'g^2H^2C^{-1} \qquad (84)$$

The transfer of energy due to tangential stress (37) can be neglected:

$$R_T=0 \qquad (84)$$

Therefore, from equation (57)

$$R_H=-\left(1+\frac{r}{\alpha}\right)R_N,\ R_C=\frac{r}{\alpha}R_N \qquad (86a, b)$$

and the decay of waves is derived from (86) without any further assumptions.

Wave period and distance from the generating area.—Substitution of (49a), (50a), and (84) in equation (86b) gives

$$\frac{1}{2}E\frac{dC}{dx}=\frac{r}{\alpha}\frac{1}{8}s\rho'g^2H^2C^{-1}$$

From (4), (6), and (78), again writing $A=2\gamma^2\rho'/\rho$, one obtains

$$\frac{dC}{dx}=2AgrC^{-1},\ \frac{dT}{dx}=\frac{8\pi^2}{g}ArT^{-1} \qquad (87)$$

Integrating from $x=F$ (end of fetch) to $x=F+D$ (end of decay distance):

$$\frac{T_D}{T_F}=\sqrt{1+16\pi^2Ar\left(\frac{D}{gT_F^2}\right)} \qquad (88)$$

where T_D and T_F are the wave periods in seconds, respectively, at the end of the decay distance and at the end of the fetch.

Travel time and distance from a generating area.—To obtain the travel time, t_D, from the end of the fetch to the end of the decay distance, it is assumed that the disturbance travels at half the wave velocity. The significance of this assumption is examined in the next section. It follows

$$t_D=\int_F^{F+D}\frac{dx}{C/2}=\frac{4\pi}{g}\int_F^{F+D}\frac{dx}{T} \tag{89}$$

According to (87)

$$\frac{dx}{T}=\frac{g}{8\pi^2Ar}dT$$

and

$$\frac{t_D}{T_F}=\frac{1}{2\pi Ar}\left(\frac{T_D}{T_F}-1\right) \tag{90}$$

Wave height and distance from a generating area.—From equation (86a), considering that $dE/E=2dH/H$, one obtains

$$\frac{1}{H}\frac{dH}{dx}=-(r+\alpha)A\frac{4\pi^2}{g}T^{-2} \tag{91}$$

Substituting for T from (87):

$$\frac{dH}{H}=-\frac{r+\alpha}{2r}\frac{dT}{T}$$

and integrating again from $x=F$ to $x=F+D$,

$$\frac{H_D}{H_F}=\left(\frac{T_D}{T_F}\right)^{-\frac{r+\alpha}{2r}} \tag{92}$$

Equations (88), (90), and (92) are presented in figure 12, the nondimensional decay graph.

Intensity and distance of storm from which swell comes.—Sometimes it is desirable to estimate properties of the storm from wave observations in the decay region. Equations (88), (90), and (92) are not suitable because the unknowns D and T_F appear on both sides of the equation. Rearranging terms and solving for D, t_D, and U^2 in terms of the observed quantities H_D, T_D, and

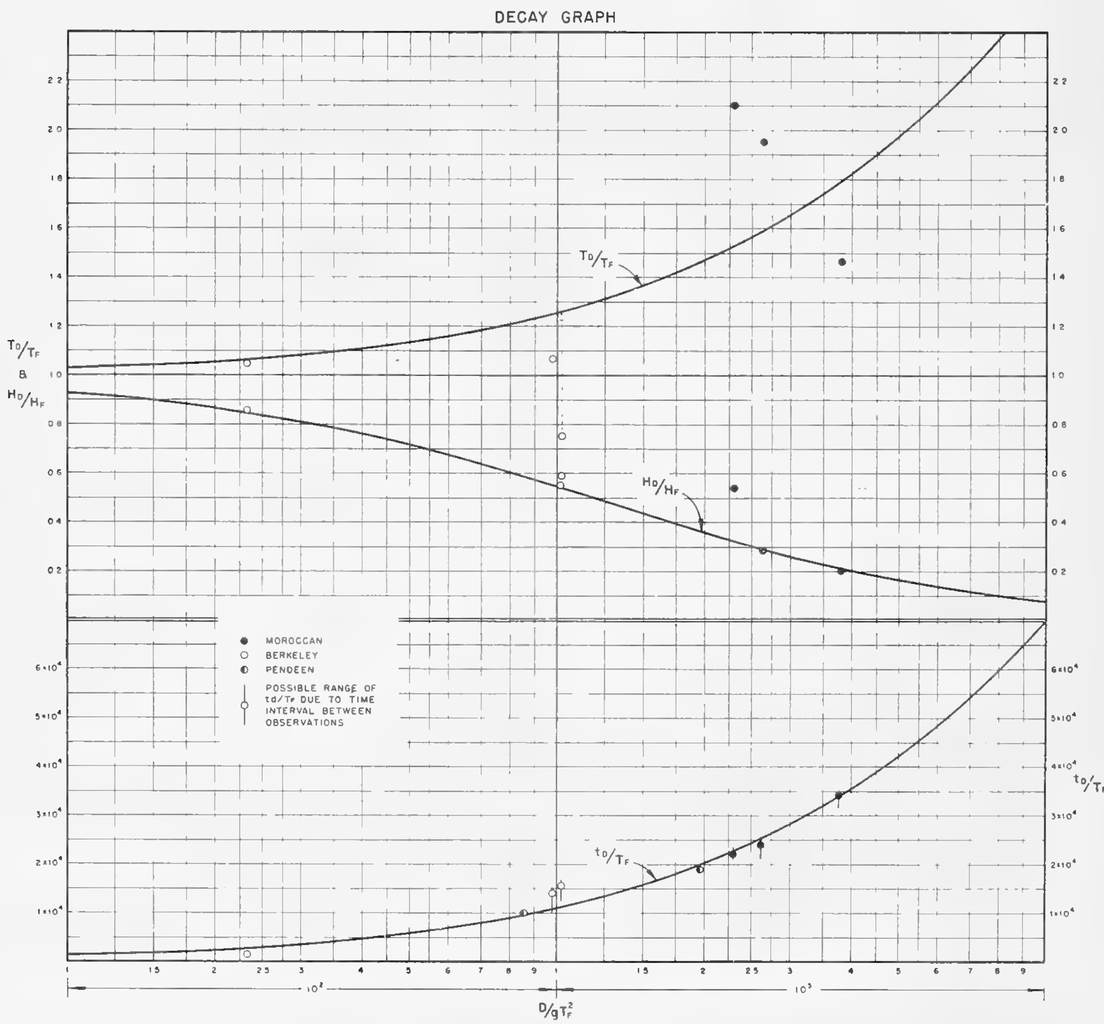

Figure 12.—Period and height of swell at end of distance of decay, and travel time represented by nondimensional parameters. Theoretical relationships shown by curves; observations by symbols.

taking into account that $\delta=H/L=2\pi H/gT^2$, one obtains:

$$\frac{D}{gT_D^2}=\frac{1}{16\pi^2Ar}\left[1-\left(\frac{\delta_D}{\delta_F}\right)^{\frac{4r}{5r+\alpha}}\right]. \tag{93}$$

$$\frac{t_D}{T_D}=\frac{1}{2\pi Ar}\left[1-\left(\frac{\delta_D}{\delta_F}\right)^{\frac{2r}{5r+\alpha}}\right] \tag{94}$$

$$\frac{U^2}{gH_D}=\frac{1}{2\pi}\frac{1}{\delta_F\beta_F^2}\left(\frac{\delta_F}{\delta_D}\right)^{\frac{r+\alpha}{5r+\alpha}} \tag{95}$$

Even in this form the equations are not satisfactory because it is impossible to eliminate all the unknowns. The parameters β_F and $\delta_F=f(\beta_F)$ appear on the right-hand side of the equations which relate decay distance, travel time, and wind velocity to the observed quantities H_D and T_D, and the unknown quantity β_F.

In order to use equations (93) to (95) it is necessary to assume values of β_F. This can be done by assuming a relationship between wind velocity and duration based on the common experience that high winds are usually of short duration while weaker winds may blow for a long time. Such a relationship determines β_F because when t and U are known the parameter gt/U can be computed and β_F read from figure 7. In figure 13 two specific relationships are shown in the inset, and in the upper and the lower parts of the figure the corresponding values of D, t_D, and U are represented as functions of the height and period of the swell, H_D and T_D.

A comparison of values read off from the two parts of figure 13 reveals that long period, high swell can occur only if the wind duration in the generating area has been long. Thus, a wind of velocity 20 m/sec and duration 24 hours (curve *A*, inset) gives a swell of period 16 seconds and height 2 feet at a distance of 2,800 km. (upper part of fig. 13), but if the duration were only 12 hours (curve *B*, inset) the swell at a distance of 2,800 km. would have a period of 14.8 seconds and a height of only 0.9 feet. It is also found that for given values of H_D and T_D conclusions as to distance of decay and travel time can be drawn with greater certainty than conclusions as to wind velocity in the generating area.

The Energy Front

The preceding discussion deals with steady state conditions in the area of decay. At any given locality in the area of decay significant waves will attain their maximum steady state height after waves have been emitted from the area of generation for a long time. Wave heights will be lower at an earlier time, when the waves advance into an area of decay which is relatively undisturbed. The time required to reach, for example, 50 or 90 percent of the wave height corresponding to steady state conditions must be found from a study of the transient state.

Consider the fundamental equations (13) or (48a) as applied to the simple case of waves in vacuum, traveling with constant wave velocity in a nonviscous fluid:

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}(VE)=0 \tag{13}$$

This hypothetical case is dealt with in an earlier section of this paper (p. 6). Equation (27) which is a solution of (13) gives the wave height at any given time and distance from the generating area with good approximation. According to (27) the wave height increases from a minute percentage to very nearly 100 percent of its maximum value within a very short distance (fig. 3). The "region of sharp increase in wave height" has at any instant traveled only half as far as the leading wave; its velocity is half the velocity of the leading wave. For purposes of forecasting, therefore, the rate at which the disturbance advances in the area of calm should be taken at one-half the wave velocity, since waves further advanced than the center wave are very low.

The following numerical example will serve as illustration. Let waves of 10-m. height and 12-sec. period be generated in a storm area 4,000 km. from the point of observation. According to figure 12 the wave height at 4,000 km. will have decreased to 270 cm., the wave period will equal 19.5 sec., and the "center wave" will arrive about 90 hours after the first waves left the generating area. According to equation (27) a height equal to 10 percent of the steady-state wave height, or 27 cm., will be attained 295 wave periods, or 96 minutes, before the arrival of the central wave; a height of 242 cm., 90 percent of the steady-state wave height, will be attained 112 periods or 36 minutes after the arrival of the "center wave." Therefore, the wave height increases from 10 to 90 percent of its maximum value in 96+36 minutes, or in about 2¼ hours, while it took 90 hours for the "center wave" to arrive.

So far, a hypothetical case has been treated for which a sudden increase from zero to full wave

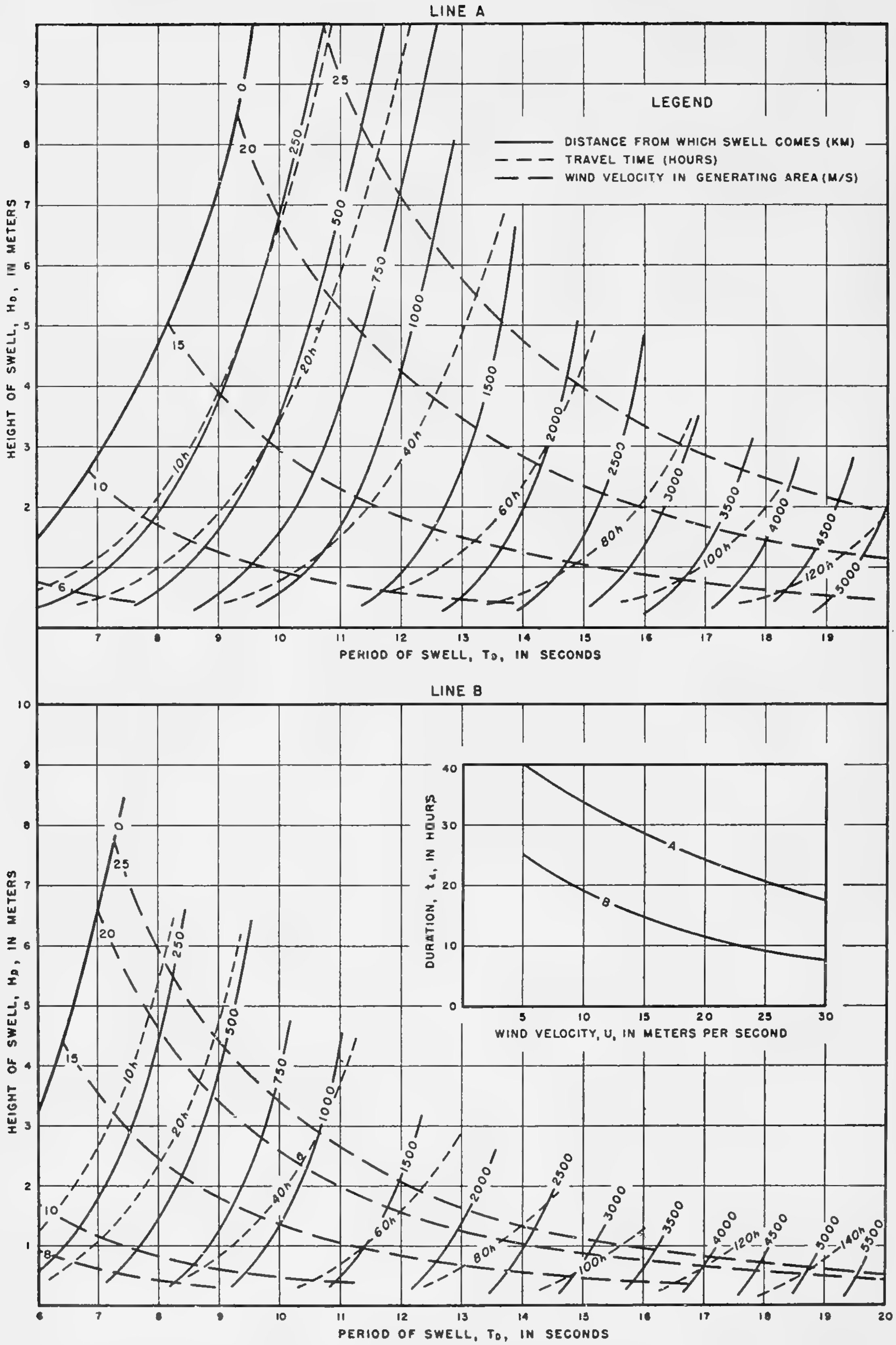

Figure 13.—Distance from which swell comes, travel time, and wind velocity in fetch as functions of height and period at swell at end of distance of decay for two assumed relationships between wind velocity in fetch and duration.

height has been assumed at the end of the generating area. Actually, this increase will not be sudden but will depend upon the fetch and the wind velocity (see figs. 6 and 7). It has been shown that the sudden increase in wave height at the end of the generating area retains much of its sharpness as the disturbance travels through the area of decay; it is reasonable to assume, therefore, that a gradual increase at the end of the fetch will appear as a similar gradual increase in wave height at any point in the area of decay.

If a storm should increase in violence, then high and long waves may arrive simultaneously with or may overtake lower and slower waves which were generated at an earlier time. If the higher waves overtake the lower and slower waves, a more abrupt increase of wave height can be expected at large distances from the generating area.

One inconsistency remains to be explained. According to equation (12), the wave period remains constant near a geometric point travelling at half the wave velocity. Therefore no increase in wave period should be observed at the energy front, and only the low waves arriving ahead of the energy front should have periods larger than the ones at the end of the fetch. Observations do indicate, however, that the period of the significant waves increases with distance from the generating area.

The explanation can probably be found in the limitations imposed by the Stokes theory, and by the assumption of the constant wave velocity made in the section dealing with the propagation of a disturbance through a region previously undisturbed. Rossby (1945) has shown that for his generalized Stokes waves a dispersion of energy from short-period to long-period waves does, indeed, take place. At present it has not been found possible to take some of these recent theoretical studies into account when dealing with significant waves in the decay region, but the theory presented here, although incomplete, has been found consistent with observations.

OBSERVATIONS OF THE DECAY OF WAVES

Little factual infomation is available concerning the behavior of waves which are no longer subject to wind action. It appears that the wave period and, therefore, the wave length continues to increase while the wave height decreases, and both these changes contribute toward reducing the wave steepness. Furthermore, shorter waves decay much more rapidly than longer waves, and in the area of calm the irregular pattern of steep wind waves is transformed into a pattern of regular, low undulations known as swell.

Changes in wave characteristics with distance from the generating area depend also upon the period of the wave at the end of the generating area. No observations are available from which all these factors can be derived. A rough check of the theory can be obtained by plotting in figure 12 the nondimensional ratios of (88), (90), and (92) as derived from observed values of T_D, t_D, and H_D and values of H_F and T_F computed from weather maps. The validity of such a check rests upon the assumption that the relationships in the generating area are well established (figs. 6 and 7) and that the weather situations have been analyzed correctly. Observations at two widely separated localities in the region of decay are desirable for a study of the decay of waves without reference to the processes of generation.

Wave height and distance from generating area.—Six observations have been plotted in figure 12. Three observations marked "MOROCCO" were taken by the French Meteorological Service. These consisted of wave height and period about three times daily. The points in figure 12 represent averages of observations at Casablanca, Rabat, and Mehedya. Three observations marked "BERKELEY" were taken from weather ships, and for those the distance of decay is somewhat shorter. The agreement between theory and observations is fair.

The following quotation from the British forecasting report is of interest (British Admiralty 1942):

> But as the forces mentioned in the first paragraph as acting on the surface now act against the motion of the water, energy is continuously removed from the waves and their height diminishes. The rate of diminution of height is greater for the shorter waves; it appears, from such evidence as is available, that the waves lose roughly one-third of their height each time they travel a distance in miles equal to their length in feet, e. g., a swell 600 feet long and 30 feet high is 20 feet high after 600 miles, 13.3 feet high after 1,200 miles, 9 feet high after 1,800 miles, 6 feet high after 2,400 miles, and so on.

According to this rule:

$$\frac{H_D}{H_F}=0.67 \text{ for } \frac{D}{L_F}=6{,}080 \text{ or } \frac{D}{gT_F^2}=\frac{6{,}080}{2\pi}=975,$$

and a comparison can be made with the curve in figure 12. One obtains:

D/L_F (D in miles, L_F in feet)	1	2	3	4	5
Dg/T_F^2	975	1, 950	2, 920	3, 900	4, 900
H_D/H_F British rule	0. 67	0. 44	0. 30	0. 20	0. 13
H_D/H_F Fig. 12	. 55	. 37	. 27	. 21	. 17

The values of H_D/H_F agree for $D/L_F=4$. Assuming T_F equal to 8 seconds, the distance of decay equals 1,300 nautical miles. For smaller distances of decay, the British rule gives larger wave heights, for longer distances, smaller wave heights, but the general agreement is quite satisfactory.

Travel time and distance from the generating area.—In additon to the Morocco and Berkeley observations, two observations at Pendeen, England, have been included. The character of the observation at Pendeen will be discussed further when dealing with wave periods. In determining the arrival time, attention was focused on an abrupt change in observed height of swell, usually a peak. At Pendeen records of the swell were obtained at intervals of 2 hours and the arrival time of the peak could be accurately established. In the other examples, for which observations were widely spaced, the possible error in arrival time is indicated by the length of the line drawn through the point of observation. The agreement between theory and observations is remarkably good.

Certain observations by Berkeley, where the winds decreased gradually, have been omitted and only observations where the duration was fairly definite are included in figure 12. Among the omitted observations there was some evidence that the theoretical travel time was too long. The problem of travel time warrants further theoretical study and examination of reliable, continuous observations.

Wave period and distance from the generating area.—The observed period increase confirms the order of magnitude of the theoretical period increase, but individual points are too scattered to draw any further conclusions. Since the period increase is implicitly contained in the relationship dealing with travel time and, to a lesser extent, in the relationship dealing with wave height, the fair agreement between theory and observations for these quantities may be regarded as indirect evidence in favor of the theoretical relationship for wave period.

The scatter of the observations is partly due to the fact that in the ratio, T_D/T_F, both numerator and denominator are subject to considerable uncertainty. This uncertainty arises because observations of period are difficult and because in given localities the periods are not constant but vary over wide limits.

Observations at Pendeen are well suited to study the increase in wave period. They were made by means of a pressure recorder placed at a depth of about 100 feet and were subjected to a harmonic analysis by means of specially designed equipment. Twenty-minute records at two-hourly intervals were examined for two periods, March 14 to 18 and April 17 to 19, 1945. For these two periods the weather situations were similar, each being characterized by one well defined low-pressure system traveling toward NE across the North Atlantic. Both storms could be expected to produce heavy swell that would reach Pendeen from nearly the same direction.

In the upper parts of figures 14 and 15 the ranges of the periods recorded at Pendeen are shown by the bands enclosed between the full-drawn and the dashed lines. The full-drawn lines include all recorded periods, the dashed lines include the more frequent ones. The reason for the great variability of the observed periods is probably that waves arriving at Pendeen did not come from a "line source" but from different portions of a large generating area and had used different travel times over different distances of decay.

The computed periods at the end of the fetch are indicated by the lower circles and the forecast wave periods at Pendeen by the circles directly above them. The numbers indicate the travel time from the end of the fetch to Pendeen. The computed fetch at the ends of the periods are not arranged in chronological order because sometimes waves which left the fetch earlier than other waves arrived at Pendeen at a later time. The forecast periods generally lie inside the band of observed periods except on March 14 and early on March 15 and on April 18, when the first "forerunner" of the swell appeared at Pendeen. These initial waves were low and may represent swell arriving ahead of the energy front.

The forecast height in feet and the observed height on an arbitrary scale are also shown in the lower parts of figures 14 and 15. The changes in computed and observed heights are in good agreement. The peaks in the figures have been used to check the theoretical travel time and are shown by the points marked "PENDEEN" in figure 12.

The concept of an increasing wave period in the

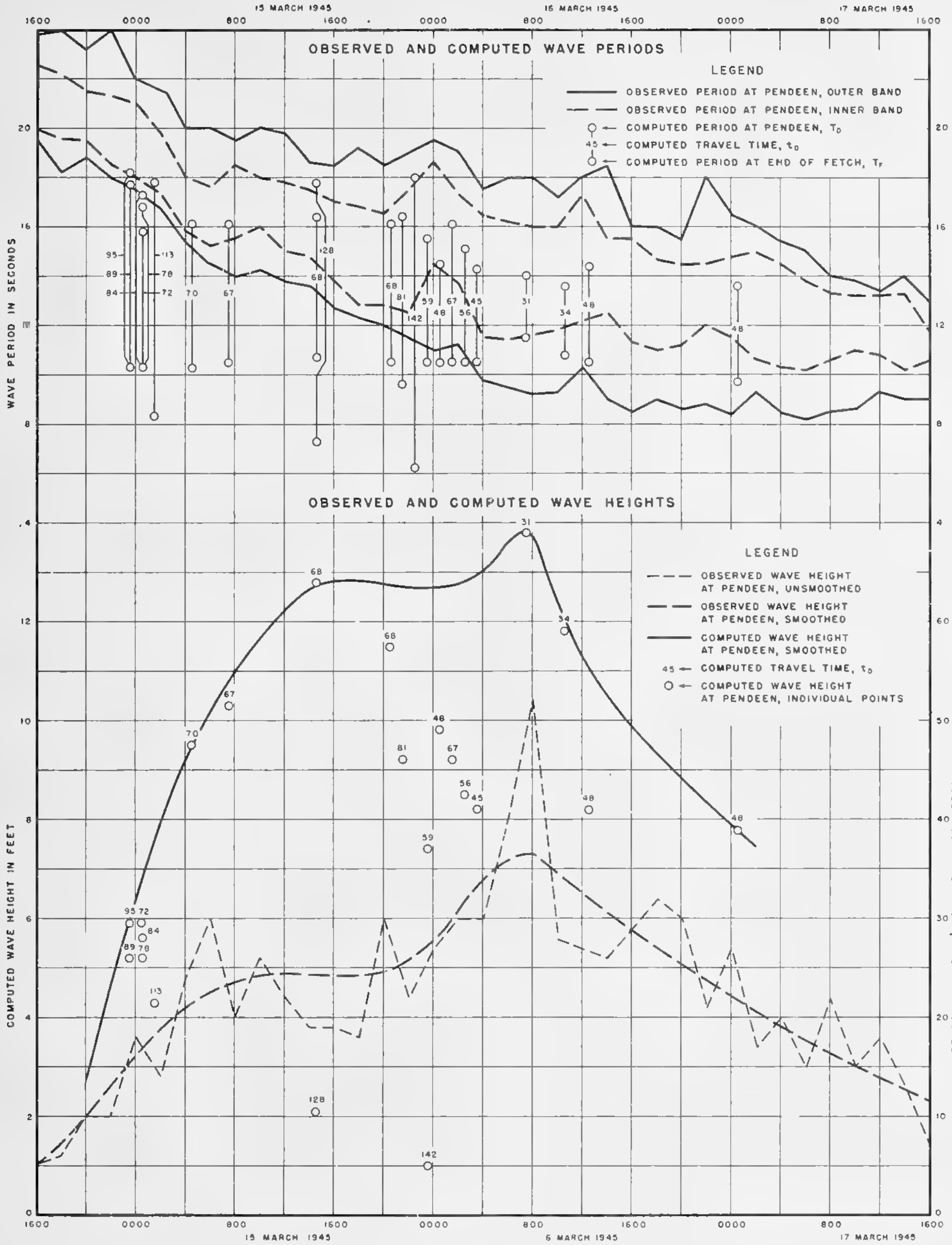

Figure 14.—Observed and computed wave periods and wave heights at Pendeen, England, March 14 to 17, 1945. For explanation see text.

region of decay has long been a controversial subject. According to the classical theory waves should proceed with constant velocity into the area of calm as long as the depth to the bottom remains greater than about one-half the wave length. In recent years Cornish (1934) has emphasized that such is the case. He has quoted a number of examples to show that the periods of swell observed on coasts were in agreement with periods which might have been produced by strong winds over the adjacent ocean. However, in the discussion dealing with observations of wave velocities and wind velocities Cornish has overestimated wave periods in the generating area since he has paid attention only to the maximum wind velocity. He has omitted any discussion of the length of time the strong wind blew in a uniform direction, as well as any estimate of the fetch of the wind. Since these factors are of great importance, Cornish's arguments lose much of their weight.

Krümmel (1911) compiled considerable evidence to show that the swell has longer periods than the originally produced waves and therefore, corre-

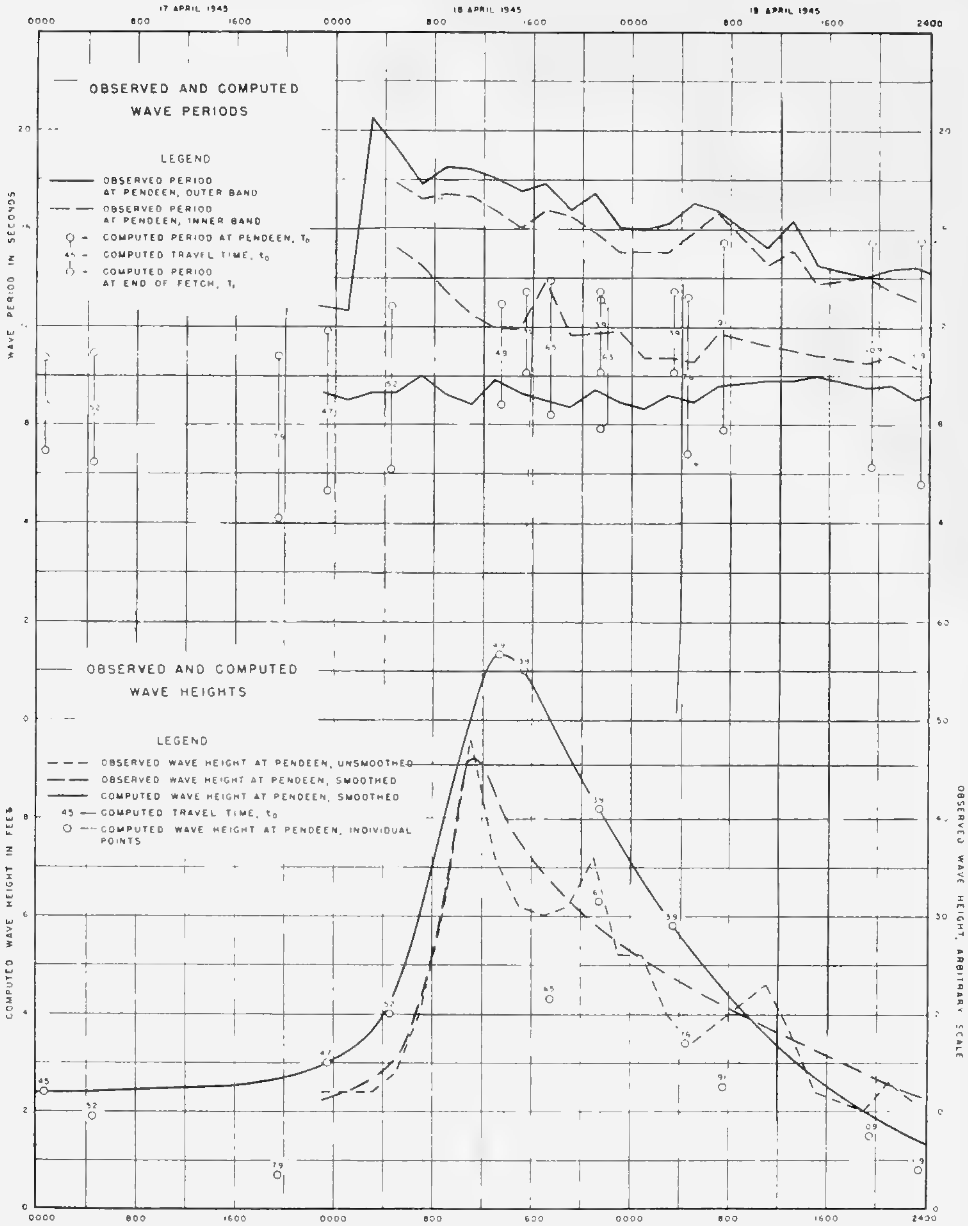

Figure 15.—Observed and computed wave periods and wave heights at Pendeen, England, April 17 to 19, 1945. For explanation see text.

spondingly greater velocity of progress and greater length. He quotes (1911, p. 94) one particular instance in the Atlantic Ocean when swell from one source was observed by a number of vessels which reported longer and longer periods the greater their distance from the source. At a distance of 6,700 km. from the source the period had increased by a factor of 1.7. Krümmel therefore concluded that the waves increase in period, velocity of progress, and length as they move away from their source.

This conclusion is substantiated by an examination of the frequencies of different wind velocities over the North Pacific and the North Atlantic and of the frequencies of swell of different periods on the coasts of southern California and of northwest Africa. Table 5 shows the periods of waves produced by different wind velocities and wind durations. The wave heights and periods for wind durations of 20, 30, and 40 hours, are taken from the curves in figure 7 which are based upon theoretical considerations but which have been shown to be in agreement with empirical evidence.

Table 5

Wave Periods for Wind of Given Velocities and Duration

Wind velocity		Wave periods, seconds		
Beaufort scale	m/sec	20 hours	30 hours	40 hours
4	5.3– 8.5	3.9– 4.9	4.5– 5.9	5.0–6.5
5	8.6–11.0	5.0– 5.9	6.0– 6.9	6.6–7.7
6	11.1–14.1	6.0– 6.9	7.0– 8.0	7.8–9.0
7	14.2–17.2	7.0– 7.8	8.1– 9.2	-------
8	17.3–20.8	7.9– 8.8	9.3–10.3	-------
9	20.9–24.4	8.9– 9.7	10.4–11.4	-------
10	24.5–28.5	9.8–10.7	---------	-------
11	28.6–32.7	10.8–11.6	---------	-------

According to the Atlas of Climatological Charts of the Oceans (U. S. Weather Bureau, 1938), wind velocities exceeding 7 Beaufort are not frequent over the oceans. For the stormy part of the North Pacific in winter the frequency of such high wind velocities exceeds 15 percent in only a few regions, and in the North Atlantic it reaches 25 percent in only one area. The duration of these winds is usually between 20 and 30 hours. Therefore, waves of periods longer than 8.9 seconds would not be produced on more than 15 to 25 percent of the days in winter; waves of periods exceeding 12 seconds would never be formed because wind velocities of 10 to 11 Beaufort occur rarely and are of short duration. It follows that if the waves proceeded without change in period the periods of the swells on the coasts of southern California and of northwest Africa would exceed 8.9 seconds in 15 to 25 percent of the cases and would never exceed 12 seconds.

Gutenberg (1929) has commented upon the problem of the increase of wave period and calls it a truly geophysical phenomenon noticeable only over long distances and periods of time. He finds that earthquake waves, microseismic and ordinary sound waves increase in period in a manner similar to the one characteristic of water waves. For seismic waves the increase might be associated with the internal viscosity of the transmitting medium.

Although the "kinematics" of the increase in wave period is now fairly well understood, due mostly to studies by Rossby (1945), the physical causes of the phenomenon remain obscure. The explanation may lie in an analogy with the Cauchy-Poisson problem of impulsive wave generation (Lamb, 1932, p. 384). Due to the impulsive generation a spectrum of wave periods is present at all times and the longer and faster waves will run ahead of the shorter and slower waves. The entire wave train stretches, resulting in an increase of wave period. However, the Cauchy-Poisson problem deals with a momentary generation of waves as, for example, waves caused by a pebble dropped in water. In storm areas the process of wave formation must be a semicontinuous one and it represents a much more difficult problem. This paper may serve to clarify the relationships between wind, sea, and swell, but it leaves a number of fundamental questions unanswered.

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APPENDIX I

To prove that

$$ {}^{n}R_m = \frac{n!}{2^n} \sum_{r=0}^{r=n-m} \frac{1}{r!(n-r)!} \qquad (22) $$

is a solution to the difference equation

$$ {}^{n+1}R_m - \frac{1}{2}{}^{n}R_m - \frac{1}{2}{}^{n}R_{m-1} = 0 \qquad (25) $$

Substituting (22) into (25)

$$ \frac{(n+1)!}{2^{n+1}} \sum_{r=0}^{r=n+1-m} \frac{1}{r!(n+1-r)!} - \frac{n!}{2\times 2^n} \sum_{r=0}^{r=n-m} \frac{1}{r!(n-r)!} - \frac{n!}{2\times 2^n} \sum_{r=0}^{r=n+1-m} \frac{1}{r!(n-r)!} = 0 $$

Multiplying by $2^{n+1}/n!$, bringing all terms on a common denominator, and combining the first and third terms, one obtains

$$ \sum_{r=0}^{r=n+1-m} \frac{r}{r!(n+1-r)!} - \sum_{r=0}^{r=n-m} \frac{n+1-r}{r!(n+1-r)!} = 0 $$

or

$$ \frac{1}{(n-m)!(m)!} + \sum_{r=0}^{r=n-m} \frac{2r-(n+1)}{r!(n+1-r)} = 0 \qquad (I) $$

Equation (I) is satisfied for any values of n and m, as will be proven by the method of mathematical induction. First, it will be shown to hold for $n-m=0$, $n-m=1$, and $n-m=2$. Then it will be shown that if we assume (I) to hold for $n-m=k$, it must also hold for $n-m=k+1$, and hence for any value of $n-m$.

As $n-m=0$, $m=n$:

$$ \frac{1}{0!m!} - \frac{n+1}{0!(n+1)!} = \frac{1}{n!} - \frac{1}{n!} = 0 $$

As $n-m=1$, $m=n-1$:

$$ \frac{1}{1!(n-1)!} - \frac{n+1}{0!(n+1)!} + \frac{2-(n+1)}{1!n!} = \frac{1}{(n-1)!} - \frac{1}{n!} + \frac{1-n}{n!} = \frac{n}{n!} - \frac{1}{n!} + \frac{1-n}{n!} = 0 $$

As $n-m=2$, $m=n-2$:

$$ \frac{1}{2!(n-2)!} - \frac{1}{n!} + \frac{1-n}{n!} + \frac{3-n}{2(n-1)!} = \frac{n-1}{2(n-1)!} - \frac{1}{(n-1)!} - \frac{(n-3)}{2(n-1)!} = \frac{n-1}{2(n-1)!} - \frac{n-1}{2(n-1)!} = 0 $$

As $n-m=k$, $m=n-k$:

$$ \frac{1}{k!(n-k)!} + \sum_{r=0}^{r=k} \frac{2r-(n+1)}{r!(n+1-r)} = 0 \qquad (II) $$

As $n-m=k+1$, $m=n-k-1$:

$$ \frac{1}{(k+1)!(n-k-1)!} + \sum_{r=0}^{r=k+1} \frac{2r-(n+1)}{r!(n+1-r)!} = \frac{n-k}{(k+1)!(n-k)!} + \sum_{r=0}^{r=k} \frac{2r-(n+1)}{r!(n+1-r)!} + \frac{2(k+1)-(n+1)}{(k+1)!(n-k)!} = $$

$$ \frac{n-2k-1}{(k+1)!(n-k)!} - \frac{n-2k-1}{(k+1)!(n-k)!} \text{ (in view of Equation II)} = 0 $$

q. e. d.

APPENDIX II. COLLECTION OF EMPIRICAL WAVE DATA

Table I

WAVE AGE, β, AND STEEPNESS, δ

No.	Wind velocity, U (m/sec)	Wave velocity, C (m/sec)	Wave length, L (m)	Wave height, H (m)	Wave age $\beta = C/U$ (ratio)	Wave steepness, $\delta = H/L$ (percent)	Reference*	Legend used in fig. 5	Manner of measurement†
1	8.8	16.4	172	2.2	1.87	1.3		Berkeley	a
2	11.3	16.4	172	2.7	1.44	1.6		do	a
3	11.3	16.6	175	2.9	1.46	1.7		do	a
4	17.0	22.6	328	5.5	1.33	1.7	X (1923)	Zimmermann	f
5	10.8	17.0	186	3.2	1.57	1.7		Berkeley	a
6	19.0	22.8	336	6.0	1.20	1.8	Z (1923)	Zimmermann	f
7	17.0	26.8	461	8.5	1.58	1.8	Z (1923)	do	f
8	7.0	11.1	79	1.5	1.58	1.9	Z (1923)	do	f
9	5.0	8.9	50	1.0	1.78	2.0	Z (1923)	do	f
10	17.0	20.5	270	5.5	1.21	2.0	Z (1923)	do	f
11	6.7	9.4	56	1.2	1.40	2.1		U. S. S. *Augusta*	a
12	12.9	20.3	264	6.1	1.57	2.3		H. M. S. *Forrester*	a
13	17.0	18.2	214	5.0	1.07	2.3	Z (1923)	Zimmermann	f
14	12.9	15.6	156	3.7	1.21	2.4		H. M. S. *Forrester*	a
15	5.0	8.9	51	1.2	1.78	2.4	Z (1923)	Zimmermann	f
16	6.0	8.1	42	1.0	1.35	2.4	Z (1923)	do	f
17	6.7	7.8			1.16	2.4	Sc (1939)	Meteor	c
18	6.2	6.2	25	.6	1.00	2.4		U. S. S. *Augusta*	a
19	5.7	6.3	25	.6	1.10	2.4		do	a
20	12.9	12.5	100	2.4	.97	2.4		do	a
21	12.9	12.5	100	2.4	.97	2.4		do	a
22	10.0	12.5	100	2.5	1.25	2.5	Z (1923)	Zimmermann	f
23	11.3	14.1	135	3.4	1.24	2.5		Berkeley	a
24	5.0	8.6	47	1.2	1.72	2.6	Z (1923)	Zimmermann	f
25	9.3	13.1	110	2.8	1.41	2.6		Berkeley	a
26	10.8	12.5	100	2.6	1.16	2.6		U. S. S. *Augusta*	a
27	5.0	8.4	45	1.2	1.68	2.7	Z (1923)	Zimmermann	f
28	14.0	14.1	127	3.5	1.01	2.8	Z (1923)	do	f
29	9.8	10.9	76	2.1	1.11	2.8		U. S. S. *Augusta*	a
30	5.0	8.1	42	1.2	1.62	2.9	Z (1923)	Zimmermann	f
31	15.0	17.0	185	3.5	1.13	2.9	Z (1923)	do	f
32	9.4	8.6			.93	2.9	Sc (1939)	Meteor	a
33	5.9	11.2			1.90	2.9	K (1911)	Paris	c
34	7.2	12.6			1.75	2.9	K (1911)	do	c
35	15.0	19.2	237	7.0	1.28	3.0	Z (1923)	Zimmermann	f
36	16.0	19.8	235	7.0	1.20	3.0	K (1911)	Paris	c
37	11.3	10.9	76	2.3	.96	3.0		U. S. S. *Augusta*	a
38	9.3	10.9	76	2.3	1.17	3.0		U. S. S. *Augusta*	a
39	8.6	12.4			1.44	3.0	K (1911)	Paris	c
40	17.0	16.1	165	5.0	.95	3.0	Z (1923)	Zimmermann	f

See footnotes at end of table.

Table I—Continued

WAVE AGE, β, AND STEEPNESS, δ—Continued

No.	Wind velocity, U (m/sec)	Wave velocity, C (m/sec)	Wave length, L (m)	Wave height, H (m)	Wave age $=\beta C/U$ (ratio)	Wave steepness, $\delta = H/L$ (percent)	Reference*	Legend used in fig. 5	Manner of measurement†
41	7.0	9.0	45	1.4	1.22	3.1	G (1935)	Gassenmayr	e
42	13.0	14.0	140	4.4	1.12	3.1	G (1935)	do	e
43	13.0	13.2	111	3.5	1.01	3.2	Z (1923)	Zimmermann	f
44	19.0	19.2	237	7.5	1.01	3.2	Z (1923)	do	f
45	8.2	10.9	76	2.4	1.33	3.2		U. S. S. *Augusta*	a
46	8.0	8.6	47	1.5	1.08	3.2	Z (1923)	Zimmermann	f
47	12.4	14.0			1.13	3.2	K (1911)	Paris	c
48	9.8	10.9	76	2.4	1.11	3.2		Berkeley	a
49	13.0	16.3	171	5.5	1.25	3.2	Z (1923)	Zimmermann	f
50	7.7	9.4	56	1.8	1.22	3.3		U. S. S. *Augusta*	a
51	6.0	8.4	45	1.5	1.40	3.3	Z (1923)	Zimmermann	f
52	7.0	9.6	59	2.0	1.37	3.4	Z (1923)	do	f
53	10.3	7.8	39	1.4	.76	3.5		U. S. S. *Augusta*	a
54	17.0	15.6	156	5.5	.92	3.6	Z (1923)	Zimmermann	f
55	11.0	11.0	78	2.8	1.00	3.6	Z (1923)	do	f
56	14.0	13.1	110	4.0	.94	3.6	Z (1923)	do	f
57	7.0	7.2	33	1.2	1.03	3.6	Z (1923)	do	f
58	5.2	6.2	25	.9	1.19	3.6		U. S. S. *Augusta*	a
59	15.0	15.1	147	5.5	1.01	3.8	Z (1923)	Zimmermann	f
60	14.0	15.0	120	4.5	1.08	3.8	G (1935)	Gassenmayr	e
61	6.0	8.0	40	1.5	1.25	3.8	G (1935)	do	e
62	10.8	9.4	56	2.1	.87	3.8		U. S. S. *Augusta*	a
63	12.9	15.6	156	6.1	1.21	3.9		H. M. S. *Forrester*	a
64	9.0	7.0	35	1.4	.78	4.0	G (1935)	Gassenmayr	e
65	10.0	8.0	40	1.6	.80	4.0	G (1935)	do	e
66	11.0	12.4	99	4.0	1.13	4.0	Z (1923)	Zimmermann	f
67	13.0	13.9	123	5.0	1.07	4.1	Z (1923)	do	f
68	12.0	13.9	123	5.0	1.16	4.1	Z (1923)	do	f
69	17.0	14.5	134	5.5	.85	4.1	Z (1923)	do	f
70	11.0	9.5	57	2.4	.85	4.2	G (1935)	Officers U. S. Navy	b
71	13.0	11.4			.88	4.2	K (1911)	Paris	c
72	14.0	12.8	105	4.5	.91	4.3	Z (1923)	Zimmermann	f
73	6.0	5.5	20	.85	.92	4.3	E (1940)	Ehring	a
74	11.0	9.4	56	2.5	.85	4.5	Z (1923)	Zimmermann	f
75	14.9	15.0			1.01	4.5	K (1911)	Paris	c
76	17.0	13.1	110	5.0	.77	4.6	Z (1923)	Zimmermann	f
77	14.0	11.7	88	4.0	.84	4.6	Z (1923)	do	f
78	23.0	18.3	214	9.8	.78	4.6	S (1893)	Schott	a
79	10.0	10.3	68	3.2	1.04	4.7	S (1893)	do	a
80	17.0	17.0	185	9.0	1.00	4.8	Z (1923)	Zimmermann	f

See footnotes at end of table.

Table I—Continued

WAVE AGE, β, AND STEEPNESS, δ—Continued

No.	Wind velocity, U (m/sec)	Wave velocity, C (m/sec)	Wave length, L (m)	Wave height, H (m)	Wave age $\beta = C/U$ (ratio)	Wave steepness, $\delta = H/L$ (percent)	Reference*	Legend used in fig. 5	Manner of measurement†
81	17.0	10.2	64	3.2	.61	5.0	G (1935)	Officers U. S. Navy	b
82	9.0	6.7	30	1.5	.79	5.0	G (1935)	Gassenmayr	e
83	9.0	7.8	39	2.0	.91	5.1	S (1893)	Schott	a
84	18.0	13.7	121	6.2	.74	5.2	S (1893)	do	a
85	23.0	19.7	248	13.0	.86	5.2	Z (1923)	Zimmermann	f
86	6.0	6.2	25	1.3	1.03	5.2	G (1935)	Gassenmayr	e
87	23.0	12.7	140	7.5	.54	5.4	G (1935)	do	e
88	21.0	20.2	262	14.0	.97	5.4	Z (1923)	Zimmermann	f
89	20.0	12.0	120	6.5	.60	5.4	G (1935)	Gassenmayr	e
90	21.0	18.5	218	12.0	.88	5.5	Z (1923)	Zimmermann	f
91	17.0	15.1	145	8.0	.89	5.5	Z (1923)	do	f
92	14.4	9.4	56	3.1	.65	5.5		H. M. S. *Forrester*	a
93	16.0	13.3	113	6.0	.83	5.5	K (1911)	Paris	c
94	16.5	3.6	8	.45	.22	5.6		Gibson	a
95	11.0	8.4	45	2.5	.76	5.6	S (1893)	Schott	a
96	11.0	8.4	45	2.5	.76	5.6	Z (1923)	Zimmermann	f
97	10.0	8.4	45	2.5	.84	5.6	Z (1923)	do	f
98	14.0	11.5	98	5.5	.83	5.6	G (1935)	Officers U. S. Navy	b
99	11.0	6.7	40	2.3	.61	5.7	G (1935)	Gassenmayr	e
100	11.0	6.2	40	2.5	.55	6.2	G (1935)	do	e
101	16.0	12.9	107	6.7	.83	6.2	C (1934)	Cornish	a
102	17.0	12.5	100	6.4	.74	6.4	G (1935)	Officers U. S. Navy	b
103	12.9	9.4	56	3.7	.73	6.6		H. M. S. *Forrester*	a
104	11.0	9.4	56	3.7	.84	6.6	S (1893)	Schott	a
105	11.0	8.0	45	3.0	.73	6.6	G (1935)	Gassenmayr	e
106	23.0	12.0	113	7.6	.52	6.7	G (1935)	Officers U. S. Navy	b
107	16.0	4.5	13	.9	.28	6.9		Gibson	a
108	20.0	14.2	129	9.0	.71	7.0	S (1893)	Schott	a
109	17.0	14.6	137	10.0	.86	7.3	Z (1923)	Zimmermann	f
110	20.0	13.3	113	8.3	.67	7.4	S (1893)	Schott	a
111	20.0	8.5	80	6.0	.43	7.5	G (1935)	Gassenmayr	e
112	26.0	12.0	92	6.9	.46	7.5	C (1934)	Cornish	a
113	14.0	9.5	58	4.5	.68	7.5	Z (1923)	Zimmermann	f
114	25.0	1.6	1.7	.13	.06	7.7	C (1934)	Cornish	d
115	25.0	5.6	20	1.5	.22	7.7	C (1934)	do	c
116	11.0	5.8	35	2.7	.52	7.7	G (1935)	Gassenmayr	e
117	17.0	6.4	45	3.5	.38	7.7	G (1935)	do	e
118	8.8	6.2	25	2.0	.71	8.0		U. S. S. *Augusta*	a
119	9.6	5.4	18	1.5	.56	8.3		Dover	c
120	8.2	6.7	29	2.4	.82	8.3		U. S. Engineers	a

See footnotes at end of table.

Table I—Continued

WAVE AGE, β, AND STEEPNESS, δ—Continued

No.	Wind velocity, U (m/sec)	Wave velocity, C (m/sec)	Wave length, L (m)	Wave height, H (m)	Wave age $\beta=C/U$ (ratio)	Wave steepness, $\delta=H/L$ (percent)	Reference*	Legend used in fig. 5	Manner of measurement†
121	11.0	6.8	30	2.6	.62	8.7	K (1911)	Krümmel	c
122	11.3	6.7	29	2.7	.59	9.3		U. S. Engineers	a
123	12.4	4.6	14	1.4	.37	10.0		Dover	c
124	23.0	13.1	110	11.0	.57	10.0	Z (1923)	Zimmermann	f
125	9.6	5.2	17	1.8	.54	10.6		Dover	c
126	12.4	4.9	15	1.7	.39	11.3		do	c
127	12.4	4.9	15	1.8	.39	12.0		do	c
128	9.8	3.8	9	1.1	.39	12.2		do	c

*Reference: C=Cornish; E=Ehring; G=Gaillard; K=Krümmel; S=Schott; Sc=Schumacher; Z=Zimmermann.

†Manner of measurement: a—C and L computed from observed T. b—C computed from observed T and L averaged. c—C computed from observed L. d—C observed, L computed from observed T. e—C and L observed. f—Manner in which C, L, and T were obtained not known.

Table II

WAVE HEIGHT AND WAVE AGE AS FUNCTIONS OF FETCH

No.	Wind velocity U (m/sec)	Wave height H (m)	Fetch F (km)	Wave velocity C (m/sec)	$\frac{gF}{U^2}$	$\frac{gH}{U^2}$	$\frac{C}{U}$	Reference*	Legend used in fig. 6	Manner of measurement †
11	6.7	1.2	157	9.4	3.4×10^4	0.27	1.40	----------	U. S. S. *Augusta*	a
18	6.2	.6	157	6.2	4.1×10^4	.16	1.00	----------	do	a
20	12.9	2.4	167	12.5	9.9×10^3	.14	.97	----------	do	a
21	12.9	2.4	167	12.5	9.9×10^3	.14	.97	----------	do	a
26	10.8	2.6	232	12.5	1.9×10^4	.22	1.16	----------	do	a
38	9.3	2.3	167	10.9	1.9×10^4	.26	1.17	----------	do	a
45	8.2	2.4	167	10.9	2.4×10^4	.35	1.33	----------	do	a
50	7.7	1.8	167	9.4	2.8×10^4	.30	1.22	----------	do	a
58	5.2	.9	157	6.2	5.7×10^4	.33	1.19	----------	do	a
94	16.5	.5	1.3	3.6	4.7×10	.016	.22	----------	Gibson	a
107	16.0	.9	3.5	4.5	1.3×10^2	.032	.28	----------	do	a
120	8.2	2.4	363	6.7	5.3×10^4	.35	.82	----------	U. S. Engineers	a
122	11.3	2.7	152	6.7	1.2×10^4	.21	.59	----------	do	a
129	7.7	1.1	156	------	2.6×10^4	.18	------	----------	do	a
130	7.7	1.7	156	------	2.6×10^4	.28	------	----------	do	a
131	6.7	1.5	161	------	3.5×10^4	.33	------	----------	do	a
132	8.8	2.1	161	------	2.0×10^4	.27	------	----------	do	a
133	7.7	2.1	156	------	2.6×10^4	.35	------	----------	do	a
134	16.0	4.0	130	------	5.0×10^3	.15	------	----------	do	a
135	16.0	4.3	120	------	4.6×10^3	.16	------	----------	do	a
136	12.9	3.4	152	------	9.0×10^3	.20	------	----------	do	a
137	11.8	2.7	152	------	1.1×10^4	.19			do	a
138	9.3	2.9	156	------	1.8×10^4	.33	------	----------	do	a
139	6.2	.9	152	------	3.9×10^4	.23	------	----------	do	a
140	9.3	1.5	145	------	1.6×10^4	.17	------	----------	do	a
141	11.8	2.9	156	------	1.1×10^4	.20	------	----------	do	a
142	6.7	.9	363	------	7.9×10^4	.20	------	----------	do	a
143	5.7	.8	363	------	1.1×10^5	.24	------	----------	do	a
144	9.3	1.2	120	------	[illegible].4×10⁴	.14	------	----------	do	a
145	9.3	1.7	152	------	1.7×10^4	.19	------	----------	do	a
146	8.2	1.8	363	------	5.3×10^4	.26	------	----------	do	a
147	10.3	2.1	156	------	1.4×10^4	.19	------	----------	do	a
148	9.8	2.1	156	------	1.6×10^4	.21	------	----------	do	a
149	15.0	6.7	482	12.4	2.1×10^4	.29	.80	C (1934)	Cornish	a
150	20.6	11.0	1,112	16.5	2.5×10^4	.25	.80	C (1934)	do	a
151	4.0	------	.0135	.50	8.3	-----	.12	S (1937)	Stanton	c
152	5.0	------	.0135	.57	5.3	-----	.11	S (1937)	do	c
153	6.2	------	.0135	.67	3.4	-----	.11	S (1937)	do	c
154	7.2	------	.0135	.77	2.6	-----	.11	S (1937)	do	c

*C=Cornish; S=Stanton.

†See footnote of table I.

APPENDIX II. COLLECTION OF EMPIRICAL WAVE DATA—Continued

Table III

WAVE HEIGHT AND WAVE AGE AS FUNCTIONS OF DURATION

No.	Wind velocity U (m/sec)	Duration t (sec)	Wave height H (m)	Wave velocity C (m/sec)	$\frac{gt}{U}$	$\frac{gH}{U^2}$	$\frac{C}{U}$	Reference*	Legend used in fig. 7	Manner of measurement†
2	11.3	8.93×10^4	2.7	16.4	7.8×10^4	0.21	1.44	----------	Berkeley	a
3	11.3	6.92×10^4	2.9	16.6	6.0×10^4	.22	1.46	----------	do	a
5	10.8	1.80×10^5	3.2	17.0	1.6×10^5	.27	1.57	----------	do	a
19	5.7	3.24×10^4	.6	6.3	5.6×10^4	.18	1.10	----------	U. S. S. *Augusta*	a
23	11.3	9.28×10^4	3.4	14.1	8.8×10^4	.26	1.24	----------	Berkeley	a
29	9.8	6.12×10^4	2.1	10.9	6.1×10^4	.21	1.11	----------	U. S. S. *Augusta*	a
36	16.0	3.60×10^5	7.0	19.8	2.2×10^5	.27	1.20	K (1911)	Paris	c
37	11.3	5.40×10^4	2.3	10.9	4.7×10^4	.14	.96	----------	U. S. S. *Augusta*	a
48	9.8	7.57×10^4	2.4	10.9	7.6×10^4	.25	1.11	----------	Berkeley	a
53	10.3	3.60×10^4	1.4	7.8	3.4×10^4	.13	.76	----------	U. S. S. *Augusta*	a
62	10.8	3.78×10^4	2.1	9.4	3.4×10^4	.18	.87	----------	do	a
93	16.0	8.64×10^4	6.0	13.3	5.4×10^4	.23	.83	K (1911)	Paris	c
118	8.8	3.06×10^4	2.0	6.2	3.4×10^4	.25	.71	----------	U. S. S. *Augusta*	a
121	11.0	6.47×10^4	2.6	6.8	5.8×10^4	.21	.62	K (1911)	Krümmel	c
128	9.8	1.80×10^4	1.1	3.8	1.8×10^4	.11	.39	----------	Dover	a
155	9.3	4.18×10^4	2.4	------	4.4×10^4	.27	-----	----------	Berkeley	a
156	18.5	2.51×10^4	4.4	------	1.3×10^4	.13	-----	----------	Emmons & Clarke	a

*K = Krümmel.

†See footnote of table I.

APPENDIX II. COLLECTION OF EMPIRICAL WAVE DATA—Continued

Table IV

HEIGHT, PERIOD, AND TRAVEL TIME OF SWELL

No.	Wind velocity U (m/sec)	Fetch F (km)	Duration t_d (km)	Decay distance D (km)	Computed wave height at end of fetch H_F (m)	Computed wave period at end of fetch T_F (sec)	Observed height of swell H_D (m)	Observed period of swell T_D (sec)	Observed travel time of swell t_D (sec)	$\frac{H_D}{H_F}$	$\frac{T_D}{T_F}$	$\frac{t_D}{T_F}$	$\frac{D}{gT_F^2}$	Reference	Legend used in fig. 12	Manner of measurement*
1	18.5	835	6.48×10^4	2,320	6.8	7.9	1.4	11.5	2.70×10^5	0.21	1.46	3.42×10^4	3.79×10^3	------	Moroccan	a
2	14.4	1,110	1.06×10^5	1,480	4.8	8.1	2.6	17.0	1.80×10^5	.54	2.10	2.22×10^4	2.30×10^3	------	do	a
3	18.0	1,390	8.65×10^4	1,950	6.9	8.7	2.0	17.0	2.09×10^5	.29	1.95	2.40×10^4	2.63×10^3	------	do	a
4	13.9	1.300	2.27×10^5	1,110	4.9	10.5	2.7	11.2	1.51×10^5	.55	1.07	1.44×10^4	1.03×10^3	------	Berkeley	a
5	13.9	2,225	----------	1,480	5.1	12.2	3.0	9.1	1.95×10^5	.59	.75	1.60×10^4	1.02×10^3	------	do	a
6	9.8	2,225	2.43×10^5	185	2.5	9.0	2.1	9.5	2.88×10^4	.84	1.06	3.20×10^3	2.33×10^2	------	do	a
7	20.1	930	1.04×10^5	1,110	8.8	9.9	------	------	1.13×10^5	------	------	1.14×10^4	1.16×10^3	------	Pendeen	a
8	22.7	740	6.48×10^4	1,480	9.2	8.8	------	------	1.68×10^5	------	------	1.91×10^4	1.95×10^3	------	do	a

* See footnote of table I.